Question 1:

$$Au = \frac{1}{h^2}\begin{bmatrix} 2 & -1 & 0 & & \\ -1 & 2 & -1 & & \\ & -1 & 2 & -1 & \\ & & \ddots & \ddots & \\ & & & -1 & 2 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_{m-1} \end{bmatrix} = \begin{bmatrix} f_0 \\ f_1 \\ \vdots \\ f_{m-1} \end{bmatrix}$$

$u_0 = 0 \quad u_m = 0$

(a) Find the eigenvalues of A.

Let $\vec{v}$ be an eigenvector of A. Applying A on $\vec{v}$ gives us the relation $A\vec{v} = \lambda\vec{v}$.

$$\Rightarrow \frac{1}{h^2}\left(-v_{k+1} + 2v_k - v_{k-1}\right) = \lambda v_k$$

(v_k is the k^{th} component of the vector $\vec{v}$).

or $\quad (2 - h^2\lambda)v_k = v_{k+1} + v_{k-1}$

Let $2\mu = 2 - h^2\lambda$

So we have $\quad v_{k+1} = 2\mu v_k - v_{k-1}$ ——— $\circledast$

This is similar to the Chebyshev polynomial of the 2^{nd} kind
$$U_{k+1}(z) = 2zU_k(z) - U_{k-1}(z)$$
$$U_0(z) = 1 \quad U_1(z) = 2z.$$

If we consider μ to be unknown in $\circledast$, then we have
$$U_{n+1}(\mu) = v_m = 0 \text{ (Boundary condition).}$$

So μ can be considered as the roots of the Chebyshev polynomial $U_k(\mu) = \dfrac{\sin((k+1)\cos^{-1}(\mu))}{\sqrt{1-\mu^2}}$

The roots of this polynomial are given by
$$\mu_k = \cos\left(\frac{k\pi}{m}\right) \quad \left[\text{The } k^{th} \text{ root is given by } \mu_k\right].$$

So, we have $2\mu_k = 2 - h^2\lambda_k$

$$2\cos\left(\frac{k\pi}{n+1}\right) = 2 - h^2\lambda_k$$

$$h^2\lambda_k = 2 - 2\cos\left(\frac{k\pi}{m}\right)$$

$$\lambda_k = \frac{2}{h^2}\left[1-\cos\left(\frac{k\pi}{m}\right)\right]$$

$$\boxed{\lambda_k = \frac{4}{h^2}\left[\sin^2\left(\frac{k\pi}{2m}\right)\right]}$$

So the k^{th} eigenvalue of A is given as $\underline{\underline{\frac{4}{h^2}\sin^2\left(\frac{k\pi}{2m}\right)}}$.

(b) We know A is symmetric since the elements on either side of the diagonal are equal.

ie, $a_{ij} = a_{ji}$ $\forall$ $i \neq j$ and $i, j \in \{1, 2 \cdots m\}$.

Further the eigenvalues of A given by

$\lambda_k = \frac{4}{h^2}\sin^2\left(\frac{k\pi}{2(n+1)}\right)$ is always positive (since

$\sin^2(x) > 0$ $\forall$ x.

A matrix is positive definite iff all its eigenvalues are positive.

$\Rightarrow$ The matrix A is symmetric positive definite.

(c) 2-norm

$$\|A\|_2^2 = \sup_{\substack{x \in \mathbb{R} \\ \|x\|_2 = 1}} \|Ax\|_2^2 = \sup_{\substack{x \in \mathbb{R} \\ \|x\|_2 = 1}} x^T A^T A x$$

If $v_1, \cdots v_m$ are the m eigenvectors of A, we know that v_i and v_j are orthogonal $(i \neq j)$ $\forall$ i, j.

(since A is symmetric).

