Question 1:

Given: Triangle with vertices $a^1, a^2, a^3 \in \mathbb{R}^2$

To show: Elemental stiffness matrix a_K

$[a_K]_{ij} = a(\phi_i, \phi_j)$ where $\phi_j(a_i) = \delta_{ij}$, $\phi_i \in P_1(K)$

$a(u, v) = \int_K \nabla u \cdot \nabla v \, dS$ is given by

$$a_K = \frac{1}{4|K|} \begin{pmatrix} u \cdot u & u \cdot v & u \cdot w \\ v \cdot u & v \cdot v & v \cdot w \\ w \cdot u & w \cdot v & w \cdot w \end{pmatrix}$$

$|K|$ = area of the triangle.

$u = a^2 - a^1 \qquad v = a^3 - a^1 \qquad w = a^1 - a^2$

Solution:

For each $i = 1, 2, 3$, we can write ϕ_i as

$$\phi_i = \begin{vmatrix} 1 & x & y \\ 1 & a_1^{(i+1)} & a_2^{(i+1)} \\ 1 & a_1^{(i+2)} & a_2^{(i+2)} \end{vmatrix} \Big/ \begin{vmatrix} 1 & a_1^1 & a_2^1 \\ 1 & a_1^2 & a_2^2 \\ 1 & a_1^3 & a_2^3 \end{vmatrix}$$

where $\begin{vmatrix} 1 & a_1^1 & a_2^1 \\ 1 & a_1^2 & a_2^2 \\ 1 & a_1^3 & a_2^3 \end{vmatrix} = 2|T|$ and $a^{(i+1)} = a^{(i+1)\%3}$ $\forall i = 1, 2, 3.$

or $\phi_i = \frac{1}{2|T|} \left(x - a_1^{(i+2)}\right)\left(a_2^{(i+1)} - a_2^{(i+2)}\right) - \left(y - a_2^{(i+2)}\right)\left(a_1^{(i+1)} - a_1^{(i+2)}\right)$

$$\frac{1}{2|T|}\left(\left(x - a_1^{i+2}\right)\left(a_2^{i+1} - a_2^{i+2}\right) - \left(y - a_2^{i+2}\right)\left(a_1^{i+1} - a_1^{i+2}\right)\right)$$

$$\nabla\phi_i = \frac{1}{2|T|}\begin{pmatrix} a_2^{(i+1)} - a_2^{(i+2)} \\ a_1^{(i+1)} - a_1^{(i+2)} \end{pmatrix}^T$$

$$u = a^2 - a^3$$
$$v = a^3 - a^1$$
$$w = a^1 - a^2$$

$$a(\phi_i, \phi_j) = \int_K \nabla\phi_i \cdot \nabla\phi_j \cdot ds = \int_K \nabla\phi_i (\nabla\phi_j)^T \cdot ds$$

$$= \frac{|T|}{4|T|^2}\left[\left(a_2^{(i+1)} - a_2^{(i+2)}\right)\left(a_2^{(j+1)} - a_2^{(j+2)}\right) + \left(a_1^{(i+1)} - a_1^{(i+2)}\right)\left(a_1^{(j+1)} - a_1^{(j+2)}\right)\right]$$

$$\boxed{= \frac{1}{4|T|}\left[\left(a_2^{(i+1)} - a_2^{(i+2)}\right)\left(a_2^{(j+1)} - a_2^{(j+2)}\right) + \left(a_1^{(i+1)} - a_1^{(i+2)}\right)\left(a_1^{(j+1)} - a_1^{(j+2)}\right)\right]}$$

Substituting for $i, j = 1, 2, 3$ we have,

$$a(\phi_1, \phi_1) = \frac{1}{4|T|}(u_2 u_2 + u_1 u_1) = \frac{u \cdot u}{4|T|}$$

$$a(\phi_1, \phi_2) = \frac{1}{4|T|}(u_2 v_2 + u_1 v_1) = \frac{u \cdot v}{4|T|} = a(\phi_2, \phi_1)$$

$$a(\phi_1, \phi_3) = \frac{1}{4|T|}(u_2 w_2 + u_1 w_1) = \frac{u \cdot w}{4|T|} = a(\phi_3, \phi_1)$$

$$a(\phi_2, \phi_2) = \frac{1}{4|T|}(v_2 v_2 + v_1 v_1) = \frac{v \cdot v}{4|T|}$$

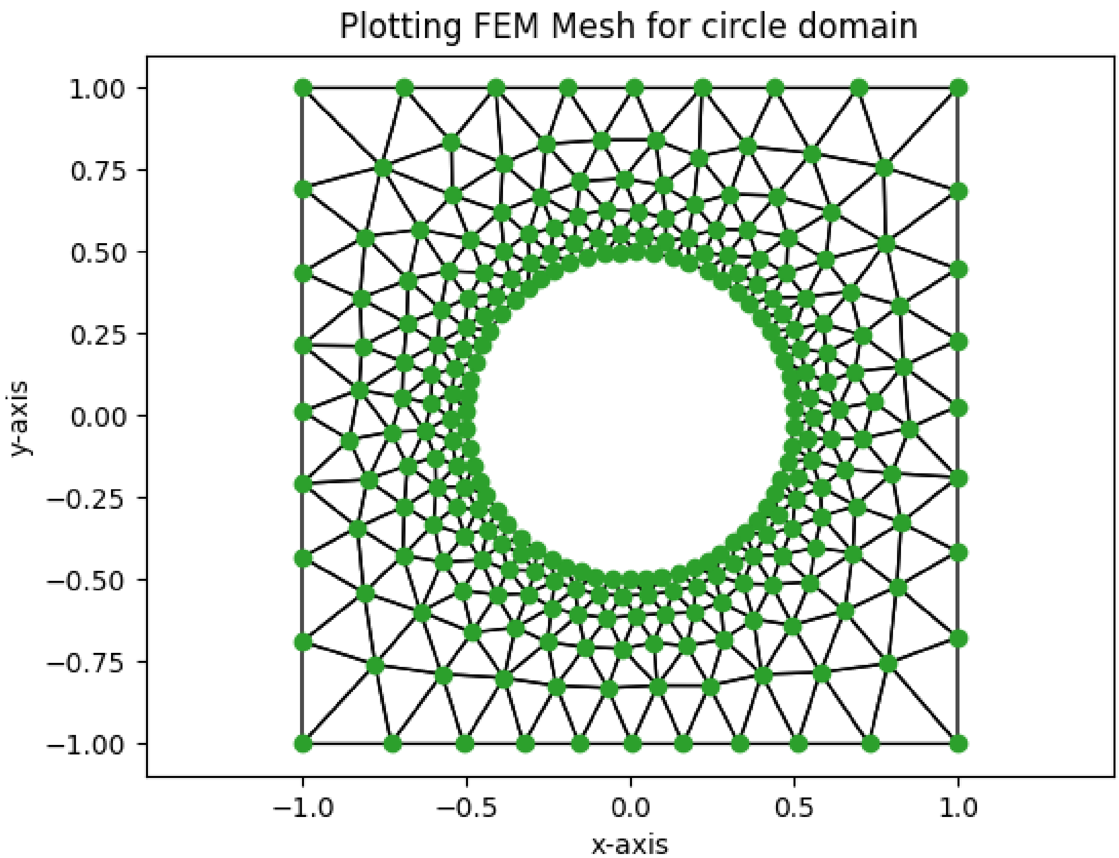
$$a(\phi_2, \phi_3) = \frac{1}{4|T|}(v_2 w_2 + v_1 w_1) = \frac{v \cdot w}{4|T|} = a(\phi_3, \phi_2)$$

