Homework-2

1. Householder Reflector $F = I - \frac{2vv^*}{v^*v}$

where $v = \|x\|_2 e_1 - x$.

ⓐ $Fy = \lambda y$ if y is an eigenvector and λ is the associated eigenvalue.

$$y - \frac{2vv^*}{v^*v} y = \lambda y$$

$$(1-\lambda) y = \left(\frac{2vv^*}{v^*v}\right) y$$

$$(1-\lambda) y = \frac{2(v^*y)}{v^*v} v$$

So one possible solution is $\boxed{\lambda = 1}$ and the corresponding eigenvector y is perpendicular to v (or the RHS, $v^*y = 0$)

If $\lambda \neq 1$, we can see that y is a scalar multiple of v.

If $y = \alpha v$, $\alpha \in \mathbb{C}$, then $(1-\lambda)\alpha = 2\alpha$ or $\boxed{\lambda = -1}$

The corresponding set of eigenvectors are scalar multiples of v.

ⓑ We know $F^2 = I$

$$(I - 2q_1 q_1^*)(I - 2q_1 q_1^*)$$

$$= I - 4q_1q_1^* + 4q_1(q_1^*q_1)q_1^* = I$$

So $\det(F) = \pm 1$

Now, consider $\det(I + xy^*)$ can be written as $\det\left(\begin{bmatrix} 1 & -y^* \\ x & I \end{bmatrix}\right)$. (where $xy^* = I$)

$= I - x^* y$ (using the determinant expression for a 2×2 matrix).

So $\det(F) = \det\left(I - 2\dfrac{vv^*}{v^*v}\right)$

If $\dfrac{v}{\sqrt{v^*v}} = q$, then

$$\boxed{\det(F) = \det(I - 2qq^*) = \det(I - 2q^*q) = -1}$$

(c) $F^* = \left(I - 2\dfrac{vv^*}{v^*v}\right)^* = I - 2\dfrac{vv^*}{v^*v} = F$

Since F is Hermitian, it has real eigenvalues and the eigendecomposition of $F = Q \Lambda Q^*$ yields a orthonormal matrix Q ($Q^* = Q^{-1}$)

So, we can write $F = Q|\Lambda| \operatorname{sign}(\Lambda) Q^*$, $\operatorname{sign}(\Lambda)$ is a diagonal matrix with signs of each of the eigenvalues in Λ.

Since $\operatorname{Sign}(\Lambda) Q^*$ is unitary, we have ourselves a singular value decomposition for F.

$\boxed{\Rightarrow \text{The singular values of } F = |\pm 1| = 1}$

(since the eigenvalues of $F = \pm 1$).