

# **Exercise of Supervised Learning: Regularization Part 2**

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# Exercise 1

For a design matrix  $\mathbf{X} \in \mathbb{R}^{n \times p}$  and the vector of targets  $\mathbf{y} \in \mathbb{R}^n$ , consider Lasso regression, i.e.,

$$\arg \min_{\boldsymbol{\theta} \in \mathbb{R}^p} 0.5 \|\mathbf{X}\boldsymbol{\theta} - \mathbf{y}\|_2^2 + \lambda \|\boldsymbol{\theta}\|_1. \quad \triangleleft$$

where  $\lambda > 0$  is the regularization parameter.

(a) Since there exists no analytical solution to Lasso regression in general, we want to find a procedure similar to gradient descent that should converge to the true solution.

## Exercise 1 (a) (i)

**Question (a) (i):** Explain why  $\mathcal{R}_{\text{reg}}$  is not differentiable.

**Solution:** Because  $\lambda \|\theta\|_1$  is not differentiable at  $\theta = \mathbf{0}$ .

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**Question (a) (i):** Explain why  $\mathcal{R}_{\text{reg}}$  is not differentiable.

**Solution:** Because  $\lambda \|\boldsymbol{\theta}\|_1$  is not differentiable at  $\boldsymbol{\theta} = \mathbf{0}$ .

## Exercise 1 (a) (ii)

**Question (a) (ii):** Show that  $\mathcal{R}_{\text{reg}}$  is convex. *Hint: The sum of convex function is convex.*

**Solution:**

1.  $0.5\|\mathbf{X}\boldsymbol{\theta} - \mathbf{y}\|_2^2$  is convex, since it is quadratic.
2.  $\lambda\|\boldsymbol{\theta}_1\|_1$  is also convex (since it is a norm).
3. Sum of convex functions is convex. So  $\mathcal{R}_{\text{reg}}$  is convex.

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## Exercise 1 (a) (iii)

**Question (a) (iii):** Find  $\rho_j, z_j \in \mathbb{R}$  for which

$$0.5 \frac{\partial}{\partial \theta_j} \|\mathbf{X}\boldsymbol{\theta} - \mathbf{y}\|_2^2 = -\rho_j + \theta_j z_j. \quad \triangleleft$$

**Solution:**

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**Solution:**

$$\frac{\partial}{\partial \theta_j} \|\mathbf{x}\boldsymbol{\theta} - \mathbf{y}\|_2^2 = \frac{\partial}{\partial \theta_j} 0.5 \sum_{i=1}^n \left( y^{(i)} - \sum_{k=1}^p \mathbf{x}_k^{(i)} \theta_k \right)^2$$

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**Solution:**

$$\begin{aligned} \frac{\partial}{\partial \theta_j} \|\mathbf{X}\boldsymbol{\theta} - \mathbf{y}\|_2^2 &= \frac{\partial}{\partial \theta_j} 0.5 \sum_{i=1}^n \left( y^{(i)} - \sum_{k=1}^p \mathbf{x}_k^{(i)} \theta_k \right)^2 \\ &= - \sum_{i=1}^n \mathbf{x}_j^{(i)} \left( y^{(i)} - \sum_{k=1}^p \mathbf{x}_k^{(i)} \theta_k \right) \end{aligned}$$

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**Solution:**

$$\begin{aligned} \frac{\partial}{\partial \theta_j} \|\mathbf{X}\boldsymbol{\theta} - \mathbf{y}\|_2^2 &= \frac{\partial}{\partial \theta_j} 0.5 \sum_{i=1}^n \left( y^{(i)} - \sum_{k=1}^p \mathbf{x}_k^{(i)} \theta_k \right)^2 \\ &= - \sum_{i=1}^n \mathbf{x}_j^{(i)} \left( y^{(i)} - \sum_{k=1}^p \mathbf{x}_k^{(i)} \theta_k \right) \\ &= - \underbrace{\sum_{i=1}^n \mathbf{x}_j^{(i)} \left( y^{(i)} - \sum_{k \neq j}^p \mathbf{x}_k^{(i)} \theta_k \right)}_{=\rho_j} + \underbrace{\theta_j \sum_{i=1}^n \left( \mathbf{x}_j^{(i)} \right)^2}_{=z_j} \end{aligned}$$