$$a(\phi_3, \phi_3) = \frac{1}{4|T|}(w_2 w_2 + w_1 w_1) = \frac{w \cdot w}{4|T|}$$

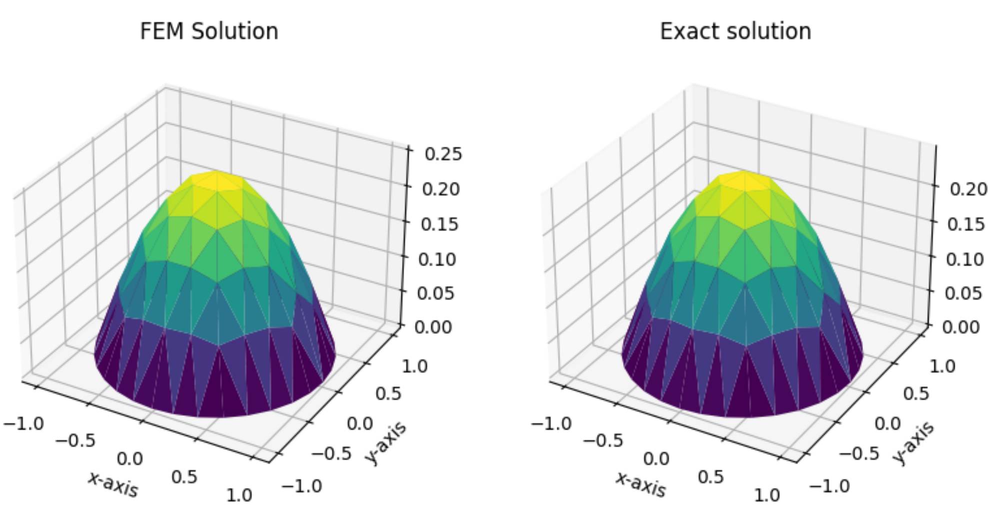
$\therefore$

$$a_K = \frac{1}{4|T|}\begin{bmatrix} u \cdot u & u \cdot v & u \cdot w \\ v \cdot u & v \cdot v & v \cdot w \\ w \cdot u & w \cdot v & w \cdot u \end{bmatrix}$$

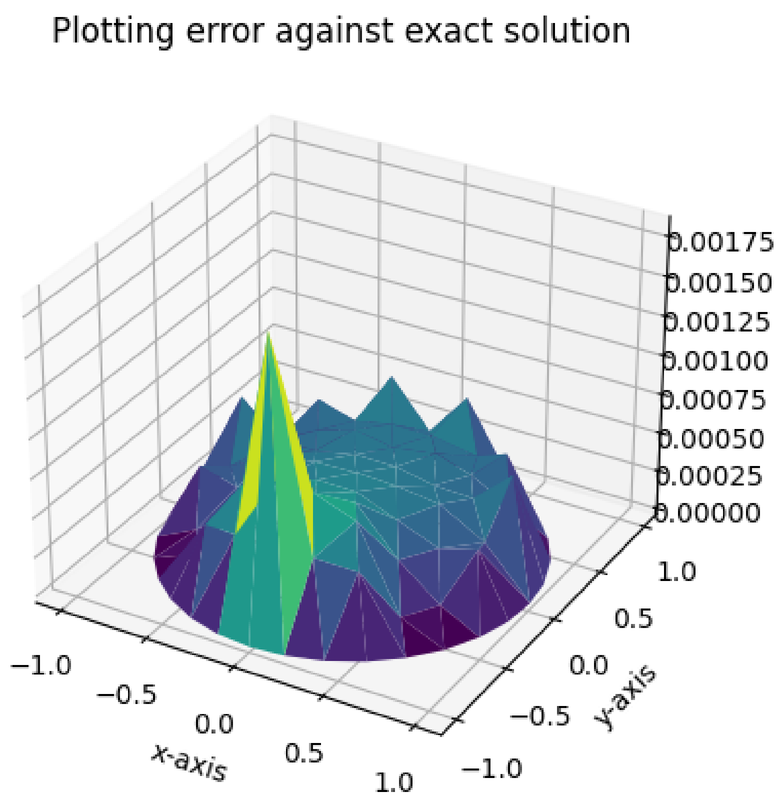
Question 2a



Question 2c



Error plot:



Question 2d

Question 2: (c) Exact Solution of $-\Delta u = 1$.

