Black Hole Mechanics

Sandy Auttelet and Bailey Conrad

Outline

1. Black Hole Metrics
2. What is Black Hole Mechanics
3. Zeroth Law
4. First Law
5. Third Law
6. Second Law

Kerr Metric

- Solution has form of

$$ds^2 = g_{tt}dt^2 + g_{rr}dr^2 + g_{\theta\theta}d\theta^2 + g_{\phi\phi}d\phi^2 + 2g_{t\phi}dtd\phi$$

- Can't assume
  - $g_{\theta\theta} = r^2$
  - $g_{\phi\phi} = r^2 \sin^2 \theta$
  - $g_{t\phi} = 0$
- Can assume
  - g_{tr} and $g_{t\theta}$ are zero

Kerr Metric

$$ds^2 = -\frac{\Delta - a^2 \sin^2\theta}{\rho^2} dt^2 - \boxed{2a \frac{2Mr\sin^2\theta}{\rho^2} dt d\phi}$$

$$+ \frac{(r^2 + a^2)^2 - a^2 \Delta \sin^2\theta}{\rho^2} \sin^2\theta \, d\phi^2 + \frac{\rho^2}{\Delta} dr^2 + \rho^2 d\theta^2$$

where

$$a \equiv \frac{J}{M} \qquad \Delta \equiv r^2 - 2Mr + a^2 \qquad \rho^2 = r^2 + a^2 \cos^2\theta$$

*Reduces to Schwarzschild metric when $a = 0$.

Metrics (non-rotating)

Schwarzschild

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta \, d\phi^2$$

Reissner-Nordström

$$ds^2 = -\left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right) dt^2 + \left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)^{-1} dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2$$

Kerr-Newman most general metric

$$ds^2 = -\frac{\Delta - a^2\sin^2\theta}{\rho^2}dt^2 - 2a\frac{2Mr\sin^2\theta}{\rho^2}dtd\phi + \frac{(r^2+a^2)^2 - a^2\Delta\sin^2\theta}{\rho^2}\sin^2\theta\, d\phi^2$$

$$+\frac{\rho^2}{\Delta}dr^2 + \rho^2 d\theta^2$$

where

<u>Limiting cases:</u>

$a = 0 \rightarrow$ Reissner-Nordström

$Q = 0 \rightarrow$ Kerr

$a = Q = 0 \rightarrow$ Schwarzschild

$a = Q = M = 0 \rightarrow$ Minkowski

What is Black Hole Mechanics?

Black holes follow laws of general relativity, but also

- Maxwell Electrodynamics
- Hydrodynamics
- Quantum Mechanics
- Other matter and radiation physics laws

Analogy of black holes to thermodynamics

Zeroth Law

- The surface gravity, κ, of a stationary black hole is constant over the event horizon
- Acceleration needed to keep an object at the horizon
- Zeroth Law of Thermodynamics:
  Temperature is constant in thermal equilibrium

First Law – Conservation of Energy

- Energy changes are related to changes in area, momentum and charge of a black hole

$$dE = \frac{\kappa}{8\pi}\, dA + \Omega\, dJ + \Phi\, dQ$$

- First Law of Thermodynamics

$$dU = Tds + PdV + \mu dN$$

Third Law

It is impossible to reduce the surface gravity to zero with a finite sequence of operations.

Third Law of Thermodynamics:
Entropy cannot go to zero.

Second Law

- The area of the event horizon doesn't decrease over time
$$dA \geq 0$$
- Second Law of Thermodynamics
$$dS \geq 0$$
- Suggests area and entropy of black hole are related which violates Second Law of Thermodynamics

Generalized Second

- Hawking Radiation – area and mass of a black hole must decrease over time

$$\tau = (2.095 \times 10^{67}\mathrm{yr})\left(\frac{M}{M_{\odot}}\right)^{3}$$

- Hawking Temperature

$$T_H = \frac{\hbar}{8\pi k_B M}$$

- Bekenstein-Hawking Entropy

$$S_{BH} = \frac{k_b A}{4\hbar}$$

- Thermodynamic Entropy

$$\frac{1}{T} = \frac{\partial S}{\partial U} \qquad S = k_B \ln \Omega$$

Thermal Equilibrium

- Thermodynamics
  - Heat flows from hot to cold
  - Object in contact with reservoir will be in thermal equilibrium
- Black holes
  - Colder than background radiation, will absorb energy and cool
  - Hotter than background radiation, will emit more energy and get hotter

LAW	BLACK HOLE	THERMODYNAMICS
Zeroth	κ constant	T constant
First	$dE = \frac{\kappa}{8\pi}\, dA + \Omega\, dJ + \Phi\, dQ$	$dU = TdS + PdV + \mu dN$
Second	$dA \geq 0$	$dS \geq 0$
Third	$\kappa \neq 0$	$S \neq 0$
Entropy	$S_{BH} = \frac{k_b A}{4\hbar}$	$S = k_B \ln \Omega$

References

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- Bernard Schutz, "A First Course in General Relativity"