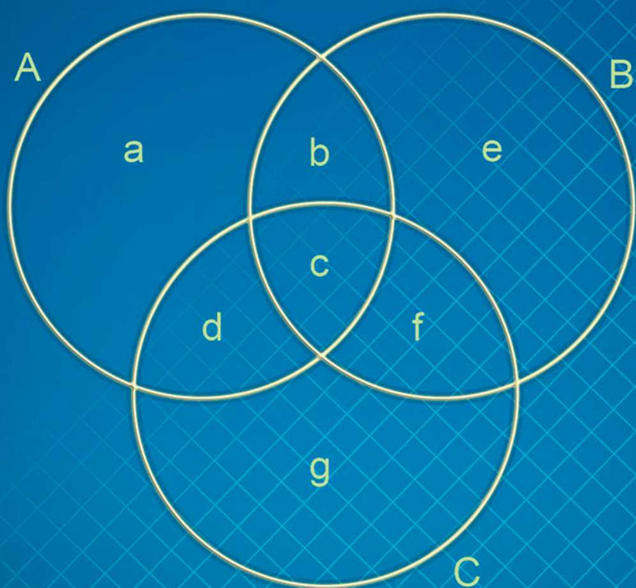
R S AGGARWAL

SENIOR SECONDARY SCHOOL

MATHEMATICS

FOR CLASS 11



BHARATI
BHAWAN
(P & D)

SENIOR SECONDARY SCHOOL MATHEMATICS

FOR CLASS 11

[In accordance with the latest CBSE syllabus]

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Preface

It gives me great pleasure in presenting the new edition of this book. In this edition, the modifications have been dictated by the changes in the CBSE syllabus. The structure and the methods used in the previous editions, which have been appreciated by teachers using the book in classroom conditions, remain unchanged.

In this edition certain topics that are no longer a part of the CBSE syllabus have been retained so that students do not face difficulty in competitive examinations, etc. These topics include conditional identities involving the angles of a triangle, infinite geometric series and exponential series.

The main consideration in writing the book was to present the considerable requirements of the syllabus in as simple a manner as possible. Special attention has been paid to the gradation of problems. This will help students gain confidence in problem-solving.

One problem faced by students is the lack of a comprehensive and carefully selected set of solved problems in textbooks of this kind. I have given due weightage to this aspect. Each set of solved examples is followed by a comprehensive exercise section in which students will get enough questions for practice. Hints have been given to the more difficult questions. Students should take their help as a last resort.

I have received many suggestions and letters of appreciation from teachers all over the country. I thank them all for contributing in the improvement of the book and for their encouragement. I hope they will like this edition as well. And as always, I would like to hear their views on the book.

R S Aggarwal

Mathematics Syllabus

For Class 11

UNIT I. Sets and Functions

1. Sets **20 Periods**

Sets and their representations. Empty set. Finite and Infinite sets. Equal sets. Subsets. Subsets of a set of real numbers especially intervals (with notation). Power set. Universal set. Venn diagrams. Union and Intersection of sets. Difference of sets. Complement of a set. Properties of complement.

2. Relations and Functions **20 Periods**

Ordered pairs, Cartesian product of sets. Number of elements in the cartesian product of two finite sets. Cartesian product of the set of reals with itself (up to R × R × R). Definition of relation, pictorial diagrams, domain, co-domain and range of a relation. Function as a special type of relation. Pictorial representation of a function, domain, co-domain and range of a function. Real valued functions, domain and range of these functions, constant, identity, polynomial, rational, modulus, signum; exponential, logarithmic and greatest integer functions with their graphs. Sum, difference, product and quotient of functions.

3. Trigonometric Functions **20 Periods**

Positive and negative angles. Measuring angles in radians and in degrees and conversion from one measure to another. Definition of trigonometric functions with the help of unit circle. Truth of the identity $\sin^2 x + \cos^2 x = 1$, for all x. Signs of trigonometric functions. Domain and range of trigonometric functions and their graphs. Expressing $\sin(x \pm y)$ and $\cos(x \pm y)$ in terms of $\sin x, \sin y, \cos x, \cos y$, and their simple applications. Deducing identities like the following:

$$\tan(x \pm y) = \frac{\tan x \pm \tan y}{1 \mp \tan x \tan y},\ \cot(x \pm y) = \frac{\cot x \cot y \mp 1}{\cot y \pm \cot x},$$

$$\sin\alpha \pm \sin\beta = 2\sin\frac{1}{2}(\alpha \pm \beta)\cos\frac{1}{2}(\alpha \mp \beta),$$

$$\cos\alpha + \cos\beta = 2\cos\frac{1}{2}(\alpha + \beta)\cos\frac{1}{2}(\alpha - \beta),$$

$$\cos\alpha - \cos\beta = -2\sin\frac{1}{2}(\alpha + \beta)\sin\frac{1}{2}(\alpha - \beta).$$

Identities related to $\sin 2x$, $\cos 2x$, $\tan 2x$, $\sin 3x$, $\cos 3x$ and $\tan 3x$. General solution of trigonometric equations of the type $\sin y = \sin a$, $\cos y = \cos a$ and $\tan y = \tan a$.

UNIT II. Algebra

1. **Principle of Mathematical Induction** **10 Periods**

   Process of the proof by induction, motivating the application of the method by looking at natural numbers as the least inductive subset of real numbers. The principle of mathematical induction and simple applications.

2. **Complex Numbers and Quadratic Equations** **15 Periods**

   Need for complex numbers, especially $\sqrt{-1}$, to be motivated by inability to solve some of the quadratic equations. Algebraic properties of complex numbers. Argand plane and polar representation of complex numbers. Statement of Fundamental Theorem of Algebra, solution of quadratic equations (with real coefficients) in the complex number system. Square root of a complex number.

3. **Linear Inequalities** **15 Periods**

   Linear inequalities. Algebraic solutions of linear inequalities in one variable and their representation on the number line. Graphical representation of linear inequalities in two variables. Graphical method of finding a solution of system of linear inequalities in two variables.

4. **Permutations and Combinations** **10 Periods**

   Fundamental principle of counting. Factorial $n(n!)$. Permutations and combinations, derivation of formulae for ${}^nP_r$ and ${}^nC_r$ and their connections, simple applications.

5. **Binomial Theorem** **10 Periods**

   History, statement and proof of the binomial theorem for positive integral indices. Pascal's triangle, general and middle term in binomial expansion, simple applications.

6. **Sequence and Series** **10 Periods**

   Sequence and Series. Arithmetic progression (AP), Arithmetic mean (AM), Geometric progression (GP), general terms of a GP, sum of first n terms of a GP, infinite GP and its sum, geometric mean (GM), relation between AM and GM. Formulae for the following special sums

   $$\sum_{k=1}^{n} k, \quad \sum_{k=1}^{n} k^2 \text{ and } \sum_{k=1}^{n} k^3.$$

UNIT III. Coordinate Geometry

1. **Straight Lines** **10 Periods**

   Brief recall of two dimensional geometry from earlier classes. Shifting of origin. Slope of a line and angle between two lines. Various forms of equations of a line: parallel to axis, point–slope form, slope–intercept form, two-point form, intercept form and normal form. General equation of a line. Equation of family of lines passing through the point of intersection of two lines. Distance of a point from a line.

2. **Conic Sections** **20 Periods**

   Sections of a cone: circle, ellipse, parabola, hyperbola, a point, a straight line and a pair of intersecting lines as a degenerated case of a conic section. Standard equations and simple properties of parabola, ellipse and hyperbola. Standard equation of a circle.

3. **Introduction to Three-dimensional Geometry** **10 Periods**

   Coordinate axes and coordinate planes in three dimensions. Coordinates of a point. Distance between two points and section formula.

UNIT IV. Calculus

1. **Limits and Derivatives** **30 Periods**

   Derivative introduced as rate of change both as that of distance function and geometrically.

   Intutive idea of limit. Limits of polynomials and rational functions; trigonometric, exponential and logarithmic functions. Definition of derivative, relate it to slope of tangent of the curve, derivative of sum, difference, product and quotient of functions. Derivatives of polynomial and trigonometric functions.

UNIT V. Mathematical Reasoning

1. **Mathematical Reasoning** **10 Periods**

   Mathematically acceptable statements. Connecting words/phrases— consolidating the understanding of "if and only if (necessary and sufficient) condition", "implies", "and/or", "implied by", "and", "or", "there exists" and their use through variety of examples related to real life and mathematics. Validating the statements involving the connecting words, difference between contradiction, converse and contrapositive.

UNIT VI. Statistics and Probability

1. **Statistics** **15 Periods**

   Measures of dispersion: Range, mean deviation, variance and standard deviation of ungrouped/grouped data. Analysis of frequency distributions with equal means but different variances.

2. **Probability** **15 Periods**

   Random experiments: outcomes, sample spaces (set representation). Events: occurrence of events, 'not', 'and' and 'or' events, exhaustive events, mutually exclusive events, Axiomatic (set theoretic) probability, connections with other theories studied in earlier classes. Probability of an event, probability of 'not', 'and' and 'or' events.

Contents

❑

1
Sets

INTRODUCTION

In our mathematical language, *everything in this universe, whether living or non-living, is called an object.*

A given collection of objects is said to be well defined, if we can definitely say whether a given particular object belongs to the collection or not.

SET *A well-defined collection of objects is called a set.*

The objects in a set are called its ***members*** *or* ***elements*** *or* ***points.***

We denote sets by capital letters A, B, C, X, Y, Z, etc.

If a is an element of a set A, we write, $\boldsymbol{a \in A}$, which means that ***a belongs to A*** or that ***a is an element of A.***

If a does not belong to A, we write, $\boldsymbol{a \notin A}$.

ILLUSTRATIONS

(i) The collection of all vowels in the English alphabet contains five elements, namely a, e, i, o, u.
So, this collection is well defined and therefore, it is a set.

(ii) The collection of all odd natural numbers less than 10 contains the numbers 1, 3, 5, 7, 9.
So, this collection is well defined and therefore, it is a set.

(iii) The collection of all prime numbers less than 20 contains the numbers 2, 3, 5, 7, 11, 13, 17, 19.
So, this collection is well defined and therefore, it is a set.

(iv) All possible roots of the quadratic equation $x^2 - x - 6 = 0$ are -2 and 3.
So, the collection of all possible roots of $x^2 - x - 6 = 0$ is well defined and therefore, it is a set.

(v) The collection of all rivers of India, is clearly well defined and therefore, it is a set.
Clearly, river Ganga belongs to this set while river Nile does not belong to it.

(vi) The collection of five most talented writers of India is not a set, since no rule has been given for deciding whether a given writer is talented or not.

(vii) The collection of most dangerous animals of the world is not a set, since no rule has been given for deciding whether a given animal is dangerous or not.

(viii) The collection of five most renowned mathematicians of the world is not a set, since there is no criterion for deciding whether a mathematician is renowned or not.

(ix) The collection of all beautiful girls of India is not a set, since the term '*beautiful*' is vague and it is not well defined.

Similarly, *'rich persons', 'honest persons', 'good players', 'old people', 'young men'*, etc., do not form sets.

However, *'blind persons', 'dumb persons', 'illiterate persons', 'retired persons'*, etc., form sets.

HOW TO DESCRIBE OR SPECIFY A SET?

There are two methods of describing a set.

I. ROSTER FORM, OR TABULATION METHOD *Under this method, we list all the members of the set within braces { } and separate them by commas.*

Note that the order in which the elements are listed, is immaterial.

EXAMPLES *Write each of the following sets in the roster form:*

(i) A = *set of all factors of* 24.

(ii) B = *set of all prime numbers between* 50 *and* 70.

(iii) C = *set of all integers between* $-\frac{3}{2}$ *and* $\frac{11}{2}$.

(iv) D = *set of all consonants in the English alphabet which precede k.*

(v) E = *set of all letters in the word 'TRIGONOMETRY'.*

(vi) F = *set of all months having* 30 *days.*

SOLUTION (i) All factors of 24 are 1, 2, 3, 4, 6, 8, 12, 24.

$\therefore \quad A = \{1, 2, 3, 4, 6, 8, 12, 24\}$.

(ii) All prime numbers between 50 and 70 are 53, 59, 61, 67.

$\therefore \quad B = \{53, 59, 61, 67\}$.

(iii) All integers between $-\frac{3}{2}$ and $\frac{11}{2}$ are –1, 0, 1, 2, 3, 4, 5.

$\therefore \quad C = \{-1, 0, 1, 2, 3, 4, 5\}$.

(iv) All consonants preceding k are b, c, d, f, g, h, j.

$\therefore \quad D = \{b, c, d, f, g, h, j\}$.

(v) It may be noted here that the repeated letters are taken only once each.

$\therefore \quad E = \{T, R, I, G, O, N, M, E, Y\}$.

(vi) We know that the months having 30 days are April, June, September, November.

$\therefore \quad F = \{\text{April, June, September, November}\}$.

NOTE We denote the sets of *all natural numbers, all integers, all rational numbers* and *all real numbers* by N, Z, Q and R respectively.

II. SET-BUILDER FORM *Under this method of describing a set, we list the property or properties satisfied by all the elements of the set.*

We write, $\{\boldsymbol{x : x}$ **has properties** $\boldsymbol{P}\}$.

We read it as, '*the set of all those x such that each x satisfies properties P*'.

ILLUSTRATIONS

EXAMPLE 1 *Write the set* $A = \{1, 2, 3, 4, 5, 6, 7\}$ *in the set-builder form.*

SOLUTION Clearly, A = set of all natural numbers less than 8.

Thus, in the set-builder form, we write it as

$$A = \{x : x \in N \text{ and } x < 8\}.$$

EXAMPLE 2 *Write the set* $B = \{1, 2, 4, 7, 14, 28\}$ *in the set-builder form.*

SOLUTION Clearly, B = set of all factors of 28.

Thus, in the set-builder form, we write it as

$$B = \{x : x \in N \text{ and } x \text{ is a factor of } 28\}.$$

EXAMPLE 3 *Write the set* $C = \{2, 4, 8, 16, 32\}$ *in the set-builder form.*

SOLUTION Clearly, $C = \{2^1, 2^2, 2^3, 2^4, 2^5\}$.

Thus, in the set-builder form, we write it as

$$C = \{x : x = 2^n, \text{ where } n \in N \text{ and } 1 \leq n \leq 5\}.$$

EXAMPLE 4 *Write the set* $D = \{-6, -4, -2, 0, 2, 4, 6\}$ *in the set-builder form.*

SOLUTION Clearly, D = set of even integers from -6 to 6.

Thus, in the set-builder form, we write it as

$$D = \{x : x = 2n, \text{ where } n \in Z \text{ and } -3 \leq n \leq 3\}.$$

EXAMPLE 5 *Write the set* $E = \{3, 6, 9, 12, 15, 18\}$ *in the set-builder form.*

SOLUTION Clearly, $E = \{3 \times 1, 3 \times 2, 3 \times 3, 3 \times 4, 3 \times 5, 3 \times 6\}$.

Thus, in the set-builder form, we write it as

$$E = \{x : x = 3n, \text{ where } n \in N \text{ and } 1 \leq n \leq 6\}.$$

EXAMPLE 6 *Write the set* $F = \left\{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \frac{6}{7}, \frac{7}{8}, \frac{8}{9}\right\}$ *in the set-builder form.*

SOLUTION Clearly, we have

$$F = \left\{x : x = \frac{n}{(n+1)}, \text{ where } n \in N \text{ and } 1 \leq n \leq 8\right\}.$$

EXAMPLE 7 *Write the set* $G = \{1, 3, 5, 7, 9, 11, \ldots\}$ *in the set-builder form.*

SOLUTION Clearly, G = set of all odd natural numbers.

Thus, in the set-builder form, we write it as

$$G = \{x : x \in N \text{ and } x \text{ is odd}\}.$$

EXAMPLE 8 *Write the set* $H = \{1, 4, 9, 16, 25, 36, \ldots\}$ *in the set-builder form.*

SOLUTION Clearly, H is the set of the squares of all natural numbers.

So, in the set-builder form, we write it as

$$H = \{x : x = n^2, \text{ where } n \in N\}.$$

EXAMPLE 9 *Match each of the sets on the left in the roster form with the same set on the right given in set-builder form:*

(i) $\{23, 29\}$	*(a)* $\{x : x = 3^n, n \in N \text{ and } 1 \le n \le 5\}$
(ii) $\{B, E, T, R\}$	*(b)* $\{x : x = n^3, n \in N \text{ and } 2 \le n \le 6\}$
(iii) $\{3, 9, 27, 81, 243\}$	*(c)* $\{x : x \text{ is prime}, 20 < x < 30\}$
(iv) $\{8, 27, 64, 125, 216\}$	*(d)* $\{x : x \text{ is a letter of the word 'BETTER'}\}$

SOLUTION (i) $\{23, 29\}$ = set of prime numbers between 20 and 30

$= \{x : x \text{ is prime}, 20 < x < 30\}$.

$\therefore$ (i) $\leftrightarrow$ (c).

(ii) $\{B, E, T, R\}$ = set of letters in the word 'BETTER'}

$= \{x : x \text{ is a letter in the word 'BETTER'}\}$.

$\therefore$ (ii) $\leftrightarrow$ (d).

(iii) $\{3, 9, 27, 81, 243\} = \{3^1, 3^2, 3^3, 3^4, 3^5\}$

$= \{x : x = 3^n, n \in N \text{ and } 1 \le n \le 5\}$.

$\therefore$ (iii) $\leftrightarrow$ (a).

(iv) $\{8, 27, 64, 125, 216\} = \{2^3, 3^3, 4^3, 5^3, 6^3\}$

$= \{x : x = n^3, n \in N, 2 \le n \le 6\}$.

$\therefore$ (iv) $\leftrightarrow$ (b).

EXERCISE 1A

1. Which of the following are sets? Justify your answer.

(i) The collection of all whole numbers less than 10.

(ii) The collection of good hockey players in India.

(iii) The collection of all questions in this chapter.

(iv) The collection of all difficult chapters in this book.

(v) A collection of Hindi novels written by Munshi Prem Chand.

(vi) A team of 11 best cricket players of India.

(vii) The collection of all the months of the year whose names begin with the letter M.

(viii) The collection of all interesting books.

(ix) The collection of all short boys of your class.

(x) The collection of all those students of your class whose ages exceed 15 years.

(xi) The collection of all rich persons of Kolkata.

(xii) The collection of all persons of Kolkata whose assessed annual incomes exceed (say) ₹ 20 lakh in the financial year 2016–17.

(xiii) The collection of all interesting dramas written by Shakespeare.

2. Let A be the set of all even whole numbers less than 10.

(a) Write A in roster form.

(b) Fill in the blanks with the approximate symbol $\in$ or $\notin$:

(i) 0 ..... A (ii) 10 ..... A (iii) 3 ..... A (iv) 6 ..... A

3. Write the following sets in roster form:

(i) $A = \{x : x \text{ is a natural number}, 30 \leq x < 36\}$.

(ii) $B = \{x : x \text{ is an integer and } -4 < x < 6\}$.

(iii) $C = \{x : x$ is a two-digit number such that the sum of its digits is 9$\}$.

(iv) $D = \{x : x \text{ is an integer}, x^2 \leq 9\}$.

(v) $E = \{x : x$ is a prime number, which is a divisor of 42$\}$.

(vi) $F = \{x : x$ is a letter in the word 'MATHEMATICS'$\}$.

(vii) $G = \{x : x \text{ is a prime number and } 80 < x < 100\}$.

(viii) $H = \{x : x \text{ is a perfect square and } x < 50\}$.

(ix) $J = \{x : x \in R \text{ and } x^2 + x - 12 = 0\}$.

(x) $K = \{x : x \in N, x \text{ is a multiple of 5 and } x^2 < 400\}$.

4. List all the elements of each of the sets given below:

(i) $A = \{x : x = 2n, n \in N \text{ and } n \leq 5\}$.

(ii) $B = \{x : x = 2n + 1, n \in W \text{ and } n < 5\}$.

(iii) $C = \left\{x : x = \frac{1}{n}, n \in N \text{ and } n < 6\right\}$.

(iv) $D = \{x : x = n^2, n \in N \text{ and } 2 \leq n \leq 5\}$.

(v) $E = \{x : x \in Z \text{ and } x^2 = x\}$.

(vi) $F = \left\{x : x \in Z \text{ and } -\frac{1}{2} < x < \frac{13}{2}\right\}$.

(vii) $G = \left\{x : x = \frac{1}{(2n-1)}, n \in N \text{ and } 1 \leq n \leq 5\right\}$.

(viii) $H = \{x : x \in Z, |x| \leq 2\}$.

5. Write each of the sets given below in set-builder form:

(i) $A = \left\{1, \frac{1}{4}, \frac{1}{9}, \frac{1}{16}, \frac{1}{25}, \frac{1}{36}, \frac{1}{49}\right\}$ (ii) $B = \left\{\frac{1}{2}, \frac{2}{5}, \frac{3}{10}, \frac{4}{17}, \frac{5}{26}, \frac{6}{37}, \frac{7}{50}\right\}$

(iii) $C = \{53, 59, 61, 67, 71, 73, 79\}$ (iv) $D = \{-1, 1\}$

(v) $E = \{14, 21, 28, 35, 42, \ldots, 98\}$

6. Match each of the sets on the left described in roster form with the same set on the right described in the set-builder form:

(i) $\{-5, 5\}$ (a) $\{x : x \in Z \text{ and } x^2 < 16\}$

(ii) $\{1, 2, 3, 6, 9, 18\}$ (b) $\{x : x \in N \text{ and } x^2 = x\}$

(iii) $\{-3, -2, -1, 0, 1, 2, 3\}$ (c) $\{x : x \in Z \text{ and } x^2 = 25\}$

(iv) {P, R, I, N, C, A, L} (d) $\{x : x \in N \text{ and } x \text{ is a factor of } 18\}$

(v) $\{1\}$ (e) $\{x : x$ is a letter in the word 'PRINCIPAL'$\}$

ANSWERS (EXERCISE 1A)

1. (i), (iii), (v), (vii), (x), (xii)

2. (a) $A = \{0, 2, 4, 6, 8\}$ (b) (i) $\in$ (ii) $\notin$ (iii) $\notin$ (iv) $\in$

3. (i) $A = \{30, 31, 32, 33, 34, 35\}$ (ii) $B = \{-3, -2, -1, 0, 1, 2, 3, 4, 5\}$

(iii) $C = \{18, 81, 27, 72, 36, 63, 45, 54, 90\}$

(iv) $D = \{-3, -2, -1, 0, 1, 2, 3\}$ (v) $E = \{2, 3, 7\}$

(vi) $F = \{$M, A, T, H, E, I, C, S$\}$ (vii) $G = \{83, 89, 97\}$

(viii) $H = \{1, 4, 9, 16, 25, 36, 49\}$ (ix) $J = \{-4, 3\}$

(x) $K = \{5, 10, 15\}$

4. (i) $A = \{2, 4, 6, 8, 10\}$ (ii) $B = \{1, 3, 5, 7, 9\}$

(iii) $C = \left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}\right\}$ (iv) $D = \{4, 9, 16, 25\}$

(v) $E = \{0, 1\}$ (vi) $F = \{0, 1, 2, 3, 4, 5, 6\}$

(vii) $G = \left\{1, \frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \frac{1}{9}\right\}$ (viii) $H = \{-2, -1, 0, 1, 2\}$

5. (i) $A = \{x : x = \frac{1}{n^2}, n \in N \text{ and } 1 \le n \le 7\}$

(ii) $B = \left\{x : x = \frac{n}{(n^2+1)}, n \in N \text{ and } 1 \le n \le 7\right\}$

(iii) $C = \{x : x \text{ is prime}, 50 < x < 80\}$

(iv) $D = \{x : x \in Z, x^2 = 1\}$

(v) $E = \{x : x = 7n, n \in N, 2 \le n \le 14\}$

6. (i) $\leftrightarrow$ (c), (ii) $\leftrightarrow$ (d), (iii) $\leftrightarrow$ (a), (iv) $\leftrightarrow$ (e), (v) $\leftrightarrow$ (b)

HINTS TO SOME SELECTED QUESTIONS

1. (i) Clearly, all whole numbers less than 10 are 0, 1, 2, 3, 4, 5, 6, 7, 8, 9.

(ii) Clearly, there is no specific criterion to decide whether a given hockey player of India, is good or not.

So, the given collection is not a set.

(iii) The collection of all questions in this chapter is a set, because, if a question is given then we can easily decide whether it is a question of the chapter or not.

(iv) Clearly, the term '*difficult*' is vague. So, the given collection is not well defined and therefore, it is not a set.

(v) Suppose we are given a collection of Hindi novels written by Munshi Prem Chand. Now, if we take any Hindi novel then we can clearly decide whether it belongs to our collection or not.

So, the given collection is a set.

(vi) The term '*best*' is vague. So, the given collection is not a set.

(vii) The given collection has definite members, namely March and May. So, this collection is a set.
(viii) The term '*interesting*' is vague. So, the given collection is not a set.
(ix) The term '*short*' is vague. So, the given collection is not a set.
(x) Clearly, it contains definite members. So, it is a set.
(xi) The term '*rich*' is vague. So, the given collection is not a set.
(xii) The given collection is clearly well defined. So, it is a set.
(xiii) The term '*interesting*' is vague. So, the given collection is not a set.

SOME TERMS RELATED TO SETS

EMPTY SET *A set containing no element at all is called* ***the empty set,*** *or* ***the null set,*** *or* ***the void set,*** denoted by ϕ, or { }.

A set which has at least one element is called a ***nonempty set.***

Examples of empty sets

(i) $\{x : x \in N \text{ and } 2 < x < 3\} = \phi$, since there is no natural number lying between 2 and 3.
(ii) $\{x : x \text{ is a number}, x \neq x\} = \phi$, since there is no number which is not equal to itself.
(iii) $\{x : x \in N, x < 5 \text{ and } x > 7\} = \phi$, since there is no natural number which is less than 5 and greater than 7.
(iv) $\{x : x \in R \text{ and } x^2 = -1\} = \phi$, since there is no real number whose square is -1.
(v) $\{x : x \text{ is rational and } x^2 - 2 = 0\} = \phi$, since there is no rational number whose square is 2.
(vi) $\{x : x \text{ is an even prime number greater than } 2\} = \phi$, since there is no prime number which is even and greater than 2.
(vii) $\{x : x \text{ is a point common to two parallel lines}\} = \phi$, since there is no point common to two parallel lines.

SINGLETON SET *A set containing exactly one element is called a singleton set.*

Examples of singleton sets

(i) {0} is a singleton set whose only element is 0.
(ii) {15} is a singleton set whose only element is 15.
(iii) {–8} is a singleton set whose only element is –8.
(iv) $\{x : x \in N \text{ and } x^2 = 4\} = \{2\}$, since 2 is the only natural number whose square is 4.
However, $\{x : x \in Z \text{ and } x^2 = 4\} = \{-2, 2\}$, which is not a singleton set.
(v) Consider the set $\{x : x \in R \text{ and } x^3 - 1 = 0\}$.
Now, $x^3 - 1 = 0 \Rightarrow (x-1)(x^2 + x + 1) = 0$

$$\Rightarrow x = 1 \text{ or } x = \frac{-1 \pm \sqrt{1-4}}{2} = \frac{-1 \pm \sqrt{-3}}{2}.$$

Thus, the given equation has one real root, namely $x = 1$.

$\therefore$ $\{x : x \in R \text{ and } x^3 - 1 = 0\} = \{1\}$, which is a singleton set.

FINITE AND INFINITE SETS

An empty set or a nonempty set in which the process of counting of elements surely comes to an end is called a finite set. A set which is not finite is called an infinite set.

The number of distinct elements contained in a finite set A is denoted by $n(A)$.

Examples of finite sets

(i) Let $A = \{2, 4, 6, 8, 10, 12\}$.
Then, A is clearly a finite set and $n(A) = 6$.

(ii) Let B = set of all letters in the English alphabet.
Then, $n(B) = 26$ and therefore, B is finite.

(iii) Let $C = \{x : x \in Z \text{ and } x^2 - 36 = 0\}$.
Then, $C = \{-6, 6\}$, which is clearly a finite set and $n(C) = 2$.

(iv) The set of all persons on earth is a finite set.

(v) The set of all animals on earth is a finite set.

Examples of infinite sets

(i) The set of all points on the arc of a circle is an infinite set.

(ii) The set of all points on a line segment is an infinite set.

(iii) The set of all circles passing through a given point is an infinite set.

(iv) The set of all straight lines parallel to a given line, say the x-axis, is an infinite set.

(v) The set of all positive integral multiples of 5 is an infinite set.
Let Z^+ be the set of all positive integers.
Then, $\{5x : x \in Z^+\} = \{5, 10, 15, 20, 25, \ldots\}$ is an infinite set.

(vi) Each of the sets N, Z, Q and R is an infinite set.

NOTE All infinite sets cannot be described in roster form.

For example, the set R of all real numbers cannot be described in this form, since the elements of this set do not follow a particular pattern.

EQUAL SETS *Two nonempty sets A and B are said to be equal, if they have exactly the same elements and we write, $A = B$.*

Otherwise, the sets are unequal and we write, $A \neq B$.

REMARKS (i) The elements of a set may be listed in any order.
Thus, $\{1, 2, 3\} = \{2, 1, 3\} = \{3, 2, 1\}$.

(ii) The repetition of elements in a set has no meaning.
Thus, $\{1, 1, 2, 2, 3\} = \{1, 2, 3\}$.

Some examples of equal sets

EXAMPLE 1 *Let A = set of letters in the word 'follow', and B = set of letters in the word 'wolf'. Show that $A = B$.*

SOLUTION Clearly, we have

$$A = \{\text{f, o, l, w}\} \text{ and } B = \{\text{w, o, l, f}\}.$$

Clearly, A and B have exactly same elements.

$\therefore \quad A = B$.

EXAMPLE 2 *Let $A = \{p, q, r, s\}$ and $B = \{q, r, p, s\}$. Are A and B equal?*

SOLUTION Since A and B have exactly the same elements, so $A = B$.

EXAMPLE 3 *Show that ϕ, $\{0\}$ and 0 are all different.*

SOLUTION We know that ϕ is a set containing no element at all.

And, $\{0\}$ is a set containing one element, namely 0.

Also, 0 is a number, not a set.

Hence, ϕ, $\{0\}$ and 0 are all different.

EXAMPLE 4 *Let $A = \{x : x \in N, x^2 - 9 = 0\}$ and $B = \{x : x \in Z, x^2 - 9 = 0\}$. Show that $A \neq B$.*

SOLUTION $x^2 - 9 = 0 \Rightarrow (x+3)(x-3) = 0 \Rightarrow x = -3$ or $x = 3$.

$\therefore \quad A = \{x : x \in N, x^2 - 9 = 0\} = \{3\}$ $\quad [\because -3 \notin N]$

and $\quad B = \{x : x \in Z, x^2 - 9 = 0\} = \{-3, 3\}$.

Hence, $A \neq B$.

EQUIVALENT SETS *Two finite sets A and B are said to be equivalent, if $n(A) = n(B)$.*

Equal sets are always equivalent. But, equivalent sets need not be equal.

EXAMPLE 1 Let $A = \{1, 3, 5\}$ and $B = \{2, 4, 6\}$.

Then, $n(A) = n(B) = 3$.

So, A and B are equivalent.

Clearly, $A \neq B$.

Hence, A and B are equivalent sets but not equal.

EXAMPLE 2 *Show that $\{0\}$ and ϕ are not equivalent sets.*

SOLUTION Let $A = \{0\}$ and $B = \phi$.

Then, clearly $n(A) = 1$ and $n(B) = 0$.

$\therefore \quad n(A) \neq n(B)$ and hence A and B are not equivalent sets.

EXERCISE 1B

1. Which of the following are examples of the null set?

(i) Set of odd natural numbers divisible by 2.

(ii) Set of even prime numbers.

(iii) $A = \{x : x \in N, 1 < x \leq 2\}$
(iv) $B = \{x : x \in N, 2x + 3 = 4\}$
(v) $C = \{x : x \text{ is prime}, 90 < x < 96\}$
(vi) $D = \{x : x \in N, x^2 + 1 = 0\}$
(vii) $E = \{x : x \in W, x + 3 \leq 3\}$
(viii) $F = \{x : x \in Q, 1 < x < 2\}$
(ix) $G = \{0\}$

2. Which of the following are examples of the singleton set?
(i) $\{x : x \in Z, x^2 = 4\}$
(ii) $\{x : x \in Z, x + 5 = 0\}$
(iii) $\{x : x \in Z, |x| = 1\}$
(iv) $\{x : x \in N, x^2 = 16\}$
(v) $\{x : x \text{ is an even prime number}\}$

3. Which of the following are pairs of equal sets?
(i) A = set of letters in the word, 'ALLOY'
B = set of letters in the word, 'LOYAL'
(ii) C = set of letters in the word, 'CATARACT'
D = set of letters in the word 'TRACT'
(iii) $E = \{x : x \in Z, x^2 \leq 4\}$ and $F = \{x : x \in Z, x^2 = 4\}$
(iv) $G = \{-1, 1\}$ and $H = \{x : x \in Z, x^2 - 1 = 0\}$
(v) $J = \{2, 3\}$ and $K = \{x : x \in Z, (x^2 + 5x + 6) = 0\}$

4. Which of the following are pairs of equivalent sets?
(i) $A = \{-2, -1, 0\}$ and $B = \{1, 2, 3\}$
(ii) $C = \{x : x \in N, x < 3\}$ and $D = \{x : x \in W, x < 3\}$
(iii) $E = \{\text{a, e, i, o, u}\}$ and $F = \{\text{p, q, r, s, t}\}$

5. State whether the given set is finite or infinite:
(i) A = set of all triangles in a plane
(ii) B = set of all points on the circumference of a circle
(iii) C = set of all lines parallel to the y-axis
(iv) D = set of all leaves on a tree
(v) E = set of all positive integers greater than 500
(vi) $F = \{x \in R: 0 < x < 1]$
(vii) $G = \{x \in Z: x < 1\}$
(viii) $H = \{x \in Z: -15 < x < 15\}$
(ix) $J = \{x : x \in N \text{ and } x \text{ is prime}\}$
(x) $K = \{x : x \in N \text{ and } x \text{ is odd}\}$
(xi) L = set of all circles passing through the origin (0, 0)

6. Rewrite the following statements using set notation:
(i) a is an element of set A.
(ii) b is not an element of A.
(iii) A is an empty set and B is a nonempty set.

(iv) Number of elements in A is 6.
(v) 0 is a whole number but not a natural number.

ANSWERS (EXERCISE 1B)

1. (i), (iv), (v), (vi) **2.** (ii), (iv), (v)
3. (i) $A = B$ (ii) $C = D$ (iv) $G = H$
4. (i) A and B are equivalent sets (iii) E and F are equivalent sets
5. (i) infinite (ii) infinite (iii) infinite (iv) finite (v) infinite
(vi) infinite (vii) infinite (viii) finite (ix) infinite (x) infinite
(xi) infinite
6. (i) $a \in A$ (ii) $b \notin A$ (iii) $A = \phi$ and $B \neq \phi$ (iv) $n(A) = 6$
(v) $0 \in W$ but $0 \notin N$

SUBSETS

SUBSET *A set A is said to be a subset of set B if every element of A is also an element of B, and we write, $A \subseteq B$.*

SUPERSET *If $A \subseteq B$, then B is called a superset of A, and we write, $B \supseteq A$.*

PROPER SUBSET *If $A \subseteq B$ and $A \neq B$ then A is called a proper subset of B and we write, $A \subset B$.*

REMARK If there exists even a single element in A which is not in B, then A is not a subset of B, and we write, $A \not\subset B$.

Examples of subsets

EXAMPLE 1 Let $A = \{2, 3, 5\}$ and $B = \{2, 3, 5, 7, 9\}$.
Then, every element of A is an element of B.
$\therefore \quad A \subseteq B$ but $A \neq B$.
Hence, A is a proper subset of set B, i.e., $A \subset B$.

EXAMPLE 2 Let $A = \{1, 2\}$ and $B = \{2, 3, 5\}$.
Then, $1 \in A$ but $1 \notin B$.
$\therefore \quad A \nsubseteq B$.
Again, $3 \in B$ but $3 \notin A$.
$\therefore \quad B \nsubseteq A$.
Thus, $A \nsubseteq B$ and $B \nsubseteq A$.

EXAMPLE 3 Clearly, $N \subset W \subset Z \subset Q \subset R$.
But, $0 \in W$ and $0 \notin N$.
$\therefore \quad W \nsubseteq N$.

EXAMPLE 4 *Let* $A = \{1, \{2, 3\}, 4\}$.
Then, which of the following statements is true?
(i) $\{2, 3\} \in A$ *(ii)* $\{2, 3\} \subset A$
Rectify the wrong statement.

SOLUTION Clearly, A is a set containing three elements, namely 1, $\{2, 3\}$ and 4.

(i) $\{2, 3\} \in A$ is a true statement.

(ii) $\{2, 3\} \subset A$ is wrong.
On rectifying this statement, we get $\{\{2, 3\}\} \subset A$ as true statement.

SOME RESULTS ON SUBSETS

THEOREM 1 *Every set is a subset of itself.*

PROOF Let A be any set.
Then, each element of A is in A.
$\therefore \quad A \subseteq A$.
Hence, every set is a subset of itself.

THEOREM 2 *The empty set is a subset of every set.*

PROOF Let A be any set and ϕ be the empty set.
Since ϕ contains no element at all, so there is no element of ϕ which is not contained in A.
Hence, $\phi \subset A$.

THEOREM 3 *The total number of subsets of a set containing n elements is* 2^n.

PROOF Let A be a finite set containing n elements. Then,

number of subsets of A each containing no element $= 1 = {}^nC_0$.
number of subsets of A each containing 1 element $= {}^nC_1$.
number of subsets of A each containing 2 elements $= {}^nC_2$.
....
....
number of subsets of A each containing n elements $= {}^nC_n$.

$\therefore$ total number of subsets of A

$$= ({}^nC_0 + {}^nC_1 + {}^nC_2 + \ldots + {}^nC_n)$$
$$= (1 + 1)^n = 2^n \qquad \text{[using binomial theorem].}$$

UNIVERSAL SET *If there are some sets under consideration then there happens to be a set which is a superset of each one of the given sets. Such a set is known as the universal set for those sets. We shall denote a universal set by U.*

EXAMPLE 1 Let $A = \{1, 2, 3\}$, $b = \{2, 3, 4, 5\}$ and $C = \{6, 7\}$.
If we consider the set $U = \{1, 2, 3, 4, 5, 6, 7\}$ then clearly, U is a superset of each of the given sets.
Hence, U is the universal set.

EXAMPLE 2 When we discuss sets of lines, triangles or circles in two-dimensional geometry, the plane in which these lines, triangles or circles lie, is the universal set.

SUBSETS OF THE SET *R* OF ALL REAL NUMBERS

(i) $N = \{1, 2, 3, 4, 5, \ldots\}$ is the set of all natural numbers.

(ii) $W = \{0, 1, 2, 3, 4, 5, \ldots\}$ is the set of all whole numbers.

(iii) $Z = \{\ldots, -4, -3, -2, -1, 0, 1, 2, 3, 4, \ldots\}$ is the set of all integers.

$Z^+ = \{1, 2, 3, 4, 5, \ldots\}$ is the set of all positive integers.

$Z^- = \{-1, -2, -3, -4, \ldots\}$ is the set of all negative integers.

Sometimes, we denote the set of all integers by I.

(iv) $Q = \left\{x : x = \frac{p}{q}, \text{ where, } p, q \in Z \text{ and } q \neq 0\right\}$ is the set of all rational numbers.

The set of all positive rational numbers is denoted by Q^+.

(v) $T = \{x : x \in R \text{ and } x \notin Q\}$ is the set of all irrational numbers.

INTERVALS AS SUBSETS OF *R*

Let $a, b \in R$ and $a < b$. Then, we define:

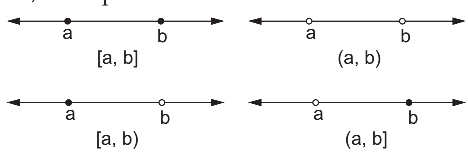
(i) *Closed Interval* $[a, b] = \{x \in R : a \leq x \leq b\}$.

(ii) *Open Interval* (a, b) or $]a, b[= \{x \in R : a < x < b\}$.

(iii) *Right Half Open Interval* $[a, b)$ or $[a, b[= \{x \in R : a \leq x < b\}$.

(iv) *Left Half Open Interval* $(a, b]$ or $]a, b] = \{x \in R : a < x \leq b\}$.

On the real line, we represent these intervals as shown below:



LENGTH OF AN INTERVAL The length of each of the intervals $[a, b]$, (a, b) $[a, b)$ and $(a, b]$ is $(b - a)$.

Examples on intervals

(i) $[-2, 3] = \{x \in R : -2 \leq x \leq 3\}$

(ii) $(-2, 3) = \{x \in R : -2 < x < 3\}$

(iii) $[-2, 3) = \{x \in R : -2 \leq x < 3\}$

(iv) $(-2, 3] = \{x \in R : -2 < x \leq 3\}$

POWER SET *The set of all subsets of a given set A is called the power set of A, denoted by $P(A)$.*

If $n(A) = m$ then $n[P(A)] = 2^m$.

SOLVED EXAMPLES ON SUBSETS, POWER SET AND INTERVALS

EXAMPLE 1 *Write down all possible subsets of $A = \{4\}$.*

SOLUTION All possible subsets of A are $\phi, \{4\}$.

$\therefore \quad P(A) = \{\phi, \{4\}\}$.

Here, $n(A) = 1$ and $n[P(A)] = 2 = 2^1$.

EXAMPLE 2 *Write down all possible subsets of* $A = \{2, 3\}$.

SOLUTION All possible subsets of A are

$\phi, \{2\}, \{3\}, \{2, 3\}$.

$\therefore \quad P(A) = \{\phi, \{2\}, \{3\}, \{2, 3\}\}$

Thus, $n(A) = 2$ and $n\{P(A)\} = 4 = 2^2$.

EXAMPLE 3 *Write down all possible subsets of* $A = \{-1, 0, 1\}$.

SOLUTION All possible subsets of A are

$\phi, \{-1\}, \{0\}, \{1\}, \{-1, 0\}, \{0, 1\}, \{-1, 1\}$, and $\{-1, 0, 1\}$.

$\therefore \quad P(A) = \{\phi, \{-1\}, \{0\}, \{1\}, \{-1, 0\}, \{0, 1\}, \{-1, 1\}, \{-1, 0, 1\}\}$.

Thus, $n(A) = 3$ and $n\{P(A)\} = 8 = 2^3$.

EXAMPLE 4 *Write down all possible subsets of* $A = \{1, \{2, 3\}\}$.

SOLUTION Here, A contains two elements, namely 1 and $\{2, 3\}$.

Let $\{2, 3\} = B$, then $A = \{1, B\}$.

$\therefore \quad P(A) = \{\phi, \{1\}, \{B\}, \{1, B\}\}$

$\Rightarrow \quad P(A) = \{\phi, \{1\}, \{\{2, 3\}\}, \{1, \{2, 3\}\}\}$.

EXAMPLE 5 *Write down all possible subsets of* ϕ.

SOLUTION ϕ has only one subset, namely ϕ.

$\therefore \quad P(\phi) = \{\phi\}$.

EXAMPLE 6 *Write each of the following subsets of R as an interval:*

(i) $A = \{x : x \in R, -3 < x \leq 5\}$ *(ii)* $B = \{x : x \in R, -5 < x < -1\}$

(iii) $C = \{x : x \in R, -2 \leq x < 0\}$ *(iv)* $D = \{x : x \in R, -1 \leq x \leq 4\}$

Find the length of each of the above intervals.

SOLUTION We have

(i) $A = \{x : x \in R, -3 < x \leq 5\} = (-3, 5]$. Length $(A) = 5 - (-3) = 8$.

(ii) $B = \{x : x \in R, -5 < x < -1\} = (-5, -1)$. Length $(B) = -1 - (-5) = 4$.

(iii) $C = \{x : x \in R, -2 \leq x < 0\} = [-2, 0)$. Length $(C) = 0 - (-2) = 2$.

(iv) $D = \{x : x \in R, -1 \leq x \leq 4\} = [-1, 4]$. Length $(D) = 4 - (-1) = 5$.

EXAMPLE 7 *Write each of the following intervals in the set-builder form:*

(i) $A = (2, 5)$ *(ii)* $B = [-4, 7]$ *(iii)* $C = [-8, 0)$ *(iv)* $D = (5, 9]$

SOLUTION We have

(i) $A = (2, 5) = \{x : x \in R, 2 < x < 5\}$.

(ii) $B = [-4, 7] = \{x : x \in R, -4 \leq x \leq 7\}$.

(iii) $C = [-8, 0) = \{x : x \in R, -8 \leq x < 0\}$.

(iv) $D = (5, 9] = \{x : x \in R, 5 < x \leq 9\}$.

EQUAL SETS *Two sets A and B are said to be equal, if every element of A is in B and every element of B is in A, and we write, A = B.*

REMARKS (i) The elements of a set may be listed in any order.

Thus, $\{1, 2, 3\} = \{3, 1, 2\} = \{2, 3, 1\}$, etc.

(ii) The repetition of elements in a set is meaningless.

Thus, $[2, 4, 6\} = \{2, 2, 4, 6, 6\}$.

THEOREM 4 *Let A and B be two sets. Then, prove that $A = B \Leftrightarrow A \subseteq B$ and $B \subseteq A$.*

PROOF Let $A = B$.

Then, by definition of equal sets, every element of A is in B and every element of B is in A.

$\therefore \quad A \subseteq B$ and $B \subseteq A$.

Thus, $(A = B) \Rightarrow (A \subseteq B$ and $B \subseteq A)$.

Again, let $A \subseteq B$ and $B \subseteq A$.

Then, by the definition of a subset, it follows that every element of A is in B and every element of B is in A.

Consequently, $A = B$.

Thus, $(A \subseteq B$ and $B \subseteq A) \Rightarrow A = B$.

Hence, $A = B \Leftrightarrow (A \subseteq B$ and $B \subseteq A)$.

EXAMPLE 8 *Let $A = \{1, \{2\}, \{3, 4\}, 5\}$. Which of the following are incorrect statements? Rectify each:*

(i) $2 \in A$ *(ii)* $\{2\} \subset A$ *(iii)* $\{1, 2\} \subset A$

(iv) $\{3, 4\} \subset A$ *(v)* $\{1, 5\} \subset A$ *(vi)* $\{\phi\} \subset A$

(vii) $1 \subset A$ *(viii)* $\{1, 2, 3, 4\} \subset A$

SOLUTION Clearly, A contains four elements, namely 1, $\{2\}$, $\{3, 4\}$ and 5.

(i) $2 \in A$ is incorrect. The correct statement would be $\{2\} \in A$.

(ii) $\{2\} \subset A$ is incorrect. The correct statement is $\{\{2\}\} \subset A$.

(iii) Clearly, $2 \notin A$ and therefore, $\{1, 2\} \subset A$ is incorrect.
The correct statement would be $\{1, \{2\}\} \subset A$.

(iv) Clearly, $\{3, 4\}$ is an element of A.
So, $\{3, 4\} \subset A$ is incorrect and $\{\{3, 4\}\} \subset A$ is correct.

(v) Since 1 and 5 are both elements of A, so $\{1, 5\} \subset A$ is correct.

(vi) Since $\phi \notin A$, so $\{\phi\} \subset A$ is incorrect while $\phi \subset A$ is correct.

(vii) Since $1 \in A$, so $1 \subset A$ is incorrect and therefore, $\{1\} \subset A$ is correct.

(viii) Since $2 \notin A$ and $3 \notin A$, so $\{1, 2, 3, 4\} \subset A$ is incorrect.
The correct statement would be $\{1, \{2\}, \{3, 4\}\} \subset A$.

EXERCISE 1C

1. State in each case whether $A \subset B$ or $A \not\subset B$.

(i) $A = \{0, 1, 2, 3\}$, $B = \{1, 2, 3, 4, 5\}$

(ii) $A = \phi$, $B = \{0\}$

(iii) $A = \{1, 2, 3\}$, $B = \{1, 2, 4\}$

(iv) $A = \{x : x \in Z, x^2 = 1\}$, $B = \{x : x \in N, x^2 = 1\}$

(v) $A = \{x : x$ is an even natural number$\}$, $B = \{x : x$ is an integer$\}$

(vi) $A = \{x : x$ is an integer$\}$, $B = \{x : x$ is a rational number$\}$

(vii) $A = \{x : x \text{ is a real number}\}$, $B = \{x : x \text{ is a complex number}\}$

(viii) $A = \{x : x \text{ is an isosceles triangle in a plane}\}$,
$B = \{x : x \text{ is an equilateral triangle in the same plane}\}$

(ix) $A = \{x : x \text{ is a square in a plane}\}$,
$B = \{x : x \text{ is a rectangle in the same plane}\}$

(x) $A = \{x : x \text{ is a triangle in a plane}\}$,
$B = \{x : x \text{ is a rectangle in the same plane}\}$

(xi) $A = \{x : x \text{ is an even natural number less than } 8\}$,
$B = \{x : x \text{ is a natural number which divides } 32\}$

2. Examine whether the following statements are true or false:

(i) $\{a, b\} \not\subset \{b, c, a\}$ (ii) $\{a\} \in \{a, b, c\}$

(iii) $\{\phi\} \subset \{a, b, c\}$

(iv) $\{\text{a, e}\} \subset \{x : x \text{ is a vowel in the English alphabet}\}$

(v) $\{x : x \in W, x + 5 = 5\} = \phi$ (vi) $a \in \{\{a\}, b\}$

(vii) $\{a\} \subset \{\{a\}, b\}$ (viii) $\{b, c\} \subset \{a, \{b, c\}\}$

(ix) $\{a, a, b, b\} = \{a, b\}$ (x) $\{a, b, a, b, a, b, \ldots\}$ is an infinite set.

(xi) If A = set of all circles of unit radius in a plane and B = set of all circles in the same plane then $A \subset B$.

3. If $A = \{1\}$ and $B = \{\{1\}, 2\}$ then show that $A \not\subset B$.

Hint $1 \in A$ but $1 \notin B$.

4. Write down all subsets of each of the following sets:

(i) $A = \{a\}$ (ii) $B = \{a, b\}$ (iii) $C = \{-2, 3\}$

(iv) $D = \{-1, 0, 1\}$ (v) $E = \phi$ (vi) $F = \{2, \{3\}\}$

(vii) $G = \{3, 4, \{5, 6\}\}$

5. Express each of the following sets as an interval:

(i) $A = \{x : x \in R, -4 < x < 0\}$ (ii) $B = \{x : x \in R, 0 \le x < 3\}$

(iii) $C = \{x : x \in R, 2 < x \le 6\}$ (iv) $D = \{x : x \in R, -5 \le x \le 2\}$

(v) $E = \{x : x \in R, -3 \le x < 2\}$ (vi) $F = \{x : x \in R, -2 \le x < 0\}$

6. Write each of the following intervals in the set-builder form:

(i) $A = (-2, 3)$ (ii) $B = [4, 10]$ (iii) $C = [-1, 8)$

(iv) $D = (4, 9]$ (v) $E = [-10, 0)$ (vi) $F = (0, 5]$

7. If $A = \{3, \{4, 5\}, 6\}$, find which of the following statements are true.

(i) $\{4, 5\} \subseteq A$ (ii) $\{4, 5\} \in A$ (iii) $\{\{4, 5\}\} \subseteq A$

(iv) $4 \in A$ (v) $\{3\} \subseteq A$ (vi) $\{\phi\} \subseteq A$

(vii) $\phi \subseteq A$ (viii) $\{3, 4, 5\} \subseteq A$ (ix) $\{3, 6\} \subseteq A$

8. If $A = \{a, b, c\}$, find $P(A)$ and $n\{P(A)\}$.

9. If $A = \{1, \{2, 3\}\}$, find $P(A)$ and $n\{P(A)\}$.

10. If $A = \phi$ then find $n\{P(A)\}$.

11. If $A = \{1, 3, 5\}$, $B = \{2, 4, 6\}$ and $C = \{0, 2, 4, 6, 8\}$ then find the universal set.

12. Prove that $A \subseteq B$, $B \subseteq C$ and $C \subseteq A \Rightarrow A = C$.

13. For any set A, prove that $A \subseteq \phi \Leftrightarrow A = \phi$.

14. State whether the given statement is true or false:

(i) If $A \subset B$ and $x \notin B$ then $x \notin A$.

(ii) If $A \subseteq \phi$ then $A = \phi$.

(iii) If A, B and C are three sets such that $A \in B$ and $B \subset C$ then $A \subset C$.

Hint Let $A = \{a\}$, $B = \{\{a\}, b\}$ and $C = \{\{a\}, b, c\}$.
Then, $\{a\} \in B$ and $B \subset C$. But, $\{a\} \not\subset C$.

(iv) If A, B and C are three sets such that $A \subset B$ and $B \in C$ then $A \in C$.

Hint Let $A = \{a\}$, $B = \{a, b\}$ and $C = \{\{a, b\}, c\}$.
Then, $A \subset B$ and $B \in C$. But, $A \notin C$.

(v) If A, B and C are three sets such that $A \not\subset B$ and $B \not\subset C$ then $A \not\subset C$.

Hint Let $A = \{a\}$, $B = \{b, c\}$ and $C = \{a, c\}$.
Then, $A \not\subset B$ and $B \not\subset C$. But $A \subset C$.

(vi) If A and B are sets such that $x \in A$ and $A \in B$ then $x \in B$.

Hint Let $A = \{x\}$, $B = \{\{x\}, y\}$.
Then, $x \in A$ and $A \in B$. But, $x \notin B$.

ANSWERS (EXERCISE 1C)

1. (i) $A \not\subset B$ (ii) $A \subset B$ (iii) $A \not\subset B$ (iv) $A \not\subset B$ (v) $A \subset B$
(vi) $A \subset B$ (vii) $A \subset B$ (viii) $A \not\subset B$ (ix) $A \subset B$ (x) $A \not\subset B$
(xi) $A \not\subset B$

2. (i) False (ii) False (iii) False (iv) True (v) False
(vi) False (vii) False (viii) False (ix) True (x) False
(xi) True

4. (i) $\phi, \{a\}$ (ii) $\phi, \{a\}, \{b\}, \{a, b\}$ (iii) $\phi, \{-2\}, \{3\}, \{-2, 3\}$
(iv) $\phi, \{-1\}, \{0\}, \{1\}, \{-1, 0\}, \{0, 1\}, \{-1, 1\}, \{-1, 0, 1\}$
(v) ϕ (vi) $\phi, \{2\}, \{\{3\}\}, \{2, \{3\}\}$
(vii) $\phi, \{3\}, \{4\}, \{\{5, 6\}\}, \{3, 4\}, \{3, \{5, 6\}\}, \{4, \{5, 6\}\}, \{3, 4, \{5, 6\}\}$

5. (i) $A = (-4, 0)$ (ii) $B = [0, 3)$ (iii) $C = (2, 6]$
(iv) $D = [-5, 2]$ (v) $E = [-3, 2)$ (vi) $F = [-2, 0)$

6. (i) $A = \{x : x \in R, -2 < x < 3\}$ (ii) $B = \{x : x \in R, 4 \leq x \leq 10\}$
(iii) $C = \{x : x \in R, -1 \leq x < 8\}$ (iv) $D = \{x : x \in R, 4 < x \leq 9\}$
(v) $E = \{x : x \in R, -10 \leq x < 0\}$ (vi) $F = \{x : x \in R, 0 < x \leq 5\}$

7. (i) False (ii) True (iii) True (iv) False (v) True
(vi) False (vii) True (viii) False (ix) True

8. $P(A) = \{\phi, \{a\}, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{a, c\}, \{a, b, c\}\}$ and $n\{P(A)\} = 2^3 = 8$

9. $P(A) = \{\phi, \{1\}, \{\{2, 3\}\}, \{1, \{2, 3\}\}\}$ and $n\{P(A)\} = 2^2 = 4$

10. $P(A) = \{\phi\} \Rightarrow n\{P(A)\} = 1$ **11.** $U = \{0, 1, 2, 3, 4, 5, 6, 8\}$

14. (i) True (ii) True (iii) False (iv) False (v) False
(vi) False

HINTS TO SOME SELECTED QUESTIONS

1. (iv) $A = \{-1, 1\}$ and $B = \{1\}$. So, $A \not\subset B$.

(v) $A = \{2, 4, 6, 8, \ldots\}$ and $B = \{\ldots, -1, 0, 1, 2, 3, 4, 5, 6, 7, 8, \ldots]$. So, $A \subset B$.

(vi) Every integer is a rational number.

(vii) We may write a real number x as $x + 0i$.
So, every real number is a complex number.

(ix) Every square is a rectangle.

(xi) $A = \{2, 4, 6\}$ and $B = \{1, 2, 4, 8, 16, 32\}$. So, $A \not\subset B$.

2. (ii) $a \in \{a, b, c\}$ is true. So, $\{a\} \in \{a, b, c\}$ is false.

(iii) $\phi \subset \{a, b, c\}$ is true and $\{\phi\} \subset \{a, b, c\}$ is false.

(iv) {a, e} ⊂ {a, e, i, o, u} is true.

(v) $\{x : x \in W, x + 5 = 5\} = \{0\}$.

(vi) $\{\{a\}, b\}$ has two elements, namely $\{a\}$ and b.

(vii) $\{a\} \in \{\{a\}, b\}$. So, $\{a\} \subset \{\{a\}, b\}$ is false.

(viii) $\{b, c\} \in \{a, \{b, c\}\}$. So, $\{b, c\} \subset \{a, \{b, c\}\}$ is false.

(ix) Repetition of elements in a set has no meaning.

(x) Given set $= \{a, b\}$, which is finite.

9. Let $A = \{1, a\}$. Then, $P(A) = \{\phi, \{1\}, \{a\}, \{1, a\}\}$, where $a = \{2, 3\}$.

$\therefore \quad P(A) = \{\phi, \{1\}, \{\{2, 3\}\}, \{1, \{2, 3\}\}\}$.

OPERATIONS ON SETS

UNION OF SETS *The union of two sets A and B, denoted by $A \cup B$, is the set of all those elements which are either in A or in B or in both A and B.*

Thus, $A \cup B = \{x : x \in A \text{ or } x \in B\}$

$\therefore \quad x \in A \cup B \Leftrightarrow x \in A \text{ or } x \in B$.

EXAMPLE 1 *If $A = \{3, 4, 5, 6\}$ and $B = \{4, 6, 8, 10\}$, find $A \cup B$.*

SOLUTION Clearly, $A \cup B = \{3, 4, 5, 6, 8, 10\}$.

EXAMPLE 2 *Let $A = \{x : x$ is a prime number less than $10\}$ and $B = \{x : x \in N, x$ is a factor of $12\}$. Find $A \cup B$.*

SOLUTION We have

$$A = \{2, 3, 5, 7\} \text{ and } B = \{1, 2, 3, 4, 6, 12\}.$$

$$\therefore \quad A \cup B = \{2, 3, 5, 7\} \cup \{1, 2, 3, 4, 6, 12\} = \{1, 2, 3, 4, 5, 6, 7, 12\}.$$

EXAMPLE 3 *Let $A = \{x : x$ is a positive integer$\}$ and let $B = \{x : x$ is a negative integer$\}$. Find $A \cup B$.*

SOLUTION Clearly, $A \cup B = \{x : x \text{ is an integer and } x \neq 0\}$.

EXAMPLE 4 *If $A = \{x : x = 2n + 1, n \in Z\}$ and $B = \{x : x = 2n, n \in Z\}$ then find $A \cup B$.*

SOLUTION We have

$$A \cup B = \{x : x \text{ is an odd integer }\} \cup \{x : x \text{ is an even integer}\}$$
$$= \{x : x \text{ is an integer}\} = Z.$$

REMARK The union of n sets $A_1, A_2, A_3, \ldots, A_n$ is denoted by

$$(A_1 \cup A_2 \cup A_3 \cup \ldots \cup A_n) = \bigcup_{i=1}^{n} A_i.$$

INTERSECTION OF SETS *The intersection of two sets A and B, denoted by $A \cap B$, is the set of all elements which are common to both A and B.*

Thus, $A \cap B = \{x : x \in A \text{ and } x \in B\}$.

$\therefore \quad x \in A \cap B \Leftrightarrow x \in A \text{ and } x \in B;$

$x \notin A \cap B \Rightarrow x \notin A \text{ or } x \notin B.$

EXAMPLE 5 *Let $A = \{1, 3, 5, 7, 9\}$ and $B = \{2, 3, 5, 7, 11, 13\}$. Find $A \cap B$.*

SOLUTION We have

$$A \cap B = \{1, 3, 5, 7, 9\} \cap \{2, 3, 5, 7, 11, 13\} = \{3, 5, 7\}.$$

EXAMPLE 6 *If $A = \{x : x \in N, x \text{ is a factor of } 12\}$ and $B = \{x : x \in N, x \text{ is a factor of } 18\}$, find $A \cap B$.*

SOLUTION We have

$A = \{x : x \in N, x \text{ is a factor of } 12\} = \{1, 2, 3, 4, 6, 12\}$,

$B = \{x : x \in N, x \text{ is a factor of } 18\} = \{1, 2, 3, 6, 9, 18\}$.

$\therefore \quad A \cap B = \{1, 2, 3, 4, 6, 12\} \cap \{1, 2, 3, 6, 9, 18\} = \{1, 2, 3, 6\}$.

EXAMPLE 7 *If $A = \{x : x = 3n, n \in Z\}$ and $B = \{x : x = 4n, n \in Z\}$ then find $A \cap B$.*

SOLUTION We have

$A = \{x : x \in Z \text{ and } x \text{ is a multiple of } 3\}$

and $B = \{x : x \in Z \text{ and } x \text{ is a multiple of } 4\}$.

$\therefore \quad A \cap B = \{x : x \in Z \text{ and } x \text{ is a multiple of both 3 and 4}\}$

$= \{x : x \in Z \text{ and } x \text{ is a multiple of } 12\}$

$= \{x : x = 12n, n \in Z\}$.

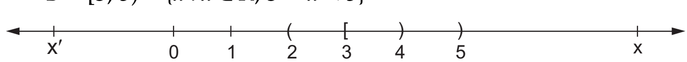
Hence, $A \cap B = \{x : x = 12n, n \in Z\}$.

EXAMPLE 8 *If $A = (2, 4)$ and $B = [3, 5)$, find $A \cap B$.*

SOLUTION We have

$A = (2, 4) = \{x : x \in R, 2 < x < 4\}$

$B = [3, 5) = \{x : x \in R, 3 \le x < 5\}$



Clearly, $A \cap B = \{x : x \in R, 3 \le x < 4\} = [3, 4)$.

REMARK The intersection of n sets $A_1, A_2, A_3, \ldots, A_n$ is denoted by

$$(A_1 \cap A_2 \cap A_3 \cap \ldots \cap A_n) = \bigcap_{i=1}^{n} A_i.$$

DISJOINT SETS *Two sets A and B are said to be disjoint if $A \cap B = \phi$.*

INTERSECTING SETS *Two sets A and B are said to be intersecting if $A \cap B \neq \phi$.*

EXAMPLE 9 *If* $A = \{1, 3, 5, 7, 9\}$, $B = \{2, 4, 6, 8\}$ *and* $C = \{2, 3, 5, 7, 11\}$, *find* $(A \cap B)$ *and* $(A \cap C)$. *What do you conclude?*

SOLUTION We have

$$A \cap B = \{1, 3, 5, 7, 9\} \cap \{2, 4, 6, 8\} = \phi$$

and $A \cap C = \{1, 3, 5, 7, 9\} \cap \{2, 3, 5, 7, 11\} = \{3, 5, 7\} \neq \phi$.

Thus, A and B are disjoint sets while A and C are intersecting sets.

EXAMPLE 10 *Give examples of three sets A, B, C such that* $(A \cap B) \neq \phi$, $(B \cap C) \neq \phi$, $(A \cap C) \neq \phi$ *and* $(A \cap B \cap C) = \phi$.

SOLUTION Consider the sets $A = \{1, 2\}$, $B = \{2, 3, 4\}$ and $C = \{1, 3, 5\}$.

Then, $(A \cap B) = \{1, 2\} \cap \{2, 3, 4\} = \{2\} \neq \phi$;

$(B \cap C) = \{2, 3, 4\} \cap \{1, 3, 5\} = \{3\} \neq \phi$;

$(A \cap C) = \{1, 2\} \cap \{1, 3, 5\} = \{1\} \neq \phi$;

$(A \cap B \cap C) = \{1, 2\} \cap \{2, 3, 4\} \cap \{1, 3, 5\} = \phi$.

Thus, $A \cap B \neq \phi$, $B \cap C \neq \phi$; $A \cap C \neq \phi$ and $A \cap B \cap C = \phi$.

EXAMPLE 11 *Give an example of three sets A, B, C such that* $A \cap B = A \cap C$ *but* $B \neq C$.

SOLUTION Consider the sets $A = \{1, 2, 3\}$, $B = \{3, 4\}$ and $C = \{3, 5, 7\}$.

Then, $A \cap B = \{1, 2, 3\} \cap \{3, 4\} = \{3\}$.

And, $A \cap C = \{1, 2, 3\} \cap \{3, 5, 7\} = \{3\}$.

Thus, $A \cap B = A \cap C$, and clearly, $B \neq C$.

DIFFERENCE OF SETS *For any sets A and B, their difference* $(A - B)$ *is defined as*

$$(A - B) = \{x : x \in A \text{ and } x \notin B\}.$$

Thus, $x \in (A - B) \Rightarrow x \in A$ *and* $x \notin B$.

EXAMPLE 12 *If* $A = \{x : x \in N, x \text{ is a factor of } 6\}$ *and* $B = \{x \in N : x \text{ is a factor of } 8\}$ *then find (i)* $A \cup B$, *(ii)* $A \cap B$, *(iii)* $A - B$, *(iv)* $B - A$.

SOLUTION We have

$$A = \{x : x \in N, x \text{ is a factor of } 6\} = \{1, 2, 3, 6\}$$

and $B = \{x : x \in N, x \text{ is a factor of } 8\} = \{1, 2, 4, 8\}$.

(i) $A \cup B = \{1, 2, 3, 6\} \cup \{1, 2, 4, 8\} = \{1, 2, 3, 4, 6, 8\}$.

(ii) $A \cap B = \{1, 2, 3, 6\} \cap \{1, 2, 4, 8\} = \{1, 2\}$.

(iii) $A - B = \{1, 2, 3, 6\} - \{1, 2, 4, 8\} = \{3, 6\}$.

(iv) $B - A = \{1, 2, 4, 8\} - \{1, 2, 3, 6\} = \{4, 8\}$.

SYMMETRIC DIFFERENCE OF TWO SETS

The symmetric difference of two sets A and B, denoted by $A \Delta B$, *is defined as*

$$A \Delta B = (A - B) \cup (B - A).$$

EXAMPLE 13 *Let* $A = \{a, b, c, d\}$ *and* $B = \{b, d, f, g\}$. *Find* $A \Delta B$.

SOLUTION We have

$$(A - B) = \{a, b, c, d\} - \{b, d, f, g\} = \{a, c\}.$$

$$(B - A) = \{b, d, f, g\} - \{a, b, c, d\} = \{f, g\}.$$

$$\therefore \quad A \,\Delta\, B = (A - B) \cup (B - A) = \{a, c\} \cup \{f, g\} = \{a, c, f, g\}.$$

COMPLEMENT OF A SET

Let U be the universal set and let $A \subset U$. Then, the complement of A, denoted by A' or $(U - A)$, is definied as

$$A' = \{x \in U : x \notin A\}.$$

Clearly, $x \in A' \Leftrightarrow x \notin A$.

EXAMPLE 14 *If $U = \{1, 2, 3, 4, 5, 6, 7, 8\}$ and $A = \{2, 4, 6, 8\}$, find (i) A' (ii) $(A')'$.*

SOLUTION We have

(i) $A' = U - A$
$= \{1, 2, 3, 4, 5, 6, 7, 8\} - \{2, 4, 6, 8\} = \{1, 3, 5, 7\}.$

(ii) $(A')' = U - A'$
$= \{1, 2, 3, 4, 5, 6, 7, 8\} - \{1, 3, 5, 7\} = \{2, 4, 6, 8\} = A.$

EXAMPLE 15 *Let N be the universal set.*

(i) If $A = \{x : x \in N$ and x is odd$\}$, find A'.

(ii) If $B = \{x : x \in N, x$ is divisible by 3 and 5$\}$, find B'.

SOLUTION We have

(i) $A' = \{x : x \in N$ and x is not odd$\} = \{x : x \in N$ and x is even$\}$.

(ii) $B' = \{x : x \in N, x$ is not divisible by 3 or x is not divisible by 5$\}$.

SOME RESULTS ON COMPLEMENTATION

EXAMPLE 16 *If $A \subset U$, prove that:*

(i) $U' = \phi$ (ii) $\phi' = U$ (iii) $(A')' = A$ (iv) $A \cup A' = U$ (v) $A \cap A' = \phi$

SOLUTION We have

(i) $U' = \{x \in U : x \notin U\} = \phi$.

(ii) $\phi' = \{x \in U : x \notin \phi\} = U$.

(iii) $(A')' = \{x \in U : x \notin A'\} = [x \in U : x \in A\} = A$.

(iv) $A \cup A' = \{x \in U : x \in A \cup A'\}$
$= \{x \in U : x \in A$ or $x \in A'\}$
$= \{x \in U : x \in A$ or $x \notin A\} = U$.

(v) $A \cap A' = \{x \in U : x \in A \cap A'\}$
$= \{x \in U : x \in A$ and $x \in A'\}$
$= \{x \in U : x \in A$ and $x \notin A\} = \phi$.

EXERCISE 1D

1. If $A = \{a, b, c, d, e, f\}$, $B = \{c, e, g, h\}$ and $C = \{a, e, m, n\}$, find:

(i) $A \cup B$ (ii) $B \cup C$ (iii) $A \cup C$

(iv) $B \cap C$ (v) $C \cap A$ (vi) $A \cap B$

2. If $A = \{1, 2, 3, 4, 5\}$, $B = \{4, 5, 6, 7, 8\}$, $C = \{7, 8, 9, 10, 11\}$ and $D = \{10, 11, 12, 13, 14\}$, find:

(i) $A \cup B$ (ii) $B \cup C$ (iii) $A \cup C$
(iv) $B \cup D$ (v) $(A \cup B) \cup C$ (vi) $(A \cup B) \cap C$
(vii) $(A \cap B) \cup D$ (viii) $(A \cap B) \cup (B \cap C)$ (ix) $(A \cup C) \cap (C \cup D)$

3. If $A = \{3, 5, 7, 9, 11\}$, $B = \{7, 9, 11, 13\}$, $C = \{11, 13, 15\}$ and $D = \{15, 17\}$, find:

(i) $A \cap B$ (ii) $A \cap C$ (iii) $B \cap C$
(iv) $B \cap D$ (v) $B \cap (C \cup D)$ (vi) $A \cap (B \cup C)$

4. If $A = \{x : x \in N\}$, $B = \{x : x \in N \text{ and } x \text{ is even}\}$, $C = \{x : x \in N \text{ and } x \text{ is odd}\}$ and $D = \{x : x \in N \text{ and } x \text{ is prime}\}$ then find:

(i) $A \cap B$ (ii) $A \cap C$ (iii) $A \cap D$
(iv) $B \cap C$ (v) $B \cap D$ (vi) $C \cap D$

5. If $A = \{2x : x \in N \text{ and } 1 \le x < 4\}$, $B = \{(x + 2) : x \in N \text{ and } 2 \le x < 5\}$ and $C = \{x : x \in N \text{ and } 4 < x < 8\}$, find:

(i) $A \cap B$ (ii) $A \cup B$ (iii) $(A \cup B) \cap C$

6. If $A = \{2, 4, 6, 8, 10, 12\}$ and $B = \{3, 4, 5, 6, 7, 8, 10\}$, find:

(i) $(A - B)$ (ii) $(B - A)$ (iii) $(A - B) \cup (B - A)$

7. If $A = \{a, b, c, d, e\}$, $B = \{a, c, e, g\}$ and $C = \{b, e, f, g\}$, find:

(i) $A \cap (B - C)$ (ii) $A - (B \cup C)$ (iii) $A - (B \cap C)$

8. If $A = \left\{\frac{1}{x} : x \in N \text{ and } x < 8\right\}$ *and* $B = \left\{\frac{1}{2x} : x \in N \text{ and } x \le 4\right\}$, find:

(i) $A \cup B$ (ii) $A \cap B$ (iii) $A - B$ (iv) $B - A$

9. If R is the set of all real numbers and Q is the set of all rational numbers then what is the set $(R - Q)$?

10. If $A = \{2, 3, 5, 7, 11\}$ and $B = \phi$, find:

(i) $A \cup B$ (ii) $A \cap B$

11. If A and B are two sets such that $A \subseteq B$ then find:

(i) $A \cup B$ (ii) $A \cap B$ (iii) $A - B$

12. Which of the following sets are pairs of disjoint sets? Justify your answer:

(i) $A = \{3, 4, 5, 6\}$ and $B = \{2, 5, 7, 9\}$
(ii) $C = \{1, 2, 3, 4, 5\}$ and $D = \{6, 7, 9, 11\}$
(iii) $E = \{x : x \in N, x \text{ is even and } x < 8\}$
$F = \{x : x = 3n, n \in N \text{ and } n < 4\}$
(iv) $G = \{x : x \in N, x \text{ is even}\}$ and $H = \{x : x \in N, x \text{ is prime}\}$
(v) $J = \{x : x \in N, x \text{ is even}\}$ and $K = \{x : x \in N, x \text{ is odd}\}$

13. If $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$, $A = \{1, 2, 3, 4\}$, $B = \{2, 4, 6, 8\}$ and $C = \{1, 4, 5, 6\}$, find:

(i) A' (ii) B' (iii) C' (iv) $(B')'$
(v) $(A \cup B)'$ (vi) $(A \cap C)'$ (vii) $(B - C)'$

14. If $U = \{a, b, c, d, e\}$, $A = \{a, b, c\}$ *and* $B = \{b, c, d, e\}$ then verify that:

(i) $(A \cup B)' = (A' \cap B')$ (ii) $(A \cap B)' = (A' \cup B')$

15. If U is the universal set and $A \subset U$ then fill in the blanks:

(i) $A \cup A' = \ldots\ldots$ (ii) $A \cap A' = \ldots\ldots$

(iii) $\phi' \cap A = \ldots\ldots$ (iv) $U' \cap A = \ldots\ldots$

ANSWERS (EXERCISE 1D)

1. (i) $\{a, b, c, d, e, f, g, h\}$ (ii) $\{a, c, e, g, h, m, n\}$ (iii) $\{a, b, c, d, e, f, m, n\}$ (iv) $\{e\}$ (v) $\{a, e\}$ (vi) $\{c, e\}$

2. (i) $\{1, 2, 3, 4, 5, 6, 7, 8\}$ (ii) $\{4, 5, 6, 7, 8, 9, 10, 11\}$ (iii) $\{1, 2, 3, 4, 5, 7, 8, 9, 10, 11\}$ (iv) $\{4, 5, 6, 7, 8, 10, 11, 12, 13, 14\}$ (v) $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11\}$ (vi) $\{7, 8\}$ (vii) $\{4, 5, 10, 11, 12, 13, 14\}$ (viii) $\{4, 5, 7, 8\}$ (ix) $\{7, 8, 9, 10, 11\}$

3. (i) $\{7, 9\}$ (ii) $\{11\}$ (iii) $\{11, 13\}$ (iv) ϕ (v) $\{11, 13\}$ (vi) $\{7, 9, 11\}$

4. (i) B (ii) C (iii) D (iv) ϕ (v) $\{2\}$ (vi) $D - \{2\}$

5. (i) $\{4, 6\}$ (ii) $\{2, 4, 5, 6\}$ (iii) $\{5, 6\}$

6. (i) $\{2, 12\}$ (ii) $\{3, 5, 7\}$ (iii) $\{2, 3, 5, 7, 12\}$

7. (i) $\{a, c\}$ (ii) $\{d\}$ (iii) $\{a, b, c, d\}$

8. (i) $\left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \frac{1}{7}, \frac{1}{8}\right\}$ (ii) $\left\{\frac{1}{2}, \frac{1}{4}, \frac{1}{6}\right\}$ (iii) $\left\{1, \frac{1}{3}, \frac{1}{5}, \frac{1}{7}\right\}$ (iv) $\left\{\frac{1}{8}\right\}$

9. $(R - Q) = \{x : x \in R, x \text{ is irrational}\}$

10. (i) 2, 3, 5, 7, 11} (ii) ϕ **11.** (i) B (ii) A (iii) ϕ

12. (ii) C and D, since $C \cap D = \phi$ (v) J and K, since $J \cap K = \phi$

13. (i) $\{5, 6, 7, 8, 9\}$ (ii) $\{1, 3, 5, 7, 9\}$ (iii) $\{2, 3, 7, 8, 9\}$ (iv) $\{2, 4, 6, 8\}$ (v) $\{5, 7, 9\}$ (vi) $\{2, 3, 5, 6, 7, 8, 9\}$ (vii) $\{1, 3, 4, 5, 6, 7, 9\}$

15. (i) U (ii) ϕ (iii) A (iv) ϕ

LAWS OF OPERATIONS ON SETS

THEOREM 1 (Idempotent laws) *For any set A, prove that:*

(i) $A \cup A = A$ *(ii)* $A \cap A = A$

PROOF We have

(i) $A \cup A = \{x : x \in A \text{ or } x \in A\} = \{x : x \in A\} = A$.

(ii) $A \cap A = \{x : x \in A \text{ and } x \in A\} = \{x : x \in A\} = A$.

THEOREM 2 (Identity laws) *For any set A, prove that:*

(i) $A \cup \phi = A$ *(ii)* $A \cap U = A$, *where U is the universal set*

PROOF We have

(i) $A \cup \phi = \{x : x \in A \text{ or } x \in \phi\} = \{x : x \in A\} = A$ $[\because x \notin \phi]$

(ii) $A \cap U = \{x : x \in A \text{ and } x \in U\} = \{x : x \in A\} = A$ $[\because A \subset U]$

NOTE ϕ and U are the identity elements for union and intersection of sets respectively.

THEOREM 3 (Commutative laws) *For any two sets A and B, prove that:*

I. $A \cup B = B \cup A$ [*Commutative law for union of sets*]

II. $A \cap B = B \cap A$ [*Commutative law for intersection of sets*]

PROOF I. Let x be an arbitrary element of $A \cup B$. Then,

$$\begin{aligned} x \in A \cup B &\Rightarrow x \in A \text{ or } x \in B \\ &\Rightarrow x \in B \text{ or } x \in A \\ &\Rightarrow x \in B \cup A. \end{aligned}$$

$\therefore \quad A \cup B \subseteq B \cup A.$ … (i)

Again, let y be an arbitrary element of $B \cup A$. Then,

$$\begin{aligned} y \in B \cup A &\Rightarrow y \in B \text{ or } y \in A \\ &\Rightarrow y \in A \text{ or } y \in B \\ &\Rightarrow y \in A \cup B. \end{aligned}$$

$\therefore \quad B \cup A \subseteq A \cup B.$ … (ii)

From (i) and (ii), we get $A \cup B = B \cup A$.

II. Let x be an arbitrary element of $A \cap B$. Then,

$$\begin{aligned} x \in A \cap B &\Rightarrow x \in A \text{ and } x \in B \\ &\Rightarrow x \in B \text{ and } x \in A \\ &\Rightarrow x \in B \cap A. \end{aligned}$$

$\therefore \quad A \cap B \subseteq B \cap A.$ … (iii)

Again, let y be an arbitray element of $B \cap A$. Then,

$$\begin{aligned} y \in B \cap A &\Rightarrow y \in B \text{ and } y \in A \\ &\Rightarrow y \in A \text{ and } y \in B \\ &\Rightarrow y \in A \cap B. \end{aligned}$$

$\therefore \quad B \cap A \subseteq A \cap B.$ … (iv)

From (iii) and (iv), we get $A \cap B = B \cap A$.

THEOREM 4 (Associative laws) *For any sets A, B, C, prove that:*

I. $(A \cup B) \cup C = A \cup (B \cup C)$ [*Associative law for union of sets*]

II. $(A \cap B) \cap C = A \cap (B \cap C)$ [*Associative law for intersection of sets*]

PROOF I. Let x be an arbitrary element of $(A \cup B) \cup C$. Then,

$$\begin{aligned} x \in (A \cup B) \cup C &\Rightarrow x \in (A \cup B) \text{ or } x \in C \\ &\Rightarrow (x \in A \text{ or } x \in B) \text{ or } x \in C \\ &\Rightarrow x \in A \text{ or } (x \in B \text{ or } x \in C) \\ &\Rightarrow x \in A \text{ or } x \in (B \cup C) \\ &\Rightarrow x \in A \cup (B \cup C). \end{aligned}$$

$\therefore \quad (A \cup B) \cup C \subseteq A \cup (B \cup C)$. ... (i)

Again, let y be an arbitrary element of $A \cup (B \cup C)$. Then,

$$\begin{aligned} y \in A \cup (B \cup C) &\Rightarrow y \in A \text{ or } y \in (B \cup C) \\ &\Rightarrow y \in A \text{ or } (y \in B \text{ or } y \in C) \\ &\Rightarrow (y \in A \text{ or } y \in B) \text{ or } y \in C \\ &\Rightarrow y \in (A \cup B) \text{ or } y \in C \\ &\Rightarrow y \in (A \cup B) \cup C. \end{aligned}$$

$\therefore \quad A \cup (B \cup C) \subseteq (A \cup B) \cup C$. ... (ii)

From (i) and (ii), we get $(A \cup B) \cup C = A \cup (B \cup C)$.

II. Let x be an arbitrary element of $(A \cap B) \cap C$. Then,

$$\begin{aligned} x \in (A \cap B) \cap C &\Rightarrow x \in (A \cap B) \text{ and } x \in C \\ &\Rightarrow (x \in A \text{ and } x \in B) \text{ and } x \in C \\ &\Rightarrow x \in A \text{ and } (x \in B \text{ and } x \in C) \\ &\Rightarrow x \in A \text{ and } x \in (B \cap C) \\ &\Rightarrow x \in A \cap (B \cap C). \end{aligned}$$

$\therefore \quad (A \cap B) \cap C \subseteq A \cap (B \cap C)$. ... (iii)

Again, let y be an arbitrary element of $A \cap (B \cap C)$. Then,

$$\begin{aligned} y \in A \cap (B \cap C) &\Rightarrow y \in A \text{ and } y \in (B \cap C) \\ &\Rightarrow y \in A \text{ and } (y \in B \text{ and } y \in C) \\ &\Rightarrow (y \in A \text{ and } y \in B) \text{ and } y \in C \\ &\Rightarrow y \in (A \cap B) \text{ and } y \in C \\ &\Rightarrow y \in (A \cap B) \cap C \end{aligned}$$

$\therefore \quad A \cap (B \cap C) \subseteq (A \cap B) \cap C$. ... (iv)

From (iii) and (iv), we get $(A \cap B) \cap C = A \cap (B \cap C)$.

THEOREM 5 (Distributive laws) *For any three sets A, B, C prove that:*

I. $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

[*Distributive law of union over intersection*]

II. $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

[*Distributive law of intersection over union*]

PROOF I. Let x be an arbitrary element of $A \cup (B \cap C)$. Then,

$$\begin{aligned} x \in A \cup (B \cap C) &\Rightarrow x \in A \text{ or } x \in (B \cap C) \\ &\Rightarrow x \in A \text{ or } (x \in B) \text{ and } x \in C \\ &\Rightarrow (x \in A \text{ or } x \in B) \text{ and } (x \in A \text{ or } x \in C)) \\ &\qquad [\because \text{ 'or' distributes 'and'}] \\ &\Rightarrow x \in (A \cup B) \text{ and } x \in (A \cup C). \\ &\Rightarrow x \in (A \cup B) \cap (A \cup C). \end{aligned}$$

$\therefore \quad A \cup (B \cap C) \subseteq (A \cup B) \cap (A \cup C)$. ... (i)

Again, let y be an arbitrary element of $(A \cup B) \cap (A \cup C)$. Then,

$$y \in (A \cup B) \cap (A \cup C) \Rightarrow y \in (A \cup B) \text{ and } y \in (A \cup C)$$
$$\Rightarrow (y \in A \text{ or } y \in B) \text{ and } (y \in A \text{ or } y \in C)$$
$$\Rightarrow y \in A \text{ or } (y \in B \text{ and } y \in C)$$
[$\because$ 'or' distributes 'and']
$$\Rightarrow y \in A \text{ or } y \in (B \cap C)$$
$$\Rightarrow y \in A \cup (B \cap C).$$

$\therefore \quad (A \cup B) \cap (A \cup C) \subseteq A \cup (B \cap C)$. … (ii)

From (i) and (ii), we get $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

II. Let x be an arbitrary element of $A \cap (B \cup C)$. Then,

$$x \in A \cap (B \cup C) \Rightarrow x \in A \text{ and } x \in (B \cup C)$$
$$\Rightarrow x \in A \text{ and } (x \in B \text{ or } x \in C)$$
$$\Rightarrow (x \in A \text{ and } x \in B) \text{ or } (x \in A \text{ and } x \in C)$$
[$\because$ 'and' distributes 'or']
$$\Rightarrow x \in (A \cap B) \text{ or } x \in (A \cap C)$$
$$\Rightarrow x \in (A \cap B) \cup (A \cap C).$$

$\therefore \quad A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$. … (iii)

Again, let y be an arbitrary element of $(A \cap B) \cup (A \cap C)$. Then,

$$y \in (A \cap B) \cup (A \cap C) \Rightarrow y \in (A \cap B) \text{ or } y \in (A \cap C)$$
$$\Rightarrow (y \in A \text{ and } y \in B) \text{ or } (y \in A \text{ and } y \in C)$$
$$\Rightarrow y \in A \text{ and } (y \in B \text{ or } y \in C)$$
[$\because$ 'and' distributes 'or']
$$\Rightarrow y \in A \text{ and } y \in (B \cup C)$$
$$\Rightarrow y \in A \cap (B \cup C).$$

$\therefore \quad (A \cap B) \cup (A \cap C) \subseteq A \cap (B \cup C)$. … (iv)

From (iii) and (iv), we get $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

THEOREM 6 (De Morgan's laws) *For any two sets A and B, prove that:*

I. $(A \cup B)' = (A' \cap B')$ *II.* $(A \cap B)' = (A' \cup B')$

PROOF I. Let x be an arbitrary element of $(A \cup B)'$. Then,

$$x \in (A \cup B)' \Rightarrow x \notin A \cup B$$
$$\Rightarrow x \notin A \text{ and } x \notin B \quad [\textit{note this point}]$$
$$\Rightarrow x \in A' \text{ and } x \in B'$$
$$\Rightarrow x \in (A' \cap B').$$

$\therefore \quad (A \cup B)' \subseteq (A' \cap B')$. … (i)

Again, let y be an arbitrary element of $(A' \cap B')$. Then,

$$y \in (A' \cap B') \Rightarrow y \in A' \text{ and } y \in B'$$
$$\Rightarrow y \notin A \text{ and } y \notin B$$
$$\Rightarrow y \notin (A \cup B) \quad [\textit{note this point}]$$
$$\Rightarrow y \in (A \cup B)'.$$

$\therefore \quad (A' \cap B') \subseteq (A \cup B)'$. … (ii)

From (i) and (ii), we get $(A \cup B)' = (A' \cap B')$.

II. Let x be an arbitrary element of $(A \cap B)'$. Then,

$$\begin{aligned} x \in (A \cap B)' &\Rightarrow x \notin (A \cap B) \\ &\Rightarrow x \notin A \text{ or } x \notin B \quad \textit{[note this point]} \\ &\Rightarrow x \in A' \text{ or } x \in B' \\ &\Rightarrow x \in (A' \cup B') \end{aligned}$$

$\therefore \quad (A \cap B)' \subseteq (A' \cup B')$. … (iii)

Again, let y be an arbitrary element of $(A' \cup B')$. Then,

$$\begin{aligned} y \in (A' \cup B') &\Rightarrow y \in A' \text{ or } y \in B' \\ &\Rightarrow y \notin A \text{ or } y \notin B \\ &\Rightarrow y \notin (A \cap B) \quad \textit{[note this point]} \\ &\Rightarrow y \in (A \cap B)'. \end{aligned}$$

$\therefore \quad (A' \cup B') \subseteq (A \cap B)'$. … (iv)

From (iii) and (iv), we get $(A \cap B)' = (A' \cup B')$.

SOME MORE RESULTS

THEOREM 7 *For any two sets A and B, prove that:*

I. $A \subseteq B \Rightarrow B' \subseteq A'$ *II.* $A - B = A \cap B'$

III. $(A - B) \cap B = \phi$ *IV.* $(A - B) \cup (B - A) = (A \cup B) - (A \cap B)$

V. $(A - B) = A \Leftrightarrow A \cap B = \phi$

PROOF I. Let $A \subseteq B$ be given and let x be an arbitrary element of B'. Then,

$$\begin{aligned} x \in B' &\Rightarrow x \notin B \\ &\Rightarrow x \notin A \quad [\because A \subseteq B] \\ &\Rightarrow x \in A'. \end{aligned}$$

$\therefore \quad B' \subseteq A'$.

Hence, $A \subseteq B \Rightarrow B' \subseteq A'$.

II. Let x be an arbitrary element of $(A - B)$. Then,

$$\begin{aligned} x \in (A - B) &\Rightarrow x \in A \text{ and } x \notin B \\ &\Rightarrow x \in A \text{ and } x \in B' \\ &\Rightarrow x \in A \cap B'. \end{aligned}$$

$\therefore \quad (A - B) \subseteq A \cap B'$. … (i)

Again, let y be an arbitrary element of $(A \cap B')$. Then,

$$\begin{aligned} x \in (A \cap B') &\Rightarrow x \in A \text{ and } x \in B' \\ &\Rightarrow x \in A \text{ and } x \notin B \\ &\Rightarrow x \in (A - B). \end{aligned}$$

$\therefore \quad (A \cap B') \subseteq (A - B)$. … (ii)

Hence, from (i) and (ii), we get $(A - B) = (A \cap B')$.

III. If possible, let $(A - B) \cap B \neq \phi$ and let $x \in (A - B) \cap B$. Then,

$$x \in (A - B) \cap B \Rightarrow x \in (A - B) \text{ and } x \in B$$

$$\Rightarrow (x \in A \text{ and } x \notin B) \text{ and } x \in B$$
$$\Rightarrow x \in A \text{ and } (x \notin B \text{ and } x \in B).$$

But, $x \notin B$ and $x \in B$ can never hold simultaneously.

Thus, we arrive at a contradiction.

Since the contradiction arises by assuming that $(A - B) \cap B \neq \phi$ and hence $(A - B) \cap B = \phi$.

IV. Let x be an arbitrary element of $(A - B) \cup (B - A)$. Then,

$$\begin{aligned} x \in (A - B) \cup (B - A) &\Rightarrow x \in (A - B) \text{ or } x \in (B - A) \\ &\Rightarrow (x \in A \text{ and } x \notin B) \text{ or } (x \in B \text{ and } x \notin A) \\ &\Rightarrow (x \in A \text{ or } x \in B) \text{ and } (x \notin A \text{ or } x \notin B) \quad [\textit{note it}] \\ &\Rightarrow x \in (A \cup B) \text{ and } x \notin (A \cap B) \\ &\Rightarrow x \in \{(A \cup B) - (A \cap B)\}. \end{aligned}$$

$\therefore \quad (A - B) \cup (B - A) \subseteq \{(A \cup B) - (A \cap B)\}$. … (iii)

Again, let y be an arbitrary element of $(A \cup B) - (A \cap B)$. Then,

$$\begin{aligned} y \in (A \cup B) - (A \cap B) &\Rightarrow y \in (A \cup B) \text{ and } y \notin (A \cap B) \\ &\Rightarrow (y \in A \text{ or } y \in B) \text{ and } (y \notin A \text{ or } y \notin B) \quad [\textit{note it}] \\ &\Rightarrow (y \in A \text{ and } y \notin B) \text{ or } (y \in B \text{ and } y \notin A) \quad [\textit{note it}] \\ &\Rightarrow y \in (A - B) \text{ or } y \in (B - A) \\ &\Rightarrow y \in (A - B) \cup (B - A). \end{aligned}$$

$\therefore \quad (A \cup B) - (A \cap B) \subseteq (A - B) \cup (B - A)$. … (iv)

From (iii) and (iv), we get $(A - B) \cup (B - A) = (A \cup B) - (A \cap B)$.

V. Let $(A - B) = A$ be given and we have to show that $A \cap B = \phi$.

If possible, let $A \cap B \neq \phi$ and let $x \in A \cap B$. Then,

$$\begin{aligned} x \in A \cap B &\Rightarrow x \in A \text{ and } x \in B \\ &\Rightarrow x \in (A - B) \text{ and } x \in B \quad [\because A = (A - B) \text{ (given)}] \\ &\Rightarrow (x \in A \text{ and } x \notin B) \text{ and } x \in B \\ &\Rightarrow x \in A \text{ and } (x \notin B \text{ and } x \in B). \end{aligned}$$

But, $x \notin B$ and $x \in B$ both can never hold simultaneously.

Thus, we arrive at a contradiction.

Since the contradiction arises by assuming that $A \cap B \neq \phi$, so $A \cap B = \phi$.

Thus, $(A - B) = A \Rightarrow A \cap B = \phi$.

Again, let $(A \cap B) = \phi$ and we have to show that $(A - B) = A$.

Now, $x \in (A - B) \Rightarrow x \in A$ and $x \notin B$

$\Rightarrow x \in A$ (surely).

$\therefore \quad (A - B) \subseteq A$. … (v)

Again, $y \in A \Rightarrow y \notin B$ $[\because A \cap B = \phi]$

$\Rightarrow y \in A$ and $y \notin B$

$\Rightarrow y \in (A - B)$.

$\therefore \quad A \subseteq (A - B)$. ... (vi)

Thus, from (v) and (vi), we get $(A - B) = A$.

$\therefore \quad A \cap B = \phi \Rightarrow (A - B) = A$.

Hence, $(A - B) = A \Leftrightarrow (A \cap B) = \phi$.

THEOREM 8 *If* $(A \cup B) = (A \cap B)$ *then prove that* $A = B$.

PROOF Let $(A \cup B) = (A \cap B)$ be given.

Let x be an arbitrary element of A. Then,

$x \in A \Rightarrow x \in A \cup B$ $[\because A \subseteq A \cup B]$

$\Rightarrow x \in A \cap B$ $[\because A \cup B = A \cap B]$

$\Rightarrow x \in A$ and $x \in B$

$\Rightarrow x \in B$ (surely).

$\therefore \quad A \subseteq B$. ... (i)

Again, let $y \in B$. Then,

$y \in B \Rightarrow y \in A \cup B$ $[\because B \subseteq A \cup B]$

$\Rightarrow y \in A \cap B$ $[\because A \cup B = A \cap B]$

$\Rightarrow y \in A$ and $y \in B$

$\Rightarrow y \in A$ (surely).

$\therefore \quad B \subseteq A$. ... (ii)

Thus, from (i) and (ii), we get $A = B$.

THEOREM 9 *If* $A \subseteq B$ *then for any set* C, *prove that* $(C - B) \subseteq (C - A)$.

PROOF Let $A \subseteq B$ be given.

Let $x \in (C - B)$. Then,

$x \in (C - B) \Rightarrow x \in C$ and $x \notin B$

$\Rightarrow x \in C$ and $x \notin A$ $[\because A \subseteq B]$

$\Rightarrow x \in (C - A)$.

$\therefore \quad (C - B) \subseteq (C - A)$.

Hence, $A \subseteq B \Rightarrow (C - B) \subseteq (C - A)$.

THEOREM 10 *For any sets A and B, prove that:*

(i) $A \cup (A \cap B) = A$ *(ii)* $A \cap (A \cup B) = A$

PROOF (i) Since $(A \cap B) \subseteq A$, we have $A \cup (A \cap B) = A$ $[\because X \subseteq Y \Rightarrow X \cup Y = Y]$.

(ii) Since $A \subseteq (A \cup B)$, we have $A \cap (A \cup B) = A$ $[\because X \subseteq Y \Rightarrow X \cap Y = X]$.

THEOREM 11 *For any sets A and B, prove that:*

(i) $(A \cap B) \cup (A - B) = A$ *(ii)* $A \cup (B - A) = (A \cup B)$

PROOF (i) We have

$$(A \cap B) \cup (A - B) = (A \cap B) \cup (A \cap B') \quad [\because (A - B) = (A \cap B')]$$
$$= A \cap (B \cup B') \quad \text{[by distributive law]}$$
$$= A \cap U \quad [\because B \cup B' = U]$$
$$= A.$$

Hence, $(A \cap B) \cup (A - B) = A$.

(ii) We have

$$A \cup (B - A) = A \cup (B \cap A') \quad [\because (B - A) = (B \cap A')]$$
$$= (A \cup B) \cap (A \cup A') \quad \text{[by distributive law]}$$
$$= (A \cup B) \cap U \quad [\because A \cup A' = U]$$
$$= (A \cup B).$$

Hence, $A \cup (B - A) = (A \cup B)$.

THEOREM 12 *If $A \cap B' = \phi$ then prove that $A = A \cap B$ and hence show that $A \subseteq B$.*

PROOF Let $A \cap B' = \phi$ be given. Then,

$A = (A \cap U)$, where U is the universal set

$$= A \cap (B \cup B') \quad [\because B \cup B' = U]$$
$$= (A \cap B) \cup (A \cap B')$$
$$= (A \cap B) \cup \phi \quad [\because A \cap B' = \phi]$$
$$= (A \cap B).$$

Hence, $A = (A \cap B)$.

Further, let $A = A \cap B$ and let $x \in A$. Then,

$$x \in A \Rightarrow x \in A \cap B \quad [\because A = A \cap B]$$
$$\Rightarrow x \in A \text{ and } x \in B$$
$$\Rightarrow x \in B \text{ (surely).}$$

$\therefore \quad A \subseteq B$.

THEOREM 13 *If A, B and C be the sets such that $A \cup B = A \cup C$ and $A \cap B = A \cap C$ then prove that $B = C$.*

PROOF Let $A \cup B = A \cup C$ and $A \cap B = A \cap C$ be given. Then,

$A \cup B = A \cup C$

$\Rightarrow \quad (A \cup B) \cap B = (A \cup C) \cap B$ and $(A \cup B) \cap C = (A \cup C) \cap C$

$\Rightarrow \quad B = (A \cap B) \cup (C \cap B)$ and $(A \cap C) \cup (B \cap C) = C$

$[\because B \subseteq (A \cup B) \text{ and } C \subseteq (A \cup C)]$

$\Rightarrow \quad B = (A \cap B) \cup (B \cap C)$ and $(A \cap B) \cup (B \cap C) = C \quad [\because A \cap C = A \cap B]$

$\Rightarrow \quad B = C$.

Hence, $B = C$.

THEOREM 14 *For any sets A, B and C, prove that:*

(i) $A - (B \cup C) = (A - B) \cap (A - C)$

(ii) $A - (B \cap C) = (A - B) \cup (A - C)$

(iii) $(A \cup B) - C = (A - C) \cup (B - C)$

(iv) $(A \cap B) - C = (A - C) \cap (B - C)$

PROOF We have

(i) $A-(B\cup C) = A\cap(B\cup C)'$ $[\because\ X-Y = X\cap Y']$

$= A\cap(B'\cap C')$ [by De Morgan's law]

$= (A\cap B')\cap(A\cap C')$

$= (A-B)\cap(A-C)$.

$\therefore\quad A-(B\cup C) = (A-B)\cap(A-C)$.

(ii) $A-(B\cap C) = A\cap(B\cap C)'$ $[\because\ X-Y = X\cap Y']$

$= A\cap(B'\cup C')$ [by De Morgan's law]

$= (A\cap B')\cup(A\cap C')$ [by distributive law]

$= (A-B)\cup(A-C)$ $[\because\ X\cap Y' = X-Y]$.

$\therefore\quad A-(B\cap C) = (A-B)\cup(A-C)$.

(iii) $(A\cup B)-C = (A\cup B)\cap C'$ $[\because\ X-Y = X\cap Y']$

$= (A\cap C')\cup(B\cap C')$ [by distributive law]

$= (A-C)\cup(B-C)$. $[\because\ X\cap Y' = X-Y]$

$\therefore\quad (A\cup B)-C = (A-C)\cup(B-C)$.

(iv) $(A\cap B)-C = (A\cap B)\cap C'$ $[\because\ X-Y = X\cap Y']$

$= (A\cap C')\cap(B\cap C')$ [*note*]

$= (A-C)\cap(B-C)$. $[\because\ X\cap Y' = X-Y]$

$\therefore\quad (A\cap B)-C = (A-C)\cap(B-C)$.

THEOREM 15 *Let A and B be sets. If $A\cap X = B\cap X = \phi$ and $A\cup X = B\cup X$ for some set X, show that $A = B$.*

PROOF $A\cup X = B\cup X$ [given]

$\Rightarrow\quad A\cap(A\cup X) = A\cap(B\cup X)$

$\Rightarrow\quad (A\cap A)\cup(A\cap X) = (A\cap B)\cup(A\cap X)$ [by distributive law]

$\Rightarrow\quad A\cup\phi = (A\cap B)\cup\phi$ $[\because\ A\cap X = \phi]$

$\Rightarrow\quad A = (A\cap B)$

$\Rightarrow\quad A\subseteq B$ … (i) $[\because\ A\cap B = A\ \Rightarrow\ A\subseteq B]$.

Again, $A\cup X = B\cup X$

$\Rightarrow\quad B\cap(A\cup X) = B\cap(B\cup X)$

$\Rightarrow\quad (B\cap A)\cup(B\cap X) = (B\cap B)\cup(B\cap X)$ [by distributive law]

$\Rightarrow\quad (A\cap B)\cup\phi = B\cup\phi$ $[\because\ B\cap X = \phi$ and $B\cap A = A\cap B]$

$\Rightarrow\quad A\cap B = B$

$\Rightarrow\quad B\subseteq A$ … (ii) $[\because\ A\cap B = B\ \Rightarrow\ B\subseteq A]$.

From (i) and (ii), we get $A = B$.

THEOREM 16 *Show that the following four conditions are equivalent:*

(i) $A\subset B$ *(ii)* $A-B = \phi$ *(iii)* $A\cup B = B$ *(iv)* $A\cap B = A$

PROOF In order to prove the required result, we will show that:

(i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) $\Rightarrow$ (i).

