(1a) $-u''(x) = f(x) = \sin(4\pi x)$

$$-\int u''(x) \cdot dx = \int \sin(4\pi x) \cdot dx$$

$$-u'(x) = \frac{\cos(4\pi x)}{4\pi} + C_0$$

$$-\int u'(x) \cdot dx = \frac{\cos(4\pi x)}{4\pi} + C_0$$

$$-u(x) = \frac{-\sin(4\pi x)}{(4\pi)^2} + C_0 x + C_1$$

$$\Rightarrow u(x) = \frac{\sin(4\pi x)}{(4\pi)^2} - C_0 x - C_1$$

Using periodic boundary conditions, we have $u(0) = u(1)$

$$\Rightarrow -C_1 = -C_0 - C_1 \Rightarrow C_0 = 0.$$

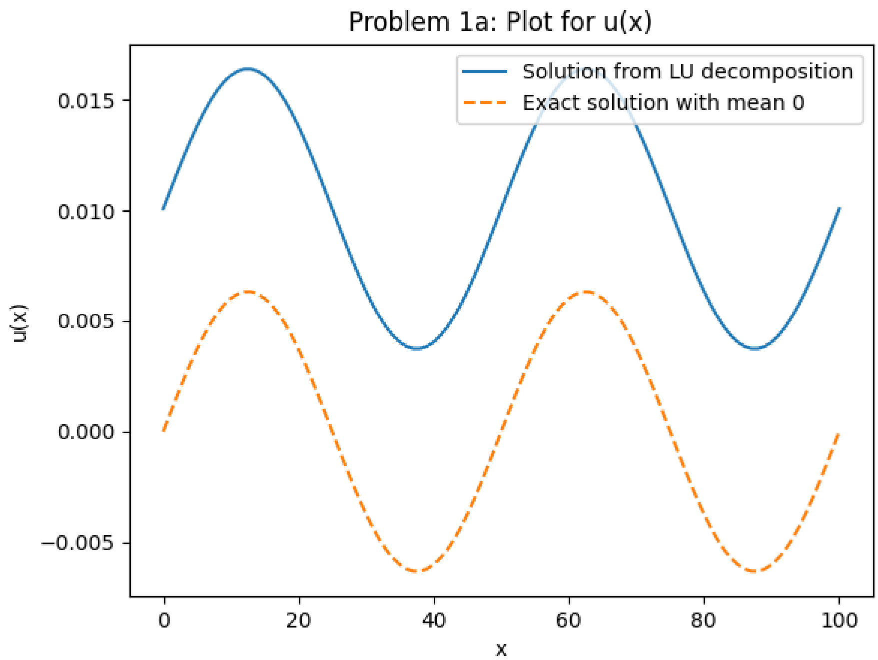
$$\boxed{\therefore u(x) = \frac{\sin(4\pi x)}{(4\pi)^2} - C_1}$$

The exact solution with mean 0 is when $C_1 = 0$.

