Black-Scholes Option Pricing Model: Mathematical Derivation and Implementation

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March 27, 2025

Abstract

This document provides a comprehensive analysis of the Black-Scholes option pricing model, including its rigorous mathematical derivation from first principles and efficient computational implementation. We begin by establishing the stochastic differential equation governing the underlying asset price, introduce the risk-neutral valuation framework, and derive the celebrated Black-Scholes partial differential equation. The closed-form solutions for European call and put options are derived in detail, followed by a discussion of the put-call parity relationship. We also examine the sensitivity measures known as the "Greeks" and provide algorithmic implementations for all components of the model. Throughout, we maintain mathematical rigor while providing intuitive explanations of the key concepts in option pricing theory.

Contents

1 Introduction

The Black-Scholes model, developed by Fischer Black, Myron Scholes, and Robert Merton in the early 1970s, revolutionized the field of financial economics by providing a theoretical framework for pricing European options (1). The model's elegant mathematical formulation and closed-form solutions made it possible to quickly calculate option prices based on observable parameters, leading to its widespread adoption in financial markets.

The key insight of the Black-Scholes model is that, under certain assumptions, it is possible to create a risk-free portfolio consisting of an option and its underlying asset. This portfolio must earn the risk-free rate of return to prevent arbitrage opportunities, which leads to a partial differential equation that the option price must satisfy.

1.1 Model Assumptions

The Black-Scholes model relies on several key assumptions:

1. The price of the underlying asset follows a geometric Brownian motion with constant drift and volatility.
2. There are no transaction costs or taxes, and all securities are perfectly divisible.
3. The risk-free interest rate is constant and the same for all maturities.
4. Short selling is allowed, with full use of proceeds.
5. There are no arbitrage opportunities.
6. Trading is continuous.
7. The underlying asset pays no dividends during the life of the option.

While these assumptions simplify the mathematical treatment, they also represent limitations of the model. Subsequent research has extended the model to address some of these limitations, such as incorporating dividends, stochastic volatility, and jumps in the price process.

2 Mathematical Derivation of the Black-Scholes Model

2.1 Geometric Brownian Motion

The Black-Scholes model assumes that the price of the underlying asset $S(t)$ follows a geometric Brownian motion, which can be described by the stochastic differential equation (SDE):

$$dS(t) = \mu S(t)dt + \sigma S(t)dW(t) \tag{1}$$

where:

- μ is the expected return (drift) of the asset
- σ is the volatility of the asset
- $W(t)$ is a standard Wiener process (Brownian motion)

This stochastic process ensures that the asset price remains positive and that returns are normally distributed over short time intervals, which aligns with empirical observations of many financial assets.

The solution to this stochastic differential equation, using Itô's lemma (which we will discuss in detail), is:

$$S(t) = S(0) \exp\left(\left(\mu - \frac{\sigma^2}{2}\right) t + \sigma W(t)\right) \tag{2}$$

This shows that the logarithm of the price ratio $\ln(S(t)/S(0))$ follows a normal distribution with mean $(\mu - \sigma^2/2)t$ and variance $\sigma^2 t$, making $S(t)$ lognormally distributed.

2.2 Rigorous Application of Itô's Lemma

Before deriving the Black-Scholes equation, we need to carefully understand Itô's lemma, which is the stochastic calculus counterpart of the chain rule for ordinary calculus.

2.2.1 Statement of Itô's Lemma

For a function $f(t, X_t)$ of time t and a stochastic process X_t that follows the stochastic differential equation (SDE):

$$dX_t = a(X_t, t)dt + b(X_t, t)dW_t \tag{3}$$

where W_t is a standard Wiener process, Itô's lemma states that:

$$df(t, X_t) = \left(\frac{\partial f}{\partial t} + a(X_t, t)\frac{\partial f}{\partial x} + \frac{1}{2}b(X_t, t)^2\frac{\partial^2 f}{\partial x^2}\right) dt + b(X_t, t)\frac{\partial f}{\partial x}dW_t \tag{4}$$

The key insight is that the second-order term in the Taylor expansion does not vanish in the limit, due to the quadratic variation property of the Wiener process: $dW_t^2 = dt$.

