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# A Catalogue of Complete Group Presentations

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A complete group presentation consists of a set of generators and a set of replacement rules generating a well-founded and confluent relation on words, thereby solving the word problem for this presentation. Complete presentations for surface, Coxeter, Dyck and symmetric groups are discussed. These complete presentations possess interesting combinatorial properties and provide uniform algorithms for the word problem.

## 1. Introduction

The notion of rewriting system has been of interest to computer scientists for a considerable time (Knuth & Bendix, 1970; Huet, 1980; Le Chenadec, 1986). More recently such systems have been studied in group theory by Gilman (1979), Bücken (1979), Bauer (1981), and Le Chenadec (1983). It is of interest to determine whether or not these techniques can be successfully applied to some common groups. The present catalogue provides evidence for an affirmative answer. Concerning efficiency, Bauer (1981) has established the existence of word rewriting systems of arbitrary complexity. Further, the computation of some complete systems analysed in this paper required several hours of CPU time on a Honeywell DPS68 computer. However, the completion procedure must be thought of as a compilation process, in contrast to the coset enumeration method of Todd & Coxeter (1936). Moreover complete and possibly infinite presentations may encode efficient solutions to the word problem. These presentations frequently have a simple interpretation in Cayley graphs. From a constructive viewpoint, the good presentations are the complete ones.

The presentations are taken from Coxeter (1980). Their completions were obtained with a Lisp implementation of the Knuth & Bendix procedure (Le Chenadec, 1983). In this paper, the reader will find complete presentations of:

- The fundamental groups of the orientable and non-orientable surfaces. These groups led to Dehn's study of groups with small cancellation.
- The Coxeter groups, which are discrete transformation groups generated by reflections. Owing to the partial commutativity of some generators, we were not successful in all cases. This class possesses infinite rewriting systems where a single parametrised rule describes the complete presentation.
- The Dyck groups, which are generated by rotations in Euclidean, spherical or hyperbolic geometry. They are the rotation subgroups of Coxeter groups. As length-increasing rules appear in some complete presentations, termination remains open in such cases.

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- The symmetric groups $S_n$. Depending upon the choice of the generating set, several complete systems may exist. While these systems possess many rules, we found some of size $n!$, the order of $S_n$, the number of rules is still well below the theoretical upper bound of $2|G| \times |\mathbf{G}|$ for a group $\mathbf{G}$ with $|G|$ generators (Le Chenadec, 1986). This bound should be compared with $|\mathbf{G}|^2$, which is the number of entries in the multiplication table. The complete presentations for these groups are closely related to some sorting algorithms.

Let $G$ be a set of generators and define $\mathscr{G} = G \cup G^{-1}$. A *rule* is a pair of words $(u, v)$ in the free monoid $\mathscr{G}^*$, denoted $u \to v$. A set $R$ of rules defines on $\mathscr{G}^*$ a binary relation of *reduction*, denoted $\to_R^*$, which is the reflexive-transitive closure of the relation $\to_R$ defined by $w \to_R w'$ if and only if $w = aub$ and $w' = avb$, $a, b \in \mathscr{G}^*$, $u \to v \in R$. The word $u$ is called a *R-redex* of the word $w$. A word is *R-irreducible* or in *R-normal form* if it does not possess any $R$-redex, the prefix $R$ will be omitted when clear from the context.

- The set of rules $R$ is *well-founded*, or possesses the *termination* property, if there is no infinite sequence of words $(w_i)_{i \geq 0}$ such that $w_0 \to_R w_1 \to_R \cdots \to_R w_n \to_R \cdots$.
- The set of rules $R$ is *confluent* if

$$\forall\, u, v, v' \in \mathscr{G}^*,\ u \to_R^* v,\ u \to_R^* v' \Rightarrow \exists\, w \in \mathscr{G}^*,\ v \to_R^* w,\ v' \to_R^* w. \tag{1}$$

Given two rules $ua \to v$ and $au' \to v'$, $a \in \mathscr{G}^*$, $a \neq 1$, we say that they *superpose* on $a$, that is, the word $uau'$ reduces to $vu'$ and $uv'$. This pair of words is called a *critical pair*. A critical pair is *resolved* when there exists a word $w$ satisfying the condition (1). A well-founded and confluent set of rules is called a *complete presentation*. Given a complete presentation $R$, all words possess a *unique* $R$-normal form. The leftmost (resp. rightmost) reduction is the relation defined by $u \to_R v$ and the redex contracted in $u$ is the leftmost (resp. rightmost) one. When the number of rules is finite, the set of irreducible words defines a regular language. Termination of $R$ is usually proved by means of *reduction orders*, which are well-founded partial orders $>$ such that:

- $\forall\, u \to v \in R, \quad u > v.$
- $\forall\, a, b, u, v \in \mathscr{G}^*, \quad a \neq 1 \Rightarrow a > 1 \quad \text{and} \quad u > v \Rightarrow aub > avb.$

We shall use two classical orders. The lexicographic order is defined as follows: words are ordered first by length, then words of equal length are ordered lexicographically using a linear order on $\mathscr{G}$. This order can be refined by a weight function $\pi$ on $\mathscr{G}$ so that the weight of a word is the sum of the weights of its generators. Words of equal weights are ordered lexicographically. Both orders are well-founded, linear, and satisfy the last condition above. Confluence is checked by the following theorem (Knuth & Bendix, 1970; Huet, 1980):

THEOREM 1. *A well-founded set of rules $R$ is confluent if and only if its critical pairs are resolved.*

Given a finite group presentation $\langle G, E\rangle$ and a linear reduction order $>$, the defining relations are transformed into rules $r \to 1$, $r \in E$, $r > 1$. If the resulting well-founded system $R$ is not confluent, add to $R$ a new rule $u \to v$, $u > v$, where $(u, v)$ is some unresolved critical pair. The *completion procedure* iterates this basic step. The set $R$ of rules is *interreduced* if for every rule $u \to v$ in $R$, $v$ is $R$-irreducible and the only redex of $u$

is $u$ itself. We restrict ourselves to such interreduced systems. The free complete set over $G$,

$$F_G = \{aa^{-1} \to 1,\ a^{-1}a \to 1 | a \in G\},$$

defines the free group on $G$ as a quotient of the free monoid $\mathscr{G}^*$. The induced relation will be called *F-reduction*. $F$-irreducible words are the usual freely reduced words. Implicit in all presentations is the set $F_G$. A cyclic permutation of the word $w$ is either $w$ itself or a word $vu$ such that $w = uv$, $u$ (resp. $v$) is a *prefix* (resp. *suffix*) of $w$. The *reverse* of $w = a_1 \cdots a_n$, $a_i \in \mathscr{G}$, $1 \leqslant i \leqslant n$ is the word $a_n \cdots a_1$. A word is *cyclically reduced* if its cyclic permutations are $F$-reduced. A *normal pair* is either a critical pair between a rule in $F_G$ and a rule $u \to v$, or the pair $(u^{-1}, v^{-1})$. This latter pair is created by the completion procedure in order to take care of the inverse operation. Given a complete group presentation, the normal forms provide unique representatives for the abstract elements of the group.

Let the two rules $ua \to v$ and $au' \to v'$ superpose on $a \neq 1$. If the two words $av^{-1}u$ and $au'v'^{-1}$ or $u^{-1}va^{-1}$ and $v'u'^{-1}a^{-1}$ are distinct cyclic permutations of two (possibly equal) cyclically reduced defining relations, the word $a$ is called a *piece* of the relations. For example, with the relation $abab^{-1}$ and the two rules $ab \to ba^{-1}$, $ba \to a^{-1}b$, $a$ is a piece while $b$ is not. If we resolve only the normal pairs and if rules do not increase length, we obtain an algorithm similar to Dehn's algorithm in small cancellation theory (Dehn, 1911; Greendlinger, 1960; Lyndon, 1966; Schupp, 1973). This restricted completion algorithm will be called the *symmetrisation* of a presentation. While a completion procedure may continue indefinitely as a consequence of the undecidability of the word problem for groups, this latter procedure always halts. Using non-length-increasing rules, superpositions on words that are not pieces result in resolved critical pairs. A more complete account of the above facts may be found in Le Chenadec (1986).

