Understanding Simple Harmonic Motion Through Damped and Undamped Spring-Mass Systems

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The goal of this experiment is to identify and understand the differences between damped and undamped simple harmonic oscillators as well as to calculate the correct value of spring constant within uncertainty. To do this, we run a series of trials using LoggerPro and a motion sensor in order to understand how displacement changes with time between the damped and undamped cases. We use the same spring throughout and insert a piece of cardboard as our source of drag for the damped measurements. We analyze the best set of trials in Python for each case by calculating the amplitude, natural frequency, damping frequency, phase constant, damping parameter, and the uncertainties in each. Using these values, we are then able to calculate the spring constant of our spring and determine if our values are reasonable. Not only does the analysis of oscillatory motion tell us about variables like natural frequency and spring constant, but it gives us insight into how the world around us works.

I. INTRODUCTION

Oscillators are found everywhere in our daily lives, from spring-mass systems and pendulums to molecules and crystals. They play an essential role in understanding the most common type of motion in physics. Simple harmonic motion (SHM) is a form of oscillatory motion subject to a retarding /restoring force that is proportional to the amount of displacement from its equilibrium position. For this particular lab, we focus on classical simple harmonic oscillators (SHO) and damped, undriven oscillators. SHO's can be modeled by the equation: $\ddot{x} + \Omega^2 x = 0$ where $\ddot{x}$ is the acceleration of the system, Ω is the natural frequency ($\sqrt{\frac{k}{m}}$), and x is the position. The general solution to this differential equation is: $x(t) = Acos(\Omega t + \phi_0)$ where A is the amplitude and ϕ_0 is the phase constant. Damped, undriven oscillators are subject to a damping force, but no driving force. They can be modeled by the equation: $\ddot{x} + 2\beta\dot{x} + \Omega^2 x = 0$ where β is the damping parameter which is equal to $b/2m$ where b is the damping constant and m is the mass of the system. The general solution to this differential equation is: $x(t) = Ae^{-\beta t}cos(\Omega_D t - \phi_0)$ where Ω_D is the damping frequency which is equal to $\sqrt{\Omega^2 - \beta^2}$. In our experiment, we use a spring-mass system as our simple harmonic oscillator and cardboard as the source that provides us with a damping force. This experiment is necessary in order to understand that damped and undamped oscillators yield different types of motion, however, can be analyzed to obtain accurate spring constant values within uncertainty.

II. PROCEDURE

To prepare our experiment, we gather a scale, meter stick, one spring, three small masses, three sheets of cardboard, a counterweight, a mass hanger, a ring stand, a crossbar with a clamp, a motion sensor, a LabQuest device, and a laptop with LoggerPro. We begin by selecting three masses and measuring the total mass (each mass plus the hanging mass). Next, we set up our apparatus by connecting the crossbar to our ring stand and clipping the base of the stand to the table for support. We tape the meter stick to the crossbar to make easier measurements. We then select a spring to use for the entire experiment. In our case, we use a yellow spring with a spring constant of 35 N/m.

Next, we place the motion sensor directly below our apparatus and connect it to the LabQuest device and LoggerPro. Make sure that the motion sensor is at least 15 cm away from the hanging mass at its closest approach. In LoggerPro, select Data Collection and set the duration to ten seconds or greater and enable triggering. The triggering is what sets the phase so LabQuest will start collecting the data at the distance you select above the motion sensor. This ensures that the phase of our displacement curve is reproducible. The triggering will vary based on what mass is used. Prior to recording data, make sure to run some tests by sending the mass oscillating and checking the graph of displacement vs. time in LoggerPro. Once the graphs appear to show constant cyclical motion, you can begin data recording.

We start with the undamped case first and obtain three different hanging masses. For each of the three spring-mass combinations, it is ideal to conduct three-five trials. The more data the better. After each trial, be sure to export your data and save the file with a meaningful name. Once all of the undamped data is complete, collect the three pieces of cardboard and poke holes through the center so that they can easily be hung on the hanging mass. Using the same spring and only one of the masses from the undamped case, conduct three-five trials for each spring-mass-cardboard system. Make sure to use three separate pieces of cardboard but the same spring and mass. It may be necessary to increase the duration for the damped case in order to see the motion change over time. Once data collection is complete, import all of the data into Python for analysis.

