

Exercise of Supervised Learning: Advanced Risk Minimization Part 1

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Exercise 1: Risk Minimizers for Generalized L2-Loss

Consider the regression learning setting, i.e., $\mathcal{Y} = \mathbb{R}$, and assume that your loss function of interest is $L(y, f(\mathbf{x})) = (m(y) - m(f(\mathbf{x})))^2$, where: $m : \mathbb{R} \rightarrow \mathbb{R}$ is a continuous strictly monotone function.

Disclaimer: In the following we always assume that $\text{Var}(m(Y))$ exists.

(a) Consider the hypothesis space of a constant models

$\mathcal{H} = \{f : \mathcal{X} \rightarrow \mathbb{R} \mid f(\mathbf{x}) = \theta \ \forall \mathbf{x} \in \mathcal{X}\}$, where \mathcal{X} is the feature space. Show that

$$\hat{f}(\mathbf{x}) = m^{-1} \left(\frac{1}{n} \sum_{i=1}^n m(y^{(i)}) \right)$$

is the optimal constant model for the loss function above, where m^{-1} is the inverse function of m .

Solution to Question (a)

1. f is a **constant model**: $f(\mathbf{x}) = \theta$ for all \mathbf{x} .
2. The empirical risk can be formulated as:

$$\mathcal{R}_{\text{emp}}(f) = \sum_{i=1}^n \left(m(y^{(i)}) - m(f(\mathbf{x}^{(i)})) \right)^2 = \sum_{i=1}^n \left(m(y^{(i)}) - m(\theta) \right)^2.$$

3. $\mathcal{R}_{\text{emp}}(f)$ is **strictly convex** (because MSE loss and m is strictly monotone). So the minimum is unique, and can be computed by solving $\partial \mathcal{R}_{\text{emp}}(f) / \partial \theta = \mathbf{0}$.

Solution to Question (a): Continued

Goal: Compute the optimal θ by solving $\partial \mathcal{R}_{\text{emp}}(f)/\partial \theta = \mathbf{0}$.

1. Compute the derivative:

$$\frac{\partial \mathcal{R}_{\text{emp}}(f)}{\partial \theta} = 2 \sum_{i=1}^n (m(y^{(i)}) - m(\theta)) \cdot \frac{\partial m(\theta)}{\partial \theta} = 0$$

2. Using the fact that $\frac{\partial m(\theta)}{\partial \theta}$ is constant for all i , we obtain:

$$\sum_{i=1}^n (m(y^{(i)}) - m(\theta)) = 0$$

$$\Rightarrow \sum_{i=1}^n m(y^{(i)}) = \sum_{i=1}^n m(\theta)$$

$$\Rightarrow m(\theta) = \frac{1}{n} \sum_{i=1}^n m(y^{(i)})$$

$$\Rightarrow \theta^* = m^{-1} \left(\frac{1}{n} \sum_{i=1}^n m(y^{(i)}) \right) \quad \triangleleft$$

Question (b)

(b) Verify that the risk of the optimal constant model is $\mathcal{R}_L(\hat{f}) = (1 + \frac{1}{n})\text{Var}(m(y))$.

Recall that the risk of \hat{f} is defined as

$$\mathcal{R}_L(\hat{f}) = \mathbb{E}_{xy}[L(y, \hat{f}(\mathbf{x}))]$$

Solution to Question (b)

$$\begin{aligned}\mathcal{R}_L(\hat{f}) &= \mathbb{E}_{xy}[L(y, \hat{f}(\mathbf{x}))] \\ &= \mathbb{E}_{xy}[(m(y) - m(\hat{f}(\mathbf{x})))^2] \\ &= \mathbb{E}_{xy} \left[\left(m(y) - \frac{1}{n} \sum_{i=1}^n m(y^{(i)}) \right)^2 \right] \\ &= \mathbb{E}_{xy}[m(y)^2] - 2 \cdot \mathbb{E}_{xy} \left[m(y) \cdot \frac{1}{n} \sum_{i=1}^n m(y^{(i)}) \right] + \mathbb{E}_{xy} \left[\left(\frac{1}{n} \sum_{i=1}^n m(y^{(i)}) \right) \left(\frac{1}{n} \sum_{i=1}^n m(y^{(i)}) \right) \right]\end{aligned}$$

Solution to Question (b): Continued

Now take a look at the second term: $-2 \cdot \mathbb{E}_{xy} \left[m(y) \cdot \frac{1}{n} \sum_{i=1}^n m(y^{(i)}) \right]$.

