Discrete Mathematics (ITPC-309)

Graphs – Part IV

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Recap

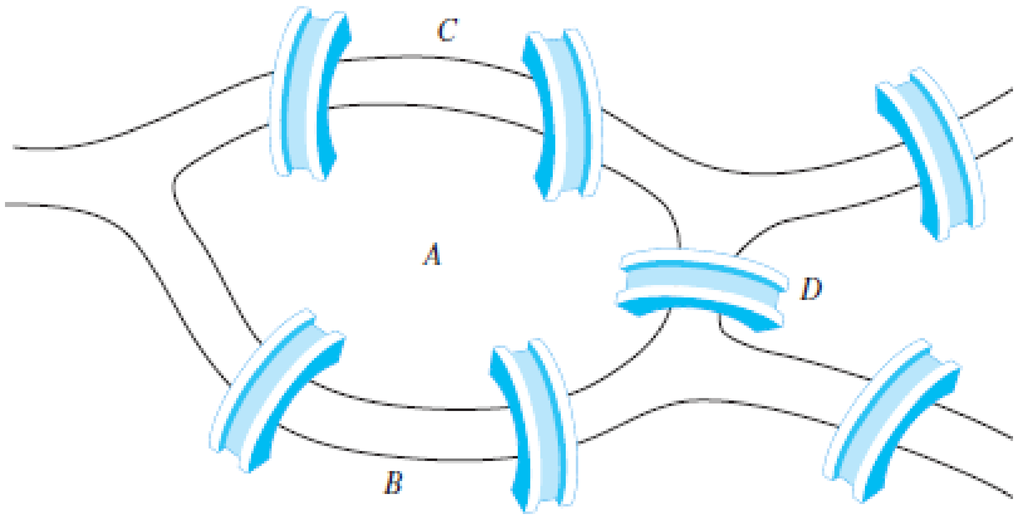
1. Regular graphs
2. Connected graphs,
3. Connectivity
4. Connected components in a graph

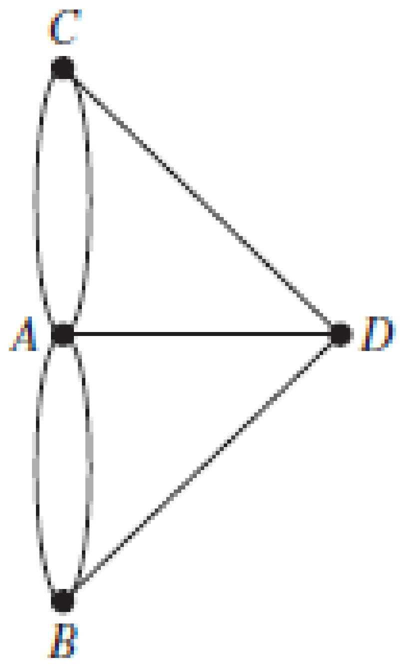
Contents

Euler Paths and Circuits

Euler Paths and Circuits - Origin

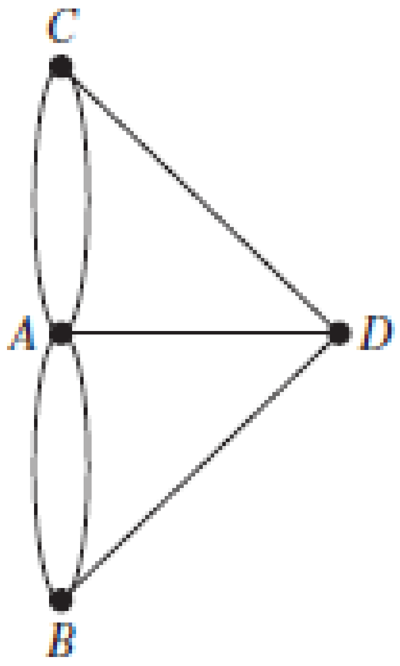
1. There was a town divided into four regions by some rivers.
2. These regions were connected by bridges.
3. **The problem: Is it possible to start at some location in the town, travel across all the bridges once (without crossing any bridge twice), and return to the starting point?**
4. The Swiss mathematician Euler solved this problem using the **multigraph** obtained when the **four regions are represented by vertices and the bridges by edges**





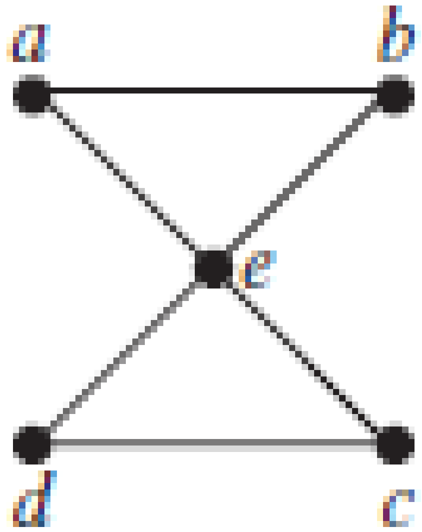
Euler Paths and Circuits

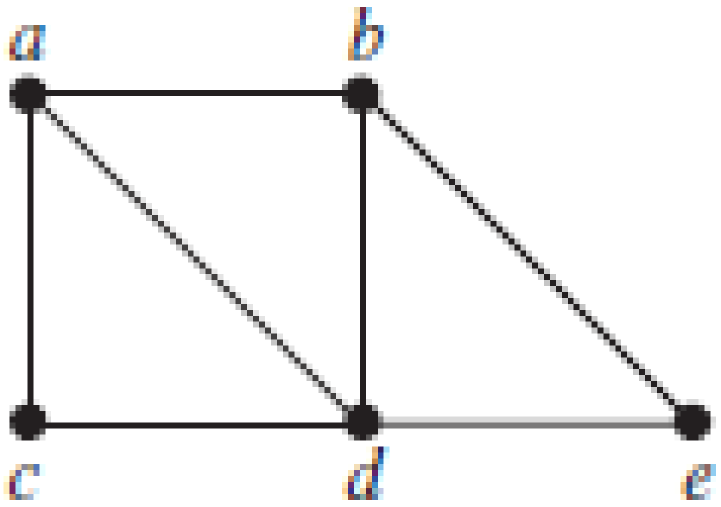
1. In graph theory terms, the question becomes:
   - **Is there a simple circuit in this multigraph that contains every edge?**
2. To solve this problem, we introduce the concepts of Euler Paths and Circuits.



Euler Paths and Circuits

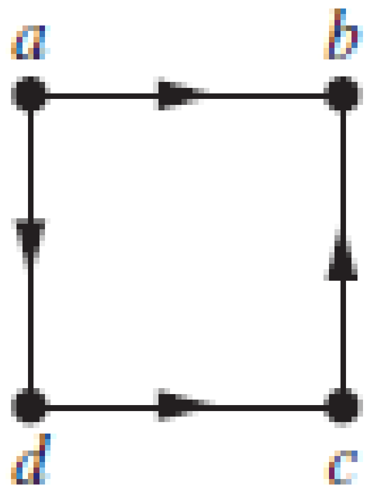
1. An **Euler circuit** in a graph G is a simple circuit containing **every edge** of G.
2. An **Euler path** in G is a simple path containing **every edge** of G.
3. Examples: Which of the undirected graphs have an Euler circuit? Which only has an Euler path?
   - **First one has Euler Circuit: a, e, c, d, e, b, a.**
   - **Second one has a Euler path: a, c, d, e, b, d, a, b.**
   - It does not have a Euler circuit.



