WEEKLY JOURNAL-06

SUMMARY

Discussed about conditions of Markov process. The discrete Markov chain was modelled for a simple 4 state system.

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Markov Chain

Week-06

Summary: Discussed about conditions of Markov process. The discrete Markov chain was modelled for a simple 4 state system.

In real world most of the systems are repairable. Methods like fault tree, cut set, conditional probability are not applicable to repairable system. Markov process is very useful in reliability evaluation of repairable systems.

Following conditions should be met by the system to apply basic Markov Process:

- A component should operate at one state. And, the state of the component will change when there is change in system condition. This type of system is called state dependent system.
- But it is a memoryless system. moving from one state to another state does not depend on previous state.
- Behavior of the system should remain same regardless of time difference at a state.
- Probability of state change from one to another should be constant at all times. It is basically a constant hazard rate system.

Analyze the reliability of the system in below figure. Assume the system meets all the necessary conditions for the Markov chain. And the infinite bus is 100% reliable.

Step I

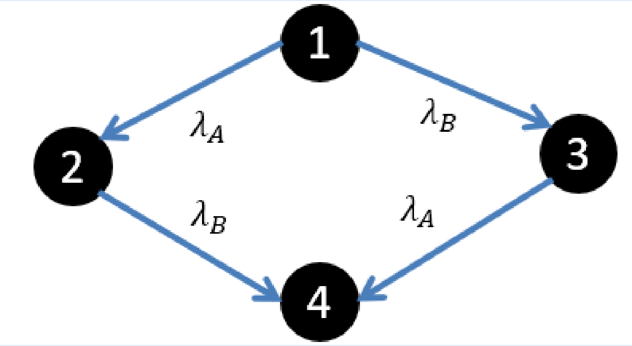
Possible states: Here we have 2 components, each component can take 2 possible states. Therefore, there are 4 possible states for the system.

State	Component A	Component B
1	Operating	Operating
2	Failed	Operating
3	Operating	Failed
4	Failed	Failed

The system will be down only when component A and Component B are in failed state. Hence, this is a parallel system from reliability perspective.

Step II

Draw the state diagram based on hazard rate.



No arrows go out from state four, therefore it is called as absorbing state. Once the system reaches this state, the system has to be restarted. Catastrophic events can be modeled by this state. System designers pay attention to minimize this minimize absorbing states.

Step III

Estimating the reliability: since these is parallel system from reliability perspective. The reliability of the system can be written as:

$$R_{sys}(t) = P_1(t) + P_2(t) + P_3(t)$$

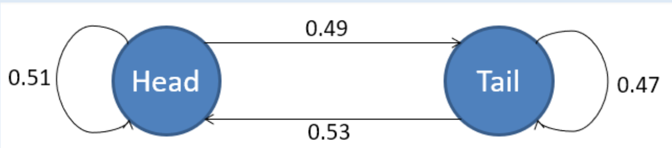
And we know that, system being any of the 4 states is equal to 1. Therefore,

$$P_1(t) + P_2(t) + P_3(t) + P_4(t) = 1$$

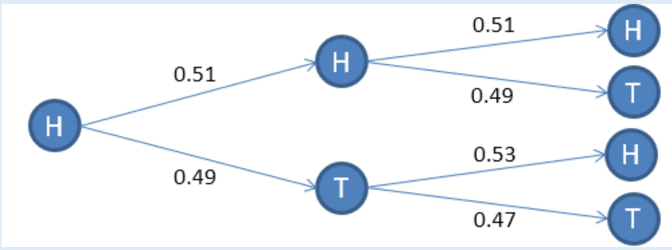
Discrete Markov Chain

Discrete Markov chain can be applied to the system with random behavior that varies with time at discrete steps.

Consider the example of tossing a special coin (not a fair coin). State diagram of the coin can be drawn as in below figure.