III. ANALYSIS

The analytical goal of this lab is to use Python to determine the key differences between damped and undamped oscillatory motion, to calculate the correct spring constant within uncertainty for both cases, and to determine the uncertainty in our measurements. We use Python to overplot our repeated trials on top of one another to see how closely they overlap with each other. We plot the trajectories, normalized trajectories, and trajectories factoring in uncertainty for both in the subsections below.

A. Undamped Oscillators

This subsection focuses primarily on undamped oscillators. The plots and uncertainty calculations are detailed below.

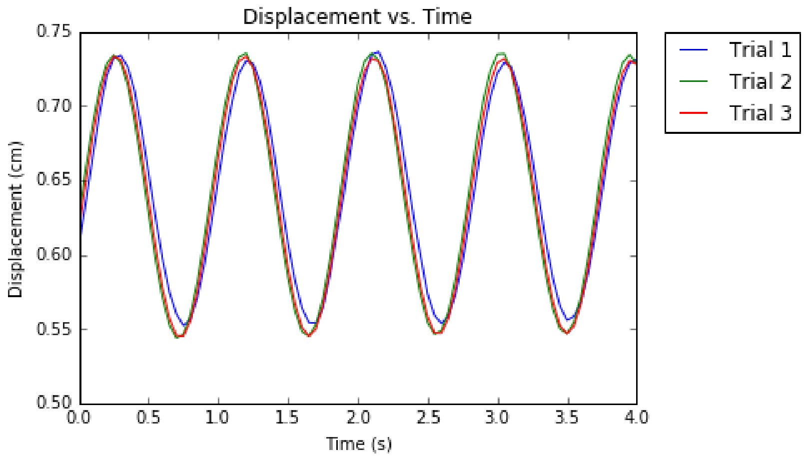

FIG. 1: Depicted above is a plot of displacement (cm) vs time (s) for our second set of undamped trials. We use a mass of 751g and a yellow spring with a spring constant of 35 N/m. The spring is stretched about 10.6 cm from equilibrium in each trial. For these measurements, our triggering is 0.60 m.

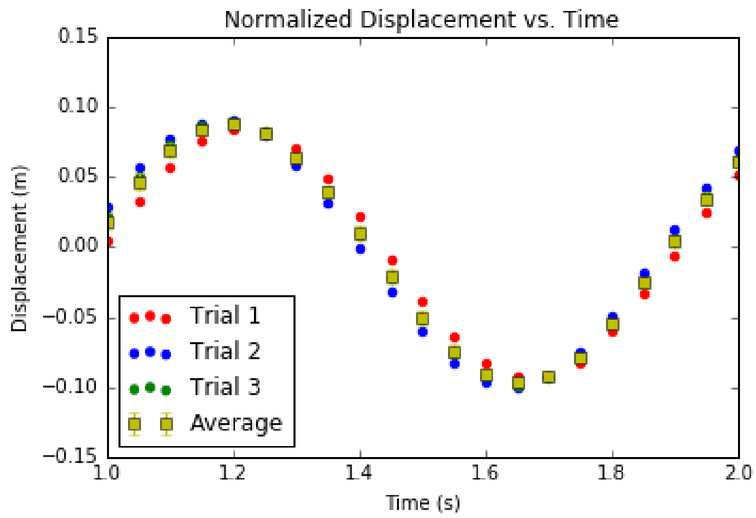

FIG. 2: Depicted above is a plot of normalized displacement (cm) vs time (s) for our second set of undamped trials with error bars. We normalize our data by taking each displacement value and subtracting it from the mean of our displacement measurements for each trial. This centers our graph on zero and allows us to get a better view of how well our three trials line up. We calculate the uncertainty in the amplitude at each time by finding the mean and standard deviation between the three trials. We then plot the error bars taking into account all of these factors.