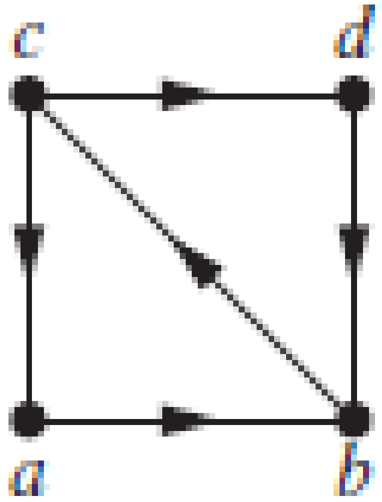


Euler Paths and Circuits

1. Which of the directed graphs have an Euler circuit? Of those that do not, which have an Euler path?
2. Neither H1 nor H3 has an Euler circuit – look at the directions.
3. H3 has an Euler path, **c, a, b, c, d, b**
4. **B**ut H1 does not



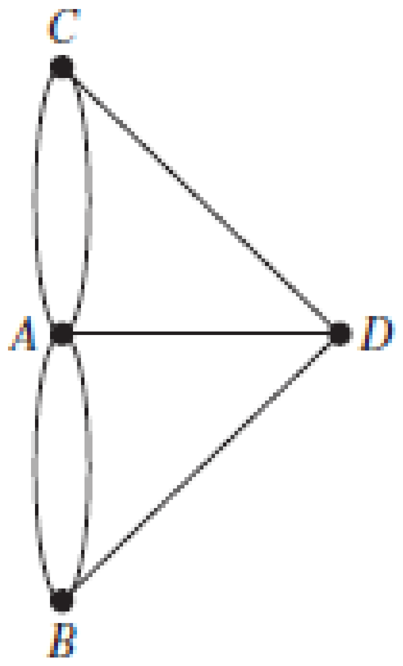
H_1



H_3

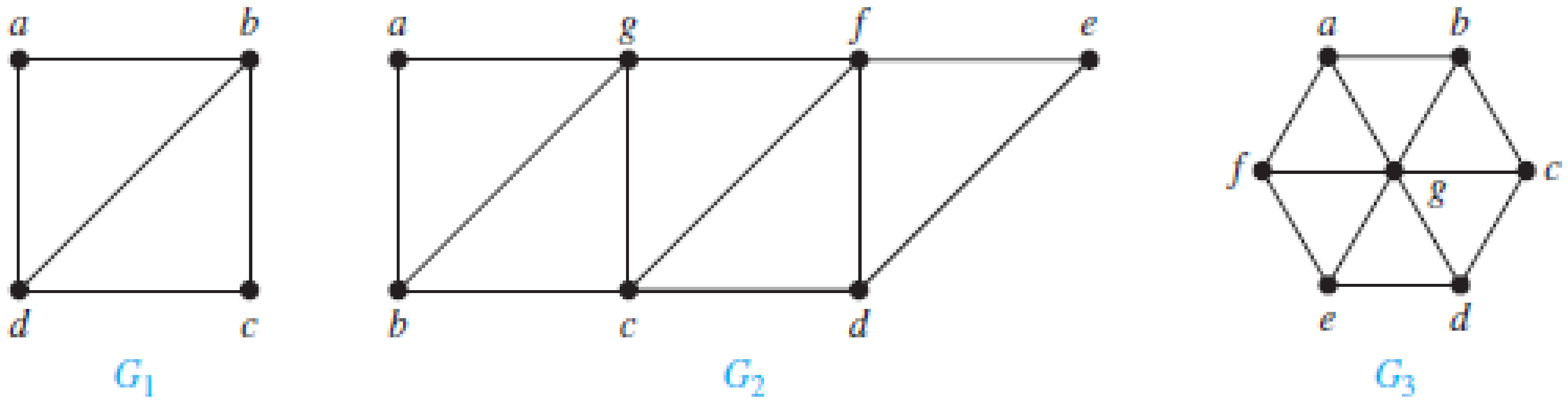
Necessary And Sufficient Conditions For Euler Circuits And Paths

1. To find if a graph has Euler paths/circuits:
2. THEOREM 1: A connected multigraph with at least two vertices has an **Euler circuit** if and only if each of its vertices has even degree.
3. Using this theorem, we can solve the original problem:
   - Because the multigraph representing the city and bridges, has four vertices of odd degree, it does not have an Euler circuit.
   - There is no way to start at a given point, cross each bridge exactly once, and return to the starting point.



Necessary And Sufficient Conditions For Euler Circuits And Paths

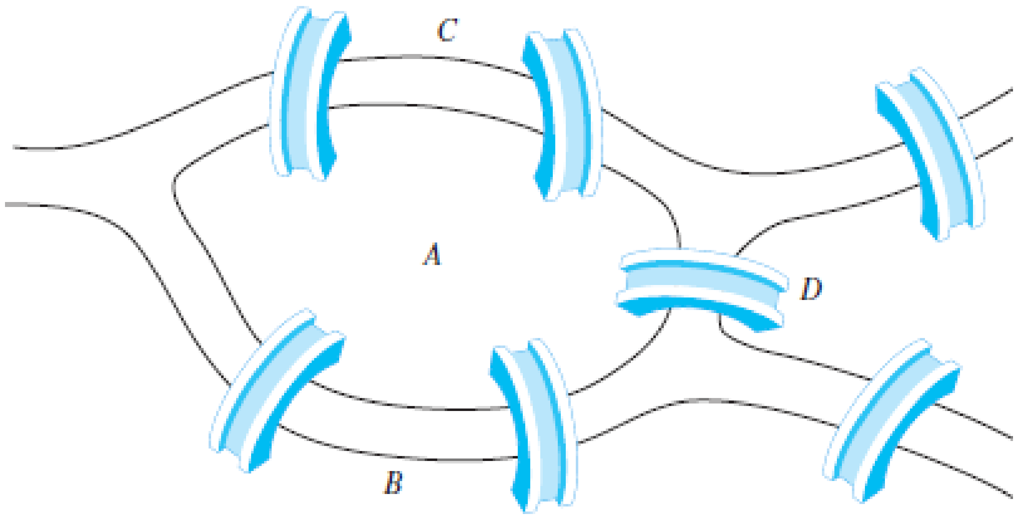
1. THEOREM 2: A connected multigraph has an Euler path but not an Euler circuit if and only if it has exactly two vertices of odd degree.
2. Which graphs have an Euler path?

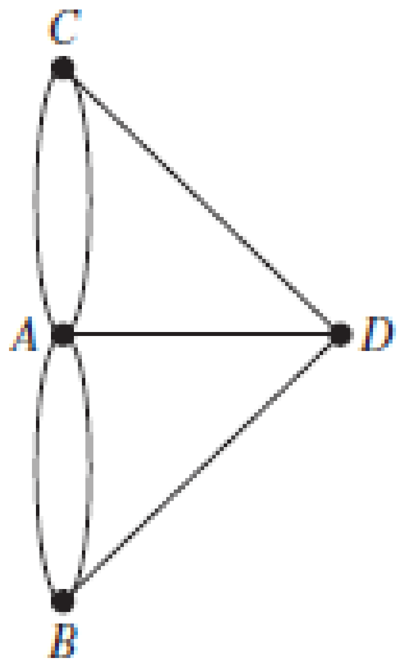


3. G1 contains exactly two vertices of odd degree, b and d. Hence, it has an Euler path that must have b and d as its endpoints. One such Euler path is d, a, b, c, d, b
4. G3 has no Euler path because it has six vertices of odd degree.

Example: Modification of the original problem

1. **The original problem**: Is it possible to start at some location in the town, travel across all the bridges once (without crossing any bridge twice), and return to the starting point? – No Euler circuit, so not possible.
2. **The modified problem**: Is it possible to start at some point in the town, travel across all the bridges once, and end up at some other point in town? – is there any Euler path?





