String Rewriting – A Survey for Group Theorists

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Term rewriting can be described as the theory of normal forms in algebraic systems. The related notion of *string rewriting* is the theory of normal forms in presentations of monoids. Both these theories, especially the latter, should therefore be of interest to group theorists. However, they are better known to computer scientists, who use them to discuss such matters as automatic theorem proving.

With the present employment situation, as jobs are easier to find in computer science than in mathematics, young mathematicians may be glad to find branches of computer science with an algebraic flavour, and rewriting systems are of this nature. One of the experts in the field, Ursula Martin, began her mathematical career as a group theorist, and others who started in group theory have worked with her.

Recall that a *monoid* is a set with an associative multiplication and an identity (thus monoids differ from groups because elements need not have inverses). For any set X, the *free monoid* X^* on X is the set of all finite sequences of elements of X (including the empty sequence) with the obvious multiplication. The empty sequence is the identity for this multiplication, so we usually denote it by 1. The elements of X^* are called *strings* or *words*.

A *rewriting system* $\mathcal{R}$ on X is a subset of $X^* \times X^*$. If $(l,r) \in \mathcal{R}$ then, for any strings u and v, we say that the string ulv *rewrites to* the string urv, and write $ulv \to urv$. For any string w, we say that w is *reducible* if there is a string z such that $w \to z$; if there is no such z we call w *irreducible*. We write $\xrightarrow{*}$ for the reflexive transitive closure of $\to$ and $\equiv$ for the equivalence relation generated by $\to$. We say that the strings u and v are *joinable*, written $u \downarrow v$, if there is a string w such that $u \xrightarrow{*} w$ and $v \xrightarrow{*} w$. We add the subscript $\mathcal{R}$ if it is necessary to look at two or more rewriting systems.

The quotient $X^*/\equiv$ is a monoid, which we call the monoid *presented by* $\langle X; \mathcal{R}\rangle$. The distinction between regarding $\mathcal{R}$ as giving a presentation and as being a rewriting system is that in the former case our attention is on $\equiv$ while in the latter it is on $\xrightarrow{*}$.

The example of a rewriting system most familiar in group theory comes in the definition of a free group. The free group on X can be regarded as the monoid $(X \cup X^{-1})^*/\equiv$, where $\equiv$ comes from the rewriting system $\{(xx^{-1}, 1), (x^{-1}x, 1);$ all $x \in X\}$. This example should be borne in mind when considering the results which follow.

If we are looking for normal forms for the elements of the monoid presented by $\langle X; \mathcal{R}\rangle$ the obvious choices are the irreducible elements. For this to be satisfactory, we would require that there is exactly one irreducible element in each equivalence class.

But there is an even more fundamental requirement; namely, that the normal form corresponding to an element can be obtained by repeated rewriting. Thus we call $\mathcal{R}$ *terminating* if there is no infinite sequence $w_1 \to w_2 \to \ldots \to w_n \to \ldots$ ($\mathcal{R}$ is also called *well-founded* or *noetherian*; we are using the notation in [14]). This is most easily achieved by requiring that $|l| > |r|$ for all $(l, r) \in \mathcal{R}$, in which case we call $\mathcal{R}$ *length-reducing*. It can also be achieved if $|l| \geq |r|$ for all $(l, r) \in \mathcal{R}$ and, further, if $|l| = |r|$ then r precedes l in the lexicographic order induced by some well-order on X; when this happens, we shall refer to $\mathcal{R}$ as a *lexicographic* rewriting system. The study of other sufficient conditions for termination in string rewriting and the more general term rewriting is a major research topic in the theory; see [13] for more on this.

A rewriting system is called a *Church-Rosser* system if $u \equiv v$ implies $u \downarrow v$, and is called *complete* if it is both terminating and Church-Rosser (such a system is sometimes called "canonical"; [14] suggests calling such a system "convergent", but I prefer the traditional word "complete"). Plainly an equivalence class in a Church-Rosser system contains at most one irreducible element. Also a terminating system in which each equivalence class contains only one irreducible element is Church-Rosser.

