Discrete Mathematics (ITPC-309)

Recursion and Recurrence Relations

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Sequences

1. A sequence is a discrete structure used to represent an ordered list.
2. For example, 1, 2, 3, 5, 8 is a sequence with five terms – **finite sequence**
3. 1, 3, 9, 27, 81 , . . . , 3^n, . . . is an **infinite sequence**.
4. Arithmetic progression
5. Geometric progression
6. Example: Consider the sequence $\{a_n\}$, where $a_n = 1/n$.
7. The terms of this sequence are:

$$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots.$$

8. Example: The sequences $\{b_n\}$ with $b_n = (-1)^n$
   - If we start at n = 0, the terms are:

$$1, -1, 1, -1, 1, \ldots;$$

Sequences

1. In all the above cases, we specified sequences by providing explicit formulas for their terms
2. **Another way to specify a sequence is to provide**
   - **one or more initial terms and**
   - **a rule for determining subsequent terms from those that precede them.**
3. This is the concept of recurrence relations

Recurrence Relations

1. A **recurrence relation** for the sequence $\{a_n\}$ is an equation that expresses a_n in terms of one or more of the initial terms of the sequence, namely, $a_0, a_1, \ldots, a_{n-1}$, for all non-negative integers n

2. A sequence is called a **solution of a recurrence relation** if its terms satisfy the recurrence relation.

3. A recurrence relation is said to **recursively define a sequence**.

Recurrence Relations

1. Example: Let $\{a_n\}$ be a sequence that satisfies the recurrence relation

   $\mathbf{a_n = a_{n-1} + 3}$ for $n = 1, 2, 3, \ldots,$

   and suppose that $a_0 = 2$. What are a_1, a_2, and a_3?

- $a_1 = a_0 + 3 = 2 + 3 = 5.$
- $a_2 = 5 + 3 = 8$
- $a_3 = 8 + 3 = 11.$

Recurrence Relations

1. The **initial conditions** for a recursively defined sequence specify the terms that precede the first term where the recurrence relation takes effect.
2. For instance, the initial condition in the previous example: $a_0 = 2$
3. The Fibonacci sequence, $f_0, f_1, f_2, \ldots,$ is defined by the **initial conditions**
    - $f_0 = 0, f_1 = 1,$
4. and the **recurrence relation**
    - $\mathbf{f_n = f_{n-1} + f_{n-2}}$ **for n = 2, 3, 4, ...**

Recurrence Relations

1. We say that we have solved the recurrence relation together with the initial conditions when we find an **explicit formula**, called a **closed formula**, for the terms of the sequence.
2. A closed formula for a sequence is independent of the previous terms in the sequence

Recurrence Relations

1. **Example**: Determine whether the sequence $\{a_n\}$, where $\mathbf{a_n = 3n}$ for every nonnegative integer n, is a solution of the recurrence relation $\mathbf{a_n = 2a_{n-1} - a_{n-2}}$ for n = 2, 3, 4…

- Suppose that $a_n = 3n$ for every nonnegative integer n.
- Then, for any $n \geq 2$, we see that
- $\mathbf{a_n = 3n, a_{n-1} = 3(n-1)}$ **and** $\mathbf{a_{n-2} = 3(n-2)}$ → replace them in the recurrent relation
- $2a_{n-1} - a_{n-2} = 2(3(n - 1)) - 3(n - 2) = 3n = a_n$.
- **Therefore, $\{a_n\}$, where $a_n = 3n$, is a solution of the given recurrence relation**

Recurrence Relations

1. **Example**: Determine whether the sequence $\{a_n\}$, where $\mathbf{a_n = 2^n}$ for every nonnegative integer n, is a solution of the recurrence relation $\mathbf{a_n = 2a_{n-1} - a_{n-2}}$ for n = 2, 3, 4… **Given: $\mathbf{a_0 = 1, a_1 = 2}$**,

- Suppose that $a_n = 2^n$ for every nonnegative integer n.
- By the closed formula $\mathbf{a_n = 2^n}$, $\mathbf{a_2 = 4.}$
- But, As it is given that $a_0 = 1$, $a_1 = 2$
- Therefore, replacing these in the recurrent relation:
- $\mathbf{2a_1 - a_0 = 2 \cdot 2 - 1 = 3 \neq a_2}$
- So, $\{a_n\}$, where $a_n = 2^n$, is not a solution of the recurrence relation.