Using change of coordinate $\rightarrow r = \sqrt{x^2+y^2}$.

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial x} = u'\cdot\frac{x}{r}$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial y} = u'\cdot\frac{y}{r}.$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial x}\left(u'\cdot\frac{x}{r}\right)$$

$$= u''\left(\frac{x}{r}\right)^2 + u'\left(\frac{1}{r} - \frac{x^2}{r^3}\right)$$

$$\frac{\partial^2 u}{\partial y^2} = u''\left(\frac{y}{r}\right)^2 + u'\left(\frac{1}{r} - \frac{y^2}{r^3}\right)$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = u'' + u'\left(\frac{2}{r}\right) - u'\left(\frac{1}{r}\right) = u'' + \frac{u'}{r}$$

Solving the PDE that now becomes an ODE:

$$-u'' - \frac{u'}{r} - 1 = 0$$

$$ru'' + u' = -r$$

$$\frac{d}{dr}(u'r) = -r$$

Let $v = u'r$

$$dv = -r.dr$$

$$\Rightarrow v = \frac{-r^2}{2} + C_1$$

So, $u'r = \frac{-r^2}{2} + C_1$

$$u' = \frac{-r}{2} + \frac{C_1}{r}$$

$$u = -\frac{r^2}{4} + C_1 \ln|r| + C_2$$

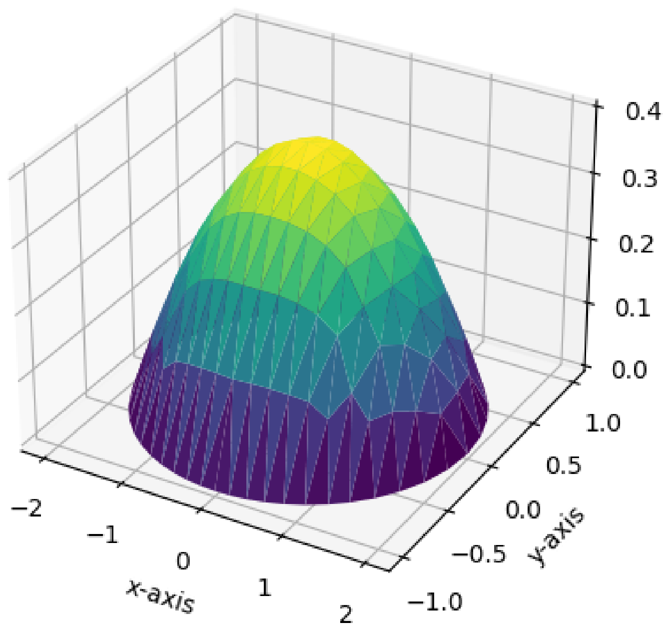
Since u is defined at $r=0$, $C_1 = 0$.

Further, $u=0$ when $r=1 \Rightarrow C_2 = +\frac{1}{4}$

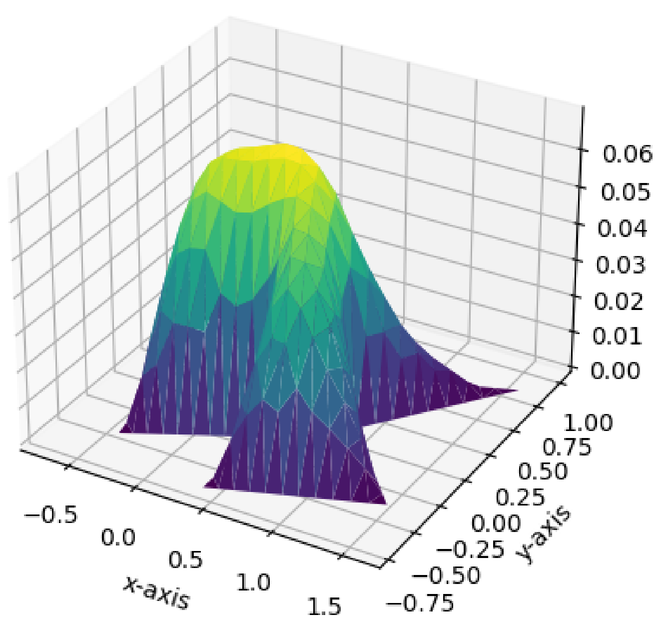
$$\boxed{\therefore\ u = -\frac{r^2}{4} + \frac{1}{4}.}$$

is the EXACT SOLUTION to $-\Delta u = 1$.

