1) (a)

Let $\vec{v}$ be an eigenvector of A

$$A\vec{v} = \lambda \vec{v}$$

$$A\vec{v} = \frac{1}{h^2}\begin{pmatrix} 2 & -1 & 0 & \cdots & 0 \\ -1 & 2 & -1 & \cdots & 0 \\ 0 & -1 & 2 & \cdots & 0 \\ \cdots & \cdots & -1 & 2 & -1 \\ 0 & 0 & 0 & -1 & 2 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ \vdots \\ v_{m-1} \end{pmatrix}$$

$$= \frac{1}{h^2}\begin{pmatrix} 2v_1 - v_2 \\ -v_1 + 2v_2 - v_3 \\ -v_2 + 2v_3 - v_4 \\ \vdots \\ -v_{m-2} + 2v_{m-1} \end{pmatrix} = \lambda\vec{v} = \begin{pmatrix} \lambda_1 v_1 \\ \lambda_2 v_2 \\ \vdots \\ \lambda_{m-1} v_{m-1} \end{pmatrix}$$

Now, we have the recurrence relation

$$\frac{1}{h^2}\left(-v_{i-1} + 2v_i - v_{i+1}\right) = \lambda_i v_i$$

$$-v_{i-1} + 2v_i - v_{i+1} = h^2 \lambda_i v_i$$

$$v_{i+1} = (2 - h^2\lambda_i) v_i - v_{i-1} \qquad (1)$$

$$\Rightarrow v_{i+1} = 2\left(1 - \tfrac{1}{2}h^2\lambda_i\right) v_i - v_{i-1}$$

From the hint, we know that Chebyshev poly

Satisfies the same recurrence relation:

$$U_{k+1}(z) = 2z\, U_k(z) - U_{k-1}(z)$$

Now, if we try to solve ① for $2-h^2\lambda_i$,

We can try finding $V_{i+1} = 0$ (equation root)

Back to Chebyshev poly, we know that its

root is:

$$z = \cos\left(\frac{k}{n+1}\pi\right) \quad k = 1, 2, \cdots n$$

In our original problem,

$$2 - h^2\lambda_i = 2\underbrace{\left(1 - \frac{1}{2}h^2\lambda_i\right)}_{z}$$

$$1 - \frac{1}{2}h^2\lambda_i = \cos\left(\frac{i}{m-1+1}\pi\right)$$

$$\frac{1}{2}h^2\lambda_i = 1 - \cos\left(\frac{i}{m}\pi\right)$$

$$\boxed{\begin{aligned}\lambda_i &= \frac{2}{h^2}\left(1-\cos\frac{i}{m}\pi\right)\\ &= \frac{4}{h^2}\left(\sin^2\left(\frac{i\pi}{2m}\right)\right)\end{aligned}}$$

(b) is A PSD?

- a PSD matrix has all its eigenvalues ≥ 0

$\sin^2 x \geq 0$ for all x

Thus $\lambda_i = \underbrace{\frac{4}{h^2}}_{\text{always } (+) \text{ too!}} \sin^2\left(\frac{i\pi}{2(m+1)}\right) \geq 0$ for all i,

- Also, we know that A is symmetric since the elements on the tridiagonal are equal

(c) • $\|A\|_2 = \sigma_{max}(A) = \sqrt{\lambda_{max}}(A)$

since all λ are positive

$$\lambda_i = \frac{4}{h^2}\left(\sin^2\left(\frac{i\pi}{2(m+1)}\right)\right)$$

- max value of $\lambda_i = \sqrt{\frac{4}{h^2}\max\left(\sin^2\left(\frac{i\pi}{2m+1}\right)\right)}$

$$= \sqrt{\frac{4}{h^2}\times 1} = 2/h$$

• $\|A\|_F = \sqrt{\sum \sigma_i^2(A)} = \sqrt{\sum \lambda_i(A)}$

$$= \frac{4}{h^2}\left[\sin^2\left(\frac{\pi}{2(m+1)}\right) + \sin^2\left(\frac{\pi}{m+1}\right) + \sin^2\left(\frac{3\pi}{2(m+1)}\right) + \dots \sin^2\left(\frac{m\pi}{2(m+1)}\right)\right]$$

$$= \frac{2}{h}\sum_i\left(\sin^2\left(\frac{i\pi}{2(m+1)}\right)\right)^{1/2}$$

- $\kappa(A) = \|A\| \, \|A^{-1}\|$

We know that A is hermitian, thus

$A = A^{-1}$ $\quad \|A\| = \|A^{-1}\|$

$$\kappa(A) = \frac{4}{h^2}$$

- For what values of m does $\kappa(A) > 10^8$?