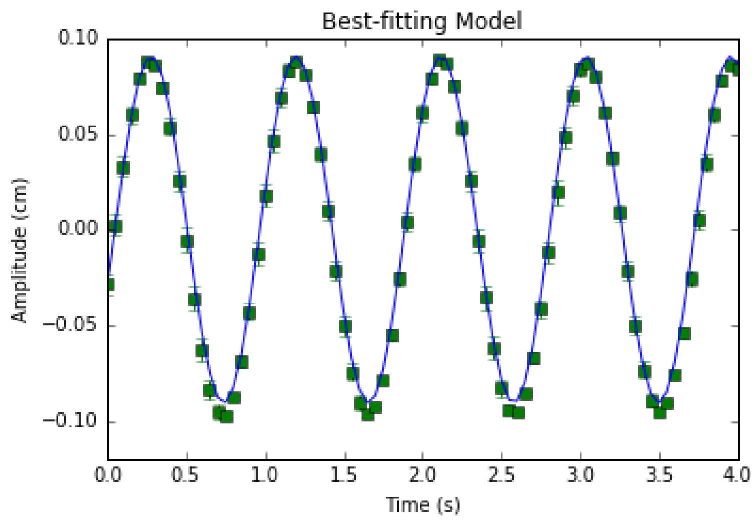

FIG. 3: Depicted above is a plot of our best-fitting parameter values and uncertainties. The error is plotted in green and the true curve neglecting error is plotted in blue. We use the general solution to the simple harmonic motion equation to determine what the motion of the spring-mass system should look like, and overplot it with our actual data using "curve fit" in Python. The general solution is: $x(t) = Acos(\Omega t - \phi_0)$ where A is the amplitude, Ω is the natural frequency, t is time, and ϕ_0 is the phase shift.

B. Uncertainty Calculations

This section details the uncertainty calculations of the phase, amplitude and natural frequency of our undamped spring-mass system. We find our values of phase, amplitude, and natural frequency by using "curve fit" in Python. Our inputs are our simple harmonic motion function, time, average displacement, the error in the displacement, and input values for our SHM function. To calculate the uncertainty in these values, we find the diagonal of the covariance matrix scaled by the χ^2. χ^2 tests to see whether distributions of variables differ from each another.

We find that the amplitude is 0.0906 ± 0.000653 cm, the natural frequency is 6.831 ± 0.0120 s^{-1}, and the phase shift is 1.902 ± 0.0323 radians

Now that we know the natural frequency value and its corresponding uncertainty, we can calculate the spring constant and its uncertainty. The spring constant is calculated as follows:

$$k = \Omega^2 m = (6.831 s^{-1})^2 (0.751 g) = 35.048 N/m$$

The uncertainty in the spring constant can then be calculated as follows:

$$\begin{aligned} \delta\Omega =& \sqrt{\frac{\delta\Omega}{\delta k}^2 \delta k^2 + \frac{\delta\Omega}{\delta m}^2 \delta m^2} \\ \delta k^2 =& \frac{\delta\Omega^2}{\frac{\delta\Omega}{\delta k}^2} \\ \delta k =& 2m\Omega\delta\Omega \end{aligned} \tag{1}$$

$$\delta k = 2(0.751 g)(6.831 s^{-1})(0.0120 s^{-1}) = 0.123 N/m$$

Therefore, our value of the spring constant, k, taking into account uncertainty for the undamped case is 35.048 ± 0.123 N/m.

C. Damped Oscillators

This subsection focuses primarily on damped oscillators. The plots and uncertainty calculations are detailed below.

