Discrete Mathematics (ITPC-309)

Algebraic Structures - Part IV

Dr. Sanga Chaki

Department of Information Technology

Dr. B. R. Ambedkar National Institute of Technology, Jalandhar

Contents

1. Cosets
2. Factor group
3. Permutation groups
4. Normal subgroup

Cosets

1. Let H be a subgroup of the group G whose operation is written multiplicatively.
2. Given an element g of G, the left cosets of H in G are the sets obtained by multiplying each element of H by a fixed element g of G

$$gH = \{gh : h \text{ an element of } H\} \text{ for } g \text{ in } G.$$

3. This is the left coset, the element g is a left factor
4. The right cosets are defined similarly, the element g is now a right factor

$$Hg = \{hg : h \text{ an element of } H\} \text{ for } g \text{ in } G.$$

5. Any two left cosets (respectively right cosets) are either disjoint or are identical as sets

Cosets - Example

1. Let G be {I, a, a^2, b, ab, a^2b}.
2. In this group, $a^3 = b^2 = I$ and $ba = a^2b$. The Cayley table (a Cayley table describes the structure of a finite group by arranging all the possible products of all the group's elements in a square table):

$*$	I	a	a^2	b	ab	a^2b
I	I	a	a^2	b	ab	a^2b
a	a	a^2	I	ab	a^2b	b
a^2	a^2	I	a	a^2b	b	ab
b	b	a^2b	ab	I	a^2	a
ab	ab	b	a^2b	a	I	a^2
a^2b	a^2b	ab	b	a^2	a	I

1. Let T be the subgroup {I, b}.
2. The (distinct) left cosets of T are:

IT = T = {I, b}, aT = {a, ab}, and a^2T = {a^2, a^2b}.

1. Since all the elements of G have appeared in one of these cosets, generating any more can not give new cosets.
2. For eg, abT = {ab, a} = aT.

Cosets

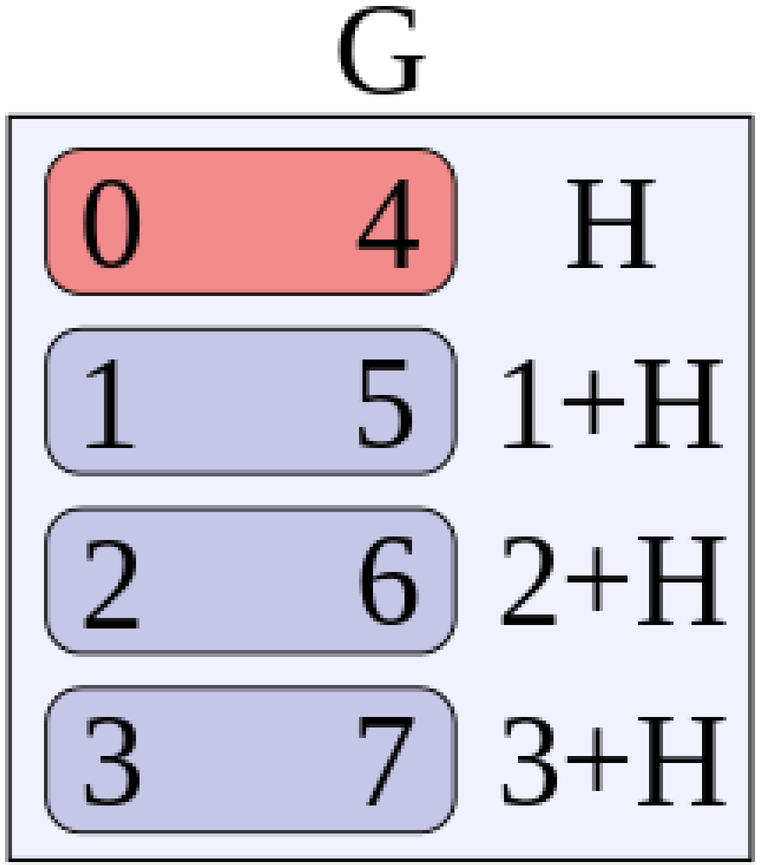
1. The right cosets of T are:
   - TI = T = {I, b},
   - Ta = {a, ba} = {a, a2b} , and
   - Ta2 = {a2, ba2} = {a2, ab}.

Some properties of cosets

1. A subgroup H of a group G may be used to decompose the underlying set of G into disjoint, equal-size subsets called cosets.
2. There are left cosets and right cosets.
3. Cosets (both left and right) have the same number of elements (cardinality) as does H.
4. Furthermore, H itself is both a left coset and a right coset.
5. The number of left cosets of H in G is equal to the number of right cosets of H in G. This common value is called the index of H in G and is usually denoted by [G : H].

Coset Example:

1. For the group $(Z_8, +)$ where the binary operation is addition modulo 8, and if H = {0, 4}
2. There are four left cosets of H: H itself, 1 + H, 2 + H, and 3 + H (written using additive notation since this is the additive group).
3. Together they partition the entire group G into equal-size, non-overlapping sets. The index [G : H] is 4.
4. Find the right cosets.



Normal subgroup

1. A subgroup H of a group G is normal in G if gH=Hg for all g∈G.
2. That is, a normal subgroup of a group G is one in which the right and left cosets are precisely the same.
3. Example:

Let G be an abelian group. Every subgroup H of G is a normal subgroup.

Solution

Since $gh = hg$ for all $g \in G$ and $h \in H$, it will always be the case that $gH = Hg$.

Factor groups and Permutation groups

1. If N is a normal subgroup of a group G, then the cosets of N in G form a group G/N under the operation (aN)(bN)=abN. [This is pronounced G mod N]
2. This group is called the factor or quotient group of G and N.
3. A permutation group is a group G whose elements are permutations of a given set M and whose group operation is the composition of permutations in G (which are thought of as bijective functions from the set M to itself).
4. For instance, a particular permutation of the set {1, 2, 3, 4, 5} can be written as follows, where σ satisfies σ(1) = 2, σ(2) = 5, σ(3) = 4, σ(4) = 3, and σ(5) = 1

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 5 & 4 & 3 & 1 \end{pmatrix}$$

5. The elements of M need not appear in any special order in the first row, so the same permutation could also be written as

$$\sigma = \begin{pmatrix} 3 & 2 & 5 & 1 & 4 \\ 4 & 5 & 1 & 2 & 3 \end{pmatrix}$$