We can write $x = \sum_{i=1}^{m} a_i v_i \Rightarrow Ax = \sum_{i=1}^{m} a_i \lambda_i v_i$

$x^T A^T A x = \sum_{i=1}^{m} a_i^2 \lambda_i^2$ (Since v_i & v_j are orthonormal $\forall\ 1 \le i, j \le m$)

$$\therefore \|A\|_2^2 = \sup_{\substack{x \in \mathbb{R} \\ \|x\|_2 = 1}} x^T A^T A x = \sup_{\substack{x \in \mathbb{R} \\ \|x\|_2 = 1}} \sum_{i=1}^{m} a_i^2 \lambda_i^2$$

$$\le \lambda_{max}^2 \sum_{i=1}^{m} a_i^2$$

Since $\|x\|_2 = 1 \Rightarrow x^T x = 1 \Rightarrow \sum_{i=1}^{m} \sum_{j=1}^{m} (a_i v_i^T)(a_j v_j) = \sum_{i=1}^{m} a_i^2 = 1$

$$\therefore \|A\|_2^2 = \sup_{\substack{x \in \mathbb{R} \\ \|x\|_2 = 1}} x^T A^T A x \le \lambda_{max}^2$$

$$\Rightarrow \boxed{\|A\|_2 = \lambda_{max} = \rho(A)}$$

Since the eigenvalues of A are given by

$$\lambda_k = \frac{4}{h^2} \left[\sin^2\left(\frac{k\pi}{2m} \right) \right]$$

$$\lambda_{max} = \frac{4}{h^2} \sin^2\left(\frac{(m-1)\pi}{2m} \right)$$

$$\boxed{\Rightarrow \|A\|_2 = \frac{4}{h^2} \sin^2\left(\frac{(m-1)\pi}{2m} \right)}$$

Frobenius norm:

$$\|A\|_F = tr(A^TA) \qquad (A = V\Lambda V^T \text{ since } A \text{ is s.p.d})$$

$$= tr(V\Lambda V^T V\Lambda V^T)$$

$$= tr(V\Lambda^2 V^T) = tr(\Lambda^2) \qquad \text{(cyclic shift inside trace)}$$

$$\Rightarrow \boxed{\|A\|_F = \sum_{k=1}^{m-1} \frac{4}{h^2}\sin^2\left(\frac{k\pi}{2m}\right)}$$

Condition number:

$$K(A) = \frac{\lambda_{max}(A)}{\lambda_{min}(A)} = \sin^2\left(\frac{(m-1)\pi}{2m}\right) \cdot \frac{1}{\sin^2\left(\frac{\pi}{2m}\right)}$$

Using Taylor expansion for $\sin^2 x = x^2 - \frac{x^4}{3} + O(x^6)$

As m increases, $\frac{1}{m} \to 0. \Rightarrow h \to 0.$

$$\Rightarrow \sin^2\left(\frac{(m-1)\pi}{2m}\right) = \frac{(m-1)^2\pi^2}{(2m)^2} + O\left(\frac{1}{m^4}\right) \approx \frac{\cancel{m^2}\pi^2}{4\cancel{m^2}} - \frac{2\cancel{m}\pi^2}{4m^{\cancel{2}}} + \frac{\pi^2}{4m^2}$$

$$= \frac{\pi^2}{4} - \frac{\pi^2}{2m} + \frac{\pi^2}{4m^2} \approx \frac{\pi^2}{4}$$

$$\sin^2\left(\frac{\pi}{2m}\right) = \frac{\pi^2}{4m^2} + O(m^4) \approx \frac{\pi^2}{4m^2} \approx \frac{h^2\pi^2}{4}$$

$$\Rightarrow K(A) = O\left(\frac{1}{h^2}\right) = O(m^2)$$

$$\boxed{\text{For } K(A) > 10^8, \quad m > 10^4}$$

Problem 1d

The eigenvalues of A are given by

$\lambda_k = 4/h^2 * sin^2(k * \pi / (2m))$ $for\ k = 1, 2, 3 .... m - 1$

To get an eigenvalue of 0, we need to substitute k = 0 or m. But this corresponds to the boundary condition, which is already known (u0 = um = 0).

Further, if 0 were an eigenvalue for A, then A would not be invertible, which is not the case for A => contradiction! Hence, 0 cannot be an eigenvalue for A.

