Exercise of Supervised Learning: Curse of Dimensionality

Yawei Li

yawei.li@stat.uni-muenchen.de

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Exercise 1

Consider a random vector $X = (X_1, \dots, X_p)^T \sim \mathcal{N}(0, \boldsymbol{I})$, i.e., a multivariate normally distributed vector with mean vector zero and covariance matrix beging the identity matrix of dimension $p \times p$. In this case, the coordinates $X_1, \dots, X_p$ are i.i.d. each with distribution $\mathcal{N}(0, 1)$.

(a) Show that $\mathbb{E}[||X||_2^2] = p$ and $\mathrm{Var}(||X||_2^2) = 2p$, where $|| \cdot ||_2$ is the Euclidean norm.

Hint: $\mathbb{E}_{Y \sim \mathcal{N}(0,1)}(Y^4) = 3$.

Solution to Exercise 1 (a)

Note that

$$||X||_2^2 = \sum_{i=1}^{p} X_i^2.$$

Then,

$$\mathbb{E}[||X||_2^2] = \mathbb{E}\left[\sum_{i=1}^{p} X_i^2\right]$$

Solution to Exercise 1 (a)

Note that

$$||X||_2^2 = \sum_{i=1}^{p} X_i^2.$$

Then,

$$\begin{aligned}\mathbb{E}[||X||_2^2] &= \mathbb{E}\left[\sum_{i=1}^{p} X_i^2\right] \\ &= \sum_{i=1}^{p} \mathbb{E}[X_i^2]\end{aligned}$$

Solution to Exercise 1 (a)

Note that

$$||X||_2^2 = \sum_{i=1}^{p} X_i^2.$$

Then,

$$\begin{aligned}
\mathbb{E}[||X||_2^2] &= \mathbb{E}\left[\sum_{i=1}^{p} X_i^2\right] \\
&= \sum_{i=1}^{p} \mathbb{E}[X_i^2] \\
&= \sum_{i=1}^{p} (\underbrace{\mathbb{E}[X_i]^2}_{=0} + \mathrm{Var}(X_i))
\end{aligned}$$

Solution to Exercise 1 (a)

Note that

$$||X||_2^2 = \sum_{i=1}^{p} X_i^2.$$

Then,

$$\begin{aligned}
\mathbb{E}[||X||_2^2] &= \mathbb{E}\left[\sum_{i=1}^{p} X_i^2\right] \\
&= \sum_{i=1}^{p} \mathbb{E}[X_i^2] \\
&= \sum_{i=1}^{p} (\underbrace{\mathbb{E}[X_i]^2}_{=0} + \mathrm{Var}(X_i)) \\
&= \sum_{i=1}^{p} 1
\end{aligned}$$

Solution to Exercise 1 (a)

Note that

$$||X||_2^2 = \sum_{i=1}^{p} X_i^2.$$

Then,

$$\begin{aligned}
\mathbb{E}[||X||_2^2] &= \mathbb{E}\left[\sum_{i=1}^{p} X_i^2\right] \\
&= \sum_{i=1}^{p} \mathbb{E}[X_i^2] \\
&= \sum_{i=1}^{p} (\underbrace{\mathbb{E}[X_i]^2}_{=0} + \mathrm{Var}(X_i)) \\
&= \sum_{i=1}^{p} 1 \\
&= p.
\end{aligned}$$

Solution to Question 1 (a): Continued

$$
\begin{aligned}
\mathrm{Var}(||X||_2^2) &= \mathrm{Var}(\sum_{i=1}^{p} X_i^2) \\
&= \sum_{i=1}^{p} \mathrm{Var}(X_i^2) \\
&= \sum_{i=1}^{p} (\underbrace{\mathbb{E}[X_i^4]}_{=3} - \underbrace{\mathbb{E}[X_i^2]^2}_{=1}) \qquad \triangleright \\
&= \sum_{i=1}^{p} (3-1) \\
&= 2p.
\end{aligned}
$$

Exercise 1 (b)

(b) Use (a) to infer that $|\mathbb{E}[||X||_2 - \sqrt{p}]| \leq \frac{1}{\sqrt{p}}$ by using the following steps:

(i) Write $||X||_2 - \sqrt{p} = \underbrace{\frac{||X||_2 - p}{2\sqrt{p}}}_{:=(1)} - \underbrace{\frac{(||X||_2^2 - p)^2}{2\sqrt{p}(||X||_2 + \sqrt{p})^2}}_{:=(2)}$.