Now, **(i) $\Rightarrow$ (ii):**

Let $A \subset B$ be given.

Then, there is no element of A which is not in B.

$\therefore \quad A - B = \{x : x \in A \text{ and } x \notin B\} = \phi$

[$\because$ there is no element of A which is not in B].

Hence, $A \subset B \Rightarrow A - B = \phi$ and therefore, (i) $\Rightarrow$ (ii).

(ii) $\Rightarrow$ (iii):

Let $A - B = \phi$ be given. Then,

$$\begin{aligned} A - B = \phi &\Rightarrow \text{every element of } A \text{ is in } B \\ &\Rightarrow A \subset B \\ &\Rightarrow A \cup B = B. \end{aligned}$$

Thus, $A - B = \phi \Rightarrow A \cup B = B$ and therefore, (ii) $\Rightarrow$ (iii).

(iii) $\Rightarrow$ (iv):

Let $A \cup B = B$ be given. Then,

$$A \cup B = B \Rightarrow A \subset B \Rightarrow A \cap B = A.$$

Thus, $A \cup B = B \Rightarrow A \cap B = A$ and therefore, (iii) $\Rightarrow$ (iv).

(iv) $\Rightarrow$ (i):

Let $A \cap B = A$ be given. Then,

$$\begin{aligned} x \in A &\Rightarrow x \in A \cap B && [\because A = A \cap B] \\ &\Rightarrow x \in A \text{ and } x \in B \\ &\Rightarrow x \in B \text{ (surely).} \end{aligned}$$

$\therefore \quad A \subseteq B$.

Thus, $A \cap B = A \Rightarrow A \subseteq B$ and therefore, (iv) $\Rightarrow$ (i).

$\therefore \quad$ (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) $\Rightarrow$ (i).

Hence, the given four conditions are equivalent.

THEOREM 17 *For any sets A and B, prove that*

$$P(A \cap B) = P(A) \cap P(B).$$

PROOF Let $X \in P(A \cap B)$. Then,

$$\begin{aligned} X \in P(A \cap B) &\Rightarrow X \subseteq A \cap B \\ &\Rightarrow X \subseteq A \text{ and } X \subseteq B \\ &\Rightarrow X \in P(A) \text{ and } X \in P(B) \\ &\Rightarrow X \in P(A) \cap P(B). \end{aligned}$$

$\therefore \quad P(A \cap B) \subseteq P(A) \cap P(B)$. ... (i)

Again, let $Y \in P(A) \cap P(B)$. Then,

$$\begin{aligned} Y \in P(A) \cap P(B) &\Rightarrow Y \in P(A) \text{ and } Y \in P(B) \\ &\Rightarrow Y \subseteq A \text{ and } Y \subseteq B \\ &\Rightarrow Y \subseteq A \cap B \\ &\Rightarrow Y \in P(A \cap B). \end{aligned}$$

$\therefore \quad P(A) \cap P(B) \subseteq P(A \cap B)$. ... (ii)

From (i) and (ii), we get $P(A \cap B) = P(A) \cap P(B)$.

THEOREM 18 *For any two sets A and B, prove that*

$$P(A) \cup P(B) \subset P(A \cup B).$$

But, $P(A \cup B)$ is not necessarily a subset of $P(A) \cup P(B)$.

PROOF Let X be an arbitrary element of $P(A) \cup P(B)$. Then,

$$\begin{aligned} X \in P(A) \cup P(B) &\Rightarrow X \in P(A) \text{ or } X \in P(B) \\ &\Rightarrow X \subset A \text{ or } X \subset B \\ &\Rightarrow X \subset (A \cup B) \\ &\Rightarrow X \in P(A \cup B). \end{aligned}$$

$\therefore \quad P(A) \cup P(B) \subset P(A \cup B)$.

However, $P(A \cup B) \subset P(A) \cup P(B)$ is not always true.

For example, let $A = \{1\}$ and $B = \{2\}$. Then, $A \cup B = \{1, 2\}$.

$\therefore \quad P(A) = \{\phi, \{1\}\}, P(B) = \{\phi, \{2\}\}$

and $\quad P(A \cup B) = \{\phi, \{1\}, \{2\}, \{1, 2\}\}$.

Also, $P(A) \cup P(B) = \{\phi, \{1\}, \{2\}\}$.

$\therefore \quad P(A \cup B) \not\subset P(A) \cup P(B)$.

Hence, in general, $P(A \cup B) \neq P(A) \cup P(B)$.

THEOREM 19 *If $P(A) = P(B)$, prove that $A = B$.*

PROOF Let $P(A) = P(B)$. Then,

$$\begin{aligned} A \subseteq A &\Rightarrow A \in P(A) \\ &\Rightarrow A \in P(B) && [\because P(A) = P(B)] \\ &\Rightarrow A \subseteq B. && \dots \text{(i)} \end{aligned}$$

$$\begin{aligned} \text{Again, } B \subseteq B &\Rightarrow B \in P(B) \\ &\Rightarrow B \in P(A) && [\because P(B) = P(A)] \\ &\Rightarrow B \subseteq A. && \dots \text{(ii)} \end{aligned}$$

From (i) and (ii), we get $A \subseteq B$ and $B \subseteq A$.

Hence, $A = B$.

EXERCISE 1E

1. If $A = \{a, b, c, d, e\}$, $B = \{a, c, e, g\}$ and $C = \{b, e, f, g\}$, verify that:

(i) $A \cup B = B \cup A$ (ii) $A \cup C = C \cup A$ (iii) $B \cup C = C \cup B$

(iv) $A \cap B = B \cap A$ (v) $B \cap C = C \cap B$ (vi) $A \cap C = C \cap A$

(vii) $(A \cup B) \cup C = A \cup (B \cup C)$ (viii) $(A \cap B) \cap C = A \cap (B \cap C)$

2. If $A = \{a, b, c, d, e\}$, $B = \{a, c, e, g\}$ and $C = \{b, e, f, g\}$, verify that:

(i) $A \cap (B - C) = (A \cap B) - (A \cap C)$ (ii) $A - (B \cap C) = (A - B) \cup (A - C)$

3. If $A = \{x : x \in N, x \leq 7\}$, $B = \{x : x \text{ is prime}, x < 8\}$ and $C = \{x : x \in N, x \text{ is odd and } x < 10\}$, verify that:

(i) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ (ii) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

4. If $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$, $A = \{2, 4, 6, 8\}$ and $B = \{2, 3, 5, 7\}$, verify that:

(i) $(A \cup B)' = (A' \cap B')$ (ii) $(A \cap B)' = (A' \cup B')$

5. Let $A = \{a, b, c\}$, $B = \{b, c, d, e\}$ and $C = \{c, d, e, f\}$ be subsets of $U = \{a, b, c, d, e, f\}$. Then, verify that:

(i) $(A')' = A$ (ii) $(A \cup B)' = (A' \cap B')$ (iii) $(A \cap B)' = (A' \cup B')$

6. Given an example of three sets A, B, C such that $A \cap B \neq \phi$, $B \cap C \neq \phi$, $A \cap C \neq \phi$ and $A \cap B \cap C = \phi$.

7. For any sets A and B, prove that:

(i) $(A - B) \cap B = \phi$ (ii) $A \cup (B - A) = A \cup B$

(iii) $(A - B) \cup (A \cap B) = A$ (iv) $(A \cup B) - B = A - B$

(v) $A - (A \cap B) = A - B$

8. For any sets A and B, prove that:

(i) $A \cap B' = \phi \Rightarrow A \subset B$ (ii) $A' \cup B = U \Rightarrow A \subset B$

HINTS TO SOME SELECTED QUESTIONS

6. Take $A = \{1, 2\}$, $B = \{2, 3\}$ and $C = \{1, 3\}$.

7. (i) $(A - B) \cap B = (A \cap B') \cap B = A \cap (B' \cap B) = A \cap \phi = \phi$.

(ii) $A \cup (B - A) = A \cup (B \cap A') = (A \cup B) \cap (A \cup A') = (A \cup B) \cap U = (A \cup B)$.

(iii) $(A - B) \cup (A \cap B) = (A \cap B') \cup (A \cap B) = A \cap (B' \cup B) = A \cap U = A$.

(iv) $(A \cup B) - B = (A \cup B) \cap B' = (A \cap B') \cup (B \cap B') = (A \cap B') \cup \phi = A \cap B' = A - B$.

(v) $A - (A \cap B) = A \cap (A \cap B)' = A \cap (A' \cup B') = (A \cap A') \cup (A \cap B') = \phi \cup (A \cap B')$
$= (A \cap B') = (A - B)$.

8. (i) $A \cap B' = \phi \Rightarrow A - B = \phi \Rightarrow A \subset B$.

(ii) $A' \cup B = U \Rightarrow A \cap (A' \cup B) = A \cap U \Rightarrow (A \cap A') \cup (A \cap B) = A$
$\Rightarrow \phi \cup (A \cap B) = A \Rightarrow A \cap B = A \Rightarrow A \subset B$.

VENN DIAGRAMS

In order to express the relationship among sets in perspective, we represent them pictorially by means of diagrams, called *Venn diagrams*.

In these diagrams, the universal set is represented by a rectangular region and its subsets by circles inside the rectangle. We represent disjoint sets by disjoint circles and intersecting sets by intersecting circles.

VENN DIAGRAMS IN DIFFERENT SITUATIONS

CASE 1 *When the universal set and its subset are given*

Let U be the universal set and let $A \subseteq U$.

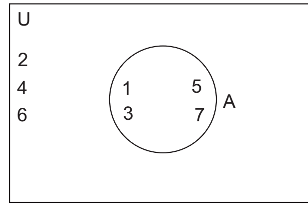
We draw a circle inside a rectangle.

The rectangular region represents U and the circular region represents A.

EXAMPLE Let $U = \{1, 2, 3, 4, 5, 6, 7\}$ and
$A = \{1, 3, 5, 7\}$.

Then, we draw the Venn diagram, as shown in the given figure.

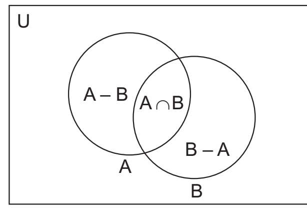
Clearly, $A' = \{2, 4, 6\}$.



CASE 2 *When two intersecting subsets of U are given*

For representing two intersecting subsets A and B of U, we draw two intersecting circles within the rectangle.

The common region of these circles represents $A \cap B$.



Excluding the region of B from that of A shows $(A - B)$.

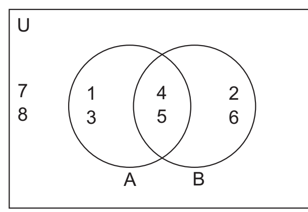
Excluding the region of A from that of B shows $(B - A)$.

EXAMPLE Let $U = \{1, 2, 3, 4, 5, 6, 7, 8\}$ be the universal set, and let

$A = \{1, 3, 4, 5\}$ and $B = \{2, 4, 5, 6\}$ be its subsets.

Then, $A \cap B = \{4, 5\}$.

We draw the Venn diagram, as shown in the given figure.



Clearly, $(A - B) = \{1, 3\}$ and $(B - A) = \{2, 6\}$.

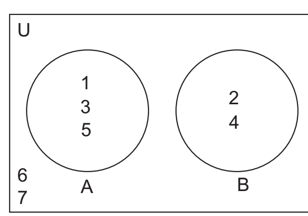
CASE 3 *When two disjoint subsets of a set be given*

In order to represent two disjoint subsets A and B of the universal set U, we draw two disjoint circles within a rectangle.

EXAMPLE Let $U = \{1, 2, 3, 4, 5, 6, 7\}$ be the universal set, and let $A = \{1, 3, 5\}$ and $B = \{2, 4\}$ be two of its disjoint subsets.

Clearly, $A \cap B = \phi$.

So, we may draw the Venn diagram, as shown in the adjoining figure.



Clearly, $A \cap B = \phi$, $(A - B) = \{1, 3, 5\} = A$ and $(B - A) = \{2, 4\} = B$.

$A' = \{2, 4, 6, 7\}$ and $B' = \{1, 3, 5, 6, 7\}$.

CASE 4 *When $B \subseteq A \subseteq U$*

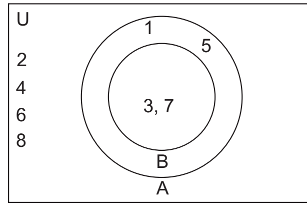
In this case, we draw two concentric circles within a rectangular region.

The inner circle represents B and the outer circle represents A.

EXAMPLE Let $U = \{1, 2, 3, 4, 5, 6, 7, 8\}$ be the universal set, and let $A = \{1, 3, 5, 7\}$ and $B = \{3, 7\}$ be its subsets.

Then, clearly $B \subseteq A$.

Now, we may draw the Venn diagram, as shown in the given figure.



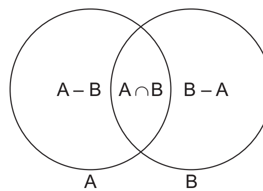
$A \cap B = B = \{3, 7\}$, $A \cup B = A = \{1, 3, 5, 7\}$,

$(A - B) = \{1, 5\}$, $(B - A) = \phi$,

$A' = \{2, 4, 6, 8\}$ and $B' = \{1, 5, 2, 4, 6, 8\}$.

IMPORTANT RESULTS FROM VENN DIAGRAMS

Let A and B be two intersecting subsets of U.



In counting the elements of $(A \cup B)$, the elements of $A \cap B$ are counted twice, once in counting the elements of A and second time in counting the elements of B.

$\therefore \quad n(A \cup B) = n(A) + n(B) - n(A \cap B)$.

If $A \cap B = \phi$, then $n(A \cap B) = 0$ and therefore, in this case, we have

$n(A \cup B) = n(A) + n(B)$.

From the Venn diagram, it is also clear that

(i) $n(A - B) + n(A \cap B) = n(A)$

(ii) $n(B - A) + n(A \cap B) = n(B)$

(iii) $n(A - B) + n(A \cap B) + n(B - A) = n(A \cup B)$.

EXERCISE 1F

1. Let $A = \{a, b, c, e, f\}$, $B = \{c, d, e, g\}$ and $C = \{b, c, f, g\}$ be subsets of the set $U = \{a, b, c, d, e, f, g, h\}$.

Draw Venn diagrams to represent the following sets:

(i) $A \cap B$ (ii) $A \cup (B \cap C)$ (iii) $A - B$

(iv) $B - A$ (v) $A - (B \cap C)$ (vi) $(B - C) \cup (C - B)$

2. Let $A = \{2, 4, 6, 8, 10\}$, $B = \{4, 8, 12, 16\}$ and $C = \{6, 12, 18, 24\}$.

Using Venn diagrams, verify that:

(i) $(A \cup B) \cup C = A \cup (B \cup C)$ (ii) $(A \cap B) \cap C = A \cap (B \cap C)$

3. Let $A = \{a, e, i, o, u\}$, $B = \{a, d, e, o, v\}$ and $C = \{e, o, t, m\}$.

Using Venn diagrams, verify the following:

(i) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ (ii) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

4. Let $A \subset B \subset U$. Exhibit it in a Venn diagram.

5. Let $A = \{2, 3, 5, 7, 11, 13\}$ and $B = \{5, 7, 9, 11, 15\}$ be subsets of $U = \{2, 3, 5, 7, 9, 11, 13, 15\}$.

Using Venn diagrams, verify that:

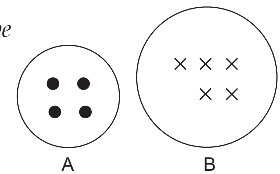
(i) $(A \cup B)' = (A' \cap B')$ (ii) $(A \cap B)' = (A' \cup B')$

6. Using Venn diagrams, show that $(A - B), (A \cap B)$ and $(B - A)$ are disjoint sets, taking $A = \{2, 4, 6, 8, 10, 12\}$ and $B = \{3, 6, 9, 12, 15\}$.

SOME RESULTS DERIVED FROM VENN DIAGRAMS

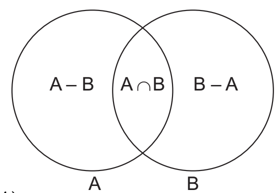
RESULT 1 *When A and B are disjoint sets, then we have*

$$n(A \cup B) = n(A) + n(B).$$



RESULT 2 *For any sets A and B, prove that*

$$n(A \cup B) = n(A) + n(B) - n(A \cap B).$$



PROOF From the given Venn diagram, it is clear that the sets $(A - B), (A \cap B)$ and $(B - A)$ are disjoint and their union is $(A \cup B)$.

$\therefore \quad n(A \cup B) = n(A - B) + n(A \cap B) + n(B - A)$

$= n(A - B) + n(A \cap B) + n(B - A) + n(A \cap B) - n(A \cap B)$

[adding and subtracting $n(A \cap B)$]

$= n(A) + n(B) - n(A \cap B)$

$[\because\ n(A - B) + n(A \cap B) = n(A)$ and $n(B - A) + n(A \cap B) = n(B)]$

Hence, $n(A \cup B) = n(A) + n(B) - n(A \cap B)$.

Corollary 1 *Prove that* $n(A - B) + n(A \cap B) = n(A)$.

PROOF Clearly, $(A - B)$ and $(A \cap B)$ are disjoint sets and their union is A.

$\therefore \quad n(A - B) + n(A \cap B) = n(A)$.

Corollary 2 *Prove that* $n(B - A) + n(A \cap B) = n(B)$.

PROOF Clearly, $(B - A)$ and $(A \cap B)$ are disjoint sets whose union is B.

$\therefore \quad n(B - A) + n(A \cap B) = n(B)$.

RESULT 3 *For any sets A, B, C prove that*

$$n(A \cup B \cup C) = [n(A) + n(B) + n(C) + n(A \cap B \cap C)] - [n(A \cap B) + n(B \cap C) + n(A \cap C)].$$

PROOF We have

$n(A \cup B \cup C) = n[(A \cup B) \cup C]$

$= n(A \cup B) + n(C) - n[(A \cup B) \cap C]$

$= \{n(A) + n(B) - n(A \cap B)\} + n(C) - n[(A \cap C) \cup (B \cap C)$

$= \{n(A) + n(B) + n(C) - n(A \cap B)\} - \{n(A \cap C) + n(B \cap C)$

$-n(A \cap C \cap B \cap C)\}$

$$= n(A) + n(B) + n(C) - n(A \cap B) - n(A \cap C) - n(B \cap C) + n(A \cap B \cap C)$$

$$= \{n(A) + n(B) + n(C) + n(A \cap B \cap C)\} - \{n(A \cap B) + n(B \cap C) + n(A \cap C)\}.$$

Hence, the result follows.

SUMMARY

For any sets A, B, C we have:

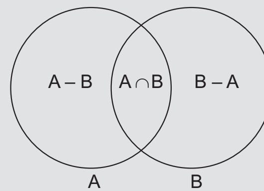
(i) $n(A \cup B) = n(A) + n(B) - n(A \cap B)$.

(ii) If $A \cap B = \phi$, then $n(A \cup B) = n(A) + n(B)$.

(iii) $n(A - B) + n(A \cap B) = n(A)$.

(iv) $n(B - A) + n(A \cap B) = n(B)$.

(v) $n(A \cup B \cup C) = \{n(A) + n(B) + n(C) + n(A \cap B \cap C)\} - \{n(A \cap B) + n(B \cap C) + n(A \cap C)\}$.



SOLVED EXAMPLES

EXAMPLE 1 *If A and B are two sets such that* $n(A) = 27$, $n(B) = 35$ *and* $n(A \cup B) = 50$, *find* $n(A \cap B)$.

SOLUTION We know that

$$n(A \cup B) = n(A) + n(B) - n(A \cap B)$$

$$\Rightarrow \quad n(A \cap B) = n(A) + n(B) - n(A \cup B) = (27 + 35 - 50) = 12.$$

Hence, $n(A \cap B) = 12$.

EXAMPLE 2 *If A and B are two sets containing* 3 *and* 6 *elements respectively, what can be the maximum number of elements in* $A \cup B$?

Find also the minimum number of elements in $A \cup B$.

SOLUTION We know that

$$n(A \cup B) = n(A) + n(B) - n(A \cap B). \qquad \text{... (i)}$$

CASE 1 From (i), it is clear that $n(A \cup B)$ will be maximum when $n(A \cap B) = 0$.

In that case, $n(A \cup B) = n(A) + n(B) = (3 + 6) = 9$.

$\therefore$ maximum number of elements in $(A \cup B) = 9$.

CASE 2 From (i), it is clear that $n(A \cup B)$ will be minimum when $n(A \cap B)$ is maximum, i.e., when $n(A \cap B) = 3$.

In this case, $n(A \cup B) = n(A) + n(B) - n(A \cap B) = (3 + 6 - 3) = 6$.

$\therefore$ minimum number of elements in $A \cup B = 6$.

EXAMPLE 3 *A survey shows that 73% of the Indians like apples, whereas 65% like oranges. What percentage of Indians like both apples and oranges?*

SOLUTION Let A = set of Indians who like apples

and B = set of Indians who like oranges.

Then, $n(A) = 73$, $n(B) = 65$ and $n(A \cup B) = 100$.

$$n(A \cap B) = n(A) + n(B) - n(A \cup B) = (73 + 65 - 100) = 38.$$

Hence, 38% of the Indians like both apples and oranges.

EXAMPLE 4 *In a survey of* 425 *students in a school, it was found that* 115 *drink apple juice,* 160 *drink orange juice and* 80 *drink both apple as well as orange juice. How many drink neither apple juice nor orange juice?*

SOLUTION Let U = set of all students surveyed;

A = set of all students who drink apple juice

and B = set of all students who drink orange juice.

Then, $n(U) = 425$, $n(A) = 115$, $n(B) = 160$ and $n(A \cap B) = 80$.

$\therefore \quad n(A \cup B) = n(A) + n(B) - n(A \cap B) = (115 + 160 - 80) = 195.$

Set of students who drink neither apple juice nor orange juice

$$= (A' \cap B') = (A \cup B)'$$

$\Rightarrow \quad n\{(A \cup B)'\} = n(U) - n(A \cup B) = (425 - 195) = 230.$

Hence, 230 students drink neither apple juice nor orange juice.

EXAMPLE 5 *In a group of* 850 *persons,* 600 *can speak Hindi and* 340 *can speak Tamil. Find (i) how many can speak both Hindi and Tamil,*

(ii) how many can speak Hindi only,

(iii) how many can speak Tamil only.

SOLUTION Let A = set of persons who can speak Hindi

and B = set of persons who can speak Tamil.

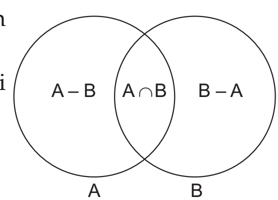
$\therefore \quad n(A) = 600$, $n(B) = 340$ and $n(A \cup B) = 850$.

(i) Set of persons who can speak both Hindi and Tamil = $(A \cap B)$.

Now, $n(A \cap B) = n(A) + n(B) - n(A \cup B)$

$= (600 + 340 - 850) = 90.$

Thus, 90 persons can speak both Hindi and Tamil.



(ii) Set of persons who can speak Hindi only = $(A - B)$.

Now, $n(A - B) + n(A \cap B) = n(A)$

$\Rightarrow \quad n(A - B) = n(A) - n(A \cap B)$

$= (600 - 90) = 510.$

Thus, 510 persons can speak Hindi only.

(iii) Set of persons who can speak Tamil only = $(B - A)$.

Now, $n(B - A) + n(A \cap B) = n(B)$

$\Rightarrow \quad n(B - A) = n(B) - n(A \cap B) = (340 - 90) = 250.$

Hence, 250 persons can speak Tamil only.

EXAMPLE 6 *A market research group conducted a survey of* 1000 *consumers and reported that* 745 *consumers like product A and* 430 *consumers like product B. What is the least number that must have liked both products?*

SOLUTION Let P and Q be the sets of consumers who like product A and product B respectively.

Then, $n(P) = 745$ and $n(Q) = 430$.

Now, $n(P \cup Q) = n(P) + n(Q) - n(P \cap Q)$

$\Rightarrow \quad n(P \cup Q) = 745 + 430 - n(P \cap Q)$

$\Rightarrow \quad n(P \cup Q) = 1175 - n(P \cap Q)$.

Clearly, $n(P \cap Q)$ is least when $n(P \cup Q)$ is maximum and therefore, $n(P \cup Q) = 1000$.

So, $1000 = 1175 - n(P \cap Q) \Rightarrow n(P \cap Q) = (1175 - 1000) = 175$.

Hence, the least number of consumers liking both the products is 175.

EXAMPLE 7 *Out of 600 car owners investigated, 500 owned car A; 200 owned car B and 50 owned both A and B cars. Verify whether the given data is correct or not.*

SOLUTION Let P and Q be the sets of those who own car A and car B respectively. Then,

$$n(P) = 500,\ n(Q) = 200 \text{ and } n(P \cap Q) = 50.$$

Now, $n(P \cup Q) = n(P) + n(Q) - n(P \cap Q)$

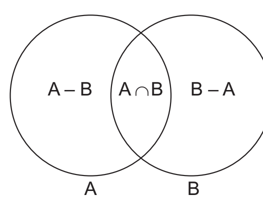
$\Rightarrow \quad n(P \cup Q) = (500 + 200 - 50) = 650$.

This is a contradiction, since the maximum value of $n(P \cup Q)$ is 600.

Hence, the given data is incorrect.

EXAMPLE 8 *In a group of 52 persons, 16 drink tea but not coffee and 33 drink tea. Find (i) how many drink tea and coffee both;*
(ii) how many drink coffee but not tea.

SOLUTION Let A = set of persons who drink tea and B = set of persons who drink coffee.



Then, $(A - B)$ = set of persons who drink tea but not coffee.

And, $(B - A)$ = set of persons who drink coffee but not tea.

Given: $n(A \cup B) = 52$, $n(A - B) = 16$ and $n(A) = 33$.

(i) Set of persons who drink tea and coffee both = $(A \cap B)$.

Now, $n(A - B) + n(A \cap B) = n(A)$

$\Rightarrow \quad n(A \cap B) = n(A) - n(A - B) = (33 - 16) = 17$.

Thus, 17 persons drink tea and coffee both.

(ii) $n(A \cup B) = n(A) + n(B) - n(A \cap B)$

$\Rightarrow \quad n(B) = n(A \cup B) + n(A \cap B) - n(A) = (52 + 17 - 33) = 36$.

Now, $n(B - A) + n(A \cap B) = n(B)$.

$\Rightarrow \quad n(B - A) = n(B) - n(A \cap B) = (36 - 17) = 19$.

Thus, number of persons who drink coffee but not tea = 19.

EXAMPLE 9 *A school awarded 58 medals in three sports, namely 38 in football; 15 in basketball and 20 in cricket. If 3 students got medals in all the three sports, how many received medals in exactly two sports?*

SOLUTION Let A, B and C denote the sets of students who won medals in football, basketball and cricket respectively.

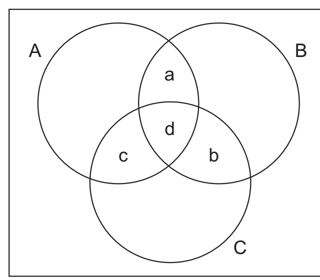
Then, $n(A) = 38$, $n(B) = 15$, $n(C) = 20$, $n(A \cap B \cap C) = 3$ and $n(A \cup B \cup C) = 58$.

We know that

$$n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(B \cap C) - n(A \cap C) + n(A \cap B \cap C)$$

$$\Rightarrow \quad n(A \cap B) + n(B \cap C) + n(A \cap C)$$
$$= n(A) + n(B) + n(C) + n(A \cap B \cap C) - n(A \cup B \cup C)$$
$$= \{(38 + 15 + 20 + 3) - 58\} = (76 - 58) = 18.$$

Let a, b, c and d denote respectively the number of students who won medals in football and basketball both; basketball and cricket both; football and cricket both and all the 3 sports.



Then, $n(A \cap B) + n(B \cap C) + n(A \cap C) = 18$

$$\Rightarrow \quad (a + d) + (b + d) + (c + d) = 18$$
$$\Rightarrow \quad (a + b + c) + 3d = 18$$
$$\Rightarrow \quad (a + b + c) + 3 \times 3 = 18 \Rightarrow a + b + c = 9 \qquad [\because d = 3].$$

Hence, 9 students received medals in exactly two sports.

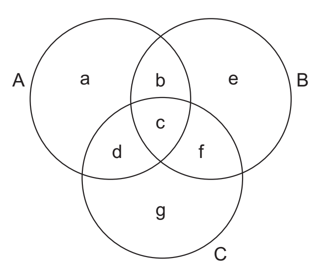
AN EASY APPROACH

EXAMPLE 10 *In a survey it is found that 21 people like product A, 26 people like product B and 29 like product C. If 14 people like products A and B; 15 people like products B and C; 12 people like products C and A; and 8 people like all the three products, find*

(i) how many people are surveyed in all;

(ii) how many like product C only.

SOLUTION Let A, B, C denote respectively the sets of people who like product A, B and C respectively, as shown in the given figure.



Let us denote the number of elements contained in bounded regions by a, b, c, d, e, f, g as shown in the given figure.

Then, we have

$a+b+c+d=21,$
$b+c+e+f=26,$
$c+d+f+g=29,$
$b+c=14, c+f=15, c+d=12$

and $c=8.$

On solving these equations, we get

$c=8, d=4, f=7, b=6, g=10, e=5, a=3.$

(i) Total number of surveyed people $=(a+b+c+d+e+f+g)=43.$

(ii) Number of persons who like product C only $=g=10.$

EXAMPLE 11 *In a survey of* 25 *students, it was found that* 12 *have taken physics,* 11 *have taken chemistry and* 15 *have taken mathematics;* 4 *have taken physics and chemistry;* 9 *have taken physics and mathematics;* 5 *have taken chemistry and mathematics while* 3 *have taken all the three subjects. Find the number of students who have taken*

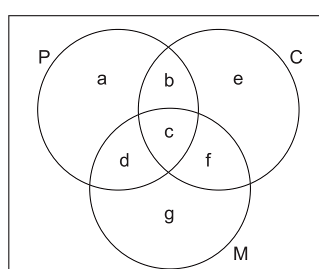
(i) physics only;
(ii) chemistry only;
(iii) mathematics only;
(iv) physics and chemistry but not mathematics;
(v) physics and mathematics but not chemistry;
(vi) only one of the subjects;
(vii) at least one of the three subjects;
(viii) none of the three subjects.

SOLUTION Let P, C and M be the sets of students who have taken physics, chemistry and mathematics respectively.

Let a, b, c, d, e, f and g denote the number of students in the respective regions, as shown in the adjoining Venn diagram.

As per data given, we have

$$\begin{cases} a+b+c+d=12, \\ b+c+e+f=11, \\ c+d+f+g+15, \\ b+c=4, \\ c+d=9, \\ c+f=5, \\ c=3. \end{cases}$$



From these equations, we get

$c=3, f=2, d=6, b=1.$

Now, $c+d+f+g=15 \Rightarrow 3+6+2+g=15 \Rightarrow g=4;$

$b+c+e+f=11 \Rightarrow 1+3+e+2=11 \Rightarrow e=5;$

$$a + b + c + d = 12 \Rightarrow a + 1 + 3 + 6 = 12 \Rightarrow a = 2;$$

$$\therefore \quad a = 2, b = 1, c = 3, d = 6, e = 5, f = 2 \text{ and } g = 4.$$

So, we have:

(i) Number of students who offered physics only $= a = 2$.

(ii) Number of students who offered chemistry only $= e = 5$.

(iii) Number of students who offered mathematics only $g = 4$.

(iv) Number of students who offered physics and chemistry but not mathematics $= b = 1$.

(v) Number of students who offered physics and mathematics but not chemistry $= d = 6$.

(vi) Number of students who offered only one of the given subjects $= (a + e + g) = (2 + 5 + 4) = 11$.

(vii) Number of students who offered at least one of the given subjects $= (a + b + c + d + e + f + g) = (2 + 1 + 3 + 6 + 5 + 2 + 4) = 23$.

(viii) Number of students who offered none of the three given subjects $= (25 - 23) = 2$.

EXERCISE 1G

1. If A and B are two sets such that $n(A) = 37$, $n(B) = 26$ and $n(A \cup B) = 51$, find $n(A \cap B)$.

2. If P and Q are two sets such that $n(P \cup Q) = 75$, $n(P \cap Q) = 17$ and $n(P) = 49$, find $n(Q)$.

3. If A and B are two sets such that $n(A) = 24$, $n(B) = 22$ and $n(A \cap B) = 8$, find:

(i) $n(A \cup B)$ (ii) $n(A - B)$ (iii) $n(B - A)$

4. If A and B are two sets such that $n(A - B) = 24$, $n(B - A) = 19$ and $n(A \cap B) = 11$, find:

(i) $n(A)$ (ii) $n(B)$ (iii) $n(A \cup B)$

5. In a committee, 50 people speak Hindi, 20 speak English and 10 speak both Hindi and English. How many speak at least one of these two languages?

6. In a group of 50 persons, 30 like tea, 25 like coffee and 16 like both. How many like

(i) either tea or coffee? (ii) neither tea nor coffee?

7. There are 200 individuals with a skin disorder, 120 had been exposed to the chemical C_1, 50 to chemical C_2, and 30 to both the chemicals C_1 and C_2. Find the number of individuals exposed to

(i) chemical C_1 but not chemical C_2

(ii) chemical C_2 but not chemical C_1

(iii) chemical C_1 or chemical C_2

8. In a class of a certain school, 50 students offered mathematics, 42 offered biology and 24 offered both the subjects. Find the number of students offering

(i) mathematics only,

(ii) biology only,

(iii) any of the two subjects.

9. In an examination, 56% of the candidates failed in English and 48% failed in science. If 18% failed in both English and science, find the percentage of those who passed in both the subjects.

10. In a group of 65 people, 40 like cricket and 10 like both cricket and tennis. How many like tennis only and not cricket? How many like tennis?

11. A school awarded 42 medals in hockey, 18 in basketball and 23 in cricket. If these medals were bagged by a total of 65 students and only 4 students got medals in all the three sports, how many students received medals in exactly two of the three sports?

12. In a survey of 60 people, it was found that 25 people read newspaper H, 26 read newspaper T, 26 read newspaper I, 9 read both H and I, 11 read both H and T, 8 read both T and I, and 3 read all the three newspapers. Find

(i) the number of people who read at least one of the newspapers,

(ii) the number of people who read exactly one newspaper.

13. In a survey of 100 students, the number of students studying the various languages is found as: English only 18; English but not Hindi 23; English and Sanskrit 8; Sanskrit and Hindi 8; English 26; Sanskrit 48 and no language 24. Find

(i) how many students are studying Hindi,

(ii) how many students are studying English and Hindi both.

14. In a town of 10,000 families, it was found that 40% of the families buy newspaper A, 20% buy newspaper B, 10% buy newspaper C, 5% buy A and B; 3% buy B and C and 4% buy A and C. If 2% buy all the three newspapers, find the number of families which buy

(i) A only,

(ii) B only,

(iii) none of A, B and C.

15. A class has 175 students. The following description gives the number of students studying one or more of the subjects in this class: mathematics 100, physics 70, chemistry 46; mathematics and physics 30; mathematics and chemistry 28; physics and chemistry 23; mathematics, physics and chemistry 18. Find

(i) how many students are enrolled in mathematics alone, physics alone and chemistry alone,

(ii) the number of students who have not offered any of these subjects.

ANSWERS (EXERCISE 1G)

1. 12 **2.** 43 **3.** (i) 38 (ii) 16 (iii) 14 **4.** (i) 35 (ii) 30 (iii) 54

5. 60 **6.** (i) 39 (ii) 11 **7.** (i) 90 (ii) 20 (iii) 140 **8.** (i) 26 (ii) 18 (iii) 68

9. 14%　**10.** 25, 35　**11.** 22　**12.** (i) 52 (ii) 30

13. (i) 18 (ii) 3　**14.** (i) 3300 (ii) 1400 (iii) 4000　**15.** (i) 60, 35, 13 (ii) 32

HINTS TO SOME SELECTED QUESTIONS

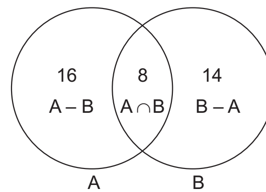
1. $n(A \cap B) = n(A) + n(B) - n(A \cup B)$.

2. $n(P \cup Q) = n(P) + n(Q) - n(P \cap Q)$.

3. (i) $n(A \cup B) = n(A) + n(B) - n(A \cap B)$.

(ii) $n(A - B) + n(A \cap B) = n(A)$.

(iii) $n(B - A) + n(A \cap B) = n(B)$.



4. (i) $n(A - B) + n(A \cap B) = n(A)$

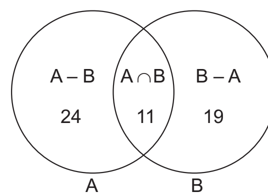
$\Rightarrow \quad n(A) = n(A - B) + n(A \cap B)$

$\Rightarrow \quad n(A) = (24 + 11) = 35$.

(ii) $n(B - A) + n(A \cap B) = n(B)$

$\Rightarrow \quad n(B) = (19 + 11) = 30$.

(iii) $n(A \cup B) = n(A) + n(B) - n(A \cap B)$.



5. $n(A \cup B) = n(A) + n(B) - n(A \cap B) = (50 + 20 - 10) = 60$.

6. $n(A) = 30$, $n(B) = 25$ and $n(A \cap B) = 16$.

(i) $n(A \cup B) = n(A) + n(B) - n(A \cap B) = 39$.

(ii) $n(A' \cap B') = n\{(A \cup B)'\} = n(U) - n(A \cup B) = (50 - 39) = 11$.

7. $n(U) = 200$, $n(C_1) = 120$, $n(C_2) = 50$ and $n(C_1 \cap C_2) = 30$.

(i) $n(C_1) = n(C_1 - C_2) + n(C_1 \cap C_2)$

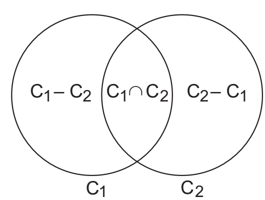
$\Rightarrow \quad n(C_1 - C_2) = n(C_1) - n(C_1 \cap C_2) = (120 - 30) = 90$.

(ii) $n(C_2) = n(C_2 - C_1) + n(C_1 \cap C_2)$

$\Rightarrow \quad n(C_2 - C_1) = n(C_2) - n(C_1 \cap C_2) = (50 - 30) = 20$.

(iii) $n(C_1 \cup C_2) = n(C_1) + n(C_2) - n(C_1 \cap C_2)$

$= (120 + 50 - 30) = 140$.



8. $n(A) = 50$, $n(B) = 42$ and $n(A \cap B) = 24$.

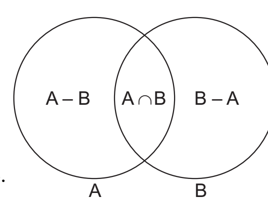
(i) $n(A) = n(A - B) + n(A \cap B)$

$\Rightarrow \quad n(A - B) = n(A) - n(A \cap B) = (50 - 24) = 26$.

(ii) $n(B) = n(B - A) + n(A \cap B)$

$\Rightarrow \quad n(B - A) = n(B) - n(A \cap B) = (42 - 24) = 18$.

(iii) $n(A \cup B) = n(A) + n(B) - n(A \cap B) = (50 + 42 - 24) = 68$.



9. Failed in English only = $(56 - 18) = 38$.

Failed in science only = $(48 - 18) = 30$.

Failed in both English and science = 18.

Failed in one or both of the subjects = $(38 + 30 + 18) = 86$.

Passed in both the subjects = $(100 - 86) = 14$.

10. Let A = set of people who like cricket.

B = set of people who like tennis.

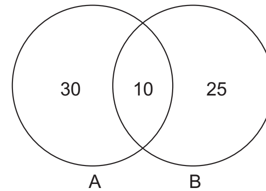
$n(A) = 40$, $n(A \cap B) = 10$.

$\therefore\quad n(A - B) = (40 - 10) = 30$

$\therefore\quad n(B - A) = 65 - (30 + 10) = 25.$

Number of people who like tennis only = 25.

Number of people who like tennis = $(25 + 10) = 35$.



11. Let A, B, C denote the sets of students who bagged medals in hockey, basketball and cricket respectively.

Then, $n(A) = 42$, $n(B) = 18$, $n(C) = 23$, $n(A \cup B \cup C) = 65$ and $n(A \cap B \cap C) = 4$. Then,

$$n(A \cup B \cup C) = \{n(A) + n(B) + n(C) + n(A \cap B \cap C)\} - \{n(A \cap B) + n(B \cap C) + n(A \cap C)\}$$

$\Rightarrow\quad 65 = (42 + 18 + 23 + 4) - x \Leftrightarrow x = (87 - 65) = 22.$

12. $a + b + c + d = 25$, $b + c + e + f = 26$, $c + d + f + g = 26$,

$c + d = 9$, $b + c = 11$, $c + f = 8$ and $c = 3$.

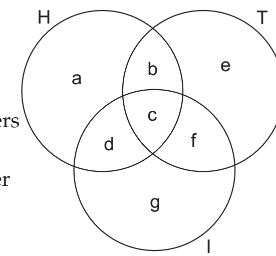
$\therefore\quad f = 5$, $b = 8$, $d = 6$, $c = 3$, $g = 12$, $e = 10$ and $a = 8$.

(i) Number of people who read at least one of the papers

$= (a + b + c + d + e + f + g) = 52.$

(ii) Number of people who read exactly one newspaper

$= (a + e + g) = (8 + 10 + 12) = 30.$



13. We have

$a = 18$, $a + d = 23$, $c + d = 8$, $c + f = 8$,

$a + b + c + d = 26$, $c + d + f + g = 48$,

$a + b + c + d + e + f + g = 100 - 24 = 76.$

$\therefore\quad a = 18$, $d = 5$, $c = 3$, $f = 5$ and $b = 0$.

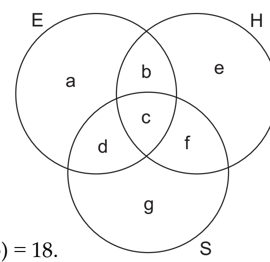
$\therefore\quad g = 48 - (3 + 5 + 5) = 35$

and $e = 76 - (18 + 0 + 3 + 5 + 5 + 35) = 76 - 66 = 10.$

(i) Number of students studying Hindi $= (b + c + e + f)$

$= (0 + 3 + 10 + 5) = 18.$

(ii) Number of students studying English and Hindi both $= (b + c) = (0 + 3) = 3.$



14. $n(A) = (40\% \text{ of } 10000) = \left(\frac{40}{100} \times 10000\right) = 4000,$

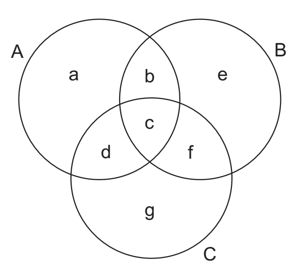
$n(B) = (20\% \text{ of } 10000) = 2000$, $n(C) = (10\% \text{ of } 10000) = 1000$,

$n(A \cap B) = (5\% \text{ of } 10000) = 500$, $n(B \cap C) = (3\% \text{ of } 10000) = 300$,

$n(A \cap C) = (4\% \text{ of } 10000) = 400$, $n(A \cap B \cap C) = (2\% \text{ of } 10000) = 200$,

and $n(U) = 10000$.

$$\begin{cases} a + b + c + d = 4000, \\ b + c + e + f = 2000, \\ c + d + f + g = 1000, \\ b + c = 500, \\ c + f = 300, \\ c + d = 400, \\ c = 200. \end{cases}$$



On solving these equations, we get

$$c = 200, d = 200, f = 100, b = 300, g = 500, e = 1400, a = 3300.$$

(i) $a = 3300$, (ii) $e = 1400$

(iii) $10000 - (3300 + 300 + 200 + 200 + 1400 + 100 + 500) = (10000 - 6000) = 4000$.

EXERCISE 1H

Very-Short-Answer Questions

1. If a set A has n elements then find the number of elements in its power set $P(A)$.

2. If $A = \phi$ then write $P(A)$.

3. If $n(A) = 3$ and $n(B) = 5$, find:

(i) the maximum number of elements in $A \cup B$,

(ii) the minimum number of elements in $A \cup B$.

4. If A and B are two sets such that $n(A) = 8$, $n(B) = 11$ and $n(A \cup B) = 14$ then find $n(A \cap B)$.

5. If A and B are two sets such that $n(A) = 23$, $n(B) = 37$ and $n(A - B) = 8$ then find $n(A \cup B)$.

Hint $n(A) = n(A - B) + n(A \cap B) \Rightarrow n(A \cap B) = (23 - 8) = 15$.

6. If A and B are two sets such that $n(A) = 54$, $n(B) = 39$ and $n(B - A) = 13$ then find $n(A \cup B)$.

Hint $n(B) = n(B - A) + n(A \cap B) \Rightarrow n(A \cap B) = (39 - 13) = 26$.

7. If $A \subset B$, prove that $B' \subset A'$.

8. If $A \subset B$, show that $(B' - A') = \phi$.

Hint $A \subset B \Rightarrow B' \subset A' \Rightarrow B' - A' = \phi$.

9. Let $A = \{x : x = 6n, n \in N\}$ and $B = \{x : x = 9n, n \in N\}$, find $A \cap B$.

10. If $A = \{5, 6, 7\}$, find $P(A)$.

11. If $A = \{2, \{2\}\}$, find $P(A)$.

12. Prove that $A \cap (A \cup B)' = \phi$.

13. Find the symmetric difference $A \Delta B$, when $A = \{1, 2, 3\}$ and $B = \{3, 4, 5\}$.

14. Prove that $A - B = A \cap B'$.

15. If $A = \{x : x \in R, x < 5\}$ and $B = \{x : x \in R, x > 4\}$, find $A \cap B$.

Hint $A = (-\infty, 5)$ and $B = (4, \infty)$. So, $A \cap B = (4, 5)$.

ANSWERS (EXERCISE 1H)

1. 2^n **2.** $\{\phi\}$ **3.** (i) 8 (ii) 5 **4.** 5 **5.** 45 **6.** 67

9. $A \cap B = \{x : x = 18n, n \in N\}$

10. $P(A) = \{\phi, \{5\}, \{6\}, \{7\}, \{5, 6\}, \{5, 7\}, \{6, 7\}, \{5, 6, 7\}\}$

11. $P(A) = \{\phi, \{2\}, \{\{2\}\}, \{2, \{2\}\}\}$ **13.** $\{1, 2, 4, 5\}$ **15.** $(4, 5)$

SUMMARY OF KEY FACTS

1. A well-defined collection of objects is called a set.

2. The objects in a set are called its *elements*, or *members*, or *points*.

3. Usually, we denote sets by capital letters A, B, C, X, Y, Z, etc.

4. A set is usually described in *tabular form* or *set-builder form*.

5. In tabulation method, we make a list of all the objects of the set and put them within braces { }. In set-builder form, we write $\{x : x \text{ satisfies properties } P\}$ which means, the set of all those x such that each x satisfies properties P.

6. A set having no element at all is called a *null set*, or a *void set* and it is denoted by ϕ.

7. A set having a single element is called a *singleton set*. For example, {3}.

8. A set having finite number of elements is called a *finite set*, otherwise it is called an *infinite set*.
The number of elements in a finite set A is denoted by $n(A)$.

9. Two sets A and B having exactly the same elements are known as equal sets and we write, $A = B$.

10. A set A is called a *subset* of a set B, if every element of A is in B and we write, $A \subseteq B$.

11. If A is a subset of set B and $A \neq B$ then A is called a *proper subset* of set B and we write, $A \subset B$.

12. The *total number of subsets of a set A* containing n elements is 2^n.

13. The collection of all subsets of a set A is called the *power set of A*, to be denoted by $P(A)$.

14. Let a and b be real numbers such that $a < b$ then

(i) *closed interval* $[a, b] = \{x : x \in R, a \leq x \leq b\}$,

(ii) *open interval* $(a, b) = \{x : x \in R, a < x < b\}$,

(iii) *right half open interval* $[a, b) = \{x : x \in R, a \leq x < b\}$,

(iv) *left half open interval* $= (a, b] = \{x : x \in R, a < x \leq b\}$.

15. The *union* of two sets A and B, denoted by $A \cup B$, is the set of all those elements which are either in A or in B or in both A and B.

$\therefore \quad A \cup B = \{x : x \in A \text{ or } x \in B\}$.

16. The *intersection* of two sets A and B, denoted by $A \cap B$, is the set of all those elements which are common to both A and B.

$\therefore \quad A \cap B = \{x : x \in A \text{ and } x \in B\}$.

17. The difference between two sets A and B, denoted by $(A - B)$, is defined as $(A - B) = \{x : x \in A \text{ and } x \notin B\}$.
Similarly, $(B - A) = \{x : x \in B \text{ and } x \notin A\}$.

18. The *symmetric difference between* the sets A and B, denoted by $A \Delta B$, is defined as $A \Delta B = (A - B) \cup (B - A)$.

19. Let A be a subset of the universal set U. Then the *complement* of A, denoted by A', or A^c, or $U - A$, is defined as $A' = \{x : x \in U \text{ and } x \notin A\}$.

20. Various laws of operations on sets

(i) $A \cup A = A$ and $A \cap A = A$ [Idempotent laws]

(ii) $A \cup \phi = A$ and $A \cap U = A$ [Identity law]

(iii) $A \cup B = B \cup A$ and $A \cap B = B \cap A$ [Commutative laws]

(iv) $(A \cup B) \cup C = A \cup (B \cup C)$ and $(A \cap B) \cap C = A \cap (B \cap C)$ [Associative laws]

(v) I. $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ [Distributive law]

II. $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ [Distributive law]

(vi) I. $(A \cup B)' = (A' \cap B')$ [De Morgan's laws]

II. $(A \cap B)' = (A' \cup B')$ [De Morgan's laws]

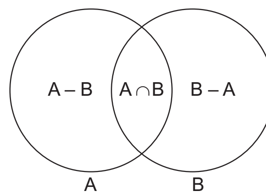
21. For any two sets A and B, we have

(i) $n(A \cup B) = n(A) + n(B) - n(A \cap B)$

(ii) $n(A - B) + n(A \cap B) = n(A)$

(iii) $n(B - A) + n(A \cap B) = n(B)$

(iv) $n(A \Delta B) = n(A - B) + n(B - A)$



22. For any three sets A, B and C, we have

$$n(A \cup B \cup C) = n(A) + n(B) + n(C) - \{n(A \cap B) + n(B \cap C) + n(A \cap C)\} + n(A \cap B \cap C).$$

❑

2
Relations

ORDERED PAIR *Two numbers a and b listed in a specific order and enclosed in parentheses form an ordered pair* (a, b).

In the ordered pair (a, b), we call a as *first member* (or *first component*) and b as *second member* (or *second component*).

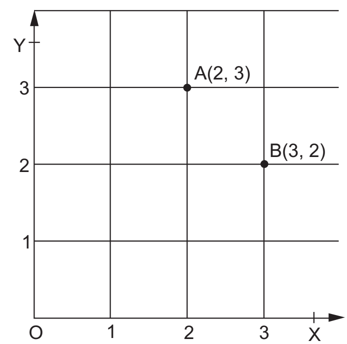
By interchanging the positions of the components, the ordered pair is changed.

Thus, $(a, b) \neq (b, a)$.

EXAMPLE In coordinate geometry, the position of a point in a plane is determined by an ordered pair.

In the given figure, the ordered pairs (2, 3) and (3, 2) represent two different points A and B respectively.

Thus, $(2, 3) \neq (3, 2)$.



EQUALITY OF TWO ORDERED PAIRS

We have $(a, b) = (c, d) \Leftrightarrow a = c$ and $b = d$.

EXAMPLE 1 *Find a and b, when* $(a-1, b+5) = (2, 3)$.

SOLUTION Using the definition of equality of two ordered pairs, we have

$$(a-1, b+5) = (2, 3) \Rightarrow a-1 = 2 \text{ and } b+5 = 3$$
$$\Rightarrow a = 3 \text{ and } b = -2.$$

Hence, $a = 3$ and $b = -2$.

EXAMPLE 2 *Find a and b, when* $(2a+b, 11) = (1, a-3b)$.

SOLUTION Using the definition of equality of two ordered pairs, we have

$$(2a+b, 11) = (1, a-3b)$$
$$\Rightarrow \begin{cases} 2a+b = 1 & \text{... (i)} \\ a-3b = 11. & \text{... (ii)} \end{cases}$$

On solving (i) and (ii), we get $a = 2$ and $b = -3$.

Hence, $a = 2$ and $b = -3$.

EXAMPLE 3 *If* $\left(\frac{x}{3}+1,\, y-\frac{2}{3}\right)=\left(\frac{5}{3},\, \frac{1}{3}\right)$*, find the values of x and y.*

SOLUTION Since the given two ordered pairs are equal, we have

$$\frac{x}{3}+1=\frac{5}{3} \Rightarrow \frac{x}{3}=\left(\frac{5}{3}-1\right)=\frac{2}{3} \Rightarrow \frac{x}{3}=\frac{2}{3} \Rightarrow x=2$$

and $$y-\frac{2}{3}=\frac{1}{3} \Rightarrow y=\left(\frac{1}{3}+\frac{2}{3}\right)=\frac{3}{3}=1 \Rightarrow y=1.$$

Hence, $x=2$ and $y=1$.

EXAMPLE 4 *Express* $\{(x, y): x^2+y^2=25$, *where* $x, y \in W\}$ *as a set of ordered pairs.*

SOLUTION It is easy to verify that each of the following ordered pairs of whole numbers satisfies the given relation $x^2+y^2=25$:

$$(5, 0), (0, 5), (3, 4) \text{ and } (4, 3).$$

Hence, the set of required ordered pairs is

$$\{(5, 0), (0, 5), (3, 4), (4, 3)\}.$$

CARTESIAN PRODUCT OF TWO SETS

Let A and B be two nonempty sets. Then, the Cartesian product of A and B is the set denoted by $(A \times B)$*, consisting of all ordered pairs (a, b) such that* $a \in A$ *and* $b \in B$.

$\therefore \quad A \times B = \{(a, b): a \in A \text{ and } b \in B\}$.

If $A=\phi$ or $B=\phi$, we define $A \times B=\phi$.

REMARKS (i) *If* $n(A)=p$ *and* $n(B)=q$ *then* $n(A \times B)=pq$ *and* $n(B \times A)=pq$.

(ii) *If at least one of A and B is infinite then* $(A \times B)$ *is infinite and* $(B \times A)$ *is infinite.*

SOLVED EXAMPLES

EXAMPLE 1 *If* $A=\{1, 3, 5\}$ *and* $B=\{2, 3\}$ *then find:*

(i) $A \times B$ *(ii)* $B \times A$ *(iii)* $(A \times B) \cap (B \times A)$

SOLUTION We have

(i) $A \times B = \{1, 3, 5\} \times \{2, 3\}$
$= \{(1, 2), (1, 3), (3, 2), (3, 3), (5, 2), (5, 3)\}$.

(ii) $B \times A = \{2, 3\} \times \{1, 3, 5\}$
$= \{(2, 1), (2, 3), (2, 5), (3, 1), (3, 3), (3, 5)\}$.

(iii) $(A \times B) \cap (B \times A) = \{(3, 3)\}$.

EXAMPLE 2 *If* $A=\{1, 2, 3\}$, $B=\{3, 4\}$ *and* $C=\{4, 5, 6\}$ *then find:*

(i) $A \times (B \cap C)$ *(ii)* $(A \times B) \cap (A \times C)$

(iii) $A \times (B \cup C)$ *(iv)* $(A \times B) \cup (A \times C)$

SOLUTION We have

(i) $B \cap C = \{3, 4\} \cap \{4, 5, 6\} = \{4\}$.

$\therefore \quad A \times (B \cap C) = \{1, 2, 3\} \times \{4\} = \{(1, 4), (2, 4), (3, 4)\}$.

(ii) $(A \times B) = \{1, 2, 3\} \times \{3, 4\}$
$= \{(1, 3), (1, 4), (2, 3), (2, 4), (3, 3), (3, 4)\}$.

$(A \times C) = \{1, 2, 3\} \times \{4, 5, 6\}$
$= \{(1, 4), (1, 5), (1, 6), (2, 4), (2, 5), (2, 6), (3, 4), (3, 5), (3, 6)\}$

$\therefore \quad (A \times B) \cap (A \times C) = \{(1, 4), (2, 4), (3, 4)\}$.

(iii) $B \cup C = \{3, 4\} \cup \{4, 5, 6\} = \{3, 4, 5, 6\}$.

$\therefore \quad A \times (B \cup C) = \{1, 2, 3\} \times \{3, 4, 5, 6\}$
$= \{(1, 3), (1, 4), (1, 5), (1, 6), (2, 3), (2, 4), (2, 5), (2, 6), (3, 3), (3, 4), (3, 5), (3, 6)\}$.

(iv) Also, from (ii), we get
$(A \times B) \cup (A \times C) = \{(1, 3), (1, 4), (1, 5), (1, 6), (2, 3), (2, 4), (2, 5), (2, 6), (3, 3), (3, 4), (3, 5), (3, 6)\}$.

EXAMPLE 3 *Let* $A = \{x \in N : x^2 - 5x + 6 = 0\}$, $B = \{x \in W : 0 \leq x < 2\}$ *and* $C = \{x \in N : x < 3\}$. *Verify that*

(i) $A \times (B \cup C) = (A \times B) \cup (A \times C)$

(ii) $A \times (B \cap C) = (A \times B) \cap (A \times C)$

SOLUTION We have

$A = \{x \in N : x^2 - 5x + 6 = 0\} = \{x \in N : (x-2)(x-3) = 0\} = \{2, 3\}$;
$B = \{x \in W : 0 \leq x < 2\} = \{0, 1\}$ and $C = \{x \in N : x < 3\} = \{1, 2\}$.

$\therefore \quad A = \{2, 3\}$, $B = \{0, 1\}$ and $C = \{1, 2\}$.

(i) $(B \cup C) = \{0, 1\} \cup (1, 2\} = \{0, 1, 2\}$.

$\therefore \quad A \times (B \cup C) = \{2, 3\} \times \{0, 1, 2\}$
$= \{(2, 0), (2, 1), (2, 2), (3, 0), (3, 1), (3, 2)\}$.

Now, $(A \times B) = \{2, 3\} \times \{0, 1\}$
$= \{(2, 0), (2, 1), (3, 0), (3, 1)\}$

and $(A \times C) = \{2, 3\} \times \{1, 2\}$
$= \{(2, 1), (2, 2), (3, 1), (3, 2)\}$.

$\therefore \quad (A \times B) \cup (A \times C) = \{(2, 0), (2, 1), (2, 2), (3, 0), (3, 1), (3, 2)\}$.

Hence, $A \times (B \cup C) = (A \times B) \cup (A \times C)$.

(ii) $(B \cap C) = \{0, 1\} \cap \{1, 2\} = \{1\}$.

$\therefore \quad A \times (B \cap C) = \{2, 3\} \times \{1\} = \{(2, 1), (3, 1)\}$.

And,
$(A \times B) \cap (A \times C) = \{(2, 0), (2, 1), (3, 0), (3, 1)\} \cap \{(2, 1), (2, 2), (3, 1), (3, 2)\}$
$= \{(2, 1), (3, 1)\}$.

Hence, $A \times (B \cap C) = (A \times B) \cap (A \times C)$.

EXAMPLE 4 *If* $(A \times B) = \{(3, 2), (3, 4), (5, 2), (5, 4)\}$, *find A and B.*

SOLUTION Clearly, we have

$$A = \text{set of all first coordinates of the elements of } (A \times B) = \{3, 5\}.$$

$$B = \text{set of all second coordinates of the elements of } (A \times B) = \{2, 4\}.$$

Thus, $A = \{3, 5\}$ and $B = \{2, 4\}$.

EXAMPLE 5 *A and B are two sets given in such a way that* $(A \times B)$ *contains* 6 *elements. If three elments of* $(A \times B)$ *be* (1, 3), (2, 5) *and* (3, 3), *find its remaining elements.*

SOLUTION Since (1, 3), (2, 5) and (3, 3) are in $(A \times B)$, it follows that 1, 2, 3 are elements of A and 3, 5 are elements of B.

$$\therefore \quad (A \times B) = \{1, 2, 3\} \times \{3, 5\}$$
$$= \{(1, 3), (1, 5), (2, 3), (2, 5), (3, 3), (3, 5)\}.$$

Hence, the remaining elements of $(A \times B)$ are (1, 5), (2, 3) and (3, 5).

EXAMPLE 6 *If* $A = \{a, b\}$, *find* $(A \times A)$.

SOLUTION We have

$$(A \times A) = \{a, b\} \times \{a, b\}$$
$$= \{(a, a), (a, b), (b, a), (b, b)\}.$$

EXAMPLE 7 *Let R be the set of all real numbers. What does* $(R \times R)$ *represent?*

SOLUTION We have $(R \times R) = \{(x, y) : x, y \in R\}$.

Thus, $(R \times R)$ represents the set of all coordinates of points in two-dimensional space.

EXAMPLE 8 *If* $(A \times A)$ *has* 9 *elements two of which are* $(-1, 0)$ *and* (0, 1), *find the set A and the remaining elements of* $(A \times A)$.

SOLUTION Clearly, –1, 0 and 1 are elements of A.

$$\therefore \quad A = \{-1, 0, 1\}$$
$$A \times A = \{-1, 0, 1\} \times \{-1, 0, 1\}$$
$$= \{(-1, -1), (-1, 0), (-1, 1), (0, -1), (0, 0), (0, 1), (1, -1), (1, 0), (1, 1)\}.$$

Hence, the remaining elements of $(A \times A)$ are

(–1, –1), (–1, 1), (0, –1), (0, 0), (1, –1), (1, 0), (1, 1).

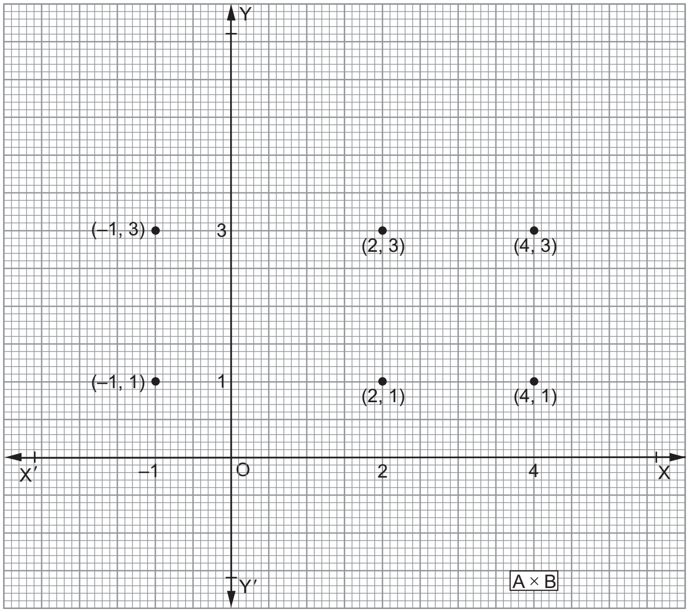
GRAPHICAL REPRESENTATION OF $A \times B$ AND $B \times A$

Let $X'OX$ and YOY' be the x-axis and y-axis respectively, drawn on a graph paper.

Let $A = \{-1, 2, 4\}$ and $B = \{1, 3\}$. Then,

$$A \times B = \{(-1, 1), (-1, 3), (2, 1), (2, 3), (4, 1), (4, 3)\}.$$

These points may be plotted on the graph paper, as shown below:



This is the graphical representation of $A \times B$.

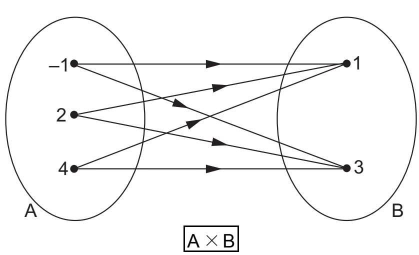
For $B \times A$, we take $B = \{1, 3\}$ along the x-axis and $A = \{-1, 2, 4\}$ along the y-axis and plot the points $(1, -1), (1, 2), (1, 4), (3, -1), (3, 2), (3, 4)$ on the graph paper.

Arrow diagram of $A \times B$

Let $A = \{-1, 2, 4\}$ and $B = \{1, 3\}$. Then,

$$A \times B = \{(-1, 1), (-1, 3), (2, 1), (2, 3), (4, 1), (4, 3)\}.$$

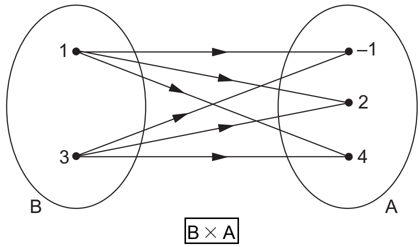
Then, $A \times B$ may be represented by arrow diagram, as shown below:



Arrow diagram of $B \times A$

Again, $B \times A = \{(1, -1), (1, 2), (1, 4), (3, -1), (3, 2), (3, 4)\}$.

We may exhibit $B \times A$ by arrow diagram, as shown below:



ORDERED TRIPLET Three numbers a, b and c listed in a specific order and enclosed in parentheses form an ordered triplet (a, b, c).

Thus, $(1, 2, 3) \neq (2, 1, 3) \neq (3, 2, 1)$, etc.

For any nonempty set A, we define:

$$(\boldsymbol{A \times A \times A}) = \{(\boldsymbol{a, b, c}) : \boldsymbol{a, b, c} \in \boldsymbol{A}\}.$$

EXAMPLE 9 *If* $A = \{1, 2\}$, *find* $(A \times A \times A)$.

SOLUTION $A \times A = \{1, 2\} \times \{1, 2\} = \{(1, 1), (1, 2), (2, 1), (2, 2)\}$.

$$\begin{aligned} \therefore \quad A \times A \times A &= (A \times A) \times A \\ &= \{(1, 1), (1, 2), (2, 1), (2, 2)\} \times \{1, 2\} \\ &= \{(1, 1, 1), (1, 1, 2), (1, 2, 1), (1, 2, 2), (2, 1, 1), \\ &\qquad (2, 1, 2), (2, 2, 1), (2, 2, 2)\}. \end{aligned}$$

EXAMPLE 10 *Let R be the set of all real numbers. What does* $(R \times R \times R)$ *represent?*

SOLUTION $(R \times R \times R) = \{(a, b, c) : a, b, c \in R\}$.

Thus, $(R \times R \times R)$ represents the set of all points in three-dimensional space.

REMARK For any nonempty sets A, B and C, we always have

$$(A \times B) \times C = A \times (B \times C) = A \times B \times C.$$

EXAMPLE 11 *Let* $A = \{1, 2\}$, $B = \{3, 4\}$ *and* $C = \{4, 5\}$.

Verify that $(A \times B) \times C = A \times (B \times C)$ *and hence find* $A \times B \times C$.

SOLUTION We have

$$A \times B = \{1, 2\} \times \{3, 4\} = \{(1, 3), (1, 4), (2, 3), (2, 4)\}$$

$$\begin{aligned} \Rightarrow \quad (A \times B) \times C &= \{(1, 3), (1, 4), (2, 3), (2, 4)\} \times \{4, 5\} \\ &= \{(1, 3, 4), (1, 3, 5), (1, 4, 4), (1, 4, 5), (2, 3, 4), \\ &\qquad (2, 3, 5), (2, 4, 4), (2, 4, 5)\}. \end{aligned}$$

Again, $B \times C = \{3, 4\} \times \{4, 5\} = \{(3, 4), (3, 5), (4, 4), (4, 5)\}$

$$\begin{aligned} \Rightarrow \quad A \times (B \times C) &= \{1, 2\} \times \{(3, 4), (3, 5), (4, 4), (4, 5)\} \\ &= \{(1, 3, 4), (1, 3, 5), (1, 4, 4), (1, 4, 5), (2, 3, 4), \\ &\qquad (2, 3, 5), (2, 4, 4), (2, 4, 5)\}. \end{aligned}$$

$$\therefore \quad (A \times B) \times C = A \times (B \times C) = A \times B \times C.$$

Hence, $(A \times B \times C) = \{(1, 3, 4), (1, 3, 5), (1, 4, 4), (1, 4, 5), (2, 3, 4),$
$(2, 3, 5), (2, 4, 4), (2, 4, 5)\}$.

EXERCISE 2A

1. Find the values of a and b, when:

(i) $(a+3, b-2) = (5, 1)$ (ii) $(a+b, 2b-3) = (4, -5)$

(iii) $\left(\frac{a}{3}+1, b-\frac{1}{3}\right) = \left(\frac{5}{3}, \frac{2}{3}\right)$ (iv) $(a-2, 2b+1) = (b-1, a+2)$

2. If $A = \{0, 1\}$ and $B = \{1, 2, 3\}$, show that $A \times B \neq B \times A$.

3. If $P = \{a, b\}$ and $Q = \{x, y, z\}$, show that $P \times Q \neq Q \times P$.

4. If $A = \{2, 3, 5\}$ and $B = \{5, 7\}$, find:

(i) $A \times B$ (ii) $B \times A$ (iii) $A \times A$ (iv) $B \times B$

5. If $A = \{x \in N : x \leq 3\}$ and $B = \{x \in W, x < 2\}$, find $(A \times B)$ and $(B \times A)$. Is $(A \times B) = (B \times A)$?

6. If $A = \{1, 3, 5\}$, $B = \{3, 4\}$ and $C = \{2, 3\}$, verify that:

(i) $A \times (B \cup C) = (A \times B) \cup (A \times C)$

(ii) $A \times (B \cap C) = (A \times B) \cap (A \times C)$

7. Let $A = \{x \in W : x < 2\}$, $B = \{x \in N : 1 < x \leq 4\}$ and $C = \{3, 5\}$. Verify that:

(i) $A \times (B \cup C) = (A \times B) \cup (A \times C)$

(ii) $A \times (B \cap C) = (A \times B) \cap (A \times C)$

Hint $A = \{0, 1\}$, $B = \{2, 3, 4\}$ and $C = \{3, 5\}$.

8. If $A \times B = \{(-2, 3), (-2, 4), (0, 3), (0, 4), (3, 3), (3, 4)\}$, find A and B.

9. Let $A = \{2, 3\}$ and $B = \{4, 5\}$. Find $(A \times B)$. How many subsets will $(A \times B)$ have?

10. Let $A \times B = \{(a, b) : b = 3a - 2\}$. If $(x, -5)$ and $(2, y)$ belong to $A \times B$, find the values of x and y.

11. Let A and B be two sets such that $n(A) = 3$ and $n(B) = 2$.
If $a \neq b \neq c$ and $(a, 0)$, $(b, 1)$, $(c, 0)$ are in $A \times B$, find A and B.

12. Let $A = \{-2, 2\}$ and $B = \{0, 3, 5\}$. Find:

(i) $A \times B$ (ii) $B \times A$ (iii) $A \times A$ (iv) $B \times B$

Represent each of the above (a) graphically and (b) by arrow diagram.

13. If $A = \{5, 7\}$, find (i) $A \times A$ and (ii) $A \times A \times A$.

14. Let $A = \{-3, -1\}$, $B = \{1, 3\}$ and $C = \{3, 5\}$. Find:

(i) $A \times B$ (ii) $(A \times B) \times C$ (iii) $B \times C$ (iv) $A \times (B \times C)$

ANSWERS (EXERCISE 2A)

1. (i) $a = 2, b = 3$ (ii) $a = 5, b = -1$ (iii) $a = 2, b = 1$ (iv) $a = 3, b = 2$

4. (i) $A \times B = \{(2, 5), (2, 7), (3, 5), (3, 7), (5, 5), (5, 7)\}$

(ii) $B \times A = \{(5, 2), (5, 3), (5, 5), (7, 2), (7, 3), (7, 5)\}$

(iii) $A \times A = \{(2, 2), (2, 3), (2, 5), (3, 2), (3, 3), (3, 5), (5, 2), (5, 3), (5, 5)\}$

(iv) $B \times B = \{(5, 5), (5, 7), (7, 5), (7, 7)\}$

5. $A \times B = \{(1, 0), (1, 1), (2, 0), (2, 1), (3, 0), (3, 1)\}$;
$B \times A = \{(0, 1), (0, 2), (0, 3), (1, 1), (1, 2), (1, 3)\}$;
No

8. $A = \{-2, 0, 3\}$ and $B = \{3, 4\}$

9. $A \times B = \{(2, 4), (2, 5), (3, 4), (3, 5)\}$; Number of subsets of $(A \times B) = 2^4 = 16$

10. $x = -1, y = 4$ **11.** $A = \{a, b, c\}$ and $B = \{0, 1\}$

12. (i) $A \times B = \{(-2, 0), (-2, 3), (-2, 5), (2, 0), (2, 3), (2, 5)\}$
(ii) $B \times A = \{(0, -2), (0, 2), (3, -2), (3, 2), (5, -2), (5, 2)\}$
(iii) $A \times A = \{(-2, -2), (-2, 2), (2, -2), (2, 2)\}$
(iv) $B \times B = \{(0, 0), (0, 3), (0, 5), (3, 0), (3, 3), (3, 5), (5, 0), (5, 3), (5, 5)\}$

13. (i) $A \times A = \{(5, 5), (5, 7), (7, 5), (7, 7)\}$
(ii) $A \times A \times A = \{(5, 5, 5), (5, 7, 5), (7, 5, 5), (7, 7, 5), (5, 5, 7), (5, 7, 7), (7, 5, 7), (7, 7, 7)\}$

14. (i) $A \times B = \{(-3, 1), (-3, 3), (-1, 1), (-1, 3)\}$
(ii) $(A \times B) \times C = \{(-3, 1, 3), (-3, 3, 3), (-1, 1, 3), (-1, 3, 3), (-3, 1, 5), (-3, 3, 5), (-1, 1, 5), (-1, 3, 5)\}$
(iii) $B \times C = \{(1, 3), (1, 5), (3, 3), (3, 5)\}$
(iv) $A \times (B \times C) = \{(-3, 1, 3), (-3, 1, 5), (-3, 3, 3), (-3, 3, 5), (-1, 1, 3), (-1, 1, 5), (-1, 3, 3), (-1, 3, 5)\}$

SOME USEFUL RESULTS

THEOREM 1 *If $A \subseteq B$ then prove that $A \times C \subseteq B \times C$ for any set C.*

PROOF Let $A \subseteq B$ and let (a, c) be an arbitrary element of $A \times C$. Then,

$$\begin{aligned} (a, c) \in A \times C &\Rightarrow a \in A \text{ and } c \in C \\ &\Rightarrow a \in B \text{ and } c \in C \qquad [\because A \subseteq B] \\ &\Rightarrow (a, c) \in B \times C. \end{aligned}$$

Thus, every element of $A \times C$ is contained in $B \times C$.

$\therefore \quad A \times C \subseteq B \times C$.

Hence, $A \subseteq B \Rightarrow A \times C \subseteq B \times C$ for any set C.

THEOREM 2 *If $A \subseteq B$ and $C \subseteq D$ then prove that $A \times C \subseteq B \times D$.*

PROOF Let $A \subseteq B$ and $C \subseteq D$. Then, we have to show that $A \times C \subseteq B \times D$.

Let (a, c) be an arbitrary element of $A \times C$. Then,

$$\begin{aligned} (a, c) \in A \times C &\Rightarrow a \in A \text{ and } c \in C \\ &\Rightarrow a \in B \text{ and } c \in D \qquad [\because A \subseteq B \text{ and } C \subseteq D] \\ &\Rightarrow (a, c) \in B \times D. \end{aligned}$$

Thus, every element of $A \times C$ is contained in $B \times D$.

Hence, $A \times C \subseteq B \times D$.

THEOREM 3 *If A and B are any two nonempty sets, prove that* $A \times B = B \times A \Leftrightarrow A = B$.

PROOF Let us first assume that $A = B$. Then,

$$A \times B = B \times B \text{ and } B \times A = B \times B \quad [\because A = B].$$

$$\therefore \quad A \times B = B \times A.$$

Thus, $A = B \Rightarrow A \times B = B \times A$.

Conversely, let $A \times B = B \times A$ and we have to show that $A = B$.

Let a be an arbitrary element of A. Then,

$$\begin{aligned} a \in A &\Rightarrow (a, b) \in A \times B \text{ for some } b \in B \\ &\Rightarrow (a, b) \in B \times A \qquad [\because A \times B = B \times A] \\ &\Rightarrow a \in B \text{ and } b \in A \\ &\Rightarrow a \in B \quad \text{(surely)}. \end{aligned}$$

$$\therefore \quad A \subseteq B.$$

Again, let b be an arbitrary element of B. Then,

$$\begin{aligned} b \in B &\Rightarrow (b, a) \in B \times A \text{ for some } a \in A \\ &\Rightarrow (b, a) \in A \times B \qquad [\because B \times A = A \times B] \\ &\Rightarrow b \in A \text{ and } a \in B \\ &\Rightarrow b \in A \quad \text{(surely)}. \end{aligned}$$

$$\therefore \quad B \subseteq A.$$

Thus, $A \subseteq B$ and $B \subseteq A \Rightarrow A = B$.

$$\therefore \quad A \times B = B \times A \Rightarrow A = B.$$

Hence, $A \times B = B \times A \Leftrightarrow A = B$.

THEOREM 4 *If* $A \subseteq B$, *prove that* $A \times A \subseteq (A \times B) \cap (B \times A)$.

PROOF Let $A \subseteq B$ and let (a, a) be an arbitrary element of $A \times A$.

Then,
$$\begin{aligned} (a, a) \in A \times A &\Rightarrow a \in A \text{ and } a \in A \\ &\Rightarrow a \in A \text{ and } a \in B \qquad [\because A \subseteq B] \\ &\Rightarrow (a \in A, a \in B) \text{ and } (a \in B, a \in A) \\ &\Rightarrow (a, b) \in A \times B \text{ and } (a, b) \in B \times A \\ &\Rightarrow (a, b) \in (A \times B) \cap (B \times A). \end{aligned}$$

$$\therefore \quad A \times A \subseteq (A \times B) \cap (B \times A).$$

Hence, $A \subseteq B \Rightarrow A \times A \subseteq (A \times B) \cap (B \times A)$.

THEOREM 5 *For any sets A, B and C, prove that*

$$A \times (B \cup C) = (A \times B) \cup (A \times C).$$

PROOF Let (a, b) be an arbitrary element of $A \times (B \cup C)$. Then,

$$\begin{aligned} (a, b) \in A \times (B \cup C) &\Rightarrow a \in A \text{ and } b \in B \cup C \\ &\Rightarrow a \in A \text{ and } (b \in B \text{ or } b \in C) \\ &\Rightarrow (a \in A \text{ and } b \in B) \text{ or } (a \in A \text{ and } b \in C) \\ &\qquad [\because p \text{ and } (q \text{ or } r) \equiv (p \text{ and } q) \text{ or } (p \text{ and } r)] \\ &\Rightarrow (a, b) \in A \times B \text{ or } (a, b) \in A \times C \\ &\Rightarrow (a, b) \in (A \times B) \cup (A \times C). \end{aligned}$$

$\therefore \quad A \times (B \cup C) \subseteq (A \times B) \cup (A \times C)$. ... (i)

Again, let (x, y) be an arbitrary element of $(A \times B) \cup (A \times C)$. Then,

$$\begin{aligned}(x, y) \in (A \times B) \cup (A \times C) &\Rightarrow (x, y) \in A \times B \text{ or } (x, y) \in A \times C \\ &\Rightarrow (x \in A \text{ and } y \in B) \text{ or } (x \in A \text{ and } y \in C) \\ &\Rightarrow x \in A \text{ and } (y \in B \text{ or } y \in C) \\ &\qquad\qquad [\text{by distributive law}] \\ &\Rightarrow x \in A \text{ and } y \in (B \cup C) \\ &\Rightarrow (x, y) \in A \times (B \cup C).\end{aligned}$$

$\therefore \quad (A \times B) \cup (A \times C) \subseteq A \times (B \cup C)$. ... (ii)

Hence, from (i) and (ii), we get

$$A \times (B \cup C) = (A \times B) \cup (A \times C).$$

THEOREM 6 *For any sets A, B and C, prove that*

$$A \times (B \cap C) = (A \times B) \cap (A \times C).$$

PROOF Let (a, b) be an arbitrary element of $A \times (B \cap C)$. Then,

$$\begin{aligned}(a, b) \in A \times (B \cap C) &\Rightarrow a \in A \text{ and } b \in (B \cap C) \\ &\Rightarrow a \in A \text{ and } (b \in B \text{ and } b \in C) \\ &\Rightarrow (a \in A \text{ and } b \in B) \text{ and } (a \in A \text{ and } b \in C) \\ &\Rightarrow (a, b) \in A \times B \text{ and } (a, b) \in A \times C \\ &\Rightarrow (a, b) \in (A \times B) \cap (A \times C).\end{aligned}$$

$\therefore \quad A \times (B \cap C) \subseteq (A \times B) \cap (A \times C)$. ... (i)

Again, let (x, y) be an arbitrary element of $(A \times B) \cap (A \times C)$. Then,

$$\begin{aligned}(x, y) \in (A \times B) \cap (A \times C) &\Rightarrow (x, y) \in A \times B \text{ and } (x, y) \in A \times C \\ &\Rightarrow (x \in A \text{ and } y \in B) \text{ and } (x \in A \text{ and } y \in C) \\ &\Rightarrow x \in A \text{ and } (y \in B \text{ and } y \in C) \\ &\Rightarrow x \in A \text{ and } y \in (B \cap C) \\ &\Rightarrow (x, y) \in A \times (B \cap C).\end{aligned}$$

$\therefore \quad (A \times B) \cap (A \times C) \subseteq A \times (B \cap C)$. ... (ii)

Hence, from (i) and (ii), we get

$$A \times (B \cap C) = (A \times B) \cap (A \times C).$$

THEOREM 7 *For any sets A, B and C, prove that*

$$A \times (B - C) = (A \times B) - (A \times C).$$

PROOF Let (a, b) be an arbitrary element of $A \times (B - C)$. Then,

$$\begin{aligned}(a, b) \in A \times (B - C) &\Rightarrow a \in A \text{ and } b \in (B - C) \\ &\Rightarrow a \in A \text{ and } (b \in B \text{ and } b \notin C) \\ &\Rightarrow (a \in A \text{ and } b \in B) \text{ and } (a \in A \text{ and } b \notin C) \\ &\Rightarrow (a, b) \in A \times B \text{ and } (a, b) \notin A \times C \\ &\Rightarrow (a, b) \in (A \times B) - (A \times C).\end{aligned}$$

$\therefore \quad A \times (B - C) \subseteq (A \times B) - (A \times C)$. ... (i)

Again, let (x, y) be an arbitrary element of $(A \times B) - (A \times C)$. Then,

$$\begin{aligned}(x, y) \in (A \times B) - (A \times C) &\Rightarrow (x, y) \in A \times B \text{ and } (x, y) \notin A \times C\\ &\Rightarrow (x \in A \text{ and } y \in B) \text{ and } (x \in A \text{ and } y \notin C)\\ &\Rightarrow x \in A \text{ and } (y \in B \text{ and } y \notin C)\\ &\Rightarrow x \in A \text{ and } y \in (B - C)\\ &\Rightarrow (x, y) \in A \times (B - C).\end{aligned}$$

$\therefore \quad (A \times B) - (A \times C) \subseteq A \times (B - C)$. … (ii)

From (i) and (ii), we get

$$A \times (B - C) = (A \times B) - (A \times C).$$

THEOREM 8 *For any sets A, B, C and D, prove that*

$$(A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D).$$

PROOF Let (a, b) be an arbitrary element of $(A \times B) \cap (C \times D)$. Then,

$$\begin{aligned}(a, b) \in (A \times B) \cap (C \times D) &\Rightarrow (a, b) \in (A \times B) \text{ and } (a, b) \in (C \times D)\\ &\Rightarrow (a \in A \text{ and } b \in B) \text{ and } (a \in C \text{ and } b \in D)\\ &\Rightarrow (a \in A \text{ and } a \in C) \text{ and } (b \in B \text{ and } b \in D)\\ &\Rightarrow a \in (A \cap C) \text{ and } b \in (B \cap D)\\ &\Rightarrow (a, b) \in (A \cap C) \times (B \cap D). \qquad \ldots \text{(i)}\end{aligned}$$

Again, let (x, y) be an arbitrary element of $(A \cap C) \times (B \cap D)$. Then,

$$\begin{aligned}(x, y) \in (A \cap C) \times (B \cap D) &\Rightarrow x \in A \cap C \text{ and } y \in B \cap D\\ &\Rightarrow (x \in A \text{ and } x \in C) \text{ and } (y \in B \text{ and } y \in D)\\ &\Rightarrow (x \in A \text{ and } y \in B) \text{ and } (x \in C \text{ and } y \in D)\\ &\Rightarrow (x, y) \in A \times B \text{ and } (x, y) \in C \times D\\ &\Rightarrow (x, y) \in (A \times B) \cap (C \times D).\end{aligned}$$

$\therefore \quad (A \cap C) \times (B \cap D) \subseteq (A \times B) \cap (C \times D)$. … (ii)

From (i) and (ii), we get

$$(A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D).$$

THEOREM 9 *For any sets A and B, prove that*

$$(A \times B) \cap (B \times A) = (A \cap B) \times (B \cap A).$$

SOLUTION Let (a, b) be an arbitrary element of $(A \times B) \cap (B \times A)$. Then,

$$\begin{aligned}(a, b) \in (A \times B) \cap (B \times A) &\Rightarrow (a, b) \in A \times B \text{ and } (a, b) \in B \times A\\ &\Rightarrow (a \in A \text{ and } b \in B) \text{ and } (a \in B \text{ and } b \in A)\\ &\Rightarrow (a \in A \text{ and } a \in B) \text{ and } (b \in B \text{ and } b \in A)\\ &\Rightarrow a \in (A \cap B) \text{ and } b \in (B \cap A)\\ &\Rightarrow (a, b) \in (A \cap B) \times (B \cap A).\end{aligned}$$

$\therefore \quad (A \times B) \cap (B \times A) \subseteq (A \cap B) \times (B \cap A)$. … (i)

Again, let (x, y) be an arbitrary element of $(A \cap B) \times (B \cap A)$. Then,

$$\begin{aligned}(x, y) \in (A \cap B) \times (B \cap A) &\Rightarrow x \in A \cap B \text{ and } y \in B \cap A\\ &\Rightarrow (x \in A \text{ and } x \in B) \text{ and } (y \in B \text{ and } y \in A)\end{aligned}$$

$\Rightarrow (x \in A \text{ and } y \in B) \text{ and } (x \in B \text{ and } y \in A)$

$\Rightarrow (x, y) \in A \times B \text{ and } (x, y) \in B \times A$

$\Rightarrow (x, y) \in (A \times B) \cap (B \times A)$.

$\therefore \quad (A \cap B) \times (B \cap A) \subseteq (A \times B) \cap (B \times A)$. … (ii)

Thus, from (i) and (ii), we get

$$(A \times B) \cap (B \times A) = (A \cap B) \times (B \cap A).$$

AN IMPORTANT RESULT *If A and B are two nonempty sets having n elements in common then* $(A \times B)$ *and* $(B \times A)$ *have* n^2 *elements in common.*

EXAMPLE *Let A and B be two nonempty sets such that* $n(A) = 5$, $n(B) = 6$ *and* $n(A \cap B) = 3$.

Find (i) $n(A \times B)$, *(ii)* $n(B \times A)$ *and (iii)* $n\{(A \times B) \cap (B \times A)\}$.

SOLUTION (i) $n(A \times B) = n(A) \times n(B) = (5 \times 6) = 30$.

(ii) $n(B \times A) = n(B) \times n(A) = (6 \times 5) = 30$.

(iii) Given: $n(A \cap B) = 3$.

$\therefore \quad A$ and B have 3 elements in common.

So, $(A \times B)$ and $(B \times A)$ have $3^2 = 9$ elements in common.

Hence, $n\{(A \times B) \cap (B \times A)\} = 9$.

THEOREM 10 *If A, B and C be three nonempty sets given in such a way that* $A \times B = A \times C$ *then prove that* $B = C$.

PROOF Let $A \times B = A \times C$ and we have to prove that $B = C$.

Let $b \in B$. Then,

$b \in B \Rightarrow (a, b) \in A \times B$ for every $a \in A$

$\Rightarrow (a, b) \in A \times C$ for every $a \in A$ $\quad [\because A \times B = A \times C]$

$\Rightarrow b \in C$.

$\therefore \quad B \subseteq C$. … (i)

Again, let $c \in C$. Then,

$c \in C \Rightarrow (a, c) \in A \times C$ for every $a \in A$

$\Rightarrow (a, c) \in A \times B$ for every $a \in A$ $\quad [\because A \times C = A \times B]$

$\Rightarrow c \in B$.

$\therefore \quad C \subseteq B$. … (ii)

From (i) and (ii), we get $B = C$.

Hence, $A \times B = A \times C \Rightarrow B = C$.

EXERCISE 2B

1. For any sets A, B and C, prove that:

(i) $A \times (B \cup C) = (A \times B) \cup (A \times C)$

(ii) $A \times (B \cap C) = (A \times B) \cap (A \times C)$

(iii) $A \times (B - C) = (A \times B) - (A \times C)$

2. For any sets A and B, prove that
$(A \times B) \cap (B \times A) = (A \cap B) \times (B \cap A)$.

3. If A and B are nonempty sets, prove that
$A \times B = B \times A \Leftrightarrow A = B$.

4. (i) If $A \subseteq B$, prove that $A \times C \subseteq B \times C$ for any set C.
(ii) If $A \subseteq B$ and $C \subseteq D$ then prove that $A \times C \subseteq B \times D$.

5. If $A \times B \subseteq C \times D$ and $A \times B \neq \phi$, prove that $A \subseteq C$ and $B \subseteq D$.

6. If A and B be two sets such that $n(A) = 3$, $n(B) = 4$ and $n(A \cap B) = 2$ then find:
(i) $n(A \times B)$ (ii) $n(B \times A)$ (iii) $n\{(A \times B) \cap (B \times A\}$

7. For any two sets A and B, show that $A \times B$ and $B \times A$ have an element in common if and only if A and B have an element in common.

8. Let $A = \{1, 2\}$ and $B = \{2, 3\}$. Then, write down all possible subsets of $A \times B$.

9. Let $A = \{a, b, c, d\}$, $B = \{c, d, e\}$ and $C = \{d, e, f, g\}$. Then verify each of the following identities:
(i) $A \times (B \cap C) = (A \times B) \cap (A \times C)$ (ii) $A \times (B - C) = (A \times B) - (A \times C)$
(iii) $(A \times B) \cap (B \times A) = (A \cap B) \times (A \cap B)$

ANSWERS (EXERCISE 2B)

6. (i) $n(A \times B) = 12$ (ii) $n(B \times A) = 12$ (iii) $n\{(A \times B) \cap (B \times A)\} = 4$

8. ϕ, $\{(1, 2)\}$, $\{(1, 3)\}$, $\{(2, 2)\}$, $\{(2, 3)\}$, $\{(1, 2), (1, 3)\}$, $\{(1, 2), (2, 2)\}$, $\{(1, 2), (2, 3)\}$, $\{(1, 3), (2, 2)\}$, $\{(1, 3), (2, 3)\}$, $\{(2, 2), (2, 3)\}$, $\{(1, 2), (1, 3), (2, 2)\}$, $\{(1, 2), (1, 3), (2, 3)\}$, $\{(1, 3), (2, 2), (2, 3)\}$, $\{(1, 2), (2, 2), (2, 3)\}$, $\{(1, 2), (1, 3), (2, 2), (2, 3)\}$

HINTS TO SOME SELECTED QUESTIONS

6. (i) We know that $n(A \times B) = n(A) \times n(B)$ (ii) $n(B \times A) = n(B) \times n(A)$.
(iii) Since A and B have 2 elements in common, so $(A \times B)$ and $(B \times A)$ will have $2^2 = 4$ elements in common.

7. We know that $(A \times B) \cap (B \times A) = (A \cap B) \times (B \cap A) = (A \cap B) \times (A \cap B)$.
Now, $n(A \cap B) = 1 \Leftrightarrow n\{(A \cap B) \times (A \cap B)\} = 1^2 = 1$
$\Leftrightarrow n\{(A \times B) \cap (B \times A)\} = 1 [\because (A \cap B) \times (A \cap B) = (A \times B) \cap (B \times A)]$.

8. $A \times B = \{(1, 2), (1, 3), (2, 2), (2, 3)\}$. So, it will have $2^4 = 16$ subsets.

RELATIONS

RELATION *Let A and B be two nonempty sets. Then, a relation R from A to B is a subset of* $(A \times B)$.

Thus, R is a relation from A to B $\Leftrightarrow R \subseteq (A \times B)$.

If $(a, b) \in R$ then we say that '*a is related to b*' and we write, $a\,R\,b$.

If $(a, b) \notin R$ then '*a is not related to b*' and we write, $a \not R\, b$.

DOMAIN, RANGE AND CO-DOMAIN OF A RELATION

Let R be a relation from A to B. Then, $R \subseteq (A \times B)$.

(i) *The set of all first coordinates of elements of R is called the domain of R, written as dom* (R).

(ii) *The set of all second coordinates of elements of R is called the range of R, denoted by range* (R).

(iii) *The set B is called the co-domain of R.*

Dom $(R) = \{a : (a, b) \in R\}$ and *Range* $(R) = \{b : (a, b) \in R\}$.

Total Number of Relations from *A* to *B*

Let $n(A) = p$ and $n(B) = q$. Then, $n(A \times B) = pq$.

We know that every subset of $A \times B$ is a relation from A to B.

Total number of subsets of $A \times B$ is 2^{pq}.

$\therefore$ *total number of relations from A to B* $= 2^{pq}$.

REPRESENTATION OF A RELATION

Let A and B be two given sets. Then, a relation $R \subseteq A \times B$ can be represented in any of the forms, given below.

(i) ROSTER FORM

In this form, R is given as a set of ordered pairs.

EXAMPLE 1 *Let* $A = \{-2, -1, 0, 1, 2\}$ *and* $B = \{0, 1, 4, 9\}$.

Let $R = \{(-2, 4), (-1, 1), (0, 0), (1, 1), (2, 4)\}$.

(i) *Show that R is a relation from A to B.*

(ii) *Find dom (R), range (R) and co-domain of R.*

SOLUTION (i) Since $R \subset A \times B$, so R is a relation from A to B.

Note that $-2R4$, $-1R1$, $0R0$, $1R1$ and $2R4$.

(ii) Dom (R) = set of first coordinates of elements of R

$= \{-2, -1, 0, 1, 2\}$.

Range (R) = set of second coordinates of elements of R

$= \{0, 1, 4\}$.

Co-domain of $R = \{0, 1, 4, 9\} = B$.

(ii) SET-BUILDER FORM

Under this method, for every $(a, b) \in R$, *a general relation is being given between a and b.*

Using this relation, all the elements of R can be obtained.

EXAMPLE 2 *Let* $A = \{1, 2, 3, 5\}$ *and* $B = \{4, 6, 9\}$.

Define a relation from A to B, given by

$$R = \{(a, b) : a \in A, b \in B \text{ and } (a - b) \text{ is odd}\}.$$

(i) *Write R in roster form.*

(ii) *Find dom (R) and range (R).*

SOLUTION (i) Clearly, we have

$$R = \{(1, 4), (1, 6), (2, 9), (3, 4), (3, 6), (5, 4), (5, 6)\}.$$

(ii) Dom (R) = set of first coordinates of elements of R

$= \{1, 2, 3, 5\}$.

Range (R) = set of second coordinates of elements of R

$= \{4, 6, 9\}$.

EXAMPLE 3 *Let* $R = \{(x, x^3) : x$ *is a prime number less than* $10\}$.

(i) Write R in roster form.

(ii) Find dom (R) and range (R).

SOLUTION Prime numbers less than 10 are 2, 3, 5, 7.

(i) $R = \{(2, 2^3), (3, 3^3), (5, 5^3), (7, 7^3)\}$

$= \{(2, 8), (3, 27), (5, 125), (7, 343)\}$.

(ii) Dom $(R) = \{2, 3, 5, 7\}$.

Range $(R) = \{8, 27, 125, 343\}$.

EXAMPLE 4 *Let* $R = \{(x, y) : x$ *and* y *are integers and* $xy = 4\}$.

(i) Write R in roster form.

(ii) Find dom (R) and range (R).

SOLUTION Clearly, we have

(i) $R = \{(-4, -1), (-2, -2), (-1, -4), (1, 4), (2, 2), (4, 1)\}$.

(ii) Dom $(R) = \{-4, -2, -1, 1, 2, 4\}$.

Range $(R) = \{-4, -2, -1, 1, 2, 4\}$.

(iii) ARROW DIAGRAM

Let R be a relation from A to B. First, we draw two bounded figures to represent A and B respectively. We mark the elements of A and B in these figures. For each $(a, b) \in R$, we draw an arrow from a to b. This gives us the required arrow diagram.

EXAMPLE 5 *Let* $A = \{1, 2, 3, 4, 5\}$. *Define a relation R from A to A by* $R = \{(x, y) : y = 2x - 3\}$.

(i) Depict R using an arrow diagram.

(ii) Find dom (R) and range (R).

SOLUTION $x = 1 \Rightarrow y = (2 - 3) = -1 \notin A$

$x = 2 \Rightarrow y = (4 - 3) = 1,$

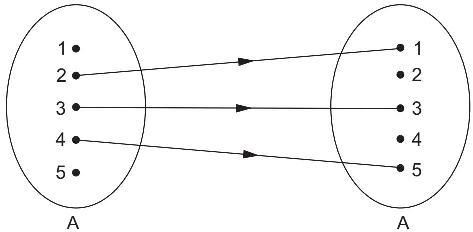
$x = 3 \Rightarrow y = (6 - 3) = 3,$

$x = 4 \Rightarrow y = (8 - 3) = 5,$

$x = 5 \Rightarrow y = (10 - 3) = 7 \notin A.$

$\therefore \quad R = \{(2, 1), (3, 3), (4, 5)\}$.

(i) We may depict it by arrow diagram, as shown below:



(ii) We have, dom $(R) = \{2, 3, 4\}$ and range $(R) = \{1, 3, 5\}$.

EXAMPLE 6 *Let $A = \{1, 2, 3, 4, 5\}$ and $B = \{1, 4, 5\}$.*
Let R be a relation 'is less than' from A to B.

(i) List the elements of R.

(ii) Find the domain, co-domain and range of R.

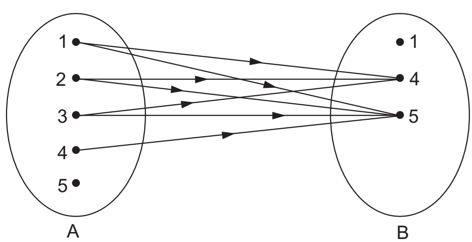
(iii) Depict the above relation by an arrow diagram.

SOLUTION Here, $A = \{1, 2, 3, 4, 5\}$ and $B = \{1, 4, 5\}$.

(i) $R = \{(1, 4), (1, 5), (2, 4), (2, 5), (3, 4), (3, 5), (4, 5)\}$.

(ii) Dom $(R) = \{1, 2, 3, 4\}$, range $(R) = \{4, 5\}$
and co-domain $(R) = \{1, 4, 5\}$.

(iii) We may represent the above relation by an arrow diagram, shown below.



REMARK Let A and B be two nonempty sets. Then, every subset of $A \times B$ is a relation from A to B.

Since $\phi \subset A \times B$, so ϕ is also a relation from A to B, called an *empty* or *void* relation.

EXAMPLE 7 *Let $A = \{x, y, z\}$ and $B = \{1, 2\}$. Find the number of all possible relations that can be defined from A to B.*

SOLUTION Here $n(A) = 3$ and $n(B) = 2$. So, $n(A \times B) = (3 \times 2) = 6$.

Since a set containing n elements has 2^n subsets.

$\therefore$ $(A \times B)$ will have $2^6 = 64$ subsets.

But, every subset of $A \times B$ is a relation from A to B.

Hence, there are in all 64 relations from A to B.

EXERCISE 2C

1. Let A and B be two nonempty sets.

(i) What do you mean by a relation from A to B?

(ii) What do you mean by the domain and range of a relation?

2. Find the domain and range of each of the relations given below:

(i) $R = \{(-1, 1), (1, 1), (-2, 4), (2, 4), (3, 9)\}$

(ii) $R = \left\{\left(x, \frac{1}{x}\right) : x \text{ is an integer, } 0 < x < 5\right\}$

(iii) $R = \{(x, y) : x + 2y = 8 \text{ and } x, y \in N\}$

(iv) $R = \{(x, y) : y = |x - 1| , x \in Z \text{ and } |x| \leq 3\}$

3. Let $A = \{1, 3, 5, 7\}$ and $B = \{2, 4, 6, 8\}$.

Let $R = \{(x, y) : x \in A, y \in B \text{ and } x > y\}$.

(i) Write R in roster form.

(ii) Find dom (R) and range (R).

(iii) Depict R by an arrow diagram.

4. Let $A = \{2, 4, 5\}$ and $B = \{1, 2, 3, 4, 6, 8\}$.

Let $R = \{(x, y)\ \ x \in A, y \in B \text{ and } x \text{ divides } y\}$.

(i) Write R in roster form.

(ii) Find dom (R) and range (R).

5. Let $A = \{2, 3, 4, 5\}$ and $B = \{3, 6, 7, 10\}$.

Let $R = \{(x, y) : x \in A, y \in B \text{ and } x \text{ is relatively prime to } y\}$.

(i) Write R in roster form.

(ii) Find dom (R) and range (R).

6. Let $A = \{1, 2, 3, 5\}$ and $B = \{4, 6, 9\}$.

Let $R = \{(x, y) : x, \in A, y \in B \text{ and } (x - y) \text{ is odd}\}$.

Write R in roster form.

7. Let $R = \{(x, y) : x + 3y = 12, x \in N \text{ and } y \in N\}$.

(i) Write R in roster form.

(ii) Find dom (R) and range (R).

8. Let $A = \{1, 2, 3, 4, 5, 6\}$.

Define a relation R from A to A by $R = \{(x, y) : y = x + 1\}$.

(i) Write R in roster form.

(ii) Find dom (R) and range (R).

(iii) What is its co-domain?

(iv) Depict R by using arrow diagram.

9. Let $R = \{(x, x + 5) : x \in \{0, 1, 2, 3, 4, 5\}\}$.

(i) Write R in roster form.

