(1a) $-u''(x) = f(x) = \sin(4\pi x)$

$$-\int u''(x)\cdot dx = \int \sin(4\pi x)\cdot dx$$

$$-u'(x) = \frac{\cos(4\pi x)}{4\pi} + C_0$$

$$-\int u'(x)\cdot dx = \frac{\cos(4\pi x)}{4\pi} + C_0$$

$$-u(x) = \frac{-\sin(4\pi x)}{(4\pi)^2} + C_0 x + C_1$$

$$\Rightarrow u(x) = \frac{\sin(4\pi x)}{(4\pi)^2} - C_0 x - C_1$$

Using periodic boundary conditions, we have $u(0) = u(1)$

$$\Rightarrow -C_1 = -C_0 - C_1 \Rightarrow C_0 = 0.$$

$$\boxed{\therefore u(x) = \frac{\sin(4\pi x)}{(4\pi)^2} - C_1}$$

The exact solution with mean 0 is when $C_1 = 0$.

$$\boxed{u(x) = \frac{\sin(4\pi x)}{(4\pi)^2}}$$

(1c) $A = \frac{1}{h^2}\begin{bmatrix} 2 & -1 & 0 & & & -1 \\ -1 & 2 & -1 & & & 0 \\ 0 & -1 & 2 & -1 & & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 \\ & & & & & \\ -1 & 0 & 0 & 0 \cdots & -1 & 2 \end{bmatrix}$

- We see that the sum of all rows/columns of A is 0, which implies that columns of A are not linearly independent $\Rightarrow A$ is not full rank.

Note that any $c \cdot e^T = c \cdot [1\ 1 \cdots 1]^T$ belongs to null(A) for any constant $c \Rightarrow c \cdot Ae = 0$.
or $Ae = 0 \Rightarrow e$ is the basis for the nullspace of A.
The nullspace is not 2 dimensional since we only have 1 degree of freedom [ie, only one of u_i's can be varied without an effect on the other u_i's. This can also be seen from the exact solution, $u_i = u(x_i) = \frac{\sin(4\pi x_i)}{(4\pi)^2} + C_1$.

If we choose $u_{m-1} = u(x_{m-1}) = \frac{\sin(4\pi(m-1))}{(4\pi)^2} + C_1 = k$

then, we can eliminate the last row & column of A, making it a full rank matrix.
$\Rightarrow$ Nullspace of A is 1 Dimensional.

(3a) We know the following:

$$(k+1)P_{k+1} = (2k+1)xP_k - kP_{k-1}$$

$$(k+1)P'_{k+1} = (2k+1)xP'_k + (2k+1)P_k - kP'_{k-1}$$

$$(k+1)P''_{k+1} = (2k+1)xP''_k + 2(2k+1)P'_k - kP''_{k-1}$$

We want to show:

$$(1-x^2)P''_{n+1} - 2xP'_{n+1} + (n+2)(n+1)P_{n+1} = 0.$$

$\boxed{\text{Answer:}}$ We show this using induction and one other identity, given by

$$P'_{n+1} - P'_{n-1} = (2n+1)P_n$$

Base Case : $P_0 = 1$ $P_1 = x$ $P_2 = \frac{1}{2}(3x^2-1)$

$k=1$:

$$(1-x^2)P''_1 - 2xP'_1 + 2P_1 = -2x + 2x = 0.$$

$k=2$:

$$(1-x^2)P''_2 - 2xP'_2 + 6P_2 = (1-x^2)3 - 2x(3x) + 9x^2 - 3$$
$$= 0$$

Now assume $(1-x^2)P''_p - 2xP'_p + p(p+1)P_p = 0$ for all $p \le n$.

Consider the following expression for $n+1$:

$$\left((1-x^2)P'_{n+1}\right)'$$
$$= \left[(1-x^2)\left(P'_{n-1} + (2n+1)P_n\right)\right]'$$
$$= \left((1-x^2)P'_{n-1}\right)' + (2n+1)\left(-2xP_n + (1-x^2)P'_n\right)$$

$$= -(n-1)n P_{n-1} - 2\left[(n+1)P_{n+1} + nP_{n-1}\right] + n(n+1)\left(P_{n-1} - P_{n+1}\right)$$

$$= \left(-(n-1)n - 2n + n(n+1)\right) P_{n-1} + \left(-2(n+1) - n(n+1)\right) P_{n+1}$$

$$= n\left[-(n-1) - 2 + (n+1)\right] P_{n-1} - (n+1)(n+2) P_{n+1}$$

$$= -(n+1)(n+2) P_{n+1}$$

$$\therefore \left((1-x^2) P'_{n+1}\right)' = (1-x^2) P''_{n+1} - 2x P'_{n+1}$$

$$= -(n+1)(n+2) P_{n+1}$$

$$\boxed{\Rightarrow (1-x^2) P''_{n+1} - 2x P'_{n+1} + (n+1)(n+2) P_{n+1} = 0.}$$

(3b) We know the following:

$T_{k+1} = 2xT_k - T_{k-1}$ $\qquad$ $(1-x^2)T_k'' - xT_k' + k^2 T_k = 0$

$T_{k+1}' = 2xT_k' + 2T_k - T_{k-1}'$ $\qquad$ $(1-x^2)T_k'' = xT_k' - k^2 T_k$

$T_{k+1}'' = 2xT_k'' + 4T_k' - T_{k-1}''$ $\qquad$ $(1-x^2)T_{k-1}'' = xT_{k-1}' - (k-1)^2 T_{k-1}$

We want to prove this: $(1-x^2)T_{k+1}'' = xT_{k+1}' - (k+1)^2 T_{k+1}$

Answer: We can use induction to prove this.