(ii) Compute $\mathbb{E}[(1)]$.

(iii) Note that $0 \leq \mathbb{E}[(2)] \leq \frac{\mathrm{Var}(||X||_2^2)}{2p^{3/2}}$ holds due to $||X||_2 \geq 0$.

(iv) Put (i)- (iii) together.

Solution to Exercise 1 (b)

Step (i): Write $||X||_2 - \sqrt{p} = \underbrace{\frac{||X||_2 - p}{2\sqrt{p}}}_{:=(1)} - \underbrace{\frac{(||X||_2^2 - p)^2}{2\sqrt{p}(||X||_2 + \sqrt{p})^2}}_{:=(2)}$. $\triangleleft$

Step (ii):

$$
\begin{aligned}
\mathbb{E}[(1)] &= \mathbb{E}\left[\frac{||X||_2^2 - p}{2\sqrt{p}}\right] \\
&= \frac{1}{2\sqrt{p}}\Big(\underbrace{\mathbb{E}[||X||_2^2]}_{=p \text{ (from Question(a))}} - p\Big) \\
&= 0.
\end{aligned}
$$

Solution to Exercise 1 (b)

Step (i): Write $||X||_2 - \sqrt{p} = \underbrace{\frac{||X||_2 - p}{2\sqrt{p}}}_{:=(1)} - \underbrace{\frac{(||X||_2^2 - p)^2}{2\sqrt{p}(||X||_2 + \sqrt{p})^2}}_{:=(2)}$. $\triangleleft$

Step (ii):

$$
\begin{aligned}
\mathbb{E}[(1)] &= \mathbb{E}\left[\frac{||X||_2^2 - p}{2\sqrt{p}}\right] \\
&= \frac{1}{2\sqrt{p}}\Big(\underbrace{\mathbb{E}[||X||_2^2]}_{=p \text{ (from Question(a))}} - p\Big) \\
&= 0.
\end{aligned}
$$

Solution to Exercise 1(b): Continued

Step (iii): Prove that $0 \leq \mathbb{E}[(2)] \leq \frac{\mathrm{Var}(||X||_2^2)}{2p^{3/2}}$ holds due to $||X||_2 \geq 0$.

$$\mathbb{E}[(2)] = \mathbb{E}\left[\frac{(||X||_2^2 - p)^2}{2\sqrt{p}(||X||_2 + \sqrt{p})^2}\right] \geq 0, \qquad \text{since all the terms are non-negative.}$$

Besides, since $||X||_2 \geq 0$, it follows that

$$\begin{aligned}
&(2) \leq \frac{(||X||_2^2 - p)^2}{2p^{3/2}} \\
\Rightarrow \quad &\mathbb{E}[(2)] \leq \mathbb{E}\left[\frac{(||X||_2^2 - p)^2}{2p^{3/2}}\right] = \frac{1}{2p^{3/2}} \cdot \mathbb{E}\left[(||X||_2^2 - \mathbb{E}[||X||_2^2])^2\right] = \frac{\mathrm{Var}(||X||_2^2)}{2p^{3/2}} \\
&= \frac{2p}{2p^{3/2}} = \frac{1}{\sqrt{p}},
\end{aligned}$$

where we utilize the lemma from (a) that $\mathbb{E}[||X||_2^2] = p$ and $\mathrm{Var}(||X||_2^2) = 2p$.

Solution to Exercise 1(b): Continued

Step (iii): Prove that $0 \leq \mathbb{E}[(2)] \leq \frac{\mathrm{Var}(||X||_2^2)}{2p^{3/2}}$ holds due to $||X||_2 \geq 0$.

$$\mathbb{E}[(2)] = \mathbb{E}\left[\frac{(||X||_2^2 - p)^2}{2\sqrt{p}(||X||_2 + \sqrt{p})^2}\right] \geq 0, \qquad \text{since all the terms are non-negative.}$$

Besides, since $||X||_2 \geq 0$, it follows that

$$\begin{aligned}
& (2) \leq \frac{(||X||_2^2 - p)^2}{2p^{3/2}} \\
\Rightarrow \quad & \mathbb{E}[(2)] \leq \mathbb{E}\left[\frac{(||X||_2^2 - p)^2}{2p^{3/2}}\right] = \frac{1}{2p^{3/2}} \cdot \mathbb{E}\left[(||X||_2^2 - \mathbb{E}[||X||_2^2])^2\right] = \frac{\mathrm{Var}(||X||_2^2)}{2p^{3/2}} \\
& = \frac{2p}{2p^{3/2}} = \frac{1}{\sqrt{p}},
\end{aligned}$$

where we utilize the lemma from (a) that $\mathbb{E}[||X||_2^2] = p$ and $\mathrm{Var}(||X||_2^2) = 2p$.

