1. $A = \begin{pmatrix} 3 & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}$

$$A = \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}_{U} \underbrace{\begin{pmatrix} 3 & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}}_{\Sigma} \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}_{V^*}$$

$$\|A\|_F = \left(tr(A^*A)\right)^{1/2}$$
$$= \left(tr\left((U\Sigma V^*)^*(U\Sigma V^*)\right)\right)^{1/2}$$
$$= \left(tr\left(V\Sigma^*U^*U\Sigma V^*\right)\right)^{1/2}$$
$$= \left(tr\left(V\Sigma^*\Sigma V^*\right)\right)^{1/2}$$
$$= \left(tr\left(\Sigma^*\Sigma V^*V\right)\right)^{1/2}$$
$$= \left(tr\left(\Sigma^*\Sigma\right)\right)^{1/2}$$
$$= \sqrt{\sigma_1^2 + \sigma_2^2}$$

$$\Rightarrow \boxed{\|A\|_F = \sqrt{9 + \frac{1}{4}} = \sqrt{\frac{14}{4}} = \sqrt{3.5}}$$

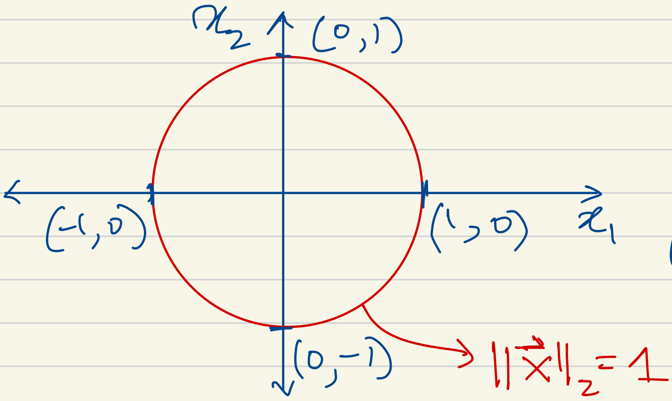
$$\|A\|_2 = \|U\Sigma V^*\|_2 = \|\Sigma\|_2 = \sigma_1$$

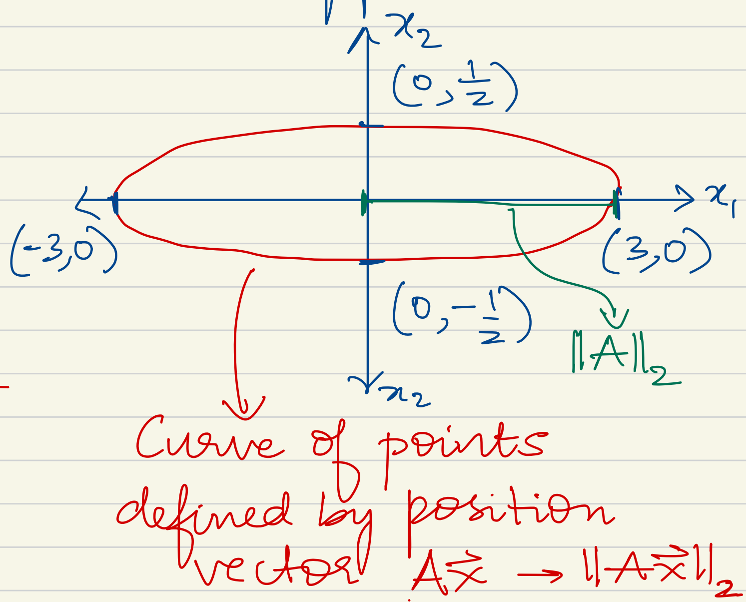
$$\Rightarrow \boxed{\|A\|_2 = 3}$$

$$A\vec{x} = x_1 \begin{bmatrix} 3 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ -\frac{1}{2} \end{bmatrix} \qquad \text{where } \vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$= \begin{bmatrix} 3x_1 \\ -\frac{x_2}{2} \end{bmatrix}$$

