Black Hole Mechanics

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Contents

1 Introduction

Black holes are singular or nonsinglur dynamic objects in spacetime that fall out from the general solutions to the Einstein equations. Their laws of energy and momentum are analogous to the laws of thermodynamics in that their variable relationships can be compared directly, allowing for crossover in mathematical models and simulations. This paper includes a discussion on the mechanical properties of black holes and the comparison between the zeroth, first, second and third laws of black hole mechanics and thermodynamics. After defining the metrics below as solutions to the Einstein equations, we can discuss how these solutions set variable relations that compare directly to the laws of thermodynamics for general quantum systems. By exploring these relations, Stephen Hawking was able to deduce the existence of Hawking Radiation. After his discovery, changes were made to the second law of black hole mechanics and its comparison to the second law of thermodynamics.

2 Metrics

The following metrics are solutions to the Einstein equations. When factoring in the charge of a black hole, Q, we now have solutions to the Einstein-Maxwell equations. For this reason, the following Reissner-Nordström Metric and the Kerr-Newman Metric have different stress-energy tensors than the Kerr and Schwarzschild Metrics. [1] Throughout this paper, we will use the geometrized unit system. This system simplifies our solutions to Einstein's equations by setting constants,

$$G = c = k = 1.$$

When looking for a solution to the Einstein equations, we need to first limit ourselves to a vacuum solution to account for the behavior of massive objects like black holes or neutron stars. We also need time independent solutions. [1] Then we can begin with the Kerr-Newman metric described below.

2.1 Kerr-Newman

This metric is not the most general solution to the Einstein equations as here we have set the cosmological constant equal to zero. However, this metric is the most general case for the scope of this paper. Here, we are generalizing the solution because there are terms in the metric factoring in the charge and momentum of the black hole. It is important to note that this solution is for stationary and asymptotically flat manifolds. Therefore, we must limit the curvature of our manifold to zero when $r \to \infty$. Then, our manifold becomes flat and our metric goes to the Minkowski metric as we approach large distances from any region. With this concept in mind, our most general metric is,

$$
\begin{aligned}
\mathrm{d}s^2 = &-\frac{\Delta - a^2 \sin^2\theta}{\rho^2}\ \mathrm{d}t^2 - 2a\frac{2Mr\sin^2\theta}{\rho^2}\ \mathrm{d}t\ \mathrm{d}\phi \\
&+ \frac{(r^2+a^2) - a^2\Delta\sin^2\theta}{\rho^2}\sin^2\theta\ \mathrm{d}\phi^2 + \frac{\rho^2}{\Delta}\ \mathrm{d}r^2 + \rho^2\ \mathrm{d}\theta^2.
\end{aligned} \tag{1}
$$

Here, we define the parameters,

$$
\Delta := r^2 - 2Mr + a^2 + Q^2 \tag{2}
$$

where Q is the charge of the black hole, M is the mass and a is the spin parameter $\frac{J}{M}$. Lastly, we can set our energy density ρ here such that

$$
\rho^2 = r^2 + a^2\cos^2\theta. \tag{3}
$$

This solution has been widely claimed as an improbable physical solution to the Einstein equations and instead simply a mathematical solution due to the instability of the Cauchy horizon. Imagine you are falling into a black hole. When you pass the event horizon, Einstein's equations still work, allowing for seamless time evolution of events. However, once you pass the Cauchy horizon, Einstein's equations predict many different configurations of the time evolution of events. To resolve this issue, Roger Penrose suggested an alternative theory called the Strong Cosmic Censorship Conjecture, claiming that the Cauchy horizon was just a mathematical result and did not describe real results. Therefore, if hit by a gravitational wave, the Cauchy horizon would collapse, allowing for singular results to appear. However, this conjecture was disproved recently by mathematicians, Dafermos and Luk. The correction is subtle in that Penrose expected a strong space-like singularity, while our new solution is a result of a weak light-like singularity. Therefore, we can expect space-time to extend past the Cauchy horizon, and there are many ways for this extension to play out. Because there are still non-unique results, this metric cannot be considered as a physical solution to the Einstein equations. However, it is interesting to consider the Cauchy horizon and how it can extend our understanding of what happens in a black hole beyond the event horizon. [2]