Solution to Exercise 1(b): Continued

Step (iii): Prove that $0 \leq \mathbb{E}[(2)] \leq \frac{\mathrm{Var}(||X||_2^2)}{2p^{3/2}}$ holds due to $||X||_2 \geq 0$.

$$\mathbb{E}[(2)] = \mathbb{E}\left[\frac{(||X||_2^2 - p)^2}{2\sqrt{p}(||X||_2 + \sqrt{p})^2}\right] \geq 0, \qquad \text{since all the terms are non-negative.}$$

Besides, since $||X||_2 \geq 0$, it follows that

$$\begin{aligned}
&(2) \leq \frac{(||X||_2^2 - p)^2}{2p^{3/2}} \\
\Rightarrow \quad & \mathbb{E}[(2)] \leq \mathbb{E}\left[\frac{(||X||_2^2 - p)^2}{2p^{3/2}}\right] = \frac{1}{2p^{3/2}} \cdot \mathbb{E}\left[(||X||_2^2 - \mathbb{E}[||X||_2^2])^2\right] = \frac{\mathrm{Var}(||X||_2^2)}{2p^{3/2}} \\
&= \frac{2p}{2p^{3/2}} = \frac{1}{\sqrt{p}},
\end{aligned}$$

where we utilize the lemma from (a) that $\mathbb{E}[||X||_2^2] = p$ and $\mathrm{Var}(||X||_2^2) = 2p$.

Solution to Exercise 1(b): Continued

Step (iv): Putting everything together:

$$|\mathbb{E}[||X||_2 - \sqrt{p}]| = |\underbrace{\mathbb{E}[(1)]}_{=0} - \underbrace{\mathbb{E}[(2)]}_{\geq 0}| = \mathbb{E}[(2)] \leq \frac{1}{\sqrt{p}}.$$

Exercise 1 (c)

(c) Use (b) to infer that $\mathrm{Var}(||X||_2) \leq 2$ by using the following steps:

(i) Write $\mathrm{Var}(||X||_2) = \mathrm{Var}(||X||_2 - \sqrt{p})$.

(ii) For any random variable Y it holds that $\mathrm{Var}(Y) \leq \mathbb{E}[Y^2]$.

(iii) If you encounter the term $E[||X||_2]$ write it as $\mathbb{E}[\underbrace{||X||_2 - \sqrt{p}}_{=(*)} + \sqrt{p}]$ and use (b) for $(*)$.

Solution to Exercise 1 (c)

Step (i): Write $\mathrm{Var}(||X||_2) = \mathrm{Var}(||X||_2 - \sqrt{p})$.

It holds because **variance does not change by constant shifts**.

Step (ii): For any random variable Y it holds that $\mathrm{Var}(Y) \leq \mathbb{E}[Y^2]$.

It holds because $\mathrm{Var}(Y) + \mathbb{E}[Y]^2 = \mathbb{E}[Y^2]$ and $\mathbb{E}[Y]^2 \geq 0$. Later we will use this inequality.

Solution to Exercise 1 (c): Continued

$$
\begin{aligned}
\mathrm{Var}(||X||_2) &= \mathrm{Var}(||X||_2 - \sqrt{p}) && \text{Step (i)} \\
&\leq \mathbb{E}[(||X||_2 - \sqrt{p})^2] && \text{Step (ii)} \\
&= \mathbb{E}[||X||_2^2 - 2\sqrt{p}||X||_2 + p] \\
&= \underbrace{\mathbb{E}[||X||_2^2]}_{=p} - 2\sqrt{p} \cdot \mathbb{E}[||X||_2] + p \\
&= 2p - 2\sqrt{p} \cdot \mathbb{E}[||X||_2] \\
&= 2p - 2\sqrt{p} \cdot \mathbb{E}[||X||_2 - \sqrt{p} + \sqrt{p}] && \text{Step (iv)} \\
&= 2p - 2p - 2\sqrt{p} \cdot \mathbb{E}[||X||_2 - \sqrt{p}] \\
&= -2\sqrt{p} \cdot \underbrace{\mathbb{E}[||X||_2 - \sqrt{p}]}_{\leq \frac{1}{\sqrt{p}} \quad \text{(from (b))}} \\
&\leq 2\sqrt{p} \cdot \frac{1}{\sqrt{p}} = 2.
\end{aligned}
$$