A rewriting system is called *confluent* if $w \xrightarrow{*} u$ and $w \xrightarrow{*} v$ implies $u \downarrow v$. It is easy to see that a system is Church-Rosser iff it is confluent. The neatest way of showing this is to observe that if the system is confluent then $\downarrow$ is transitive.

A rewriting system is called *locally confluent* if $w \to u$ and $w \to v$ implies $u \downarrow v$. A terminating locally confluent system is confluent. This result is sometimes known as the Diamond Lemma; it was first proved in [28] (see also [11] for a version of the proof, and [4] for further results). The proof is fairly easy, using an inductive principle which can be formulated for terminating systems.

Note that the results of the previous paragraphs apply to an arbitrary relation $\to$ on an arbitrary set, and do not require that $\to$ comes from a string rewriting system (or a term rewriting system). It is interesting to observe that this theory was used in [28] to obtain the Normal Form Theorem for free groups and also to obtain normal forms in the λ-calculus; the latter is closely related to the theory of functional programming in computer science.

Jantzen's book [20] contains numerous interesting results on string rewriting, and is a valuable reference. Complete rewriting systems for presenting various interesting groups are given in [25]. Small cancellation theory is also looked at in [25] and [3], where it is shown that the main results of that theory can be obtained in the current context. Jantzen's book contains a long list of references to results in the theory. Conferences on Rewriting Techniques and Applications are held regularly; their Proceedings are published in the Springer Lecture Notes in Computer Science, and usually include surveys as well as more technical material. Interesting papers have appeared in many computer science journals, such as *Journal of Symbolic Computation, Journal of Computer and System Sciences*, and *Information and Computation.*

There are many special results in the theory which are of interest to group theorists, and I mention only two of them. It is easy to see that the free abelian group of rank n cannot be generated as a monoid by n generators but it has a finite presentation on m generators if $m > n$. However [15], this group is presented by a finite complete rewriting system on m generators iff $m \geq 2n$.

An abelian subgroup of a free group is infinite cyclic, and an abelian subgroup of the free product of finite groups is either infinite cyclic or finite. Both these groups may be presented by a finite complete length-reducing rewriting system, namely $\{(xx^{-1}, 1), (x^{-1}x, 1) \mid x \in X\}$ for the free group on X, and $\{(ab, c) \mid a, b, c \in G_i$ for some i and $ab = c$ in $G_i\}$ on the alphabet $\bigcup G_i$ for the free product $* G_i$. Now let G be any group which can be presented by a finite complete length-reducing rewriting system. According to [26], any finitely generated abelian subgroup of G is either infinite cyclic or finite. If G has non-trivial centre then G itself is either infinite cyclic or finite. Also, if G has a non-trivial finite normal subgroup then G is finite.

It is not possible to decide whether or not a finite rewriting system is terminating, confluent, locally confluent, or complete (see [20] for details). However, if a finite system is known to be terminating we can decide whether or not it is complete, using a criterion due to Knuth and Bendix [23] given below.

Let $\mathcal{R}$ be a terminating system. For each string w choose an irreducible string $S(w)$ such that $w \xrightarrow{*} S(w)$. This can be done arbitrarily, but if we wish to perform an algorithmic process we could require $S(w)$ to be obtained by always rewriting the leftmost substring possible.

We call the triple of non-empty strings u, v, w an *overlap ambiguity* if there are r_1 and r_2 such that (uv, r_1) and (vw, r_2) are in $\mathcal{R}$; we then say that $r_1 w$ and ur_2 are the corresponding *critical pair.* The triple u, v, w of possibly empty strings is called an *inclusion ambiguity* if there are r_1 and r_2 (which must be distinct if both u and w are empty, but otherwise may be equal) such that (v, r_1) and (uvw, r_2) are in $\mathcal{R}$; we then say that ur_1w and r_2 are the corresponding critical pair. If p and q are the critical pair corresponding to an (overlap or inclusion) ambiguity u, v, w then $uvw \to p$ and $uvw \to q$.