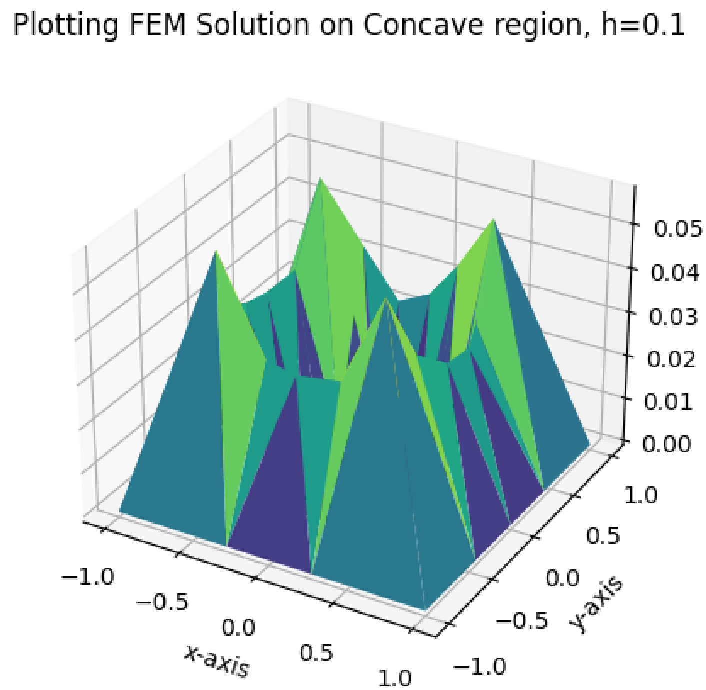
Plotting FEM Solution on Ellipse



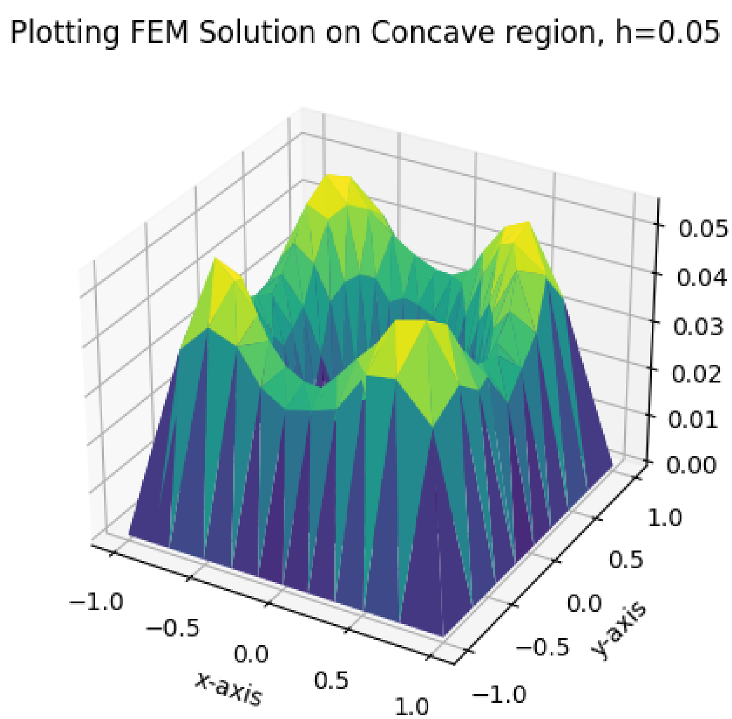
Plotting FEM Solution on Polygon



Question 2e



Plotting FEM Solution on Concave region, h=0.1
0.05
0.04
0.03
0.02
0.01
0.00
−1.0
−0.5
0.0
0.5
1.0
x-axis
1.0
0.5
0.0
−0.5
−1.0
y-axis



Plotting FEM Solution on Concave region, h=0.05
0.05
0.04
0.03
0.02
0.01
0.00
−1.0
−0.5
0.0
0.5
1.0
x-axis
1.0
0.5
0.0
−0.5
−1.0
y-axis

Resolution Study:

Consider the concave region:
For h = 0.01, the L2 norm of the solution = 0.18239867646815075
For h = 0.05, the L2 norm of the solution = 0.45766497250426924

Now, consider the circular region:
For h=0.1, the L2 norm of u = 2.610957571554937
For h=0.5, the L2 norm of u = 0.4458913952006164

We can say that in the concave region, as h reduces or the mesh becomes more fine, the L2 norm of the solution increases.
However, for the same factor of decrease in h, the L2 norm of u reduces.

Since the exact solution for the concave case is not known, this study is inconclusive for the behavior of the solution u on a concave region as h is refined.
The right way to do it would be to compute the exact solution for the concave region and compare the FEM solution with this exact solution by refining the mesh.

$$\int v\,du = vu - \int u\,dv$$

Question 3:

$$\Delta^2 u = \nabla^4 u = f \quad \text{in } \Omega \subset \mathbb{R}^2 \qquad f \in L^2(\Omega)$$

$$\frac{\partial u}{\partial n} = 0 \quad \text{on } \partial\Omega$$

Consider a test function $v \in H_0^2(\Omega)$.

$$\int_\Omega \Delta^2 u\, v . dS = \int_\Omega \nabla\cdot(\nabla \Delta u) v . dS$$

$$= \underbrace{\int_{\partial\Omega} \nabla\cdot(\Delta u) v . d\Omega}_{\substack{\| \\ 0}} - \int_\Omega \nabla(\Delta u)\cdot \nabla v \, . dS$$

since $v \in H_0^2(\Omega) \Rightarrow v = 0$ on $\partial\Omega$.

$$-\int_\Omega \nabla(\Delta u)\cdot\nabla v . dS = \underbrace{-\int_{\partial\Omega} \Delta u \cdot \nabla v \cdot \hat{n}\, d\Omega}_{} + \int_\Omega \Delta u \cdot \Delta v . dS$$

$= 0$ since $\frac{\partial u}{\partial n} = \nabla u \cdot \hat{n} = 0$ on $\partial\Omega$.

$$= \int_\Omega \Delta u \cdot \Delta v . dS$$

$$\boxed{\Rightarrow \int_\Omega f v . dS = \int_\Omega \Delta u \cdot \Delta v . dS \quad \forall v \in H_0^2(\Omega).}$$

To show that $l(v) = (f, v)$ is continuous, we show that it is bounded.

$$l(v) = (f,v) = \int_\Omega f v . dS = \|fv\|_{H_0^2(\Omega)} \le \underbrace{\|f\|_{H_0^2(\Omega)}}_{\Lambda} \|v\|_{H_0^2(\Omega)}$$

$$\therefore l(v) = (f,v) \le \Lambda \|v\|_{H_0^2(\Omega)}$$

$\Rightarrow l(v)$ is bounded and hence continuous.