2.2.2 Application to the Underlying Asset Price

Recall that the underlying asset price follows geometric Brownian motion:

$$dS(t) = \mu S(t)dt + \sigma S(t)dW(t) \tag{5}$$

Here, $a(S,t) = \mu S$ and $b(S,t) = \sigma S$. Note that by computing $(dS)^2$, we get:

$$\begin{aligned}(dS)^2 &= (\mu Sdt)^2 + 2(\mu Sdt)(\sigma SdW) + (\sigma SdW)^2 &(6)\\ &= \mu^2 S^2 (dt)^2 + 2\mu\sigma S^2 dtdW + \sigma^2 S^2 (dW)^2 &(7)\end{aligned}$$

Since $(dt)^2 \approx 0$, $dtdW \approx 0$, and $(dW)^2 \approx dt$ as $dt \to 0$, we have:

$$(dS)^2 \approx \sigma^2 S^2 dt \tag{8}$$

2.2.3 Application to Option Pricing

Let $V(S,t)$ be the price of an option as a function of the underlying asset price S and time t. We will apply Itô's lemma to determine the dynamics of $V(S,t)$.

Since S follows the SDE $dS = \mu Sdt + \sigma SdW$, we have $a(S,t) = \mu S$ and $b(S,t) = \sigma S$. By Itô's lemma:

$$dV = \frac{\partial V}{\partial t}dt + \frac{\partial V}{\partial S}dS + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}(dS)^2 \tag{9}$$

$$\begin{aligned}dV &= \frac{\partial V}{\partial t}dt + \frac{\partial V}{\partial S}(\mu Sdt + \sigma SdW) + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}(\sigma^2 S^2 dt) &(10)\\ &= \left(\frac{\partial V}{\partial t} + \mu S\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}\right)dt + \sigma S\frac{\partial V}{\partial S}dW &(11)\end{aligned}$$

This gives us the complete stochastic differential equation for the option price process.

2.3 Construction of a Risk-Free Portfolio

To derive the Black-Scholes partial differential equation, we construct a risk-free portfolio Π consisting of:

- A long position in one option: $+V$

- A short position in $\frac{\partial V}{\partial S}$ shares of the underlying asset: $-\frac{\partial V}{\partial S} S$

The value of this portfolio at time t is:

$$\Pi(t) = V(S,t) - \frac{\partial V}{\partial S} S \tag{12}$$

Now, we need to determine the change in the portfolio value over a small time interval dt. We use the differential form:

$$d\Pi = dV - \frac{\partial V}{\partial S} dS - d\left(\frac{\partial V}{\partial S}\right) S \tag{13}$$

Since we are rebalancing the portfolio continuously, the number of shares $\frac{\partial V}{\partial S}$ is adjusted instantaneously. Therefore, we can simplify to:

$$d\Pi = dV - \frac{\partial V}{\partial S} dS \tag{14}$$

Substituting the expressions for dV from Itô's lemma and dS from the geometric Brownian motion:

$$\begin{align}
d\Pi &= \left(\frac{\partial V}{\partial t} + \mu S \frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}\right) dt + \sigma S \frac{\partial V}{\partial S} dW - \frac{\partial V}{\partial S}(\mu S dt + \sigma S dW) \tag{15}\\
&= \left(\frac{\partial V}{\partial t} + \mu S \frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}\right) dt + \sigma S \frac{\partial V}{\partial S} dW - \mu S \frac{\partial V}{\partial S} dt - \sigma S \frac{\partial V}{\partial S} dW \tag{16}\\
&= \left(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}\right) dt \tag{17}
\end{align}$$

Observe that the stochastic terms involving dW cancel out exactly, resulting in a deterministic differential $d\Pi$. This means that the portfolio Π is instantaneously risk-free.

2.4 No-Arbitrage Condition and the Black-Scholes Equation

According to the no-arbitrage principle, any risk-free portfolio must earn exactly the risk-free interest rate r. Hence:

$$d\Pi = r\Pi dt \tag{18}$$

That is, the instantaneous return on the portfolio must equal the risk-free rate times the portfolio value. Substituting our expressions:

$$\left(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}\right) dt = r\left(V - \frac{\partial V}{\partial S} S\right) dt \tag{19}$$

Dividing both sides by dt and rearranging terms:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} = rV - r\frac{\partial V}{\partial S} S \tag{20}$$

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0 \tag{21}$$

This results in the Black-Scholes PDE (equation (22)), which is the fundamental equation governing option prices:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0 \tag{22}$$

Notably, the drift parameter μ of the underlying asset does not appear in the equation. This is a consequence of the construction of the risk-free portfolio and reflects the principle of risk-neutral valuation: the expected return on the underlying asset does not affect option prices; only the volatility matters.

2.5 Risk-Neutral Valuation

An alternative approach to deriving option prices is through risk-neutral valuation. Under the risk-neutral measure $\mathbb{Q}$, the expected return of all assets is the risk-free rate r, and the price of any derivative is the discounted expected value of its future payoff.

In the risk-neutral world, the stochastic differential equation for the asset price becomes:

$$dS(t) = rS(t)dt + \sigma S(t)dW^{\mathbb{Q}}(t) \tag{23}$$

where $W^{\mathbb{Q}}(t)$ is a Wiener process under the risk-neutral measure $\mathbb{Q}$.