Let $\mathcal{R}$ be a terminating system. For any critical pair p, q we have $S(p) \equiv S(q)$, and so, if $\mathcal{R}$ is complete then $S(p) = S(q)$. Conversely, if $S(p) = S(q)$ for all critical pairs p, q it is easy to see that $\mathcal{R}$ is locally confluent, and hence complete. Evidently, when $\mathcal{R}$ is finite we can decide whether or not this condition holds. Note that it would be enough to know that for any critical pair p, q we have $p \downarrow q$, which is sometimes easier to check.

The rewriting systems $\mathcal{R}$ and $\mathcal{S}$ are called *equivalent* if $\equiv_{\mathcal{R}}$ is the same as $\equiv_{\mathcal{S}}$; this condition is stronger than saying that the monoids presented by $\langle X; \mathcal{R}\rangle$ and $\langle X; \mathcal{S}\rangle$ are isomorphic. Any rewriting system is equivalent to a complete one, as is shown by the Knuth-Bendix procedure [23] given below.

Let $\mathcal{R}$ be a lexicographic system (and hence a terminating system). Let $\mathcal{R}'$ be obtained from $\mathcal{R}$ by considering all critical pairs p, q such that $S(p) \neq S(q)$ and adding to $\mathcal{R}$ for such a critical pair either $(S(p), S(q))$ or $(S(q), S(p))$; the choice of pairs to add is to be made so that $\mathcal{R}'$ remains lexicographic. Evidently $\mathcal{R}$ is complete iff $\mathcal{R}' = \mathcal{R}$. Because $S(p) \equiv_{\mathcal{R}} S(q)$, $\mathcal{R}'$ is equivalent to $\mathcal{R}$.

Now let $\mathcal{R}$ be an arbitrary system. Let $\mathcal{R}_0$ be obtained from $\mathcal{R}$ by replacing some of the pairs (l, r) by (r, l) in such a way that $\mathcal{R}_0$ is lexicographic. Inductively, define $\mathcal{R}_n$ for all n by $\mathcal{R}_{n+1} = (\mathcal{R}_n)'$, and let $\mathcal{R}_\infty = \bigcup_n \mathcal{R}_n$. It is easy to check that $\mathcal{R}_\infty$ is lexicographic and equivalent to $\mathcal{R}$. Consider a critical pair p, q for $\mathcal{R}_\infty$. Then p, q is a critical pair for $\mathcal{R}_n$ for some n. Letting $S_n(p)$ and $S_n(q)$ be the chosen irreducibles for $\mathcal{R}_n$ corresponding to p and q, we know that either $S_n(p) = S_n(q)$ or one of $(S_n(p), S_n(q))$ and $(S_n(q), S_n(p))$ is in $\mathcal{R}_{n+1}$. Hence $p \downarrow q$ for $\mathcal{R}_{n+1}$ and so also for $\mathcal{R}_\infty$. Thus, by the Knuth-Bendix criterion, $\mathcal{R}_\infty$ is complete.

The restriction to lexicographic systems is made to avoid problems with termination. If we simply know that $\mathcal{R}_n$ is terminating, we have to decide which of the two pairs to add in each case, and then try to show that $\mathcal{R}_{n+1}$ is also terminating. This process (in the more general case of term rewriting) has been much investigated, and there are computer programs aimed at performing this process either automatically or interactively, with backtracking as necessary (that is, if $\mathcal{R}_{n+1}$ is not terminating — or one cannot easily see that it is terminating — then we may reconsider the choices made for $\mathcal{R}_n$ or at earlier stages).

When $\mathcal{R}$ is finite then each $\mathcal{R}_n$ is finite, but $\mathcal{R}_\infty$ may be infinite. If, for some n, $\mathcal{R}_{n+1} = \mathcal{R}_n$ then $\mathcal{R}_n$ is complete and $\mathcal{R}_\infty = \mathcal{R}_n$. It follows that $\mathcal{R}$ is equivalent to the finite complete system $\mathcal{R}_n$. Conversely, if $\mathcal{R}$ is equivalent to some finite complete system then [25] there is some n such that $\mathcal{R}$ is equivalent to $\mathcal{R}_n$.