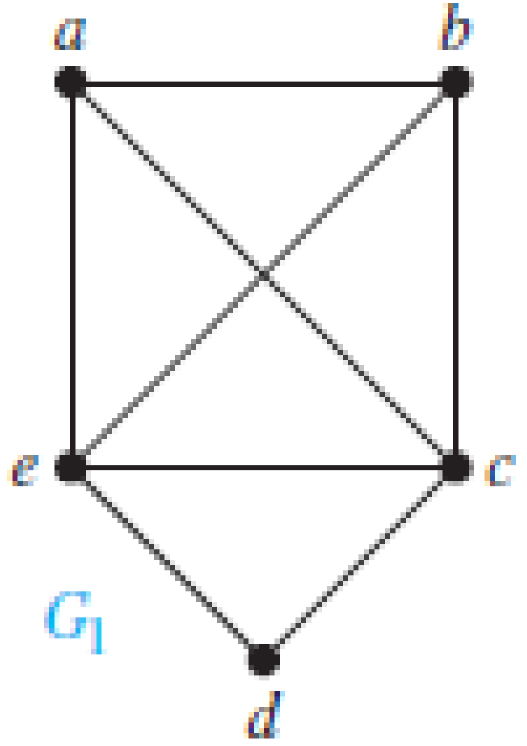
Applications

1. Chinese postman problem: The problem of finding the shortest circuit that traverses every edge at least once.
2. Layout of circuits
3. In network multicasting
4. Molecular biology - in the sequencing of DNA.

Hamilton Paths and Circuits

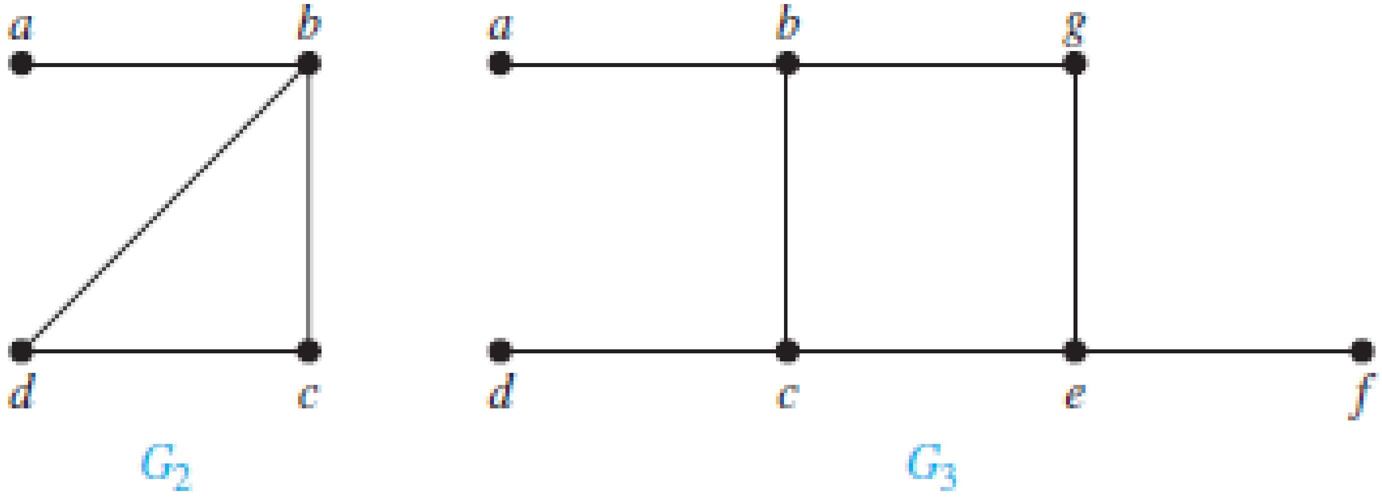
Hamilton Paths and Circuits

1. **Hamilton path**: A simple path in a graph G that passes through **every vertex** exactly once is called a Hamilton path.
2. **Hamilton circuit**: A simple circuit in a graph G that passes through every vertex exactly once is called a Hamilton circuit.
3. Example: Hamilton Path = a, b, c, d, e
4. Example: Hamilton Circuit = a, b, c, d, e, a



Hamilton Paths and Circuits

1. Which of the simple graphs have a Hamilton circuit or, a Hamilton path?



2. There is no Hamilton circuit in G2 - any circuit containing every vertex must contain b and a twice
3. G2 has a Hamilton path, namely, a, b, c, d.
4. G3 has neither a Hamilton circuit nor a Hamilton path,

Conditions for the Existence Of Hamilton Circuits

1. How to know if a graph has Hamilton paths/circuits?
2. There are no known simple necessary and sufficient criteria for the existence of Hamilton circuits.
3. However, many theorems are known that give sufficient conditions for the existence of Hamilton circuits.
4. Also, certain properties can be used to show that a graph has no Hamilton circuit.

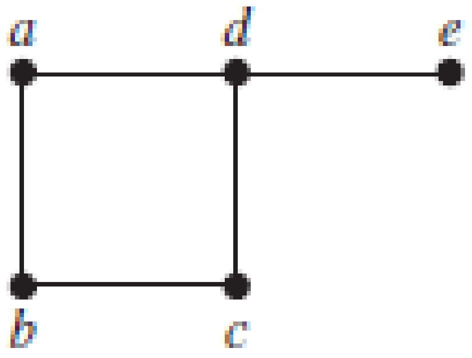
Sufficient (But Not Necessary) Conditions for the Existence Of Hamilton Circuits

1. Dirac’s Theorem: If G is a simple graph with n vertices with n ≥ 3 such that the degree of every vertex in G is at least n/2, then G has a Hamilton circuit.

2. If G is a simple graph with n vertices with n ≥ 3 such that deg(u) + deg(v) ≥ n for every pair of nonadjacent vertices u and v in G, then G has a Hamilton circuit.
   - Example: Draw such a graph and see if this holds.

Properties of Hamilton Circuit.

1. A graph with a vertex of degree one cannot have a Hamilton circuit.
    - Because in a Hamilton circuit, each vertex is incident with two edges in the circuit.
    - There is no Hamilton circuit in G because G has a vertex of degree one, e.
    - How to modify this graph so that it has a Hamilton Circuit?

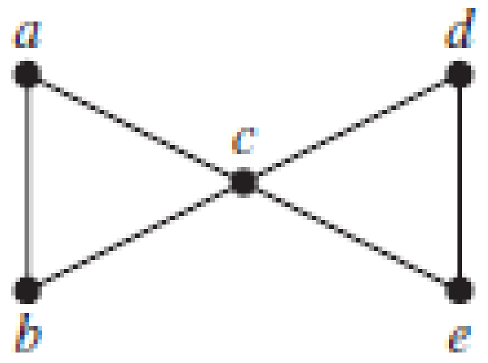


G

2. If a vertex in the graph has degree two, then both edges that are incident with this vertex must be part of any Hamilton circuit.

Hamilton Paths and Circuits

1. When a Hamilton circuit is being constructed and this circuit has passed through a vertex, then all remaining edges incident with this vertex, other than the two used in the circuit, can be removed from consideration.

