Assignment 6

Q1

$$u(x) = \frac{x - \alpha x^2}{2}, \quad x \sim N(\mu, \sigma^2)$$

$$\Rightarrow \mathbb{E}[u(x)] = \mathbb{E}\left[x - \frac{\alpha x^2}{2}\right]$$

$$= \mathbb{E}[x] - \frac{\alpha}{2}\mathbb{E}[x^2]$$

$$= \boxed{\mu - \frac{\alpha}{2}\cdot(\sigma^2 + \mu^2)}$$

$\Rightarrow$ From lecture (via Taylor expansion and $\mathbb{E}[u(x)] = u(X_{CE})$),

$$u'(\bar{x})\cdot(X_{CE} - \bar{x}) \approx \frac{1}{2}u''(\bar{x})\cdot\sigma_x^2$$

$$u'(x) = 1 - \alpha x$$
$$u''(x) = -\alpha$$

$$\Rightarrow X_{CE} \approx \bar{x} + \frac{1}{2}\frac{u''(x)\,\sigma_x^2}{u'(x)}$$

$$\approx \mu + \frac{1}{2}\left(\frac{-\alpha\cdot\sigma^2}{1-\alpha x}\right)$$

$$\approx \boxed{\mu - \frac{\alpha\sigma^2}{2(1-\alpha)}}$$

$$\Rightarrow \pi_A = \bar{x} - X_{CE} = \mu - \left(\mu - \frac{\alpha\sigma^2}{2(1-\alpha)}\right) = \boxed{\frac{\alpha\sigma^2}{2(1-\alpha)}}$$

Portfolio

$$\Rightarrow W \sim N(1 + r + z(\mu - r),\ z^2\sigma^2)$$

We need to maximize X_{CE} (same as maximizing utility)

$$X_{CE} \approx 1 + r + z(\mu - r) - \frac{\alpha z^2\sigma^2}{2(1-\alpha)}$$

$$\frac{\partial X_{CE}}{\partial z} = \mu - r - \frac{\alpha\sigma^2}{(1-\alpha)}z = 0$$

$$z^* = \boxed{\frac{(\mu - r)(1-\alpha)}{\alpha\sigma^2}}$$

Q2

$\Rightarrow$ $u(x) = \log(x)$ is a special case of $u(x) = \frac{x^{1-\gamma}-1}{1-\gamma}$ when $\gamma = 1$

Therefore,

$$X_{CE} = e^{\mu + \frac{\sigma^2}{2}(1-\gamma)}$$

$$= e^{\mu} \quad \text{when } \gamma = 1$$

$\Rightarrow$ We have from lecture,

$$\log W \sim N\left(r + \pi(\mu - r) - \frac{\pi^2 \sigma^2}{2}, \pi^2\sigma^2\right)$$

$\Rightarrow$ we need to maximize X_{CE}

$\Rightarrow$ mamimizing. mean

$\Rightarrow$ maximize $r + \pi(\mu - r) - \frac{\pi^2\sigma^2}{2}$

find critical pt $\Rightarrow$ $\mu - r - \sigma^2\pi = 0$

$$\pi^* = \boxed{\frac{\mu - r}{\sigma^2}}$$

Q3

Two outcomes for wealth W

$\Rightarrow f \cdot W_0 (1+\alpha) + W_0(1-f) = W_0(1+f\alpha)$

$f \cdot W_0 (1-B) + W_0 (1-f) = W_0 (1+fB)$

Two outcomes for log (utility) of W

$\Rightarrow \log(W_0(1+f\alpha))$

$\cdot \log(W_0(1-fB))$

$$\Rightarrow E[\log(W)] = p \cdot \log(W_0(1+f\alpha)) + q \cdot \log(W_0(1-fB))$$

$$\Rightarrow \frac{\partial E[\log(W)]}{\partial f} = \frac{p \cancel{W_0} \alpha}{\cancel{W_0}(1+f\alpha)} + \frac{-q \cancel{W_0} B}{\cancel{W_0}(1-fB)} = 0$$

$$\Rightarrow \frac{p}{1+f\alpha} = \frac{q}{1-fB}$$

$$\therefore f^* = \boxed{\frac{\alpha p - Bq}{2\alpha B}}$$

$\rightarrow$ f^* makes intuitive sense. For some α, B, you will bet more ($\uparrow f^*$) if p or probability of positive return (winning) is higher.

The denominator has αB which means it also considers the relationship (relative payout) between positive return and negative return.