$h = \frac{1}{m}$ $\quad \kappa(A) = \frac{4}{(1/m)^2} = 4m^2$

$$4m^2 > 10^8$$

$$m^2 > \frac{1}{4} 10^8$$

$$m > \frac{1}{2} 10^4$$

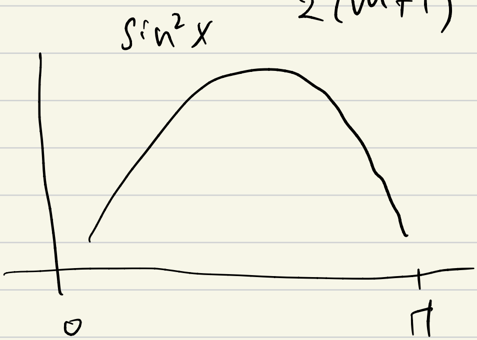
(d) We solve eigenvalues of A by finding the recurrence relation of

$$Au = A \begin{pmatrix} u_1 \\ \vdots \\ u_{m-1} \end{pmatrix}$$

and found

$$\lambda_i = \frac{4}{h^2}\left(\sin^2\left(\frac{i\pi}{2(m+1)}\right)\right) \quad i = 1, 2, \ldots, m$$

Thus $0 < \dfrac{i\pi}{2(m+1)} < 1$ and $\lambda_i > 0$

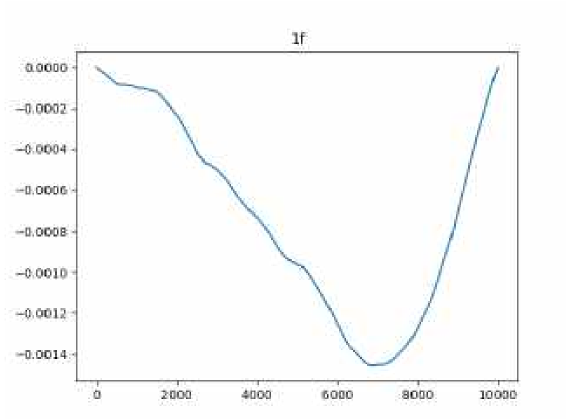


My best guess: The 2-dimensional space of linear function in Null (A) was that that correspond to $u(0) = 0$ and $u(\pi) = 0$ (boundary conditions).

(g) Thomas algorithm is a simplified version of Gaussian elimination that can solve tridiagonal system in $O(n)$ instead of $O(n^3)$.

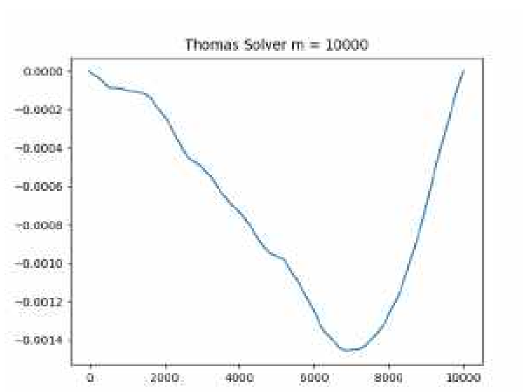
(h) With large m values, h becomes very small and thus each entry of A and f becomes very close to each other.

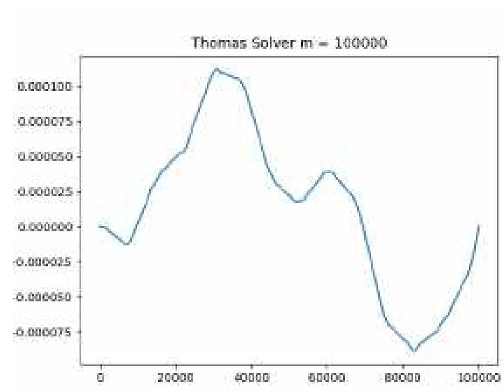
1e) R = 4.0
1f) Time elapsed 8.30s



1h)
M = 10k
Norm diff with regular solver 1.5477194987643645e-20
Time elapsed 0.0069 s

