Advanced Counting Techniques

Chapter 8

RECURRENCE RELATION

A recurrence relation for a sequence $a_0, a_1, a_2, \ldots,$ is a formula that relates each term a_k to certain of its predecessors $a_{k-1}, a_{k-2}, \ldots, a_{k-i}$, where i is a fixed integer and k is any integer greater than or equal to i.

The initial conditions for such a recurrence relation specify the values of

$$a_0, a_1, a_2, \ldots, a_{i-1}.$$

EXAMPLE

Let $\{a_n\}$ be a sequence that satisfies the recurrence relation $a_n = a_{n-1} + 3$ for n = 1, 2, 3, …, and suppose that $a_0 = 2$. What are a_1 , a_2 , a_3 ?

Solution:

$$a_n = a_{n-1} + 3$$

$$a_1 = a_0 + 3$$

$$a_1 = 2 + 3 = 5$$

$$a_2 = a_1 + 3$$

$$a_2 = 5 + 3$$

$$a_2 = 8$$

$$a_3 = a_2 + 3$$

$$a_3 = 8 + 3$$

$$a_3 = 11$$

EXAMPLE

Let $\{a_n\}$ be a sequence that satisfies the recurrence relation $\mathbf{a_n = a_{n-1} - a_{n-2}}$ for n = 2, 3, 4, … and suppose that $a_0 = 3$ and $a_1 = 5$. What are a_2 , a_3 ?

Solution:

$$\mathbf{a_n = a_{n-1} - a_{n-2}}$$

$$\mathbf{a_2 = a_1 - a_0}$$

$$\mathbf{a_2 = 5 - 3 = 2}$$

$$\mathbf{a_3 = a_2 - a_1}$$

$$\mathbf{a_3 = 2 - 5 = -3}$$

THE FIBONACCI SEQUENCE

The **Fibonacci sequence** is defined as follows.

- BASE

$$F_0 = 0,\ F_1 = 1$$

- Recursion

$$F_k = F_{k-1} + F_{k-2} \qquad \text{for all integers } k \geq 2$$

$$F_2 = F_1 + F_0 = 1 + 0 = 1$$

$$F_3 = F_2 + F_1 = 1 + 1 = 2$$

$$F_4 = F_3 + F_2 = 2 + 1 = 3$$

$$F_5 = F_4 + F_3 = 3 + 2 = 5$$

$$\ldots$$

EXAMPLE

Suppose that the number of bacteria in a colony triples every hour.

a) Set up a **recurrence relation** for the number of bacteria after **n hours** have elapsed.

Solution:

a_n is the number of bacteria after n hours

a_0 is the number of bacteria used to begin a new colony.

$$a_n = 3a_{n-1}$$

b) If **100 bacteria** are used to begin a new colony, how many bacteria will be in the colony in **10 hours**?

$$
\begin{aligned}
a_{n} &= 3a_{n-1} \\
&= 3*(3a_{n-2}) = 3^{2}\, a_{n-2} \\
&= 3^{2} * 3a_{n-3} = 3^{3}\, a_{n-3} \\
&. \\
&. \\
&. \\
&= 3^{n}\, a_{n-n} \\
&= 3^{n}\, a_{0}
\end{aligned}
$$

Where $a_0 = 100$, so $a_{10} = 3^{10}\, a_0 = 3^{10} * 100 = 5{,}904{,}900$

EXAMPLE

Assume that the population of the world in 1995 is 7 billion and is growing 3% a year.
a) Set up a recurrence relation for the population of the world n years after 1995.

Solution:

$\mathbf{a_n}$ is the population of world n years after 1995

$\mathbf{a_0}$ is the population of world in 1995, which is 7 billion

$$\mathbf{a_n = a_{n-1} + 3\%\ a_{n-1}}$$

$$\mathbf{a_n = 1.03a_{n-1}}$$

b) Find an explicit formula for the population of the world n years after 1995.

Solution:

$$a_n = 1.03a_{n-1}$$
$$= 1.03*(1.03a_{n-2}) = 1.03^2\, a_{n-2}$$
$$= 1.03^2 * 1.03a_{n-3} = 1.03^3\, a_{n-3}$$

.

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$$= 1.03^n\, a_{n-n}$$
$$= 1.03^n\, a_0$$
$$= 7 * 1.03^n$$

Where $a_0 = 7$ billion.

c) What will be the population of the world be in 2010?