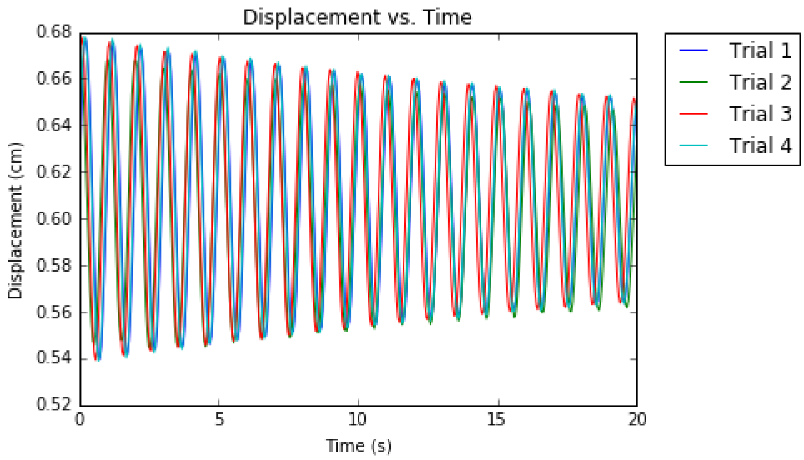

FIG. 4: Depicted above is a plot of displacement (cm) vs time (s) for our first set of damped trials. We use a mass of 852 g, a yellow spring with a spring constant of 35 N/m, and a piece of cardboard with a cross-sectional area of 3.24 cm^2. The spring is stretched about 7.5 cm from equilibrium in each trial. For these measurements, our triggering is 0.60 m.

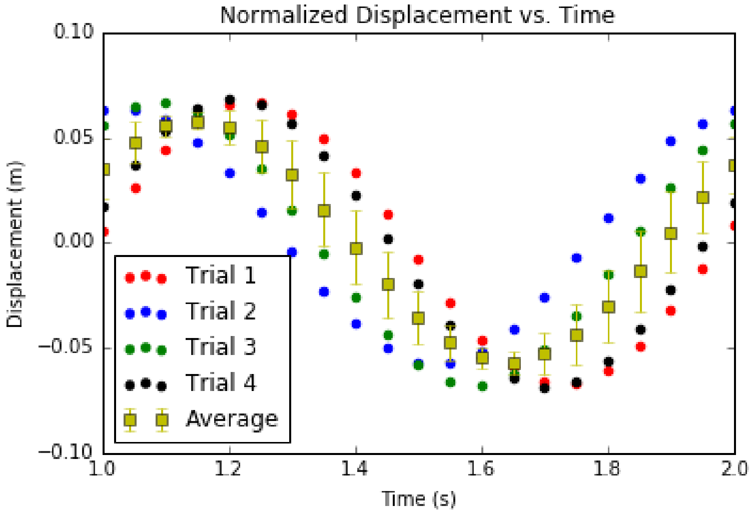

FIG. 5: Depicted above is a plot of normalized displacement (cm) vs time (s) for our first set of damped trials with error bars. We normalize our data by taking each displacement value and subtracting it from the mean of our displacement measurements for each trial. This centers our graph on zero and allows us to get a better view of how well our three trials line up. We calculate the uncertainty in the amplitude at each time by finding the mean and standard deviation between the three trials. We then plot the error bars taking into account all of these factors.