## 2. Surface Groups

### 2.1. Orientable Surfaces

A defining presentation of a torus with $p$ holes is:

$$T_p = \langle A_1, \ldots, A_{2p};\ A_1 \cdots A_{2p} A_1^{-1} \cdots A_{2p}^{-1} \rangle.$$

The pieces of the presentation are the generators and their inverses, so that when $p \geqslant 2$, the word problem relative to this presentation can be solved by Dehn's algorithm (Dehn, 1911). The case $p = 1$ defines the group $\mathbb{Z} \times \mathbb{Z}$. The complete presentation in this case expresses the commutativity between elements of $\mathscr{G}$. The case $p = 2$ gives the following complete system:

$$T_2 \left\{ \begin{array}{cc} DCBA \to ABCD & D^{-1}C^{-1}B^{-1}A^{-1} \to A^{-1}B^{-1}C^{-1}D^{-1} \\ BCDA^{-1} \to A^{-1}DCB & B^{-1}C^{-1}D^{-1}A \to AD^{-1}C^{-1}B^{-1} \\ B^{-1}A^{-1}DC \to CDA^{-1}B^{-1} & BAD^{-1}C^{-1} \to C^{-1}D^{-1}AB \\ DA^{-1}B^{-1}C^{-1} \to C^{-1}B^{-1}A^{-1}D & D^{-1}ABC \to CBAD^{-1} \end{array} \right.$$

In the general case, the completion leads to a system $T_p$ of $4p$ rules composed of words of length $2p$:

$$T_p \begin{cases} A_{2k} \cdots A_{2p} A_1^{-1} \cdots A_{2k-1}^{-1} \to A_{2k-1}^{-1} \cdots A_1^{-1} A_{2p} \cdots A_{2k} \\ A_{2k} \cdots A_1 A_{2p}^{-1} \cdots A_{2k+1}^{-1} \to A_{2k+1}^{-1} \cdots A_{2p}^{-1} A_1 \cdots A_{2k} \\ A_{2k}^{-1} \cdots A_1^{-1} A_{2p} \cdots A_{2k+1} \to A_{2k+1} \cdots A_{2p} A_1^{-1} \cdots A_{2k}^{-1} \\ A_{2k}^{-1} \cdots A_{2p}^{-1} A_1 \cdots A_{2k-1} \to A_{2k-1} \cdots A_1 A_{2p}^{-1} \cdots A_{2k}^{-1} \end{cases} \quad k = 1, \ldots, p.$$

All rules have the form $\lambda \to \bar{\lambda}$, where $\bar{\lambda}$ is the reverse of $\lambda$. This fact is of practical importance for the speed-up of reductions. The proof that a system of rules defines a complete presentation involves three steps, i.e. (i) termination, (ii) all rules are consequences of the definition, and (iii) critical pairs are resolved. Termination is proved with the lexicographic order such that:

$$A_{2p} > A_{2p}^{-1} > \cdots > A_2 > A_2^{-1} > A_1 > A_1^{-1} > \cdots > A_{2p-1} > A_{2p-1}^{-1}.$$

For each rule $\lambda \to \rho$, the word $\lambda\rho^{-1}$ or its inverse is a cyclic permutation of the defining relation, i.e. the complete system corresponds to the symmetrised system. Consequently, the induced relation defines an algorithm which refines Dehn's reduction algorithm in two ways. Firstly, we have general confluence instead of confluence over the unit element and secondly, redexes are sought among $4p$ words of length $2p$ instead of $8p$ words of length $2p+1$. In the case of (iii), superpositions cannot occur with common subwords of length one. As pieces have length one, by the last result quoted in §1, all critical pairs are resolved.

Rewriting sets provide three algorithms, one for reducing a word to its normal form, and two others performing the two group operations on normal forms: multiplication and inversion. Let us mention an initial estimate of the complexity of the computation of normal forms by giving an upper bound to the number of $T_p$-reductions. Book (1982) has shown that a linear-time reduction algorithm exists for word rewriting systems which possess length- reducing rules only. His analysis has been applied to Dehn's algorithm by Domanski & Anskel (1985). Bauer and Otto (1984) exhibit a finite, length-preserving, complete system with a PSPACE-complete word problem. For torus groups, we have a more precise result:

PROPOSITION 2. *Let $T_p$ be a complete torus presentation as defined above. Then there exists a linear-time algorithm which does not involve backtracking and which computes the normal form of a word $M$ with no more than $|M|/2p$ reductions.*

PROOF. We sketch the proof. Let $a_k$ denote either the generator $A_k$ or its inverse, with the obvious meaning for $a_k^{-1}$. The first letter of any rule left-hand side is an even-numbered generator $a_{2k}$. Let $M = Wa_{2k} \ldots a_{2k+1}^{-1} W'$ be a word with leftmost redex as displayed, the case of redexes with prefix $a_{2k-1}$ is similar. Assuming that $Wa_{2k}$ is $F$-reduced, we have $M \to Wa_{2k+1}^{-1} \ldots a_{2k} W'$. What are the $F$ or $T_p$-rules reducing a suffix of $W$?

1. $F$-reduction using the rule $a_{2k+1}a_{2k+1}^{-1} \to 1$ results in the reduction:

$$M \to^* W_0 a_{2k+2}^{-1} a_{2k+3}^{-1} \ldots a_{2k-1} a_{2k} W'.$$

   No further reduction is possible, for a $F$-redex implies $W = W_0 a_{2k+1} = W_1 a_{2k+2} a_{2k+1}$ and the initial $T_p$-redex would not be the leftmost one. Any $T_p$-redex implies $W = W_0 a_{2k+1} = W_1 a_{2k}^{-1} a_{2k+1}^{-1} a_{2k+1}$ and the subword $W$ would be $F$-reducible.

2. $T_p$-reduction. Every $T_p$-redex using a suffix of $W$ either must include the letter $a_{2k}^{-1}$, but this is impossible since the word $Wa_{2k} = W_0 a_{2k}^{-1} a_{2k}$ would be $F$-reducible, or we have:

$$W = W_1 a_{2j} a_{2j+1} \dots a_{2p} a_1^{-1} \dots a_{2p}^{-1},$$

but once more the initial word would be $F$-reducible by $a_{2p} a_{2p}^{-1} \to 1$.

Thus, the leftmost reduction needs at most one $F$-reduction on the right-hand side of $W$ after a $T_p$-reduction. Now, at what index do we resume the search of a new redex? If the next $T_p$-redex has a common subword with the right-hand side just introduced, we have the following case:

$$M \to W a_{2k+1}^{-1} \dots a_{2j}^{-1} \dots a_{2p}^{-1} a_1 \dots a_{2k} a_{2k+1} \dots a_{2j-1} W_0' \quad \text{with} \quad a_{2k+1} \dots a_{2j-1} W_0' = W'.$$

This implies that $M$ would be $F$-reducible where $W'$ and the initial redex meet. Thus, we must $T_p$-reduce the leftmost redex only if we cannot $F$-reduce on both sides of this $T_p$-redex, i.e. free cancellations have higher priority. The redex search can be resumed in $W'$. But since $F$-reductions affecting $W'$ are possible after the $T_p$-reduction, this leads to a special case where the new right-hand side disappears entirely:

$$M \to W a_{2k+1}^{-1} \dots a_{2p}^{-1} a_1 \dots a_{2k} a_{2k}^{-1} a_{2k-1}^{-1} \dots a_1^{-1} a_{2p} \dots a_{2k+1} W_0' \to_F^* W W_0'$$

and in this case we move back along $2p-1$ letters to resume the reductions. However, since this special case reduces the length of $M$ by $4p$ letters, moving back along $2p-1$ letters is equivalent to moving forward along $2p+1$ letters. Hence the computation of the normal form for $M$ does not involve any more than $|M|/2p$ $T_p$-reductions. This analysis summarizes the reduction algorithm. □