(ii) Find dom (R) and range (R).

10. Let $A = \{1, 2, 3, 4, 6\}$ and let $R = \{(a, b) : a, b \in A \text{ and } a \text{ divides } b\}$.

(i) Write R in roster form.

(ii) Find dom (R) and range (R).

11. Define a relation R from Z to Z, given by

$$R = \{(a, b) : a, b \in Z \text{ and } (a - b) \text{ is an integer}\}.$$

Find dom (R) and range (R).

Hint The difference of two integers is always an integer.

12. Let $R = \{(x, y) : x, y \in Z \text{ and } x^2 + y^2 \le 4\}$.

(i) Write R in roster form.

(ii) Find dom (R) and range (R).

13. Let $A = \{2, 3\}$ and $B = \{3, 5\}$.

(i) Find $(A \times B)$ and $n(A \times B)$.

(ii) How many relations can be defined from A to B?

14. Let $A = \{3, 5\}$ and $B = \{7, 9\}$. Let $R = \{(a, b) : a \in A, b \in B \text{ and } (a - b) \text{ is odd}\}$. Show that R is an empty relation from A to B.

Hint The difference of two odd numbers cannot be odd.

ANSWERS (EXERCISE 2C)

2. (i) dom $(R) = \{-2, -1, 1, 2, 3\}$ and range $(R) = \{1, 4, 9\}$

(ii) dom $(R) = \{1, 2, 3, 4\}$ and range $(R) = \left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}\right\}$

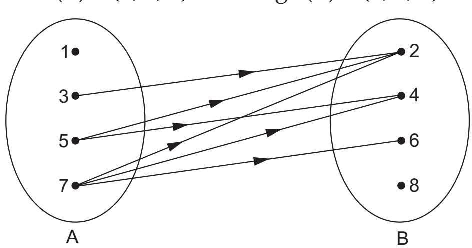
(iii) dom $(R) = \{2, 4, 6\}$ and range $(R) = \{3, 2, 1\}$

(iv) dom $(R) = \{-3, -2, -1, 0, 1, 2, 3\}$ and range $(R) = \{0, 1, 2, 3, 4\}$

3. (i) $R = \{(3, 2), (5, 2), (5, 4), (7, 2), (7, 4), (7, 6)\}$

(ii) dom $(R) = \{3, 5, 7\}$ and range $(R) = \{2, 4, 6\}$

(iii)



4. (i) $R = \{(2, 2), (2, 4), (2, 6), (2, 8), (4, 4), (4, 8)\}$

(ii) dom $(R) = \{2, 4\}$ and range $(R) = \{2, 4, 6, 8\}$

5. (i) $R = \{(2, 3), (2, 7), (3, 7), (3, 10), (4, 3), (4, 7), (5, 3), (5, 6), (5, 7)\}$

(ii) dom $(R) = \{2, 3, 4, 5\}$ and range $(R) = \{3, 6, 7, 10\}$

6. $R = \{(1, 4), (1, 6), (2, 9), (3, 4), (3, 6), (5, 4), (5, 6)\}$

7. (i) $R = \{(3, 3), (6, 2), (9, 1)\}$

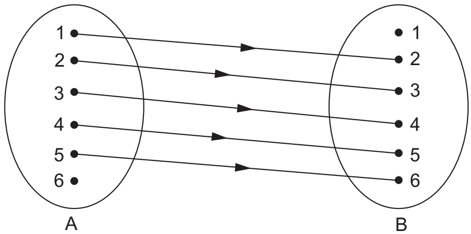
(ii) dom $(R) = \{3, 6, 9\}$ and range $(R) = \{3, 2, 1\}$

8. (i) $R = \{(1, 2), (2, 3), (3, 4), (4, 5), (5, 6)\}$

(ii) dom $(R) = \{1, 2, 3, 4, 5\}$, range $(R) = \{2, 3, 4, 5, 6\}$

(iii) co-domain of $R = \{1, 2, 3, 4, 5, 6\}$

(iv)



9. (i) $R = \{(0, 5), (1, 6), (2, 7), (3, 8), (4, 9), 5, 10)\}$

(ii) dom $(R) = \{0, 1, 2, 3, 4, 5\}$ and range $(R) = \{5, 6, 7, 8, 9, 10\}$

10. (i) $R = \{(1, 1), (1, 2), (1, 3), (1, 4), (1, 6), (2, 2), (2, 4), (2, 6), (3, 3), (3, 6), (4, 4), (6, 6)\}$

(ii) dom $(R) = \{1, 2, 3, 4, 6\}$ and range $(R) = \{1, 2, 3, 4, 6\}$

11. dom $(R) = Z$ and range $(R) = Z$

12. (i) $R = \{(-2, 0), (0, -2), (2, 0), (0, 2), (-1, 0), (0, -1), (1, 0), (0, 1), (1, 1), (1, -1), (-1, 1), (-1, -1)\}$

(ii) dom $(R) = \{-2, -1, 0, 1, 2\}$ and range $(R) = \{-2, -1, 0, 1, 2\}$

13. (i) $A \times B = \{(2, 3), (2, 5), (3, 3), (3, 5)\}$ and $n(A \times B) = 4$

(ii) number of relations from A to $B = 2^4 = 16$

BINARY RELATION ON A SET

Let A be a nonempty set. Then, every subset of $(A \times A)$ is called a binary relation or simply a relation on A.

REMARKS (i) Since $\phi \subset A \times A$ so ϕ is relation on A, called the *empty* or *void relation* on A.

(ii) Since $A \times A \subseteq A \times A$ so $(A \times A)$ is a relation on A, called the *universal relation* on A.

(iii) Let $I_A = \{(a, a) : a \in A\}$. Then, clearly, $I_A \subseteq (A \times A)$ and therefore, it is a relation on A, called the *identity relation* on A.

SOLVED EXAMPLES

EXAMPLE 1 *Let $A = \{1, 2, 3\}$ and $R = \{(1, 2), (2, 2), (3, 1), (3, 2)\}$. Show that R is a binary relation on A. Find its domain and range.*

SOLUTION $R = \{(1, 2), (2, 2), (3, 1), (3, 2)\}$.

Clearly, $R \subset A \times A$ and so R is a relation on A.

Dom (R) = set of first coordinates of elements of $R = \{1, 2, 3\}$.

Range (R) = set of second coordinates of elements of $R = \{1, 2\}$.

Hence, dom (R) = $\{1, 2, 3\}$ and range (R) = $\{1, 2\}$.

EXAMPLE 2 *Let N be the set of all natural numbers. Let $R = \{(a, b) : a, b \in N$ and $2a + b = 10\}$. Show that R is a binary relation on N. Find its domain, range and co-domain.*

SOLUTION Here $R = \{(a, b) : a, b \in N \text{ and } 2a + b = 10\}$.

Now, $2a + b = 10 \Rightarrow b = (10 - 2a)$.

$\therefore \quad (a = 1 \Rightarrow b = 8), (a = 2 \Rightarrow b = 6), (a = 3 \Rightarrow b = 4), (a = 4 \Rightarrow b = 2)$.

$\therefore \quad R = \{(1, 8), (2, 6), (3, 4), (4, 2)\}$.

Since $R \subset N \times N$, so R is a binary relation on N.

Dom (R) = set of 1st coordinates of elements of R

$= \{1, 2, 3, 4\}$.

Range (R) = set of 2nd coordinates of elements of R

$= \{8, 6, 4, 2\}$.

Co-domain of $R = N$.

INVERSE RELATION

Let R be a binary relation on a set A. Then, the inverse of R, denoted by R^{-1} is a binary relation on A, defined by $R^{-1} = \{(b, a) : (a, b) \in R\}$.

Clearly, $(a, b) \in R \Leftrightarrow (b, a) \in R^{-1}$.

Also, ***dom* (R) = *range* (R^{-1}) *and range* (R) = *dom* (R^{-1}).**

EXAMPLE 3 *Let A be the set of first ten natural numbers. Let R be a binary relation on A, defined by*

$$R = \{(a, b) : a, b \in A \text{ and } a + 2b = 10\}.$$

Express R and R^{-1} as sets of ordered pairs.

Show that (i) dom (R) = range (R^{-1}) (ii) range (R) = dom (R^{-1}).

SOLUTION $a + 2b = 10 \Rightarrow b = \dfrac{(10 - a)}{2}$.

Now, $(a = 2 \Rightarrow b = 4), (a = 4 \Rightarrow b = 3), (a = 6 \Rightarrow b = 2),$
$(a = 8 \Rightarrow b = 1)$.

$\therefore \quad R = \{(2, 4), (4, 3), (6, 2), (8, 1)\}$

$\Rightarrow \quad R^{-1} = \{(4, 2), (3, 4), (2, 6), (1, 8)\}$.

$\therefore \quad$ (i) dom (R) = $\{2, 4, 6, 8\}$ = range (R^{-1}).

(ii) range (R) = $\{4, 3, 2, 1\}$ = dom (R^{-1}).

VARIOUS TYPES OF RELATIONS

Let A be a nonempty set. Then, a relation R on A is said to be:

(i) *reflexive*, if $(a, a) \in R$ for all $a \in A$, i.e., $a\,R\,a$ for all $a \in A$;

(ii) *symmetric*, if $(a, b) \in R \Rightarrow (b, a) \in R$ for all $a, b \in A$,
i.e., $a\,R\,b \Rightarrow b\,R\,a$ for all $a, b \in A$;

(iii) *transitive,* if $(a, b) \in R$ and $(b, c) \in R \Rightarrow (a, c) \in R$ for all $a, b, c \in A$.
i.e., $a\,R\,b,\; b\,R\,c \Rightarrow a\,R\,c$ for all $a, b, c \in A$.

EQUIVALENCE RELATION

A relation which is reflexive, symmetric and transitive is called an equivalence relation.

EXAMPLE 4 *Let R be a relation on the set Q of all rationals defined by* $R = \{(a, b) : a, b \in Q$ *and* $a - b \in Z\}$. *Show that R is an equivalence relation.*

SOLUTION Given: $R = \{(a, b) : a, b \in Q$ and $a - b \in Z\}$.

(i) Let $a \in Q$. Then, $a - a = 0 \in Z$.
$\therefore \quad (a, a) \in R$ for all $a \in Q$.
So, R is reflexive.

(ii) $(a, b) \in R \Rightarrow (a - b) \in Z$, i.e., $(a - b)$ is an integer
$\Rightarrow -(a - b)$ is an integer
$\Rightarrow (b - a)$ is an integer
$\Rightarrow (b, a) \in R$.
Thus, $(a, b) \in R \Rightarrow (b, a) \in R$.
$\therefore \quad R$ is symmetric.

(iii) $(a, b) \in R$ and $(b, c) \in R$
$\Rightarrow \quad (a - b)$ is an integer and $(b - c)$ is an integer
$\Rightarrow \quad \{(a - b) + (b - c)\}$ is an integer
$\Rightarrow \quad (a - c)$ is an integer
$\Rightarrow \quad (a, c) \in R$.
Thus, $(a, b) \in R$ and $(b, c) \in R \Rightarrow (a, c) \in R$.
$\therefore \quad R$ is transitive.
Thus, R is reflexive, symmetric and transitive.
So, R is an equivalence relation.

EXAMPLE 5 *Let m be a given fixed positive integer.*
Let $R = \{(a, b) : a, b \in Z$ *and* $(a - b)$ *is divisible by* $m\}$.
Show that R is an equivalence relation on Z.

SOLUTION $R = \{(a, b) : a, b \in Z$ and $(a - b)$ is divisible by $m\}$.

(i) Let $a \in Z$. Then,
$a - a = 0$, which is divisible by m.
$\therefore \quad (a, a) \in R$ for all $a \in Z$.
So, R is reflexive.

(ii) Let $(a, b) \in R$. Then,
$(a, b) \in R \Rightarrow (a - b)$ is divisible by m
$\Rightarrow -(a - b)$ is divisible by m
$\Rightarrow (b - a)$ is divisible by m
$\Rightarrow (b, a) \in R$.

Thus, $(a, b) \in R \Rightarrow (b, a) \in R$.

So, R is symmetric.

(iii) Let $(a, b) \in R$ and $(b, c) \in R$. Then,

$(a, b) \in R$ and $(b, c) \in R$.

$\Rightarrow$ $(a-b)$ is divisible by m and $(b-c)$ is divisible by m

$\Rightarrow$ $\{(a-b)+(b-c)\}$ is divisible by m

$\Rightarrow$ $(a-c)$ is divisible by m

$\Rightarrow$ $(a, c) \in R$.

$\therefore$ $(a, b) \in R$ and $(b, c) \in R \Rightarrow (a, c) \in R$.

So, R is transitive.

Thus, R is reflexive, symmetric and transitive.

Hence, R is an equivalence relation on Z.

EXAMPLE 6 *Show that the relation 'is parallel to' on the set S of all straight lines in a plane is an equivalence relation.*

SOLUTION Let S be the set of all straight lines in a plane.

Then, the relation, 'is parallel to' on S is

(i) *reflexive*, since every line is parallel to itself,

i.e., $L \parallel L$ for all L in S;

(ii) *symmetric*, since $L \parallel M \Rightarrow M \parallel L$ for all $L, M \in S$;

(iii) *transitive*, since for all L, M, N in S, we have

$$L \parallel M \text{ and } M \parallel N \Rightarrow L \parallel N.$$

Thus, the given relation is reflexive, symmetric and transitive.

Hence, it is an equivalence relation on S.

EXAMPLE 7 *Show that the relation 'is congruent to' on the set of all triangles in a plane is an equivalence relation.*

SOLUTION Let S be the set of all triangles in a plane.

Then, the congruence relation on S is

(i) *reflexive*, since $\triangle \cong \triangle$ for every $\triangle \in S$;

(ii) *symmetric*, since $\triangle_1 \cong \triangle_2 \Rightarrow \triangle_2 \cong \triangle_1$ for all $\triangle_1, \triangle_2 \in S$

(iii) *transitive*, since $\triangle_1 \cong \triangle_2$ and $\triangle_2 \cong \triangle_3 \Rightarrow \triangle_1 \cong \triangle_3$ for all $\triangle_1, \triangle_2, \triangle_3 \in S$.

Hence, the given relation is an equivalence relation.

EXAMPLE 8 *Let $R = \{(a, b) : a, b \in N \text{ and } a = b^2\}$. Show that R satisfies none of reflexivity, symmetry and transitivity.*

SOLUTION (i) R is not reflexive, since $2 \neq 2^2$ and therefore $(2, 2) \notin R$.

(ii) Since $4 = 2^2$, so $(4, 2) \in R$.

But, $2 \neq 4^2$. So, $(2, 4) \notin R$.

Thus, $(4, 2) \in R$ but $(2, 4) \notin R$.

$\therefore$ R is not symmetric.

(iii) Since $16 = 4^2$, so $(16, 4) \in R$.
Also, $4 = 2^2$, so $(4, 2) \in R$.
But, $16 \neq 2^2 \Rightarrow (16, 2) \notin R$.
Thus, $(16, 4) \in R$ and $(4, 2) \in R$. But $(16, 2) \notin R$.
$\therefore$ R is not transitive.
Hence, R satisfies none of reflexivity, symmetry and transitivity.

EXAMPLE 9 *Let S be the set of all real numbers and let R be a binary relation on S defined by $(a, b) \in R \Leftrightarrow 1 + ab > 0$ for all $a, b \in S$. Show that R is reflexive as well as symmetric. Give an example to show that R is not transitive.*

SOLUTION (i) Let a be an arbitrary real number.
Then, $(1 + a^2) > 0 \Rightarrow (a, a) \in R$ for every $a \in S$.
$\therefore$ R is reflexive.

(ii) Let $(a, b) \in R$. Then,
$$(a, b) \in R \Rightarrow 1 + ab > 0$$
$$\Rightarrow 1 + ba > 0$$
$$\Rightarrow (b, a) \in R.$$
$\therefore$ $(a, b) \in R \Rightarrow (b, a) \in R$.
So, R is symmetric.

(iii) In order to show that R is not transitive, consider the real numbers $\left(\frac{-2}{3}\right)$, 1 and 2.
Now, $\left(\frac{-2}{3}, 1\right) \in R$, since $\left\{1 + \left(\frac{-2}{3}\right) \times 1\right\} = \left(1 - \frac{2}{3}\right) = \frac{1}{3} > 0$.
And, $(1, 2) \in R$, since $\{1 + (1 \times 2)\} = 3 > 0$.
But, $\left(\frac{-2}{3}, 2\right) \notin R$, since $\left\{1 + \left(\frac{-2}{3}\right) \times 2\right\} = \left(1 - \frac{4}{3}\right) = -\frac{1}{3} < 0$.
Thus, $\left(\frac{-2}{3}, 1\right) \in R$, $(1, 2) \in R$ and $\left(\frac{-2}{3}, 2\right) \notin R$.
This shows that R is not transitive.

EXAMPLE 10 *Let R be a relation on $N \times N$, defined by*
$$(a, b)\, R\, (c, d) \Leftrightarrow a + d = b + c \text{ for all } (a, b), (c, d) \in N \times N.$$
Show that R is an equivalence relation.

SOLUTION Here R is a relation on $N \times N$, defined by
$$(a, b)\, R\, (c, d) \Leftrightarrow a + d = b + c \text{ for all } (a, b), (c, d) \in N \times N.$$
We shall show that R satisfies the following properties.

(i) **Reflexivity:**
We know that $a + b = b + a$ for all $a, b \in N$.
$\therefore$ $(a, b)\, R\, (a, b)$ for all $(a, b) \in (N \times N)$.
So, R is reflexive.

(ii) **Symmetry:**

Let $(a, b)\ R\ (c, d)$. Then,

$$(a, b)\ R\ (c, d) \Rightarrow a + d = b + c$$
$$\Rightarrow c + b = d + a$$
$$\Rightarrow (c, d)\ R\ (a, b).$$

$\therefore \quad (a, b)\ R\ (c, d) \Rightarrow (c, d)\ R\ (a, b)$ for all $(a, b), (c, d) \in N \times N$.

This shows that R is symmetric.

(iii) **Transitivity:**

Let $(a, b)\ R\ (c, d)$ and $(c, d)\ R\ (e, f)$. Then,

$(a, b)\ R\ (c, d)$ and $(c, d)\ R\ (e, f)$

$\Rightarrow \quad a + d = b + c$ and $c + f = d + e$

$\Rightarrow \quad a + d + c + f = b + c + d + e$

$\Rightarrow \quad a + f = b + e$

$\Rightarrow \quad (a, b)\ R\ (e, f)$.

Thus, $(a, b)\ R\ (c, d)$ and $(c, d)\ R\ (e, f) \Rightarrow (a, b)\ R\ (e, f)$.

This shows that R is transitive.

$\therefore \quad R$ is reflexive, symmetric and transitive.

Hence, R is an equivalence relation on $N \times N$.

EXERCISE 2D

1. What do you mean by a binary relation on a set A?
Define the domain and range of a relation on A.

2. Let $A = \{2, 3, 5\}$ and $R = \{(2, 3), (2, 5), (3, 3), (3, 5)\}$.
Show that R is a binary relation on A. Find its domain and range.

3. Let $A = \{0, 1, 2, 3, 4, 5, 6, 7, 8\}$ and let $R = \{(a, b) : a, b \in A \text{ and } 2a + 3b = 12\}$.
Express R as a set of ordered pairs. Show that R is a binary relation on A.
Find its domain and range.

4. If R is a binary relation on a set A, define R^{-1} on A.
Let $R = \{(a, b) : a, b \in W \text{ and } 3a + 2b = 15\}$, where W is the set of whole numbers.
Express R and R^{-1} as sets of ordered pairs.
Show that (i) dom (R) = range (R^{-1}) (ii) range (R) = dom (R^{-1}).

5. What is an equivalence relation?
Show that the relation of 'similarity' on the set S of all triangles in a plane is an equivalence relation.

6. Let $R = \{(a, b) : a, b \in Z \text{ and } (a - b) \text{ is even}\}$.
Then, show that R is an equivalence relation on Z.

7. Let $A = \{1, 2, 3\}$ and $R = \{(a, b) : a, b \in A \text{ and } |a^2 - b^2| \le 5$.
Write R as set of ordered pairs.

Mention whether R is (i) reflexive (ii) symmetric (iii) transitive.
Give reason in each case.

8. Let $R = \{(a, b): a, b \in Z \text{ and } b = 2a - 4\}$. If $(a, -2) \in R$ and $(4, b^2) \in R$. Then, write the values of a and b.

9. Let R be a relation on Z, defined by $(x, y) \in R \Leftrightarrow x^2 + y^2 = 9$. Then, write R as set of ordered pairs. What is its domain?

10. Let A be the set of first five natural numbers and let R be a relation on A, defined by $(x, y) \in R \Leftrightarrow x \le y$.
Express R and R^{-1} as sets of ordered pairs.
Find: dom (R^{-1}) and range (R).

11. Let $R = \{(x, y): x, y \in Z \text{ and } x^2 + y^2 = 25\}$.
Express R and R^{-1} as sets of ordered pairs. Show that $R = R^{-1}$.

12. Find R^{-1}, when
(i) $R = \{(1, 2), (1, 3), (2, 3), (3, 2), (4, 5)\}$,
(ii) $R = \{(x, y): x, y \in N, x + 2y = 8\}$.

13. Let $A = \{a, b\}$. List all relations on A and find their number.

Hint $A \times A = \{(a, a), (a, b), (b, a), (b, b)\}$ and every subset of $A \times A$ is a relation on A.
So, their number $= 2^4 = 16$.

14. Let $R = \{(a, b): a, b \in N \text{ and } a < b\}$.
Show that R is a binary relation on N, which is neither reflexive nor symmetric. Show that R is transitive.

Hint Since $R \subset N \times N$, so it is a binary relation on N.

ANSWERS (EXERCISE 2D)

2. dom $(R) = \{2, 3\}$ and range $(R) = \{3, 5\}$

3. $R = \{(0, 4), (3, 2), (6, 0)\}$, dom $(R) = \{0, 3, 6\}$ and range $(R) = \{0, 2, 4\}$

4. $R = \{(1, 6), (3, 3), (5, 0)\}$, $R^{-1} = \{(6, 1), (3, 3), (0, 5)\}$

7. $R = \{(1, 1), (2, 2), (3, 3), (2, 1), (1, 2), (3, 2), (2, 3)\}$.
R is reflexive and symmetric but not transitive.

8. $a = 1, b = 4$

9. $R = \{(-3, 0), (0, -3), (3, 0), (0, 3)$, dom $(R) = \{-3, 0, 3\}$

10. $R = \{(1, 1), (1, 2), (1, 3), (1, 4), (1, 5), (2, 2), (2, 3), (2, 4), (2, 5), (3, 3), (3, 4), (3, 5), (4, 4), (4, 5), (5, 5)\}$
$R^{-1} = \{(1, 1), (2, 1), (3, 1), (4, 1), (5, 1), (2, 2), (3, 2), (4, 2), (5, 2), (3, 3), (4, 3), (5, 3), (4, 4), (5, 4), (5, 5)\}$
dom $(R^{-1}) = \{1, 2, 3, 4, 5\} =$ range (R)

11. $R = \{(-5, 0), (0, -5), (5, 0), (0, 5), (3, 4), (4, 3), (-3, 4), (4, -3), (-3, -4), (-4, -3)\}$
$R^{-1} = \{(0, -5), (-5, 0), (0, 5), (5, 0), (4, 3), (3, 4), (4, -3), (-3, 4), (-4, -3), (-3, -4)\}$

12. (i) $R^{-1} = \{(2, 1), (3, 1), (3, 2), (2, 3), (5, 4)\}$ (ii) $R^{-1} = \{(3, 2), (2, 4), (1, 6)\}$

13. 16

EXERCISE 2E

Very-Short-Answer Questions

1. Let A and B be two sets such that $n(A) = 5$, $n(B) = 3$ and $n(A \cap B) = 2$.
(i) $n(A \cup B)$ (ii) $n(A \times B)$ (iii) $n\{(A \times B) \cap (B \times A)\}$

Hint (i) $n(A \cup B) = n(A) + n(B) - n(A \cap B)$.
(ii) $n(A \times B) = n(A) \cdot n(B)$
(iii) If $n(A \cap B) = m$ then $n\{(A \times B) \cap (B \times A)\} = 2^m$.

2. Find a and b when $(a - 2b, 13) = (7, 2a - 3b)$.

3. If $A = \{1, 2\}$, find $A \times A \times A$.

4. If $A = \{2, 3, 4\}$ and $B = \{4, 5\}$, draw an arrow diagram to represent $(A \times B)$.

5. If $A = \{3, 4\}$, $B = \{4, 5\}$ and $C = \{5, 6\}$, find $A \times (B \times C)$.

6. If $A \subseteq B$, prove that $A \times C = B \times C$.

7. Prove that $A \times B = B \times A \Rightarrow A = B$.

8. If $A = \{5\}$ and $B = \{5, 6\}$, write down all possible subsets of $A \times B$.

9. Let $R = \{(x, x^2) : x$ is a prime number less than $10\}$.
(i) Write R in roster form.
(ii) Find dom (R) and range (R).

10. Let $A = \{1, 2, 3\}$ and $B = \{4\}$.
How many relations can be defined from A to B?

11. Let $A = \{3, 4, 5, 6\}$ and $R = \{(a, b) : a, b \in A$ and $a > b\}$.
Write R in roster form.
Find: dom (R) and range (R).
Write R^{-1} in roster form.

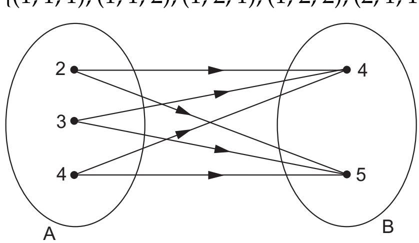
12. Let $R = \{(a, b) : a, b \in N, a > b\}$.
Show that R is a binary relation which is neither reflexive, nor symmetric.
Show that R is transitive.

ANSWERS (EXERCISE 2E)

1. (i) 6 (ii) 15 (iii) 4 **2.** $a = 5, b = -1$

3. $\{(1, 1, 1), (1, 1, 2), (1, 2, 1), (1, 2, 2), (2, 1, 1), (2, 1, 2), (2, 2, 1), (2, 2, 2)\}$

4.



$\boxed{A \times B}$

5. {(3, 4, 5), (3, 4, 6), (3, 5, 5), (3, 5, 6), (4, 4, 5), (4, 4, 6), (4, 5, 5), (4, 5, 6)}

8. ϕ, {(5, 5)}, {(5, 6)}, {(5, 5), (5, 6)}

9. (i) $R = \{(2, 4), (3, 9), (5, 25), (7, 49)\}$

(ii) dom $(R) = \{2, 3, 5, 7\}$ and range $(R) = \{4, 9, 25, 49\}$

10. 8

11. $R = \{(4, 3), (5, 3), (5, 4), (6, 3), (6, 4), (6, 5)$

(i) dom $(R) = \{4, 5, 6\}$ and range $(R) = \{3, 4, 5\}$

$R^{-1} = \{(3, 4), (3, 5), (4, 5), (3, 6), (4, 6), (5, 6)\}$

SUMMARY OF KEY FACTS

1. Two numbers a and b listed in a specific order and enclosed in parentheses form an ordered pair (a, b).
Here a is the first component and b is the second component.
In general, $(a, b) \neq (b, a)$.

2. $(a, b) = (c, d) \Leftrightarrow a = c$ and $b = d$.

3. Let A and B be two nonempty sets. Then, the Cartesian product of A and B is defined as

$$A \times B = \{(a, b) : a \in A \text{ and } b \in B\}.$$

If $A = \phi$ or $B = \phi$ then $A \times B = \phi$.

4. If $n(A) = p$ and $n(B) = q$ then $n(A \times B) = pq$.

5. Three numbers a, b, c listed in a specific order and enclosed in parentheses form an ordered triplet (a, b, c).

$$(a, b, c) \neq (b, a, c) \neq (c, a, b), \text{ etc.}$$

6. For any nonempty sets $A, B, C,$ we have

$$(A \times B) \times C = A \times (B \times C), \text{ each denoted by } A \times B \times C.$$

7. For any sets A, B and C, we have:

(i) $A \times (B \cup C) = (A \times B) \cup (A \times C)$

(ii) $A \times (B \cap C) = (A \times B) \cap (A \times C)$

(iii) $A \times (B - C) = (A \times B) - (A \times C)$

(iv) $(A \times B) \cap (B \times A) = (A \cap B) \times (B \cap A) = (A \cap B) \times (A \cap B)$

(v) $A \times B = A \times C \Rightarrow B = C$

(vi) $A \subset B \Rightarrow (A \times A) \subset (A \times B) \cap (B \times A)$

(vii) $A \subset B \Rightarrow (A \times C) \subset (B \times C)$

(viii) $A \subset B$ and $C \subset D \Rightarrow (A \times C) \subset (B \times D)$

(ix) $A \times B = B \times A \Leftrightarrow A = B$

8. Let A and B be two nonempty sets and let $R \subseteq A \times B$.
Then, R is called *a* relation from A to B.
If $(a, b) \in R$, we say that '*a is related to b*' and we write, $a\,R\,b$.

If $(a, b) \notin R$, we say that '*a is not related to b*' and we write, $a \not{R}\, b$.
Dom $(R) = \{a : (a, b) \in R\}$, range $(R) = \{b : (a, b) \in R\}$.

9. We define, $R^{-1} = \{(b, a) : (a, b) \in R\}$,

dom (R) = range (R^{-1}) and range (R) = dom (R^{-1}).

10. Let A be a nonempty set. Then, every subset of $A \times A$ is called a binary relation on A.

11. Let A be a nonempty set and R be a binary relation on A.

Then, R is said to be:

(i) ***reflexive,*** if $(a, a) \in R$ for all $a \in A$;

i.e., $a\, R\, a$ for all $a \in A$.

(ii) ***symmetric,*** if $(a, b) \in R \Rightarrow (b, a) \in R$ for all $a, b \in A$;

i.e., $a\, R\, b \Rightarrow b\, R\, a$ for all $a, b \in A$.

(iii) ***transitive,*** if $(a, b) \in R$ and $(b, c) \in R \Rightarrow (a, c) \in R$ for all $a, b, c \in A$;

i.e., $a\, R\, b$ and $b\, R\, c \Rightarrow a\, R\, c$ for all $a, b, c \in R$.

12. A relation R on A, which is reflexive, symmetric and transitive, is called an equivalence relation on A.

❑

3
Functions

FUNCTION *Let X and Y be two nonempty sets. Then, a relation f from X to Y is called a function, if every element in X has a unique image in Y, and we write, $f: X \to Y$.*

Thus, a relation f from X to Y is a function, if dom $(f) = X$ and no two distinct ordered pairs in f have the same first coordinate.

If $(x, y) \in f$, we write, $f(x) = y$.

Here, y is called the *image of x under f* and x is called the *pre-image* of y.

If $f: X \to Y$ then *dom* $(f) = X$ and *range* $(f) \subseteq Y$.

Also, Y is called the *co-domain* of f.

IMAGE OF A SUBSET

Let $f: X \to Y$ and let $A \subset X$.

Then, the image of A under f is defined as

$$f(A) = \{f(x) : x \in A\}.$$

Clearly, $f(A) \subseteq Y$.

EXAMPLE 1 *Let* $X = \{1, 2, 3, 4\}$ *and* $Y = \{1, 4, 9, 16, 25\}$.

Let $f = \{(x, y) : x \in X, y \in Y$ *and* $y = x^2\}$.

(i) Show that f is a function from X to Y. Find its domain and range.

(ii) Draw a pictorial representation of the above function.

(iii) If $A = \{2, 3, 4\}$, *find* $f(A)$.

SOLUTION (i) We have, $f = \{(x, y) : x \in X, y \in Y$ and $y = x^2\}$.

Giving different values to x from the set X and getting the corresponding value of $y = x^2$, we get

$$f = \{(1, 1), (2, 4), (3, 9), (4, 16)\}.$$

Clearly, every element in X has a unique image in Y.

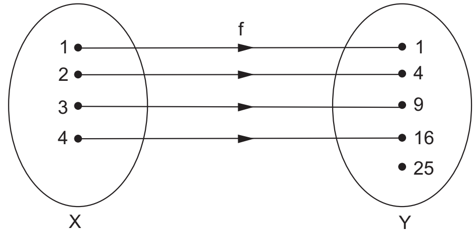
Hence, f is a function from X to Y.

Dom $(f) = \{1, 2, 3, 4\} = X$.

Range $(f) = \{1, 4, 9, 16\} \subset Y$.

Clearly, $25 \in Y$ does not have its pre-image in X.

(ii) A pictorial representation of the above mapping f is given below.



(iii) Now, let $A = \{2, 3, 4\}$. Then,

$$f(2) = 2^2 = 4, \ \ f(3) = 3^2 = 9 \ \text{ and } \ f(4) = 4^2 = 16.$$

$\therefore \quad f(A) = \{f(x) : x \in A\} = \{4, 9, 16\}$.

RELATION AS A PARTICULAR CASE OF A FUNCTION

A relation f from X to Y is a function if dom $(f) = X$ and no two distinct ordered pairs in f have the same first coordinate.

If $(x, y) \in f$, we write, $f(x) = y$.

EXAMPLE 2 *Let $X = \{2, 3, 4, 5\}$ and $Y = \{7, 9, 11, 13, 15, 17\}$.*
Define a relation f from X to Y by:

$$f = \{(x, y) : x \in X, y \in Y \text{ and } y = 2x + 3\}.$$

(i) Write f in roster form.
(ii) Find dom (f) and range (f).
(iii) Show that f is a function from X to Y.

SOLUTION Here $X = \{2, 3, 4, 5\}$ and $Y = 2x + 3$.

Now, $x = 2 \Rightarrow y = (2 \times 2 + 3) = 7,$
$x = 3 \Rightarrow y = (2 \times 3 + 3) = 9,$
$x = 4 \Rightarrow y = (2 \times 4 + 3) = 11,$
$x = 5 \Rightarrow y = (2 \times 5 + 3) = 13.$

(i) $\therefore \quad f = \{(2, 7), (3, 9), (4, 11), (5, 13)\}$.

(ii) Clearly, dom $(f) = \{2, 3, 4, 5\}$ and range $(f) = \{7, 9, 11, 13\} \subset Y$.

(iii) It is clear that no two distinct ordered pairs in f have the same first coordinate.

$\therefore \quad f$ is a function from X to Y.

EXAMPLE 3 *Which of the following relations are functions? Give reasons. In case of a function, find its domain and range.*

(i) $f = \{(1, 3), (1, 5), (2, 3), (2, 5)\}$
(ii) $g = \{(2, 1), (5, 1), (8, 1), (11, 1)\}$
(iii) $h = \{(2, 1), (4, 2), (6, 3), (8, 4), (10, 5), (12, 6)\}$

SOLUTION (i) $f = \{(1, 3), (1, 5), (2, 3), (2, 5)\}$.

Here, one element, namely 1 has two images 3 and 5 under f.

$\therefore \quad f$ is not a function.

(ii) $g = \{(2, 1), (5, 1), (8, 1), (11, 1)\}$.

Clearly, no two distinct ordered pairs in g have the same first coordinate. So, g is a function.

$\therefore$ dom $(g) = \{2, 5, 8, 11\}$ and range $g = \{1\}$.

(iii) $h = \{(2, 1), (4, 2), (6, 3), (8, 4), (10, 5), (12, 6)\}$.

Clearly, no two distinct ordered pairs in h have the same first coordinate. So, h is a function.

$\therefore$ dom$(h) = \{2, 4, 6, 8, 10, 12\}$ and range$(h) = \{1, 2, 3, 4, 5, 6\}$.

EXAMPLE 4 *Let* $A = \{-2, -1, 0, 1, 2\}$ *and* $f : A \to Z : f(x) = x^2 - 2x - 3$.
Find (i) range (f) *(ii) pre-images of* 6, –3 *and* 5.

SOLUTION We have

$$f(-2) = (-2)^2 - 2 \times (-2) - 3 = (4 + 4 - 3) = 5;$$
$$f(-1) = (-1)^2 - 2 \times (-1) - 3 = 0;$$
$$f(0) = -3;$$
$$f(1) = (1 - 2 - 3) = -4 \text{ and } f(2) = (2^2 - 2 \times 2 - 3) = -3.$$

$\therefore$ $f = \{(-2, 5), (-1, 0), (0, -3), (1, -4), (2, -3)\}$.

(i) Range $(f) = \{5, 0, -3, -4\}$.

(ii) $f(x) = 6 \Rightarrow x^2 - 2x - 3 = 6$

$$\Rightarrow x^2 - 2x - 9 = 0 \Rightarrow x = \frac{2 \pm \sqrt{4 + 36}}{2} = (1 \pm \sqrt{10}) \notin A.$$

$\therefore$ 6 has no pre-image in A.

$f(x) = -3 \Rightarrow x^2 - 2x - 3 = -3$

$$\Rightarrow x^2 - 2x = 0 \Rightarrow x(x - 2) = 0 \Rightarrow x = 0 \text{ or } x = 2.$$

Clearly, $0, 2 \in A$.

So, the pre-images of –3 are 0 and 2.

$f(x) = 5 \Rightarrow x^2 - 2x - 3 = 5$

$$\Rightarrow x^2 - 2x - 8 = 0 \Rightarrow x^2 - 4x + 2x - 8 = 0$$
$$\Rightarrow x(x - 4) + 2(x - 4) = 0 \Rightarrow (x - 4)(x + 2) = 0$$
$$\Rightarrow x = -2 \text{ or } x = 4.$$

Now, $-2 \in A$ but $4 \notin A$.

So, the pre-image of 5 is –2.

EQUAL FUNCTIONS

Two functions f and g are said to be equal if

(i) dom (f) = *dom* (g) *(ii) co-domain of* f = *co-domain of* g

(iii) $f(x) = g(x)$ *for every x in their common domain.*

EXAMPLE 5 *Let* $A = \{1, 2\}$ *and* $B = \{3, 6\}$ *and f and g be functions from A to B, defined by* $f(x) = 3x$ *and* $g(x) = x^2 + 2$. *Show that* $f = g$.

SOLUTION Clearly, we have

$$\text{dom}(f) = \text{dom}(g) = \{1, 2\}.$$

Co-domain of f = co-domain of $g = \{3, 6\}$.

Also, $f(1) = (3 \times 1) = 3$, $f(2) = (3 \times 2) = 6$;

$$g(1) = (1^2 + 2) = 3, \; g(2) = (2^2 + 2) = 6.$$

$\therefore \quad f(1) = g(1)$ and $f(2) = g(2)$.

Thus, $f(x) = g(x)$ for all $x \in A$.

Hence, $f = g$.

EXAMPLE 6 *Let $f : Z \to Z : f(x) = x^2$ and $g : Z \to Z : g(x) = |x|^2$ for all $x \in Z$. Show that $f = g$.*

SOLUTION We have

$$\text{dom}(f) = \text{dom}(g) = Z$$

and co-domain of f = co-domain of $g = Z$.

Also, for all $x \in Z$, we have

$$f(x) = x^2 \text{ and } g(x) = |x|^2 = x^2.$$

$\therefore \quad f(x) = g(x)$ for all $x \in Z$.

Hence, $f = g$.

EXAMPLE 7 *Let $f : R \to R : f(x) = x + 2$ and $g : R - \{2\} \to R : g(x) = \dfrac{x^2 - 4}{x - 2}$.*

Show that $f \neq g$. Re-define f and g such that $f = g$.

SOLUTION We have, $\text{dom}(f) = R$ and $\text{dom}(g) = R - \{2\}$.

Since $\text{dom}(f) \neq \text{dom}(g)$, so we have $f \neq g$.

For every real number $x \neq 2$, we have

$$g(x) = \frac{x^2 - 4}{x - 2} = \frac{(x-2)(x+2)}{(x-2)} = (x + 2) \qquad [\because \; (x - 2) \neq 0].$$

Thus, $f(x) = g(x)$ for all $x \in R - \{2\}$.

$\therefore \quad f = g$ only when, they are re-defined as under:

$$f : R - \{2\} \to R : f(x) = x + 2 \text{ and } g : R - \{2\} \to R : g(x) = \frac{x^2 - 4}{x - 2}.$$

EXAMPLE 8 *Let $f = \{(-1, -3), (0, -1), (1, 1), (2, 3)\}$ be a function, described by the formula, $f(x) = \alpha x + \beta$. Then, find the values of α and β. Also, find the formula.*

SOLUTION Here $f(x) = \alpha x + \beta$. ... (i)

Also, $f(-1) = -3$, $f(0) = -1$, $f(1) = 1$ and $f(2) = 3$. [given]

Putting $x = -1$ and $f(-1) = -3$ in (i), we get

$$-\alpha + \beta = -3 \Rightarrow \alpha - \beta = 3. \qquad \text{... (ii)}$$

Putting $x = 0$ and $f(0) = -1$ in (i), we get

$$\alpha \times 0 + \beta = -1 \Rightarrow \beta = -1. \qquad \text{... (iii)}$$

Putting $\beta = -1$ from (iii) in (ii), we get $\alpha = 2$.

$\therefore \quad \alpha = 2$ and $\beta = -1$.

Hence, $f(x) = 2x - 1$ is the required formula.

EXAMPLE 9 *Let* $f : R \to R : f(x) = x^2 + 3$.

Find the pre-images of each of the following under f:

(i) 19 *(ii)* 28 *(iii)* 2

SOLUTION Given: $f(x) = x^2 + 3$.

(i) Let x be the pre-image of 19. Then,

$$f(x) = 19 \Rightarrow x^2 + 3 = 19 \Rightarrow x^2 = 16 \Rightarrow x = \pm 4.$$

$\therefore$ 4 and –4 are the pre-images of 19.

(ii) Let x be the pre-image of 28. Then,

$$f(x) = 28 \Rightarrow x^2 + 3 = 28 \Rightarrow x^2 = 25 \Rightarrow x = \pm 5.$$

$\therefore$ 5 and –5 are the pre-images of 28.

(iii) Let x be the pre-image of 2. Then,

$$f(x) = 2 \Rightarrow x^2 + 3 = 2 \Rightarrow x^2 = -1.$$

But, no real value of x satisfies the equation, $x^2 = -1$.

$\therefore$ 2 does not have any pre-image under f.

INVERSE OF A FUNCTION

Let $f : X \to Y$ and let $y \in Y$.

Then, we define, $f^{-1}(y) = \{x \in X : f(x) = y\}$ = *set of pre-images of y.*

f^{-1} *is called the inverse of f.*

REMARK In above example 9, we have

(i) $f^{-1}\{19\} = \{-4, 4\}$ (ii) $f^{-1}\{28\} = \{-5, 5\}$ (iii) $f^{-1}\{2\} = \phi$.

EXAMPLE 10 *Let* $f : R \to R : f(x) = x^2 + 1$.

Find (i) $f^{-1}\{-4\}$ *(ii)* $f^{-1}\{10\}$ *(iii)* $f^{-1}\{5, 17\}$.

SOLUTION It is given that $f(x) = x^2 + 1$.

(i) Let $f^{-1}(-4) = x$. Then,

$$f(x) = -4 \Rightarrow x^2 + 1 = -4 \Rightarrow x^2 = -5.$$

But, there is no real value of x whose square is –5.

$\therefore$ $f^{-1}\{-4\} = \phi$.

(ii) Let $f^{-1}(10) = x$. Then,

$$f(x) = 10 \Rightarrow x^2 + 1 = 10 \Rightarrow x^2 = 9 \Rightarrow x = \pm 3.$$

$\therefore$ $f^{-1}\{10\} = \{-3, 3\}$.

(iii) Let $f^{-1}(5) = x$. Then,

$$f(x) = 5 \Rightarrow x^2 + 1 = 5 \Rightarrow x^2 = 4 \Rightarrow x = \pm 2.$$

$\therefore$ $f^{-1}\{5\} = \{-2, 2\}$.

Let $f^{-1}(17) = x$. Then,

$$f(x) = 17 \Rightarrow x^2 + 1 = 17 \Rightarrow x^2 = 16 \Rightarrow x = \pm 4.$$

$\therefore$ $f^{-1}\{17\} = \{-4, 4\}$.

Hence, $f^{-1}\{5, 17\} = \{-2, 2, -4, 4\}$.

EXAMPLE 11 *Let $f : R \to R$, defined by*

$$f(x) = \begin{cases} 1, \text{ if } x \in Q \\ -1, \text{ if } x \notin Q. \end{cases}$$

Find (i) $f\left(\frac{1}{2}\right)$ *(ii)* $f(0.34)$ *(iii)* $f(\sqrt{2})$ *(iv)* $f(\pi)$

(v) range (f) *(vi)* $f^{-1}\{1\}$ *(vii)* $f^{-1}\{-1\}$

SOLUTION Since each one of $\frac{1}{2}$ and 0.34 is rational, we have

(i) $f\left(\frac{1}{2}\right) = 1$ and (ii) $f(0.34) = 1$.

Since each one of $\sqrt{2}$ and π is irrational, we have

(iii) $f(\sqrt{2}) = -1$ and (iv) $f(\pi) = -1$.

(v) range $(f) = \{f(x) : x \in R\}$

$= \{f(x) : x \in Q\} \cup \{f(x) : x \in R - Q\} = \{1, -1\}$.

(vi) $f^{-1}\{1\} = \{x : f(x) = 1\} = Q$.

(vii) $f^{-1}\{-1\} = \{x : f(x) = -1\} = (R - Q)$.

EXAMPLE 12 *Let $f : R \to R$, defined by*

$$f(x) = \begin{cases} 3x - 2, & x < 0 \\ 1, & x = 0 \\ 4x + 1, & x > 0. \end{cases}$$

Find (i) $f(2)$ *(ii)* $f(-2)$ *(iii)* $f(0)$ *(iv)* $f(3.5)$.

SOLUTION Clearly, we have

(i) $f(2) = (4 \times 2 + 1) = 9$ $[\because\ 2 > 0]$.

(ii) $f(-2) = \{3 \times (-2) - 2\} = -8$ $[\because\ -2 < 0]$.

(iii) $f(0) = 1$.

(iv) $f(3.5) = (4 \times 3.5 + 1) = 15$ $[\because\ 3.5 > 0]$.

EXAMPLE 13 *Consider the relation*

$$f(x) = \begin{cases} x^2, 0 \le x \le 2 \\ 3x, 2 \le x \le 10. \end{cases}$$

Show that f is not a function.

SOLUTION We have, dom $(f) = [0, 10] \subset R$.

Now, $f(2) = 2^2 = 4$ and also, $f(2) = (3 \times 2) = 6$.

As such, a single element, namely 2 has two distinct images, namely 4 and 6.

$\therefore$ f is not a function.

EXAMPLE 14 *Consider the relation*

$$g(x) = \begin{cases} x^2, 0 \le x \le 3 \\ 3x, 3 \le x \le 10. \end{cases}$$

Show that g is a function.

SOLUTION We have, dom $(g) = [0, 10] \subset R$.

It is easy to verify that every element in [0, 10] is associated with a unique real number under g.

In particular $g(3) = 3^2 = 9$ and also $g(3) = 3 \times 3 = 9$, i.e., 3 has a unique image, namely 9.

$\therefore$ g is a function from [0, 10] into R.

EXERCISE 3A

1. Define a function as a set of ordered pairs.
2. Define a function as a correspondence between two sets.
3. What is the fundamental difference between a relation and a function? Is every relation a function?
4. Let $X = \{1, 2, 3, 4\}$, $Y = \{1, 5, 9, 11, 15, 16\}$ and $f = \{(1, 5), (2, 9), (3, 1), (4, 5), (2, 11)\}$.
   Are the following true?
   (i) f is a relation from X to Y (ii) f is a function from X to Y. Justify your answer in each case.
5. Let $X = \{-1, 0, 3, 7, 9\}$ and $f : X \to R : f(x) = x^3 + 1$.
   Express the function f as set of ordered pairs.
6. Let $A = \{-1, 0, 1, 2\}$ and $B = \{2, 3, 4, 5\}$. Find which of the following are functions from A to B. Give reason.
   (i) $f = \{(-1, 2), (-1, 3), (0, 4), (1, 5)\}$
   (ii) $g = \{(0, 2), (1, 3), (2, 4)\}$
   (iii) $h = \{(-1, 2), (0, 3), (1, 4), (2, 5)\}$
7. Let $A = \{1, 2\}$ and $B = \{2, 4, 6\}$. Let $f = \{(x, y) : x \in A, y \in B \text{ and } y > 2x + 1\}$.
   Write f as a set of ordered pairs.
   Show that f is a relation but not a function from A to B.
8. Let $A = \{0, 1, 2\}$ and $B = \{3, 5, 7, 9\}$. Let $f = \{(x, y) : x \in A, y \in B \text{ and } y = 2x + 3\}$.
   Write f as a set of ordered pairs.
   Show that f is a function from A to B.
   Find dom (f) and range (f).
9. Let $A = \{2, 3, 5, 7\}$ and $B = \{3, 5, 9, 13, 15\}$.
   Let $f = \{(x, y) : x \in A, y \in B \text{ and } y = 2x - 1\}$.
   Write f in roster form. Show that f is a function from A to B.
   Find the domain and range of f.
10. Let $g = \{(1, 2), (2, 5), (3, 8), (4, 10), (5, 12), (6, 12)\}$.
    Is g a function? If yes, find its domain and range.
    If no, give reason.
11. Let $f = \{(0, -5), (1, -2), (2, 1), (3, 4), (4, 7)\}$ be a linear function from Z into Z. Write an expression for f.
    Hint Let $f(x) = ax + b$.

12. If $f(x) = x^3$, find the value of $\dfrac{\{f(5) - f(1)\}}{(5-1)}$.

13. If $f(x) = x^2$, find the value of $\dfrac{\{f(1.1) - f(1)\}}{(1.1)-1}$.

14. Let $A = \{12, 13, 14, 15, 16, 17\}$ and $f : A \to Z : f(x) =$ highest prime factor of x. Find range (f).

15. Let R^+ be the set of all positive real numbers.
Let $f : R^+ \to R : f(x) = \log_e x$.
Find (i) range (f) (ii) $\{x : x \in R^+ \text{ and } f(x) = -2\}$.
(iii) Find out whether $f(xy) = f(x) + f(y)$ for all $x, y \in R^+$.

16. Let $f : R \to R : f(x) = 2^x$.
Find (i) range (f) (ii) $\{x : f(x) = 1\}$.
(iii) Find out whether $f(x+y) = f(x) \cdot f(y)$ for all $x, y \in R$.

17. Let $f : R \to R : f(x) = x^2$ and $g : C \to C : g(x) = x^2$, where C is the set of all complex numbers.
Show that $f \neq g$.

18. f, g and h are three functions defined from R to R as follows:
(i) $f(x) = x^2$ (ii) $g(x) = x^2 + 1$ (iii) $h(x) = \sin x$
Then, find the range of each function.

19. Let $f : R \to R : f(x) = x^2 + 1$. Find (i) $f^{-1}\{10\}$ (ii) $f^{-1}\{-3\}$.

20. The function $F(x) = \dfrac{9x}{5} + 32$ is the formula to convert x°C to Fahrenheit units. Find
(i) $F(0)$, (ii) $F(-10)$, (iii) the value of x when $f(x) = 212$.
Interpret the result in each case.

Hints (i) $f(0) = \left\{\dfrac{9 \times 0}{5} + 32\right\} = 32 \Rightarrow 0°\text{C} = 32°\text{F}$.

(ii) $F(-10) = \left\{\dfrac{9 \times (-10)}{5} + 32\right\} = 14 \Rightarrow (-10)°\text{C} = 14°\text{F}$.

(iii) $F(x) = 212 \Rightarrow \dfrac{9x}{5} + 32 = 212 \Rightarrow x = 100 \Rightarrow 212°\text{F} = 100°\text{C}$.

ANSWERS (EXERCISE 3A)

4. (i) Yes (ii) No **5.** $f = \{(-1, 0), (0, 1), (3, 28), (7, 344), (9, 730)\}$

6. (i) No, since one element, namely –1, has two different images.
(ii) No, since dom $(g) \neq A$.
(iii) Yes, since each element in A has a unique image in B.

7. $f = \{(1, 4), (1, 6), (2, 6)\}$

8. $f = \{(0, 3), (1, 5), (2, 7)\}$, dom $(f) = \{0, 1, 2\}$, range $(f) = \{3, 5, 7\}$

9. $f = \{(2, 3), (3, 5), (5, 9), (7, 13)\}$, dom $(f) = \{2, 3, 5, 7\}$, range $(f) = \{3, 5, 9, 13\}$

10. Yes, dom $(f) = \{1, 2, 3, 4, 5, 6\}$ and range $(f) = \{2, 5, 8, 10, 12\}$

11. $f(x) = 3x - 5$ **12.** 31 **13.** 2.1 **14.** range $(f) = \{3, 13, 7, 5, 2, 17\}$

15. (i) R (ii) $\{e^{-2}\}$ (iii) Yes **16.** (i) R^+ (ii) $\{0\}$ (iii) Yes

18. (i) $R^+ = \{x \in R : x \geq 0\}$ (ii) $\{x \in R : x \geq 1\}$ (iii) $\{x \in R : -1 \leq x \leq 1\}$

19. (i) $\{-3, 3\}$ (ii) ϕ **20.** (i) 32 (ii) 14 (iii) $x = 100$

HINTS TO SOME SELECTED QUESTIONS

4. (i) Since $f \subset X \times Y$, so f is a relation from X to Y.

(ii) Dom $(f) = X$ and range $(f) = \{1, 5, 9, 11\} \subset Y$.

But, two different ordered pairs namely (2, 9) and (2, 11) have the same first coordinate. So, f is not a function.

6. (i) No, since two different ordered pairs (–1, 2) and (–1, 3) have the same first coordinate.

(ii) No, since dom $(g) = \{0, 1, 2\} \neq A$.

(iii) Yes, dom $(h) = A$ and no two different ordered pairs have the same first coordinate.

11. $f(0) = -5 \Rightarrow a \times 0 + b = -5 \Rightarrow b = -5.$

$f(1) = -2 \Rightarrow a \times 1 + b = -2 \Rightarrow a - 5 = -2 \Rightarrow a = 3.$

12. $\dfrac{\{f(5) - f(1)\}}{(5-1)} = \dfrac{\{(5)^3 - 1^3\}}{4} = \dfrac{(125-1)}{4} = \dfrac{124}{4} = 31.$

13. $\dfrac{\{f(1.1) - f(1)\}}{(1.1-1)} = \dfrac{\{(1.1)^2 - 1^2\}}{(1.1-1)} = (1.1 + 1) = 2.1.$

14. Highest prime factors of 12, 13, 14, 15, 16, 17 are 3, 13, 7, 5, 2, 17 respectively.

15. (i) For every $x \in R^+$, we have $\log_e x = R$. So, range $(f) = R$.

(ii) $f(x) = -2 \Rightarrow \log_e x = -2 \Rightarrow x = e^{-2}$. So, $\{x : x \in R^+ \text{ and } f(x) = -2\} = \{e^{-2}\}$.

(iii) $f(xy) = \log_e (xy) = \log_e x + \log_e y = f(x) + f(y)$.

16. (i) $f(x) = 2^x > 0$ for every $x \in R$.

If $x \in R^+$, there exists $\log_2 x$ such that $f(\log_2 x) = 2^{\log_2 x} = x$.

$\therefore$ range $(f) = R^+$.

(ii) $f(x) = 1 \Rightarrow 2^x = 1 = 2^0 \Rightarrow x = 0$. So, $\{x : f(x) = 1\} = \{0\}$.

(iii) $f(x + y) = 2^{x+y} = 2^x \times 2^y = f(x) \cdot f(y)$.

17. Dom $(f) = R$ and dom $(g) = C \Rightarrow \text{dom}(f) \neq \text{dom}(g)$

18. (i) Clearly, $f(x) = x^2 \geq 0$ for all $x \in R$.

(ii) $f(x) = x^2 + 1 \geq 1$ for all $x \in R$.

(iii) $\sin x \in [-1, 1]$.

19. (i) $x^2 + 1 = 10 \Rightarrow x^2 = 9 \Rightarrow x = \pm 3.$

(ii) $x^2 + 1 = -3 \Rightarrow x^2 = -4$ and there is no real number whose square is –4.

REAL FUNCTIONS

REAL-VALUED FUNCTIONS *A function which has either R or one of its subsets as its range, is called a real-valued function.*

REAL FUNCTIONS *A function $f: X \to Y$ is called a real function, if $x \subseteq R$ and $y \subseteq R$.*

In general, real functions are described as general expressions or formulae, without mentioning their domains and co-domains. Following are some examples of real functions.

SOLVED EXAMPLES

EXAMPLE 1 *If $f(x) = 3x^3 - 5x^2 + 10$, find $f(x-1)$.*

SOLUTION We have, $f(x) = 3x^3 - 5x^2 + 10$. ... (i)

Replacing x by $(x-1)$ in (i), we get

$$\begin{aligned} f(x-1) &= 3(x-1)^3 - 5(x-1)^2 + 10 \\ &= 3\{x^3 - 1 - 3x(x-1)\} - 5(x^2 - 2x + 1) + 10 \\ &= 3(x^3 - 3x^2 + 3x - 1) - 5(x^2 - 2x + 1) + 10 \\ &= 3x^3 - 14x^2 + 19x + 2. \end{aligned}$$

$\therefore \quad f(x-1) = 3x^3 - 14x^2 + 19x + 2.$

EXAMPLE 2 *If $f(x) = x + \frac{1}{x}$, show that $\{f(x)\}^3 = f(x^3) + 3f\left(\frac{1}{x}\right)$.*

SOLUTION We have, $f(x) = x + \frac{1}{x}$. ... (i)

On cubing both sides of (i), we get

$$\begin{aligned} \{f(x)\}^3 &= x^3 + \frac{1}{x^3} + 3 \times x \times \frac{1}{x} \times \left(x + \frac{1}{x}\right) \\ &= \left(x^3 + \frac{1}{x^3}\right) + 3\left(\frac{1}{x} + x\right) \\ &= f(x^3) + 3f\left(\frac{1}{x}\right) \qquad \left[\because\ f\left(\frac{1}{x}\right) = \left\{\frac{1}{x} + \frac{1}{\frac{1}{x}}\right\} = \left(\frac{1}{x} + x\right)\right]. \end{aligned}$$

Hence, $\{f(x)\}^3 = f(x^3) + 3f\left(\frac{1}{x}\right)$.

EXAMPLE 3 *If $f(x) = \frac{x-1}{x+1}$, $x \neq -1$ then show that $f\{f(x)\} = \frac{-1}{x}$, where $x \neq 0$.*

SOLUTION We have, $f(x) = \frac{x-1}{x+1}$, where $x \neq -1$.

$$\therefore \quad f\{f(x)\} = f\left(\frac{x-1}{x+1}\right) = \frac{\left\{\frac{x-1}{x+1} - 1\right\}}{\left\{\frac{x-1}{x+1} + 1\right\}}$$

$$= \frac{\{(x-1)-(x+1)\}}{(x+1)} \times \frac{(x+1)}{\{(x-1)+(x+1)\}} = \frac{-2}{2x} = \frac{-1}{x}.$$

Hence, $f\{f(x)\} = \frac{-1}{x}$, where $x \neq 0$.

EXAMPLE 4 *If $y = f(x) = \frac{ax-b}{bx-a}$ and $a^2 \neq b^2$ then prove that $x = f(y)$.*

SOLUTION We have, $y = f(x) = \dfrac{ax-b}{bx-a}$.

$$\therefore \quad f(y) = f\{f(x)\} = f\left(\frac{ax-b}{ax-a}\right) = \frac{\left\{a\left(\frac{ax-b}{bx-a}\right)-b\right\}}{\left\{b\left(\frac{ax-b}{bx-a}\right)-a\right\}}$$

$$= \frac{\{(a^2x-ab)-(b^2x-ab)\}}{(bx-a)} \times \frac{(bx-a)}{\{(abx-b^2)-(abx-a^2)\}}$$

$$= \frac{(a^2x-b^2x)}{(a^2-b^2)} = \frac{(a^2-b^2)x}{(a^2-b^2)} = x.$$

Hence, $x = f(y)$.

EXAMPLE 5 *If $f(x) = \dfrac{x-1}{x+1}$ then prove that $f(2x) = \dfrac{3f(x)+1}{f(x)+3}$.*

SOLUTION We have, $f(x) = \dfrac{x-1}{x+1}$.

$$\therefore \quad \frac{3f(x)+1}{f(x)+3} = \frac{3\left(\frac{x-1}{x+1}\right)+1}{\frac{(x-1)}{(x+1)}+3}$$

$$= \frac{(3x-3)+(x+1)}{(x+1)} \times \frac{(x+1)}{(x-1)+(3x+3)} = \frac{2x-1}{2x+1} = f(2x).$$

Hence, $f(2x) = \dfrac{3f(x)+1}{f(x)+3)}$.

EXAMPLE 6 *Let the functions f and g be defined by*

$$f(x) = (x-3) \text{ and } g(x) = \begin{cases} \dfrac{x^2-9}{x+3}, & \text{when } x \neq -3 \\ k, & \text{when } x = -3. \end{cases}$$

Find the value of k such that $f(x) = g(x)$ for all $x \in R$.

SOLUTION We have, $f(x) = g(x)$ for all $x \in R$.

$\therefore \quad f(-3) = g(-3)$

$\Rightarrow \quad (-3-3) = k \qquad [\because f(-3) = (-3-3) \text{ and } g(-3) = k]$

$\Rightarrow \quad k = -6.$

Hence, $k = -6$.

EXAMPLE 7 *If $af(x) + bf\left(\dfrac{1}{x}\right) = \dfrac{1}{x}$, where $a \neq b$ and $x \neq 0$, find $f(x)$.*

SOLUTION We have, $af(x) + bf\left(\dfrac{1}{x}\right) = \dfrac{1}{x}$. ... (i)

Replacing x by $\dfrac{1}{x}$ in (i), we get

$$af\left(\frac{1}{x}\right) + bf(x) = x. \qquad \text{... (ii)}$$

Adding (i) and (ii), we get

$$a\left\{f(x)+f\left(\frac{1}{x}\right)\right\}+b\left\{f(x)+f\left(\frac{1}{x}\right)\right\}=\left(x+\frac{1}{x}\right)$$

$$\Rightarrow \quad (a+b)\left\{f(x)+f\left(\frac{1}{x}\right)\right\}=\left(x+\frac{1}{x}\right)$$

$$\Rightarrow \quad f(x)+f\left(\frac{1}{x}\right)=\frac{1}{(a+b)}\left(x+\frac{1}{x}\right). \qquad \dots \text{(iii)}$$

On subtracting (ii) from (i), we get

$$a\left\{f(x)-f\left(\frac{1}{x}\right)\right\}-b\left\{f(x)-f\left(\frac{1}{x}\right)\right\}=\left(\frac{1}{x}-x\right)$$

$$\Rightarrow \quad (a-b)\left\{f(x)-f\left(\frac{1}{x}\right)\right\}=\left(\frac{1}{x}-x\right)$$

$$\Rightarrow \quad \left\{f(x)-f\left(\frac{1}{x}\right)\right\}=\frac{1}{(a-b)}\left(\frac{1}{x}-x\right). \qquad \dots \text{(iv)}$$

On adding (iii) and (iv), we get

$$2f(x)=\left\{\frac{x}{(a+b)}-\frac{x}{(a-b)}\right\}+\left\{\frac{1}{(a+b)x}+\frac{1}{(a-b)x}\right\}$$

$$=\frac{(a-b)x-(a+b)x}{(a^2-b^2)}+\frac{(a-b)+(a+b)}{(a^2-b^2)x}$$

$$=\frac{-2bx^2+2d}{(a^2-b^2)x}=\frac{2}{(a^2-b^2)}\left(\frac{a}{x}-bx\right).$$

Hence, $f(x)=\dfrac{1}{(a^2-b^2)}\left(\dfrac{a}{x}-bx\right).$

EXERCISE 3B

1. If $f(x)=x^2-3x+4$ and $f(x)=f(2x+1)$, find the values of x.

2. If $f(x)=\dfrac{x-1}{x+1}$ then show that (i) $f\left(\dfrac{1}{x}\right)=-f(x)$ (ii) $f\left(\dfrac{-1}{x}\right)=\dfrac{-1}{f(x)}.$

3. If $f(x)=x^3-\dfrac{1}{x^3}$ then show that $f(x)+f\left(\dfrac{1}{x}\right)=0.$

4. If $f(x)=\dfrac{x+1}{x-1}$ then show that $f\{f(x)\}=x.$

5. If $f(x)=\dfrac{1}{(2x+1)}$ and $x\neq\dfrac{-1}{2}$ then prove that $f\{f(x)\}=\dfrac{2x+1}{2x+3}$, when it is given that $x\neq\dfrac{-3}{2}.$

6. If $f(x)=\dfrac{1}{(1-x)}$ then show that $f[f\{f(x)\}]=x.$

7. If $f(x)=\dfrac{2x}{(1+x^2)}$ then show that $f(\tan\theta)=\sin 2\theta.$

8. If $y=f(x)=\dfrac{3x+1}{5x-3}$, prove that $x=f(y)$.

ANSWERS (EXERCISE 3B)

1. $x=-1$ or $x=\dfrac{2}{3}$

HINTS TO SOME SELECTED QUESTIONS

1. $f(x)=f(2x+1) \Rightarrow x^2-3x+4=(2x+1)^2-3(2x+1)+4$

$\Rightarrow x^2-3x+4=4x^2-2x+2 \Rightarrow 3x^2+x-2=0 \Rightarrow (x+1)(3x-2)=0.$

5. $f(x)=\dfrac{1}{(2x+1)} \Rightarrow f\{f(x)\}=\dfrac{1}{\left\{2\times\dfrac{1}{(2x+1)}+1\right\}}=\dfrac{2x+1}{2x+3}.$

6. $f(x)=\dfrac{1}{(1-x)} \Rightarrow f\{f(x)\}=\dfrac{1}{\left\{1-\dfrac{1}{(1-x)}\right\}}=\dfrac{1-x}{-x}=\dfrac{x-1}{x}$ [replace x by $\dfrac{1}{(1-x)}$]

$$\Rightarrow f[f\{f(x)\}]=\frac{\left\{\dfrac{1}{(1-x)}-1\right\}}{\dfrac{1}{(1-x)}}=\frac{x}{(1-x)}\times\frac{(1-x)}{1}=x.$$

7. $f(\tan\theta)=\dfrac{2\tan\theta}{(1+\tan^2\theta)}=\dfrac{2\tan\theta}{\sec^2\theta}=\dfrac{2\sin\theta}{\cos\theta}\times\cos^2\theta=2\sin\theta\cdot\cos\theta=\sin 2\theta.$

8. $f(y)=\dfrac{3y+1}{5y-3}=\dfrac{3\left(\dfrac{3x+1}{5x-3}\right)+1}{5\left(\dfrac{3x+1}{5x-3}\right)-3}=\dfrac{9x+3+5x-3}{15x+5-15x+9}=\dfrac{14x}{14}=x.$

PROBLEMS BASED ON DOMAINS AND RANGES OF REAL FUNCTIONS

SOLVED EXAMPLES

EXAMPLE 1 *Find the domain and the range of the real function,* $f(x)=\dfrac{1}{x+3}$.

SOLUTION We have, $f(x)=\dfrac{1}{(x+3)}$.

Clearly, $f(x)$ is defined for all real values of x except that at which $x+3=0$, i.e., $x=-3$.

$\therefore$ dom $(f)=R-\{-3\}$.

Let $y=f(x)$. Then,

$$y=\frac{1}{x+3} \Rightarrow x+3=\frac{1}{y}$$

$$\Rightarrow x=\left(\frac{1}{y}-3\right). \quad \ldots \text{(i)}$$

It is clear from (i) that x assumes real values for all real values of y except $y=0$.

$\therefore$ range $(f)=R-\{0\}$.

Hence, dom $(f)=R-\{-3\}$ and range $(f)=R-\{0\}$.

EXAMPLE 2 *Find the domain and the range of the real function,* $f(x) = \frac{x-3}{x-5}\cdot$

SOLUTION We have, $f(x) = \frac{x-3}{x-5}\cdot$

Clearly, $f(x)$ is defined for all real values of x except that at which $x-5=0$, i.e., $x=5$.

$\therefore$ dom $(f) = R - \{5\}$.

Let $y = f(x)$. Then,

$$y = \frac{x-3}{x-5} \Rightarrow xy - 5y = x - 3$$

$$\Rightarrow x(y-1) = 5y - 3 \Rightarrow x = \frac{5y-3}{y-1}\cdot \quad \ldots \text{(i)}$$

It is clear from (i) that x is not defined when $y-1=0$, i.e., when $y=1$.

$\therefore$ range $(f) = R - \{1\}$.

Hence, dom $(f) = R - \{5\}$ and range $(f) = R - \{1\}$.

EXAMPLE 3 *Find the domain and the range of the real function,* $f(x) = \frac{x^2+1}{x^2-1}\cdot$

SOLUTION We have, $f(x) = \frac{x^2+1}{x^2-1}\cdot$

Clearly, $f(x)$ is defined for all real values of x except those for which $x^2-1=0$, i.e., $x=\pm 1$.

$\therefore$ dom $(f) = R - \{-1, 1\}$.

Let $y = f(x)$. Then,

$$y = \frac{x^2+1}{x^2-1} \Rightarrow x^2y - y = x^2 + 1 \Rightarrow x^2(y-1) = (y+1)$$

$$\Rightarrow x^2 = \frac{y+1}{y-1} \Rightarrow x = \pm\sqrt{\frac{y+1}{y-1}}\cdot \quad \ldots \text{(i)}$$

It is clear from (i) that x is not defined when $y-1=0$ or when $\frac{y+1}{y-1} < 0$.

Now, $y - 1 = 0 \Rightarrow y = 1$. $\quad \ldots$ (ii)

And $\frac{y+1}{y-1} < 0 \Rightarrow (y+1>0 \text{ and } y-1<0) \text{ or } (y+1<0 \text{ and } y-1>0)$

$$\Rightarrow (y > -1 \text{ and } y < 1) \text{ or } (y < -1 \text{ and } y > 1)$$

$$\Rightarrow -1 < y < 1. \quad \ldots \text{(iii)}$$

[$\because$ $y < -1$ and $y > 1$ is not possible]

Thus, x is not defined when $-1 < y \le 1$. [using (ii) and (iii)]

$\therefore$ range $(f) = R - (-1, 1]$.

Hence, dom $(f) = R - \{-1, 1\}$ and range $(f) = R - (-1, 1]$.

EXAMPLE 4 *Find the domain of the real-valued function:*

$$f(x) = \frac{x^2 - x + 1}{x^2 - 5x + 4}.$$

SOLUTION We have, $f(x) = \dfrac{x^2 - x + 1}{x^2 - 5x + 4}$.

Clearly, $f(x)$ is defined for all real values of x except those at which $x^2 - 5x + 4 = 0$.

But, $x^2 - 5x + 4 = 0 \Rightarrow (x-1)(x-4) = 0 \Rightarrow x = 1$ or $x = 4$.

$\therefore$ dom $(f) = R - \{1, 4\}$.

EXAMPLE 5 *Find the domain of each of the following real functions:*

(i) $f(x) = \sqrt{x-3}$ *(ii)* $g(x) = \sqrt{4 - x^2}$ *(iii)* $h(x) = \dfrac{1}{\sqrt{1-x}}$.

SOLUTION (i) We have, $f(x) = \sqrt{x-3}$.

Clearly, $f(x)$ is defined for all real values of x for which $x - 3 \geq 0$, i.e., $x \geq 3$.

$\therefore$ dom $(f) = [3, \infty)$.

(ii) We have, $g(x) = \sqrt{4 - x^2}$.

Clearly, $g(x)$ is defined for all real values of x for which $(4 - x^2) \geq 0$.

But, $(4 - x^2) \geq 0 \Rightarrow -(4 - x^2) \leq 0$

$\Rightarrow x^2 - 4 \leq 0 \Rightarrow (x+2)(x-2) \leq 0$

$\Rightarrow -2 \leq x \leq 2 \Rightarrow x \in [-2, 2]$.

$\therefore$ dom $(g) = [-2, 2]$.

(iii) We have, $h(x) = \dfrac{1}{\sqrt{1-x}}$.

Clearly, $h(x)$ is defined for all real values of x for which $(1 - x) > 0$.

But, $1 - x > 0 \Rightarrow 1 > x \Rightarrow x < 1 \Rightarrow x \in (-\infty, 1)$.

$\therefore$ dom $(h) = (-\infty, 1)$.

EXAMPLE 6 *Find the domain of the function,* $f(x) = \sqrt{3-x} + \dfrac{1}{\sqrt{x^2 - 1}}$.

SOLUTION We have, $f(x) = \sqrt{3-x} + \dfrac{1}{\sqrt{x^2-1}}$.

Clearly, $f(x)$ is defined for all real values of x for which

$3 - x \geq 0$ and $x^2 - 1 > 0$

$\Rightarrow$ $x - 3 \leq 0$ and $(x+1)(x-1) > 0$

$\Rightarrow$ $x \leq 3$ and $(x < -1$ or $x > 1)$

$\Rightarrow$ $(x \leq 3$ and $x < -1)$ or $(x \leq 3$ and $x > 1)$

$\Rightarrow \quad (x < -1)$ or $(1 < x \leq 3)$

$\Rightarrow \quad x \in (-\infty, -1) \cup (1, 3]$.

$\therefore \quad \text{dom}(f) = (-\infty, -1) \cup (1, 3]$.

EXAMPLE 7 *Find the range of each of the following functions:*

(i) $f(x) = 2 - 3x$, $x \in R$ *and* $x > 0$ *(ii)* $g(x) = x^2 + 2$, $x \in R$.

SOLUTION (i) We have, $f(x) = 2 - 3x$, where $x \in R$ and $x > 0$.

Now, $x > 0 \Rightarrow 3x > 0 \Rightarrow -3x < 0$

$\Rightarrow -3x + 2 < 0 + 2 \Rightarrow 2 - 3x < 2$

$\Rightarrow f(x) < 2 \Rightarrow f(x) \in (-\infty, 2)$.

Hence, range $(f) = (-\infty, 2)$.

(ii) We have, $g(x) = x^2 + 2$, $x \in R$.

Now, $x \in R \Rightarrow x^2 \geq 0 \Rightarrow x^2 + 2 \geq 0 + 2$

$\Rightarrow x^2 + 2 \geq 2 \Rightarrow g(x) \geq 2$

$\Rightarrow g(x) \in [2, \infty)$.

Hence, range $(g) = [2, \infty)$.

EXAMPLE 8 *Find the domain and the range of the function,* $f(x) = \dfrac{x-2}{x-3}$.

SOLUTION We have, $f(x) = \dfrac{x-2}{x-3}$.

Clearly, $f(x)$ is defined for all real values of x for which $x - 3 \neq 0$, i.e., $x \neq 3$.

$\therefore \quad \text{dom}(f) = R - \{3\}$.

Let $y = f(x)$. Then,

$$y = \frac{x-2}{x-3} \Rightarrow xy - 3y = x - 2$$

$$\Rightarrow x(y-1) = 3y - 2$$

$$\Rightarrow x = \frac{3y-2}{y-1}. \qquad \ldots \text{(i)}$$

It follows from (i) that x assumes real values for all y except that for which $y - 1 = 0$, i.e., $y = 1$.

$\therefore \quad \text{range}(f) = R - \{1\}$.

Hence, dom $(f) = R - \{3\}$ and range $(f) = R - \{1\}$.

EXAMPLE 9 *Find the domain and the range of the real function,* $f(x) = \sqrt{x-3}$.

SOLUTION We have, $f(x) = \sqrt{x-3}$.

Clearly, $f(x)$ is defined for all real values of x for which $x - 3 \geq 0$, i.e., $x \geq 3$.

$\therefore \quad \text{dom}(f) = [3, \infty)$.

Also, $x \geq 3 \Rightarrow f(x) = \sqrt{x-3} \geq 0$.

$\therefore \quad \text{range}(f) = [0, \infty)$.

Hence, dom $(f) = [3, \infty)$ and range $(f) = [0, \infty)$.

EXAMPLE 10 *Find the domain and the range of each of the functions given below.*

(i) $f(x) = |x-1|$ *(ii)* $g(x) = -|x|$

SOLUTION (i) We have, $f(x) = |x-1|$.

Clearly, $f(x)$ is defined for all $x \in R$. So, dom $(f) = R$.

For all $x \in R$, we have

$$|x-1| \geq 0 \Rightarrow f(x) \geq 0.$$

$\therefore$ range $(f) = [0, \infty)$.

Hence, dom $(f) = R$ and range $(f) = [0, \infty)$.

(ii) We have, $g(x) = -|x|$.

Clearly, $g(x)$ is defined for all $x \in R$. So, dom $(g) = R$.

For all $x \in R$, we have

$$|x| \geq 0 \Rightarrow -|x| \leq 0 \Rightarrow g(x) \leq 0.$$

$\therefore$ range $(g) = (-\infty, 0]$.

Hence, dom $(g) = R$ and range $(g) = (-\infty, 0]$.

EXAMPLE 11 *Find the domain and the range of the real function,* $f(x) = \dfrac{1}{\sqrt{x-2}}$.

SOLUTION We have, $f(x) = \dfrac{1}{\sqrt{x-2}}$.

Clearly, $f(x)$ is defined for all real values of x for which $x-2 > 0$, i.e., $x > 2$.

$\therefore$ dom $(f) = (2, \infty)$.

For any real value of $x > 2$, we have

$$x > 2 \Rightarrow x-2 > 0 \Rightarrow \sqrt{x-2} > 0$$

$$\Rightarrow \frac{1}{\sqrt{x-2}} > 0 \Rightarrow f(x) > 0.$$

$\therefore$ range $(f) = (0, \infty)$.

Hence, dom $(f) = (2, \infty)$ and range $(f) = (0, \infty)$.

EXAMPLE 12 *Find the domain and the range of the real function,* $f(x) = \sqrt{9-x^2}$.

SOLUTION We have, $f(x) = \sqrt{9-x^2}$.

Clearly, $f(x)$ is defined for all real values of x for which $(9-x^2) \geq 0$.

And, $9-x^2 \geq 0 \Rightarrow x^2-9 \leq 0 \Rightarrow (x-3)(x+3) \leq 0$

$$\Rightarrow -3 \leq x \leq 3 \Rightarrow x \in [-3, 3].$$

$\therefore$ dom $(f) = [-3, 3]$.

Let $y = f(x)$. Then,

$$y = \sqrt{9-x^2} \Rightarrow y^2 = 9-x^2 \Rightarrow x^2 = 9-y^2 \Rightarrow x = \sqrt{9-y^2}.$$

Clearly, x will take real values only when $9-y^2 \geq 0$.

Now, $9-y^2 \geq 0 \Rightarrow y^2-9 \leq 0 \Rightarrow (y+3)(y-3) \leq 0$

$$\Rightarrow -3 \leq y \leq 3 \Rightarrow y \in [-3, 3]$$

$\Rightarrow\ y \in [0, 3] \quad \{\because\ y = \sqrt{9 - x^2} \geq 0 \text{ for all } x \in [-3, 3]\}.$

$\therefore\quad$ range $(f) = [0, 3]$.

Hence, dom $(f) = [-3, 3]$ and range $(f) = [0, 3]$.

EXAMPLE 13 *Find the domain and the range of the real function,* $f(x) = \dfrac{x}{(1 + x^2)}$.

SOLUTION We have, $f(x) = \dfrac{x}{(1 + x^2)}$.

Clearly, $f(x)$ is defined for all $x \in R$. So, dom $(f) = R$.

Let $y = f(x)$. Then,

$$y = \frac{x}{(1 + x^2)} \Rightarrow x^2 y - x + y = 0$$

$$\Rightarrow x = \frac{1 \pm \sqrt{1 - 4y^2}}{2y}. \qquad \ldots \text{(i)}$$

It is clear from (i) that x will take real values, when

$(1 - 4y^2) \geq 0$ and $y \neq 0$

$\Rightarrow\quad (4y^2 - 1) \leq 0$ and $y \neq 0$

$\Rightarrow\quad (2y + 1)(2y - 1) \leq 0$ and $y \neq 0$

$\Rightarrow\quad \left(y + \frac{1}{2}\right)\left(y - \frac{1}{2}\right) \leq 0$ and $y \neq 0$

$\Rightarrow\quad -\frac{1}{2} \leq y \leq \frac{1}{2}$ and $y \neq 0$

$\Rightarrow\quad y \in \left[\frac{-1}{2}, \frac{1}{2}\right] - \{0\}$.

Also, $x = 0 \Rightarrow y = 0$.

$\therefore\quad$ range $(f) = \left[\frac{-1}{2}, \frac{1}{2}\right]$.

Hence, dom $(f) = R$ and range $(f) = \left[\frac{-1}{2}, \frac{1}{2}\right]$.

EXAMPLE 14 *Find the domain and the range of the real function,* $f(x) = \dfrac{3}{(2 - x^2)}$.

SOLUTION We have, $f(x) = \dfrac{3}{(2 - x^2)}$.

Clearly, $f(x)$ is defined for all real values of x except those for which $2 - x^2 = 0$, i.e., $x = \pm\sqrt{2}$.

$\therefore\quad$ dom $(f) = R - \{-\sqrt{2}, \sqrt{2}\}$.

Let $y = f(x)$. Then,

$$y = \frac{3}{(2 - x^2)} \Rightarrow 2y - x^2 y = 3 \Rightarrow x^2 y = 2y - 3$$

$$\Rightarrow x^2 = \frac{2y - 3}{y} \Rightarrow x = \pm\sqrt{\frac{2y - 3}{y}}. \qquad \ldots \text{(i)}$$

It is clear from (i) that x will take real values only when $\frac{2y-3}{y} \geq 0$.

$$\text{Now, } \frac{2y-3}{y} \geq 0 \Leftrightarrow (2y-3 \leq 0 \text{ and } y<0) \text{ or } (2y-3 \geq 0 \text{ and } y>0)$$

$$\Leftrightarrow \left(y \leq \frac{3}{2} \text{ and } y<0\right) \text{ or } \left(y \geq \frac{3}{2} \text{ and } y>0\right)$$

$$\Leftrightarrow (y<0) \text{ or } \left(y \geq \frac{3}{2}\right)$$

$$\Leftrightarrow y \in (-\infty, 0) \text{ or } y \in \left[\frac{3}{2}, \infty\right)$$

$$\Leftrightarrow y \in (-\infty, 0) \cup \left[\frac{3}{2}, \infty\right).$$

$\therefore$ range $(f) = (-\infty, 0) \cup \left[\frac{3}{2}, \infty\right)$.

Hence, dom $(f) = R - \{-\sqrt{2}, \sqrt{2}\}$ and range $(f) = (-\infty, 0) \cup \left[\frac{3}{2}, \infty\right)$.

EXAMPLE 15 *Find the domain and the range of the real function* $f = \left\{\left(x, \frac{x^2}{x^2+1}\right) : x \in R\right\}$ *from R into R.*

SOLUTION We have, $f(x) = \frac{x^2}{x^2+1}$, $x \in R$.

Clearly, $x^2+1 \neq 0$ for any real value of x.

$\therefore$ $f(x)$ is defined for all real values of x.

$\therefore$ dom $(f) = R$.

Let $y = f(x)$. Then,

$$y = \frac{x^2}{(x^2+1)} \Rightarrow x^2y + y = x^2 \Rightarrow x^2(1-y) = y$$

$$\Rightarrow x^2 = \frac{y}{1-y} \Rightarrow x = \pm\sqrt{\frac{y}{(1-y)}}. \qquad \ldots \text{(i)}$$

It is clear from (i) that x will take real values only when $\frac{y}{(1-y)} \geq 0$.

$$\text{Now, } \frac{y}{(1-y)} \geq 0 \Leftrightarrow (y \leq 0 \text{ and } 1-y<0) \text{ or } (y \geq 0 \text{ and } 1-y>0)$$

$$\Leftrightarrow (y \leq 0 \text{ and } y>1) \text{ or } (y \geq 0 \text{ and } y<1)$$

$$\Leftrightarrow (y \geq 0 \text{ and } y<1)$$

[$\because$ $y \leq 0$ and $y > 1$ is not possible]

$$\Leftrightarrow y \in [0, 1).$$

$\therefore$ range $(f) = [0, 1)$.

Hence, dom $(f) = R$ and range $(f) = [0, 1)$.

EXAMPLE 16 *Find the domain and the range of the function*

$$f = \left\{\left(x, \frac{1}{1-x^2}\right) : x \in R \text{ and } x \neq \pm 1\right\}.$$

SOLUTION We have, $f(x) = \dfrac{1}{(1-x^2)}, x \in R.$

Clearly, $f(x)$ is defined for all real values of x for which $(1-x^2) \neq 0$.

Now, $(1-x^2) = 0 \Rightarrow (1+x)(1-x) = 0 \Rightarrow x = -1$ or $x = +1$.

Thus, $f(x)$ is defined for all values of $x \in R$ except ± 1.

$\therefore \quad \text{dom}(f) = R - \{-1, 1\}$.

Let $y = f(x)$. Then,

$$y = \frac{1}{1-x^2} \Rightarrow y - x^2 y = 1 \Rightarrow x^2 y = y - 1$$

$$\Rightarrow x^2 = \frac{y-1}{y} \Rightarrow x = \pm\sqrt{\frac{y-1}{y}}. \quad \ldots \text{(i)}$$

It is clear from (i) that x will take real values only when $\dfrac{y-1}{y} \geq 0$.

$$\text{Now, } \frac{y-1}{y} \geq 0 \Leftrightarrow (y-1 \leq 0 \text{ and } y < 0) \text{ or } (y-1 \geq 0 \text{ and } y > 0)$$

$$\Leftrightarrow (y \leq 1 \text{ and } y < 0) \text{ or } (y \geq 1 \text{ and } y > 0)$$

$$\Leftrightarrow (y < 0) \text{ or } (y \geq 1)$$

$$\Leftrightarrow y \in (-\infty, 0) \text{ or } [1, \infty).$$

$\therefore \quad \text{range}(f) = (-\infty, 0) \cup [1, \infty)$.

Hence, $\text{dom}(f) = R - \{-1, 1\}$ and $\text{range}(f) = (-\infty, 0) \cup [1, \infty)$.

EXAMPLE 17 *Find the domain and the range of the real function,* $f(x) = \dfrac{x^2 - 25}{x-5}$.

SOLUTION We have, $f(x) = \dfrac{x^2-25}{x-5}$.

Clearly, $f(x)$ is defined for all real values of x for which $x - 5 \neq 0$, i.e., $x \neq 5$.

$\therefore \quad \text{dom}(f) = R - \{5\}$.

Let $y = f(x)$. Then,

$$y = \frac{x^2-25}{x-5} \Rightarrow y = x + 5, \text{ when } x - 5 \neq 0$$

$$\Rightarrow y = x + 5, \text{ when } x \neq 5$$

$$\Rightarrow y \neq 5 + 5 \Rightarrow y \neq 10.$$

Then, y can be assigned any real value except 10.

$\therefore \quad \text{range}(f) = R - \{10\}$.

Hence, $\text{dom}(f) = R - \{5\}$ and $\text{range}(f) = R - \{10\}$.

EXAMPLE 18 *Find the domain and the range of the real function,* $f(x) = \dfrac{3-x}{x-3}$.

SOLUTION We have, $f(x) = \dfrac{3-x}{x-3}$.

Clearly, $f(x)$ is defined for all real values of x for which $x-3 \neq 0$, i.e., $x \neq 3$.

$\therefore \quad \text{dom}(f) = R - \{3\}$.

Let $y = f(x)$. Then,

$$y = \frac{3-x}{x-3} \Rightarrow y = -1, \text{ when } x - 3 \neq 0$$

$$\Rightarrow y = -1, \text{ when } x \neq 3.$$

$\therefore \quad \text{range}(f) = \{-1\}$.

Hence, $\text{dom}(f) = R - \{3\}$ and $\text{range}(f) = \{-1\}$.

EXAMPLE 19 *Find the domain and the range of the real function,* $f(x) = \dfrac{|x-3|}{(x-3)}$.

SOLUTION We have, $f(x) = \dfrac{|x-3|}{(x-3)}$.

Clearly, $f(x)$ is defined for all real values of x for which $x-3 \neq 0$, i.e., $x \neq 3$.

$\therefore \quad \text{dom}(f) = R - \{3\}$.

Now, when $x \neq 3$, we have

$$f(x) = \begin{cases} 1, \text{ when } x - 3 > 0 & \{\because |x-3| = (x-3)\} \\ -1, \text{ when } x - 3 < 0 & \{\because |x-3| = -(x-3)\} \end{cases}$$

$\therefore \quad \text{range}(f) = \{1, -1\}$.

Hence, $\text{dom}(f) = R - \{3\}$ and $\text{range}(f) = \{1, -1\}$.

EXAMPLE 20 *Find the domain of the real function,* $f(x) = \dfrac{1}{\sqrt{x+|x|}}$.

SOLUTION We have, $f(x) = \dfrac{1}{\sqrt{x+|x|}}$.

Now, $\quad |x| = \begin{cases} x, \text{ when } x \geq 0 \\ -x, \text{ when } x < 0 \end{cases}$

$\Rightarrow \quad x + |x| = \begin{cases} x + x, \text{ when } x \geq 0 \\ x - x, \text{ when } x < 0 \end{cases}$

$\Rightarrow \quad x + |x| = \begin{cases} 2x, \text{ when } x \geq 0 \\ 0, \text{ when } x < 0 \end{cases}$

$\Rightarrow \quad x + |x| > 0$, when $x > 0$

$\Rightarrow \quad f(x) = \dfrac{1}{\sqrt{x+|x|}}$ assumes real values only when $x + |x| > 0$ and this happens only when $x > 0$.

$\therefore \quad \text{dom}(f) = (0, \infty)$.

EXAMPLE 21 *Show that* $f(x) = \dfrac{1}{\sqrt{x-|x|}}$ *is not defined for any* $x \in R$. *How will you define dom* (f) *and range* (f)?

SOLUTION We have, $f(x) = \dfrac{1}{\sqrt{x - |x|}}$.

Now, $|x| = \begin{cases} x, \text{ when } x \geq 0 \\ -x, \text{ when } x < 0 \end{cases}$

$\Rightarrow \quad x - |x| = \begin{cases} x - x, \text{ when } x \geq 0 \\ x + x, \text{ when } x < 0 \end{cases}$

$\Rightarrow \quad x - |x| = \begin{cases} 0, \text{ when } x \geq 0 \\ 2x, \text{ when } x < 0 \end{cases}$

$\Rightarrow \quad x - |x| \leq 0$ for all $x \in R$

$\Rightarrow \quad \dfrac{1}{\sqrt{x - |x|}}$ is not defined for any $x \in R$.

$\therefore \quad$ dom $(f) = \phi$ and range $(f) = \phi$.

GREATEST INTEGER FUNCTION (STEP FUNCTION)

The function $f : R \to R : f(x) = [x]$, where $[x]$ denotes the greatest integer less than or equal to x, is called the Greatest Integer Function.

EXAMPLES [3.65] = 3, [3.02] = 3, [5] = 5, [–2.6] = –3, [–4] = –4.

NOTE $x - [x] = 0$ for $x \in Z$ and $0 < x - [x] < 1$ for $x \in R - Z$.

EXAMPLE 22 *Find the domain and the range of the real function,* $f(x) = \dfrac{1}{\sqrt{x + [x]}}$.

SOLUTION We have, $f(x) = \dfrac{1}{\sqrt{x + [x]}}$.

We know that

$$\begin{cases} x + [x] > 0 \text{ for all } x > 0 \\ x + [x] = 0, \text{ when } x = 0 \\ x + [x] < 0 \text{ for all } x < 0. \end{cases}$$

$\therefore \quad f(x) = \dfrac{1}{\sqrt{x + [x]}}$ is defined only when $x + [x] > 0$ and this happens only when $x > 0$.

$\therefore \quad$ dom $(f) = (0, \infty)$.

Let $y = f(x)$. Then,

$$y = \frac{1}{\sqrt{x + [x]}} \Rightarrow \sqrt{x + [x]} = \frac{1}{y}. \quad \ldots \text{(i)}$$

Now, $x > 0 \Rightarrow x + [x] > 0 \Rightarrow \sqrt{x + [x]} > 0 \Rightarrow \dfrac{1}{y} > 0 \Rightarrow y > 0$.

$\therefore \quad$ range $(f) = (0, \infty)$.

Hence, dom $(f) = (0, \infty)$ and range $(f) = (0, \infty)$.

EXAMPLE 23 *Find the domain and the range of the real function,* $f(x) = \dfrac{1}{\sqrt{x - [x]}}$.

SOLUTION We have, $f(x) = \dfrac{1}{\sqrt{x-[x]}}$.

We know that

$0 \le x - [x] < 1$ for all $x \in R$

and $x - [x] = 0$ for all $x \in Z$.

$\therefore$ $0 < x - [x] < 1$ for all $x \in R - Z$

$\Rightarrow$ $f(x) = \dfrac{1}{\sqrt{x-[x]}}$ exists for all $x \in R - Z$

$\Rightarrow$ dom $(f) = R - Z$.

Also, $0 < x - [x] < 1$ for all $x \in R - Z$

$\Rightarrow$ $0 < \sqrt{x-[x]} < 1$ for all $x \in R - Z$

$\Rightarrow$ $1 < \dfrac{1}{\sqrt{x-[x]}} < \infty$ for all $x \in R - Z$

$\Rightarrow$ $1 < f(x) < \infty$ for all $x \in R - Z$

$\Rightarrow$ range $(f) = (1, \infty)$.

Hence, dom $(f) = R - Z$ and range $(f) = (1, \infty)$.

EXERCISE 3C

1. Find the domain of each of the following real functions:

(i) $f(x) = \dfrac{3x+5}{x^2-9}$ (ii) $f(x) = \dfrac{2x-3}{x^2+x-2}$

(iii) $f(x) = \dfrac{x^2-2x+1}{x^2-8x+12}$ (iv) $f(x) = \dfrac{x^3-8}{x^2-1}$

Find the domain and the range of each of the following real functions:

2. $f(x) = \dfrac{1}{x}$ **3.** $f(x) = \dfrac{1}{(x-5)}$ **4.** $f(x) = \dfrac{x-3}{2-x}$

5. $f(x) = \dfrac{3x-2}{x+2}$ **6.** $f(x) = \dfrac{x^2-16}{x-4}$ **7.** $f(x) = \dfrac{1}{\sqrt{2x-3}}$

8. $f(x) = \dfrac{ax-b}{cx-d}$ **9.** $f(x) = \sqrt{3x-5}$ **10.** $f(x) = \sqrt{\dfrac{x-5}{3-x}}$

11. $f(x) = \dfrac{1}{\sqrt{x^2-1}}$ **12.** $f(x) = 1 - |x-2|$ **13.** $f(x) = \dfrac{|x-4|}{x-4}$

14. $f(x) = \dfrac{x^2-9}{x-3}$ **15.** $f(x) = \dfrac{1}{2-\sin 3x}$

ANSWERS (EXERCISE 3C)

1. (i) $R - \{3, -3\}$ (ii) $R - \{1, -2\}$ (iii) $R - \{2, 6\}$ (iv) $R - \{1, -1\}$

2. dom $(f) = R - \{0\}$, range $(f) = R - \{0\}$

3. dom $(f) = R - \{5\}$, range $(f) = R - \{0\}$

4. dom $(f) = R - \{2\}$, range $(f) = R - \{-1\}$

5. dom $(f) = R - \{-2\}$, range $(f) = R - \{3\}$

6. dom $(f) = R - \{4\}$, range $(f) = R - \{8\}$

7. dom $(f) = R - \left(-\infty, \frac{3}{2}\right]$, range $(f) = R - \{0\}$

8. dom $(f) = R - \left\{\frac{d}{c}\right\}$, range $(f) = R - \left\{\frac{a}{c}\right\}$

9. dom $(f) = \left[\frac{5}{3}, \infty\right)$, range $(f) = R - [0, \infty)$

10. dom $(f) = (3, 5]$, range $(f) = [0, \infty)$

11. dom $(f) = (-\infty, -1) \cup (1, \infty)$, range $(f) = R - \{0\}$

12. dom $(f) = R$, range $(f) = (-\infty, 1]$

13. dom $(f) = R - \{4\}$, range $(f) = \{-1, 1\}$

14. dom $(f) = R - \{3\}$, range $(f) = R - \{6\}$

15. dom $(f) = R$, range $(f) = \left[\frac{1}{3}, 1\right]$

HINTS TO SOME SELECTED QUESTIONS

2. $f(x) = \frac{1}{x}$ is not defined when $x = 0$. So, dom $(f) = R - \{0\}$.

Let $y = f(x)$. Then, $y = \frac{1}{x} \Rightarrow x = \frac{1}{y}$.

$\therefore$ x is not defined when $y = 0$. So, range $(f) = R - \{0\}$.

3. $f(x) = \frac{1}{(x-5)}$ is not defined when $x = 5$. So, dom $(f) = R - \{5\}$.

Let $y = f(x)$. Then, $y = \frac{1}{(x-5)} \Rightarrow x - 5 = \frac{1}{y} \Rightarrow x = \left(\frac{1}{y} + 5\right)$.

$\therefore$ x is not defined when $y = 0$. So, range $(f) = R - \{0\}$.

4. $f(x) = \frac{x-3}{2-x}$ is not defined when $x = 2$. So, dom $(f) = R - \{2\}$.

Let $y = f(x)$. Then,

$$y = \frac{x-3}{2-x} \Rightarrow 2y - xy = x - 3 \Rightarrow xy + x = 2y + 3$$

$$\Rightarrow x(y+1) = 2y + 3 \Rightarrow x = \frac{2y+3}{y+1}.$$

$\therefore$ x is not defined when $y = -1$. So, range $(f) = R - \{-1\}$.

5. $f(x) = \frac{3x-2}{x+2}$ is not defined when $x = -2$. So, dom $(f) = R - \{-2\}$.

Let $y = f(x)$. Then,

$$y = \frac{3x-2}{x+2} \Rightarrow xy + 2y = 3x - 2 \Rightarrow 3x - xy = 2y + 2$$

$$\Rightarrow x(3-y) = 2y + 2 \Rightarrow x = \frac{2y+2}{3-y}. \quad \ldots \text{(i)}$$

It follows from (i) that x is not defined when $y = 3$.

$\therefore$ range $(f) = R - \{3\}$.

6. $f(x) = \dfrac{x^2 - 16}{x - 4}$ is not defined when $x = 4$. So, dom $(f) = R - \{4\}$.

Let $y = f(x)$. Then,

$$y = \frac{x^2 - 16}{x - 4} = (x + 4), \text{ when } x \neq 4$$

$\Rightarrow \quad y \neq (4 + 4) \Rightarrow y \neq 8$. So, range $(f) = R - \{8\}$.

7. $f(x) = \dfrac{1}{\sqrt{2x - 3}}$ is not defined when $2x - 3 \leq 0$, i.e., when $x \leq \dfrac{3}{2}\cdot$

$\therefore \quad$ dom $(f) = R - \left(-\infty, \dfrac{3}{2}\right]\cdot$

Let $y = f(x)$. Then,

$$y = \frac{1}{\sqrt{2x - 3}} \Rightarrow y^2 = \frac{1}{(2x - 3)} \Rightarrow (2x - 3) = \frac{1}{y^2} \Rightarrow x = \frac{1}{2}\left(\frac{1}{y^2} + 3\right)\cdot$$

$\therefore \quad x$ is not defined when $y = 0$. So, range $(f) = R - \{0\}$.

8. $f(x) = \dfrac{ax - b}{cx - d}$ is not defined when $cx - d = 0 \Rightarrow x = \dfrac{d}{c}\cdot$

$\therefore \quad$ dom $(f) = R - \left\{\dfrac{d}{c}\right\}\cdot$

Let $y = f(x)$. Then,

$$y = \frac{ax - b}{cx - d} \Rightarrow cxy - dy = ax - b \Rightarrow cxy - ax = dy - b$$

$$\Rightarrow x(cy - a) = dy - b \Rightarrow x = \frac{dy - b}{cy - a}\cdot$$

Clearly, x is not defined when $cy - a = 0 \Rightarrow y = \dfrac{a}{c}\cdot$

$\therefore \quad$ range $(f) = R - \left\{\dfrac{a}{c}\right\}\cdot$

9. $f(x) = \sqrt{3x - 5}$ is defined only when $3x - 5 \geq 0$, i.e., when $3x - 5 \geq 0$, i.e., when $x \geq \dfrac{5}{3}\cdot$

$\therefore \quad$ dom $(f) = \left[\dfrac{5}{3}, \infty\right)\cdot$

$x \geq \dfrac{5}{3} \Rightarrow f(x) \geq 0 \Rightarrow$ range $(f) = R - [0, \infty)$.

10. $f(x) = \sqrt{\dfrac{x - 5}{3 - x}}$ is defined only when $\dfrac{x - 5}{3 - x} \geq 0$.

Now, $\dfrac{x - 5}{3 - x} = 0 \Rightarrow x - 5 = 0 \Rightarrow x = 5$.

And, $\dfrac{x - 5}{3 - x} > 0 \Rightarrow (x - 5 < 0 \text{ and } 3 - x < 0)$ or $(x - 5 > 0 \text{ and } 3 - x > 0)$

$\Rightarrow (x < 5 \text{ and } 3 < x)$ or $(x > 5 \text{ and } 3 > x)$

$\Rightarrow (3 < x \text{ and } x < 5)$ or $(x > 5 \text{ and } x < 3)$

$\Rightarrow 3 < x < 5 \quad [\because x > 5 \text{ and } x < 3 \text{ cannot hold}]$.

Thus, $f(x)$ is defined when $3 < x \leq 5$.

$\therefore \quad$ dom $(f) = (3, 5]$.

Clearly, $f(x) = \sqrt{\dfrac{x - 5}{3 - x}} \geq 0$.

Let $y = f(x)$. Then,

$$y = \sqrt{\frac{x-5}{3-x}} \Rightarrow y^2 = \frac{x-5}{3-x} \Rightarrow 3y^2 - xy^2 = x - 5$$

$$\Rightarrow x + xy^2 = 3y^2 + 5 \Rightarrow x(1+y^2) = (3y^2+5) \Rightarrow x = \frac{3y^2+5}{1+y^2}. \quad \dots \text{(i)}$$

It follows from (i) that x is defined for each real value of y.

$\therefore$ range $(f) = [0, \infty)$ $\quad [\because f(x) \geq 0]$.

