Discrete Mathematics (ITPC-309)

The Foundations: Logic and Proofs

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Why logic and proofs?

1. To explain what makes up a correct mathematical argument and
2. To introduce tools to construct these arguments
3. importance of logic in **understanding mathematical reasoning**
4. Use: rules are used in the
    - design of computer circuits,
    - the construction of computer programs,
    - the verification of the correctness of programs

Propositional Logic

1. Propositions:
   - A proposition is a declarative sentence that is either true or false, but not both.
   - declares a fact
2. Examples:
   - Jalandhar is the capital of India
   - $1 + 1 = 2$.
   - $2 + 2 = 3$.
3. Examples of not propositions:
   - Did you pass the exam? - not declarative sentences
   - $x + y = z$. - neither true nor false

Propositional Logic

1. We use letters to denote **propositional variables/statement variables**, that is, variables that represent propositions. Eg. ***p, q, r, s, . . .***
2. The **truth value** of a proposition is true (T), if it is a true proposition.
3. The truth value of a proposition is false, denoted by F, if it is a false proposition.
4. The area of logic that deals with propositions is called the propositional calculus or **propositional logic**

Propositional Logic

1. There are some methods for producing new propositions from known propositions – developed by George Boole

2. Many mathematical statements are constructed by combining one or more propositions.

3. New propositions, called **compound propositions**, are formed from existing propositions using **logical operators**

Propositional Logic - Negation

1. Let p be a proposition.
2. The negation of p, denoted by ¬p (also denoted by p), is the statement "It is not the case that p."
3. The proposition ¬p is read "not p."
4. The truth values of p and ¬p are opposites of each other
5. Example:
   - P = You have received 90% marks
   - ¬p = It is not the case that you have received 90% marks or
   - You have received less than 90% marks

Propositional Logic - Negation

1. **Truth Table**:

p	$\neg p$
T	F
F	T

2. The negation of a proposition can also be considered the result of the operation of the **negation operator** on a proposition.
3. It constructs a new proposition from a single existing proposition.
4. **Connectives**: logical operators that are used to form new propositions from two or more existing propositions

Propositional Logic - Connectives

1. Let p and q be propositions.
2. Conjunction:
   1. The conjunction of p and q, denoted by $p \land q$, is the proposition “p and q.”
   2. The conjunction $p \land q$ is true when both p and q are true and is false otherwise

p	q	$p \land q$
T	T	T
T	F	F
F	T	F
F	F	F

p	q	$p \lor q$
T	T	T
T	F	T
F	T	T
F	F	F

3. Disjunction:
   - The disjunction of p and q, denoted by $p \lor q$, is the proposition “p or q.”
   - The disjunction $p \lor q$ is false when both p and q are false and is true otherwise

Propositional Logic - Connectives

1. Exclusive or:
   1. The exclusive or of p and q, denoted by p $\oplus$ q, is the proposition that is true when exactly one of p and q is true and is false otherwise.

p	q	$p \oplus q$
T	T	F
T	F	T
F	T	T
F	F	F

Propositional Logic - Conditional Statements

1. The conditional statement $p \rightarrow q$ is the proposition "if p, then q."
2. false when p is true and q is false, and true otherwise
3. P is called the hypothesis
4. q is called the conclusion
5. Example: If I am elected, then I will lower taxes – what are the two propositions?

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

Propositional Logic - Converse, contrapositive, and inverse

1. We can form some new conditional statements starting with a conditional statement $p \rightarrow q$
2. The proposition $q \rightarrow p$ is called the **converse** of $p \rightarrow q$.
3. The **contrapositive** of $p \rightarrow q$ is the proposition $\neg q \rightarrow \neg p$.
4. The proposition $\neg p \rightarrow \neg q$ is called the **inverse** of $p \rightarrow q$.
5. Do: draw the truth tables for these, and find which has same truth table as the proposition $p \rightarrow q$
6. only the contrapositive always has the same truth value as $p \rightarrow q$

Propositional Logic - Biconditionals

1. The biconditional statement $p \leftrightarrow q$ is the proposition "p if and only if q."
2. The biconditional statement $p \leftrightarrow q$ is true when p and q have the same truth values, and is false otherwise.
3. Biconditional statements are also called bi-implications.
4. $p \leftrightarrow q$ has exactly the same truth value as $(p \rightarrow q) \land (q \rightarrow p)$.