How to solve Recurrence Relations

1. **Solving a recurrence relation = finding the closed formula**
2. Many methods are there
3. Simplest: **iteration**
4. **Example**: Solve the recurrence relation and initial condition
    - $\mathbf{a_n = a_{n-1} + 3}$ **for n = 1, 2, 3, . . . , and suppose that** $\mathbf{a_1 = 2}$.

- We can successively apply the recurrence relation,
- starting with the initial condition $a_1 = 2$, and
- working upward until we reach a_n
- to deduce a closed formula for the sequence:

How to solve Recurrence Relations

$$a_2 = 2 + 3$$
$$a_3 = (2 + 3) + 3 = 2 + 3 \cdot 2$$
$$a_4 = (2 + 2 \cdot 3) + 3 = 2 + 3 \cdot 3$$
$$\vdots$$
$$a_n = a_{n-1} + 3 = (2 + 3 \cdot (n - 2)) + 3 = \underline{2 + 3(n - 1)}.$$

This is the closed formula for the given recurrence relation and the given initial condition – Iteration approach

How to solve Recurrence Relations

1. **Example**: Suppose that a person deposits INR 10,000 in a savings account at a bank yielding 11% per year with interest compounded annually. How much will be in the account after 30 years?
2. To solve this problem, let P_n denote the amount in the account after n years.
3. Because the amount in the account after n years equals the **amount in the account after (n−1) years + interest for the nth year**, we see that the sequence $\{P_n\}$ satisfies the recurrence relation

$$P_n = P_{n-1} + 0.11 P_{n-1} = (1.11) P_{n-1}.$$

How to solve Recurrence Relations

1. The initial condition is $P_0 = 10{,}000$.
2. We can use an iterative approach to find a formula for P_n

$$
\begin{aligned}
P_1 &= (1.11)P_0 \\
P_2 &= (1.11)P_1 = (1.11)^2 P_0 \\
P_3 &= (1.11)P_2 = (1.11)^3 P_0 \\
&\vdots \\
P_n &= (1.11)P_{n-1} = (1.11)^n P_0.
\end{aligned}
$$

3. So, after 30 years the account contains

$$P_{30} = (1.11)^{30} 10{,}000$$

How to solve Recurrence Relations

1. This method is called **forward substitution** – find successive terms beginning with the initial condition and ending with a_n.
2. The opposite is also possible - **backward substitution** - begin with a_n and iterated to express it in terms of falling terms of the sequence until you reach a_1.
3. Note using iteration, essentially we guess a formula for the terms of the sequence.
4. To prove that our guess is correct, we need to use the concept of **mathematical induction**

Mathematical Induction

1. **Example**: Suppose that we have an infinite ladder, and we want to know whether we can reach every step on this ladder.
2. **Given**:
   a) We can reach the first rung of the ladder.
   b) If we can reach a particular rung of the ladder, we can also reach the next rung.
3. **Can we conclude that we can reach every rung?**

Mathematical Induction

1. **Proof**: By (a), we know that we can reach the first rung of the ladder.
2. Moreover, because we can reach the first rung, by (2), we can also reach the second rung; it is the next rung after the first rung.
3. Applying (2) again, because we can reach the second rung, we can also reach the third rung and so on.
4. That is, we can show that P(n) is true for every positive integer n, where P(n) = we can reach the n^{th} rung of the ladder.
5. So we can reach every rung of the ladder – thus proved.
6. This is a proof using **Mathematical induction**.