11. $f(x) = \dfrac{1}{\sqrt{x^2-1}}$ is defined only when $x^2 - 1 > 0$.

Now, $x^2 - 1 > 0 \Rightarrow (x+1)(x-1) > 0 \Rightarrow (x < -1)$ or $(x > 1)$

$\Rightarrow x \in (-\infty, -1)$ or $x \in (1, \infty)$.

$\therefore$ dom $(f) = (-\infty, -1) \cup (1, \infty)$.

Let $y = f(x)$. Then,

$$y = \frac{1}{\sqrt{x^2-1}} \Rightarrow y^2 = \frac{1}{(x^2-1)} \Rightarrow x^2 - 1 = \frac{1}{y^2} \Rightarrow x = \sqrt{\frac{1}{y^2}+1} = \sqrt{\frac{1+y^2}{y^2}}.$$

Clearly, x is not defined when $y = 0$.

$\therefore$ range $(f) = R - \{0\}$.

12. $f(x) = 1 - |x-2|$ is defined for all $x \in R$. So, dom $(f) = R$.

Now, $0 \leq |x-2| < \infty \Rightarrow |x-2| \geq 0$ and $|x-2| < \infty$

$\Rightarrow -|x-2| \leq 0$ and $-|x-2| > -\infty$

$\Rightarrow 1 - |x-2| \leq 1$ and $1 - |x-2| > -\infty$

$\Rightarrow -\infty < 1 - |x-2| \leq 1 \Rightarrow f(x) \in (-\infty, 1]$.

$\therefore$ range $(f) = (-\infty, 1]$.

13. $f(x) = \dfrac{|x-4|}{x-4}$ is not defined at $x = 4$. So, dom $(f) = R - \{4\}$.

Now, when $x \neq 4$, we have

$$f(x) = \begin{cases} 1, \text{ when } x - 4 > 0 & \{\because |x-4| = x-4\} \\ -1, \text{ when } x - 4 < 0 & \{\because |x-4| = -(x-4)\}. \end{cases}$$

$\therefore$ range $(f) = [-1, 1]$.

14. $f(x) = \dfrac{x^2-9}{x-3}$ is defined for all real values of x for which $x \neq 3$.

$\therefore$ dom $(f) = R - \{3\}$.

Let $y = f(x)$. Then,

$$y = \frac{x^2-9}{x-3} \Rightarrow y = x + 3, \text{ when } x \neq 3$$

$$\Rightarrow y \neq 3 + 3 \Rightarrow y \neq 6.$$

$\therefore$ range $(f) = R - \{6\}$.

15. Clearly, $-1 \leq -\sin 3x \leq 1$ for all $x \in R$

$\Rightarrow$ $2 + (-1) \leq 2 + (-\sin 3x) \leq 2 + 1$ for all $x \in R$

$\Rightarrow$ $1 \leq 2 - \sin 3x \leq 3$ for all $x \in R$

$\Rightarrow$ $2 - \sin 3x \neq 0$ for any $x \in R$.

$\therefore$ $f(x) = \dfrac{1}{(2 - \sin 3x)}$ is defined for all $x \in R$.

We have, $2-\sin 3x \geq 1$ and $2-\sin 3x \leq 3$

$\therefore \quad \frac{1}{2-\sin 3x} \leq 1$ and $\frac{1}{2-\sin 3x} \geq \frac{1}{3} \Rightarrow \frac{1}{3} \leq \frac{1}{2-\sin 3x} \leq 1.$

$\therefore \quad$ range $(f) = \left[\frac{1}{3}, 1\right]$.

GRAPHS OF SOME STANDARD REAL FUNCTIONS

1. IDENTITY FUNCTION

The function $f: R \to R: f(x) = x$ for all $x \in R$ is called an *identity function* on R.

Dom $(f) = R$ and range $(f) = R$.

EXAMPLE 1 *Draw the graph of the identity function $f: R \to R: f(x) = x$ for all $x \in R$.*

SOLUTION Here $f(x) = x$ for all $x \in R$.

We may prepare its table as under.

x	–2	–1	0	1	2	3
$f(x) = x$	–2	–1	0	1	2	3

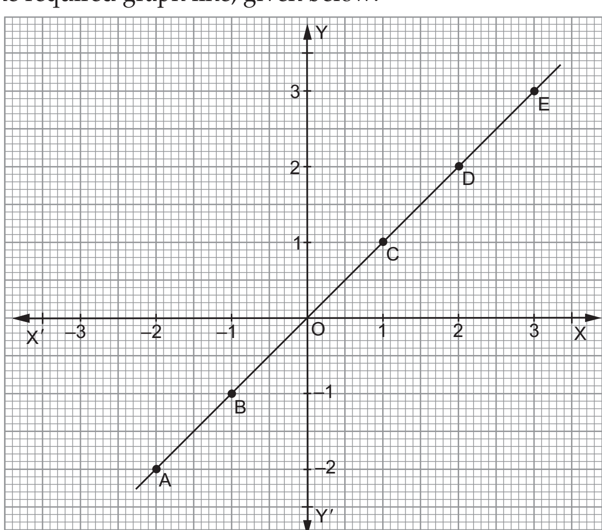
On a graph paper, we draw $X'OX$ and YOY' as the x-axis and the y-axis respectively.

We take 10 small divisions = 1 unit.

Now, on this graph paper, we plot the points

$$A(-2, -2), B(-1, -1), O(0, 0), C(1, 1), D(2, 2) \text{ and } E(3,3).$$

Join these points successively to obtain the straight line $ABOCDE$ as the required graph line, given below.



Graph of the identity function, $f(x) = x$

2. CONSTANT FUNCTION

Let c be a fixed real number.

Then, the function $f : R \to R : f(x) = c$ for all $x \in R$ is called the *constant function*.

For the constant function $f(x) = c$ for all $x \in R$, we have

dom $(f) = R$ and range $(f) = \{c\}$.

EXAMPLE 2 *Draw the graph of each of the following constant functions:*

(i) $f(x) = 2$ *for all* $x \in R$

(ii) $f(x) = 0$ *for all* $x \in R$

(iii) $f(x) = -2$ *for all* $x \in R$

SOLUTION (i) Let $f(x) = 2$ for all $x \in R$.

Then, dom $(f) = R$ and range $(f) = \{2\}$.

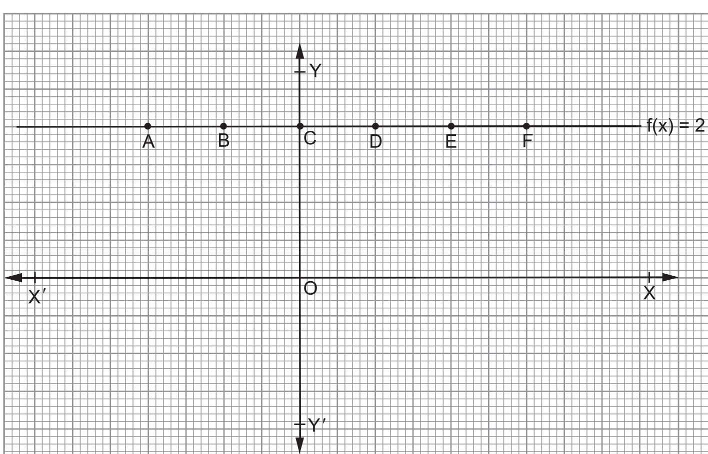
We may prepare the table given below.

x	–2	–1	0	1	2	3
$f(x) = 2$	2	2	2	2	2	2

On a graph paper, we draw $X'OX$ and YOY' as the x-axis and the y-axis respectively.

We take the scale: 10 small divisions = 1 unit.

On this graph paper, we plot the points $A(-2, 2)$, $B(-1, 2)$, $C(0, 2)$, $D(1, 2)$, $E(2, 2)$ and $F(3, 2)$. Join A, B, C, D, E and F successively to obtain the required graph line $ABCDEF$, whose equation is $y = 2$.



Graph of the function, $f(x) = 2$

(ii) Let $f(x) = 0$ for all $x \in R$. Then,

$$\text{dom}(f) = R \text{ and range}(f) = \{0\}.$$

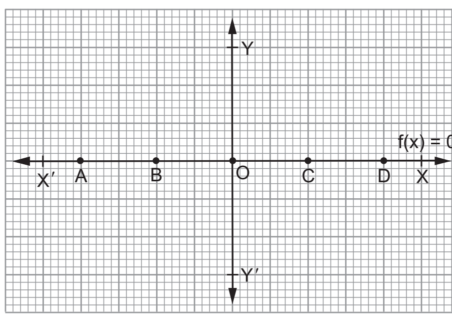
Now, we have:

x	–2	–1	0	1	2
$f(x) = 0$	0	0	0	0	0

On a graph paper, we draw the horizontal line $X'OX$ as the x-axis and the vertical line YOY' as the y-axis.

We take the scale: 10 small divisions = 1 unit.

Now, on this graph paper, we plot the points $A(-2, 0)$, $B(-1, 0)$, $O(0, 0)$, $C(1, 0)$ and $D(2, 0)$ and join them successively to get the graph line $ABOCD$, whose equation is $y = 0$.



Graph of the function, $f(x) = 0$

(iii) Let $f(x) = -2$ for all $x \in R$. Then,

$$\text{dom}(f) = R \text{ and range}(f) = -2.$$

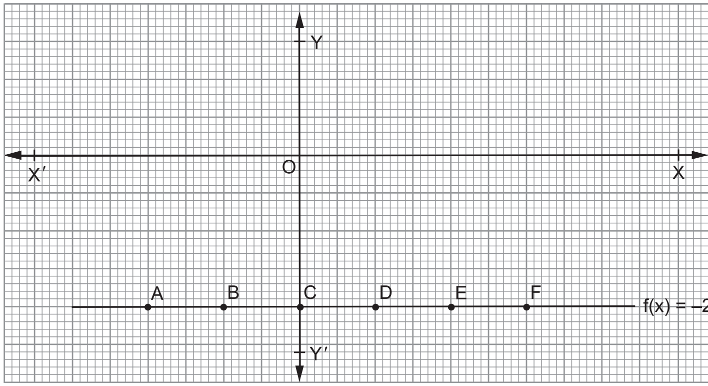
Now, we have:

x	–2	–1	0	1	2	3
$f(x) = -2$	–2	–2	–2	–2	–2	–2

On a graph paper, we draw the horizontal line $X'OX$ as the x-axis and the vertical line YOY' as the y-axis.

We take the scale: 10 small divisions = 1 unit.

On this graph paper, we plot the points $A(-2, -2)$, $B(-1, -2)$, $C(0, -2)$, $D(1, -2)$, $E(2, -2)$ and $F(3, -2)$ and join these points successively to get the required graph line $ABCDE$, as shown below. Its equation is $y = -2$.



Graph of the function, $f(x) = -2$

3. THE MODULUS FUNCTION

The function f, defined by $f(x) = |x| = \begin{cases} x, \text{ when } x \geq 0 \\ -x, \text{ when } x < 0 \end{cases}$ *is called the modulus function.*

We know that $|x| \geq 0$ for all $x \in R$.

$\therefore$ dom $(f) = R$ and range $(f) = [0, \infty)$.

EXAMPLE 3 *Draw the graph of the modulus function, defined by*

$$f : R \to R : f(x) = |x| = \begin{cases} x, \text{ when } x \geq 0 \\ -x, \text{ when } x < 0. \end{cases}$$

SOLUTION We have

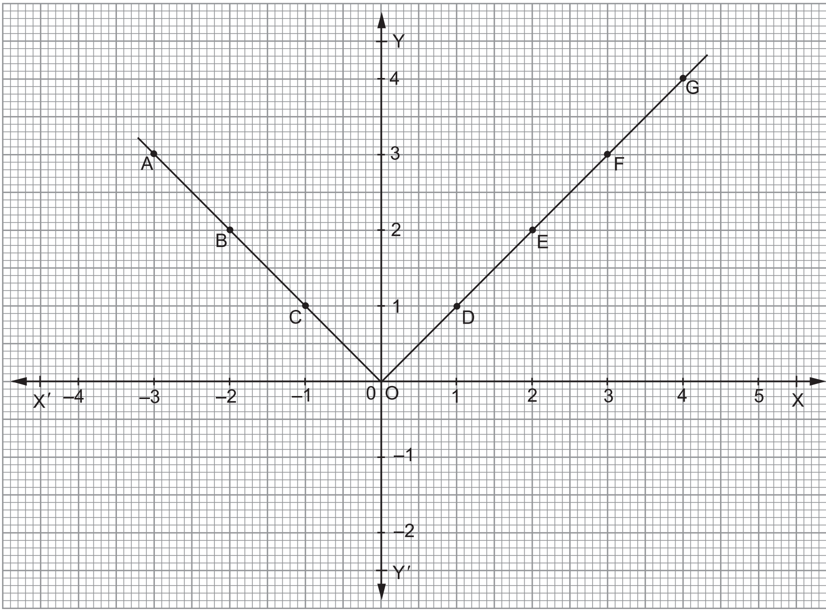
x	–3	–2	–1	0	1	2	3	4
$f(x) = \|x\|$	3	2	1	0	1	2	3	4

On a graph paper draw the horizontal line $X'OX$ as the x-axis and the vertical line YOY' as the y-axis.

Take the scale: 10 small divisions = 1 unit.

On this graph paper, plot the points $A(-3, 3)$, $B(-2, 2)$, $C(-1, 1)$, $O(0, 0)$, $D(1, 1)$, $E(2, 2)$, $F(3, 3)$ and $G(4, 4)$.

Join these points successively to obtain the graph lines $ABCO$ and $ODEFG$, as shown below.



Graph of $f(x) = |x|$

4. THE GREATEST INTEGER FUNCTION, OR STEP FUNCTION

For any real number x, the symbol $[x]$ *denotes the greatest integer less than or equal to x.*

EXAMPLES (i) $[6.85] = 6$ (ii) $[8] = 8$ (iii) $[0.536] = 0$ (iv) $[-5.8] = -6$ (v) $[-5] = -5$

The function $f: R \to R: f(x) = [x]$ *for all* $x \in R$ is called the *greatest integer function, or step function.*

Clearly, dom $(f) = R$ and range $(f) = Z$.

EXAMPLE 4 *Draw the graph of the greatest integer function:*

$$f: R \to R: f(x) = [x] \text{ for all } x \in R.$$

SOLUTION Using the definition of $[x]$, we have

$[x] = -2$ for $-2 \le x < -1$

$[x] = -1$ for $-1 \le x < 0$

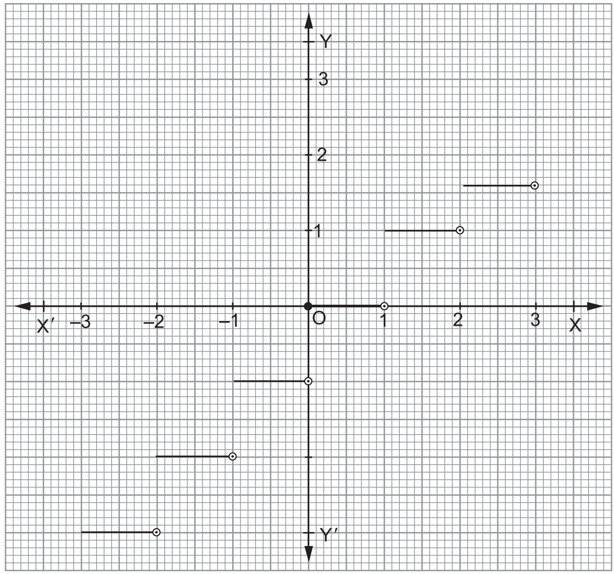
$[x] = 0$ for $0 \le x < 1$

$[x] = 1$ for $1 \le x < 2$

$[x] = 2$ for $2 \le x < 3$

$[x] = 3$ for $3 \le x < 4$.

Taking the scale: 10 small divisions = 1 unit, the graph of the function $f(x) = [x]$ may be drawn as shown below.



Graph of greatest integer function [*x*]

5. THE SMALLEST INTEGER FUNCTION

For any real number x, the smallest integer greater than or equal to x is denoted by $\lceil x \rceil$.

For example: $\lceil 3.8 \rceil = 4$, $\lceil 6 \rceil = 6$, $\lceil 0.76 \rceil = 1$, $\lceil -6.2 \rceil = -6$, etc.

The function $f : R \to R : f(x) = \lceil x \rceil$ *for all* $x \in R$ is called the *smallest integer function.*

Clearly, dom $(f) = R$, range $(f) = Z$.

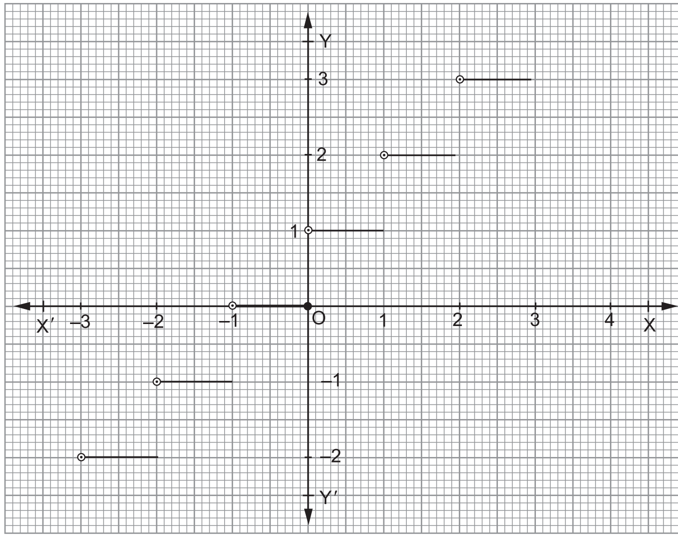
EXAMPLE 5 *Draw the graph of the smallest integer function* $f : R \to R : f(x) = \lceil x \rceil$ *for all* $x \in R$.

SOLUTION Using the definition of $\lceil x \rceil$, we have

x	$f(x) = \lceil x \rceil$
$-3 < x \leq -2$	-2
$-2 < x \leq -1$	-1
$-1 < x \leq 0$	0
$0 < x \leq 1$	1
$1 < x \leq 2$	2
$2 < x \leq 3$	3

SCALE: 10 small divisions = 1 unit.

Now, the graph may be drawn as shown below.



Graph of smallest integer function $\lceil x \rceil$

6. THE FRACTIONAL PART FUNCTION

Let $f : R \to R : f(x) = x - [x] = \{x\}$.

Here $f(x) = \{x\}$ denotes the fractional part of x.

For example: $\{2.35\} = 0.35$, $\{6\} = 0$, $\{-7\} = 0$, $\{-0.65\} = 0.35$, $\{-3.75\} = 0.25$, etc.

Clearly, dom $(f) = R$ and range $(f) = [0, 1)$.

EXAMPLE 6 *Draw the graph of the fractional part function:*

$$f : R \to R : f(x) = x - [x] = \{x\}$$

SOLUTION We have, $f(x) = x - [x] = \{x\}$.

$f(3) = 0$, $f(3.1) = 0.1$, $f(3.2) = 0.2, \ldots, f(3.9) = 0.9$, $f(4) = 0$.

$f(2) = 0$, $f(2.1) = 0.1$, $f(2.2) = 0.2, \ldots, f(2.9) = 0.9$, $f(3) = 0$.

$f(1) = 0$, $f(1.1) = 0.1$, $f(1.2) = 0.2, \ldots, f(1.9) = 0.9$, $f(2) = 0$.

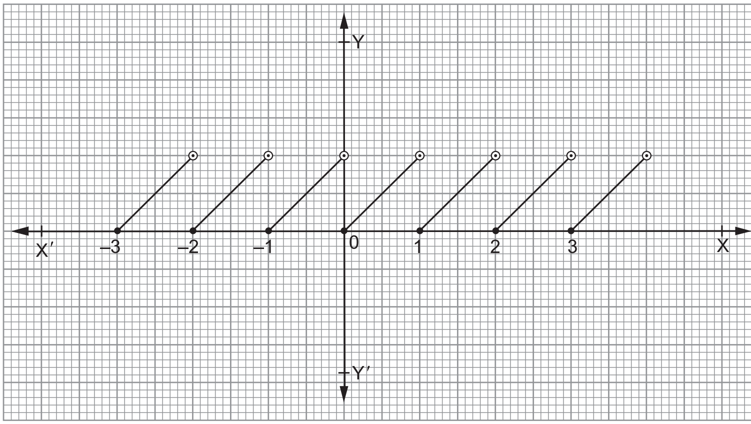
$f(0) = 0$, $f(0.1) = 0.1$, $f(0.2) = 0.2, \ldots, f(0.9) = 0.9$, $f(1) = 0$.

$f(-1) = 0$, $f(-0.9) = 0.1$, $f(-0.8) = 0.2, \ldots, f(-0.1) = 0.9$, $f(0) = 0$.

$f(-2) = 0$, $f(-1.9) = 0.1$, $f(-1.8) = 0.2, \ldots, f(-1.1) = 0.9$, $f(-1) = 0$.

$f(-3) = 0$, $f(-2.9) = 0.1$, $f(-2.8) = 0.2, \ldots, f(-2.1) = 0.9$, $f(-2) = 0$.

Plotting these points on a graph paper, we may get the graph as shown below.



Graph of fractional part function, $f(x) = \{x\}$

7. THE SIGNUM FUNCTION

The function $f : R \to R : f(x) = \begin{cases} \dfrac{x}{|x|}, \text{ when } x \neq 0 \\ 0, \text{ when } x = 0 \end{cases}$ *is called the* ***signum function.***

The above function may be simplified as under:

$$f(x) = \begin{cases} 1, \text{ if } x > 0 \\ 0, \text{ if } x = 0 \\ -1, \text{ if } x < 0. \end{cases}$$

Clearly, dom $(f) = R$ and range $(f) = \{-1, 0, 1\}$.

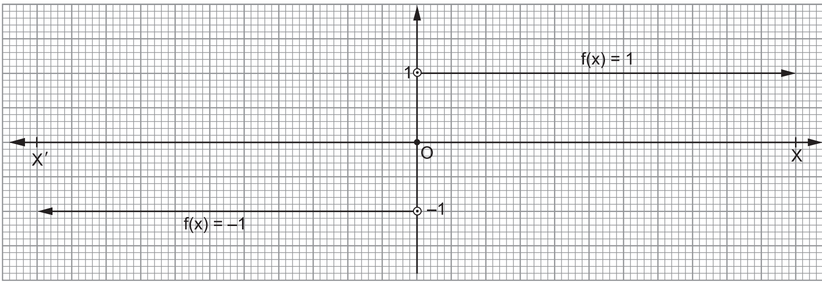
EXAMPLE 7 *Draw the graph of the signum function,* $f : R \to R$, *defined by*

$$f(x) = \begin{cases} \dfrac{x}{|x|}, \text{ when } x \neq 0 \\ 0, \text{ when } x = 0 \end{cases} \quad or \quad f(x) = \begin{cases} 1, \text{ if } x > 0 \\ 0, \text{ if } x = 0 \\ -1, \text{ if } x < 0 \end{cases}$$

SOLUTION Clearly, we have

$$x < 0 \Rightarrow f(x) = -1$$
$$x = 0 \Rightarrow f(x) = 0$$
$$x > 0 \Rightarrow f(x) = 1.$$

We may now draw the graph as shown below.



Graph of signum function

8. POLYNOMIAL FUNCTION

A function of the form $f(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n$, where n is a non-negative integer and $a_0, a_1, a_2, \ldots, a_n \in R$, is called a *polynomial function*.

EXAMPLES (i) $f(x) = x^3 - 2x^2 + 5$ is a polynomial function.

(ii) $g(x) = 4x^2 - x + 2$ is a polynomial function.

(iii) $h(x) = 2x - 3\sqrt{x} + 5$ is not a polynomial function.

EXAMPLE 8 *Let $f : R \to R : f(x) = x^2$ for all $x \in R$.*

Find its domain and range.

Also, draw its graph.

SOLUTION Here, $f : R \to R : f(x) = x^2$ for all $x \in R$.

Dom $(f) = R$ and range $(f) = \{x \in R : x \geq 0\} = [0, \infty)$.

Now, we have:

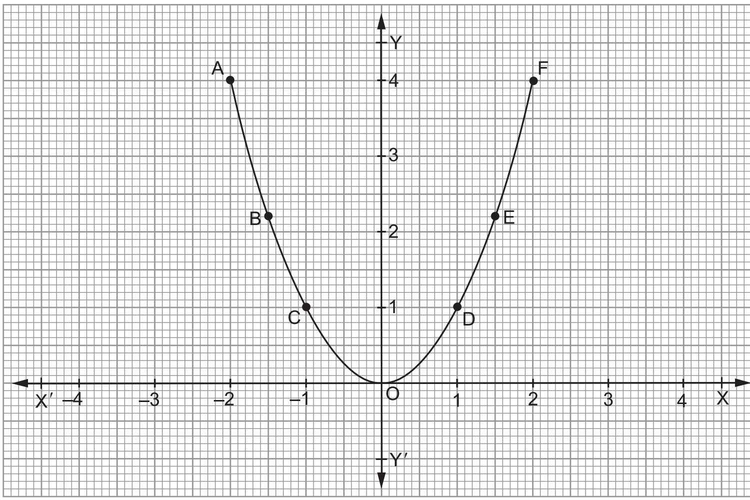
x	–2	–1.5	–1	0	1	1.5	2
$f(x) = x^2$	4	2.25	1	0	1	2.25	4

On a graph paper, we draw $X'OX$ and YOY' as the x-axis and the y-axis respectively.

Now, we plot the points $A(-2, 4)$, $B(-1.5, 2.25)$, $C(-1, 1)$, $O(0, 0)$, $D(1, 1)$, $E(1.5, 2.25)$ and $F(2, 4)$.

Join these points freehand successively to obtain the required curve.

Scale: 10 small divisions = 1 unit.



Graph of $f(x) = x^2$

EXAMPLE 9 *Let $f: R \to R: f(x) = x^3$ for all $x \in R$. Find its domain and range. Also, draw its graph.*

SOLUTION Let $f: R \to R: f(x) = x^3$ for all $x \in R$. Then,

dom $(f) = R$ and range $(f) = R$.

Now, we have:

x	–2	–1.5	–1	0	1	1.5	2
$f(x) = x^3$	–8	–3.375	–1	0	1	3.375	8

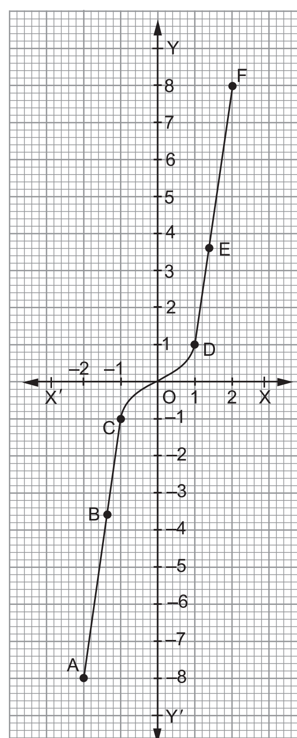
On a graph paper, we draw $X'OX$ and YOY' as the x-axis and the y-axis respectively.

We take the scale as

5 small divisions = 1 unit.

Now, we plot the points $A(-2, -8)$, $B(-1.5, -3.375)$, $C(-1, -1)$, $O(0, 0)$, $D(1, 1)$, $E(1.5, 3.375)$ and $F(2, 8)$.

We join these points freehand successively to obtain the required curve.



9. RATIONAL FUNCTION

The functions of the type $\dfrac{f(x)}{g(x)}$, where $f(x)$ and $g(x)$ are polynomial functions of x, defined in a domain, where $g(x) \neq 0$.

EXAMPLE 10 *Let* $f:(R-\{0\}) \rightarrow R: f(x)=\dfrac{1}{x}$ *for all values of* $x \in R-\{0\}$. *Find its domain and range. Also, draw its graph.*

SOLUTION Let $f:(R-\{0\}) \rightarrow R: f(x)=\dfrac{1}{x}$ · Then,

$$\text{dom}(f)=R-\{0\} \text{ and } \text{range}(f)=R-\{0\}.$$

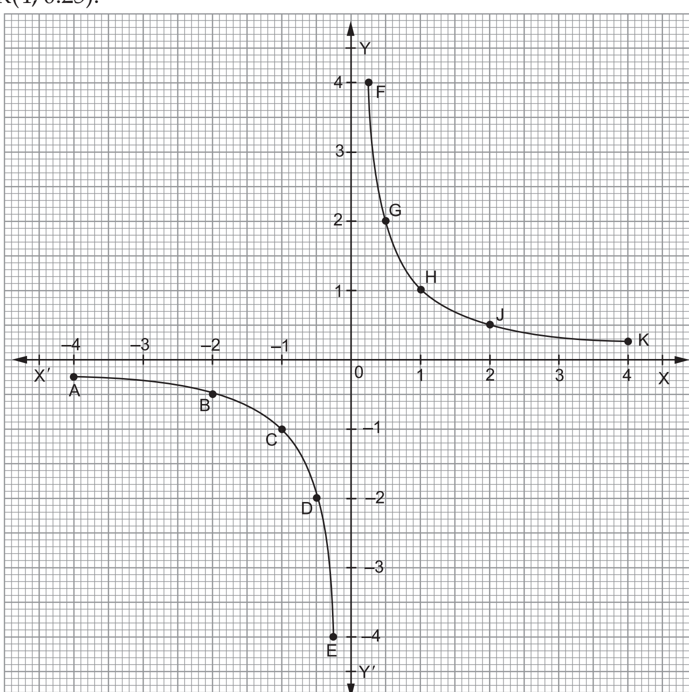
We have:

x	–4	–2	–1	–0.5	–0.25	0.25	0.5	1	2	4
$f(x)=\dfrac{1}{x}$	–0.25	–0.5	–1	–2	–4	4	2	1	0.5	0.25

On a graph paper, let $X'OX$ and YOY' be the x-axis and the y-axis respectively.

We take the scale: 10 small divisions = 1 unit.

Now, we plot the points $A(-4, -0.25)$, $B(-2, -0.5)$, $C(-1, -1)$, $D(-0.5, -2)$, $E(-0.25, -4)$ and $F(0.25, 4)$, $G(0.5, 2)$, $H(1, 1)$, $J(2, 0.5)$ and $K(4, 0.25)$.



Graph of the function, $f(x)=\dfrac{1}{x}$

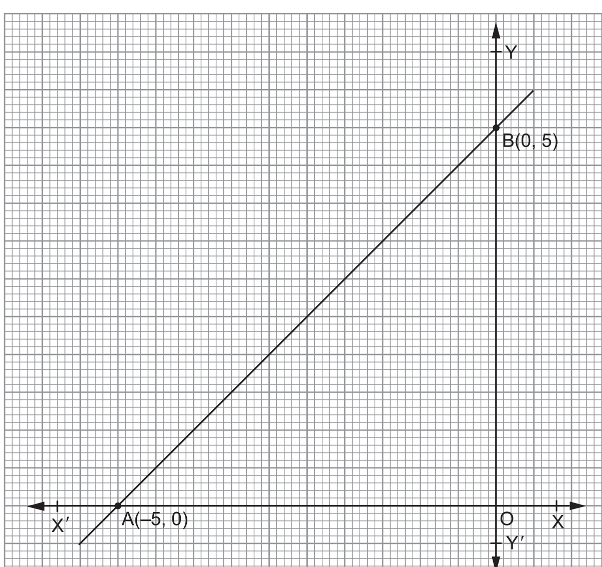
We join A, B, C, D and E freehand successively to obtain a curve. Also, we join F, G, H, J and K freehand successively to obtain another curve, as shown above.

EXERCISE 3D

1. Consider the real function $f : R \to R : f(x) = x + 5$ for all $x \in R$.
   Find its domain and range.
   Draw the graph of this function.
2. Consider the function $f : R \to R$, defined by
   $$f(x) = \begin{cases} 1 - x, \text{ when } x < 0 \\ 1, \text{ when } x = 0 \\ x + 1, \text{ when } x > 0. \end{cases}$$
   Write its domain and range.
   Also, draw the graph of $f(x)$.
3. Find the domain and the range of the square root function,
   $f : R^+ \cup \{0\} \to R : f(x) = \sqrt{x}$ for all non-negative real numbers.
   Also, draw its graph.
4. Find the domain and the range of the cube root function,
   $f : R \to R : f(x) = x^{1/3}$ for all $x \in R$.
   Also, draw its graph.

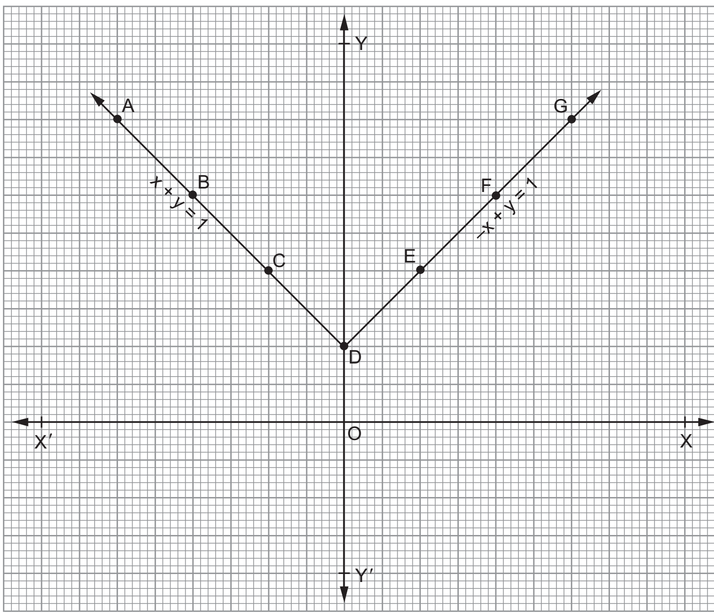
HINTS TO SOME SELECTED QUESTIONS

1. Clearly, dom $(f) = R$ and range $(f) = R$.



Now, $y = x + 5$.
Putting $x = 0$, we get $y = 5$.
Putting $y = 0$, we get $x = -5$.
On the graph paper, plot the points $A(-5, 0)$ and $B(0, 5)$.
Join AB to get the required graph.

2. $x < 0 \Rightarrow y = 1 - x$.
$x = 1 \Rightarrow y = 1$.
$x > 1 \Rightarrow y = x + 1$.



(i) For $x < 0$, we have $y = 1 - x$.

x	–3	–2	–1
y	4	3	2

(ii) For $x = 0$, we have $y = 1$.

x	0
y	1

$[\because\ x = 0 \Rightarrow y = 1]$

(iii) For $x > 0$, we have $y = x + 1$.

x	1	2	3
y	2	3	4

On a graph paper, draw horizontal line $X'OX$ and the vertical line YOY'. Now, plot the points $A(-3, 4)$, $B(-2, 3)$, $C(-1, 2)$, $D(0, 1)$, $E(1, 2)$, $F(2, 3)$ and $G(3, 4)$.
Join $ABCD$ and $DEFG$ to get the required graph.

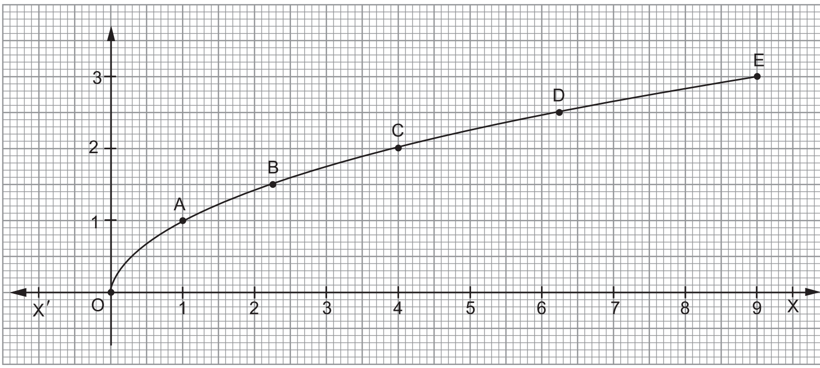
3. Let $f : R^+ \cup \{0\} \to R : f(x) = \sqrt{x}$. Then,

$\text{dom}(f) = [0, \infty)$ and range $(f) = [0, \infty)$.

We have

$\sqrt{0} = 0, \sqrt{1} = 1, \sqrt{2.25} = 1.5, \sqrt{4} = 2, \sqrt{6.25} = 2.5$ and $\sqrt{9} = 3$.

On a graph paper, we plot the points $O(0, 0)$, $A(1, 1)$, $B(2.25, 1.5)$, $C(4, 2)$, $D(6.25, 2.5)$, and $E(9, 3)$. Join these points freehand successively to get the required curve.



Graph of the curve, $f(x) = \sqrt{x}$

4. Let $f : R \to R : f(x) = x^{1/3}$ for all $x \in R$.

Clearly, dom $(f) = R$ and range $(f) = R$.

We have

$\sqrt[3]{8} = 2, \sqrt[3]{5} = 1.7, \sqrt[3]{3} = 1.4, \sqrt[3]{1} = 1, \sqrt[3]{0} = 0, \sqrt[3]{-1} = -1, \sqrt[3]{-3} = -1.4, \sqrt[3]{-5} = -1.7,$ and $\sqrt[3]{-8} = -2$.

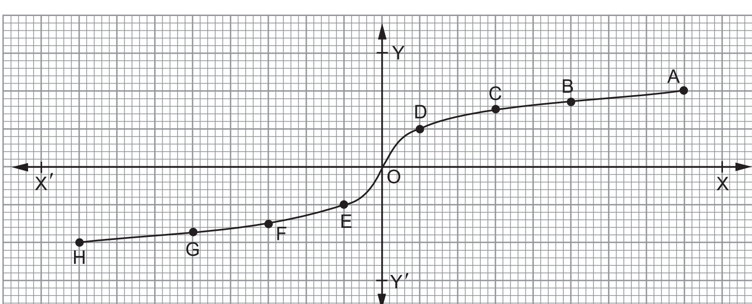
On a graph paper, take the horizontal line $X'OX$ as the x-axis and the vertical line YOY' as the y-axis.

Take the scale: 5 small divisions = 1 unit.

Now, plot the points $A(8, 2)$, $B(5, 1.7)$, $C(3, 1.4)$, $D(1, 1)$, $O(0, 0)$, $E(-1, -1)$, $F(-3, -1.4)$, $G(-5, -1.7)$ and $H(-8, -2)$.

Join $ABCDOEFGH$ freehand to get the required curve.

Remember: $\sqrt[3]{3} = 1.4$ and $\sqrt[3]{5} = 1.7$.



OPERATIONS ON FUNCTIONS

(i) SUM OF TWO REAL FUNCTIONS

Let $f: D_1 \to R$ and $g: D_2 \to R$, where $D_1 \subseteq R$ and $D_2 \subseteq R$. Then,

$$(f+g):(D_1 \cap D_2) \to R:(f+g)(x) = f(x)+g(x) \text{ for all } x \in (D_1 \cap D_2).$$

(ii) DIFFERENCE OF TWO REAL FUNCTIONS

Let $f: D_1 \to R$ and $g: D_2 \to R$, where $D_1 \subseteq R$ and $D_2 \subseteq R$. Then,

$$(f-g):(D_1 \cap D_2) \to R:(f-g)(x) = f(x)-g(x) \text{ for all } x \in (D_1 \cap D_2).$$

(iii) MULTIPLICATION OF TWO REAL FUNCTIONS

Let $f: D_1 \to R$ and $g: D_2 \to R$, where $D_1 \subseteq R$ and $D_2 \subseteq R$. Then,

$$(fg):(D_1 \cap D_2) \to R:(fg)(x) = f(x) \cdot g(x) \text{ for all } x \in (D_1 \cap D_2).$$

NOTE This is known as pointwise multiplication of functions.

(iv) QUOTIENT OF TWO REAL FUNCTIONS

Let $f: D_1 \to R$ and $g: D_2 \to R$, where $D_1 \subseteq R$ and $D_2 \subseteq R$.

Let $D = (D_1 \cap D_2) - \{x : g(x) = 0\}$. Then,

$$\left(\frac{f}{g}\right): D \to R : \left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} \text{ for all } x \in D.$$

(v) RECIPROCAL OF A FUNCTION

Let $f: D_1 \to R$ and let $D = D_1 - \{x : f(x) = 0\}$.

Then, the reciprocal of f is the function $\frac{1}{f}$, defined by

$$\frac{1}{f}: D \to R : \left(\frac{1}{f}\right)(x) = \frac{1}{f(x)} \text{ for all } x \in D.$$

(vi) SCALAR MULTIPLE OF A FUNCTION

Let $f: D \to R$ and let α be a scalar (a real number). Then,

$$(\alpha f): D \to R : (\alpha f)(x) = \alpha \cdot f(x) \text{ for all } x \in D.$$

SOLVED EXAMPLES

EXAMPLE 1 *Let* $f: R \to R : f(x) = x^2$ *and* $g: R \to R : g(x) = 2x+1$.

Find (i) $(f+g)(x)$ *(ii)* $(f-g)(x)$ *(iii)* $(fg)(x)$ *(iv)* $\left(\frac{f}{g}\right)(x)$

SOLUTION Here, dom $(f) = R$ and dom $(g) = R$.

$\therefore$ dom $(f) \cap$ dom $(g) = (R \cap R) = R$.

(i) $(f+g): R \to R$ is given by

$(f+g)(x) = f(x)+g(x) = x^2+(2x+1) = (x+1)^2$.

(ii) $(f-g): R \to R$ is given by

$(f-g)(x) = f(x)-g(x) = x^2-(2x+1) = (x^2-2x-1)$.

(iii) $(fg): R \to R$ is given by

$(fg)(x) = f(x) \cdot g(x) = x^2 \cdot (2x+1) = (2x^3 + x^2)$.

(iv) $\{x : g(x) = 0\} = \{x : 2x + 1 = 0\} = \left\{\frac{-1}{2}\right\}$.

$\therefore \quad \text{dom}\left(\frac{f}{g}\right) = R \cap R - \left\{\frac{-1}{2}\right\} = R - \left\{\frac{-1}{2}\right\}$.

The function $\frac{f}{g} : R - \left\{\frac{-1}{2}\right\} \to R$ is given by

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} = \frac{x^2}{2x+1},\ x \neq \frac{-1}{2}.$$

EXAMPLE 2 *Let $f(x) = \sqrt{x}$ and $g(x) = x$ be two functions defined over the set of non-negative real numbers. Find:*

(i) $(f+g)(x)$ (ii) $(f-g)(x)$ (iii) $(fg)(x)$ (iv) $\frac{f}{g}(x)$

SOLUTION Here $f : [0, \infty) \to R : f(x) = \sqrt{x}$ and $g : [0, \infty) \to R : g(x) = x$.

$\therefore \quad \text{dom}(f) = [0, \infty)$ and $\text{dom}(g) = [0, \infty)$.

So, $\text{dom}(f) \cap \text{dom}(g) = [0, \infty) \cap [0, \infty) = [0, \infty)$.

(i) $(f+g) : [0, \infty) \to R$ is given by

$(f+g)(x) = f(x) + g(x) = (\sqrt{x} + x)$.

(ii) $(f-g) : [0, \infty) \to R$ is given by

$(f-g)(x) = f(x) - g(x) = (\sqrt{x} - x)$.

(iii) $(fg) : [0, \infty) \to R$ is given by

$(fg)(x) = f(x) \cdot g(x) = (\sqrt{x} \times x) = x^{3/2}$.

(iv) $\{x : g(x) = 0\} = \{0\}$.

$$\therefore \quad \text{dom}\left(\frac{f}{g}\right) = \text{dom}(f) \cap \text{dom}(g) - \{x : g(x) = 0\}$$
$$= [0, \infty) \cap [0, \infty) - \{0\} = (0, \infty).$$

So, $\frac{f}{g} : (0, \infty) \to R$ is given by

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} = \frac{\sqrt{x}}{x} = \frac{1}{\sqrt{x}},\ x \neq 0.$$

EXAMPLE 3 *Let f and g be real functions, defined by $f(x) = \frac{1}{(x+4)}$ and $g(x) = (x+4)^3$.*

Find (i) $(f+g)(x)$ (ii) $(f-g)(x)$ (iii) $(fg)(x)$ (iv) $\left(\frac{f}{g}\right)(x)$ (v) $\left(\frac{1}{f}\right)(x)$

SOLUTION Clearly, $f(x) = \frac{1}{(x+4)}$ is defined for all real values of x except that at which $x + 4 = 0$, i.e., $x = -4$.

$\therefore \quad \text{dom}(f) = R - \{-4\}$.

And, $g(x) = (x+4)^3$ is defined for all $x \in R$. So, dom $(g) = R$.

$\therefore$ dom $(f) \cap$ dom $(g) = R - \{-4\} \cap R = R - \{-4\}$.

(i) $(f+g) : R - \{-4\} \to R$ is given by

$$(f+g)(x) = f(x) + g(x) = \frac{1}{(x+4)} + (x+4)^3 = \frac{1+(x+4)^4}{(x+4)}.$$

(ii) $(f-g) : R - \{-4\} \to R$ is given by

$$(f-g)(x) = f(x) - g(x) = \frac{1}{(x+4)} - (x+4)^3 = \frac{1-(x+4)^4}{(x+4)}.$$

(iii) $(fg) : R - \{-4\} \to R$ is given by

$$(fg)(x) = f(x) \cdot g(x) = \frac{1}{(x+4)} \times (x+4)^3 = (x+4)^2.$$

(iv) $\{x : g(x) = 0\} = \{x : (x+4)^3 = 0\} = \{x : x+4 = 0\} = \{-4\}$.

$$\therefore \quad \text{dom}\left(\frac{f}{g}\right) = \text{dom}(f) \cap \text{dom}(g) - \{x : g(x) = 0\} = R - \{-4\}.$$

$\left(\frac{f}{g}\right) : R - \{-4\} \to R$ is given by

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} = \frac{\frac{1}{(x+4)}}{(x+4)^3} = \frac{1}{(x+4)^4},\ x \neq -4.$$

(v) Clearly, $f(x) \neq 0$ for any $x \in R - \{-4\}$.

$\therefore$ $\frac{1}{f} : R - \{-4\} \to R$ is given by

$$\left(\frac{1}{f}\right)(x) = \frac{1}{f(x)} = \frac{1}{\frac{1}{(x+4)}} = (x+4).$$

EXAMPLE 4 *Let f and g be real functions defined by* $f(x) = \sqrt{x-1}$ *and* $g(x) = \sqrt{x+1}$.

Find (i) $(f+g)(x)$ *(ii)* $(f-g)(x)$ *(iii)* $(fg)(x)$ *(iv)* $\left(\frac{f}{g}\right)(x)$.

SOLUTION Clearly, $f(x) = \sqrt{x-1}$ is defined for all real values of x for which $x - 1 \geq 0$, i.e., $x \geq 1$. So, dom $(f) = [1, \infty)$.

Also, $g(x) = \sqrt{x+1}$ is defined for all real values of x for which $x + 1 \geq 0$, i.e., $x \geq -1$. So, dom $(g) = [-1, \infty)$.

$\therefore$ dom $(f) \cap$ dom $(g) = \{1, \infty) \cap [-1, \infty) = [1, \infty)$.

(i) $(f+g) : [1, \infty) \to R$ is given by

$(f+g)(x) = f(x) + g(x) = (\sqrt{x-1} + \sqrt{x+1})$.

(ii) $(f-g) : [1, \infty) \to R$ is given by

$(f-g)(x) = f(x) - g(x) = (\sqrt{x-1} - \sqrt{x+1})$.

(iii) $(fg) : [1, \infty) \to R$ is given by

$(fg)(x) = f(x) \cdot g(x) = \sqrt{x-1} \times \sqrt{x+1} = \sqrt{x^2-1}$.

(iv) $\{x : g(x) = 0\} = \{x : \sqrt{x+1} = 0\} = \{x : x+1 = 0\} = \{-1\}$.

$\therefore$ $\quad \text{dom}(f) \cap \text{dom}(g) - \{x : g(x) = 0\}$
$$= [1, \infty) \cap [-1, \infty) - \{-1\} = [1, \infty).$$

$\therefore$ $\quad \dfrac{f}{g} \to [1, \infty) \to R$ is given by
$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} = \frac{\sqrt{x-1}}{\sqrt{x+1}}.$$

EXAMPLE 5 *Let f and g be real functions, defined by* $f(x) = \sqrt{x+2}$ *and* $g(x) = \sqrt{4-x^2}$. *Find (i)* $(f+g)(x)$ *(ii)* $(f-g)(x)$ *(iii)* $(fg)(x)$ *(iv)* $(ff)(x)$

(v) $(gg)(x)$ *(vi)* $\left(\dfrac{f}{g}\right)(x)$.

SOLUTION Clearly, $f(x) = \sqrt{x+2}$ is defined for all $x \in R$ such that

$x + 2 \geq 0$, i.e., $x \geq -2$.

$\therefore$ $\quad \text{dom}(f) = [-2, \infty)$.

Again, $g(x) = \sqrt{4-x^2}$ is defined for all $x \in R$ such that

$4 - x^2 \geq 0$.

But, $4 - x^2 \geq 0 \Rightarrow x^2 - 4 \leq 0 \Rightarrow (x+2)(x-2) \leq 0 \Rightarrow x \in [-2, 2]$.

$\therefore$ $\quad \text{dom}(g) = [-2, 2]$.

$\therefore$ $\quad \text{dom}(f) \cap \text{dom}(g) = [-2, \infty) \cap [-2, 2] = [-2, 2]$.

(i) $(f+g) : [-2, 2] \to R$ is given by
$$(f+g)(x) = f(x) + g(x) = \sqrt{x+2} + \sqrt{4-x^2}.$$

(ii) $(f-g) : [-2, 2] \to R$ is given by
$$(f-g)(x) = f(x) - g(x) = \sqrt{x+2} - \sqrt{4-x^2}.$$

(iii) $(fg) : [-2, 2] \to R$ is given by
$$(fg)(x) = f(x) \cdot g(x) = (\sqrt{x+2})(\sqrt{4-x^2})$$
$$= \sqrt{(x+2)^2(2-x)} = (x+2)\sqrt{(2-x)}.$$

(iv) $(ff) : [-2, 2] \to R$ is given by
$$(ff)(x) = f(x) \cdot f(x) = (\sqrt{x+2})(\sqrt{x+2}) = (x+2).$$

(v) $(gg) : [-2, 2] \to R$ is given by
$$(gg)(x) = g(x) \cdot g(x) = (\sqrt{4-x^2})(\sqrt{4-x^2}) = (4-x^2).$$

(vi) $\{x : g(x) = 0\} = \{x : 4 - x^2 = 0] = \{x : (2-x)(2+x) = 0\} = \{-2, 2\}$.

$\therefore$ $\quad \text{dom}\left(\dfrac{f}{g}\right) = \text{dom}(f) \cap \text{dom}\{g\} - \{x : g(x) = 0\}$
$$= [-2, 2] - \{-2, 2\} = (-2, 2).$$

$\therefore$ $\quad \dfrac{f}{g} : (-2, 2) \to R$ is given by

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} = \frac{\sqrt{x+2}}{\sqrt{4-x^2}} = \frac{\sqrt{2+x}}{(\sqrt{2+x})(\sqrt{2-x})} = \frac{1}{(\sqrt{2-x})}.$$

EXAMPLE 6 *Let f be the exponential function and g be the logarithmic function. Then, find:*

(i) $(f+g)(1)$ *(ii)* $(fg)(1)$ *(iii)* $(4f)(1)$ *(iv)* $(3g)(1)$

SOLUTION Let $f: R \to R: f(x) = e^x$ and $g: R^+ \to R: g(x) = \log_e x$.

Then, $\text{dom}(f) \cap \text{dom}(g) = R \cap R^+ = R^+$.

(i) $(f+g): R^+ \to R$ is given by

$$(f+g)(x) = f(x) + g(x) = (e^x + \log_e x).$$

$$\therefore \quad (f+g)(1) = (e^1 + \log_e 1) = (e+0) = e.$$

(ii) $(fg): R^+ \to R$ is given by

$$(fg)(x) = f(x) \cdot g(x) = e^x (\log_e x).$$

$$\therefore \quad (fg)(1) = e^1 (\log_e 1) = (e \times 0) = 0.$$

(iii) $(4f): R^+ \to R$ is given by

$$(4f)(x) = 4 \times f(x) = 4e^x.$$

$$\therefore \quad (4f)(1) = (4 \times e^1) = 4e.$$

(iv) $(3g): R^+ \to R$ is given by

$$(3g)(x) = 3 \times g(x) = 3 \times (\log_e x)$$

$$\therefore \quad (3g)(1) = 3 \times g(1) = 3 \times (\log_e 1) = (3 \times 0) = 0.$$

EXAMPLE 7 *If* $f(x) = \log_e (1-x)$ *and* $g(x) = [x]$ *then find:*

(i) $(f+g)(x)$ *(ii)* $(fg)(x)$ *(iii)* $\left(\frac{f}{g}\right)(x)$ *(iv)* $\left(\frac{g}{f}\right)(x)$.

Also find $(f+g)(-1)$, $(fg)(0)$, $\left(\frac{f}{g}\right)(-1)$, $\left(\frac{g}{f}\right)\left(\frac{1}{2}\right)$.

SOLUTION Clearly, $\log_e (1-x)$ is defined only when $1-x>0$, i.e., $x<1$.

$\therefore \quad \text{dom}(f) = (-\infty, 1)$.

Also, $\text{dom}(g) = R$.

$\therefore \quad \text{dom}(f) \cap \text{dom}(g) = (-\infty, 1) \cap R = (-\infty, 1)$.

(i) $(f+g): (-\infty, 1) \to R$ is given by

$$(f+g)(x) = f(x) + g(x) = \log_e (1-x) + [x]$$

(ii) $(fg): (-\infty, 1) \to R$ is given by

$$(fg)(x) = f(x) \times g(x) = \{\log_e (1-x)\} \times [x].$$

(iii) $\{x : g(x) = 0\} = \{x : [x] = 0\} = [0, 1)$.

$$\therefore \quad \text{dom}\left(\frac{f}{g}\right) = \text{dom}(f) \cap \text{dom}(g) - \{x : g(x) = 0\}$$

$$= (-\infty, 1) \cap R - [0, 1) = (-\infty, 0).$$

$\therefore \quad \dfrac{f}{g} : (-\infty, 0) \to R$ is given by

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} = \frac{\log_e(1-x)}{[x]}.$$

(iv) $\{x : f(x) = 0\} = \{x : \log_e(1-x) = 0\} = \{0\}$.

$$\therefore \quad \text{dom}\left(\frac{g}{f}\right) = \text{dom}\,(g) \cap \text{dom}\,(f) - \{x : f(x) = 0\}$$
$$= R \cap (-\infty, 1) - \{0\} = (-\infty, 0) \cup (0, 1).$$

$\therefore \quad \dfrac{g}{f} : (-\infty, 0) \cup (0, 1) \to R$ is given by

$$\left(\frac{g}{f}\right)(x) = \frac{g(x)}{f(x)} = \frac{[x]}{\log_e(1-x)}.$$

Now, we have:

$(f+g)(-1) = f(-1) + g(-1) = [-1] + \log_e(1+1) = (\log_e 2) - 1.$

$(fg)(0) = f(0) \times g(0) = \log_e(1-0) \times [0] = (\log_e 1 \times 0) = (0 \times 0) = 0.$

$$\left(\frac{f}{g}\right)(-1) = \frac{f(-1)}{g(-1)} = \frac{[-1]}{\log_e(1+1)} = \frac{-1}{\log_e 2}.$$

$$\left(\frac{g}{f}\right)\left(\frac{1}{2}\right) = \frac{g\left(\frac{1}{2}\right)}{f\left(\frac{1}{2}\right)} = \frac{\left[\frac{1}{2}\right]}{\log_e\left(1-\frac{1}{2}\right)} = \frac{[0.5]}{\log_e\left(\frac{1}{2}\right)} = 0.$$

EXAMPLE 8 *Find the sum and the difference of the identity function and the modulus function.*

SOLUTION Let $f : R \to R : f(x) = x$ be the identity function.

And, let $g : R \to R : g(x) = |x|$ be the modulus function.

Then, dom $(f) = R$ and dom $(g) = R$.

$\therefore \quad \text{dom}\,(f) \cap \text{dom}\,(g) = R \cap R = R.$

(i) dom $(f+g) =$ dom $(f) \cap$ dom $(g) = R$.

Now, $(f+g) : R \to R$ is given by

$$(f+g)(x) = f(x) + g(x)$$
$$= x + |x| = x + \begin{cases} x, \text{ when } x \geq 0 \\ -x, \text{ when } x < 0 \end{cases}$$
$$= \begin{cases} x + x, \text{ when } x \geq 0 \\ x - x, \text{ when } x < 0 \end{cases} = \begin{cases} 2x, \text{ when } x \geq 0 \\ 0, \text{ when } x < 0. \end{cases}$$

Hence, $(f+g)(x) = \begin{cases} 2x, \text{ when } x \geq 0 \\ 0, \text{ when } x < 0. \end{cases}$

(ii) dom $(f-g) =$ dom $(f) \cap$ dom $(g) = R$.

$$\therefore \quad (f-g)(x) = f(x) - g(x)$$
$$= x - |x| = x - \begin{cases} x, \text{ when } x \geq 0 \\ -x, \text{ when } x < 0 \end{cases}$$

$$= \begin{cases} x - x, \text{ when } x \geq 0 \\ x + x, \text{ when } x < 0 \end{cases} = \begin{cases} 0, \text{ when } x \geq 0 \\ 2x, \text{ when } x < 0 \end{cases}$$

$$\therefore \qquad (f - g)(x) = \begin{cases} 0, \text{ when } x \geq 0 \\ 2x, \text{ when } x < 0. \end{cases}$$

EXAMPLE 9 *Find the sum and the difference of the identity function and the reciprocal function.*

SOLUTION Let $f : R \to R : f(x) = x$ and $g : R - \{0\} \to R : g(x) = \frac{1}{x}$ be the identity function and the reciprocal function respectively.

Then, dom $(f) \cap$ dom $(g) = R \cap R - \{0\} = R - \{0\}$.

$$\therefore \qquad (f + g) : R - \{0\} \to R : (f + g)(x) = f(x) + g(x) = \left(x + \frac{1}{x}\right).$$

Hence, $(f + g)(x) = \left(x + \frac{1}{x}\right)$ for all $x \in R - \{0\}$.

And, $(f - g) : R - \{0\} \to R : (f - g)(x) = f(x) - g(x) = \left(x - \frac{1}{x}\right)$.

Hence, $(f - g)(x) = \left(x - \frac{1}{x}\right)$ for all $x \in R - \{0\}$.

EXAMPLE 10 *Find the product of the identity function by the modulus function.*

SOLUTION Let $f : R \to R : f(x) = x$ and $g : R \to R : g(x) = |x|$ be the identity function and the modulus function respectively.

Then, dom (fg) = dom $(f) \cap$ dom $(g) = R \cap R = R$.

$$\therefore \qquad (fg) : R \to R : (fg)(x) = f(x) \cdot g(x).$$

Now, $(fg)(x) = f(x) \cdot g(x) = x \cdot |x|$

$$= x \cdot \begin{cases} x, \text{ when } x \geq 0 \\ -x, \text{ when } x < 0 \end{cases} = \begin{cases} x^2, \text{ when } x \geq 0 \\ -x^2, \text{ when } x < 0. \end{cases}$$

Hence, $(fg)(x) = \begin{cases} x^2, \text{ when } x \geq 0 \\ -x^2, \text{ when } x < 0. \end{cases}$

EXAMPLE 11 *Find the product of the identity function by the reciprocal function.*

SOLUTION Let $f : R \to R : f(x) = x$ and $g : R - \{0\} \to R : g(x) = \frac{1}{x}$ be the identity function and the reciprocal function respectively.

Then, dom (fg) = dom $(f) \cap$ dom $(g) = R \cap R - \{0\} = R - \{0\}$.

$$\therefore \qquad (fg) : R - \{0\} \to R : (fg)(x) = f(x) \cdot g(x) = \left(x \times \frac{1}{x}\right) = 1.$$

Hence, $(fg)(x) = 1$ for all $x \in R - \{0\}$.

EXAMPLE 12 *Find the quotient of the identity function by the modulus function.*

SOLUTION Let $f : R \to R : f(x) = x$ and $g : R \to R : g(x) = |x|$ be the identity function and the modulus function respectively.

Now, $\text{dom}\left(\frac{f}{g}\right) = \text{dom}(f) \cap \text{dom}(g) - \{x : g(x) = 0\}$

and $\{x : g(x) = 0\} = \{x : |x| = 0\} = \{0\}$.

$\therefore \quad \text{dom}\left(\frac{f}{g}\right) = [R \cap R - \{0\}] - \{0\} = R - \{0\}$.

So, $\frac{f}{g} : R - \{0\} \to R : \left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} = \frac{x}{|x|} = \begin{cases} 1, \text{ when } x > 0 \\ -1, \text{ when } x < 0. \end{cases}$

Hence, $\left(\frac{f}{g}\right)(x) = \begin{cases} 1, \text{ when } x > 0 \\ -1, \text{ when } x < 0. \end{cases}$

EXAMPLE 13 *Find the quotient of the identity function by the reciprocal function.*

SOLUTION Let $f : R \to R : f(x)$ and $g : R - \{0\} \to R : g(x) = \frac{1}{x}$ be the identity function and the reciprocal function respectively.

Now, $\text{dom}\left(\frac{f}{g}\right) = \text{dom}(f) \cap \text{dom}(g) - \{x : g(x) = 0\}$

and $\{x : g(x) = 0\} = \left\{x : \frac{1}{x} = 0\right\} = \phi$.

$\therefore \quad \text{dom}\left(\frac{f}{g}\right) = [R \cap R - \{0\}] - \phi = R - \{0\}$.

So, $\frac{f}{g} : R - \{0\} \to R : \left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} = \frac{x}{\frac{1}{x}} = x^2$.

Hence, $\left(\frac{f}{g}\right)(x) = x^2$ for all $x \in R - \{0\}$.

EXERCISE 3E

1. Let $f : R \to R : f(x) = x + 1$ and $g : R \to R : g(x) = 2x - 3$.

Find (i) $(f + g)(x)$ (ii) $(f - g)(x)$ (iii) $(fg)(x)$ (iv) $\left(\frac{f}{g}\right)(x)$.

2. Let $f : R \to R : f(x) = 2x + 5$ and $g : R \to R : g(x) = x^2 + x$.

Find (i) $(f + g)(x)$ (ii) $(f - g)(x)$ (iii) $(fg)(x)$ (iv) $\left(\frac{f}{g}\right)(x)$.

3. Let $f : R \to R : f(x) = x^3 + 1$ and $g : R \to R : g(x) = (x + 1)$.

Find (i) $(f + g)(x)$ (ii) $(f - g)(x)$ (iii) $\left(\frac{1}{f}\right)(x)$ (iv) $\left(\frac{f}{g}\right)(x)$.

4. Let $f : R \to R : f(x) = \frac{x}{c}$, where c is a constant.

Find (i) $(cf)(x)$ (ii) $(c^2 f)(x)$ (iii) $\left(\frac{1}{c} f\right)(x)$.

5. Let $f : [2, \infty) \to R : f(x) = \sqrt{x - 2}$ and $g : [2, \infty) \to R : g(x) = \sqrt{x + 2}$.

Find (i) $(f + g)(x)$ (ii) $(f - g)(x)$ (iii) $(fg)(x)$.

ANSWERS (EXERCISE 3E)

1. (i) $3x-2$ (ii) $-x+4$ (iii) $2x^2-x-3$ (iv) $\dfrac{x+1}{2x-3}$

2. (i) x^2+3x+5 (ii) $-x^2+x+5$ (iii) $2x^3+7x^2+5x$ (iv) $\dfrac{2x+5}{x^2+x}$

3. (i) x^3+x+2 (ii) x^3-x (iii) $\dfrac{1}{x^3+1}$ (iv) x^2+x+1

4. (i) x (ii) cx (iii) $\dfrac{x}{c^2}$

5. (i) $\sqrt{x-2}+\sqrt{x+2}$ for all $x \in [2, \infty)$ (ii) $\sqrt{x-2}-\sqrt{x+2}$ for all $x \in [2, \infty)$
(iii) $\sqrt{x^2-4}$ for all $x \in [2, \infty)$

HINTS TO SOME SELECTED QUESTIONS

1. (iv) $\text{dom}\left(\dfrac{f}{g}\right) = \text{dom}(f) \cap \text{dom}(g) - \{x : g(x) = 0\}$
$= R \cap R - \left\{\dfrac{3}{2}\right\} = R - \left\{\dfrac{3}{2}\right\}$.

2. (iv) $\text{dom}\left(\dfrac{f}{g}\right) = \text{dom}(f) \cap \text{dom}(g) - \{x : g(x) = 0\}$
$= R \cap R - \{0, -1\} = R - \{0, -1\}$.

3. (iii) $\text{dom}\left(\dfrac{1}{f}\right) = \text{dom}(f) - \{x : f(x) = 0\} = R - \{-1\}$.

(iv) $\text{dom}\left(\dfrac{f}{g}\right) = \text{dom}(f) \cap \text{dom}(g) - \{x : g(x) = 0\}$
$= R \cap R - \{-1\} = R - \{-1\}$.

EXERCISE 3F

Very-Short-Answer Questions

1. Find the set of values for which the function $f(x) = 1 - 3x$ and $g(x) = 2x^2 - 1$ are equal.
2. Find the set of values for which the functions $f(x) = x + 3$ and $g(x) = 3x^2 - 1$ are equal.
3. Let $X = \{-1, 0, 2, 5\}$ and $f : X \to R : f(x) = x^3 + 1$. Then, write f as a set of ordered pairs.
4. Let $A = \{-2, -1, 0, 1, 2\}$ and $f : A \to Z : f(x) = x^2 - 2x - 3$. Find $f(A)$.
5. Let $f : R \to R : f(x) = x^2$.
   Determine (i) range (f) (ii) $\{x : f(x) = 4\}$
6. Let $f : R \to R : f(x) = x^2 + 1$. Find $f^{-1}\{10\}$.
7. Let $f : R^+ \to R : f(x) = \log_e x$. Find $\{x : f(x) = -2\}$.

8. Let $A = \{6, 10, 11, 15, 21\}$ and let $f : A \to N : f(n)$ is the highest prime factor of n. Find range (f).

9. Find the range of the function $f(x) = \sin x$.

10. Find the range of the function $f(x) = |x|$.

11. Write the domain and the range of the function, $f(x) = \sqrt{x - [x]}$.

12. If $f(x) = \dfrac{x-5}{5-x}$ then find dom (f) and range (f).

13. Let $f = \{(1, 6), (2, 5), (4, 3), (5, 2), (8, -1), (10, -3)\}$ and $g = \{(2, 0), (3, 2), (5, 6), (7, 10), (8, 12), (10, 16)\}$.

Find (i) dom $(f+g)$ (ii) dom $\left(\dfrac{f}{g}\right)$.

14. If $f(x) = \dfrac{x-1}{x}$, find the value of $f\left\{f\left(\dfrac{1}{x}\right)\right\}$.

15. If $f(x) = \dfrac{kx}{x+1}$, where $x \neq -1$ and $f\{f(x)\} = x$ for $x \neq -1$ then find the value of k.

16. Find the range of the function, $f(x) = \dfrac{x}{|x|}$.

17. Find the domain of the function, $f(x) = \log |x|$.

18. If $f\left(x + \dfrac{1}{x}\right) = \left(x^2 + \dfrac{1}{x^2}\right)$ for all $x \in R - \{0\}$ then write an expression for $f(x)$.

19. Write the domain and the range of the function, $f(x) = \dfrac{ax+b}{bx-a}$.

20. Write the domain and the range of the function, $f(x) = \sqrt{x-1}$.

21. Write the domain and the range of the function, $f(x) = -|x|$.

ANSWERS (EXERCISE 3F)

1. $\left\{-2, \dfrac{1}{2}\right\}$ **2.** $\left\{\dfrac{4}{3}, -1\right\}$ **3.** $f = \{(-1, 0), (0, 1), (2, 9), (5, 126)\}$

4. $f(A) = \{5, 0, -3, -4\}$ **5.** (i) $[0, \infty)$ (ii) $\{-2, 2\}$ **6.** $\{-3, 3\}$

7. $\{e^{-2}\}$ **8.** $\{3, 5, 7, 11\}$ **9.** $[-1, 1]$ **10.** $[0, \infty)$

11. dom $(f) = R$, range $(f) = [0, 1)$ **12.** dom $(f) = R - \{5\}$, range $(f) = \{-1\}$

13. (i) $\{2, 5, 8, 10\}$ (ii) $\{5, 8, 10\}$ **14.** $\dfrac{x}{x-1}$ **15.** $k = -1$

16. $\{1, -1\}$ **17.** $R - \{0\}$ **18.** $f(x) = x^2 - 2, x \in R$

19. dom $(f) = R - \left\{\dfrac{a}{b}\right\}$, range $(f) = R - \left\{\dfrac{a}{b}\right\}$ **20.** dom $(f) = [1, \infty)$, range $(f) = R$

21. dom $(f) = R$, range $(f) = (-\infty, 0]$

HINTS TO SOME SELECTED QUESTIONS

1. $2x^2 - 1 = 1 - 3x \Rightarrow 2x^2 + 3x - 2 = 0 \Rightarrow (x+2)(2x-1) = 0.$

2. $3x^2 - 1 = x + 3 \Rightarrow 3x^2 - x - 4 = 0 \Rightarrow (3x-4)(x+1) = 0.$

5. (i) $f(x) = x^2 \geq 0$. So, range $(f) = [0, \infty)$.

(ii) $f(x) = 4 \Rightarrow x^2 = 4 \Rightarrow x = \pm 2.$

6. $f(x) = 10 \Rightarrow x^2 + 1 = 10 \Rightarrow x^2 = 9 \Rightarrow x = \pm 3.$

7. $f(x) = -2 \Rightarrow \log_e x = -2 \Rightarrow x = e^{-2}.$

8. $f(6) = 3,\ f(10) = 5,\ f(11) = 11,\ f(15) = 5,\ f(21) = 7.$

$\therefore$ range $(f) = \{3, 5, 7, 11\}$.

10. $f(x) = |x| \geq 0$. So, range $(f) = [0, \infty)$.

11. $x - [x] = 0$ for $x \in Z$ and $x - [x] > 0$ for $x \in R - Z$.

$\therefore$ $x - [x] \geq 0$ for $x \in R$. So, dom $(f) = R$.

Also, $0 \leq x - [x] < 1 \Rightarrow 0 \leq \sqrt{x - [x]} < 1$. So, range $(f) = [0, 1)$.

12. $f(x)$ is not defined when $x = 5$. So, dom $(f) = R - \{5\}$.

$f(x) = \dfrac{(x-5)}{(5-x)} = -1$. So, range $(f) = \{-1\}$.

13. dom $(f+g)$ = dom $(f) \cap$ dom $(g) = \{2, 5, 8, 10\}$.

dom $\left(\dfrac{f}{g}\right)$ = dom $(f) \cap$ dom $(g) - \{x : g(x) = 0\} = \{2, 5, 8, 10\} - \{2\} = \{5, 8, 10\}$.

14. $f\left(\dfrac{1}{x}\right) = \dfrac{\dfrac{1}{x} - 1}{\dfrac{1}{x}} = (1 - x) \Rightarrow f\left\{f\left(\dfrac{1}{x}\right)\right\} = \dfrac{(1-x)-1}{1-x} = \dfrac{-x}{1-x} = \dfrac{x}{x-1}.$

15. $f(f(x)) = f\left(\dfrac{kx}{x+1}\right) = \dfrac{k \times \dfrac{kx}{x+1}}{\dfrac{kx}{x+1} + 1} = \dfrac{k^2x}{kx + x + 1}$. So, $\dfrac{k^2x}{kx+x+1} = x.$

$\therefore$ $k^2 = kx + x + 1 \Rightarrow k^2 - kx - (x+1) = 0 \Rightarrow k = \dfrac{x \pm \sqrt{x^2 + 4(x+1)}}{2} = \dfrac{x \pm (x+2)}{2}$

$\Rightarrow$ $k = x + 1$ or $k = -1 \Rightarrow k = -1$ [$\because$ k is a constant].

18. $f\left(x + \dfrac{1}{x}\right) = \left(x + \dfrac{1}{x}\right)^2 - 2 \Rightarrow f(x) = x^2 - 2,\ x \in R.$

SUMMARY OF KEY FACTS

1. **(i) Function or Mapping**

Let X and Y be two nonempty sets and let $f \subseteq X \times Y$ be a relation from X to Y such that

(i) dom $(f) = X$ and

(ii) no two different ordered pairs in f have the same first coordinate.

Then, f is called a ***function*** or a ***mapping*** from X to Y and we write, $f : X \to Y$.

(ii) Dom $(f) = \{x : (x, y) \in f\} = X$, range $(f) = \{y : (x, y) \in f\} \subseteq Y$ and co-domain $(f) = Y$.

(iii) If $(x, y) \in f$, we write, $f(x) = y$.

Here, y is called the ***image*** of x under f, while x is called the ***pre-image*** of y.

(iv) Two functions f and g having same domain X are said to be equal only when $f(x) = g(x)$ for all $x \in X$.

2. (i) Function or Mapping

Let X and Y be two nonempty sets. Then, a rule f which associates to each $x \in X$, a unique element $f(x)$ of Y, is called a function from X to Y. We write, $f : X \to Y$. Dom $(f) = X$, range $(f) = \{f(x) : x \in X\} \subseteq Y$ and co-domain $(f) = Y$.

(ii) A mapping $f : X \to Y$ is one-one if $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$.

(iii) Real Functions

Let R be the set of all real numbers. Then, a function $f : R \to R$ is called a real function.

3. (i) Constant Function

Let k be a fixed real number. Then, a function $f : R \to R : f(x) = k$ for all $x \in R$ is called a constant function. Dom $(f) = R$ and range $(f) = \{k\}$.

(ii) Identity Function

The function $I : R \to R : I(x) = x$ for all $x \in R$ is called the identity function. Dom (I) = Range $(I) = R$.

(iii) Modulus Function

The function $f : R \to R : f(x) = |x|$ for all $x \in R$ is called the modulus function. Dom $(f) = R$ and range $(f) = \{x : x \in R \text{ and } x \geq 0\} = [0, \infty)$.

(iv) Greatest Integer Function, or Step Function

The function $f : R \to R : f(x) = [x]$, where $[x]$ denotes the greatest integer less than or equal to x, is called the greatest integer function.

Dom $(f) = R$, range $(f) = Z$.

EXAMPLES $[3.85] = 3$, $[0.94] = 0$, $[5.0] = 5$, $[-7.35] = -8$ and $[-6.0] = -6$, etc.

(v) Smallest Integer Function

The function $f : R \to R : f(x) = \lceil x \rceil$, where $\lceil x \rceil$ denotes the smallest integer greater than or equal to x. Dom $(f) = R$ and range $(f) = Z$.

EXAMPLES $\lceil 6.8 \rceil = 7$, $\lceil 5 \rceil = 5$, $\lceil 0.85 \rceil = 1$, $\lceil -4.5 \rceil = -4$, etc.

(vi) Fractional Part Function

The function $f : R \to R : f(x) = \{x\}$, where $\{x\}$ denotes the fractional part or decimal part of x. Note that $\{x\} = x - [x]$.

Dom $(f) = R$ and range $(f) = [0, 1)$.

EXAMPLES $\{6.83\} = 0.83$, $\{-3.65\} = 0.35$, $\{8\} = 0$, $\{-3\} = 0$, etc.

(vii) Signum Function

The function $f : R \to R : f(x) = \begin{cases} \dfrac{|x|}{x}, & \text{when } x \neq 0 \\ 0, & \text{when } x = 0 \end{cases}$

is called the signum function.

Dom $(f) = R$ and range $(f) = \{-1, 0, 1\}$.

(viii) Exponential Function

Let a be a real number such that $a > 0$ and $a \neq 1$.

Then, $f : R \to R : f(x) = a^x$ for all $x \in R$ is called the exponential function.

Dom $(f) = R$ and range $(f) = (0, \infty)$.

In particular, $f : R \to R : f(x) = e^x$ for all $x \in R$.

(ix) Logarithmic Function

The function $f : R^+ \to R : f(x) = \log_e x$ for all $x \in R^+$ is called the logarithmic function. Dom $(f) = (0, \infty)$ and range $(f) = R$.

(x) Reciprocal Function

The function $f : R - \{0\} \to R : f(x) = \dfrac{1}{x}$ for all $x \in R - \{0\}$ is called the reciprocal function. Dom $(f) = R - \{0\}$ and range $(f) = R - \{0\}$.

(xi) Square Root Function

The function $f : R^+ \cup \{0\} \to R : f(x) = \sqrt{x}$ for all $x \in R^+ \cup \{0\}$ is called the square root function. Dom $(f) = R^+ \cup \{0\} = [0, \infty)$, range $(f) = [0, \infty)$.

(xii) Cube Root Function

The function $f : R \to R : f(x) = x^{1/3}$ for all $x \in R$ is called the cube root function. Dom $(f) = R$ and range $(f) = R$.

(xiii) Rational Function

The function $f : R \to R : f(x) = \dfrac{p(x)}{q(x)}$ where $p(x)$ and $q(x)$ are polynomials and $q(x) \neq 0$, is called a rational function. Dom $(f) = R - \{x : q(x) = 0\}$.

4. OPERATIONS ON REAL FUNCTIONS

Let f and g be real functions having domains D_1 and D_2 respectively. Then,

(i) $(f+g) : D_1 \cap D_2 \to R : (f+g)(x) = f(x) + g(x)$.

(ii) $(f-g) : D_1 \cap D_2 \to R : (f-g)(x) = f(x) - g(x)$.

(iii) $(fg) : D_1 \cap D_2 \to R : (fg)(x) = f(x) \cdot g(x)$

(iv) $(\alpha f) : D_1 \to R : (\alpha f)(x) = \alpha \cdot f(x)$.

(v) $\dfrac{f}{g} : (D_1 \cap D_2) - \{x : g(x) = 0\} \to R : \left(\dfrac{f}{g}\right)(x) = \dfrac{f(x)}{g(x)}, \; g(x) \neq 0$.

❑

4

Principle of Mathematical Induction

PRINCIPLE OF MATHEMATICAL INDUCTION

Let $P(n)$ be a statement involving the natural number n such that

(i) $P(1)$ *is true and* (ii) $P(k+1)$ *is true, whenever* $P(k)$ *is true*

then $P(n)$ *is true for all* $n \in N$.

SOLVED EXAMPLES

EXAMPLE 1 *Using the principle of mathematical induction, prove that*

$$1+3+5+7+\ldots+(2n-1)=n^2 \text{ for all } n \in N.$$

SOLUTION Let the given statement be $P(n)$. Then,

$$P(n):\ 1+3+5+7+\ldots+(2n-1)=n^2.$$

Putting $n=1$ in the given statement, we get

LHS = 1 and RHS = $1^2 = 1$.

$\therefore$ LHS = RHS.

Thus, $P(1)$ is true.

Let $P(k)$ be true. Then,

$$P(k):\ 1+3+5+7+\ldots+(2k-1)=k^2. \qquad \ldots \text{(i)}$$

Now, $1+3+5+7+\ldots+(2k-1)+\{2(k+1)-1\}$

$= \{1+3+5+7+\ldots+(2k-1)\}+(2k+1)$

$= k^2+(2k+1)$ [using (i)]

$= (k+1)^2$.

$\therefore$ $P(k+1):\ 1+3+5+7+\ldots(2k-1)+\{2(k+1)-1\}=(k+1)^2$.

This shows that $P(k+1)$ is true, whenever $P(k)$ is true.

Thus, $P(1)$ is true and $P(k+1)$ is true, whenever $P(k)$ is true.

Hence, by the principle of mathematical induction, we have

$$1+3+5+7+\ldots+(2n-1)=n^2 \text{ for all } n \in N.$$

EXAMPLE 2 *Using the principle of mathematical induction, prove that*

$$1+4+7+10+\ldots+(3n-2)=\frac{1}{2}n(3n-1) \text{ for all } n \in N.$$

SOLUTION Let the given statement be $P(n)$. Then,

$$P(n):\ 1+4+7+10+\ldots+(3n-2)=\frac{1}{2}n(3n-1).$$

Putting $n = 1$ in the given statement, we get

$$\text{LHS} = 1 \text{ and RHS} = \frac{1}{2} \times 1 \times (3 \times 1 - 1) = 1.$$

$\therefore$ LHS = RHS.

Thus, $P(1)$ is true.

Let $P(k)$ be true. Then,

$$P(k): \ 1 + 4 + 7 + 10 + \ldots + (3k - 2) = \frac{1}{2}k(3k - 1). \qquad \ldots \text{(i)}$$

Now, $1 + 4 + 7 + \ldots + (3k - 2) + \{3(k + 1) - 2\}$

$$= \{1 + 4 + 7 + \ldots (3k - 2)\} + (3k + 1)$$

$$= \frac{1}{2}k(3k - 1) + (3k + 1) \qquad \text{[using (i)]}$$

$$= \frac{1}{2}(3k^2 - k + 6k + 2) = \frac{1}{2}(3k^2 + 5k + 2)$$

$$= \frac{1}{2}(3k^2 + 3k + 2k + 2) = \frac{1}{2} \cdot \{3k(k + 1) + 2(k + 1)\}$$

$$= \frac{1}{2}(k + 1)(3k + 2) = \frac{1}{2}(k + 1)\{3(k + 1) - 1\}.$$

$$\therefore \quad P(k + 1): \ 1 + 4 + 7 + \ldots + \{3(k + 1) - 2\} = \frac{1}{2}(k + 1)\{3(k + 1) - 1\}.$$

This shows that $P(k + 1)$ is true, whenever $P(k)$ is true.

Thus, $P(1)$ is true and $P(k + 1)$ is true, whenever $P(k)$ is true.

Hence, by the principle of mathematical induction, we have

$$1 + 4 + 7 + \ldots + (3n - 2) = \frac{1}{2}n(3n - 1) \text{ for all } n \in N.$$

EXAMPLE 3 *Using the principle of mathematical induction, prove that*

$$1^2 + 2^2 + 3^2 + \ldots + n^2 = \frac{1}{6}n(n + 1)(2n + 1) \textit{ for all } n \in N.$$

SOLUTION Let the given statement be $P(n)$. Then,

$$P(n): \ 1^2 + 2^2 + 3^2 + \ldots + n^2 = \frac{1}{6}n(n + 1)(2n + 1).$$

Putting $n = 1$ in the given statement, we get

$$\text{LHS} = 1^2 = 1 \text{ and RHS} = \frac{1}{6} \times 1 \times 2 \times (2 \times 1 + 1) = 1.$$

$\therefore$ LHS = RHS.

Thus, $P(1)$ is true.

Let $P(k)$ be true. Then,

$$P(k): \ 1^2 + 2^2 + 3^2 + \ldots + k^2 = \frac{1}{6}k(k + 1)(2k + 1). \qquad \ldots \text{(i)}$$

Now, $1^2 + 2^2 + 3^2 + \ldots + k^2 + (k + 1)^2$

$$= \{1^2 + 2^2 + 3^2 + \ldots + k^2\} + (k + 1)^2$$

$$= \frac{1}{6}k(k + 1)(2k + 1) + (k + 1)^2 \qquad \text{[using (i)]}$$

$$= \frac{1}{6}(k+1)\cdot\{k(2k+1)+6(k+1)\}$$
$$= \frac{1}{6}(k+1)(2k^2+7k+6) = \frac{1}{6}(k+1)\{(2k^2+4k)+(3k+6)\}$$
$$= \frac{1}{6}(k+1)(k+2)(2k+3) = \frac{1}{6}(k+1)(k+1+1)[2(k+1)+1].$$

$\therefore\quad P(k+1):\ 1^2+2^2+3^2+\ldots+(k+1)^2$
$$= \frac{1}{6}(k+1)(k+1+1)[2(k+1)+1].$$

This shows that $P(k+1)$ is true, whenever $P(k)$ is true.

Thus, $P(1)$ is true and $P(k+1)$ is true, whenever $P(k)$ is true.

Hence, by the principle of mathematical induction, we have

$$1^2+2^2+3^2+\ldots+n^2 = \frac{1}{6}n(n+1)(2n+1) \text{ for all } n \in N.$$

EXAMPLE 4 *Using the principle of mathematical induction, prove that*

$$1^3+2^3+3^3+\ldots+n^3 = \left\{\frac{n(n+1)}{2}\right\}^2 \textit{ for all } n \in N.$$

SOLUTION Let the given statement be $P(n)$. Then,

$$P(n):\ 1^3+2^3+3^3+\ldots+n^3 = \left\{\frac{n(n+1)}{2}\right\}^2.$$

Putting $n = 1$ in the given statement, we get

$$\text{LHS} = 1^3 = 1 \text{ and RHS} = \left(\frac{1\times 2}{2}\right)^2 = 1^2 = 1.$$

$\therefore\quad$ LHS = RHS.

Thus, $P(1)$ is true.

Let $P(k)$ be true for some $k \in N$. Then,

$$P(k):\ 1^3+2^3+3^3+\ldots k^3 = \left\{\frac{k(k+1)^2}{2}\right\}^2. \qquad \ldots \text{(i)}$$

Now, $1^3+2^3+3^3+\ldots+k^3+(k+1)^3$

$$= \{1^3+2^3+3^3+\ldots+k^3\}+(k+1)^3$$
$$= \left\{\frac{k(k+1)}{2}\right\}^2+(k+1)^3 \qquad \text{[using (i)]}$$
$$= (k+1)^2\left\{\frac{k^2}{4}+(k+1)\right\} = (k+1)^2\left\{\frac{k^2+4k+4}{4}\right\}$$
$$= \frac{(k+1)^2(k+2)^2}{4} = \left\{\frac{(k+1)\{(k+1)+1\}}{2}\right\}^2$$

$$\therefore\quad P(k+1):\ 1^3+2^3+3^3+\ldots+(k+1)^3 = \left[\frac{(k+1)\{(k+1)+1\}}{2}\right]^2.$$

This shows that $P(k+1)$ is true, whenever $P(k)$ is true.

$\therefore$ $P(1)$ is true and $P(k+1)$ is true, whenever $P(k)$ is true.

Hence, by the principle of mathematical induction, we have

$$1^3 + 2^3 + 3^3 + \ldots + n^3 = \left\{\frac{n(n+1)}{2}\right\}^2 \text{ for all } n \in N.$$

EXAMPLE 5 *Using the principle of mathematical induction, prove that*

$$1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \ldots + n(n+1) = \frac{1}{3}n(n+1)(n+2).$$

SOLUTION Let the given statement be $P(n)$. Then,

$$P(n)\text{: } 1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \ldots + n(n+1) = \frac{1}{3}n(n+1)(n+2).$$

When $n = 1$, $\quad \text{LHS} = 1 \cdot 2 = 2 \quad$ and $\quad \text{RHS} = \frac{1}{3} \times 1 \times 2 \times (1+2) = 2.$

$\therefore$ LHS = RHS.

Thus, the given statement is true for $n = 1$, i.e., $P(1)$ is true.

Let $P(k)$ be true. Then,

$$P(k)\text{: } 1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \ldots + k(k+1) = \frac{1}{3}k(k+1)(k+2). \qquad \ldots \text{(i)}$$

Now, $1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \ldots + k(k+1) + (k+1)(k+2)$

$$= \{1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \ldots + k(k+1) + (k+1)(k+2)$$

$$= \frac{1}{3}k(k+1)(k+2) + (k+1)(k+2) \qquad \text{[using (i)]}$$

$$= \frac{1}{3} \cdot [k(k+1)(k+2) + 3(k+1)(k+2)] = \frac{1}{3}(k+1)(k+2)(k+3).$$

$\therefore$ $P(k+1)\text{: } 1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \ldots + (k+1)(k+2)$

$$= \frac{1}{3}(k+1)(k+2)(k+3).$$

This shows that $P(k+1)$ is true, whenever $P(k)$ is true.

$\therefore$ $P(1)$ is true and $P(k+1)$ is true, whenever $P(k)$ is true.

Hence, by the principle of mathematical induction, we have

$$1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \ldots + n(n+1) = \frac{1}{3}n(n+1)(n+2) \text{ for all } n \in N.$$

EXAMPLE 6 *Using the principle of mathematical induction, prove that*

$$1 \cdot 3 + 3 \cdot 5 + 5 \cdot 7 + \ldots + (2n-1)(2n+1) = \frac{1}{3}n(4n^2 + 6n - 1).$$

SOLUTION Let the given statement be $P(n)$. Then,

$$P(n)\text{: } 1 \cdot 3 + 3 \cdot 5 + 5 \cdot 7 + \ldots + (2n-1)(2n+1) = \frac{1}{3}n(4n^2 + 6n - 1).$$

When $n = 1$, we have

$\text{LHS} = 1 \cdot 3 = 3$ and $\text{RHS} = \frac{1}{3} \times 1 \times (4 \times 1^2 + 6 \times 1 - 1) = \frac{1}{3} \times 1 \times 9 = 3.$

$\therefore$ LHS = RHS.

Thus, $P(1)$ is true.

Let $P(k)$ be true. Then,

$$P(k):\ 1\cdot 3+3\cdot 5+5\cdot 7+\ldots+(2k-1)(2k+1)=\frac{1}{3}k(4k^2+6k-1).\quad \ldots \text{(i)}$$

Now, $1\cdot 3+3\cdot 5+5\cdot 7+\ldots+(2k-1)(2k+1)+\{2(k+1)-1\}\{2(k+1)+1\}$

$$=\{1\cdot 3+3\cdot 5+5\cdot 7+\ldots+(2k-1)(2k+1)\}+(2k+1)(2k+3)$$

$$=\frac{1}{3}k(4k^2+6k-1)+(2k+1)(2k+3) \qquad \text{[using (i)]}$$

$$=\frac{1}{3}[(4k^3+6k^2-k)+3(4k^2+8k+3)]=\frac{1}{3}(4k^3+18k^2+23k+9)$$

$$=\frac{1}{3}(k+1)(4k^2+14k+9)=\frac{1}{3}(k+1)[4(k+1)^2+6(k+1)-1].$$

$\therefore$ $\quad P(k+1):\ 1\cdot 3+3\cdot 5+5\cdot 7+\ldots+\{2(k+1)-1\}\{2(k+1)+1\}$

$$=\frac{1}{3}(k+1)\{4(k+1)^2+6(k+1)-1\}.$$

Thus, $P(k+1)$ is true, whenever $P(k)$ is true.

$\therefore$ $\quad P(1)$ is true and $P(k+1)$ is true, whenever $P(k)$ is true.

Hence, by the principle of mathematical induction, we have

$1\cdot 3+3\cdot 5+5\cdot 7+\ldots+(2n-1)(2n+1)=\frac{1}{3}n(4n^2+6n-1)$ for all $n\in N$.

EXAMPLE 7 *Using the principle of mathematical induction, prove that*

$$1\cdot 2\cdot 3+2\cdot 3\cdot 4+\ldots+n(n+1)(n+2)=\frac{1}{4}n(n+1)(n+2)(n+3)$$

for all $n\in N$.

SOLUTION Let the given statement be $P(n)$. Then,

$$P(n):\ 1\cdot 2\cdot 3+2\cdot 3\cdot 4+\ldots+n(n+1)(n+2)=\frac{1}{4}n(n+1)(n+2)(n+3).$$

When $n=1$, we have

$$\text{LHS}=1\cdot 2\cdot 3=6 \ \text{ and } \ \text{RHS}=\frac{1}{4}\times 1\times 2\times 3\times 4=6.$$

$\therefore$ $\quad$ LHS = RHS.

Thus, the given statement is true for $n=1$, i.e., $P(1)$ is true.

Let $P(k)$ be true. Then,

$$P(k):\ 1\cdot 2\cdot 3+2\cdot 3\cdot 4+\ldots+k(k+1)(k+2)$$

$$=\frac{1}{4}k(k+1)(k+2)(k+3). \qquad \ldots \text{(i)}$$

Now, $1\cdot 2\cdot 3+2\cdot 3\cdot 4+\ldots+k(k+1)(k+2)+(k+1)(k+2)(k+3)$

$$=\{1\cdot 2\cdot 3+2\cdot 3\cdot 4+\ldots+k(k+1)(k+2)\}+(k+1)(k+2)(k+3)$$

$$=\frac{1}{4}k(k+1)(k+2)(k+3)+\frac{4(k+1)(k+2)(k+3)}{4} \qquad \text{[using (i)]}$$

$$=\frac{1}{4}(k+1)(k+2)(k+3)(k+4).$$

$\therefore \quad P(k+1)$: $1 \cdot 2 \cdot 3 + 2 \cdot 3 \cdot 4 + \ldots + (k+1)(k+2)(k+3)$

$$= \frac{1}{4}(k+1)(k+2)(k+3)(k+4).$$

Thus, $P(k+1)$ is true, whenever $P(k)$ is true.

$P(1)$ is true and $P(k+1)$ is true, whenever $P(k)$ is true.

Hence, by the principle of mathematical induction, we have

$$1 \cdot 2 \cdot 3 + 2 \cdot 3 \cdot 4 + \ldots + n(n+1)(n+2) = \frac{1}{4}n(n+1)(n+2)(n+3)$$

for all values of $n \in N$.

EXAMPLE 8 *Using the principle of mathematical induction, prove that*

$$\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \ldots + \frac{1}{n(n+1)} = \frac{n}{(n+1)}.$$

SOLUTION Let the given statement be $P(n)$. Then,

$$P(n):\ \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \ldots + \frac{1}{n(n+1)} = \frac{n}{(n+1)}.$$

Putting $n = 1$ in the given statement, we get

$$\text{LHS} = \frac{1}{1 \cdot 2} = \frac{1}{2} \text{ and } \text{RHS} = \frac{1}{(1+1)} = \frac{1}{2}.$$

$\therefore \quad$ LHS = RHS.

Thus, $P(1)$ is true.

Let $P(k)$ be true. Then,

$$P(k):\ \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \ldots + \frac{1}{k(k+1)} = \frac{k}{(k+1)}. \qquad \ldots \text{(i)}$$

Now, $\dfrac{1}{1 \cdot 2} + \dfrac{1}{2 \cdot 3} + \dfrac{1}{3 \cdot 4} + \ldots + \dfrac{1}{k(k+1)} + \dfrac{1}{(k+1)(k+2)}$

$$= \left\{\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \ldots + \frac{1}{k(k+1)}\right\} + \frac{1}{(k+1)(k+2)}$$

$$= \frac{k}{(k+1)} + \frac{1}{(k+1)(k+2)} \qquad \text{[using (i)]}$$

$$= \frac{k(k+2)+1}{(k+1)(k+2)} = \frac{(k+1)^2}{(k+1)(k+2)} = \frac{(k+1)}{(k+2)}.$$

$$\therefore \quad P(k+1):\ \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} = \frac{1}{(k+1)(k+2)} = \frac{(k+1)}{(k+2)}.$$

Thus, $P(k+1)$ is true, whenever $P(k)$ is true.

$\therefore \quad P(1)$ is true and $P(k+1)$ is true, whenever $P(k)$ is true.

Hence, by the principle of mathematical induction, we have

$$\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \ldots + \frac{1}{n(n+1)} = \frac{n}{n+1} \text{ for all } n \in N.$$

EXAMPLE 9 *Using the principle of mathematical induction, prove that*

$$\frac{1}{3 \cdot 5} + \frac{1}{5 \cdot 7} + \frac{1}{7 \cdot 9} + \ldots + \frac{1}{(2n+1)(2n+3)} = \frac{n}{3(2n+3)}.$$

SOLUTION Let the given statement be $P(n)$. Then,

$$P(n):\ \frac{1}{3\cdot 5}+\frac{1}{5\cdot 7}+\frac{1}{7\cdot 9}+\ldots+\frac{1}{(2n+1)(2n+3)}=\frac{n}{3(2n+3)}.$$

Putting $n = 1$ in the given statement, we get

$$\text{LHS}=\frac{1}{3\cdot 5}=\frac{1}{15}\ \text{ and }\ \text{RHS}=\frac{1}{3(2\times 1+3)}=\frac{1}{15}.$$

$\therefore$ LHS = RHS.

Thus, $P(1)$ is true.

Let $P(k)$ be true. Then,

$$P(k):\ \frac{1}{3\cdot 5}+\frac{1}{5\cdot 7}+\frac{1}{7\cdot 9}+\ldots+\frac{1}{(2k+1)(2k+3)}=\frac{k}{3(2k+3)}. \quad \ldots \text{(i)}$$

Now, $\dfrac{1}{3\cdot 5}+\dfrac{1}{5\cdot 7}+\ldots+\dfrac{1}{(2k+1)(2k+3)}+\dfrac{1}{\{2(k+1)+1\}\{2(k+1)+3\}}$

$$=\left\{\frac{1}{3\cdot 5}+\frac{1}{5\cdot 7}+\ldots+\frac{1}{(2k+1)(2k+3)}\right\}+\frac{1}{(2k+3)(2k+5)}$$

$$=\frac{k}{3(2k+3)}+\frac{1}{(2k+3)(2k+5)} \qquad \text{[using (i)]}$$

$$=\frac{k(2k+5)+3}{3(2k+3)(2k+5)}=\frac{(2k^2+5k+3)}{3(2k+3)(2k+5)}=\frac{(k+1)(2k+3)}{3(2k+3)(2k+5)}$$

$$=\frac{(k+1)}{3(2k+5)}=\frac{(k+1)}{3\{2(k+1)+3\}}.$$

$$\therefore \quad P(k+1):\ \frac{1}{3\cdot 5}+\frac{1}{5\cdot 7}+\frac{1}{7\cdot 9}+\ldots+\frac{1}{\{2(k+1)+1\}\{2(k+1)+3\}}=\frac{(k+1)}{3\{2(k+1)+3\}}.$$

This shows that $P(k+1)$ is true, whenever $P(k)$ is true.

$\therefore$ $P(1)$ is true and $P(k+1)$ is true, whenever $P(k)$ is true.

Hence, by the principle of mathematical induction, we have

$$\frac{1}{3\cdot 5}+\frac{1}{5\cdot 7}+\frac{1}{7\cdot 9}+\ldots+\frac{1}{(2n+1)(2n+3)}=\frac{n}{3(2n+3)}$$

for all values of $n \in N$.

EXAMPLE 10 *Using the principle of mathematical induction, prove that*

$$\frac{1}{1\cdot 2\cdot 3}+\frac{1}{2\cdot 3\cdot 4}+\ldots+\frac{1}{n(n+1)(n+2)}=\frac{n(n+3)}{4(n+1)(n+2)}\ \textit{for all } n \in N.$$

SOLUTION Let the given statement be $P(n)$. Then,

$$P(n):\ \frac{1}{1\cdot 2\cdot 3}+\frac{1}{2\cdot 3\cdot 4}+\ldots+\frac{1}{n(n+1)(n+2)}=\frac{n(n+3)}{4(n+1)(n+2)}.$$

Putting $n = 1$ in the given statement, we get

$$\text{LHS}=\frac{1}{1\cdot 2\cdot 3}=\frac{1}{6}\ \text{ and }\ \text{RHS}=\frac{1\times(1+3)}{4\times(1+1)(1+2)}=\frac{1\times 4}{4\times 2\times 3}=\frac{1}{6}.$$

$\therefore$ LHS = RHS.

Thus, the given statement is true for $n = 1$, i.e., $P(1)$ is true.

Let $P(k)$ be true. Then,

$$P(k)\text{: } \frac{1}{1\cdot2\cdot3}+\frac{1}{2\cdot3\cdot4}+\ldots+\frac{1}{k(k+1)(k+2)}=\frac{k(k+3)}{4(k+1)(k+2)}. \quad \ldots\text{(i)}$$

Now, $\frac{1}{1\cdot2\cdot3}+\frac{1}{2\cdot3\cdot4}+\ldots+\frac{1}{k(k+1)(k+2)}+\frac{1}{(k+1)(k+2)(k+3)}$

$$=\left\{\frac{1}{1\cdot2\cdot3}+\frac{1}{2\cdot3\cdot4}+\ldots+\frac{1}{k(k+1)(k+2)}\right\}+\frac{1}{(k+1)(k+2)(k+3)}$$

$$=\left\{\frac{k(k+3)}{4(k+1)(k+2)}+\frac{1}{(k+1)(k+2)(k+3)}\right\} \quad \text{[using (i)]}$$

$$=\frac{k(k+3)^2+4}{4(k+1)(k+2)(k+3)}=\frac{(k^3+6k^2+9k+4)}{4(k+1)(k+2)(k+3)}$$

$$=\frac{(k+1)(k+1)(k+4)}{4(k+1)(k+2)(k+3)}=\frac{(k+1)(k+4)}{4(k+2)(k+3)}.$$

$\therefore$ $P(k+1)$: $\frac{1}{1\cdot2\cdot3}+\frac{1}{2\cdot3\cdot4}+\ldots+\frac{1}{(k+1)(k+2)(k+3)}$

$$=\frac{(k+1)(k+4)}{4(k+2)(k+3)}.$$

This shows that $P(k+1)$ is true, whenever $P(k)$ is true.

$\therefore$ $P(1)$ is true and $P(k+1)$ is true, whenever $P(k)$ is true.

Hence, by the principle of mathematical induction, we have

$$\frac{1}{1\cdot2\cdot3}+\frac{1}{2\cdot3\cdot4}+\frac{1}{3\cdot4\cdot5}+\ldots+\frac{1}{n(n+1)(n+2)}=\frac{n(n+3)}{4(n+1)(n+2)}$$

for all values of $n \in N$.

EXAMPLE 11 *Using the principle of mathematical induction, prove that*

$$1\cdot3+2\cdot3^2+3\cdot3^3+\ldots+n\cdot3^n=\frac{(2n-1)3^{n+1}+3}{4} \text{ for all } n \in N.$$

SOLUTION Let the given statement be $P(n)$. Then,

$$P(n)\text{: } 1\cdot3+2\cdot3^2+3\cdot3^3+\ldots+n\cdot3^n=\frac{(2n-1)3^{n+1}+3}{4}.$$

Putting $n = 1$ in the given statement, we get

$$\text{LHS}=1\cdot3=3 \text{ and } \text{RHS}=\frac{(2\times1-1)3^{1+1}+3}{4}=\frac{1\times9+3}{4}=\frac{12}{4}=3.$$

$\therefore$ LHS = RHS.

Thus, $P(1)$ is true.

