Discrete Mathematics (ITPC-309)

Graphs – Part III

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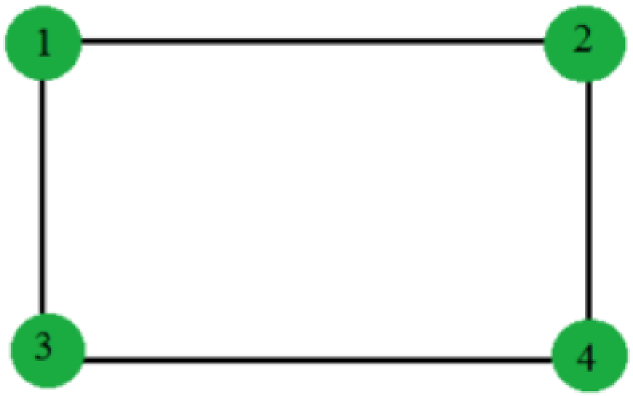
- Regular graphs
- Connected graphs,
- Connectivity
- Connected components in a graph

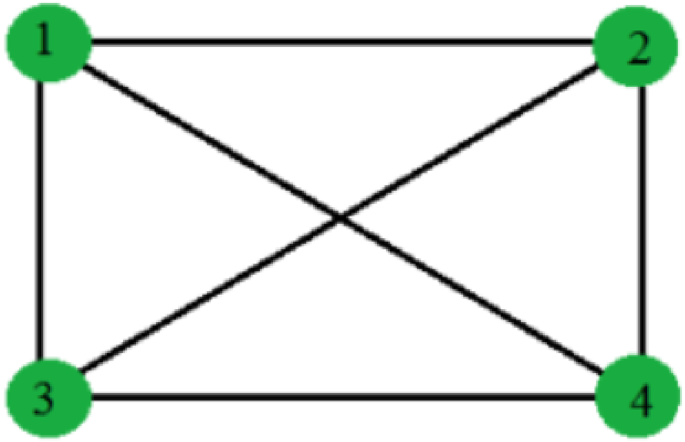
Regular Graphs

Regular Graphs

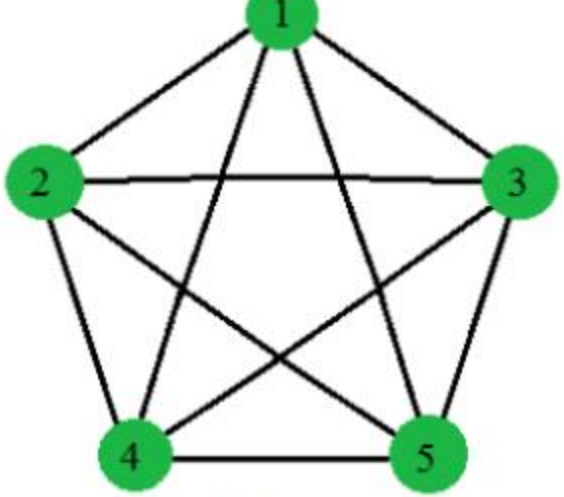
1. A regular graph is a graph where **each vertex has the same number of neighbors**
2. Every vertex has the same degree
3. A graph is called K regular if degree of each vertex in the graph is K.
4. In regular directed graph, the indegree and outdegree of each vertex are equal to each other.



2 Regular



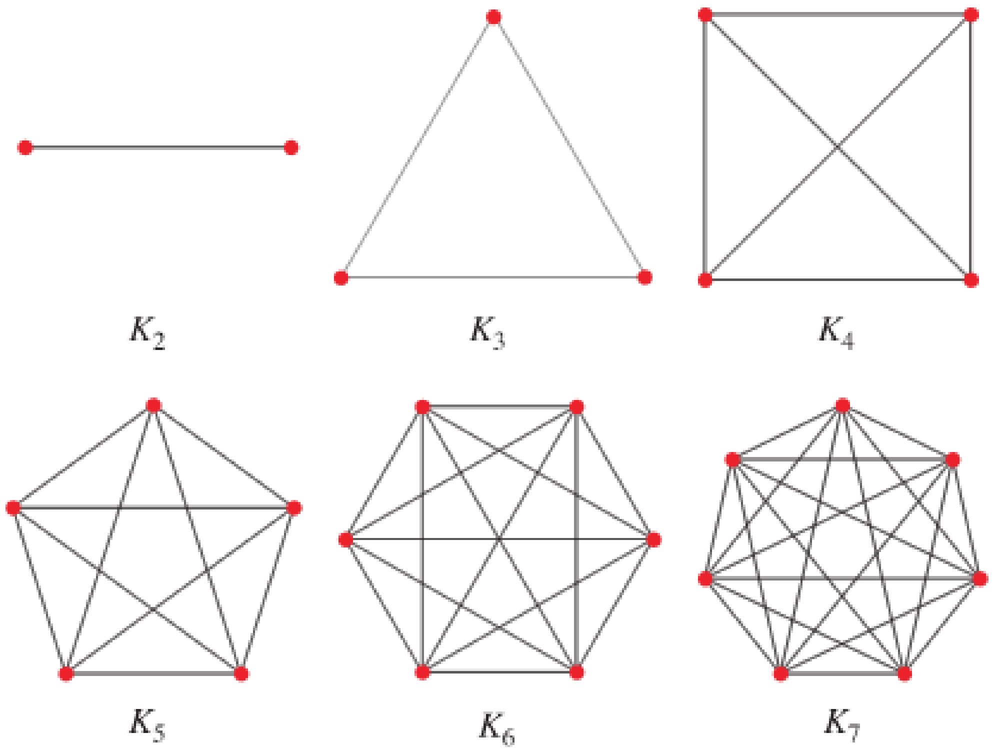
3 Regular



4 Regular

Regular Graphs - Properties

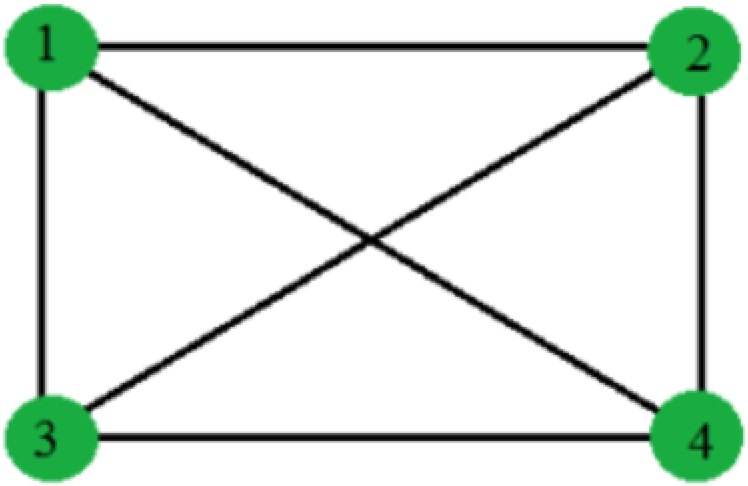
1. **A complete graph with N vertices is (N-1) regular.**
   - Proof: In a complete graph of N vertices, each vertex is connected to all (N-1) remaining vertices.
   - So, degree of each vertex is (N-1).
   - So the graph is (N-1) Regular.



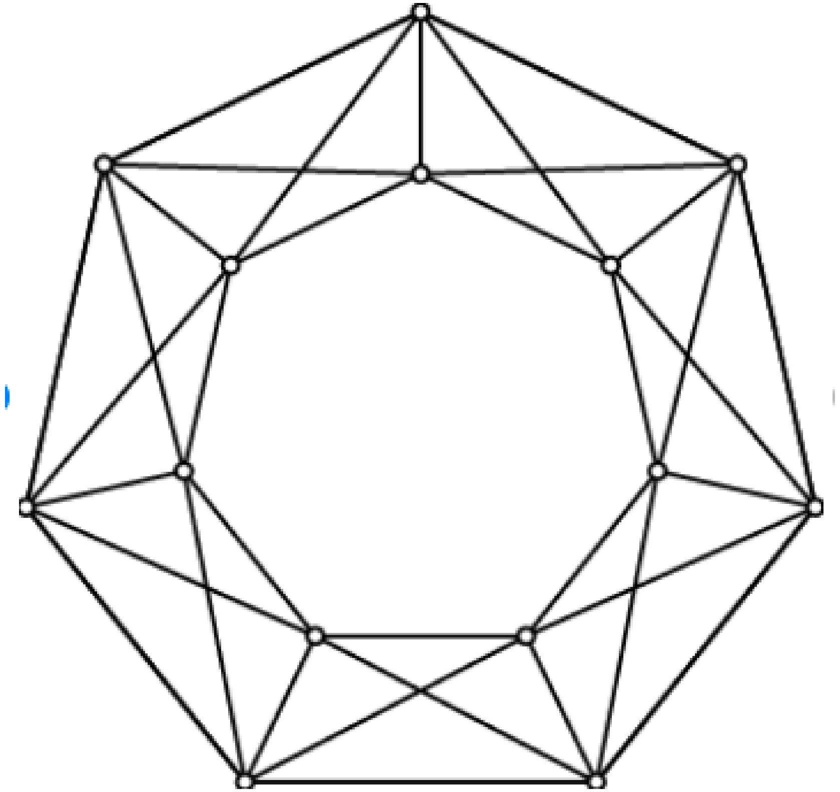
Regular Graphs - Properties

1. **For a K Regular graph, if K is odd, then the number of vertices of the graph must be even.**
   - Proof: Lets assume, number of vertices, N is odd.
   - From Handshaking Theorem we know,
   - Sum of degree of all the vertices = 2 * Number of edges of the graph …….(1)
   - The R.H.S of the equation (1) is a even number.
   - For a K regular graph, each vertex is of degree K.
   - Sum of degree of all the vertices = K * N, where K and N both are odd.
   - So their product (sum of degree of all the vertices) must be odd.
   - This makes L.H.S of the equation (1) is a odd number.
   - So L.H.S not equals R.H.S.
   - So our initial assumption that N is odd, was wrong.
   - So, number of vertices(N) must be even.