Let us give some details of the completion of the groups presented by the above defining relation, with an odd number of generators: $\langle A_1, \dots, A_{2p+1}; A_1 \cdots A_{2p+1} A_1^{-1} \cdots A_{2p+1}^{-1} \rangle$. The group $\perp_1$ has a symmetrised set which is also complete:

$$\perp_1 \begin{cases} CBA \to ABC & A^{-1}B^{-1}C^{-1} \to C^{-1}B^{-1}A^{-1} \\ BCA^{-1} \to A^{-1}CB & AC^{-1}B^{-1} \to B^{-1}C^{-1}A \\ B^{-1}A^{-1}C \to CA^{-1}B^{-1} & C^{-1}AB \to BAC^{-1} \end{cases}$$

But its termination does not follow from a classical order. We exhibit a complete presentation $\perp_p$ having $4p+2$ rules, where the common length of the words is $2p+1$:

$$\perp_p \begin{cases} A_{2k+1} \cdots A_1 A_{2p+1}^{-1} \cdots A_{2k+2}^{-1} \to A_{2k+2}^{-1} \cdots A_{2p+1}^{-1} A_1 \cdots A_{2k+1} \\ A_{2k+1}^{-1} \cdots A_{2p+1}^{-1} A_1 \cdots A_{2k} \to A_{2k} \cdots A_1 A_{2p+1}^{-1} \cdots A_{2k+1}^{-1} \\ A_{2k} \cdots A_{2p+1} A_1^{-1} \cdots A_{2k-1}^{-1} \to A_{2k-1}^{-1} \cdots A_1^{-1} A_{2p+1} \cdots A_{2k} \\ A_{2k}^{-1} \cdots A_1^{-1} A_{2p+1} \cdots A_{2k+1} \to A_{2k+1} \cdots A_{2p+1} A_1^{-1} \cdots A_{2k}^{-1} \end{cases} \quad k = 0, \dots, p.$$

The termination of $\perp_p$ follows from the observation that each member of $\mathcal{G}$ is the prefix of exactly one rule. Since both sides of each rule have the same length, we may restrict ourselves to reduction chains of words of the same length.

LEMMA 3. *If $U$ and $V$ are two words having the same length and if $b_1 \ldots b_{2p}$, where $b_i \in \mathcal{G}$, is a left-hand side prefix, then the reduction $b_1 \ldots b_{2p} U \to^*_{\perp_p} b_{2p}^{-1} \ldots b_1^{-1} V$ never arises.*

PROOF. The proof proceeds by induction on $|U|$. The proposition is immediate when $|U| = 0$ or $1$ from the irreducibility of the right-hand sides. Set $P = b_1 \ldots b_{2p} = A_{2p+1} \cdots A_2$; because of the symmetry of the rules the other cases are similar. Any reduction from $PU$ to $P^{-1}V$ uses the first rule $A_{2p+1} \cdots A_1 \to A_1 \cdots A_{2p+1}$ since (i) $A_{2p+1}$ must be reduced, (ii) the rule $A_{2p+1}A_{2p+1}^{-1} \to 1$ cannot arise since reductions preserve length and (iii) this rule is the only one in $\perp_p$ with a left-hand side prefix equals to $A_{2p+1}$. Consequently, $PU \to^* A_{2p+1} \cdots A_2 A_1 U' \to A_1 A_2 \cdots A_{2p+1} U'$. Afterwards, having $A_2^{-1}$ as prefix is possible only if $A_1$ disappears, which in turn is possible only using the rule $A_1 A_{2p+1}^{-1} \cdots A_2^{-1} \to A_2^{-1} \cdots A_{2p+1}^{-1} A_1$. We necessarily have $A_2 \cdots A_{2p+1} U' \to^* A_{2p+1}^{-1} \cdots A_2^{-1} V'$, which contradicts the induction hypothesis. □

COROLLARY 4. *If $U$ and $V$ are two words having the same length, then there is no reduction of the form $\rho U \to^*_{\perp_p} \lambda V$, where $\rho$ (resp. $\lambda$) is a right (resp. left) hand side of a rule.*

PROOF. By symmetry, we can restrict our attention to the case where $\rho = A_1^{-1} A_{2p+1} \cdots A_2$. If $\lambda \neq \mu = A_1^{-1} \cdots A_{2p+1}^{-1}$, the generator $A_1^{-1}$ must be reduced by a rule in $\perp_p$, i.e. the redex $\mu$ is contracted before the redex $\lambda$. Therefore, we have $A_{2p+1} \cdots A_2 U \to^*_{\perp_p} A_2^{-1} \cdots A_{2p+1}^{-1} V$, which is impossible by lemma 3. □

Thus the $\perp_p$-reductions must halt as a prefix can be reduced once only and length remains constant. Geometrically, the two families of complete presentations possess a concise description in terms of $4p$ and $(4p+2)$-gons (Figs. 1 & 2). Each polygon represents an elementary circuit in the Cayley graph of the groups. The arrows represent irreducible paths. All rules are embodied on these graphs.

### 2.2. NON-ORIENTABLE SURFACES

The non-orientable surface groups are defined by $\langle A_1, \ldots, A_p; A_1^2 \cdots A_p^2 \rangle$. In general, the complete set of rules $R_p$ depends upon the parity of $p$. Let $n = [p/2]$ and $A_i = A_{i+p}$ if

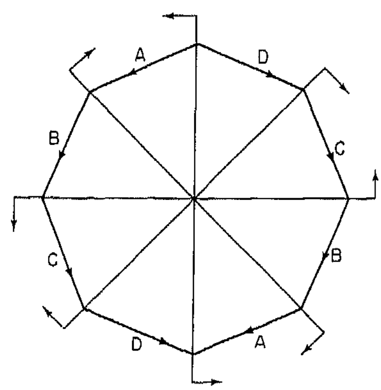


**Fig. 1.** $T_2$.

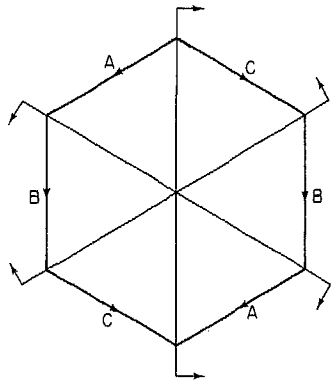


Fig. 2. $\perp_1$.

$i = 1-p, \ldots, 0$, $A_i = A_{i-p}$ if $i = p+1, \ldots, 2p$. Notice that $A_{k-n} = A_{k+n+1}$. Both cases split into two sets of rules:

$$p = 2n+1 \left\{ \begin{array}{l} \left\{ \begin{array}{c} A_k^{-1} A_{k-1}^{-2} \cdots A_{k-n}^{-2} \rightarrow A_k A_{k+1}^2 \cdots A_{k+n}^2 \\ A_k^{-2} A_{k-1}^{-2} \cdots A_{k-n+1}^{-2} A_{k-n}^{-1} \rightarrow A_{k+1}^2 \cdots A_{k+n}^2 A_{k+n+1} \\ A_k^2 \cdots A_{k+n}^2 \rightarrow A_{k-1}^{-2} \cdots A_{k-n}^{-2} \\ A_k A_{k+1}^2 \cdots A_{k+n}^2 A_{k+n+1} \rightarrow A_k^{-1} A_{k-1}^{-2} \cdots A_{k-n+1}^{-2} A_{k-n}^{-2} \end{array} \right. \\ \left\{ \begin{array}{l} A_k^{-2} \cdots A_{k-n+1}^{-2} A_{k-n} A_{k-n+1}^2 \cdots A_k^2 \\ \qquad \rightarrow A_{k+1}^2 \cdots A_{k+n}^2 A_{k+n+1} A_{k+n}^{-2} \cdots A_{k+1}^{-2} \\ A_k^{-1} A_{k-1}^{-2} \cdots A_{k-n+1}^{-2} A_{k-n}^{-1} A_{k-n+1}^2 \cdots A_k^2 A_{k+1} \\ \qquad \rightarrow A_k A_{k+1}^2 \cdots A_{k+n}^2 A_{k+n-1}^{-1} A_{k+n}^{-2} \cdots A_{k+2}^{-2} A_{k+1}^{-1} \end{array} \right. \end{array} \right.$$