Let $P(k)$ be true. Then,

$$P(k)\text{: } 1\cdot3+2\cdot3^2+3\cdot3^3+\ldots+k\cdot3^k=\frac{(2k-1)3^{k+1}+3}{4}. \quad \ldots\text{(i)}$$

$\therefore$ $1\cdot3+2\cdot3^2+3\cdot3^3+\ldots+k\cdot3^k+(k+1)3^{(k+1)}$

$$= \{1\cdot 3 + 2\cdot 3^2 + 3\cdot 3^3 + \ldots + k\cdot 3^k\} + (k+1)\,3^{(k+1)}$$

$$= \frac{(2k-1)\,3^{k+1}+3}{4} + (k+1)\,3^{k+1} \qquad \text{[using (i)]}$$

$$= \frac{(2k-1)\,3^{k+1}+3+4\,(k+1)\,3^{k+1}}{4} = \frac{(2k-1+4k+4)\,3^{k+1}+3}{4}$$

$$= \frac{(6k+3)\,3^{k+1}+3}{4} = \frac{\{2\,(k+1)-1\}\,3^{(k+1)+1}+3}{4}.$$

$\therefore \quad P(k+1)\colon\ 1\cdot 3 + 2\cdot 3^2 + 3\cdot 3^3 + \ldots + (k+1)\cdot 3^{k+1}$

$$= \frac{\{2(k+1)-1\}\,3^{(k+1)+1}+3}{4}.$$

This shows that $P(k+1)$ is true, whenever $P(k)$ is true.
Thus, $P(1)$ is true and $P(k+1)$ is true, whenever $P(k)$ is true.
Hence, by the principle of mathematical induction, we have

$$1\cdot 3 + 2\cdot 3^2 + 3\cdot 3^3 + \ldots + n\cdot 3^n = \frac{(2n-1)\,3^{n+1}+3}{4}$$

for all values of $n \in N$.

EXAMPLE 12 *Using the principle of mathematical induction, prove that*

$$\left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\left(1-\frac{1}{4}\right)\ldots\left(1-\frac{1}{n+1}\right) = \frac{1}{(n+1)} \textit{ for all } n \in N.$$

SOLUTION Let the given statement be $P(n)$. Then,

$$P(n)\colon\ \left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\left(1-\frac{1}{4}\right)\ldots\left(1-\frac{1}{n+1}\right) = \frac{1}{(n+1)}.$$

When $n = 1$, $\text{LHS} = \left(1-\frac{1}{2}\right) = \frac{1}{2}$ and $\text{RHS} = \frac{1}{(1+1)} = \frac{1}{2}$.

$\therefore \quad$ LHS = RHS.

Thus, $P(1)$ is true.
Let $P(k)$ be true. Then,

$$P(k)\colon\ \left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\left(1-\frac{1}{4}\right)\ldots\left(1-\frac{1}{k+1}\right) = \frac{1}{(k+1)}. \qquad \ldots \text{(i)}$$

Now, $\left\{\left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\left(1-\frac{1}{4}\right)\ldots\left(1-\frac{1}{k+1}\right)\right\}\cdot\left(1-\frac{1}{k+2}\right)$

$$= \frac{1}{(k+1)}\cdot\left[\frac{(k+2)-1}{(k+2)}\right] = \frac{1}{(k+1)}\cdot\frac{(k+1)}{(k+2)} = \frac{1}{(k+2)} \qquad \text{[using (i)].}$$

$$\therefore \quad P(k+1)\colon\ \left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\left(1-\frac{1}{4}\right)\ldots\left(1-\frac{1}{k+2}\right) = \frac{1}{(k+2)}.$$

This shows that $P(k+1)$ is true, whenever $P(k)$ is true.
Thus, $P(1)$ is true and $P(k+1)$ is true, whenever $P(k)$ is true.
Hence, by the principle of mathematical induction, we have

$$\left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\left(1-\frac{1}{4}\right)\ldots\left(1-\frac{1}{n+1}\right) = \frac{1}{(n+1)} \text{ for all } n \in N.$$

EXAMPLE 13 *Using the principle of mathematical induction, prove that*

$$a + ar + ar^2 + \ldots + ar^{n-1} = \frac{a(r^n - 1)}{(r-1)} \text{ for } r > 1 \text{ and all } n \in N.$$

SOLUTION Let the given statement be $P(n)$. Then,

$$P(n):\ a + ar + ar^2 + \ldots + ar^{n-1} = \frac{a(r^n - 1)}{(r-1)}.$$

When $n = 1$, we have

$$\text{LHS} = a \text{ and RHS} = \frac{a(r^1 - 1)}{(r-1)} = a.$$

$\therefore$ LHS = RHS.

Thus, $P(1)$ is true.

Let $P(k)$ be true. Then,

$$P(k):\ a + ar + ar^2 + \ldots + ar^{k-1} = \frac{a(r^k - 1)}{(r-1)}. \qquad \ldots \text{(i)}$$

Now, $(a + ar + ar^2 + \ldots + ar^{k-1}) + ar^k = \dfrac{a(r^k - 1)}{(r-1)} + ar^k$ [using (i)]

$$= \frac{a(r^{k+1} - 1)}{(r-1)}.$$

$\therefore$ $P(k+1):\ a + ar + ar^2 + \ldots + ar^{k-1} + ar^k = \dfrac{a(r^{k+1} - 1)}{(r-1)}.$

This shows that $P(k+1)$ is true, whenever $P(k)$ is true.

$\therefore$ $P(1)$ is true and $P(k+1)$ is true, whenever $P(k)$ is true.

Hence, by the principle of mathematical induction, we have

$$a + ar + ar^2 + \ldots + ar^{n-1} = \frac{a(r^n - 1)}{(r-1)} \text{ for } r > 1 \text{ and all } n \in N.$$

EXAMPLE 14 *Let a and b be arbitrary real numbers. Using the principle of mathematical induction, prove that*

$$(ab)^n = a^n b^n \text{ for all } n \in N.$$

SOLUTION Let the given statement be $P(n)$. Then,

$$P(n):\ (ab)^n = a^n b^n.$$

When $n = 1$, we have

$$\text{LHS} = (ab)^1 = ab \text{ and RHS} = a^1 b^1 = ab.$$

$\therefore$ LHS = RHS.

Thus, the given statement is true for $n = 1$, i.e., $P(1)$ is true.

Let $P(k)$ be true. Then,

$$P(k):\ (ab)^k = a^k b^k. \qquad \ldots \text{(i)}$$

Now, $(ab)^{k+1} = (ab)^k (ab) = (a^k b^k)(ab)$ [using (i)]

$= (a^k \cdot a)(b^k \cdot b)$ [by commutativity and associativity of multiplication on real numbers]

$= (a^{k+1} \cdot b^{k+1}).$

$\therefore \quad P(k+1): (ab)^{k+1} = (a^{k+1} \cdot b^{k+1})$.

This shows that $P(k+1)$ is true, whenever $P(k)$ is true.

$\therefore \quad P(1)$ is true and $P(k+1)$ is true, whenever $P(k)$ is true.

Hence, by the principle of mathematical induction, we have

$(ab)^n = a^n b^n$ for all $x \in N$.

EXAMPLE 15 *Using the principle of mathematical induction, prove that* (n^2+n) *is even for all* $n \in N$.

SOLUTION Let $P(n)$: (n^2+n) is even.

For $n=1$, the given expression becomes $(1^2+1)=2$, which is even.

So, the given statement is true for $n=1$, i.e., $P(1)$ is true.

Let $P(k)$ be true. Then,

$P(k)$: (k^2+k) is even

$\Rightarrow \quad (k^2+k)=2m$ for some natural number m. ... (i)

Now, $(k+1)^2+(k+1) = k^2+3k+2 = (k^2+k)+2(k+1)$

$= 2m+2(k+1)$ [using (i)]

$= 2[m+(k+1)]$, which is clearly even.

$\therefore \quad P(k+1)$: $(k+1)^2+(k+1)$ is even.

Thus, $P(k+1)$ is true, whenever $P(k)$ is true.

$\therefore \quad P(1)$ is true and $P(k+1)$ is true, whenever $P(k)$ is true.

Hence, by the principle of induction, it follows that (n^2+n) is even for all $n \in N$.

EXAMPLE 16 *Using the principle of mathematical induction, prove that* $n(n+1)(n+5)$ *is a multiple of* 3 *for all* $n \in N$.

SOLUTION Let $P(n)$: $n(n+1)(n+5)$ is a multiple of 3.

For $n=1$, the given expression becomes $(1 \times 2 \times 6) = 12$, which is a multiple of 3.

So, the given statement is true for $n=1$, i.e., $P(1)$ is true.

Let $P(k)$ be true. Then,

$P(k)$: $k(k+1)(k+5)$ is a multiple of 3

$\Rightarrow \quad k(k+1)(k+5)=3m$ for some natural number m. ... (i)

Now, $(k+1)(k+2)(k+6)$

$= (k+1)(k+2)k + 6(k+1)(k+2)$

$= k(k+1)(k+5-3) + 6(k+1)(k+2)$

$= k(k+1)(k+5) - 3k(k+1) + 6(k+1)(k+2)$

$= 3m + 3(k+1)(2k+4-k)$

$= 3m + 3(k+1)(k+4)$

$= 3\{m+(k+1)(k+4)\}$, which is a multiple of 3.

$\therefore \quad P(k+1)$: $(k+1)(k+2)(k+6)$ is a multiple of 3.

Thus, $P(k+1)$ is true, whenever $P(k)$ is true.

$\therefore$ $P(1)$ is true and $P(k+1)$ is true, whenever $P(k)$ is true.

Hence, by the principle of mathematical induction, it follows that $n(n+1)(n+5)$ is a multiple of 3 for all $n \in N$.

EXAMPLE 17 *Using the principle of mathematical induction, prove that* $(7^n - 3^n)$ *is divisible by* 4 *for all* $n \in N$.

SOLUTION Let $P(n)$: $(7^n - 3^n)$ is divisible by 4.

For $n = 1$, the given expression becomes $(7^1 - 3^1) = 4$, which is divisible by 4.

So, the given statement is true for $n = 1$, i.e., $P(1)$ is true.

Let $P(k)$ be true. Then,

$P(k)$: $(7^k - 3^k)$ is divisible by 4.

$\Rightarrow$ $(7^k - 3^k) = 4m$ for some natural number m. ... (i)

Now, $\{7^{(k+1)} - 3^{(k+1)}\}$

$= 7^{(k+1)} - 7 \cdot 3^k + 7 \cdot 3^k - 3^{(k+1)}$ [subtracting and adding $7 \cdot 3^k$]

$= 7(7^k - 3^k) + 3^k(7-3)$

$= (7 \times 4m) + 4 \cdot 3^k$ [using (i)]

$= 4(7m + 3^k)$, which is clearly divisible by 4.

$\therefore$ $P(k+1)$: $\{7^{(k+1)} - 3^{(k+1)}\}$ is divisible by 4.

Thus, $P(k+1)$ is true, whenever $P(k)$ is true.

$\therefore$ $P(1)$ is true and $P(k+1)$ is true, whenever $P(k)$ is true.

Hence, by the principle of mathematical induction, it follows that $(7^n - 3^n)$ is divisible by 4 for all values of $n \in N$.

EXAMPLE 18 *Using the principle of mathematical induction, prove that* $(10^{2n-1} + 1)$ *is divisible by* 11 *for all* $n \in N$.

SOLUTION Let $P(n)$: $(10^{2n-1} + 1)$ is divisible by 11.

For $n = 1$, the given expression becomes $\{10^{(2 \times 1 - 1)} + 1\} = 11$, which is divisible by 11.

So, the given statement is true for $n = 1$, i.e., $P(1)$ is true.

Let $P(k)$ be true. Then,

$P(k)$: $(10^{2k-1} + 1)$ is divisible by 11

$\Rightarrow$ $(10^{2k-1} + 1) = 11m$, for some natural number m. ... (i)

Now, $\{10^{2(k+1)-1} + 1\} = \{10^2 \cdot 10^{(2k-1)} + 1\}$

$= 100 \times \{10^{2k-1} + 1\} - 99$

$= (100 \times 11m) - 99$ [using (i)]

$= 11 \times (100m - 9)$, which is divisible by 11.

$\therefore$ $P(k+1)$: $\{10^{2(k+1)-1} + 1\}$ is divisible by 11.

Thus, $P(k+1)$ is true, whenever $P(k)$ is true.

$\therefore$ $P(1)$ is true and $P(k+1)$ is true, whenever $P(k)$ is true.

Hence, by the principle of mathematical induction, it follows that $\{10^{2n-1}+1\}$ is divisble by 11 for all $n \in N$.

EXAMPLE 19 *Using the principle of mathematical induction, prove that* $(2\cdot 7^n + 3\cdot 5^n - 5)$ *is divisible by 24 for all* $n \in N$.

SOLUTION Let $P(n)$: $(2\cdot 7^n + 3\cdot 5^n - 5)$ is divisible by 24.

For $n=1$, the given expression becomes $(2\cdot 7^1 + 3\cdot 5^1 - 5) = 24$, which is clearly divisible by 24.

So, the given statement is true for $n=1$, i.e., $P(1)$ is true.

Let $P(k)$ be true. Then,

$P(k)$: $(2\cdot 7^k + 3\cdot 5^k - 5)$ is divisible by 24

$\Rightarrow$ $(2\cdot 7^k + 3\cdot 5^k - 5) = 24m$, for some $m \in N$. ... (i)

Now, $(2\cdot 7^{k+1} + 3\cdot 5^{k+1} - 5)$

$= (2\cdot 7^k \cdot 7 + 3\cdot 5^k \cdot 5 - 5)$

$= 7(2\cdot 7^k + 3\cdot 5^k - 5) - 6\cdot 5^k + 30$

$= (7\times 24m) - 6(5^k - 5)$ [using (i)]

$= (24\times 7m) - 6\times 5\times(5^{k-1}-1)$

$= (24\times 7m) - 5\times 24p$ $\left[\begin{array}{l}\because\ (5^{k-1}-1) \text{ is divisible by } (5-1), \text{ i.e., } 4\\ \Rightarrow\ (5^{k-1}-1) = 4p\end{array}\right]$

$= 24\times(7m - 5p)$, which is clearly divisible by 24.

$\therefore$ $P(k+1)$: $(2\cdot 7^{k+1} + 3\cdot 5^{k+1} - 5)$ is divisible by 24.

Thus, $P(k+1)$ is true, whenever $P(k)$ is true.

$\therefore$ $P(1)$ is true and $P(k+1)$ is true, whenever $P(k)$ is true.

Hence, by the principle of mathematical induction, it follows that $(2\cdot 7^n + 3\cdot 5^n - 5)$ is divisible by 24 for all $n \in N$.

EXAMPLE 20 *Using the principle of mathematical induction, prove that* $(x^n - y^n)$ *is divisible by* $(x-y)$ *for all* $n \in N$.

SOLUTION Let the given statement be $P(n)$. Then,

$P(n)$: $(x^n - y^n)$ is divisible by $(x-y)$.

When $n=1$, the given statement becomes: $(x^1 - y^1)$ is divisible by $(x-y)$, which is clearly true.

$\therefore$ $P(1)$ is true.

Let $P(k)$ be true. Then,

$P(k)$: $(x^k - y^k)$ is divisible by $(x-y)$. ... (i)

Now, $(x^{k+1} - y^{k+1})$

$= \{x^{k+1} - x^k y + x^k y - y^{k+1}\}$ [on adding and subtracting $x^k y$]

$= x^k(x-y) + y(x^k - y^k)$, which is divisible by $(x-y)$ [using (i)].

Thus, $P(k+1)$ is true, whenever $P(k)$ is true.

$\therefore$ $P(1)$ is true and $P(k+1)$ is true, whenever $P(k)$ is true.

Hence, by the principle of mathematical induction, it follows that

$(x^n - y^n)$ is divisible by $(x-y)$ for all $n \in N$.

EXAMPLE 21 *Using the principle of mathematical induction, prove that* $(1+x)^n \geq (1+nx)$ *for all* $n \in N$, *where* $x > -1$.

SOLUTION Let the given statement be $P(n)$. Then,

$P(n)$: $(1+x)^n \geq (1+nx)$, where $x > -1$.

When $n = 1$, we have

LHS $= (1+x)^1 = (1+x)$ and RHS $= (1+1 \times x) = (1+x)$.

$\therefore$ LHS $\geq$ RHS is true.

Thus, $P(1)$ is true.

Let $P(k)$ be true. Then,

$P(k)$: $(1+x)^k \geq (1+kx)$, where $x > -1$. ... (i)

Now, $(1+x)^{k+1} = (1+x)^k(1+x)$

$\geq (1+kx)(1+x)$ [using (i)]

$= 1+(k+1)x+kx^2$

$\geq 1+(k+1)x$ [$\because$ $kx^2 \geq 0$].

$\therefore$ $P(k+1)$: $(1+x)^{k+1} \geq 1+(k+1)x$.

Thus, $P(k+1)$ is true, whenever $P(k)$ is true.

$\therefore$ $P(1)$ is true and $P(k+1)$ is true, whenever $P(k)$ is true.

Hence, by the principle of mathematical induction, we have

$(1+x)^n \geq (1+nx)$, where $x > -1$ for all $n \in N$.

EXAMPLE 22 *Using the principle of mathematical induction, prove that* $n < 2^n$ *for all* $n \in N$.

SOLUTION Let the given statement be $P(n)$. Then,

$P(n)$: $n < 2^n$.

When $n = 1$, we get LHS $= 1$ and RHS $= 2^1 = 2$.

Clearly, $1 < 2$.

$\therefore$ $P(n)$ is true for $n = 1$, i.e., $P(1)$ is true.

Let $P(k)$ be true. Then,

$P(k)$: $k < 2^k$.

Now, $k < 2^k \Rightarrow 2k < 2^{k+1}$

$\Rightarrow (k+k) < 2^{k+1}$

$\Rightarrow (k+1) \leq (k+k) < 2^{k+1}$ [$\because$ $1 \leq k$]

$\Rightarrow (k+1) < 2^{k+1}$.

$\therefore$ $P(k+1)$: $(k+1) < 2^{k+1}$.

Thus, $P(k+1)$ is true, whenever $P(k)$ is true.

$\therefore$ $P(1)$ is true and $P(k+1)$ is true, whenever $P(k)$ is true.

Hence, by the principle of mathematical induction, it follows that $n < 2^n$ for all $n \in N$.

EXAMPLE 23 *Using the principle of mathematical induction, prove that*

$$(2n+7) < (n+3)^2 \text{ for all values of } n \in N.$$

SOLUTION Let the given statement be $P(n)$. Then,

$$P(n):\ (2n+7) < (n+3)^2.$$

When $n = 1$, we have

$$\text{LHS} = (2 \times 1 + 7) = 9 \text{ and } \text{RHS} = (1+3)^2 = 4^2 = 16.$$

$\therefore$ LHS < RHS [$\because$ $9 < 16$].

Thus, $P(n)$ is true for $n = 1$, i.e., $P(1)$ is true.

Let $P(k)$ be true. Then,

$$P(k):\ (2k+7) < (k+3)^2. \quad \ldots \text{(i)}$$

Now, $2(k+1)+7 = (2k+7)+2$

$$< (k+3)^2 + 2 = (k^2+6k+11) \quad \text{[using (i)]}$$

$$< (k^2+8k+16) = (k+4)^2 = (k+1+3)^2.$$

$\therefore$ $P(k+1)$: $2(k+1)+7 < (k+1+3)^2$.

Thus, $P(k+1)$ is true, whenever $P(k)$ is true.

$\therefore$ $P(1)$ is true and $P(k+1)$ is true, whenever $P(k)$ is true.

Hence, by the principle of mathematical induction, it follows that $(2n+7) < (n+3)^2$ for all $n \in N$.

EXAMPLE 24 *Using the principle of mathematical induction, prove that*

$$(1^2+2^2+\ldots+n^2) > \frac{n^3}{3} \text{ for all values of } n \in N.$$

SOLUTION Let $P(n)$: $(1^2+2^2+\ldots+n^2) > \dfrac{n^3}{3}\cdot$

When $n = 1$, LHS $= 1^2 = 1$ and RHS $= \dfrac{1^3}{3} = \dfrac{1}{3}\cdot$

Since $1 > \frac{1}{3}$, it follows that $P(1)$ is true.

Let $P(k)$ be true. Then,

$$P(k):\ (1^2+2^2+\ldots+k^2) > \frac{k^3}{3}\cdot \quad \ldots \text{(i)}$$

Now, $1^2+2^2+\ldots+k^2+(k+1)^2$

$$= \{1^2+2^2+\ldots+k^2\}+(k+1)^2$$

$$> \frac{k^3}{3}+(k+1)^2 \quad \text{[using (i)]}$$

$$= \frac{1}{3}\cdot\{k^3+3(k+1)^2\} = \frac{1}{3}\cdot\{k^3+3k^2+6k+3\}$$

$$= \frac{1}{3}[\{(k^3 + 1 + 3k(k+1)\} + (3k+2)] = \frac{1}{3} \cdot [(k+1)^3 + (3k+2)]$$

$$> \frac{1}{3}(k+1)^3.$$

$\therefore$ $P(k+1)$: $1^2 + 2^2 + \ldots + k^2 + (k+1)^2 > \frac{1}{3}(k+1)^3$.

Thus, $P(k+1)$ is true, whenever $P(k)$ is true.

$\therefore$ $P(1)$ is true and $P(k+1)$ is true, whenever $P(k)$ is true.

Hence, by the principle of mathematical induction, it follows that

$$(1^2 + 2^2 + 3^2 + \ldots + n^2) > \frac{n^3}{3} \text{ for all } n \in N.$$

EXAMPLE 25 *Using the principle of mathematical induction, prove that*

$$(1 + 2 + 3 + \ldots + n) < \frac{1}{8}(2n+1)^2 \textit{ for all values of } n \in N.$$

SOLUTION Let the given statement be $P(n)$. Then,

$$P(n): (1 + 2 + 3 + \ldots + n) < \frac{1}{8}(2n+1)^2.$$

When $n = 1$, we have

$$\text{LHS} = 1 \text{ and RHS} = \frac{1}{8}(2 \times 1 + 1)^2 = \frac{9}{8}.$$

Clearly, $1 < \frac{9}{8}$.

$\therefore$ $P(1)$ is true.

Let $P(k)$ be true. Then,

$$P(k): (1 + 2 + 3 + \ldots + k) < \frac{1}{8}(2k+1)^2. \qquad \ldots \text{(i)}$$

Now, $1 + 2 + 3 + \ldots + k + (k+1)$

$$= \{1 + 2 + 3 + \ldots + k\} + (k+1)$$

$$< \frac{1}{8}(2k+1)^2 + (k+1) = \frac{(2k+1)^2 + 8(k+1)}{8} \qquad \text{[using (i)]}$$

$$= \frac{(4k^2 + 12k + 9)}{8} = \frac{(2k+3)^2}{8} = \frac{\{2(k+1)+1\}^2}{8}.$$

$\therefore$ $\{1 + 2 + 3 + \ldots + k + (k+1)\} < \frac{\{2(k+1)+1\}^2}{8}$.

This shows that $P(k+1)$ is true.

Thus, $P(k+1)$ is true, whenever $P(k)$ is true.

$\therefore$ $P(1)$ is true and $P(k+1)$ is true, whenever $P(k)$ is true.

Hence, by the principle of mathematical induction, it follows that

$$(1 + 2 + 3 + \ldots + n) < \frac{1}{8}(2n+1)^2 \text{ for all values of } n \in N.$$

EXERCISE 4

Using the principle of mathematical induction, prove each of the following for all $n \in N$:

1. $1+2+3+4+\ldots+n=\frac{1}{2}n(n+1)$.

2. $2+4+6+8+\ldots+2n=n(n+1)$.

3. $1+3+3^2+3^3+\ldots+3^{n-1}=\frac{1}{2}(3^n-1)$.

4. $2+6+18+\ldots+2\cdot 3^{n-1}=(3^n-1)$.

5. $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\ldots+\frac{1}{2^n}=\left(1-\frac{1}{2^n}\right)$.

6. $1^2+3^2+5^2+7^2+\ldots+(2n-1)^2=\frac{n(2n-1)(2n+1)}{3}$.

7. $1\cdot 2+2\cdot 2^2+3\cdot 2^3+\ldots+n\cdot 2^n=(n-1)2^{n+1}+2$.

8. $3\cdot 2^2+3^2\cdot 2^3+3^3\cdot 2^4+\ldots+3^n\cdot 2^{n+1}=\frac{12}{5}(6^n-1)$.

9. $1+\frac{1}{(1+2)}+\frac{1}{(1+2+3)}+\ldots+\frac{1}{(1+2+3+\ldots+n)}=\frac{2n}{(n+1)}$.

10. $\frac{1}{2\cdot 5}+\frac{1}{5\cdot 8}+\frac{1}{8\cdot 11}+\ldots+\frac{1}{(3n-1)(3n+2)}=\frac{n}{(6n+4)}$.

11. $\frac{1}{1\cdot 4}+\frac{1}{4\cdot 7}+\frac{1}{7\cdot 10}+\ldots+\frac{1}{(3n-2)(3n+1)}=\frac{n}{(3n+1)}$.

12. $\frac{1}{1\cdot 3}+\frac{1}{3\cdot 5}+\frac{1}{5\cdot 7}+\ldots+\frac{1}{(2n-1)(2n+1)}=\frac{n}{(2n+1)}$.

13. $\frac{1}{2\cdot 5}+\frac{1}{5\cdot 8}+\frac{1}{8\cdot 11}+\ldots+\frac{1}{(3n-1)(3n+2)}=\frac{n}{(6n+4)}$.

14. $\left(1+\frac{3}{1}\right)\left(1+\frac{5}{4}\right)\left(1+\frac{7}{9}\right)\ldots\left\{1+\frac{(2n+1)}{n^2}\right\}=(n+1)^2$.

15. $\left(1+\frac{1}{1}\right)\left(1+\frac{1}{2}\right)\left(1+\frac{1}{3}\right)\ldots\left(1+\frac{1}{n}\right)=(n+1)$.

16. $n(n+1)(n+2)$ is a multiple of 6.

17. $(x^{2n}-y^{2n})$ is divisible by $(x+y)$.

18. $(x^{2n}-1)$ is divisible by $(x-1)$, where $x\neq 1$.

19. $\{(41)^n-(14)^n\}$ is divisible by 27.

20. $(4^n+15n-1)$ is divisible by 9.

21. $(3^{2n+2}-8n-9)$ is divisible by 8.

22. $(2^{3n}-1)$ is a multiple of 7.

23. $3^n \geq 2^n$

HINTS TO SOME SELECTED QUESTIONS

6. $P(k)$: $1^2+3^2+5^2+\ldots+(2k-1)^2=\dfrac{k(2k-1)(2k+1)}{3}$.

Now, $\{1^2+3^2+5^2+\ldots+(2k-1)^2\}+\{2(k+1)-1\}^2$

$$=\frac{k(2k-1)(2k+1)}{3}+(2k+1)^2=\frac{1}{3}\cdot\{k(2k-1)(2k+1)+3(2k+1)^2\}$$
$$=\frac{1}{3}(2k+1)\{k(2k-1)+3(2k+1)\}=\frac{1}{3}(2k+1)(2k^2+5k+3)$$
$$=\frac{1}{3}(k+1)(2k+1)(2k+3).$$

7. $P(k)$: $1\cdot 2+2\cdot 2^2+3\cdot 2^3+\ldots+k\cdot 2^k=(k-1)\cdot 2^{k+1}+2$. ... (i)

Now, $\{1\cdot 2+2\cdot 2^2+3\cdot 2^3+\ldots+k\cdot 2^k\}+(k+1)\cdot 2^{k+1}$

$$=\{(k-1)\cdot 2^{k+1}+2\}+(k+1)\cdot 2^{k+1}=2k\cdot 2^{k+1}+2=k\cdot 2^{k+2}+2.$$

8. $P(k)$: $3\cdot 2^2+3^2\cdot 2^3+\ldots+3^k\cdot 2^{k+1}=\dfrac{12}{5}(6^k-1)$.

Now, $(3\cdot 2^2+3^2\cdot 2^3+\ldots+3^k\cdot 2^{k+1})+3^{k+1}\cdot 2^{k+2}$

$$=\left\{\frac{12}{5}(6^k-1)+3^{k+1}\cdot 2^{k+1}\cdot 2\right\}=\frac{1}{5}\cdot\{12(6^k-1)+10\times 6^{k+1}\}$$
$$=\frac{1}{5}\cdot\{2(6^{k+1}-6)+10\times 6^{k+1}\}=\frac{12}{5}(6^{k+1}-1).$$

14. $P(k)$: $(1+3)\left(1+\frac{5}{4}\right)\left(1+\frac{7}{9}\right)\ldots\left\{1+\frac{(2k+1)}{k^2}\right\}=(k+1)^2$.

Now, $(1+3)\left(1+\frac{5}{4}\right)\ldots\left(1+\frac{2k+1}{k^2}\right)\left\{1+\frac{2(k+1)+1}{(k+1)^2}\right\}$

$$=(k+1)^2\times\frac{(k+1)^2+(2k+3)}{(k+1)^2}=(k^2+4k+4)=(k+2)^2.$$

15. $P(k)$: $(1+1)\left(1+\frac{1}{2}\right)\left(1+\frac{1}{3}\right)\ldots\left(1+\frac{1}{k}\right)=(k+1)$.

Now, $\left\{(1+1)\left(1+\frac{1}{2}\right)\left(1+\frac{1}{3}\right)\ldots\left(1+\frac{1}{k}\right)\right\}\left(1+\frac{1}{k+1}\right)$

$$=(k+1)\left(1+\frac{1}{k+1}\right)=(k+1)\cdot\frac{(k+2)}{(k+1)}=(k+2).$$

16. Let $k(k+1)(k+2)=6p$ ... (i). Then,

$(k+1)(k+2)(k+3)=(k+1)(k+2)k+3(k+1)(k+2)=k(k+1)(k+2)+3(k+1)(k+2)$

$=6p+3(k+1)(k+2)$ [using (i)]

$=6p+6q=6(p+q)$, where $(k+1)(k+2)=2q$

{$\because$ $(k+1)$ and $(k+2)$ being consecutive integers, one of them must be even and so their product is even}.

17. Let $\dfrac{(x^{2n}-y^{2n})}{(x+y)}=p$ ... (i). Then,

$\{x^{2(n+1)}-y^{2(n+1)}\}=(x^{2n}\cdot x^2-y^{2n}\cdot y^2)=x^{2n}\cdot x^2-x^2y^{2n}+x^2y^{2n}-y^{2n}\cdot y^2$

$= x^2(x^{2n} - y^{2n}) + y^{2n}(x^2 - y^2) = x^2 p(x + y) + y^{2n}(x^2 - y^2)$ [using (i)]

$= (x + y)[x^2 p + y^{2n}]$, which is divisible by $(x + y)$.

18. Let $\dfrac{(x^{2n} - 1)}{(x - 1)} = p$. ... (i). Then,

$\{x^{2(n+1)} - 1\} = (x^{2n} \cdot x^2 - x^2 + x^2 - 1) = x^2(x^{2n} - 1) + (x^2 - 1)$

$= x^2 \cdot p(x - 1) + (x^2 - 1)$ [using (i)]

$= (x - 1)(px^2 + x + 1)$, which is divisible by $(x - 1)$.

19. Let $\dfrac{(41)^k - (14)^k}{27} = p$. ... (i). Then,

$\{(41)^{k+1} - (14)^{k+1}\} = (41)^{k+1} - (41)^k \cdot 14 + (41)^k \cdot 14 - (14)^{k+1}$

$= (41)^k(41 - 14) + 14\{(41)^k - (14)^k\} = 27 \times (41)^k + 14 \times 27p$

$= 27 \times [(41)^k + 14]$.

20. Let $(4^k + 15k - 1) = 9p$ ... (i). Then,

$\{4^{(k+1)} + 15(k + 1) - 1\} = \{4^{(k+1)} + 15k + 14\} = (4 \cdot 4^k + 60k - 4) - 45k + 18$

$= 4(4^k + 15k - 1) - 9(5k - 2) = (4 \times 9p) - 9(5k - 2)$ [using (i)]

$= 9 \times (4p - 5k + 2)$, which is divisible by 9.

21. Let $(3^{2k+2} - 8k - 9) = 8p$ ... (i). Then,

$\{3^{2(k+1)+2} - 8(k + 1) - 9\} = \{3^{(2k+2)} \cdot 3^2 - 8k - 17\} = (9 \cdot 3^{(2k+2)} - 72k - 81) + (64k + 64)$

$= 9 \cdot \{3^{(2k+2)} - 8k - 9\} + 64(k + 1)$

$= (9 \times 8p) + 64(k + 1)$ [using (i)]

$= 8 \times [9p + 8(k + 1)]$, which is divisible by 8.

22. Let $(2^{3k} - 1) = 7p$ ... (i). Then,

$\{2^{3(k+1)} - 1\} = (2^{3k} \cdot 2^3 - 1) = (8 \cdot 2^{3k} - 8) + 7 = \{8(2^{3k} - 1) + 7\} = (8 \times 7p) + 7$ [using (i)]

$= (56p + 7) = 7(8p + 1)$, which is divisible by 7.

23. Clearly, $3^1 \geq 2^1$. So, the result is true for $n = 1$.

Let it be true for $n = k$. Then,

$3^k \geq 2^k$ and $3 > 2 \;\Rightarrow\; 3^k \cdot 3 \geq 2^k \cdot 2 \;\Rightarrow\; 3^{k+1} \geq 2^{k+1}$.

So, whenever the result is true for k, then it is also true for $(k + 1)$.

SUMMARY OF KEY FACTS

PRINCIPLE OF MATHEMATICAL INDUCTION

Let $P(n)$ be a statement involving the natural number n such that

(i) *$P(1)$ is true and (ii) $P(k + 1)$ is true, whenever $P(k)$ is true then $P(n)$ is true for all $n \in N$.*

❑

5

Complex Numbers and Quadratic Equations

IMAGINARY NUMBERS

If the square of a given number is negative then such a number is called an imaginary number.

For example, $\sqrt{-1}$, $\sqrt{-2}$, etc., are imaginary numbers.

We denote $\sqrt{-1}$ *by the Greek letter* iota 'ι', which is transliterated as '*i*'.

Thus, $\sqrt{-4} = 2i$, $\sqrt{-9} = 3i$ and $\sqrt{-5} = i\sqrt{5}$, etc.

POWERS OF *i* We have

$$i^0 = 1, i^1 = i, i^2 = -1, i^3 = i^2 \times i = (-1) \times i = -i$$

and $I^4 = i^2 \times i^2 = (-1) \times (-1) = 1$.

Thus, we have

$$\boxed{i^0 = 1, i^1 = i, i^2 = -1, i^3 = -i, i^4 = 1.}$$

Let us consider i^n, where n is a positive integer and $n > 4$.

On dividing n by 4, let the quotient be m and the remainder be r. Then,

$$n = 4m + r, \text{ where } 0 \le r < 4.$$

$$\therefore \quad i^n = i^{4m+r} = i^{4m} \times i^r = (i^4)^m \times i^r = i^r. \quad [\because\ i^4 = 1]$$

EXAMPLES (i) $i^{98} = i^{4 \times 24 + 2} = (i^4)^{24} \times i^2 = i^2 = -1. \quad [\because\ i^4 = 1]$

(ii) $i^{-98} = \dfrac{1}{i^{98}} = \dfrac{1}{i^{98}} \times \dfrac{i^2}{i^2} = \dfrac{i^2}{i^{(98+2)}} = \dfrac{-1}{1} = -1. \quad [\because\ i^{100} = 1 \text{ and } i^2 = -1]$

SOLVED EXAMPLES

EXAMPLE 1 *Evaluate:*

(i) i^{23} *(ii)* i^{998} *(iii)* i^{-998} *(iv)* i^{-71}

(v) $(\sqrt{-1})^{91}$ *(vi)* $(i^{37} \times i^{-61})$ *(vii)* i^{-1}

SOLUTION We have

(i) $i^{23} = i^{(4 \times 5)+3} = (i^4)^5 \times i^3 = i^3 = -i. \quad [\because\ i^4 = 1]$

(ii) $i^{998} = i^{4 \times 249 + 2} = (i^4)^{249} \times i^2 = (1 \times i^2) = i^2 = -1. \quad [\because\ i^4 = 1]$

(iii) $i^{-998} = \frac{1}{i^{998}} \times \frac{i^2}{i^2} = \frac{i^2}{i^{1000}} = \frac{-1}{1} = -1.$ $[\because\ i^{1000} = (i^4)^{250} = 1]$

(iv) $i^{-71} = \frac{1}{i^{71}} \times \frac{i}{i} = \frac{i}{i^{72}} = \frac{i}{(i^4)^{18}} = \frac{i}{1} = i.$ $[\because\ i^4 = 1]$

(v) $(\sqrt{-1})^{91} = i^{91} = i^{4\times 22+3} = (i^4)^{22} \times i^3 = 1 \times (-i) = -i.$ $[\because\ i^3 = i]$

(vi) $i^{37} = i^{4\times 9+1} = (i^4)^9 \times i = 1 \times i = i.$

$$i^{-61} = \frac{1}{i^{61}} \times \frac{i^3}{i^3} = \frac{i^3}{i^{64}} = \frac{-i}{(i^4)^{16}} = \frac{-i}{1} = -i.$$

$\therefore\ (i^{37} + i^{-61}) = i + (-i) = 0.$

(vii) $i^{-1} = \frac{1}{i} = \frac{1}{i} \times \frac{i^3}{i^3} = \frac{-i}{i^4} = \frac{-i}{1} = -i.$

EXAMPLE 2 *Prove that:*

(i) $i^n + i^{n+1} + i^{n+2} + i^{n+3} = 0$

(ii) $i^{107} + i^{112} + i^{117} + i^{122} = 0$

(iii) $(1+i)^4 \times \left(1 + \frac{1}{i}\right)^4 = 16.$

SOLUTION We have

(i) $i^n + i^{n+1} + i^{n+2} + i^{n+3}$

$= i^n(1 + i + i^2 + i^3)$

$= i^n(1 + i - 1 - i) = (i^n \times 0) = 0.$ $[\because\ i^2 = -1 \text{ and } i^3 = -i]$

(ii) $i^{107} + i^{112} + i^{117} + i^{122}$

$= i^{107}(1 + i^5 + i^{10} + i^{15}) = i^{107}(1 + i^4 \times i + i^8 \times i^2 + i^{12} \times i^3)$

$= i^{107}(1 + i + i^2 + i^3)$ $[\because\ i^4 = 1, i^8 = 1 \text{ and } i^{12} = 1]$

$= i^{107}(1 + i - 1 - i) = (i^{107} \times 0) = 0.$ $[\because\ i^2 = -1, i^3 = -i]$

(iii) $(1+i)^4 \times \left(1 + \frac{1}{i}\right)^4$

$= (1+i)^4 \times \left(1 + \frac{1}{i} \times \frac{i}{i}\right)^4 = (1+i)^4(1-i)^4$ $[\because\ i^2 = -1]$

$= \{(1+i)(1-i)\}^4 = (1-i^2)^4 = \{1-(-1)\}^4 = 2^4 = 16.$

EXAMPLE 3 *Show that* $\left\{i^{23} + \left(\frac{1}{i}\right)^{29}\right\}^2 = -4.$

SOLUTION We have

$i^{23} = i^{(4\times 5+3)} = (i^4)^5 \times i^3 = 1 \times (-i) = -i.$ $[\because\ i^4 = 1 \text{ and } i^3 = -i]$

$\left(\frac{1}{i}\right)^{29} = \frac{1}{i^{29}} = \frac{1}{i^{29}} \times \frac{i^3}{i^3} = \frac{i^3}{i^{32}} = \frac{-i}{1} = -i.$ $[\because\ i^3 = -i \text{ and } i^{32} = 1]$

$\therefore\ \left\{i^{23} + \left(\frac{1}{i}\right)^{29}\right\}^2 = (-i-i)^2 = (-2i)^2 = 4i^2 = 4 \times (-1) = -4.$

EXAMPLE 4 *Simplify:*

(i) $(-2i)\left(\frac{1}{6}i\right)$ (ii) $(-i)(3i)\left(\frac{-1}{6}i\right)^3$ (iii) $4\sqrt{-4}+5\sqrt{-9}-3\sqrt{-16}$

SOLUTION We have

(i) $(-2i)\left(\frac{1}{6}i\right)=\left(-2\times\frac{1}{6}\right)\times i^2=\frac{-1}{3}\times(-1)=\frac{1}{3}$.

(ii) $(-i)(3i)\left(\frac{-1}{6}i\right)^3=(-3i^2)\left(\frac{-1}{216}i^3\right)$

$=(-3)\times(-1)\left[\frac{-1}{216}\times(-i)\right]$ $[\because\ i^3=-i]$

$=\left(3\times\frac{1}{216}\times i\right)=\frac{1}{72}i.$

(iii) $4\sqrt{-4}+5\sqrt{-9}-3\sqrt{-16}$

$=(4\times 2i)+(5\times 3i)-(3\times 4i)$ $[\because\ \sqrt{-4}=2i, \sqrt{-9}=3i, \sqrt{-16}=4i]$

$=(8i+15i-12i)=(11)i.$

EXAMPLE 5 *Show that* $(-\sqrt{-1})^{4n+3}=i$, *where n is a positive integer.*

SOLUTION We have

$(-\sqrt{-1})^{4n+3}=(-i)^{4n+3}$ $[\because\ \sqrt{-1}=i]$

$=(-i)^{4n}\times(-i)^3$

$=\{(-i)^4\}^n\times(-i^3)=(1\times i)=i.$

$[\because\ (-i)^4=1 \text{ and } -i^3=-(-i)=i]$

Hence, $(-\sqrt{-1})^{4n+3}=i.$

EXAMPLE 6 *Show that the sum* $(1+i^2+i^4+\ldots+i^{2n})$ *is* 0 *when n is odd and* 1 *when n is even.*

SOLUTION Let $S=1+i^2+i^4+\ldots+i^{2n}$.

This is clearly a GP having $(n+1)$ terms with $a=1$ and $r=i^2=-1$.

$$\therefore\quad S=\frac{a(1-r^{n+1})}{(1-r)}=\frac{1\times\{1-(i^2)^{n+1}\}}{(1-i^2)}$$

$$=\frac{\{1-(-1)^{n+1}\}}{1-(-1)}=\frac{\{1-(-1)^{n+1}\}}{2}$$

$$=\begin{cases}\frac{1}{2}(1-1)=0, \text{ when } n \text{ is odd}\\ \frac{1}{2}(1+1)=1, \text{ when } n \text{ is even.}\end{cases}$$

AN IMPORTANT RESULT

For any two real numbers a and b, the result $\sqrt{a}\times\sqrt{b}=\sqrt{ab}$ is true only when at least one of the given numbers is either zero or positive.

Thus, $\sqrt{-2}\times\sqrt{-3}=\sqrt{(-2)\times(-3)}=\sqrt{6}$ is wrong.

In fact, $\sqrt{-2}\times\sqrt{-3}=(i\sqrt{2})(i\sqrt{3})=i^2\times\sqrt{6}=-\sqrt{6}$.

EXAMPLE 7 *Explain the fallacy:*

$$-1 = (i \times i) = \sqrt{-1} \times \sqrt{-1} = \sqrt{(-1) \times (-1)} = \sqrt{1} = 1.$$

SOLUTION We know that for any real numbers a and b, $\sqrt{a} \times \sqrt{b} = \sqrt{ab}$ is true only when at least one of a and b is either 0 or positive.

$\therefore \quad \sqrt{-1} \times \sqrt{-1} \neq \sqrt{(-1) \times (-1)}$.

EXAMPLE 8 *Evaluate:*

(i) $\sqrt{-25} \times \sqrt{-49}$ (ii) $\sqrt{-36} \times \sqrt{16}$

SOLUTION We have

(i) $\sqrt{-25} \times \sqrt{-49} = (5i) \times (7i) = (35 \times i^2) = 35 \times (-1) = -35.$

(ii) $\sqrt{-36} \times \sqrt{16} = (6i) \times 4 = 24i.$

EXAMPLE 9 *Evaluate* $\sqrt{-16} + 3\sqrt{-25} + \sqrt{-36} - \sqrt{-625}$.

SOLUTION We have

$$\sqrt{-16} + 3\sqrt{-25} + \sqrt{-36} - \sqrt{-625}$$
$$= (4i + 3 \times 5i + 6i - 25i) = (4i + 15i + 6i - 25i) = 0.$$

EXERCISE 5A

1. Evaluate: (i) i^{19} (ii) i^{62} (iii) i^{373}
2. Evaluate: (i) $(\sqrt{-1})^{192}$ (ii) $(\sqrt{-1})^{93}$ (iii) $(\sqrt{-1})^{30}$
3. Evaluate: (i) i^{-50} (ii) i^{-9} (iii) i^{-131}
4. Evaluate: (i) $\left(i^{41} + \dfrac{1}{i^{71}}\right)$ (ii) $\left(i^{53} + \dfrac{1}{i^{53}}\right)$

Prove that:

5. $1 + i^2 + i^4 + i^6 = 0.$
6. $6i^{50} + 5i^{33} - 2i^{15} + 6i^{48} = 7i.$
7. $\dfrac{1}{i} - \dfrac{1}{i^2} + \dfrac{1}{i^3} - \dfrac{1}{i^4} = 0.$
8. $(1 + i^{10} + i^{20} + i^{30})$ is a real number.
9. $\left\{i^{21} - \left(\dfrac{1}{i}\right)^{46}\right\}^2 = 2i.$
10. $\left\{i^{18} + \left(\dfrac{1}{i}\right)^{25}\right\}^3 = 2(1 - i).$
11. $(1 - i)^n \left(1 - \dfrac{1}{i}\right)^n = 2^n$ for all values of $n \in N$.
12. $\sqrt{-16} + 3\sqrt{-25} + \sqrt{-36} - \sqrt{-625} = 0.$
13. $(1 + i^2 + i^4 + i^6 + i^8 + \ldots + i^{20}) = 1.$
14. $i^{53} + i^{72} + i^{93} + i^{102} = 2i.$
15. $\sum\limits_{n=1}^{13} (i^n + i^{n+1}) = (-1 + i), n \in N.$

ANSWERS (EXERCISE 5A)

1. (i) $-i$ (ii) -1 (iii) i
2. (i) 1 (ii) i (iii) -1
3. (i) -1 (ii) $-i$ (iii) i
4. (i) $2i$ (ii) 0

HINTS TO SOME SELECTED QUESTIONS

7. $\frac{1}{i}=\frac{1}{i}\times\frac{i^3}{i^3}=i^3=-i,\ \frac{1}{i^2}=\frac{1}{-1}=-1,\ \frac{1}{i^3}=\frac{1}{i^3}\times\frac{i}{i}=i$ and $\frac{1}{i^4}=1.$

$\therefore\quad \left\{\frac{1}{i}-\frac{1}{i^2}+\frac{1}{i^3}-\frac{1}{i^4}\right\}=(-i+1+i-1)=0.$

8. $(1+i^{10}+i^{20}+i^{30})=\{1+i^{4\times 2+2}+i^{4\times 5}+i^{4\times 7+2}\}$

$=\{1+(i^4)^2\times i^2+(i^4)^5+(i^4)^7\times i^2\}$

$=(1+i^2+1+i^2)=(1-1+1-1)=0.$

11. $\frac{1}{i}=\frac{1}{i}\times\frac{i^3}{i^3}=\frac{i^3}{i^4}=i^3=-i.$

$\therefore\quad (1-i)^n\left(1-\frac{1}{i}\right)^n=(1-i)^n(1+i)^n \qquad \left[\because\ \frac{1}{i}=-i\right]$

$=\{(1-i)(1+i)\}^n=(1-i^2)^n=2^n. \qquad [\because\ i^2=-1]$

13. This is a GP in which $a=1,\ r=i^2=-1$ and $n=11$.

$\therefore\quad S=\frac{a(1-r^n)}{(1-r)}=\frac{1\times\{1-(-1)^{11}\}}{\{1-(-1)\}}=\frac{2}{2}=1.$

14. $i^{53}+i^{72}+i^{93}+i^{102}=(i^4)^{13}\times i+(i^4)^{18}+(i^4)^{23}\times i+(i^4)^{25}\times i^2$

$=(1\times i)+1+(1\times i)+1\times(-1)=i+1+i-1=2i.$

15. $\sum_{n=1}^{13}(i^n+i^{n+1})=\sum_{n=1}^{13}i^n(1+i)=(1+i)\cdot\sum_{n=1}^{13}i^n$

$=(1+i)\cdot\{i+i^2+i^3+\ldots+i^{13}\}$

$=(1+i)\frac{a(1-r^n)}{(1-r)}$, where $a=i,\ r=i$ and $n=13$

$=(1+i)\cdot\frac{i(1-i^{13})}{(1-i)}=i(1+i)\cdot\frac{(1-i)}{(1-i)}=(-1+i).\ [\because\ i^{13}=(i^4)^3\times i=1\times i=i]$

COMPLEX NUMBERS

COMPLEX NUMBERS *The numbers of the form* $(a+ib)$, *where a and b are real numbers and* $i=\sqrt{-1}$, *are known as complex numbers.* The set of all complex numbers is denoted by C.

$\therefore\quad C=\{(a+ib): a, b\in R\}.$

EXAMPLES Each of the numbers $(5+8i)$, $(-3+\sqrt{2}\,i)$ and $\left(\frac{2}{3}-\frac{5}{7}i\right)$ is a complex number.

For a complex number, $z=(a+ib)$, *we have*

a = *real part of* z, written as $\text{Re}(z)$

and b = *imaginary part of* z, written as $\text{Im}(z)$.

EXAMPLES (i) If $z = (5 + 9i)$ then $\text{Re}(z) = 5$ and $\text{Im}(z) = 9$.

(ii) If $z = \left(\frac{2}{3} - 3i\right)$ then $\text{Re}(z) = \frac{2}{3}$ and $\text{Im}(z) = -3$.

(iii) If $z = \left(-7 + \frac{5}{8}i\right)$ then $\text{Re}(z) = -7$ and $\text{Im}(z) = \frac{5}{8}\cdot$

PURELY REAL AND PURELY IMAGINARY NUMBERS

A complex number z is said to be

(i) *purely real*, if $\text{Im}(z) = 0$,

(ii) *purely imaginary*, if $\text{Re}(z) = 0$.

Thus, each of the numbers $2, -7, \sqrt{3}$ is *purely real*.

And, each of the numbers $2i$, $(\sqrt{3}\,i)$, $\left(\frac{-3}{2}i\right)$ is *purely imaginary*.

CONJUGATE OF A COMPLEX NUMBER

Conjugate of a complex number $z = (a + ib)$ *is defined as,* $\bar{z} = (a - ib)$.

EXAMPLES (i) $\overline{(3+8i)} = (3 - 8i)$ (ii) $\overline{(-6-2i)} = (-6 + 2i)$ (iii) $-\bar{3} = -3$.

MODULUS OF A COMPLEX NUMBER

Modulus of a complex number $z = (a + ib)$, *denoted by* $|z|$, *is defined as* $|z| = \sqrt{a^2 + b^2}$.

EXAMPLES (i) If $z = (2 + 3i)$ then $|z| = \sqrt{2^2 + 3^2} = \sqrt{13}$.

(ii) If $z = (-5 - 4i)$ then $|z| = \sqrt{(-5)^2 + (-4)^2} = \sqrt{41}$.

EQUALITY OF COMPLEX NUMBERS If $z_1 = a_1 + ib_1$ and $z_2 = a_2 + ib_2$ then

$$z_1 = z_2 \Leftrightarrow a_1 = a_2 \text{ and } b_1 = b_2.$$

EXAMPLE *If* $2y + (3x - y)i = (5 - 2i)$, *find the values of* x *and* y.

SOLUTION Equating the real and imaginary parts, we get

$$2y + (3x - y)i = (5 - 2i)$$

$$\Leftrightarrow \quad 2y = 5 \quad \text{and} \quad 3x - y = -2$$

$$\Leftrightarrow \quad y = \frac{5}{2} \quad \text{and} \quad 3x - \frac{5}{2} = -2$$

$$\Leftrightarrow \quad y = \frac{5}{2} \quad \text{and} \quad x = \frac{1}{6}\cdot$$

Hence, $x = \frac{1}{6}$ and $y = \frac{5}{2}\cdot$

SUM AND DIFFERENCE OF COMPLEX NUMBERS

If $z_1 = (a_1 + ib_1)$ *and* $z_2 = (a_2 + ib_2)$ *then we define:*

(i) $z_1 + z_2 = (a_1 + a_2) + i(b_1 + b_2)$

(ii) $z_1 - z_2 = (a_1 - a_2) + i(b_1 - b_2)$.

EXAMPLES (i) If $z_1 = (3+5i)$ and $z_2 = (-5+2i)$ then

$$z_1 + z_2 = \{3+(-5)\} + i\{5+2\} = (-2+7i)$$

and $z_1 - z_2 = \{3-(-5)\} + i(5-2) = (8+3i)$.

(ii) If $z_1 = (-6-2i)$ and $z_2 = (-3-5i)$ then

$$z_1 + z_2 = \{(-6)+(-3)\} + i\{(-2)+(-5)\} = (-9-7i)$$

and $z_1 - z_2 = \{-6-(-3)\} + i\{(-2)-(-5)\} = (-3+3i)$.

REMARKS (i) $\text{Re}(z_1+z_2) = \text{Re}(z_1) + \text{Re}(z_2)$ and $\text{Im}(z_1+z_2) = \text{Im}(z_1) + \text{Im}(z_2)$.

(ii) $\text{Re}(z_1-z_2) = \text{Re}(z_1) - \text{Re}(z_2)$ and $\text{Im}(z_1-z_2) = \text{Im}(z_1) - \text{Im}(z_2)$.

PROPERTIES OF ADDITION OF COMPLEX NUMBERS

(i) CLOSURE PROPERTY

The sum of two complex numbers is always a complex number.

Let $z_1 = (a_1 + ib_1)$ and $z_2 = (a_2 + ib_2)$ be any two complex numbers.

Then, $z_1 + z_2 = (a_1 + ib_1) + (a_2 + ib_2)$

$= (a_1 + a_2) + i(b_1 + b_2)$, which is a complex number.

Thus, if z_1 and z_2 are any two complex numbers then $(z_1 + z_2)$ is also a complex number.

(ii) COMMUTATIVE LAW

For any two complex numbers z_1 and z_2, prove that

$$z_1 + z_2 = z_2 + z_1.$$

PROOF Let $z_1 = (a_1 + ib_1)$ and $z_2 = (a_2 + ib_2)$, where a_1, a_2, b_1, b_2 are real numbers.

$$\begin{aligned}\therefore \quad z_1 + z_2 &= (a_1 + ib_1) + (a_2 + ib_2)\\ &= (a_1 + a_2) + i(b_1 + b_2)\\ &= (a_2 + a_1) + i(b_2 + b_1) \quad \text{[by commutative law of addition in } R]\\ &= (a_2 + ib_2) + (a_1 + ib_1) = z_2 + z_1.\end{aligned}$$

Thus, $z_1 + z_2 = z_2 + z_1$ for all $z_1, z_2 \in C$.

Hence, addition of complex numbers is commutative.

(iii) ASSOCIATIVE LAW

For any complex numbers z_1, z_2 and z_3, prove that

$$(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3).$$

PROOF Let $z_1 = (a_1 + ib_1)$, $z_2 = (a_2 + ib_2)$ and $z_3 = (a_3 + ib_3)$, where a_1, a_2, a_3 and b_1, b_2, b_3 are all real numbers.

$$\begin{aligned}\therefore \quad (z_1 + z_2) + z_3 &= \{(a_1 + ib_1) + (a_2 + ib_2)\} + (a_3 + ib_3)\\ &= \{(a_1 + a_2) + i(b_1 + b_2)\} + (a_3 + ib_3)\\ &= \{(a_1 + a_2) + a_3\} + i\{(b_1 + b_2) + b_3\}\\ &= \{a_1 + (a_2 + a_3)\} + i\{b_1 + (b_2 + b_3\}\\ &\qquad \text{[by associative law of addition in } R]\\ &= (a_1 + ib_1) + \{(a_2 + a_3) + i(b_2 + b_3)\}\end{aligned}$$

$$= (a_1 + ib_1) + \{(a_2 + ib_2) + (a_3 + ib_3)\}$$
$$= z_1 + (z_2 + z_3).$$

$\therefore \quad (z_1 + z_2) + z_3 = z_1 + (z_2 + z_3)$ for all $z_1, z_2, z_3 \in C$.

Hence, addition of complex numbers is associative.

(iv) EXISTENCE OF ADDITIVE IDENTITY

For any complex number z, prove that

$$z + 0 = 0 + z = z.$$

PROOF Let $z = (a + ib)$ and we may write, $0 = (0 + i0)$. Then,

$$z + 0 = (a + ib) + (0 + i0) = (a + 0) + i(b + 0) = (a + ib)$$

and $0 + z = (0 + i0) + (a + ib) = (0 + a) + i(0 + b) = (a + ib)$.

$\therefore \quad z + 0 = 0 + z = z$ for all values of $x \in C$.

Thus, 0 is the additive identity for complex numbers.

(v) EXISTENCE OF ADDITIVE INVERSE

For every complex number z, prove that

$$z + (-z) = (-z) + z = 0.$$

PROOF Let $z = (a + ib)$. Then, $(-z) = (-a) + i(-b)$.

$\therefore \quad z + (-z) = (a + ib) + \{(-a) + i(-b)\}$
$$= \{a + (-a)\} + i\{b + (-b)\} = (0 + i0).$$

Similarly, $(-z) + z = (0 + i0) = 0$.

Hence, $z + (-z) = (-z) + z = 0$.

Thus, every complex number z has $(-z)$ as its additive inverse.

MULTIPLICATION OF COMPLEX NUMBERS

Let $z_1 = (a_1 + ib_1)$ and $z_2 = (a_2 + ib_2)$. Then, we define

$$z_1 z_2 = (a_1 + ib_1)(a_2 + ib_2) = (a_1 a_2 - b_1 b_2) + i(a_1 b_2 + b_1 a_2).$$

$$\therefore \quad z_1 z_2 = \{\text{Re}(z_1) \cdot \text{Re}(z_2) - \text{Im}(z_1) \cdot \text{Im}(z_2)\} + i\{\text{Re}(z_1) \cdot \text{Im}(z_2) + \text{Re}(z_2) \cdot \text{Im}(z_1)\}.$$

EXAMPLES (i) Let $z_1 = (3 + 2i)$ and $z_2 = (5 + 4i)$. Then,

$$z_1 z_2 = \{(3 \times 5) - (2 \times 4)\} + i\{(3 \times 4) + (2 \times 5)\}$$
$$= (15 - 8) + i(12 + 10) = (7 + 22i).$$

(ii) Let $z_1 = (-2 + 3i)$ and $z_2 = (7 - 5i)$. Then,

$$z_1 z_2 = \{(-2) \times 7 - 3 \times (-5)\} + i\{(-2) \times (-5) + 3 \times 7\}$$
$$= (-14 + 15) + i(10 + 21) = (1 + 31i).$$

PROPERTIES OF MULTIPLICATION OF COMPLEX NUMBERS

(i) CLOSURE PROPERTY *The product of two complex numbers is always a complex number.*

PROOF Let $z_1 = (a_1 + ib_1)$ and $z_2 = (a_2 + ib_2)$ be any two complex numbers. Then, a_1, a_2 and b_1, b_2 are real numbers.

$\therefore \quad z_1 z_2 = (a_1 + ib_1)(a_2 + ib_2)$

$= (a_1 a_2 - b_1 b_2) + i(a_1 b_2 + b_1 a_2)$, which is a complex number.

Thus, the product of two complex numbers is always a complex number.

(ii) COMMUTATIVE LAW

For any two complex numbers z_1 and z_2, *prove that*

$$z_1 z_2 = z_2 z_1.$$

PROOF Let $z_1 = (a_1 + ib_1)$ and $z_2 = (a_2 + ib_2)$. Then,

$z_1 z_2 = (a_1 + ib_1)(a_2 + ib_2) = (a_1 a_2 - b_1 b_2) + i(a_1 b_2 + b_1 a_2)$

$= (a_2 a_1 - b_2 b_1) + i(b_2 a_1 + a_2 b_1)$

[by commutativity of multiplication on R]

$= (a_2 + ib_2)(a_1 + ib_1) = z_2 z_1.$

$\therefore \quad z_1 z_2 = z_2 z_1$ for all $z_1, z_2 \in C$.

Hence, multiplication of complex numbers is commutative.

(iii) ASSOCIATIVE LAW

For any three complex numbers z_1, z_2 *and* z_3, *prove that*

$$(z_1 z_2) z_3 = z_1 (z_2 z_3).$$

PROOF Let $z_1 = (a_1 + ib_1)$, $z_2 = (a_2 + ib_2)$ and $z_3 = (a_3 + ib_3)$ be any three complex numbers. Then a_1, a_2, a_3 and b_1, b_2, b_3 are all real numbers.

$\therefore \quad (z_1 z_2) z_3 = \{(a_1 + ib_1)(a_2 + ib_2)\}(a_3 + ib_3)$

$= \{(a_1 a_2 - b_1 b_2) + i(a_1 b_2 + b_1 a_2)\}(a_3 + ib_3)$

$= \{(a_1 a_2 - b_1 b_2) a_3 - (a_1 b_2 + b_1 a_2) b_3\}$
$\quad + i\{(a_1 a_2 - b_1 b_2) b_3 + (a_1 b_2 + b_1 a_2) a_3\}$

$= \{a_1 (a_2 a_3 - b_2 b_3) - b_1 (a_2 b_3 + b_2 a_3)\}$
$\quad + i\{a_1 (a_2 b_3 + a_3 b_2) + b_1 (a_2 a_3 - b_2 b_3)\}$

$= (a_1 + ib_1)\{(a_2 a_3 - b_2 b_3) + i(a_2 b_3 + a_3 b_2)\}$

$= (a_1 + ib_1)\{(a_2 + ib_2)(a_3 + ib_3)\} = z_1 (z_2 z_3).$

Thus, $(z_1 z_2) z_3 = z_1 (z_2 z_3)$ for all $z_1, z_2, z_3 \in C$.

Hence, multiplication of complex numbers is associative.

(iv) EXISTENCE OF MULTIPLICATIVE IDENTITY

The complex number $(1 + i0)$ is the multiplicative identity in C.

Let $z = (a + ib)$. Then,

$z \times 1 = (a + ib)(1 + i0) = \{(a \times 1) - (b \times 0)\} + i\{(a \times 0) + (b \times 1)\} = (a + ib) = z.$

Similarly, $1 \times z = z$.

Thus, $(z \times 1) = (1 \times z) = z$ for all $z \in C$.

Hence, the complex number $1 = (1 + i0)$ is the multiplicative identity.

(v) EXISTENCE OF MULTIPLICATIVE INVERSE

Let $z = (a + ib)$. Then,

$$z^{-1} = \frac{1}{z} = \frac{1}{(a+ib)} = \frac{1}{(a+ib)} \times \frac{(a-ib)}{(a-ib)} = \frac{(a-ib)}{(a^2+b^2)}.$$

Clearly, $z \times z^{-1} = z^{-1} \times z = 1$.

Thus, every $z = (a + ib)$ has its multiplicative inverse, given by

$$z^{-1} = \frac{1}{z} = \frac{(a-ib)}{(a^2+b^2)} = \frac{\bar{z}}{|z|^2}.$$ $\left[\text{Remember: } z^{-1} = \frac{\bar{z}}{|z|^2}\right]$

$\therefore \quad z\bar{z} = |z|^2$.

THINGS TO REMEMBER

(i) $z = (a + ib) \Rightarrow \bar{z} = (a - ib)$ and $|z|^2 = (a^2 + b^2)$.

(ii) $z = (a + ib) \Rightarrow z^{-1} = \frac{z}{|z|^2} = \frac{(a-ib)}{(a^2+b^2)}$.

(vi) DISTRIBUTIVE LAWS

For complex numbers z_1, z_2, z_3, prove that

$z_1 \cdot (z_2 + z_3) = z_1z_2 + z_1z_3$.

$(z_1 + z_2) \cdot z_3 = z_1z_3 + z_2z_3$.

PROOF Let $z_1 = (a_1 + ib_1)$, $z_2 = (a_2 + ib_2)$ and $z_3 = (a_3 + ib_3)$ be any three complex numbers. Then, a_1, a_2, a_3 and b_1, b_2, b_3 are real numbers.

$\therefore \quad z_1 \cdot (z_2 + z_3)$

$= (a_1 + ib_1)[(a_2 + ib_2) + (a_3 + ib_3)]$

$= (a_1 + ib_1)[(a_2 + a_3) + i(b_2 + b_3)]$

$= \{a_1(a_2 + a_3) - b_1(b_2 + b_3)\} + i\{a_1(b_2 + b_3) + b_1(a_2 + a_3)\}$

$= \{(a_1a_2 - b_1b_2) + i(a_1b_2 + b_1a_2)\} + \{(a_1a_3 - b_1b_3) + i(a_1b_3 + b_1a_3)\}$

$= z_1z_2 + z_1z_3$.

Thus, $z_1(z_2 + z_3) = z_1z_2 + z_1z_3$ for all $z_1, z_2, z_3 \in C$.

Similarly, we can prove that

$(z_1 + z_2)z_3 = z_1z_3 + z_2z_3$.

DIVISION OF TWO COMPLEX NUMBERS

Let z_1 and z_2 be complex numbers such that $z_2 \neq 0$.

Then, $\frac{z_1}{z_2} = z_1 \cdot \frac{1}{z_2} = z_1z_2^{-1}$.

EXAMPLE *Find $\frac{z_1}{z_2}$, when $z_1 = (6 + 3i)$ and $z_2 = (3 - i)$.*

SOLUTION We have $\frac{z_1}{z_2} = z_1z_2^{-1}$.

$$\text{Now, } z_2^{-1} = \frac{\bar{z}_2}{|z_2|^2} = \frac{\overline{(3-i)}}{|3-i|^2} = \frac{(3+i)}{\{3^2+(-1)^2\}} = \frac{(3+i)}{10}.$$

$$\therefore \quad \frac{z_1}{z_2} = z_1 \cdot z_2^{-1} = (6+3i) \cdot \frac{(3+i)}{10} = \frac{(6+3i)(3+i)}{10}$$

$$= \frac{15+15i}{10} = \frac{15(1+i)}{10} = \frac{3(1+i)}{2}.$$

SOME IDENTITIES ON COMPLEX NUMBERS

THEOREM 1 *For any complex numbers z_1 and z_2, prove that:*

(i) $(z_1+z_2)^2 = z_1^2+z_2^2+2z_1z_2$

(ii) $(z_1-z_2)^2 = z_1^2+z_2^2-2z_1z_2$

(iii) $(z_1^2-z_2^2) = (z_1+z_2)(z_1-z_2)$

(iv) $(z_1+z_2)^3 = z_1^3+z_2^3+3z_1z_2(z_1+z_2)$

(v) $(z_1-z_2)^3 = z_1^3-z_2^3-3z_1z_2(z_1-z_2)$

PROOF We have

(i) $(z_1+z_2)^2 = (z_1+z_2)(z_1+z_2)$

$= (z_1+z_2)z_1 + (z_1+z_2)z_2$ [by distributive law]

$= z_1^2 + z_2z_1 + z_1z_2 + z_2^2$ [by distributive law]

$= z_1^2 + 2z_1z_2 + z_2^2$ [$\because\ z_2z_1 = z_1z_2$]

$= z_1^2 + z_2^2 + 2z_1z_2.$

$\therefore \quad (z_1+z_2)^2 = z_1^2+z_2^2+2z_1z_2.$

(ii) $(z_1-z_2)^2 = (z_1-z_2)(z_1-z_2)$

$= (z_1-z_2)z_1 - (z_1-z_2)z_2$ [by distributive law]

$= z_1^2 - z_2z_1 - z_1z_2 + z_2^2$ [by distributive law]

$= z_1^2 - 2z_1z_2 + z_2^2$ [$\because\ z_2z_1 = z_1z_2$]

$= z_1^2 + z_2^2 - 2z_1z_2.$

$\therefore \quad (z_1-z_2)^2 = z_1^2+z_2^2-2z_1z_2.$

(iii) RHS $= (z_1+z_2)(z_1-z_2)$

$= (z_1+z_2)z_1 - (z_1+z_2)z_2$ [by distributive law]

$= z_1^2 + z_2z_1 - z_1z_2 - z_2^2$ [by distributive law]

$= (z_1^2 - z_2^2) =$ LHS. [$\because\ z_2z_1 = z_1z_2$]

$\therefore \quad (z_1^2-z_2^2) = (z_1+z_2)(z_1-z_2).$

(iv) $(z_1+z_2)^3=(z_1+z_2)^2\cdot(z_1+z_2)$

$=(z_1^2+z_2^2+2z_1z_2)(z_1+z_2)$ [expanding $(z_1+z_2)^2$]

$=(z_1^2+z_2^2+2z_1z_2)z_1+(z_1^2+z_2^2+2z_1z_2)z_2$

[by distributive law]

$=z_1^3+z_2^2z_1+2z_1^2z_2+z_1^2z_2+z_2^3+2z_1z_2^2$

$=z_1^3+z_2^3+3z_1^2z_2+3z_1z_2^2=z_1^3+z_2^3+3z_1z_2(z_1+z_2)$.

$\therefore\quad (z_1+z_2)^3=z_1^3+z_2^3+3z_1z_2(z_1+z_2)$.

(v) LHS $=(z_1-z_2)^3$

$=(z_1-z_2)^2(z_1-z_2)=(z_1^2-2z_1z_2+z_2^2)(z_1-z_2)$

$=(z_1^2-2z_1z_2+z_2^2)z_1-(z_1^2-2z_1z_2+z_2^2)z_2$

$=z_1^3-2z_1^2z_2+z_2^2z_1-z_1^2z_2+2z_1z_2^2-z_2^3$

$=z_1^3-3z_1^2z_2+3z_1z_2^2-z_2^3=$ RHS.

This may be written as $(z_1-z_2)^3=z_1^3-z_2^3-3z_1z_2(z_1-z_2)$.

SOLVED EXAMPLES

EXAMPLE 1 *Simplify:*

(i) $3(6+6i)+i(6+6i)$ *(ii)* $(1-i)-(-3+6i)$

(iii) $\left(\frac{1}{3}-\frac{2}{3}i\right)-\left(4+\frac{3}{2}i\right)$ *(iv)* $\left\{\left(\frac{1}{5}+\frac{7}{5}i\right)-\left(6+\frac{1}{5}i\right)\right\}-\left(\frac{-4}{5}+i\right)$

SOLUTION We have

(i) $3(6+6i)+i(6+6i)=18+18i+6i+6i^2=18+24i-6=12+24i$.

(ii) $(1-i)-(-3+6i)=1-i+3-6i=4-7i$.

(iii) $\left(\frac{1}{3}-\frac{2}{3}i\right)-\left(4+\frac{3}{2}i\right)=\frac{1}{3}-\frac{2}{3}i-4-\frac{3}{2}i$

$=\left(\frac{1}{3}-4\right)-\left(\frac{2}{3}+\frac{3}{2}\right)i=\frac{-11}{3}-\frac{13}{6}i.$

(iv) $\left\{\left(\frac{1}{5}+\frac{7}{5}i\right)-\left(6+\frac{1}{5}i\right)\right\}-\left(\frac{-4}{5}+i\right)$

$=\left(\frac{1}{5}+\frac{7}{5}i-6-\frac{1}{5}i\right)-\left(\frac{-4}{5}+i\right)=\left\{\left(\frac{1}{5}-6\right)+\left(\frac{7}{5}-\frac{1}{5}\right)i\right\}-\left(\frac{-4}{5}+i\right)$

$=\frac{-29}{5}+\frac{6}{5}i+\frac{4}{5}-i=\left(\frac{-29}{5}+\frac{4}{5}\right)+\left(\frac{6}{5}i-i\right)=\left(-5+\frac{1}{5}i\right)$.

EXAMPLE 2 *Express* $(-\sqrt{3}+\sqrt{-2})(2\sqrt{3}-i)$ *in the form* $(a+ib)$.

SOLUTION We have

$(-\sqrt{3}+\sqrt{-2})(2\sqrt{3}-i)=(-\sqrt{3}+i\sqrt{2})(2\sqrt{3}-i)$

$=-6+i\sqrt{3}+2\sqrt{6}\,i-\sqrt{2}\,i^2$

$=(-6+\sqrt{2})+\sqrt{3}(1+2\sqrt{2})i.$

EXAMPLE 3 *Express each of the following in the form* $(a+ib)$:

(i) $(3+\sqrt{-5})(3-\sqrt{-5})$ *(ii)* $(-2+\sqrt{-3})(-3+2\sqrt{-3})$

(iii) $(-2+3i)^2$ *(iv)* $(\sqrt{5}-7i)^2$

SOLUTION We have

(i) $(3+\sqrt{-5})(3-\sqrt{-5}) = (3+\sqrt{5}\,i)(3-\sqrt{5}\,i) = \{3^2-(\sqrt{5}\,i)^2\}$

$= (9-5i^2) = (9+5) = 14+0i.$

$[\because (z_1+z_2)(z_1-z_2) = (z_1^2-z_2^2)]$

(ii) $(-2+\sqrt{-3})(-3+2\sqrt{-3}) = (-2+\sqrt{3}\,i)(-3+2\sqrt{3}\,i)$

$= (6-4\sqrt{3}\,i-3\sqrt{3}\,i+6i^2)$

$= -7\sqrt{3}\,i = 0-7\sqrt{3}\,i.$

(iii) $(-2+3i)^2 = (-2)^2+(3i)^2+2\times(-2)\times 3i$

$= (4+9i^2-12i) = (4-9-12i) = (-5-12i).$

(iv) $(\sqrt{5}-7i)^2 = (5)^2+(7i)^2-2\times\sqrt{5}\times 7i$

$= (25+49i^2-14\sqrt{5}\,i) = (25-49-14\sqrt{5}\,i)$

$= (-24-14\sqrt{5}\,i).$

EXAMPLE 4 *Prove that* $(1-i)^4 = -4.$

SOLUTION We have

$(1-i)^4 = (1-i)^2\times(1-i)^2 = (1+i^2-2i)\times(1+i^2-2i)$

$= (-2i)(-2i) = 4i^2 = 4\times(-1) = -4.$ $[\because i^2=-1]$

EXAMPLE 5 *Express* $(2+3i)^3$ *in the form* $(a+ib)$.

SOLUTION We know that $(z_1+z_2)^3 = z_1^3+z_2^3+3z_1z_2(z_1+z_2)$.

$\therefore$ $(2+3i)^3 = 2^3+(3i)^3+3\times 2\times 3i\times(2+3i)$

$= 8+27i^3+36i+54i^2$

$= (8-27i+36i-54)$ $[\because i^3=-i \text{ and } i^2=-1]$

$= (-46+9i).$

Hence, $(2+3i)^3 = (-46+9i)$.

EXAMPLE 6 *Express* $(3-5i)^3$ *in the form* $(a+ib)$.

SOLUTION We know that $(z_1-z_2)^3 = z_1^3-z_2^3-3z_1z_2(z_1-z_2)$.

$\therefore$ $(3-5i)^3 = 3^3-(5i)^3-3\times 3\times 5i\times(3-5i)$

$= 27-125i^3-135i+225i^2$

$= (27+125i-135i-225)$ $[\because i^3=-i \text{ and } i^2=-1]$

$= (-198-10i).$

EXAMPLE 7 *If* $z = (\sqrt{2} - \sqrt{-3})$, *find* $\mathrm{Re}(z)$, $\mathrm{Im}(z)$, $\bar{z}$ *and* $|z|$.

SOLUTION The given number may be written as

$$z = \sqrt{2} - i\sqrt{3}.$$

$\therefore$ $\mathrm{Re}(z) = \sqrt{2}$, $\mathrm{Im}(z) = -\sqrt{3}$, $\bar{z} = \overline{(\sqrt{2} - i\sqrt{3})} = (\sqrt{2} + i\sqrt{3})$

and $|z|^2 = \{(\sqrt{2})^2 + (-\sqrt{3})^2\} = (2+3) = 5 \Rightarrow |z| = \sqrt{5}$.

EXAMPLE 8 *Write down the modulus of:*

(i) $4 + \sqrt{-3}$ *(ii)* $2 - 5i$ *(iii)* $-i$ *(iv)* $(-1+3i)^2$

SOLUTION We know that, if $z = (a + ib)$ then $|z| = \sqrt{a^2 + b^2}$.

(i) Let $z = 4 + \sqrt{-3}$. Then, $z = 4 + i\sqrt{3}$.

$\therefore$ $|z|^2 = \{(4)^2 + (\sqrt{3})^2\} = (16+3) = 19 \Rightarrow |z| = \sqrt{19}$.

(ii) Let $z = 2 - 5i$. Then,

$|z|^2 = \{2^2 + (-5)^2\} = (4+25) = 29 \Rightarrow |z| = \sqrt{29}$.

(iii) Let $z = 0 - i$. Then,

$|z|^2 = \{0^2 + (-1)^2\} = (0+1) = 1 \Rightarrow |z| = \sqrt{1} = 1$.

(iv) Let $z = (-1+3i)^2$. Then,

$z = (-1)^2 + (3i)^2 + 2 \times (-1) \times 3i = (-8 - 6i)$.

$\therefore$ $|z|^2 = \{(-8)^2 + (-6)^2\} = (64+36) = 100 \Rightarrow |z| = \sqrt{100} = 10$.

EXAMPLE 9 *Write down the conjugate of each of the following:*

(i) $(-5 + \sqrt{-1})$ *(ii)* $(-6 - \sqrt{-3})$ *(iii)* i^3 *(iv)* $(4+5i)^2$

SOLUTION We know that the conjugate of $z = (a + ib)$ is given as $\bar{z} = (a - ib)$.

(i) Let $z = -5 + \sqrt{-1} = (-5 + i)$. Then,

$\bar{z} = \overline{(-5+i)} = (-5 - i)$.

(ii) Let $z = -6 - \sqrt{-3} = -6 - i\sqrt{3}$. Then,

$\bar{z} = \overline{(-6 - i\sqrt{3})} = (-6 + i\sqrt{3})$.

(iii) Let $z = i^3 = -i = 0 - i$. Then,

$\bar{z} = \overline{(0 - i)} = (0 + i) = i$.

(iv) Let $z = (4+5i)^2 = (4)^2 + (5i)^2 + 2 \times 4 \times 5i$

$= (16 - 25 + 40i) = (-9 + 40i)$.

$\therefore$ $\bar{z} = \overline{(-9+40i)} = (-9 - 40i)$.

EXAMPLE 10 *Find the multiplicative inverse of each of the following:*

(i) $\sqrt{5} + 3i$ *(ii)* $4 - 3i$ *(iii)* $(3i - 1)^2$ *(iv)* $-i$

SOLUTION We know that the multiplicative inverse of the complex number z is given by $z^{-1} = \dfrac{\bar{z}}{|z|^2}$.

(i) Let $z = (\sqrt{5} + 3i)$. Then,

$$\bar{z} = (\sqrt{5} - 3i) \text{ and } |z|^2 = (\sqrt{5})^2 + 3^2 = (5+9) = 14.$$

$$\therefore \quad z^{-1} = \frac{\bar{z}}{|z|^2} = \frac{(\sqrt{5} - 3i)}{14} = \left(\frac{\sqrt{5}}{14} - \frac{3}{14}i\right).$$

(ii) Let $z = 4 - 3i$. Then,

$$\bar{z} = \overline{(4-3i)} = (4+3i) \text{ and } |z|^2 = \{(4)^2 + (-3)^2\} = (16+9) = 25.$$

$$\therefore \quad z^{-1} = \frac{\bar{z}}{|z|^2} = \frac{(4+3i)}{25} = \left(\frac{4}{25} + \frac{3}{25}i\right).$$

(iii) Let $z = (3i-1)^2 = (9i^2 + 1 - 6i) = (-9 + 1 - 6i) = (-8 - 6i)$.

$$\therefore \quad \bar{z} = \overline{(-8-6i)} = (-8+6i)$$

and $|z|^2 = \{(-8)^2 + (-6)^2\} = (64+36) = 100$.

Hence, $z^{-1} = \dfrac{\bar{z}}{|z|^2} = \dfrac{(-8+6i)}{100} = \left(\dfrac{-8}{100} + \dfrac{6}{100}i\right) = \left(\dfrac{-2}{25} + \dfrac{3}{50}i\right)$.

(iv) Let $z = (0 - i)$. Then,

$$\bar{z} = \overline{(0-i)} = i \text{ and } |z|^2 = \sqrt{0^2 + (-1)^2} = 1.$$

$$\therefore \quad z^{-1} = \frac{\bar{z}}{|z|^2} = \frac{i}{1} = i.$$

EXAMPLE 11 *Express each of the following in the form* $(a+ib)$:

(i) $\dfrac{i}{(1+i)}$ *(ii)* $(-1+\sqrt{3}i)^{-1}$ *(iii)* $\dfrac{5+\sqrt{2}i}{1-\sqrt{2}i}$

SOLUTION We have

(i) $$\frac{i}{(1+i)} = \frac{i}{(1+i)} \times \frac{(1-i)}{(1-i)} = \frac{i(1-i)}{(1^2 - i^2)}$$

$$= \frac{(i - i^2)}{1-(-1)} = \frac{(i+1)}{2} = \frac{(1+i)}{2} = \left(\frac{1}{2} + \frac{1}{2}i\right).$$

(ii) $$(-1+\sqrt{3}i)^{-1} = \frac{1}{(-1+\sqrt{3}i)} \times \frac{(-1-\sqrt{3}i)}{(-1-\sqrt{3}i)}$$

$$= \frac{(-1-\sqrt{3}i)}{(-1+\sqrt{3}i)(-1-\sqrt{3}i)} = \frac{(-1-\sqrt{3}i)}{(-1)^2 - (\sqrt{3}i)^2}$$

$$= \frac{(-1-\sqrt{3}i)}{(1-3i^2)} = \frac{(-1-\sqrt{3}i)}{4} = \left(\frac{-1}{4} - \frac{\sqrt{3}}{4}i\right).$$

(iii) $$\frac{5+\sqrt{2}i}{1-\sqrt{2}i} = \frac{(5+\sqrt{2}i)}{(1-\sqrt{2}i)} \times \frac{(1+\sqrt{2}i)}{(1+\sqrt{2}i)} = \frac{(5+\sqrt{2}i)(1+\sqrt{2}i)}{(1-2i^2)}$$

$$= \frac{(3+6\sqrt{2}i)}{3} = (1+2\sqrt{2}i).$$

EXAMPLE 12 *Reduce* $\left(\frac{1}{1+2i}+\frac{3}{1-i}\right)\left(\frac{3-2i}{1+3i}\right)$ *to the form* $(a+ib)$.

SOLUTION We have

$$\left(\frac{1}{1+2i}+\frac{3}{1-i}\right)\left(\frac{3-2i}{1+3i}\right)$$

$$=\left\{\frac{(1-i)+(3+6i)}{(1+2i)(1-i)}\right\}\left(\frac{3-2i}{1+3i}\right)=\frac{(4+5i)}{(3+i)}\times\frac{(3-2i)}{(1+3i)}$$

$$=\frac{(4+5i)(3-2i)}{(3+i)(1+3i)}=\frac{(12+10)+(15-8)i}{(3-3)+(9i+i)}=\frac{(22+7i)}{10i}\times\frac{i}{i}$$

$$=\frac{(7i^2+22i)}{10i^2}=\frac{-7+22i}{-10}=\left(\frac{7}{10}-\frac{22}{10}i\right)=\left(\frac{7}{10}-\frac{11}{5}i\right).$$

EXAMPLE 13 *Reduce* $\left(\frac{1+i}{1-i}-\frac{1-i}{1+i}\right)$ *to the form* $(a+ib)$ *and hence find its modulus.*

SOLUTION Let $z=\left(\frac{1+i}{1-i}-\frac{1-i}{1+i}\right)=\frac{(1+i)^2-(1-i)^2}{(1-i)(1+i)}=\frac{(4\times1\times i)}{(1-i^2)}=\frac{4i}{2}=2i.$

Thus, $z=0+2i \Rightarrow |z|=\sqrt{0^2+2^2}=\sqrt{4}=2$.

Hence, $z=0+2i$ and $|z|=2$.

EXAMPLE 14 *If* $\sqrt{\frac{1+i}{1-i}}=(a+ib)$ *then show that* $(a^2+b^2)=1$.

SOLUTION We have

$$(a+ib)=\sqrt{\frac{1+i}{1-i}}=\frac{\sqrt{1+i}}{\sqrt{1-i}}\times\frac{\sqrt{1+i}}{\sqrt{1+i}}=\frac{(1+i)}{\sqrt{1-i^2}}$$

$$=\frac{(1+i)}{\sqrt{2}}=\left(\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}i\right)$$

$$\Rightarrow\quad |a+ib|^2=\left(\frac{1}{\sqrt{2}}\right)^2+\left(\frac{1}{\sqrt{2}}\right)^2=\left(\frac{1}{2}+\frac{1}{2}\right)=1.$$

$$\Rightarrow\quad (a^2+b^2)=1.$$

Hence, $(a^2+b^2)=1$.

EXAMPLE 15 *Find the least positive integer m for which* $\left(\frac{1+i}{1-i}\right)^m=1$.

SOLUTION We have

$$\left(\frac{1+i}{1-i}\right)=\frac{(1+i)}{(1-i)}\times\frac{(1+i)}{(1+i)}=\frac{(1+i)^2}{(1-i^2)}=\frac{(1+2i+i^2)}{2}=\frac{2i}{2}=i.$$

$$\therefore\quad \left(\frac{1+i}{1-i}\right)^m=1 \Rightarrow i^m=1.$$

And, we know that 4 is the least positive integer such that $i^4=1$ and therefore, $m=4$.

EXAMPLE 16 *Separate* $\left(\dfrac{3+\sqrt{-1}}{2-\sqrt{-1}}\right)$ *into real and imaginary parts and hence find its modulus.*

SOLUTION Let $z = \left(\dfrac{3+\sqrt{-1}}{2-\sqrt{-1}}\right) = \left(\dfrac{3+i}{2-i}\right) = \dfrac{(3+i)}{(2-i)} = \dfrac{(3+i)}{(2-i)} \times \dfrac{(2+i)}{(2+i)}$

$$= \frac{(3+i)(2+i)}{(2-i)(2+i)} = \frac{(6+i^2)+5i}{(4-i^2)} = \frac{5+5i}{\{4-(-1)\}} = \frac{5(1+i)}{5} = (1+i).$$

$\therefore \quad |z| = \sqrt{1^2+1^2} = \sqrt{2}.$

Hence, $z = (1+i)$ and $|z| = \sqrt{2}$.

EXAMPLE 17 *Reduce* $\left\{\dfrac{\sqrt{5+12i}+\sqrt{5-12i}}{\sqrt{5+12i}-\sqrt{5-12i}}\right\}$ *to the form* $(a+ib)$ *and hence find its conjugate.*

SOLUTION We have

$$z = \frac{\sqrt{5+12i}+\sqrt{5-12i}}{\sqrt{5+12i}-\sqrt{5-12i}}$$

$$= \frac{(\sqrt{5+12i}+\sqrt{5-12i})}{(\sqrt{5+12i}-\sqrt{5-12i})} \times \frac{(\sqrt{5+12i}+\sqrt{5-12i})}{(\sqrt{5+12i}+\sqrt{5-12i})}$$

$$= \frac{(\sqrt{5+12i}+\sqrt{5-12i})^2}{(5+12i)-(5-12i)} = \frac{(5+12i)+(5-12i)+2\sqrt{(5)^2-(12i)^2}}{24i}$$

$$= \frac{10+2\sqrt{25+144}}{24i} = \frac{10+2\sqrt{169}}{24i} = \frac{(10+2\times 13)}{24i}$$

$$= \frac{36}{24i} = \frac{3}{2i} = \frac{3}{2i} \times \frac{i}{i} = \frac{3i}{2i^2} = -\frac{3}{2}i.$$

Hence, $z = \left(0 - \dfrac{3}{2}i\right)$ and $\bar{z} = \overline{\left(0-\dfrac{3}{2}i\right)} = \left(0+\dfrac{3}{2}i\right)$.

EXAMPLE 18 *If* $(x+iy)^{1/3} = (a+ib)$ *then prove that:*

(i) $\dfrac{x}{a} + \dfrac{y}{b} = 4(a^2-b^2)$ (ii) $\dfrac{x}{a} - \dfrac{y}{b} = -2(a^2+b^2)$

SOLUTION We have

$$(x+iy)^{1/3} = (a+ib)$$

$\Rightarrow \quad (x+iy) = (a+ib)^3$ [on cubing both sides]

$\Rightarrow \quad (x+iy) = a^3 + i^3b^3 + 3iab(a+ib)$

$$= a^3 - ib^3 + 3a^2bi - 3ab^2 = (a^3-3ab^2) + i(3a^2b-b^3)$$

$\Rightarrow \quad x = a^3 - 3ab^2$ and $y = 3a^2b - b^3$

[on equating real and imaginary parts separately]

$\Rightarrow \quad \frac{x}{a} = (a^2 - 3b^2)$ and $\frac{y}{b} = (3a^2 - b^2)$

$\Rightarrow \quad \left(\frac{x}{a} + \frac{y}{b}\right) = 4(a^2 - b^2)$ and $\left(\frac{x}{a} - \frac{y}{b}\right) = -2(a^2 + b^2)$.

Hence, (i) $\frac{x}{a} + \frac{y}{b} = 4(a^2 - b^2)$ and (ii) $\frac{x}{a} - \frac{y}{b} = -2(a^2 + b^2)$.

EXAMPLE 19 *Let z be a complex number such that $z \neq 1$ and $|z| = 1$. Then, prove that $\left(\frac{z-1}{z+1}\right)$ is purely imaginary. What will be your conclusion when $z = 1$?*

SOLUTION Let $z = (x + iy)$ be the given complex number such that $z \neq 1$ and $|z| = 1$. Then,

$$|z| = 1 \Rightarrow |z|^2 \Rightarrow x^2 + y^2 = 1 \Rightarrow x^2 + y^2 - 1 = 0. \qquad \ldots \text{(i)}$$

$$\therefore \quad \left(\frac{z-1}{z+1}\right) = \frac{(z-1)}{(z+1)} = \frac{x+iy-1}{x+iy+1} = \frac{(x-1)+iy}{(x+1)+iy} \times \frac{(x+1)-iy}{(x+1)-iy}$$

$$= \frac{\{(x-1)+iy\}\{(x+1)-iy\}}{\{(x+1)^2+y^2\}}$$

$$= \frac{(x^2-1+y^2)+i\{(x+1)y-(x-1)y\}}{\{(x+1)^2+y^2\}}$$

$$= \frac{(x^2+y^2-1)+(2y)i}{\{(x+1)^2+y^2\}}$$

$$= \left\{\frac{2yi}{(x+1)^2+y^2}\right\} \quad \text{[using (i)]},$$

which is purely imaginary.

Particular Case *When $z = 1$.*

In this case, $z = 1 \Rightarrow x + iy = 1$

$\Rightarrow (x-1) + iy = 0$

$\Rightarrow x - 1 = 0, y = 0 \Rightarrow x = 1, y = 0.$

$$\therefore \quad \frac{z-1}{z+1} = \frac{(x+iy)-1}{(x+iy)+1} = \frac{(x-1)+iy}{(x+1)+iy} = \frac{(1-1)+i\times 0}{(1+1)+i\times 0} = \frac{0}{2} = 0.$$

Thus, $z = 1 \Rightarrow \frac{z-1}{z+1}$ is purely real.

EXAMPLE 20 *Find the real value of θ for which $\left(\frac{3+2i\sin\theta}{1-2i\sin\theta}\right)$ is purely real.*

SOLUTION We have

$$\left(\frac{3+2i\sin\theta}{1-2i\sin\theta}\right) = \frac{(3+2i\sin\theta)}{(1-2i\sin\theta)} \times \frac{(1+2i\sin\theta)}{(1+2i\sin\theta)}$$

$$= \frac{(3+2i\sin\theta)(1+2i\sin\theta)}{(1-4i^2\sin^2\theta)}$$

$$= \frac{(3-4\sin^2\theta)+i(6\sin\theta+2\sin\theta)}{(1+4\sin^2\theta)}$$

$$= \frac{(3-4\sin^2\theta)+i(8\sin\theta)}{(1+4\sin^2\theta)}.$$

Now, $\left(\frac{3+2i\sin\theta}{1-2i\sin\theta}\right)$ will be purely real only when $\frac{8\sin\theta}{(1+4\sin^2\theta)}=0$.

This happens only when $8\sin\theta=0 \Leftrightarrow \sin\theta=0 \Leftrightarrow \theta=n\pi, n\in N$.

Hence, the required value of θ is $n\pi$, where $n\in N$.

EXAMPLE 21 *Show that* $|1-i|^x=2^x$ *has no nonzero integral solution.*

SOLUTION $|1-i|^x=2^x \Rightarrow (\sqrt{2})^x=2^x$ $\quad [\because |1-i|=\sqrt{1^2+(-1)^2}=\sqrt{2}]$

$$\Rightarrow 2^{x/2}=2^x \Rightarrow \frac{2^x}{2^{x/2}}=1$$

$$\Rightarrow 2^{x/2}=1=2^0 \Rightarrow \frac{x}{2}=0 \Rightarrow x=0.$$

Thus, $x=0$ is the only solution of the given equation.

Hence, the given equation has no nonzero integral solution.

EXAMPLE 22 *Solve for x and y:*

(i) $(3x-7)+2iy=-5y+(5+x)i$

(ii) $(x+iy)(2-3i)=4+i$

SOLUTION (i) The given equation is

$$(3x-7)+2iy=-5y+(5+x)i.$$

Equating the real parts and the imaginary parts of the given equation separately, we get

$3x-7=-5y$ and $2y=5+x$

$\Rightarrow$ $3x+5y=7$ and $x-2y=-5$

$\Rightarrow$ $x=-1$ and $y=2$ $\quad$ [on solving $3x+5y=7$, $x-2y=-5$].

Hence, $x=-1$ and $y=2$.

(ii) The given equation is

$(x+iy)(2-3i)=4+i$

$\Rightarrow$ $(2x+3y)+i(2y-3x)=4+i$

$\Rightarrow$ $2x+3y=4$ and $2y-3x=1$

[equating real parts and imaginary parts separately]

$\Rightarrow$ $2x+3y=4$ and $-3x+2y=1$

$\Rightarrow$ $x=\frac{5}{13}$ and $y=\frac{14}{13}$ $\quad$ [on solving $2x+3y=4$, $-3x+2y=1$].

Hence, $x=\frac{5}{13}$ and $y=\frac{14}{13}$ is the required solution.

EXAMPLE 23 *Find the real values of x and y for which* $\left(\frac{x-1}{3+i}+\frac{y-1}{3-i}\right)=i.$

SOLUTION We have

$$\frac{x-1}{3+i}+\frac{y-1}{3-i}=i$$

$$\Rightarrow \quad \frac{(x-1)}{(3+i)}\times\frac{(3-i)}{(3-i)}+\frac{(y-1)}{(3-i)}\times\frac{(3+i)}{(3+i)}=i$$

$$\Rightarrow \quad \frac{(x-1)(3-i)}{(9-i^2)}+\frac{(y-1)(3+i)}{(9-i^2)}=i$$

$$\Rightarrow \quad \frac{3(x-1)-i(x-1)}{10}+\frac{3(y-1)+i(y-1)}{10}=i$$

$$\Rightarrow \quad 3(x-1)-i(x-1)+3(y-1)+i(y-1)=10i$$

$$\Rightarrow \quad (3x-3+3y-3)+i(y-1-x+1)=10i$$

$$\Rightarrow \quad (3x+3y-6)+i(y-x)=10i$$

$$\Rightarrow \quad 3(x+y-2)=0 \text{ and } y-x=10$$

[equating real parts and the imaginary parts separately]

$$\Rightarrow \quad x+y-2=0 \text{ and } y-x=10$$

$$\Rightarrow \quad x+y=2 \text{ and } -x+y=10$$

$$\Rightarrow \quad x=-4 \text{ and } y=6 \qquad [\text{on solving } x+y=2,\ -x+y=10].$$

Hence, $x=-4$ and $y=6$ are the required values.

EXAMPLE 24 *Find the complex number z for which* $|z+1|=z+2(1+i)$.

SOLUTION Let the required complex number be $z=(x+iy)$. Then,

$$|z+1|=z+2(1+i)$$

$$\Rightarrow \quad |(x+iy)+1|=(x+iy)+2(1+i)$$

$$\Rightarrow \quad \sqrt{(x+1)^2+y^2}=(x+2)+i(y+2)$$

$$\Rightarrow \quad \sqrt{(x+1)^2+y^2}=(x+2) \text{ and } y+2=0$$

[equating real parts and imaginary parts separately]

$$\Rightarrow \quad y=-2 \text{ and } \sqrt{(x+1)^2+(-2)^2}=(x+2)$$

$$\Rightarrow \quad y=-2 \text{ and } \sqrt{x^2+2x+5}=(x+2)$$

$$\Rightarrow \quad y=-2 \text{ and } (x^2+2x+5)=(x+2)^2$$

$$\Rightarrow \quad x^2+2x+5=x^2+4x+4 \text{ and } y=-2$$

$$\Rightarrow \quad 2x=1 \text{ and } y=-2 \Rightarrow x=\frac{1}{2} \text{ and } y=-2.$$

Hence, the required complex number is $z=\left(\frac{1}{2}-2i\right)\cdot$

EXAMPLE 25 *Solve the equation* $|z| + z = (2 + i)$ *for complex value of* z.

SOLUTION Let $z = (x + iy)$. Then,

$$|z| + z = 2 + i$$

$$\Rightarrow \quad |x + iy| + (x + iy) = 2 + i$$

$$\Rightarrow \quad \{\sqrt{x^2 + y^2} + x\} + iy = (2 + i)$$

$$\Rightarrow \quad \sqrt{x^2 + y^2} + x = 2 \text{ and } y = 1$$

[equating real parts and imaginary parts separately]

$$\Rightarrow \quad y = 1 \text{ and } \sqrt{x^2 + 1^2} + x = 2 \Rightarrow y = 1 \text{ and } \sqrt{x^2 + 1} = (2 - x)$$

$$\Rightarrow \quad y = 1 \text{ and } x^2 + 1 = (2 - x)^2 \Rightarrow y = 1 \text{ and } x^2 + 1 = x^2 - 4x + 4$$

$$\Rightarrow \quad 4x = 3 \text{ and } y = 1 \Rightarrow x = \frac{3}{4} \text{ and } y = 1.$$

Hence, $z = \left(\frac{3}{4} + i\right)$ is the desired solution.

EXAMPLE 26 *Solve the equation* $z + \sqrt{2}|z + 1| + i = 0$ *for complex value of* z.

SOLUTION Let the required complex number be $z = x + iy$. Then,

$$z + \sqrt{2}|z + 1| + i = 0$$

$$\Rightarrow \quad (x + iy) + \sqrt{2}|(x + iy) + 1| + i = 0$$

$$\Rightarrow \quad x + \sqrt{2}\{\sqrt{(x + 1)^2 + y^2}\} + (y + 1)i = 0$$

$$\Rightarrow \quad x + \sqrt{2}\{\sqrt{(x + 1)^2 + y^2}\} = 0 \text{ and } y + 1 = 0$$

[equating real parts and imaginary parts separately on both sides]

$$\Rightarrow \quad y = -1 \text{ and } \sqrt{2}\{\sqrt{(x + 1)^2 + (-1)^2}\} = (-x)$$

$$\Rightarrow \quad y = -1 \text{ and } \sqrt{2} \cdot \sqrt{x^2 + 2x + 2} = (-x)$$

$$\Rightarrow \quad y = -1 \text{ and } 2(x^2 + 2x + 2) = x^2$$

$$\Rightarrow \quad y = -1 \text{ and } x^2 + 4x + 4 = 0 \Rightarrow (x + 2)^2 = 0 \text{ and } y = -1$$

$$\Rightarrow \quad x + 2 = 0 \text{ and } y = -1 \Rightarrow x = -2 \text{ and } y = -1.$$

Hence, the required complex number is $z = (-2 - i)$.

EXAMPLE 27 *If* $z = 3 + 2i$, *prove that* $z^2 - 6z + 13 = 0$ *and hence deduce that* $3z^3 - 13z^2 + 9z + 65 = 0$.

SOLUTION $z = 3 + 2i \Rightarrow z - 3 = 2i \Rightarrow (z - 3)^2 = 4i^2$

$$\Rightarrow z^2 - 6z + 13 = 0. \qquad \ldots \text{(i)}$$

Thus, $z^2 - 6z + 13 = 0$.

Now, $3z^3 - 13z^2 + 9z + 65 = 3z(z^2 - 6z + 13) + 5(z^2 - 6z + 13)$

$= (3z \times 0) + (5 \times 0) = 0$ [using (i)].

Hence, $3z^3 - 13z^2 + 9z + 65 = 0$.

EXAMPLE 28 *If* $z=-5+3i$, *find the value of* $(z^4+9z^3+26z^2-14z+8)$.

SOLUTION We have

$$z=-5+3i \Rightarrow (z+5)=3i$$
$$\Rightarrow (z+5)^2=9i^2 \Rightarrow z^2+10z+34=0. \quad \ldots \text{(i)}$$

Now, $z^4+9z^3+26z^2-14z+8$

$$=z^2(z^2+10z+34)-z(z^2+10z+34)+2(z^2+10z+34)-60$$
$$=(z^2\times 0)-(z\times 0)+(2\times 0)-60=-60 \quad \text{[using (i)]}.$$

Hence, the value of $z^4+9z^3+26z^2-14z+8$ is -60.

EXAMPLE 29 *If* $z=2+i$, *prove that* $z^3+3z^2-9z+8=(1+14i)$.

SOLUTION We have

$$z=2+i \Rightarrow z-2=i$$
$$\Rightarrow (z-2)^2=i^2 \Rightarrow z^2-4z+5=0. \quad \ldots \text{(i)}$$

$\therefore$ z^3+3z^2-9z+8

$$=z(z^2-4z+5)+7(z^2-4z+5)+14z-27$$
$$=(z\times 0)+(7\times 0)+14z-27=(14z-27) \quad \text{[using (i)]}$$
$$=14(2+i)-27=(1+14i).$$

Hence, $z^3+3z^2-9z+8=(1+14i)$.