Regular Graphs - Properties



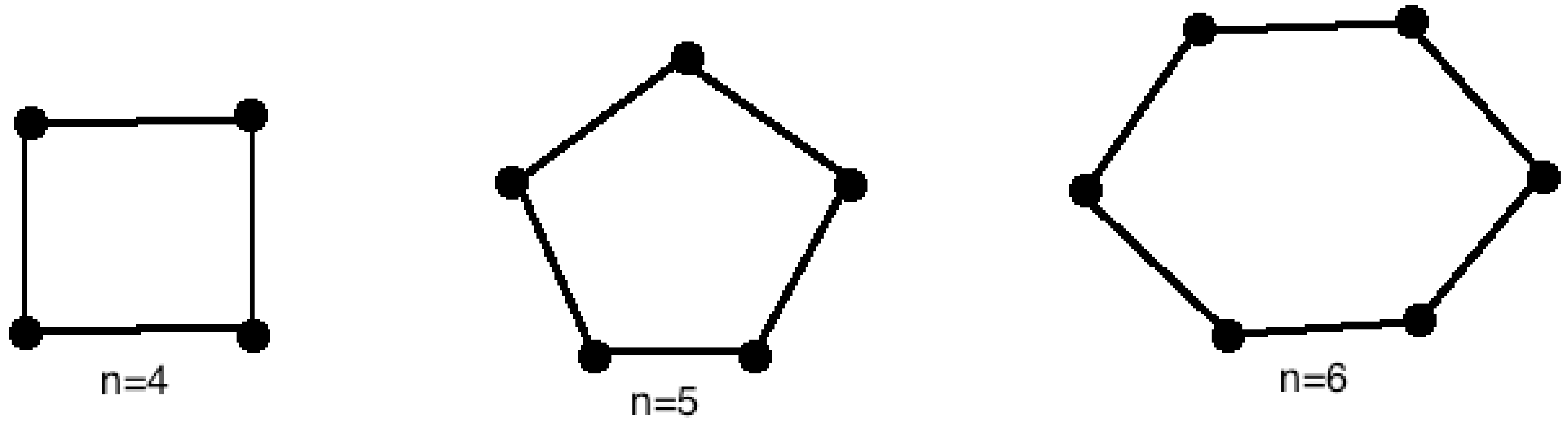
3 Regular



5 Regular graph, #vertices = 14

Regular Graphs - Properties

1. **A cycle (Cn) is always 2 Regular.**
    - Proof:
    - In Cycle (Cn) each vertex has two neighbors. So, they are 2 Regular.

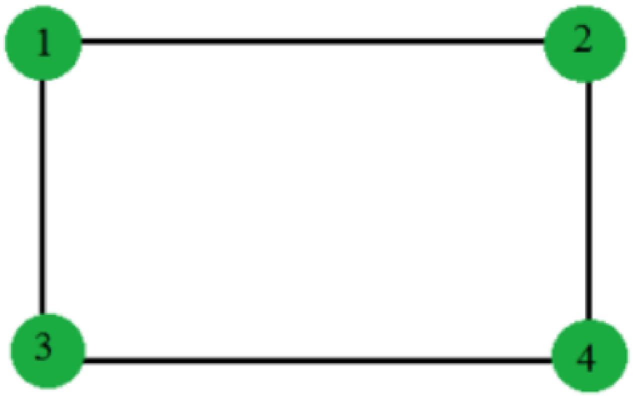


Regular Graphs - Properties

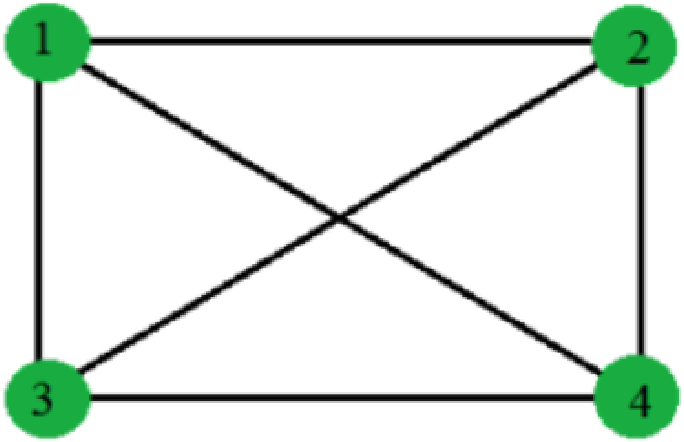
1. **Number of edges of a K Regular graph with N vertices = (N*K)/2.**
    - Proof: Let, the number of edges of a K Regular graph with N vertices be E.
    - From Handshaking Theorem we know,
    - Sum of degree of all the vertices = 2 * E
    - So, N * K = 2 * E
    - or, E = (N*K)/2