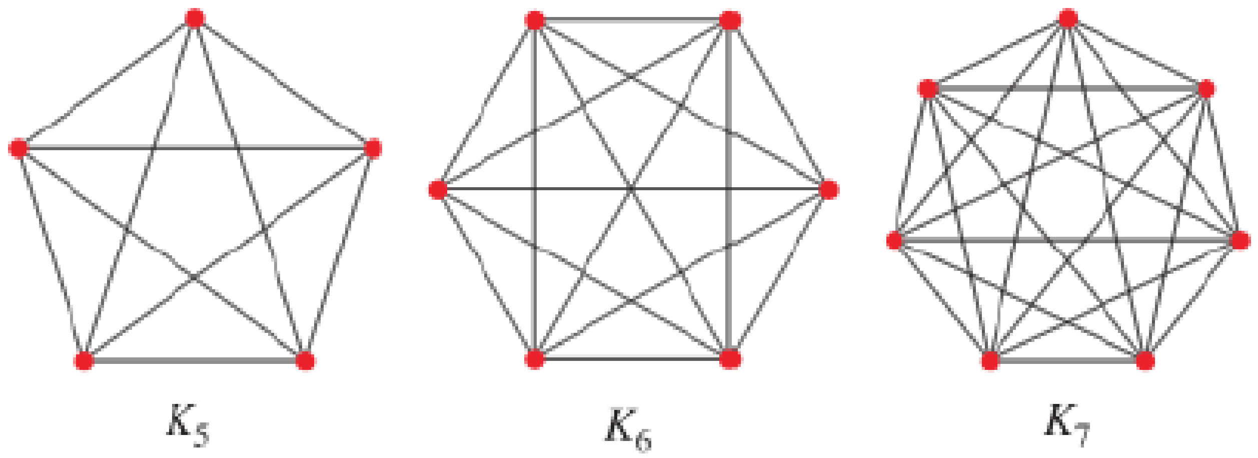


H

Example: In H, because the degrees of the vertices a, b, d, and e are all two, every edge incident with these vertices must be part of any Hamilton circuit. So, no Hamilton circuit can exist in H, for any Hamilton circuit would have to contain four edges incident with c, which is impossible

Hamilton Paths and Circuits

1. Show that K_n - complete graph with N vertices - has a Hamilton circuit whenever $n \geq 3$.



We can form a Hamilton circuit in Kn beginning at any vertex.

Such a circuit can be built by visiting vertices in any order we choose, as long as the path begins and ends at the same vertex and visits each other vertex exactly once.

This is possible because there are edges in Kn between any two vertices.

Applications

1. Traveling salesperson problem or TSP : asks for the shortest route a traveling salesperson should take to visit a set of cities.

Planar Graphs

Planar Graphs

1. A graph is called planar if it can be drawn in the plane without any edges crossing
2. Such a drawing is called a planar representation of the graph.

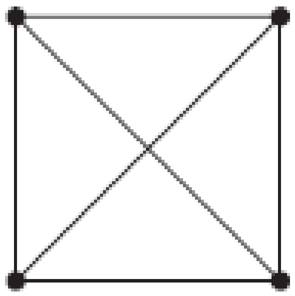


FIGURE 2 The Graph K_4.

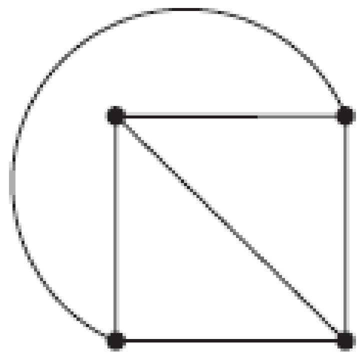


FIGURE 3 K_4 Drawn with No Crossings.

Planar Graphs

1. Is Q3, planar? - Q3 is planar, because it can be drawn without any edges crossing

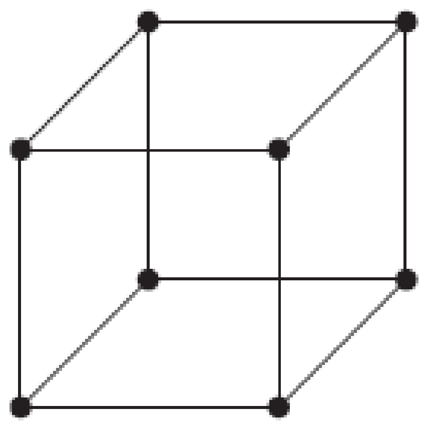


FIGURE 4 The Graph Q_3.

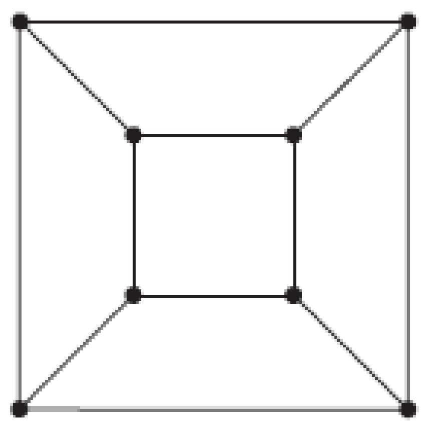
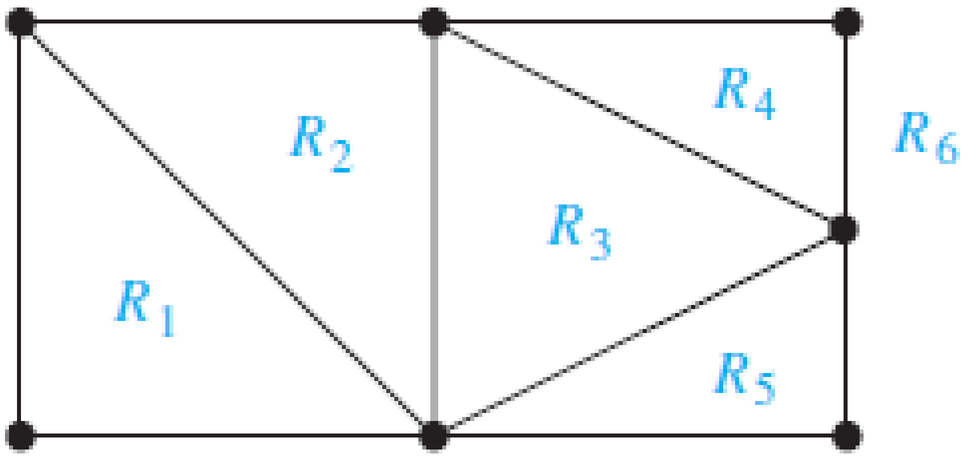


FIGURE 5 A Planar Representation of Q_3.

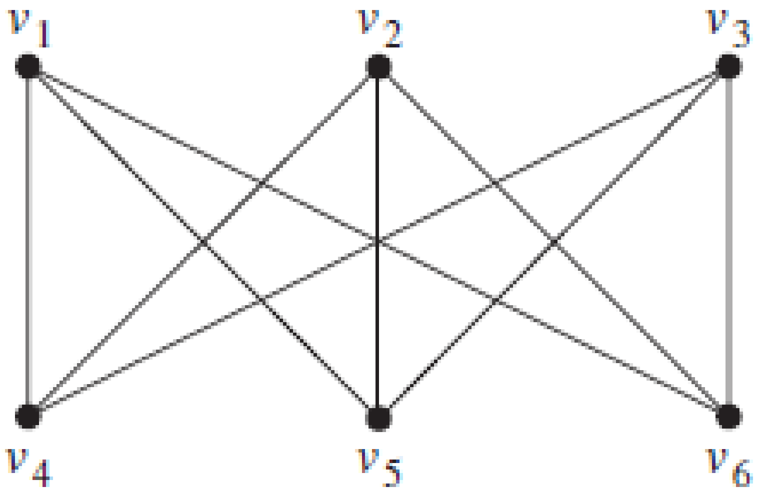
Planar Graphs

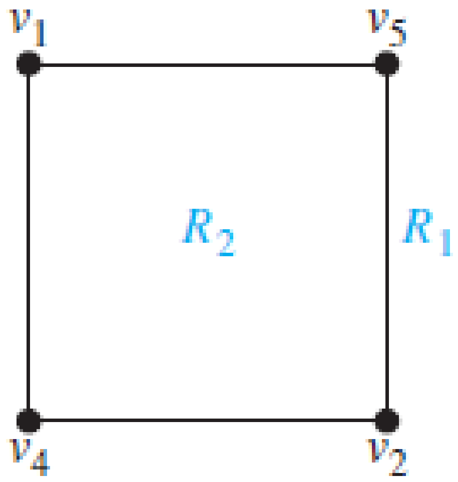
1. A planar representation of a graph splits the plane into regions, including an unbounded region.
2. Euler showed that all planar representations of a graph split the plane into the same number of regions.
3. He also found a relationship among the number of regions, the number of vertices, and the number of edges of a planar graph.