Solution:

$$2010 - 1995 = 15$$

$$a_{15} = 7 * 1.03^{15} = 10.9 \text{ billion}$$

Chapter Summary

- Applications of Recurrence Relations
- Solving Linear Recurrence Relations
  - Homogeneous Recurrence Relations
  - Nonhomogeneous Recurrence Relations
- Divide-and-Conquer Algorithms and Recurrence Relations
- Generating Functions
- Inclusion-Exclusion
- Applications of Inclusion-Exclusion

Applications of Recurrence Relations

Section 8.1

Recurrence Relations
(recalling definitions from Chapter 2)

Definition: A *recurrence relation* for the sequence $\{a_n\}$ is an equation that expresses a_n in terms of one or more of the previous terms of the sequence, namely, $a_0, a_1, \ldots, a_{n-1}$, for all integers n with $n \geq n_0$, where n_0 is a nonnegative integer.

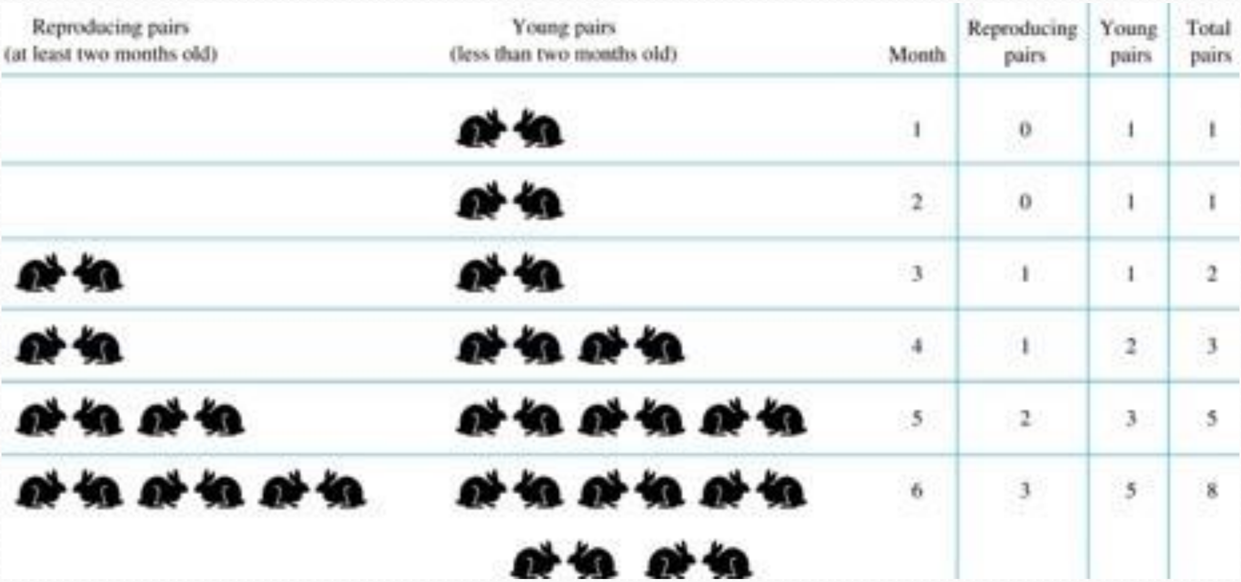
- A sequence is called a *solution* of a recurrence relation if its terms satisfy the recurrence relation.
- The *initial conditions* for a sequence specify the terms that precede the first term where the recurrence relation takes effect.

Rabbits and the Fiobonacci Numbers

Example: A young pair of rabbits (one of each gender) is placed on an island. A pair of rabbits does not breed until they are 2 months old. After they are 2 months old, each pair of rabbits produces another pair each month. Find a recurrence relation for the number of pairs of rabbits on the island after n months, assuming that rabbits never die.

This is the original problem considered by Leonardo Pisano (Fibonacci) in the thirteenth century.

Rabbits and the Fiobonacci Numbers (*cont.*)



Reproducing pairs (at least two months old)	Young pairs (less than two months old)	Month	Reproducing pairs	Young pairs	Total pairs
		1	0	1	1
		2	0	1	1
		3	1	1	2
		4	1	2	3
		5	2	3	5
		6	3	5	8

Modeling the Population Growth of Rabbits on an Island

Rabbits and the Fibonacci Numbers (*cont.*)

Solution: Let f_n be the the number of pairs of rabbits after n months.