M = 100k
Time elapsed m = 100000 : 0.03486800193786621
Using regular solver kept getting killed in my computer





```
from re import M
import numpy as np
import time
import matplotlib.pyplot as plt

from scipy.sparse import csc_matrix, diags
from scipy.sparse.linalg import spsolve

def generate_A(m, h=None):
    if h == None:
        h = 1/m
    A = np.zeros((m-1,m-1))
    for i in range(A.shape[0]):
        for j in range(A.shape[1]):
            if i == j:
                A[i][j] = (1/h)**2 * 2
                # print(A[i][j])
            elif j == i+1 or i == j+1:
                A[i][j] = (1/h)**2 * -1
    return A

def generate_f(m):
    f = np.arange(1, m, 1, dtype=int)
    f = np.array([np.sin(1000 * x**2) for x in f])
    return f


def get_calculated_eig(m):
    h = 1/m
    w = [(4/h**2) * (np.sin((i*np.pi)/((2*m))))**2 for i in
range(1,m)]
    return w

def p_e():
    m=1000
    h = 1/m

    A = generate_A(m, h)
    f = [np.sin(100 * x**2) for x in range(1, m)]
    u_h = np.linalg.solve(A, f)
    M_uh = np.amax(np.abs(u_h))

    A_h2 = generate_A(m, h/2)
    u_h2 = np.linalg.solve(A_h2, f)
    M_uh2 = np.amax(np.abs(u_h2))

    A_h4 = generate_A(m, h/4)
    u_h4 = np.linalg.solve(A_h4, f)
    M_uh4 = np.amax(np.abs(u_h4))

    r = (M_uh - M_uh2)/(M_uh2 - M_uh4)
    return r

def p_f(m):
```

```python
    f = generate_f(m)
    A = generate_A(m)
    start = time.time()
    soln = np.linalg.solve(A, f)
    print(f"time elapsed {time.time() - start}")
    plt.plot(soln,)
    plt.title("1f")
    plt.show()
    return soln

def p_h(m):
    h=1/m
    A_sparse = csc_matrix(diags([-1, 2, -1], [-1, 0, 1], shape=(m-1,
m-1))/h**2)
    f = generate_f(m)
    start = time.time()
    u = spsolve(A_sparse,f)
    print(f"time elapsed m = {m} : {time.time() - start}")
    plt.plot(u)
    plt.title(f"Thomas Solver m = {m}")
    plt.show()
    regular_soln = p_f(m)
    norm_diff = np.linalg.norm(regular_soln - u)
    print("Norm diff with regular solver", norm_diff)
```

2.0 Proof of Gershgorin circle Theorem

- Lets start w/ the definition of eigenvalue

① $A\vec{v} = \lambda \vec{v}$ where $\vec{v}$ its a corresponding eigenvector

Now, lets choose an index i of $\vec{v}$ entries, where V_i is the maximum entry.

if we write ith operation of ① in Sumation form:

$$\sum_{i} A_{ij} V_i = \lambda V_i$$

$$\Rightarrow A_{ii} V_i + \sum_{i \neq j} A_{ij} V_i = \lambda V_i$$

$$\Rightarrow A_{ii} + \sum_{i \neq j} A_{ij} = \lambda$$

$$\Rightarrow \sum_{i \neq j} A_{ij} = \lambda - A_{ii} \Rightarrow |\lambda - A_{ii}| = \left|\sum_{i \neq j} A_{ij}\right|$$

$$\Rightarrow \underbrace{|\lambda - A_{ii}|}_{\text{eigenvalue} - \text{circle centre}} \leq \sum_{i \neq j} |A_{ij}| \leftarrow R$$

(sum off non-diagonal row entries)

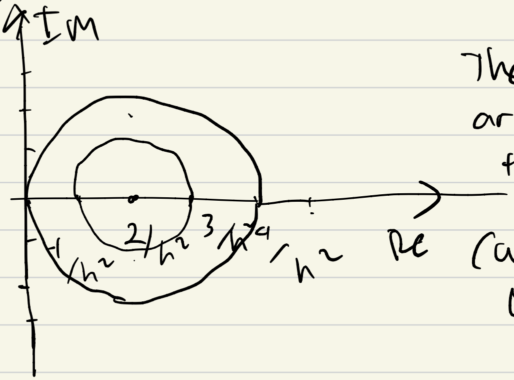
PROVEN

- Result of 1(a)

$$\lambda_i = \frac{4}{h^2}\left(\sin^2\left(\frac{i\pi}{2(m+1)}\right)\right)$$

$$A = \frac{1}{h^2}\begin{pmatrix} 2 & -1 & 0 & \cdots & 0 \\ -1 & 2 & -1 & \cdots & 0 \\ 0 & -1 & 2 & \cdots & 0 \\ \cdots & \cdots & -1 & 2 & -1 \\ 0 & 0 & 0 & -1 & 2 \end{pmatrix}$$

The Gershgorin circles are:



The eigenvalues are in one of the 2 circles. (we dont know which one tho)

Since we already know that A is PSD:

$0 < \lambda_i \leq 4/h^2$ (no imaginary component)

$$\lambda_i = \frac{4}{h^2}\left(\sin^2\left(\frac{i\Pi}{2m}\right)\right)$$

Plugging this to the circle equation:

$$|\lambda_i - A_{ii}| \leq \sum_{i \neq j} |A_{ij}|$$

$$\underbrace{\left|\frac{4}{h^2}\sin^2\left(\frac{i\Pi}{2m}\right) - \frac{2}{h^2}\right|}_{LHS} \underset{\uparrow \text{ prove this!}}{\leq} \underbrace{\left|\frac{2}{h^2}\right|}_{RHS} \quad i = 1, 2, \ldots, m-1$$

① $$\frac{4}{h^2}\sin^2\left(\frac{i\Pi}{2m}\right) - \frac{2}{h^2} \leq \frac{2}{h^2}$$

$$\frac{4}{h^2}\sin^2\left(\frac{i\Pi}{2m}\right) \leq \frac{4}{h^2}$$

$$\sin^2\left(\frac{i\Pi}{2m}\right) \leq 1 \implies \text{True}$$

since $\sin^2(x) \leq 1$ $\forall x$

② $\frac{4}{n^2}\sin^2\left(\frac{i\pi}{2m}\right)-\frac{2}{n^2} \geq -\frac{2}{n^2}$

$\sin^2\left(\frac{i\pi}{2m}\right) \geq 0 \Rightarrow$ True since $\sin^2(x) \geq 0 \ \forall x$

$\Rightarrow$ $\boxed{\text{We connected Gershgorin circle theorem with } \underline{1}\ (a)}$

3) - Show that for a non-Hermitian matrix $A \in \mathbb{C}^{m \times m}$ the Rayleigh quotient gives an eigenvalue estimate whose accuracy is generally linear

$$A = \frac{1}{2}(A + A^T) + \frac{1}{2}(A - A^T)$$

$$r(x) = \frac{x^T A x}{x^+ x}$$

Let $x^+ A x = N$ and $\underset{1 \times m}{x^+} \underset{m \times 1}{x} = D$

$$r'(x) = \frac{N'D - ND'}{D^2}$$

- $D^2 = (x^+ x)^2 = (\|x\|^2)^2 = \|x\|^4$
- $\frac{d x^T A x}{dx} = x^+ (A + A^T) \Rightarrow N' = x^T (A + A^T)$
- $D' = 2x$

$$\Rightarrow r'(x) = \frac{x^+ (A + A^T) x^+ x - x^+ A x (2x)}{\|x\|^4}$$

- In the Hermitian case, since $A = A^T$

$$r'(x) = \frac{x^T 2A\, x^T x - x^T A x\, 2x}{\|x\|^4}$$

$$= \frac{2x^T A\, x^T x - 2\, x^T A x x}{(x^T x)^2}$$

$$= \frac{2\, x^T x\, x^T A - 2\, x^T A x\, x}{(x^T x)^2}$$

$$= \frac{2\, x^T A}{x^T x} - \frac{2\, r(x)\, x}{x^T x}$$

$$= \frac{2\, Ax}{x^T x} - \frac{2\, r(x)\, x}{x^T x}$$

$$= \frac{2}{x^T x}\left(Ax - r(x)\, x \right)$$

result in book

Thus we can have $r'(x) = 0$ at $x = q_j$ q_j is eigenvec

Now, if A is not Hermitian,

$$r'(x) = \frac{x^{T}(A+A^{T})x^{T}x - x^{T}Ax(2\cancel{x})}{\|x\|^4}$$

$A \neq A^{T}$ Thus $A + A^{T} \neq 2A$

$$A + A^{T} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ \vdots & & & \\ a_{m1} & & & \end{bmatrix} \begin{bmatrix} a_{11} & \cdots & a_{m1} \\ a_{12} & & \\ \vdots & & \\ a_{1m} & & \end{bmatrix}$$

$$= A + A^{T} \begin{cases} 2a_{ij} & i == j \\ a_{ij} + a_{ji} & i \neq j \end{cases}, \text{ lets call this } B$$

$$r'(x) = \frac{x^{T}Bx^{T}x - x^{T}Ax(2x)}{(x^{T}x)^2}$$

$$= \frac{x^{T}B}{x^{T}x} - \frac{2x^{T}Ax\,x}{(x^{T}x)^2} = \frac{x^{T}B}{x^{T}x} - \frac{2r(x)\,x}{x^{T}x}$$

$$= \frac{1}{x^{T}x}\left(x^{T}(A+A^{T}) - 2r(x)\,x\right)$$

$\boxed{r'(q_j) \neq 0 \text{ when } q_j \text{ is the eigenvalue}}$

Thus, eigenvectors of non-Hermitian A are NOT the stationary points of $r(x)$

$$r(x) - r(q_j) = O(\|x - q_j\|)$$

as $x \to q_j$

- if the current convergence rate was derived from $r(x) - r(q_j) = O(\|x - q_j\|)^2$,