## Exercise 1 (a) (iv)

**Question (a) (iv):** In this situation, we can use the so-called sub-derivative which we denote with  $\partial f$  for a real-valued convex continuous function  $f$ . The subderivative maps a point  $\theta \in \mathbb{R}$  to an interval

► and if  $f$  is differentiable at  $\tilde{\theta} \in \mathbb{R}$ , then  $\partial f(\tilde{\theta}) = \{\frac{d}{d\theta} f(\tilde{\theta})\}$ ,

► and for  $f(\theta) = \lambda|\theta|$  and  $\lambda > 0$ , it holds that  $\partial f(\theta) = \begin{cases} \{-\lambda\} & \text{for } \theta < 0, \\ [-\lambda, \lambda] & \text{for } \theta = 0, \\ \{\lambda\} & \text{for } \theta > 0 \end{cases}$

► and for  $f, g$  real-valued convex functions with  $\partial f(\tilde{\theta}) = [a, b]$ ,  $\partial g(\tilde{\theta}) = [c, d]$ ,

$$\partial(f + g)(\tilde{\theta}) = [a + c, b + d]$$

where  $b \geq a$  and  $d \geq c$ .

With this compute the sub-derivative of  $\mathcal{R}_{\text{reg}, \theta_{\neq j}}$  w.r.t.  $\theta_j$ , i.e.,  $\partial_{\theta_j} \mathcal{R}_{\text{reg}, \theta_{\neq j}}$  where

$\mathcal{R}_{\text{reg}, \theta_{\neq j}} : \mathbb{R} \rightarrow \mathbb{R}_{\geq 0}$ ,  $\theta_j \mapsto \mathcal{R}_{\text{reg}}(\theta_1, \dots, \theta_j, \dots, \theta_p)$  for constants

$\theta_{\neq j} = (\theta_1, \dots, \theta_{j-1}, \theta_{j+1}, \dots, \theta_p)^T$ . *Hint: Use (a) (iii).*

## Exercise 1 (a) (iv): Continued

**Solution:** Recall that  $\mathcal{R}_{\text{reg}} = 0.5\|\mathbf{X}\boldsymbol{\theta} - \mathbf{y}\|_2^2 + \lambda\|\boldsymbol{\theta}\|_1$  and  $\frac{\partial}{\partial\theta_j}0.5\|\mathbf{X}\boldsymbol{\theta} - \mathbf{y}\|_2^2 = -\rho_j + \theta_j z_j$ . Therefore,

$$\partial_{\theta_j}\mathcal{R}_{\text{reg},\boldsymbol{\theta}_{\neq j}} = \begin{cases} \{-\rho_j + \theta_j z_j - \lambda\} & \text{for } \theta_j < 0, \\ [-\rho_j - \lambda, -\rho_j + \lambda] & \text{for } \theta_j = 0; \\ \{-\rho_j + \theta_j z_j + \lambda\} & \text{for } \theta_j > 0. \end{cases}$$

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**Solution:** Recall that  $\mathcal{R}_{\text{reg}} = 0.5\|\mathbf{X}\boldsymbol{\theta} - \mathbf{y}\|_2^2 + \lambda\|\boldsymbol{\theta}\|_1$  and  $\frac{\partial}{\partial\theta_j}0.5\|\mathbf{X}\boldsymbol{\theta} - \mathbf{y}\|_2^2 = -\rho_j + \theta_j z_j$ . Therefore,

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## Exercise 1 (a) (v)

**Question (a) (v):** For a real-valued convex function  $f$ , the global minimum (if it exists) can be characterized in the following way:

A point  $\theta^* \in \mathbb{R}$  is the global minimum of  $f$  if and only if  $0 \in \partial f(\theta^*)$ .