- There are is $f_1 = 1$ pairs of rabbits on the island at the end of the first month.
- We also have $f_2 = 1$ because the pair does not breed during the first month.
- To find the number of pairs on the island after n months, add the number on the island after the previous month, f_{n-1}, and the number of newborn pairs, which equals f_{n-2}, because each newborn pair comes from a pair at least two months old.

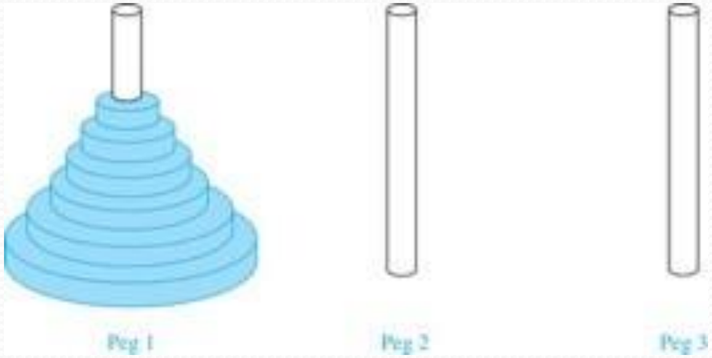
Consequently the sequence $\{f_n\}$ satisfies the recurrence relation $f_n = f_{n-1} + f_{n-2}$ for $n \geq 3$ with the initial conditions $f_1 = 1$ and $f_2 = 1$.
The number of pairs of rabbits on the island after n months is given by the nth Fibonacci number.

The Tower of Hanoi

In the late nineteenth century, the French mathematician Édouard Lucas invented a puzzle consisting of three pegs on a board with disks of different sizes. Initially all of the disks are on the first peg in order of size, with the largest on the bottom.

Rules: You are allowed to move the disks one at a time from one peg to another as long as a larger disk is never placed on a smaller.
Goal: Using allowable moves, end up with all the disks on the second peg in order of size with largest on the bottom.

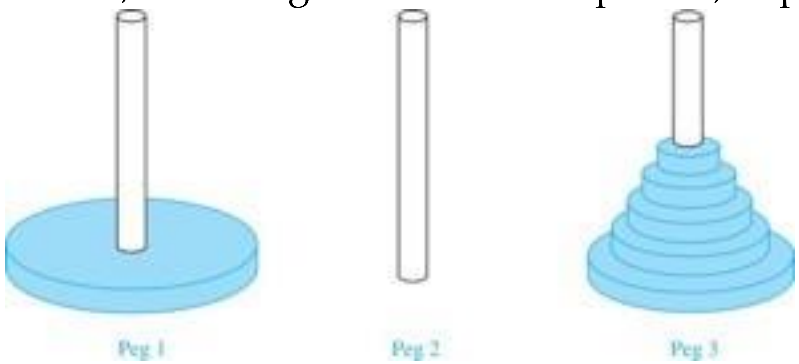
The Tower of Hanoi (*continued*)



The Initial Position in the Tower of Hanoi Puzzle

The Tower of Hanoi (*continued*)

Solution: Let $\{H_n\}$ denote the number of moves needed to solve the Tower of Hanoi Puzzle with n disks. Set up a recurrence relation for the sequence $\{H_n\}$. Begin with n disks on peg 1. We can transfer the top $n-1$ disks, following the rules of the puzzle, to peg 3 using H_{n-1} moves.



First, we use 1 move to transfer the largest disk to the second peg. Then we transfer the $n-1$ disks from peg 3 to peg 2 using H_{n-1} additional moves. This can not be done in fewer steps. Hence,

$$H_n = 2H_{n-1} + 1.$$

The initial condition is $H_1 = 1$ since a single disk can be transferred from peg 1 to peg 2 in one move.

The Tower of Hanoi (*continued*)

- We can use an iterative approach to solve this recurrence relation by repeatedly expressing H_n in terms of the previous terms of the sequence.

$$H_n = 2H_{n-1} + 1$$
$$= 2(2H_{n-2} + 1) + 1 = 2^2 H_{n-2} + 2 + 1$$
$$= 2^2(2H_{n-3} + 1) + 2 + 1 = 2^3 H_{n-3} + 2^2 + 2 + 1$$
$$\vdots$$
$$= 2^{n-1}H_1 + 2^{n-2} + 2^{n-3} + \ldots + 2 + 1$$
$$= 2^{n-1} + 2^{n-2} + 2^{n-3} + \ldots + 2 + 1 \qquad \text{because } H_1 = 1$$
$$= 2^n - 1 \qquad \text{using the formula for the sum of the terms of a geometric series}$$

