Discrete Mathematics (ITPC-309)

Ordered Sets and Lattices – Part II

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Contents

Lattices

1. There are two ways to define a lattice L.
    - In terms of Posets
    - In terms of axioms
2. A lattice L may be defined as a partially ordered set in which inf(a,b) and sup(a, b) exist for any pair of elements a, b $\in$ L.

Lattices

1. Let L be a nonempty set closed under two binary operations called meet and join, denoted respectively by ∧ and ∨.
2. Then (L, ∧, ∨) is called lattice if the following axioms hold where a, b, c are elements in L:

[L$_1$] Commutative law:

(1a) $a \wedge b = b \wedge a$ (1b) $a \vee b = b \vee a$

[L$_2$] Associative law:

(2a) $(a \wedge b) \wedge c = a \wedge (b \wedge c)$ (2b) $(a \vee b) \vee c = a \vee (b \vee c)$

[L$_3$] Absorption law:

(3a) $a \wedge (a \vee b) = a$ (3b) $a \vee (a \wedge b) = a$

Duality

1. The dual of any statement in a lattice $(L, \wedge, \vee)$ is defined to be the statement that is obtained by interchanging $\wedge$ and $\vee$.
2. For example, the dual of $a \wedge (b \vee a) = a \vee a$ is $a \vee (b \wedge a) = a \wedge a$
3. Notice that the dual of each axiom of a lattice is also an axiom.
4. Accordingly, the principle of duality holds: **The dual of any theorem in a lattice is also a theorem**

Idempotent Law

1. Important property of lattices following directly from the absorption laws: the idempotent laws:

$$\text{(i) } a \wedge a = a; \quad \text{(ii) } a \vee a = a.$$

2. Proof:

$$\begin{aligned} a \wedge a &= a \wedge (a \vee (a \wedge b)) && \text{(using (3}b\text{))} \\ &= a && \text{(using (3}a\text{))} \end{aligned}$$