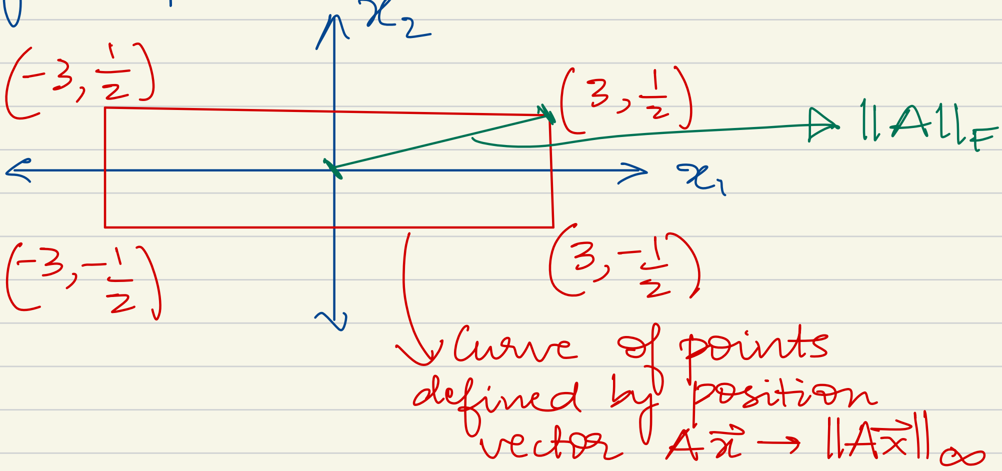
For all $\vec{x}$ such that $\|\vec{x}\|_2 = 1$, Ax triples the first dimension of $\vec{x}$ and halves the second dimension of $\vec{x}$ in the opposite direction.





$\|A\|_2 = 3$ and it geometrically signifies the length of the "longest" axis of the ellipse formed by the curve of points defined by position vector $A\vec{x}$ in 2 norm.

$\|A\|_F = \sqrt{3.5}$ and it geometrically signifies the length of the half diagonal of the rectangle defined by the curve of points defined by the position vector $A\vec{x}$ in the infinity norm.



2. Determine SVD for the following matrices by hand:

(a) $\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$

$$\boxed{A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}$$

(b) $A = \begin{pmatrix} 0 & 2 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$

$$AA^T = \begin{pmatrix} 0 & 2 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 2 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 4 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\det(AA^T - \lambda I) = (4-\lambda)\lambda^2 = 0$$

Eigenvalues of AA^T $\Rightarrow$ $\lambda_1 = 4$, $\lambda_2 = \lambda_3 = 0$

$$(AA^T - \lambda_1 I)\vec{u}_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & -4 \end{pmatrix} \begin{pmatrix} u_{11} \\ u_{12} \\ u_{13} \end{pmatrix} = \begin{pmatrix} 0 \\ -4u_{12} \\ -4u_{13} \end{pmatrix}$$

$$\vec{u}_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad \vec{u}_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$(AA^T - \lambda_2 I)\vec{u}_2 = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} u_{21} \\ u_{22} \\ u_{23} \end{pmatrix} = \begin{pmatrix} 4u_{21} \\ 0 \\ 0 \end{pmatrix}$$

$$\vec{u}_3 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$A^TA = \begin{pmatrix} 0 & 0 & 0 \\ 2 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 2 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 4 \end{pmatrix}$$

$$\det(A^TA - \lambda I) = (4-\lambda)(-\lambda) = 0$$

$$\Rightarrow \quad \lambda_1 = 4 \text{ and } \lambda_2 = 0$$

$$(A^TA - \lambda_1 I)\vec{v}_1 = \begin{pmatrix} -4 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} v_{11} \\ v_{12} \end{pmatrix} = \begin{pmatrix} -4v_{11} \\ 0 \end{pmatrix}$$

$$\vec{v}_1 = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$$

$$(A^TA - \lambda_2 I)\vec{v}_2 = \begin{pmatrix} 0 & 0 \\ 0 & 4 \end{pmatrix} \begin{pmatrix} v_{21} \\ v_{22} \end{pmatrix} = \begin{pmatrix} 0 \\ 4v_{22} \end{pmatrix}$$

$$\vec{v}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\boxed{A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}}$$