The price of a European option with payoff function $\Phi(S(T))$ at maturity T is then:

$$V(S,t) = e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}\left[\Phi(S(T)) \mid S(t) = S\right] \tag{24}$$

where $\mathbb{E}^{\mathbb{Q}}[\cdot]$ denotes expectation under the risk-neutral measure.

2.6 Deriving the Closed-Form Solutions

We now derive the closed-form solutions for European options by solving the Black-Scholes PDE directly. This rigorous derivation demonstrates the powerful connection between the PDE approach and the risk-neutral valuation method.

2.6.1 The PDE Boundary Conditions

For a European call option with strike price E and maturity T, we need to solve the Black-Scholes PDE:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0 \tag{25}$$

subject to the terminal condition:

$$V(S,T) = \max(S-E,0) \tag{26}$$

and the boundary conditions:

$$V(0,t) = 0 \quad \text{(since if } S=0\text{, it remains 0 and the call expires worthless)} \tag{27}$$

$$\lim_{S\to\infty} V(S,t) = S - Ee^{-r(T-t)} \quad \text{(asymptotic behavior for deep in-the-money calls)} \tag{28}$$

2.6.2 Transformation to the Heat Equation

To solve this PDE, we transform it into the standard heat equation through a series of variable substitutions. First, we define:

$$\tau = T - t \quad \text{(time to maturity)} \tag{29}$$

$$x = \ln(S/E) \quad \text{(log-moneyness)} \tag{30}$$

$$V(S,t) = EF(x,\tau) \quad \text{(normalized option price)} \tag{31}$$

Computing the partial derivatives:

$$\frac{\partial V}{\partial t} = E\frac{\partial F}{\partial t} = -E\frac{\partial F}{\partial \tau} \tag{32}$$

$$\frac{\partial V}{\partial S} = E\frac{\partial F}{\partial x}\frac{\partial x}{\partial S} = E\frac{\partial F}{\partial x}\frac{1}{S} \tag{33}$$

$$\frac{\partial^2 V}{\partial S^2} = E\frac{\partial}{\partial S}\left(\frac{\partial F}{\partial x}\frac{1}{S}\right) = E\left(\frac{\partial^2 F}{\partial x^2}\frac{1}{S^2} - \frac{\partial F}{\partial x}\frac{1}{S^2}\right) \tag{34}$$

Substituting into the Black-Scholes PDE:

$$-E\frac{\partial F}{\partial \tau} + \frac{1}{2}\sigma^2 S^2 E\left(\frac{\partial^2 F}{\partial x^2}\frac{1}{S^2} - \frac{\partial F}{\partial x}\frac{1}{S^2}\right) + rSE\frac{\partial F}{\partial x}\frac{1}{S} - rEF = 0 \tag{35}$$

$$-\frac{\partial F}{\partial \tau} + \frac{1}{2}\sigma^2\frac{\partial^2 F}{\partial x^2} - \frac{1}{2}\sigma^2\frac{\partial F}{\partial x} + r\frac{\partial F}{\partial x} - rF = 0 \tag{36}$$

$$-\frac{\partial F}{\partial \tau} + \frac{1}{2}\sigma^2\frac{\partial^2 F}{\partial x^2} + \left(r - \frac{1}{2}\sigma^2\right)\frac{\partial F}{\partial x} - rF = 0 \tag{37}$$

Now, we make a further transformation:

$$F(x,\tau) = e^{\alpha x + \beta\tau}u(x,\tau) \tag{38}$$

where α and β are constants to be determined. Computing the derivatives:

$$\frac{\partial F}{\partial \tau} = \beta e^{\alpha x+\beta\tau}u + e^{\alpha x+\beta\tau}\frac{\partial u}{\partial \tau} \tag{39}$$

$$\frac{\partial F}{\partial x} = \alpha e^{\alpha x+\beta\tau}u + e^{\alpha x+\beta\tau}\frac{\partial u}{\partial x} \tag{40}$$

$$\frac{\partial^2 F}{\partial x^2} = \alpha^2 e^{\alpha x+\beta\tau}u + 2\alpha e^{\alpha x+\beta\tau}\frac{\partial u}{\partial x} + e^{\alpha x+\beta\tau}\frac{\partial^2 u}{\partial x^2} \tag{41}$$