Problem 1e

The residual norms for the different values of h are:

- h = 0.001, res = 2.5870416919815398e-12
- h = 0.0005, res = 1.2190581877291606e-11
- h = 0.00025, res = 4.2633674368630636e-11

The convergence rate = 4.0006112759774055 (~= 4)

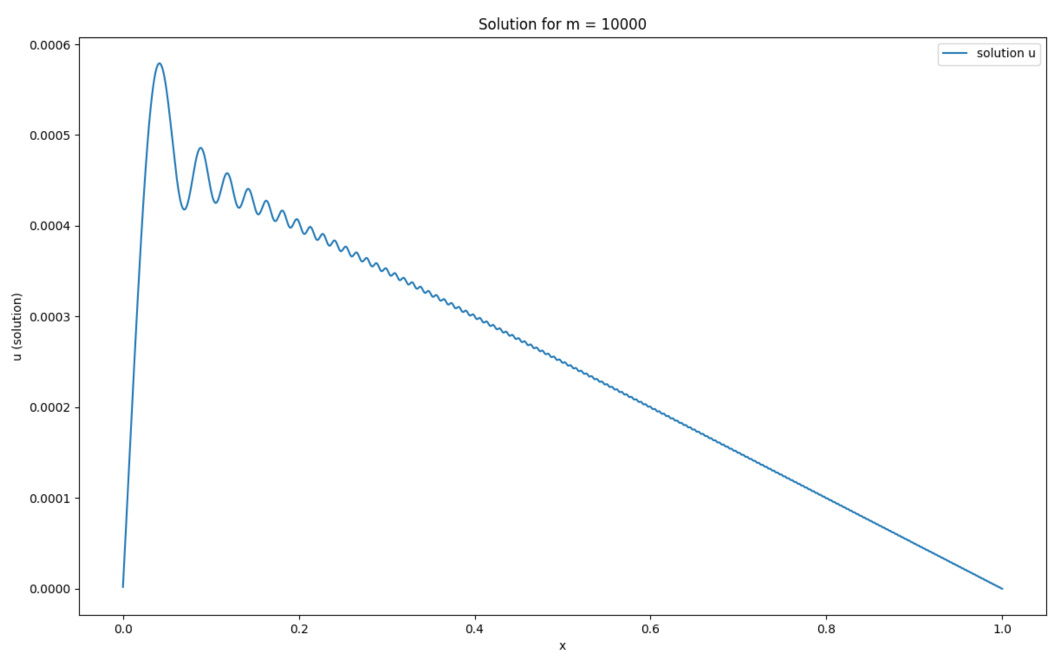
Problem 1f

f = sin(10000*x*x)

m = 10000

Time taken for solve = 4.859262943267822 s

The plot of the solution is:



Problem 1g

Thomas algorithm

Thomas algorithm is a simplified form of Gaussian elimination that can be used to solve tridiagonal systems of equations. For a tridiagonal system with n equations, the solution can be obtained in O(n) time using the Thomas algorithm as opposed to the O(n^3) time complexity required by Gaussian elimination. In this algorithm, the values below the diagonal are first eliminated, followed by a back substitution using the resulting upper triangular matrix to solve the system.

Problem 1h

f = sin(1000*x*x)

m = 100000

Residual norm for sparse solve = 2.718582026162153e-09

Time taken for sparse solve = 0.04828000068664551 s

The 1D laplace matrix A has eigenvectors where each element is a sin function. And the eigenvalues are multiples of the sin^2 function.

$$\lambda_k = 4/h^2 * sin^2(k * \pi / (2m)) \; for \; k = 1, 2, 3 .... m$$

When m increases, the m eigenvalues and eigenvectors are not estimated with high accuracy for the higher modes.

In this problem, since f is a sin function. So solving Au=f can be interpreted as estimating the eigenvalues of A for the eigenvector f. But since the value of m is very high, the frequency of f is very high. So estimating eigenvalues for higher modes (ie, higher values of k) is not very accurate. So, since the accuracy suffers, the value of R is not very informative with large values in this case.