Regular Graphs - Properties

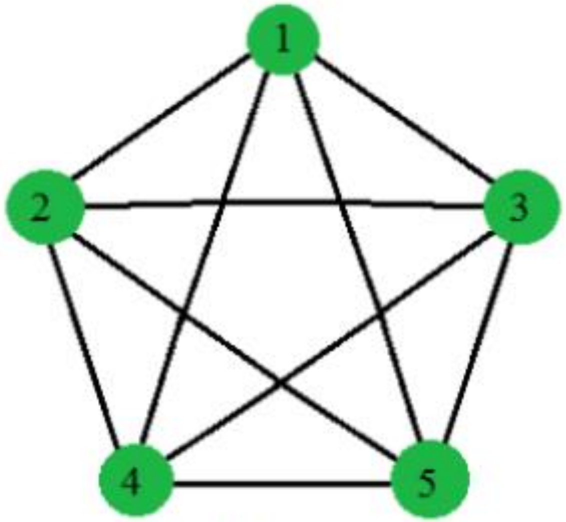
1. **Check the number of edges**



2 Regular



3 Regular

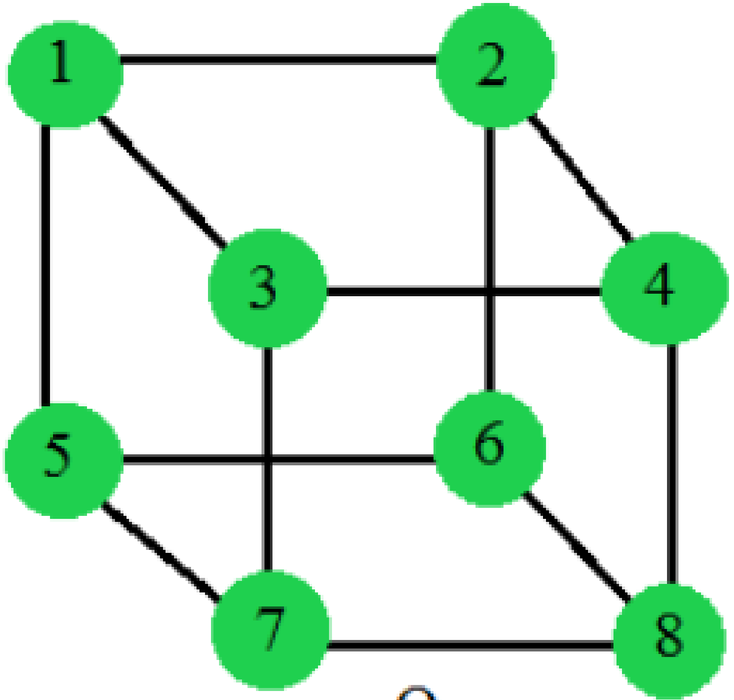


4 Regular

Regular Graphs - Properties

1. A K-dimensional Hyper cube (Qk) is a K Regular graph.

Below is a 3-dimensional Hyper cube(Q3) which is a 3 Regular graph

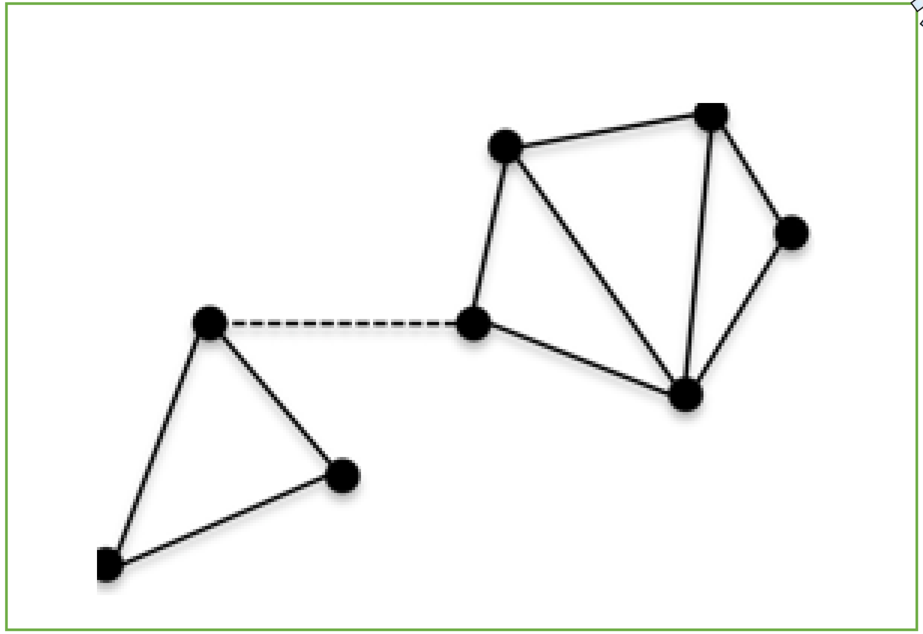


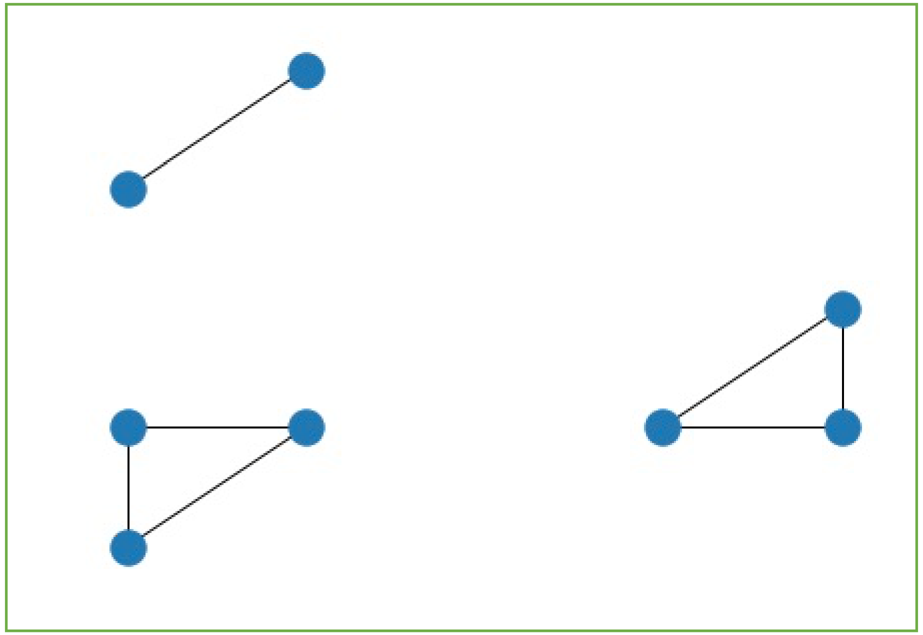
Q_3

Connectivity in Graphs

Connectivity

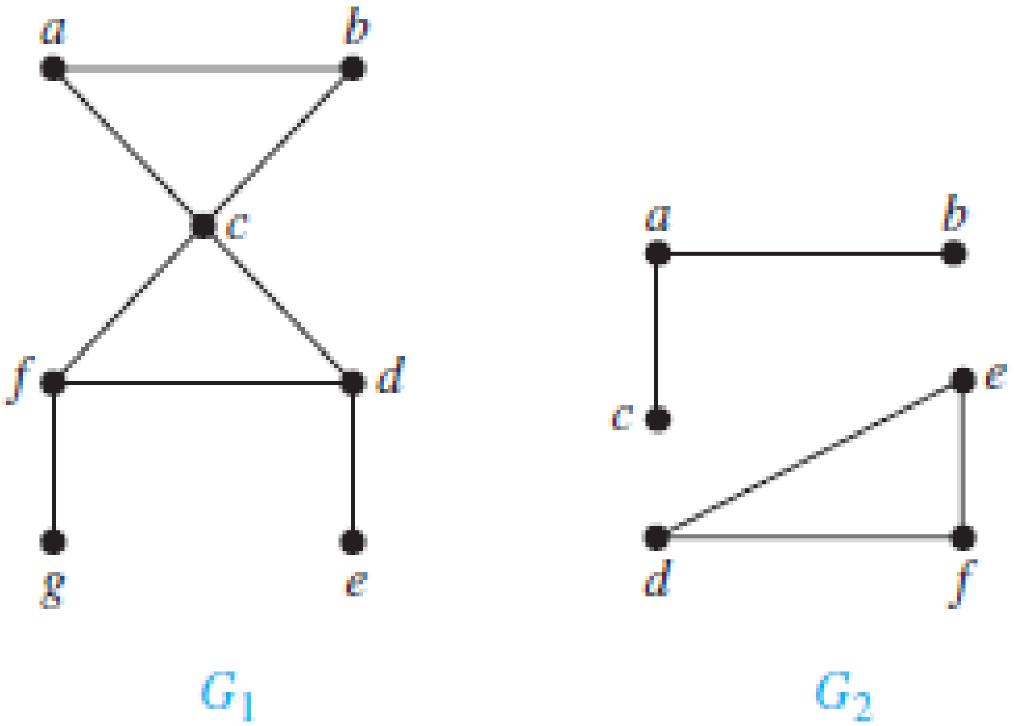
1. Connectivity of a graph defines whether a graph is connected or disconnected.
2. **A graph is said to be connected if every pair of vertices in the graph is connected**.
3. This means that there is a path between every pair of vertices.
4. An undirected graph that is not connected is called disconnected.
5. Are these graphs connected? How can they be made disconnected?