EXERCISE 5B

1. Simplify each of the following and express it in the form $a+ib$:

(i) $2(3+4i)+i(5-6i)$
(ii) $(3+\sqrt{-16})-(4-\sqrt{-9})$
(iii) $(-5+6i)-(-2+i)$
(iv) $(8-4i)-(-3+5i)$
(v) $(1-i)^2(1+i)-(3-4i)^2$
(vi) $(5+\sqrt{-3})(5-\sqrt{-3})$
(vii) $(3+4i)(2-3i)$
(viii) $(-2+\sqrt{-3})(-3+2\sqrt{-3})$

2. Simplify and express each of the following in the form $(a+ib)$:

(i) $(2+\sqrt{-3})^2$
(ii) $(5-2i)^2$
(iii) $(-3+5i)^3$
(iv) $\left(-2-\frac{1}{3}i\right)^3$
(v) $(4-3i)^{-1}$
(vi) $(-2+\sqrt{-3})^{-1}$
(vii) $(2+i)^{-2}$
(viii) $(1+2i)^{-3}$
(ix) $(1+i)^3-(1-i)^3$

3. Express each of the following in the form $(a+ib)$:

(i) $\dfrac{1}{(4+3i)}$
(ii) $\dfrac{(3+4i)}{(4-5i)}$
(iii) $\dfrac{(5+\sqrt{2}\,i)}{(1-\sqrt{2}\,i)}$
(iv) $\dfrac{(-2+5i)}{(3-6i)}$
(v) $\dfrac{(3-4i)}{(4-2i)(1+i)}$
(vi) $\dfrac{(3-2i)(2+3i)}{(1+2i)(2-i)}$
(vii) $\dfrac{(2+3i)^2}{(2-i)}$
(viii) $\dfrac{(1-i)^3}{(1-i^3)}$
(ix) $\dfrac{(1+2i)^3}{(1+i)(2-i)}$

4. Simplify and express each of the following in the form $(a+ib)$:

(i) $\left(\frac{5}{-3+2i}+\frac{2}{1-i}\right)\left(\frac{4-5i}{3+2i}\right)$ (ii) $\left(\frac{1}{1-4i}-\frac{2}{1+i}\right)\left(\frac{1-i}{5+3i}\right)$

5. Show that

(i) $\left\{\frac{(3+2i)}{(2-3i)}+\frac{(3-2i)}{(2+3i)}\right\}$ is purely real,

(ii) $\left\{\frac{(\sqrt{7}+i\sqrt{3})}{(\sqrt{7}-i\sqrt{3})}+\frac{(\sqrt{7}-i\sqrt{3})}{(\sqrt{7}+i\sqrt{3})}\right\}$ is purely real.

6. Find the real values of θ for which $\left(\frac{1+i\cos\theta}{1-2i\cos\theta}\right)$ is purely real.

7. If $|z+i| = |z-i|$, prove that z is real.

8. Give an example of two complex numbers z_1 and z_2 such that $z_1 \neq z_2$ and $|z_1| = |z_2|$.

9. Find the conjugate of each of the following:

(i) $(-5-2i)$ (ii) $\frac{1}{(4+3i)}$ (iii) $\frac{(1+i)^2}{(3-i)}$ (iv) $\frac{(1+i)(2+i)}{(3+i)}$

(v) $\sqrt{-3}$ (vi) $\sqrt{2}$ (vii) $-\sqrt{-1}$ (viii) $(2-5i)^2$

10. Find the modulus of each of the following:

(i) $(3+\sqrt{-5})$ (ii) $(-3-4i)$ (iii) $(7+24i)$ (iv) $3i$

(v) $\frac{(3+2i)^2}{(4-3i)}$ (vi) $\frac{(2-i)(1+i)}{(1+i)}$ (vii) 5 (viii) $(1+2i)(i-1)$

11. Find the multiplicative inverse of each of the following:

(i) $(1-\sqrt{3}\,i)$ (ii) $(2+5i)$ (iii) $\frac{(2+3i)}{(1+i)}$ (iv) $\frac{(1+i)(1+2i)}{(1+3i)}$

12. If $\left(\frac{1-i}{1+i}\right)^{100} = (a+ib)$, find the values of a and b.

13. If $\left(\frac{1+i}{1-i}\right)^3 - \left(\frac{1-i}{1+i}\right)^3 = x+iy$, find x and y.

14. If $(x+iy) = \frac{a+ib}{a-ib}$, prove that $x^2+y^2=1$.

15. If $(a+ib) = \frac{c+i}{c-i}$, where c is real, prove that $a^2+b^2=1$ and $\frac{b}{a} = \frac{2c}{c^2-1}$.

16. Show that $(1-i)^n\left(1-\frac{1}{i}\right)^n = 2^n$ for all $n \in N$.

17. Find the smallest positive integer n for which $(1+i)^{2n} = (1-i)^{2n}$.

18. Prove that $(x+1+i)(x+1-i)(x-1+i)(x-1-i) = (x^4+4)$.

19. If $a = (\cos\theta + i\sin\theta)$, prove that $\frac{1+a}{1-a} = \left(\cot\frac{\theta}{2}\right)i$.

20. If $z_1 = (2 - i)$ and $z_2 = (1 + i)$, find $\left| \dfrac{z_1 + z_2 + 1}{z_1 - z_2 + i} \right|$.

21. Find the real values of x and y for which:

(i) $(1 - i)x + (1 + i)y = 1 - 3i$ (ii) $(x + iy)(3 - 2i) = (12 + 5i)$

(iii) $x + 4yi = ix + y + 3$ (iv) $(1 + i)y^2 + (6 + i) = (2 + i)x$

(v) $\dfrac{(x + 3i)}{(2 + iy)} = (1 - i)$ (vi) $\dfrac{(1 + i)x - 2i}{(3 + i)} + \dfrac{(2 - 3i)y + i}{(3 - i)} = i$

22. Find the real values of x and y for which $(x - iy)(3 + 5i)$ is the conjugate of $(-6 - 24i)$.

23. Find the real values of x and y for which the complex numbers $(-3 + iyx^2)$ and $(x^2 + y + 4i)$ are conjugates of each other.

24. If $z = (2 - 3i)$, prove that $z^2 - 4z + 13 = 0$ and hence deduce that $4z^3 - 3z^2 + 169 = 0$.

25. If $(1 + i)z = (1 - i)\bar{z}$ then prove that $z = -i\bar{z}$.

26. If $\left(\dfrac{z - 1}{z + 1}\right)$ is purely an imaginary number and $z \neq -1$ then find the value of $|z|$.

27. Solve the system of equations, $\mathrm{Re}(z^2) = 0$, $|z| = 2$.

28. Find the complex number z for which $|z| = z + 1 + 2i$.

ANSWERS (EXERCISE 5B)

1. (i) $(12 + 13i)$ (ii) $(-1 + 7i)$ (iii) $(-3 + 5i)$ (iv) $(11 - 9i)$
(v) $(9 + 22i)$ (vi) 28 (vii) $(18 - i)$ (viii) $-7\sqrt{3}\,i$

2. (i) $(1 + 4\sqrt{3}\,i)$ (ii) $(21 - 20i)$ (iii) $(198 + 10i)$ (iv) $\left(\dfrac{-22}{3} - \dfrac{107}{27}i\right)$
(v) $\left(\dfrac{4}{25} + \dfrac{3}{25}i\right)$ (vi) $\left(\dfrac{-2}{7} - \dfrac{\sqrt{3}}{7}i\right)$ (vii) $\left(\dfrac{3}{25} - \dfrac{4}{25}i\right)$ (viii) $\left(\dfrac{-11}{125} + \dfrac{2}{125}i\right)$
(ix) $4i$

3. (i) $\left(\dfrac{4}{25} - \dfrac{3}{25}i\right)$ (ii) $\left(\dfrac{-8}{41} + \dfrac{31}{41}i\right)$ (iii) $(1 + 2\sqrt{2}\,i)$ (iv) $\left(\dfrac{-4}{5} + \dfrac{1}{15}i\right)$
(v) $\left(\dfrac{1}{4} - \dfrac{3}{4}i\right)$ (vi) $\left(\dfrac{63}{25} - \dfrac{16}{25}i\right)$ (vii) $\left(\dfrac{-22}{5} + \dfrac{19}{5}i\right)$ (viii) $(-2 + 0i)$
(ix) $\left(\dfrac{-7}{2} + \dfrac{1}{2}i\right)$

4. (i) $\left(\dfrac{5}{13} + \dfrac{4}{13}i\right)$ (ii) $\left(\dfrac{4}{17} + \dfrac{5}{17}i\right)$ **6.** $\theta = (2n + 1)\dfrac{\pi}{2}$, $n \in I$

9. (i) $(-5 + 2i)$ (ii) $\left(\dfrac{4}{25} + \dfrac{3}{25}i\right)$ (iii) $\left(\dfrac{-1}{5} - \dfrac{3}{5}i\right)$ (iv) $\left(\dfrac{3}{5} - \dfrac{4}{5}i\right)$
(v) $-\sqrt{3}\,i$ (vi) $\sqrt{2}$ (vii) i (viii) $(-21 + 20i)$

10. (i) $\sqrt{14}$ (ii) 5 (iii) 25 (iv) 3
(v) $\frac{13}{5}$ (vi) $\sqrt{5}$ (vii) 5 (viii) $\sqrt{10}$

11. (i) $\left(\frac{1}{4}, \frac{\sqrt{3}}{4}i\right)$ (ii) $\left(\frac{2}{29} - \frac{5}{29}i\right)$ (iii) $\left(\frac{5}{13} - \frac{1}{13}i\right)$ (iv) $\left(\frac{4}{5} - \frac{3}{5}i\right)$

12. $a = 1, b = 0$ **13.** $x = 0, y = -2$ **17.** $n = 2$ **20.** $2\sqrt{2}$

21. (i) $x = 2, y = -1$ (ii) $x = 2, y = 3$ (iii) $x = 4, y = 1$
(iv) $(x = 5, y = 2)$ or $(x = 5, y = -2)$ (v) $x = 7, y = 5$ (vi) $x = 3, y = -1$

22. $x = 3, y = -3$ **23.** $(x = 1, y = -4)$ or $(x = -1, y = -4)$

26. $|z| = 1$ **27.** $z = \sqrt{2}(1 \pm i)$ or $z = \sqrt{2}(-1 \pm i)$ **28.** $\left(\frac{3}{2} - 2i\right)$

HINTS TO SOME SELECTED QUESTIONS

2. (iii) $(-3+5i)^3 = (5i-3)^3 = (5i)^3 - 3^3 - (3 \times 5i \times 3)(5i-3)$
$= (-125i - 27 + 225 + 135i) = (198 + 10i)$.

(iv) $\left(-2 - \frac{1}{3}i\right)^3 = (-1)^3\left(2 + \frac{1}{3}i\right)^3 = -\left\{8 + \frac{1}{27}i^3 + 2i\left(2 + \frac{1}{3}i\right)\right\}$.

(v) $(4-3i)^{-1} = \frac{1}{(4-3i)} \times \frac{(4+3i)}{(4+3i)} = \frac{(4+3i)}{25}$.

(vi) $(-2+\sqrt{3}\,i)^{-1} = \frac{1}{(-2+\sqrt{3}\,i)} \times \frac{(-2-\sqrt{3}\,i)}{(-2-\sqrt{3}\,i)}$.

(vii) $(2+i)^{-2} = \frac{1}{(2+i)^2} = \frac{1}{(4+i^2+4i)} = \frac{1}{(3+4i)} \times \frac{(3-4i)}{(3-4i)}$.

(viii) $(1+2i)^{-3} = \frac{1}{(1+2i)^3} = \frac{1}{\{1+8i^3+6i(1+2i)\}} = \frac{1}{(-11-2i)} \times \frac{(-11+2i)}{(-11+2i)}$.

4. (i) $\left(\frac{5}{-3+2i} + \frac{2}{1-i}\right)\left(\frac{4-5i}{3+2i}\right) = \frac{5(1-i)+2(-3+2i)}{(-3+2i)(1-i)} \times \frac{(4-5i)}{(3+2i)}$
$= \frac{(-1-i)(4-5i)}{(-1+5i)(3+2i)} = \frac{(1+i)(4-5i)}{(1-5i)(3+2i)}$
$= \frac{(9-i)}{13(1-i)} \times \frac{(1+i)}{(1+i)} = \frac{(10+8i)}{26} = \left(\frac{5}{13} + \frac{4}{13}i\right)$.

5. (i) Given expression $= \frac{(3+2i)(2+3i)+(3-2i)(2-3i)}{(2-3i)(2+3i)} = \frac{0}{13} = 0$.

(ii) Given expression $= \frac{(\sqrt{7}+i\sqrt{3})^2+(\sqrt{7}-i\sqrt{3})^2}{(\sqrt{7}-i\sqrt{3})(\sqrt{7}+i\sqrt{3})} = \frac{2\{(\sqrt{7})^2+(i\sqrt{3})^2\}}{(7+3)} = \frac{8}{10} = \frac{4}{5}$.

6. Given expression $= \frac{(1+i\cos\theta)}{(1-2i\cos\theta)} \times \frac{(1+2i\cos\theta)}{(1+2i\cos\theta)} = \frac{(1-2\cos^2\theta)+i(3\cos\theta)}{(1+4\cos^2\theta)}$.

This expression will be purely real when $3\cos\theta = 0$.

Now, $3\cos\theta = 0 \Leftrightarrow \cos\theta = 0 \Leftrightarrow \theta = (2n+1)\frac{\pi}{2}$, where $n \in I$.

7. Let $z=(x+iy)$. Then,

$$|z+i|^2=|z-i|^2$$

$$\Rightarrow \quad |(x+iy)+i|^2=|(x+iy)-i|^2$$

$$\Rightarrow \quad |x+(y+1)i|^2=|x+(y-1)i|^2 \Rightarrow x^2+(y+1)^2=x^2+(y-1)^2$$

$$\Rightarrow \quad (y+1)^2-(y-1)^2=0 \Rightarrow 4y=0 \Rightarrow y=0.$$

Hence, z is purely real.

8. Let $z_1=(3+2i)$ and $z_2=(3-2i)$. Then, $z_1 \neq z_2$.

But, $|z|=|z_2|=\sqrt{13}$.

10. (v) $z=\dfrac{(3+2i)^2}{(4-3i)}=\dfrac{(9+4i^2+12i)}{(4-3i)}=\dfrac{(5+12i)}{(4-3i)}\times\dfrac{(4+3i)}{(4+3i)}=\dfrac{-16+63i}{25}$

$$\Rightarrow \quad |z|^2=\left\{\left(\frac{-16}{25}\right)^2+\left(\frac{63}{25}\right)^2\right\}=\left(\frac{256}{625}+\frac{3969}{625}\right)=\frac{4225}{625}$$

$$\Rightarrow \quad |z|=\sqrt{\frac{4225}{625}}=\frac{65}{25}=\frac{13}{5}.$$

(vi) $z=\dfrac{(2-i)(1+i)}{(1-i)}=\dfrac{(3+i)}{(1-i)}\times\dfrac{(1+i)}{(1+i)}=\dfrac{(2+4i)}{2}=(1+2i).$

11. Use the formula, $z^{-1}=\dfrac{\bar{z}}{|z|^2}$.

(iii) $z=\dfrac{(2+3i)}{(1+i)}\times\dfrac{(1-i)}{(1-i)}=\dfrac{(5+i)}{2}=\left(\dfrac{5}{2}+\dfrac{1}{2}i\right)$.

$$|z|^2=\left(\frac{25}{4}+\frac{1}{4}\right)=\frac{26}{4}=\frac{13}{2} \text{ and } \bar{z}=\left(\frac{5}{2}-\frac{1}{2}i\right)=\frac{(5-i)}{2}.$$

$$\therefore \quad z^{-1}=\frac{\bar{z}}{|z|^2}=\frac{(5-i)}{2}\times\frac{2}{13}=\frac{(5-i)}{13}=\left(\frac{5}{13}-\frac{1}{13}i\right).$$

(iv) $z=\dfrac{(1+i)(1+2i)}{(1+3i)}=\dfrac{(-1+3i)}{(1+3i)}\times\dfrac{(1-3i)}{(1-3i)}=\dfrac{(8+6i)}{10}=\left(\dfrac{4}{5}+\dfrac{3}{5}i\right)$.

$$\therefore \quad \bar{z}=\left(\frac{4}{5}-\frac{3}{5}i\right)=\frac{(4-3i)}{5} \text{ and } |z|^2=\left(\frac{16}{25}+\frac{9}{25}\right)=\frac{25}{25}=1.$$

$$\therefore \quad z^{-1}=\frac{\bar{z}}{|z|^2}=\left(\frac{4}{5}-\frac{3}{5}i\right).$$

12. $\dfrac{(1-i)}{(1+i)}=\dfrac{(1-i)}{(1+i)}\times\dfrac{(1-i)}{(1-i)}=\dfrac{(1-i)^2}{2}=\dfrac{(1+i^2-2i)}{2}=-i.$

$$\therefore \quad \left(\frac{1-i}{1+i}\right)^{100}=(-i)^{100}=\{(-i)^4\}^{25}=(1)^{25}=1 \qquad [\because\ (-i)^4=i^4=1].$$

Hence, $a=1$ and $b=0$.

13. $\left(\dfrac{1+i}{1-i}\right)=\dfrac{(1+i)}{(1-i)}\times\dfrac{(1+i)}{(1+i)}=\dfrac{(1+i)^2}{2}=\dfrac{(1+i^2+2i)}{2}=\dfrac{2i}{2}=i.$

Similarly, $\left(\dfrac{1-i}{1+i}\right)=-i.$

Given expression $=i^3-(-i)^3=2i^3=-2i=0+(-2)i.$

14. $(x+iy)=\dfrac{a+ib}{a-ib} \Rightarrow \overline{(x+iy)}=\dfrac{\overline{(a+ib)}}{\overline{(a-ib)}} \Rightarrow (x-iy)=\dfrac{(a-ib)}{(a+ib)}\cdot$

$\therefore \quad (x+iy)(x-iy) = \dfrac{(a+ib)}{(a-ib)} \times \dfrac{(a-ib)}{(a+ib)} = 1 \Rightarrow x^2+y^2 = 1.$

15. $(a+ib) = \dfrac{(c+i)}{(c-i)} \times \dfrac{(c+i)}{(c+i)} = \dfrac{(c+i)^2}{(c^2+1)} = \dfrac{c^2+i^2+2ci}{(c^2+1)} = \dfrac{(c^2-1)}{(c^2+1)} + \dfrac{2c}{(c^2+1)}i.$

$\Rightarrow \quad |a+ib|^2 = \left\{\left(\dfrac{c^2-1}{c^2+1}\right)^2 + \dfrac{4c^2}{(c^2+1)^2}\right\} = \dfrac{(c^2-1)^2+4c^2}{(c^2+1)^2} = \dfrac{(c^2+1)^2}{(c^2+1)^2} = 1.$

$\Rightarrow \quad a^2+b^2 = 1.$

Now, $a = \dfrac{(c^2-1)}{(c^2+1)}$ and $b = \dfrac{2c}{(c^2+1)} \Rightarrow \dfrac{b}{a} = \dfrac{2c}{(c^2-1)}\cdot$

16. $\left(1-\dfrac{1}{i}\right) = \dfrac{(i-1)}{i} \times \dfrac{i}{i} = \dfrac{(i^2-i)}{i^2} = \dfrac{(-1-i)}{-1} = (1+i).$

$\therefore \quad$ given expression $= (1-i)^n \times (1+i)^n = \{(1-i)+(1+i)\}^2 = (1-i^2)^n = 2^n.$

17. $\dfrac{(1+i)}{(1-i)} = \dfrac{(1+i)}{(1-i)} \times \dfrac{(1+i)}{(1+i)} = \dfrac{(1+i)^2}{2} = \dfrac{(1+i^2+2i)}{2} = \dfrac{2i}{2} = i.$

$\therefore \quad (1+i)^{2n} = (1-i)^{2n} \Rightarrow \dfrac{(1+i)^{2n}}{(1-i)^{2n}} = 1 \Rightarrow \left(\dfrac{1+i}{1-i}\right)^{2n} = 1 \Rightarrow i^{2n} = 1.$

Least value of $2n$ is 4 and so the least value of n is 2.

18. Given expression $= \{(x+1)+i\}\{(x+1)-i\}\{(x-1)+i\}\{(x-1)-i\}$

$= \{(x+1)^2-i^2\}\{(x-1)^2-i^2\} = \{(x+1)^2+1\}\{(x-1)^2+1\}$

$= \{(x^2+2)+2x\}\{(x^2+2)-2x\}$

$= (x^2+2)^2-4x^2 = (x^4+4).$

19. $\dfrac{1+a}{1-a} = \dfrac{(1+\cos\theta)+i\sin\theta}{(1-\cos\theta)-i\sin\theta} \times \dfrac{(1-\cos\theta)+i\sin\theta}{(1-\cos\theta)+i\sin\theta}$

$= \dfrac{(1-\cos^2\theta-\sin^2\theta)+2i\sin\theta}{(1-\cos\theta)^2+\sin^2\theta} = \dfrac{1-(\cos^2\theta+\sin^2\theta)+2i\sin\theta}{1+(\cos^2\theta+\sin^2\theta)-2\cos\theta}$

$= \dfrac{2i\sin\theta}{2(1-\cos\theta)} = \dfrac{i\sin\theta}{(1-\cos\theta)} = \dfrac{2\sin(\theta/2)\cos(\theta/2)}{2\sin^2(\theta/2)}\cdot i = \left(\cot\dfrac{\theta}{2}\right)i.$

20. $\dfrac{z_1+z_2+1}{z_1-z_2+i} = \dfrac{(2-i)+(1+i)+1}{(2-i)-(1+i)+i} = \dfrac{4}{(1+i)} \times \dfrac{(1-i)}{(1-i)} = 2(1-i)$

$\Rightarrow \quad \left|\dfrac{z_1+z_2+1}{z_1-z_2+i}\right| = 2 \times \sqrt{1+(-1)^2} = 2\sqrt{2}.$

21. (iv) $(y^2+6-2x)+(y^2-x+1)i = 0 \Rightarrow y^2+6-2x = 0, y^2-x+1 = 0$

$\Rightarrow \quad y^2-2x = -6$ and $y^2-x = -1.$

On subtracting, we get $x = 5$.

$\therefore \quad y^2-5 = -1 \Rightarrow y^2 = 4 \Rightarrow y = \pm 2.$

(v) $(x+3i) = (1-i)(2+iy)$

$\Rightarrow \quad (x+3i) = (2+y)+(y-2)i \Rightarrow (x-y-2)+i(5-y) = 0$

$\Rightarrow \quad x-y-2 = 0$ and $5-y = 0.$

(vi) $\{x+(x-2)i\}(3-i)+\{2y+(1-3y)i\}(3+i)=i(3+i)(3-i)$

$\Rightarrow \quad \{3x+(x-2)\}+\{3(x-2)-x\}i+\{6y-(1-3y)\}+\{2y+(3-9y\}i=10i$

$\Rightarrow \quad (4x-2+9y-1)+(2x-6+3-7y)i=10i$

$\Rightarrow \quad (4x+9y-3)+(2x-7y-3)i=10i$

$\Rightarrow \quad 4x+9y=3$ and $2x-7y=13$.

22. $(x-iy)(3+5i)=\overline{(-6-24i)}=(-6+24i)$

$\Rightarrow \quad (3x+5y)+(5x-3y)i=-6+24i$

$\Rightarrow \quad 3x+5y=-6$ and $5x-3y=24$.

23. $(-3+iyx^2)=\overline{(x^2+y+4i)}$

$\Rightarrow \quad (-3+iyx^2)=(x^2+y-4i)$

$\Rightarrow \quad (x^2+y+3)-(4+yx^2)i=0$

$\Rightarrow \quad x^2+y+3=0$ and $4+yx^2=0$

$\Rightarrow \quad x^2+y=-3$ ... (i) and $yx^2=-4$ ... (ii).

Putting $x^2=(-3-y)$ from (i) in (ii), we get

$y(-3-y)=-4 \Rightarrow y^2+3y-4=0 \Rightarrow (y+4)(y-1)=0 \Rightarrow y=-4$ or $y=1$.

Now, $y=1 \Rightarrow x^2=-4 \Rightarrow x$ is imaginary. So, $y \neq 1$.

When $y=-4$, we get $x^2=(-3+4)=1 \Rightarrow x=\pm 1$.

$\therefore \quad (x=1, y=-4)$ or $(x=-1, y=-4)$.

25. $(1+i)z=(1-i)\bar{z} \Rightarrow \dfrac{z}{\bar{z}}=\dfrac{(1-i)}{(1+i)}\times\dfrac{(1-i)}{(1-i)}=\dfrac{(1-i)^2}{2}=\dfrac{(1+i^2-2i)}{2}=-i.$

Hence, $z=-i\bar{z}$.

26. Let $z=x+iy$. Then,

$$\frac{z-1}{z+1}=\frac{(x-1)+iy}{(x+1)+iy}\times\frac{(x+1)-iy}{(x+1)-iy}=\frac{(x^2+y^2-1)+(x+y)i}{(x+1)^2+y^2}.$$

Now, $\dfrac{z-1}{z+1}$ is purely imaginary $\Leftrightarrow x^2+y^2-1=0 \Leftrightarrow x^2+y^2=1 \Leftrightarrow |z|=1$.

27. Let $z=(x+iy)$. Then, $z^2=(x^2-y^2)+2ixy$.

Now, $\text{Re}(z^2)=0 \Rightarrow x^2-y^2=0$. ... (i)

$|z|=2 \Rightarrow |z|^2=4 \Rightarrow x^2+y^2=4$. ... (ii)

On solving (i) and (ii), we get

$x^2=2$ and $y^2=2$.

$\therefore \quad x=\pm\sqrt{2}$ and $y=\pm\sqrt{2}$.

$\therefore \quad z=\sqrt{2}(1\pm i)$ or $z=\sqrt{2}(-1\pm i)$.

28. Let the required complex number be $z=(x+iy)$. Then,

$|z|=z+1+2i$

$\Rightarrow \quad |x+iy|=(x+iy)+1+2i$

$\Rightarrow \quad \sqrt{x^2+y^2}=(x+1)+(y+2)i$

$\Rightarrow \quad \sqrt{x^2+y^2} = (x+1)$ and $y+2=0$

$\Rightarrow \quad y=-2$ and $\sqrt{x^2+(-2)^2} = (x+1)$

$\Rightarrow \quad y=-2$ and $x^2+4=(x+1)^2 \Rightarrow x=\frac{3}{2}$ and $y=-2$.

SOME RESULTS ON MODULUS AND CONJUGATE OF COMPLEX NUMBERS

THEOREM 1 *For any complex numbers z, z_1 and z_2 prove that:*

(i) $\overline{(\overline{z})} = z$

(ii) $z\overline{z} = |z|^2 = \{\text{Re } z\}^2 + \{\text{Im}(z)\}^2$

(iii) $z+\overline{z} = 2\,\text{Re}(z)$

(iv) $z-\overline{z} = 2i\cdot\text{Im}(z)$

(v) $z=\overline{z} \Leftrightarrow$ *z is purely real*

(vi) $z+\overline{z}=0 \Leftrightarrow$ *z is purely imaginary*

(vii) $\overline{z_1+z_2} = \overline{z}_1+\overline{z}_2$

(viii) $\overline{z_1-z_2} = \overline{z}_1-\overline{z}_2$

(ix) $\overline{z_1z_2} = (\overline{z}_2)(\overline{z}_2)$

(x) $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\overline{z}_1}{\overline{z}_2}$, $z_2 \neq 0$

PROOF Let $z=(a+ib)$, $z_1=(a_1+ib_1)$ and $z_2=(a_2+ib_2)$. Then,

(i) $z=(a+ib) \Rightarrow \overline{z} = \overline{(a+ib)} = (a-ib)$

$\Rightarrow \overline{(\overline{z})} = \overline{(a-ib)} = (a+ib) = z.$

Hence, $\overline{(\overline{z})} = z$.

(ii) $z\overline{z} = (a+ib)\overline{(a+ib)} = (a+ib)(a-ib)$

$= (a^2+b^2) = \{\text{Re}(z)\}^2 + \{\text{Im}(z)\}^2 = |z|^2.$

$\therefore \quad z\overline{z} = \{\text{Re}(z)\}^2 + \{\text{Im}(z)\}^2 = |z|^2.$

(iii) $z+\overline{z} = (a+ib)+\overline{(a+ib)}$

$= (a+ib)+(a-ib) = 2a = 2\text{Re}(z).$

$\therefore \quad z+\overline{z} = 2\,\text{Re}(z).$

(iv) $z-\overline{z} = (a+ib)-\overline{(a+ib)}$

$= (a+ib)-(a-ib) = 2ib = 2i\cdot\text{Im}(z).$

$\therefore \quad z-\overline{z} = 2i\cdot\text{Im}(z).$

(v) $z=\overline{z} \Leftrightarrow (a+ib) = \overline{(a+ib)}$

$\Leftrightarrow (a+ib)=(a-ib) \Leftrightarrow 2ib=0$

$\Leftrightarrow b=0 \Leftrightarrow \text{Im}(z)=0.$

$\therefore \quad z=\overline{z} \Leftrightarrow \text{Im}(z)=0 \Leftrightarrow z$ is purely real.

(vi) $z+\overline{z}=0 \Leftrightarrow (a+ib)+\overline{(a+ib)}=0$

$\Leftrightarrow (a+ib)+(a-ib)=0 \Leftrightarrow 2a=0 \Leftrightarrow a=0$

$\Leftrightarrow z$ is purely imaginary.

$\therefore \quad z+\overline{z}=0 \Leftrightarrow z$ is purely imaginary.

(vii) $\overline{z_1+z_2} = \overline{(a_1+ib_1)+(a_2+ib_2)}$

$$= \overline{(a_1+a_2)+i(b_1+b_2)} = (a_1+a_2)-i(b_1+b_2)$$

$$= (a_1-ib_1)+(a_2-ib_2) = \bar{z}_1+\bar{z}_2.$$

$\therefore \quad \overline{z_1+z_2} = \bar{z}_1+\bar{z}_2.$

(viii) $\overline{z_1-z_2} = \overline{(a_1+ib_1)-(a_2+ib_2)}$

$$= \overline{(a_1-a_2)+i(b_1-b_2)} = (a_1-a_2)-i(b_1-b_2)$$

$$= (a_1-ib_1)-(a_2-ib_2) = \bar{z}_1-\bar{z}_2.$$

$\therefore \quad \overline{z_1-z_2} = \bar{z}_1-\bar{z}_2.$

(ix) $z_1z_2 = (a_1+ib_1)(a_2+ib_2) = (a_1a_2-b_1b_2)+i(a_1b_2+b_1a_2)$

$\Rightarrow \quad \overline{z_1z_2} = \overline{(a_1a_2-b_1b_2)+i(a_1b_2+b_1a_2)}$

$$= (a_1a_2-b_1b_2)-i(a_1b_2+b_1a_2)$$

$$= (a_1-ib_1)(a_2-ib_2) = \bar{z}_1\,\bar{z}_2.$$

$\therefore \quad \overline{z_1z_2} = \bar{z}_1\,\bar{z}_2.$

(x) $\dfrac{z_1}{z_2} = \dfrac{(a_1+ib_1)}{(a_2+ib_2)} = \dfrac{(a_1+ib_1)}{(a_2+ib_2)} \times \dfrac{(a_2-ib_2)}{(a_2-ib_2)}$

$$= \frac{(a_1+ib_1)(a_2-ib_2)}{(a_2^2+b_2^2)} = \frac{(a_1a_2+b_1b_2)+i(b_1a_2-a_1b_2)}{(a_2^2+b_2^2)}$$

$$= \frac{(a_1a_2+b_1b_2)}{(a_2^2+b_2^2)} + i\frac{(b_1a_2-a_1b_2)}{(a_2^2+b_2^2)}$$

$$\therefore \quad \overline{\left(\frac{z_1}{z_2}\right)} = \frac{(a_1a_2+b_1b_2)}{(a_2^2+b_2^2)} - i\frac{(b_1a_2-a_1b_2)}{(a_2^2+b_2^2)}. \qquad \ldots \text{(i)}$$

Also, $\dfrac{\bar{z}_1}{\bar{z}_2} = \bar{z}_1 \cdot \dfrac{1}{\bar{z}_2} = (a_1-ib_1)\cdot\left\{\dfrac{1}{(a_2-ib_2)} \times \dfrac{(a_2+ib_2)}{(a_2+ib_2)}\right\}$

$$= \frac{(a_1-ib_1)(a_2+ib_2)}{(a_2^2+b_2^2)} = \frac{(a_1a_2+b_1b_2)}{(a_2^2+b_2^2)} - i\frac{(b_1a_2-a_1b_2)}{(a_2^2+b_2^2)}. \qquad \ldots \text{(ii)}$$

From (i) and (ii), we get

$$\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\bar{z}_1}{\bar{z}_2}.$$

THEOREM 2 *For all complex numbers z, z_1 and z_2, prove that:*

(i) $z\bar{z} = |z|^2$ *(ii)* $|z| = |\bar{z}| = |-z|$

(iii) $|z| = 0 \Leftrightarrow z = 0$ *(iv)* $-|z| \le \mathrm{Re}(z) \le |z|$

(v) $-|z| \le \mathrm{Im}(z) \le |z|$ *(vi)* $|z_1z_2| = |z_1||z_2|$

(vii) $\left|\dfrac{z_1}{z_2}\right| = \dfrac{|z_1|}{|z_2|}$, $z_2 \ne 0$

PROOF Let $z = (a+ib)$, $z_1 = (a_1 + ib_1)$ and $z_2 = (a_2 + ib_2)$. Then,

(i) $z\bar{z} = (a+ib)\overline{(a+ib)} = (a+ib)(a-ib) = (a^2+b^2) = |z|^2.$

$\therefore \quad z\,\bar{z} = |z|^2.$

(ii) $|z| = |a+ib| = \sqrt{a^2+b^2}$, $|\bar{z}| = |a-ib| = \sqrt{a^2+(-b)^2} = \sqrt{a^2+b^2}$

and $|-z| = |-(a+ib)| = |(-a)+i(-b)| = \sqrt{(-a)^2+(-b)^2} = \sqrt{a^2+b^2}.$

$\therefore \quad |z| = |\bar{z}| = |-z|.$

(iii) $|z| = 0 \Leftrightarrow |z|^2 = 0 \Leftrightarrow |a+ib|^2 = 0$

$\Leftrightarrow a^2+b^2 = 0 \Leftrightarrow a = 0$ and $b = 0 \Leftrightarrow z = 0.$

(iv) Let $z = (a+ib)$. Then, $|z| = \sqrt{a^2+b^2}$.

Clearly, $-\sqrt{a^2+b^2} \le a \le \sqrt{a^2+b^2}$.

$\therefore \quad -|z| \le \mathrm{Re}(z) \le |z|.$

(v) Let $z = (a+ib)$. Then, $|z| = \sqrt{a^2+b^2}$.

Clearly, $-\sqrt{a^2+b^2} \le b \le \sqrt{a^2+b^2}$.

$\therefore \quad -|z| \le \mathrm{Im}(z) \le |z|.$

(vi) $z_1 z_2 = (a_1+ib_1)(a_2+ib_2) = (a_1a_2 - b_1b_2) + i(a_1b_2 + b_1a_2).$

$$\begin{aligned}\therefore \quad |z_1z_2| &= \sqrt{(a_1a_2-b_1b_2)^2+(a_1b_2+b_1a_2)^2}\\ &= \sqrt{a_1^2a_2^2+b_1^2b_2^2+a_1^2b_2^2+b_1^2a_2^2}\\ &= \sqrt{a_1^2(a_2^2+b_2^2)+b_1^2(a_2^2+b_2^2)} = \sqrt{(a_1^2+b_1^2)(a_2^2+b_2^2)}\\ &= \left\{\sqrt{a_1^2+b_1^2}\right\}\left\{\sqrt{a_2^2+b_2^2}\right\} = |z_1|\cdot|z_2|\end{aligned}$$

$\therefore \quad |z_1z_2| = |z_1|\cdot|z_2|.$

(vii) $$\frac{z_1}{z_2} = z_1\cdot\frac{1}{z_2} = (a_1+ib_1)\cdot\left\{\frac{1}{(a_2+ib_2)}\times\frac{(a_2-ib_2)}{(a_2-ib_2)}\right\}$$

$$= \frac{(a_1+ib_1)(a_2-ib_2)}{(a_2+ib_2)(a_2-ib_2)} = \frac{(a_1a_2+b_1b_2)+i(b_1a_2-a_1b_2)}{(a_2^2+b_2^2)}$$

$$= \frac{(a_1a_2+b_1b_2)}{(a_2^2+b_2^2)} + i\frac{(a_2b_1-a_1b_2)}{(a_2^2+b_2^2)}$$

$$\Rightarrow \quad \left|\frac{z_1}{z_2}\right| = \sqrt{\left\{\frac{a_1a_2+b_1b_2}{a_2^2+b_2^2}\right\}^2+\left\{\frac{a_2b_1-a_1b_2}{a_2^2+b_2^2}\right\}^2}$$

$$= \sqrt{\frac{(a_1a_2+b_1b_2)^2+(a_2b_1-a_1b_2)^2}{(a_2^2+b_2^2)^2}}$$

$$= \sqrt{\frac{a_1^2a_2^2+b_1^2b_2^2+a_2^2b_1^2+a_1^2b_2^2}{(a_2^2+b_2^2)^2}}$$

$$= \sqrt{\frac{a_1^2(a_2^2+b_2^2)+b_1^2(a_2^2+b_2^2)}{(a_2^2+b_2^2)^2}}$$

$$= \sqrt{\frac{(a_2^2+b_2^2)(a_1^2+b_1^2)}{(a_2^2+b_2^2)^2}} = \sqrt{\frac{(a_1^2+b_1^2)}{(a_2^2+b_2^2)}} = \frac{\sqrt{a_1^2+b_1^2}}{\sqrt{a_2^2+b_2^2}} = \frac{|z_1|}{|z_2|}.$$

Hence, $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$, $z_2 \neq 0$.

THEOREM 3 *For all complex numbers z_1 and z_2, prove that:*

(i) $\text{Re}(z_1 z_2) = \text{Re}(z_1)\cdot\text{Re}(z_2) - \text{Im}(z_1)\cdot\text{Im}(z_2)$

(ii) $\text{Im}(z_1 z_2) = \text{Re}(z_1)\cdot\text{Im}(z_2) + \text{Im}(z_1)\cdot\text{Re}(z_2)$

PROOF Let $z_1 = (a_1 + ib_1)$ *and* $z_2 = (a_2 + ib_2)$. Then,

$$z_1 z_2 = (a_1 + ib_1)(a_2 + ib_2)$$

$$\Rightarrow \quad z_1 z_2 = (a_1 a_2 - b_1 b_2) + i(a_1 b_2 + b_1 a_2).$$

(i) $\text{Re}(z_1 z_2) = (a_1 a_2 - b_1 b_2)$

$= \text{Re}(z_1)\cdot\text{Re}(z_2) - \text{Im}(z_1)\cdot\text{Im}(z_2)$.

(ii) $\text{Im}(z_1 z_2) = (a_1 b_2 + b_1 a_2)$

$= \text{Re}(z_1)\cdot\text{Im}(z_2) + \text{Im}(z_1)\cdot\text{Re}(z_2)$.

THEOREM 4 *For all $a, b \in R$ and $z_1, z_2 \in C$, prove that*

$$|az_1 - bz_2|^2 + |bz_1 + az_2|^2 = (a^2 + b^2)(|z_1|^2 + |z_2|^2).$$

PROOF We have

$$|az_1 - bz_2|^2 = (az_1 - bz_2)\overline{(az_1 - bz_2)}$$

$$= (az_1 - bz_2)(a\bar{z}_1 - b\bar{z}_2) \qquad [\because\ a, b \in R]$$

$$= a^2 z_1\bar{z}_1 + b^2 z_2\bar{z}_2 - abz_1\bar{z}_2 - abz_2\bar{z}_1$$

$$= a^2|z_1|^2 + b^2|z_2|^2 - ab(z_1\bar{z}_2 + z_2\bar{z}_1). \qquad \dots \text{(i)}$$

$$|bz_1 + az_2|^2 = (bz_1 + az_2)\overline{(bz_1 + az_2)}$$

$$= (bz_1 + az_2)(b\bar{z}_1 + a\bar{z}_2) \qquad [\because\ a, b \in R]$$

$$= b^2 z_1\bar{z}_1 + a^2 z_2\bar{z}_2 + abz_1\bar{z}_2 + abz_2\bar{z}_1$$

$$= b^2|z_1|^2 + a^2|z_2|^2 + ab(z_1\bar{z}_2 + z_2\bar{z}_1). \qquad \dots \text{(ii)}$$

Adding the corresponding sides of (i) and (ii), we get

$$|az_1 - bz_2|^2 + |bz_1 + az_2|^2 = (a^2 + b^2)(|z_1|^2 + |z_2|^2).$$

SOLVED EXAMPLES

EXAMPLE 1 *If* $\left|\frac{z-5i}{z+5i}\right| = 1$, *show that z is a real number.*

SOLUTION Let $z = (x + iy)$. Then,

$$\left|\frac{z-5i}{z+5i}\right| = 1 \Rightarrow \frac{|z-5i|}{|z+5i|} = 1 \qquad \left[\because\ \left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}\right]$$

$\Rightarrow \quad |z-5i| = |z+5i| \Rightarrow |z-5i|^2 = |z+5i|^2$

$\Rightarrow \quad |(x+iy)-5i|^2 = |(x+iy)+5i|^2 \qquad [\because\ z=(x+iy)]$

$\Rightarrow \quad |x+i(y-5)|^2 = |x+i(y+5)|^2$

$\Rightarrow \quad x^2+(y-5)^2 = x^2+(y+5)^2 \qquad [\because\ |x+iy|^2 = (x^2+y^2)]$

$\Rightarrow \quad (y+5)^2-(y-5)^2 = 0 \Rightarrow 4\times y\times 5 = 0 \Rightarrow y=0.$

$\therefore \quad z = x+i0 \Rightarrow z = x$, where x is real.

Hence, z is a real number.

EXAMPLE 2 *If* $(a+ib) = \dfrac{(x+i)^2}{(2x^2+1)}$ *then prove that* $(a^2+b^2) = \dfrac{(x^2+1)^2}{(2x^2+1)^2}\cdot$

SOLUTION We have

$$(a+ib) = \frac{(x+i)^2}{(2x^2+1)} = \frac{(x^2+i^2+2ix)}{(2x^2+1)} = \frac{(x^2-1)+i(2x)}{(2x^2+1)}$$

$$\Rightarrow \quad (a+ib) = \frac{(x^2-1)}{(2x^2+1)} + i\cdot\frac{2x}{(2x^2+1)}$$

$$\Rightarrow \quad |a+ib|^2 = \left|\frac{(x^2-1)}{(2x^2+1)} + i\cdot\frac{2x}{(2x^2+1)}\right|^2$$

$$\Rightarrow \quad (a^2+b^2) = \left\{\frac{(x^2-1)^2}{(2x^2+1)^2} + \frac{4x^2}{(2x^2+1)^2}\right\}$$

$$= \frac{(x^2-1)^2+4x^2}{(2x^2+1)^2} = \frac{(x^2+1)^2}{(2x^2+1)^2}\cdot$$

Hence, $(a^2+b^2) = \dfrac{(x^2+1)^2}{(2x^2+1)^2}\cdot$

EXAMPLE 3 *If* $(p+iq) = \dfrac{(a+i)^2}{(2a-i)}$ *then prove that* $(p^2+q^2) = \dfrac{(a^2+1)^2}{(4a^2+1)}\cdot$

SOLUTION We have

$$(p+iq) = \frac{(a+i)^2}{(2a-i)} = \frac{(a^2+i^2+2ai)}{(2a-i)} = \frac{(a^2-1)+2ai}{(2a-i)}$$

$$\Rightarrow \quad |p+iq|^2 = \frac{|(a^2-1)+2ai|^2}{|2a-i|^2} = \frac{(a^2-1)^2+4a^2}{\{4a^2+(-1)^2\}} = \frac{(a^2+1)^2}{(4a^2+1)}$$

$$\Rightarrow \quad (p^2+q^2) = \frac{(a^2+1)^2}{(4a^2+1)}\cdot$$

Hence, $(p^2+q^2) = \dfrac{(a^2+1)^2}{(4a^2+1)}\cdot$

EXAMPLE 4 *Let z be a complex number such that* $\left|\frac{1-iz}{z-i}\right| = 1$. *Show that z is purely real.*

SOLUTION Let $z = x + iy$. Then,

$$\left|\frac{1-iz}{z-i}\right| = 1 \Rightarrow \frac{|1-iz|}{|z-i|} = 1 \qquad \left[\because \left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}\right]$$

$\Rightarrow \quad |1-iz| = |z-i| \Rightarrow |1-iz|^2 = |z-i|^2$

$\Rightarrow \quad |1-i(x+iy)|^2 = |(x+iy)-i|^2$

$\Rightarrow \quad |(1+y)-ix|^2 = |x+(y-1)i|^2$

$\Rightarrow \quad (1+y)^2 + (-x)^2 = x^2 + (y-1)^2$

$\Rightarrow \quad 1 + y^2 + 2y + x^2 = x^2 + y^2 - 2y + 1$

$\Rightarrow \quad 4y = 0 \Rightarrow y = 0.$

$\therefore \quad z = x + 0y$ and hence z is purely real.

EXAMPLE 5 *If* z_1 *and* z_2 *are complex numbers such that* $\frac{2z_1}{3z_2}$ *is purely imaginary then prove that* $\left|\frac{z_1-z_2}{z_1+z_2}\right| = 1$.

SOLUTION Let $\frac{2z_1}{3z_2} = ki$ for some real number k. Then,

$$\frac{2z_1}{3z_2} = ki \Rightarrow \frac{z_1}{z_2} = \frac{3ki}{2}. \qquad \ldots \text{(i)}$$

$$\therefore \quad \left|\frac{z_1-z_2}{z_1+z_2}\right| = \frac{|z_1-z_2|}{|z_1+z_2|} = \frac{\left|\frac{z_1}{z_2}-1\right|}{\left|\frac{z_1}{z_2}+1\right|} \qquad \left[\because \left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}\right]$$

$$= \frac{\left|\frac{3ki}{2}-1\right|}{\left|\frac{3ki}{2}+1\right|} = \frac{|3ki-2|}{|3ki+2|} = \frac{\sqrt{9k^2+4}}{\sqrt{9k^2+4}} = 1 \qquad \text{[using (i)]}.$$

Hence, $\left|\frac{z_1-z_2}{z_1+z_2}\right| = 1.$

EXAMPLE 6 *If* $|z_1| = |z_2| = |z_3| = \ldots = |z_n| = 1$ *then prove that*

$$\left|\frac{1}{z_1} + \frac{1}{z_2} + \frac{1}{z_3} + \ldots + \frac{1}{z_n}\right| = |z_1 + z_2 + z_3 + \ldots + z_n|.$$

SOLUTION We have

$|z_1| = |z_2| = |z_3| = \ldots = |z_n| = 1$

$\Rightarrow \quad |z_1|^2 = |z_2|^2 = |z_3|^2 = \ldots = |z_n|^2 = 1$

$\Rightarrow \quad z_1\bar{z}_1 = 1, z_2\bar{z}_2 = 1, z_3\bar{z}_3 = 1, \ldots, z_n\bar{z}_n = 1$

$\Rightarrow \quad \frac{1}{z_1} = \bar{z}_1, \frac{1}{z_2} = \bar{z}_2, \frac{1}{z_3} = \bar{z}_3, \ldots, \frac{1}{z_n} = \bar{z}_n$

$$\Rightarrow \quad \left|\frac{1}{z_1}+\frac{1}{z_2}+\frac{1}{z_3}+\ldots+\frac{1}{z_n}\right| = |\bar{z}_1+\bar{z}_2+\bar{z}_3+\ldots+\bar{z}_n|$$

$$= |\overline{z_1+z_2+z_3+\ldots+z_n}|$$

$$= |z_1+z_2+z_3+\ldots+z_n| \qquad [\because |\bar{z}| = |z|].$$

$$\therefore \quad \left|\frac{1}{z_1}+\frac{1}{z_2}+\frac{1}{z_3}+\ldots+\frac{1}{z_n}\right| = |z_1+z_2+z_3+\ldots+z_n|.$$

EXAMPLE 7 *If* $\dfrac{a+ib}{c+id} = x+iy$, *prove that*

(i) $\dfrac{a-ib}{c-id} = x-iy$ *and* (ii) $\dfrac{a^2+b^2}{c^2+d^2} = x^2+y^2$.

SOLUTION (i) $\left(\dfrac{a+ib}{c+id}\right) = (x+iy) \Rightarrow \overline{\left(\dfrac{a+ib}{c+id}\right)} = \overline{(x+iy)}$

$$\Rightarrow \frac{\overline{(a+ib)}}{\overline{(c+id)}} = (x-iy) \Rightarrow \frac{(a-ib)}{(c-id)} = (x-iy).$$

(ii) We have

$$\frac{(a+ib)}{(c+id)} = (x+iy) \text{ and } \frac{(a-ib)}{(c-id)} = (x-iy)$$

$$\Rightarrow \frac{(a+ib)}{(c+id)} \times \frac{(a-ib)}{(c-id)} = (x+iy)(x-iy)$$

$$\Rightarrow \frac{(a+ib)(a-ib)}{(c+id)(c-id)} = (x+iy)(x-iy)$$

$$\Rightarrow \frac{(a^2+b^2)}{(c^2+d^2)} = (x^2+y^2).$$

EXAMPLE 8 *If* $(x+iy) = \sqrt{\dfrac{a+ib}{c+id}}$, *prove that* $(x^2+y^2)^2 = \dfrac{a^2+b^2}{c^2+d^2}$.

SOLUTION We have

$$(x+iy) = \sqrt{\frac{a+ib}{c+id}} = \frac{\sqrt{a+ib}}{\sqrt{c+id}}$$

$$\Rightarrow \quad (x-iy) = \frac{\sqrt{a-ib}}{\sqrt{c-id}}$$

$$\Rightarrow \quad (x+iy)(x-iy) = \frac{\sqrt{a+ib}}{\sqrt{c+id}} \times \frac{\sqrt{a-ib}}{\sqrt{c-id}} = \frac{\sqrt{(a+ib)(a-ib)}}{\sqrt{(c+id)(c-id)}}$$

$$\Rightarrow \quad (x^2+y^2) = \frac{\sqrt{a^2+b^2}}{\sqrt{c^2+d^2}} \Rightarrow (x^2+y^2)^2 = \frac{(a^2+b^2)}{(c^2+d^2)}.$$

Hence, $(x^2+y^2)^2 = \dfrac{(a^2+b^2)}{(c^2+d^2)}$.

EXAMPLE 9 *If* $(1+i)(1+2i)(1+3i)\dots(1+ni) = (x+iy)$ *then prove that*

$$2 \times 5 \times 10 \times \dots \times (1+n^2) = (x^2+y^2).$$

SOLUTION $(1+i)(1+2i)(1+3i)\dots(1+ni) = (x+iy)$

$\Rightarrow \quad |(1+i)(1+2i)(1+3i)\dots(1+ni)|^2 = |x+iy|^2$

$\Rightarrow \quad |1+i|^2 \times |1+2i|^2 \times |1+3i|^2 \times \dots \times |1+ni|^2 = (x^2+y^2)$

$\Rightarrow \quad (\sqrt{2})^2 \times (\sqrt{5})^2 \times (\sqrt{10})^2 \times \dots \times (\sqrt{1+n^2})^2 = (x^2+y^2)$

$\Rightarrow \quad 2 \times 5 \times 10 \times \dots \times (1+n^2) = (x^2+y^2).$

Hence, $2 \times 5 \times 10 \times \dots \times (1+n^2) = (x^2+y^2)$.

EXAMPLE 10 *If* $(a+ib)(c+id)(e+if)(g+ih) = (A+iB)$ *then show that*

$$(a^2+b^2)(c^2+d^2)(e^2+f^2)(g^2+h^2) = (A^2+B^2).$$

SOLUTION $(a+ib)(c+id)(e+if)(g+ih) = (A+iB)$

$\Rightarrow \quad |(a+ib)(c+id)(e+if)(g+ih)| = |A+iB|$

$\Rightarrow \quad |a+ib| \cdot |c+id| \cdot |e+if| \cdot |g+ih| = |A+iB|$

$\Rightarrow \quad |a+ib|^2 \cdot |c+id|^2 \cdot |e+if|^2 \cdot |g+ih|^2 = |A+iB|^2$

$\Rightarrow \quad (a^2+b^2)(c^2+d^2)(e^2+f^2)(g^2+h^2) = (A^2+B^2).$

Hence, $(a^2+b^2)(c^2+d^2)(e^2+f^2)(g^2+h^2) = (A^2+B^2)$.

EXAMPLE 11 *If* α *and* β *are different complex numbers such that* $|\beta| = 1$, *show that* $\left|\dfrac{\beta-\alpha}{1-\bar{\alpha}\beta}\right| = 1.$

SOLUTION We have

$$\left|\frac{\beta-\alpha}{1-\bar{\alpha}\beta}\right|^2 = \left\{\frac{\beta-\alpha}{1-\bar{\alpha}\beta}\right\}\overline{\left\{\frac{\beta-\alpha}{1-\bar{\alpha}\beta}\right\}} \qquad [\because |z|^2 = z\bar{z}]$$

$$= \frac{(\beta-\alpha)}{(1-\bar{\alpha}\beta)} \cdot \frac{(\bar{\beta}-\bar{\alpha})}{(1-\alpha\bar{\beta})} = \frac{(\beta-\alpha)(\bar{\beta}-\bar{\alpha})}{(1-\bar{\alpha}\beta)(1-\alpha\bar{\beta})}$$

$$= \frac{\beta\bar{\beta}+\alpha\bar{\alpha}-\beta\bar{\alpha}-\alpha\bar{\beta}}{1-\alpha\bar{\beta}-\bar{\alpha}\beta+(\alpha\bar{\alpha})(\beta\bar{\beta})}$$

$$= \frac{|\beta|^2+|\alpha|^2-(\alpha\bar{\beta}+\beta\bar{\alpha})}{1-(\alpha\bar{\beta}+\beta\bar{\alpha})+|\alpha|^2\,|\beta|^2} \qquad [\because \alpha\bar{\alpha} = |\alpha|^2, \beta\bar{\beta} = |\beta|^2]$$

$$= \frac{1+|\alpha|^2-(\alpha\bar{\beta}+\beta\bar{\alpha})}{1+|\alpha|^2-(\alpha\bar{\beta}+\beta\bar{\alpha})} = 1 \qquad [\because |\beta| = 1].$$

Hence, $\left|\dfrac{\beta-\alpha}{1-\bar{\alpha}\beta}\right| = 1.$

EXAMPLE 12 *If* $|z^2-1| = |z|^2+1$ *then show that z is imaginary.*

SOLUTION Let $z=(x+iy)$. Then, $z^2=(x^2-y^2)+2ixy$ and $|z|^2=(x^2+y^2)$.

Now, $|z^2-1| = |z|^2+1$

$\Rightarrow \quad |(x^2-y^2-1)+i(2xy)| = (x^2+y^2+1)$

$\Rightarrow \quad |(x^2-y^2-1)+i(2xy)|^2 = (x^2+y^2+1)^2$

$\Rightarrow \quad (x^2-y^2-1)^2+4x^2y^2 = (x^2+y^2+1)^2$

$\Rightarrow \quad [x^2+(y^2+1)]^2-[x^2-(y^2+1)]^2 = 4x^2y^2$

$\Rightarrow \quad 4x^2(y^2+1) = 4x^2y^2 \Rightarrow 4x^2\{(y^2+1)-y^2\}=0$

$\Rightarrow \quad 4x^2=0 \Rightarrow x=0.$

Hence, $z=0+iy$, which shows that z is imaginary.

EXAMPLE 13 *If* $iz^3+z^2-z+i=0$ *then show that* $|z|=1$.

SOLUTION We have

$iz^3+z^2-z+i=0$

$\Rightarrow \quad z^3-iz^2+iz+1=0$ [on dividing both sides by i]

$\Rightarrow \quad z^2(z-i)+i(z-i)=0$

$\Rightarrow \quad (z-i)(z^2+i)=0$

$\Rightarrow \quad z=i \quad \text{or} \quad z^2=-i.$

Now, $z=i \Rightarrow |z|=|i| \Rightarrow |z|=1$.

And, $z^2=-i \Rightarrow |z^2|=|-i|=1 \Rightarrow |z|^2=1 \Rightarrow |z|=1$.

Hence, in either case, we have $|z|=1$.

EXAMPLE 14 *Find all nonzero complex numbers z satisfying* $\bar{z}=iz^2$.

SOLUTION Let $z=x+iy$. Then, $\bar{z}=x-iy$ and $z^2=(x^2-y^2)+2ixy$.

$\therefore \quad \bar{z}=iz^2 \Rightarrow x-iy=i(x^2-y^2)-2xy$

$\Rightarrow (x+2xy)-i(x^2-y^2+y)=0.$ … (i)

On equating real parts and imaginary parts on both sides of (i) separately, we get

$x+2xy=0$ … (ii) and $x^2-y^2+y=0$ … (iii).

From (ii), we get

$x+2xy=0 \Rightarrow x(1+2y)=0$

$\Rightarrow x=0 \text{ or } 1+2y=0 \Rightarrow x=0 \text{ or } y=\frac{-1}{2}\cdot$

Case I *When* $x=0$.

Putting $x=0$ in (iii), we get

$-y^2+y=0 \Rightarrow -y(y-1)=0 \Rightarrow y=0 \text{ or } y=1.$

Thus, $(x=0, y=0)$ or $(x=0, y=1)$

$\therefore \quad z=(0+i0)$ or $z=(0+1\cdot i)$.

Case II *When* $y=\frac{-1}{2}\cdot$

Putting $y=\frac{-1}{2}$ in (iii), we get

$$x^2-\left(\frac{-1}{2}\right)^2+\left(\frac{-1}{2}\right)=0 \Rightarrow x^2-\frac{1}{4}-\frac{1}{2}=0$$

$$\Rightarrow \quad x^2=\left(\frac{1}{4}+\frac{1}{2}\right)=\frac{3}{4} \Rightarrow x=\pm\frac{\sqrt{3}}{2}\cdot$$

$$\therefore \quad \left(x=\frac{\sqrt{3}}{2}, y=\frac{-1}{2}\right) \text{ or } \left(x=\frac{-\sqrt{3}}{2}, y=\frac{-1}{2}\right)$$

$$\therefore \quad z=\left(\frac{\sqrt{3}}{2}-\frac{1}{2}i\right) \text{ or } z=\left(\frac{-\sqrt{3}}{2}-\frac{1}{2}i\right)\cdot$$

Hence, the required nonzero complex numbers are

$$i, \left(\frac{\sqrt{3}}{2}-\frac{1}{2}i\right) \text{ and } \left(\frac{-\sqrt{3}}{2}-\frac{1}{2}i\right)\cdot$$

EXAMPLE 15 *Solve the equation,* $z^2=\bar{z}$, *where* z *is a complex number.*

SOLUTION Let $z=x+iy$. Then,

$$z^2=\bar{z} \Rightarrow (x+iy)^2=x-iy$$

$$\Rightarrow (x^2-y^2)+2ixy=x-iy. \qquad \ldots \text{(i)}$$

Equating real parts and imaginary parts on both sides of (i) separately, we get

$$x^2-y^2=x \quad \ldots \text{(ii)} \quad \text{and} \quad 2xy=-y \quad \ldots \text{(iii)}.$$

From (iii), we get

$$2xy+y=0 \Rightarrow y(2x+1)=0 \Rightarrow y=0 \text{ or } x=\frac{-1}{2}\cdot$$

Case I *When* $y=0$.

Putting $y=0$ in (ii), we get

$$x^2-x=0 \Rightarrow x(x-1)=0 \Rightarrow x=0 \text{ or } x=1.$$

$\therefore \quad (x=0, y=0)$ or $(x=1, y=0)$

Thus, $z=(0+i0)$ or $z=(1+i0)$.

Case II *When* $x=\frac{-1}{2}\cdot$

Putting $x=\frac{-1}{2}$ in (ii), we get

$$\left(\frac{-1}{2}\right)^2-y^2=\left(\frac{-1}{2}\right) \Rightarrow y^2=\left(\frac{1}{4}+\frac{1}{2}\right)=\frac{3}{4} \Rightarrow y=\pm\frac{\sqrt{3}}{2}\cdot$$

$$\therefore \quad \left(x=\frac{-1}{2}, y=\frac{\sqrt{3}}{2}\right) \text{ or } \left(x=\frac{-1}{2}, y=\frac{-\sqrt{3}}{2}\right).$$

$$\text{Thus, } z=\left(\frac{-1}{2}+\frac{\sqrt{3}}{2}i\right) \text{ or } z=\left(\frac{-1}{2}-\frac{\sqrt{3}}{2}i\right).$$

Hence, $z=0, 1, \left(\frac{-1}{2}+\frac{\sqrt{3}}{2}i\right)$ and $\left(\frac{-1}{2}-\frac{\sqrt{3}}{2}i\right)$ are the required roots of the given equation.

EXAMPLE 16 *For all complex numbers z_1 and z_2, prove that*

$$|z_1+z_2|^2+|z_1-z_2|^2=2(|z_1|^2+|z_2|^2).$$

SOLUTION We have

$$|z_1+z_2|^2=(z_1+z_2)\overline{(z_1+z_2)}$$
$$=(z_1+z_2)(\bar{z}_1+\bar{z}_2) \qquad [\because \overline{(z_1+z_2)}=\bar{z}_1+\bar{z}_2]$$
$$=z_1\bar{z}_1+z_2\bar{z}_2+z_1\bar{z}_2+z_2\bar{z}_1$$
$$=|z_1|^2+|z_2|^2+z_1\bar{z}_2+z_2\bar{z}_1. \qquad \dots \text{(i)}$$
$$|z_1-z_2|^2=(z_1-z_2)\overline{(z_1-z_2)}=(z_1-z_2)(\bar{z}_1-\bar{z}_2)$$
$$=z_1\bar{z}_1+z_2\bar{z}_2-z_1\bar{z}_2-z_2\bar{z}_1$$
$$=|z_1|^2+|z_2|^2-z_1\bar{z}_2-z_2\bar{z}_1. \qquad \dots \text{(ii)}$$

On adding the corresponding sides of (i) and (ii), we get

$$|z_1+z_2|^2+|z_1-z_2|^2=2(|z_1|^2+|z_2|^2).$$

EXERCISE 5C

1. Express each of the following in the form $(a+ib)$ and find its conjugate:

(i) $\frac{1}{(4+3i)}$ (ii) $(2+3i)^2$ (iii) $\frac{(2-i)}{(1-2i)^2}$

(iv) $\frac{(1+i)(1+2i)}{(1+3i)}$ (v) $\left(\frac{1+2i}{2+i}\right)^2$ (vi) $\frac{(2+i)}{(3-i)(1+2i)}$

2. Express each of the following in the form $(a+ib)$ and find its multiplicative inverse:

(i) $\frac{1+2i}{1-3i}$ (ii) $\frac{(1+7i)}{(2-i)^2}$ (iii) $\frac{-4}{(1+i\sqrt{3})}$

3. If $(x+iy)^3=(u+iv)$ then prove that $\left(\frac{u}{x}+\frac{v}{y}\right)=4(x^2-y^2)$.

4. If $(x+iy)^{1/3}=(a+ib)$ then prove that $\left(\frac{x}{a}+\frac{y}{b}\right)=4(a^2-b^2)$.

5. Express $(1-2i)^{-3}$ in the form $(a+ib)$.

6. Find real values of x and y for which

$$(x^4 + 2xi) - (3x^2 + iy) = (3 - 5i) + (1 + 2iy).$$

7. If $z^2 + |z|^2 = 0$, show that z is purely imaginary.

8. If $\dfrac{z-1}{z+1}$ is purely imaginary and $z \neq -1$, show that $|z| = 1$.

9. If z_1 is a complex number other than –1 such that $|z_1| = 1$ and $z_2 = \dfrac{z_1 - 1}{z_1 + 1}$ then show that z_2 is purely imaginary.

10. For all $z \in C$, prove that

(i) $\dfrac{1}{2}(z + \bar{z}) = \text{Re}(z)$ (ii) $\dfrac{1}{2i}(z - \bar{z}) = \text{Im}(z)$ (iii) $z\bar{z} = |z|^2$

(iv) $(z + \bar{z})$ is real (v) $(z - \bar{z})$ is 0 or imaginary.

11. If $z_1 = (1 + i)$ and $z_2 = (-2 + 4i)$, prove that $\text{Im}\left(\dfrac{z_1 z_2}{\bar{z}_1}\right) = 2.$

12. If a and b are real numbers such that $a^2 + b^2 = 1$ then show that a real value of x will satisfy the equation, $\dfrac{1 - ix}{1 + ix} = (a - ib)$.

ANSWERS (EXERCISE 5C)

1. (i) $z = \left(\dfrac{4}{25} - \dfrac{3}{25}i\right), \bar{z} = \left(\dfrac{4}{25} + \dfrac{3}{25}i\right)$ (ii) $z = (-5 + 12i), \bar{z} = (-5 - 12i)$

(iii) $z = \left(\dfrac{-2}{25} + \dfrac{11}{25}i\right), \bar{z} = \left(\dfrac{-2}{25} - \dfrac{11}{25}i\right)$ (iv) $z = \left(\dfrac{4}{5} + \dfrac{3}{5}i\right), \bar{z} = \left(\dfrac{4}{5} - \dfrac{3}{5}i\right)$

(v) $z = \left(\dfrac{7}{25} + \dfrac{24}{25}i\right), \bar{z} = \left(\dfrac{7}{25} - \dfrac{24}{25}i\right)$ (vi) $z = \left(\dfrac{3}{10} - \dfrac{1}{10}i\right), \bar{z} = \left(\dfrac{3}{10} + \dfrac{1}{10}i\right)$

2. (i) $z = \left(\dfrac{-1}{2} + \dfrac{1}{2}i\right), z^{-1} = (-1 - i)$ (ii) $z = (-1 + i), z^{-1} = \left(\dfrac{-1}{2} - \dfrac{1}{2}i\right)$

(iii) $z = (-1 + i\sqrt{3}), z^{-1} = \left(\dfrac{-1}{4} - \dfrac{\sqrt{3}}{4}i\right)$

5. $\left(\dfrac{-11}{125} - \dfrac{2}{125}i\right)$ **6.** $(x = 2, y = 3)$ or $\left(x = -2, y = \dfrac{1}{3}\right)$

HINTS TO SOME SELECTED QUESTIONS

3. $(u + iv) = (x + iy)^3 = x^3 - iy^3 + 3ixy(x + iy)$

$\Rightarrow \quad (u + iv) = (x^3 - 3xy^2) + i(3x^2y - y^3)$

$\Rightarrow \quad u = x^3 - 3x^2y$ and $v = 3x^2y - y^3$

$\Rightarrow \quad \left(\dfrac{u}{x} + \dfrac{v}{y}\right) = (x^2 - 3y^2) + (3x^2 - y^2) = 4(x^2 - y^2).$

4. $(x + iy) = (a + ib)^3 = a^3 - ib^3 + 3iab(a + ib)$

$\Rightarrow \quad (x + iy) = (a^3 - 3ab^2) + i(3a^2b - b^3)$

$\Rightarrow \quad x = a^3 - 3ab^2$ and $y = 3a^2b - b^3$

$\Rightarrow \quad \left(\frac{x}{a} + \frac{y}{b}\right) = (a^2 - 3b^2) + (3a^2 - b^2) = 4(a^2 - b^2)$.

5. $(1-2i)^{-3} = \dfrac{1}{(1-2i)^3} = \dfrac{1}{\{1 - 8i^3 - 6i(1-2i)\}}$

$$= \frac{1}{(-11+2i)} \times \frac{(-11-2i)}{(-11-2i)} = \frac{(-11-2i)}{125}.$$

6. $(x^4 - 3x^2) + i(2x - y) = 4 + (2y - 5)i$

$\Rightarrow \quad (x^4 - 3x^2 - 4) + i(2x - y - 2y + 5) = 0$

$\Rightarrow \quad x^4 - 3x^2 - 4 = 0$ and $2x - 3y + 5 = 0$.

Now, $x^4 - 3x^2 - 4 = 0 \Rightarrow (x^2 - 4)(x^2 + 1) = 0 \Rightarrow x = 2$ or $x = -2$.

$(x = 2 \Rightarrow y = 3)$ and $\left(x = -2 \Rightarrow y = \frac{1}{3}\right)$.

7. Let $z = (x + iy)$. Then,

$z^2 + |z|^2 = 0 \Rightarrow x^2 - y^2 + 2ixy + x^2 + y^2 = 0$

$\Rightarrow x^2 + ixy = 0 \Rightarrow x^2 = 0$ and $xy = 0 \Rightarrow x = 0$.

$\therefore \quad z$ is purely imaginary.

8. Let $z = (x + iy)$. Then,

$$\frac{z-1}{z+1} = \frac{(x+iy)-1}{(x+iy)+1} = \frac{(x-1)+iy}{(x+1)+iy} \times \frac{(x+1)-iy}{(x+1)-iy} = \frac{(x^2+y^2-1)+2iy}{(x+1)^2+y^2}.$$

$\dfrac{z-1}{z+1}$ is purely imaginary $\Leftrightarrow x^2 + y^2 - 1 = 0 \Leftrightarrow x^2 + y^2 = 1 \Leftrightarrow |z| = 1$.

9. Let $z_1 = x_1 + iy_1$. Then, $|z_1| = 1 \Rightarrow |z_1|^2 = 1 \Rightarrow x_1^2 + y_1^2 = 1$.

$$z_2 = \frac{z_1 - 1}{z_1 + 1} = \frac{(x_1 + iy_1) - 1}{(x_1 + iy_1) + 1} = \frac{(x_1 - 1) + iy_1}{(x_1 + 1) + iy_1} \times \frac{(x_1 + 1) - iy_1}{(x_1 + 1) - iy_1}$$

$= \dfrac{(x_1^2 + y_1^2 - 1) + 2iy_1}{(x_1+1)^2 + y_1^2} = \dfrac{2iy_1}{(x_1+1)^2 + y_1^2}$, which is purely imaginary. $\quad [\because |z_1|^2 = 1]$

12. We have

$\dfrac{1-ix}{1+ix} = \dfrac{a-ib}{1} \Rightarrow \dfrac{(1-ix)+(1+ix)}{(1-ix)-(1+ix)} = \dfrac{(a-ib)+1}{(a-ib)-1}$ [by componendo and dividendo]

$\Rightarrow \dfrac{2}{-2ix} = \dfrac{(a+1)-ib}{(a-1)-ib} \Rightarrow -ix = \dfrac{(a-1)-ib}{(a+1)-ib} \times \dfrac{(a+1)+ib}{(a+1)+ib}$

$\Rightarrow -ix = \dfrac{(a^2 - 1 + b^2) + \{(a-1)b - (a+1)b\}i}{(a+1)^2 + b^2}$

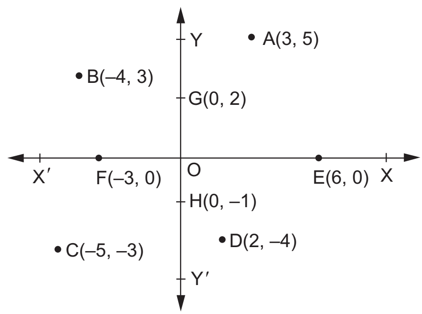
$= \dfrac{-2bi}{(a+1)^2 + b^2} \quad [\because a^2 + b^2 = 1]$

$\Rightarrow x = \dfrac{2b}{(a+1)^2 + b^2}$, which is purely real.

POLAR REPRESENTATION OF COMPLEX NUMBERS

COMPLEX PLANE OR ARGAND PLANE

Let $X'OX$ and YOY' be the mutually perpendicular lines, known as the x-axis and the y-axis respectively. The complex number $(x+iy)$ corresponds to the ordered pair (x, y) and it can be represented by the point $P(x, y)$ in the x-y plane. The x-y plane is known as the *complex plane*, or the *Argand plane*. x-axis is called the *real axis* and y-axis is called the *imaginary axis*.



Note that every number on the x-axis is a *real number*, while each on the y-axis is an *imaginary number*.

The complex numbers represented geometrically in the above diagram are

$$(3+5i), (-4+3i), (-5-3i), (2-4i),$$
$$(6+0i), (-3+0i), (0+2i) \text{ and } (0-i),$$

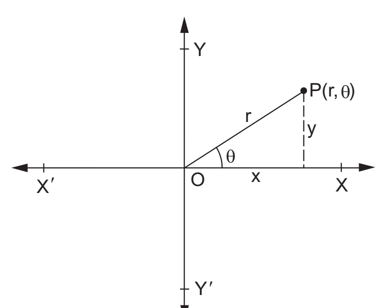
represented by the points

$$A(3, 5), B(-4, 3), C(-5, -3), D(2, -4), E(6, 0),$$
$$F(-3, 0), G(0, 2) \text{ and } H(0, -1) \text{ respectively.}$$

POLAR FORM OF A COMPLEX NUMBER

Let the complex number $z = x + iy$ be represented by the point $P(x, y)$ in the complex plane.

Let $\angle XOP = \theta$ and $|OP| = r > 0$. Then, $P(r, \theta)$ are called the *polar coordinates* of P.



We call the origin O as *pole*.

Clearly, $x = r\cos\theta$ and $y = r\sin\theta$

$\therefore \quad z = r(\cos\theta + i\sin\theta)$.

This is called the *polar form,* or *trigonometric form,* or *modulus-amplitude form,* of z.

Here, $r = \sqrt{x^2 + y^2} = |z|$ is called the *modulus of* z.

And θ is called the *argument,* or *amplitude of* z, written as $\arg(z)$, or $\text{amp}(z)$.

The value of θ such that $-\pi < \theta \le \pi$ is called the *principal argument of* z.

METHOD FOR FINDING THE PRINCIPAL ARGUMENT OF A COMPLEX NUMBER

Case I *When* $z = (x + iy)$ *lies on one of the axes:*

I. *When* z *is purely real.*

In this case, z lies on the x-axis.

(i) If z lies on positive side of the x-axis then $\theta = 0$.

(ii) If z lies on negative side of the x-axis then $\theta = \pi$.

II. *When* z *is purely imaginary.*

In this case, z lies on the y-axis.

(i) If z lies on the y-axis and above the x-axis then $\theta = \frac{\pi}{2}$.

(ii) If z lies on the y-axis and below the x-axis then $\theta = \frac{-\pi}{2}$.

Case II *When* $z = (x + iy)$ *does not lie on any axes:*

Step 1. Find the acute angle α given by $\tan\alpha = \left|\frac{\text{Im}(z)}{\text{Re}(z)}\right|$.

Step 2. Find the quadrant in which $P(x, y)$ lies.

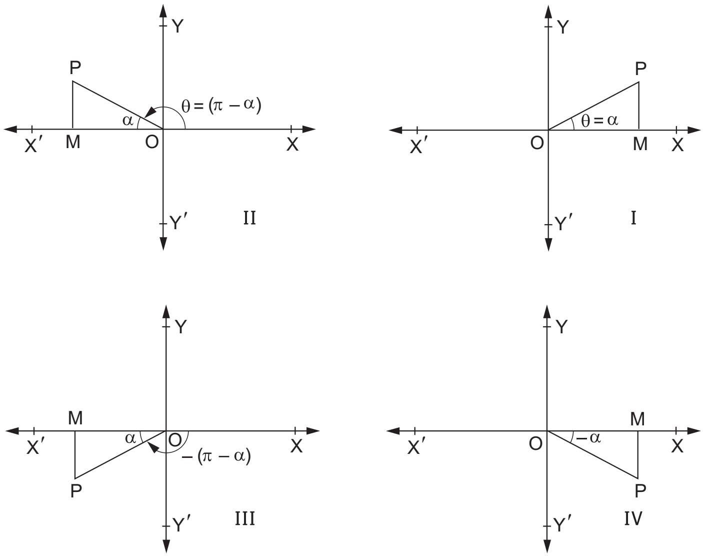
Then, $\theta = \arg(z)$ or $\text{amp}(z)$ may be obtained as under.

(i) *When* z *lies in quad. I;*
then, $\theta = \alpha \Rightarrow \arg(z) = \alpha$.

(ii) *When* z *lies in quad. II;*
then, $\theta = (\pi - \alpha) \Rightarrow \arg(z) = (\pi - \alpha)$.

(iii) *When* z *lies in quad. III;*
then, $\theta = (\alpha - \pi)$ or $(\pi + \alpha)$
$\Rightarrow \arg(z) = (\alpha - \pi)$ or $(\pi + \alpha)$.

(iv) *When* z *lies in quad. IV;*
then, $\theta = -\alpha$ or $\theta = (2\pi - \alpha)$
$\Rightarrow \arg(z) = -\alpha$ or $(2\pi - \alpha)$.



SUMMARY

Polar form of $z = x + iy$ is $r(\cos\theta + i\sin\theta)$.

(i) $r = |z| = \sqrt{x^2 + y^2}$.

(ii) $\tan\alpha = \left|\frac{y}{x}\right| = \left|\frac{\text{Im}(z)}{\text{Re}(z)}\right|$.

(iii) When $-\pi < \theta \le \pi$, then θ is the principal argument of z.

(iv)

Quad. in which z lies	$\arg(z)$
I	$\theta = \alpha$
II	$\theta = \pi - \alpha$
III	$\theta = -(\pi - \alpha)$
IV	$\theta = -\alpha$ or $(2\pi - \alpha)$

(v) *When z is purely real;*

then, z lies on the x-axis.

$(x > 0 \Rightarrow \theta = 0)$ and $(x < 0 \Rightarrow \theta = \pi)$.

(vi) *When z is purely imaginary;*

then, z lies on the y-axis.

$\left(y > 0 \Rightarrow \theta = \frac{\pi}{2}\right)$ and $\left(y < 0 \Rightarrow \theta = \frac{-\pi}{2}\right)$.

SOLVED EXAMPLES

EXAMPLE 1 *Convert each of the following complex numbers into polar form:*

(i) 3 *(ii)* –5 *(iii)* i *(iv)* $-2i$

SOLUTION (i) The given complex number is $z = 3 + 0i$.

Let its polar form be $z = r(\cos\theta + i\sin\theta)$.

Now, $r = |z| = \sqrt{3^2 + 0^2} = \sqrt{9} = 3$.

Clearly, $z = 3 + 0i$ is represented by the point $P(3, 0)$, which lies on the positive side of the x-axis.

$\therefore\quad \arg(z) = \theta = 0$.

Thus, $r = 3$ and $\theta = 0$.

Hence, the required polar form of $z = 3 + 0i$ is $3(\cos 0 + i\sin 0)$.

(ii) The given complex number is $z = -5 + 0i$.

Let its polar form be $z = r(\cos\theta + i\sin\theta)$.

Now, $r = |z| = \sqrt{(-5)^2 + 0^2} = \sqrt{25} = 5$.

Clearly, $z = -5 + 0i$ is represented by the point $P(-5, 0)$, which lies on the negative side of the x-axis.

$\therefore\quad \arg(z) = \pi \Rightarrow \theta = \pi$.

Thus, $r = 5$ and $\theta = \pi$.

Hence, the required polar form of $z = -5 + 0i$ is $5(\cos\pi + i\sin\pi)$.

(iii) The given complex number is $z = 0 + i$.

Let its polar form be $z = r(\cos\theta + i\sin\theta)$.

Now, $r = |z| = \sqrt{0^2 + 1^2} = 1$.

Clearly, $z = 0 + i$ is represented by the point $P(0, 1)$, which lies on the y-axis and above the x-axis.

$\therefore\quad \arg(z) = \frac{\pi}{2} \Rightarrow \theta = \frac{\pi}{2}\cdot$

Thus, $r = 1$ and $\theta = \frac{\pi}{2}\cdot$

Hence, the required polar form of $z = 0 + i$ is

$$1\cdot\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right), \text{ i.e., } \left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)\cdot$$

(iv) The given complex number is $z = 0 - 2i$.

Let its polar form be $z = r(\cos\theta + i\sin\theta)$.

Now, $r = |z| = \sqrt{0^2 + (-2)^2} = \sqrt{4} = 2$.

Clearly, $z = (0 - 2i)$ is represented by the point $P(0, -2)$, which lies on the y-axis and below the x-axis.

$\therefore\quad \arg(z) = \frac{-\pi}{2} \Rightarrow \theta = \frac{-\pi}{2}\cdot$

Thus, $r = 2$ and $\theta = \frac{-\pi}{2}\cdot$

Hence, the required polar form of $z = 0 - 2i$ is

$$z = 2\left\{\cos\left(\frac{-\pi}{2}\right) + i\sin\left(\frac{-\pi}{2}\right)\right\}, \text{ i.e., } 2\left(\cos\frac{\pi}{2} - i\sin\frac{\pi}{2}\right).$$

EXAMPLE 2 *Find the modulus and argument of each of the complex numbers given below:*
(i) $1+i$ *(ii)* $-\sqrt{3}+i$ *(iii)* $-1-i\sqrt{3}$

SOLUTION (i) Let $z = 1+i$. Then, $|z| = \sqrt{1^2+1^2} = \sqrt{2}$.

Let α be the acute angle given by

$$\tan\alpha = \left|\frac{\text{Im}(z)}{\text{Re}(z)}\right| = \left|\frac{1}{1}\right| = 1 \Rightarrow \alpha = \frac{\pi}{4}.$$

Clearly, the point representing $z = 1+i$ is $P(1, 1)$, which lies in the first quadrant.

$$\therefore \quad \arg(z) = \theta = \alpha = \frac{\pi}{4}.$$

Hence, $|z| = \sqrt{2}$ and $\arg(z) = \frac{\pi}{4}$.

(ii) Let $z = -\sqrt{3}+i$. Then, $|z| = \sqrt{(-\sqrt{3})^2+1^2} = \sqrt{4} = 2$.

Let α be the acute angle given by

$$\tan\alpha = \left|\frac{\text{Im}(z)}{\text{Re}(z)}\right| = \left|\frac{1}{-\sqrt{3}}\right| = \frac{1}{\sqrt{3}} \Rightarrow \alpha = \frac{\pi}{6}.$$

Now, $z = (-\sqrt{3}+i)$ is represented by the point $P(-\sqrt{3}, 1)$, which lies in the second quadrant.

$$\therefore \quad \arg(z) = \theta = (\pi - \alpha) = \left(\pi - \frac{\pi}{6}\right) = \frac{5\pi}{6}.$$

Hence, $|z| = 2$ and $\arg(z) = \frac{5\pi}{6}$.

(iii) Let $z = -1-i\sqrt{3}$. Then, $|z| = \sqrt{(-1)^2+(-\sqrt{3})^2} = \sqrt{4} = 2$.

Let α be the acute angle given by

$$\tan\alpha = \left|\frac{\text{Im}(z)}{\text{Re}(z)}\right| = \left|\frac{-\sqrt{3}}{-1}\right| = \sqrt{3} \Rightarrow \alpha = \frac{\pi}{3}.$$

Now, $z = (-1-i\sqrt{3})$ is represented by the point $P(-1, -\sqrt{3})$, which lies in the third quadrant.

$$\therefore \quad \arg(z) = \theta = (\alpha - \pi) = \left(\frac{\pi}{3} - \pi\right) = \frac{-2\pi}{3}.$$

Hence, $|z| = 2$ and $\arg(z) = \frac{-2\pi}{3}$.

EXAMPLE 3 *Convert the complex number* $(1+i\sqrt{3})$ *into polar form.*

SOLUTION The given complex number is $z = (1+i\sqrt{3})$.

Let its polar form be $z = r(\cos\theta + i\sin\theta)$.

Now, $r = |z| = \sqrt{1^2+(\sqrt{3})^2} = \sqrt{4} = 2$.

Let α be the acute angle, given by

$$\tan\alpha = \left|\frac{\operatorname{Im}(z)}{\operatorname{Re}(z)}\right| = \left|\frac{\sqrt{3}}{1}\right| = \sqrt{3} \Rightarrow \alpha = \frac{\pi}{3}.$$

Clearly, the point representing $z = (1 + i\sqrt{3})$ is $P(1, \sqrt{3})$, which lies in the first quadrant.

$\therefore \quad \arg(z) = \theta = \alpha = \frac{\pi}{3}.$

Thus, $r = |z| = 2$ and $\theta = \arg(z) = \frac{\pi}{3}.$

Hence, the required polar form of $z = (1 + i\sqrt{3})$ is $2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right).$

EXAMPLE 4 *Convert the complex number* $(2\sqrt{3} - 2i)$ *into polar form.*

SOLUTION The given complex number is $z = (2\sqrt{3} - 2i)$.

Let its polar form be $z = r(\cos\theta + i\sin\theta)$.

Now, $r = |z| = \sqrt{(2\sqrt{3})^2 + (-2)^2} = \sqrt{12 + 4} = \sqrt{16} = 4.$

Let α be the acute angle, given by

$$\tan\alpha = \left|\frac{\operatorname{Im}(z)}{\operatorname{Re}(z)}\right| = \left|\frac{-2}{2\sqrt{3}}\right| = \frac{1}{\sqrt{3}} \Rightarrow \alpha = \frac{\pi}{6}.$$

Clearly, the point representing $z = (2\sqrt{3} - 2i)$ is $P(2\sqrt{3}, -2)$, which lies in the fourth quadrant.

$\therefore \quad \arg(z) = \theta = -\alpha = \frac{-\pi}{6}.$

Thus, $r = |z| = 4$ and $\theta = \arg(z) = \frac{-\pi}{6}.$

Hence, the required polar form of $z = (2\sqrt{3} - 2i)$ is

$$4\left[\cos\left(\frac{-\pi}{6}\right) + i\sin\left(\frac{-\pi}{6}\right)\right] = 4\left(\cos\frac{\pi}{6} - i\sin\frac{\pi}{6}\right).$$

EXAMPLE 5 *Convert the complex number* $(-2 + 2i\sqrt{3})$ *into polar form.*

SOLUTION The given complex number is $z = (-2 + 2i\sqrt{3})$.

Let its polar form be $z = r(\cos\theta + i\sin\theta)$.

Now, $r = |z| = \sqrt{(-2)^2 + (2\sqrt{3})^2} = \sqrt{4 + 12} = \sqrt{16} = 4.$

Let α be the acute angle, given by

$$\tan\alpha = \left|\frac{\operatorname{Im}(z)}{\operatorname{Re}(z)}\right| = \left|\frac{2\sqrt{3}}{-2}\right| = \sqrt{3} \Rightarrow \alpha = \frac{\pi}{3}.$$

Clearly, the point representing $z = (-2 + 2i\sqrt{3})$ is $P(-2, 2\sqrt{3})$, which lies in the second quadrant.

$\therefore \quad \arg(z) = \theta = (\pi - \alpha) = \left(\pi - \frac{\pi}{3}\right) = \frac{2\pi}{3}.$

Thus, $r = |z| = 4$ and $\theta = \frac{2\pi}{3}.$

Hence, the required polar form is $z = 4\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right).$

EXAMPLE 6 *Convert* $\dfrac{(1+7i)}{(2-i)^2}$ *into polar form.*

SOLUTION Let $z = \dfrac{(1+7i)}{(2-i)^2} = \dfrac{(1+7i)}{(4+i^2-4i)} = \dfrac{(1+7i)}{(3-4i)} \times \dfrac{(3+4i)}{(3+4i)}$

$$\Rightarrow \qquad z = \frac{(1+7i)(3+4i)}{(9+16)} = \frac{-25+25i}{25} = (-1+i).$$

Let its polar form be $z = r(\cos\theta + i\sin\theta)$.

Now, $r = |z| = \sqrt{(-1)^2+1^2} = \sqrt{2}$.

Let α be the acute angle, given by

$$\tan\alpha = \left|\frac{\text{Im}(z)}{\text{Re}(z)}\right| = \left|\frac{1}{-1}\right| = 1 \Rightarrow \alpha = \frac{\pi}{4}\cdot$$

Clearly, the point representing $z = (-1+i)$ is $P(-1, 1)$, which lies in the second quadrant.

$$\therefore \qquad \arg(z) = \theta = (\pi - \alpha) = \left(\pi - \frac{\pi}{4}\right) = \frac{3\pi}{4}\cdot$$

Thus, $r = |z| = \sqrt{2}$ and $\theta = \dfrac{3\pi}{4}\cdot$

Hence, the required polar form is $z = \sqrt{2}\left(\cos\dfrac{3\pi}{4} + i\sin\dfrac{3\pi}{4}\right)\cdot$

EXAMPLE 7 *Convert the complex number* $\dfrac{1}{(1+i)}$ *into polar form.*

SOLUTION Let $z = \dfrac{1}{(1+i)} \times \dfrac{(1-i)}{(1-i)} = \dfrac{(1-i)}{(1-i^2)} = \dfrac{(1-i)}{2} = \left(\dfrac{1}{2} - \dfrac{1}{2}i\right)\cdot$

Let its polar form be $z = r(\cos\theta + i\sin\theta)$.

Now, $r = |z| = \sqrt{\left(\dfrac{1}{2}\right)^2 + \left(\dfrac{-1}{2}\right)^2} = \sqrt{\dfrac{1}{4}+\dfrac{1}{4}} = \sqrt{\dfrac{1}{2}} = \dfrac{1}{\sqrt{2}}\cdot$

Let α be the acute angle, given by

$$\tan\alpha = \left|\frac{\text{Im}(z)}{\text{Re}(z)}\right| = \left|\frac{(-1/2)}{(1/2)}\right| = 1 \Rightarrow \alpha = \frac{\pi}{4}\cdot$$

Clearly, the point representing $z = \left(\dfrac{1}{2} - \dfrac{1}{2}i\right)$ is $P\left(\dfrac{1}{2}, \dfrac{-1}{2}\right)$, which lies in the fourth quadrant.

$$\therefore \qquad \arg(z) = \theta = -\alpha = -\frac{\pi}{4}\cdot$$

Thus, $r = |z| = \dfrac{1}{\sqrt{2}}$ and $\theta = \dfrac{-\pi}{4}\cdot$

Hence, the required polar form is

$$z = \frac{1}{\sqrt{2}}\left\{\cos\left(\frac{-\pi}{4}\right) + i\sin\left(\frac{-\pi}{4}\right)\right\}, \text{ i.e., } \frac{1}{\sqrt{2}}\left(\cos\frac{\pi}{4} - i\sin\frac{\pi}{4}\right)\cdot$$

EXAMPLE 8 *Convert the complex number* $\dfrac{(i-1)}{\left(\cos\dfrac{\pi}{3}+i\sin\dfrac{\pi}{3}\right)}$ *into polar form.*

SOLUTION Let $z=\dfrac{(i-1)}{\left(\cos\dfrac{\pi}{3}+i\sin\dfrac{\pi}{3}\right)}=\dfrac{(i-1)}{\left(\dfrac{1}{2}+i\dfrac{\sqrt{3}}{2}\right)}=\dfrac{2(i-1)}{(1+i\sqrt{3})}$

$$\Rightarrow \quad z=\frac{2(i-1)}{(1+i\sqrt{3})}\times\frac{(1-i\sqrt{3})}{(1-i\sqrt{3})}=\frac{(i-1)(1-i\sqrt{3})}{2}$$

$$=\left\{\frac{(\sqrt{3}-1)}{2}+\frac{(\sqrt{3}+1)}{2}i\right\}.$$

Let its polar form be $z=r(\cos\theta+i\sin\theta)$.

Now, $r^2=|z|^2=\left\{\dfrac{(\sqrt{3}-1)^2}{4}+\dfrac{(\sqrt{3}+1)^2}{4}\right\}=\dfrac{8}{4}=2 \Rightarrow r=\sqrt{2}$.

Let α be the acute angle, given by

$$\tan\alpha=\left|\frac{\text{Im}(z)}{\text{Re}(z)}\right|=\left|\frac{(\sqrt{3}+1)}{2}\times\frac{2}{(\sqrt{3}-1)}\right|=\frac{(\sqrt{3}+1)}{(\sqrt{3}-1)}=\frac{\left(1+\dfrac{1}{\sqrt{3}}\right)}{\left(1-\dfrac{1}{\sqrt{3}}\right)}$$

$$\Rightarrow \quad \tan\alpha=\frac{\left(\tan\dfrac{\pi}{4}+\tan\dfrac{\pi}{6}\right)}{\left(1-\tan\dfrac{\pi}{4}\cdot\tan\dfrac{\pi}{6}\right)}=\tan\left(\frac{\pi}{4}+\frac{\pi}{6}\right)=\tan\frac{5\pi}{12}$$

$$\Rightarrow \quad \alpha=\frac{5\pi}{12}.$$

Clearly, the point representing the given complex number is $P\left(\dfrac{\sqrt{3}-1}{2},\dfrac{\sqrt{3}+1}{2}\right)$, which lies in the first quadrant.

$$\therefore \quad \arg(z)=\theta=\alpha=\frac{5\pi}{12} \Rightarrow \theta=\frac{5\pi}{12}.$$

Thus, $r=|z|=\sqrt{2}$ and $\theta=\dfrac{5\pi}{12}$.

Hence, the required polar form is $z=\sqrt{2}\left(\cos\dfrac{5\pi}{12}+i\sin\dfrac{5\pi}{12}\right)$.

EXAMPLE 9 *Express the complex number* $(-1-i)$ *in polar form.*

SOLUTION The given complex number is $z=-1-i$.

Let its polar form be $z=r(\cos\theta+i\sin\theta)$.

Now, $r=|z|=\sqrt{(-1)^2+(-1)^2}=\sqrt{2}$.

Let α be the acute angle, given by

$$\tan\alpha=\left|\frac{\text{Im}(z)}{\text{Re}(z)}\right|=\left|\frac{-1}{-1}\right|=1 \Rightarrow \alpha=\frac{\pi}{4}.$$

Clearly, the point representing the complex number $z=-1-i$ is $P(-1,-1)$, which lies in the third quadrant.

$\therefore \quad \arg(z)=\theta=-(\pi-\alpha)=-\left(\pi-\frac{\pi}{4}\right)=\frac{-3\pi}{4}\cdot$

Thus, $r=|z|=\sqrt{2}$ and $\theta=\frac{-3\pi}{4}\cdot$

Hence, the required polar form of $z=(-1-i)$ is given by

$$z=\sqrt{2}\left\{\cos\left(\frac{-3\pi}{4}\right)+i\sin\left(\frac{-3\pi}{4}\right)\right\}, \text{ i.e., } \sqrt{2}\left(\cos\frac{3\pi}{4}-i\sin\frac{3\pi}{4}\right)\cdot$$

EXAMPLE 10 *Express the complex number* $(-\sqrt{3}-i)$ *in polar form.*

SOLUTION The given complex number is $z=(-\sqrt{3}-i)$.

Let its polar form be $z=r(\cos\theta+i\sin\theta)$.

Now, $r=|z|=\sqrt{(-\sqrt{3})^2+(-1)^2}=\sqrt{4}=2$.

Let α be the acute angle, given by

$$\tan\alpha=\left|\frac{\text{Im}(z)}{\text{Re}(z)}\right|=\left|\frac{-1}{-\sqrt{3}}\right|=\frac{1}{\sqrt{3}} \Rightarrow \alpha=\frac{\pi}{6}\cdot$$

Clearly, the point representing the complex number $z=(-\sqrt{3}-i)$ is $P(-\sqrt{3},-1)$, which lies in the third quadrant.

$\therefore \quad \arg(z)=\theta=-(\pi-\alpha)=-\left(\pi-\frac{\pi}{6}\right)=\frac{-5\pi}{6}\cdot$

Thus, $r=|z|=2$ and $\theta=\frac{-5\pi}{6}\cdot$

Hence, the polar form of $z=(-\sqrt{3}-i)$ is given by

$$z=2\left\{\cos\left(\frac{-5\pi}{6}\right)+i\sin\left(\frac{-5\pi}{6}\right)\right\}, \text{ i.e., } 2\left(\cos\frac{5\pi}{6}-i\sin\frac{5\pi}{6}\right)\cdot$$

EXAMPLE 11 *Convert the complex number* $\frac{1+2i}{1-3i}$ *into polar form.*

SOLUTION Let $z=\frac{(1+2i)}{(1-3i)}\times\frac{(1+3i)}{(1+3i)}=\frac{(1+2i)(1+3i)}{(1+9)}=\frac{-5+5i}{10}=\left(\frac{-1}{2}+\frac{1}{2}i\right)\cdot$

Let its polar form be $z=r(\cos\theta+i\sin\theta)$.

Now, $r=|z|=\sqrt{\left(\frac{-1}{2}\right)^2+\left(\frac{1}{2}\right)^2}=\sqrt{\frac{1}{2}}=\frac{1}{\sqrt{2}}\cdot$

Let α be the acute angle, given by

$$\tan\alpha=\left|\frac{\text{Im}(z)}{\text{Re}(z)}\right|=\left|\frac{(1/2)}{-(1/2)}\right|=1 \Rightarrow \alpha=\frac{\pi}{4}\cdot$$

Clearly, the point representing $z=\left(\frac{-1}{2}+\frac{1}{2}i\right)$ is $P\left(\frac{-1}{2},\frac{1}{2}\right)$, which lies in the second quadrant.

$\therefore \quad \arg(z) = \theta = (\pi - \alpha) = \left(\pi - \frac{\pi}{4}\right) = \frac{3\pi}{4}\cdot$

Thus, $r = |z| = \frac{1}{\sqrt{2}}$ and $\theta = \frac{3\pi}{4}\cdot$

Hence, the required polar form is $z = \frac{1}{\sqrt{2}}\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)\cdot$

EXAMPLE 12 *Express the complex number* $\sin\frac{\pi}{5} + i\left(1 - \cos\frac{\pi}{5}\right)$ *in polar form.*

SOLUTION Let $z = \sin\frac{\pi}{5} + i\left(1 - \cos\frac{\pi}{5}\right)\cdot$

Let its polar form be $z = r(\cos\theta + i\sin\theta)$.

Now, $r^2 = |z|^2 = \sin^2\frac{\pi}{5} + \left(1 - \cos\frac{\pi}{5}\right)^2 = \left(\sin^2\frac{\pi}{5} + \cos^2\frac{\pi}{5}\right) + 1 - 2\cos\frac{\pi}{5}$

$\Rightarrow \quad r^2 = 2\left(1 - \cos\frac{\pi}{5}\right) = 4\sin^2\frac{\pi}{10} \Rightarrow r = 2\sin\frac{\pi}{10}\cdot$

Let α be the acute angle, given by

$$\tan\alpha = \left|\frac{\operatorname{Im}(z)}{\operatorname{Re}(z)}\right| = \left|\frac{1 - \cos\frac{\pi}{5}}{\sin\frac{\pi}{5}}\right| = \frac{2\sin^2\frac{\pi}{10}}{2\sin\frac{\pi}{10}\cdot\cos\frac{\pi}{10}} = \tan\frac{\pi}{10} \Rightarrow \alpha = \frac{\pi}{10}\cdot$$

Clearly, the point representing z lies in the first quadrant as $x > 0$ and $y > 0$.

$\therefore \quad \arg(z) = \theta = \alpha = \frac{\pi}{10}\cdot$

Thus, $r = 2\sin\frac{\pi}{10}$ and $\theta = \frac{\pi}{10}\cdot$

Hence, the required polar form is $2\sin\frac{\pi}{10}\left(\cos\frac{\pi}{10} + i\sin\frac{\pi}{10}\right)\cdot$

EXAMPLE 13 *Convert* $4(\cos 300^\circ + i\sin 300^\circ)$ *into Cartesian form.*

SOLUTION

$$\begin{aligned} 4(\cos 300^\circ + i\sin 300^\circ) &= 4[\cos(360^\circ - 60^\circ) + i\sin(360^\circ - 60^\circ)] \\ &= 4[\cos(-60^\circ) + i\sin(-60^\circ)] \\ &= 4[\cos 60^\circ - i\sin 60^\circ] = 4\left(\frac{1}{2} - i\cdot\frac{\sqrt{3}}{2}\right) \\ &= (2 - 2i\sqrt{3}). \end{aligned}$$

EXAMPLE 14 *For any complex numbers z, z_1 and z_2 prove that:*

(i) $\arg(\bar{z}) = -\arg(z)$

(ii) $\arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$

(iii) $\arg(z_1 \bar{z}_2) = \arg(z_1) - \arg(z_2)$

(iv) $\arg\left(\frac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2)$

SOLUTION (i) Let $z = r(\cos\theta + i\sin\theta)$. Then, $|z| = r$ and $\arg(z) = \theta$.

Now, $z = r(\cos\theta + i\sin\theta)$

$\Rightarrow \quad z = r\cos\theta + i(r\sin\theta)$

$\Rightarrow \quad \bar{z} = r\cos\theta - i(r\sin\theta) = r(\cos\theta - i\sin\theta)$

$\quad = r\{\cos(-\theta) + i\sin(-\theta)\}$

$\Rightarrow \quad |\bar{z}| = r$ and $\arg(\bar{z}) = -\theta = -\arg(z)$.

Hence, $\arg(\bar{z}) = -\arg(z)$.

(ii) Let $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$. Then,

$|z_1| = r_1$, $\arg(z_1) = \theta$ and $|z_2| = r_2$, $\arg(z_2) = \theta_2$.

$$\begin{aligned}
\therefore \quad z_1 z_2 &= r_1(\cos\theta_1 + i\sin\theta_1)\cdot r_2(\cos\theta_2 + i\sin\theta_2)\\
&= r_1 r_2\{(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2)\\
&\qquad + i(\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2)\}\\
&= r_1 r_2\{\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)\}
\end{aligned}$$

$\Rightarrow \quad \arg(z_1 z_2) = (\theta_1 + \theta_2) = \arg(z_1) + \arg(z_2)$.

REMARKS (I) Note here that $|z_1 z_2| = r_1 r_2 = |z_1|\ |z_2|$.

(II) In general, we have

$$|z_1 z_2 \ldots z_n| = |z_1|\cdot|z_2|\ldots|z_n|$$

and $\arg(z_1 z_2 \ldots z_n) = \arg(z_1) + \arg(z_2) + \ldots + \arg(z_n)$.