Question 1 (d)

Now let $X' = (X'_1, \ldots, X'_p)^T \sim \mathcal{N}(0, \boldsymbol{I})$ be another multivariate normally distributed vector with mean vector zero and covariance matrix being the identity matrix of dimension $p \times p$. Further, assume that X and X' are independent, so that $Z := \frac{X - X'}{\sqrt{2}} \sim \mathcal{N}(0, \boldsymbol{I})$. Conclude from the previous that

$$\left| \mathbb{E}\left[||X - X'||_2 - \sqrt{2p} \right] \right| \leq \frac{2}{p} \qquad \text{and} \qquad \mathrm{Var}(||X - X'||_2) \leq 4.$$

Solution to Exercise 1 (d)

We first investigate Z. Since $Z = \frac{X-X'}{\sqrt{2}} \sim \mathcal{N}(0, \boldsymbol{I})$, it follows from (b) and (c) that

$$|\mathbb{E}[||Z||_2 - \sqrt{p}]| \leq \sqrt{\frac{1}{p}}, \tag{1}$$

$$\mathrm{Var}(||Z||_2) \leq 2 \tag{2}$$

But the norm of Z

$$||Z||_2 = \sqrt{\sum_{i=1}^{p}\left(\frac{X_i - X_i'}{\sqrt{2}}\right)^2} = \sqrt{\frac{1}{2}\sum_{i=1}^{p}(X_i - X_i')^2} = \sqrt{\frac{1}{2}}\sqrt{\sum_{i=1}^{p}(X_i - X_i')^2} = \sqrt{\frac{1}{2}}||X - X'||_2. \tag{3}$$

It follows from (1) that

$$\sqrt{2} \cdot |\mathbb{E}[||Z||_2 - \sqrt{p}]| \leq \frac{2}{p} \Rightarrow |\mathbb{E}[\sqrt{2} \underbrace{||Z||_2}_{||X-X'||_2} - \sqrt{2p}]| \leq \sqrt{\frac{2}{p}}.$$

Solution to Exercise 1 (d)

We first investigate Z. Since $Z = \frac{X-X'}{\sqrt{2}} \sim \mathcal{N}(0, \boldsymbol{I})$, it follows from (b) and (c) that

$$|\mathbb{E}[||Z||_2 - \sqrt{p}]| \leq \sqrt{\frac{1}{p}}, \tag{1}$$

$$\mathrm{Var}(||Z||_2) \leq 2 \tag{2}$$

But the norm of Z

$$||Z||_2 = \sqrt{\sum_{i=1}^{p}\left(\frac{X_i - X_i'}{\sqrt{2}}\right)^2} = \sqrt{\frac{1}{2}\sum_{i=1}^{p}(X_i - X_i')^2} = \sqrt{\frac{1}{2}}\sqrt{\sum_{i=1}^{p}(X_i - X_i')^2} = \sqrt{\frac{1}{2}}||X - X'||_2. \tag{3}$$

It follows from (1) that

$$\sqrt{2} \cdot |\mathbb{E}[||Z||_2 - \sqrt{p}]| \leq \frac{2}{p} \Rightarrow |\mathbb{E}[\sqrt{2} \underbrace{||Z||_2}_{||X-X'||_2} - \sqrt{2p}]| \leq \sqrt{\frac{2}{p}}.$$

Solution to Exercise 1 (d)

We first investigate Z. Since $Z = \frac{X-X'}{\sqrt{2}} \sim \mathcal{N}(0, \boldsymbol{I})$, it follows from (b) and (c) that

$$|\mathbb{E}[||Z||_2 - \sqrt{p}]| \leq \sqrt{\frac{1}{p}}, \tag{1}$$

$$\mathrm{Var}(||Z||_2) \leq 2 \tag{2}$$

But the norm of Z

$$||Z||_2 = \sqrt{\sum_{i=1}^{p} \left(\frac{X_i - X_i'}{\sqrt{2}}\right)^2} = \sqrt{\frac{1}{2}\sum_{i=1}^{p}(X_i - X_i')^2} = \sqrt{\frac{1}{2}}\sqrt{\sum_{i=1}^{p}(X_i - X_i')^2} = \sqrt{\frac{1}{2}}||X - X'||_2. \tag{3}$$

