Discrete Mathematics (ITPC-309)

Ordered Sets and Lattices – Part I

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Contents

Ordered Sets

1. Suppose R is a relation on a set S satisfying the following three properties:
   a) (Reflexive) For any $a \in S$, we have aRa.
   b) (Antisymmetric) If aRb and bRa, then $a = b$.
   c) (Transitive) If aRb and bRc, then aRc.
2. Then R is called a **partial order** or, an order relation
3. R is said to define a **partial ordering** of S
4. The set S with the partial order is called a **partially ordered set** or, an ordered set or **poset**.
5. We write (S,R) when we want to specify the relation R.

Usual order

The most familiar order relation, called the *usual order*, is the relation $\leq$ (read "less than or equal") on the positive integers $\mathbf{N}$ or, more generally, on any subset of the real numbers $\mathbf{R}$. For this reason, a partial order relation is usually denoted by $\precsim$; and

$$a \precsim b$$

is read "a precedes b." In this case we also write:

$a \prec b$ means $a \precsim b$ and $a \neq b$;	read "a strictly precedes b."
$b \succsim a$ means $a \precsim b$;	read "b succeeds a."
$b \succ a$ means $a \prec b$;	read "b strictly succeeds a."

$\not\precsim, \nprec, \not\succsim$, and $\nsucc$ are self-explanatory.

When there is no ambiguity, the symbols $\leq$, $<$, $>$, and $\geq$ are frequently used instead of $\precsim$, $\prec$, $\succ$, and $\succsim$, respectively.

Examples:

(a) Let S be any collection of sets. The relation $\subseteq$ of set inclusion is a partial ordering of S. Specifically, $A \subseteq A$ for any set A; if $A \subseteq B$ and $B \subseteq A$ then $A = B$; and if $A \subseteq B$ and $B \subseteq C$ then $A \subseteq C$.

(b) Consider the set **N** of positive integers. We say "a divides b," written $a \mid b$, if there exists an integer c such that $ac = b$. For example, $2 \mid 4$, $3 \mid 12$, $7 \mid 21$, and so on. This relation of divisibility is a partial ordering of **N**.

(c) The relation "$\mid$" of divisibility is not an ordering of the set **Z** of integers. Specifically, the relation is not antisymmetric. For instance, $2 \mid -2$ and $-2 \mid 2$, but $2 \neq -2$.

(d) Consider the set **Z** of integers. Define aRb if there is a positive integer r such that $b = a^r$. For instance, $2\,R\,8$ since $8 = 2^3$. Then R is a partial ordering of **Z**.

Dual Order

1. Let $\precsim$ is be any partial ordering of a set S.
2. The relation $\succsim$ is also a partial ordering of S and it is called the **dual order**.
3. Observe that a $\precsim$ b if and only if b $\succsim$ a
4. Hence the dual order $\succsim$ is the inverse of the relation $\precsim$

Ordered Subsets

1. Let A be a subset of an ordered set S, and suppose a, b $\in$ A.
2. Define a $\precsim$ b as elements of A whenever a $\precsim$ b are elements of S.
3. This defines a partial ordering of A called the induced order on A.
4. The subset A with the induced order is called an **ordered subset of S**.

Quasi-Order

1. Suppose $\prec$ is a relation on a set S satisfying the following two properties:
   - [Q1] (Irreflexive) For any $a \in A$, we have $a \nprec a$.
   - [Q2] (Transitive) If $a \prec b$, and $b \prec c$, then $a \prec c$.

Then $\prec$ is called a quasi-order on S.

Comparability

1. Suppose a and b are elements in a partially ordered set S.
2. We say a and b are comparable if $a \precsim b$ or $b \precsim a$
3. That is, if one of them precedes the other.
4. Thus a and b are noncomparable, written $a \parallel b$ if neither $a \precsim b$ or $b \precsim a$
5. The word “partial” is used in defining a partially ordered set S since some of the elements of S need not be comparable

Linearly Ordered Sets

1. If every pair of elements of S are comparable, then S is said to be **totally ordered or linearly ordered**, and S is called a **chain**.
2. An ordered set S may not be linearly ordered
3. But, it is possible for a subset A of S to be linearly ordered.
4. Every subset of a linearly ordered set S must also be linearly ordered.