$$p = 2n \left\{ \begin{array}{l} \left\{ \begin{array}{c} A_k^{-2} A_{k-1}^{-2} \cdots A_{k-n+1}^{-2} \rightarrow A_{k+1}^2 \cdots A_{k+n}^2 \\ A_k^{-1} A_{k-1}^{-2} \cdots A_{k-n+1}^{-2} A_{k-n}^{-1} \rightarrow A_k A_{k+1}^2 \cdots A_{k+n-1}^2 A_{k+n} \\ A_k^2 \cdots A_{k+n-1}^2 A_{k+n} \rightarrow A_{k-1}^{-2} \cdots A_{k-n+1}^{-2} A_{k-n}^{-1} \\ A_k A_{k+1}^2 \cdots A_{k+n}^2 \rightarrow A_k^{-1} A_{k-1}^{-2} \cdots A_{k-n+1}^{-2} \end{array} \right. \\ \left\{ \begin{array}{l} A_k^{-2} \cdots A_{k-n+2}^{-2} A_{k-n+1} A_{k-n+2}^2 \cdots A_{k+1}^2 \\ \qquad \rightarrow A_{k+1}^2 \cdots A_{k+n}^2 A_{k+n+1}^{-1} A_{k+n}^{-2} \cdots A_{k+2}^{-2} \\ A_k^{-1} A_{k-1}^{-2} \cdots A_{k-n+1}^{-2} A_{k-n} A_{k-n+1}^2 \cdots A_{k-1}^2 A_k \\ \qquad \rightarrow A_k A_{k+1}^2 \cdots A_{k+n-1}^2 A_{k+n} A_{k+n-1}^{-2} \cdots A_{k+1}^{-2} A_k^{-1} \end{array} \right. \end{array} \right.$$

In each of these rules, $k$ ranges from 1 to $p$.

*Termination*: lexicographic order with $A_1^{-1} > \cdots > A_p^{-1} > A_1 > \cdots > A_p$.

Every set consists of $6p$ rules. In both cases, the first four rules correspond to symmetrised presentations, while the remaining two rules arise from critical pairs with the pieces of length one. For example, if $p = 4$, the two rules $A_3 A_4 A_4 A_1 A_1 \rightarrow A_3^{-1} A_2^{-1} A_2^{-1}$ and $A_1 A_1 A_2 A_2 A_3 \rightarrow A_4^{-1} A_4^{-1} A_3^{-1}$, superposed on the piece $A_1$, create the new rule:

$$A_3^{-1} A_2^{-1} A_2^{-1} A_1 A_2 A_2 A_3 \rightarrow A_3 A_4 A_4 A_1 A_4^{-1} A_4^{-1} A_3^{-1}.$$

## 3. Coxeter Groups

### 3.1. COMPLETION

The word problem for Coxeter groups has been proved decidable by Tits (1969). The completion of these group presentations is one of the most convincing examples of the power of rewriting systems. It proves the solvability of the word problem using elementary syntactical methods, while the traditional proof employs geometrical methods (Tits, 1969; Bourbaki, 1978). Although the family is parametrised by $n \times n$ symmetric integer matrices, a concise complete presentation is found that leads to an efficient word problem algorithm. However, a drawback is encountered in that the partial commutativity of a presentation may lead to a failure.

Let $I$ be a (not necessarily finite) set of generators. A Coxeter matrix on $I$ is a function $M: I \times I \mapsto \mathbb{N} \cup \{\infty\}$ such that for all $i, j$ in $I$, $M(i, i) = 1$ and $M(i, j) = M(i, i) \geqslant 2$ if $i \neq j$. The value $M(i, j)$ will be denoted by $m_{ij}$. The Coxeter group $C(M)$ is presented as $\langle I, E \rangle$ where $E$ is the set of equations $(ij)^{m_{ij}} = 1$, $m_{ij} \neq \infty$. As $m_{ii} = 1$ implies $i^{-1} = i$, we may represent the elements of $C(M)$ by words from the free monoid $I^*$ on $I$.

Throughout this section, $[ij]^k$ will denote the product $ijij \ldots$ of $k$ generators alternatively equal to $i$ and $j$; $\alpha$ will denote $[ij]^{m_{ij}-1}$. The generators $i$ and $j$ will be denoted by $f(\alpha)$ and $s(\alpha)$ respectively, and $l(\alpha)$ denotes the last generator of $\alpha$, which is equal to $i$ (resp. $j$) when $m_{ij}$ is even (resp. odd). With the word $\alpha$ we associate the word $\bar{\alpha} = [ji]^{m_{ij}-1}$. Finally, $m_{ij}$ will be abbreviated as $m_\alpha$. The same conventions apply with $\beta = [ij]^{m_{ij}}$ and $\gamma = [ij]^{m_{ij}-2}$.

A solution to the word problem first appeared in a theorem of Tits (Bourbaki, 1978, p. 93). Suppose $I$ is finite with $n$ generators. Generators are represented by linear transformations of a real vector space with basis $e_1, \ldots, e_n$:

$$s_i: e_j \to e_j - 2\left(\cos \frac{\pi}{m_{ij}}\right) e_i.$$

A word $w = i_1 \ldots i_k$ in $I^*$ is equal to 1 in $C(M)$ if and only if $s_{i_1} \ldots s_{i_k}(\sum_{j=1}^n e_j) = \sum_{j=1}^n e_j$. Involving matrices computations, this solution is not efficient. Another solution proposed by Tits is based on a reduction in $I^*$ defined by the following rules:

$$\begin{cases} wiiw' \to ww', & i \in I, \quad w, w' \in I^*, \\ w\beta w' \to w\bar{\beta}w', & w, w' \in I^*. \end{cases}$$

Confluence of the rewriting system is proved using the above linear representation. Since the reduction does not increase length, an enumeration of the words derived from a particular one halts, thereby solving the word problem. A completion based on any lexicographic order may be used to significantly improve this algorithm. We restrict ourselves to matrices having no entry equal to 2. The completion procedure starts by symmetrising the given presentation:

LEMMA 5. *Given a Coxeter matrix $M$ on the set $I$ which possesses a linear order $>$, the completion procedure generates the following two sets of rules:*

$$R_I = \{i^{-1} \to i,\ ii \to 1 \mid i \in I\} \quad \textit{and} \quad S_I = \{\beta \to \bar{\beta} \mid f(\beta) > s(\beta)\}.$$

PROOF. Straightforward. $\square$

The rules $R_I$ imply that we can restrict to words in $I^*$. Notice that a word $\alpha$ is both a left-hand side suffix (resp. prefix) and a right-hand side prefix (resp. suffix) of a rule in $S_I$ when $s(\alpha) > f(\alpha)$ (resp. $f(\alpha) > s(\alpha)$), and that both $\alpha\alpha$ and $\alpha\bar{\alpha}$ reduce to 1.

