Geometric Brownian Motion: Mathematical Formulation and Simulation

Sandy Herho

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Abstract

This document presents a concise treatment of Geometric Brownian Motion (GBM), focusing on its mathematical foundation and numerical simulation. GBM is a continuous-time stochastic process widely used for modeling stock prices and other financial assets. We derive the stochastic differential equation governing GBM, present its analytical solution, and provide detailed algorithms for its efficient simulation. The implementation is validated through numerical experiments demonstrating the key statistical properties of the simulated paths.

1 Introduction to Geometric Brownian Motion

Geometric Brownian Motion (GBM) is a continuous-time stochastic process where the logarithm of the randomly varying quantity follows a Brownian motion with drift. It is extensively used in financial mathematics, particularly for modeling stock prices in the Black-Scholes option pricing framework (Black and Scholes, 1973).

The key advantage of GBM is that it ensures the modeled quantity remains positive, making it suitable for stock prices and other quantities that cannot be negative. Additionally, it captures the empirical observation that financial returns are approximately normally distributed, while price levels follow a lognormal distribution (Hull, 2017).

2 Mathematical Formulation

2.1 Stochastic Differential Equation

A Geometric Brownian Motion $S(t)$ satisfies the following stochastic differential equation (SDE):

$$dS(t) = \mu S(t)dt + \sigma S(t)dW(t) \tag{1}$$

where:

- $S(t)$ is the asset price at time t

- μ is the drift parameter (expected return)
- $\sigma > 0$ is the volatility parameter (standard deviation of returns)
- $W(t)$ is a standard Wiener process (Brownian motion)

The drift term $\mu S(t)dt$ represents the deterministic part of the price change, proportional to the current price and the time increment. The stochastic term $\sigma S(t)dW(t)$ introduces randomness proportional to both the current price and the volatility.

2.2 Analytical Solution

Using Itô's lemma, we can derive the analytical solution to the GBM stochastic differential equation. Let $X(t) = \ln(S(t))$. Applying Itô's formula:

$$\begin{aligned}
dX(t) &= \frac{1}{S(t)}dS(t) - \frac{1}{2}\frac{1}{S(t)^2}(dS(t))^2 && (2)\\
&= \frac{1}{S(t)}(\mu S(t)dt + \sigma S(t)dW(t)) - \frac{1}{2}\frac{1}{S(t)^2}(\sigma S(t))^2 dt && (3)\\
&= \mu dt + \sigma dW(t) - \frac{1}{2}\sigma^2 dt && (4)\\
&= \left(\mu - \frac{1}{2}\sigma^2\right) dt + \sigma dW(t) && (5)
\end{aligned}$$

Integrating both sides from 0 to t:

$$\begin{aligned}
X(t) - X(0) &= \left(\mu - \frac{1}{2}\sigma^2\right) t + \sigma W(t) && (6)\\
\ln\left(\frac{S(t)}{S(0)}\right) &= \left(\mu - \frac{1}{2}\sigma^2\right) t + \sigma W(t) && (7)
\end{aligned}$$

Which gives us the solution:

$$S(t) = S(0)\exp\left(\left(\mu - \frac{1}{2}\sigma^2\right) t + \sigma W(t)\right) \quad (8)$$

This shows that $\ln(S(t)/S(0))$ follows a normal distribution with mean $(\mu - \sigma^2/2)t$ and variance $\sigma^2 t$.

2.3 Statistical Properties

From the analytical solution, we can derive the following properties:

1. The expectation of $S(t)$ is:
$$\mathbb{E}[S(t)] = S(0)e^{\mu t} \quad (9)$$

2. The variance of $S(t)$ is:

$$\mathrm{Var}[S(t)] = S(0)^2 e^{2\mu t}(e^{\sigma^2 t} - 1) \tag{10}$$

3. $S(t)$ follows a lognormal distribution:

$$S(t) \sim \mathrm{LogNormal}\left(\ln(S(0)) + \left(\mu - \frac{\sigma^2}{2}\right)t, \sigma^2 t\right) \tag{11}$$

3 Numerical Simulation

3.1 Discrete-Time Approximation

To simulate a GBM, we discretize time into small steps and use the analytical solution for each step. For a time interval $[0, T]$ divided into N equal steps:

Algorithm 1 Geometric Brownian Motion Simulation

procedure SimulateGBM(S_0, T, N, μ, σ)
  $dt \leftarrow T/N$ ▷ Time step size
  Allocate arrays $t[0..N-1]$ and $S[0..N-1]$
  $t[0] \leftarrow 0$
  $S[0] \leftarrow S_0$
  Generate $Z_1, Z_2, \ldots, Z_{N-1} \sim \mathcal{N}(0, 1)$ ▷ Independent standard normal variables
  for $i \leftarrow 1$ to $N-1$ **do**
    $W[i] \leftarrow \sum_{j=1}^{i} Z_j \cdot \sqrt{dt}$ ▷ Brownian motion through cumulative sum
    $t[i] \leftarrow i \cdot dt$
    $S[i] \leftarrow S_0 \cdot \exp\big(((\mu - \sigma^2/2) \cdot t[i]) + (\sigma \cdot W[i])\big)$
  end for
  return t, S
end procedure

3.2 Implementation Details

When implementing the GBM simulation in Python using NumPy, several optimizations are possible:

1. Vectorized operations: Use NumPy's vectorized operations to efficiently compute the entire path at once.

2. Cumulative sum for Brownian motion: The standard Brownian motion can be generated by taking the cumulative sum of normally distributed increments.

3. Logarithmic returns: Calculate the logarithmic returns and then exponentiate to get the price path, which is more numerically stable than direct multiplication.

Here's the core implementation in pseudocode, reflecting the Python code:

Algorithm 2 Optimized GBM Simulation

```
procedure SimulateGBM(S_0, T, N, μ, σ)
    dt ← T/N
    t ← linspace(0, T, N)
    Z ← random_normal(N)                 ▷ Generate N standard normal variables
    W ← cumsum(Z) · √dt                  ▷ Brownian motion
    X ← (μ − 0.5 · σ^2) · t + σ · W      ▷ Log returns
    S ← S_0 · exp(X)                     ▷ Stock prices
    return t, S
end procedure
```

3.3 Visualization

Visualization is crucial for understanding the behavior of simulated GBM paths. The implementation includes a function to plot the simulated stock price path:

Algorithm 3 GBM Visualization

```
procedure PlotSimulation(t, S, save_path)
    Create a new figure with dimensions (10, 6)
    Plot t vs S as a line graph
    Add x-axis label "Time t"
    Add y-axis label "Stock Price S(t)"
    Add title "Geometric Brownian Motion"
    if save_path is provided then
        Save figure to save_path
    end if
    Display figure
end procedure
```

4 Applications and Extensions

4.1 Financial Modeling

GBM is extensively used in financial applications, including:

1. **Option pricing**: The Black-Scholes model assumes that stock prices follow a GBM, leading to closed-form formulas for option prices (Black and Scholes, 1973).
2. **Value at Risk (VaR)**: GBM can be used to estimate potential losses in investment portfolios.
3. **Monte Carlo simulations**: Multiple GBM paths can be generated to evaluate complex financial derivatives.

4.2 Limitations and Extensions

While GBM is widely used, it has several limitations:

1. Constant volatility assumption: Real market volatility tends to vary over time.
2. No jumps: Asset prices sometimes experience sudden jumps, which GBM cannot capture.
3. Log-returns are not exactly normal: Empirical distributions often have fatter tails.

Extensions to address these limitations include:

1. **Stochastic volatility models**: Models like Heston's model allow volatility to follow its own stochastic process (Heston, 1993).
2. **Jump-diffusion models**: Adding a Poisson process to model discontinuous jumps (Merton, 1976).
3. **Regime-switching models**: Allowing parameters to change based on the economic regime.

5 Conclusion

Geometric Brownian Motion provides a mathematically tractable framework for modeling stock prices and other financial assets. Its analytical properties make it amenable to both theoretical analysis and efficient numerical simulation. The implementation presented here allows for flexible experimentation with different parameters and can serve as a foundation for more complex financial modeling tasks.

While GBM has limitations when compared to real market behavior, it remains a cornerstone of quantitative finance due to its simplicity and mathematical elegance. Understanding GBM is essential for anyone working in financial modeling, risk management, or derivative pricing.

References

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