With this show that

$$\theta_j^* \in \arg \min_{\theta_j \in \mathbb{R}} \mathcal{R}_{\text{reg}, \theta_{\neq j}} \Leftrightarrow \theta_j^* = \begin{cases} \frac{\rho_j + \lambda}{z_j} & \text{for } \rho_j < -\lambda \\ 0 & \text{for } -\lambda \leq \rho_j \leq \lambda \\ \frac{\rho_j - \lambda}{z_j} & \text{for } \rho_j > \lambda \end{cases}$$

## Exercise 1 (a) (v): Continued

**Solution:** we need to show that

$$0 \in \partial_{\theta_j} \mathcal{R}_{\text{reg}, \theta_{\neq j}} = \begin{cases} \{-\rho_j + \theta_j z_j - \lambda\} & \text{for } \theta_j < 0, \\ [-\rho_j - \lambda, -\rho_j + \lambda] & \text{for } \theta_j = 0; \\ \{-\rho_j + \theta_j z_j + \lambda\} & \text{for } \theta_j > 0. \end{cases}$$

where  $z_j = \sum_{i=1}^n \left( \mathbf{x}_j^{(i)} \right)^2 \geq 0$ .

- ▶ If  $\theta_j < 0$ , then  $-\rho_j + \theta_j z_j - \lambda = 0$  leads to  $\theta_j^* = \frac{\rho_j + \lambda}{z_j}$ . Since  $\theta_j^* < 0$  and  $z_j \geq 0$ , we have  $\rho_j < -\lambda$ ;
- ▶ If  $\theta_j = 0$ , then  $\theta_j^* = 0$ , and  $-\rho_j - \lambda \leq 0 \leq -\rho_j + \lambda$  leads to  $-\lambda \leq \rho_j \leq \lambda$  ;
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**Solution:** we need to show that

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**Solution:** we need to show that

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## Exercise 1 (a) (v): Continued

Summarize everything up:

$$\theta_j^* \in \arg \min_{\theta_j \in \mathbb{R}} \mathcal{R}_{\text{reg}, \theta_{\neq j}} \Leftrightarrow \theta_j^* = \begin{cases} \frac{\rho_j + \lambda}{z_j} & \text{for } \rho_j < -\lambda \\ 0 & \text{for } -\lambda \leq \rho_j \leq \lambda \\ \frac{\rho_j - \lambda}{z_j} & \text{for } \rho_j > \lambda \end{cases}$$

## Exercise 1 (a) (vi)

**Question (a) (vi):** Plot  $\theta_j^*$  as a function of  $\rho_j$  for  $\rho_j \in [-5, 5]$ ,  $\lambda = 1$ ,  $z_j = 1$ . (This function is called soft thresholding operator).

**Solution:** Show the standard solution.

## Exercise 1 (b)

**Question (b):** For non-singular  $\mathbf{X}^T \mathbf{X}$ , find the matrix  $\mathbf{A} \in \mathbb{R}^{p \times p}$  for which

$$\mathbf{A}^T \mathbf{X}^T \mathbf{X} \mathbf{A} = \mathbf{I}$$

*Hint:  $\mathbf{X}^T \mathbf{X}$  is positive definite.*

**Solution:** Since  $\mathbf{X}^T \mathbf{X}$  is positive definite, there exist an orthogonal matrix  $\mathbf{V}$  and a diagonal matrix  $\mathbf{D}$  with  $D_{ii} > 0$  such that

$$\mathbf{V} \mathbf{D} \mathbf{V}^T = \mathbf{X}^T \mathbf{X}.$$

So

$$\mathbf{A}^T \mathbf{X}^T \mathbf{X} \mathbf{A} = \mathbf{A}^T \mathbf{V} \mathbf{D} \mathbf{V}^T \mathbf{A} = \mathbf{I}$$

To solve the equation for  $\mathbf{A}$ , we substitute the variable  $\mathbf{V}^T \mathbf{A} = \mathbf{P}$ . Then,

$$\mathbf{P}^T \mathbf{D} \mathbf{P} = \mathbf{I}$$

Since  $\mathbf{D}$  is a diagonal matrix, it can be seen that  $\mathbf{P} = \mathbf{D}^{-0.5}$ . Hence  $\mathbf{V}^T \mathbf{A} = \mathbf{P}$ . Because  $\mathbf{V}$  is orthogonal, we have  $\mathbf{A} = \mathbf{V} \mathbf{D}^{-0.5}$ .



