Discrete Mathematics (ITPC-309)

Graphs Part I

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- What are graphs
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- Basic Terminology
- Finite and infinite graph
- Simple graph, multi graph,
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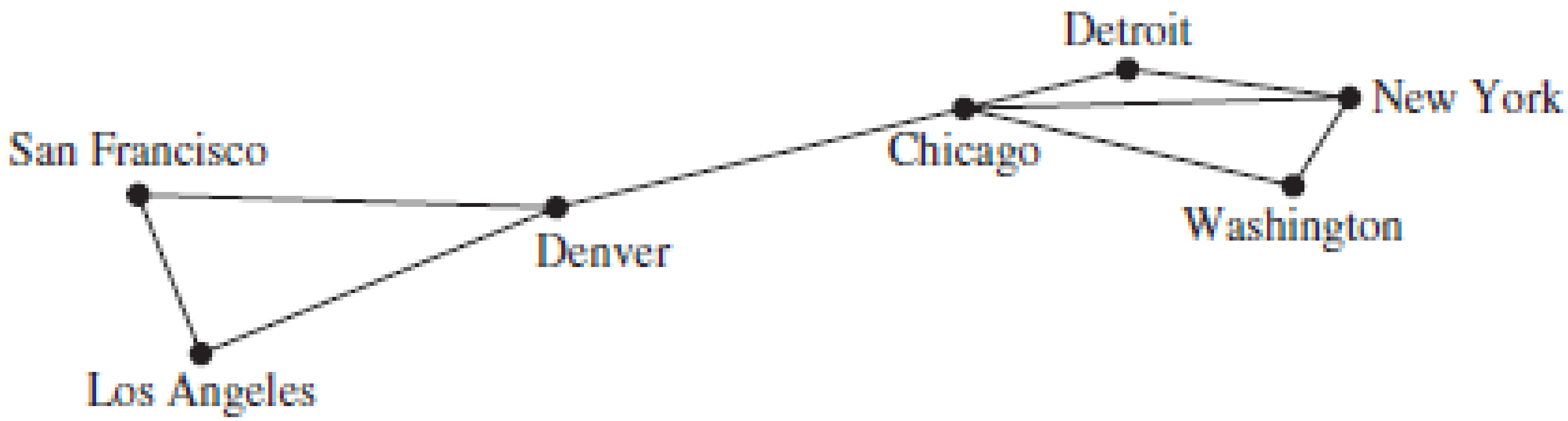
Graphs

1. Graphs are discrete structures consisting of vertices and edges that connect these vertices.
2. Where used? everywhere:
   - to model social media connections
   - Model telephone calls between telephone numbers,
   - model links between websites.
   - model roadmaps
   - finding the shortest path between two cities in a transportation network.
   - Schedule exams
   - to represent who influences whom in an organization,
   - to represent the outcomes of round-robin tournaments.

Graph Terminology

1. Formally, A graph G = (V ,E) consists of
   - V , a nonempty set of vertices (or nodes) and
   - E, a set of edges.
2. Each edge has either one or two vertices associated with it, called its endpoints.
3. An edge is said to connect its endpoints.