Code for Problem1:

```
import numpy as np
from scipy.sparse import csc_matrix, csr_matrix, diags
from scipy.sparse.linalg import spsolve
import time
import matplotlib.pyplot as plt

def construct_1D_laplace(m):
   h = 1./m
   diagonal = 2. * np.ones(m-1)
   off_diag = -1. * np.ones(m-2)
   return (1./(h*h)) * (np.diag(diagonal, 0) + np.diag(off_diag, 1) +
```

```
np.diag(off_diag, -1))

def sample_points_1D(m):
   h = 1./m
   x = np.linspace(h, 1-h, num=m-1)
   return x, np.zeros_like(x)

def get_rhs(x, const):
   return np.sin(const*x*x)

def compute_residual_norm(A, u, f):
   return np.linalg.norm(f - np.dot(A, u), np.inf)

def direct_solve(A, f):
   return np.linalg.solve(A, f)

def build_sparse_A(m):
   h = 1./m
   A_sparse = csc_matrix(diags([-1, 2, -1], [-1, 0, 1], shape=(m-1, m-1)))/(h**2)
   return A_sparse

def sparse_solve(A_sparse, f):
   return spsolve(A_sparse,f)

def verify_eigen_values(A):
   n = A.shape[0]
   h = 1./(n+1)
   v_num, _ = np.linalg.eig(A)
   v_num = np.sort(v_num)
   k = np.arange(1, n+1)
   v_an = (4/(h*h)) * np.sin((k*np.pi)/(2*n+2)) * np.sin((k*np.pi)/(2*n+2))
   print("Inf Norm of diff in eigenvalues for m = {} is {}".format(n+1,
np.linalg.norm(v_an-v_num, np.inf)))

m = 8
A = construct_1D_laplace(m)
verify_eigen_values(A)

ms = [1000, 2000, 4000]
rn = []
for m in ms:
```

```
    const = 100
    A = construct_1D_laplace(m)
    x, u = sample_points_1D(m)
    f = get_rhs(x, const)
    u_ds = direct_solve(A, f)
    rn.append(np.linalg.norm(u_ds, np.inf))
    print("\t h = {}, res = {}".format( 1./m, compute_residual_norm(A, u_ds, f)))

print("Convergence rate = {}".format((rn[0]-rn[1])/(rn[1]-rn[2])))

m = 10000
const = 1000
A = construct_1D_laplace(m)
x, u = sample_points_1D(m)
f = get_rhs(x, const)
start = time.time()
u_ds = direct_solve(A, f)
end = time.time() - start
print("Time taken for solve = {}".format(end))

plt.plot(x, u_ds, label='solution u')
plt.title("Solution for m = {}".format(m))
plt.ylabel("u (solution)")
plt.xlabel("x")
plt.legend()
plt.show()

m = 100000
const = 1000

A_sparse = build_sparse_A(m)
x, u = sample_points_1D(m)
f = get_rhs(x, const)

start = time.time()
u_ss = sparse_solve(A_sparse, f)
end = time.time() - start
print("Residual norm for sparse solve = {} for m =
{}".format(np.linalg.norm(f-A_sparse.dot(u_ss), np.inf), m))
print("Time taken for sparse solve = {} for m = {}".format(end, m))
```

Question 2: Gershgorin circle Theorem

If $A \in \mathbb{C}^{n\times n}$ and $r_i(A) = \sum_{i\neq j} |a_{ij}|$, every eigenvalue of A lies within atleast one Gershgorin disc.

Proof: Let λ be an eigenvalue of A and $\vec{x}$ be the corresponding non-zero eigenvector.