  - There was a myth created with the puzzle. Monks in a tower in Hanoi are transferring 64 gold disks from one peg to another following the rules of the puzzle. They move one disk each day. When the puzzle is finished, the world will end.
  - Using this formula for the 64 gold disks of the myth,

    $$2^{64} - 1 = 18{,}446{,}744{,}073{,}709{,}551{,}615$$

    days are needed to solve the puzzle, which is more than 500 billion years.
  - Reve's puzzle (proposed in 1907 by Henry Dudeney) is similar but has 4 pegs. There is a well-known unsettled conjecture for the the minimum number of moves needed to solve this puzzle. (*see Exercises* 38-45)

Counting Bit Strings

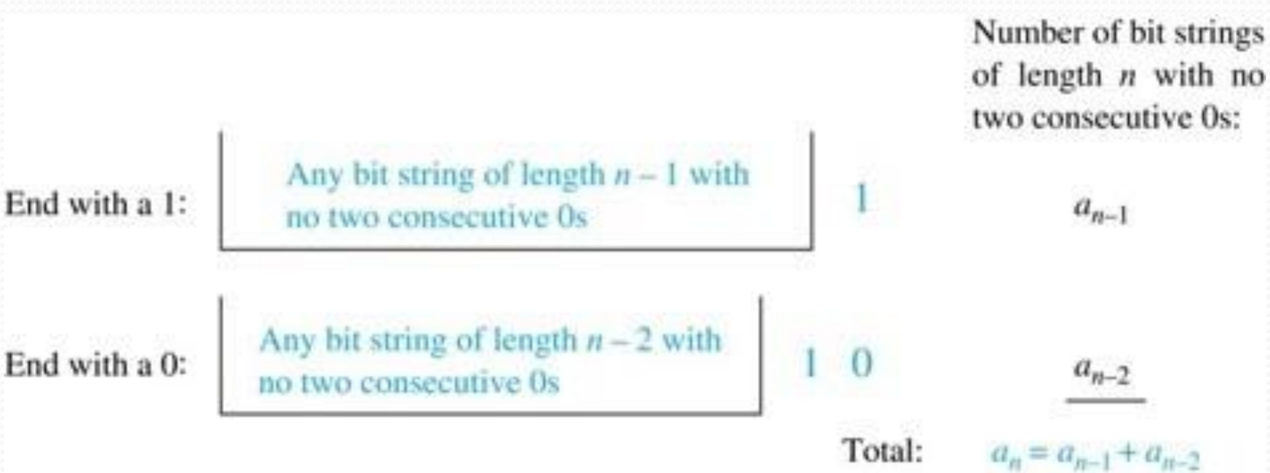
Example 3: Find a recurrence relation and give initial conditions for the number of bit strings of length n without two consecutive 0s. How many such bit strings are there of length five?

Solution: Let a_n denote the number of bit strings of length n without two consecutive 0s. Toobtain a recurrence relation for $\{a_n\}$ note that the number of bit strings of length n that do not have two consecutive 0s is the number of bit strings ending with a 0 plus the number of such bit strings ending with a 1.

Now assume that $n \geq 3$.

- The bit strings of length n ending with 1 without two consecutive 0s are the bit strings of length n −1 with no two consecutive 0s with a 1 at the end. Hence, there are a_{n-1} such bit strings.
- The bit strings of length n ending with 0 without two consecutive 0s are the bit strings of length n −2 with no two consecutive 0s with 10 at the end. Hence, there are a_{n-2} such bit strings.

We conclude that $a_n = a_{n-1} + a_{n-2}$ for $n \geq 3$.



Bit Strings (*continued*)

The initial conditions are:

- $a_1 = 2$, since both the bit strings 0 and 1 do not have consecutive 0s.
- $a_2 = 3$, since the bit strings 01, 10, and 11 do not have consecutive 0s, while 00 does.

To obtain a_5, we use the recurrence relation three times to find that:

- $a_3 = a_2 + a_1 = 3 + 2 = 5$
- $a_4 = a_3 + a_2 = 5 + 3 = 8$
- $a_5 = a_4 + a_3 = 8 + 5 = 13$

Note that $\{a_n\}$ satisfies the same recurrence relation as the Fibonacci sequence. Since $a_1 = f_3$ and $a_2 = f_4$, we conclude that $a_n = f_{n+2}$.