Mathematical Induction – Formal Description

1. Mathematical induction is used to prove statements that assert that $P(n)$ is true for all positive integers n, where $P(n)$ is a propositional function.
2. A proof by mathematical induction has two parts,
   a) **a basis step**, where we show that $P(1)$ is true, and
   b) **an inductive step**, where we show that for all positive integers k,
      - if $P(k)$ is true, then $P(k + 1)$ is true.

Mathematical Induction – Formal Description

1. **Principle Of Mathematical Induction**: To prove that P(n) is true for all positive integers n, where P(n) is a propositional function, we complete two steps:
   I. BASIS STEP: We verify that P(1) is true.
   II. INDUCTIVE STEP: We show that the conditional statement $P(k) \rightarrow P(k + 1)$ is true for all positive integers k.
2. **To complete the inductive step of a proof using the principle of mathematical induction, we assume that P(k) is true for an arbitrary positive integer k and show that under this assumption, P(k + 1) must also be true**.
3. The assumption that P(k) is true is called the **inductive hypothesis**.

Mathematical Induction – Formal Description

Using quantifiers:

1. In the inductive step, we show that **$\forall k\,(P(k) \rightarrow P(k+1))$** is true, where the domain is the set of positive integers.
2. Once we complete both steps in a proof by mathematical induction, we have shown that P(n) is true for all positive integers, **$\forall n\, P(n)$** is true where the quantification is over the set of positive integers.
3. Expressed as a rule of inference, this proof technique can be stated as

$$(P(1) \wedge \forall k\,(P(k) \rightarrow P(k+1))) \rightarrow \forall n\, P(n)$$

Where the domain is the set of positive integers

Mathematical Induction

1. **Example**: Show that if n is a positive integer, then

$$1 + 2 + \cdots + n = \frac{n(n+1)}{2}.$$

2. Solution: Let P(n) be the proposition that the sum of the first n positive integersis n(n + 1)/2
3. BASIS STEP: P(1) is true, because 1 = 1(1 + 1)/2
4. INDUCTIVE STEP: For the **inductive hypothesis** we assume that P(k) holds for an arbitrary positive integer k. So

$$1 + 2 + \cdots + k = \frac{k(k+1)}{2}.$$

Mathematical Induction

1. Under this assumption, it must be shown that P(k + 1) is true, or that the following holds: (Use the formula and apply (k+1) there)

$$1 + 2 + \cdots + k + (k+1) = \frac{(k+1)[(k+1)+1]}{2} = \frac{(k+1)(k+2)}{2}$$

2. When we add k + 1 to both sides of the equation in P(k), we obtain

$$\begin{aligned} 1 + 2 + \cdots + k + (k+1) &\overset{\text{IH}}{=} \frac{k(k+1)}{2} + (k+1) \\ &= \frac{k(k+1) + 2(k+1)}{2} \\ &= \frac{(k+1)(k+2)}{2}. \end{aligned}$$

3. This last equation shows that P(k + 1) is true under the assumption that P(k) is true. This completes the inductive step.

Mathematical Induction

1. We have completed the basis step and the inductive step, so by mathematical induction we know that $P(n)$ is true for all positive integers n.
2. That is, we have proven that $1 + 2 + \cdots + n = n(n + 1)/2$ for all positive integers n.

Mathematical Induction

1. **Example**: Use mathematical induction to show that

$$1 + 2 + 2^2 + \cdots + 2^n = 2^{n+1} - 1$$

2. Solution: For the integer n Let P(n) be the proposition that

$$1 + 2 + 2^2 + \cdots + 2^n = 2^{n+1} - 1$$

3. BASIS STEP: P(0) is true because $2^0 = 1 = 2^1 - 1$. This completes the basis step.
4. INDUCTIVE STEP: For the inductive hypothesis, we assume that P(k) is true for an arbitrary nonnegative integer k. That is, we assume that

$$1 + 2 + 2^2 + \cdots + 2^k = 2^{k+1} - 1$$

Mathematical Induction

1. To carry out the inductive step using this assumption, we must show that when we assume that P(k) is true, then P(k + 1) is also true. That is, we must show that

$$1 + 2 + 2^2 + \cdots + 2^k + 2^{k+1} = 2^{(k+1)+1} - 1 = 2^{k+2} - 1$$

2. Under the assumption of P(k), we see that

$$\begin{aligned} 1 + 2 + 2^2 + \cdots + 2^k + 2^{k+1} &= (1 + 2 + 2^2 + \cdots + 2^k) + 2^{k+1} \\ &\overset{\text{IH}}{=} (2^{k+1} - 1) + 2^{k+1} \\ &= 2 \cdot 2^{k+1} - 1 \\ &= 2^{k+2} - 1. \end{aligned}$$