The rewriting system $\{(aba, bab)\}$ on the alphabet $\{a, b\}$ has no equivalent finite complete rewriting system [21]. However, an isomorphic monoid may be presented on the alphabet $\{a, b, c\}$ by the rewriting system $\{(ab, c), (ca, bc)\}$. This system is not complete, but the completion process gives the equivalent finite complete system $\{(ab, c), (ca, bc), (bcb, cc), (ccb, acc)\}$.

This example leads us to ask what properties are satisfied by a monoid which has (with respect to some set of generators) a presentation by a finite complete rewriting system. One necessary condition is that the monoid has a sovable word problem (recall that if the word problem is solvable in one finite presentation then it is solvable in every finite presentation). For let the monoid be presented by $\langle X; \mathcal{R} \rangle$, where $\mathcal{R}$ is a finite complete system. Since $\mathcal{R}$ is terminating and finite, we can calculate for each string u an irreducible string $S(u)$ such that $u \xrightarrow{*} S(u)$. Then the word problem for this presentation is solvable, since $u \equiv v$ iff $S(u) = S(v)$.

Groves and Smith [17] investigate how the property of being presented by a finite complete rewriting system behaves under various group-theoretic constructions (subgroups, quotient groups, wreath products, HNN extensions, etc.). In particular, they show that constructable soluble groups (see [2]) are presented by finite complete rewriting sytems; conversely, metabelian groups presented by finite complete rewriting systems are constructable, and they ask whether any soluble group presented by a finite complete rewriting system is constructable. They show that if A is a subgroup of the group G and A can be presented by a finite complete rewriting system then so can G if A has finite index in G and also if A is normal in G and such that G/A can be presented by a finite complete rewriting system. When A has finite index in G and G can be presented by a finite complete rewriting system it is not known whether A must be presentable by a finite complete rewriting system.

A break-through occurred in 1987 when Squier [29] showed that a monoid with a presentation by a finite complete rewriting system satisfies a homological condition, from which he was able to give examples of monoids with no such presentation. It was subsequently noticed that an earlier paper by Anick [1] had, using very different terminology, obtained a stronger homological condition.

A *free resolution* of a monoid M is an exact sequence

$$\ldots P_n \to P_{n-1} \to \ldots \to P_1 \to \mathbb{Z}M \to \mathbb{Z} \to 0$$

of free $\mathbb{Z}M$-modules. M is called FP_∞ if there is a free resolution such that P_i is finitely generated for all i, and it is called FP_n if there is a free resolution with P_i finitely generated for all $i \leq n$. Squier proved that a monoid which can be presented by a finite complete rewriting system is FP_3, while Anick's result shows that such a monoid is FP_∞. For a monoid, as distinct from a group, we have to specify whether we are using left modules or right modules (an example of a monoid which is right FP_∞ but not left FP_1 is given in [10]), but since there is left-right symmetry in the definition of a finite complete rewriting system, these monoids are both left FP_∞ and right FP_∞.

Anick's result is much stronger, and obtains a free resolution from a (not necessarily finite) complete rewriting system. We say that a rewriting system is *reduced* if it has no inclusion ambiguities and for each $(l, r) \in \mathcal{R}$ the string r is irreducible. It is *strongly reduced* if it is reduced and each element of X is irreducible. It is easy [22] (se also [29]) to obtain a reduced complete rewriting system equivalent to a given complete rewriting system (if the original system is finite then the reduced system will also be finite), and we can immediately obtain a strongly reduced complete system which presents an isomorphic monoid (this system will not be equivalent to the previous one, since the set of generators is different).