(iii) Let $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$. Then,

$\bar{z}_2 = \overline{r_2\cos\theta_2 + i(r_2\sin\theta_2)} = r_2\cos\theta_2 - i(r_2\sin\theta_2)$

$\Rightarrow \quad \bar{z}_2 = r_2\{\cos(-\theta_2) + i\sin(-\theta_2)\}$.

$$\begin{aligned}
\therefore \quad z_1\bar{z}_2 &= r_1(\cos\theta_1 + i\sin\theta_1)\cdot r_2\{\cos(-\theta_2) + i\sin(-\theta_2)\}\\
&= r_1 r_2(\cos\theta_1 + i\sin\theta_1)\{\cos(-\theta_2) + i\sin(-\theta_2)\}\\
&= r_1 r_2[\cos\{\theta_1 + (-\theta_2)\} + i\sin\{\theta_1 + (-\theta_2)\}]\\
&= r_1 r_2\{\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)\}.
\end{aligned}$$

Hence, $\arg(z_1\bar{z}_2) = (\theta_1 - \theta_2) = \arg(z_1) - \arg(z_2)$.

(iv) Let $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$. Then,

$|z_1| = r_1$, $|z_2| = r_2$, $\arg(z_1) = \theta_1$ and $\arg(z_2) = \theta_2$.

$$\begin{aligned}
\therefore \quad \frac{z_1}{z_2} &= \frac{r_1(\cos\theta_1 + i\sin\theta_1)}{r_2(\cos\theta_2 + i\sin\theta_2)} \times \frac{(\cos\theta_2 - i\sin\theta_2)}{(\cos\theta_2 - i\sin\theta_2)}\\
&= \frac{r_1}{r_2}\cdot\left\{\frac{(\cos\theta_1\cdot\cos\theta_2 - \sin\theta_1\cdot\sin\theta_2) + i(\sin\theta_1\cdot\cos\theta_2 - \cos\theta_1\cdot\sin\theta_2)}{(\cos^2\theta_2 + \sin^2\theta_2)}\right\}\\
&= \frac{r_1}{r_2}\cdot\{\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)\}
\end{aligned}$$

$\Rightarrow \quad \arg\left(\dfrac{z_1}{z_2}\right) = (\theta_1 - \theta_2) = \arg(z_1) - \arg(z_2)$.

Hence, $\arg\left(\dfrac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2)$.

EXERCISE 5D

Find the modulus and argument of each of the following complex numbers and hence express each of them in polar form:

1. 4 **2.** -2 **3.** $-i$ **4.** $2i$

5. $1-i$ **6.** $-1+i$ **7.** $\sqrt{3}+i$ **8.** $-1+\sqrt{3}\,i$

9. $1-\sqrt{3}\,i$ **10.** $2-2i$ **11.** $-4+4\sqrt{3}\,i$ **12.** $-3\sqrt{2}+3\sqrt{2}\,i$

13. $\dfrac{1+i}{1-i}$ **14.** $\dfrac{1-i}{1+i}$ **15.** $\dfrac{1+3i}{1-2i}$ **16.** $\dfrac{1-3i}{1+2i}$

17. $\dfrac{5-i}{2-3i}$ **18.** $\dfrac{-16}{1+\sqrt{3}\,i}$ **19.** $\dfrac{2+6\sqrt{3}\,i}{5+\sqrt{3}\,i}$ **20.** $\sqrt{\dfrac{1+i}{1-i}}$

21. $-\sqrt{3}-i$ **22.** $(i^{25})^3$ **23.** $\dfrac{(1-i)}{\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)}$

24. $(\sin 120^\circ - i\cos 120^\circ)$

ANSWERS (EXERCISE 5D)

1. $4, 0, 4(\cos 0 + i\sin 0)$

2. $2, \pi, 2(\cos\pi + i\sin\pi)$

3. $1, \frac{-\pi}{2}, \left\{\cos\left(\frac{-\pi}{2}\right)+i\sin\left(\frac{-\pi}{2}\right)\right\}$

4. $2, \frac{\pi}{2}, 2\left(\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}\right)$

5. $\sqrt{2}, \frac{-\pi}{4}, \sqrt{2}\left\{\cos\left(\frac{-\pi}{4}\right)+i\sin\left(\frac{-\pi}{4}\right)\right\}$

6. $\sqrt{2}, \frac{3\pi}{4}, \sqrt{2}\left(\cos\frac{3\pi}{4}+i\sin\frac{3\pi}{4}\right)$

7. $2, \frac{\pi}{6}, 2\left(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}\right)$

8. $2, \frac{2\pi}{3}, 2\left(\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3}\right)$

9. $2, \frac{-\pi}{3}, 2\left\{\cos\left(\frac{-\pi}{3}\right)+i\sin\left(\frac{-\pi}{3}\right)\right\}$

10. $2\sqrt{2}, \frac{-\pi}{4}, 2\sqrt{2}\left\{\cos\left(\frac{-\pi}{4}\right)+i\sin\left(\frac{-\pi}{4}\right)\right\}$

11. $8, \frac{2\pi}{3}, 8\left\{\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3}\right\}$

12. $6, \frac{3\pi}{4}, 6\left(\cos\frac{3\pi}{4}+i\sin\frac{3\pi}{4}\right)$

13. $1, \frac{\pi}{2}, \left(\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}\right)$

14. $1, \frac{-\pi}{2}, \left\{\cos\left(\frac{-\pi}{2}\right)+i\sin\left(\frac{-\pi}{2}\right)\right\}$

15. $\sqrt{2}, \frac{3\pi}{4}, \sqrt{2}\left(\cos\frac{3\pi}{4}+i\sin\frac{3\pi}{4}\right)$

16. $\sqrt{2}, \frac{-3\pi}{4}, \left\{\cos\left(\frac{-3\pi}{4}\right)+i\sin\left(\frac{-3\pi}{4}\right)\right\}$

17. $\sqrt{2}, \frac{\pi}{4}, \sqrt{2}\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)$

18. $8, \frac{2\pi}{3}, 8\left(\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3}\right)$

19. $2, \frac{\pi}{3}, 2\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)$

20. $1, \frac{\pi}{4}, \left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)$

21. $2, \frac{-5\pi}{6}, 2\left\{\cos\left(\frac{-5\pi}{6}\right)+i\sin\left(\frac{-5\pi}{6}\right)\right\}$

22. $1, \frac{-\pi}{2}, \left\{\cos\left(\frac{-\pi}{2}\right)+i\sin\left(\frac{-\pi}{2}\right)\right\}$

23. $\sqrt{2}, \frac{7\pi}{12}, \sqrt{2}\left(\cos\frac{7\pi}{12}-i\sin\frac{7\pi}{12}\right)$

24. $(\cos 30^\circ + i\sin 30^\circ)$

HINTS TO SOME SELECTED QUESTIONS

21. Let $z=-\sqrt{3}-i \Rightarrow r^2=|z|^2=(-\sqrt{3})^2+(-1)^2=4 \Rightarrow r=|z|=2.$

$$\tan\alpha=\left|\frac{-1}{-\sqrt{3}}\right|=\frac{1}{\sqrt{3}} \Rightarrow \alpha=\frac{\pi}{6}.$$

The given number represents the point $P(-\sqrt{3}, -1)$ which lies in the third quadrant.

$\therefore \quad \arg(z) = \theta = -(\pi - \alpha) = -\left(\pi - \frac{\pi}{6}\right) = \frac{-5\pi}{6}$.

$\therefore \quad z = 2\left[\cos\left(\frac{-5\pi}{6}\right) + i\sin\left(\frac{-5\pi}{6}\right)\right]$.

23. Given number $= \dfrac{2(1-i)}{(1+\sqrt{3}i)} \times \dfrac{(1-\sqrt{3}i)}{(1-\sqrt{3}i)} = \left\{\dfrac{(1-\sqrt{3})}{2} - \dfrac{(1+\sqrt{3})}{2}i\right\}$

$$\therefore \quad |z|^2 = \frac{(1-\sqrt{3})^2}{4} + \frac{(1+\sqrt{3})^2}{4} = \frac{2(1+3)}{4} = 2 \Rightarrow |z| = \sqrt{2}.$$

$$\tan\alpha = \left|\frac{\frac{-(1+\sqrt{3})}{2}}{\frac{(1-\sqrt{3})}{2}}\right| = \frac{(\sqrt{3}+1)}{(\sqrt{3}-1)} = \frac{\left(1+\frac{1}{\sqrt{3}}\right)}{\left(1-\frac{1}{\sqrt{3}}\right)} = \frac{\tan\frac{\pi}{4} + \tan\frac{\pi}{6}}{1 - \tan\frac{\pi}{4}\cdot\tan\frac{\pi}{6}}$$

$$= \tan\left(\frac{\pi}{4} + \frac{\pi}{6}\right) = \tan\frac{5\pi}{12}.$$

The given number z is represented by the point $P\left(\dfrac{-(\sqrt{3}-1)}{2}, \dfrac{-(\sqrt{3}+1)}{2}\right)$.

So, it lies in the third quadrant.

$\therefore \quad \arg(z) = \theta = -(\pi - \alpha) = -\left(\pi - \frac{5\pi}{12}\right) = \frac{-7\pi}{12}$.

Hence, $z = \sqrt{2}\left\{\cos\left(\frac{-7\pi}{12}\right) + i\sin\left(\frac{-7\pi}{12}\right)\right\}$.

QUADRATIC EQUATIONS (With Complex Roots)

FUNDAMENTAL THEOREM OF ALGEBRA

A polynomial equation of degree n has at the most n roots.

SOLVING A QUADRATIC EQUATION

Let the given equation be $ax^2 + bx + c = 0$, where $a, b, c \in R$ and $a \neq 0$.

Then, $ax^2 + bx + c = 0, a \neq 0$

$\Rightarrow \quad ax^2 + bx = -c$

$\Rightarrow \quad x^2 + \frac{b}{a}\cdot x = \frac{-c}{a}$

$\Rightarrow \quad x^2 + \frac{b}{a}\cdot x + \left(\frac{b}{2a}\right)^2 = \frac{-c}{a} + \left(\frac{b}{2a}\right)^2 \qquad$ [adding $\left(\frac{b}{2a}\right)^2$ on both sides]

$\Rightarrow \quad \left(x + \frac{b}{2a}\right)^2 = \frac{(b^2 - 4ac)}{4a^2}$

$\Rightarrow \quad x + \frac{b}{2a} = \pm\frac{\sqrt{b^2 - 4ac}}{2a}$

$$\Rightarrow \quad x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}$$

$$\Rightarrow \quad x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Hence, the roots of the equation $ax^2 + bx + c = 0$, $a \neq 0$ are

$$\frac{-b + \sqrt{b^2 - 4ac}}{2a} \text{ and } \frac{-b - \sqrt{b^2 - 4ac}}{2a}.$$

The expression $(b^2 - 4ac)$ is called the *discriminant*.

If $(b^2 - 4ac) < 0$, the given quadratic equation will have complex roots.

Here, we shall consider only the quadratic equations having complex roots.

SOLVED EXAMPLES

EXAMPLE 1 *Solve:* $x^2 + 3 = 0$.

SOLUTION We have

$$x^2 + 3 = 0 \Rightarrow x^2 = -3 \Rightarrow x = \pm\sqrt{-3} = \pm i\sqrt{3}.$$

$\therefore$ solution set $= \{i\sqrt{3}, -i\sqrt{3}\}$.

EXAMPLE 2 *Solve:* $x^2 + 3x + 9 = 0$.

SOLUTION The given equation is $x^2 + 3x + 9 = 0$.

This is of the form $ax^2 + bx + c = 0$, where $a = 1$, $b = 3$ and $c = 9$.

$\therefore$ $(b^2 - 4ac) = (3^2 - 4 \times 1 \times 9) = (9 - 36) = -27 < 0$.

So, the given equation has complex roots.

These roots are given by

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-3 \pm \sqrt{-27}}{2 \times 1} \qquad [\because\ b^2 - 4ac = -27]$$

$$= \frac{-3 \pm i\sqrt{27}}{2} = \frac{-3 \pm i3\sqrt{3}}{2}.$$

$$\therefore \quad \text{solution set} = \left\{\frac{-3 + i3\sqrt{3}}{2}, \frac{-3 - i3\sqrt{3}}{2}\right\}$$

$$= \left\{\frac{-3}{2} + \frac{3\sqrt{3}}{2}i, \frac{-3}{2} - \frac{3\sqrt{3}}{2}i\right\}.$$

EXAMPLE 3 *Solve:* $9x^2 + 10x + 3 = 0$.

SOLUTION The given equation is $9x^2 + 10x + 3 = 0$.

This is of the form $ax^2 + bx + c = 0$, where $a = 9$, $b = 10$ and $c = 3$.

$\therefore$ $(b^2 - 4ac) = \{(10)^2 - 4 \times 9 \times 3\} = (100 - 108) = -8 < 0$.

So, the given equation has complex roots.

These roots are given by

$$\frac{-b \pm \sqrt{b^2-4ac}}{2a} = \frac{-10 \pm \sqrt{-8}}{2 \times 9} \qquad [\because\ (b^2-4ac) = -8]$$

$$= \frac{-10 \pm i2\sqrt{2}}{18} = \frac{-5 \pm i\sqrt{2}}{9}.$$

$\therefore$ solution set $= \left\{\frac{-5+i\sqrt{2}}{9}, \frac{-5-i\sqrt{2}}{9}\right\} = \left\{\frac{-5}{9}+\frac{\sqrt{2}}{9}i, \frac{-5}{9}-\frac{\sqrt{2}}{9}i\right\}.$

EXAMPLE 4 *Solve:* $\sqrt{2}x^2 + x + \sqrt{2} = 0.$

SOLUTION The given equation is $\sqrt{2}x^2 + x + \sqrt{2} = 0.$

This is of the form $ax^2 + bx + c = 0$, where $a = \sqrt{2}$, $b = 1$ and $c = \sqrt{2}$.

$\therefore$ $(b^2-4ac) = (1^2 - 4 \times \sqrt{2} \times \sqrt{2}) = (1-8) = -7 < 0.$

So, the given equation has complex roots.

These roots are given by

$$\frac{-b \pm \sqrt{b^2-4ac}}{2a} = \frac{-1 \pm \sqrt{-7}}{2\sqrt{2}} \qquad [\because\ b^2-4ac = -7]$$

$$= \frac{-1 \pm i\sqrt{7}}{2\sqrt{2}}.$$

$\therefore$ solution set $= \left\{\frac{-1+i\sqrt{7}}{2\sqrt{2}}, \frac{-1-i\sqrt{7}}{2\sqrt{2}}\right\}$

$$= \left\{\frac{-1}{2\sqrt{2}}+\frac{\sqrt{7}}{2\sqrt{2}}i, \frac{-1}{2\sqrt{2}}-\frac{\sqrt{7}}{2\sqrt{2}}i\right\}.$$

EXAMPLE 5 *Solve:* $\sqrt{3}x^2 - \sqrt{2}x + 3\sqrt{3} = 0.$

SOLUTION The given equation is

$$\sqrt{3}x^2 - \sqrt{2}x + 3\sqrt{3} = 0.$$

This is of the form $ax^2 + bx + c = 0$, where $a = \sqrt{3}$, $b = -\sqrt{2}$ and $c = 3\sqrt{3}$.

$\therefore$ $(b^2-4ac) = \{(-\sqrt{2})^2 - 4 \times \sqrt{3} \times 3\sqrt{3}\} = (2-36) = -34 < 0.$

So, the given equation has complex roots.

These roots are given by

$$\frac{-b \pm \sqrt{b^2-4ac}}{2a} = \frac{\sqrt{2} \pm \sqrt{-34}}{2 \times \sqrt{3}} \qquad [\because\ b^2-4ac = -34]$$

$$= \frac{\sqrt{2} \pm i\sqrt{34}}{2\sqrt{3}}.$$

$\therefore$ solution set $= \left\{\frac{\sqrt{2}+i\sqrt{34}}{2\sqrt{3}}, \frac{\sqrt{2}-i\sqrt{34}}{2\sqrt{3}}\right\}$

$$= \left\{\frac{1}{\sqrt{6}}+\frac{\sqrt{34}}{2\sqrt{3}}i, \frac{1}{\sqrt{6}}-\frac{\sqrt{34}}{2\sqrt{3}}i\right\}.$$

EXAMPLE 6 *Solve:* $3x^2+8ix+3=0$.

SOLUTION The given equation is

$$3x^2+8ix+3=0.$$

This is of the form $ax^2+bx+c=0$, where $a=3$, $b=8i$ and $c=3$.

$\therefore \quad (b^2-4ac)=\{(8i)^2-4\times3\times3\}=(-64-36)=-100<0.$

So, the given equation has complex roots.

These roots are given by

$$\frac{-b\pm\sqrt{b^2-4ac}}{2a}=\frac{-8i\pm\sqrt{-100}}{2\times3} \qquad [\because\ b^2-4ac=-100]$$

$$=\frac{-8i\pm10i}{6}=\frac{-4i\pm5i}{3}.$$

Thus, the roots of the given equation are

$$\frac{-4i+5i}{3}=\frac{i}{3} \text{ and } \frac{-4i-5i}{3}=\frac{-9i}{3}=-3i.$$

$\therefore \quad$ solution set $=\left\{\frac{i}{3},-3i\right\}$.

EXERCISE 5E

Solve:

1. $x^2+2=0$ **2.** $x^2+5=0$ **3.** $2x^2+1=0$

4. $x^2+x+1=0$ **5.** $x^2-x+2=0$ **6.** $x^2+2x+2=0$

7. $2x^2-4x+3=0$ **8.** $x^2+3x+5=0$ **9.** $\sqrt{5}x^2+x+\sqrt{5}=0$

10. $25x^2-30x+11=0$ **11.** $8x^2+2x+1=0$ **12.** $27x^2+10x+1=0$

13. $2x^2-\sqrt{3}x+1=0$ **14.** $17x^2-8x+1=0$ **15.** $3x^2+5=7x$

16. $3x^2-4x+\frac{20}{3}=0$ **17.** $3x^2+7ix+6=0$ **18.** $21x^2-28x+10=0$

19. $x^2+13=4x$ **20.** $x^2+3ix+10=0$ **21.** $2x^2+3ix+2=0$

ANSWERS (EXERCISE 5E)

1. $\{i\sqrt{2},-i\sqrt{2}\}$ **2.** $\{i\sqrt{5},-i\sqrt{5}\}$ **3.** $\left\{\frac{i}{\sqrt{2}},\frac{-i}{\sqrt{2}}\right\}$

4. $\left\{\frac{-1}{2}+\frac{\sqrt{3}}{2}i,\frac{-1}{2}-\frac{\sqrt{3}}{2}i\right\}$ **5.** $\left\{\frac{1}{2}+\frac{\sqrt{7}}{2}i,\frac{1}{2}-\frac{\sqrt{7}}{2}i\right\}$ **6.** $\{-1+i,-1-i\}$

7. $\left\{1+\frac{1}{\sqrt{2}}i,1-\frac{1}{\sqrt{2}}i\right\}$ **8.** $\left\{\frac{-3}{2}+\frac{\sqrt{11}}{2}i,\frac{-3}{2}-\frac{\sqrt{11}}{2}i\right\}$

9. $\left\{\frac{-1}{2\sqrt{5}}+\frac{\sqrt{19}}{2\sqrt{5}}i,\frac{-1}{2\sqrt{5}}-\frac{\sqrt{19}}{2\sqrt{5}}i\right\}$ **10.** $\left\{\frac{3}{5}+\frac{\sqrt{2}}{5}i,\frac{3}{5}-\frac{\sqrt{2}}{5}i\right\}$

11. $\left\{\frac{-1}{8}+\frac{\sqrt{7}}{8}i, \frac{-1}{8}-\frac{\sqrt{7}}{8}i\right\}$

12. $\left\{\frac{-5}{27}+\frac{\sqrt{2}}{27}i, \frac{-5}{27}-\frac{\sqrt{2}}{27}i\right\}$

13. $\left\{\frac{\sqrt{3}}{4}+\frac{\sqrt{5}}{4}i, \frac{\sqrt{3}}{4}-\frac{\sqrt{5}}{4}i\right\}$

14. $\left\{\frac{4}{17}+\frac{1}{17}i, \frac{4}{17}-\frac{1}{17}i\right\}$

15. $\left\{\frac{7}{6}+\frac{\sqrt{11}}{6}i, \frac{7}{6}-\frac{\sqrt{11}}{6}i\right\}$

16. $\left\{\frac{2}{3}+\frac{4}{3}i, \frac{2}{3}-\frac{4}{3}i\right\}$

17. $\left\{\frac{2}{3}i, -3i\right\}$

18. $\left\{\frac{2}{3}+\frac{\sqrt{14}}{21}i, \frac{2}{3}-\frac{\sqrt{14}}{21}i\right\}$

19. $\{2+3i, 2-3i\}$

20. $\{2i, -5i\}$

21. $\left\{\frac{1}{2}i, -2i\right\}$

HINTS TO SOME SELECTED QUESTIONS

17. $a = 3$, $b = 7i$ and $c = 6$.

$\therefore \quad (b^2 - 4ac) = \{(7i)^2 - 4 \times 3 \times 6\} = (-49 - 72) = (-121) < 0.$

So, the given equation has complex roots, given by

$$\frac{-b \pm \sqrt{b^2-4ac}}{2a} = \frac{-7i \pm \sqrt{-121}}{2 \times 3} = \frac{-7i \pm 11i}{6}.$$

Solution set $= \left\{\frac{-7i+11i}{6}, \frac{-7i-11i}{6}\right\} = \left\{\frac{2}{3}i, -3i\right\}.$

SQUARE ROOTS OF A COMPLEX NUMBER

To Evaluate $\sqrt{a+ib}$:

SOLUTION Let $\sqrt{a+ib} = (x+iy)$. … (i)

On squaring both sides of (i), we get

$$(a+ib) = (x+iy)^2$$

$$\Rightarrow \quad (a+ib) = (x^2-y^2) + i(2xy). \quad \text{… (ii)}$$

On comparing real parts and imaginary parts on both sides of (ii), we get

$$(x^2-y^2) = a \text{ and } 2xy = b$$

$$\Rightarrow \quad (x^2+y^2) = \sqrt{(x^2-y^2)^2+4x^2y^2} = \sqrt{a^2+b^2}.$$

$$\therefore \quad (x^2-y^2) = a \quad \text{… (iii)} \quad \text{and} \quad (x^2+y^2) = \sqrt{a^2+b^2} \quad \text{… (iv)}$$

On solving (iii) and (iv), we get

$$x^2 = \frac{1}{2}\left\{\sqrt{a^2+b^2}+a\right\} \text{ and } y^2 = \frac{1}{2}\left\{\sqrt{a^2+b^2}-a\right\}.$$

Hence, $\sqrt{a+ib} = \pm\left[\sqrt{\frac{1}{2}\left\{\sqrt{a^2+b^2}+a\right\}} + i\sqrt{\frac{1}{2}\left\{\sqrt{a^2+b^2}-a\right\}}\,\right].$

REMARK Similarly, by assuming that $\sqrt{a-ib} = (x-iy)$, we may find $\sqrt{a-ib}$.

SOLVED EXAMPLES

EXAMPLE 1 *Evaluate* $\sqrt{6+8i}$.

SOLUTION Let $\sqrt{6+8i}=(x+iy)$. ... (i)

On squaring both sides of (i), we get

$$6+8i=(x+iy)^2 \Rightarrow 6+8i=(x^2-y^2)+i(2xy). \quad \text{... (ii)}$$

On comparing real parts and imaginary parts on both sides of (ii), we get

$$x^2-y^2=6 \text{ and } 2xy=8$$

$\Rightarrow$ $x^2-y^2=6$ and $xy=4$

$\Rightarrow$ $(x^2+y^2)=\sqrt{(x^2-y^2)^2+4x^2y^2}=\sqrt{6^2+4\times 16}=\sqrt{100}=10$

$\Rightarrow$ $x^2-y^2=6$ and $x^2+y^2=10$

$\Rightarrow$ $2x^2=16$ and $2y^2=4$

$\Rightarrow$ $x^2=8$ and $y^2=2$

$\Rightarrow$ $x=\pm 2\sqrt{2}$ and $y=\pm\sqrt{2}$.

Since $xy>0$, so x and y are of the same sign.

$\therefore$ $(x=2\sqrt{2}$ and $y=\sqrt{2})$ or $(x=-2\sqrt{2}$ and $y=-\sqrt{2})$.

Hence, $\sqrt{6+8i}=(2\sqrt{2}+\sqrt{2}i)$ or $(-2\sqrt{2}-\sqrt{2}i)$.

EXAMPLE 2 *Evaluate* $\sqrt{-5+12i}$.

SOLUTION Let $\sqrt{-5+12i}=(x+iy)$. ... (i)

On squaring both sides of (i), we get

$$-5+12i=(x+iy)^2 \Rightarrow -5+12i=(x^2-y^2)+i(2xy). \quad \text{... (ii)}$$

On comparing real parts and imaginary parts on both sides of (ii), we get

$$x^2-y^2=-5 \text{ and } 2xy=12$$

$\Rightarrow$ $x^2-y^2=-5$ and $xy=6$

$\Rightarrow$ $(x^2+y^2)=\sqrt{(x^2-y^2)^2+4x^2y^2}=\sqrt{(-5)^2+4\times 36}=\sqrt{169}=13$

$\Rightarrow$ $x^2-y^2=-5$ and $x^2+y^2=13$

$\Rightarrow$ $2x^2=8$ and $2y^2=18$

$\Rightarrow$ $x^2=4$ and $y^2=9$

$\Rightarrow$ $x=\pm 2$ and $y=\pm 3$.

Since $xy>0$, so x and y are of the same sign.

$\therefore$ $(x=2$ and $y=3)$ or $(x=-2$ and $y=-3)$.

Hence, $\sqrt{-5+12i}=(2+3i)$ or $(-2-3i)$.

EXAMPLE 3 *Evaluate* $\sqrt{8-15i}$.

SOLUTION Let $\sqrt{8-15i} = (x-iy)$. ... (i)

On squaring both sides of (i), we get

$$(8-15i) = (x-iy)^2 \Rightarrow (8-15i) = (x^2-y^2) - i(2xy). \quad \text{... (ii)}$$

On comparing real parts and imaginary parts on both sides of (ii), we get

$$x^2-y^2 = 8 \text{ and } 2xy = 15$$

$$\Rightarrow \quad x^2-y^2 = 8 \text{ and } xy = \frac{15}{2}$$

$$\Rightarrow \quad (x^2+y^2) = \sqrt{(x^2-y^2)^2+4x^2y^2} = \sqrt{64+225} = \sqrt{289} = 17$$

$$\Rightarrow \quad x^2-y^2 = 8 \text{ and } x^2+y^2 = 17$$

$$\Rightarrow \quad 2x^2 = 25 \text{ and } 2y^2 = 9$$

$$\Rightarrow \quad x^2 = \frac{25}{2} \text{ and } y^2 = \frac{9}{2}$$

$$\Rightarrow \quad x = \pm\frac{5}{\sqrt{2}} \text{ and } y = \pm\frac{3}{\sqrt{2}}.$$

Since $xy > 0$, so x and y are of the same sign.

Hence, $\sqrt{8-15i} = \left(\frac{5}{\sqrt{2}} - \frac{3}{2}i\right)$ or $\left(\frac{-5}{\sqrt{2}} + \frac{3}{\sqrt{2}}i\right)$.

EXAMPLE 4 *Evaluate* $\sqrt{-24-10i}$.

SOLUTION Let $\sqrt{-24-10i} = (x-iy)$. ... (i)

On squaring both sides of (i), we get

$$(-24-10i) = (x^2-y^2) - i(2xy). \quad \text{... (ii)}$$

On comparing real parts and imaginary parts on both sides of (ii), we get

$$x^2-y^2 = -24 \text{ and } 2xy = 10$$

$$\Rightarrow \quad x^2-y^2 = -24 \text{ and } xy = 5$$

$$\Rightarrow \quad (x^2+y^2) = \sqrt{(x^2-y^2)^2+4x^2y^2} = \sqrt{(-24)^2+4\times 25} = \sqrt{676} = 26$$

$$\Rightarrow \quad x^2-y^2 = -24 \text{ and } x^2+y^2 = 26$$

$$\Rightarrow \quad 2x^2 = 2 \text{ and } 2y^2 = 50$$

$$\Rightarrow \quad x^2 = 1 \text{ and } y^2 = 25 \Rightarrow x = \pm 1 \text{ and } y = \pm 5.$$

Since $xy > 0$, so x and y are of the same sign.

$$\therefore \quad \sqrt{-24-10i} = (1-5i) \text{ or } (-1+5i).$$

EXAMPLE 5 *Evaluate* $\sqrt{-i}$.

SOLUTION Let $\sqrt{-i} = (x-iy)$. ... (i)

On squaring both sides of (i), we get

$$(-i) = (x-iy)^2 \Rightarrow -i = (x^2-y^2) - i(2xy). \quad \text{... (ii)}$$

On comparing real parts and imaginary parts on both sides of (ii), we get

$$(x^2 - y^2) = 0 \text{ and } 2xy = 1.$$

$$\therefore \quad (x^2 + y^2) = \sqrt{(x^2 - y^2)^2 + 4x^2y^2} = \sqrt{0^2 + 1^2} = \sqrt{0+1} = \sqrt{1} = 1.$$

Thus, $(x^2 - y^2) = 0$ … (iii) and $(x^2 + y^2) = 1$ … (iv).

On solving (iii) and (iv), we get $x^2 = \frac{1}{2}$ and $y^2 = \frac{1}{2}$.

$$\therefore \quad x = \pm\frac{1}{\sqrt{2}} \text{ and } y = \pm\frac{1}{\sqrt{2}}.$$

Since $xy > 0$, so x and y are of the same sign.

$$\therefore \quad \left(x = \frac{1}{\sqrt{2}}, y = \frac{1}{\sqrt{2}}\right) \text{ or } \left(x = \frac{-1}{\sqrt{2}}, y = \frac{-1}{\sqrt{2}}\right).$$

Hence, $\sqrt{-i} = \left(\frac{1}{\sqrt{2}} - i \cdot \frac{1}{\sqrt{2}}\right)$ or $\left(\frac{-1}{\sqrt{2}} + i \cdot \frac{1}{\sqrt{2}}\right)$.

EXAMPLE 6 *Evaluate* $\sqrt{4 + 3\sqrt{-20}} + \sqrt{4 - 3\sqrt{-20}}$.

SOLUTION $\sqrt{4 + 3\sqrt{-20}} = \sqrt{4 + 6\sqrt{5}\,i}$ and $\sqrt{4 - 3\sqrt{-20}} = \sqrt{4 - 6\sqrt{5}\,i}$.

Let $\sqrt{4 + 6\sqrt{5}\,i} = (x + iy)$. … (i)

On squaring both sides of (i), we get

$$(4 + 6\sqrt{5}\,i) = (x + iy)^2 \Rightarrow (4 + 6\sqrt{5}\,i) = (x^2 - y^2) + i(2xy). \quad \text{… (ii)}$$

On comparing real parts and imaginary parts on both sides of (ii), we get

$$(x^2 - y^2) = 4 \text{ and } 2xy = 6\sqrt{5}.$$

$$\therefore \quad (x^2 + y^2) = \sqrt{(x^2 - y^2)^2 + 4x^2y^2} = \sqrt{(4)^2 + (6\sqrt{5})^2}$$

$$= \sqrt{16 + 180} = \sqrt{196} = 14.$$

Thus, $(x^2 - y^2) = 4$ … (iii) and $(x^2 + y^2) = 14$ … (iv).

On solving (iii) and (iv), we get $x^2 = 9$ and $y^2 = 5$.

$$\therefore \quad x = \pm 3 \text{ and } y = \pm\sqrt{5}.$$

Since $xy > 0$, so x and y are of the same sign.

$$\therefore \quad (x = 3, y = \sqrt{5}) \text{ or } (x = -3, y = -\sqrt{5}).$$

$$\therefore \quad \sqrt{4 + 3\sqrt{-20}} = \sqrt{4 + 6\sqrt{5}\,i} = (3 + \sqrt{5}\,i) \text{ or } (-3 - \sqrt{5}\,i).$$

Similarly, $\sqrt{4 - 3\sqrt{-20}} = \sqrt{4 - 6\sqrt{5}\,i} = (3 - \sqrt{5}\,i)$ or $(-3 + \sqrt{5}\,i)$.

$$\therefore \quad \sqrt{4 + 3\sqrt{-20}} + \sqrt{4 - 3\sqrt{-20}} = \begin{cases} \{(3 + \sqrt{5}\,i) + (3 - \sqrt{5}\,i)\} = 6 \\ \text{or} \\ \{-3 - \sqrt{5}\,i\} + \{-3 + \sqrt{5}\,i\} = -6. \end{cases}$$

Hence, $\sqrt{4 + 3\sqrt{-20}} + \sqrt{4 - 3\sqrt{-20}} = 6$ or -6.

EXERCISE 5F

Evaluate:

1. $\sqrt{5+12i}$ **2.** $\sqrt{-7+24i}$ **3.** $\sqrt{-2+2\sqrt{3}\,i}$

4. $\sqrt{1+4\sqrt{-3}}$ **5.** $\sqrt{i}$ **6.** $\sqrt{4i}$

7. $\sqrt{3+4\sqrt{-7}}$ **8.** $\sqrt{16-30i}$ **9.** $\sqrt{-4-3i}$

10. $\sqrt{-15-8i}$ **11.** $\sqrt{-11-60i}$ **12.** $\sqrt{7-30\sqrt{-2}}$

13. $\sqrt{-8i}$ **14.** $\sqrt{1-i}$

ANSWERS (EXERCISE 5F)

1. $(3+2i)$ or $(-3-2i)$ **2.** $(3+4i)$ or $(-3-4i)$

3. $(1+\sqrt{3}\,i)$ or $(-1-\sqrt{3}\,i)$ **4.** $(2+\sqrt{3}\,i)$ or $(-2-\sqrt{3}\,i)$

5. $\frac{1}{\sqrt{2}}(1+i)$ or $\frac{1}{\sqrt{2}}(-1-i)$ **6.** $\sqrt{2}(1+i)$ or $\sqrt{2}(-1-i)$

7. $(\sqrt{7}+2i)$ or $(-\sqrt{7}-2i)$ **8.** $(5-3i)$ or $(-5+3i)$

9. $\frac{1}{\sqrt{2}}(1-3i)$ or $\frac{1}{\sqrt{2}}(-1+3i)$ **10.** $(1-4i)$ or $(-1+4i)$

11. $(5-6i)$ or $(-5+6i)$ **12.** $(5-3\sqrt{2}\,i)$ or $(-5+3\sqrt{2}\,i)$

13. $(2-2i), (-2+2i)$

14. $\left\{\sqrt{\frac{\sqrt{2}+1}{2}}\right\}-\left\{\sqrt{\frac{\sqrt{2}-1}{2}}\right\}i$ or $-\left\{\sqrt{\frac{\sqrt{2}+1}{2}}\right\}+\left\{\sqrt{\frac{\sqrt{2}-1}{2}}\right\}i$

HINTS TO SOME SELECTED QUESTIONS

14. Let $\sqrt{1-i}=x-iy$. … (i)

On squaring both sides, we get

$$(1-i)=(x-iy)^2=(x^2-y^2)-i(2xy)$$

$\therefore\quad (x^2-y^2)=1$ and $2xy=1$

$\Rightarrow\quad (x^2+y^2)=\sqrt{(x^2-y^2)^2+4x^2y^2}=\sqrt{1^2+1^2}=\sqrt{2}$

$\Rightarrow\quad x^2-y^2=1$ and $x^2+y^2=\sqrt{2}$

$\Rightarrow\quad x^2=\frac{\sqrt{2}+1}{2}$ and $y^2=\frac{\sqrt{2}-1}{2}$

$\Rightarrow\quad x=\pm\sqrt{\frac{\sqrt{2}+1}{2}}$ and $y=\pm\sqrt{\frac{\sqrt{2}-1}{2}}$.

Since $xy>0$, so x and y have the same sign.

$\therefore\quad \left\{x=\sqrt{\frac{\sqrt{2}+1}{2}},\ y=\sqrt{\frac{\sqrt{2}-1}{2}}\right\}$ or $\left\{x=-\sqrt{\frac{\sqrt{2}+1}{2}},\ y=-\sqrt{\frac{\sqrt{2}-1}{2}}\right\}$.

$\therefore\quad \sqrt{1-i}=\left\{\sqrt{\frac{\sqrt{2}+1}{2}}-\sqrt{\frac{\sqrt{2}-1}{2}}\,i\right\}$ or $\left\{-\sqrt{\frac{\sqrt{2}+1}{2}}+i\sqrt{\frac{\sqrt{2}-1}{2}}\right\}$.

EXERCISE 5G

Very-Short-Answer Questions

1. Evaluate $\frac{1}{i^{78}}$.

2. Evaluate $(i^{57}+i^{70}+i^{91}+i^{101}+i^{104})$.

3. Evaluate $\left(\frac{i^{180}+i^{178}+i^{176}+i^{174}+i^{172}}{i^{170}+i^{168}+i^{166}+i^{164}+i^{162}}\right)$.

4. Evaluate $(i^{4n+1}-i^{4n-1})$.

5. Evaluate $(\sqrt{-36}\times\sqrt{-25})$.

6. Find the sum $(i^n+i^{n+1}+i^{n+2}+i^{n+3})$, where $n\in N$.

7. Find the sum $(i+i^2+i^3+i^4+\ldots$ up to 400 terms$)$.

8. Evaluate $(1+i^{10}+i^{20}+i^{30})$.

9. Evaluate $\left(i^{41}+\frac{1}{i^{71}}\right)$.

10. Find the least positive integer n for which $\left(\frac{1+i}{1-i}\right)^n=1$.

11. Express $(2-3i)^3$ in the form $m(a+ib)$.

12. Express $\frac{(3+i\sqrt{5})(3-i\sqrt{5})}{(\sqrt{3}+\sqrt{2}\,i)-(\sqrt{3}-\sqrt{2}\,i)}$ in the form $(a+ib)$.

13. Express $\left(\frac{3-\sqrt{-16}}{1-\sqrt{-9}}\right)$ in the form $(a+ib)$.

14. Solve for x: $(1-i)x+(1+i)y=1-3i$.

15. Solve for x: $x^2-5ix-6=0$.

16. Find the conjugate of $\frac{1}{(3+4i)}$.

17. If $z=(1-i)$, find z^{-1}.

18. If $z=(\sqrt{5}+3i)$, find z^{-1}.

19. Prove that $\arg(z)+\arg(\bar{z})=0$.

20. If $|z|=6$ and $\arg(z)=\frac{3\pi}{4}$, find z.

21. Find the principal argument of $(-2i)$.

22. Write the principal argument of $(1+i\sqrt{3})^2$.

23. Write -9 in polar form.

24. Write $2i$ in polar form.

25. Write $-3i$ in polar form.

26. Write $z=(1-i)$ in polar form.

27. Write $z = (-1 + i\sqrt{3})$ in polar form.

28. If $|z| = 2$ and $\arg(z) = \dfrac{\pi}{4}$, find z.

ANSWERS (EXERCISE 5G)

1. -1 **2.** $-i$ **3.** -1 **4.** $2i$ **5.** -30 **6.** 0 **7.** 0 **8.** 0 **9.** $2i$

10. $n = 4$ **11.** $46 - 9i$ **12.** $a = 0, b = \dfrac{-7}{\sqrt{2}}$ **13.** $\left(\dfrac{3}{2} + \dfrac{1}{2}i\right)$

14. $x = 2, y = -1$ **15.** $x = 3i$ or $x = 2i$ **16.** $\left(\dfrac{3}{25} + \dfrac{4}{25}i\right)$ **17.** $\left(\dfrac{1}{2} + \dfrac{1}{2}i\right)$

18. $\left(\dfrac{\sqrt{5}}{14} - \dfrac{3}{14}i\right)$ **20.** $z = 3\sqrt{2}(-1 + i)$ **21.** $\dfrac{-\pi}{2}$ **22.** $\dfrac{2\pi}{3}$

23. $9(\cos \pi + i \sin \pi)$ **24.** $2\left(\cos \dfrac{\pi}{2} + i \sin \dfrac{\pi}{2}\right)$

25. $3\left\{\cos\left(\dfrac{-\pi}{2}\right) + i \sin\left(\dfrac{-\pi}{2}\right)\right\}$ **26.** $\sqrt{2}\left\{\cos\left(\dfrac{-\pi}{4}\right) + i \sin\left(\dfrac{-\pi}{4}\right)\right\}$

27. $2\left(\cos \dfrac{2\pi}{3} + i \sin \dfrac{2\pi}{3}\right)$ **28.** $z = \sqrt{2}(1 + i)$

HINTS TO SOME SELECTED QUESTIONS

1. $\dfrac{1}{i^{78}} = \dfrac{1}{i^{78}} \times \dfrac{i^2}{i^2} = -1.$ $\quad [\because\ i^{80} = (i^4)^{20} = 1]$

2. Given expression $= (i^{48} \times i^3 + i^{68} \times i^2 + i^{88} \times i^3 + i^{100} \times i + i^{104})$
$= (i^3 + i^2 + i^3 + i + 1) = (-i - 1 - i + i + 1) = -i.$

3. Given expression $= \dfrac{i^{172}(i^8 + i^6 + i^4 + i^2 + 1)}{i^{162}(i^8 + i^6 + i^4 + i^2 + 1)} = i^{(172-162)} = i^{10} = i^8 \times i^2 = -1.$

4. Given expression $= (i^{4n} \times i - i^{4n} \times i^{-1}) = \left(i - \dfrac{1}{i}\right) = \dfrac{(i^2 - 1)}{i} \times \dfrac{i}{i} = \dfrac{-2i}{-1} = 2i.$

5. $(\sqrt{-36} \times \sqrt{-25}) = (6i \times 5i) = 30i^2 = -30.$

6. Given sum $= i^n \times (1 + i + i^2 + i^3) = i^n(1 + i - 1 - i) = 0.$

7. Given sum $= (i + i^2 + i^3 + i^4) + (i^5 + i^6 + i^7 + i^8) + \ldots\ldots$
$= (i + i^2 + i^3 + 1) + i^5(1 + i + i^2 + i^3) + \ldots = 0.$ $\quad [\because\ (1 + i + i^2 + i^3) = 0]$

8. Given sum $= (1 + i^2 + 1 + i^2) = 0.$

9. $\left(i^{41} + \dfrac{1}{i^{71}}\right) = \left(i + \dfrac{1}{i^3}\right) = \left(i + \dfrac{1}{i^3} \times \dfrac{i}{i}\right) = (i + i) = 2i.$

10. $\left(\dfrac{1+i}{1-i}\right) = \dfrac{(1+i)}{(1-i)} \times \dfrac{(1+i)}{(1+i)} = \dfrac{(1+i)^2}{(1-i^2)} = \dfrac{(1 + i^2 + 2i)}{2} = \dfrac{2i}{2} = i.$

$\therefore \quad \left(\dfrac{1+i}{1-i}\right)^n = 1 \Rightarrow i^n = 1 \Rightarrow i^n = i^4 \Rightarrow n = 4.$

11. $(2-3i)^3 = 2^3 - (3i)^3 - (3\times 2\times 3i)(2-3i)$
$= (8-27i^3-36i-54) = (8+27i-36i-54) = (46-9i)$.

12. Given expression $= \dfrac{(9-5i^2)}{2\sqrt{2}\,i} = \dfrac{14}{2\sqrt{2}\,i}\times\dfrac{i}{i} = \dfrac{-7i}{\sqrt{2}} \Rightarrow a=0, b=\dfrac{-7}{\sqrt{2}}$.

13. Given expression $= \dfrac{3-4i}{1-3i}\times\dfrac{1+3i}{1+3i} = \dfrac{(3-4i)(1+3i)}{(1+9)} = \dfrac{15+5i}{10} = \dfrac{1}{2}(3+i)$.

14. $(1-i)x+(1+i)y = 1-3i \Leftrightarrow (x+y)+(y-x)i = 1-3i$
$\therefore\quad x+y=1$ and $y-x=-3 \Rightarrow x=2, y=-1$.

15. $x^2-5ix-6=0 \Rightarrow x^2-3ix-2ix-6=0 \Rightarrow x(x-3i)-2i(x-3i)=0$
$\Rightarrow (x-3i)(x-2i)=0 \Rightarrow x=3i$ or $x=2i$.

16. $z = \dfrac{1}{(3+4i)}\times\dfrac{(3-4i)}{(3-4i)} = \dfrac{3-4i}{25} = \left(\dfrac{3}{25}-\dfrac{4}{25}i\right) \Rightarrow \bar{z} = \left(\dfrac{3}{25}+\dfrac{4}{25}i\right)$.

17. $z^{-1} = \dfrac{\bar{z}}{|z|^2} = \dfrac{(1+i)}{(1^2+1^2)} = \left(\dfrac{1}{2}+\dfrac{1}{2}i\right)$.

18. $z^{-1} = \dfrac{\bar{z}}{|z|^2} = \dfrac{(\sqrt{5}-3i)}{(5+9)} = \left(\dfrac{\sqrt{5}}{14}-\dfrac{3}{14}i\right)$.

20. $z = 6\left(\cos\dfrac{3\pi}{4}+i\sin\dfrac{3\pi}{4}\right) = 6\left\{\cos\left(\pi-\dfrac{\pi}{4}\right)+i\sin\left(\pi-\dfrac{\pi}{4}\right)\right\}$

$\Rightarrow\quad z = 6\left(-\cos\dfrac{\pi}{4}+i\sin\dfrac{\pi}{4}\right) = 6\left(\dfrac{-1}{\sqrt{2}}+i\cdot\dfrac{1}{\sqrt{2}}\right) = \dfrac{6}{\sqrt{2}}(-1+i)$

$\Rightarrow\quad z = 3\sqrt{2}(-1+i)$.

22. $z = (1+i\sqrt{3})^2 = (-2+2i\sqrt{3})$.

REVIEW OF FACTS AND FORMULAE

1. (i) **Imaginary Numbers** *A given number is said to be imaginary, if its square is negative,* e.g., $\sqrt{-1}, \sqrt{-2}, \sqrt{-3}, \sqrt{-4}$, etc.

 (ii) We denote $\sqrt{-1}$ by the Greek letter iota 'ι' transliterated as 'i'.

 (iii) $i^0 = 1, i^1 = i, i^2 = -1, i^3 = -i$ and $i^4 = 1$.

 (iv) Let n be a positive integer. On dividing n by 4, let m be the quotient and r be the remainder.

 Then, $n = 4m+r$, where $0 \le r < 4$.

 $\therefore\quad i^n = i^{4m+r} = (i^4)^m\cdot i^r = i^r. \qquad [\because\ i^4 = 1]$

 (v) $\sqrt{-16}\times\sqrt{-9} = (4i)\times(3i) = 12i^2 = -12$.

2. **Complex Numbers**

 (i) The numbers of the form $z = (a+ib)$ are called complex numbers, where $a, b \in R$. Here, $\text{Re}(z) = a$ and $\text{Im}(z) = b$.

If $b = 0$ then z is purely real.

If $a = 0$ then z is purely imaginary.

(ii) Let $z_1 = (a_1 + ib_1)$ and $z_2 = (a_2 + ib_2)$ then

I. $(z_1 + z_2) = (a_1 + a_2) + i(b_1 + b_2)$

II. $(z_1 - z_2) = (a_1 - a_2) + i(b_1 - b_2)$

III. $z_1 z_2 = (a_1 a_2 - b_1 b_2) + i(a_1 b_2 + a_2 b_1)$.

(iii) The conjugate of $z = (a + ib)$ is $\overline{z} = \overline{(a + ib)} = (a - ib)$.

(iv) The modulus of $z = (a + ib)$ is $|z| = \sqrt{a^2 + b^2}$.

(v) If $z_1 = (a_1 + ib_1)$ and $z_2 = (a_2 + ib_2)$ then $z_1 = z_2 \Leftrightarrow a_1 = a_2$ and $b_1 = b_2$.

(vi) If $z = (a + ib)$ then $z^{-1} = \dfrac{\overline{z}}{|z|^2} = \dfrac{(a - ib)}{(a^2 + b^2)}$.

3. (i) $\overline{z_1 + z_2} = \overline{z}_1 + \overline{z}_2$ (ii) $\overline{z_1 - z_2} = \overline{z}_1 - \overline{z}_2$ (iii) $\overline{z_1 z_2} = \overline{z}_1\, \overline{z}_2$

(iv) $\overline{\left(\dfrac{z_1}{z_2}\right)} = \dfrac{\overline{z}_1}{\overline{z}_2}$ (v) $z + \overline{z} = 2\,\mathrm{Re}(z)$ and $z - \overline{z} = 2\,\mathrm{Im}(z)$

(vi) $z\overline{z} = |z|^2$

4. Polar form of $z = (x + iy)$ is $z = r(\cos\theta + i\sin\theta)$.

Here, $r = \sqrt{x^2 + y^2} = |z|$ and the value of θ such that $-\pi < \theta \le \pi$ is called the principal argument or amplitude of z.

5. *When z is purely real;* then, it lies on the x-axis.

(i) If z lies on positive side of the x-axis then $\theta = 0$.

(ii) If z lies on negative side of the x-axis then $\theta = \pi$.

6. *When z is purely imaginary*; then, it lies on the y-axis.

(i) If z lies on the y-axis and above the x-axis then $\theta = \dfrac{\pi}{2}$.

(ii) If z lies on the y-axis and below the x-axis then $\theta = \dfrac{-\pi}{2}$.

7. When $z = (x + iy)$ does not lie on any axes:

Step 1. Find α for which $\tan\alpha = \left|\dfrac{\mathrm{Im}(z)}{\mathrm{Re}(z)}\right|$.

Step 2. Find the quadrant in which $P(x, y)$ lies.

(i) *When z lies in quadrant I;*
then, $\theta = \alpha \Rightarrow \arg(z) = \alpha$.

(ii) *When z lies in quadrant II;*
then, $\theta = \pi - \alpha \Rightarrow \arg(z) = \pi - \alpha$.

(iii) *When z lies in quadrant III;*
then, $\theta = (\alpha - \pi) \Rightarrow \arg(z) = \alpha - \pi$.

(iv) *When z lies in quadrant IV;*
then, $\theta = -\alpha \Rightarrow \arg(z) = -\alpha$.

8. (i) *Fundamental theorem of algebra*

A polynomial equation of degree n has at most n roots.

(ii) Let $ax^2 + bx + c = 0$, where $a, b, c \in R$ and $a \neq 0$.

Its roots are given by

$$\alpha = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \text{ and } \beta = \frac{-b - \sqrt{b^2 - 4ac}}{2a}.$$

Here, $D = (b^2 - 4ac)$ is called the *discriminant*.

If $D < 0$, the given quadratic equation has complex roots.

9. $\sqrt{a + ib} = \pm(x + iy)$, where $x^2 = \frac{1}{2}\left\{\sqrt{a^2 + b^2} + a\right\}$ and $y^2 = \frac{1}{2}\left\{\sqrt{a^2 + b^2} - a\right\}$.

Similarly, by assuming that $\sqrt{a - ib} = \pm(x - iy)$, we may find $\sqrt{a - ib}$.

❑

6

Linear Inequations (In one variable)

Linear Inequations in One Variable

Inequalities of the form:

(i) $ax+b<c$ (ii) $ax+b\le c$ (iii) $ax+b>c$ (iv) $ax+b\ge c$,

where a, b, c are real numbers, $a\neq 0$ and x is a variable, are called inequations in x.

Thus, each of the following is a linear inequation:

(i) $2x+3<7$ (ii) $3x-5\le 10$ (iii) $3x+2>8$ (iv) $4x-3\ge 9$

Replacement Set or Domain of the Variable

A set given to us from which the values of x are replaced in an inequation in x, is called the *replacement set*.

SOLUTION SET The set of all those values of x taken from the replacement set which satisfy the given inequation is called the *solution set of the inequation*.

EXAMPLE *Write down the solution set of the inequation $x<6$, when the replacement set is (i) N, (ii) W, (iii) Z.*

SOLUTION (i) Solution set $=\{x\in N : x<6\}=\{1, 2, 3, 4, 5\}$.

(ii) Solution set $=\{x\in W : x<6\}=\{0, 1, 2, 3, 4, 5\}$.

(iii) Solution set $=\{x\in Z : x<6\}=\{5, 4, 3, 2, 1, 0, -1, -2, -3, \ldots\}$.

Rules for Solving an Inequation

RULE 1 Adding the same number or expression to each side of an inequation does not change the inequality.

RULE 2 Subtracting the same number or expression from each side of an inequation does not change the inequality.

RULE 3 Multiplying (or dividing) each side of an inequation by the same positive number does not change the inequality.

RULE 4 Multiplying (or dividing) each side of an inequation by the same negative number reverses the inequality.

Thus, $-x>3 \Rightarrow x<-3$ [on multiplying both sides by –1].

And, $-3x\le -12 \Rightarrow x\ge 4$ [on dividing both sides by –3].

TRANSPOSITION Using Rule 1 and Rule 2, we can drop any term from one side of an inequation and put it on the other side with the opposite sign.

This process is called *transposition*.

Thus, $x+3<2 \Rightarrow x<2-3 \Rightarrow x<-1$.

And, $2x-3 \geq 6 \Rightarrow 2x \geq 6+3 \Rightarrow 2x \geq 9 \Rightarrow x \geq \frac{9}{2}$.

SOLVED EXAMPLES

EXAMPLE 1 *Solve* $5x < 24$ *when (i)* $x \in N$, *(ii)* $x \in Z$.

In each case, represent the solution set on the number line.

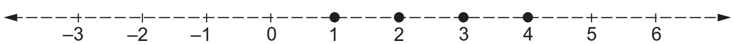
SOLUTION $5x < 24 \Rightarrow x < \frac{24}{5}$

$\Rightarrow x < 4.8.$

(i) Solution set $= \{x \in N : x < 4.8\}$

$= \{1, 2, 3, 4\}$.

On the number line, we may represent it as shown below.



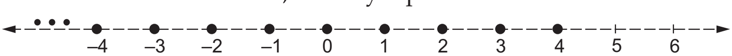
The darkened circles indicate the natural numbers contained in the set.

(ii) Solution set $= \{x \in Z : x < 4.8\}$

$= \{4, 3, 2, 1, 0, -1, -2, -3, \ldots\}$

$= \{\ldots, -3, -2, -1, 0, 1, 2, 3, 4\}$.

On the number line, we may represent it as shown below.



The darkened circles show the integers contained in the set and three dark dots above the left part of the line show that the negative integers are continued idefinitely.

EXAMPLE 2 *Solve* $12 + 1\frac{5}{6}x \leq 5 + 3x$ *when (i)* $x \in N$, *(ii)* $x \in R$.

Draw the graph of the solution set in each case.

SOLUTION $12 + 1\frac{5}{6}x \leq 5 + 3x$

$\Rightarrow \quad 12 + \frac{11}{6}x \leq 5 + 3x$

$\Rightarrow \quad 72 + 11x \leq 30 + 18x$ [multiplying both sides by 6]

$\Rightarrow \quad 11x \leq 18x - 42$ [adding -72 to both sides]

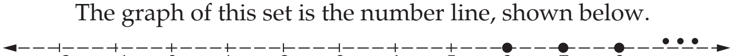
$\Rightarrow \quad -7x \leq -42$ [adding $-18x$ to both sides]

$\Rightarrow \quad x \geq 6$ [dividing both sides by -7].

(i) Solution set $= \{x \in N : x \geq 6\}$

$= \{6, 7, 8, 9, \ldots\}$.

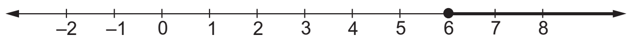
The graph of this set is the number line, shown below.



The darkened circles indicate the natural numbers contained in the set. Three dots above the right part of the line show that the natural numbers are continued indefinitely.

(ii) Solution set $= \{x \in R : x \geq 6\} = [6, \infty[$.

The graph of this set is shown below.



This graph consists of 6 and all real numbers greater than 6.

EXAMPLE 3 *Solve* $\dfrac{x+4}{x-2} > 0$ *and draw the graph of the solution set.*

SOLUTION Note that $\dfrac{a}{b} > 0$ when $(a > 0$ and $b > 0)$ or $(a < 0$ and $b < 0)$.

$\therefore$ either $(x+4 > 0$ and $x-2 > 0)$ or $(x+4 < 0$ and $x-2 < 0)$.

Case I *When* $x+4 > 0$ *and* $x-2 > 0$.

In this case, $x+4 > 0$ and $x-2 > 0$

$\Rightarrow \quad x > -4$ and $x > 2$

$\Rightarrow \quad x > 2$

$\Rightarrow \quad x \in (2, \infty)$.

Case II *When* $x+4 < 0$ *and* $x-2 < 0$.

In this case, $x+4 < 0$ and $x-2 < 0$

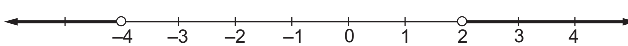
$\Rightarrow \quad x < -4$ and $x < 2$

$\Rightarrow \quad x < -4$

$\Rightarrow \quad x \in (-\infty, -4)$.

$\therefore$ solution set $= (-\infty, -4) \cup (2, \infty)$.

The graph of the solution set is given below.



EXAMPLE 4 *Solve* $\dfrac{x-3}{x+4} < 0$ *and draw the graph of the solution set.*

SOLUTION Note that $\dfrac{a}{b} < 0$ when $(a > 0$ and $b < 0)$ or $(a < 0$ and $b > 0)$.

$\therefore$ either $(x-3 > 0$ and $x+4 < 0)$ or $(x-3 < 0$ and $x+4 > 0)$.

Case I *When* $x-3 > 0$ *and* $x+4 < 0$.

In this case, $x-3 > 0$ and $x+4 < 0$

$\Rightarrow \quad x > 3$ and $x < -4$.

This is not possible, as we can never find a real number which is greater than 3 and less than −4.

Case II *When* $x-3 < 0$ *and* $x+4 > 0$.

In this case, $x-3 < 0$ and $x+4 > 0$

$\Rightarrow \quad x < 3$ and $x > -4$

$\Rightarrow \quad -4 < x < 3 \Rightarrow x \in (-4, 3)$.

$\therefore$ solution set $= (-4, 3)$.

The graph of the solution set is given below.

EXAMPLE 5 *Solve* $\dfrac{x-2}{x+5} > 2.$

SOLUTION We have

$$\frac{x-2}{x+5} > 2$$

$$\Rightarrow \quad \frac{x-2}{x+5} - 2 > 0$$

$$\Rightarrow \quad \frac{x-2-2x-10}{x+5} > 0$$

$$\Rightarrow \quad \frac{-(x+12)}{(x+5)} > 0$$

$$\Rightarrow \quad \frac{x+12}{x+5} < 0 \qquad \text{[on multiplying both sides by } -1\text{]}.$$

$\therefore$ either $(x+12 < 0$ and $x+5 > 0)$ or $(x+12 > 0$ and $x+5 < 0)$.

Case I *When* $x+12 < 0$ *and* $x+5 > 0$.

In this case, $x+12 < 0$ and $x+5 > 0$

$\Rightarrow$ $x < -12$ and $x > -5$

$\Rightarrow$ $-5 < x < -12$, which is impossible [$\because$ $-12 < -5$].

Case II *When* $x+12 > 0$ *and* $x+5 < 0$.

In this case, $x+12 > 0$ and $x+5 < 0$

$\Rightarrow$ $x > -12$ and $x < -5$

$\Rightarrow$ $-12 < x < -5 \Rightarrow x \in (-12, -5)$.

$\therefore$ solution set $= (-12, -5)$.

EXAMPLE 6 *Solve* $\dfrac{5}{x-2} > 3$ *and represent the solution set on the number line.*

SOLUTION $\dfrac{5}{x-2} > 3 \Rightarrow \dfrac{5}{x-2} - 3 > 0$ [adding -3 to both sides]

$$\Rightarrow \frac{5-3x+6}{x-2} > 0$$

$$\Rightarrow \frac{11-3x}{x-2} > 0.$$

$\therefore$ either $(11-3x > 0$ and $x-2 > 0)$ or $(11-3x < 0$ and $x-2 < 0)$.

Case I *When* $11-3x > 0$ *and* $x-2 > 0$.

Now, $11-3x > 0$ and $x-2 > 0$

$\Rightarrow$ $-3x > -11$ and $x > 2$

$\Rightarrow$ $x < \dfrac{11}{3}$ and $x > 2$

$$\Rightarrow \quad 2 < x < \frac{11}{3}. \qquad \dots \text{(i)}$$

Case II *When* $11-3x<0$ *and* $x-2<0$.

Now, $11-3x<0$ and $x-2<0$

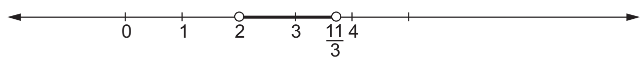
$\Rightarrow \quad -3x<-11$ and $x<2$

$\Rightarrow \quad x>\dfrac{11}{3}$ and $x<2$.

This is not possible, as we cannot find a real number which is greater than $\dfrac{11}{3}$ and less than 2.

$\therefore \quad$ solution set $=\left\{x\in R: 2<x<\dfrac{11}{3}\right\}=\left(2,\dfrac{11}{3}\right)$.

We can represent this set on the number line, as given below.



SOME USEFUL RESULTS

Let a be a positive real number. Then,

(i) $|x|<a \Leftrightarrow -a<x<a \Leftrightarrow x\in(-a,a)$.

(ii) $|x|\le a \Leftrightarrow -a\le x\le a \Leftrightarrow x\in[-a,a]$.

(iii) $|x|>a \Leftrightarrow x<-a$ or $x>a$.

(iv) $|x|\ge a \Leftrightarrow x\le -a$ or $x\ge a$.

EXAMPLE 7 *Solve* $|4-x|<2$.

SOLUTION We have, $|x|<a \Leftrightarrow -a<x<a$.

$\therefore \quad |4-x|<2$

$\Leftrightarrow \quad -2<4-x<2$

$\Leftrightarrow \quad -2<4-x$ and $4-x<2$

$\Leftrightarrow \quad -2-4<-x$ and $-x<2-4$

$\Leftrightarrow \quad -6<-x$ and $-x<-2$

$\Leftrightarrow \quad 6>x$ and $x>2$

$\Leftrightarrow \quad 2<x<6$.

$\therefore \quad$ solution set $=\{x\in R: 2<x<6\}=(2,6)$.

EXAMPLE 8 *Solve* $|3x-2|\le\dfrac{1}{2}$, $x\in R$.

SOLUTION We have, $|x|\le a \Leftrightarrow -a\le x\le a$.

$\therefore \quad |3x-2|\le\dfrac{1}{2} \Leftrightarrow -\dfrac{1}{2}\le 3x-2\le\dfrac{1}{2}$

$\Leftrightarrow \dfrac{-1}{2}\le 3x-2$ and $3x-2\le\dfrac{1}{2}$

$\Leftrightarrow \dfrac{-1}{2}+2\le 3x$ and $3x\le\dfrac{1}{2}+2$

$$\Leftrightarrow \frac{3}{2} \le 3x \text{ and } 3x \le \frac{5}{2}$$

$$\Leftrightarrow \frac{1}{2} \le x \text{ and } x \le \frac{5}{6}$$

$$\frac{1}{2} \le x \le \frac{5}{6}.$$

$\therefore$ solution set $= \left\{x \in R : \frac{1}{2} \le x \le \frac{5}{6}\right\} = \left[\frac{1}{2}, \frac{5}{6}\right]$.

EXAMPLE 9 *Solve* $|x+1| > 4$, $x \in R$.

SOLUTION We have, $|x| > a \Leftrightarrow x < -a$ or $x > a$.

$$\therefore \quad |x+1| > 4 \Leftrightarrow x+1 < -4 \text{ or } x+1 > 4$$
$$\Leftrightarrow x < -4-1 \text{ or } x > 4-1$$
$$\Leftrightarrow x < -5 \text{ or } x > 3$$
$$\Leftrightarrow x \in (-\infty, -5) \text{ or } x \in (3, \infty).$$

$\therefore$ solution set $= (-\infty, -5) \cup (3, \infty)$.

EXAMPLE 10 *Solve* $|3-4x| \ge 9$, $x \in R$.

SOLUTION We have, $|x| \ge a \Leftrightarrow x \le -a$ or $x \ge a$.

$$\therefore \quad |3-4x| \ge 9 \Leftrightarrow 3-4x \le -9 \text{ or } 3-4x \ge 9$$
$$\Leftrightarrow -4x \le -9-3 \text{ or } -4x \ge 9-3$$
$$\Leftrightarrow -4x \le -12 \text{ or } -4x \ge 6$$
$$\Leftrightarrow x \ge 3 \text{ or } x \le \frac{-3}{2}$$
$$\Leftrightarrow x \le \frac{-3}{2} \text{ or } x \ge 3$$
$$\Leftrightarrow x \in \left(-\infty, \frac{-3}{2}\right] \text{ or } x \in [3, \infty).$$

$\therefore$ solution set $= \left(-\infty, \frac{-3}{2}\right] \cup [3, \infty)$.

EXAMPLE 11 *Solve* $\frac{2}{|x-3|} > 5$, $x \in R$.

SOLUTION Clearly, $x - 3 \neq 0$ and therefore, $x \neq 3$.

We have, $\frac{2}{|x-3|} > 5$. … (i)

Since $|x-3|$ is positive, we may multiply both sides of (i) by $|x-3|$. This gives

$$2 > 5\,|x-3|$$

$$\Leftrightarrow \quad \frac{2}{5} > |x-3|$$

$$\Leftrightarrow \quad |x-3| < \frac{2}{5}$$

$$\Leftrightarrow \quad \frac{-2}{5} < x-3 < \frac{2}{5} \qquad [\because\ |x| < a \Leftrightarrow -a < x < a]$$

$\Leftrightarrow \quad \frac{-2}{5} < x - 3$ and $x - 3 < \frac{2}{5}$

$\Leftrightarrow \quad \frac{-2}{5} + 3 < x$ and $x < \frac{2}{5} + 3$

$\Leftrightarrow \quad \frac{13}{5} < x$ and $x < \frac{17}{5}$.

$\Leftrightarrow \quad \frac{13}{5} < x < \frac{17}{5}$.

Also, as shown above, $x \neq 3$.

$$\therefore \quad \text{solution set} = \left\{x \in R : \frac{13}{5} < x < \frac{17}{5}\right\} - \{3\}$$
$$= (2.6, 3.4) - \{3\} = (2.6, 3) \cup (3, 3.4).$$

EXAMPLE 12 *Solve* $\frac{|x-2|-1}{|x-2|-2} \leq 0, x \in R$.

SOLUTION We have either $x - 2 \geq 0$ or $x - 2 < 0$.

Case I *When* $x - 2 \geq 0$.

In this case, $x \geq 2$ and $|x-2| = x - 2$.

So, the given inequation becomes

$$\frac{x-2-1}{x-2-2} \leq 0 \Rightarrow \frac{x-3}{x-4} \leq 0.$$

$\therefore \quad (x - 3 \geq 0$ and $x - 4 < 0)$ or $(x - 3 \leq 0$ and $x - 4 > 0)$

$\Rightarrow \quad (x \geq 3$ and $x < 4)$ or $(x < 3$ and $x > 4)$

$\Rightarrow \quad 3 \leq x < 4 \qquad$ [$\because$ $x < 3$ and $x > 4$ is not possible]

$\Rightarrow \quad x \in [3, 4) \qquad$ [this includes $x \geq 2$].

Case II *When* $x - 2 < 0$.

In this case, $x < 2$ and $|x-2| = -(x-2) = -x + 2$.

So, the given inequation becomes

$$\frac{-x+2-1}{-x+2-2} \leq 0 \Rightarrow \frac{-x+1}{-x} \leq 0$$

$$\Rightarrow \frac{x-1}{x} \leq 0$$

[multiplying num. and denom. by –1]

$\therefore \quad (x - 1 \leq 0$ and $x > 0)$ or $(x - 1 \geq 0$ and $x < 0)$

$\Rightarrow \quad (x \leq 1$ and $x > 0)$ or $(x \geq 1$ and $x < 0)$

$\Rightarrow \quad 0 < x \leq 1 \qquad$ [$\because$ $x \geq 1$ and $x < 0$ is not possible]

$\Rightarrow \quad x \in (0, 1] \qquad$ [this includes $x < 2$].

Hence, from the above two cases, we get

solution set $= (0, 1] \cup [3, 4)$.

EXAMPLE 13 *Solve* $\frac{|x+3|+x}{(x+2)} > 1, x \in R$.

SOLUTION We have

$$\frac{|x+3|+x}{(x+2)} > 1$$

$$\Rightarrow \quad \frac{|x+3|+x}{(x+2)} - 1 > 0$$

$$\Rightarrow \quad \frac{|x+3|+x-x-2}{(x+2)} > 0$$

$$\Rightarrow \quad \frac{|x+3|-2}{(x+2)} > 0. \qquad \ldots \text{(i)}$$

Now, we have either $x+3 \geq 0$ or $x+3 < 0$.

Case I *When* $x+3 \geq 0$.

In this case, $x \geq -3$ and $|x+3| = x+3$.

$\therefore$ (i) becomes

$$\frac{x+3-2}{x+2} > 0 \Rightarrow \frac{x+1}{x+2} > 0.$$

$\therefore$ $(x+1 > 0$ and $x+2 > 0)$ or $(x+1 < 0$ and $x+2 < 0)$

$\Rightarrow$ $(x > -1$ and $x > -2)$ or $(x < -1$ and $x < -2)$

$\Rightarrow$ $(x > -1)$ or $(x < -2)$

$\Rightarrow$ $(x < -2)$ or $(x > -1)$

$\Rightarrow$ $x \in (-\infty, -2)$ or $x \in (-1, \infty)$

$\Rightarrow$ $x \in [-3, -2)$ or $x \in (-1, \infty)$ $\qquad [\because\ x \geq -3]$

$\Rightarrow$ $x \in [-3, -2) \cup (-1, \infty)$.

Case II *When* $x+3 < 0$.

In this case, $x < -3$ and $|x+3| = -(x+3) = -x-3$.

$\therefore$ (i) becomes

$$\frac{-x-3-2}{x+2} > 0 \Rightarrow \frac{-(x+5)}{(x+2)} > 0 \Rightarrow \frac{x+5}{x+2} < 0.$$

$\therefore$ $(x+5 < 0$ and $x+2 > 0)$ or $(x+5 > 0$ and $x+2 < 0)$

$\Rightarrow$ $(x < -5$ and $x > -2)$ or $(x > -5$ and $x < -2)$

$\Rightarrow$ $-5 < x < -2$ $\qquad [\because\ x < -5$ and $x > -2$ is not possible$]$

$\Rightarrow$ $-5 < x < -3$ $\qquad [\because\ x < -3]$

$\Rightarrow$ $x \in (-5, -3)$.

$\therefore$ solution set $= (-5, -3) \cup [-3, -2) \cup (-1, \infty)$

$= (-5, -2) \cup (-1, \infty)$.

EXAMPLE 14 *Solve* $|x-1| + |x-2| \geq 4,\ x \in R$.

SOLUTION Putting $x-1=0$ and $x-2=0$, we get $x=1$ and $x=2$ as the critical points. These points divide the whole real line into three parts, namely $(-\infty, 1), [1, 2)$ and $[2, \infty)$. So, we consider the following three cases.

Case I *When* $-\infty < x < 1$.

In this case, $x-1<0$ and $x-2<0$.

$\therefore \quad |x-1| = -(x-1) = -x+1$ and $|x-2| = -(x-2) = -x+2$.

Now, $|x-1|+|x-2| \geq 4$

$\Rightarrow \quad -x+1-x+2 \geq 4$

$\Rightarrow \quad -2x+3 \geq 4 \Rightarrow -2x \geq 4-3 \Rightarrow -2x \geq 1 \Rightarrow x \leq \frac{-1}{2}$

$\Rightarrow \quad x \in \left(-\infty, \frac{-1}{2}\right]$.

But, $-\infty < x < 1$.

$\therefore$ solution set in this case $= \left(-\infty, \frac{-1}{2}\right] \cap (-\infty, 1) = \left(-\infty, \frac{-1}{2}\right]$.

Case II *When* $1 \leq x < 2$.

In this case, $x-1 \geq 0$ and $x-2<0$.

$\therefore \quad |x-1| = x-1$ and $|x-2| = -(x-2) = -x+2$.

Now, $|x-1|+|x-2| \geq 4$

$\Rightarrow \quad x-1-x+2 \geq 4 \Rightarrow -1 \geq 4$, which is absurd.

So, the given inequation has no solution in $[1, 2)$.

Case III *When* $2 \leq x < \infty$.

In this case, $x-2 \geq 0$ and $x-1>0$.

$\therefore \quad |x-2| = x-2$ and $|x-1| = x-1$.

Now, $|x-1|+|x-2| \geq 4$

$\Rightarrow \quad x-1+x-2 \geq 4 \Rightarrow 2x-3 \geq 4 \Rightarrow 2x \geq 7 \Rightarrow x \geq \frac{7}{2}$.

Also, in this case, we have $x \geq 2$.

$\therefore \quad$ solution set in this case $= \left[\frac{7}{2}, \infty\right) \cap [2, \infty) = \left[\frac{7}{2}, \infty\right)$.

Hence, from all the above cases, we have

$$\text{solution set} = \left(-\infty, \frac{-1}{2}\right] \cup \left[\frac{7}{2}, \infty\right).$$

SOLVING SIMULTANEOUS INEQUATIONS IN ONE VARIABLE

METHOD Suppose we have to solve two inequations simultaneously. Find the solution set of each of them. Then, the intersection of these solution sets is the required solution set.

EXAMPLE 15 *Solve the inequations* $-3 \leq 3-2x < 9$, $x \in R$. *Represent the solution set on the real line.*

SOLUTION We have

$$-3 \leq 3-2x < 9 \Rightarrow -3 \leq 3-2x \text{ and } 3-2x < 9.$$

Now, $-3 \leq 3-2x \Rightarrow -3+2x \leq 3$ [adding $2x$ on both sides]

$\Rightarrow 2x \leq 6$ [adding 3 on both sides]

$\Rightarrow x \leq 3$ [dividing both sides by 2]

$\Rightarrow x \in (-\infty, 3]$.

And, $3-2x<9 \Rightarrow -2x<6$ [subtracting 3 from both sides]

$\Rightarrow x>-3$ [dividing both sides by -2]

$\Rightarrow x \in (-3, \infty)$.

$\therefore$ solution set $= (-\infty, 3] \cap (-3, \infty) = (-3, 3]$.

We may represent it on the number line as shown below.

EXAMPLE 16 *Solve the inequations* $2x-3<x+2 \leq 3x+5,\ x \in R$. *Draw the graph of the solution set.*

SOLUTION We have

$$2x-3<x+2 \leq 3x+5 \Rightarrow 2x-3<x+2 \text{ and } x+2 \leq 3x+5.$$

Now, $2x-3<x+2 \Rightarrow x-3<2$ [adding $-x$ on both sides]

$\Rightarrow x<5$ [adding 3 on both sides]

$\Rightarrow x \in (-\infty, 5)$.

And, $x+2 \leq 3x+5 \Rightarrow 3x+5 \geq x+2$ [$\because a \leq b \Rightarrow b \geq a$]

$\Rightarrow 2x+5 \geq 2$ [adding $-x$ on both sides]

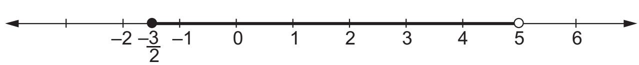
$\Rightarrow 2x \geq -3$ [adding -5 on both sides]

$\Rightarrow x \geq \dfrac{-3}{2}$ [dividing both sides by 2]

$\Rightarrow x \in \left[\dfrac{-3}{2}, \infty\right)$.

$\therefore$ solution set $= (-\infty, 5) \cap \left[\dfrac{-3}{2}, \infty\right) = \left[\dfrac{-3}{2}, 5\right)$.

We may represent it on the number line, as shown below.



EXAMPLE 17 *Solve the system of inequations* $2x-1>x+\dfrac{7-x}{3}>2,\ x \in R$.

Represent the solution set on the number line.

SOLUTION We have

$$2x-1>x+\frac{7-x}{3}>2 \Rightarrow 2x-1>x+\frac{7-x}{3} \text{ and } x+\frac{7x}{3}>2.$$

Now, $2x-1>x+\dfrac{7-x}{3} \Rightarrow 6x-3>3x+7-x$

[multiplying both sides by 3]

$\Rightarrow 6x-3>2x+7$

$\Rightarrow 4x-3>7$ [adding $-2x$ on both sides]

$\Rightarrow 4x>10$ [adding 3 on both sides]

$\Rightarrow x>\dfrac{5}{2}$ [dividing both sides by 4]

$\Rightarrow x \in \left(\dfrac{5}{2}, \infty\right)$.

And, $x+\dfrac{7-x}{3}>2 \Rightarrow 3x+7-x>6$ [multiplying both sides by 3]

$\Rightarrow 2x+7>6$

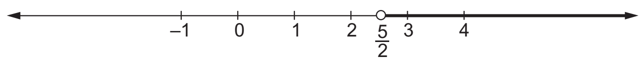
$\Rightarrow 2x>-1$ [adding –7 on both sides]

$\Rightarrow x>\dfrac{-1}{2}$ [dividing both sides by 2]

$\Rightarrow x \in \left(\dfrac{-1}{2}, \infty\right)$.

$\therefore$ solution set $=\left(\dfrac{5}{2}, \infty\right) \cap \left(\dfrac{-1}{2}, \infty\right)=\left(\dfrac{5}{2}, \infty\right)$.

The solution set on the number line may be represented as shown below.



EXAMPLE 18 *Solve the system of inequations* $-5 \le \dfrac{2-3x}{4} \le 9$.

SOLUTION We have

$$-5 \le \frac{2-3x}{4} \le 9 \Rightarrow -5 \le \frac{2-3x}{4} \text{ and } \frac{2-3x}{4} \le 9.$$

Now, $-5 \le \dfrac{2-3x}{4} \Rightarrow -20 \le 2-3x$ [multiplying both sides by 4]

$\Rightarrow -22 \le -3x$ [adding -2 on both sides]

$\Rightarrow 22 \ge 3x$ [multiplying both sides by -1]

$\Rightarrow 3x \le 22$

$\Rightarrow x \le \dfrac{22}{3}$ [dividing both sides by 3]

$\Rightarrow x \in \left(-\infty, \dfrac{22}{3}\right]$.

And, $\dfrac{2-3x}{4} \le 9 \Rightarrow 2-3x \le 36$ [multiplying both sides by 4]

$\Rightarrow -3x \le 34$

$\Rightarrow x \ge \dfrac{-34}{3}$ [dividing both sides by -3]

$\Rightarrow x \in \left[\dfrac{-34}{3}, \infty\right)$.

$\therefore$ solution set $=\left(-\infty, \dfrac{22}{3}\right] \cap \left[\dfrac{-34}{3}, \infty\right)=\left[\dfrac{-34}{3}, \dfrac{22}{3}\right]$.

EXAMPLE 19 *Solve the system of inequations:*

$$\frac{6x}{4x-1}<\frac{1}{2},\ \frac{x}{2x+1} \ge \frac{1}{4}.$$

SOLUTION The first inequation of the given system is

$$\frac{6x}{4x-1}<\frac{1}{2}$$

$$\Rightarrow \quad \frac{6x}{4x-1}-\frac{1}{2}<0$$

$$\Rightarrow \quad \frac{12x-4x+1}{2(4x-1)}<0$$

$$\Rightarrow \quad \frac{8x+1}{2(4x-1)}<0 \Rightarrow \frac{8x+1}{4x-1}<0. \qquad \dots \text{(i)}$$

$$\therefore \quad (8x+1>0 \text{ and } 4x-1<0) \text{ or } (8x+1<0 \text{ and } 4x-1>0)$$

$$\Rightarrow \quad (8x>-1 \text{ and } 4x<1) \text{ or } (8x<-1 \text{ and } 4x>1)$$

$$\Rightarrow \quad \left(x>\frac{-1}{8} \text{ and } x<\frac{1}{4}\right) \text{ or } \left(x<\frac{-1}{8} \text{ and } x>\frac{1}{4}\right)$$

$$\Rightarrow \quad \frac{-1}{8}<x<\frac{1}{4} \qquad \left[\because\ x<\frac{-1}{8} \text{ and } x>\frac{1}{4} \text{ is not possible}\right]$$

$$\Rightarrow \quad x \in \left(\frac{-1}{8}, \frac{1}{4}\right).$$

The second inequation of the given system is

$$\frac{x}{2x+1} \geq \frac{1}{4}$$

$$\Rightarrow \quad \frac{x}{2x+1}-\frac{1}{4} \geq 0$$

$$\Rightarrow \quad \frac{4x-2x-1}{4(2x+1)} \geq 0 \Rightarrow \frac{2x-1}{2x+1} \geq 0. \qquad \dots \text{(ii)}$$

$$\therefore \quad (2x-1 \leq 0 \text{ and } 2x+1<0) \text{ or } (2x-1 \geq 0 \text{ and } 2x+1>0)$$

$$\Rightarrow \quad (2x \leq 1 \text{ and } 2x<-1) \text{ or } (2x \geq 1 \text{ and } 2x>-1)$$

$$\Rightarrow \quad \left(x \leq \frac{1}{2} \text{ and } x<\frac{-1}{2}\right) \text{ or } \left(x \geq \frac{1}{2} \text{ and } x>\frac{-1}{2}\right)$$

$$\Rightarrow \quad \left(x<\frac{-1}{2}\right) \text{ or } \left(x \geq \frac{1}{2}\right)$$

$$\Rightarrow \quad x \in \left(-\infty, \frac{-1}{2}\right) \text{ or } x \in \left[\frac{1}{2}, \infty\right)$$

$$\Rightarrow \quad x \in \left(-\infty, \frac{-1}{2}\right) \cup \left[\frac{1}{2}, \infty\right)$$

$$\therefore \quad \text{solution set} = \left(\frac{-1}{8}, \frac{1}{4}\right) \cap \left\{\left(-\infty, \frac{-1}{2}\right) \cup \left[\frac{1}{2}, \infty\right)\right\} = \phi.$$

Hence, the given system of inequations has no solution.

EXAMPLE 20 *Solve the system of inequations:* $|x-1| \leq 5,\ |x| \geq 2.$

SOLUTION The first inequation is $|x-1| \leq 5$.

Using $|x| \leq a \Leftrightarrow -a \leq x \leq a$, we get

$$|x-1| \leq 5 \Rightarrow -5 \leq x-1 \leq 5$$

$$\Rightarrow -4 \leq x \leq 6 \Rightarrow x \in [-4, 6].$$

The second inequation is $|x| \geq 2$.

Using $|x| \geq a \Leftrightarrow x \leq -a$ or $x \geq a$, we get

$|x| \geq 2 \Rightarrow x \leq -2$ or $x \geq 2$

$\Rightarrow x \in (-\infty, -2] \cup [2, \infty)$.

$\therefore$ solution set for the given system is

$$\{(-\infty, -2] \cup [2, \infty)\} \cap [-4, 6] = [-4, -2] \cup [2, 6].$$

WORD PROBLEMS

EXAMPLE 1 *Find all pairs of consecutive odd positive integers both of which are smaller than* 10 *such that their sum is more than* 11.

SOLUTION Let the required consecutive odd positive integers be x and $x+2$. Then,

$x + 2 < 10$ and $x + (x+2) > 11$

$\Rightarrow$ $x < 8$ and $2x + 2 > 11$

$\Rightarrow$ $x < 8$ and $2x > 9$

$\Rightarrow$ $x < 8$ and $x > \dfrac{9}{2}$

$\Rightarrow$ $\dfrac{9}{2} < x < 8 \Rightarrow 4.5 < x < 8$.

$\therefore$ x can take the odd integral values 5 and 7.

Hence, the required pairs of odd integers are (5, 7) and (7, 9).

EXAMPLE 2 *Find all pairs of consecutive even positive integers both of which are larger than* 5 *such that their sum is less than* 23.

SOLUTION Let the required consecutive even positive integers be x and $x+2$. Then,

$x > 5$ and $x + (x+2) < 23$

$\Rightarrow$ $x > 5$ and $2x < 21$

$\Rightarrow$ $5 < x$ and $x < 10.5$

$\Rightarrow$ $5 < x < 10.5$.

$\therefore$ x can take the even integral values 6, 8 and 10.

Hence, the required pairs of even integers are (6, 8), (8, 10) and (10, 12).

EXAMPLE 3 *The longest side of a triangle is* 3 *times the shortest side and the third side is* 2 cm *shorter than the longest side. If the perimeter of the triangle is at least* 61 cm, *find the minimum length of the shortest side.*

SOLUTION Let the minimum length of the shortest side be x cm.

Then, the longest side = $3x$ cm and the third side = $(3x - 2)$ cm.

$\therefore$ $x + 3x + (3x - 2) \geq 61 \Rightarrow 7x \geq 63 \Rightarrow x \geq 9$.

Hence, the minimum length of the shortest side is 9 cm.

EXAMPLE 4 *The cost and revenue functions of a product are given by* $C(x) = 20x + 4000$ *and* $R(x) = 60x + 2000$ *respectively, where* x *is the number of items produced and sold. How many items must be sold to realise some profit?*

SOLUTION Let the required number of items sold be x and let $P(x)$ be the profit function. Then,

profit function = (revenue function) – (cost function)

$\Rightarrow \quad P(x) = (60x + 2000) - (20x + 4000)$

$\Rightarrow \quad P(x) = 40x - 2000.$

For some profit, we have $P(x) > 0$.

$\therefore \quad 40x - 2000 > 0$

$\Rightarrow \quad 40x > 2000$

$\Rightarrow \quad x > 50.$

Hence, the manufacturer must sell more than 50 items to realise some profit.

EXAMPLE 5 *A man wants to cut three lengths from a single piece of cloth of length* 91 cm. *The second length is to be* 3 cm *longer than the shortest and the third length is to be twice as long as the shortest. What are the possible lengths of the shortest piece of cloth if the third piece is to be at least* 5 cm *longer than the second.*

SOLUTION Let the length of the shortest piece be x cm. Then, second length $= (x + 3)$ cm and third length $= 2x$ cm.

$\therefore \quad x + (x + 3) + 2x \leq 91$ and $2x \geq (x + 3) + 5$

$\Rightarrow \quad 4x + 3 \leq 91$ and $2x \geq x + 8$

$\Rightarrow \quad 4x \leq 91 - 3$ and $2x - x \geq 8$

$\Rightarrow \quad 4x \leq 88$ and $x \geq 8$

$\Rightarrow \quad x \leq 22$ and $x \geq 8$

$\Rightarrow \quad 8 \leq x \leq 22.$

Hence, the length of the shortest piece is to be greater than or equal to 8 but less than or equal to 22.

EXAMPLE 6 *In drilling world's deepest hole it was found that the temperature* $T(x)$ *in degree Celsius,* x km *below the earth's surface was given by* $T(x) = 30 + 25(x - 3)$, $3 \leq x \leq 15$. *At what depth will the temperature be between* 155°C *and* 205°C*?*

SOLUTION We have,

$T(x) = 30 + 25(x - 3)$

$\Rightarrow \quad T(x) = 30 + 25x - 75$

$\Rightarrow \quad T(x) = 25x - 45. \qquad \ldots \text{(i)}$

Let x km below the earth's surface, the temperature lies between 155°C and 205°C. Then,

$155 < T(x) < 205$

$\Rightarrow \quad 155 < 25x - 45 < 205$

$\Rightarrow \quad 155 < 25x - 45$ and $25x - 45 < 205$

$\Rightarrow \quad 155 + 45 < 25x$ and $25x < 205 + 45$

$\Rightarrow \quad 200 < 25x$ and $25x < 250$

$\Rightarrow \quad 8 < x$ and $x < 10$

$\Rightarrow \quad 8 < x < 10.$

Clearly, $8 < x < 10$ lies in the range $3 \le x \le 15$.

Hence, the required depth is more than 8 km and less than 10 km.

EXAMPLE 7 *A manufacturer has 460 litres of a 9% acid solution. How many litres of a 3% acid solution must be added to it so that the acid content in the resulting mixture be more than 5% but less than 7%?*

SOLUTION Let x litres of a 3% acid solution be added to 460 litres of 9% acid solution. Then,

total quantity of mixture $= (460 + x)$ litres.

Total acid content in $(460 + x)$ litres of mixture

$$= \left\{\left(460 \times \frac{9}{100}\right) + \left(x \times \frac{3}{100}\right)\right\} \text{ litres} = \left(\frac{207}{5} + \frac{3x}{100}\right) \text{ litres.}$$

Now, the acid content in the resulting mixture must be more than 5% and less than 7%.

$\therefore \quad 5\%$ of $(460 + x) < \left(\frac{207}{5} + \frac{3x}{100}\right) < 7\%$ of $(460 + x)$

$\Rightarrow \quad \frac{5}{100} \times (460 + x) < \frac{4140 + 3x}{100} < \frac{7}{100} \times (460 + x)$

$\Rightarrow \quad 5(460 + x) < 4140 + 3x < 7(460 + x)$

$\Rightarrow \quad 2300 + 5x < 4140 + 3x < 3220 + 7x$

$\Rightarrow \quad 2300 + 5x < 4140 + 3x$ and $4140 + 3x < 3220 + 7x$

$\Rightarrow \quad 5x - 3x < 4140 - 2300$ and $4140 - 3220 < 7x - 3x$

$\Rightarrow \quad 2x < 1840$ and $920 < 4x$

$\Rightarrow \quad x < 920$ and $230 < x$

$\Rightarrow \quad 230 < x < 920.$

Hence, the required quantity of 3% acid solution to be added must be more than 230 litres and less than 920 litres.

EXAMPLE 8 *A solution is to be kept between 40°C and 45°C. What is the range of temperature in degree Fahrenheit, if the conversion formula is* $F = \frac{9}{5}C + 32$?

SOLUTION $F = \frac{9}{5}C + 32 \Rightarrow C = \frac{5}{9}(F - 32).$ … (i)

$\therefore \quad 40 < C < 45$

$\Rightarrow \quad 40 < \frac{5}{9}(F - 32) < 45$

$\Rightarrow \quad 40 < \frac{5}{9}(F - 32)$ and $\frac{5}{9}(F - 32) < 45$

$\Rightarrow \quad 40 \times \frac{9}{5} < F - 32$ and $F - 32 < 45 \times \frac{9}{5}$

$\Rightarrow \quad 72 < F - 32$ and $F - 32 < 81$

$\Rightarrow \quad 72 + 32 < F$ and $F < 81 + 32$

$\Rightarrow \quad 104 < F$ and $F < 113$

$\Rightarrow \quad 104 < F < 113.$

Hence, the solution is to be kept between 104°F and 113°F.

EXAMPLE 9 *The IQ of a person is given by the formula,* $IQ = \frac{m}{c} \times 100$, *where m is the mental age and c is the chronological age. If* $80 \le IQ \le 140$ *for a group of 12-year children, find the range of their mental age.*

SOLUTION When $c = 12$, we have $IQ = \left(\frac{m}{12} \times 100\right) = \frac{25m}{3}$.

$$\therefore \quad 80 \le IQ \le 140 \Rightarrow 80 \le \frac{25m}{3} \le 140$$

$$\Rightarrow 80 \le \frac{25m}{3} \text{ and } \frac{25m}{3} \le 140$$

$$\Rightarrow \frac{3}{25} \times 80 \le m \text{ and } m \le 140 \times \frac{3}{25}$$

$$\Rightarrow \frac{48}{5} \le m \text{ and } m \le \frac{84}{5}$$

$$\Rightarrow 9.6 \le m \le 16.8.$$

Hence, the required mental age for a group of 12-year children is 9.6 years or more and 16.8 years or less.

EXERCISE 6A

1. Fill in the blanks with correct inequality sign ($>, <, \ge, \le$).

(i) $5x < 20 \Rightarrow x \ldots\ldots 4$

(ii) $-3x > 9 \Rightarrow x \ldots\ldots -3$

(iii) $4x > -16 \Rightarrow x \ldots\ldots -4$

(iv) $-6x \le -18 \Rightarrow x \ldots\ldots 3$

(v) $x > -3 \Rightarrow -2x \ldots\ldots 6$

(vi) $a < b$ and $c < 0 \Rightarrow \frac{a}{c} \ldots\ldots \frac{b}{c}$

(vii) $p - q = -3 \Rightarrow p \ldots\ldots q$

(viii) $u - v = 2 \Rightarrow u \ldots\ldots v$

Solve each of the following inequations and represent the solution set on the number line.

2. $6x \le 25$, where (i) $x \in N$, (ii) $x \in Z$.

3. $-2x > 5$, where (i) $x \in Z$, (ii) $x \in R$.

4. $3x + 8 > 2$, where (i) $x \in Z$, (ii) $x \in R$.

5. $5x + 2 < 17$, where (i) $x \in Z$, (ii) $x \in R$.

6. $3x-4>x+6$, where $x\in R$.

7. $3-2x\geq 4x-9$, where $x\in R$.

8. $\frac{5x-8}{3}\geq\frac{4x-7}{2}$, where $x\in R$.

9. $\frac{5x}{4}-\frac{4x-1}{3}>1$, where $x\in R$.

10. $\frac{1}{4}\left(\frac{2}{3}x+1\right)\geq\frac{1}{3}(x-2)$, where $x\in R$.

11. $\frac{2x-1}{12}-\frac{x-1}{3}<\frac{3x+1}{4}$, where $x\in R$.

12. $\frac{x}{4}<\frac{(5x-2)}{3}-\frac{(7x-3)}{5}$, where $x\in R$.

13. $\frac{(2x-1)}{3}\geq\frac{(3x-2)}{4}-\frac{(2-x)}{5}$, where $x\in R$.

Solve:

14. $\frac{x-3}{x+1}<0,\ x\in R$

15. $\frac{x-3}{x+4}>0,\ x\in R$

16. $\frac{2x-3}{3x-7}>0,\ x\in R$

17. $\frac{x-7}{x-2}\geq 0,\ x\in R$

18. $\frac{3}{x-2}>2,\ x\in R$

19. $\frac{1}{x-1}\leq 2,\ x\in R$

20. $\frac{5x+8}{4-x}<2,\ x\in R$

21. $|3x-7|>4,\ x\in R$

22. $|5-2x|\leq 3,\ x\in R$

23. $|4x-5|\leq\frac{1}{3},\ x\in R$

24. $\frac{1}{|x|-3}\leq\frac{1}{2},\ x\in R$

25. $\frac{|x+2|-x}{x}<2,\ x\in R$

26. $\left|\frac{2x-1}{x-1}\right|>2,\ x\in R$

27. $\frac{|x-3|}{x-3}>0,\ x\in R$

28. $\frac{|x|-1}{|x|-2}\geq 0,\ x\in R-\{-2,2\}$

29. $\frac{1}{2-|x|}\geq 1,\ x\in R-\{-2,2\}$

30. $|x+1|+|x|>3,\ x\in R$

31. $\left|\frac{2}{x-4}\right|>1,\ x\neq 4$

Solve the following systems of linear inequations:

32. $\frac{4}{x+1}\leq 3\leq\frac{6}{x+1},\ x>0$

33. $-11\leq 4x-3\leq 13$

34. $5x-7<3(x+3),\ 1-\frac{3x}{2}\geq x-4$

35. $-2<\frac{6-5x}{4}<7$

36. $3x-2>x+\frac{4-x}{3}>3$

37. $\frac{7x-1}{2}<-3,\ \frac{3x+8}{5}+11<0$

38. $-12<4-\frac{3x}{-5}\leq 2$

39. $1\leq|x-2|\leq 3$

40. Find all pairs of consecutive odd positive integers, both of which are smaller than 18 such that their sum is more than 20.

41. Find all pairs of consecutive even positive integers both of which are larger than 8 such that their sum is less than 25.

42. A company manufactures cassettes. Its cost and revenue functions are $C(x) = 26000 + 30x$ and $R(x) = 43x$ respectively, where x is the number of cassettes produced and sold in a week. How many cassettes must be sold by the company to realise some profit?

43. The water acidity in a pool is considered normal when the average pH reading of three daily measurements is between 8.2 and 8.5. If the first two pH readings are 8.48 and 8.35, find the range of the pH values for the third reading that will result in the acidity level being normal.

44. A manufacturer has 640 litres of a 8% solution of boric acid. How many litres of a 2% boric acid solution be added to it so that the boric acid content in the resulting mixture will be more than 4% but less than 6%?

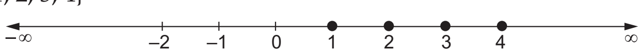
45. How many litres of water will have to be added to 600 litres of the 45% solution of acid so that the resulting mixture will contain more than 25%, but less than 30% acid content?

46. To receive grade A in a course one must obtain an average of 90 marks or more in five papers, each of 100 marks. If Tanvy scored 89, 93, 95 and 91 marks in first four papers, find the minimum marks that she must score in the last paper to get grade A in the course.

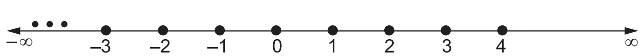
ANSWERS (EXERCISE 6A)

1. (i) $<$ (ii) $<$ (iii) $>$ (iv) $\geq$ (v) $<$ (vi) $>$ (vii) $<$ (viii) $>$

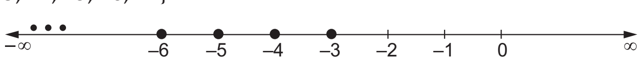
2. (i) {1, 2, 3, 4}



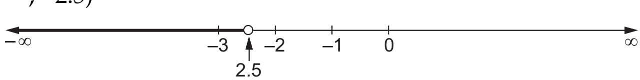
(ii) {…, –3, –2, –1, 0, 1, 2, 3, 4}



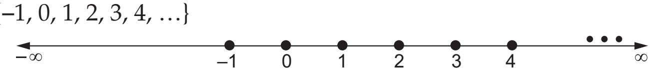
3. (i) {–3, –4, –5, –6, …}



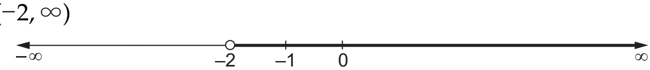
(ii) $(-\infty, -2.5)$



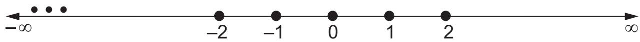
4. (i) {–1, 0, 1, 2, 3, 4, …}



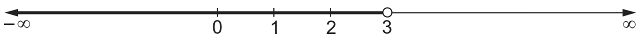
(ii) $(-2, \infty)$



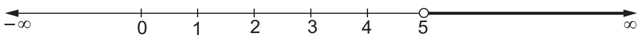
5. (i) $\{2, 1, 0, -1, -2, \ldots\}$



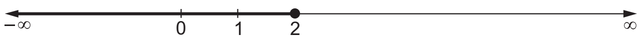
(ii) $(-\infty, 3)$



6. $(5, \infty)$

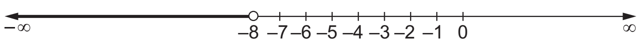


7. $(-\infty, 2]$

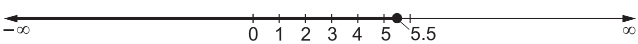


8. $\left(-\infty, \frac{5}{2}\right]$

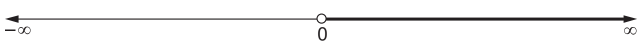
9. $(-\infty, -8)$



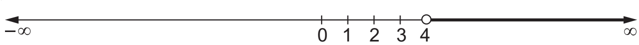
10. $(-\infty, 5.5]$



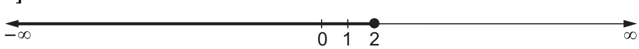
11. $(0, \infty)$



12. $(4, \infty)$



13. $(-\infty, 2]$



14. $(-1, 3)$ **15.** $(-\infty, -4) \cup (3, \infty)$ **16.** $\left(-\infty, \frac{3}{2}\right) \cup \left(\frac{7}{3}, \infty\right)$

17. $(-\infty, 2) \cup [7, \infty)$ **18.** $\left(2, \frac{7}{2}\right)$ **19.** $(-\infty, 1) \cup \left[\frac{3}{2}, \infty\right)$

20. $(-\infty, 0) \cup (4, \infty)$ **21.** $(-\infty, 1) \cup \left(\frac{11}{3}, \infty\right)$ **22.** $[1, 4]$

23. $\left[\frac{7}{6}, \frac{4}{3}\right]$ **24.** $(-\infty, -5] \cup (-3, 3) \cup [5, \infty)$

25. $(-\infty, -2) \cup (1, \infty)$ **26.** $\left(\frac{3}{4}, 1\right) \cup (1, \infty)$ **27.** $(3, \infty)$

28. $[-1, 1] \cup (-\infty, -2) \cup (2, \infty)$ **29.** $(-2, -1] \cup [1, 2)$

30. $(-\infty, -2) \cup (1, \infty)$ **31.** $(2, 4) \cup (4, 6)$ **32.** $\frac{1}{3} \leq x \leq 1$

33. $-2 \leq x \leq 4$ **34.** $-\infty < x \leq 2$ **35.** $-4.4 < x < 2.8$

36. $\frac{5}{2} < x < \infty$ **37.** $-\infty < x < -21$ **38.** $\frac{-80}{3} < x \le \frac{-10}{3}$

39. $[-1, 1] \cup [3, 5]$ **40.** (11, 13), (13, 15), (15, 17)

41. (10, 12) **42.** More than 2000 **43.** Between 7.77 and 8.67

44. More than 320 litres and less than 1280 litres

45. More than 300 litres but less than 480 litres **46.** 82 marks

HINTS TO SOME SELECTED QUESTIONS

1. (ii) $-3x > 9 \Rightarrow x < -3$. (iii) $4x > -16 \Rightarrow x > -4$.

(iv) $-6x \le -18 \Rightarrow x \ge 3$. (v) $x > -3 \Rightarrow -2x < 6$.

(vi) $a < b$ and $c < 0 \Rightarrow \frac{a}{c} > \frac{b}{c}$. (vii) $p - q = -3 \Rightarrow p < q$.

3. $-2x > 5 \Rightarrow x < \frac{-5}{2} \Rightarrow x < -2.5$.

(i) All integers less than -2.5 are $-3, -4, -5, -6, \ldots$.

(ii) Set of all real numbers less than -2.5 are $(-\infty, -2.5)$.

4. $3x > -6 \Rightarrow x > -2$.

(i) All integers greater than -2 are $-1, 0, 1, 2, 3, 4, \ldots$.

(ii) Set of all real numbers greater than -2 is $(-2, \infty)$.

7. $3 - 2x \ge 4x - 9 \Rightarrow -6x \ge -12 \Rightarrow x \le 2$.

$\therefore$ solution set $= (-\infty, 2]$.

8. $10x - 16 \ge 12x - 21 \Rightarrow -2x \ge -5 \Rightarrow x \le \frac{5}{2}$.

$\therefore$ solution set $= \left(-\infty, \frac{5}{2}\right]$.

