

Exercise of Supervised Learning: Advanced Risk Minimization Part 3

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Exercise 3: Connection between MLE and ERM

Suppose we are facing a regression task, i.e., $\mathcal{Y} = \mathbb{R}$, and the feature space $\mathcal{X} \subseteq \mathbb{R}^p$. Let us assume that the relationship between the features and labels is specified by

$$y = m^{-1}(m(f_{\text{true}}(\mathbf{x})) + \epsilon), \quad \triangleleft$$

where $m : \mathbb{R} \rightarrow \mathbb{R}$ is a continuous strictly monotone function with m^{-1} being its inverse function, and the errors are Gaussian, i.e., $\epsilon \sim \mathcal{N}(0, \sigma^2)$. In particular, for the data points $(\mathbf{x}^{(1)}, y^{(1)}, \dots, \mathbf{x}^{(n)}, y^{(n)})$ it holds that

$$y^{(i)} = m^{-1}(m(f_{\text{true}}(\mathbf{x}^{(i)})) + \epsilon^{(i)}),$$

where $\epsilon^{(1)}, \dots, \epsilon^{(n)}$ are i.i.d. with distribution $\mathcal{N}(0, \sigma^2)$.

Disclaimer: We assume in the following that $m(y)$ and $m(f(\mathbf{x}))$ is well-defined for any $y \in \mathcal{Y}$, $f \in \mathcal{H}$ and $\mathbf{x} \in \mathcal{X}$.

(a) How can we transform the labels $y^{(1)}, \dots, y^{(n)}$ to “new” labels $z^{(1)}, \dots, z^{(n)}$ such that $z^{(i)} \mid \mathbf{x}$ is normally distributed? What are the parameters of this normal distribution?

Solution: Question (a)

We can use

$$z^{(i)} = m(y^{(i)}).$$

We know that $y^{(i)} = m^{-1}(m(f_{\text{true}}(\mathbf{x}^{(i)})) + \epsilon^{(i)})$, therefore $m(y^{(i)}) = m(f_{\text{true}}(\mathbf{x}^{(i)})) + \epsilon^{(i)}$.

Hence,

$$z^{(i)} = m(f_{\text{true}}(\mathbf{x}^{(i)})) + \epsilon^{(i)}, \quad \text{with } \epsilon^{(i)} \sim \mathcal{N}(0, \sigma^2).$$

Therefore, $z^{(i)} \mid \mathbf{x}^{(i)}$ is normally distributed, i.e.,

$$z^{(i)} \mid \mathbf{x}^{(i)} \sim \mathcal{N}(m(f_{\text{true}}(\mathbf{x}^{(i)})), \sigma^2)$$

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Question (b)

Assume that the hypothesis space is

$\mathcal{H} = \{f(\cdot, \boldsymbol{\theta}) : \mathcal{X} \rightarrow \mathbb{R} \mid f(\cdot | \boldsymbol{\theta}) \text{ belongs to a certain functional family parameterized by } \boldsymbol{\theta} \in \Theta\}$,

where $\boldsymbol{\theta} = (\theta_1, \theta_2, \dots, \theta_d)$ is a parameter vector, which is an element of a **parameter space** Θ . Based on your findings (a), establish a relationship between minimizing the negative log-likelihood for $(\mathbf{x}^{(1)}, z^{(1)}), \dots, (\mathbf{x}^{(n)}, z^{(n)})$ and empirical loss minimizing over \mathcal{H} of the generalized L2-loss function of Exercise sheet 1, i.e.,
 $L(y, f(\mathbf{x})) = (m(y) - m(f(\mathbf{x})))^2$.

Solution to Question (b)

The likelihood for $(\mathbf{x}^{(1)}, z^{(1)}), \dots, (\mathbf{x}^{(n)}, z^{(n)})$ is

$$\mathcal{L}(\theta) = \prod_{i=1}^n p\left(z^{(i)} \mid f(\mathbf{x}^{(i)} \mid \theta), \sigma^2\right)$$

So the negative log-likelihood (NLL) is

Therefore, the NLL is proportional to the empirical risk of $f(\cdot \mid \theta)$ w.r.t. the generalized L2-Loss.

Solution to Question (b)

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$$\begin{aligned}\mathcal{L}(\theta) &= \prod_{i=1}^n p\left(z^{(i)} \mid f(\mathbf{x}^{(i)} \mid \theta), \sigma^2\right) \\ &\propto \exp\left(-\frac{1}{2\sigma^2} \sum_{i=1}^n \left[z^{(i)} - m(f(\mathbf{x}^{(i)} \mid \theta))\right]^2\right).\end{aligned}$$

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$$-\ell(\boldsymbol{\theta}) = -\log(\mathcal{L}(\boldsymbol{\theta}))$$

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Therefore, the NLL is proportional to the empirical risk of $f(\cdot \mid \boldsymbol{\theta})$ w.r.t. the generalized L2-Loss.

Question (c)

(c) In many practical applications, one often observed statistical property is that the label y given \mathbf{x} follows a *log-normal distribution*. Note that we can obtain such a relationship by using $m(x) = \log(x)$ above. In the following we want to consider the conjecture of James D. Forbes, who conjectured in the year 1857 that the relationship between the air pressure y and the boiling point of water x is given by

$$y = \theta_1 \exp(\theta_2 x + \epsilon),$$

for some specific values $\theta_1 \in \mathbb{R}_+$, $\theta_2 \in \mathbb{R}$ and some error terms ϵ (of course, we assume that this error term is stochastic and normally distributed).

What would be a suitable hypothesis space \mathcal{H} if this conjecture holds?

Solution to Question (c)

- ▶ The goal is to introduce a proper form for $f(\cdot|\theta)$ so that $y = m^{-1}(m(f(x)) + \epsilon)$.
- ▶ We use the transformation $m(\cdot) = \log(\cdot)$. So $m^{-1}(\cdot) = \exp(\cdot)$.
- ▶ The Forbes' conjectured model $y = \theta_1 \exp(\theta_2 x + \epsilon)$ can be written as

Therefore, $f(x|\theta) = \theta_1 \exp(\theta_2 x)$ is a suitable functional form for the hypothesis. So

$$\mathcal{H} = \{f(x|\theta) = \theta_1 \exp(\theta_2 x) \mid \theta \in \Theta\}.$$

(The standard solution uses $\mathcal{X} = 1 \times \mathbb{R}$, i.e., $\mathbf{x} = (x_1, x_2)^T = (1, x_2)^T$.)

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Therefore, $f(x | \boldsymbol{\theta}) = \theta_1 \exp(\theta_2 x)$ is a suitable functional form for the hypothesis. So

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Solution to Question (c): Continued

Show the code in the standard solution.