(c) $A = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$

$$AA^T = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\det(AA^T - \lambda I) = (2-\lambda)(-\lambda) = 0$$

$$\lambda_1 = 2 \qquad \lambda_2 = 0$$

$$(AA^T - 2I)\vec{u}_1 = \begin{pmatrix} 0 & 0 \\ 0 & -2 \end{pmatrix} \begin{pmatrix} u_{11} \\ u_{12} \end{pmatrix} = \begin{pmatrix} 0 \\ -2u_{12} \end{pmatrix} \qquad \vec{u}_1 = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$$

$$(AA^T - 0I)\vec{u}_2 = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} u_{21} \\ u_{22} \end{pmatrix} = \begin{pmatrix} 2u_{21} \\ 0 \end{pmatrix} \qquad \vec{u}_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$A^TA = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

$$\det(A^TA - \lambda I) = (1-\lambda)^2 - 1 = 0$$

$$\lambda_1 = 2 \qquad \lambda_2 = 0$$

$$(A^TA - 2I)\vec{v}_1 = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} v_{11} \\ v_{12} \end{pmatrix} = \begin{pmatrix} -v_{11} + v_{12} \\ v_{11} - v_{12} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\vec{v}_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$(A^TA - 0I)\vec{v}_2 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} v_{21} \\ v_{22} \end{pmatrix} = \begin{pmatrix} v_{21} + v_{22} \\ v_{21} + v_{22} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\vec{v}_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$\boxed{A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \sqrt{2} & 0 \\ 0 & 0 \end{pmatrix} \cdot \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}}$$

(d) $A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$

$$AA^T = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}$$

$$\det(AA^T - \lambda I) = (2-\lambda)^2 - 4 = 0$$

$$\lambda_1 = 4 \qquad \lambda_2 = 0$$

$$(AA^T - 4I)\vec{v}_1 = \begin{pmatrix} -2 & 2 \\ 2 & -2 \end{pmatrix} \begin{pmatrix} v_{11} \\ v_{12} \end{pmatrix} = \begin{pmatrix} -2v_{11} + 2v_{12} \\ 2v_{11} - 2v_{12} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\vec{v}_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$(AA^T - 0I)\vec{v}_2 = \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix} \begin{pmatrix} v_{21} \\ v_{22} \end{pmatrix} = \begin{pmatrix} 2v_{21} + 2v_{22} \\ 2v_{21} + 2v_{22} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\vec{v}_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$A^TA = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}$$

$$\Rightarrow \boxed{A = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} \cdot \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}}$$

3. Suppose $A \in \mathbb{C}^{m \times m}$ has $A = U\Sigma V^*$.

Then, $A^* = V\Sigma^* U^*$

Let $B = \begin{pmatrix} 0 & A^* \\ A & 0 \end{pmatrix}$

$$\begin{bmatrix} 0 & A^* \\ A & 0 \end{bmatrix} = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}\begin{bmatrix} A & 0 \\ 0 & A^* \end{bmatrix}$$

$$= \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}\begin{bmatrix} U\Sigma V^* & 0 \\ 0 & V\Sigma U^* \end{bmatrix}$$

$$= \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}\begin{bmatrix} U & 0 \\ 0 & V \end{bmatrix}\begin{bmatrix} \Sigma & 0 \\ 0 & \Sigma \end{bmatrix}\begin{bmatrix} V^* & 0 \\ 0 & U^* \end{bmatrix}$$

$$\begin{bmatrix} 0 & V \\ U & 0 \end{bmatrix}\begin{bmatrix} \Sigma & 0 \\ 0 & \Sigma \end{bmatrix}\begin{bmatrix} V^* & 0 \\ 0 & U^* \end{bmatrix}$$

We can substitute the LHS of the below equation for the RHS Matrix.

$$\begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}\begin{bmatrix} 0 & U^* \\ V^* & 0 \end{bmatrix} = \begin{bmatrix} V^* & 0 \\ 0 & U^* \end{bmatrix}$$

$$= \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} \begin{bmatrix} U & 0 \\ 0 & V \end{bmatrix} \begin{bmatrix} \Sigma & 0 \\ 0 & \Sigma \end{bmatrix} \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} \begin{bmatrix} 0 & U^* \\ V^* & 0 \end{bmatrix}$$

$$= \underbrace{\begin{bmatrix} 0 & V \\ U & 0 \end{bmatrix}}_{X} \underbrace{\begin{bmatrix} 0 & \Sigma \\ \Sigma & 0 \end{bmatrix}} \underbrace{\begin{bmatrix} 0 & U^* \\ V^* & 0 \end{bmatrix}}_{X^{-1}}$$

$$\begin{bmatrix} 0 & \Sigma \\ \Sigma & 0 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} I & -I \\ I & I \end{bmatrix} \begin{bmatrix} \Sigma & 0 \\ 0 & -\Sigma \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} I & I \\ -I & I \end{bmatrix}$$