Base case: $T_0 = 1$ $\quad T_1 = x$ $\quad T_2 = 2x^2 - 1$

$k=1$:

$$(1-x^2)T_1'' - xT_1' + T_1 = -x + x = 0.$$

$k=2$:

$$(1-x^2)T_2'' - xT_2' + 4T_2$$
$$= (1-x^2)(4) - x(4x) + 4(2x^2-1)$$
$$= 4 - 4x^2 - 4x^2 + 8x^2 - 4$$
$$= 0.$$

Now, assume $(1-x^2)T_p'' - xT_p' + k^2 T_p = 0$ is true for all $p < k$.

We show that it is true for $k+1$.

$$(1-x^2)T_{k+1}''$$
$$= (1-x^2)\left[2xT_k'' + 4T_k' - T_{k-1}''\right]$$
$$= 2x\left[xT_k' - k^2 T_k\right] + 4(1-x^2)T_k' - \left[xT_{k-1}' - (k-1)^2 T_{k-1}\right]$$
$$= -2x^2 T_k' + 4T_k' - xT_{k-1}' - 2xk^2 T_k + (k-1)^2 T_{k-1}$$

Adding $-xT_{k+1}' + (k+1)^2 T_{k+1}$, we have

$$-2x^2T_k' + 4T_k' - x\left(T_{k-1}' + T_{k+1}'\right) - 2xk^2T_k + (k-1)^2T_{k-1} + (k+1)^2T_{k+1}$$

$$= -2x^2T_k' + 4T_k' - x\left(2xT_k' + 2T_k\right) - k^2\left(2xT_k - T_{k-1} - T_{k+1}\right) - 2kT_{k-1} + 2kT_{k+1} + T_{k-1} + T_{k+1}$$

$$= 4(1-x^2)T_k' - \cancel{2xT_k} - 2kT_{k-1} + 2k\left(2xT_k - T_{k-1}\right) + \cancel{2xT_k}$$

$$= 4(1-x^2)T_k' + 4kxT_k - 4kT_{k-1}$$

In order to simplify this, we use the identity that

$$(1-x^2)T_k' = kT_{k-1} - kxT_k$$

$$\boxed{\Rightarrow (1-x^2)T_{k+1}'' - xT_{k+1}' + (k+1)^2T_{k+1} = 0.}$$

(3c) $\{p_n\}$ family of orthogonal polynomials.

To get p_{n+1}, we can orthogonalise $x p_n$ using Gram Schmidt orthogonalisation.

$$p_{n+1} = x p_n - \sum_{k=0}^{n} \frac{\langle x p_n, p_k \rangle}{\langle p_k, p_k \rangle} p_k$$

Consider $\langle p_k, x p_n \rangle = \int_a^b x \cdot p_k \cdot p_n \cdot dx$

$$= \langle p_n, x p_k \rangle.$$

$x p_k$ is a polynomial of degree $\leq k+1$.

$\Rightarrow \quad x p_k = \sum_{j=0}^{k+1} c_j p_j$

$\Rightarrow \langle p_n, x p_k \rangle = \sum_{j=0}^{k+1} c_j \langle p_n, p_j \rangle$

We can see that when $k < n-2$, $k+1 < n-1$,

$\sum_{j=0}^{k+1} c_j \langle p_n, p_j \rangle = 0.$ ie, $\langle p_n, x p_k \rangle = 0$ if $k < n-2$

$$\Rightarrow \quad p_{n+1} = x p_n - \sum_{k=0}^{n} \frac{\langle p_n, x p_k \rangle}{\langle p_k, p_k \rangle} p_k$$

$$= x p_n - \frac{\langle p_n, x p_n \rangle}{\langle p_n, p_n \rangle} p_n - \frac{\langle p_n, x p_{n-1} \rangle}{\langle p_{n-1}, p_{n-1} \rangle} p_{n-1}$$

$$\boxed{\Rightarrow p_{n+1} = \left(x - \frac{\langle p_n, x p_n \rangle}{\langle p_n, p_n \rangle} \right) p_n - \frac{\langle p_n, x p_{n-1} \rangle}{\langle p_{n-1}, p_{n-1} \rangle} p_{n-1}}$$

$\therefore$ There is a 3-term recurrence relation for a family of orthogonal polynomials.

Problem 4:

$\{p_n\}$ family of monic orthogonal polynomials

Consider q: a monic polynomial of degree n.

$$q = p_n + \sum_{k=0}^{n-1} a_k p_k$$

$$\langle q, q\rangle = \langle p_n + \sum_{k=0}^{n-1} a_k p_k, p_n + \sum_{k=0}^{n-1} a_k p_k\rangle$$

$$= \langle p_n, p_n\rangle + \underbrace{2\langle \sum_{k=0}^{n-1} a_k p_k, p_n\rangle}_{=0}$$

$$+ \langle \sum_{k=0}^{n-1} a_k p_k, \sum_{k=0}^{n-1} a_k p_k\rangle$$

$$= \langle p_n, p_n\rangle + \sum_{k=0}^{n-1} \sum_{l=0}^{n-1} a_k a_l \underbrace{\langle p_k, p_l\rangle}_{=\delta_{kl}}$$

$$= \langle p_n, p_n\rangle + \sum_{k=0}^{n-1} a_k^2 \langle p_k, p_k\rangle$$

We know $\langle p_k, p_k\rangle > 0 \quad \forall\ k = 0, 1, \cdots n-1$.

$\therefore \langle q, q\rangle = \langle p_n, p_n\rangle + C$ where $C \geq 0$ for all n.

$\Rightarrow \langle q, q\rangle \geq \langle p_n, p_n\rangle$.

The equality is achieved when $q = p_n$. or $\sum_{k=0}^{n-1} a_k p_k = 0$.