Homework 4: Root Finding and Derivative Estimations

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1 Root Finding

1.1

You can use the bisection algorithm to estimate the value of some constants to a specified accuracy.
(A) Find a function that has $\sqrt{2}$ as a root, but does not explicitly have $\sqrt{2}$ in its definition. (otherwise you would need to know $\sqrt{2}$ to define it).
(B) Use the bisection method on your function with tolerance 10^{-12} and sufficient iterations for convergence to estimate $\sqrt{2}$.

```
b = 2
a = 0
def f(x):
    return -x**2 + 2

def bisect(a,b,f,tol=10**-12,max_iter=1000):
    top_b = b
    bot_b = a
    if not f(a)*f(b) < 0:
        raise Exception("Zero not in interval")
    for i in range(max_iter):
        xm = (b+a)/2
        if np.abs(f(xm)) < tol:
            error = (top_b-bot_b)/2**i
            return xm, error
        if f(a)*f(xm) < 0:
            b = xm
        else:
            a = xm
    raise Exception("Max iteration reached")

x_approx, error = bisect(a,b,f)
```

1.2

You can use Newton's method to estimate the reciprocal of a number a, $\frac{1}{a}$, without performing any divisions. You can do this with the funtion $f(x) = a - \frac{1}{x}$.
. (A) Prove by hand from the definition of Newton's that for the given function f above no divisions are necessary to calculate x_{n+1} from x_n.

$$\begin{aligned}
f(x) &= a - \frac{1}{x} \\
df(x) &= \frac{1}{x^2} \\
x_{n+1} &= x_n - \frac{f(x_n)}{df(x_n)} \\
x_{n+1} &= x_n - \frac{a - \frac{1}{x_n}}{1} \cdot \frac{x_n^2}{1} \\
x_{n+1} &= x_n - ax_n^2 - x_n \\
x_{n+1} &= -ax_n^2
\end{aligned}$$

Therefore, no division is required if we know x_n.

(B) With a tolerance of 10^{-14} and initial guess of $x_0 = 0.1$, use the above to calculate the reciprocal of 12.

```
        c = 12

        def g(x):
            return c - (1/x)

        def dg(x):
            return (1/x**2)

        def newton(x0,f,df,tol=10**-14, max_iter=1000):
            x = x0
            for i in range(0,max_iter):
                if np.abs(f(x)) < tol:
                    return x
                x = x - f(x)/df(x)
            raise Exception("Max iteration reached")

        print(newton(0.1,g,dg))
```

2 Derivative Approximations

2.1

We are going to use the finite difference formula $f'(\bar{x}) \approx af(\bar{x}+2h) + bf(\bar{x}+h) + cf(\bar{x}) + df(\bar{x}-h) + ef(\bar{x}-2h)$ to estimate an unknown function value from the data. The constants are given below:

$$a = \frac{-1}{12h}, b = \frac{2}{3h}, c = 0, d = \frac{-2}{3h}, e = \frac{1}{12h}$$

You are given the following population data for a small town:

Year(x)	1880	1890	1900	1910	1920
Population f(x)	362	391	?	420	490

We would like to use this data and our formula to estimate the unknown population in the year 1900.
(A) If we wanted to use our formula with this data to estimate $f'(1900)$, what would h have to be?
(B) Use the formula to estimate $f'(1900)$.
(C) Add $h \cdot f'(1900)$ to $f(1890)$ or subtract to approximate $f(1900)$.
(This is actually making use of something akin to linear approximations).

```
h = 10
constants = np.array([-1/12,2/3,-2/3,1/12])
fun_vals = np.array([362,391,420,490])
edf = (1/h)*np.dot(constants,fun_vals)
fun_x = h*edf + 391

output:
fun_x = 382.333333333

```

2.2

Consider the following ODE:

$$y'(x) = 3y(x) + 2$$

And suppose that we have an initial value of $y(0) = 2$.
(A) Substitute in by hand a forward difference approximation

$$y'(x) \approx \frac{y(x+h) - y(x)}{h}$$

to get an approximate solution $y(x+h) = \dots$ to the ODE.

$$\begin{aligned} y'(x) &\approx \frac{y(x+h) - y(x)}{h} \\ 3y(x) + 2 &\approx \frac{y(x+h) - y(x)}{h} \\ y(x+h) &\approx h(3y(x) + 2) + y(x) \\ for\ x &= 0 \\ y(h) &\approx 8h + 2 \end{aligned}$$

(B) Using your approximation above with $h = 0.1$ and the given initial value, find $y(0.1)$, $y(0.2)$ and $y(0.3)$. You can do this with code or by hand.

$$\begin{aligned}
&for\ h = 0.1\ and\ x = 0.1\\
&y(0.1 + 0.1) \approx 8(0.2) + 2 = 3.6\\
&\qquad for\ x = 0.2\\
&y(0.2 + 0.1) \approx 8(0.3) + 2 = 4.4\\
&\qquad for\ x = 0.3\\
&y(0.3 + 0.1) \approx 8(0.4) + 2 = 5.2
\end{aligned}$$