Substituting into our transformed PDE:

$$-\left(\beta e^{\alpha x+\beta\tau}u + e^{\alpha x+\beta\tau}\frac{\partial u}{\partial \tau}\right) + \frac{1}{2}\sigma^2\left(\alpha^2 e^{\alpha x+\beta\tau}u + 2\alpha e^{\alpha x+\beta\tau}\frac{\partial u}{\partial x} + e^{\alpha x+\beta\tau}\frac{\partial^2 u}{\partial x^2}\right) \tag{42}$$

$$+\left(r - \frac{1}{2}\sigma^2\right)\left(\alpha e^{\alpha x+\beta\tau}u + e^{\alpha x+\beta\tau}\frac{\partial u}{\partial x}\right) - re^{\alpha x+\beta\tau}u = 0 \tag{43}$$

Dividing through by $e^{\alpha x+\beta\tau}$:

$$-\beta u - \frac{\partial u}{\partial \tau} + \frac{1}{2}\sigma^2\alpha^2 u + \sigma^2\alpha\frac{\partial u}{\partial x} + \frac{1}{2}\sigma^2\frac{\partial^2 u}{\partial x^2} \tag{44}$$

$$+\left(r - \frac{1}{2}\sigma^2\right)\alpha u + \left(r - \frac{1}{2}\sigma^2\right)\frac{\partial u}{\partial x} - ru = 0 \tag{45}$$

Collecting terms:

$$\frac{\partial u}{\partial \tau} = \frac{1}{2}\sigma^2\frac{\partial^2 u}{\partial x^2} + \left[\sigma^2\alpha + \left(r - \frac{1}{2}\sigma^2\right)\right]\frac{\partial u}{\partial x} + \left[\frac{1}{2}\sigma^2\alpha^2 + \left(r - \frac{1}{2}\sigma^2\right)\alpha - r - \beta\right]u \tag{46}$$

We choose α and β to eliminate the coefficient of $\frac{\partial u}{\partial x}$ and the term multiplying u:

$$\sigma^2\alpha + \left(r - \frac{1}{2}\sigma^2\right) = 0 \tag{47}$$

$$\frac{1}{2}\sigma^2\alpha^2 + \left(r - \frac{1}{2}\sigma^2\right)\alpha - r - \beta = 0 \tag{48}$$

Solving the first equation for α:

$$\alpha = -\frac{r - \frac{1}{2}\sigma^2}{\sigma^2} = -\frac{r}{\sigma^2} + \frac{1}{2} \tag{49}$$

Substituting into the second equation to find β:

$$\frac{1}{2}\sigma^2\left(-\frac{r}{\sigma^2} + \frac{1}{2}\right)^2 + \left(r - \frac{1}{2}\sigma^2\right)\left(-\frac{r}{\sigma^2} + \frac{1}{2}\right) - r - \beta = 0 \tag{50}$$

$$\frac{1}{2}\sigma^2\left(\frac{r^2}{\sigma^4} - \frac{r}{\sigma^2} + \frac{1}{4}\right) + \left(r - \frac{1}{2}\sigma^2\right)\left(-\frac{r}{\sigma^2} + \frac{1}{2}\right) - r - \beta = 0 \tag{51}$$

After algebraic manipulation:

$$\beta = -\frac{r}{2} - \frac{\sigma^2}{8} \tag{52}$$

With these values of α and β, our PDE reduces to:

$$\frac{\partial u}{\partial \tau} = \frac{1}{2}\sigma^2\frac{\partial^2 u}{\partial x^2} \tag{53}$$

This is the standard heat equation, for which the solution is well-known.

2.6.3 Transforming the Boundary Conditions

We need to transform the terminal condition for the call option. At maturity:

$$V(S,T) = \max(S-E,0) \tag{54}$$
$$EF(x,0) = \max(Ee^x - E,0) \tag{55}$$
$$F(x,0) = \max(e^x - 1,0) \tag{56}$$

In terms of u, with $\tau = 0$ and $F(x,0) = e^{\alpha x + \beta \cdot 0} u(x,0) = e^{\alpha x} u(x,0)$:

$$e^{\alpha x} u(x,0) = \max(e^x - 1, 0) \tag{57}$$

Therefore:

$$u(x,0) = e^{-\alpha x} \max(e^x - 1, 0) = \max(e^{(1-\alpha)x} - e^{-\alpha x}, 0) \tag{58}$$

Substituting the value of $\alpha = -\frac{r}{\sigma^2} + \frac{1}{2}$:

$$u(x,0) = \max(e^{(1+\frac{r}{\sigma^2}-\frac{1}{2})x} - e^{(\frac{r}{\sigma^2}-\frac{1}{2})x}, 0) \tag{59}$$

2.6.4 Solving the Heat Equation

The solution to the heat equation with initial condition $u(x,0) = g(x)$ is:

$$u(x,\tau) = \frac{1}{\sqrt{2\pi\sigma^2\tau}} \int_{-\infty}^{\infty} g(y) e^{-\frac{(x-y)^2}{2\sigma^2\tau}} \, dy \tag{60}$$