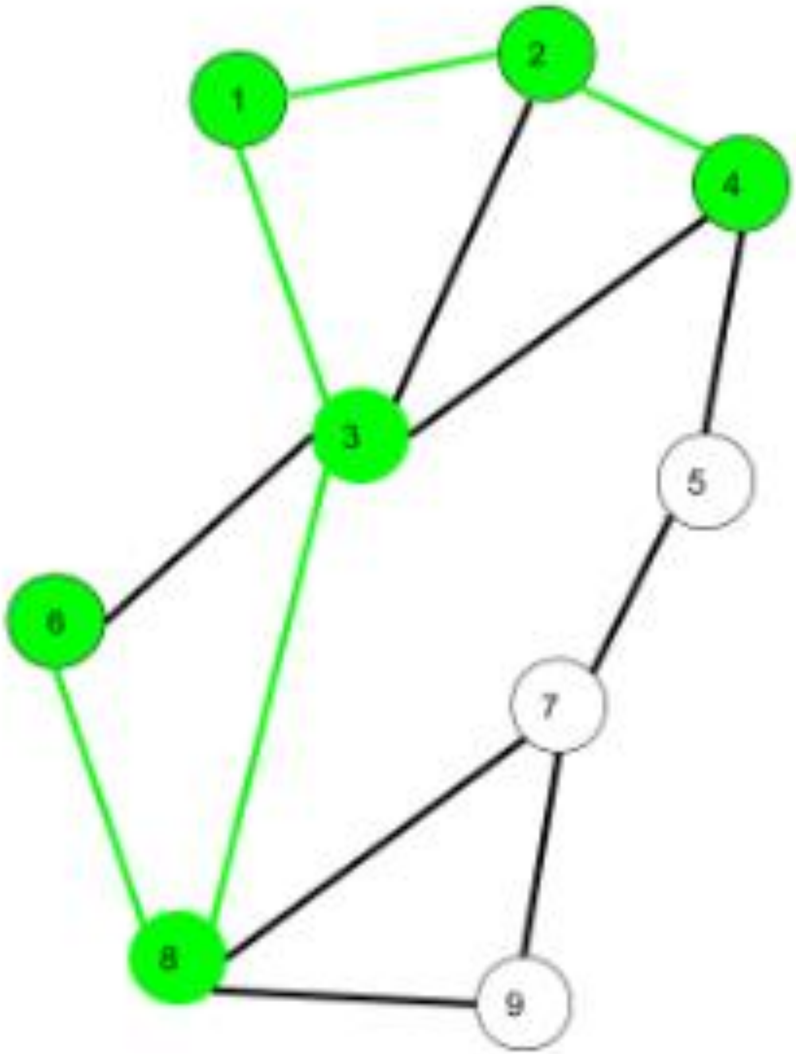
Connectedness

1. Are these connected or disconnected?



Connectivity – Some related terms

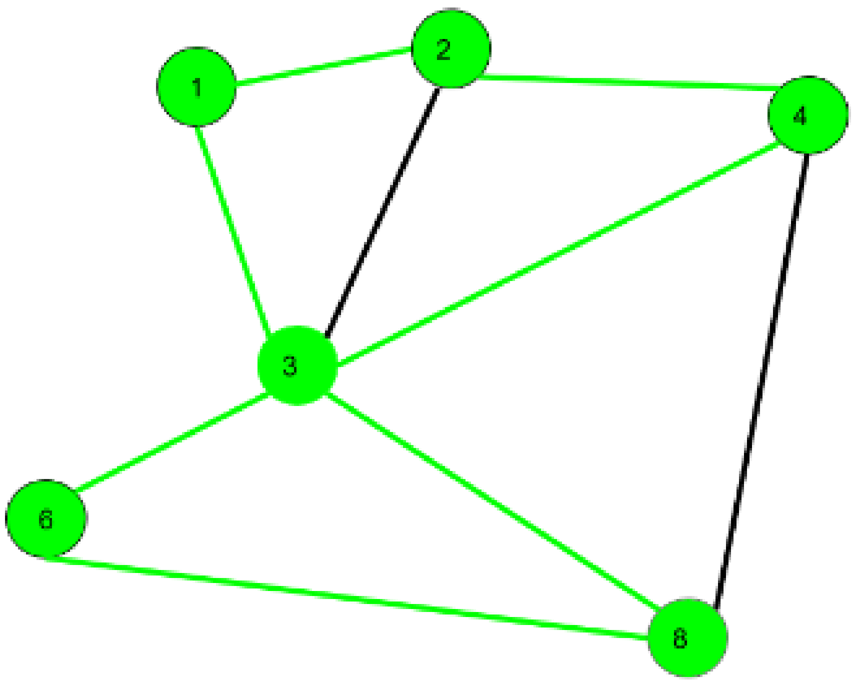
1. **Path**: A path is a sequence of non-repeated nodes connected through edges present in a graph.
2. Neither vertices, nor edges are repeated.
3. For a simple graph, the path is denoted by a sequence of vertices.
4. The length of the path = number of edges traversed.



Here 6->8->3->1->2->4 is a Path

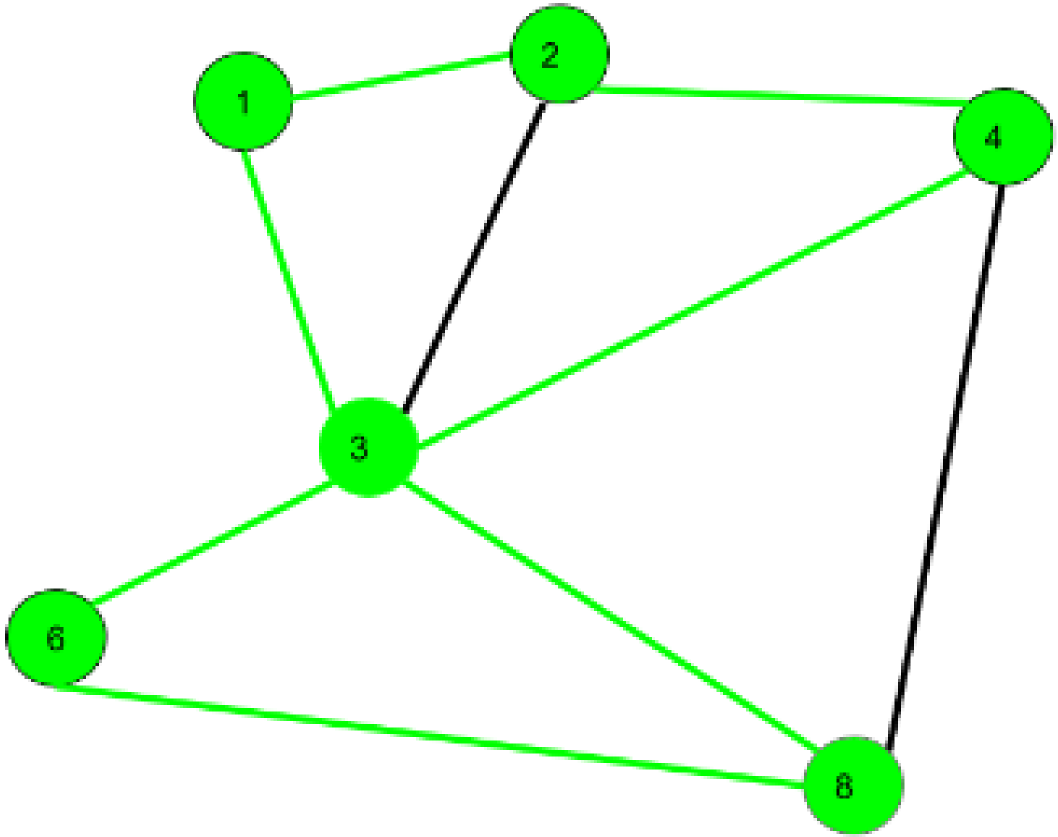
Connectivity

1. The path is a **circuit** if it begins and ends at the same vertex and has length greater than zero.
   - Edges not repeated
   - Vertices can be repeated



Here 1->2->4->3->6->8->3->1 is a circuit.

Connectivity



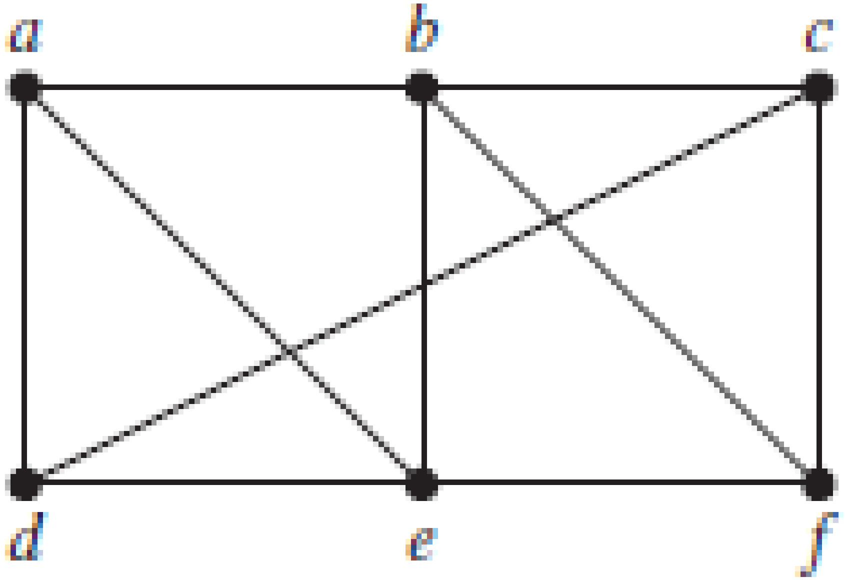
1. Example: In this graph, is this sequence forming a circuit?
   1 – 2 – 4 – 3 – 6 – 8 – 3
2. Is it a path?

Connectivity

1. A path or circuit is **simple** if it does not contain the same edge more than once
2. A path of length zero consists of a single vertex

Connectivity

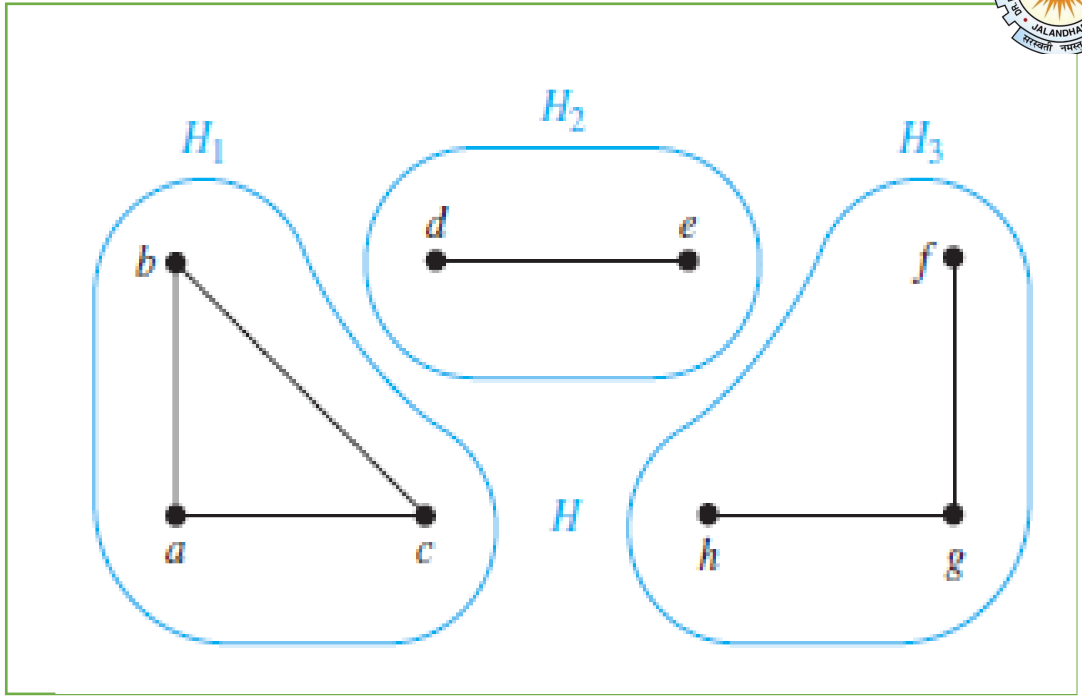
1. Example:

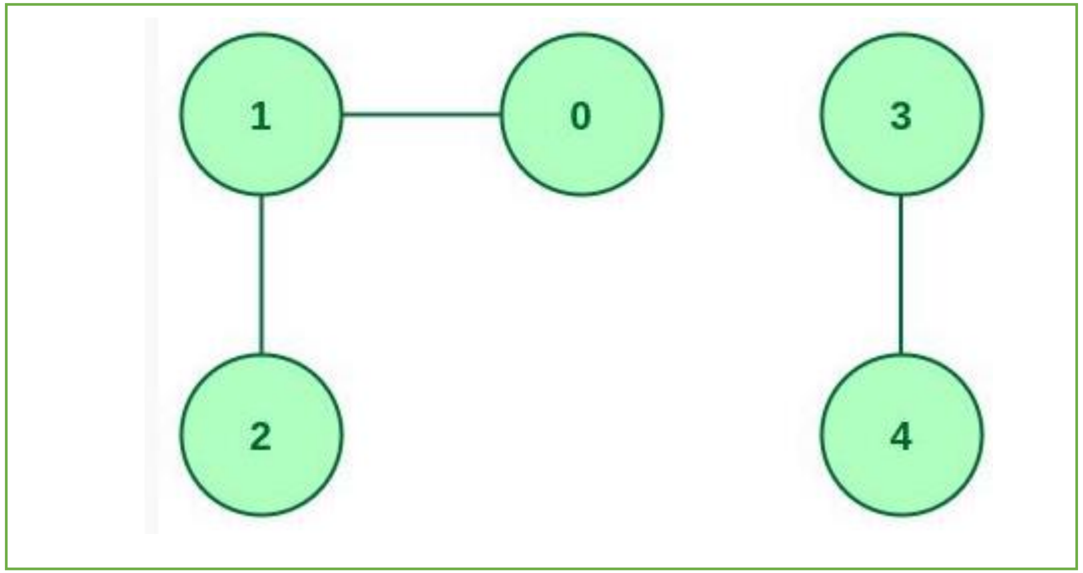


2. a, d, c, f , e is a simple path of length 4
3. d, e, c, a is not a path
4. b, c, f , e, b is a simple circuit of length 4
5. The path a, b, e, d, a, b, which is of length 5, is it a path? Is it a circuit?

Connected Components

1. A set of nodes forms a **connected component** in an undirected graph if, starting from one node in this set, it is possible to reach any other node in this set, by traversing edges.
2. A graph G that is not connected has two or more connected components that are disjoint and have G as their union.
3. A **connected component** of a graph G is a connected subgraph of G that is not a proper subgraph of another connected subgraph of G

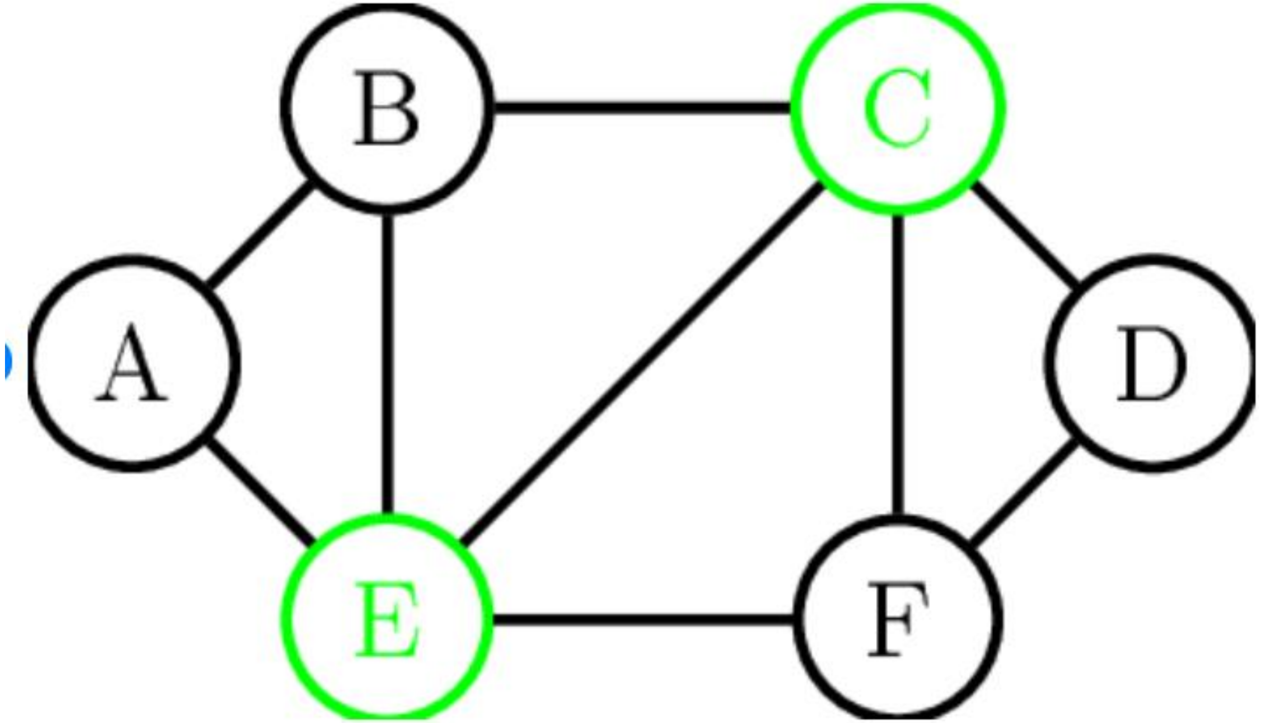




How Connected is a Graph?

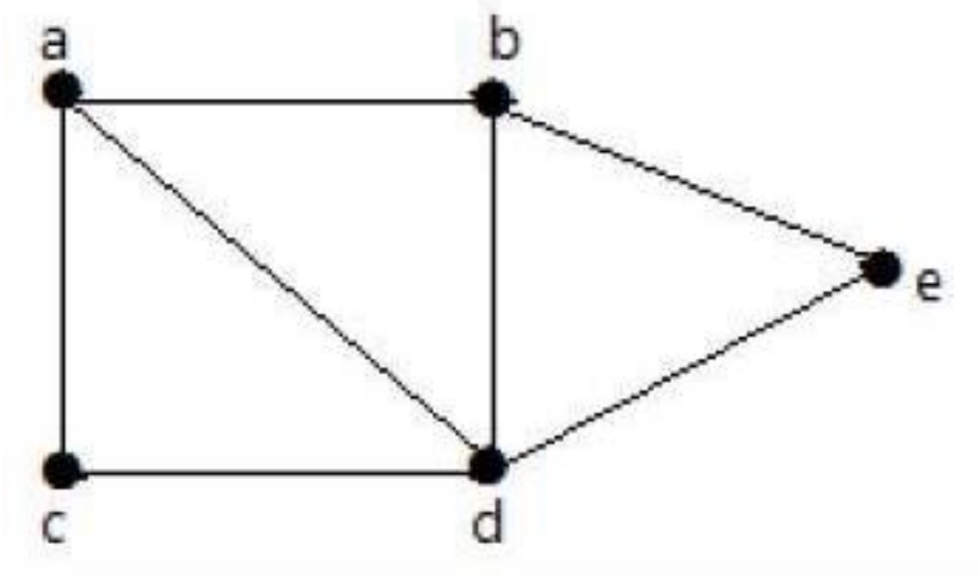
1. We can measure the connectivity of a graph – how well connected the graph is.
2. Why this measure is needed?
3. The graph connectivity is the measure of the robustness of the graph as a network.
4. A connected graph may demand a minimum number of edges or vertices which are required to be removed to separate one set of vertices from another.

How Connected is a Graph?



1. In a connected graph, if any of the vertices are removed, and the graph gets disconnected, then the graph is called a **vertex-connected** graph.