If $a(u,v)$ is continuous and coercive, we should satisfy the following conditions:

Continuity: $|a(u,v)| \leq \gamma \|u\|_V \|v\|_V$

and $|\ell(v)| \leq \Lambda \|v\|_V$

Coercivity: $\alpha \|v\|_V^2 \leq a(v,v) \qquad \forall u,v \in H_0^2(\Omega)$

Consider,

$$|a(u,v)| = \left| \int_\Omega \Delta u \cdot \Delta v \cdot dS \right|$$

Using Frederick's Inequality, we have,

$$v^2 \leq (\nabla v)^2 \leq (\nabla^2 v)^2$$

So, $$\|v\|_{H_0^2}^2 = \int_\Omega v^2 + (\nabla v)^2 + (\nabla^2 v)^2 \cdot dS$$

$$\|v\|_{H_0^2}^2 \leq 3 \int_\Omega (\Delta v)^2 \cdot dS = 3\, a(v,v).$$

$$a(v,v) \geq \frac{1}{3} \|v\|_{H_0^2}^2 \Rightarrow a(v,v) \text{ is coercive. } \forall v \in H_0^2(\Omega).$$

Also,

$$|a(u,v)| = \left| \int_\Omega \Delta u \cdot \Delta v \cdot dS \right|$$

$$\leq \|\Delta u\|_{L^2(\Omega)} \|\Delta v\|_{L^2(\Omega)}$$

$$\leq \|u\|_{H_0^2(\Omega)} \|v\|_{H_0^2(\Omega)}$$

$$\Rightarrow |a(u,v)| \leq 1 \cdot \|u\|_{H_0^2(\Omega)} \|v\|_{H_0^2(\Omega)}$$

$\Rightarrow$ $a(u,v)$ is continuous (since it is bounded) $\forall u,v \in H_0^2(\Omega)$

Question 4:

K: Tetrahedron with vertices a^i, $i = 1,2,3,4$.

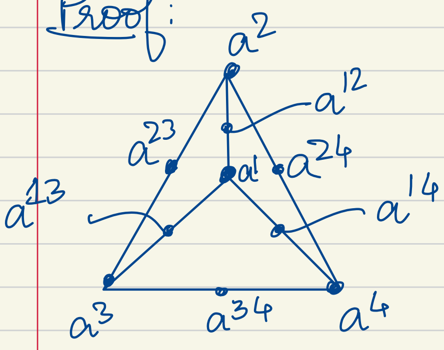
a^{ij}: midpoint on line between a^i and a^j. $i<j$

To Prove:

$v \in P_2(K)$ is uniquely determined by $v(a^i)$ & $v(a^{ij})$

Show that the corresponding finite element space V_h satisfies $V_h \subset C^0(h)$.

Proof:



Dimension of $P_2(K) = 10$

$1 + x^2 + y^2 + z^2 + xy + yz + xz + x + y + z$

which is also the number of degrees of freedom in this setting.

So, to show that $v \in P_2(K)$ is uniquely determined by $v(a^i)$ and $v(a^{ij})$, it is sufficient to show that if $v \in P_2(K)$ & $v(a^i) = 0$ and $v(a^{ij}) = 0$, then $v \equiv 0$.

Consider the edge a^2a^3. Along this edge, v has quadratic variation and is 0 at 3 points – a^2, a^{23}, a^3.

$\Rightarrow$ v vanishes identically on the edge a^2a^3.

So we can factor out $\lambda_1(x,y,z)$ and get

$$v(x,y,z) = \lambda_1(x,y,z) \cdot w(x,y,z).$$

where $\lambda_i(x,y,z) \in P_1(K)$ is the linear lagrange basis function of the form

$$\lambda_i(x,y,z) = \gamma(d_{i1}x + d_{i2}y + d_{i3}z + d_{i4})$$

Similarly, we can see that v also vanishes at 3 points on the edge a^2a^4 — at points a^2, a^{24}, a^4. So, we can factor out $\lambda_2(x,y,z)$ to get

$$v(x,y,z) = \lambda_1(x,y,z)\,\lambda_2(x,y,z)\,w_0(x,y,z)$$

We know that $\lambda_2 \in P_1(K)$

Similarly, v is 0 on $a^3a^4 \Rightarrow v$ vanishes at a^3, a^{34}, a^4. So, we can factor out $\lambda_3(x,y,z)$.

$$\Rightarrow v(x,y,z) = \underbrace{\lambda_1(x,y,z)\,\lambda_2(x,y,z)\,\lambda_3(x,y,z)}_{\text{polynomial of degree 3}}\underbrace{w_0(x,y,z)}_{\overset{!!}{0}}$$

Since $v \in P_2(K)$, we have to have $w_0 = 0$ for this to be true. $\Rightarrow v \equiv 0$.

Consider a pair of elements that either share a node or an edge $\Rightarrow$ The 2 elements will have the same nodal values for the shared nodes.

If we define $V_h = \{v : v \in P_2(K),\ v \equiv 0 \text{ if } v(a^i) = 0 \ \&\ v(a^{ij}) = 0\}$, then, we have that $V_h \subset C^0(h)$.

ie, for $v_1, v_2 \in V_h$, if the elements e_1 & e_2 share a node a^i, $v_1(a^i) = v_2(a^i) \Rightarrow V_h \subset C^0$.

Question 5: (a)

$$-\Delta u + u = f \quad \text{in } \Omega$$

$$\frac{\partial u}{\partial n} = g \quad \text{on } \partial\Omega$$

$v \in H^1\Omega$. Derive the variational form of the PDE.

$$\int_{\Omega} f v \cdot dS = -\int_{\Omega} \Delta u v + u v \cdot dS$$

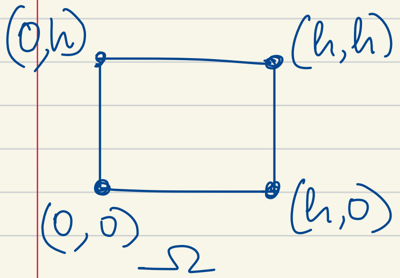
$$= -\int_{\partial\Omega} v \frac{\partial u}{\partial n} \cdot d\Omega + \int_{\Omega} (\nabla u \cdot \nabla v + u v) \, dS$$

$$= -\int_{\partial\Omega} v \cdot g \cdot d\Omega + \int_{\Omega} \nabla u \cdot \nabla v + u v \cdot dS$$

Or the variational form of the PDE is:

$$\int_{\Omega} \nabla u \cdot \nabla v + u v \cdot dS = \int_{\Omega} f v \cdot dS + \int_{\partial\Omega} v g \cdot d\Omega .$$

Q5 (b) $\int_{\Omega} \nabla u \cdot \nabla v + uv \cdot dS = \int_{\Omega} fv \cdot dS + \int_{\partial\Omega} gv \cdot d\Omega.$



