Modern Portfolio Theory: A Formalized Mathematical Approach

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Abstract

This document provides a formalized mathematical framework for Modern Portfolio Theory (MPT) based on the work of Harry Markowitz. We present the rigorous mathematical foundation of portfolio optimization, emphasizing the risk-return tradeoff, efficient frontier, and Sharpe ratio optimization. Precise notation and formal derivations are presented alongside pseudocode implementations of key algorithms, adhering to standard conventions in financial mathematics.

1 Introduction

Modern Portfolio Theory (MPT), pioneered by [1], provides a mathematical framework for constructing investment portfolios that maximize expected returns for a given level of risk. The key insight of MPT is that an asset's risk and return should not be assessed in isolation but by how it contributes to a portfolio's overall risk and return.

2 Mathematical Formulation

2.1 Notational Conventions

We adopt the following standard notations from financial mathematics:

- N: Number of assets in the investment universe
- $\boldsymbol{w} = (w_1, w_2, \ldots, w_N)^T$: Vector of portfolio weights
- $\boldsymbol{\mu} = (\mu_1, \mu_2, \ldots, \mu_N)^T$: Vector of expected returns
- $\boldsymbol{\Sigma}$: Covariance matrix of returns, where $\sigma_{ij} = \text{Cov}(r_i, r_j)$
- $r_{i,t}$: Return of asset i at time t
- $\mu_p = \boldsymbol{w}^T \boldsymbol{\mu}$: Expected return of the portfolio

- $\sigma_p^2 = \boldsymbol{w}^T \boldsymbol{\Sigma} \boldsymbol{w}$: Variance of the portfolio
- $\sigma_p = \sqrt{\boldsymbol{w}^T \boldsymbol{\Sigma} \boldsymbol{w}}$: Standard deviation (volatility) of the portfolio
- r_f: Risk-free rate
- $S(\boldsymbol{w}) = \frac{\mu_p - r_f}{\sigma_p}$: Sharpe ratio of the portfolio

2.2 Return Calculation

For empirical implementation, we calculate logarithmic returns rather than simple returns:

Definition 1 (Logarithmic Return)**.** *The logarithmic return $r_{i,t}$ of asset i at time t is defined as:*

$$r_{i,t} = \ln\left(\frac{P_{i,t}}{P_{i,t-1}}\right) \tag{1}$$

where $P_{i,t}$ represents the price of asset i at time t.

Logarithmic returns offer several advantages over simple returns:

- Time additivity: $r_{i,[t_1,t_3]} = r_{i,[t_1,t_2]} + r_{i,[t_2,t_3]}$
- Better statistical properties: Typically more normally distributed
- Unbounded below: Simple returns are bounded by -100%

2.3 Portfolio Expected Return and Risk

Definition 2 (Portfolio Expected Return)**.** *The expected return of a portfolio μ_p is the weighted sum of the expected returns of individual assets:*

$$\mu_p = \sum_{i=1}^{N} w_i \mu_i = \boldsymbol{w}^T \boldsymbol{\mu} \tag{2}$$

subject to the budget constraint:

$$\sum_{i=1}^{N} w_i = 1 \tag{3}$$

Definition 3 (Portfolio Variance)**.** *The variance of a portfolio σ_p^2 is given by:*

$$\sigma_p^2 = \sum_{i=1}^{N} \sum_{j=1}^{N} w_i w_j \sigma_{ij} = \boldsymbol{w}^T \boldsymbol{\Sigma} \boldsymbol{w} \tag{4}$$

where σ_{ij} is the covariance between assets i and j.

Definition 4 (Portfolio Volatility)**.** *The volatility (standard deviation) of a portfolio σ_p is:*

$$\sigma_p = \sqrt{\sigma_p^2} = \sqrt{\boldsymbol{w}^T \boldsymbol{\Sigma} \boldsymbol{w}} \tag{5}$$

2.4 Estimation of Parameters

In practice, we estimate the expected returns and covariance matrix from historical data:

Definition 5 (Sample Mean Return)**.** *The sample mean return $\hat{\mu}_i$ for asset i is:*

$$\hat{\mu}_i = \frac{1}{T} \sum_{t=1}^{T} r_{i,t} \tag{6}$$

where T is the number of observations.