Solving Linear Recurrence Relations

Section 8.2

Section Summary

- Linear Homogeneous Recurrence Relations
- Solving Linear Homogeneous Recurrence Relations with Constant Coefficients.
- Solving Linear Nonhomogeneous Recurrence Relations with Constant Coefficients.

Linear Homogeneous Recurrence Relations

Definition: A *linear homogeneous recurrence relation of degree* k *with constant coefficients* is a recurrence relation of the form $a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots\ldots + c_k\, a_{n-k}$, where $c_1, c_2, \ldots, c_k$ are real numbers, and $c_k \neq 0$

•it is *linear* because the right-hand side is a sum of the previous terms of the sequence each multiplied by a function of n.
• it is *homogeneous* because no terms occur that are not multiples of the a_js. Each coefficient is a constant.
• the *degree* is k because a_n is expressed in terms of the previous k terms of the sequence.

By strong induction, a sequence satisfying such a recurrence relation is uniquely determined by the recurrence relation and the k initial conditions $a_0 = C_1$, $a_0 = C_1$,… , $a_{k-1} = C_{k-1}$.

Examples of Linear Homogeneous Recurrence Relations

- $P_n = (1.11)P_{n-1}$ linear homogeneous recurrence relation of degree one
- $f_n = f_{n-1} + f_{n-2}$ linear homogeneous recurrence relation of degree two
- $a_n = a_{n-1} + a_{n-2}^2$ not linear
- $H_n = 2H_{n-1} + 1$ not homogeneous
- $B_n = nB_{n-1}$ coefficients are not constants

Solving Linear Homogeneous Recurrence Relations

- The basic approach is to look for solutions of the form $a_n = r^n$, where r is a constant.
- Note that $a_n = r^n$ is a solution to the recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}$ if and only if $r^n = c_1 r^{n-1} + c_2 r^{n-2} + \cdots + c_k r^{n-k}$.
- Algebraic manipulation yields the *characteristic equation*:
  $r^k - c_1 r^{k-1} - c_2 r^{k-2} - \cdots - c_{k-1} r - c_k = 0$
- The sequence $\{a_n\}$ with $a_n = r^n$ is a solution if and only if r is a solution to the characteristic equation.
- The solutions to the characteristic equation are called the *characteristic roots* of the recurrence relation. The roots are used to give an explicit formula for all the solutions of the recurrence relation.

Solving Linear Homogeneous Recurrence Relations of Degree Two

Theorem 1: Let c_1 and c_2 be real numbers. Suppose that $r^2 - c_1 r - c_2 = 0$ has two distinct roots r_1 and r_2. Then the sequence $\{a_n\}$ is a solution to the recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2}$ if and only if

$$a_n = \alpha r_1^n + \alpha_2 r_2^n$$

for $n = 0,1,2,\ldots$, where α_1 and α_2 are constants.

Using Theorem 1

Example: What is the solution to the recurrence relation

$$a_n = a_{n-1} + 2a_{n-2} \text{ with } a_0 = 2 \text{ and } a_1 = 7?$$

Solution: The characteristic equation is $r^2 - r - 2 = 0$.
Its roots are $r = 2$ and $r = -1$. Therefore, $\{a_n\}$ is a solution to the recurrence relation if and only if $a_n = \alpha_1 2^n + \alpha_2(-1)^n$, for some constants α_1 and α_2.

To find the constants α_1 and α_2, note that

$$a_0 = 2 = \alpha_1 + \alpha_2 \text{ and } a_1 = 7 = \alpha_1 2 + \alpha_2(-1).$$

Solving these equations, we find that $\alpha_1 = 3$ and $\alpha_2 = -1$.

Hence, the solution is the sequence $\{a_n\}$ with $a_n = 3 \cdot 2^n - (-1)^n$.

An Explicit Formula for the Fibonacci Numbers

We can use Theorem 1 to find an explicit formula for the Fibonacci numbers. The sequence of Fibonacci numbers satisfies the recurrence relation $f_n = f_{n-1} + f_{n-2}$ with the initial conditions: $f_0 = 0$ and $f_1 = 1$.

Solution: The roots of the characteristic equation $r^2 - r - 1 = 0$ are

$$r_1 = \frac{1+\sqrt{5}}{2}$$

$$r_2 = \frac{1-\sqrt{5}}{2}$$

Fibonacci Numbers (*continued*)

Therefore by Theorem 1

$$f_n = \alpha_1 \left(\frac{1+\sqrt{5}}{2}\right)^n + \alpha_2 \left(\frac{1-\sqrt{5}}{2}\right)^n$$

for some constants α_1 and α_2.

