

Exercise of Supervised Learning: Curse of Dimensionality

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Exercise 1

Consider a random vector $X = (X_1, \dots, X_p)^T \sim \mathcal{N}(0, I)$, i.e., a multivariate normally distributed vector with mean vector zero and covariance matrix being the identity matrix of dimension $p \times p$. In this case, the coordinates X_1, \dots, X_p are i.i.d. each with distribution $\mathcal{N}(0, 1)$.

(a) Show that $\mathbb{E}[\|X\|_2^2] = p$ and $\text{Var}(\|X\|_2^2) = 2p$, where $\|\cdot\|_2$ is the Euclidean norm.

Hint: $\mathbb{E}_{Y \sim \mathcal{N}(0,1)}(Y^4) = 3$.

Solution to Exercise 1 (a)

Note that

$$||X||_2^2 = \sum_{i=1}^p X_i^2.$$

Then,

$$\mathbb{E}[||X||_2^2] = \mathbb{E}\left[\sum_{i=1}^p X_i^2\right]$$

Solution to Exercise 1 (a)

Note that

$$\|X\|_2^2 = \sum_{i=1}^p X_i^2.$$

Then,

$$\begin{aligned}\mathbb{E}[\|X\|_2^2] &= \mathbb{E}\left[\sum_{i=1}^p X_i^2\right] \\ &= \sum_{i=1}^p \mathbb{E}[X_i^2]\end{aligned}$$

Solution to Exercise 1 (a)

Note that

$$||X||_2^2 = \sum_{i=1}^p X_i^2.$$

Then,

$$\begin{aligned}\mathbb{E}[||X||_2^2] &= \mathbb{E}\left[\sum_{i=1}^p X_i^2\right] \\ &= \sum_{i=1}^p \mathbb{E}[X_i^2] \\ &= \sum_{i=1}^p (\underbrace{\mathbb{E}[X_i]^2}_{=0} + \text{Var}(X_i))\end{aligned}$$

Solution to Exercise 1 (a)

Note that

$$||X||_2^2 = \sum_{i=1}^p X_i^2.$$

Then,

$$\begin{aligned}\mathbb{E}[||X||_2^2] &= \mathbb{E}\left[\sum_{i=1}^p X_i^2\right] \\ &= \sum_{i=1}^p \mathbb{E}[X_i^2] \\ &= \sum_{i=1}^p (\underbrace{\mathbb{E}[X_i]^2}_{=0} + \text{Var}(X_i)) \\ &= \sum_{i=1}^p 1\end{aligned}$$

Solution to Exercise 1 (a)

Note that

$$\|X\|_2^2 = \sum_{i=1}^p X_i^2.$$

Then,

$$\begin{aligned}\mathbb{E}[\|X\|_2^2] &= \mathbb{E}\left[\sum_{i=1}^p X_i^2\right] \\&= \sum_{i=1}^p \mathbb{E}[X_i^2] \\&= \sum_{i=1}^p (\underbrace{\mathbb{E}[X_i]^2}_{=0} + \text{Var}(X_i)) \\&= \sum_{i=1}^p 1 \\&= p.\end{aligned}$$

Solution to Question 1 (a): Continued

$$\begin{aligned}\text{Var}(\|X\|_2^2) &= \text{Var}\left(\sum_{i=1}^p X_i^2\right) \\&= \sum_{i=1}^p \text{Var}(X_i^2) \\&= \sum_{i=1}^p \left(\underbrace{\mathbb{E}[X_i^4]}_{=3} - \underbrace{\mathbb{E}[X_i^2]^2}_{=1}\right) \quad \triangleright \\&= \sum_{i=1}^p (3 - 1) \\&= 2p.\end{aligned}$$

Exercise 1 (b)

(b) Use (a) to infer that $|\mathbb{E}[\|X\|_2 - \sqrt{p}]| \leq \frac{1}{\sqrt{p}}$ by using the following steps:

(i) Write $\|X\|_2 - \sqrt{p} = \underbrace{\frac{\|X\|_2 - p}{2\sqrt{p}}}_{:= (1)} - \underbrace{\frac{(\|X\|_2^2 - p)^2}{2\sqrt{p}(\|X\|_2 + \sqrt{p})^2}}_{:= (2)}.$

(ii) Compute $\mathbb{E}[(1)]$.