Planar Graphs

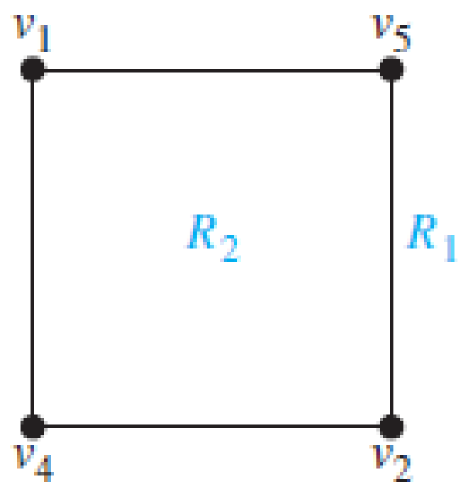
1. Is $K_{3,3}$, planar? – No.
2. How can we prove this using the concept of regions?
3. Lets start with the vertices V1 and V2.
4. In any planar representation of $K_{3,3}$, the vertices v1 and v2 must be connected to both v4 and v5.
5. These four edges form a closed curve that splits the plane into two regions, R1 and R2
6. The vertex v3 is in either R1 or R2

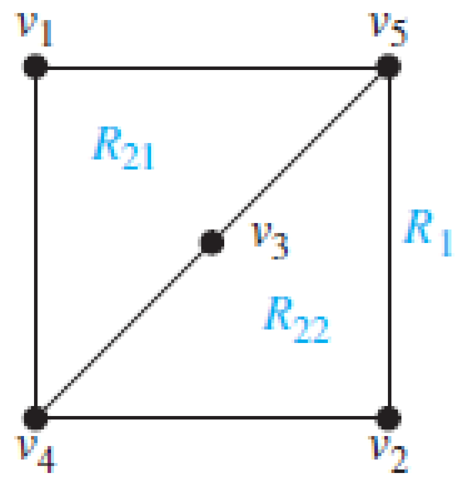




Planar Graphs

1. When v3 is in R2, the inside of the closed curve, the edges between v3 and v4 and between v3 and v5 separate R2 into two subregions, R21 and R22.
2. **Note that there is no way to place the final vertex v6 without forcing a crossing.**
3. If v6 is in R1, then the edge between v6 and v3 cannot be drawn without a crossing.
4. If v6 is in R21, then the edge between v2 and v6 cannot be drawn without a crossing.
5. If v6 is in R22, then the edge between v1 and v6 cannot be drawn without a crossing.
6. A similar argument can be used when v3 is in R1.
7. This shows that this graph is not planar.





Planar Graphs - Euler’s Formula

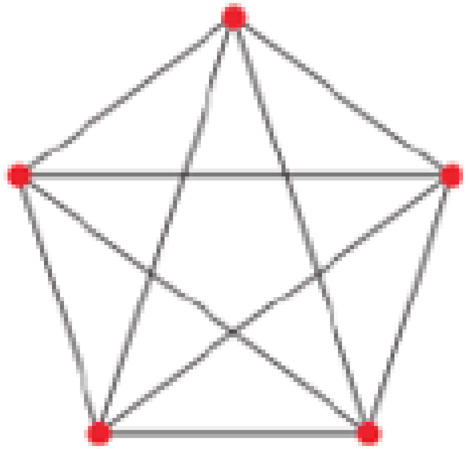
1. Let G be a connected planar simple graph with e edges and v vertices.
2. Let r be the number of regions in a planar representation of G.
3. **Then r = e − v + 2**.
4. **Example**: Suppose that a connected planar simple graph has 20 vertices, each of degree 3. Into how many regions does a representation of this planar graph split the plane?
   - Solution: This graph has 20 vertices, each of degree 3, so v = 20.
   - <u>sum of the degrees of the vertices = 2e</u>
   - Because the sum of the degrees of the vertices, 3v = 3 · 20 = 60, is equal to twice the number of edges, 2e, we have 2e = 60, or e = 30.
   - Consequently, from <u>Euler’s formula</u>, the number of regions is r = e − v + 2 = 30 − 20 + 2 = 12.

Planar Graphs - Properties

1. The following inequalities that must be satisfied by planar graphs
   a) If G is a connected planar simple graph with e edges and v vertices, where $v \geq 3$, then $e \leq 3v - 6$.
   b) If G is a connected planar simple graph, then G has a vertex of degree not exceeding five.
   c) If a connected planar simple graph has e edges and v vertices with $v \geq 3$ and no circuits of length three, then $e \leq 2v - 4$.

Planar Graphs - Properties

1. Show that K5 is nonplanar using the property that:
   - If G is a connected planar simple graph with e edges and v vertices, where $v \geq 3$, then $e \leq 3v - 6$.
2. Solution: The graph K5 has five vertices and 10 edges. However, the inequality $e \leq 3v - 6$ is not satisfied for this graph because $e = 10$ and $3v - 6 = 9$. Therefore, K5 is not planar