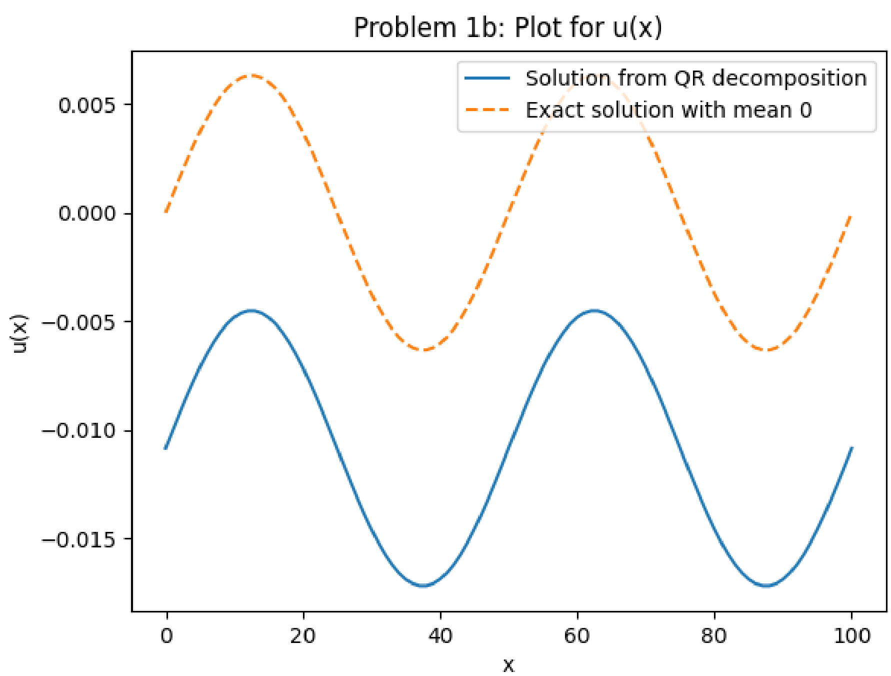
$$\boxed{u(x) = \frac{\sin(4\pi x)}{(4\pi)^2}}$$

Problem 1

a. Plot solution using LU and exact solution



b. Plot solution using QR and exact solution



c.

(1c) $A = \frac{1}{h^2}\begin{bmatrix} 2 & -1 & 0 & & & & -1 \\ -1 & 2 & -1 & & & & 0 \\ 0 & -1 & 2 & -1 & & & 0 \\ 0 & 0 & -1 & 2 & -1 & & 0 \\ & & & & & & \\ -1 & 0 & 0 & 0 & \cdots & -1 & 2 \end{bmatrix}$

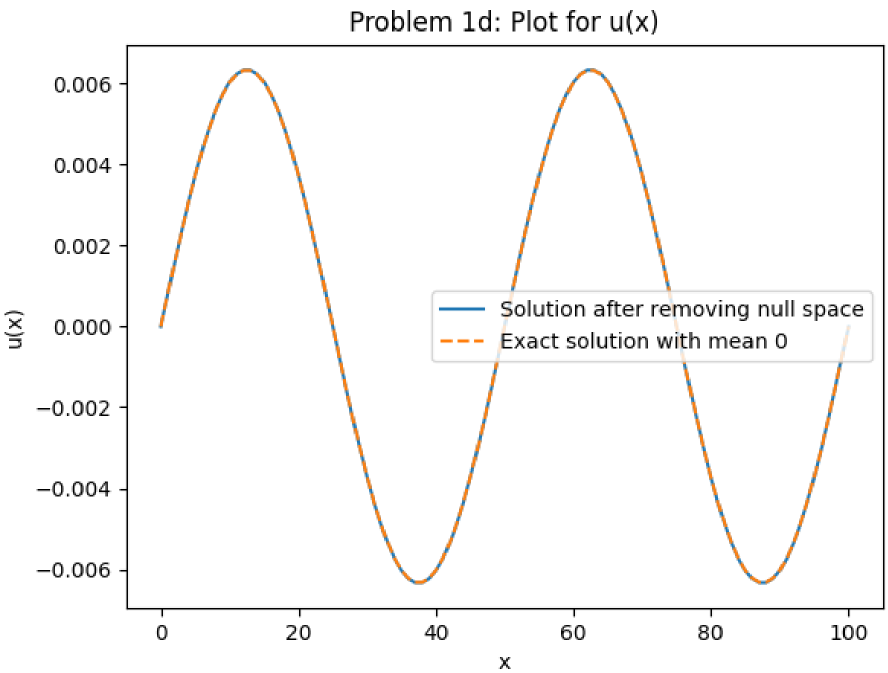
- We see that the sum of all rows/columns of A is 0, which implies that columns of A are not linearly independent $\Rightarrow$ A is not full rank.