(iii) Note that $0 \leq \mathbb{E}[(2)] \leq \frac{\text{Var}(\|X\|_2^2)}{2p^{3/2}}$ holds due to $\|X\|_2 \geq 0$.

(iv) Put (i)- (iii) together.

Solution to Exercise 1 (b)

Step (i): Write $\|X\|_2 - \sqrt{p} = \underbrace{\frac{\|X\|_2 - p}{2\sqrt{p}}}_{:= (1)} - \underbrace{\frac{(\|X\|_2^2 - p)^2}{2\sqrt{p}(\|X\|_2 + \sqrt{p})^2}}_{:= (2)}.$ \triangleleft

Step (ii):

$$\begin{aligned}\mathbb{E}[(1)] &= \mathbb{E}\left[\frac{\|X\|_2^2 - p}{2\sqrt{p}}\right] \\ &= \frac{1}{2\sqrt{p}}\left(\underbrace{\mathbb{E}[\|X\|_2^2]}_{=p \text{ (from Question(a))}} - p\right) \\ &= 0.\end{aligned}$$

Solution to Exercise 1 (b)

Step (i): Write $\|X\|_2 - \sqrt{p} = \underbrace{\frac{\|X\|_2 - p}{2\sqrt{p}}}_{:= (1)} - \underbrace{\frac{(\|X\|_2^2 - p)^2}{2\sqrt{p}(\|X\|_2 + \sqrt{p})^2}}_{:= (2)}.$ \triangleleft

Step (ii):

$$\begin{aligned}\mathbb{E}[(1)] &= \mathbb{E}\left[\frac{\|X\|_2^2 - p}{2\sqrt{p}}\right] \\ &= \frac{1}{2\sqrt{p}}\left(\underbrace{\mathbb{E}[\|X\|_2^2]}_{=p \text{ (from Question(a))}} - p\right) \\ &= 0.\end{aligned}$$

Solution to Exercise 1(b): Continued

Step (iii): Prove that $0 \leq \mathbb{E}[(2)] \leq \frac{\text{Var}(\|X\|_2^2)}{2p^{3/2}}$ holds due to $\|X\|_2 \geq 0$.

$$\mathbb{E}[(2)] = \mathbb{E} \left[\frac{(\|X\|_2^2 - p)^2}{2\sqrt{p}(\|X\|_2 + \sqrt{p})^2} \right] \geq 0, \quad \text{since all the terms are non-negative.}$$

Besides, since $\|X\|_2 \geq 0$, it follows that

$$\begin{aligned} (2) &\leq \frac{(\|X\|_2^2 - p)^2}{2p^{3/2}} \\ \Rightarrow \mathbb{E}[(2)] &\leq \mathbb{E} \left[\frac{(\|X\|_2^2 - p)^2}{2p^{3/2}} \right] = \frac{1}{2p^{3/2}} \cdot \mathbb{E} [(\|X\|_2^2 - \mathbb{E}[\|X\|_2^2])^2] = \frac{\text{Var}(\|X\|_2^2)}{2p^{3/2}} \\ &= \frac{2p}{2p^{3/2}} = \frac{1}{\sqrt{p}}, \end{aligned}$$

where we utilize the lemma from (a) that $\mathbb{E}[\|X\|_2^2] = p$ and $\text{Var}(\|X\|_2^2) = 2p$.

Solution to Exercise 1(b): Continued

Step (iii): Prove that $0 \leq \mathbb{E}[(2)] \leq \frac{\text{Var}(\|X\|_2^2)}{2p^{3/2}}$ holds due to $\|X\|_2 \geq 0$.

$$\mathbb{E}[(2)] = \mathbb{E} \left[\frac{(\|X\|_2^2 - p)^2}{2\sqrt{p}(\|X\|_2 + \sqrt{p})^2} \right] \geq 0, \quad \text{since all the terms are non-negative.}$$

Besides, since $\|X\|_2 \geq 0$, it follows that

$$\begin{aligned} (2) &\leq \frac{(\|X\|_2^2 - p)^2}{2p^{3/2}} \\ \Rightarrow \mathbb{E}[(2)] &\leq \mathbb{E} \left[\frac{(\|X\|_2^2 - p)^2}{2p^{3/2}} \right] = \frac{1}{2p^{3/2}} \cdot \mathbb{E} [(\|X\|_2^2 - \mathbb{E}[\|X\|_2^2])^2] = \frac{\text{Var}(\|X\|_2^2)}{2p^{3/2}} \\ &= \frac{2p}{2p^{3/2}} = \frac{1}{\sqrt{p}}, \end{aligned}$$

where we utilize the lemma from (a) that $\mathbb{E}[\|X\|_2^2] = p$ and $\text{Var}(\|X\|_2^2) = 2p$.