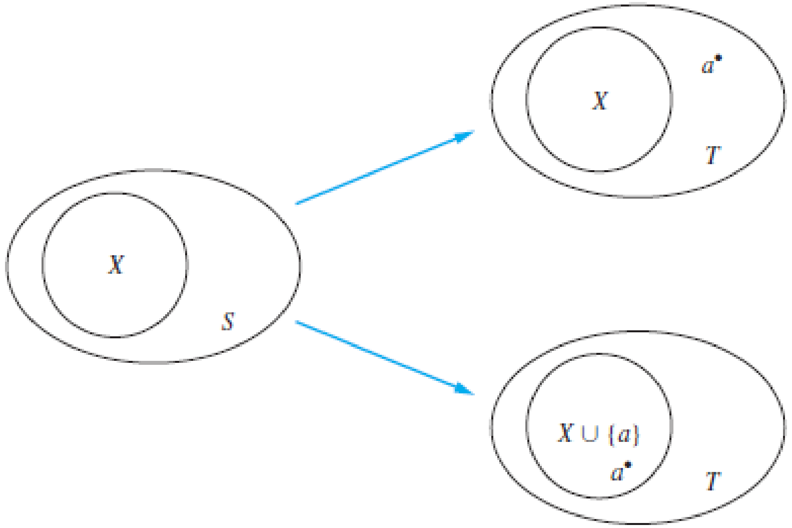
3. Because we have completed the basis step and the inductive step, by mathematical induction we know that P(n) is true for all nonnegative integers n.

Mathematical Induction

1. **Example**: Use mathematical induction to show that if S is a finite set with n elements, where n is a nonnegative integer, then S has 2^n subsets.
2. **Solution**: Let P(n) be the proposition that a set with n elements has 2^n subsets
3. **BASIS STEP**: P(0) is true, because a set with zero elements, the empty set, has exactly $2^0 = 1$ subset, namely, itself.
4. **INDUCTIVE STEP**: For the inductive hypothesis we assume that P(k) is true for an arbitrary nonnegative integer k, so that every set with k elements has 2^k subsets.
5. It must be shown that under this assumption, P(k + 1), which is the statement that every set with k + 1 elements has 2^{k+1} subsets, must also be true.

Mathematical Induction

1. To show this, let T be a set with k + 1 elements.
2. Then, it is possible to write T = S ∪ {a}, where a is one of the the elements of T and S = T − {a} (and hence |S| = k).
3. The subsets of T can be obtained in the following way.
   - For each subset X of S there are exactly two subsets of T , X and X ∪ {a}.
   - These constitute all the subsets of T and are all distinct



Mathematical Induction

1. We now use the inductive hypothesis to say that S has 2^k subsets, because it has k elements.
2. We also know that there are two subsets of T for each subset of S.
3. Therefore, there are $2 \cdot 2^k = 2^{k+1}$ subsets of T .
4. This finishes the inductive argument.

Mathematical Induction

1. **Example 1**: Use mathematical induction to prove this formula for the sum of a finite number of terms of a geometric progression with initial term a and common ratio r:

$$\sum_{j=0}^{n} ar^j = a + ar + ar^2 + \cdots + ar^n = \frac{ar^{n+1} - a}{r - 1} \qquad \text{when } r \neq 1,$$

where n is a nonnegative integer.

2. **Example 2**: Use mathematical induction to prove the following inequality for all positive integers n.

$$n < 2^n$$

Thanks!