$$B = \begin{bmatrix} 0 & A^* \\ A & 0 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 0 & V \\ U & 0 \end{bmatrix} \begin{bmatrix} I & -I \\ I & I \end{bmatrix} \begin{bmatrix} \Sigma & 0 \\ 0 & -\Sigma \end{bmatrix} \begin{bmatrix} I & I \\ -I & I \end{bmatrix} \begin{bmatrix} 0 & U^* \\ V^* & 0 \end{bmatrix}$$

$$\boxed{\begin{bmatrix} 0 & A^* \\ A & 0 \end{bmatrix} = \frac{1}{2} \underbrace{\begin{bmatrix} V & V \\ U & -U \end{bmatrix}}_{X} \underbrace{\begin{bmatrix} \Sigma & 0 \\ 0 & -\Sigma \end{bmatrix}}_{\Lambda} \underbrace{\begin{bmatrix} V^* & U^* \\ V^* & -U^* \end{bmatrix}}_{X^{-1}}}$$

4. P is an orthogonal projection. $\Rightarrow$ $P^2 = P$ & $P^* = P$

$(I-2P)^*(I-2P) = I - 2P - 2P^* + 4P^2 = I$

$\Rightarrow I - 2P$ is unitary.

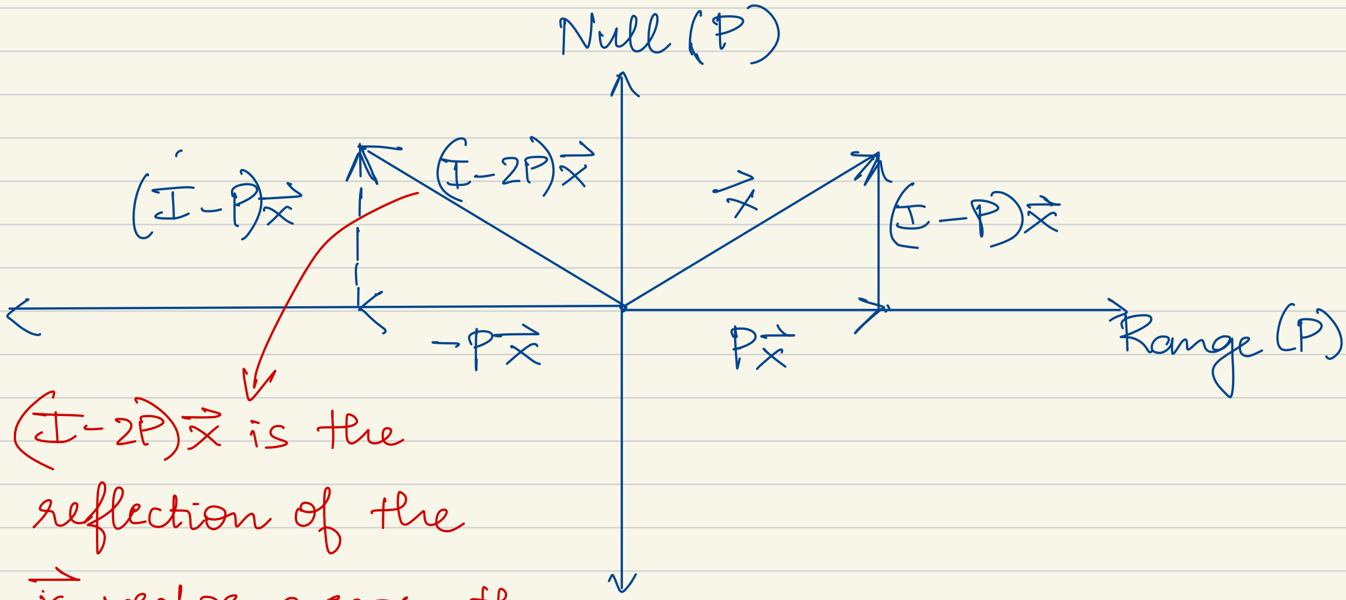
Geometric interpretation of $I-2P$:

When P is applied on a vector $\vec{x}$, $P\vec{x} \in \text{Range}(P)$

$(I-P)\vec{x} \in \text{Null}(P)$.