Solution to Exercise 1(b): Continued

Step (iii): Prove that $0 \leq \mathbb{E}[(2)] \leq \frac{\text{Var}(\|X\|_2^2)}{2p^{3/2}}$ holds due to $\|X\|_2 \geq 0$.

$$\mathbb{E}[(2)] = \mathbb{E} \left[\frac{(\|X\|_2^2 - p)^2}{2\sqrt{p}(\|X\|_2 + \sqrt{p})^2} \right] \geq 0, \quad \text{since all the terms are non-negative.}$$

Besides, since $\|X\|_2 \geq 0$, it follows that

$$\begin{aligned} (2) &\leq \frac{(\|X\|_2^2 - p)^2}{2p^{3/2}} \\ \Rightarrow \mathbb{E}[(2)] &\leq \mathbb{E} \left[\frac{(\|X\|_2^2 - p)^2}{2p^{3/2}} \right] = \frac{1}{2p^{3/2}} \cdot \mathbb{E} [(\|X\|_2^2 - \mathbb{E}[\|X\|_2^2])^2] = \frac{\text{Var}(\|X\|_2^2)}{2p^{3/2}} \\ &= \frac{2p}{2p^{3/2}} = \frac{1}{\sqrt{p}}, \end{aligned}$$

where we utilize the lemma from (a) that $\mathbb{E}[\|X\|_2^2] = p$ and $\text{Var}(\|X\|_2^2) = 2p$.

Solution to Exercise 1(b): Continued

Step (iv): Putting everything together:

$$|\mathbb{E}[||X||_2 - \sqrt{p}]| = |\underbrace{\mathbb{E}[(1)]}_{=0} - \underbrace{\mathbb{E}[(2)]}_{\geq 0}| = \mathbb{E}[(2)] \leq \frac{1}{\sqrt{p}}.$$

Exercise 1 (c)

(c) Use (b) to infer that $\text{Var}(\|X\|_2) \leq 2$ by using the following steps:

- (i) Write $\text{Var}(\|X\|_2) = \text{Var}(\|X\|_2 - \sqrt{p})$.
- (ii) For any random variable Y it holds that $\text{Var}(Y) \leq \mathbb{E}[Y^2]$.
- (iii) If you encounter the term $E[\|X\|_2]$ write it as $\mathbb{E}[\underbrace{\|X\|_2 - \sqrt{p}}_{= (*)} + \sqrt{p}]$ and use (b) for $(*)$.

Solution to Exercise 1 (c)

Step (i): Write $\text{Var}(\|X\|_2) = \text{Var}(\|X\|_2 - \sqrt{p})$.

It holds because **variance does not change by constant shifts**.

Step (ii): For any random variable Y it holds that $\text{Var}(Y) \leq \mathbb{E}[Y^2]$.

It holds because $\text{Var}(Y) + \mathbb{E}[Y]^2 = \mathbb{E}[Y^2]$ and $\mathbb{E}[Y]^2 \geq 0$. Later we will use this inequality.

Solution to Exercise 1 (c): Continued

$$\begin{aligned}\text{Var}(\|X\|_2) &= \text{Var}(\|X\|_2 - \sqrt{p}) && \text{Step (i)} \\ &\leq \mathbb{E}[(\|X\|_2 - \sqrt{p})^2] && \text{Step (ii)} \\ &= \mathbb{E}[\|X\|_2^2 - 2\sqrt{p}\|X\|_2 + p] \\ &= \underbrace{\mathbb{E}[\|X\|_2^2]}_{=p} - 2\sqrt{p} \cdot \mathbb{E}[\|X\|_2] + p \\ &= 2p - 2\sqrt{p} \cdot \mathbb{E}[\|X\|_2] \\ &= 2p - 2\sqrt{p} \cdot \mathbb{E}[\|X\|_2 - \sqrt{p} + \sqrt{p}] && \text{Step (iv)} \\ &= 2p - 2p - 2\sqrt{p} \cdot \mathbb{E}[\|X\|_2 - \sqrt{p}] \\ &= -2\sqrt{p} \cdot \underbrace{\mathbb{E}[\|X\|_2 - \sqrt{p}]}_{\leq \frac{1}{\sqrt{p}} \text{ (from (b))}} \\ &\leq 2\sqrt{p} \cdot \frac{1}{\sqrt{p}} = 2.\end{aligned}$$