Substituting our initial condition:

$$u(x,\tau) = \frac{1}{\sqrt{2\pi\sigma^2\tau}} \int_{-\infty}^{\infty} \max(e^{(1+\frac{r}{\sigma^2}-\frac{1}{2})y} - e^{(\frac{r}{\sigma^2}-\frac{1}{2})y}, 0) e^{-\frac{(x-y)^2}{2\sigma^2\tau}} \, dy \tag{61}$$
$$= \frac{1}{\sqrt{2\pi\sigma^2\tau}} \int_{0}^{\infty} (e^{(1+\frac{r}{\sigma^2}-\frac{1}{2})y} - e^{(\frac{r}{\sigma^2}-\frac{1}{2})y}) e^{-\frac{(x-y)^2}{2\sigma^2\tau}} \, dy \tag{62}$$

This integral can be evaluated by completing the square in the exponent. After careful evaluation and algebraic manipulation:

$$u(x,\tau) = e^{(1+\frac{r}{\sigma^2}-\frac{1}{2})x+(1+\frac{r}{\sigma^2}-\frac{1}{2})^2\frac{\sigma^2\tau}{2}}\mathcal{N}\left(\frac{x+(1+\frac{r}{\sigma^2}-\frac{1}{2})\sigma^2\tau}{\sigma\sqrt{\tau}}\right) \tag{63}$$

$$- e^{(\frac{r}{\sigma^2}-\frac{1}{2})x+(\frac{r}{\sigma^2}-\frac{1}{2})^2\frac{\sigma^2\tau}{2}}\mathcal{N}\left(\frac{x+(\frac{r}{\sigma^2}-\frac{1}{2})\sigma^2\tau}{\sigma\sqrt{\tau}}\right) \tag{64}$$

where $\mathcal{N}(\cdot)$ is the cumulative distribution function of the standard normal distribution.

2.6.5 Reversing the Transformations

Now we reverse our transformations to obtain the option price:

$$F(x,\tau) = e^{\alpha x+\beta\tau}u(x,\tau) \tag{65}$$

$$V(S,t) = EF(x,\tau) \tag{66}$$

After substituting and simplifying, we arrive at the Black-Scholes formula for a European call option:

$$C(S,t) = S\mathcal{N}(d_1) - Ee^{-r\tau}\mathcal{N}(d_2) \tag{67}$$

where:

$$d_1 = \frac{\ln(S/E)+(r+\sigma^2/2)\tau}{\sigma\sqrt{\tau}} \tag{68}$$

$$d_2 = d_1 - \sigma\sqrt{\tau} = \frac{\ln(S/E)+(r-\sigma^2/2)\tau}{\sigma\sqrt{\tau}} \tag{69}$$

and $\tau = T - t$ is the time to maturity.

Similarly, for a European put option with payoff function $\Phi(S(T)) = \max(E - S(T), 0)$, the Black-Scholes formula is:

$$P(S,t) = Ee^{-r\tau}\mathcal{N}(-d_2) - S\mathcal{N}(-d_1) \tag{70}$$

These closed-form solutions allow for efficient calculation of option prices based on observable parameters: the current stock price S, the strike price E, the risk-free rate r, the volatility σ, and the time to maturity τ.

3 Put-Call Parity

Put-call parity is a fundamental relationship between the prices of European put and call options with the same strike price and expiration date. It is model-independent and relies only on no-arbitrage arguments.

Consider a portfolio consisting of:

- A long position in a European call option with strike price E and maturity T
- A cash amount of $Ee^{-r\tau}$ (the present value of the strike price)

And another portfolio consisting of:

- A long position in a European put option with the same strike price E and maturity T
- A long position in the underlying asset

At maturity T, both portfolios have the same value:

- If $S(T) > E$, the first portfolio is worth $S(T) - E + E = S(T)$, and the second portfolio is worth $0 + S(T) = S(T)$.
- If $S(T) < E$, the first portfolio is worth $0 + E = E$, and the second portfolio is worth $E - S(T) + S(T) = E$.

Since both portfolios have the same value at maturity, they must have the same value at any time before maturity to prevent arbitrage opportunities. This gives us the put-call parity relationship:

$$C(S,t) + Ee^{-r\tau} = P(S,t) + S \tag{71}$$

This relationship can be rearranged to express the put option price in terms of the call option price:

$$P(S,t) = C(S,t) + Ee^{-r\tau} - S \tag{72}$$

or the call option price in terms of the put option price:

$$C(S,t) = P(S,t) + S - Ee^{-r\tau} \tag{73}$$

Put-call parity provides a useful check on option pricing models and can be used to determine if options are priced consistently relative to each other.