$\Rightarrow A\vec{x} = \lambda \vec{x}$

or $$\sum_{j=1}^{n} a_{ij} x_j = \lambda x_i$$

Subtracting $a_{ii} x_i$ from both sides,

$$\sum_{j\neq i} a_{ij} x_j = \lambda x_i - a_{ii} x_i = (\lambda_i - a_{ii}) x_i$$

Since $\vec{x} \neq 0$, we know there is some $k < n$ such that

$$0 \leq |x_k| = \max\{|x_i| : 1 \leq i \leq n\}$$

So, $$\sum_{j=1}^{n} a_{kj} x_j = \lambda x_k \Rightarrow \sum_{\substack{j=1 \\ j\neq k}}^{n} a_{kj} x_j = (\lambda_k - a_{kk}) x_k$$

Using Triangle Inequality,

$$|\lambda - a_{kk}||x_k| = \sum_{j\neq k} |a_{kj}||x_j| \leq |x_k| \sum_{j\neq k} |a_{kj}|$$
$$= |x_k| \, r_k(A)$$

Since $|x_k| > 0$,

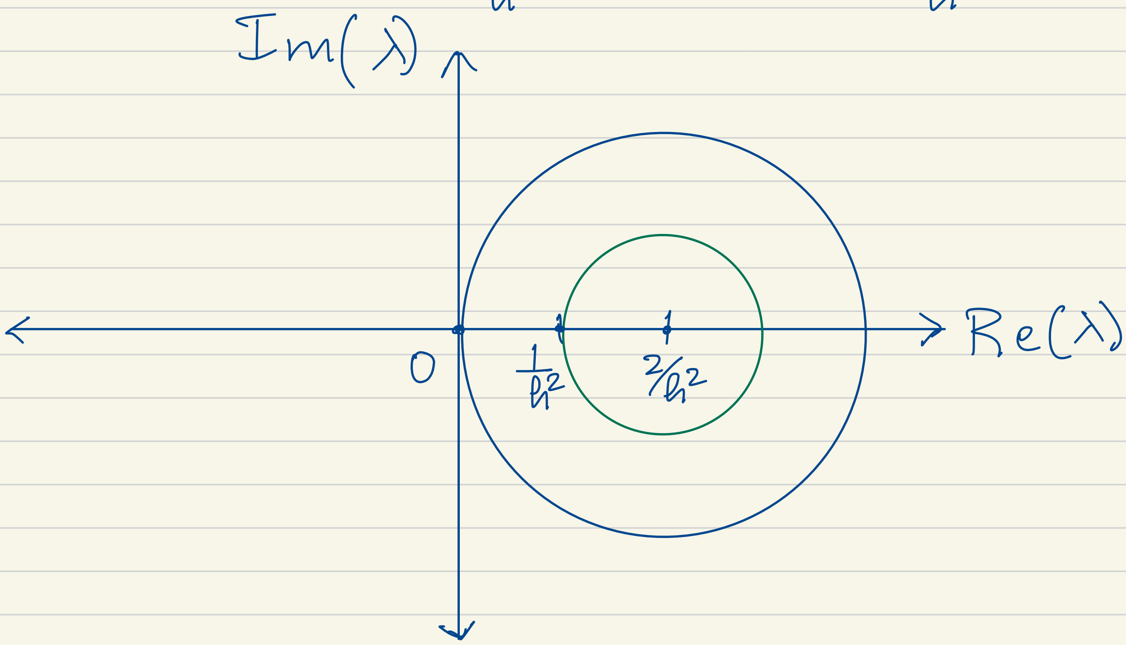
$$\boxed{|\lambda - a_{kk}| \leq r_k(A)}$$

This equation shows that the eigenvalue lies inside a disc centered at a_{kk} with radius $r_k(A) = \sum_{j\neq k} a_{kj}$.

For the given A matrix (1D laplace), the eigenvalues are given as

$$\lambda_k = \frac{4}{h^2}\sin^2\left(\frac{k\pi}{2m}\right) \quad k = 1, 2, \cdots m.$$

Using Gershgorin theorem, we get 2 distinct discs, one centered at $\frac{2}{h^2}$ with radius $\frac{2}{h^2}$ and another centered at $\frac{2}{h^2}$ with radius $\frac{1}{h^2}$.