Anick constructs a free resolution corresponding to a strongly reduced complete rewriting system. The free generators of each P_n are certain repeated overlaps, from which it is easy to see that the monoid is FP_∞ if the system is finite. Anick constructs the boundary maps and contracting homotopies in the resolution simultaneously by a complicated inductive process, involving not only boundary maps and contracting homotopies on elements of smaller degree but also their values on earlier elements of the same degree. The effect is that it is not possible to get a clear understanding of what is happening, and, although the construction can in principle be used for computation of homology, the definition of the boundary is too complicated to make his method practical. A subsequent generalisation and simplification by Koboyashi [24] still has the same problems.

The situation was elucidated by Brown [5] using a topological approach (the result was also proved by Groves [16], whose technique is intermediate between those of Anick and Brown). He showed that a strongly reduced complete rewriting system (in fact, he looked at the irreducibles rather than at the system itself) gives rise to a structure which he called a *collapsing scheme* on the bar resolution (which is a large resolution which can be obtained for any monoid), and that this collapsing scheme enables one to replace the bar resolution by a smaller resolution. The whole situation now becomes clear, and the boundary operators can be easily calculated. Thus the theorem not only enables us to use homological methods to obtain results about rewriting systems, but also enable us to use rewriting systems to prove results in the homology theory of groups. Brown's paper is very beautiful and contains some elegant calculations. In particular, the "big resolution" for the monoid $\langle x_i\ (i \in \mathbb{N});\ (x_j x_i, x_i x_{j+1})$ for $i < j\rangle$ constructed in [7] comes from the collapsing scheme corresponding to this complete rewriting system, and the proof that this monoid (and the group with the same presentation) is FP_∞ uses a new collapsing scheme on this resolution.

Brown's proof in [5] is topological, and he leaves it to the reader to translate the proof into an algebraic form. I think that there are likely to be people interested in Brown's work but unfamiliar with the topology. Such people should nonetheless have no difficulty in obtaining a general overview of the results from his account, and they should find the specific calculations straightforward and interesting. But I feel it is worthwhile to give an explicit algebraic translation of his proof, so that such readers do not have to take his main theorem on trust.

We begin with a chain complex

$$\mathbf{P}: \ldots \to P_{n+1} \overset{\partial_{n+1}}{\to} P_n \overset{\partial_n}{\to} P_{n-1} \to \ldots \to P_1 \to P_0$$

of free modules over a ring A. We require not only that each P_n is a free module but also that a specific basis is chosen for each P_n; the elements of this basis are called *n-cells*. A *collapsing scheme* on $\mathbf{P}$ is defined to consist of the following:

(1) a division of the cells into three pairwise disjoint classes, which we refer to as the *essential, redundant*, and *collapsible* cells, with all 0-cells being essential and all 1-cells being either essential or redundant;
(2) a function, called *weight*, from the set of all redundant cells into $\mathbb{N}$;
(3) a bijection, for each n, between the set of redundant n-cells and the set of collapsible $(n+1)$-cells, such that, if the collapsible cell c corresponds to the redundant cell r, then there is a unit u of A for which all redundant cells in the chain $r - u\partial c$ have weight less than the weight of r (in particular, if r has weight 0 then $r - u\partial c$ contains no redundant cells).

The motivation for this definition, and various examples, can be found in [5]. The weight is often given implictly, rather than explicitly, by the following procedure. Let S be an arbitrary set, and let $>$ be a binary relation on S such that, for all s, $\{t;\ s > t\}$ is finite (thus, we should think of $s > t$ as meaning, not "s is greater than t", but as "t is a child of s" or "t is used in s"). It is then a well-known (and easy to prove) result, sometimes called König's Lemma, that if, for some s, there are sequences $s > s_1 > \ldots > s_n$ for arbitrarily large n then there is an infinite sequence $s > s_1 > \ldots > s_n > s_{n+1} > \ldots$ It follows that if $>$ is terminating then, for each s there are only finitely many n for which there is a sequence $s > \ldots > s_n$, and the maximum such n may be called the weight of s. In particular, if $s > t$, then s has greater weight than t.