K_5

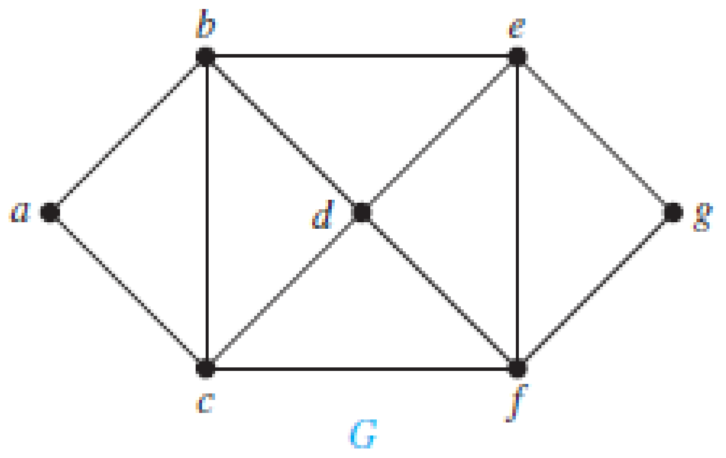
Graph Coloring

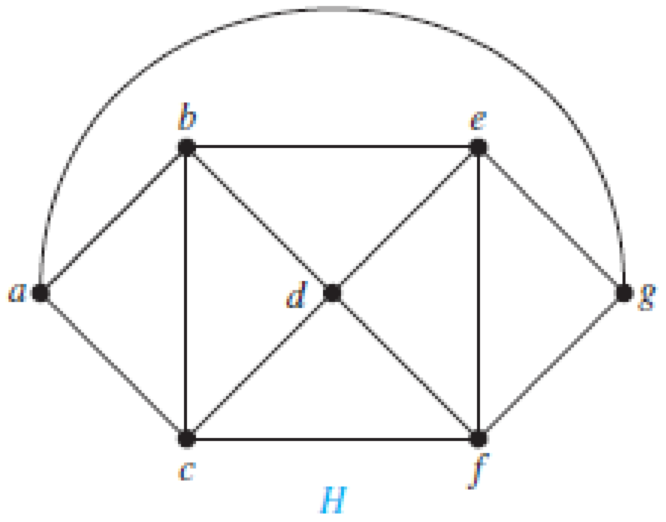
Graph Coloring

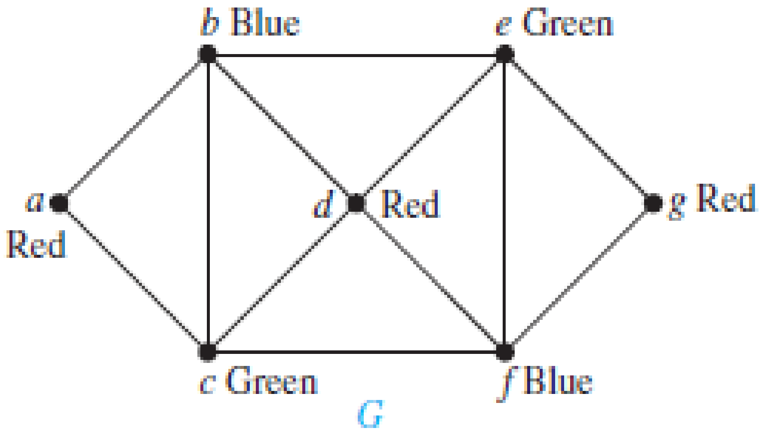
1. A coloring of a simple graph is the assignment of a color to each vertex of the graph so that no two adjacent vertices are assigned the same color.
2. The **chromatic number** of a graph is the least number of colors needed for a coloring of this graph.
3. The chromatic number of a graph G is denoted by $\chi(G)$. (Here χ is the Greek letter chi.)
4. THE FOUR COLOR THEOREM: The chromatic number of a planar graph is no greater than four.

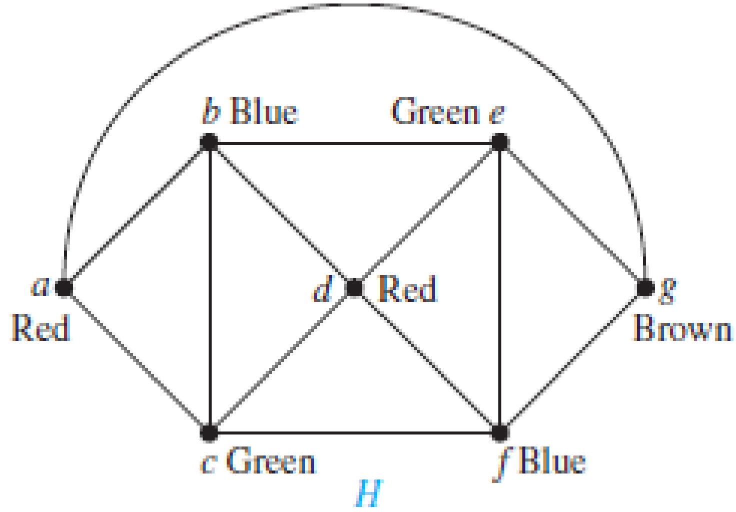
Graph Coloring

1. What are the chromatic numbers of the graphs G and H?



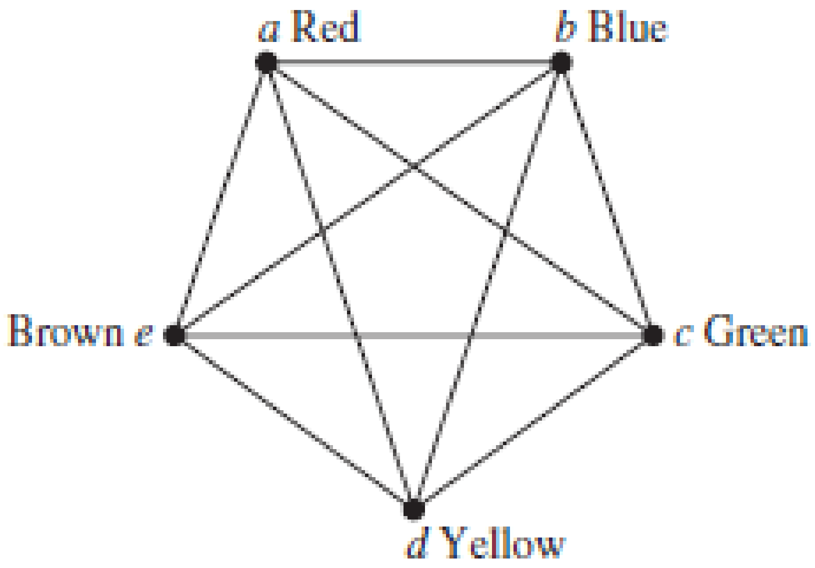






Graph Coloring

1. What is the chromatic number of K_n?
   - A coloring of K_n can be constructed using n colors by assigning a different color to each vertex.
   - No coloring using fewer colors is possible.
   - No two vertices can be assigned the same color, because every two vertices of this graph are adjacent



Homomorphism of Graphs

Homomorphism of Graphs

1. A graph homomorphism is a mapping between two graphs that respects their structure.
2. It is a function between the vertex sets of two graphs that maps adjacent vertices to adjacent vertices.
3. Most often used in constraint satisfaction problems, such as certain scheduling or frequency assignment problems.