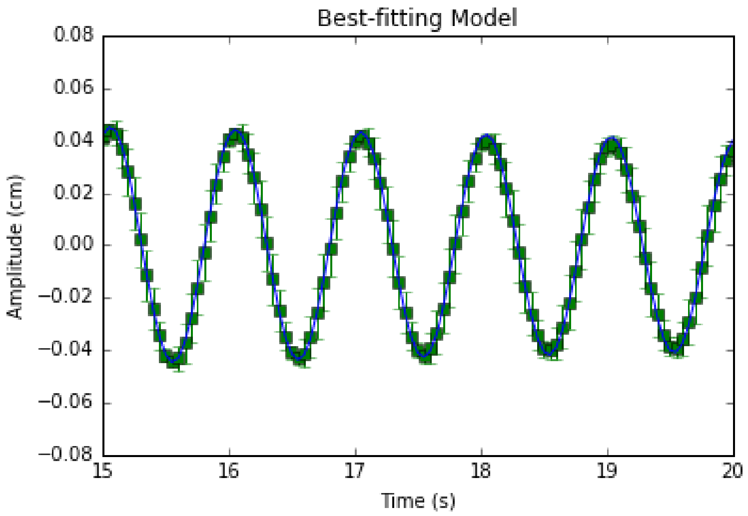

FIG. 6: Depicted above is a plot of our best-fitting parameter values and uncertainties for the damped oscillator. The error is plotted in green and the true curve neglecting error is plotted in blue. We use the general solution to the damped oscillator to determine what the motion of the spring-mass system should look like, and overplot it with our actual data using $curve_fit$ in Python. The general solution is: $x(t) = Ae^{-\beta t}cos(\Omega_D t - \phi_0)$ where A is the amplitude, β is the damping parameter, Ω_D is the damping frequency, t is time, and ϕ_0 is the phase shift. As you can see, the amplitude decreases slightly overtime.

D. Calculating Uncertainty

This section details the uncertainty calculations of the phase, amplitude, damping frequency, and damping parameter of our damped spring-mass system. Our values and uncertainties of phase, amplitude, and damping frequency are calculated in the same way as the undamped case. A new parameter that is calculated is the damping parameter, β which is what causes the amplitude to decrease over time.

We find that the amplitude is 0.0624 ± 0.000173 cm, the damping frequency is 6.320 ± 0.000668 s^{-1}, the phase shift is 0.946 ± 0.00768 radians, and the damping parameter is $0.0215 \pm 0.000262\ s^{-1}$.

Now that we know the natural frequency value and its corresponding uncertainty, we can calculate the spring constant and its uncertainty. The spring constant is calculated as follows:

$$k = \Omega^2 m = (6.320s^{-1})^2(0.852g) = 34.034N/m$$

The uncertainty in the spring constant can then be calculated as follows:

$$\begin{aligned}
\delta\Omega_D =& \sqrt{\frac{\delta\Omega_D}{\delta k}^2 \delta k^2 + \frac{\delta\Omega_D}{\delta\beta}^2 \delta\beta^2} \\
\frac{\delta\Omega_D}{\delta k} &= \frac{1}{2}(\frac{k}{m} - \beta^2)^{\frac{1}{2}}(\frac{1}{m}) \\
\frac{\delta\Omega_D}{\delta k} &= \frac{1}{2m\Omega_D} \\
\frac{\delta\Omega_D}{\delta\beta} &= \frac{1}{2\Omega_D}(-2\beta) = -\frac{\beta}{\Omega_D} \\
\delta\Omega_D^2 &= \frac{\delta k^2}{(2m\Omega_D)^2} = \frac{\delta\beta^2\beta^2}{\Omega_D^2} \\
\delta k &= \sqrt{4m^2\Omega_D^2\delta\Omega_D^2 - 4m^2\beta^2\delta\beta^2}
\end{aligned} \tag{2}$$

$$\delta k = \sqrt{4(0.852g)^2(6.320s^{-1})^2(0.000668)^2 - 4(0.852g)^2(0.0215s^{-1})^2(0.000262)^2} = 0.00718N/m$$

Therefore, our value of the spring constant, k, taking into account uncertainty for the undamped case is 34.034 ± 0.00718 N/m.

IV. DISCUSSION

Our best-fitting model curves look reasonable compared to the "true" oscillatory motion for each spring-mass system. The curve lies in between our error bars, indicating that it is a good fit to our model.