We can now state and prove the algebraic form of Brown's theorem. *If the free chain complex* $\mathbf{P}$ *has a collapsing scheme then it is chain-equivalent to a free chain complex* $\mathbf{Q}$ *for which the essential n-cells are a basis for* Q_n *for all* n. This holds for augmented complexes as well as for non-augmented ones.

We begin by defining homomorphisms $\theta_n : P_n \to P_n$ for all n as follows. For an essential cell e let $\theta e = e$ (subscripts will usually be omitted) and for a collapsible cell c let $\theta c = 0$. For a redundant cell r we define θr to be $r - u\partial c$, where c and u are as in (3) of the definition of a collapsing scheme (the definition would permit more than one suitable u; if this happens we just choose one). Let p be a chain in which all the redundant cells have weight at most k. Then θp is a chain in which all the redundant cells have weight less than k. We then see that all the redundant cells in $\theta^k p$ have weight 0, that $\theta^{k+1} p$ contains no redundant cells, and so $\theta^{k+2} p$ consists only of essential cells, and $\theta^m p = \theta^{k+2} p$ for $m > k+2$. We can therefore define homomorphisms $\phi_n : P_n \to P_n$ by $\phi p = \theta^{k+2} p$ when all the redundant cells in p have weight at most k. Plainly $\phi = \phi\theta$.

We now show that $\phi\partial = \phi\partial\theta$. Since $\theta e = e$ we have $\phi\partial e = \phi\partial\theta e$. Since $\theta c = 0$, we need to show that $\phi\partial c = 0$; this holds because $\partial c = u^{-1}(r - \theta r)$ and $\phi = \phi\theta$. Finally, $r - \theta r = u\partial c$, so $\partial(r - \theta r) = 0$, and $\phi\partial r = \phi\partial\theta r$. It follows that $\phi\partial = \phi\partial\theta^m$ for all m, and hence $\phi\partial = \phi\partial\phi$.

Let Q_n be the free module with basis the essential n-cells. We can regard ϕ_n as a homomorphism from P_n to Q_n whenever convenient. We define $\delta_n : Q_n \to Q_{n-1}$ by $\delta = \phi\partial$. Then $\delta\delta = \phi\partial\phi\partial = \phi\partial\partial$, by the previous paragraph, so $\delta\delta = 0$, and we have a chain complex $\mathbf{Q}$. Also $\delta\phi = \phi\partial\phi = \phi\partial$, so ϕ is a chain-map from $\mathbf{P}$ to $\mathbf{Q}$. Notice that δ is easy to compute from ∂ and the collapsing scheme.

We next define homomorphisms $\alpha_n : P_n \to P_{n+1}$. We let $\alpha e = 0$ and $\alpha c = 0$ for any essential cell e or collapsible cell c. For a redundant cell r, we define αr using induction on the weight. Precisely, we have $r = u\partial c + \theta r$, where the redundant cells in θr have weight less than the weight of r, so we can define αr by $\alpha r = uc + \alpha\theta r$. Since αp involves only collapsible cells for any chain p, we have $\phi\alpha p = 0$ for all p.

Define $\psi_n : P_n \to P_n$ by $\psi = \iota - \alpha\partial - \partial\alpha$, where ι is the identity. We show that $\psi\theta = \psi$, from which it will follow, as before, that $\psi\phi = \psi$. First, $\psi\theta e = \psi e$, since $\theta e = e$. Next, $\alpha c = 0$, while $\partial c = u^{-1}(r - \theta r)$, so, by the definition of α, $\alpha\partial c = c$. Hence $\psi c = 0 = \psi\theta c$, since $\theta c = 0$. Finally, $\partial\alpha(r - \theta r) = r - \theta r$, by definition, and $\partial(r - \theta r) = \partial(u\partial c) = 0$, so $\psi(r - \theta r) = 0$. Also, plainly, $\psi\partial = \partial\psi$.