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

Truth Tables of Compound Propositions

1. Construct the truth table of the compound proposition

$(p \vee \neg q) \rightarrow (p \wedge q)$.

p	q	$\neg q$	$p \vee \neg q$	$p \wedge q$	$(p \vee \neg q) \rightarrow (p \wedge q)$
T	T	F	T	T	T
T	F	T	T	F	F
F	T	F	F	F	T
F	F	T	T	F	F

1. Construct the truth table for $(p \rightarrow q) \wedge (q \rightarrow p)$ and compare it to $p \leftrightarrow q$.

Tautology, Contradiction, Contingency

1. Compound proposition: an expression formed from propositional variables using logical operators, such as p ∧ q.
2. A compound proposition that is always true, no matter what the truth values of the propositional variables that occur in it, is called a **tautology**.
3. A compound proposition that is always false is called a **contradiction**.
4. A compound proposition that is neither a tautology nor a contradiction is called a **contingency**.
5. Example:
   - p ∨¬p is always true, it is a tautology.
   - p ∧¬p is always false, it is a contradiction.

p	$\neg p$	$p \vee \neg p$	$p \wedge \neg p$
T	F	T	F
F	T	T	F

Propositional Equivalences: Logical Equivalences

1. **Compound propositions that have the same truth values in all possible cases are called logically equivalent**
2. The compound propositions p and q are called logically equivalent if $p \leftrightarrow q$ (p biconditional q) is a tautology.
3. The **notation $p \equiv q$** denotes that p and q are logically equivalent.
4. How to determine whether two compound propositions are equivalent?
   - use a truth table
   - Use a series of logical equivalences
5. Example:
   - **De Morgan laws equivalences**

De Morgan laws

1. Show that **¬(p ∨ q) and ¬p ∧¬q** are logically equivalent

p	q	$p \vee q$	$\neg(p \vee q)$	$\neg p$	$\neg q$	$\neg p \wedge \neg q$
T	T	T	F	F	F	F
T	F	T	F	F	T	F
F	T	T	F	T	F	F
F	F	F	T	T	T	T

2. Show that **¬(p ∧ q) ≡ ¬p ∨¬q** are logically equivalent
3. Prove these two are equivalent using truth tables.

Logical equivalence

1. Show that **$p \vee (q \wedge r)$ and $(p \vee q) \wedge (p \vee r)$** are logically equivalent.
2. This is the distributive law of disjunction over conjunction

p	q	r	$q \wedge r$	$p \vee (q \wedge r)$	$p \vee q$	$p \vee r$	$(p \vee q) \wedge (p \vee r)$
T	T	T	T	T	T	T	T
T	T	F	F	T	T	T	T
T	F	T	F	T	T	T	T
T	F	F	F	T	T	T	T
F	T	T	T	T	T	T	T
F	T	F	F	F	T	F	F
F	F	T	F	F	F	T	F
F	F	F	F	F	F	F	F

Logical equivalence

1. There are many such Logical Equivalences like
   - Identity laws,
   - Double negation law,
   - Commutative laws,
   - Associative laws,
   - De Morgan's laws,
   - Logical Equivalences Involving Conditional and Biconditional Statements
2. Extension of De Morgan laws:
   - $\neg(p \vee q) \equiv \neg p \wedge \neg q$ -------------→
     $\neg(p_1 \vee p_2 \vee \cdots \vee p_n) \equiv (\neg p_1 \wedge \neg p_2 \wedge \cdots \wedge \neg p_n)$
   - And $\neg(p_1 \wedge p_2 \wedge \cdots \wedge p_n) \equiv (\neg p_1 \vee \neg p_2 \vee \cdots \vee \neg p_n)$.

Constructing New Logical Equivalences

1. Established Logical Equivalences can be used to construct additional logical equivalences
2. This is allowed because a proposition in a compound proposition can be replaced by a compound proposition that is logically equivalent to it without changing the truth value of the original compound proposition
3. Show that $\neg(p \rightarrow q)$ and $p \wedge \neg q$ are logically equivalent by developing a series of logical equivalences.
   - $\neg(p \rightarrow q) \equiv \neg(\neg p \vee q)$
   - $\equiv \neg(\neg p) \wedge \neg q$ by the second De Morgan law
   - $\equiv p \wedge \neg q$ by the double negation law

Predicates and Quantifiers

1. Propositional logic cannot adequately express the meaning of all statements in mathematics and in natural language
2. We use a more powerful type of logic called predicate logic
3. used to express the meaning of a wide range of statements
4. permit us to reason and explore relationships between objects
5. For this we need predicates and quantifiers

Predicates

1. Some statements are neither true nor false when the values of the variables are not specified
   - $x > 3$
   - $x = y + 3$
   - $x + y = z$
   - computer x is under attack by an intruder
2. How propositions can be produced from such statements?
3. For the first example, The first part, the variable x, is the **subject** of the statement.
4. The second part—**the predicate**, "is greater than 3"—refers to a property that the subject of the statement can have.

Predicates

1. We denote the statement "x is greater than 3" by P(x), where P denotes the predicate "is greater than 3" and x is the variable
2. Once a value has been assigned to the variable x, the statement P(x) becomes a proposition and has a truth value
3. **Example: Let P(x) denote the statement "x > 3." What are the truth values of P(4) and P(2)? ------- true and false respectively**

Predicates

1. We can also have statements that involve more than one variable
2. Example: Let Q(x, y) denote the statement “x = y + 3.” What are the truth values of the propositions Q(1, 2) and Q(3, 0)?
   - false and true
3. Example: let R(x, y, z) denote the statement`‘x + y = z.” What are the truth values of the propositions R(1, 2, 3) and R(0, 0, 1)?
   - True and false
4. **In general, a statement involving the n variables $x_1, x_2, \ldots, x_n$ can be denoted by $P(x_1, x_2, \ldots, x_n)$.**
5. P is also called an n-place predicate or a **n-ary predicate**.