Our best-fitting parameters and their uncertainties also seem reasonable. According to Figure 2, our amplitude for our undamped curve appears to be around 0.09 cm while our calculated value is 0.0906 ± 0.000653 cm. In Figure 4, our amplitude appears to be around 0.06 cm while our calculated value is 0.0624 ± 0.000173 cm. From direct observation of our graphs, we can confirm that the amplitude seems reasonable. Our uncertainty values are small because we pull the spring-mass system to about the same length each time, which doesn't cause the amplitude to change a significant amount in each trial. To determine if the natural frequency is reasonable, we can convert it to a period of oscillations and compare our value with our graph. To confirm this, we can convert our values of natural and damped frequencies into a period. For the undamped case, the natural frequency can be converted into a period as follows: $\tau = \frac{2\pi}{\Omega} = \frac{2\pi}{6.831s^{-1}} = 0.920s = \tau$. By looking at Figure 2, we can estimate our period to be around 1 s, so we can therefore conclude that this value of natural frequency is reasonable. For the damped case, we can calculate our period as follows: $\tau = \frac{2\pi}{\Omega_D} = \frac{2\pi}{6.320s^{-1}} = 0.994s = \tau$. According to Figure 4, the period is nearly 1 s, so our value of Ω_D is reasonable. The uncertainty values for the natural and damping frequencies are relatively small which makes sense because we are running multiple trials at nearly the same distance each time which should not change the frequency value a significant amount. Both phase shift values seem to be acceptable. In Figure 2, the graph appears to be shifted by $\pi/2$ radians while the calculated value is 1.902 radians, which is relatively close to $\pi/2$. In Figure 4, the graph appears to be shifted by $\pi/4$ radians while the calculated value is 0.946 radians. Therefore, we can say that the value of phase shift is reasonable. Our uncertainty in phase shift is reasonable because a few millimeters off from our distance from equilibrium will not really alter the phase constant. Our calculated value for the damping parameter, β, is $0.0215 \pm 0.000262\ s^{-1}$. It makes sense for β to be a very small value because our drag force caused by our cardboard is very small which should not cause the amplitude to change drastically. The higher the β value, the faster the exponential, $e^{-\beta t}$, approaches zero.

Our damped and undamped SHM yield statistically consistent values for the spring constant. The spring constant on the box says that the yellow spring has a spring constant of 35 N/m $\pm$ 10%. Our k value for the undamped case is 35.048 ± 0.123 N/m which is within 10% of 35 N/m. Our k value for the damped case is 34.034 ± 0.00718 N/m which is also within 10% of 35 N/m. Therefore, we can conclude that our measured k values are consistent with the actual value.

Although we take into account some error in our data analysis, there are random and systematic uncertainties that we neglect. We ignore the mass of the spring and the mass of the cardboard because they are negligible and are not significant enough to change our values of Ω and k. Another random error that can occur is releasing the spring-mass system at an angle rather than perpendicular to the floor. This may slightly change how long it takes for the system to get back to equilibrium, but overall should not change the results significantly. A systematic error could be a miscommunication between LoggerPro and the motion sensor. There is a chance that LoggerPro doesn't measure the mass at the same fixed distance every time, however, this will likely have minimal impact on our data since this should be a small difference. All of the error described above does not change our data enough to take into consideration, however, it is important to recognize that it is still there.

It is a reasonable assumption to claim that the damped SHM in this lab is weak because our β value is very small. This means that the exponential function $e^{-\beta t}$ will approach zero slowly and won't cause the amplitude to decrease significantly. This makes sense because the drag force caused by the cardboard is very small and shouldn't cause the amplitude to change by that much.

Since our class had a lot of extra time to do more data collection, I think it would have been interesting to see how our curves changed if we changed springs. This would allow us to see how the spring constant plays a role in oscillations.

V. CONCLUSIONS

Overall, this experiment aided in my understanding of the differences between damped and undamped oscillators as well as the variables that affect them. This lab tested our ability to take our own data using LoggerPro and a motion sensor in order to interpret our results in Python. We learn how to obtain amplitude, natural frequency, phase shift, and the damping parameter just from interpreting our displacement vs. time graphs. We also gain practice in calculating uncertainties in our measurements as well as analyzing if our values are reasonable. We found that all of our measured values of amplitude, natural frequency, damping frequency, phase constant, and the damping parameter are reasonable values within uncertainty. Additionally, we backed up our reasoning through math and/or interpretations of our figures. We were successfully able to calculate the correct value of our spring constant within uncertainty, meaning that our measurements and calculations were consistent with what is expected. All in all, I learned how we can analyze the properties of oscillatory motion to better understand the world we live in.