Definition 6 (Sample Covariance)**.** *The sample covariance $\hat{\sigma}_{ij}$ between assets i and j is:*

$$\hat{\sigma}_{ij} = \frac{1}{T-1} \sum_{t=1}^{T} (r_{i,t} - \hat{\mu}_i)(r_{j,t} - \hat{\mu}_j) \tag{7}$$

2.5 Time Scaling of Parameters

For investment horizons, we typically scale daily estimates to annual figures:

Definition 7 (Annualization)**.** *Given daily return estimate $\hat{\mu}_i^d$ and covariance estimate $\hat{\sigma}_{ij}^d$, the annualized estimates are:*

$$\hat{\mu}_i^a = \hat{\mu}_i^d \cdot K \tag{8}$$

$$\hat{\sigma}_{ij}^a = \hat{\sigma}_{ij}^d \cdot K \tag{9}$$

where K is the number of trading days in a year (typically 252).

3 The Markowitz Optimization Problem

3.1 Mean-Variance Optimization

The classical Markowitz optimization problem can be formulated in two equivalent ways:

Definition 8 (Minimum Variance Problem)**.** *Find the portfolio weights $\boldsymbol{w}$ that:*

$$\min_{\boldsymbol{w}} \quad \boldsymbol{w}^T \boldsymbol{\Sigma} \boldsymbol{w} \tag{10}$$

$$\textit{subject to} \quad \boldsymbol{w}^T \boldsymbol{\mu} \geq \mu_{target} \tag{11}$$

$$\mathbf{1}^T \boldsymbol{w} = 1 \tag{12}$$

$$w_i \geq 0, \quad i = 1, 2, \ldots, N \tag{13}$$

where μ_{target} is the target return, and the last constraint represents the non-negativity constraint (no short-selling).

Definition 9 (Maximum Return Problem)**.** *Find the portfolio weights* $\boldsymbol{w}$ *that:*

$$\max_{\boldsymbol{w}} \quad \boldsymbol{w}^T\boldsymbol{\mu} \tag{14}$$

$$\text{subject to} \quad \boldsymbol{w}^T\boldsymbol{\Sigma}\boldsymbol{w} \leq \sigma^2_{target} \tag{15}$$

$$\mathbf{1}^T\boldsymbol{w} = 1 \tag{16}$$

$$w_i \geq 0, \quad i = 1, 2, \ldots, N \tag{17}$$

where σ^2_{target} *is the target variance.*

3.2 Efficient Frontier

Definition 10 (Efficient Portfolio)**.** *A portfolio is said to be efficient if it has the highest expected return for a given level of risk or the lowest risk for a given level of expected return.*

Definition 11 (Efficient Frontier)**.** *The efficient frontier is the set of all efficient portfolios in the risk-return space. Mathematically, it is the solution to the following problem for various values of* μ_{target}*:*

$$\min_{\boldsymbol{w}} \quad \boldsymbol{w}^T\boldsymbol{\Sigma}\boldsymbol{w} \tag{18}$$

$$\text{subject to} \quad \boldsymbol{w}^T\boldsymbol{\mu} = \mu_{target} \tag{19}$$

$$\mathbf{1}^T\boldsymbol{w} = 1 \tag{20}$$

$$w_i \geq 0, \quad i = 1, 2, \ldots, N \tag{21}$$

In practice, the efficient frontier can be approximated using a Monte Carlo simulation:

Algorithm 1 Monte Carlo Approximation of Efficient Frontier

```
Input: μ, Σ, M (number of simulations)
Output: Set of portfolios approximating the efficient frontier
P ← ∅ {Initialize empty set of portfolios}
for i = 1 to M do
    Generate random weights w such that Σ_{j=1}^{N} w_j = 1 and w_j ≥ 0
    Compute μ_p = w^T μ
    Compute σ_p = sqrt(w^T Σ w)
    P ← P ∪ {(σ_p, μ_p, w)}
end for
return P
```

3.3 Sharpe Ratio Optimization

Definition 12 (Sharpe Ratio)**.** *The Sharpe ratio* $S(\boldsymbol{w})$ *of a portfolio with weights* $\boldsymbol{w}$ *is defined as:*

$$S(\boldsymbol{w}) = \frac{\boldsymbol{w}^T\boldsymbol{\mu} - r_f}{\sqrt{\boldsymbol{w}^T\boldsymbol{\Sigma}\boldsymbol{w}}} \tag{22}$$

where r_f is the risk-free rate.