2.2 Reissner-Nordström

The next metric we discuss here is called the Reissner-Nordström metric. Like the Kerr-Newman metric, we restrict our solutions to asymptotically flat, stationary solutions. In addition to these constraints, we are also limited in this case to static solutions. This metric comes from (1) when we set $a = 0$, so our black hole has no momentum. Then our metric simplifies to be,

$$
\mathrm{d}s^2 = \left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)\ \mathrm{d}t^2 + \left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)^{-1}\ \mathrm{d}r^2 + r^2\ \mathrm{d}\theta^2 + r^2\sin^2\theta\ \mathrm{d}\phi^2. \tag{4}
$$

Black holes under this metric also have a Cauchy horizon. For this reason, it is expected that these results are also restricted to mathematical results instead of physical results. Because we are considering the charge of the black hole, that charge contributes to the mass of our system due to the equivalence of mass and energy. Therefore, if the black hole has a larger charge than mass, the solutions become degenerate and therefore aren't easily represented as physical solutions. Additionally, it is widely believed that the universe is electrically neutral. If this belief is true, then even if a black hole did somehow become charged, it would accrete an opposite charge effectively neutralizing itself. [3]

2.3 Kerr

The Kerr Metric falls out of the Kerr Newman Metric (1) when setting $Q = 0$ such that our new metric becomes,

$$\begin{aligned} \mathrm{d}s^2 = &-\frac{\Delta - a^2\sin^2\theta}{\rho^2}\ \mathrm{d}t^2 - 2a\frac{2Mr\sin^2\theta}{\rho^2}\ \mathrm{d}t\ \mathrm{d}\phi \\ &+ \frac{(r^2+a^2) - a^2\Delta\sin^2\theta}{\rho^2}\sin^2\theta\ \mathrm{d}\phi^2 + \frac{\rho^2}{\Delta}\ \mathrm{d}r^2 + \rho^2\ \mathrm{d}\theta^2. \end{aligned} \tag{5}$$

Here, this metric looks exactly the same as the Kerr-Newman, but it's important to keep in mind that now (2) becomes

$$\Delta = r^2 - 2Mr + a^2 \tag{6}$$

so some terms are reduced. This metric is important because it is an exact solution to the Einstein equation in empty-space. Then this solution is also valid for the empty space outside of an uncharged massive black hole. [1] Additionally, objects under the Kerr metric will experience frame-dragging effects which develops a new analogy. We can compare dragging of the inertial frame to electromagnetism as a spinning charge becomes magnetized, analogous to the gravitational effects of a spinning mass. Thus, we can blame this effect on gravitomagnetism. Another phenomenon called the Lense-Thirring effect can be attributed to gravitomagnetism. This effect was measured using Gravity Probe B by "tracking the precession of on-board gyroscopes." [4] These effects were predicted initially by general relativity, and result in attracting masses acquiring a spin. This spin occurs due to the swirling of the curvature of spacetime. First, we can define the ergosphere of the black hole to be the limit where once inside, no particle can remain fixed. Inside of this limit, all particles and photons must rotate with the black hole. We define this limit as

$$r_0 := r_{\text{ergosphere}} = M + \sqrt{M^2 - a^2\cos^2\theta}. \tag{7}$$

These ergoregions can theoretically exist for neutron stars with no horizon, but this case is unlikely due to the compactness required for these effects to occur. [Schutz]

2.4 Schwarzschild

This metric comes directly from a combination of the Kerr Metric (5) and the Reissner-Nordström metric (4) as here we set both $Q = 0$ and $a = 0$ in the Kerr-Newman metric (1). Then our new metric becomes,

$$\mathrm{d}s^2 = -\left(1 - \frac{2M}{r}\right)\ \mathrm{d}t^2 + \left(1 - \frac{2M}{r}\right)^{-1}\ \mathrm{d}r^2 + r^2\ \mathrm{d}\theta^2 + r^2 \sin^2\theta\ \mathrm{d}\phi^2. \tag{8}$$

