WEEKLY JOURNAL-09

SUMMARY

Summary: Summary: continued the discussion on continuous Markov process. This week the repairable system concept was introduced. Continuous Markov process is useful in solving this problem.

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Markov Process

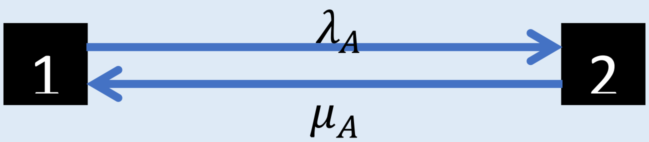
Week-09

Summary: continued the discussion on continuous Markov process. This week the repairable system concept was introduced. Continuous Markov process is useful in solving this problem.

Repairable component

So far, the modelling was done for non-repairable systems. In this section the Markov process is developed for repairable systems. Real world systems will have repairable components. So the component will have failure rate λ and repair rate μ.

For example, take below system



Being in state 1 at time t+Δt can be written as:

$$P_1(t+\Delta t) = P_1(t) - \lambda_A.\Delta \text{t}.P_1(t) + \mu_A P_2(t)$$

$$P_1(t+\Delta t) - P_1(t) = -\lambda_A.\Delta \text{t}.P_1(t) + \mu_A.\Delta \text{t}.P_2(t)$$

$$\frac{P_1(t+\Delta t) - P_1(t)}{\Delta \text{t}} = -\lambda_A.P_1(t) + \mu_A.P_2(t)$$

$$\frac{dP_1(t)}{dt} = -\lambda_A.P_1(t) + \mu_A.P_2(t)$$

Being in state 2 at time t+Δt can be written as:

$$P_2(t+\Delta t) = P_2(t) + \lambda_A.\Delta \text{t}.P_1(t) - \mu_A P_2(t)$$

$$P_2(t+\Delta t) - P_2(t) = +\lambda_A.\Delta \text{t}.P_1(t) + \mu_A.\Delta \text{t}.P_2(t)$$

$$\frac{P_2(t+\Delta t) - P_2(t)}{\Delta \text{t}} = \lambda_A.P_1(t) - \mu_A.P_2(t)$$

$$\frac{dP_2(t)}{dt} = \lambda_A.P_1(t) - \mu_A.P_2(t)$$

We can solve this by Laplace transform

$$\frac{dP_1(t)}{dt} = -\lambda_A.P_1(t) + \mu_A.P_2(t)$$

$$\frac{dP_2(t)}{dt} = \lambda_A.P_1(t) - \mu_A.P_2(t)$$

$$\frac{d}{dt}\begin{bmatrix} P_1(t) \\ P_2(t) \end{bmatrix} = \begin{bmatrix} -\lambda_A & \mu_A \\ \lambda_A & -\mu_A \end{bmatrix} \begin{bmatrix} P_1(t) \\ P_2(t) \end{bmatrix}$$

$$\begin{bmatrix} P_1(s) \\ P_2(s) \end{bmatrix} = \left(\begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} -\lambda_A & \mu_A \\ \lambda_A & -\mu_A \end{bmatrix} \right)^{-1} \begin{bmatrix} P_1(0) \\ P_2(0) \end{bmatrix}$$

$$\begin{bmatrix} P_1(s) \\ P_2(s) \end{bmatrix} = \begin{bmatrix} s+\lambda_A & -\mu_A \\ -\lambda_A & s+\mu_A \end{bmatrix}^{-1} \begin{bmatrix} P_1(0) \\ P_2(0) \end{bmatrix}$$

To find the inverse, swap diagonal elements and put negatives at 12, 21 position. Then divide this matrix by the original determinant.