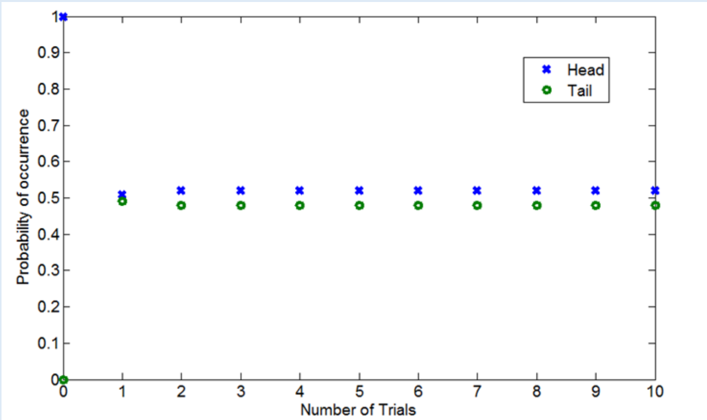
A tree diagram can be drawn for the system as in below figure.



Following results obtain when the initial state was head and 6 number of trials conducted.

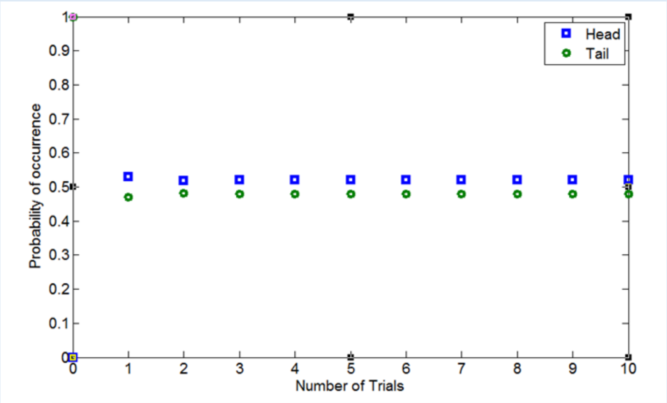
Trial	Head	Tail
0	1	0
1	0.51	0.49
2	0.5198	0.4802
3	0.5196	0.4804
4	0.5196	0.4804
5	0.5196	0.4804
6	0.5196	0.4804

This can be plotted as in below:



Below figures shows the results when the initial state was tail.

Trial	Head	Tail
0	0	1
1	0.53	0.47
2	0.5194	0.4806
3	0.5196	0.4804
4	0.5196	0.4804
5	0.5196	0.4804
6	0.5196	0.4804

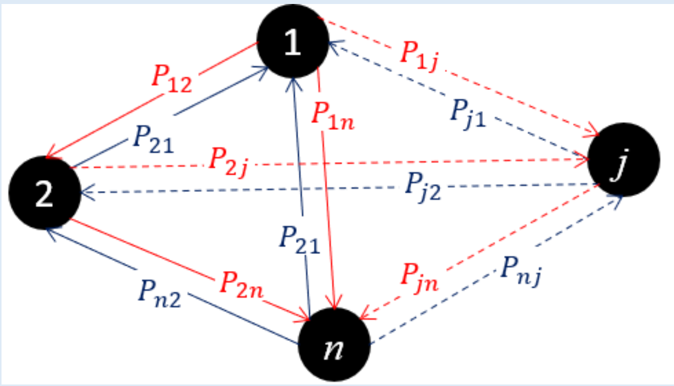


It can be seen that the transient behaviors depend on the initial state but the limiting conditions remains same in both scenarios.

These types of systems are called as ergodic systems. And there should not be any absorbing states.

Stochastic Transitional Probability Matrix

Matrix method is used when there are many states. Matrix method is scalable.



Here Pij = Probability of making a transition from state i from state j after one time period. And the transitional probability matrix is developed as in below matrix.

$$\boldsymbol{P} = \begin{array}{cc} & \textbf{To} \\ & \begin{array}{cccc} \mathbf{1} & \mathbf{2} & \cdots & \boldsymbol{n} \end{array} \\ \textbf{From} \;\; \mathbf{1} & \Bigg[\qquad\qquad\qquad \end{array}$$

After mth trial the transitional probability matrix is

$$\boldsymbol{P}(m) = \boldsymbol{P}^m$$

$$= \left(\begin{bmatrix} P_{\mathbf{11}} & P_{12} & \cdots & P_{1n} \\ P_{21} & P_{22} & \cdots & P_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ P_{n1} & P_{n2} & \cdots & P_{nn} \end{bmatrix} \right)^m$$

If the initial probability is:

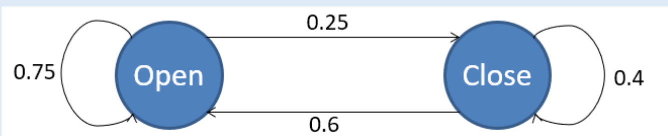
$$P(0) = \begin{array}{cccc} \mathbf{1} & \mathbf{2} & \cdots & \boldsymbol{n} \\ \left[P_1^0\right. & P_2^0 & \cdots & \left.P_n^0\right] \end{array}$$

then the general expression

$$P(n) = \boldsymbol{P}(0)\boldsymbol{P}(m)$$

Example of a two-state system

A tie-line switch can be in either open or closed position. Following state transition diagram shows the probability of transition from any state to the other



1. If the initial state in open determine the probability of being in each state after third interval

Step I transitional matrix

$$\boldsymbol{P} = \begin{bmatrix} P_{OO} & P_{OC} \\ P_{CO} & P_{CC} \end{bmatrix} = \begin{bmatrix} 0.75 & 0.25 \\ 0.60 & 0.40 \end{bmatrix}$$

Step II Initial states

$$P(0) = [P_1 \quad P_2] = [1 \quad 0]$$

Probability of being in each state after 3rd level.

$$P(3) = P(0)\boldsymbol{P}^3$$
$$= [1 \quad 0]\begin{bmatrix} 0.75 & 0.25 \\ 0.60 & 0.40 \end{bmatrix}^3$$
$$= [0.7069 \quad 0.2931]$$

2. Determine the limiting state probabilities

$$P^c(\boldsymbol{P} - \boldsymbol{I}) = [P_1^c \quad P_2^c]\left(\begin{bmatrix} 0.75 & 0.25 \\ 0.60 & 0.40 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = [0 \quad 0]$$

$$[-0.25P_1^c + 0.6P_2^c \quad 0.25P_1^c - 0.6P_2^c] = [0 \quad 0]$$

Second equation is a reduntant equation. But we can use the following fact that

$$P_1^c - P_2^c = 1$$

Therefore

$$\begin{bmatrix} -0.25 & 0.6 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} P_1^c \\ P_2^c \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} P_1^c \\ P_2^c \end{bmatrix} = \begin{bmatrix} 0.7059 \\ 0.2941 \end{bmatrix}$$

There are two types of Markov chains

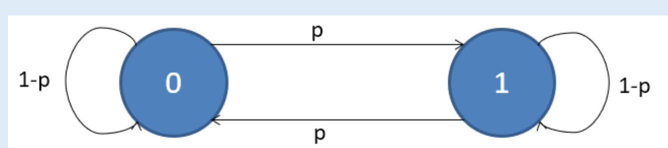
1. Time homogeneous Markov
2. Time inhomogeneous Markov chain

Finding solutions for time inhomogeneous Markov chain is harder. Because the transitional matrix varies at each trial (hazard rate depends on time such that $\boldsymbol{\lambda}(\boldsymbol{t}) = \boldsymbol{\lambda}(\boldsymbol{t})$).

Example[3]

A certain calculating machine uses only the digits 0 and 1. It is supposed to transmit one of these digits through several stages. However, at every stage, there is a probability p that the digit that enters this stage will be changed when it leaves and a probability q = 1 − p that it won't. Form a Markov chain to represent the process of transmission by taking as states the digits 0 and 1. What is the matrix of transition probabilities?

Now draw a tree and assign probabilities assuming that the process begins in state 0 and moves through two stages of transmission. What is the probability that the machine, after n stages, produces the digit 1(making error).



Step 1.

$$\boldsymbol{P} = \begin{bmatrix} P_{00} & P_{01} \\ P_{10} & P_{11} \end{bmatrix}$$
$$= \begin{bmatrix} 1-p & p \\ p & 1-p \end{bmatrix}$$

Step II Initial states

$$P(0) = [P_1 \quad P_2] = [1 \quad 0]$$

Probability that the machine will give 1, when the initial bit is 0:

$$P(2) = P(0)\boldsymbol{P}^3$$
$$= [1 \quad 0] \begin{bmatrix} 1-p & p \\ p & 1-p \end{bmatrix}^n$$

References:

[1] Power system reliability lecture notes.
[2] Problem set given for the class
[3] "stat.berkeley.edu," 2022. [Online]. Available: https://www.stat.berkeley.edu/~aldous/150/takis_exercises.pdf. [Accessed 14 10 2022].