THEOREM 6. *Let $M$ be a Coxeter matrix on the set $I$ which possesses a linear order $>$. If $m_{ij} \neq 2$, $i, j \in I$, the completion procedure generates the set of rules $R_I \cup T_I$, where $T_I$ consists of all rules of the form:*

$$\alpha_1 \ldots \alpha_k l(\overline{\alpha_k}) \to s(\alpha_1)\alpha_1 \ldots \alpha_k \tag{1}$$

*with $k$ a positive integer, and for all $p$ such that $1 < p < k$:*

$$f(\alpha_1) > s(\alpha_1), \qquad s(\alpha_p) > f(\alpha_p), \qquad f(\alpha_{p+1}) \neq l(\alpha_p), \qquad s(\alpha_{p+1}) = l(\overline{\alpha_p}). \tag{2}$$

PROOF. The proof may be found in Le Chenadec (1986). □

### 3.2. EXAMPLES

Rules belonging to $R_I$ are omitted when we give a complete set of rules.

$$G_1 \begin{cases} baba \to abab \\ bcbcb \to cbcbc \\ acacac \to cacaca \\ bcbcabab \to cbcbcaba \\ babcacaca \to ababcacac \\ bcbcabacbcbc \to cbcbcabacbcb \end{cases} \qquad G_2 \begin{cases} abab \to baba \\ cbcbc \to bcbcb \\ acacac \to cacaca \\ acacabcbcb \to cacacabcbc \end{cases} \qquad G_3 \begin{cases} dcd \to cdc \\ dbdb \to bdbd \\ dada \to adad \end{cases}$$

The sets $G_1$ and $G_2$ are both complete and define the same group. The order is defined by $b > a > c$ for $G_1$ and by $a > c > b$ for $G_2$. Notice that the number of rules depends on the order. The set of rules may be infinite. For example, in the case of $G_3$ the completion procedure creates infinitely many rules:

$$dcbdb(adabdb)^m d \to cdcbdb(adabdb)^m, \quad m \geqslant 0.$$

For reduction computations, it is noteworthy that all $T_I$-rules are in Post normal form, i.e. they are of the type $Va \to bV$, with $V = \alpha_1 \ldots \alpha_m$. If $M$ is a Coxeter matrix with infinite coefficients $m_{ij}$'s, the parametrised rule (1) still describes the complete system, and the previous theorem remains valid for $M$ with the convention that no component $\alpha$ exists for $i$ and $j$. We now give two examples of commuting pairs of generators:

$$G_4 \begin{cases} ad \to da \\ bd \to db \\ ca \to ac \\ cd \to dc \\ cbc \to bcb \\ cbac \to bcba \\ cbabcb \to bcbabc \end{cases} \qquad G_5 \begin{cases} ca \to ac \\ cb \to bc \\ dad \to ada \\ dbd \to bdb \\ dcd \to cdc \end{cases}$$

With the *complete* system $G_4$, the rule $cbadc \to bcbad$ is never created as both its sides are confluent under the commutativity rules and the rule $cbac \to bcba$. Given $G_5$, the completion procedure generates infinitely many rules:

$$dxcd[yx]^m c \to xdxcb[yx]^m, \quad m \geqslant 0, \quad \text{where } \{x, y\} = \{a, b\}.$$

We now consider finite Coxeter groups which were first described by Coxeter (1935). The notation is taken from Bourbaki (1978). The finite Coxeter groups whose Coxeter matrix has entries equal to 1, 2, 3, 4 or 6 are called crystallographic groups.

$$H_4 \begin{cases} dcd \to cdc \\ cbc \to bcb \\ cbac \to bcba \\ babab \to ababa \\ cbabcb \to bcbabc \\ cbabacbaba \to bcbabacbab \end{cases} \quad H_3 \begin{cases} cbc \to bcb \\ cbac \to bcba \\ babab \to ababa \\ cbabcb \to bcbabc \\ cbabacbaba \to bcbabacbab \end{cases} \quad F_4 \begin{cases} bab \to aba \\ dcd \to cdc \\ cbcb \to bcbc \\ cbacba \to bcbacb \end{cases}$$

The two groups $H_3$ and $H_4$ are not crystallographic. The dihedral groups $I_2(n)$, $n > 4$, among which only $I_2(6)$ is crystallographic, possess complete sets equal to $R_I \cup S_I$. The other finite groups are crystallographic. We failed to construct complete presentations for the three isolated groups $E_n$, $n = 6, 7, 8$, and the family $D_n$, $n \geqslant 4$. For the symmetric groups $A_n$, two complete presentations may be found in §5; despite the presence of commuting pairs of generators, notice that only $T_I$-rules arise. The Coxeter matrix for the group $B_n$ is:

$$\begin{pmatrix} 1 & 3 & & & & \\ 3 & 1 & \ddots & & 2 & \\ & \ddots & \ddots & \ddots & & \\ & & \ddots & 1 & 3 & 2 \\ & 2 & & 3 & 1 & 4 \\ & & & 2 & 4 & 1 \end{pmatrix}$$

The complete system includes the rules $R_I$, the commutativity rules, and the following $T_I$-rules:

$$\begin{cases} a_i a_{i-1} \ldots a_{i-k} a_i \to a_{i-1} a_i a_{i-1} \ldots a_{i-k}, & n > i > k > 0 \\ (a_n a_{n-1} \ldots a_{n-k})^2 \to a_{n-1} a_n a_{n-1} \ldots a_{n-k} a_n a_{n-1} \ldots a_{n-k+1}, & n > k > 0 \end{cases}$$

### 3.3. A REDUCTION ALGORITHM

The simplest reduction algorithm iterates the search of a left-hand side and the substitution of right-hand sides. We may improve it by using the uniform structure of the rules.

A reasonable goal is a reduction algorithm which does not resort to backward search in a word already scanned and reduced, as in the case of the groups of orientable surfaces. After a reduction, what are the possible rules overlapping the new right-hand side? We first restrict our attention to rules in $T_I$:

$$W \to^* v\alpha_1 \ldots \alpha_{k-1}\beta_k w \to v\overline{\beta_1}\alpha_2 \ldots \alpha_k w$$

Any reduction of a suffix of $v$ reduces no more than the subword $\overline{\beta_1}$ by condition (2) of theorem 6. If it reduces a $\overline{\beta_1}$ prefix of length greater than one, then condition (2) between

$\overline{\beta_1}$ and the suffix of $v$ implies that $v\alpha_1$ is reducible. Finally, if the prefix of $\overline{\beta_1}$ is a single generator, contractions may occur. For example, over $G_1$ we have:

$$acaca\ bcbcabab \to acaca\ cbcbcaba = acacac\ bcbcaba \to cacacabcbcaba. \tag{3}$$

Consequently, the algorithm must update a stack of previous redex prefixes, $\alpha_1 \cdots \alpha_i$, $i > 1$. Such a prefix becomes a redex if and only if the current generator completes the last prefix's component $\alpha_i$ into $\beta_i$.

On suffixes of the new right-hand side, condition (2) implies that there is only one possible creation of a redex: a suffix of the last component $\alpha_k$ is a prefix of the first component of this redex. It follows that before the reduction we have the configuration $v\alpha_1 \cdots \alpha_k l(\overline{\alpha_k})l(\overline{\alpha_k})w'$. Also, to overcome such overlapping reductions, the algorithm will $R_I$-reduce on the right-hand side of a $T_I$-redex before contracting this redex. Therefore, we may resume the redex search on the last generator of the new right-hand side. This is illustrated with the same example $G_1$:

$$bcbcabab\ cacac \to cbcbcaba\ cacac = cbcbcab\ acacac \to bcbcabcacaca. \tag{4}$$

Let us now look more closely at the possible $R_I$-reductions following a $T_I$-one. On prefixes of the new right-hand side, at most one $R_I$-reduction can arise. Otherwise, the $T_I$-redex would not be leftmost as $f(\alpha_1)s(\alpha_1)\alpha_1 \to \overline{\beta_1}l(\alpha_1)$. On suffixes of the new right-hand side, several $R_I$-reductions can occur. Finally, $R_I$-reductions can affect the redex stack by deleting whole or part of the topmost redex.

The basic points of the reduction algorithm are: (i) use of a stack for scanned redex prefixes, (ii) the redex prefix on top of the stack is completed into a redex by a single generator ($c$ in (3)) and (iii) redex search is resumed on the last generator of a new $T_I$-right-hand side ($a$ in (4)). These few observations suffice to outline a reduction algorithm based on leftmost reductions. As in the case of orientable surfaces, not only is this algorithm linear in time, but it does not involve any backtracking as is the case in general (Book, 1982; Domanski & Anskel, 1985; Bauer & Otto, 1984).