Because $y, y^{(1)}, \dots, y^{(n)}$ are i.i.d. with $\mathbb{E}_{xy}[m(y^{(i)})] = \mathbb{E}_{xy}[m(y)]$, we have

$$\begin{aligned} \mathbb{E}_{xy} \left[m(y) \cdot \frac{1}{n} \sum_{i=1}^n m(y^{(i)}) \right] &= \frac{1}{n} \cdot \mathbb{E}_{xy} \left[m(y) \sum_{i=1}^n m(y^{(i)}) \right] \\ &= \frac{1}{n} \cdot \mathbb{E}_{xy}[m(y)] \mathbb{E}_{xy} \left[\sum_{i=1}^n m(y^{(i)}) \right] \\ &= \frac{1}{n} \cdot \mathbb{E}_{xy}[m(y)] \cdot n \cdot \mathbb{E}_{xy}[m(y)] \\ &= \mathbb{E}_{xy}[m(y)]^2. \end{aligned}$$

Solution to Question (b): Continued

Now take a look at the third term: $\mathbb{E}_{xy} \left[\left(\frac{1}{n} \sum_{i=1}^n m(y^{(i)}) \right) \left(\frac{1}{n} \sum_{i=1}^n m(y^{(i)}) \right) \right]$.

Similarly, we have

$$\begin{aligned} & \mathbb{E}_{xy} \left[\left(\frac{1}{n} \sum_{i=1}^n m(y^{(i)}) \right) \left(\frac{1}{n} \sum_{i=1}^n m(y^{(i)}) \right) \right] \\ &= \frac{1}{n^2} \left(\sum_{i=1}^n \mathbb{E}_{xy}[m(y^{(i)})^2] + \sum_{i=1}^n \sum_{j \neq i} \mathbb{E}_{xy}[m(y^{(i)})m(y^{(j)})] \right) \\ &= \frac{1}{n^2} \left(\sum_{i=1}^n \mathbb{E}_{xy}[m(y^{(i)})^2] + \sum_{i=1}^n \sum_{j \neq i} \mathbb{E}_{xy}[m(y^{(i)})] \cdot \mathbb{E}_{xy}[m(y^{(j)})] \right) \quad \triangleright \text{(The square is within E due to dependency of (i) and (j))} \\ &= \frac{1}{n^2} (n\mathbb{E}_{xy}[m(y)^2] + n(n-1)\mathbb{E}_{xy}[m(y)]^2) \quad \triangleright \\ &= \frac{1}{n} \mathbb{E}_{xy}[m(y)^2] + \left(1 - \frac{1}{n}\right) \mathbb{E}_{xy}[m(y)]^2 \end{aligned}$$

Solution to Question (b): Continued

Combining the results so far, we get

$$\begin{aligned}\mathcal{R}_L(\hat{f}) &= \mathbb{E}_{xy}[m(y)^2] - 2 \cdot \mathbb{E}_{xy} \left[m(y) \cdot \frac{1}{n} \sum_{i=1}^n m(y^{(i)}) \right] + \mathbb{E}_{xy} \left[\left(\frac{1}{n} \sum_{i=1}^n m(y^{(i)}) \right) \left(\frac{1}{n} \sum_{i=1}^n m(y^{(i)}) \right) \right] \\&= \mathbb{E}_{xy}[m(y)^2] - 2\mathbb{E}_{xy}[m(y)]^2 + \frac{1}{n}\mathbb{E}_{xy}[m(y)^2] + \left(1 - \frac{1}{n}\right)\mathbb{E}_{xy}[m(y)]^2 \\&= \left(1 + \frac{1}{n}\right) (\mathbb{E}_{xy}[m(y)^2] - \mathbb{E}_{xy}[m(y)]^2) \\&= \left(1 + \frac{1}{n}\right) \text{Var}(m(y)). \quad \triangleleft\end{aligned}$$

Question (c)

Derive that the risk minimizer f^* is given by $f^*(\mathbf{x}) = m^{-1}(\mathbb{E}_{y|\mathbf{x}}[m(y)|\mathbf{x}])$.