Note that any $c.e^T = c.[1\ 1 \cdots 1]^T$ belongs to null(A) for any constant $c \Rightarrow c.Ae = 0$.
or $Ae = 0 \Rightarrow e$ is the basis for the nullspace of A.
The nullspace is not 2 dimensional since we only have 1 degree of freedom [ie, only one of u_i's can be varied without an effect on the other u_i's. This can also be seen from the exact solution, $u_i = u(x_i) = \frac{\sin(4\pi x_i)}{(4\pi)^2} + c_1$.

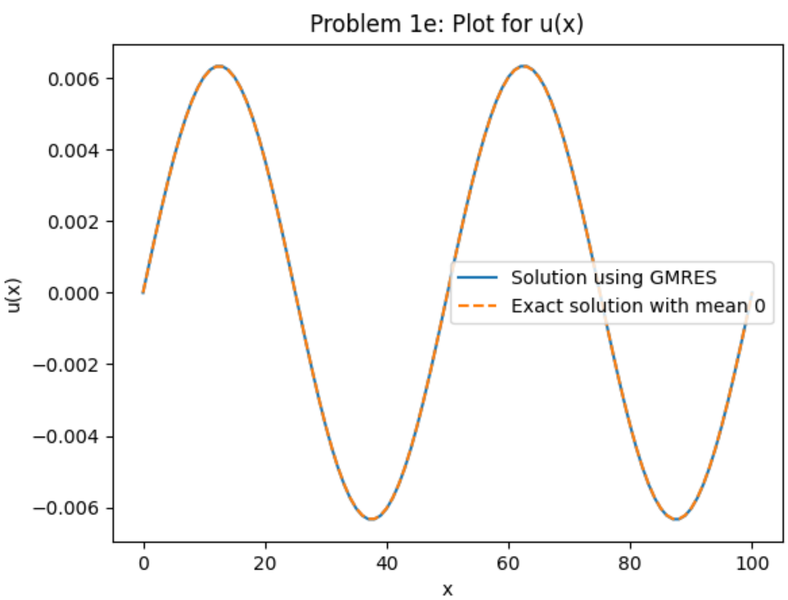
If we choose $u_{m-1} = u(x_{m-1}) = \frac{\sin(4\pi(m-1))}{(4\pi)^2} + c_1 = k$

then, we can eliminate the last row & column of A, making it a full rank matrix.
$\Rightarrow$ Nullspace of A is 1 Dimensional.

d. Plot solution after removing nullspace and exact solution



e. Plot solution using GMRES and exact solution

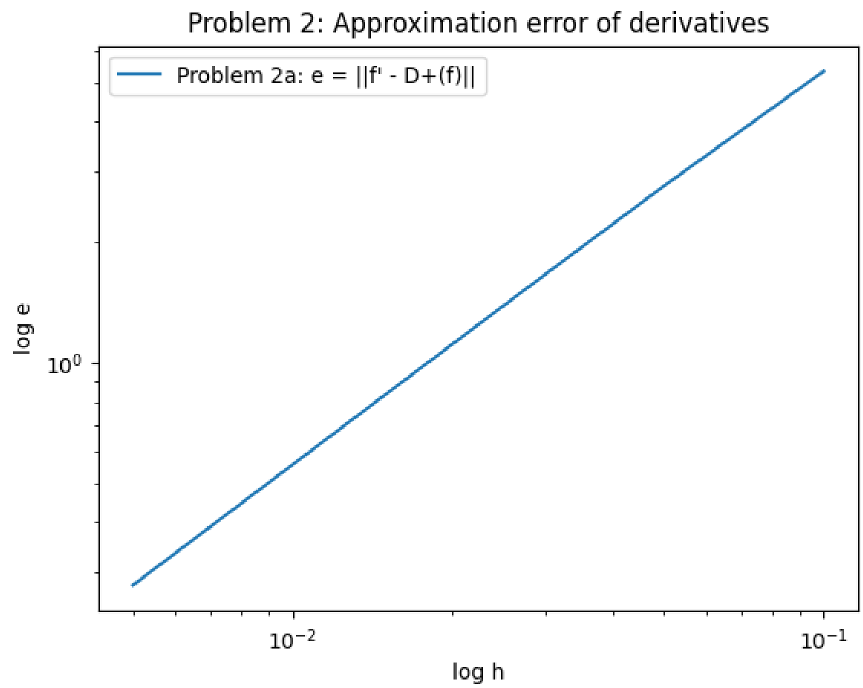


f. Summary of what is going on in (a) - (e)