## 4. Dyck Groups

### 4.1. POLYHEDRAL GROUPS

The polyhedral group $(l, m, n)$ is defined by the presentation $\langle A, B, C;\ A^l, B^m, C^n, ABC\rangle$ (Coxeter, 1980). In §4.2 and §4.3, we present complete systems for the following generalization, known as Dyck groups:

$$(p_1, \ldots, p_n) = \langle A_1, \ldots, A_n;\ A_1^{p_1}, \ldots, A_n^{p_n}, A_1 \cdots A_n\rangle, \quad n \geqslant 3.$$

Notice that these groups are the rotation subgroups of Coxeter groups. The general complete system also requires $p_i \geqslant 3$, $i = 1, \ldots, n$. Let us first examine the case $n = 3$. This presentation is redundant in that one of the generators can be eliminated. With this new presentation, we observe that, whenever $l$, $m$ and $n$ are greater than 3, the group has the small cancellation property, as the only pieces are the generators and their inverses. Thus, its word problem is solvable. For a study of Dyck and Coxeter groups from the point of view of small cancellation, see Appel & Schupp (1983). The groups are infinite when

$$\frac{1}{l} + \frac{1}{m} + \frac{1}{n} \leqslant 1.$$

Finite groups arise in the case of the triples $(2, 2, n)$, $(2, 3, 3)$, $(2, 3, 4)$, $(2, 3, 5)$. The first triple corresponds to the dihedral groups whose complete presentation is given in §3.2. The remaining finite groups are the rotation groups of the five Platonic polyhedrons. As isolated finite groups they possess many complete presentations; for example:

$$(2,3,3)\ :\ \textbf{Tetrahedron}\ \left\{\begin{array}{ll} C^{-1}\to CC & A\to C^{-1}B^{-1} \\ B^{-1}\to BB & A^{-1}\to BC \\ CBC\to BB & CCC\to 1 \\ CCBB\to BC & BBB\to 1 \\ CBBC\to BCCB & BCB\to CC \\ & BBCC\to CB \end{array}\right.$$

*Termination*: weight order with $\pi(B^{-1}) = 3$, $\pi(C^{-1}) = 3$, $\pi(B) = \pi(C) = 1$ and $C > B$.

$$(2,3,4)\ :\ \begin{array}{c}\textbf{Cube}\text{ or}\\ \textbf{Octahedron}\end{array}\left\{\begin{array}{ll} A^{-1}\to A & CCC\to ACACA \\ B^{-1}\to CA & CACAC\to A \\ C^{-1}\to ACACA & CACCAC\to ACCA \\ AA\to 1 & CCACCA\to ACCACC \\ B\to A^{-1}C^{-1} & \end{array}\right.$$

*Termination*: weight order with $\pi(C^{-1}) = 5$, $\pi(A^{-1}) = \pi(C) = \pi(A) = 1$, $C^{-1} > A^{-1} > C > A$.

$$(2,3,5)\ :\ \begin{array}{c}\textbf{Icosahedron}\text{ or}\\ \textbf{Dodecahedron}\end{array}\left\{\begin{array}{ll} A^{-1}\to A & BABBABABBABA\to ABABBABABBAB \\ B^{-1}\to BB & BBABB\to ABABABA \\ C^{-1}\to AB & C\to B^{-1}C^{-1} \\ BABABAB\to ABBA & BBB\to 1 \\ BABABBABAB\to ABBABABBA & AA\to 1 \end{array}\right.$$

*Termination*: weight order with $\pi(B^{-1}) = 6$, $\pi(B) = 3$, $\pi(A^{-1}) = \pi(A) = 1$, and $A^{-1} > B^{-1} > A > B$.

The remaining groups are infinite and we may assume that $l \leqslant m \leqslant n$. There are two distinct cases, depending upon whether some generator has even order or not. The simpler case occurs when all parameters are odd: $l = 2p+1$, $m = 2q+1$, $n = 2r+1$. An initial set results from the symmetrisation of the presentation:

$$(1)\left\{\begin{array}{ll} AB\to C^{-1} & A^{-1}C^{-1}\to B \\ BC\to A^{-1} & C^{-1}B^{-1}\to A \\ CA\to B^{-1} & B^{-1}A^{-1}\to C \\ A^{p+1}\to A^{-p} & A^{-(p+1)}\to A^{p} \\ B^{q+1}\to B^{-p} & B^{-(q+1)}\to B^{q} \\ C^{r+1}\to C^{-r} & C^{-(r+1)}\to C^{r} \end{array}\right.$$

Also we have the critical pairs rules:

$$(2)\left\{\begin{array}{ll} A^{-1}C^{r}\to BC^{-r} & A^{-p}B\to A^{p}C^{-1} \\ B^{-1}A^{p}\to CA^{-p} & B^{-q}C\to B^{q}A^{-1} \\ C^{-1}B^{q}\to AB^{-q} & C^{-r}A\to C^{r}B^{-1} \end{array}\right.$$

*Termination*: lexicographic order with $C^{-1} > B^{-1} > A^{-1} > C > B > A$.

Three cases remain, corresponding to one, two or three generators having even order. The set (1) is modified when the order of a generator becomes even. Assuming that $A$ has order $2p$, the two corresponding rules become $A^{p+1} \to A^{-(p-1)}$ and $A^{-p} \to A^{p}$. Then, in the second set of rules, the first rule becomes $A^{-(p-1)}B \to A^{p}C^{-1}$ and $B^{-1}A^{p} \to CA^{-(p-1)}$. We give four complete sets, omitting the rules arising from the symmetrization of the defining relation $ABC$:

$$(7,7,7)\left\{\begin{array}{l} A^4 \to A^{-3} \\ A^{-4} \to A^3 \\ B^4 \to B^{-3} \\ B^{-4} \to B^3 \\ C^4 \to C^{-3} \\ C^{-4} \to C^3 \\ A^{-1}C^3 \to BC^{-3} \\ A^{-3}B \to A^3C^{-1} \\ B^{-1}A^3 \to CA^{-3} \\ B^{-3}C \to B^3A^{-1} \\ C^{-1}B^3 \to AB^{-3} \\ C^{-3}A \to C^3B^{-1} \end{array}\right. \qquad (7,8,9)\left\{\begin{array}{l} A^4 \to A^{-3} \\ A^{-4} \to A^3 \\ B^5 \to B^{-3} \\ B^{-4} \to B^4 \\ C^5 \to C^{-4} \\ C^{-5} \to C^4 \\ A^{-1}C^4 \to BC^{-4} \\ A^{-3}B \to A^3C^{-1} \\ B^{-1}A^3 \to CA^{-3} \\ B^{-3}C \to B^4A^{-1} \\ C^{-1}B^4 \to AB^{-3} \\ C^{-4}A \to C^4B^{-1} \end{array}\right.$$

$$(7,8,8)\left\{\begin{array}{l} A^4 \to A^{-3} \\ A^{-4} \to A^3 \\ B^5 \to B^{-3} \\ B^{-4} \to B^4 \\ C^5 \to C^{-3} \\ C^{-4} \to C^4 \\ A^{-1}C^4 \to BC^{-3} \\ A^{-3}B \to A^3C^{-1} \\ B^{-1}A^3 \to CA^{-3} \\ B^{-3}C \to B^4A^{-1} \\ C^{-1}B^4 \to AB^{-3} \\ C^{-3}A \to C^4B^{-1} \end{array}\right. \qquad (8,8,8)\left\{\begin{array}{l} A^5 \to A^{-3} \\ A^{-4} \to A^4 \\ B^5 \to B^{-3} \\ B^{-4} \to B^4 \\ C^5 \to C^{-3} \\ C^{-4} \to C^4 \\ A^{-1}C^4 \to BC^{-3} \\ A^{-3}B \to A^4C^{-1} \\ B^{-1}A^4 \to CA^{-3} \\ B^{-3}C \to B^4A^{-1} \\ C^{-1}B^4 \to AB^{-3} \\ C^{-3}A \to C^4B^{-1} \end{array}\right.$$

At least one, and at most three rules in the even case are length increasing, and no classical order proves the termination. From hand computations, we conjecture that the reductions are well-founded. The irreducible forms of $(l, m, n)$ are conveniently represented by the finite automaton shown in Fig. 3, where the following conventions are used:

- A state labelled $A$ (resp. $B$, $C$) recognises the subwords $A^{i}$, $i = 1, \ldots, \left\lfloor \frac{i}{2} \right\rfloor$.