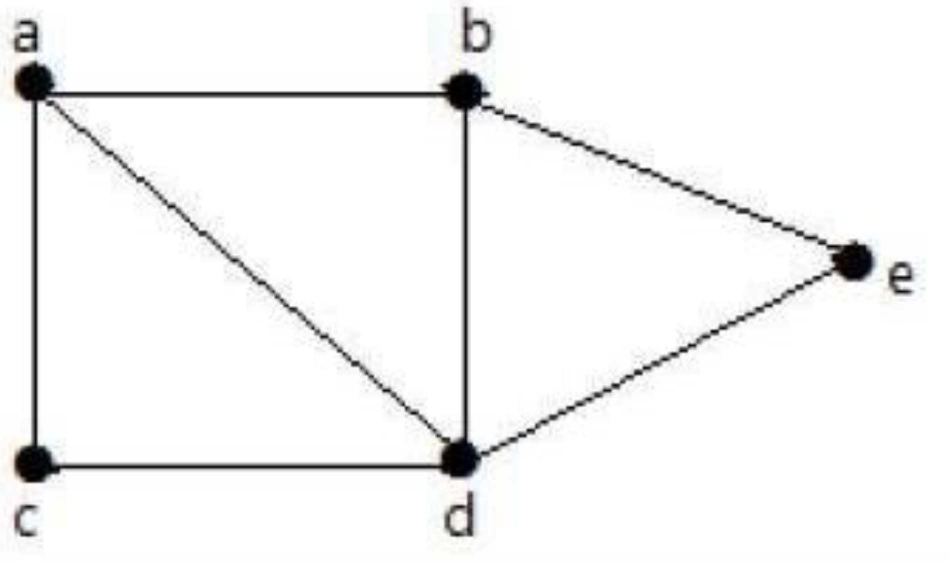
How Connected is a Graph?



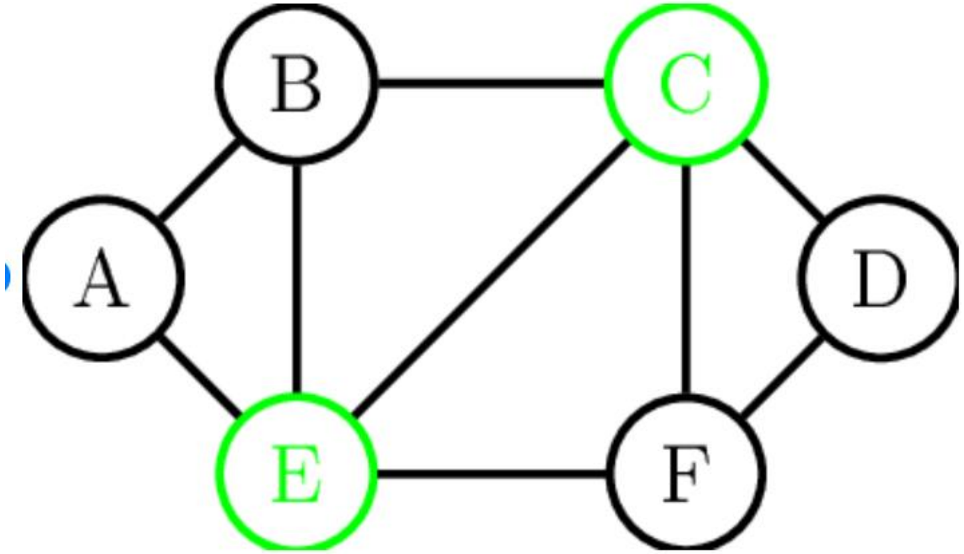
1. In a connected graph, if any of the edges are removed, and the graph gets disconnected, then the graph is called a **edge-connected** graph.

How Connected is a Graph?

1. **Is this a vertex-connected graph? Can you find a vertex, the removal of which will make this graph disconnected?**



1. **Is this an edge-connected graph? Can you find an edge, the removal of which will make this graph disconnected?**

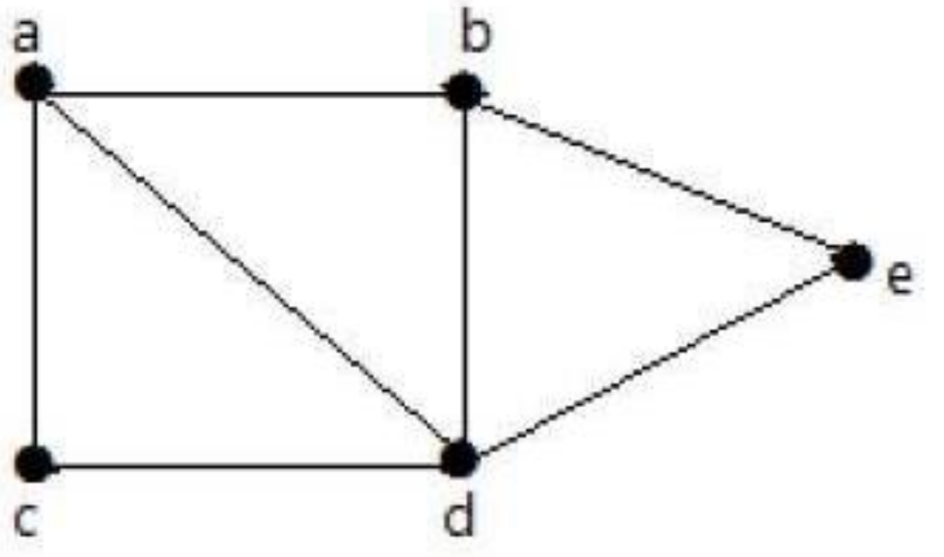


How Connected is a Graph?

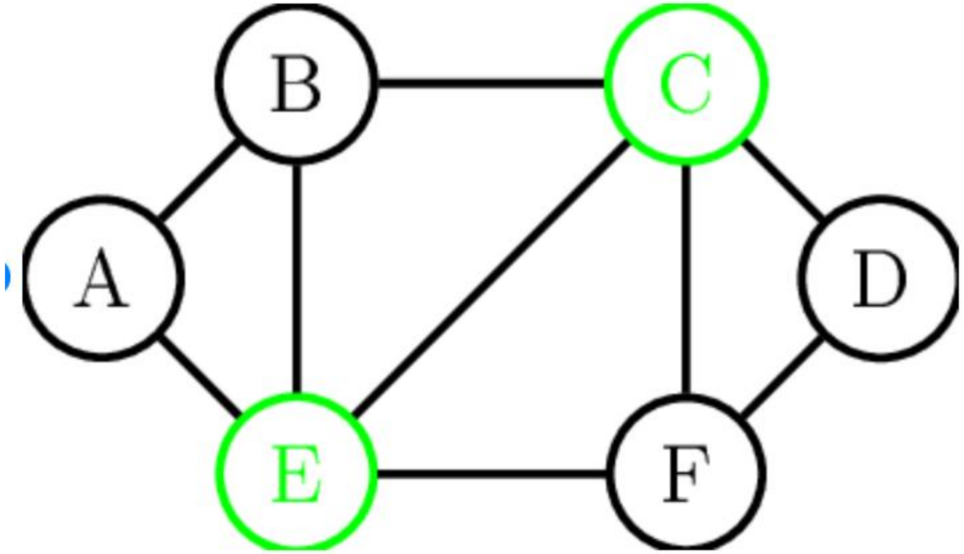
1. If there exists a set S of edges or vertices in a connected graph, such that the removal of S results in a disconnected graph
   - then that set S is called a **cut set**.
2. If S consists of vertices, then it is called a **vertex-cut set**.
3. Similarly, if it has edges, then it is called an **edge-cut set**.
4. **Cut vertices**: If the removal from a graph of a vertex and all incident edges produces a subgraph with more connected components than the original graph, such vertices are called cut vertices (or **articulation points**).The removal of a cut vertex from a connected graph produces a subgraph that is not connected.
5. **Cut edge/bridge :** an edge whose removal produces a graph with more connected components than in the original graph.

How Connected is a Graph?

1. **What is the minimum edge cut-set for this graph?**
2. **Are other cut sets possible?**

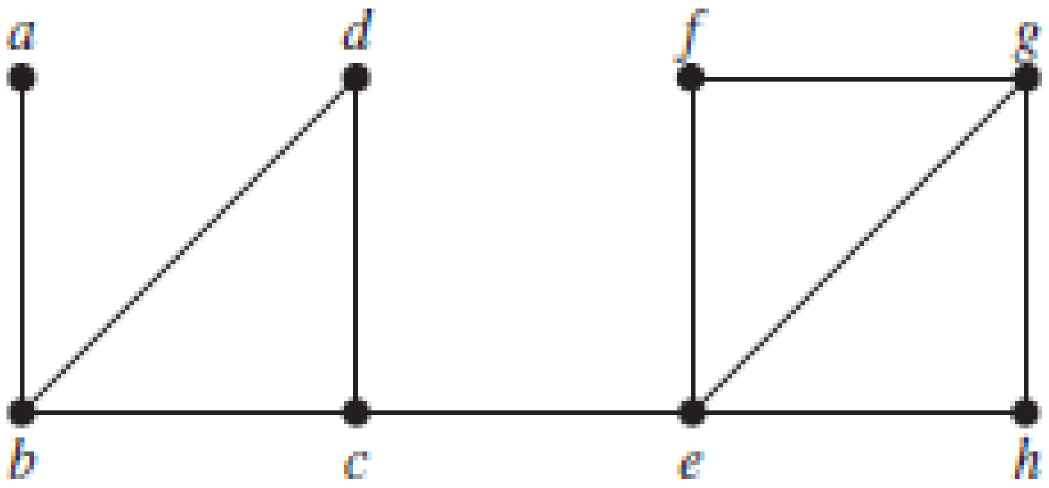


1. **What is the minimum vertex cut-set for this graph?**



How Connected is a Graph?