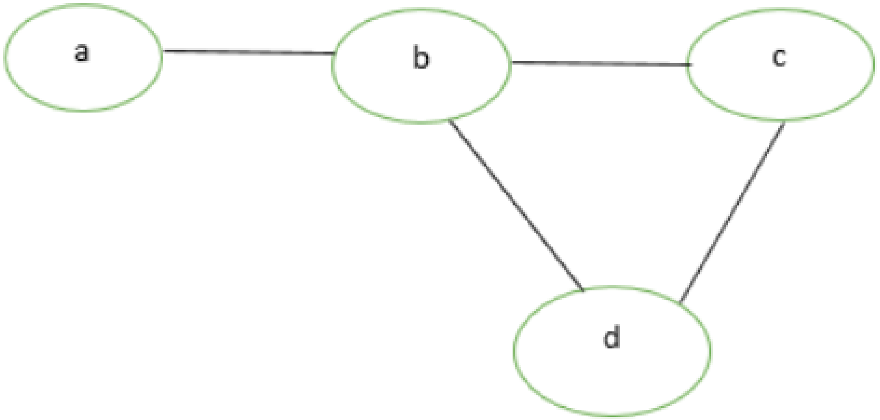


Types: Finite and Infinite Graph

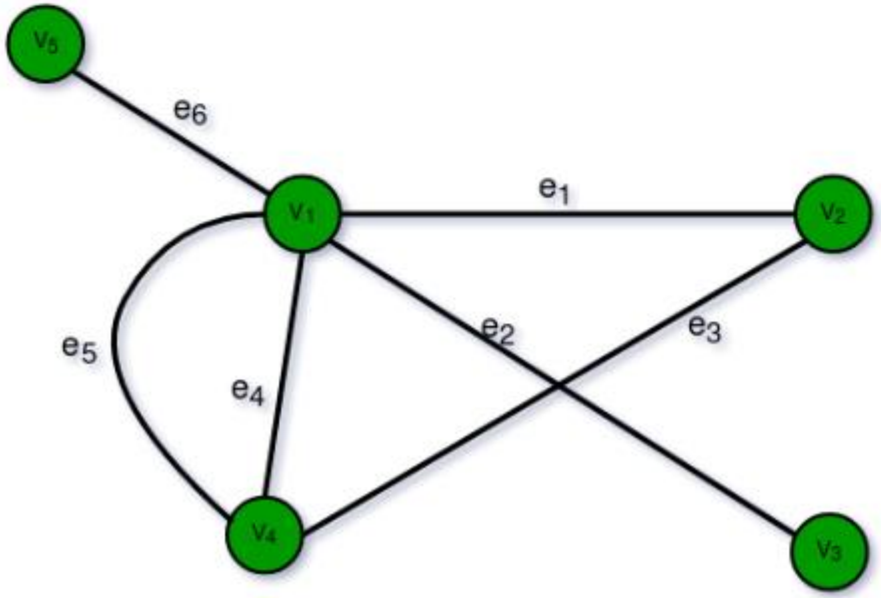
1. The set of vertices V of a graph G may be infinite.
2. A graph with an infinite vertex set or an infinite number of edges is called an infinite graph.
3. A graph with a finite vertex set and a finite edge set is called a finite graph

Simple and Multigraphs

1. A graph in which
   - each edge connects two different vertices
   - and where no two edges connect the same pair of vertices is called a simple graph

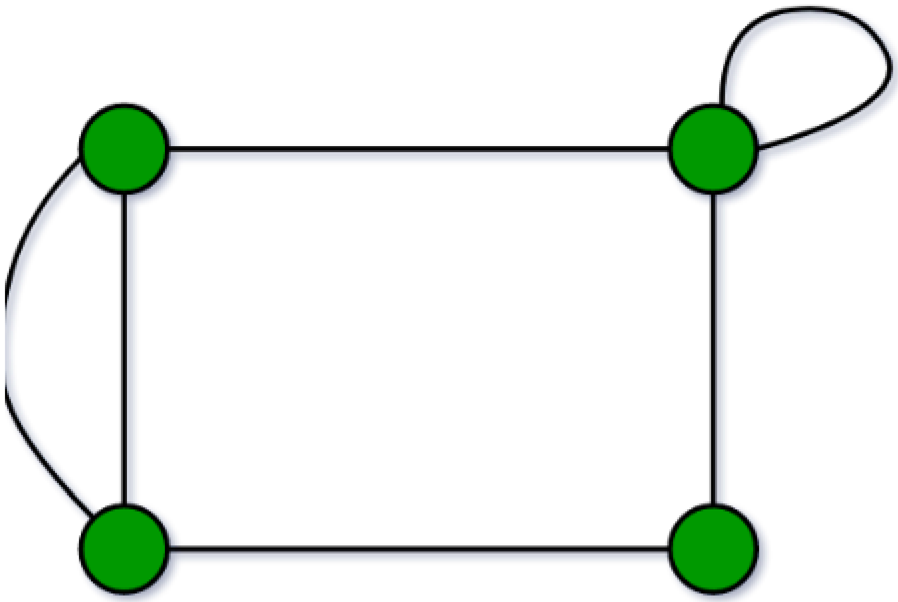


2. Graphs that may have multiple edges connecting the same vertices are called multigraphs.
   - When there are m different edges associated to the same unordered pair of vertices {u, v}, we also say that {u, v} is an edge of **multiplicity m**