Predicates

1. Where do we use these in computer programs?
   - Conditional statements
2. Example:
   - Consider the statement
   - if $x > 0$ then $x := x + 1$.
   - When this statement is encountered in a program, the value of the variable x at that point in the execution of the program is inserted into P(x), which is “$x > 0$.”
   - If P(x) is true for this value of x, the assignment statement $x := x + 1$ is executed, x is incremented
   - Otherwise not.

Quantifiers

1. **Quantification expresses the extent to which a predicate is true over a range of elements**
2. We focus on two types of quantification here:
   - **universal quantification**, which tells us that a predicate is true for every element under consideration, and
   - **existential quantification**, which tells us that there is one or more element under consideration for which the predicate is true.
3. The area of logic that deals with predicates and quantifiers is called the **predicate calculus**.

The Universal Quantifier

1. The universal quantification of $P(x)$ is the statement
2. “$P(x)$ for all values of x in the domain.”
3. The notation $\forall x\, P(x)$ denotes the universal quantification of $P(x)$.
4. Here $\forall$ is called the universal quantifier.
5. We read $\forall x\, P(x)$ as “for all x $P(x)$” or “for every x $P(x)$.”
6. **An element for which $P(x)$ is false is called a counterexample of $\forall x\, P(x)$.**

The Universal Quantifier

1. Example: Let $P(x)$ be the statement "$x + 1 > x$." What is the truth value of the quantification $\forall x\, P(x)$, where the domain consists of all real numbers?
   - Solution: Because $P(x)$ is true for all real numbers x, the quantification $\forall x\, P(x)$ is true.
2. Let $Q(x)$ be the statement "$x < 2$." What is the truth value of the quantification $\forall x\, Q(x)$, where the domain consists of all real numbers?
   - Solution: $Q(x)$ is not true for every real number x, because, for instance, $Q(3)$ is false. That is, $x = 3$ is a counterexample for the statement $\forall x\, Q(x)$. Thus $\forall x\, Q(x)$ is false.
3. Suppose that $P(x)$ is "$x^2 > 0$." To show that the statement $\forall x\, P(x)$ is false where the universe of discourse consists of all integers, we give a counterexample.
   - We see that $x = 0$ is a counterexample because $x^2 = 0$ when $x = 0$, so that x^2 is not greater than 0 when $x = 0$.

The Universal Quantifier

1. When all the elements in the domain can be listed—say, $x_1, x_2, \ldots, x_n$—it follows that the universal quantification $\forall x\, P(x)$ is the same as the conjunction $P(x_1) \wedge P(x_2) \wedge \cdots \wedge P(x_n)$, because this conjunction is true if and only if $P(x_1), P(x_2), \ldots, P(x_n)$ are all true.
2. What is the truth value of $\forall x\, P(x)$, where $P(x)$ is the statement "$x^2 < 10$" and the domain consists of the positive integers not exceeding 4?
   - Solution: The statement $\forall x\, P(x)$ is the same as the conjunction $P(1) \wedge P(2) \wedge P(3) \wedge P(4)$, because the domain consists of the integers 1, 2, 3, and 4.
   - Because $P(4)$, which is the statement "$4^2 < 10$," is false, it follows that $\forall x\, P(x)$ is false.

The Universal Quantifier

1. What is the truth value of $\forall x\, (x^2 \geq x)$ if the domain consists of all real numbers?
2. What is the truth value of this statement if the domain consists of all integers?

The Existential Quantifier

1. Many mathematical statements assert that there is an element with a certain property.
2. Such statements are expressed using existential quantification.
3. With existential quantification, we form a proposition that is true if and only if $P(x)$ is true for at least one value of x in the domain.
4. The existential quantification of $P(x)$ is the proposition "There exists an element x in the domain such that $P(x)$."
5. We use the notation $\exists x\, P(x)$ for the existential quantification of $P(x)$. Here $\exists$ is called the existential quantifier.

The Existential Quantifier

1. Let $P(x)$ denote the statement "$x > 3$." What is the truth value of the quantification $\exists x\, P(x)$, where the domain consists of all real numbers?
   - Solution: Because "$x > 3$" is sometimes true—for instance, when $x = 4$—the existential quantification of $P(x)$, which is $\exists x\, P(x)$, is true.
2. Let $Q(x)$ denote the statement "$x = x + 1$."What is the truth value of the quantification $\exists x\, Q(x)$, where the domain consists of all real numbers?
   - Solution: Because $Q(x)$ is false for every real number x, the existential quantification of $Q(x)$, which is $\exists x\, Q(x)$, is false.

The Existential Quantifier

1. What is the truth value of ∃x P(x), where P(x) is the statement "$x^2 > 10$" and the universe of discourse consists of the positive integers not exceeding 4?

   - Solution: Because the domain is {1, 2, 3, 4}, the proposition ∃x P(x) is the same as the disjunction P(1) ∨ P(2) ∨ P(3) ∨ P(4). Because P(4), which is the statement "42 > 10," is true, it follows that ∃x P(x) is true.