Examples

1. Consider the set N of positive integers ordered by divisibility.
2. Then 21 and 7 are comparable since 7 | 21.
3. 3 and 5 are noncomparable since neither 3 | 5 nor 5 | 3.
4. Thus N is not linearly ordered by divisibility.
5. A = {2, 6, 12, 36} is a linearly ordered subset of N since 2 | 6, 6 | 12 and 12 | 36.

Product Sets, Order & Lexicographical Order

1. There are a number of ways to define an order relation on the Cartesian product of given ordered sets. Two of these ways follow:

(a) ***Product Order*:** Suppose S and T are ordered sets. Then the following is an order relation on the product set $S \times T$, called the *product order*:

$$(a, b) \precsim (a', b') \quad \text{if} \quad a \leq a' \text{ and } b \leq b'$$

(b) ***Lexicographical Order*:** Suppose S and T are linearly ordered sets. Then the following is an order relation on the product set $S \times T$, called the *lexicographical* or *dictionary order*:

$$(a, b) \prec (a', b') \quad \text{if} \quad a < b \quad \text{or if} \quad a = a' \text{ and } b < b'$$

Hasse Diagrams Of Partially Ordered Sets

1. Let S be a partially ordered set, and suppose a, b belong to S.
2. We say that **a is an immediate predecessor of b**, or that **b is an immediate successor of a**, or that **b is a cover of a**, written

$$a \ll b$$

- if a < b but no element in S lies between a and b
- that is, there exists no element c in S such that a < c < b.

Hasse Diagrams Of Partially Ordered Sets

1. Suppose S is a finite partially ordered set.
2. Then the order on S is completely known once we know all pairs a, b in S such that $a \ll b$
3. that is, once we know the relation $\ll$ on S.
4. This follows from the fact that x < y if and only if x $\ll$ y or there exist elements a_1, a_2, . . . , am in S such that

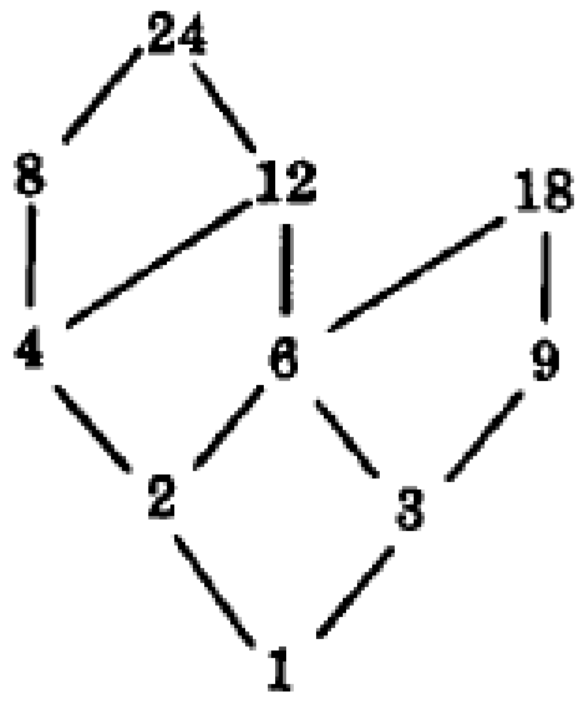
$$x \ll a_1 \ll a_2 \ll \cdots \ll a_m \ll y$$

Hasse Diagrams Of Partially Ordered Sets

1. The Hasse diagram of a finite partially ordered set S is the **directed graph** whose **vertices are the elements of S** and **there is a directed edge from a to b** whenever $a \ll b$ in S.
2. Instead of drawing an arrow from a to b, we sometimes place b higher than a and draw a line between them.
3. It is then understood that movement upwards indicates succession.
4. In the diagram thus created, there is a directed edge from vertex a to vertex b if and only if $a \ll b$
5. Also, there can be no (directed) cycles in the diagram of S since the order relation is antisymmetric.