Question 1 (d)

Now let $X' = (X'_1, \dots, X'_p)^T \sim \mathcal{N}(0, I)$ be another multivariate normally distributed vector with mean vector zero and covariance matrix being the identity matrix of dimension $p \times p$. Further, assume that X and X' are independent, so that $Z := \frac{X - X'}{\sqrt{2}} \sim \mathcal{N}(0, I)$. Conclude from the previous that

$$\left| \mathbb{E} \left[\|X - X'\|_2 - \sqrt{2p} \right] \right| \leq \frac{2}{p} \quad \text{and} \quad \text{Var}(\|X - X'\|_2) \leq 4.$$

Solution to Exercise 1 (d)

We first investigate Z . Since $Z = \frac{X-X'}{\sqrt{2}} \sim \mathcal{N}(0, I)$, it follows from (b) and (c) that

$$|\mathbb{E}[\|Z\|_2 - \sqrt{p}]| \leq \sqrt{\frac{1}{p}}, \quad (1)$$

$$\text{Var}(\|Z\|_2) \leq 2 \quad (2)$$

But the norm of Z

$$\|Z\|_2 = \sqrt{\sum_{i=1}^p \left(\frac{X_i - X'_i}{\sqrt{2}}\right)^2} = \sqrt{\frac{1}{2} \sum_{i=1}^p (X_i - X'_i)^2} = \sqrt{\frac{1}{2}} \sqrt{\sum_{i=1}^p (X_i - X'_i)^2} = \sqrt{\frac{1}{2}} \|X - X'\|_2. \quad (3)$$

It follows from (1) that

$$\sqrt{2} \cdot |\mathbb{E}[\|Z\|_2 - \sqrt{p}]| \leq \frac{2}{p} \Rightarrow |\mathbb{E}[\underbrace{\sqrt{2} \|Z\|_2}_{\|X-X'\|_2} - \sqrt{2p}]| \leq \sqrt{\frac{2}{p}}.$$

Solution to Exercise 1 (d)

We first investigate Z . Since $Z = \frac{X-X'}{\sqrt{2}} \sim \mathcal{N}(0, I)$, it follows from (b) and (c) that

$$|\mathbb{E}[\|Z\|_2 - \sqrt{p}]| \leq \sqrt{\frac{1}{p}}, \quad (1)$$

$$\text{Var}(\|Z\|_2) \leq 2 \quad (2)$$

But the norm of Z

$$\|Z\|_2 = \sqrt{\sum_{i=1}^p \left(\frac{X_i - X'_i}{\sqrt{2}}\right)^2} = \sqrt{\frac{1}{2} \sum_{i=1}^p (X_i - X'_i)^2} = \sqrt{\frac{1}{2}} \sqrt{\sum_{i=1}^p (X_i - X'_i)^2} = \sqrt{\frac{1}{2}} \|X - X'\|_2. \quad (3)$$

It follows from (1) that

$$\sqrt{2} \cdot |\mathbb{E}[\|Z\|_2 - \sqrt{p}]| \leq \frac{2}{p} \Rightarrow |\mathbb{E}[\underbrace{\sqrt{2} \|Z\|_2}_{\|X-X'\|_2} - \sqrt{2p}]| \leq \sqrt{\frac{2}{p}}.$$

Solution to Exercise 1 (d)

We first investigate Z . Since $Z = \frac{X-X'}{\sqrt{2}} \sim \mathcal{N}(0, I)$, it follows from (b) and (c) that

$$|\mathbb{E}[\|Z\|_2 - \sqrt{p}]| \leq \sqrt{\frac{1}{p}}, \quad (1)$$

$$\text{Var}(\|Z\|_2) \leq 2 \quad (2)$$

But the norm of Z

$$\|Z\|_2 = \sqrt{\sum_{i=1}^p \left(\frac{X_i - X'_i}{\sqrt{2}}\right)^2} = \sqrt{\frac{1}{2} \sum_{i=1}^p (X_i - X'_i)^2} = \sqrt{\frac{1}{2}} \sqrt{\sum_{i=1}^p (X_i - X'_i)^2} = \sqrt{\frac{1}{2}} \|X - X'\|_2. \quad (3)$$