$(I-2P)\vec{x}$ can be expressed as $(I-P)\vec{x} + (-P\vec{x})$

Thus, we can represent $(I-2P)\vec{x}$ geometrically as:



$(I-2P)\vec{x}$ is the reflection of the $\vec{x}$ vector across the hyperplane defined by the nullspace of P.

$\Rightarrow (I-2P)$ is a reflector.

5. $x \in \mathbb{R}^m$: $Ex = \frac{x + Fx}{2} = \frac{1}{2}(I + F)x$

$$F \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{pmatrix} = \begin{pmatrix} x_m \\ x_{m-1} \\ \vdots \\ x_1 \end{pmatrix} \Longrightarrow F = \begin{bmatrix} 0 & 0 & 0 & \cdots & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & \cdots 1 & 0 & 0 \\ \vdots & & & & & \\ 1 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix}$$

$$F_{ij} = \begin{cases} 1 & \text{if } i+j = m+1 \\ 0 & \text{otherwise} \end{cases}$$

$$E = \left(\frac{I + F}{2}\right) \Rightarrow E_{ij} = \frac{1}{2}\begin{cases} 1 + f_{ij} & \text{if } i == j \\ 0 + f_{ij} & \text{otherwise} \end{cases}$$

If m is odd

$$E = \frac{1}{2}\begin{bmatrix} 1 & 0 & \cdots & \cdots & 0 & 1 \\ 0 & 1 & \cdots & \cdots & 1 & 0 \\ & & \ddots & \iddots & & \\ & & & 2 & & \\ & & \iddots & \ddots & & \\ 0 & 1 & & & 1 & 0 \\ 1 & 0 & & & 0 & 1 \end{bmatrix}$$

If m is even

$$E = \frac{1}{2}\begin{bmatrix} 1 & 0 & \cdots & \cdots & 0 & 1 \\ 0 & 1 & \cdots & \cdots & 1 & 0 \\ & & \ddots & \iddots & & \\ & & \iddots & \ddots & & \\ 0 & 1 & \cdots & \cdots & 1 & 0 \\ 1 & 0 & \cdots & \cdots & 0 & 1 \end{bmatrix}$$

Note $F^2\vec{x} = \vec{x}$ since F reverses the order of elements in $\vec{x}$ and applying F twice gives the original $\vec{x}$. $\Rightarrow \boxed{F^2 = I}$

$$E^2 = \frac{1}{4}(I + 2F + F^2) = \frac{1}{2}(I + F) = E.$$

$\boxed{\Rightarrow E \text{ is a projection matrix.}}$

Also, since $F^* = F$, $E^* = E \Rightarrow$ $\boxed{E \text{ is an othogonal projector.}}$

6. Given: Matrix A with $\langle a_i, a_j \rangle = 0$ if $i \in \{1,3,5,7,\ldots\}$ and $j \in \{2,4,6,8,\ldots\}$

Also, A is full rank.

In order to compute the reduced QR factorisation of A, we will use the Gram-Schmidt Algorithm.

An entry of the R matrix R_{ij} is computed as

$R_{ij} = \langle q_i^*, a_j \rangle$ and each orthogonal vector is

$$q_j = \frac{a_j - \sum_{i=1}^{j-1} R_{ij} q_i}{R_{jj}}$$

We know that q_1 is parallel to $a_1 \Rightarrow R_{1j} = 0$ for all $j \in \{2,4,6,8,\ldots\}$.

Similarly, $q_2 = \frac{a_2 - R_{11} a_1}{R_{22}}$ and so on...

So, for any q_j where $j \in \{2,4,6,8,\ldots\}$, we can see $\langle q_j^*, a_k \rangle = 0$ when $k \in \{1,3,5,7,\ldots\}$ (can be shown by induction). $\Rightarrow R_{jk} = 0$ if j is even and k is odd or vice versa.

$\Rightarrow$ R matrix has the following structure:

$$R = \begin{bmatrix} R_{11} & 0 & R_{13} & 0 & R_{15} & 0 & \cdots \\ & R_{22} & 0 & R_{24} & 0 & R_{26} & \cdots \\ & & R_{33} & 0 & R_{35} & 0 & \cdots \\ & 0 & & \ddots & & & \\ & & & & \ddots & & \\ & & & & & \ddots & \\ & & & & & & \ddots \end{bmatrix}$$

$\rightarrow$ This is a striped upper triangular matrix or checkerboard matrix.