1. Find the cut vertices and cut edges in the graph G1

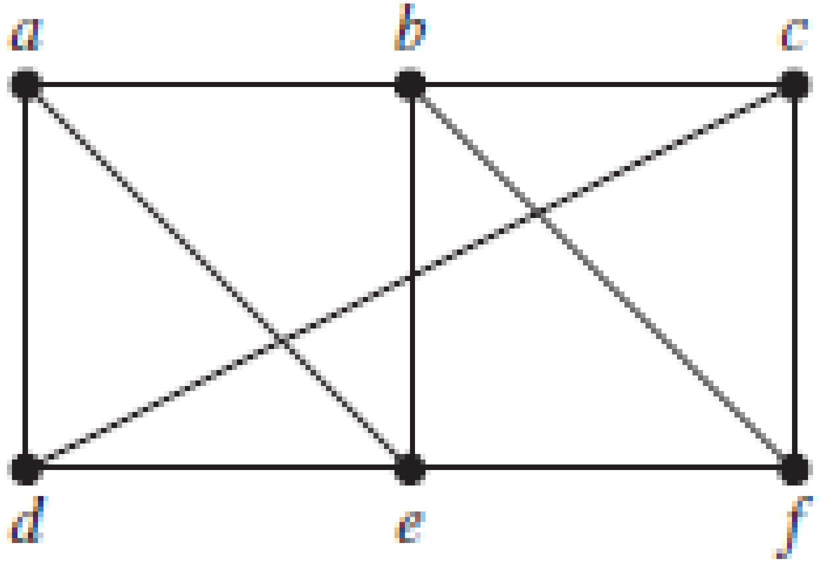


G_1

2. The cut vertices of G1 are b, c, and e. The removal of one of these vertices (and its adjacent edges) disconnects the graph.
3. The cut edges are {a, b} and {c, e}. Removing either one of these edges disconnects G1.

How Connected is a Graph?

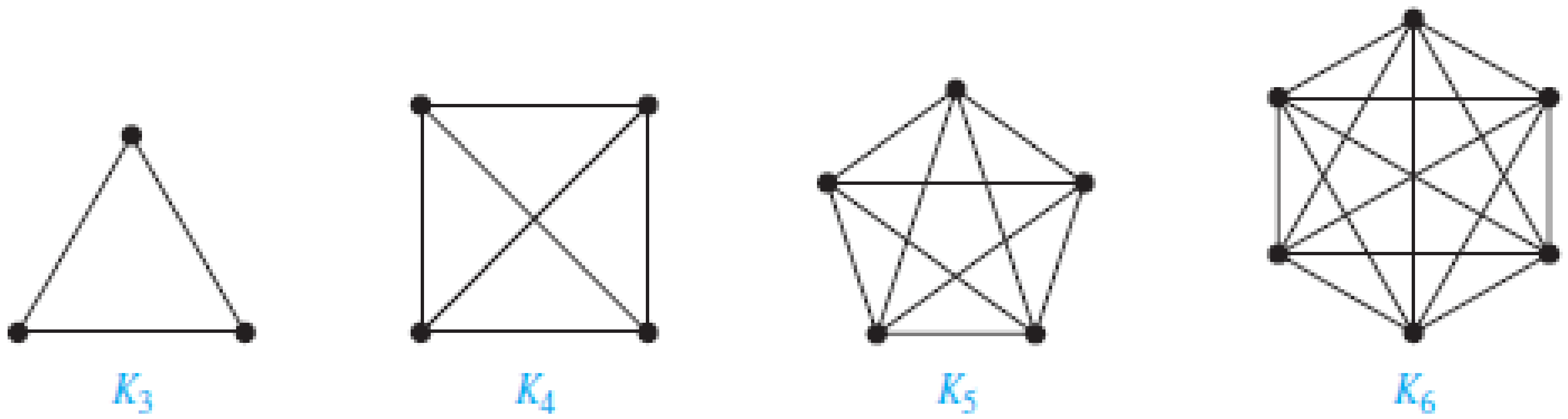
1. Find the cut vertices and cut edges in the graph



2. set {b, c, e} is a vertex cut

Vertex Connectivity – Fully Connected Graphs

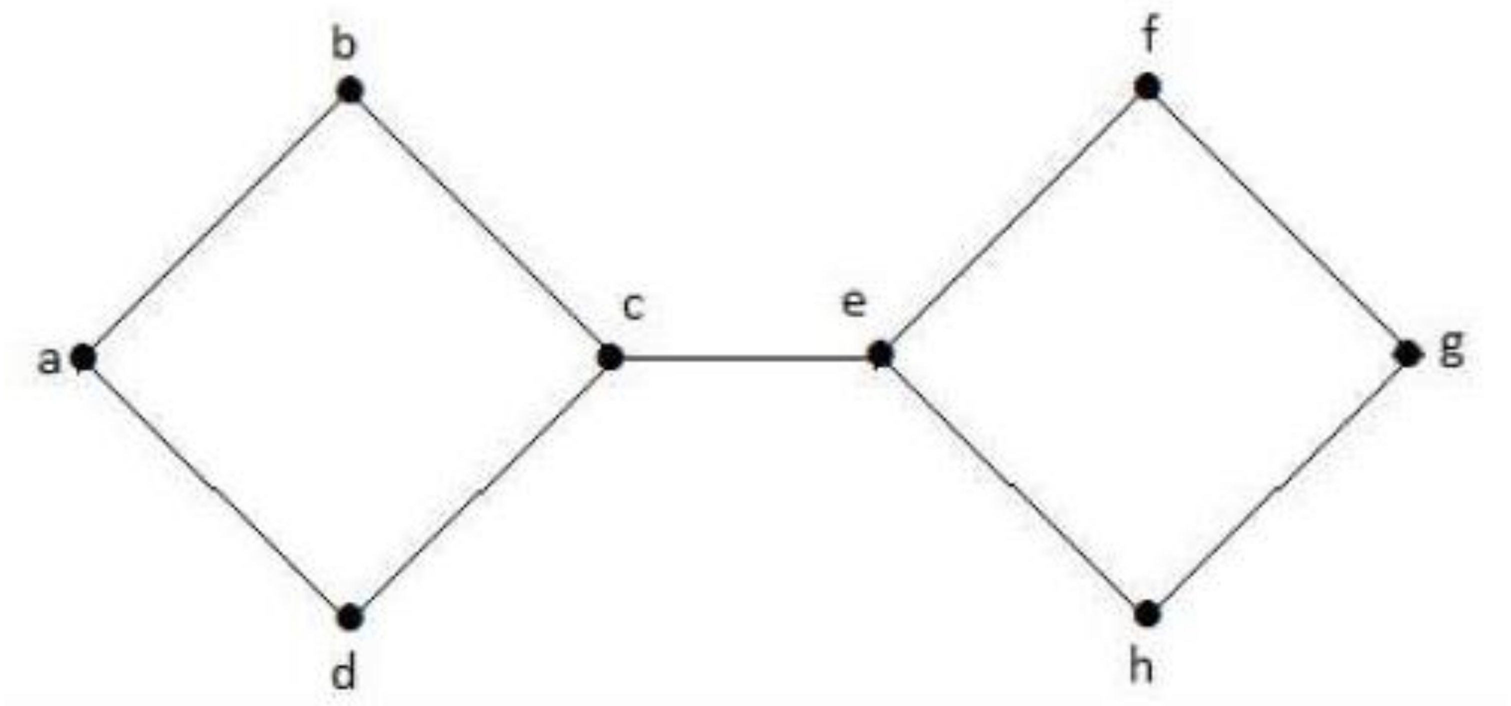
1. Not all graphs have cut vertices
2. Example: the complete graph K_n (Kuratowski) where $n \geq 3$, has no cut vertices.
3. Connected graphs without cut vertices are called **non-separable graphs/fully-connected graphs**, and can be thought of as **more connected** than those with a cut vertex.



Vertex Connectivity: k-connected graphs

1. What happens to the separable graphs?
2. We define the **vertex connectivity** of a noncomplete graph G, denoted by κ(G), as the **minimum number of vertices in a vertex cut**.
3. Consequently, for every graph G, κ(G) is minimum number of vertices that can be removed from G to either disconnect G or produce a graph with a single vertex.
4. **The larger κ(G) is, the more connected we consider G to be**
5. Disconnected graphs have $\kappa(G) = 0$
6. **A graph is k-connected** if $\kappa(G) \geq k$.