We know $0 \le \sin^2(x) \le 1$. $\frac{4}{h^2}$ is the diameter of the largest Gershgorin disc.

$$\Rightarrow \quad 0 \le \frac{4}{h^2}\sin^2\left(\frac{k\pi}{2m}\right) \le \frac{4}{h^2} \Rightarrow \lambda_k \text{ lies with}$$

atleast one Gershgorin disc of A $\forall$ $k = 1, 2, \cdots m$.

Question 3:

A is a non-hermitian matrix. $A \neq A^*$, $A \in \mathbb{C}^{m \times m}$

Consider $r(x) = \frac{\langle x, Ax \rangle}{\langle x, x \rangle}$. Let (λ, v) be the eigenvalue & eigenvector pair of A. (Assume $\|v\|_2 = 1$)

We use Taylor expansion for $r(x)$ as shown:

$$r(x) = r(v) + \nabla r(v)^* (x-v) + \frac{1}{2} (x-v)^* H_r(v) (x-v) + O((x-v)^3)$$

Here $H_r(v)$ = hessian of $r(x)$.

$$r(x) = \frac{\langle x, Ax \rangle}{\langle x, x \rangle}$$

$$\nabla x^* A x = (A^* + A) x$$

$$\nabla x^* x = 2x$$

$$\nabla r(x) = \frac{(A^* + A)x}{x^* x} - \frac{2(x^* A x) x}{(x^* x)^2}$$

$$= \frac{(A^* + A)x}{x^* x} - \frac{2 r(x) x}{(x^* x)}$$

$$= \frac{1}{x^* x} \left(A^* x + Ax - 2 r(x) x \right)$$

$$\nabla r(v) = \frac{1}{v^* v} \left(A^* v + Av - 2 \lambda v \right) = A^* v - Av$$

$$\therefore r(x) = r(v) + (A^* v - Av)^* (x - v) + \text{H.O.T}$$

$$\boxed{r(x) = r(v) + (v^* A - v^* A^*)(x - v) + \text{H.O.T}}$$

If $\|x - v\| \leq \varepsilon$, then $r(x) - r(v) = O(\varepsilon)$

So, we have that $(r(x) - r(v)) = O(\|x - v\|)$

$\Rightarrow$ The accuracy of Rayleigh Quotient Iteration on a non-Hermitian matrix is linear.

When one Rayleigh Quotient Iteration is carried out to compute $x^{(k)}$ from $x^{(k-1)}$ and $\lambda^{(k)}$ from $x^{(k)}$, we can see that the convergence is <u>Quadratic</u>.

$$\|x^{(k)} - v\| = O\left(|\lambda^{(k)} - \lambda| \, \|x^{(k-1)} - v\|\right)$$

$$= O(\varepsilon^2) \qquad \text{if } \|x^{(k-1)} - v\| = O(\varepsilon)$$

Question 4:

$A \in \mathbb{C}^{m\times m}$.

$$W(A) = \{\langle x, Ax\rangle : x \in \mathbb{C}^m \text{ with } \|x\| = 1\}$$

(We can assume $\|x\| = 1$, without loss of generality).

(a) Show that $W(A)$ contains the convex hull of the eigenvalues of A.

Say $\lambda_1, \lambda_2 \in W(A)$, $\lambda_1 \neq \lambda_2$, if $W(A)$ is a convex hull, $(1-t)\lambda_1 + t\lambda_2 \in W(A)$ $\forall\, t \in [0,1]$.

Let $\eta = t\lambda_1 + (1-t)\lambda_2$.

$$\eta \in W(A) \Longleftrightarrow t \in W(\alpha I + \beta A), \; \alpha, \beta \in \mathbb{C}.$$

To see this, consider $\lambda_2 \in W(A)$, then $\lambda_2 = \langle y, Ay\rangle$ with $\|y\| = 1$.

So,

$$t = \frac{\eta - \lambda_2}{\lambda_1 - \lambda_2} = \alpha + \beta\lambda_2 = \alpha + \beta\langle y, Ay\rangle$$

$$= \alpha + \langle y, \beta A y\rangle$$

$$= \langle y, (\alpha I + \beta A) y\rangle$$

$\Rightarrow t \in W(\alpha I + \beta A)$ where $\alpha = \dfrac{-\lambda_2}{\lambda_1 - \lambda_2}$ & $\beta = \dfrac{1}{\lambda_1 - \lambda_2}$.