4 The Greeks: Sensitivity Measures

The "Greeks" are sensitivity measures that describe how option prices change with respect to various parameters. They are essential for risk management and hedging strategies.

4.1 Delta

Delta (Δ) measures the rate of change of the option price with respect to the price of the underlying asset:

$$\Delta = \frac{\partial V}{\partial S} \tag{74}$$

For a European call option:

$$\Delta_{\text{call}} = \mathcal{N}(d_1) \tag{75}$$

For a European put option:

$$\Delta_{\text{put}} = \mathcal{N}(d_1) - 1 = -\mathcal{N}(-d_1) \tag{76}$$

Delta ranges from 0 to 1 for call options and from -1 to 0 for put options. It represents the number of shares of the underlying asset that should be held to hedge an option position.

4.2 Gamma

Gamma (Γ) measures the rate of change of delta with respect to the price of the underlying asset:

$$\Gamma = \frac{\partial^2 V}{\partial S^2} = \frac{\partial \Delta}{\partial S} \tag{77}$$

For both European call and put options:

$$\Gamma = \frac{N'(d_1)}{S\sigma\sqrt{\tau}} = \frac{e^{-d_1^2/2}}{S\sigma\sqrt{\tau}\sqrt{2\pi}} \tag{78}$$

where $N'(\cdot)$ is the probability density function of the standard normal distribution.

Gamma represents the curvature of the relationship between the option price and the underlying asset price. A high gamma indicates that delta will change rapidly with small movements in the underlying price.

4.3 Theta

Theta (Θ) measures the rate of change of the option price with respect to time (time decay):

$$\Theta = \frac{\partial V}{\partial t} \tag{79}$$

For a European call option:

$$\Theta_{\text{call}} = -\frac{S\sigma N'(d_1)}{2\sqrt{\tau}} - rEe^{-r\tau}N(d_2) \tag{80}$$

For a European put option:

$$\Theta_{\text{put}} = -\frac{S\sigma N'(d_1)}{2\sqrt{\tau}} + rEe^{-r\tau}N(-d_2) \tag{81}$$

Theta is typically negative for both call and put options, indicating that the option loses value as time passes, all else being equal.

4.4 Vega

Vega measures the rate of change of the option price with respect to the volatility of the underlying asset:

$$\text{Vega} = \frac{\partial V}{\partial \sigma} \tag{82}$$

For both European call and put options:

$$\text{Vega} = S\sqrt{\tau}N'(d_1) = S\sqrt{\tau}\frac{e^{-d_1^2/2}}{\sqrt{2\pi}} \tag{83}$$

Vega is always positive for standard options, indicating that higher volatility leads to higher option prices.

4.5 Rho

Rho (ρ) measures the rate of change of the option price with respect to the risk-free interest rate:

$$\rho = \frac{\partial V}{\partial r} \tag{84}$$

For a European call option:

$$\rho_{\text{call}} = E\tau e^{-r\tau}N(d_2) \tag{85}$$

For a European put option:

$$\rho_{\text{put}} = -E\tau e^{-r\tau}N(-d_2) \tag{86}$$

Rho is typically positive for call options and negative for put options, reflecting the impact of interest rates on the present value of the strike price.

5 Comprehensive Black-Scholes Algorithm

We now present a comprehensive algorithm that encapsulates all the functionality needed to price European options using the Black-Scholes model, calculate the Greeks, and verify put-call parity. This single algorithm captures the entire computational process in a rigorous manner.

Algorithm 1 Comprehensive Black-Scholes Option Pricing

procedure BlackScholesModel(S_0, E, T, r_f, σ)
    Input: S_0 (current stock price), E (strike price), T (time to expiration in years)
    Input: r_f (risk-free interest rate), σ (volatility)
    Output: Call price, put price, Greeks, put-call parity verification

Algorithm 2 Comprehensive Black-Scholes Option Pricing (continued)

function CalculateDParameters(S, E, r_f, σ, τ)
if $\tau \leq 0$ **then**
    raise Exception: "Time to expiration must be positive"
end if
$d_1 \leftarrow \frac{\ln(S/E)+(r_f+\sigma^2/2)\cdot\tau}{\sigma\cdot\sqrt{\tau}}$
$d_2 \leftarrow d_1 - \sigma \cdot \sqrt{\tau}$
return (d_1, d_2)

function CalculateNormalCDF(x)
return $\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-t^2/2}dt$ ▷ CDF of standard normal

function CalculateNormalPDF(x)
return $\frac{1}{\sqrt{2\pi}}e^{-x^2/2}$ ▷ PDF of standard normal
=0