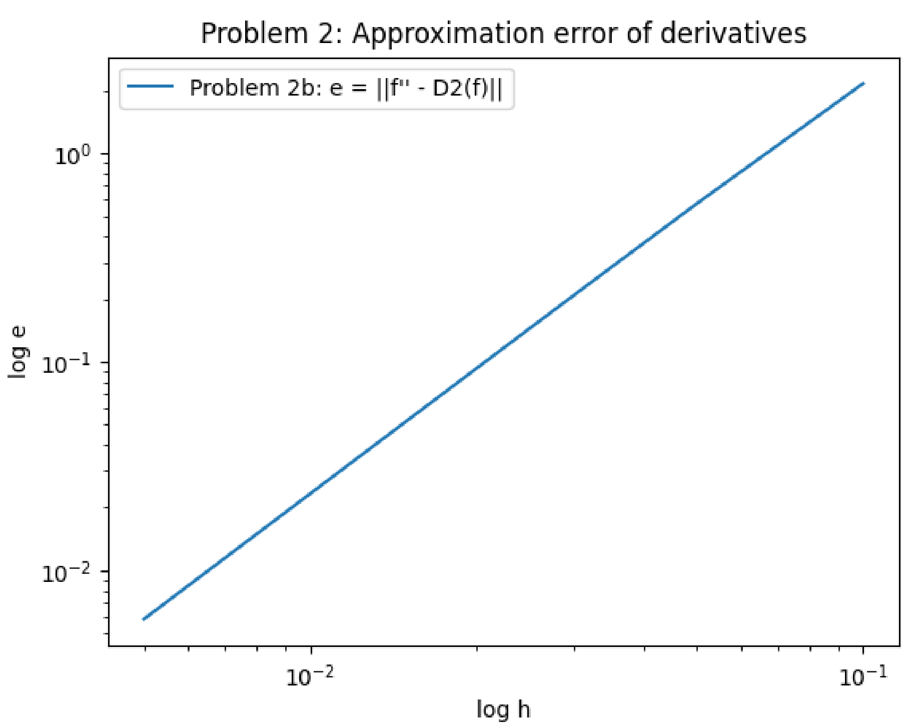
We see that there is a null space in the A matrix. This null space has 1 basis vector and hence is one dimensional. The LU and QR decomposition solutions circumvent this null space by choosing one of the many solutions. GMRES on the other hand chooses the solution that minimizes the mean and is exactly what the exact solution with mean 0 results in. Finally, after removing the null space and solving, the solution matches the exact solution with mean 0.