Theorem 1 (Tangency Portfolio)**.** *The portfolio that maximizes the Sharpe ratio (also known as the tangency portfolio) is the solution to:*

$$\max_{\boldsymbol{w}} \quad \frac{\boldsymbol{w}^T\boldsymbol{\mu} - r_f}{\sqrt{\boldsymbol{w}^T\boldsymbol{\Sigma}\boldsymbol{w}}} \tag{23}$$

$$\textit{subject to} \quad \mathbf{1}^T\boldsymbol{w} = 1 \tag{24}$$

$$w_i \geq 0, \quad i = 1, 2, \ldots, N \tag{25}$$

When $r_f = 0$, this simplifies to:

$$\max_{\boldsymbol{w}} \quad \frac{\boldsymbol{w}^T\boldsymbol{\mu}}{\sqrt{\boldsymbol{w}^T\boldsymbol{\Sigma}\boldsymbol{w}}} \tag{26}$$

$$\textit{subject to} \quad \mathbf{1}^T\boldsymbol{w} = 1 \tag{27}$$

$$w_i \geq 0, \quad i = 1, 2, \ldots, N \tag{28}$$

This optimization problem can be solved using numerical methods:

Algorithm 2 Sharpe Ratio Optimization

Input: $\boldsymbol{\mu}$, $\boldsymbol{\Sigma}$, r_f (risk-free rate)
Output: Optimal portfolio weights $\boldsymbol{w}^*$
Define objective function $f(\boldsymbol{w}) = -\frac{\boldsymbol{w}^T\boldsymbol{\mu} - r_f}{\sqrt{\boldsymbol{w}^T\boldsymbol{\Sigma}\boldsymbol{w}}}$ {Negative for minimization}
Define constraint $g(\boldsymbol{w}) = \mathbf{1}^T\boldsymbol{w} - 1 = 0$
Define bounds $0 \leq w_i \leq 1$ for $i = 1, 2, \ldots, N$
$\boldsymbol{w}^* \leftarrow$ Solve optimization problem using numerical method
return $\boldsymbol{w}^*$

4 Analytical Solutions for Special Cases

4.1 Minimum Variance Portfolio (MVP)

Theorem 2 (Minimum Variance Portfolio)**.** *The minimum variance portfolio weights $\boldsymbol{w}_{MVP}$ are given by:*

$$\boldsymbol{w}_{MVP} = \frac{\boldsymbol{\Sigma}^{-1}\mathbf{1}}{\mathbf{1}^T\boldsymbol{\Sigma}^{-1}\mathbf{1}} \tag{29}$$

where $\mathbf{1}$ is a vector of ones.

Proof. We form the Lagrangian for the constrained optimization problem:

$$L(\boldsymbol{w}, \lambda) = \boldsymbol{w}^T\boldsymbol{\Sigma}\boldsymbol{w} - \lambda(\mathbf{1}^T\boldsymbol{w} - 1) \tag{30}$$

Taking the derivative with respect to $\boldsymbol{w}$ and setting it to zero:

$$\frac{\partial L}{\partial \boldsymbol{w}} = 2\boldsymbol{\Sigma}\boldsymbol{w} - \lambda\mathbf{1} = \mathbf{0} \tag{31}$$

Solving for $\boldsymbol{w}$:

$$\boldsymbol{w} = \frac{\lambda}{2}\boldsymbol{\Sigma}^{-1}\mathbf{1} \tag{32}$$

Using the constraint $\mathbf{1}^T\boldsymbol{w} = 1$:

$$\mathbf{1}^T\boldsymbol{w} = \frac{\lambda}{2}\mathbf{1}^T\boldsymbol{\Sigma}^{-1}\mathbf{1} = 1 \tag{33}$$

Solving for $\lambda/2$:

$$\frac{\lambda}{2} = \frac{1}{\mathbf{1}^T\boldsymbol{\Sigma}^{-1}\mathbf{1}} \tag{34}$$

Therefore:

$$\boldsymbol{w}_{MVP} = \frac{\boldsymbol{\Sigma}^{-1}\mathbf{1}}{\mathbf{1}^T\boldsymbol{\Sigma}^{-1}\mathbf{1}} \tag{35}$$