Algorithm 3 Comprehensive Black-Scholes Option Pricing (continued)

function CallOptionPrice(S, E, r_f, σ, τ)
$(d_1, d_2) \leftarrow$ CalculateDParameters(S, E, r_f, σ, τ)
$N_{d1} \leftarrow$ CalculateNormalCDF(d_1)
$N_{d2} \leftarrow$ CalculateNormalCDF(d_2)
$call\_price \leftarrow S \cdot N_{d1} - E \cdot e^{-r_f \cdot \tau} \cdot N_{d2}$
return $call\_price$

function PutOptionPrice(S, E, r_f, σ, τ)
$(d_1, d_2) \leftarrow$ CalculateDParameters(S, E, r_f, σ, τ)
$N_{-d1} \leftarrow$ CalculateNormalCDF($-d_1$)
$N_{-d2} \leftarrow$ CalculateNormalCDF($-d_2$)
$put\_price \leftarrow E \cdot e^{-r_f \cdot \tau} \cdot N_{-d2} - S \cdot N_{-d1}$
return put_price

Algorithm 4 Comprehensive Black-Scholes Option Pricing (continued)

function VerifyPutCallParity($call\_price, put\_price, S, E, r_f, \tau$)
$left\_side \leftarrow call\_price + E \cdot e^{-r_f \cdot \tau}$
$right\_side \leftarrow put\_price + S$
$difference \leftarrow |left\_side - right\_side|$
return $(left\_side, right\_side, difference)$

Algorithm 5 Comprehensive Black-Scholes Option Pricing (continued)

function CalculateGreeks(S, E, r_f, σ, τ)
$(d_1, d_2) \leftarrow$ CalculateDParameters(S, E, r_f, σ, τ)
$d1\_pdf \leftarrow$ CalculateNormalPDF(d_1)
// **Delta calculations**
$call\_delta \leftarrow$ CalculateNormalCDF(d_1)
$put\_delta \leftarrow call\_delta - 1$
// **Gamma calculation (same for calls and puts)**
$gamma \leftarrow \frac{d1\_pdf}{S \cdot \sigma \cdot \sqrt{\tau}}$
// **Theta calculations (time decay, expressed as per day)**
$call\_theta \leftarrow -\frac{S \cdot \sigma \cdot d1\_pdf}{2 \cdot \sqrt{\tau}} - r_f \cdot E \cdot e^{-r_f \cdot \tau} \cdot$ CalculateNormalCDF(d_2)
$call\_theta \leftarrow call\_theta/365$ ▷ Convert to daily decay
$put\_theta \leftarrow -\frac{S \cdot \sigma \cdot d1\_pdf}{2 \cdot \sqrt{\tau}} + r_f \cdot E \cdot e^{-r_f \cdot \tau} \cdot$ CalculateNormalCDF($-d_2$)
$put\_theta \leftarrow put\_theta/365$ ▷ Convert to daily decay

Algorithm 6 Comprehensive Black-Scholes Option Pricing (continued)

// **Vega calculation (sensitivity to volatility, same for calls and puts)**
$vega \leftarrow S \cdot \sqrt{\tau} \cdot d1\_pdf/100$ ▷ For 1% change in volatility
// **Rho calculations (sensitivity to interest rate, for 1% change)**
$call\_rho \leftarrow E \cdot \tau \cdot e^{-r_f \cdot \tau} \cdot$ CalculateNormalCDF$(d_2)/100$
$put\_rho \leftarrow -E \cdot \tau \cdot e^{-r_f \cdot \tau} \cdot$ CalculateNormalCDF$(-d_2)/100$
return {call_delta, put_delta, gamma, call_theta, put_theta, vega, call_rho, put_rho}

Algorithm 7 Comprehensive Black-Scholes Option Pricing (continued)

// **Main procedure execution**
$\tau \leftarrow T$ ▷ Time to expiration (assuming current time $t = 0$)
// **Calculate option prices**
$call\_price \leftarrow$ CallOptionPrice$(S_0, E, r_f, \sigma, \tau)$
$put\_price \leftarrow$ PutOptionPrice$(S_0, E, r_f, \sigma, \tau)$
// **Verify put-call parity**
$(left, right, diff) \leftarrow$ VerifyPutCallParity$(call_price, put_price, S_0, E, r_f, \tau)$
// **Calculate Greeks**
$greeks \leftarrow$ CalculateGreeks$(S_0, E, r_f, \sigma, \tau)$
// **Return all results**
return {call_price, put_price, left, right, diff, greeks}

This comprehensive algorithm encapsulates the core functionality of the Black-Scholes model in a structured, modular approach. It begins with helper functions for calculating the d_1 and d_2 parameters and normal distribution functions. It then implements the pricing functions for call and put options, followed by a function to verify put-call parity. Finally, it calculates the Greeks, which are essential for risk management.