$$\begin{bmatrix} P_1(s) \\ P_2(s) \end{bmatrix} = \frac{\begin{bmatrix} s+\mu_A & \mu_A \\ \lambda_A & s+\lambda_A \end{bmatrix}}{(s+\lambda_A)(s+\mu_A) - \lambda_A \mu_A} \begin{bmatrix} P_1(0) \\ P_2(0) \end{bmatrix}$$

$$\begin{bmatrix} P_1(s) \\ P_2(s) \end{bmatrix} = \frac{\begin{bmatrix} s+\mu_A & \mu_A \\ \lambda_A & s+\lambda_A \end{bmatrix}}{s^2 + (\lambda_A + \mu_A)s} \begin{bmatrix} P_1(0) \\ P_2(0) \end{bmatrix}$$

$$\begin{bmatrix} P_1(t) \\ P_2(t) \end{bmatrix} = \begin{bmatrix} \frac{\mu_A}{\lambda_A+\mu_A} + \frac{\lambda_A}{\lambda_A+\mu_A}\exp(-(\lambda_A+\mu_A)t) & \frac{\mu_A}{\lambda_A+\mu_A} - \frac{\mu_A}{\lambda_A+\mu_A}\exp(-(\lambda_A+\mu_A)t) \\ \frac{\lambda_A}{\lambda_A+\mu_A} - \frac{\lambda_A}{\lambda_A+\mu_A}\exp(-(\lambda_A+\mu_A)t) & \frac{\lambda_A}{\lambda_A+\mu_A} + \frac{\mu_A}{\lambda_A+\mu_A}\exp(-(\lambda_A+\mu_A)t) \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} P_1(t) \\ P_2(t) \end{bmatrix} = \begin{bmatrix} \frac{\mu_A}{\lambda_A+\mu_A} + \frac{\lambda_A}{\lambda_A+\mu_A}\exp(-(\lambda_A+\mu_A)t) & \frac{\mu_A}{\lambda_A+\mu_A} - \frac{\mu_A}{\lambda_A+\mu_A}\exp(-(\lambda_A+\mu_A)t) \\ \frac{\lambda_A}{\lambda_A+\mu_A} - \frac{\lambda_A}{\lambda_A+\mu_A}\exp(-(\lambda_A+\mu_A)t) & \frac{\lambda_A}{\lambda_A+\mu_A} + \frac{\mu_A}{\lambda_A+\mu_A}\exp(-(\lambda_A+\mu_A)t) \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Simplify

$$\begin{bmatrix} P_1(t) \\ P_2(t) \end{bmatrix} = \begin{bmatrix} \frac{\mu_A}{\lambda_A+\mu_A} + \frac{\lambda_A}{\lambda_A+\mu_A}\exp(-(\lambda_A+\mu_A)t) \\ \frac{\lambda_A}{\lambda_A+\mu_A} - \frac{\lambda_A}{\lambda_A+\mu_A}\exp(-(\lambda_A+\mu_A)t) \end{bmatrix}$$

$$\lim_{t\to\infty} \begin{bmatrix} P_1(t) \\ P_2(t) \end{bmatrix} = \lim_{t\to\infty} \begin{bmatrix} \frac{\mu_A}{\lambda_A+\mu_A} + \frac{\lambda_A}{\lambda_A+\mu_A}\exp(-(\lambda_A+\mu_A)t) \\ \frac{\lambda_A}{\lambda_A+\mu_A} - \frac{\lambda_A}{\lambda_A+\mu_A}\exp(-(\lambda_A+\mu_A)t) \end{bmatrix}$$

Final solution

$$\begin{bmatrix} P_1 \\ P_2 \end{bmatrix} = \begin{bmatrix} \frac{\mu_A}{\lambda_A+\mu_A} \\ \frac{\lambda_A}{\lambda_A+\mu_A} \end{bmatrix}$$

State space diagram and transition rates in f/hr of a continuous Markov process is shown in Figure 2. Determine the following