Let $S = \alpha I + \beta A$.

Let us fix unit vectors x & y such that $x, y \in \mathbb{C}^m$

$0 = \langle x, Sx \rangle$ and $1 = \langle y, Sy \rangle$.

Define $g: \mathbb{R} \longrightarrow \mathbb{C}$ as

$$g(t) = \langle x, Sy \rangle e^{-it} + \langle y, Sx \rangle e^{it} \quad t \in \mathbb{R}.$$

Since $\cos \pi = -1$, we can see that $g(t+\pi) = -g(t)$ $\forall t \in \mathbb{R}$. $\Rightarrow g$ is periodic.

Moreover, $\exists\, t_0 \in [0, \pi]$ such that $\operatorname{Im} g(t_0) = 0$.

This can be shown using Intermediate Value Theorem.

Since $\operatorname{Im}(g(0)) = -\operatorname{Im}(g(\pi))$ and g is a continuous function, $\exists\, t_0 \in [0, \pi]$ such that $\operatorname{Im}(g(t_0)) = 0$.

Now, observe the vectors x and $\hat{y} = e^{it_0} y$ are linearly independent.

Otherwise $x = \alpha \hat{y}$ for some $\alpha \in \mathbb{C}$, $|\alpha| = 1$ and

$0 = \langle x, Sx \rangle = \alpha^2 \langle \hat{y}, S\hat{y} \rangle = \langle y, Sy \rangle = 1$! $\Rightarrow$ Contradiction.

So, we can define continuous functions z and f by

$$z(s) = \frac{(1-s)x + s\hat{y}}{\|(1-s)x + s\hat{y}\|}, \quad s \in [0,1].$$

and $f(s) = \langle z(s), Sz(s) \rangle$, $s \in [0,1]$.

$$f(0) = \langle z(0), Sz(0) \rangle = \langle x, Sx \rangle = 0$$

$$f(1) = \langle z(1), Sz(1) \rangle = \langle \hat{y}, S\hat{y} \rangle = \langle y, Sy \rangle = 1$$

We can also see that f is real valued.

$\boxed{\text{Thus } t \in [0,1] \subset f([0,1]) \subset W(S)}$ $\Rightarrow$ <u>$W(A)$ is convex.</u>

(b) A is normal $\Rightarrow AA^* = A^*A$, $A \in \mathbb{C}^{m\times m}$

The numerical range of A is invariant w.r.t unitary transformations, i.e, if Q is unitary,

$$W(Q^*AQ) = \{ x^*Q^*AQx : x^*x = 1\}$$

$$= \{ (Qx)^*A(Qx) : x^*x = 1\}$$

$$= \{ y^*Ay : y^*y = 1\} = W(A).$$

If A is normal, then $A = Q^*\Lambda Q$ for some unitary Q and a diagonal Λ where Λ has the eigenvalues of A on its diagonal. $\Lambda = \text{diag}\{\lambda_i\}_{i=1}^{m}$

So, $W(A) = W(Q^*\Lambda Q) = W(\Lambda) = \{x^*\Lambda x : x^*x = 1\}$

Let $x = [x_1\ x_2 \cdots x_n]^T$. We have

$$x^*\Lambda x = \sum_{i=1}^{n} \lambda_i \bar{x}_i x_i = \sum_{i=1}^{n} \lambda_i |x_i|^2 = \sum_{i=1}^{n} \lambda_i t_i$$

where $t_i = |x_i|^2$. Since $x^*x = 1$, $\sum_{i=1}^{n} |x_i|^2 = \sum_{i=1}^{n} t_i = 1$.

$\Rightarrow x^*\Lambda x$ is a convex combination of $\lambda_i \ \forall\ i = 1, 2, \ldots n$.

Consequently, W(A) consists of convex combinations of the eigenvalues of A.