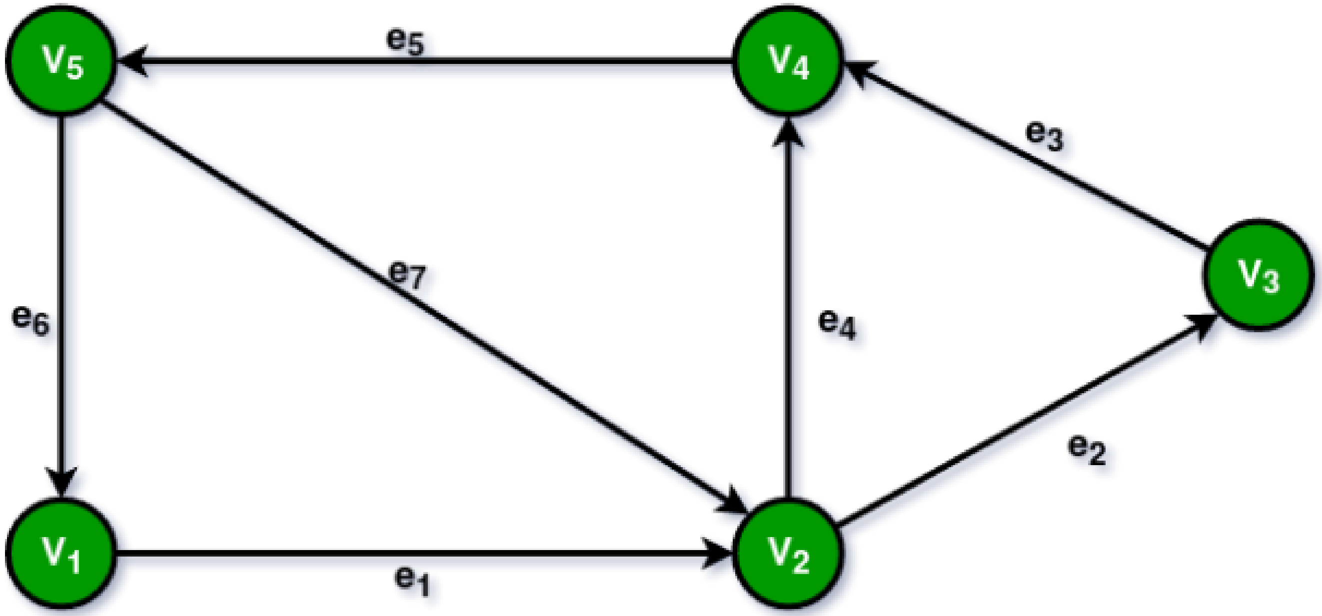
Loops and Pseudographs

1. Edges that connect a vertex to itself are called loops
2. Graphs that may include loops, and possibly multiple edges connecting the same pair of vertices or a vertex to itself, are sometimes called **pseudographs**.
3. So far the graphs we have introduced are **undirected graphs**.
4. Their edges are also said to be undirected – no particular direction to the edges are assigned.



Undirected and Directed Graphs

1. A directed graph (or digraph) (V ,E) consists of a nonempty set of vertices V and a set of directed edges (or arcs) E.
2. **Each directed edge is associated with an ordered pair of vertices**.
3. The directed edge associated with the ordered pair (u, v) is said to start at u and end at v.
4. we use an arrow pointing from u to v to indicate the direction of an edge that starts at u and ends at v



Directed simple and multigraphs, Mixed Graph

1. **Directed multigraphs**: A directed graph containing
   - loops
   - and multiple directed edges that start and end at the same vertices
2. A directed graph may also contain directed edges that connect vertices u and v in both directions
3. **Simple directed graph**: When a directed graph has no loops and has no multiple directed edges, it is called a simple directed graph.
4. **Mixed graph**: A graph with both directed and undirected edges is called a mixed graph

Adjacent Vertices and Neighborhood of a Vertex

1. Two vertices u and v in an undirected graph G are called **adjacent** (or neighbors) in G if u and v are endpoints of an edge e of G.
2. Such an **edge e is called incident with the vertices u and v** and e is said to connect u and v.
3. Formally:

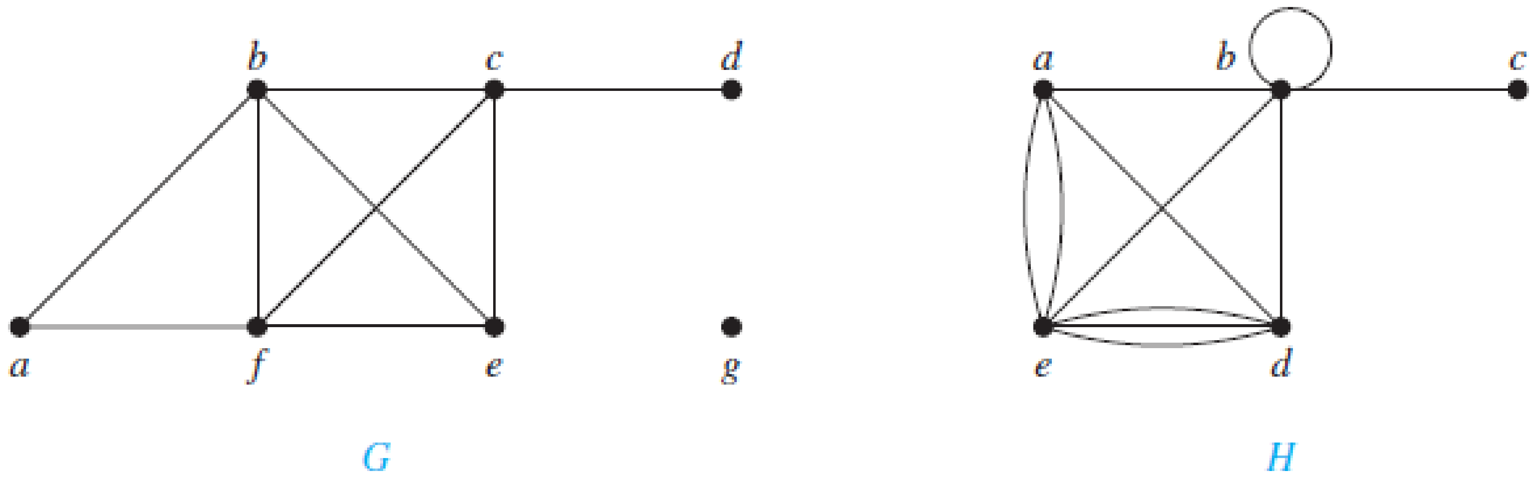
The set of all neighbors of a vertex v of $G = (V, E)$, denoted by $N(v)$, is called the *neighborhood* of v. If A is a subset of V, we denote by $N(A)$ the set of all vertices in G that are adjacent to at least one vertex in A. So, $N(A) = \bigcup_{v \in A} N(v)$.

Degree of a Vertex

1. The **degree of a vertex** in an undirected graph is the number of edges incident with it.
2. A loop at a vertex contributes twice to the degree of that vertex.
3. The degree of the vertex v is denoted by deg(v).

Graph Terminology

1. What are the degrees and what are the neighborhoods of the vertices in the graphs G and H displayed in this figure?



2. In G, deg(a) = 2, deg(b) = deg(c) = deg(f) = 4, deg(d) = 1, deg(e) = 3, deg(g) = 0.
3. The neighborhoods of these vertices are N(a) = {b, f }, N(b) = {a, c, e, f }, N(c) = {b, d, e, f }, N(d) = {c}, N(e) = {b, c, f }, N(f) = {a, b, c, e}, and N(g) = ∅.
4. In H, deg(a) = 4, deg(b) = deg(e) = 6, deg(c) = 1, and deg(d) = 5.
5. The neighborhoods of these vertices are N(a) = {b, d, e}, N(b) = {a, b, c, d, e}, N(c) = {b}, N(d) = {a, b, e}, and N(e) = {a, b, d}.

Isolated Vertex, Pendant vertex

1. A vertex of degree zero is called isolated.
2. An isolated vertex is not adjacent to any vertex
3. A vertex is pendant if and only if it has degree one.
4. So a pendant vertex is adjacent to exactly one other vertex.
5. Identify the pendant and isolated vertices in the previous two graphs.