This metric, using Birkhoff's theorem, is proved to be the only spherically symmetric asymptotically flat solution to Einstein's vacuum field equations. This fact implies that there are no gravitational waves from pulsating spherical systems, analogous to the concept that there are no electromagnetic mono-poles. [4] If we take r to be large, then this metric becomes

$$\mathrm{d}s^2 \approx -\left(1 - \frac{2M}{r}\right)\mathrm{d}t^2 + \left(1 + \frac{2M}{r}\right)\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2 \tag{9}$$

where $\mathrm{d}\Omega^2$ represents the angular component $\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2$. The metric can be transformed into Cartesian coordinates by fixing r to $R := \sqrt{x^2 + y^2 + z^2}$ such that the metric becomes

$$\mathrm{d}s^2 \approx -\left(1 - \frac{2M}{R}\right)\mathrm{d}t^2 + \left(1 + \frac{2M}{R}\right)(\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2). \tag{10}$$

2.5 Minkowski

The Minkowski metric is our simplest metric. When setting all energy components to zero such that $Q = a = M = 0$, then we are left with only,

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \mathrm{d}r^2 + r^2\ \mathrm{d}\theta^2 + r^2 \sin^2\theta\ \mathrm{d}\phi^2 \tag{11}$$

from (8). Under these conditions, our spacetime is asymptotically flat. This metric, like all others mentioned, is written in spherical coordinates. When written in Cartesian coordinates, we simplify even further to lose the r^2 and $r^2 \sin^2\theta$ terms in our metric when replacing r, θ, ϕ with x, y, z. Then the Minkowski metric becomes,

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2. \tag{12}$$

This simple metric is a good way to introduce special relativity and metrics to a new student.

3 The Four Laws of Black Hole Mechanics

In this section, we will begin the discussion with the definition of each law that defines the mechanics of a black hole. These laws have been compared to the four laws of thermodynamics. We will compare the laws of black holes to thermodynamics to attempt to draw analogies between the variables within these laws. In [Hawking], these laws were derived and compared, for which we will summarize below.

3.1 Zeroth Law

Defined in [5], the zeroth law for black hole mechanics states that the surface gravity, κ, is constant over the event horizon. First, we can define surface gravity, κ, as the gravitational acceleration experienced by the black hole's surface. The proof of this constant acceleration is shown in [Hawking]. This gravity is analogous to temperature. Therefore, this zeroth law for black hole mechanics is correlated to the zeroth law of thermodynamics, where temperature is constant in thermal equilibrium. Then we can think of the event horizon of a black hole as acceleration equilibrium.

3.2 First Law

As defined in [5], any two neighboring stationary axisymmetric solutions containing a perfect fluid with circular flow and a central black hole are related by

$$\delta M = \frac{\kappa}{8\pi}\delta A + \Omega_H \delta J_H + \int \Omega \delta \mathrm{d}J + \int \bar{\mu} \delta \mathrm{d}N + \int \bar{\theta} \delta \mathrm{d}S. \tag{13}$$

In this comparison, $\frac{\kappa}{8\pi}$ is analogous to temperature and A is analogous to entropy from our first law of thermodynamics. Here we will note that $\bar{\theta}$ represents the red-shifted effective temperature of any matter orbiting the black hole. This temperature must go to zero as this matter approaches the event horizon of the black hole. We see this effect due to the time dilatation factor also going to zero at the horizon. We also define $\bar{\mu}$ as the effective chemical potential. Additionally, Ω_H and J_H are angular momentum terms for our matter and black hole respectively. Using the mass-energy relation, we can make the claim that energy changes are related to changes in area, momentum and charge of a black hole, so (13) can be rewritten as

$$\mathrm{d}E = \frac{\kappa}{8\pi}\mathrm{d}A + \Omega \mathrm{d}J + \Phi \mathrm{d}Q. \tag{14}$$

To compare (14) with the first law of thermodynamics,

$$\mathrm{d}U = T\mathrm{d}S + P\mathrm{d}V + \mu \mathrm{d}N, \tag{15}$$

we can see the analogy between area and entropy change as well as surface gravity and temperature in the first term. Then, (14) becomes analogous to (15).