Using the initial conditions $f_0 = 0$ and $f_1 = 1$, we have

$$f_0 = \alpha_1 + \alpha_2 = 0$$

$$f_1 = \alpha_1 \left(\frac{1+\sqrt{5}}{2}\right) + \alpha_2 \left(\frac{1-\sqrt{5}}{2}\right) = 1.$$

Solving, we obtain $\alpha_1 = \frac{1}{\sqrt{5}}$, $\alpha_2 = -\frac{1}{\sqrt{5}}$.

Hence,

$$f_n = \frac{1}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}} \left(\frac{1-\sqrt{5}}{2}\right)^n$$

The Solution when there is a Repeated Root

Theorem 2: Let c_1 and c_2 be real numbers with $c_2 \neq 0$. Suppose that $r^2 - c_1 r - c_2 = 0$ has one repeated root r_0. Then the sequence $\{a_n\}$ is a solution to the recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2}$ if and only if

$$a_n = \alpha r_0^n + \alpha_2 n r_0^n$$

for $n = 0,1,2,\ldots$, where α_1 and α_2 are constants.

Using Theorem 2

Example: What is the solution to the recurrence relation
$a_n = 6a_{n-1} - 9a_{n-2}$ with $a_0 = 1$ and $a_1 = 6$?

Solution: The characteristic equation is $r^2 - 6r + 9 = 0$.
The only root is $r = 3$. Therefore, $\{a_n\}$ is a solution to the recurrence relation if and only if

$$a_n = \alpha_1 3^n + \alpha_2 n(3)^n$$

where α_1 and α_2 are constants.

To find the constants α_1 and α_2, note that

$$a_0 = 1 = \alpha_1 \quad \text{and} \quad a_1 = 6 = \alpha_1 \cdot 3 + \alpha_2 \cdot 3.$$

Solving, we find that $\alpha_1 = 1$ and $\alpha_2 = 1$.
Hence,

$$a_n = 3^n + n3^n .$$

Solving Linear Homogeneous Recurrence Relations of Arbitrary Degree

This theorem can be used to solve linear homogeneous recurrence relations with constant coefficients of any degree when the characteristic equation has distinct roots.

Theorem 3: Let $c_1, c_2, \ldots, c_k$ be real numbers. Suppose that the characteristic equation

$$r^k - c_1 r^{k-1} - \cdots - c_k = 0$$

has k distinct roots $r_1, r_2, \ldots, r_k$. Then a sequence $\{a_n\}$ is a solution of the recurrence relation

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots.. + c_k a_{n-k}$$

if and only if

$$a_n = \alpha_1 r_1^n + \alpha_2 r_2^m + \cdots + \alpha_k r_k^n$$

for $n = 0, 1, 2, \ldots$, where $\alpha_1, \alpha_2, \ldots, \alpha_k$ are constants.

The General Case with Repeated Roots Allowed

Theorem 4: Let c_1, c_2 ,…, c_k be real numbers. Suppose that the characteristic equation

$$r^k - c_1 r^{k-1} - \ldots - c_k = 0$$

has t distinct roots $r_1, r_2, \ldots, r_t$ with multiplicities $m_1, m_2, \ldots, m_t$, respectively so that $m_i \geq 1$ for $i = 0, 1, 2, \ldots, t$ and $m_1 + m_2 + \ldots + m_t = k$. Then a sequence $\{a_n\}$ is a solution of the recurrence relation

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots.. + c_k a_{n-k}$$

if and only if

$$\begin{aligned} a_n = {} & (\alpha_{1,0} + \alpha_{1,1} n + \cdots + \alpha_{1,m_1-1} n^{m_1-1}) r_1^n \\ & + (\alpha_{2,0} + \alpha_{2,1} n + \cdots + \alpha_{2,m_2-1} n^{m_2-1}) r_2^n \\ & + \cdots + (\alpha_{t,0} + \alpha_{t,1} n + \cdots + \alpha_{t,m_t-1} n^{m_t-1}) r_t^n \end{aligned}$$

for $n = 0, 1, 2, \ldots$, where $\alpha_{i,j}$ are constants for $1 \leq i \leq t$ and $0 \leq j \leq m_{i-1}$.