$\Rightarrow$ Previously, $|\lambda^k - \lambda_j| = O(\varepsilon^2)$

Now, $|\lambda^k - \lambda_j| = O(\varepsilon)$

Thus: if $\|v^{(k)} - q_j\| \leq \varepsilon$:

$$\|v^{k+1} - q_j\| = O(|\lambda^k - \lambda_j| \|v^{(k)} - q_j\|) = \underline{O(\varepsilon^2)}$$

4) (a)

$$W(A) = W(\underset{\uparrow \text{Schur Factorization}}{QTQ^*}) = W(Q^*TQ)$$

$$W(Q^*TQ) = \left\{ \frac{x^* Q^* T Q x}{x^* x} \right\}$$

$$= \left\{ \frac{(Qx)^* T Qx}{x^* x} \right\} \quad \text{let } y = Qx$$

$$= \left\{ \frac{y^* T y}{x^* x} \right\}$$

we know $T = \begin{bmatrix} \lambda_1 & t_{12} & t_{13} \cdots & t_{1m} \\ & \lambda_2 & t_{21} \cdots & \vdots \\ 0 & & \ddots & \vdots \\ & & & \lambda_m \end{bmatrix}$

$$y^* T = \underset{1\times n}{(y_1 \; y_2 \; \cdots \; y_m)} \underset{m\times n}{\begin{bmatrix} \lambda_1 & t_{12} & \cdots & t_{1m} \\ & \lambda_2 & & \\ 0 & & \ddots & \\ & & & \lambda_m \end{bmatrix}}$$

$$= (y_1\lambda_1 \quad y_1 t_{12} + y_2\lambda_2 \quad y_1 t_{13}\, y_2 t_{23}\, y_3\lambda_3 \quad \cdots \quad y_1 t_{1m} + y_2 t_{2m} + \cdots + y_m \lambda_m)$$

$$= \left(y_1\lambda_1 \quad y_2\lambda_2 + y_1 t_{12} \quad \cdots \quad y_m\lambda_m + \sum_{i=1}^{m-1} y_i t_{im} \right)$$

$$y^* T y = \left(y_1\lambda_1 \quad y_2\lambda_2 + y_1 t_{12} \quad \cdots \quad y_m\lambda_m + \sum_{i=1}^{m-1} y_i t_{im} \right) \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{pmatrix}$$

$$= y_1^2\lambda_1 + y_2^2\lambda_2 + y_1 t_{12} + \cdots + y_m^2\lambda_m + y_m \sum_{i=1}^{m} y_i t_{im}$$

$$= \sum_{i=1}^{m} y_i^2 \lambda_i + y_1 t_{12} + y_1 t_{13} + y_2 t_{23} + \cdots + y_m \sum_{i=1}^{m-1} y_i t_{im}$$

$$= \sum_{i=1}^{m} y_i^2 \lambda_i + \sum_{j=1}^{m-1} y_j \sum_{i=1}^{j} y_i t_{ij+1}$$

$$W(A) = W(Q^* T Q) = \left\{ \frac{\sum_i y_i^2 \lambda_i + \sum_{j=1}^{m-1} y_j \sum_{i=1}^{j} y_i t_{ij+1}}{x^* x} \right\}$$

From the equation above, we see that

$$\left\{ \frac{\sum y_i^2 \lambda_i}{x^* x} \right\} \in W(A) \Rightarrow$$ Thus $W(A)$ contains the convex hull of $\lambda_1, \ldots \lambda_m$

(b) When A is normal, we can do the above decomposition with $A = Q \Lambda Q^* = Q^* \Lambda Q$

where Λ is a diagonal matrix

$$\Lambda = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_m \end{bmatrix}$$

Because all off-diagonal entries are 0, ⊛ becomes:

$$\boxed{W(A) = \left\{ \frac{\sum_{i=1}^{m} y_i^2 \lambda_i}{X^* X} \right\} \Rightarrow \text{W(A) is exactly the convex hull of } \lambda_1, \dots, \lambda_m}$$