Since we may regard Q_n as a submodule of P_n, we may regard ψ_n as a homomorphism from Q_n to P_n. We then have $\partial\psi e = \psi\partial e = \psi\phi\partial e = \psi\delta e$, so that ψ is a chain-map from $\mathbf{Q}$ to $\mathbf{P}$.

Since $\alpha e = 0$, and $\phi\alpha p = 0$ for all p, we see that $\phi\psi e = e$. Also $\psi\phi = \psi = \iota - \alpha\partial - \partial\alpha$. Hence ϕ is a chain equivalence, as required.

Suppose that we have a strongly reduced complete rewriting system. We should compare the resolutions obtained by the various methods. It is easy to show that there is a natural bijection between the bases in Anick's resolution and Brown's. With more effort, using an inductive argument, we can show that the boundaries and contracting homotopies in Anick's resolution are the same as in Brown's (note that Brown's resolution has a contracting homotopy obtained from its chain-equivalence with the bar resolution and the natural contracting homotopy in the bar resolution). Because of the difficulty of explicitly constructing the boundaries in Anick's resolution, it is usually better to look at Brown's resolution when considering detailed questions of homology. However, I find Anick's description of the basis elements to be easier to follow than Brown's, so it may be preferable to use Anick's approach if the boundaries are not needed explicitly (for instance, in discussions of Euler characteristic or cohomological dimension).

Squier's construction is slightly different (and it is also necessary to interchange left and right in order to make comparisons). Squier's P_3 has as basis all overlap ambiguities u, v, w. Anick's P_3 is smaller. Its basis consists of those overlap ambiguities u, v, w for which there is no overlap ambiguity u', v', w' with $u'v'w'$ an initial segment of uvw. Also, Squier uses a more general expression for ∂_3. Squier's construction of ∂_3 depends on a choice, for each string w, of a sequence $w = w_0, w_1, \ldots, w_n$ such that w_n is irreducible and, for all $i < n$, either $w_i \to w_{i+1}$ or $w_{i+1} \to w_i$. If this choice is made so that w_{i+1} always comes from w_i by leftmost rewriting then this ∂_3 is the same as Anick's. It is interesting to note that the first hypothesis of Squier's Theorem 3.2 amounts to saying that his P_3 is the same as Anick's, while his second hypothesis ensures that Anick's P_4 is zero; thus his conclusion is immediate.

In an unpublished paper, Squier has shown that a monoid which can be presented by a finite complete rewriting system satisfies an additional condition, which he refers to as *having finite derivation type*, and he exhibits a finitely presented FP_∞ monoid with solvable word problem which is not of finite derivation type. I do not properly understand this criterion, but it appears to be of a homotopical nature, instead of being homological.

Squier's unpublished results, and Brown's work, give rise to a number of interesting problems.

Are monoids with solvable word problem and of finite derivation type necessarily presented by a finite complete rewriting system? (My guess is "No".)

Is a monoid of finite derivation type necessarily FP_∞?

Do automatic groups [9] (which are known to be FP_∞ with solvable word problem) have presentations by finite complete rewriting systems? Are they of finite derivation type?

There are finitely presented infinite simple groups (which of necessity have solvable word problem) which are FP_∞ [18, 27]. One of these is given explicitly in [6], in a form from which a presentation could easily be written down. Does this group (more generally, this family of groups) have a presentation by a finite complete rewriting sytem? Is it (are they) of finite derivation type?

The group $\langle x_i\ (i \in \mathbb{N});\ x_j x_i = x_i x_{j+1}\ (\text{all } i, j \text{ with } i < j)\rangle$ is discussed in [7], where a finite presentation is given, and the group is shown to be FP_∞. It is easy to see that this group has solvable word problem. Can it be presented by a finite complete rewriting system? Does it have finite derivation type?

Let F be a free group of rank n. According to Culler and Vogtmann [12], the group $\mathrm{Out}(F)$ of outer automorphisms of F has a subgroup of finite index whose cohomological dimension is $2n-3$. They prove the result by exhibiting a complex on which the subgroup acts. Can it be proved by applying Brown's result (or Anick's, equivalently) to some presentation?

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