Handshaking Theorem

1. What will you get if you add the degrees of all the vertices?
2. This is given by THE HANDSHAKING THEOREM:
3. Let G = (V ,E) be an undirected graph with m edges. Then

$$2m = \sum_{v \in V} \deg(v).$$

4. Each edge contributes two to the sum of the degrees of the vertices because an edge is incident with exactly two vertices
5. (Note that this applies even if multiple edges and loops are present.)
6. Derived from this, we have another theorem.

Theorem 2

1. **An undirected graph has an even number of vertices of odd degree**.
2. Proof: Let V1 and V2 be the set of vertices of even degree and the set of vertices of odd degree, respectively, in an undirected graph G = (V ,E) with m edges. Then

$$2m = \sum_{v \in V} \deg(v) = \sum_{v \in V_1} \deg(v) + \sum_{v \in V_2} \deg(v).$$

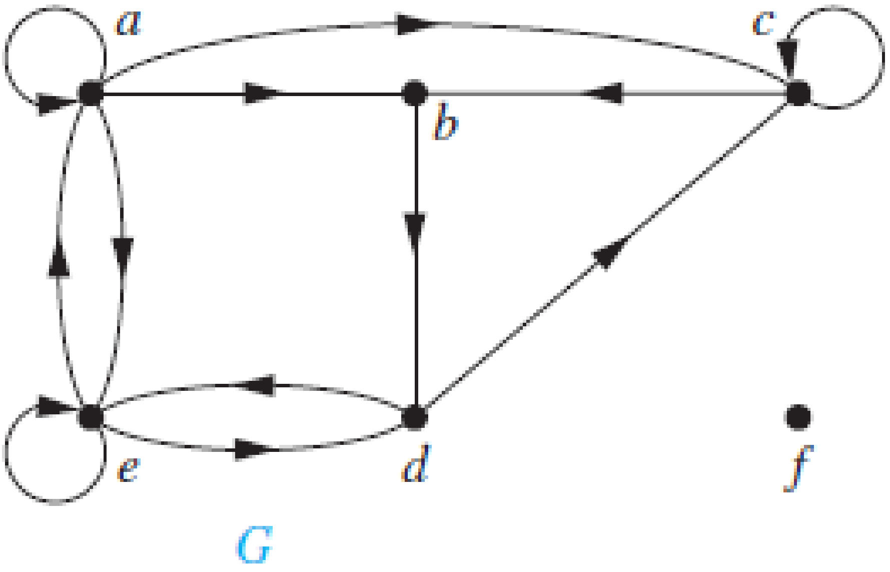
3. Because deg(v) is even for v ∈ V1, the first term in the right-hand side of the last equality is even.
4. Furthermore, the sum of the two terms on the right-hand side of the last equality is even, because this sum is 2m.
5. Hence, the second term in the sum is also even. Because all the terms in this sum are odd, there must be an even number of such terms.
6. Thus, there are an even number of vertices of odd degree.

Adjacent Vertices of Directed Graph

1. When (u, v) is an edge of the graph G with directed edges, **u is said to be adjacent to v and v is said to be adjacent from u.**
2. The vertex u is called the **initial vertex** of (u, v), and v is called the **terminal or end vertex** of (u, v).
3. The initial vertex and terminal vertex of a loop are the same.
4. In a graph with directed edges the **in-degree of a vertex v**, denoted by $deg^{-}(v)$, is the number of edges with v as their terminal vertex.
5. The **out-degree of v**, denoted by $deg^{+}(v)$, is the number of edges with v as their initial vertex.
6. Note that a loop at a vertex contributes 1 to both the in-degree and the out-degree of this vertex.

In Degree and Out Degree

1. Find the in-degree and out-degree of each vertex in the graph G with directed edges



2. The in-degrees in G are deg−(a) = 2, deg−(b) = 2, deg−(c) = 3, deg−(d) = 2, deg−(e) = 3, and deg− (f) = 0.
3. The out-degrees are deg+(a) = 4, deg+(b) = 1, deg+(c) = 2,deg+(d) = 2, deg+(e) = 3, and deg+(f) = 0.