The algorithm is designed to be both mathematically rigorous and computationally efficient. It follows directly from the theoretical derivations presented earlier and provides a complete framework for option pricing under the Black-Scholes model.

In practice, this algorithm can be implemented in various programming languages, with appropriate numerical methods for evaluating the normal CDF and PDF functions. The implementation would typically use library functions for these calculations, such as those provided by SciPy in Python, as seen in the original code.

6 Numerical Examples and Visualizations

6.1 Example Calculation

Consider a European call option with the following parameters:

- Current stock price: $S = 100$
- Strike price: $E = 100$
- Time to expiration: $T = 1$ year
- Risk-free interest rate: $r = 5\%$ per annum
- Volatility: $\sigma = 20\%$ per annum

First, we calculate the d_1 and d_2 parameters:

$$d_1 = \frac{\ln(100/100) + (0.05 + 0.2^2/2) \cdot 1}{0.2 \cdot \sqrt{1}} = \frac{0 + 0.05 + 0.02}{0.2} = 0.35 \tag{87}$$

$$d_2 = 0.35 - 0.2 \cdot \sqrt{1} = 0.35 - 0.2 = 0.15 \tag{88}$$

Next, we calculate the call option price:

$$\begin{aligned} C &= 100 \cdot \Phi(0.35) - 100 \cdot e^{-0.05\cdot 1} \cdot \Phi(0.15) & (89)\\ &= 100 \cdot 0.6368 - 100 \cdot 0.9512 \cdot 0.5596 & (90)\\ &= 63.68 - 53.23 = 10.45 & (91) \end{aligned}$$

Similarly, we can calculate the put option price:

$$\begin{aligned} P &= 100 \cdot e^{-0.05\cdot 1} \cdot \Phi(-0.15) - 100 \cdot \Phi(-0.35) & (92)\\ &= 100 \cdot 0.9512 \cdot 0.4404 - 100 \cdot 0.3632 & (93)\\ &= 41.89 - 36.32 = 5.57 & (94) \end{aligned}$$

We can verify the put-call parity relationship:

$$\begin{aligned} C + Ee^{-r\tau} &= P + S & (95)\\ 10.45 + 100 \cdot e^{-0.05\cdot 1} &= 5.57 + 100 & (96)\\ 10.45 + 95.12 &= 5.57 + 100 & (97)\\ 105.57 &= 105.57 & (98) \end{aligned}$$

The put-call parity relationship holds, confirming the consistency of our calculations.

6.2 Option Values as a Function of Stock Price

To visualize how option values change with the stock price, we can plot the call and put option values for a range of stock prices while keeping other parameters constant.

For the same option parameters as in the previous example, but varying the stock price from 50 to 150, the call option values will increase approximately linearly when the option is deep in-the-money ($S \gg E$), and approach zero when the option is deep out-of-the-money ($S \ll E$). Conversely, the put option values will decrease approximately linearly when the option is deep out-of-the-money ($S \gg E$), and approach the discounted strike price when the option is deep in-the-money ($S \ll E$).

The code provided in the implementation can be used to generate these plots.

6.3 The Greeks and Risk Management

The Greeks provide valuable information for risk management. For instance:

- Delta indicates the number of shares of the underlying asset needed to hedge an option position.
- Gamma indicates how frequently the hedge needs to be adjusted as the underlying price changes.
- Theta indicates how quickly the option loses value due to time decay.
- Vega indicates the exposure to changes in volatility.
- Rho indicates the exposure to changes in interest rates.

For the example option with at-the-money strike ($S = E = 100$), we would expect:

- A call delta close to 0.5 (actually 0.6368, reflecting the positive drift due to the risk-free rate)
- A put delta close to -0.5 (actually -0.3632)
- A relatively high gamma, indicating significant sensitivity to changes in the underlying price
- A negative theta, indicating time decay
- A positive vega, indicating that increases in volatility will increase the option value

These values can be calculated using the algorithms provided in the implementation.

7 Conclusion

The Black-Scholes model provides a powerful framework for pricing options and understanding their behavior. Despite its simplifying assumptions, it remains a cornerstone of financial theory and practice. The model's elegant mathematical formulation, based on no-arbitrage principles, leads to closed-form solutions that can be efficiently computed.

In this document, we have derived the Black-Scholes partial differential equation from first principles, presented the closed-form solutions for European call and put options, discussed the put-call parity relationship, and examined the Greeks as sensitivity measures. We have also provided algorithmic implementations for all components of the model, allowing for practical application.

While extensions and refinements of the model continue to be developed to address its limitations, the core insights of Black, Scholes, and Merton have stood the test of time and continue to influence how financial markets operate.

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