Elements of Machine Learning

Assigment 1 - Problem 4

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Curse of dimensionality

Part 1

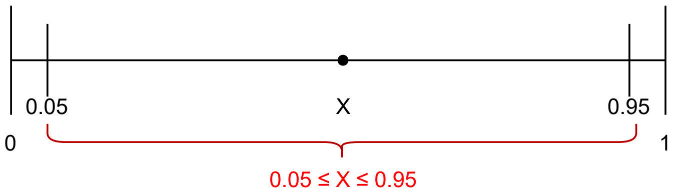
Explain when the so-called *curse of dimensionality* occurs. Describe the phenomenon in your own words. Often with lot of input features, the feature space becomes high-dimensional. That is to say, number of required samples increases exponentially with the number of dimensions. The curse of dimensionality states that if our number of dimension gets bigger, we need more data to generalize.

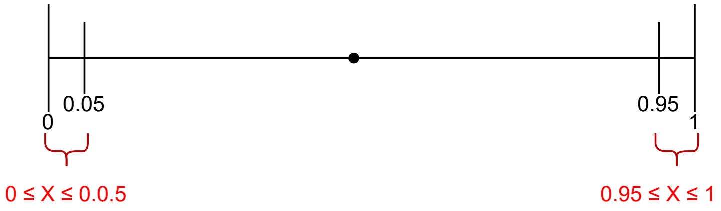
Part 2

Exercise 4.7.4 in ISLR. Please answer the following questions with regard to observations $x \in X$ taking values in the interval [0.05, 0.95].

a

Suppose that we have a set of observations, each with measurements on $p = 1$ feature, X. We assume that X is uniformly distributed in $[0, 1]$. Associated with each observation is a response value. Suppose that we wish to predict a test observation's response using only observations that are within 10% of the range of X closest to that test observation. For instance, in order to predict the response for a test observation with $X = 0.6$, we will use observations in the range $[0.55, 0.65]$. On average, what fraction of the available observations will we use to make the prediction?





1. The first figure illustrates the interval when $X \in [0.05, 0.95]$. Therefore, for each associated response corresponding to X to lie within 10% of the range of X is given by $[X - 0.05, X + 0.05]$ i.e. 0.05 units on the either side of X as to cover a length of 0.1 which is 10% of the range of X

2. The second picture illustrates when $X \in [0, 0.05)$. Therefore, for each associated response corresponding to X, it would lie in the interval $[0, X + 0.05]$. Here we have lower bound as 0, because there is no room to cover the range to the left of X. For e.g. if $X = 0.02$, it can only cover range on its right i.e. $0.02 + 0.05$, while to the left $0.02 - 0.05$ is out of the given interval.

3. Similarly for $X \in (0.95, 1]$, the interval used to predict the responses would be $[X - 0.05, 1]$. Because it can only take the range of 0.05 units towards its left.

Now in order to average fraction of available observation, we use

$$f_{ave} = \frac{1}{b-a}\int_a^b f(x)dx$$

where $f(x)$ in our case would be the length of the interval.

1. For interval: $[0.05 \leq X \leq 0.95]$: Observation interval: $[X - 0.05, X + 0.05]$.
   Length of the interval $= 0.1$

2. For interval: $[0 \leq X < 0.05]$: Observation interval: $[0, X + 0.05]$.
   Length of the interval $= X + 0.05 - 0 = X + 0.05$

3. For interval: $[0.95 < X \leq 1]$: Observation interval: $[X - 0.05, 1]$.
   Length of the interval $= 1 - (X - 0.05) = 1.05 - X$

We have

$$\begin{aligned}
f_{ave} &= \frac{1}{(1-0)}\left[\int_0^{0.05}(x+0.05)dx + \frac{1}{(0.95-0.05)}\int_{0.05}^{0.95}(0.1)dx + \frac{1}{(1-0.95)}\int_{0.95}^{1}(1.05-x)dx\right] \\
&= (0.00125 + 0.0025) + (0.09) + (0.0525 - 0.04875) \\
&= 0.0975
\end{aligned}$$

Therefore, it can be concluded that on an average 9.75% of available observations we will be using to make predictions.

b

Now suppose that we have a set of observations, each with measurements on $p = 2$ features, X_1 and X_2 . We assume that (X_1, X_2) are uniformly distributed on $[0, 1] \times [0, 1]$. We wish to predict a test observation's response using only observations that are within 10% of the range of X_1 and X_2 closest to that test observation. On average, what fraction of the available observations will we use to make the prediction?
On a similar note, for X_1 we know that on an average 9.75% of the available observation will be used. Similarly, for X_2 we will have 9.75%. Hence, for (X_1, X_2) being uniformly distributed, we will have $0.0975 \times 0.0975 = 9.5e - 3$ as the average fraction or 0.95% of available observation used to make prediction

c

Now suppose we have a set of observations on $p = m$ features. In the same scenario as before, we wish to predict a test observation's response using only observations that are within 10% of each feature's range that is closest to that test observation. On average, what fraction of the available observations will we use to make the prediction?
Following from the last part, we will have 0.0975^m as the average fraction of the available observations to make predictions.

d

Now suppose that we wish to make a prediction for a test observation by creating a p-dimensional hypercube centered around the test observation that contains, on average, 10% of the training observations. For p=1,2, and 100, what is the length of each side of the hypercube? Comment on your answer. Note: A hypercube is a generalization of a cube to an arbitrary number of dimensions. When $p = 1$, a hypercube is simply a line segment, when $p = 2$ it is a square, and when $p = 100$ it is a 100-dimensional cube.
Given that each p-dimensional hypercube would contain 10% of the training observations or 0.1 as the fraction of training observations.

1. For $p = 1$: the hypercube is simply a line. Hence length of the line=0.1 units.
2. For $p = 2$: the hyperbole is square for which the area associated is 0.1. Hence length side of each side of square is given as $\sqrt{0.1} = 0.31$ units.
3. For $p = 100$: the hyperbole is a 100-dimension cube. The volume associated is 0.1. Hence length of each side is given as $0.1^(\frac{1}{100}) = 0.977$.