In Degree and Out Degree

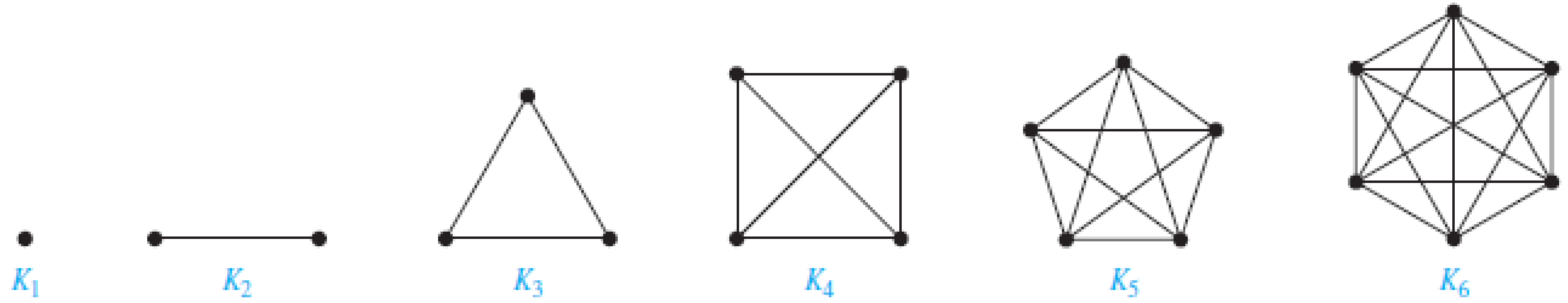
1. Because each edge has an initial vertex and a terminal vertex, **the sum of the in-degrees and the sum of the out-degrees of all vertices in a graph with directed edges are the same.**
2. Both of these sums are the number of edges in the graph.
3. This result is stated as Theorem 3:
4. Let G = (V ,E) be a graph with directed edges. Then

$$\sum_{v\in V} \deg^{-}(v) = \sum_{v\in V} \deg^{+}(v) = |E|.$$

5. The undirected graph that results from ignoring directions of edges in a directed graph is called the **underlying undirected graph**

Some Special Simple Graphs

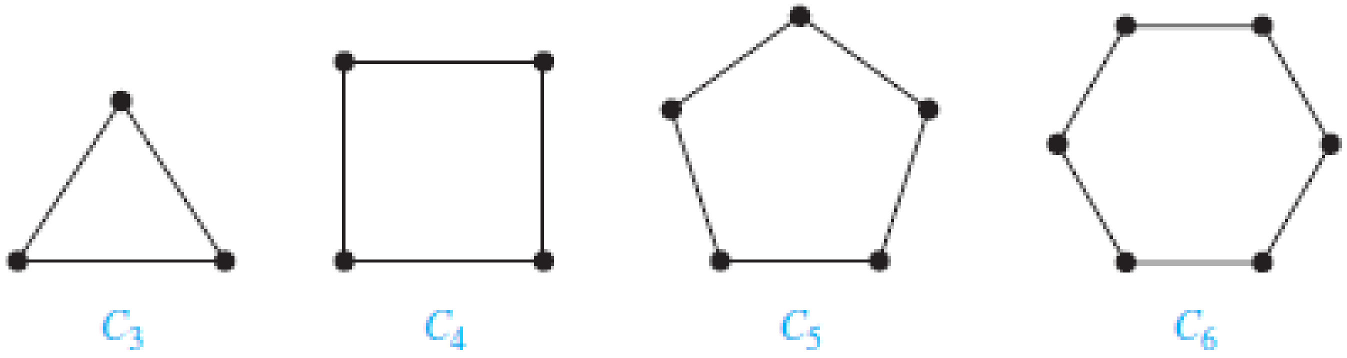
1. **Complete Graphs**: A complete graph of n vertices, denoted by K_n, is a simple graph that contains exactly one edge between each pair of distinct vertices.
2. The graphs Kn, for n = 1, 2, 3, 4, 5, 6, are displayed below.