1. Limiting probabilities of each state

Being in each state can be written as

$$\frac{dP_1(t)}{dt} = -0.02P_1(t) + 0.4 * P_2(t) + 0.6 * P_3(t)$$

$$\frac{dP_2(t)}{dt} = 0.01P_1(t) - 0.01 * P_2(t) + 0$$

$$\frac{dP_3(t)}{dt} = 0.01P_1(t) + 0.01 * P_2(t) - 0.6 * P_3(t)$$

Solution to these equations by using Laplace transformation

$$\frac{d}{dt}\begin{bmatrix} P_1(t) \\ P_2(t) \\ P_3(t) \end{bmatrix} = \begin{bmatrix} -0.02 & 0.4 & 0.6 \\ 0.01 & -0.01 & 0 \\ 0.01 & 0.01 & -0.6 \end{bmatrix} \begin{bmatrix} P_1(t) \\ P_2(t) \\ P_3(t) \end{bmatrix}$$

At limiting states $P^T p = \dot{p} = 0$

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -0.02 & 0.4 & 0.6 \\ 0.01 & -0.01 & 0 \\ 0.01 & 0.01 & -0.6 \end{bmatrix} \begin{bmatrix} P_1(t) \\ P_2(t) \\ P_3(t) \end{bmatrix}$$

However, the 3rd row is dependent row. Therefore, we can use following property

$$P_1(t) + P_2(t) + P_3(t) = 1$$

$$\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -0.02 & 0.4 & 0.6 \\ 0.01 & -0.01 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} P_1(t) \\ P_2(t) \\ P_3(t) \end{bmatrix}$$

Solution can be found by simplifying the above equation.

If state 1 is the normally operating state, and states 2 and 3 are failure states, determine:

2. Reliability of the system

$$\frac{d}{dt}\begin{bmatrix} P_1(t) \\ P_2(t) \\ P_3(t) \end{bmatrix} = \begin{bmatrix} -0.02 & 0.4 & 0.6 \\ 0.01 & -0.01 & 0 \\ 0.01 & 0.01 & -0.6 \end{bmatrix} \begin{bmatrix} P_1(t) \\ P_2(t) \\ P_3(t) \end{bmatrix}$$

Using Laplace transformation

$$\begin{bmatrix} sP_1(s) \\ sP_2(s) \\ sP_3(s) \end{bmatrix} - \begin{bmatrix} P_1(0) \\ P_2(0) \\ P_3(0) \end{bmatrix} = \begin{bmatrix} -0.02 & 0.4 & 0.6 \\ 0.01 & -0.01 & 0 \\ 0.01 & 0.01 & -0.6 \end{bmatrix} \begin{bmatrix} P_1(s) \\ P_2(s) \\ P_3(s) \end{bmatrix}$$

$$\begin{bmatrix} P_1(0) \\ P_2(0) \\ P_3(0) \end{bmatrix} = \begin{bmatrix} s-0.02 & 0.4 & 0.6 \\ 0.01 & s-0.01 & 0 \\ 0.01 & 0.01 & s-0.6 \end{bmatrix} \begin{bmatrix} P_1(s) \\ P_2(s) \\ P_3(s) \end{bmatrix}$$

System initially in state 1 therefore $P_1(0) = 1$, other values are 0.

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} s-0.02 & 0.4 & 0.6 \\ 0.01 & s-0.01 & 0 \\ 0.01 & 0.01 & s-0.6 \end{bmatrix} \begin{bmatrix} P_1(s) \\ P_2(s) \\ P_3(s) \end{bmatrix}$$

$$\begin{bmatrix} P_1(s) \\ P_2(s) \\ P_3(s) \end{bmatrix} = \begin{bmatrix} s-0.02 & 0.4 & 0.6 \\ 0.01 & s-0.01 & 0 \\ 0.01 & 0.01 & s-0.6 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Solving this will give $P_1(s), P_2(s)$ and $P_3(s)$

System reliability can be found by following equation

$$P_1(s) + P_2(s)$$

Then we can find being in state 1, state 2 probability with respect to time domain by taking inverse Laplace.

So

$$R(t) = P_1(t) + P_2(t)$$

Mean time to failure

$$R(t) = \int_0^\infty R(t)\mathrm{dt}$$

$$R(t) = \int_0^\infty (P_1(t) + P_2(t))dt$$