We can define the piecewise bilinear ϕ as

$$\phi_1(x,y) = \left(\frac{1-x}{h}\right)\left(\frac{1-y}{h}\right)$$

$$\phi_2(x,y) = \left(\frac{1-x}{h}\right)\left(\frac{y}{h}\right)$$

$$\phi_3(x,y) = \left(\frac{x}{h}\right)\left(\frac{1-y}{h}\right)$$

$$\phi_4(x,y) = \left(\frac{x}{h}\right)\left(\frac{y}{h}\right)$$

$$\nabla\phi_1 = \left(-\left(\frac{1-y}{h^2}\right) \quad -\left(\frac{1-x}{h^2}\right)\right) \qquad \nabla\phi_2 = \left(\frac{y}{h^2} \quad \frac{1-x}{h^2}\right)$$

$$\nabla\phi_3 = \left(\frac{1-y}{h^2} \quad \frac{-x}{h^2}\right) \qquad \nabla\phi_4 = \left(\frac{y}{h^2} \quad \frac{x}{h^2}\right)$$

Elemental Stiffness Matrix:

$$a^K = \begin{bmatrix} a^K_{11} & a^K_{12} & a^K_{13} & a^K_{14} \\ a^K_{21} & a^K_{22} & a^K_{23} & a^K_{24} \\ a^K_{31} & a^K_{32} & a^K_{33} & a^K_{34} \\ a^K_{41} & a^K_{42} & a^K_{43} & a^K_{44} \end{bmatrix}$$

where each $a^K_{ij} = \int_0^1\int_0^1 \nabla\phi_i \cdot \nabla\phi_j \, dx \cdot dy + \int_0^1\int_0^1 \phi_i \phi_j \, dx \cdot dy.$

When evaluated using Mathematica, we get:

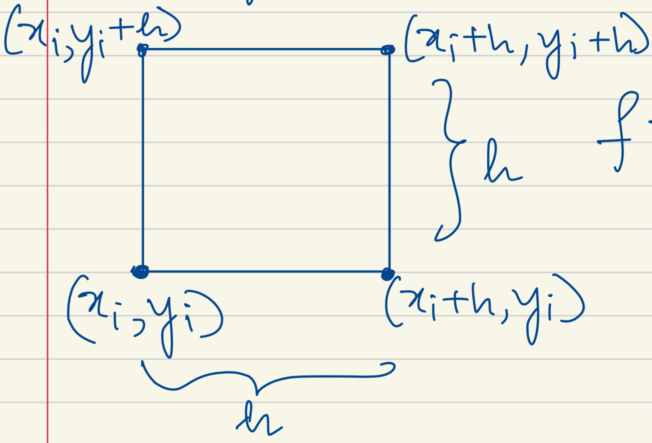
$$a^K = \begin{bmatrix} \frac{1}{9}(6+h^2) & \frac{1}{18}(-3+h^2) & \frac{1}{18}(-3+h^2) & \frac{1}{36}(-12+h^2) \\ \frac{1}{18}(-3+h^2) & \frac{1}{9}(6+h^2) & \frac{1}{36}(-12+h^2) & \frac{1}{18}(-3+h^2) \\ \frac{1}{18}(-3+h^2) & \frac{1}{36}(-12+h^2) & \frac{1}{9}(6+h^2) & \frac{1}{18}(-3+h^2) \\ \frac{1}{36}(-12+h^2) & \frac{1}{18}(-3+h^2) & \frac{1}{18}(-3+h^2) & \frac{1}{9}(6+h^2) \end{bmatrix}$$

For the RHS, we need $\int_0^1\int_0^1 f v \, dS + \int_{\partial\Omega} g v \, d\Omega$

when $f = \cos(\pi x) - \cos(\pi y)$ and $g = 0$,

$$\int_{\Omega} f v \cdot dS + \int_{\partial\Omega} g v \cdot d\Omega = \int_{\Omega} f v \cdot dS$$

We use a change of variable for f to compute it on any element in the domain:



$f = \cos(\pi(x + x_i)) - \cos(\pi(y + y_i))$ on this element.

$$\therefore \int_0^h \int_0^h f \cdot \phi_1 \cdot dx \cdot dy$$

$$= \int_0^h \int_0^h \left(1 - \frac{x}{h}\right)\left(1 - \frac{y}{h}\right) \cos(\pi(x + x_i)) - \cos(\pi(y + y_i)) \, dx \cdot dy$$

$$\int_0^h \int_0^h f \cdot \phi_2 \cdot dx \cdot dy$$

$$= \int_0^h \int_0^h \left(1 - \frac{x}{h}\right) \frac{y}{h} \cos(\pi(x + x_i)) - \cos(\pi(y + y_i)) \, dx \cdot dy$$

$$\int_0^h \int_0^h f \cdot \phi_3 \cdot dx \cdot dy$$

$$= \int_0^h \int_0^h \left(\frac{x}{h}\right)\left(1 - \frac{y}{h}\right) \cos(\pi(x + x_i)) - \cos(\pi(y + y_i)) \, dx \cdot dy$$

$$\int_0^h \int_0^h f \cdot \phi_4 \cdot dx \cdot dy$$

$$= \int_0^h \int_0^h \left(\frac{x}{h}\right)\left(\frac{y}{h}\right) \cos(\pi(x + x_i)) - \cos(\pi(y + y_i)) \, dx \cdot dy$$

In[1]:=
```
p1 = (1 - x / h) (1 - y / h)
p2 = (1 - x / h) (y / h)
p3 = (x / h) (1 - y / h)
p4 = (x / h) (y / h)
```

Out[1]= $\left(1-\frac{x}{h}\right)\left(1-\frac{y}{h}\right)$

Out[2]= $\frac{\left(1-\frac{x}{h}\right) y}{h}$

Out[3]= $\frac{x\left(1-\frac{y}{h}\right)}{h}$

Out[4]= $\frac{x y}{h^2}$

In[6]:= $\text{Integrate}\left[\nabla_{\{x,y\}} p1 . \nabla_{\{x,y\}} p1 + p1 * p1, \{x, 0, h\}, \{y, 0, h\}\right]$