Problem 2

a. Approximation error of 1st derivative
   Slope = 0.9997690254725266



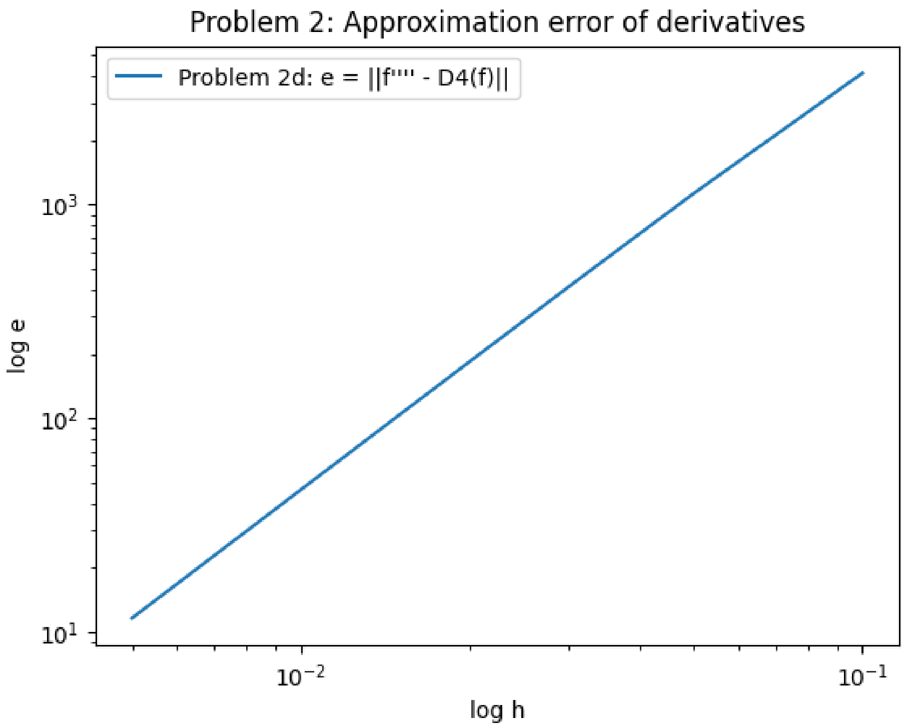
b. Approximation error of 2nd derivative
   Slope = 1.9995842513400808



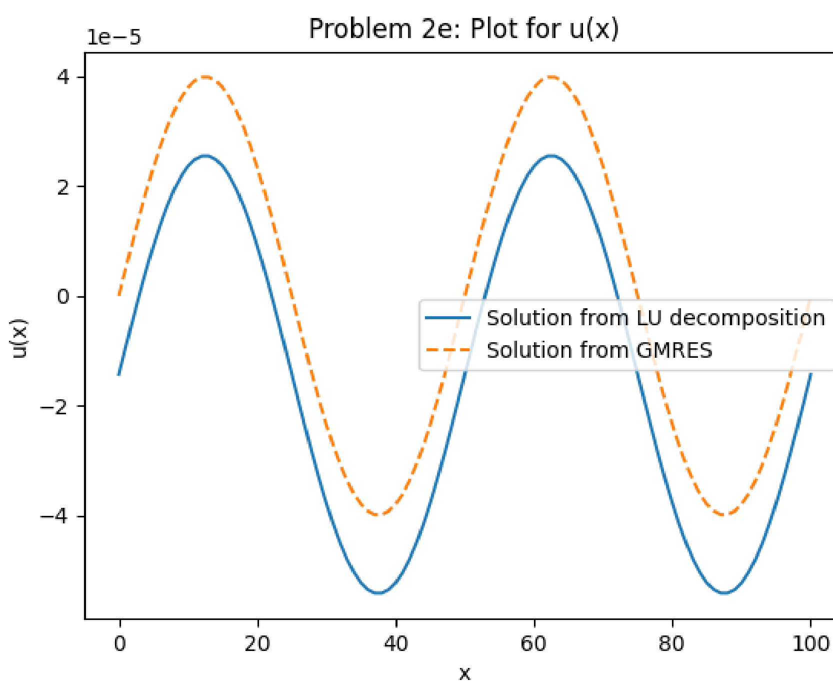
c. D2 operator: We verify that the laplace operator -A is same as D+D- and not (D1)^2.

m	||-A - (D+)(D-)||_2	||-A - (D1)(D1)||_2
10	1.10E-13	661.4378278
20	6.23E-13	3741.657387
30	0	10310.79531
40	3.52E-12	21166.01049
50	0	36975.49864
60	0	58326.66629
70	1.86E-11	85750
80	1.99E-11	119733.0364
90	0	160729.3921
100	0	209165.0066
110	4.67E-11	265442.6727
120	0	329945.45
130	0	403039.3126
140	1.05E-10	485075.2519
150	1.09E-10	576390.9806
160	1.13E-10	677312.3356
170	1.16E-10	788154.4503
180	0	909222.745
190	0	1040813.774
200	0	1183215.957