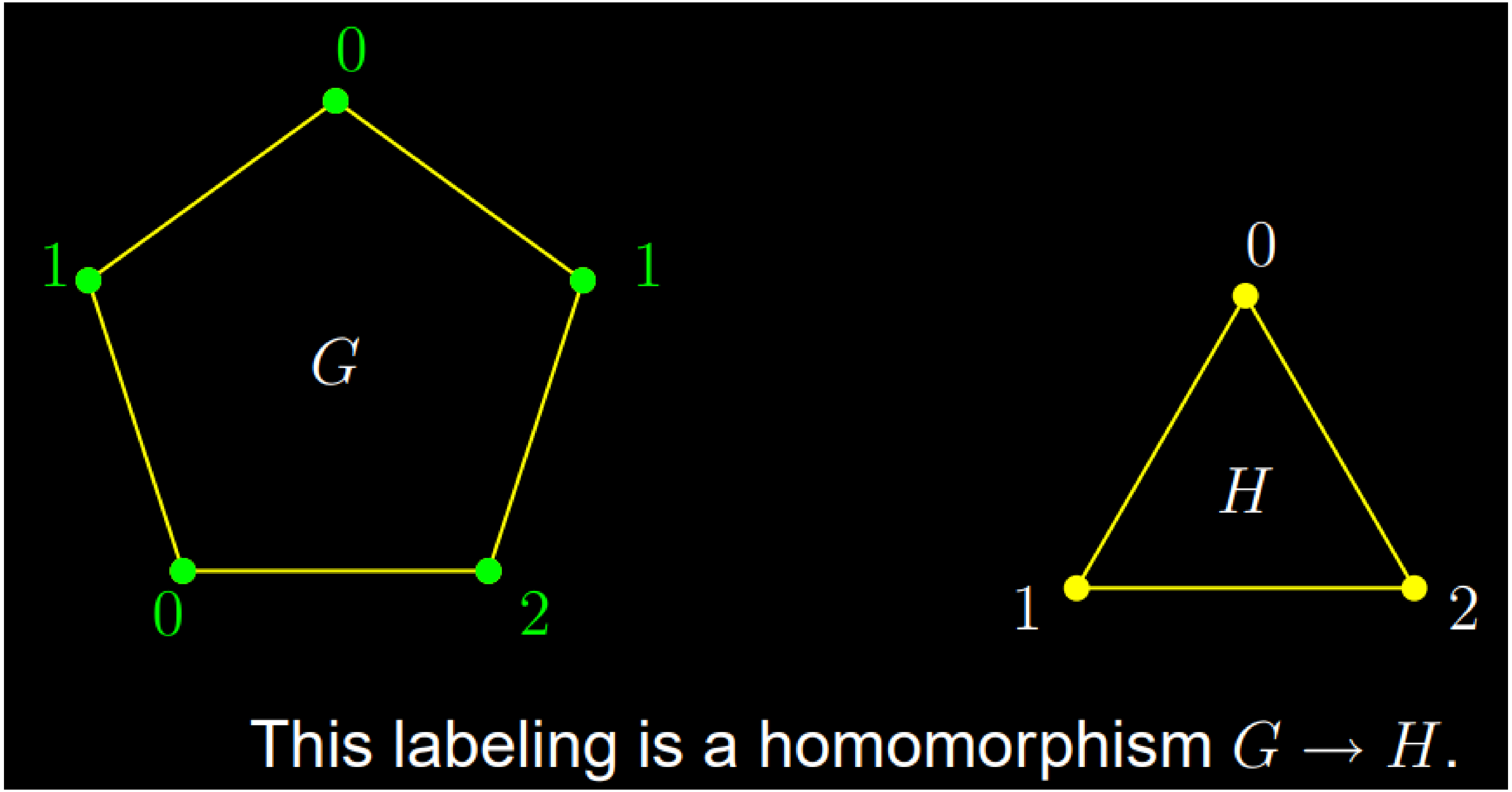
Homomorphism of Graphs

1. A graph homomorphism f from a graph G = (V(G),E(G)), to a graph H = (V(H),E(H)), written as $f: G \rightarrow H$ is a function from V(G) to V(H) that preserves edges.

$$(u,v) \in E(G) \text{ implies } (f(u),f(v)) \in E(H), \text{ for all pairs of vertices } u,v \text{ in } V(G).$$

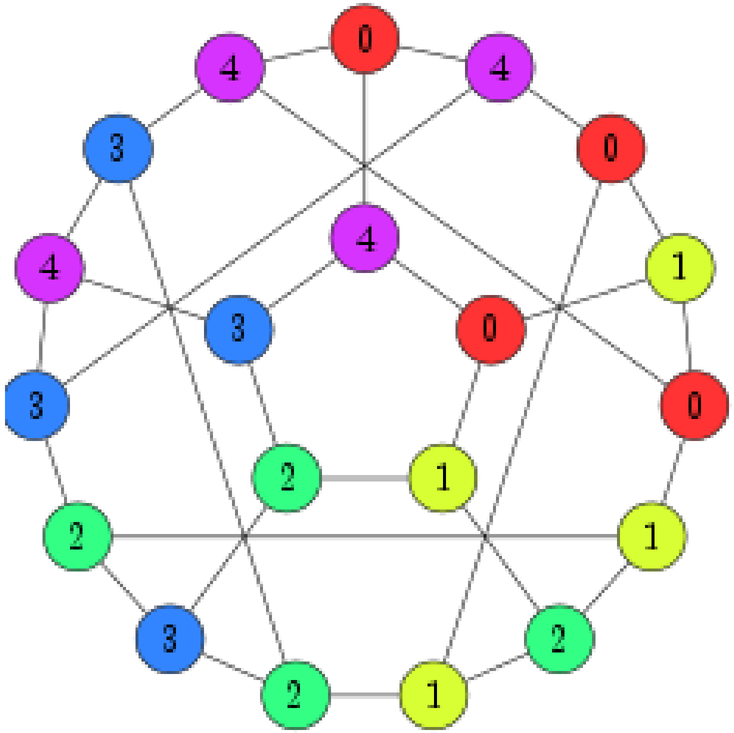
2. If there exists any homomorphism from G to H, then G is said to be homomorphic to H or H-colorable.

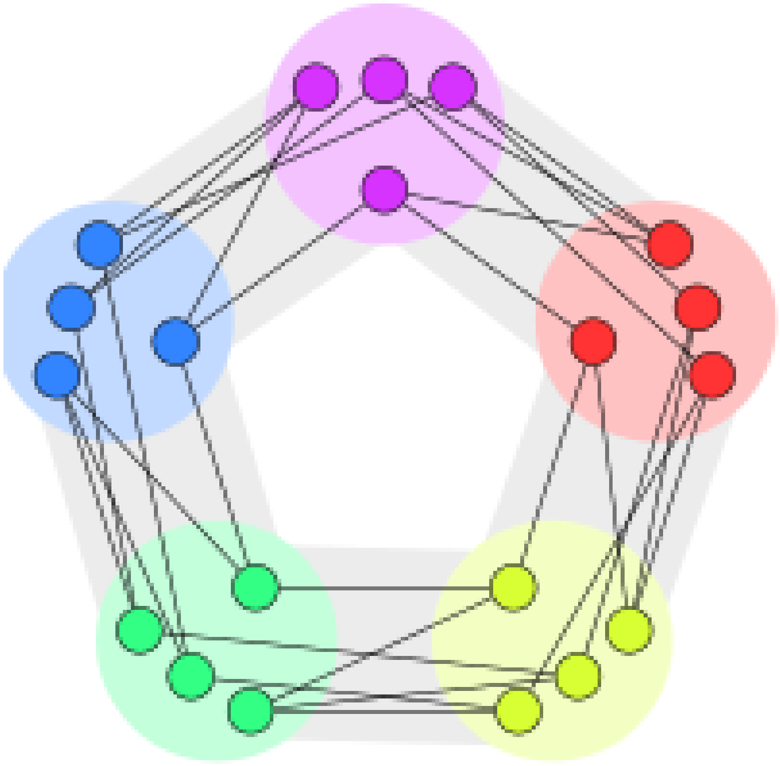
Homomorphism of Graphs



Homomorphism of Graphs

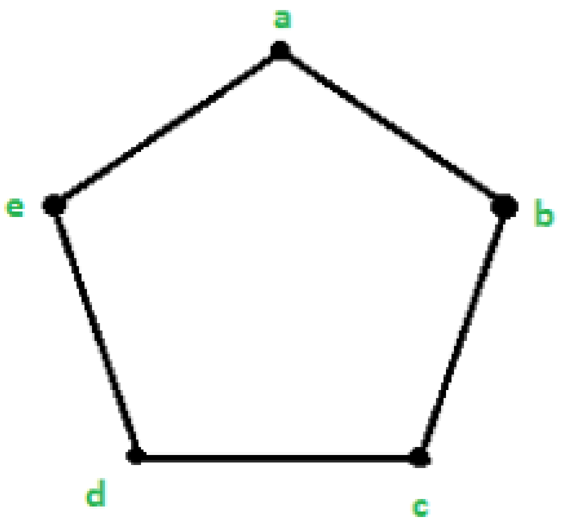
1. Example



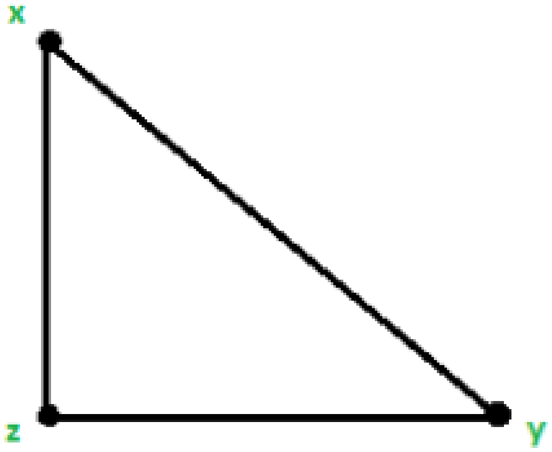


Homomorphism of Graphs

1. Below are the 2 graphs G = (V, E) with V = {a, b, c, d, e} and E = {(a, b), (b, c), (c, d), (d, e), (e, a)} and G’ = (V’, E’) with V’ = {x, y, z} and E’ = {(x, y), (y, z), (z, x)}. Are they homomorphic?