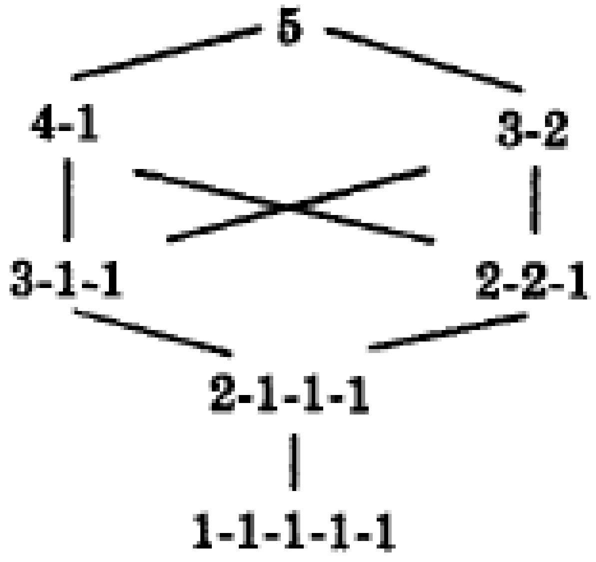
Hasse Diagrams: Examples

1. Let A = {1, 2, 3, 4, 6, 8, 9, 12, 18, 24} be ordered by the relation "x divides y." The diagram of A is given below.
2. Unlike rooted trees, the direction of a line in the diagram of a poset is always upward.



Hasse Diagrams: Examples

1. A partition of a positive integer m is a set of positive integers whose sum is m.
2. For instance, there are seven partitions of m = 5 as follows: 5, 3 − 2, 2 − 2 − 1, 1 − 1 − 1 − 1 − 1, 4 − 1, 3 − 1 − 1, 2 − 1 − 1 − 1
3. We order the partitions of an integer m as follows:
   a) A partition P1 precedes a partition P2 if the integers in P1 can be added to obtain the integers in P2 or, equivalently, if the integers in P2 can be further subdivided to obtain the integers in P1.
   b) For example, 2 − 2 − 1 precedes 3 − 2 since 2 + 1 = 3.
   c) On the other hand, 3 − 1 − 1 and 2 − 2 − 1 are noncomparable.
4. The Hasse diagram is given here.



Minimal and Maximal Elements

1. Let S be a partially ordered set.
2. An element a in S is called a minimal element if no other element of S strictly precedes (is less than) a.
3. Similarly, an element b in S is called a maximal element if no element of S strictly succeeds (is larger than) b.
4. Geometrically speaking, a is a minimal element if no edge enters a (from below), and b is a maximal element if no edge leaves b (in the upward direction).
5. Note that S can have more than one minimal and more than one maximal element.
6. If S is infinite, then S may have no minimal and no maximal element

First and Last Elements

1. An element **a** is S is called a first element if for every element x in S, $a \precsim x$
2. An element b in S is called a last element if for every element y in S, $y \precsim b$
3. S can have at most one first element, which must be a minimal element
4. S can have at most one last element, which must be a maximal element.
5. Generally, S may have neither a first nor a last element, even when S is finite.

Supremum And Infimum

1. Let A be a subset of a partially ordered set S.
2. An element M in S is called an upper bound of A if M succeeds every element of A, i.e., if, for every x in A, we have

$$x \precsim M$$

3. If an upper bound of A precedes every other upper bound of A, then it is called the supremum of A and is denoted by

$$\sup(A)$$

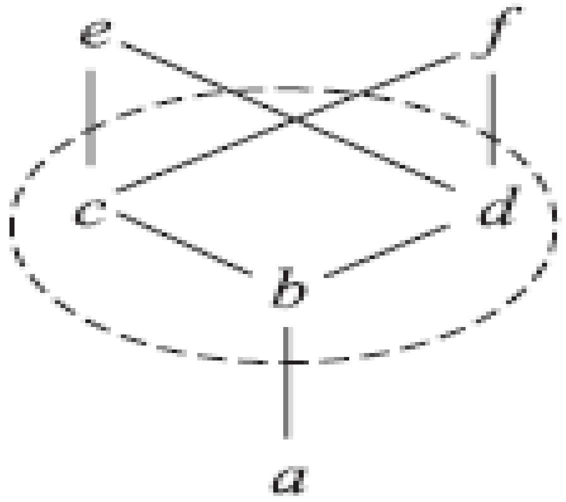
4. There can be at most one sup(A)
5. However, sup(A) may not exist.

Supremum And Infimum

1. Let A be a subset of a partially ordered set S.
2. An element m in a poset S is called a lower bound of a subset A of S if m precedes every element of A, i.e., if, for every y in A, we have
$$m \precsim y$$
3. If a lower bound of A succeeds every other lower bound of A, then it is called the infimum of A and is denoted by
$$\inf(A)$$
4. There can be at most one inf (A) although inf (A) may not exist.