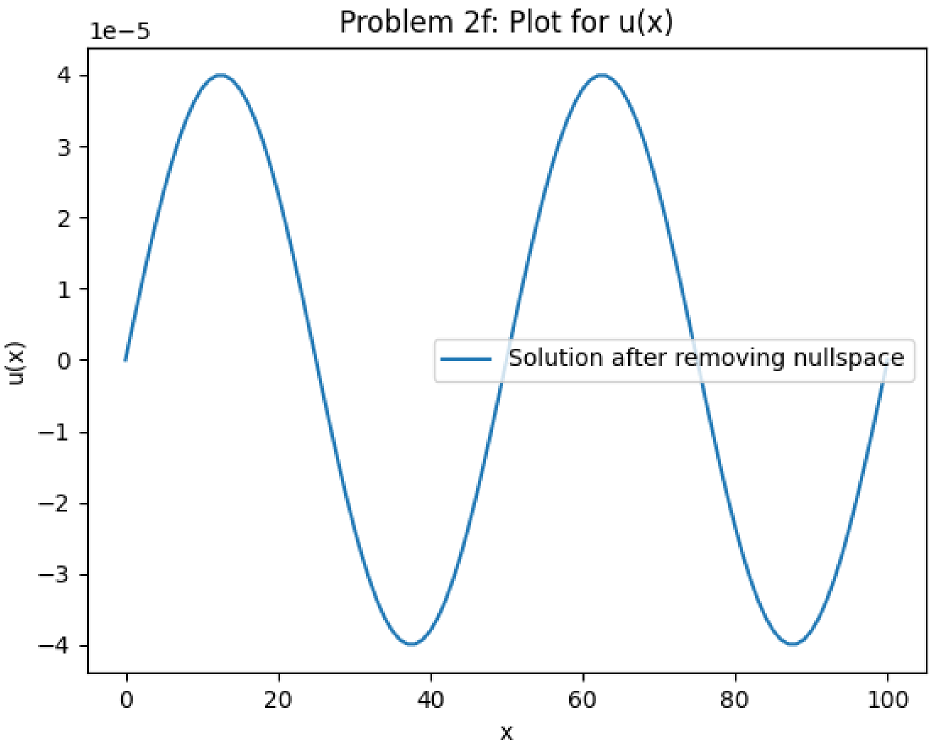
d. Approximation error of 4th derivative
   Slope = 1.9993762145951628



e. Solution from LU & GMRES



f. Solution after removing null space



(3a) We know the following:

$$(k+1)P_{k+1} = (2k+1)xP_k - kP_{k-1}$$

$$(k+1)P'_{k+1} = (2k+1)xP'_k + (2k+1)P_k - kP'_{k-1}$$

$$(k+1)P''_{k+1} = (2k+1)xP''_k + 2(2k+1)P'_k - kP''_{k-1}$$

We want to show:

$$(1-x^2)P''_{n+1} - 2xP'_{n+1} + (n+2)(n+1)P_{n+1} = 0.$$

$\boxed{\text{Answer:}}$ We show this using induction and one other identity, given by

$$P'_{n+1} - P'_{n-1} = (2n+1)P_n$$

Base Case: $P_0 = 1 \quad P_1 = x \quad P_2 = \frac{1}{2}(3x^2-1)$

$k=1$:

$$(1-x^2)P''_1 - 2xP'_1 + 2P_1 = -2x + 2x = 0.$$

$k=2$:

$$(1-x^2)P''_2 - 2xP'_2 + 6P_2 = (1-x^2)3 - 2x(3x) + 9x^2 - 3$$

$$= 0$$

Now assume $(1-x^2)P''_p - 2xP'_p + p(p+1)P_p = 0$ for all $p \le n$.