3.3 Second Law

Initially, the area of the event horizon was compared to entropy in thermodynamics in that the area of a black hole, like entropy, doesn't decrease over time. As discussed in [Hawking], while area cannot be transferred from one black hole to the other unlike entropy can in thermodynamic systems, if two black holes coalesce, the area of the final event horizon must be larger than the two areas independently. However, as Hawking discovered just two years later, there must be radiation. This radiation suggests the area and mass of a black hole must decrease over time. Then every negative-energy photon falling into the black hole reduces M so the lifetime of the black hole scales as

$$\tau = 2.095 \times 10^{67}\text{yr} \left(\frac{M}{M_\odot}\right)^3 . \tag{16}$$

With this discovery, Hawking showed that a black hole behaves like a thermodynamic black body. [4] This revelation was discovered by applying quantum field theory on the background of spacetime of a black hole.

3.4 Third Law

The third law for black hole mechanics states that it is not possible to reduce the surface gravity to zero with a finite sequence of operations. While there is no formal proof for this claim, there is an inductive way of thinking to validate this law. For example, suppose you tried to decrease the surface gravity of a Kerr black hole by increasing the angular momentum. We can increase this momentum by throwing new particles into the black hole. Define the critical ratio $J/M^2 = 1$ for which the surface gravity is zero. Because the angular momentum only converges to 1 with infinite time and infinite divisibility of matter, there is no physical way to, in finite time, reach the critical ratio of angular momentum. Thus, it would take infinite time and infinite many particles to get the angular momentum up to the critical ratio, and thus get the surface gravity to zero. We also reach a contradiction when approaching the contrapositive. Suppose we could reach zero surface gravity, then there must be operations to be carried after this point, creating a naked singularity (i.e. no acceleration). The contradiction arises as we made the claim initially that we are working with asymptotically flat space. If we obtain a naked singularity, we break our asymptotic predictability, thus causing many results in black hole theory to break down. Therefore, the third law holds such that surface gravity cannot be zero. [5]

We can compare this third law to the third law of thermodynamics where here, surface gravity compares to entropy. In thermodynamics, restricting ourselves to a closed system, as temperature goes to zero, entropy must also go to zero. However, because for something to be at absolute zero, we must again have infinite time or infinitely divisible matter, we can never actually get to absolute zero in a physical system. Therefore, entropy also can never be zero in this closed system we call the universe. We can now make the claim that surface gravity of a black hole is analogous to the entropy of a closed system, both unable to obtain a value of zero. As a result, a black hole can never be in thermal equilibrium with the cosmic microwave background because increasing energy of a black hole decreases the temperature. [1]

4 Conclusion

In this paper, we have discussed five different metrics for black holes. Each metric can be derived from the Kerr-Newman metric in (1). By setting momentum, charge, mass or a combination of these three variables equal to 0, we can derive the four other metrics:

Reissner-Nordström (4), Kerr (5), Schwarzschild (8), and Minkowski (11). These metrics describe the mathematical solutions to the Einstein equations. We also compared the four laws of black hole mechanics to the four laws of thermodynamics. Below is a table outlining these variable relations.

Table 1: Analogies for Mechanical Laws

Law	Black Hole	Thermodynamics
Zeroth	κ constant	T constant
First	$\mathrm{d}E = \frac{\kappa}{8\pi}\mathrm{d}A + \Omega \mathrm{d}J + \Phi \mathrm{d}Q$	$\mathrm{d}U = T\mathrm{d}S + P\mathrm{d}V + \mu \mathrm{d}N$
Second	$\mathrm{d}A \geq 0$	$\mathrm{d}S \geq 0$
Third	$\kappa \neq 0$	$S \neq 0$

These analogies make creating a mathematical model for quantum gravity more attainable. While the material in this paper certainly can help support a proof of quantum gravity, we are still searching for the correct model that can describe all of the features discussed in this paper. Overall, we can describe our metrics with different curvature by applying new coordinate transformations. We can use these metrics to develop analogies between thermodynamics and black hole mechanics. Apply quantum field theory on spacetime can help develop the theories we are searching for, ultimately illuminating our understanding of the universe.

References

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