Supremum And Infimum

1. Let S = {a, b, c, d, e, f } be ordered as pictured below, and let A = {b, c, d}.



2. The upper bounds of A are e and f since only e and f succeed every element in A.
3. The lower bounds of A are a and b since only a and b precede every element of A.
4. Note that e and f are noncomparable; hence sup(A) does not exist.
5. However, b also succeeds a, hence inf (A) = b.

Isomorphic (Similar) Ordered Sets

1. Suppose X and Y are partially ordered sets.
2. A one-to-one (injective) function f: X → Y is called a similarity mapping from X into Y if f preserves the order relation, that is, if the following two conditions hold for any pair a and a′ in X:

(1) If $a \precsim a'$ then $f(a) \precsim f(a')$.

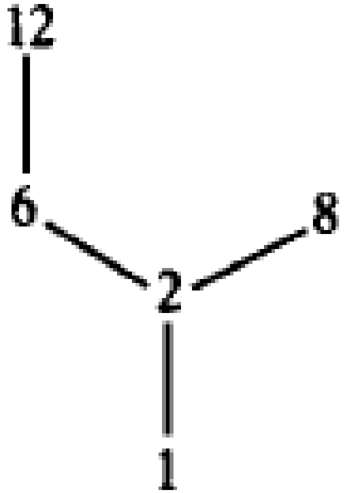
(2) If $a \parallel a'$ (noncomparable), then $f(a) \parallel f(a')$.

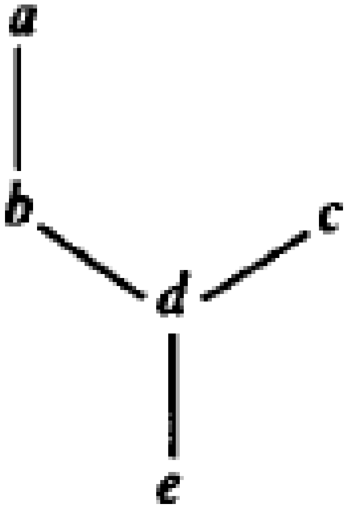
3. Two ordered sets X and Y are said to be **isomorphic** or similar: $X \simeq Y$

if there exists a one-to-one correspondence (bijective mapping) f : X → Y which preserves the order relations, i.e., which is a similarity mapping.

Isomorphic (Similar) Ordered Sets

1. Suppose X = {1, 2, 6, 8, 12} is ordered by divisibility and suppose Y = {a, b, c, d, e} is isomorphic to X; say, the following function f is a similarity mapping from X onto Y : f = {(1, e), (2, d), (6, b), (8, c), (12, a)}. Draw the Hasse diagram of Y .
2. The similarity mapping preserves the order of the initial set X and is one-to-one and onto.
3. Thus the mapping can be viewed simply as a relabeling of the vertices in the Hasse diagram of the initial set X.





Well-Ordered Sets

1. An ordered set S is said to be well-ordered if every subset of S has a first element
2. Example: set N of positive integers with the usual order ≤.
3. Some properties:
   a) A well-ordered set is **linearly ordered**. For if a, b, ∈ S, then {a, b} has a first element; hence a and b are comparable.
   b) Every subset of a well-ordered set is well-ordered.
   c) If X is well-ordered and Y is isomorphic to X, then Y is well-ordered.
   d) Every element a ∈ S, other than a last element, has an immediate successor. Let M(a) denote the set of elements which strictly succeed a. Then the first element of M(a) is the immediate successor of a.
   e) All finite linearly ordered sets with the same number n of elements are well-ordered and are all isomorphic to each other. In fact, they are all isomorphic to {1, 2, . . . , n} with the usual order ≤.

Well-Ordered Sets - Examples

1. Example 1: The set Z of integers with the usual order ≤ is linearly ordered and every element has an immediate successor and an immediate predecessor, but Z is not well-ordered.
   - For example, Z itself has no first element.
   - However, any subset of Z which is bounded from below is well-ordered.
2. Example 2: The set Q of rational numbers with the usual order ≤ is linearly ordered. Is it well-ordered? No. Why?
   - No element in Q has an immediate successor or an immediate predecessor.

For if $a, b \in Q$, say $a < b$, then $(a+b)/2 \in \mathbf{Q}$ and $a < \frac{a+b}{2} < b$