Consider the following expression for $n+1$:

$$\left((1-x^2)P'_{n+1}\right)'$$

$$= \left[(1-x^2)\left(P'_{n-1} + (2n+1)P_n\right)\right]'$$

$$= \left((1-x^2)P'_{n-1}\right)' + (2n+1)\left(-2xP_n + (1-x^2)P'_n\right)$$

$$= -(n-1)n P_{n-1} - 2\left[(n+1)P_{n+1} + nP_{n-1}\right] + n(n+1)\left(P_{n-1} - P_{n+1}\right)$$

$$= \left(-(n-1)n - 2n + n(n+1)\right)P_{n-1} + \left(-2(n+1) - n(n+1)\right)P_{n+1}$$

$$= n\left[-(n-1) - 2 + (n+1)\right]P_{n-1} - (n+1)(n+2)P_{n+1}$$

$$= -(n+1)(n+2)P_{n+1}$$

$$\therefore \left((1-x^2)P'_{n+1}\right)' = (1-x^2)P''_{n+1} - 2xP'_{n+1}$$

$$= -(n+1)(n+2)P_{n+1}$$

$$\boxed{\Rightarrow (1-x^2)P''_{n+1} - 2xP'_{n+1} + (n+1)(n+2)P_{n+1} = 0.}$$

(3b) We know the following:

$T_{k+1} = 2xT_k - T_{k-1}$

$T_{k+1}' = 2xT_k' + 2T_k - T_{k-1}'$

$T_{k+1}'' = 2xT_k'' + 4T_k' - T_{k-1}''$

$(1-x^2)T_k'' - xT_k' + k^2T_k = 0$

$(1-x^2)T_k'' = xT_k' - k^2T_k$

$(1-x^2)T_{k-1}'' = xT_{k-1}' - (k-1)^2T_{k-1}$

We want to prove this: $(1-x^2)T_{k+1}'' = xT_{k+1}' - (k+1)^2T_{k+1}$

Answer: We can use induction to prove this.

Base case: $T_0 = 1 \quad T_1 = x \quad T_2 = 2x^2 - 1$

$k=1$:

$(1-x^2)T_1'' - xT_1' + T_1 = -x + x = 0.$

$k=2$:

$$
\begin{aligned}
&(1-x^2)T_2'' - xT_2' + 4T_2 \\
&= (1-x^2)(4) - x(4x) + 4(2x^2-1) \\
&= 4 - 4x^2 - 4x^2 + 8x^2 - 4 \\
&= 0.
\end{aligned}
$$

Now, assume $(1-x^2)T_p'' - xT_p' + k^2T_p = 0$ is true for all $p \le k$.

We show that it is true for $k+1$.

$$
\begin{aligned}
&(1-x^2)T_{k+1}'' \\
&= (1-x^2)\left[2xT_k'' + 4T_k' - T_{k-1}''\right] \\
&= 2x\left[xT_k' - k^2T_k\right] + 4(1-x^2)T_k' - \left[xT_{k-1}' - (k-1)^2T_{k-1}\right] \\
&= -2x^2T_k' + 4T_k' - xT_{k-1}' - 2xk^2T_k + (k-1)^2T_{k-1}
\end{aligned}
$$

Adding $-xT_{k+1}' + (k+1)^2 T_{k+1}$, we have

$$\underline{-2x^2T_k'} + 4T_k' - \underline{x(T_{k-1}' + T_{k+1}')} - 2xk^2T_k + (k-1)^2T_{k-1} + (k+1)^2T_{k+1}$$

$$= -2x^2T_k' + 4T_k' - x(2xT_k' + 2T_k)$$
$$- k^2(2xT_k - T_{k-1} - T_{k+1}) - 2kT_{k-1} + 2kT_{k+1} + T_{k-1} + T_{k+1}$$
$$= 4(1-x^2)T_k' - \cancel{2xT_k} - 2kT_{k-1} + 2k(2xT_k - T_{k-1}) + \cancel{2xT_k}$$
$$= 4(1-x^2)T_k' + 4kxT_k - 4kT_{k-1}$$

In order to simplify this, we use the identity that

$$(1-x^2)T_k' = kT_{k-1} - kxT_k$$

$$\boxed{\Rightarrow (1-x^2)T_{k+1}'' - xT_{k+1}' + (k+1)^2T_{k+1} = 0.}$$

(3c) $\{p_n\}$ family of orthogonal polynomials.