  A state labelled $a$ (resp. $b$, $c$) recognises the subwords $A^{-i}$, $i = 1, \ldots, \left\lfloor \frac{i-1}{2} \right\rfloor$.

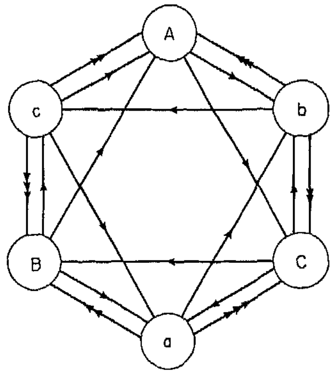


**Fig. 3.** Finite automaton defined by the normal forms of polyhedral groups.

- Simple arrows represent all transitions, whatever the subword recognised by the initial state of the arrow. Double arrows represent all transitions except the one whose initial state recognises the maximal length subword (rules with left-hand sides $B^{-3}C$). Triple arrows represent all transitions except the one whose final state recognises the maximal subword (rules with left-hand sides $B^{-1}A^3$).

We now describe the complete systems for the Dyck groups defined on at least four generators. We have two cases according to the parity of the number of generators. We restrict ourselves to generators having order greater than 2.

### 4.2. Dyck groups on an odd number of generators

Let $G = \{A_1, \ldots, A_{2n+1}\}$ be the set of generators. The set $\mathscr{G}$ is ordered such that inverses are greater than generators. Let $\alpha_1 \ldots \alpha_{n+1}$ be any subword of length $n+1$ of the word $W_G = A_1 \cdots A_{2n+1} A_1 \cdots A_n$. The word $\alpha_{n+2} \ldots \alpha_{2n+1}$ denotes its *complement*, i.e. a suffix of length $n$, or prefix of length $n$ if such a suffix does not exist. The complete system for $(2p_1+1, \ldots, 2p_{2n+1}+1)$, where $p_i > 0$, is:

$$\left\{\begin{array}{c} \alpha_1 \ldots \alpha_{n+1} \to (\alpha_{n+2} \ldots \alpha_{2n+1})^{-1} \\ (\alpha_1 \ldots \alpha_{n+1})^{-1} \to \alpha_{n+2} \ldots \alpha_{2n+1} \\ \alpha^{p_\alpha+1} \to \alpha^{-p_\alpha} \\ \alpha^{-(p_\alpha+1)} \to \alpha^{p_\alpha} \\ \alpha_{n+1}^{-1} \ldots \alpha_2^{-1} \alpha_1^{p_{\alpha_1}} \to \alpha_{n+2} \ldots \alpha_{2n+1} \alpha_1^{-p_{\alpha_1}} \\ \alpha_1^{-p_{\alpha_1}} \alpha_2 \ldots \alpha_{n+1} \to \alpha_1^{p_{\alpha_1}} (\alpha_{n+2} \ldots \alpha_{2n+1})^{-1} \end{array}\right.$$

For complete presentations with generators $\alpha$ having even order $2p_\alpha$, the third and fourth rules become $\alpha^{p_\alpha+1} \to \alpha^{-(p_\alpha-1)}$ and $\alpha^{-p_\alpha} \to \alpha^{p_\alpha}$, respectively. The other pairs of exponents $(p_\alpha, -p_\alpha)$ become $(p_\alpha, -(p_\alpha-1))$. The number of rules is $6|G|$. As in the case of surface groups, the rules are simply described geometrically by means of the Cayley graph, by directing them away from a given vertex. We exhibit the graph for (5, 5, 5) in Fig. 4. For the other groups with an odd number $2n+1$ of generators the number of polygons containing the central vertex is $4n+2$. The triangles represent the defining relation $ABC = 1$. Since the Cayley graphs are planar, they are oriented according to the accompanying arrow. The size of the other polygons depend on the order of the

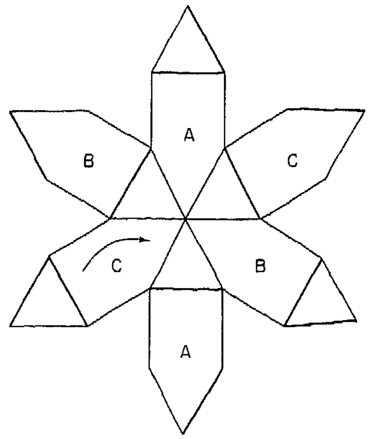


**Fig. 4.** (5, 5, 5).

generator. Thick lines denote forbidden edges for paths in normal forms starting at the central vertex.

### 4.3. Dyck groups on an even number of generators

In this case, the number of rules is $10|G|$. Let $2n$ be the number of generators and define $W_G = A_1 \cdots A_{2n} A_1 \cdots A_n$. The words $\alpha_1 \ldots \alpha_{n+1}$ and $\alpha_{n+2} \ldots \alpha_{2n}$ are defined as above. For the sake of clarity, the pairs $(\alpha_1, p_{\alpha_1})$ and $(\alpha_{n+1}, p_{\alpha_{n+1}})$ are replaced by $(\sigma, s)$ and $(\rho, r)$. We give the complete system for generators of odd order, $(2p_1+1, \ldots, 2p_{2n}+1)$, $p_i > 0$:

$$
\left\{
\begin{array}{c}
\alpha_1 \ldots \alpha_{n+1} \to (\alpha_{n+2} \ldots \alpha_{2n})^{-1} \\
(\alpha_1 \ldots \alpha_n)^{-1} \to \alpha_{n+1} \ldots \alpha_{2n} \\
\alpha^{p_\alpha+1} \to \alpha^{-p_\alpha} \\
\alpha^{-(p_\alpha+1)} \to \alpha^{p_\alpha} \\
\alpha_1 \ldots \alpha_n \rho^{-r} \to (\alpha_{n+2} \ldots \alpha_{2n})^{-1} \rho^r \\
\sigma^{-s} \alpha_2 \ldots \alpha_{n+1} \to \sigma^s (\alpha_{n+2} \ldots \alpha_{2n})^{-1} \\
\sigma^{-s} \alpha_2 \ldots \alpha_n \rho^{-r} \to \sigma^s (\alpha_{n+2} \ldots \alpha_{2n})^{-1} \rho r \\
(\alpha_{n+2} \ldots \alpha_{2n})^{-1} \rho^r (\alpha_2 \ldots \alpha_n)^{-1} \sigma^s \to \alpha_1 \ldots \alpha_n \rho^{-(r-1)} \alpha_{n+2} \ldots \alpha_{2n} \sigma^{-s} \\
\alpha_1 \ldots \alpha_n \rho^{-(r-1)} \alpha_{n+2} \ldots \alpha_{2n} \alpha_1 \to (\alpha_{n+2} \ldots \alpha_{2n})^{-1} \rho^r (\alpha_2 \ldots \alpha_n)^{-1} \\
\sigma^{-s} \alpha_2 \ldots \alpha_n \rho^{-(r-1)} \alpha_{n+2} \ldots \alpha_{2n} \alpha_1 \to \sigma^s (\alpha_{n+2} \ldots \alpha_{2n})^{-1} \rho^r (\alpha_2 \ldots \alpha_n)^{-1}
\end{array}
\right.
$$