Out[6]= $\frac{1}{9}\left(6+h^2\right)$

In[7]:= $\text{Integrate}\left[\nabla_{\{x,y\}} p1 . \nabla_{\{x,y\}} p2 + p1 * p2, \{x, 0, h\}, \{y, 0, h\}\right]$

Out[7]= $\frac{1}{18}\left(-3+h^2\right)$

In[8]:= $\text{Integrate}\left[\nabla_{\{x,y\}} p1 . \nabla_{\{x,y\}} p3 + p1 * p3, \{x, 0, h\}, \{y, 0, h\}\right]$

Out[8]= $\frac{1}{18}\left(-3+h^2\right)$

In[9]:= $\text{Integrate}\left[\nabla_{\{x,y\}} p1 . \nabla_{\{x,y\}} p4 + p1 * p4, \{x, 0, h\}, \{y, 0, h\}\right]$

Out[9]= $\frac{1}{36}\left(-12+h^2\right)$

In[10]:= $\text{Integrate}\left[\nabla_{\{x,y\}} p2 . \nabla_{\{x,y\}} p2 + p2 * p2, \{x, 0, h\}, \{y, 0, h\}\right]$

Out[10]= $\frac{2}{3}+\frac{h^2}{9}$

In[11]:= $\text{Integrate}\left[\nabla_{\{x,y\}} p2 . \nabla_{\{x,y\}} p3 + p2 * p3, \{x, 0, h\}, \{y, 0, h\}\right]$

Out[11]= $\frac{1}{36}\left(-12+h^2\right)$

In[12]:= $\text{Integrate}\left[\nabla_{\{x,y\}} p2 . \nabla_{\{x,y\}} p4 + p2 * p4, \{x, 0, h\}, \{y, 0, h\}\right]$

Out[12]= $-\frac{1}{6}+\frac{h^2}{18}$

In[13]:= `Integrate[∇{x,y} p3.∇{x,y} p3 + p3 * p3, {x, 0, h}, {y, 0, h}]`

Out[13]= $\frac{1}{9}\left(6+h^2\right)$

In[14]:= `Integrate[∇{x,y} p3.∇{x,y} p4 + p3 * p4, {x, 0, h}, {y, 0, h}]`

Out[14]= $\frac{1}{18}\left(-3+h^2\right)$

In[15]:= `Integrate[∇{x,y} p4.∇{x,y} p4 + p4 * p4, {x, 0, h}, {y, 0, h}]`

Out[15]= $\frac{1}{3}+\frac{1}{3}\left(\frac{1}{h^3}+\frac{1}{3\,h}\right)h^3$

In[16]:= `f = Cos[π (x + xi)] - Cos[π (y + yi)]`

Out[16]= Cos[π (x + xi)] - Cos[π (y + yi)]

In[17]:= `Integrate[p1 * f, {x, 0, h}, {y, 0, h}]`

Out[17]= $\frac{\text{Cos}[\pi\,\text{xi}]-\text{Cos}[\pi\,(\text{h}+\text{xi})]-\text{Cos}[\pi\,\text{yi}]+\text{Cos}[\pi\,(\text{h}+\text{yi})]-\text{h}\,\pi\,\text{Sin}[\pi\,\text{xi}]+\text{h}\,\pi\,\text{Sin}[\pi\,\text{yi}]}{2\,\pi^2}$

In[19]:= `Integrate[p2 * f, {x, 0, h}, {y, 0, h}]`

Out[19]= $-\frac{-\text{Cos}[\pi\,\text{xi}]+\text{Cos}[\pi\,(\text{h}+\text{xi})]-\text{Cos}[\pi\,\text{yi}]+\text{Cos}[\pi\,(\text{h}+\text{yi})]+\text{h}\,\pi\,\text{Sin}[\pi\,\text{xi}]+\text{h}\,\pi\,\text{Sin}[\pi\,(\text{h}+\text{yi})]}{2\,\pi^2}$

In[20]:= `Integrate[p3 * f, {x, 0, h}, {y, 0, h}]`

Out[20]= $\frac{-\text{Cos}[\pi\,\text{xi}]+\text{Cos}[\pi\,(\text{h}+\text{xi})]-\text{Cos}[\pi\,\text{yi}]+\text{Cos}[\pi\,(\text{h}+\text{yi})]+\text{h}\,\pi\,\text{Sin}[\pi\,(\text{h}+\text{xi})]+\text{h}\,\pi\,\text{Sin}[\pi\,\text{yi}]}{2\,\pi^2}$

In[21]:= `Integrate[p4 * f, {x, 0, h}, {y, 0, h}]`

Out[21]= $\frac{1}{2\,\pi^2}\Big(-\text{Cos}[\pi\,\text{xi}]+\text{Cos}[\pi\,(\text{h}+\text{xi})]+\text{Cos}[\pi\,\text{yi}]-$
$\text{Cos}[\pi\,(\text{h}+\text{yi})]+\text{h}\,\pi\,\text{Sin}[\pi\,(\text{h}+\text{xi})]-\text{h}\,\pi\,\text{Sin}[\pi\,(\text{h}+\text{yi})]\Big)$

In[◦]:= `sol1 = DSolve[{- ∇²_{x, y} u[x, y] + u[x, y] == Cos[π * x] - Cos[π * y]}, u[x, y], {x, y}]`

DSolve : General solution is not available for the given linear partial differential equation. Trying to build a particular solution.