To get p_{n+1}, we can orthogonalise $x p_n$ using Gram Schmidt orthogonalisation.

$$p_{n+1} = x p_n - \sum_{k=0}^{n} \frac{\langle x p_n, p_k\rangle}{\langle p_k, p_k\rangle} p_k$$

Consider $\langle p_k, x p_n\rangle = \int_a^b x \cdot p_k \cdot p_n \cdot dx$

$$= \langle p_n, x p_k\rangle.$$

$x p_k$ is a polynomial of degree $\leq k+1$.

$$\Rightarrow x p_k = \sum_{j=0}^{k+1} c_j p_j$$

$$\Rightarrow \langle p_n, x p_k\rangle = \sum_{j=0}^{k+1} c_j \langle p_n, p_j\rangle$$

We can see that when $k < n-2$, $k+1 < n-1$,

$\sum_{j=0}^{k+1} c_j \langle p_n, p_j\rangle = 0$. ie, $\langle p_n, x p_k\rangle = 0$ if $k < n-2$

$$\Rightarrow p_{n+1} = x p_n - \sum_{k=0}^{n} \frac{\langle p_n, x p_k\rangle}{\langle p_k, p_k\rangle} p_k$$

$$= x p_n - \frac{\langle p_n, x p_n\rangle}{\langle p_n, p_n\rangle} p_n - \frac{\langle p_n, x p_{n-1}\rangle}{\langle p_{n-1}, p_{n-1}\rangle} p_{n-1}$$

$$\boxed{\Rightarrow p_{n+1} = \left(x - \frac{\langle p_n, x p_n\rangle}{\langle p_n, p_n\rangle}\right) p_n - \frac{\langle p_n, x p_{n-1}\rangle}{\langle p_{n-1}, p_{n-1}\rangle} p_{n-1}}$$

$\therefore$ There is a 3-term recurrence relation for a family of orthogonal polynomials.

Problem 4:

$\{p_n\}$ family of monic orthogonal polynomials

Consider q: a monic polynomial of degree n.

$$q = p_n + \sum_{k=0}^{n-1} a_k p_k$$

$$\langle q, q\rangle = \langle p_n + \sum_{k=0}^{n-1} a_k p_k ,\ p_n + \sum_{k=0}^{n-1} a_k p_k \rangle$$

$$= \langle p_n, p_n\rangle + \underbrace{2\langle \sum_{k=0}^{n-1} a_k p_k , p_n\rangle}_{=0}$$

$$+ \langle \sum_{k=0}^{n-1} a_k p_k ,\ \sum_{k=0}^{n-1} a_k p_k \rangle$$

$$= \langle p_n, p_n\rangle + \sum_{k=0}^{n-1}\sum_{l=0}^{n-1} a_k a_l \underbrace{\langle p_k, p_l\rangle}_{=\delta_{kl}}$$

$$= \langle p_n, p_n\rangle + \sum_{k=0}^{n-1} a_k^2 \langle p_k, p_k\rangle$$

We know $\langle p_k, p_k\rangle > 0 \ \forall \ k = 0, 1, \dots n-1$.

$\therefore \langle q, q\rangle = \langle p_n, p_n\rangle + C$ where $C > 0$ for all n.

$\Rightarrow \langle q, q\rangle \geq \langle p_n, p_n\rangle$.

The equality is achieved when $q = p_n$. or $\sum_{k=0}^{n-1} a_k p_k = 0$.