Hints:

- ▶ Consider unstricted hypothesis space $\mathcal{H} = \{f : \mathcal{X} \rightarrow \mathbb{R}\}$.
- ▶ Since \mathcal{H} is unrestricted, for each \mathbf{x} , we can predict any value $c \in \mathbb{R}$ we want. \rightsquigarrow Point-wise prediction.

Solution to Question (c)

By the law of total expectation,

$$\begin{aligned}\mathcal{R}_L(f) &= \mathbb{E}_{xy}[L(y, f(\mathbf{x}))] \\ &= \mathbb{E}_{\mathbf{x}}[\mathbb{E}_{y|\mathbf{x}}[L(y, f(\mathbf{x})) \mid \mathbf{x}]] \\ &= \mathbb{E}_{\mathbf{x}}[\mathbb{E}_{y|\mathbf{x}}[(m(y) - m(f(\mathbf{x})))^2 \mid \mathbf{x}]].\end{aligned}$$

Since we consider a point-wise prediction, we can omit the $\mathbb{E}_{\mathbf{x}}$, and we focus on computing $f^*(\mathbf{x}) = c$ given a **fixed** \mathbf{x} . In other words, we solve the optimal c for each \mathbf{x} separately.

To solve $f^*(\mathbf{x}) = \arg \min_c \mathbb{E}_{y|\mathbf{x}}[(m(y) - m(f(\mathbf{x})))^2 \mid \mathbf{x}]$, we adopt the same way as the solution of Question (a), obtaining

$$f^*(\mathbf{x}) = m^{-1}(\mathbb{E}_{y|\mathbf{x}}[m(y) \mid \mathbf{x}]).$$

Question (d)

(d): What is the optimal **constant** model in terms of the (theoretical) risk for the loss above and what is the risk?

Note: in Question (c), we allow f outputs different values c for different \mathbf{x} . In Question (d), we aim to search an optimal $\bar{f}(\mathbf{x}) = c$ for all \mathbf{x} .

Solution to Question (d)

The (theoretical) risk for a constant model $\bar{f}(\mathbf{x}) = c$ is:

$$\begin{aligned}\mathcal{R}_L(\bar{f}) &= \mathbb{E}_{\mathbf{x}} \left[\mathbb{E}_{y|\mathbf{x}} \left[(m(y) - m(\bar{f}(\mathbf{x})))^2 \right] \right] \\&= \int_y \int_{\mathbf{x}} (m(y) - m(\bar{f}(\mathbf{x})))^2 p(\mathbf{x}, y) d\mathbf{x} dy \\&= \int_y \int_{\mathbf{x}} (m(y) - m(c))^2 p(\mathbf{x}, y) d\mathbf{x} dy \\&= \int_y (m(y) - m(c))^2 p(y) dy \quad \triangleright \\&= \mathbb{E}_y \left[(m(y) - m(c))^2 \right]\end{aligned}$$

Therefore, the optimal constant model is

$$\bar{f}(\mathbf{x}) = c = m^{-1}(\mathbb{E}_y[m(y)])$$

Solution to Question (d): Continued

The risk given $\bar{f}(\mathbf{x}) = c = m^{-1}(\mathbb{E}_y[m(y)])$ is:

$$\mathcal{R}_L(\bar{f}) = \mathbb{E}_{xy}[(m(y) - m(\bar{f}(\mathbf{x})))^2] = \mathbb{E}_y[(m(y) - \mathbb{E}_y[m(y)])^2] = \text{Var}(m(y)) \quad \triangleleft$$

Question (e)

(e): Recall the decomposition of the Bayes regret into the estimation and the approximation error. Show that the former is $\frac{1}{n} \text{Var}(m(y))$, while the latter is $\text{Var}(\mathbb{E}_{y|\mathbf{x}}[m(y) | \mathbf{x}])$ for the optimal constant model $\hat{f}(\mathbf{x})$ if the hypothesis space \mathcal{H} consists of the constant models.