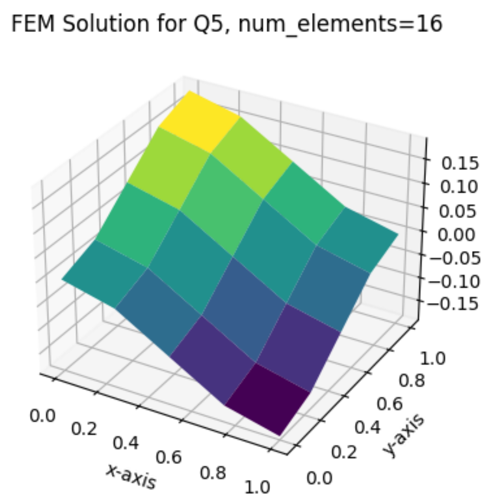
Out[◦]= $\left\{\left\{u[x, y] \to \frac{\mathrm{Cos}[\pi x] - \mathrm{Cos}[\pi y]}{1 + \pi^2}\right\}\right\}$

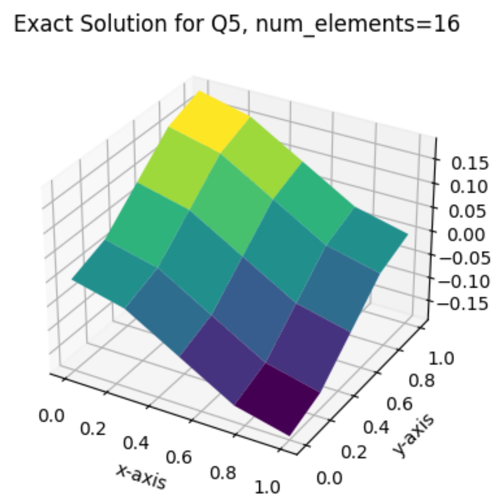
Question 5c

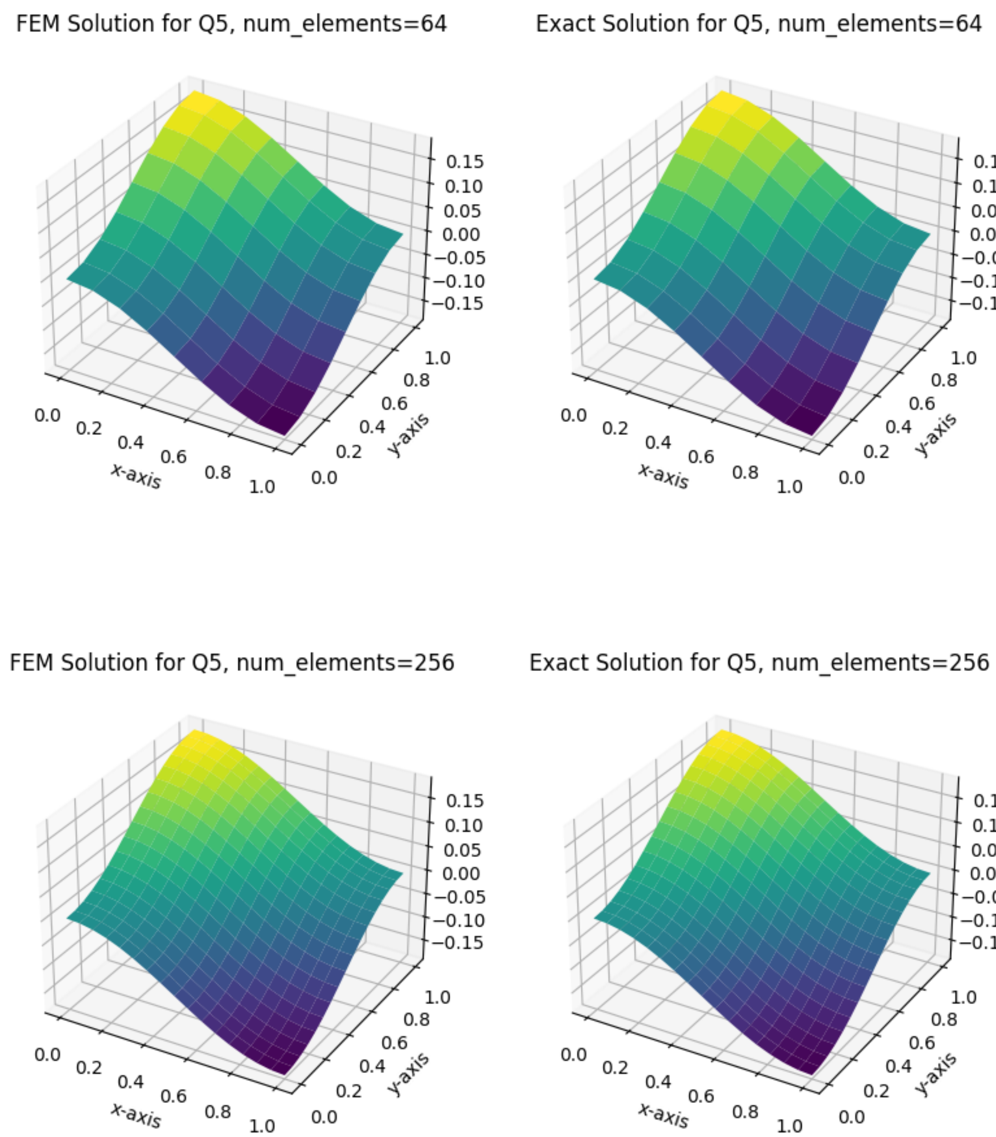
Num Elements = 16   h=0.25   Error = |u - u_exact| = 0.0008465312128414781
Num Elements = 64   h=0.125   Error = |u - u_exact| = 0.00021610678305636344
Num Elements = 256 h=0.0625 Error = |u - u_exact| = 5.429626130537546e-05

Error(h=0.25)/Error(h=0.125) = 3.9171894600859978
Error(h=0.125)/Error(h=0.0625) = 3.9171894600859978

Thus, the expected convergence of error, which is quadratic, is satisfied. When h is reduced by a factor of 2, the error reduces by a factor of 4.






FEM Solution for Q5, num_elements=64
Exact Solution for Q5, num_elements=64
FEM Solution for Q5, num_elements=256
Exact Solution for Q5, num_elements=256
x-axis
y-axis
0.0
0.2
0.4
0.6
0.8
1.0
0.15
0.10
0.05
0.00
−0.05
−0.10
−0.15