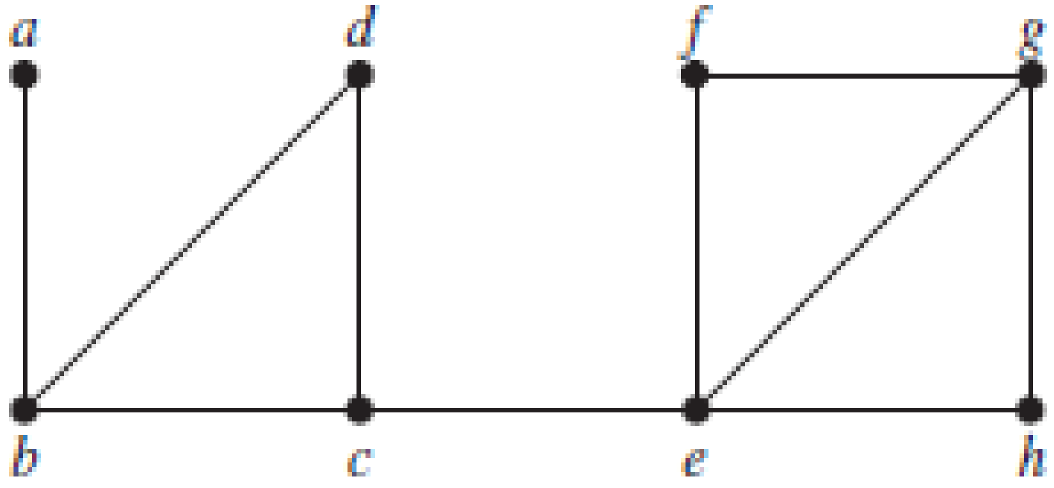
Vertex Connectivity



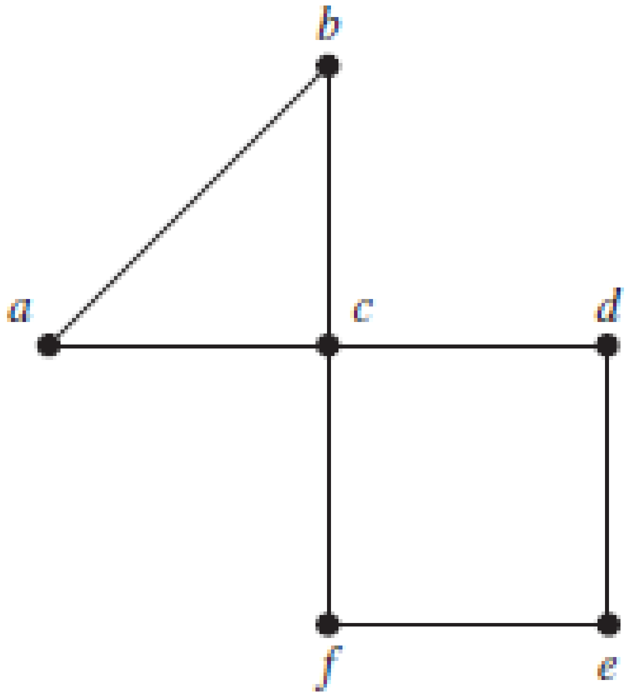
1. What is the k(G) for the following graph?
2. What are the cut vertices?

Vertex Connectivity

1. Find the vertex connectivity and the cut vertices for each of the graphs



G_1



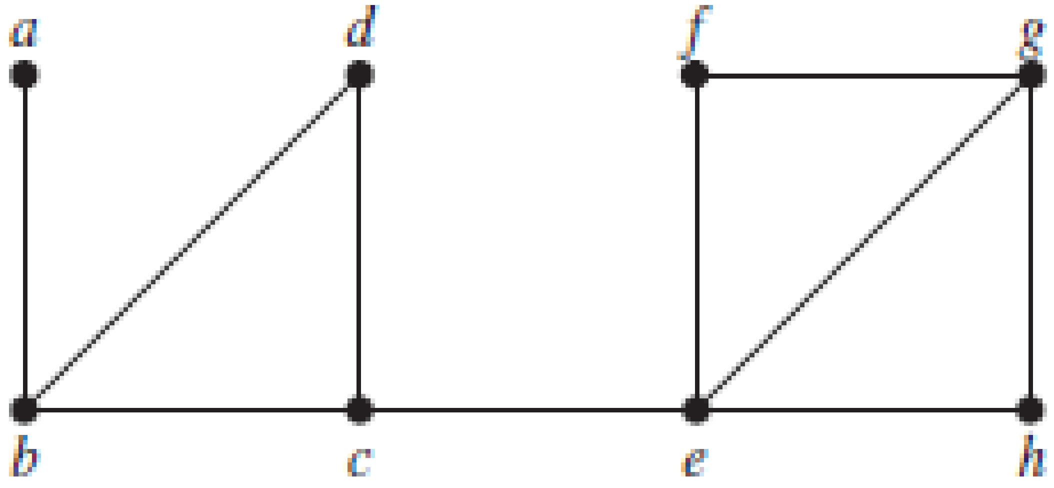
G_2

Edge Connectivity

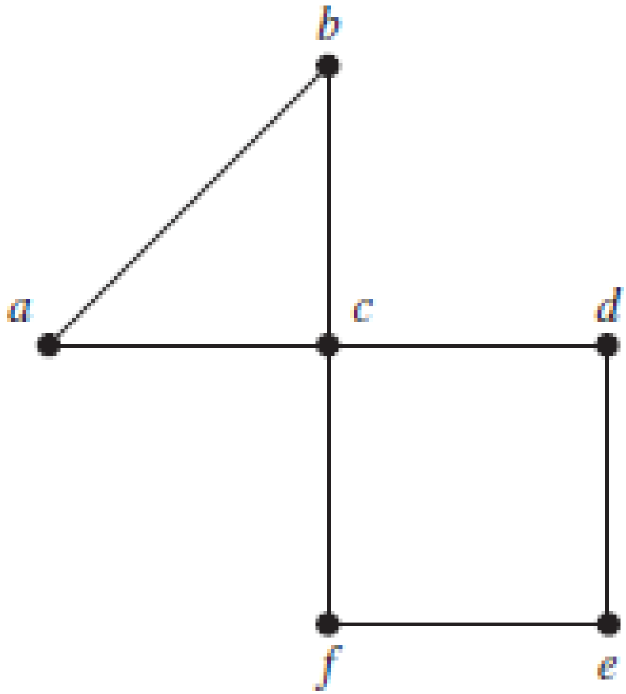
1. Measure the connectivity of a connected graph $G = (V, E)$ in terms of the minimum number of edges that we can remove to disconnect it.
2. If a graph has a cut edge, then we need only remove it to disconnect G.
3. If G does not have a cut edge, we look for the smallest set of edges that can be removed to disconnect it
4. The edge connectivity of a graph G, denoted by $\lambda(G)$, is the minimum number of edges in an edge cut of G.

Vertex Connectivity

1. Find the edge connectivity and the cut edges for each of the graphs



G_1



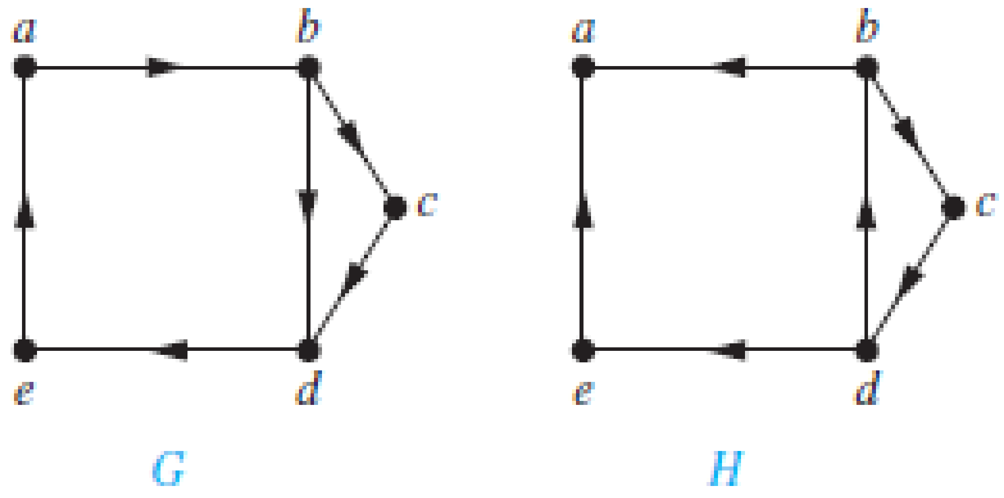
G_2

Connectedness in Directed Graphs

1. A directed graph is **strongly connected** if there is a path from a to b and from b to a whenever a and b are vertices in the graph.
2. A directed graph is **weakly connected** if and only if there is always a path between two vertices when the directions of the edges are disregarded.
3. Clearly, any strongly connected directed graph is also weakly connected.

Connectedness in Directed Graphs

1. Are the directed graphs G and H strongly connected?



2. G is strongly connected because there is a path between any two vertices in this directed graph. Hence, G is also weakly connected.
3. The graph H is not strongly connected. Which path is missing?