3. A simple graph for which there is at least one pair of distinct vertex not connected by an edge is called **noncomplete**

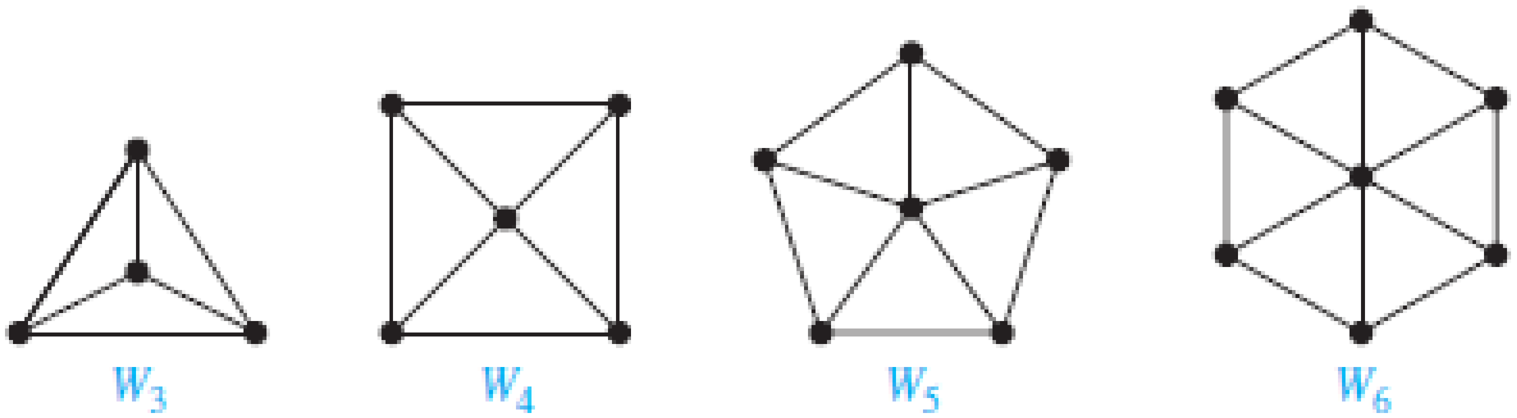
Some Special Simple Graphs

1. **Cycles**: A cycle C_n, $n \geq 3$, consists of n vertices $v_1, v_2, \ldots, v_n$ and edges $\{v_1, v_2\}, \{v_2, v_3\}, \ldots, \{v_{n-1}, v_n\}$, and $\{v_n, v_1\}$.
2. The cycles C_3, C_4, C_5, and C_6 are displayed



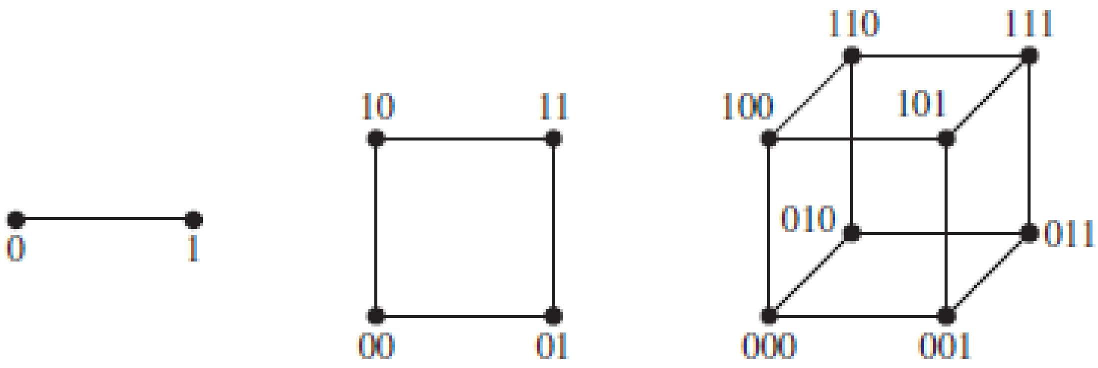
Some Special Simple Graphs

1. **Wheels**: We obtain a wheel W_n when we add an additional vertex to a cycle C_n, for $n \geq 3$, and connect this new vertex to each of the n vertices in C_n, by new edges.
2. The wheels W_3, W_4, W_5, and W_6 are displayed



Some Special Simple Graphs

1. **n-Cubes:** An n-dimensional hypercube, or n-cube, denoted by Qn, is a graph that has vertices representing the 2^n bit strings of length n.
2. Two vertices are adjacent if and only if the bit strings that they represent differ in exactly one bit position.
3. We display Q1, Q2, and Q3



Q_1 Q_2 Q_3