Solution to Question (e)

- Recall from Question (a) that $\hat{f}(\mathbf{x}) = m^{-1} \left(\frac{1}{n} \sum_{i=1}^n m(y^{(i)}) \right)$ and $\mathcal{R}_L(\hat{f}) = \left(1 + \frac{1}{n}\right) \text{Var}(m(y))$.
- Recall from Question (d) that $\bar{f}(\mathbf{x}) = \arg \min_{f \in \mathcal{H}} \mathcal{R}_L(f)$ and $\mathcal{R}_L(\bar{f}) = \text{Var}(m(y))$.
- Recall from (c) that the point-wise risk minimizer $f^*(\mathbf{x}) = m^{-1}(\mathbb{E}_{y|\mathbf{x}}[m(y) | \mathbf{x}]) = \arg \min_f \mathcal{R}_L(f)$ for an **unrestricted** function space.
- Remember to differentiate between these risk minimizers.
- The Bayes regret can be decomposed as:

$$\mathcal{R}_L(\hat{f}) - \mathcal{R}_L^* = \underbrace{\left[\mathcal{R}_L(\hat{f}) - \inf_{f \in \mathcal{H}} \mathcal{R}_L(f) \right]}_{\text{estimation error}} + \underbrace{\left[\inf_{f \in \mathcal{H}} \mathcal{R}_L(f) - \mathcal{R}_L^* \right]}_{\text{approximation error}}$$

Solution to Question (e): Continued

The estimation error is:

$$\begin{aligned}\mathcal{R}_L(\hat{f}) - \inf_{f \in \mathcal{H}} \mathcal{R}_L(f) &= \mathcal{R}_L(\hat{f}) - \mathcal{R}_L(\bar{f}) \\ &= \left(1 + \frac{1}{n}\right) \text{Var}(m(y)) - \text{Var}(m(y)) \\ &= \frac{1}{n} \text{Var}(m(y)).\end{aligned}$$

Solution to Question (e): Continued

The approximation error is:

$$\begin{aligned}\inf_{f \in \mathcal{H}} \mathcal{R}_L(f) - \mathcal{R}_L^* &= \mathcal{R}_L(\bar{f}) - \mathcal{R}_L(f^*) \\&= \text{Var}(m(y)) - \mathbb{E}_{\mathbf{x}} \left[\mathbb{E}_{y|\mathbf{x}}[(m(y) - m(f^*(\mathbf{x})))^2 | \mathbf{x}] \right] \quad (\text{plug in (d)}) \\&= \text{Var}(m(y)) - \mathbb{E}_{\mathbf{x}} \left[\mathbb{E}_{y|\mathbf{x}}[(m(y) - m(m^{-1}(\mathbb{E}_{y|\mathbf{x}}[m(y) | \mathbf{x}]))^2 | \mathbf{x}] \right] \quad (\text{plug in (c)}) \\&= \text{Var}(m(y)) - \mathbb{E}_{\mathbf{x}} \left[\mathbb{E}_{y|\mathbf{x}}[(m(y) - \mathbb{E}_{y|\mathbf{x}}[m(y) | \mathbf{x}])^2 | \mathbf{x}] \right] \\&= \text{Var}(m(y)) - \mathbb{E}_{\mathbf{x}} [\text{Var}(m(y) | \mathbf{x})] \\&= \text{Var}(\mathbb{E}_{y|\mathbf{x}}[m(y) | \mathbf{x}])\end{aligned}$$

The last step holds because of the law of total variation:

$$\text{Var}(Y) = \mathbb{E}_X[\text{Var}(Y | X)] + \text{Var}[\mathbb{E}_{Y | X}[Y | X]]$$