

# **Exercise of Supervised Learning: SVM Part 1**

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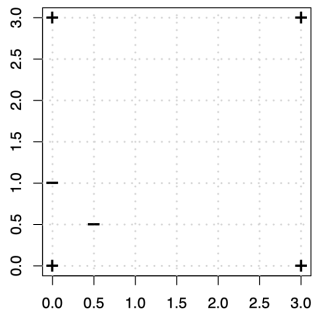
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# Exercise 1: Soft Margin Classifier

The primal optimization problem for the two-class soft margin SVM classification is given by

$$\begin{aligned} \min_{\boldsymbol{\theta}, \theta_0, \zeta^{(i)}} \quad & \frac{1}{2} \|\boldsymbol{\theta}\|^2 + \sum_{i=1}^n \zeta^{(i)} \\ \text{s.t.:} \quad & y^{(i)}(\boldsymbol{\theta}^T \mathbf{x}^{(i)} + \theta_0) \geq 1 - \zeta^{(i)}, \\ & \zeta^{(i)} \geq 0, \quad \forall i = 1, \dots, n. \end{aligned}$$

(a) Add the decision boundary to the figure for  $\hat{\boldsymbol{\theta}} = (1, 1)^T$ ,  $\hat{\theta}_0 = -2$ . (NB: This is the approximate optimum for  $C = 10$ ).



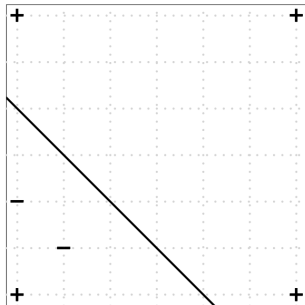
# Solution to Exercise 1 (a)

The hyperplane is given by:

$$\theta_1 x_1^{(i)} + \theta_2 x_2^{(i)} + \theta_0 = 0$$

Plugging in the values for the  $\theta$ s and solving for  $x_2$ , we get the decision boundary:

$$x_2 = -x_1 + 2$$



**Draw this figure on whiteboard.**

## Exercise 1 (b)

(b) Identify the points on the margin hyperplanes and compute the values of their slack variable  $\zeta^{(i)}$ .

# Solution to Exercise 1 (b)

To determine which points are support vectors, we will use the constraint:

$$y^{(i)}(\mathbf{x}^{(i)}\hat{\boldsymbol{\theta}} + \hat{\theta}_0) \geq 1 - \zeta^{(i)}$$

- ▶  $(0, 0)$ :  $1(0 + 0 - 2) = -2 \geq 1 - \zeta^{(1)} \rightarrow \zeta^{(1)} \geq 3$ ,  $\rightsquigarrow$  **Not** on the margin, slack variable  $\zeta^{(i)} = 3$ .
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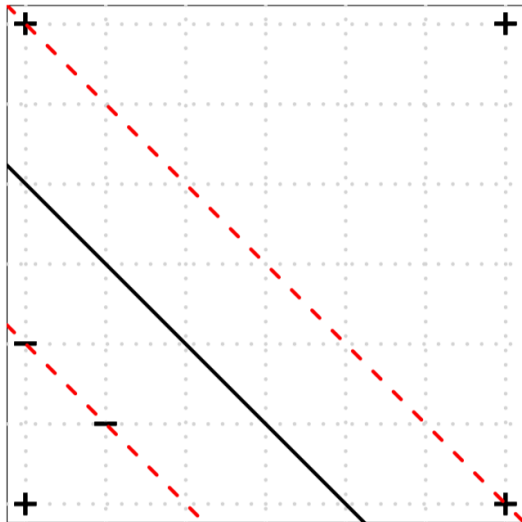
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## Solution to Exercise 1 (b): Continued



## Exercise 1 (c)

(c) Compute the Euclidean distance of the non-margin-violating support vector(s) to the decision boundary.

## Solution to Exercise 1 (c)

We can use  $\mathbf{x}^{(i)} = (0.5, 0.5)^T$ :

$$d(f, \mathbf{x}^{(i)}) = \frac{y^{(i)} f(\mathbf{x}^{(i)})}{\|\boldsymbol{\theta}\|_2} = \frac{-1(0.5 + 0.5 - 2)}{\sqrt{2}} = \frac{1}{\sqrt{2}}$$

The distance is the same for all non-margin-violating support vectors.

## Exercise 1 (d)

(d) What needs to be changed in the plot such that a hard margin SVM results into the same decision boundary?

# Solution to Exercise 1 (d)

Some alternatives are:

- ▶ Convert the  $(0, 0)^T$  into a negative class.
- ▶ Move the  $(0, 0)^T$  to  $(2, 2)^T$ .
- ▶ Delete  $(0, 0)^T$ .



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## Exercise 2: Optimization

Write your own stochastic subgradient descent routine to solve the soft-margin SVM in the primal formulation.

*Hints:*

- ▶ Use the regularized-empirical-risk-minimization formulation, i.e., an optimization criterion without constraints.
- ▶ No kernels, just a linear SVM.
- ▶ Compare your implementation with an existing implementation (e.g. `kernallab` in R. Are your results similar? Note that you might have to switch off the automatic data scaling in the already existing implementation.

**Solution: show the standard solution.**