□

4.2 Tangency Portfolio

Theorem 3 (Tangency Portfolio)**.** *When there is a risk-free asset with return r_f, the tangency portfolio weights $\boldsymbol{w}_{TP}$ are given by:*

$$\boldsymbol{w}_{TP} = \frac{\boldsymbol{\Sigma}^{-1}(\boldsymbol{\mu} - r_f\mathbf{1})}{\mathbf{1}^T\boldsymbol{\Sigma}^{-1}(\boldsymbol{\mu} - r_f\mathbf{1})} \tag{36}$$

Proof. Similar to the MVP proof, we form the Lagrangian and take derivatives. The detailed proof is omitted for brevity but follows standard optimization techniques. □

5 Implementation Considerations

5.1 Regularization and Robustness

The sample covariance matrix can be ill-conditioned, especially when the number of assets is large relative to the sample size. Several regularization approaches can be used:

- Shrinkage estimators: $\hat{\boldsymbol{\Sigma}}_{\text{shrink}} = \delta\hat{\boldsymbol{\Sigma}} + (1-\delta)\hat{\boldsymbol{\Sigma}}_{\text{target}}$
- Factor models: $\hat{\boldsymbol{\Sigma}}_{\text{factor}} = \boldsymbol{B}\boldsymbol{F}\boldsymbol{B}^T + \boldsymbol{D}$
- Robust estimators: Minimum Covariance Determinant, M-estimators

5.2 Constraints

Practical portfolio optimization often includes additional constraints:

- Sector constraints: $\sum_{i \in \text{Sector}_j} w_i \leq c_j$
- Position limits: $l_i \leq w_i \leq u_i$
- Turnover constraints: $\sum_{i=1}^{N} |w_i - w_{i,\text{current}}| \leq \tau$

6 Practical Pseudocode Implementation

Algorithm 3 Markowitz Portfolio Optimization

Input: Historical price data $\{P_{i,t}\}$, number of simulations M
Output: Optimal portfolio weights $\boldsymbol{w}^*$
Step 1: Calculate log returns
for each asset i and time t **do**
  $r_{i,t} \leftarrow \ln\left(\frac{P_{i,t}}{P_{i,t-1}}\right)$
end for
Step 2: Estimate parameters
$\hat{\boldsymbol{\mu}} \leftarrow \text{SampleMean}(\{r_{i,t}\})$
$\hat{\boldsymbol{\Sigma}} \leftarrow \text{SampleCovariance}(\{r_{i,t}\})$
Step 3: Annualize parameters
$\hat{\boldsymbol{\mu}}_{\text{annual}} \leftarrow \hat{\boldsymbol{\mu}} \cdot 252$
$\hat{\boldsymbol{\Sigma}}_{\text{annual}} \leftarrow \hat{\boldsymbol{\Sigma}} \cdot 252$
Step 4: Generate random portfolios
$\mathcal{P} \leftarrow \emptyset$
for $i = 1$ to M **do**
  Generate random weights $\boldsymbol{w}$ such that $\sum_{j=1}^{N} w_j = 1$ and $w_j \geq 0$
  $\mu_p \leftarrow \boldsymbol{w}^T \hat{\boldsymbol{\mu}}_{\text{annual}}$
  $\sigma_p \leftarrow \sqrt{\boldsymbol{w}^T \hat{\boldsymbol{\Sigma}}_{\text{annual}} \boldsymbol{w}}$
  $S_p \leftarrow \frac{\mu_p}{\sigma_p}$ {Sharpe ratio with $r_f = 0$}
  $\mathcal{P} \leftarrow \mathcal{P} \cup \{(\sigma_p, \mu_p, S_p, \boldsymbol{w})\}$
end for
Step 5: Find optimal portfolio
Define objective function $f(\boldsymbol{w}) = -\frac{\boldsymbol{w}^T \hat{\boldsymbol{\mu}}_{\text{annual}}}{\sqrt{\boldsymbol{w}^T \hat{\boldsymbol{\Sigma}}_{\text{annual}} \boldsymbol{w}}}$
Define constraint $g(\boldsymbol{w}) = \mathbf{1}^T \boldsymbol{w} - 1 = 0$
Define bounds $0 \leq w_i \leq 1$ for $i = 1, 2, \ldots, N$
$\boldsymbol{w}^* \leftarrow \text{NumericalOptimization}(f, g, \text{bounds})$
return $\boldsymbol{w}^*$

7 Conclusion

The Markowitz Portfolio Theory provides a formal mathematical framework for portfolio optimization, balancing risk and return. While we have presented the theoretical foundations, practical implementation requires careful consideration of parameter estimation, regularization, and additional constraints.

References

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