It follows from (1) that

$$\sqrt{2} \cdot |\mathbb{E}[\|Z\|_2 - \sqrt{p}]| \leq \frac{2}{p} \Rightarrow |\mathbb{E}[\underbrace{\sqrt{2} \|Z\|_2}_{\|X - X'\|_2} - \sqrt{2p}]| \leq \sqrt{\frac{2}{p}}.$$

Solution to Exercise 1 (d): Continued

Moreover, (2): $\text{Var}(\|Z\|_2) \leq 2$ implies that

$$\text{Var}(\|Z\|_2) \leq 2$$

$$\Leftrightarrow 2\text{Var}(\|Z\|_2) \leq 4$$

$$\Leftrightarrow \text{Var}(\sqrt{2}\|Z\|_2) \leq 4 \quad (\text{Var}(aY) = a^2\text{Var}(Y) \text{ for any RV } Y)$$

$$\Leftrightarrow \text{Var}(\|X - X'\|_2) \leq 4 \quad \text{Using (3) that } \|Z\|_2 = \sqrt{\frac{1}{2}}\|X - X'\|_2.$$

Exercise 1 (e)

(e) From the cosine rule we can infer that for any $x, x' \in \mathbb{R}^p$ it holds that

$$\langle x, x' \rangle = \frac{1}{2}(\|x\|_2^2 + \|x'\|_2^2 - \|x - x'\|_2^2).$$

Use this to show that $\mathbb{E}[\langle X, X' \rangle] = 0$. Moreover, derive that $\text{Var}(\langle X, X' \rangle) = p$.

Solution to Exercise 1 (e)

Since $\langle x, x' \rangle = \frac{1}{2}(\|x\|_2^2 + \|x'\|_2^2 - \|x - x'\|_2^2)$, we can infer that

$$\begin{aligned}\mathbb{E}[\langle X, X' \rangle] &= \frac{1}{2}(\mathbb{E}[\|X\|_2^2] + \mathbb{E}[\|X'\|_2^2] - \mathbb{E}[\|X - X'\|_2^2]) \\ &= \frac{1}{2} \left(p + p - 2 \cdot \mathbb{E} \left[\underbrace{\frac{1}{2} \|X - X'\|_2^2}_{=\|Z\|_2^2} \right] \right) \\ &= \frac{1}{2}(p + p - 2p) = 0. \quad (\text{From (a) we know that } \mathbb{E}[\|Z\|_2^2] = p)\end{aligned}$$

Solution to Exercise 1 (e): Continued

$$\begin{aligned}\text{Var}(\langle X, X' \rangle) &= \text{Var}\left(\sum_{i=1}^p X_i X'_i\right) \\&= \sum_{i=1}^p \text{Var}(X_i X'_i) \\&= p \text{Var}(X_1 X'_1) \\&= p \cdot (\mathbb{E}[X_1^2 (X'_1)^2] - \mathbb{E}[X_1 (X'_1)]^2) \\&= p \cdot \underbrace{(\mathbb{E}[X_1^2])}_{=1} \cdot \underbrace{\mathbb{E}[(X'_1)^2]}_{=1} - \underbrace{\mathbb{E}[X_1]^2}_{=0} \cdot \underbrace{\mathbb{E}[X'_1]^2}_{=0} \\&= p.\end{aligned}$$

Exercise 1 (f)

(f) For different dimensions p , e.g., $p \in \{1, 2, 4, 8, \dots, 1024\}$, create two sets consisting of 100 i.i.d. random observations from $\mathcal{N}(0, \mathbf{I})$, respectively and

- (i) compute the average Euclidean length of (one of) the sampled sets and compare it to \sqrt{p} ;
- (ii) compute the average Euclidean distances between the sampled sets and compare it to $\sqrt{2p}$;
- (iii) compute the average inner products between the sampled sets;
- (iv) compute in (i)-(iii) also the empirical variances of the respective terms.

Visualize your results in an appropriate manner.

Show the standard solution.