$$(3b) \quad a \vee (a \wedge b) = a$$

$$(3a) \quad a \wedge (a \vee b) = a$$

Lattices and Order

1. Given a lattice L, we can define a partial order on L as follows:

$$a \precsim b \quad \text{if} \quad a \wedge b = a$$

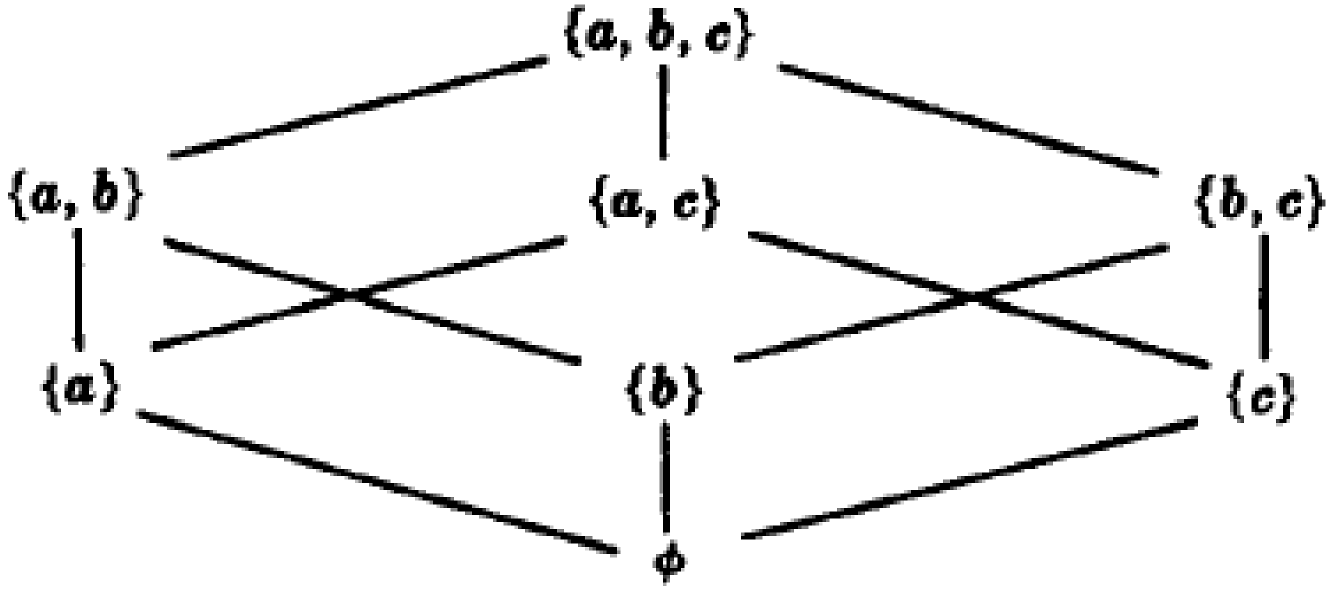
Analogously, we could define

$$a \precsim b \quad \text{if} \quad a \vee b = b$$

2. Now that we have a partial order on any lattice L, we can picture L by a diagram as was done for partially ordered sets in general – Hasse Diagrams

Lattices and Order - Example

1. Let C be a collection of sets closed under intersection and union.
2. Then (C, ∩, ∪) is a lattice.
3. In this lattice, the partial order relation is the same as the set inclusion relation.
4. Below figure shows the diagram of the lattice L of all subsets of {a, b, c}.



Bounded Lattices

1. A lattice L is said to have a lower bound 0 if for any element x in L we have $0 \precsim x$.
2. Analogously, L is said to have an upper bound I if for any x in L we have $x \precsim I$.
3. We say L is bounded if L has both a lower bound 0 and an upper bound I.
4. In such a lattice we have the following identities for any element a in L.

$$a \vee I = I, \quad a \wedge I = a, \quad a \vee 0 = a, \quad a \wedge 0 = 0$$

Bounded Lattices - Examples

1. Example 1: The nonnegative integers with the usual ordering, $0 < 1 < 2 < 3 < 4 < \cdots$ have 0 as a lower bound but have no upper bound.
2. Example 2: The lattice P(U) of all subsets of any universal set U is a bounded lattice with U as an upper bound and the empty set Φ as a lower bound.
3. Example 3: Suppose $L = \{a_1, a_2, \ldots, a_n\}$ is a finite lattice. Then $a_1 \vee a_2 \vee \cdots \vee a_n$ and $a_1 \wedge a_2 \wedge \cdots \wedge a_n$ are upper and lower bounds for L, respectively.
4. Note: Every finite lattice L is bounded

Distributive Lattices

1. A lattice L is said to be distributive if for any elements a, b, c in L we have the following:

[L_4] Distributive law:

$$(4a)\ a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c) \qquad (4b)\ a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$$

2. Otherwise, L is said to be non-distributive.

Complements

1. Let L be a bounded lattice with lower bound 0 and upper bound I .
2. Let a be an element of L.
3. An element x in L is called a complement of a if

$$a \vee x = I \quad \text{and} \quad a \wedge x = 0$$

4. Complements need not exist and need not be unique

Complements

1. Let L be a bounded distributive lattice.
2. Then complements are unique if they exist.

Proof: Suppose x and y are complements of any element a in L. Then

$$a \vee x = I, \quad a \vee y = I, \quad a \wedge x = 0, \quad a \wedge y = 0$$

Using distributivity,

$$x = x \vee 0 = x \vee (a \wedge y) = (x \vee a) \wedge (x \vee y) = I \wedge (x \vee y) = x \vee y$$

Similarly,

$$y = y \vee 0 = y \vee (a \wedge x) = (y \vee a) \wedge (y \vee x) = I \wedge (y \vee x) = y \vee x$$

Thus

$$x = x \vee y = y \vee x = y$$

Complemented Lattices

1. A lattice L is said to be complemented if L is bounded and every element in L has a complement.
2. The figure shows a complemented lattice where complements are not unique:
3. Find the complements of each of the elements of the below lattice