It follows from (1) that

$$\sqrt{2} \cdot |\mathbb{E}[||Z||_2 - \sqrt{p}]| \leq \frac{2}{p} \Rightarrow |\mathbb{E}[\underbrace{\sqrt{2}\;||Z||_2}_{||X-X'||_2} - \sqrt{2p}]| \leq \sqrt{\frac{2}{p}}.$$

Solution to Exercise 1 (d): Continued

Moreover, (2): $\mathrm{Var}(||Z||_2) \leq 2$ implies that

$$\begin{aligned}
& \mathrm{Var}(||Z||_2) \leq 2 & \\
\Leftrightarrow \quad & 2\mathrm{Var}(||Z||_2) \leq 4 & \\
\Leftrightarrow \quad & \mathrm{Var}(\sqrt{2}||Z||_2) \leq 4 & \quad (\mathrm{Var}(aY) = a^2\mathrm{Var}(Y) \text{ for any RV } Y) \\
\Leftrightarrow \quad & \mathrm{Var}(||X - X'||_2) \leq 4 & \quad \text{Using (3) that } ||Z||_2 = \sqrt{\frac{1}{2}}||X - X'||_2.
\end{aligned}$$

Exercise 1 (e)

(e) From the cosine rule we can infer that for any $x, x' \in \mathbb{R}^p$ it holds that

$$\langle x, x' \rangle = \frac{1}{2}(||x||_2^2 + ||x'||_2^2 - ||x - x'||_2^2).$$

Use this to show that $\mathbb{E}[\langle X, X' \rangle] = 0$. Moreover, derive that $\mathrm{Var}(\langle X, X' \rangle) = p$.

Solution to Exercise 1 (e)

Since $\langle x, x' \rangle = \frac{1}{2}(||x||_2^2 + ||x'||_2^2 - ||x - x'||_2^2)$, we can infer that

$$\begin{aligned}
\mathbb{E}[\langle X, X' \rangle] &= \frac{1}{2}(\mathbb{E}[||X||_2^2] + \mathbb{E}[||X'||_2^2] - \mathbb{E}[||X - X'||_2^2]) \\
&= \frac{1}{2}\left(p + p - 2 \cdot \mathbb{E}\left[\underbrace{\frac{1}{2}||X - X'||_2^2}_{=||Z||_2^2}\right]\right) \\
&= \frac{1}{2}(p + p - 2p) = 0. \qquad \text{(From (a) we know that } \mathbb{E}[||Z||_2^2] = p\text{)}
\end{aligned}$$

Solution to Exercise 1 (e): Continued

$$
\begin{aligned}
\mathrm{Var}(\langle X, X' \rangle) &= \mathrm{Var}(\sum_{i=1}^{p} X_i X_i') \\
&= \sum_{i=1}^{p} \mathrm{Var}(X_i X_i') \\
&= p\mathrm{Var}(X_1 X_1') \\
&= p \cdot (\mathbb{E}[X_1^2 (X_1')^2] - \mathbb{E}[X_1 (X_1')]^2) \\
&= p \cdot (\underbrace{\mathbb{E}[X_1^2]}_{=1} \cdot \underbrace{\mathbb{E}[(X_1')^2]}_{=1} - \underbrace{\mathbb{E}[X_1]^2}_{=0} \cdot \underbrace{\mathbb{E}[X_1']^2}_{=0}) \\
&= p.
\end{aligned}
$$

Exercise 1 (f)

(f) For different dimensions p, e.g., $p \in \{1, 2, 4, 8, \ldots, 1024\}$, create two sets consisting of 100 i.i.d. random observations from $\mathcal{N}(0, \boldsymbol{I})$, respectively and

(i) compute the average Euclidean length of (one of) the sampled sets and compare it to $\sqrt{p}$;

(ii) comptue the average Euclidean distances between the sampled sets and compare it to $\sqrt{2p}$;

(iii) compute the average inner products between the sampled sets;

(iv) compute in (i)-(iii) also the empirical variances of the respective terms.

Visualize your results in an appropriate manner.

Show the standard solution.