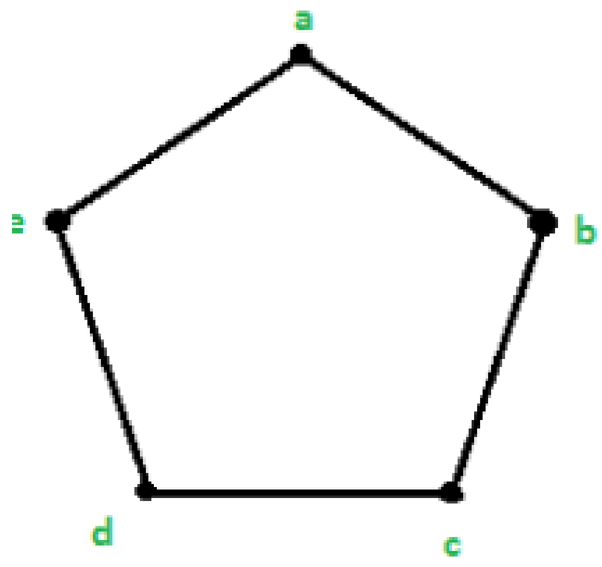
Graph G



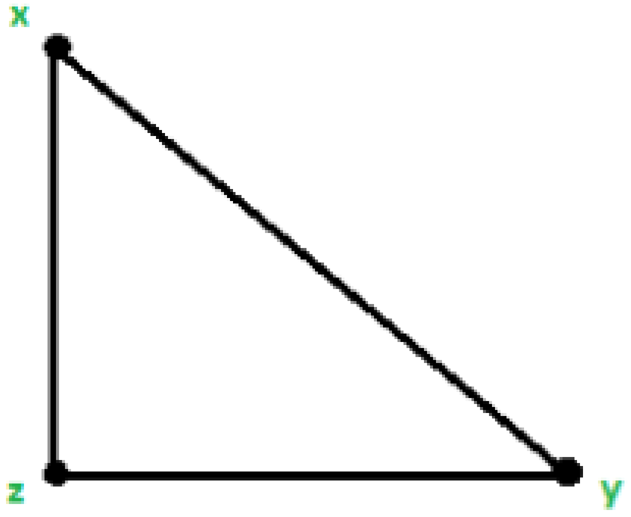
Graph G'

Homomorphism of Graphs

1. Let us assume $f(a) = x$, $f(b) = y$, $f(c) = z$, $f(d) = x$ and $f(e) = z$.
2. (a, b) is an edge in G, then $(f(a), f(b))$ must be an edge in E’. $f(a) = x$ and $f(b) = y \Rightarrow (f(a), f(b)) = (x, y) \in E'$
3. (b, c) is an edge in G, then $(f(b), f(c))$ must be an edge in E’. $f(b) = y$ and $f(c) = z \Rightarrow (f(b), f(c)) = (y, z) \in E'$
4. (c, d) is an edge in G, then $(f(c), f(d))$ must be an edge in E’. $f(c) = z$ and $f(d) = x \Rightarrow (f(c), f(d)) = (z, x) \in E'$
5. (d, e) is an edge in G, then $(f(d), f(e))$ must be an edge in E’. $f(d) = x$ and $f(e) = z \Rightarrow (f(d), f(e)) = (x, z) \in E'$
6. (e, a) is an edge in G, then $(f(e), f(a))$ must be an edge in E’. $f(e) = z$ and $f(a) = x \Rightarrow (f(c), f(d)) = (z, x) \in E'$
7. So, it can be seen that $\forall\{u, v\} \in E \Rightarrow \exists\{f(u), f(v)\} \in E'$.
8. So f is a homomorphism.

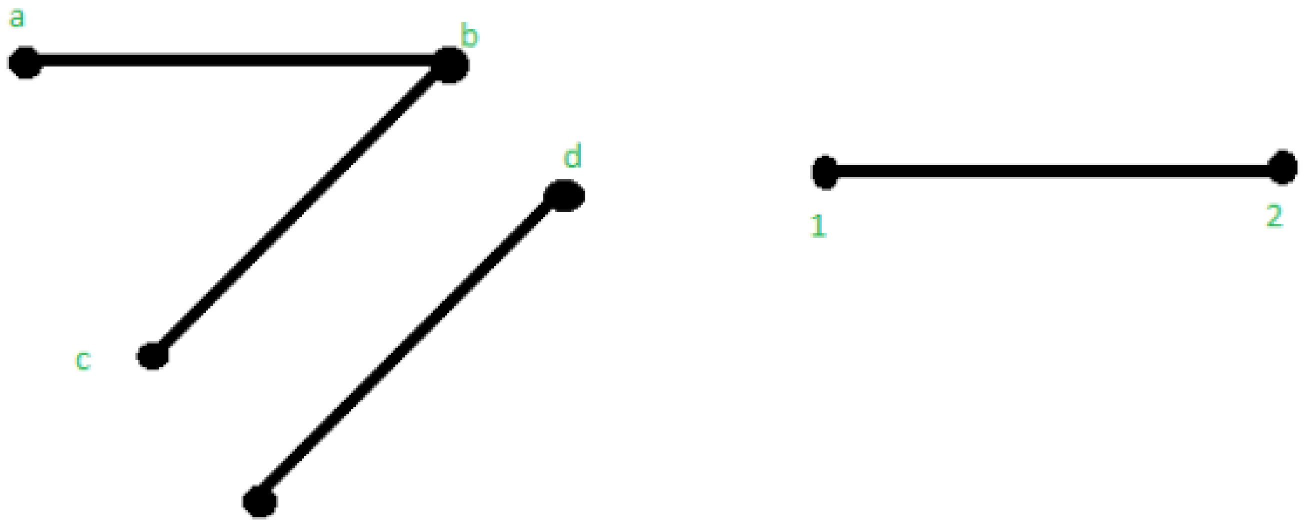


Graph G



Homomorphism of Graphs

1. Below are the 2 graphs G = (V, E) with V = {a, b, c, d, e} and E = {(a, b), (b, c), (d, e), (e, h)} and G' = (V', E') with V' = {1, 2} and E' = { (1, 2)}. Are they homomorphic?

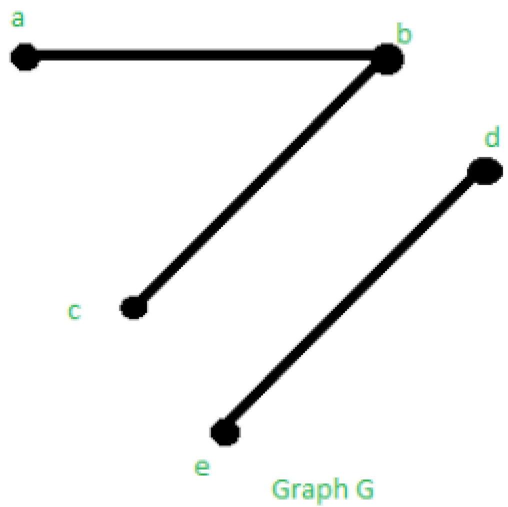


Graph G

Graph G'

Homomorphism of Graphs

1. Let us assume f(a) = 1, f(b) = 2, f(c) = 1, f(d) = 2, f(e) = 1
2. If (a, b) is an edge in G, then (f(a), f(b)) must be an edge in E'. f(a) = 1 and f(b) = 2 $\Rightarrow$ (f(a), f(b)) = (1, 2) $\in$ E'
3. If (b, c) is an edge in G, then (f(b), f(c)) must be an edge in E'. f(b) = 2 and f(c) = 1 $\Rightarrow$ (f(b), f(c)) = (2, 1) $\in$ E'
4. If (d, e) is an edge in G, then (f(d), f(e)) must be an edge in E'. f(d) = 2 and f(e) = 1 $\Rightarrow$ (f(d), f(e)) = (2, 1) $\in$ E'
5. But, (b, e) is not an edge in G, but (f(b), f(e)) = (2, 1) is an edge in the graph G'.
6. So it is not a homomorphism



Thanks!