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So

$$\mathbf{A}^T \mathbf{X}^T \mathbf{X} \mathbf{A} = \mathbf{A}^T \mathbf{V} \mathbf{D} \mathbf{V}^T \mathbf{A} = \mathbf{I}$$

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## Exercise 1 (c) (i)

(c) For a design matrix with orthonormal columns, i.e.,  $\mathbf{X}^T \mathbf{X} = \mathbf{I}$ , exists an analytical minimizer of the Lasso regression  $\hat{\boldsymbol{\theta}}_{Lasso} = (\hat{\theta}_{Lasso,1}, \dots, \hat{\theta}_{Lasso,p})^T$  that is given by

$$\hat{\theta}_{Lasso,i} = \text{sign}(\hat{\theta}_i) \max \left\{ \left| \hat{\theta}_i - \lambda, 0 \right| \right\}, \quad i = 1, \dots, p,$$

where  $\hat{\boldsymbol{\theta}} = (\hat{\theta}_1, \dots, \hat{\theta}_p)^T = (\mathbf{X}^T \mathbf{X})^{-1} \mathbf{X}^T \mathbf{y}$  is the minimizer of the unregularized empirical risk (w.r.t. the L2 loss).

Under the assumption that  $\mathbf{X}^T \mathbf{X}$  is non-singular, your colleague proposes to project  $\mathbf{X}$  with  $\mathbf{A}$  from (b), i.e., use  $\tilde{\mathbf{A}} = \mathbf{X}\mathbf{A}$  and then apply the analytical solution given here.

(i) Show that this does not generally solve the original Lasso regression.

*Hint: You only need to check under which condition*

$\nabla_{\boldsymbol{\theta}} 0.5 \|\mathbf{X}\boldsymbol{\theta} - \mathbf{y}\|_2^2 = \nabla_{\boldsymbol{\theta}} 0.5 \|\mathbf{X}\mathbf{A}\boldsymbol{\theta} - \mathbf{y}\|_2^2$ . The proof can be finished with a subgradient argument regarding stationarity, which you do not have to do.

## Exercise 1 (c) (i): Continued

**Solution:**

$$\begin{aligned}\nabla_{\theta} 0.5 \|\mathbf{X}\mathbf{A}\theta - \mathbf{y}\|_2^2 &= \mathbf{A}\mathbf{X}^T\mathbf{X}\mathbf{A}\theta - \mathbf{A}^T\mathbf{X}^T\mathbf{y} \\ \nabla_{\theta} 0.5 \|\mathbf{X}\theta - \mathbf{y}\|_2^2 &= \mathbf{X}^T\mathbf{X}\theta - \mathbf{X}^T\mathbf{y}\end{aligned}$$

To let the gradient be equal, it must be satisfied that

$$\mathbf{A} = \mathbf{I}.$$

In other words, if  $\mathbf{A} \neq \mathbf{I}$ , then the analytical solution does not solve the original Lasso regression.

## Exercise 1 (c) (i): Continued

**Solution:**

$$\nabla_{\theta} 0.5 \|\mathbf{X}\mathbf{A}\theta - \mathbf{y}\|_2^2 = \mathbf{A}\mathbf{X}^T\mathbf{X}\mathbf{A}\theta - \mathbf{A}^T\mathbf{X}^T\mathbf{y}$$

$$\nabla_{\theta} 0.5 \|\mathbf{X}\theta - \mathbf{y}\|_2^2 = \mathbf{X}^T\mathbf{X}\theta - \mathbf{X}^T\mathbf{y}$$

To let the gradient be equal, it must be satisfied that

$$\mathbf{A} = \mathbf{I}.$$

In other words, if  $\mathbf{A} \neq \mathbf{I}$ , then the analytical solution does not solve the original Lasso regression.

## Exercise 1 (c) (ii)

**Question (c) (ii):** How could you adapt the penalty term such that the solution to the projected problem is equivalent to the original Lasso regression? In this case, can we still solve for parameters independently?

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## Exercise 1 (d)

(d) You are given the following code to compare the quality of the projected Lasso regression vs. the regular Lasso regression. Complete the missing code of the algorithms and interpret the result.

**Solution: show the standard solution.**