For generators having even order, the observations of the previous section remain valid, with the convention that $-(p_\alpha - 1)$ becomes $-(p_\alpha - 2)$ for a generator $\alpha$ of order $2p_\alpha$. As in the previous section, we give a geometrical interpretation of the rules for (5, 5, 5, 5). When the number of generators increases, so does the number of branches around the central vertex. And the number of edges in polygons varies according to the exponent of the generators. Observe that the initial cycles surrounding the central vertex give two rules, the others only one. These figures give a concise construction of Cayley graphs. The critical pairs are computed by superposition on a *single* generator, as for Coxeter groups. The completion procedure stops since the remaining superposition creates a subgraph

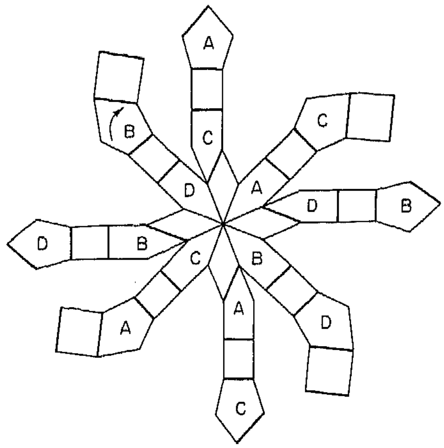


**Fig. 5.** (5, 5, 5, 5).

appearing somewhere else in the graph. Lastly, recall that for presentations including generators with even order, we did not prove the termination of the system due to length-increasing rules, such as $A^{-2}BC^{-1}DA \to A^3D^{-1}C^2B^{-1}$ in (6, 5, 5, 5).

## 5. Symmetric Groups

We give two complete presentations for the symmetric group $\mathbf{S}_n$. The first has fewer generators and relations than the second. Although we obtain a larger complete set for the second presentation, $O(n^4)$ against $O(n^2)$, this last system allows us to work with shorter words and the left-hand sides are simpler. Again, the essential point is not the number of rules, but their regularity and the complexity of the underlying algorithm solving the word problem.

### 5.1. PRESENTATION WITH ADJACENT TRANSPOSITIONS

When $\mathbf{S}_n$ is presented by adjacent transpositions, it can be defined as follows:

$$\mathbf{S}_n \begin{cases} R_i = (i\; i+1) & i = 1, \ldots, n-1 \\ R_i^2 = 1 & i = 1, \ldots, n-1 \\ R_iR_j = R_jR_i & i \leqslant j-2 \\ (R_iR_{i+1})^3 = 1 & i \leqslant n-2. \end{cases}$$

The completion procedure gives $n^2-2n+2$ rules (cf. $A_n$ of §3.2):

$$S_n \begin{cases} R_i^{-1} \to R_i & i = 1, \ldots, n \\ R_i^2 \to 1 & i = 1, \ldots, n \\ R_iR_j \to R_jR_i & j \leqslant i-2 \\ R_iR_{i-1} \ldots R_jR_i \to R_{i-1}R_iR_{i-1} \ldots R_j & j < i. \end{cases}$$

If $1 = R_0$ in $S_n$, for each rule the integer obtained by the concatenation of the indices of the generators on the left-hand side is greater than the one on the right-hand side. Thus, the system is well-founded. A noticeable feature of these systems is that $S_n \subset S_{n+1}$. The infinite set of rules $S_\infty = \bigcup_{n=1}^{\infty} S_n$ defines a normal form for a permutation of any degree. Moreover, a complete presentation of symmetric groups gives rise to a sorting algorithm. Regarding a permutation $\phi$ as an unsorted list, the normal form of $\phi^{-1}$ sorts $\phi$. The reader may check that the presentation $S_\infty$ defines insertion sorting (Knuth, 1973).

### 5.2. PRESENTATION WITH ALL TRANSPOSITIONS

We put $T_{i,j} = (i\ j)$ with $1 \leqslant i < j \leqslant n$. These new generators are related to the previous ones by the relations:

$$T_{i,j} = R_i R_{i+1} \dots R_{j-2} R_{j-1} R_{j-2} R_{j-3} \dots R_i.$$

We give the new definition of $\mathbf{S}_n$ and a possible completion $\mathscr{S}_n$:

$$\mathbf{S}_n \begin{cases} T_{i,j}^2 = 1 & \\ T_{i,j} T_{i,i+1} = T_{i,i+1} T_{i+1,j} & \\ (T_{i,i+1} T_{i+1,i+2})^3 = 1 & \\ (T_{i,i+1} T_{j,j+1})^2 = 1 & i+1 < j \\ T_{i,i+1} T_{i+1,j} T_{i,i+1} = T_{i,j} & i+1 < j. \end{cases}$$

$$\mathscr{S}_n \begin{cases} T_{i,j}^{-1} \to T_{i,j} & \\ T_{i,j} T_{i,j} \to 1 & \\ T_{i,j} T_{k,l} \to T_{k,l} T_{i,j} & i \neq k, \quad i \neq , \quad j \neq l \\ T_{i,j} T_{i,k} \to T_{i,k} T_{k,j} & i < k < j \\ T_{i,j} T_{k,j} \to T_{k,i} T_{i,j} & k < i < j \\ T_{i,j} T_{k,i} \to T_{k,i} T_{k,j} & k < i < j. \end{cases}$$

*Termination*: lexicographic order, with all the inverses greater than their corresponding generators and

$$T_{n-1,n} > T_{n-2,n} > \dots > T_{1,n} > T_{n-2,n-1} > \dots > T_{1,n-1} > \dots > T_{1,2}.$$

The number of rules is $O(n^4)$, which is far from the exponential upper bound we gave in §1, viz. $(n-1)(n-2)n!$. Once more, we have $T_n \subset T_{n+1}$. Thus $T_\infty = \bigcup_{n=1}^{\infty} T_n$ reduces a permutation of any degree to a normal form. The complete set is equal to the symmetrised set. The rules enumerate the permutation identities involving transpositions, and are sufficient to compute in $\mathbf{S}_n$. The presentation $\mathscr{S}_n$ defines max (or min) sorting. The complete system for quicksort, if it exists, seems complex. Further, the reader may check that bubble sort does not possess a complete presentation: it is well-known that the bubble sort is stupid and forgets earlier steps. For example $R_1 R_4 \to R_4 R_1$ and $R_4 R_1 \to R_1 R_4$ are both produced by the same input.

## 6. Conclusion

We briefly compare the Todd–Coxeter and Knuth–Bendix procedures for finite groups. The coset enumeration constructs the Cayley graph of the group represented by an edge table, while the completion procedure, by splitting and combining its cycles, determines a unique path (the normal form) between any two vertices. It is therefore obvious that, as quoted by Gilman (1979), the coset enumeration is generally more efficient, cf. Cannon *et al.* (1973), for a detailed analysis of this algorithm. M. F. Newman (private communication) reports that the Canberra implementation of a Todd–Coxeter procedure produced a complete coset table in less than three minutes for the group $E_6$, while we could not complete this group. The main advantage of completion over enumeration is its ability to handle parametrised classes, thereby providing a solution to the word problem for entire classes of groups. Moreover, as we have seen in the case of Coxeter groups, infinite sets of rules could be described by complete presentations. We mention the work of Pedersen (1984) for another example of an infinite set of rules solving the free word problem for the groupoid variety $(x.xy)x = y$. Another prominent feature of complete presentations lies in the regularity of the irreducible words. Standard techniques of language theory can be applied, e.g. elimination of negation, rate of growth of the group (Gilman, 1979). Therefore, these two algorithms appear to be complementary, one being well-suited for isolated groups, the other for parametrised families. We should mention that we were unable to obtain complete presentations for the alternating groups, due to combinatorial explosion in the algorithm's running time. Besides this kind of failure, it may also happen that a parametrised complete set does not handle some values of the parameters, cf. the Coxeter and Dyck groups. In such cases of failure, other complete presentations may be found, as for example was done in Le Chenadec (1983) where complete sets are proposed for some polyhedral groups having generators of order 2. Finally, it seems reasonable to believe that a complete presentation exists for the most general Fuschian group, since we have been successful in determining complete presentations for the main subclasses of Fuschians groups (Zieschang *et al.*, 1980).

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