9. $15x - 4(4x - 1) > 12 \Rightarrow 15x - 16x + 4 > 12 \Rightarrow -x > 8 \Rightarrow x < -8$.

$\therefore$ solution set $= (-\infty, -8)$.

10. $\frac{1}{4} \cdot \frac{(2x+3)}{3} \ge \frac{x-2}{3} \Rightarrow 2x + 3 \ge 4x - 8 \Rightarrow -2x \ge -11 \Rightarrow x \le \frac{11}{2}$.

$\therefore$ solution set $= (-\infty, 5.5]$.

11. $(2x - 1) - 4(x - 1) < 3(3x + 1) \Rightarrow -2x + 3 < 9x + 3 \Rightarrow -11x < 0 \Rightarrow x > 0$.

$\therefore$ solution set $= (0, \infty)$.

12. $15x < 20(5x - 2) - 12(7x - 3) \Rightarrow 15x < 100x - 40 - 84x + 36$

$\Rightarrow 15x - 16x < -4 \Rightarrow -x < -4 \Rightarrow x > 4$.

$\therefore$ solution set $= (4, \infty)$.

13. $20(2x - 1) \ge 15(3x - 2) - 12(2 - x) \Rightarrow 40x - 20 \ge 45x - 30 - 24 + 12x$

$\Rightarrow -17x \ge -34 \Rightarrow x \le 2$.

$\therefore$ solution set $= (-\infty, 2]$.

14. $(x - 3 < 0$ and $x + 1 > 0)$ or $(x - 3 > 0$ and $x + 1 < 0)$

$\Rightarrow$ $(x < 3$ and $x > -1)$ or $(x > 3$ and $x < -1)$

$\Rightarrow$ $-1 < x < 3 \Rightarrow x \in (-1, 3)$. [$x \ge 3$ and $x < -1$ is not possible]

15. $(x-3<0 \text{ and } x+4<0)$ or $(x-3>0 \text{ and } x+4>0)$

$\Rightarrow \quad (x<3 \text{ and } x<-4)$ or $(x>3 \text{ and } x>-4)$

$\Rightarrow \quad (x<-4)$ or $(x>3) \Rightarrow x \in (-\infty, -4) \cup (3, \infty)$.

16. $(2x-3<0 \text{ and } 3x-7<0)$ or $(2x-3>0 \text{ and } 3x-7>0)$

$$\Rightarrow \quad \left(x<\frac{3}{2} \text{ and } x<\frac{7}{3}\right) \text{ or } \left(x>\frac{3}{2} \text{ and } x>\frac{7}{3}\right)$$

$$\Rightarrow \quad \left(x<\frac{3}{2}\right) \text{ or } \left(x>\frac{7}{3}\right) \Rightarrow x \in \left(-\infty, \frac{3}{2}\right) \cup \left(\frac{7}{3}, \infty\right).$$

17. $(x-7 \le 0 \text{ and } x-2<0)$ or $(x-7 \ge 0 \text{ and } x-2>0)$

$\Rightarrow \quad (x \le 7 \text{ and } x<2)$ or $(x \ge 7 \text{ and } x>2)$

$\Rightarrow \quad (x<2)$ or $(x \ge 7) \Rightarrow x \in (-\infty, 2) \cup [7, \infty)$.

18. $\dfrac{3}{x-2}-2>0 \Rightarrow \dfrac{3-2x+4}{x-2}>0 \Rightarrow \dfrac{7-2x}{x-2}>0$

$\Rightarrow \quad (7-2x<0 \text{ and } x-2<0)$ or $(7-2x>0 \text{ and } x-2>0)$

$\Rightarrow \quad (7<2x \text{ and } x<2)$ or $(7>2x \text{ and } x>2)$

$$\Rightarrow \quad \left(x>\frac{7}{2} \text{ and } x<2\right) \text{ or } \left(x<\frac{7}{2} \text{ and } x>2\right)$$

$$\Rightarrow \quad 2<x<\frac{7}{2} \Rightarrow x \in \left(2, \frac{7}{2}\right). \qquad \left[x<\frac{7}{2} \text{ and } x>2 \text{ cannot hold}\right]$$

19. $\dfrac{1}{x-1}-2 \le 0 \Rightarrow \dfrac{1-2x+2}{x-1} \le 0 \Rightarrow \dfrac{3-2x}{x-1} \le 0$

$\Rightarrow \quad (3-2x \le 0 \text{ and } x-1>0)$ or $(3-2x \ge 0 \text{ and } x-1<0)$

$\Rightarrow \quad (3 \le 2x \text{ and } x>1)$ or $(-2x \ge -3 \text{ and } x<1)$

$$\Rightarrow \quad \left(x \ge \frac{3}{2} \text{ and } x>1\right) \text{ or } \left(x \le \frac{3}{2} \text{ and } x<1\right)$$

$$\Rightarrow \quad (x<1) \text{ or } \left(x \ge \frac{3}{2}\right) = x \in (-\infty, 1) \cup \left[\frac{3}{2}, \infty\right).$$

20. $\dfrac{5x+8}{4-x}-2<0 \Rightarrow \dfrac{5x+8-8+2x}{4-x}<0 \Rightarrow \dfrac{7x}{4-x}<0$

$\Rightarrow \quad (7x<0 \text{ and } 4-x>0)$ or $(7x>0 \text{ and } 4-x<0)$

$\Rightarrow \quad (x<0 \text{ and } 4>x)$ or $(x>0 \text{ and } -x<-4)$

$\Rightarrow \quad (x<0 \text{ and } x<4)$ or $(x>0 \text{ and } x>4)$

$\Rightarrow \quad (x<0)$ or $(x>4) \Rightarrow x \in (-\infty, 0) \cup (4, \infty)$.

21. Using $|x|>a \Rightarrow x<-a$ or $x>a$, we get

$$|3x-7|>4 \Rightarrow 3x-7<-4 \text{ or } 3x-7>4$$

$$\Rightarrow 3x<3 \text{ or } 3x>11 \Rightarrow x<1 \text{ or } x>\frac{11}{3}$$

$$\Rightarrow x \in (-\infty, 1) \cup \left(\frac{11}{3}, \infty\right).$$

22. Using $|x| \le a \Leftrightarrow -a \le x \le a$, we get

$$|5-2x| \le 3 \Rightarrow -3 \le 5-2x \le 3$$

$$\Rightarrow -3 \le 5-2x \text{ and } 5-2x \le 3$$

$$\Rightarrow 2x \le 8 \text{ and } 2x \ge 2 \Rightarrow x \le 4 \text{ and } x \ge 1$$

$$\Rightarrow 1 \le x \le 4 \Rightarrow x \in [1, 4].$$

23. Using $|x| \le a \Leftrightarrow -a \le x \le a$, we get

$$|4x-5| \le \frac{1}{3} \Rightarrow -\frac{1}{3} \le 4x-5 \le \frac{1}{3} \Rightarrow \frac{14}{3} \le 4x \text{ and } 4x \le \frac{16}{3}$$

$$\Rightarrow \frac{7}{6} \le x \text{ and } x \le \frac{4}{3} \Rightarrow \frac{7}{6} \le x \le \frac{4}{3} \Rightarrow x \in \left[\frac{7}{6}, \frac{4}{3}\right].$$

24. Case I *When* $x \ge 0$.

Then, $|x| = x$.

$\therefore \quad \frac{1}{|x|-3} \le \frac{1}{2} \Rightarrow \frac{1}{x-3} - \frac{1}{2} \le 0 \Rightarrow \frac{2-x+3}{2(x-3)} \le 0 \Rightarrow \frac{5-x}{x-3} \le 0.$

$\therefore \quad (5-x \le 0 \text{ and } x-3 > 0)$ or $(5-x \ge 0 \text{ and } x-3 < 0)$

$\Rightarrow \quad (x \ge 5 \text{ and } x > 3)$ or $(x \le 5 \text{ and } x < 3)$

$\Rightarrow \quad (x \ge 5)$ or $(x < 3)$

$\Rightarrow \quad (0 \le x < 3)$ or $(x \ge 5)$ $\quad [\because x \ge 0]$

$\Rightarrow \quad x \in [0, 3) \cup [5, \infty)$.

Case II *When* $x < 0$.

Then, $|x| = -x$.

$\therefore \quad \frac{1}{|x|-3} \le \frac{1}{2} \Rightarrow \frac{1}{-x-3} \le \frac{1}{2} \Rightarrow \frac{-1}{x+3} \le \frac{1}{2} \Rightarrow \frac{1}{2} + \frac{1}{x+3} \ge 0$

$$\Rightarrow \frac{x+3+2}{x+3} \ge 0 \Rightarrow \frac{x+5}{x+3} \ge 0.$$

$\therefore \quad (x+5 \ge 0 \text{ and } x+3 > 0)$ or $(x+5 \le 0 \text{ and } x+3 < 0)$

$\Rightarrow \quad (x \ge -5 \text{ and } x > -3)$ or $(x \le -5 \text{ and } x < -3)$

$\Rightarrow \quad (x > -3)$ or $(x \le -5)$.

But, $x < 0$.

$\therefore \quad (-3 < x < 0)$ or $(x \le -5) \Rightarrow x \in (-3, 0) \cup (-\infty, -5]$.

Hence, $x \in (-\infty, -5] \cup (-3, 0) \cup [0, 3) \cup [5, \infty)$

$\Rightarrow \quad x \in (-\infty, -5] \cup (-3, 3) \cup [5, \infty)$.

25. $\frac{|x+2|-x}{x} - 2 < 0 \Rightarrow \frac{|x+2|-x-2x}{x} < 0 \Rightarrow \frac{|x+2|-3x}{x} < 0.$

Case I *When* $x+2 \ge 0$.

Then, $x \ge -2$ and $|x+2| = x+2$.

$\therefore \quad \frac{|x+2|-3x}{x} < 0 \Rightarrow \frac{x+2-3x}{x} < 0 \Rightarrow \frac{2-2x}{x} < 0$

$$\Rightarrow \frac{2}{x} - 2 < 0 \Rightarrow \frac{2}{x} < 2 \Rightarrow 2x > 2 \Rightarrow x > 1.$$

$\therefore \quad (x \ge -2 \text{ and } x > 1) \Rightarrow x > 1 \Rightarrow x \in (1, \infty)$.

Case II *When* $x+2 < 0$.

Then, $x < -2$ and $|x+2| = -(x+2)$.

$\therefore \quad \frac{|x+2|-3x}{x} < 0 \Rightarrow \frac{-x-2-3x}{x} < 0 \Rightarrow \frac{-4x-2}{x} < 0$

$$\Rightarrow \frac{4x+2}{x} > 0 \Rightarrow 4 + \frac{2}{x} > 0 \Rightarrow \frac{2}{x} > -4$$

$$\Rightarrow -4x < 2 \Rightarrow x < \frac{-1}{2}.$$

$\therefore \quad x < -2 \text{ and } x < \frac{-1}{2} \Rightarrow x < -2 \Rightarrow x \in (-\infty, -2)$.

Hence, solution set $= (-\infty, -2) \cup (1, \infty)$.

26. Using $|x| > a \Leftrightarrow x < -a$ or $x > a$, we have

$$\left|\frac{2x-1}{x-1}\right| > 2 \Leftrightarrow \frac{2x-1}{x-1} < -2 \text{ or } \frac{2x-1}{x-1} > 2.$$

Case I *When* $\frac{2x-1}{x-1} < -2$.

Then, $\frac{2x-1}{x-1} + 2 < 0 \Rightarrow \frac{2x-1+2x-2}{x-1} < 0 \Rightarrow \frac{4x-3}{x-1} < 0.$

$\therefore$ $(4x-3 < 0 \text{ and } x-1 > 0)$ or $(4x-3 > 0 \text{ and } x-1 < 0)$

$\Rightarrow$ $\left(x < \frac{3}{4} \text{ and } x > 1\right)$ or $\left(x > \frac{3}{4} \text{ and } x < 1\right)$

$\Rightarrow$ $\frac{3}{4} < x < 1 \Rightarrow x \in \left(\frac{3}{4}, 1\right)$.

Case II *When* $\frac{2x-1}{x-1} > 2$.

Then, $\frac{2x-1}{x-1} - 2 > 0 \Rightarrow \frac{2x-1-2x+2}{x-1} > 0 \Rightarrow \frac{1}{x-1} > 0$

$\Rightarrow x-1 > 0 \Rightarrow x > 1 \Rightarrow x \in (1, \infty)$.

$\therefore$ solution set $= \left(\frac{3}{4}, 1\right) \cup (1, \infty)$.

27. $\frac{|x-3|}{x-3} > 0 \Leftrightarrow x-3 > 0 \Leftrightarrow x > 3 \Leftrightarrow x \in (3, \infty)$.

28. Case I *When* $x \geq 0$.

Then, $|x| = x$ and so $\frac{|x|-1}{|x|-2} \geq 0 \Rightarrow \frac{x-1}{x-2} \geq 0.$

$\therefore$ $(x-1 \geq 0 \text{ and } x-2 > 0)$ or $(x-1 \leq 0 \text{ and } x-2 < 0)$

$\Rightarrow$ $(x \geq 1 \text{ and } x > 2)$ or $(x \leq 1 \text{ and } x < 2)$

$\Rightarrow$ $(x \leq 1)$ or $(x > 2) \Rightarrow (0 \leq x \leq 1)$ or $(x > 2)$ $[\because x \geq 0]$

$\Rightarrow$ $x \in [0, 1] \cup (2, \infty)$.

Case II *When* $x < 0$.

Then, $|x| = -x$ and so $\frac{|x|-1}{|x|-2} \geq 0 \Rightarrow \frac{-x-1}{-x-2} \geq 0 \Rightarrow \frac{x+1}{x+2} \geq 0.$

$\therefore$ $(x+1 \geq 0 \text{ and } x+2 > 0)$ or $(x+1 \leq 0 \text{ and } x+2 < 0)$

$\Rightarrow$ $(x \geq -1 \text{ and } x > -2)$ or $(x \leq -1 \text{ and } x < -2)$

$\Rightarrow$ $(x \geq -1)$ or $(x < -2)$

$\Rightarrow$ $(-1 \leq x < 0)$ or $(x < -2) \Rightarrow x \in [-1, 0) \cup (-\infty, -2)$. $[\because x < 0]$

$\therefore$ solution set $= [-1, 0) \cup [0, 1] \cup (-\infty, -2) \cup (2, \infty)$

$= [-1, 1] \cup (-\infty, -2) \cup (2, \infty)$.

29. Case I *When* $x \geq 0$.

Then, $|x| = x$ and so $\frac{1}{2-|x|} \geq 1 \Rightarrow \frac{1}{2-x} - 1 \geq 0 \Rightarrow \frac{x-1}{2-x} \geq 0.$

$\therefore$ $(x-1 \geq 0 \text{ and } 2-x > 0)$ or $(x-1 \leq 0 \text{ and } 2-x < 0)$

$\Rightarrow$ $(x \geq 1 \text{ and } x < 2)$ or $(x \leq 1 \text{ and } x > 2)$

$\Rightarrow$ $(1 \leq x < 2) \Rightarrow x \in [1, 2)$.

Case II *When* $x < 0$.

Then, $|x| = -x$. So, $\frac{1}{2-|x|} \geq 1 \Rightarrow \frac{1}{2+x} - 1 \geq 0 \Rightarrow \frac{-1-x}{2+x} \geq 0 \Rightarrow \frac{1+x}{2+x} \leq 0$

$\Rightarrow \quad (1+x \le 0 \text{ and } 2+x > 0) \text{ or } (1+x \ge 0 \text{ and } 2+x < 0)$

$\Rightarrow \quad (x \le -1 \text{ and } x > -2) \text{ or } (x \ge -1 \text{ and } x < -2)$

$\Rightarrow \quad (-2 < x \le -1) \Rightarrow x \in (-2, -1]$.

$\therefore$ solution set $= (-2, -1] \cup [1, 2)$.

30. Putting $x+1=0$ and $x=0$, we get $x=-1$ and $x=0$ as the critical points. These points divide the whole real line into three parts, namely $(-\infty, -1)$, $[-1, 0)$ and $[0, \infty)$.

Case I *When* $-\infty < x < -1$.

In this case, $x+1 < 0$ and $x < 0$.

$\therefore \quad |x+1| = -(x+1) = -x-1$ and $|x| = -x$.

So, $|x+1| + |x| > 3 \Rightarrow -x-1-x > 3 \Rightarrow -2x > 4 \Rightarrow x < -2$.

$\therefore$ solution set in this case $= (-\infty, -2)$. $[\because -\infty < x < -1]$

Case II *When* $-1 \le x < 0$.

In this case, $x+1 \ge 0$ and $x < 0$.

$\therefore \quad |x+1| = x+1$ and $|x| = -x$.

So, $|x+1| + |x| > 3 \Rightarrow x+1-x > 3 \Rightarrow 1 > 3$, which is absurd.

Case III *When* $0 \le x < \infty$.

In this case, $x+1 > 0$ and $x \ge 0$.

$\therefore \quad |x+1| = x+1$ and $|x| = x$.

So, $|x+1| + |x| > 3 \Rightarrow x+1+x > 3 \Rightarrow 2x > 2 \Rightarrow x > 1$.

$\therefore$ solution set in this case $= (1, \infty)$.

Hence, solution set $= (-\infty, -2) \cup (1, \infty)$.

31. Using $|x| > a \Leftrightarrow x < -a$ or $x > a$, we get

$$\frac{2}{x-4} < -1 \text{ or } \frac{2}{x-4} > 1$$

$$\Rightarrow \quad \frac{2}{x-4} + 1 < 0 \text{ or } \frac{2}{x-4} - 1 > 0$$

$$\Rightarrow \quad \frac{x-2}{x-4} < 0 \text{ or } \frac{6-x}{x-4} > 0.$$

Case I *When* $\dfrac{x-2}{x-4} < 0$.

Then, $(x-2 < 0 \text{ and } x-4 > 0)$ or $(x-2 > 0 \text{ and } x-4 < 0)$

$\Rightarrow \quad (x < 2 \text{ and } x > 4) \text{ or } (x > 2 \text{ and } x < 4)$

$\Rightarrow \quad 2 < x < 4 \Rightarrow x \in (2, 4)$.

Case II *When* $\dfrac{6-x}{x-4} > 0$.

Then, $(6-x < 0 \text{ and } x-4 < 0)$ or $(6-x > 0 \text{ and } x-4 > 0)$

$\Rightarrow \quad (x > 6 \text{ and } x < 4) \text{ or } (x < 6 \text{ and } x > 4)$

$\Rightarrow \quad 4 < x < 6 \Rightarrow x \in (4, 6)$.

$\therefore$ solution set $= (2, 4) \cup (4, 6)$.

32. $\dfrac{4}{x+1} \le 3 \Rightarrow \dfrac{4}{x+1} - 3 \le 0 \Rightarrow \dfrac{1-3x}{x+1} \le 0$.

$\therefore \quad (1-3x \le 0 \text{ and } x+1 > 0) \text{ or } (1-3x \ge 0 \text{ and } x+1 < 0)$

$\Rightarrow \quad \left(x \ge \dfrac{1}{3} \text{ and } x > -1\right) \text{ or } \left(x \le \dfrac{1}{3} \text{ and } x < -1\right)$

$\Rightarrow \quad x \geq \frac{1}{3}$ or $x < -1 \Rightarrow x \in \left[\frac{1}{3}, \infty\right) \cup (-\infty, -1)$

Again, $\frac{6}{x+1} \geq 3 \Rightarrow \frac{6}{x+1} - 3 \geq 0 \Rightarrow \frac{3-3x}{x+1} \geq 0 \Rightarrow \frac{1-x}{1+x} \geq 0.$

$\therefore \quad (1-x \geq 0 \text{ and } 1+x > 0)$ or $(1-x \leq 0 \text{ and } 1+x < 0)$

$\Rightarrow \quad (x \leq 1 \text{ and } x > -1)$ or $(x \geq 1 \text{ and } x < -1)$

$\Rightarrow \quad -1 < x \leq 1 \Rightarrow x \in (-1, 1].$

$\therefore \quad$ solution set $= \left\{(-\infty, -1) \cup \left[\frac{1}{3}, \infty\right)\right\} \cap (-1, 1] = \left[\frac{1}{3}, 1\right].$

39. (i) $|x-2| \geq 1 \Rightarrow (x-2) \leq -1$ or $(x-2) \geq 1$ $\quad [\because |x| \geq a \Rightarrow x \leq -a \text{ or } x \geq a]$

$\Rightarrow x \leq 1$ or $x \geq 3 \Rightarrow x \in (-\infty, 1] \cup [3, \infty).$

(ii) $|x-2| \leq 3 \Rightarrow -3 \leq x-2 \leq 3 \Rightarrow -1 \leq x \leq 5 \Rightarrow x \in [-1, 5].$

$\therefore \quad$ required solution set $= \{(-\infty, 1] \cup [3, \infty)\} \cap [-1, 5] = [-1, 1] \cup [3, 5].$

40. Let the required odd integers be x and $x+2$.

Then, $x+2 < 18$ and $x+(x+2) > 20$

$\Rightarrow \quad x < 16$ and $2x+2 > 20$

$\Rightarrow \quad x < 16$ and $x+1 > 10$

$\Rightarrow \quad x > 9$ and $x < 16 \Rightarrow 9 < x < 16.$

So, the required values of x are 11, 13 and 15.

Hence, the required pairs are (11, 13), (13, 15) and (15, 17).

42. $P(x) = R(x) - C(x) \Rightarrow P(x) = 43x - (26000 + 30x)$

$\Rightarrow P(x) = (13x - 26000).$

Now, $P(x) > 0 \Rightarrow 13x - 26000 > 0 \Rightarrow x > 2000.$

Hence, more than 2000 cassettes must be sold.

43. Let the third reading be x. Then,

$$8.2 < \frac{8.48 + 8.35 + x}{3} < 8.5$$

$\Rightarrow \quad 24.6 < 16.83 + x < 25.5$

$\Rightarrow \quad 24.6 - 16.83 < x < 25.5 - 16.83 \Rightarrow 7.77 < x < 8.67.$

44. Let x litres of 2% boric acid be added to 640 litres of 8% solution of boric acid. Then, total mixture $= (640 + x)$ litres.

Boric acid content in this mixture $= \left(\frac{640 \times 8}{100} + \frac{x \times 2}{100}\right)$ litres.

$\therefore \quad \frac{4}{100} \times (640 + x) < \left\{\frac{640 \times 8}{100} + \frac{x \times 2}{100}\right\} < \frac{6}{100} \times (640 + x)$

$\Rightarrow \quad 2560 + 4x < 5120 + 2x < 3840 + 6x$

$\Rightarrow \quad 2x < 2560$ and $4x > 1280 \Rightarrow 320 < x < 1280.$

45. Let x litres of water may be added. Then,

$$25\% \text{ of } (600 + x) < \frac{600 \times 45}{100} < 30\% \text{ of } (600 + x)$$

$\Rightarrow \quad \frac{600 + x}{4} < 270 < \frac{1800 + 3x}{10}$

$\Rightarrow \quad \frac{600 + x}{4} < 270$ and $\frac{1800 + 3x}{10} > 270$

$\Rightarrow \quad 600 + x < 1080$ and $1800 + 3x > 2700 \Rightarrow 300 < x < 480.$

46. Let the required marks be x. Then,

$$(89+93+95+91+x) \geq (90 \times 5) \Rightarrow x \geq 82.$$

EXERCISE 6B

Very-Short-Answer Questions

1. Find the solution set of the inequation $\frac{1}{x-2} < 0$.

2. Find the solution set of the inequation $|x-1| < 2$.

3. Find the solution set of the inequation $|2x-3| > 1$.

4. Find the solution set of the inequation $\frac{|x-2|}{(x-2)} > 0, x \neq 2$.

5. Find the solution set of the inequation $\frac{x+1}{x+2} > 1$.

6. Solve the system of inequations $x-2 \geq 0, 2x-5 \leq 3$.

7. Solve $-4x > 16$, when $x \in Z$.

8. Solve $x+5 > 4x-10$, when $x \in R$.

9. Solve $\frac{3}{x-2} < 1$, when $x \in R$.

10. Solve $\frac{x}{x-5} > \frac{1}{2}$, when $x \in R$.

11. Solve $|x| < 4$, where $x \in R$.

12. Solve $|x| > 4$, where $x \in R$.

ANSWERS (EXERCISE 6B)

1. $(-\infty, 2)$ **2.** $(-1, 3)$ **3.** $(-\infty, 1) \cup (2, \infty)$ **4.** $(2, \infty)$
5. $(-\infty, -2)$ **6.** $x \in [2, 4]$ **7.** $x \in \{-5, -6, -7, \ldots\}$ **8.** $x \in (-\infty, 5)$
9. $x \in (-\infty, 2) \cup (5, \infty)$ **10.** $x \in (-\infty, -5) \cup (5, \infty)$ **11.** $x \in (-4, 4)$
12. $x \in (-\infty, -4) \cup (4, \infty)$

HINTS TO SOME SELECTED QUESTIONS

1. $\frac{1}{(x-2)} < 0 \Rightarrow x-2 < 0 \Rightarrow x < 2 \Rightarrow x \in (-\infty, 2)$.

2. Using $|x| < a \Leftrightarrow -a < x < a$, we get

$$|x-1| < 2 \Rightarrow -2 < x-1 < 2 \Rightarrow -1 < x < 3 \Rightarrow x \in (-1, 3).$$

3. Using $|x| > a \Leftrightarrow x < -a$ or $x > a$, we get

$$|2x-3| > 1 \Rightarrow 2x-3 < -1 \text{ or } 2x-3 > 1$$
$$\Rightarrow x < 1 \text{ or } x > 2 \Rightarrow x \in (-\infty, 1) \cup (2, \infty).$$

4. Clearly, $(x-2) > 0 \Rightarrow x > 2 \Rightarrow x \in (2, \infty)$.

5. $\dfrac{x+1}{x+2}-1>0 \Rightarrow \dfrac{x+1-x-2}{x+2}>0 \Rightarrow \dfrac{-1}{x+2}>0 \Rightarrow x+2<0$

$\Rightarrow x<-2 \Rightarrow x\in(-\infty,-2)$.

6. $(x-2\geq 0$ and $2x-5\leq 3)$

$\Rightarrow$ $(x\geq 2$ and $x\leq 4) \Rightarrow 2\leq x\leq 4 \Rightarrow x\in[2,4]$.

7. $-4x>16 \Rightarrow x<-4 \Rightarrow x\in[-5,-6,-7,-8,\ldots\}$.

8. $x+5>4x-10 \Rightarrow 3x<15 \Rightarrow x<5 \Rightarrow x\in(-\infty,5)$.

9. $\dfrac{3}{x-2}-1<0 \Rightarrow \dfrac{3-x+2}{x-2}<0 \Rightarrow \dfrac{5-x}{x-2}<0$.

$\therefore$ $(5-x<0$ and $x-2>0)$ or $(5-x>0$ and $x-2<0)$

$\Rightarrow$ $(x>5$ and $x>2)$ or $(x<5$ and $x<2)$

$\Rightarrow$ $(x>5$ or $x<2) \Rightarrow x\in(-\infty,2)\cup(5,\infty)$.

10. $\dfrac{x}{x-5}-\dfrac{1}{2}>0 \Rightarrow \dfrac{2x-x+5}{2(x-5)}>0 \Rightarrow \dfrac{x+5}{x-5}>0$.

$\therefore$ $(x+5<0$ and $x-5<0)$ or $(x+5>0$ and $x-5>0)$

$\Rightarrow$ $(x<-5$ and $x<5)$ or $(x>-5$ and $x>5)$

$\Rightarrow$ $(x<-5)$ or $(x>5) \Rightarrow x\in(-\infty,-5)\cup(5,\infty)$.

11. Using $|x|<a \Leftrightarrow -a<x<a$, we get

$|x|<4 \Rightarrow -4<x<4 \Rightarrow x\in(-4,4)$.

12. Using $|x|>a \Leftrightarrow x<-a$ or $x>a$, we get

$|x|>4 \Leftrightarrow x<-4$ or $x>4$

$\Leftrightarrow x\in(-\infty,-4)\cup(4,\infty)$.

KEY FACTS AND FORMULAE

1. (i) **Linear Inequations in One Variable:**

Inequalities of the form:

(i) $ax+b<c$ (ii) $ax+b\leq c$ (iii) $ax+b>c$ (iv) $ax+b\geq c$

where $a,b,c\in R$, $a\neq 0$ and x is a variable, are called inequations in x.

(ii) **Replacement Set** A set from which the values are replaced in an inequation in x, is called the replacement set.

(iii) **Solution Set** The set of all those values of x taken from the replacement set which satisfy the given inequation is called the solution set of the inequation.

2. Rules for Solving an Inequation

RULE 1 Adding the same number or expression to each side of an inequation does not change the inequality.

RULE 2 Subtracting the same number or expression from each side of an inequation does not change the inequality.

RULE 3 Multiplying (or dividing) each side of an inequation by the same positive number does not change the inequality.

RULE 4 Multiplying (or dividing) each side of an inequation by the same negative number reverses the inequality.

Thus, $-x > 3 \Rightarrow x < -3$ [on multiplying both sides by –1].

And, $-3x \leq -12 \Rightarrow x \geq 4$ [on dividing both sides by –3].

TRANSPOSITION Using Rule 1 and Rule 2, we can drop any term from one side of an inequation and put it on the other side with the opposite sign.

3. Some Useful Results

Let a be a positive real number. Then,

(i) $|x| < a \Leftrightarrow -a < x < a \Leftrightarrow x \in (-a, a)$.

(ii) $|x| \leq a \Leftrightarrow -a \leq x \leq a \Leftrightarrow x \in [-a, a]$.

(iii) $|x| > a \Leftrightarrow x < -a$ or $x > a$.

(iv) $|x| \geq a \Leftrightarrow x \leq -a$ or $x \geq a$.

4. Solving Two Simultaneous Linear Inequations in *x*

Suppose we have to solve two inequations in x simultaneously. Then, we find the solution set for each one of them. The intersection of these solution sets is the required solution set.

❑

7
Linear Inequations (In two variables)

LINEAR INEQUATIONS *The statements of inequalities of the form* $ax + by + c > 0$, $ax + by + c \geq 0$, $ax + by + c < 0$ *and* $ax + by + c \leq 0$ *are called linear inequations in two variables x and y.*

The set of all ordered pairs (x, y) which satisfy the given inequation is called the solution set of the inequation.

GRAPH OF A LINEAR INEQUATION Let an inequation of any of the types given above be given.

Then, in order to solve it, we proceed according to steps given below.

Step 1 Consider the equation, $ax + by + c = 0$.

Draw the graph of this equation, which is a line.

In case of strict inequation > or < draw the line dotted, otherwise make it thick.

This line divides the plane into two equal parts.

Step 2 Choose a point [*if possible* (0, 0)], not lying on this line. If this point satisfies the given inequation then shade the part of the plane containing this point, otherwise shade the other part.

The shaded portion represents the solution set of the given inequation.

The dotted line is not a part of the solution, while the thick line is a part of it.

SOLVED EXAMPLES

EXAMPLE 1 *Solve* $3x + 4y \leq 12$ *graphically.*

SOLUTION Consider the equation, $3x + 4y = 12$.

The values of (x, y) satisfying $3x + 4y = 12$ are:

x	4	0
y	0	3

Plot the points $A(4, 0)$ and $B(0, 3)$ on a graph paper.

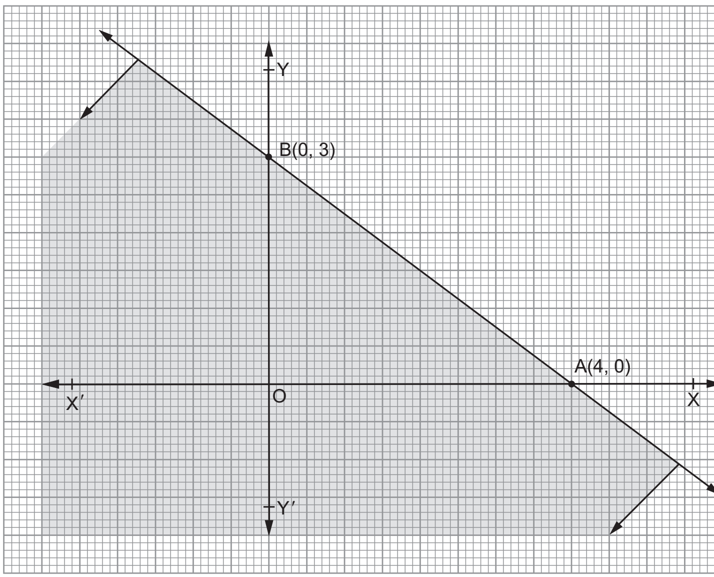
Then, line AB represents $3x + 4y = 12$.

This line divides the plane of the paper in two equal parts.

Clearly, the point (0, 0) satisfies $3x + 4y \le 12$.

So, we shade that part of the plane divided by AB which contains (0, 0).

The shaded part of the plane together with all the points on line AB represents $3x + 4y \le 12$.



EXAMPLE 2 *Draw the graph of the solution set of the inequation* $2x - y \ge 1$.

SOLUTION Consider the equation, $2x - y = 1$.

The values of (x, y) satisfying $2x - y = 1$ are:

x	2	0
y	3	–1

Take a graph paper and plot the points $A(2, 3)$ and $B(0, -1)$.

Then, line AB represents $2x - y = 1$.

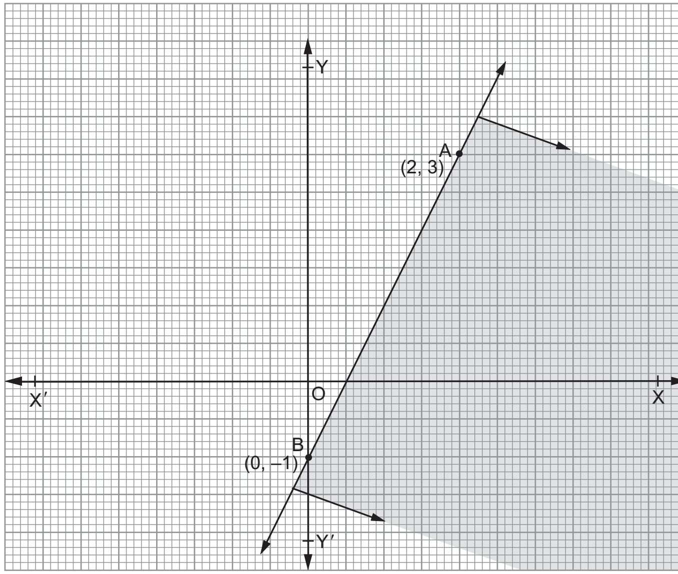
This line divides the plane of the paper in two equal parts.

Clearly, the point (0, 0) does not lie on $2x - y = 1$.

Also, (0, 0) does not satisfy $2x - y \ge 1$.

So, we shade that part of the plane divided by line AB which does not contain (0, 0), as shown below.

The shaded part of the plane together with all points on the line AB constitutes the solution set of the inequation, $2x - y \ge 1$.



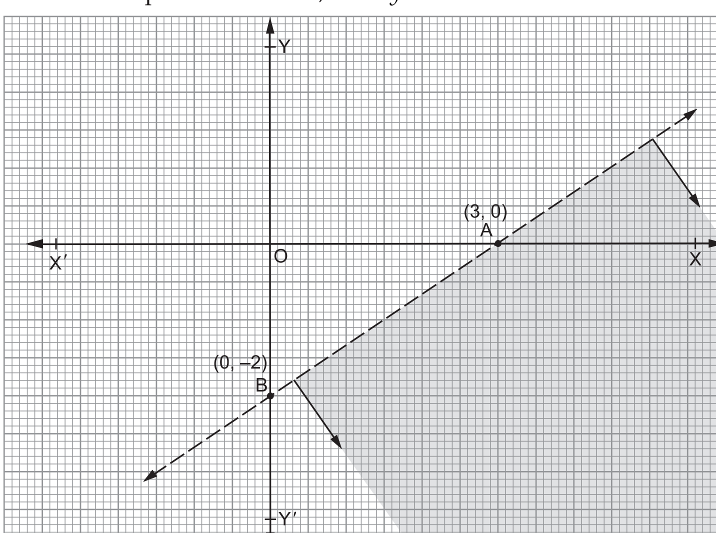
EXAMPLE 3 *Solve* $2x - 3y > 6$ *graphically.*

SOLUTION Consider the equation, $2x - 3y = 6$.

This equation may be written as, $\frac{x}{3} + \frac{y}{-2} = 1$.

Thus, it cuts the x-axis at $A(3, 0)$ and the y-axis at $B(0, -2)$.

$\therefore$ AB represents the line, $2x - 3y = 6$.



Clearly, the point (0, 0) satisfies $2x - 3y < 6$.

$\therefore$ (0, 0) does not satisfy $2x - 3y > 6$.

So, we shade that part of the plane divided by AB which does not contain $O(0, 0)$.

The shaded part of the plane excluding line AB forms the solution set of $2x - 3y > 6$, as shown above.

EXAMPLE 4 *Solve $x + y < 5$ graphically.*

SOLUTION Consider the equation, $x + y = 5$.

This equation may be written as $\frac{x}{5} + \frac{y}{5} = 1$.

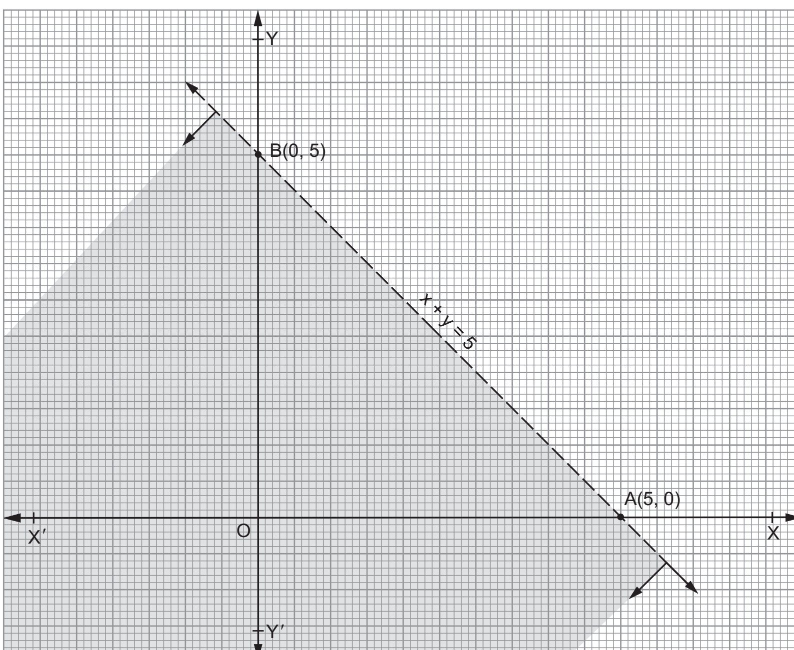
Thus, it cuts the x-axis at $A(5, 0)$ and the y-axis at $B(0, 5)$.

$\therefore$ AB represents the line $x + y = 5$.

Clearly, (0, 0) satisfies the inequation $x + y < 5$.

So, we shade that part of the plane divided by AB which contains $O(0, 0)$.

The shaded part of the plane excluding line AB forms the solution set of $x + y < 5$, as shown below.



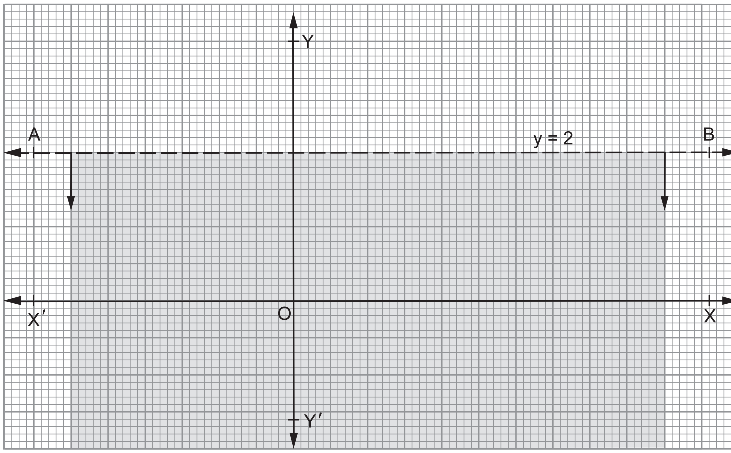
EXAMPLE 5 *Solve $y < 2$ graphically.*

SOLUTION The graph of $y = 2$ is a straight line AB drawn at a distance of 2 units above the x-axis and parallel to the x-axis.

Clearly, the point (0, 0) satisfies the inequation, $y < 2$.

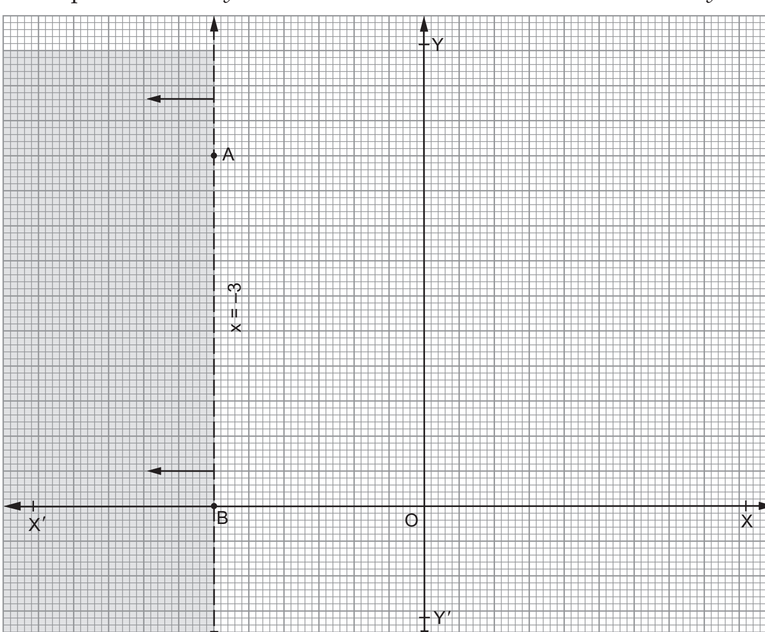
So, we shade that part of the plane divided by AB which contains the origin $O(0, 0)$, excluding AB.

The shaded part of the plane forms the solution set of the inequation, $y < 2$, as shown below.



EXAMPLE 6 *Solve the inequation $x < -3$ graphically.*

SOLUTION We know that the graph of $x = -3$ is a straight line AB drawn parallel to the y-axis at a distance of 3 units to the left of the y-axis.



Clearly, the point (0, 0) does not satisfy the inequation, $x < -3$.

So, we shade that part of the plane divided by AB which does not contain $O(0, 0)$, excluding AB.

The shaded part of the plane is the solution set of $x < -3$, as shown above.

EXAMPLE 7 *Exhibit graphically the solution set of the linear inequations:*

$$x + 2y \leq 10,\ x + y \leq 6,\ x \leq 4,\ x \geq 0 \text{ and } y \geq 0.$$

SOLUTION (i) First, we draw the graph of $x + 2y \leq 10$.

Consider the line, $x + 2y = 10$.

Clearly, the points $A(0, 5)$ and $B(10, 0)$ satisfy, $x + 2y = 10$.

Then, line AB represents $x + 2y = 10$.

Clearly, (0, 0) satisfies the inequation $x + 2y \leq 10$.

Thus, the line AB and part of the plane containing $O(0, 0)$ represent the solution set of $x + 2y \leq 10$.

(ii) Next, we draw the graph of $x + y \leq 6$.

Consider the line, $x + y = 6$.

Clearly, the points $C(0, 6)$ and $D(6, 0)$ satisfy $x + y = 6$.

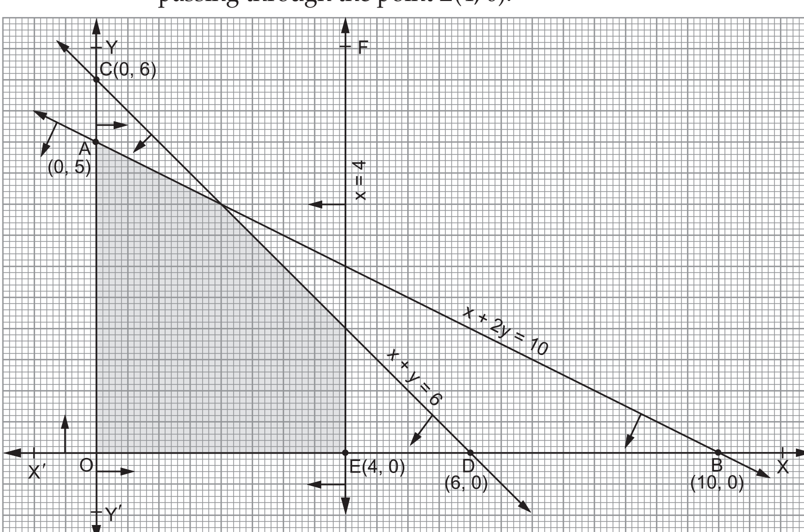
So, line CD represents $x + y = 6$.

Clearly, (0, 0) satisfies the inequation, $x + y \leq 6$.

So, the line CD and part of the plane containing $O(0, 0)$ represent the solution set of $x + y \leq 6$.

(iii) Next, we draw the graph of $x \leq 4$.

We know that $x = 4$ is the line EF parallel to the y-axis and passing through the point $E(4, 0)$.



Clearly, (0, 0) satisfies the inequation, $x \le 4$.

Thus, the line EF and part of the plane containing $O(0, 0)$ represent the solution set of $x \le 4$.

(iv) $x \ge 0$ is represented by the y-axis and the plane on its right.

(v) $y \ge 0$ is represented by the x-axis and the plane above the x-axis.

The intersection of all these planes is the shaded part, which together with its boundary represents the solution of the given system of inequations.

EXAMPLE 8 *Draw the graph of the solution set of the inequations:*

$$2x + y \ge 2,\ x - y \le 1,\ x + 2y \le 8,\ x \ge 0 \text{ and } y \ge 0.$$

SOLUTION (i) First, we draw the graph of $2x + y \ge 2$.

Consider the line, $2x + y = 2$.

Now, $2x + y = 2 \Rightarrow \frac{x}{1} + \frac{y}{2} = 1$.

This line meets the axes at $A(1, 0)$ and $B(0, 2)$.

Clearly, the line AB represents $2x + y = 2$.

Now, (0, 0) does not satisfy $2x + y \ge 2$.

Thus, the line AB and part of the plane separated by AB and not containing $O(0, 0)$, represent the solution set of $2x + y \ge 2$.

(ii) Next, we draw the graph of $x - y \le 1$.

Consider the line, $x - y = 1$.

$$x - y = 1 \Rightarrow \frac{x}{1} + \frac{y}{(-1)} = 1.$$

This line meets at the axes at $A(1, 0)$ and $C(0, -1)$.

Clearly, the line AC represents $x - y = 1$.

Also, (0, 0) satisfies, $x - y \le 1$.

Thus, the line AC and part of the plane separated by AC and containing $O(0, 0)$ represent the solution set of $x - y \le 1$.

(iii) Now, we draw the graph of $x + 2y \le 8$.

Consider the line, $x + 2y = 8$.

Now, $x + 2y = 8 \Rightarrow \frac{x}{8} + \frac{y}{4} = 1$.

This line meets the axes at $D(8, 0)$ and $E(0, 4)$.

So, the line DE represents $x + 2y = 8$.

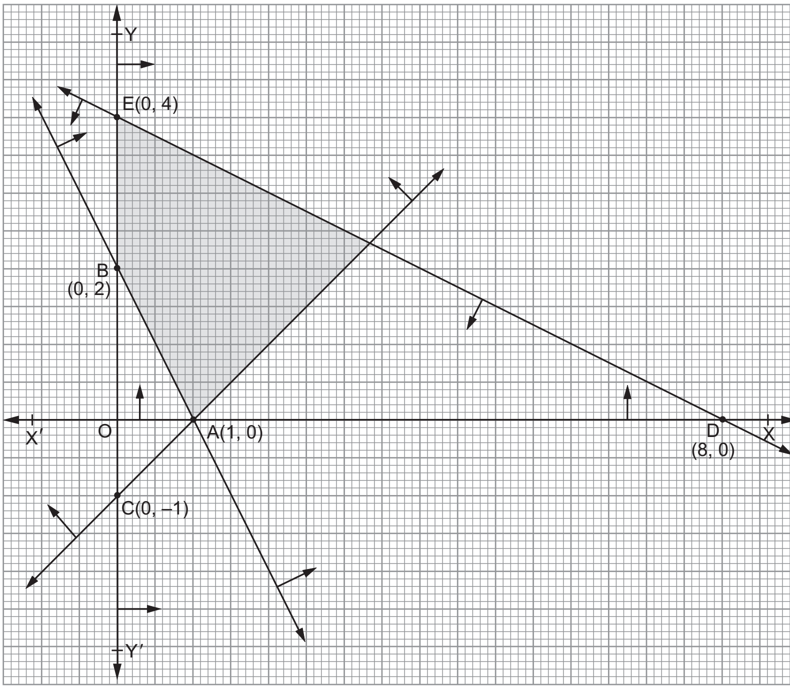
Also, (0, 0) satisfies the inequation, $x + 2y \le 8$.

Thus, the line DE and part of the plane containing $O(0, 0)$ represent the solution set of $x + 2y \le 8$.

(iv) $x \ge 0$ is represented by the y-axis and the plane on its right.

(v) $y \ge 0$ is represented by the x-axis and the plane above the x-axis.

The intersection of all these planes is the required shaded part, representing the solution of the given system.



EXAMPLE 9 *Solve the following system of inequations graphically:*

$$5x + 4y \leq 40,\ x \geq 2,\ y \geq 3.$$

SOLUTION (i) First, we draw the graph line $5x + 4y = 40$.

Now, $5x + 4y = 40 \Rightarrow \frac{x}{8} + \frac{y}{10} = 1$.

This line meets the axes at $A(8, 0)$ and $B(0, 10)$.

Clearly, the line AB represents $5x + 4y = 40$.

Now, (0, 0) satisfies the inequation $5x + 4y \leq 40$.

Thus, the line AB and part of the plane containing $O(0, 0)$ and separated by AB represent the solution set of the inequation, $5x + 4y \leq 40$.

(ii) Clearly, $x = 2$ is the line CD parallel to the y-axis at a distance of 2 units to its right.

Clearly, (0, 0) does not satisfy the inequation, $x \geq 2$.

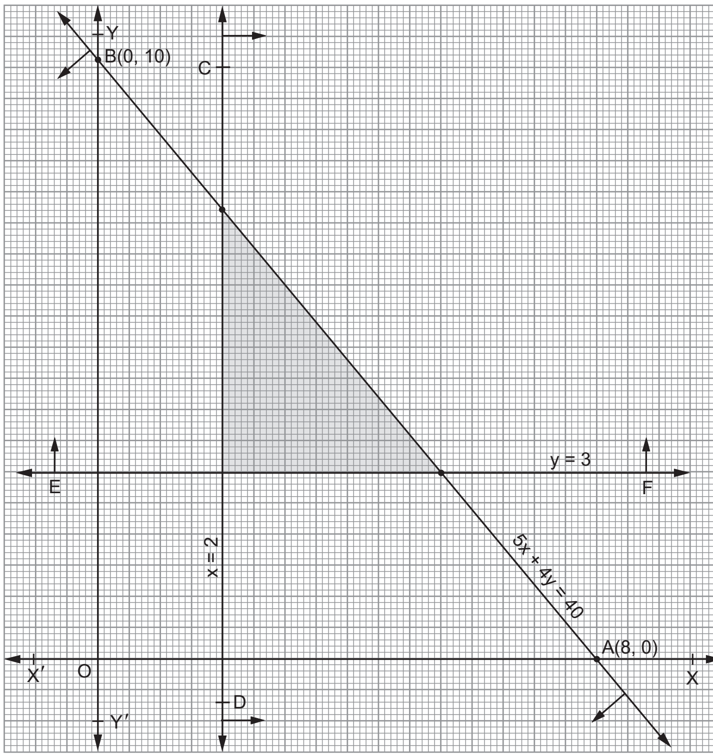
So, $x \geq 2$ consists of the line CD and part of the plane on right of CD.

(iii) Also, $y = 3$ is the line EF parallel to the x-axis lying above it at a distance of 3 units from it.

Clearly, (0, 0) does not satisfy the inequation, $y \geq 3$.

So, $y \geq 3$ consists of the line EF and part of the plane above it.

The intersection of all these planes is the required shaded part, representing the solution of the given system, as shown below.



EXAMPLE 10 *Show that the following system of linear inequations has no solution:*

$$x + 2y \leq 3,\ 3x + 4y \geq 12,\ x \geq 0,\ y \geq 1.$$

SOLUTION (i) First, we draw the graph of $x + 2y \leq 3$.

Consider the line, $x + 2y = 3$.

Clearly, the points $A(1, 1)$ and $B(-1, 2)$ lie on this line.

Then, line AB represents $x + 2y = 3$.

Now, (0, 0) satisfies $x + 2y \leq 3$.

So, the line AB and part of the plane containing $O(0,0)$ represent the solution set of $x + 2y \leq 3$.

(ii) Next, we draw the graph of $3x + 4y \geq 12$.

Consider the line, $3x + 4y = 12$.

Now, $3x + 4y = 12 \Rightarrow \dfrac{x}{4} + \dfrac{y}{3} = 1$.

So, this line meets the axes at $C(4, 0)$ and $D(0, 3)$.

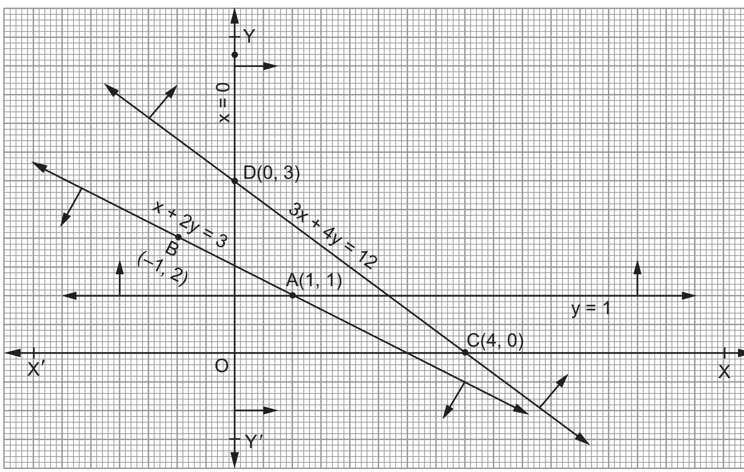
Thus, CD represents the line $3x + 4y \geq 12$.

Clearly, (0, 0) does not satisfy $3x + 4y \geq 12$.

Therefore, the line CD and part of the plane not containing $O(0, 0)$ represent the solution set of $3x + 4y \geq 12$.

(iii) Clearly, $x \geq 0$ is represented by the y-axis and the plane on its right.

(iv) $y \geq 1$ represents a line parallel to the x-axis at a distance of 1 unit above the x-axis and the plane above this line.



Clearly, all these planes have no common intersection.

Hence, the given system of inequations has no solution.

EXAMPLE 11 *Show that the solution set of the following linear inequations is an unbounded region* $2x + y \geq 8$, $x + 2y \geq 10$, $x \geq 0$, $y \geq 0$.

SOLUTION (i) First, we draw the graph of $2x + y \geq 8$.

Consider the line, $2x + y = 8$.

Now, $2x + y = 8 \Rightarrow \dfrac{x}{4} + \dfrac{y}{8} = 1$.

This line meets the axes at $A(4, 0)$ and $B(0, 8)$.

Clearly, the line AB represents $2x + y = 8$.

It is also clear that (0, 0) does not satisfy $2x + y \geq 8$.

Thus, the line AB and the part of the plane not containing $O(0, 0)$ represent the solution of the inequation $2x + y \geq 8$.

(ii) Next, we draw the graph of $x + 2y \geq 10$.

Consider the line, $x + 2y = 10$.

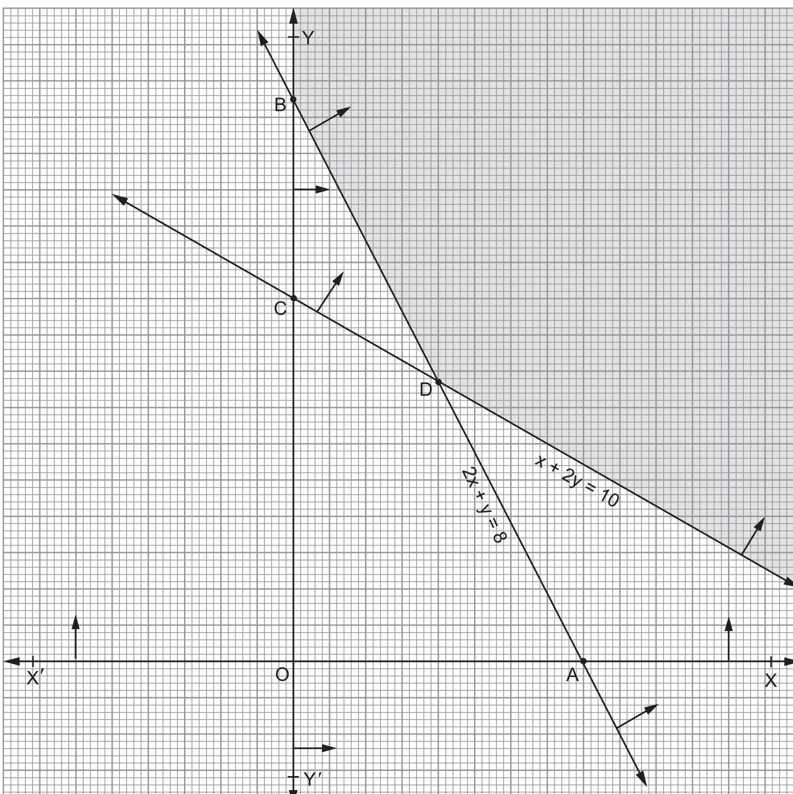
Clearly, the points $C(0, 5)$ and $D(2, 4)$ lie on this line.

Also, $(0, 0)$ does not satisfy $x + 2y \geq 10$.

So, the line CD and the part of the plane not containing $O(0, 0)$ represent the solution set of $x + 2y \geq 10$.

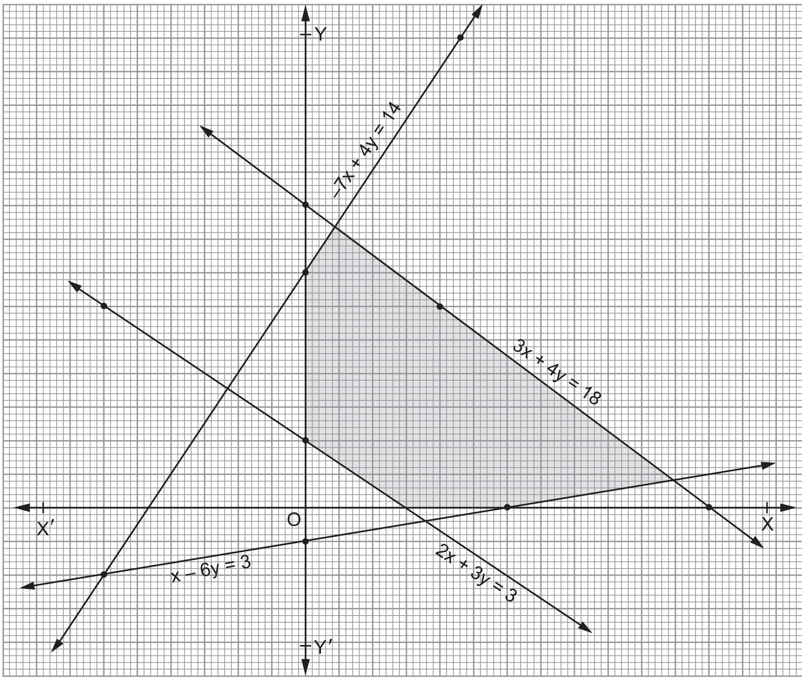
(iii) $x \geq 0$ is represented by the y-axis and the plane to its right.

(iv) $y \geq 0$ is represented by the x-axis and the plane above the x-axis.



It is clear from the above graph that the solution set of the given inequations is an unbounded region, shown by the shaded region.

EXAMPLE 12 *Find the linear inequations for which the shaded region in the figure given below, is the solution set.*



SOLUTION (i) Consider the line, $3x + 4y = 18$.

We observe here from the graph that the shaded region and the origin $O(0, 0)$ lie on the same side of this line.

But, $(0, 0)$ satisfies the inequation $3x + 4y \leq 18$.

So, the inequation corresponding to this line is $3x + 4y \leq 18$.

(ii) Consider the line, $2x + 3y = 3$.

We observe here from the graph that the shaded region and the origin $O(0, 0)$ lie on the opposite sides of this line.

Here, $(0, 0)$ satisfies the inequation $2x + 3y \leq 3$.

So, the inequation corresponding to this line is $2x + 3y \geq 3$.

(iii) Consider the line, $-7x + 4y = 14$.

We observe here from the graph that the shaded region and the origin $O(0, 0)$ lie on the same side of this line.

Here, $(0, 0)$ satisfies the inequation $-7x + 4y \leq 14$.

So, the inequation corresponding to this line is $-7x + 4y \leq 14$.

(iv) Consider the line, $x - 6y = 3$.

We observe here from the graph that the shaded region and the origin $O(0, 0)$ lie on the same side of this line.

Also, the point (0, 0) satisfies the inequation $x - 6y \le 3$.

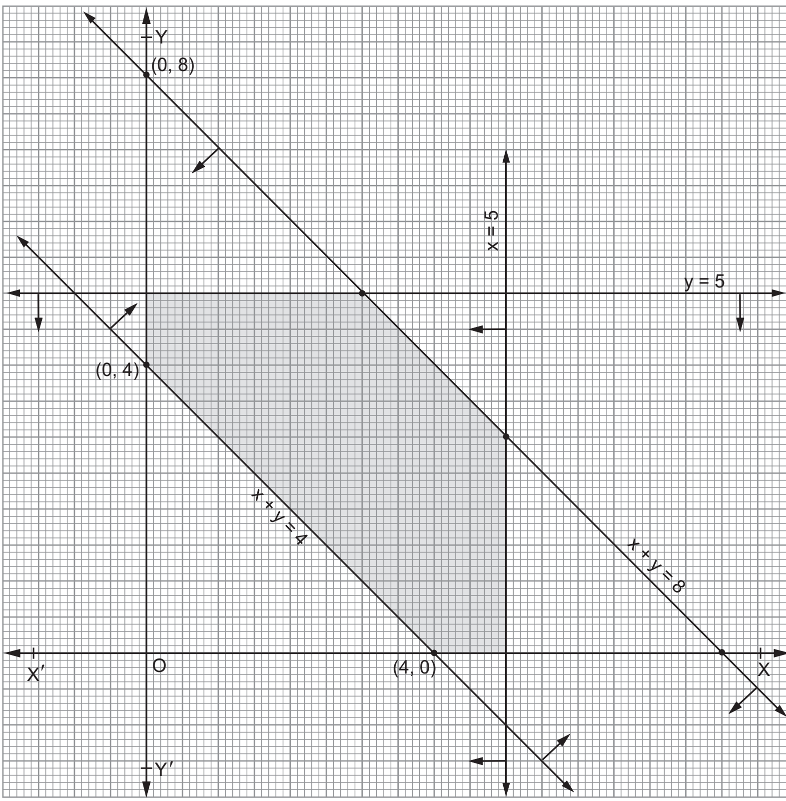
So, the inequation corresponding to this line is $x - 6y \le 3$.

(v) It is clear from the given figure that the whole shaded region lies in the first quadrant only. So, $x \ge 0$ and $y \ge 0$.

Hence, the linear inequations corresponding to the given solution set are

$$3x + 4y \le 18,\ 2x + 3y \ge 3,\ -7x + 4y \le 14,\ x - 6y \le 3,\ x \ge 0 \text{ and } y \ge 0.$$

EXAMPLE 13 *Find the linear inequations for which the shaded region of the figure given below, is the solution set.*



SOLUTION (i) Consider the graph line, $x = 5$.

It is clear that the shaded region and the origin $O(0, 0)$ lie on the same side of this line.

And, the point (0, 0) satisfies the inequation $x \le 5$.

So, the inequation corresponding to this line is $x \le 5$.

(ii) Consider the graph line, $y = 5$.

Clearly, the shaded region and the origin $O(0, 0)$ lie on the same side of this line.

And, the point (0, 0) satisfies the inequation $y \le 5$.

So, the inequation corresponding to this line is $y \le 5$.

(iii) Consider the graph line, $x + y = 8$.

Clearly, the shaded region and the origin $O(0, 0)$ lie on the same side of this line.

And, the point (0, 0) satisfies the inequation $x + y \le 8$.

So, the inequation corresponding to this line is $x + y \le 8$.

(iv) Consider the graph line, $x + y = 4$.

It is clear from the graph that the shaded region and the origin $O(0, 0)$ lie on the opposite sides of this line. Also, the point (0, 0) satisfies the inequation $x + y \le 4$.

So, the inequation corresponding to this line is $x + y \ge 4$.

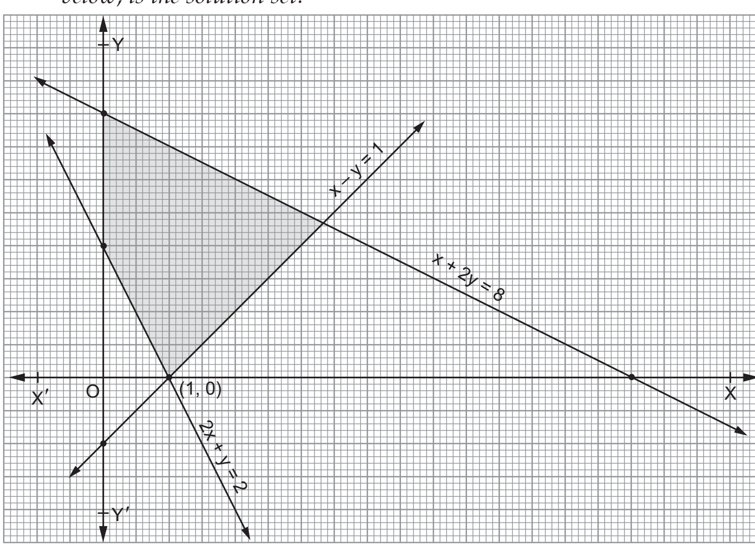
(v) It is clear from the given figure that the whole shaded region lies in the first quadrant only.

So, $x \ge 0$ and $y \ge 0$.

Hence, the linear inequations corresponding to the given solution set are

$$x \le 5,\ y \le 5,\ x + y \le 8,\ x + y \ge 4,\ x \ge 0 \text{ and } y \ge 0.$$

EXAMPLE 14 *Find the linear inequations for which the shaded region of the figure given below, is the solution set.*



SOLUTION (i) Consider the graph line, $x + 2y = 8$.

It is clear that the shaded region and the origin $O(0, 0)$ lie on the same side of the line $x + 2y = 8$ and (0, 0) satisfies the inequation $x + 2y \le 8$.

So, the inequation corresponding to this line is $x + 2y \le 8$.

(ii) Consider the graph line, $2x + y = 2$.

It is clear that the shaded region and the origin $O(0, 0)$ lie on the opposite sides of the line $2x + y = 2$.

And, (0, 0) does not satisfy the inequation $2x + y \geq 2$.

So, the inequation corresponding to this line is $2x + y \geq 2$.

(iii) Consider the graph line, $x - y = 1$.

It is clear that the shaded region and the origin $O(0, 0)$ lie on the same side of the line $x - y = 1$.

And, (0, 0) satisfies the inequation $x - y \leq 1$.

So, the inequation corresponding to this line is $x - y \leq 1$.

(iv) It may be noted here that the shaded region is above the x-axis and is on the right side of the y-axis.

So, we must have $x \geq 0$ and $y \geq 0$.

Hence, the linear inequations corresponding to the given solution set are

$$x + 2y \leq 8,\ 2x + y \geq 2,\ x - y \leq 1,\ x \geq 0,\ y \geq 0.$$

EXAMPLE 15 *A small manufacturing firm produces two types of gadgets A and B, which are first processed in the foundry and then sent to another machine for finishing. The number of man-hours of labour required in each shop for the production of each unit of A and B and the number of man-hours for the firm available per week are as follows:*

	Foundry	*Machine shop*
Man-hours for 1 unit of gadget A	10	5
Man-hours for 1 unit of gadget B	6	4
Firm's capacity per week in man-hours	1000	600

Formulate it in the form of linear inequations. Draw the graph showing the solution of all these inequations.

SOLUTION Suppose that the firm manufactures x units of A and y units of B.

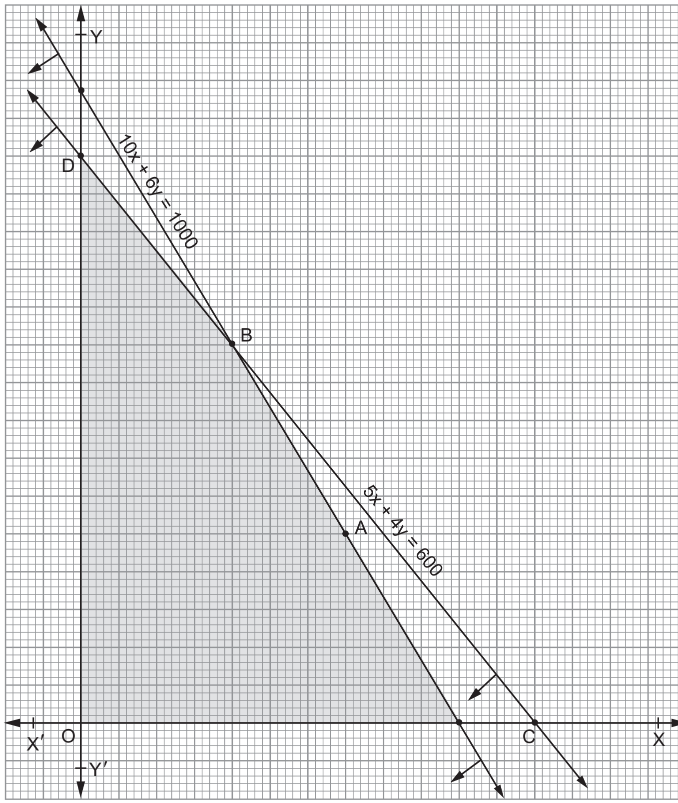
$\therefore$ number of man-hours for A and B in the foundry $= 10x + 6y$,

and number of man-hours for A and B in the machine shop $= 5x + 4y$.

$\therefore$ $10x + 6y \leq 1000$ and $5x + 4y \leq 600$.

Clearly, $x \geq 0$ and $y \geq 0$ [$\because$ number of units cannot be negative].

SCALE 1 small division = 2 units.



(i) First we draw the graph of $10x + 6y \leq 1000$.

Consider the equation, $10x + 6y = 1000$.

$$10x + 6y = 1000 \Rightarrow 5x + 3y = 500.$$

The values of (x, y) satisfying $5x + 3y = 500$ are:

x	70	40
y	50	100

Plot the points $A(70, 50)$ and $B(40, 100)$ on a graph paper and join them by the thick line AB.

Consider (0, 0). It does not lie on $5x + 3y = 500$.

Also, (0, 0) satisfies $10x + 6y \leq 1000$.

Therefore, the line AB and the part of the plane separated by AB, containing (0, 0), represent the graph of $10x + 6y \leq 1000$.

(ii) Now, we draw the graph of $5x + 4y \leq 600$.

Consider the line, $5x + 4y = 600$.

$$5x + 4y = 600 \Rightarrow \frac{x}{120} + \frac{y}{150} = 1.$$

This line meets the axes at $C(120, 0)$ and $D(0, 150)$.

Plot these points on the same graph paper as above and join them by the thick line CD.

Now, consider (0, 0). It does not lie on $5x + 4y = 600$.

Also, (0, 0) satisfies the inequation $5x + 4y \leq 600$.

So, the line CD and that part of the plane separated by CD, containing (0, 0), represent the graph of $5x + 4y \leq 600$.

$x \geq 0$ is represented by the y-axis and the part of the plane on its right side.

$y \geq 0$ is represented by the x-axis and the plane above the x-axis.

Clearly, the shaded region together with its boundary represents the solution set of the given inequations.

EXERCISE 7

Draw the graph of the solution set of each of the following inequations:

1. $x + y \geq 4$ **2.** $x - y \leq 3$ **3.** $y - 2 \leq 3x$

4. $x \geq y - 2$ **5.** $3x + 2y > 6$ **6.** $3x + 5y < 15$

Solve the following systems of inequations graphically:

7. $x \geq 2, y \geq 3$

8. $3x + 2y \leq 12, x \geq 1, y \geq 2$

9. $x + y \leq 6, x + y \geq 4$

10. $2x + y \geq 6, 3x + 4y \leq 12$

11. $x + y \leq 9, y > x, x \geq 0$

12. $2x - y > 1, x - 2y < -1$

13. $5x + 4y \leq 20, x \geq 1, y \geq 2$

14. $3x + 4y \leq 60, x + 3y \leq 30, x \geq 0, y \geq 0$

15. $2x + y \geq 4, x + y \leq 3, 2x - 3y \leq 6$

16. $x + 2y \leq 10, x + y \geq 1, x - y \leq 0, x \geq 0, y \geq 0$

17. $4x + 3y \leq 60, y \geq 2x, x \geq 3, x \geq 0, y \geq 0$

18. $x - 2y \leq 2, x + y \geq 3, -2x + y \leq 4, x \geq 0, y \geq 0$

19. $x + 2y \leq 100, 2x + y \leq 120, x + y \leq 70, x \geq 0, y \geq 0$

20. $x + 2y \leq 2000, x + y \leq 1500, y \leq 600, x \geq 0, y \geq 0$

21. Show that each of the following systems of linear inequations has no solution:

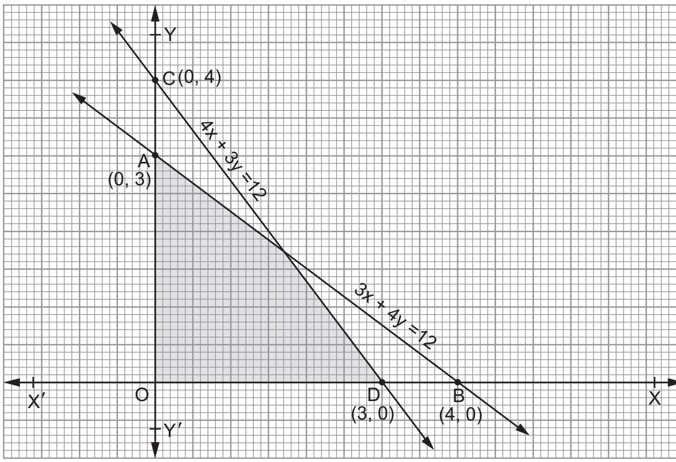
(i) $3x + 2y \geq 24, 3x + y \leq 15, x \geq 4$

(ii) $2x - y \leq -2, x - 2y \geq 0, x \geq 0, y \geq 0$

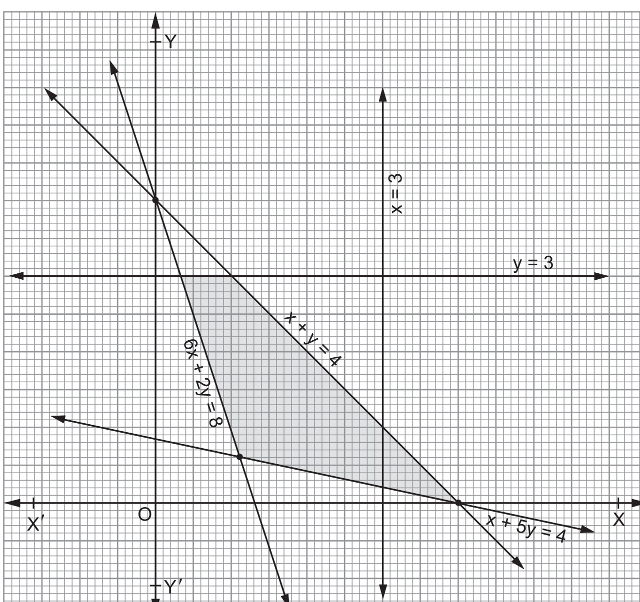
22. Show that the solution set of the following system of inequations is an unbounded set:

$$3x + y \geq 12, x + y \geq 9, x \geq 0, y \geq 0.$$

23. Find the linear inequations for which the shaded area is the solution set in the figure given below.



24. Find the linear inequations for which the shaded area in the figure given below, is the solution set.



25. A furniture dealer deals in only two items: tables and chairs. He has ₹ 30000 to invest and a space to store at most 60 pieces. A table costs him ₹ 1500 and a chair ₹ 300. Formulate the data in the form of inequations and draw a graph representing the solution of these inequations.

Hint Suppose he makes x tables and y chairs. Then,

$$1500x + 300y \leq 30000,\ x + y \leq 60,\ x \geq 0 \text{ and } y \geq 0.$$

26. If a young man rides his motorcycle at 40 km per hour, he has to spend ₹ 6 per km on petrol and if he rides it at 50 km per hour, the petrol cost rises to ₹ 10 per km. He has ₹ 500 to spend on petrol and wishes to find the maximum distance he can travel within one hour. Formulate the data in the form of inequations and draw a graph representing the solution of these inequations.

Hint Suppose he rides x km at 40 km/hr and y km at 50 km/hr.

Then, $6x + 10y \leq 500,\ \frac{x}{40} + \frac{y}{50} \leq 1,\ x \geq 0,\ y \geq 0.$

ANSWERS (EXERCISE 7)

23. $3x + 4y \leq 12,\ 4x + 3y \leq 12,\ x \geq 0,\ y \geq 0$

24. $x + y \leq 4,\ x + 5y \geq 4,\ 6x + 2y \geq 8,\ x \leq 3,\ y \leq 3,\ x \geq 0,\ y \geq 0$

❑

8
Permutations

THE FACTORIAL

FACTORIAL NOTATION *Let n be a positive integer. Then, the continued product of first n natural numbers is called factorial n, to be denoted by n !, or* $\underline{|n}$. *Also, we define* $0! = 1$.

When n is negative or a fraction, $n!$ is not defined.

Thus, $n! = n(n-1)(n-2)\ldots 3\cdot 2\cdot 1$.

EXAMPLE $6! = (6\times 5\times 4\times 3\times 2\times 1) = 720$, $3! = 3\times 2\times 1 = 6$ and $1! = 1$.

DEDUCTION
$$\begin{aligned} n! &= n(n-1)(n-2)(n-3)\ldots 3\cdot 2\cdot 1 \\ &= n[(n-1)(n-2)(n-3)\ldots 3\cdot 2\cdot 1] \\ &= n[(n-1)!]. \end{aligned}$$

Thus, $9! = 9\times(8!)$, $6! = 6\times(5!)$, $3! = 3\times(2!)$.

Also, $1! = 1\times(0!) \Rightarrow 0! = 1$.

Thus, we have

$$\boxed{\begin{aligned} &\text{(i)}\ n! = n(n-1)(n-2)\ldots 3\times 2\times 1. \\ &\text{(ii)}\ n! = n\times(n-1)!. \\ &\text{(iii)}\ 0! = 1. \end{aligned}}$$

SOLVED EXAMPLES

EXAMPLE 1 *Compute:*

(i) $\dfrac{10!}{(7!)\times(3!)}$ *(ii)* $\dfrac{30!}{28!}$ *(iii)* $\dfrac{11!-10!}{9!}$

SOLUTION We have

(i) $\dfrac{10!}{(7!)\times(3!)} = \dfrac{10\times 9\times 8\times(7!)}{(7!)\times(3\times 2\times 1)} = \dfrac{10\times 9\times 8}{3\times 2\times 1} = 120.$

(ii) $\dfrac{30!}{28!} = \dfrac{30\times 29\times(28!)}{(28!)} = 30\times 29 = 870.$

(iii) $\dfrac{11!-10!}{9!} = \dfrac{11\times 10\times(9!) - 10\times(9!)}{9!} = \dfrac{(110-10)(9!)}{(9!)} = 100.$

EXAMPLE 2 *If* $\dfrac{x}{10!} = \dfrac{1}{8!} + \dfrac{1}{9!}$, *find the value of x.*

SOLUTION On multiplying both sides by 10 !, we get

$$x = \frac{10!}{8!} + \frac{10!}{9!} \Rightarrow x = \frac{10 \times 9 \times (8!)}{(8!)} + \frac{10 \times (9!)}{(9!)}$$

$$\Rightarrow x = (10 \times 9) + 10 = (90 + 10) = 100.$$

Hence, $x = 100$.

EXAMPLE 3 *Express each of the following products in factorials:*

(i) $5 \times 6 \times 7 \times 8 \times 9$ *(ii)* $2 \times 4 \times 6 \times 8 \times 10$

SOLUTION We have

(i) $5 \times 6 \times 7 \times 8 \times 9 = \dfrac{(1 \times 2 \times 3 \times 4) \times (5 \times 6 \times 7 \times 8 \times 9)}{(1 \times 2 \times 3 \times 4)}$

[multiplying and dividing the given product by $(1 \times 2 \times 3 \times 4)$]

$$= \frac{1 \times 2 \times 3 \times 4 \times 5 \times 6 \times 7 \times 8 \times 9}{1 \times 2 \times 3 \times 4} = \frac{9!}{4!}.$$

(ii) $2 \times 4 \times 6 \times 8 \times 10 = (2 \times 1) \times (2 \times 2) \times (2 \times 3) \times (2 \times 4) \times (2 \times 5)$

$$= 2^5 \times (1 \times 2 \times 3 \times 4 \times 5) = 2^5 \times (5!).$$

EXAMPLE 4 *Find the LCM of* $(5!, 6!, 7!)$.

SOLUTION We have

$$\text{LCM}(5!, 6!, 7!) = \text{LCM}(5!, 6 \times 5!, 7 \times 6 \times 5!)$$

$$= (5!) \times \text{LCM}(1, 6, 42) = (5!) \times 42$$

$$= 7 \times 6 \times 5! = 7!.$$

EXAMPLE 5 *If* $(n+1)! = 90 \times (n-1)!$, *find* n.

SOLUTION We have

$$(n+1)! = 90 \times (n-1)!$$

$\Rightarrow$ $(n+1) \times n \times (n-1)! = 90 \times (n-1)!$

$\Rightarrow$ $(n+1)n = 90$

$\Rightarrow$ $(n+1)n = 10 \times 9$

[writing 90 as product of two consecutive integers]

$\Rightarrow$ $n = 9$.

Hence, $n = 9$.

EXAMPLE 6 *If* $(n+2)! = 60 \times (n-1)!$, *find* n.

SOLUTION We have

$$(n+2)! = 60 \times (n-1)!$$

$\Rightarrow$ $(n+2) \times (n+1) \times n \times (n-1)! = 60 \times (n-1)!$

$\Rightarrow$ $(n+2)(n+1)n = 60$

$\Rightarrow$ $(n+2)(n+1)n = 5 \times 4 \times 3$

[writing 60 as product of 3 consecutive integers]

$\Rightarrow$ $n = 3$.

Hence, $n = 3$.

EXAMPLE 7 *Prove that* $(n!+1)$ *is not divisible by any natural number between* 2 *and* n.

SOLUTION Since $n! = n(n-1)(n-2)\ldots 3\cdot 2\cdot 1$, it follows that $n!$ is divisible by every natural number between 2 and n. So, $(n!+1)$, when divided by any natural number between 2 and n, leaves 1 as remainder.

Hence, $(n!+1)$ is not divisible by any natural number between 2 and n.

EXAMPLE 8 *Prove that* $n(n-1)(n-2)\ldots(n-r+1) = \dfrac{n!}{(n-r)!}$.

SOLUTION We have

$$\begin{aligned} & n(n-1)(n-2)\ldots(n-r+1) \\ &= \frac{n(n-1)(n-2)\ldots(n-r+1)\cdot(n-r)!}{(n-r)!} \\ & \qquad\qquad [\text{multiplying num. and denom. by } (n-r)!] \\ &= \frac{n!}{(n-r)!}. \end{aligned}$$

Hence, $n(n-1)(n-2)\ldots(n-r+1) = \dfrac{n!}{(n-r)!}$.

EXAMPLE 9 *Prove that* $\dfrac{(2n)!}{n!} = 2^n \times \{1\times 3\times 5\times\ldots\times(2n-1)\}$.

SOLUTION We have

$$\begin{aligned} \frac{(2n)!}{n!} &= \frac{\{1\times 2\times 3\times 4\times 5\times\ldots\times(2n-2)\times(2n-1)\times 2n\}}{n!} \\ &= \frac{\{1\times 3\times 5\times\ldots\times(2n-1)\}\times\{2\times 4\times 6\times\ldots\times(2n-2)\times 2n\}}{n!} \\ &= \frac{\{1\times 3\times 5\times\ldots\times(2n-1)\}\times 2^n\times\{1\times 2\times 3\times\ldots\times n\}}{n!} \\ &= \frac{\{1\times 3\times 5\times\ldots\times(2n-1)\}\times 2^n\times(n!)}{n!} \\ &= 2^n\times\{1\times 3\times 5\times\ldots\times(2n-1)\}. \end{aligned}$$

Hence, $\dfrac{(2n)!}{n!} = 2^n\times\{1\times 3\times 5\times\ldots\times(2n-1)\}$.

EXAMPLE 10 *Prove that* $(n!)^2 \le (n!)\times n^n < (2n)!$ *for all* $n\in N$.

SOLUTION We have

$$\begin{aligned} (n!)^2 &= (n!)\times(n!) = (n!)\times(1\times 2\times 3\times\ldots\times n) \\ &\le (n!)\times n^n \qquad [\because\ r\le n \text{ for each } r = 1, 2, 3, \ldots, n]. \end{aligned}$$

$\therefore\quad (n!)^2 \le (n!)\times n^n.$ … (i)

Again, we have

$$\begin{aligned} (2n)! &= (1\times 2\times 3\times\ldots\times n)\times(n+1)\times(n+2)\times\ldots\times 2n \\ &> (n!)\times n^n. \qquad [\because\ (n+r) > n \text{ for each } r = 1, 2, \ldots, n]. \end{aligned}$$

Thus, $(n!)\times n^n < (2n)!$. … (ii)

Hence, from (i) and (ii), we have

$$(n!)^2 \le (n!)\times n^n < (2n)!.$$

EXERCISE 8A

1. Compute:

   (i) $\dfrac{9!}{(5!)\times(3!)}$ (ii) $\dfrac{32!}{29!}$ (iii) $\dfrac{(12!)-(10!)}{9!}$

2. Prove that LCM $\{6!, 7!, 8!\} = 8!$.

3. Prove that $\dfrac{1}{10!}+\dfrac{1}{11!}+\dfrac{1}{12!}=\dfrac{145}{12!}$.

4. If $\dfrac{1}{6!}+\dfrac{1}{7!}=\dfrac{x}{8!}$, find the value of x.

5. Write the following products in factorial notation:

   (i) $6\times7\times8\times9\times10\times11\times12$ (ii) $3\times6\times9\times12\times15$

6. Which of the following are true or false?

   (i) $(2+3)! = 2!+3!$ (ii) $(2\times3)! = (2!)\times(3!)$

7. If $(n+1)! = 12\times(n-1)!$, find the value of n.

8. If $(n+2)! = 2550\times n!$, find the value of n.

9. If $(n+3)! = 56\times(n+1)!$, find the value of n.

10. If $\dfrac{n!}{(2!)\times(n-2)!}:\dfrac{n!}{(4!)\times(n-4)!}=2:1$, find the value of n.

11. If $\dfrac{(2n)!}{(3!)\times(2n-3)!}:\dfrac{n!}{(2!)\times(n-2)!}=44:3$, find the value of n.

12. Evaluate $\dfrac{n!}{(r!)\times(n-r)!}$, when $n = 15$ and $r = 12$.

13. Prove that $(n+2)\times(n!) = (n!)+(n+1)!$.

14. Prove that:

   (i) $\dfrac{n!}{r!}=n(n-1)(n-2)\ldots(r+1)$

   (ii) $(n-r+1)\cdot\dfrac{n!}{(n-r+1)!}=\dfrac{n!}{(n-r)!}$

   (iii) $\dfrac{n!}{r!(n-r)!}+\dfrac{n!}{(r-1)!(n-r+1)!}=\dfrac{(n+1)!}{r!(n-r+1)!}$

ANSWERS (EXERCISE 8A)

1. (i) 504 (ii) 29760 (iii) 1310 **4.** $x = 64$

5. (i) $\dfrac{12!}{5!}$ (ii) $3^5(5!)$ **6.** (i) False (ii) False **7.** $n = 3$

8. $n = 49$ **9.** $n = 5$ **10.** $n = 5$ **11.** $n = 6$ **12.** 455

HINTS TO SOME SELECTED QUESTIONS

3. LHS $=\left\{\dfrac{1}{10!}+\dfrac{1}{11\times(10!)}\right\}+\dfrac{1}{12!}=\dfrac{(11+1)}{11\times(10!)}+\dfrac{1}{12\times(11!)}$

$$= \frac{12}{11!} + \frac{1}{12 \times (11!)} = \frac{(144+1)}{12 \times (11!)} = \frac{145}{12!} = \text{RHS}.$$

7. $(n+1)n \times (n-1)! = 12 \times (n-1)! \Rightarrow (n+1) \times n = 12 = 4 \times 3 \Rightarrow n = 3.$

8. $(n+2)(n+1) \times (n!) = 51 \times 50 \times (n!) \Rightarrow (n+2)(n+1) = 51 \times 50$

$\Rightarrow n+1 = 50 \Rightarrow n = 49.$

9. $(n+3)(n+2) \times (n+1)! = 8 \times 7 \times (n+1)! \Rightarrow (n+3)(n+2) = 8 \times 7 \Rightarrow n+2 = 7 \Rightarrow n = 5.$

10. $$\frac{n!}{2 \times (n-2)(n-3) \times (n-4)!} \times \frac{24 \times (n-4)!}{n!} = \frac{2}{1}$$

$\Rightarrow (n-2)(n-3) = 6 = 3 \times 2 \Rightarrow n-3 = 2 \Rightarrow n = 5.$

11. $$\frac{(2n)(2n-1)(2n-2) \times (2n-3)!}{6 \times (2n-3)!} \times \frac{2 \times (n-2)!}{n(n-1) \times (n-2)!} = \frac{44}{3}$$

$$\Rightarrow \frac{8 \times n \times (n-1) \times (2n-1)}{6 \times n \times (n-1)} = \frac{44}{3} \Rightarrow 2n-1 = \frac{44}{3} \times \frac{6}{8} \Rightarrow 2n-1 = 11 \Rightarrow n = 6.$$

12. $$\frac{15!}{(12!) \times (3!)} = \frac{15 \times 14 \times 13 \times (12!)}{6 \times (12!)} = 455.$$

13. RHS $= (n!) + (n+1) \times (n!) = (n+2) \times (n!) =$ LHS.

14. (i) RHS $= \dfrac{n(n-1)(n-2) \ldots (r+1) \times (r!)}{r!} = \dfrac{n!}{r!}.$

(ii) We know that $(n-r+1)! = (n-r+1) \cdot (n-r)!$.

(iii) LHS $= \dfrac{n!}{(n-r)! \cdot (r-1)!} \left\{ \dfrac{1}{r} + \dfrac{1}{n-r+1} \right\}$

$$= \frac{n!}{(n-r)! \cdot (r-1)!} \cdot \frac{(n+1)}{r(n-r+1)} = \frac{(n+1)!}{(r!) \cdot (n-r+1)!} = \text{RHS}.$$

FUNDAMENTAL PRINCIPLES OF COUNTING

1. FUNDAMENTAL PRINCIPLE OF MULTIPLICATION *If there are two operations such that one of them can be performed in m ways, and when it has been performed in anyone of these m ways, the second operation can be performed in n ways then the two operations in succession can be performed in* $(m \times n)$ *ways.*

EXAMPLE 1 *In a class there are 15 boys and 12 girls. The teacher wants to select 1 boy and 1 girl to represent the class for a function. In how many ways can the teacher make the selection?*

SOLUTION Here the teacher is to perform two operations:

(i) selecting 1 boy out of 15 boys,

and (ii) selecting 1 girl out of 12 girls.

The first of these can be done in 15 ways and the second in 12 ways.

So, by the fundamental principle of multiplication, the required number of ways $= (15 \times 12) = 180$.

Hence, the teacher can make the selection of 1 boy and 1 girl in 180 ways.

2. FUNDAMENTAL PRINCIPLE OF ADDITION *If there are two operations such that they can be performed independently in m and n ways respectively then either of the two operations can be performed in* $(m+n)$ *ways.*

EXAMPLE 2 *In a class there are* 16 *boys and* 9 *girls. The teacher wants to select either a boy or a girl as a class representative. In how many ways can the teacher make the selection?*

SOLUTION Here the teacher is to perform either of the following two operations:

(i) selecting 1 boy out of 16 boys,

and (ii) selecting 1 girl out of 9 girls.

The first of these can be performed in 16 ways and the second in 9 ways.

By the fundamental principle of addition, either of the two operations can be performed in $(16+9)$ ways = 25 ways.

Hence, the teacher can make the selection of 1 boy or 1 girl in 25 ways.

ILLUSTRATIVE EXAMPLES

EXAMPLE 1 *In a cinema hall, there are three entrance doors and two exit doors. In how many ways can a person enter the hall and then come out?*

SOLUTION Clearly, a person can enter the hall through any of the three entrance doors. So, there are 3 ways of entering the hall.

After entering the hall, the person can come out through any of the two exit doors. So, there are 2 ways of coming out.

Hence, by the fundamental principle of multiplication, the number of ways in which a person can enter the hall and then come out $= (3 \times 2) = 6$.

EXAMPLE 2 *Three persons enter a railway carriage, where there are* 5 *vacant seats. In how many ways can they seat themselves?*

SOLUTION Clearly, the first person can occupy any of the 5 seats.

So, there are 5 ways in which the first person can seat himself.

Now, the second person can occupy any of the remaining 4 seats. So, the second person can be seated in 4 ways.

Similarly, the third person can occupy a seat in 3 ways.

Hence, by the fundamental principle of multiplication, the required number of ways $= (5 \times 4 \times 3) = 60$.

EXAMPLE 3 *The flag of a newly formed forum is in the form* □□□ *of three blocks, each to be coloured differently. If there are six different colours on the whole to choose from, how many such designs are possible?*

SOLUTION The first block of the flag can be coloured by anyone of the 6 given colours. So, there are 6 ways to colour the first block.

Now, the second block can be coloured by anyone of the remaining five colours. So, there are 5 ways to colour the second block.

The third block can now be coloured by anyone of the remaining four colours. So, there are 4 ways to colour the third block.

Hence, by the fundamental principle of multiplication, the required number of designs = $(6 \times 5 \times 4) = 120$.

EXAMPLE 4 *There are* 4 *routes between Delhi and Patna. In how many different ways can a man go from Delhi to Patna and return, if for returning*

(i) any of the routes is taken;

(ii) the same route is taken;

(iii) the same route is not taken?

SOLUTION We have the following three cases.

Case (i) *When any of the routes is taken for returning:*

The man may take any route for going from Delhi to Patna.

So, there are 4 ways of going from Delhi to Patna.

When done so, he may return by any of the 4 routes.

So, there are 4 ways of returning from Patna to Delhi.

Hence, by the fundamental principle of multiplication, the total number of ways for going to Patna and returning back to Delhi = $(4 \times 4) = 16$.

Case (ii) *When the same route is taken for returning*:

In this case, there are 4 ways of going to Patna and only 1 way of returning, namely by the same route.

Hence, the required number of ways = $(4 \times 1) = 4$.

Case (iii) *When the same route is not taken for returning:*

In this case, there are 4 ways of going to Patna.

But, the man does not return by the same route.

So, there are 3 ways of returning back to Delhi.

Hence, the required number of ways = $(4 \times 3) = 12$.

EXAMPLE 5 *There are six multiple-choice questions in an examination. Find the total number of ways of answering these questions, if the first three questions have* 5 *choices each and the next three have* 4 *choices each.*

SOLUTION Each of the first three questions has 5 choices.

So, each one of them can be answered in 5 ways.

Each of the next three questions has 4 choices.

So, each one of them can be answered in 4 ways.

Hence, by the fundamental principle of multiplication, the total number of ways of answering the six questions = $(5 \times 5 \times 5 \times 4 \times 4 \times 4) = 8000$.

EXAMPLE 6 *Four flags of different colours are given. How many different signals can be generated, if a signal requires the use of two flags, one below the other?*

SOLUTION The total number of signals is equal to the number of ways of filling two places $\begin{array}{|c|}\hline \qquad \\ \hline \qquad \\ \hline\end{array}$ in succession by four flags of different colours.

The upper place can be filled in 4 different ways by any of the four flags. Following it, the lower space can be filled in 3 different ways by anyone of the remaining three flags.

Hence, by the fundamental principle of multiplication, the required number of signals = $(4 \times 3) = 12$.

EXAMPLE 7 *Find the number of 4-letter words, with or without meaning, which can be formed out of the letters of the word, 'NOSE', when:*

(i) the repetition of the letters is not allowed,

(ii) the repetition of the letters is allowed.

SOLUTION **Case (i)** *When the repetition of the letters is not allowed:*

In this case, the total number of words is the same as the number of ways of filling 4 places □□□□ by 4 different letters chosen from N, O, S, E.

The first place can be filled by any of the 4 letters. Thus, there are 4 ways of filling the first place. Since the repetition of letters is not allowed, so the second place can be filled by any of the remaining 3 letters in 3 different ways.

Following it, the third place can be filled by any of the two remaining letters in 2 different ways. And, the fourth place can be filled by the remaining one letter in 1 way.

So, by the fundamental principle of multiplication, the required number of 4-letter words = $(4 \times 3 \times 2 \times 1) = 24$.

Hence, required number of words = 24.

Case (ii) *When the repetition of the letters is allowed*:

In this case, each of the four different places can be filled in succession in 4 different ways.

So, the required number of 4-letter words = $(4 \times 4 \times 4 \times 4) = 256$.

EXAMPLE 8 *How many words (with or without meaning) of three distinct letters of the English alphabet are there?*

SOLUTION Clearly, we have to fill up three places by distinct letters of the English alphabet.

The first place can be filled by any of the 26 letters. Thus, there are 26 ways to fill the first place.

The second place can be filled by any of the remaining 25 letters. So, there are 25 ways to fill the second place.

The third place can be filled by any of the remaining 24 letters.

Thus, there are 24 ways to fill the third place.

Hence, by the fundamental principle of multiplication, the required number of words = $(26 \times 25 \times 24) = 15600$.

EXAMPLE 9 *How many numbers are there between* 100 *and* 1000 *in which all the digits are distinct?*

SOLUTION Clearly, every number between 100 and 1000 is a 3-digit number.

So, we have to form all possible 3-digit numbers with distinct digits. We cannot have 0 at the hundred's place.

So, the hundred's place can be filled with any of the nine digits, namely 1, 2, 3, 4, 5, 6, 7, 8, 9.

So, there are 9 ways of filling the hundred's place. After filling the hundred's place, nine digits are left including 0. So, there are 9 ways of filling the ten's place.

Now, the unit's place can be filled by any of the remaining 8 digits. So, there are 8 ways of filling the unit's place.

Hence, the required number of numbers = $(9 \times 9 \times 8) = 648$.

EXAMPLE 10 *How many 9-digit numbers of different digits can be formed?*

SOLUTION The ten-crore's place of a 9-digit number cannot be 0.

So, this place can be filled up by any of the digits from 1 to 9. So, there are 9 ways of filling the ten-crore's place.

The crore's place can now be filled with 0 or any of the remaining 8 digits. Thus, there are 9 ways of filling it.

The ten-lakh's place can now be filled with any of the remaining 8 digits. Thus, there are 8 ways of filling it.

Similarly, we can fill lakh's place in 7 ways; ten thousand's place in 6 ways; thousand's place in 5 ways; hundred's place in 4 ways; ten's place in 3 ways and unit's place in 2 ways.

Hence, by the fundamental principle of multiplication, the required number of numbers = $(9 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2) = 3265920$.

EXAMPLE 11 *How many numbers between* 2000 *and* 3000 *can be formed from the digits* 2, 3, 4, 5, 6, 7 *when repetition of digits is not allowed?*

SOLUTION Clearly, a number between 2000 and 3000 will have 2 at the thousand's place. So, this place can be filled in 1 way only.

Now, the hundred's place can be filled by any of the remaining five digits. So, there are 5 ways of filling the hundred's place.

Similarly, the ten's place can be filled in 4 ways and the unit's place can be filled in 3 ways.

Hence, by the fundamental principle of multiplication, the total number of required numbers = $(1 \times 5 \times 4 \times 3) = 60$.

EXAMPLE 12 *How many 3-digit odd numbers can be formed by using the digits* 1, 2, 3, 4, 5, 6 *when*

(i) *the repetition of digits is not allowed?*

(ii) *the repetition of digits is allowed?*

SOLUTION For a number to be odd, we must have 1, 3 or 5 at the unit's place. So, there are 3 ways of filling the unit's place.

Case (i) *When the repetition of digits is not allowed*:

In this case, after filling the unit's place, we may fill the ten's place by any of the remaining five digits. So, there are 5 ways of filling the ten's place.

Now, the hundred's place can be filled by any of the remaining 4 digits. So, there are 4 ways of filling the hundred's place.

So, by the fundamental principle of multiplication, the required number of odd numbers = $(3 \times 5 \times 4) = 60$.

Case (ii) *When the repetition of digits is allowed:*

Since the repetition of digits is allowed, so after filling the unit's place, we may fill the ten's place by any of the given six digits. So, there are 6 ways of filling the ten's place.

Similarly, the hundred's place can be filled by any of the given six digits. So, it can be filled in 6 ways.

Hence, by the fundamental principle of multiplication, the required number of odd numbers = $(3 \times 6 \times 6) = 108$.

EXAMPLE 13 *How many odd numbers less than* 1000 *can be formed by using the digits* 0, 2, 5, 7 *when the repetition of digits is allowed?*

SOLUTION Since each number is less than 1000, required numbers are the 1-digit, 2-digit and 3-digit numbers.

One-digit numbers: Clearly, there are two one-digit odd numbers, namely 5 and 7, formed of the given digits.

Two-digit numbers: Since we are to form 2-digit odd numbers, we may put 5 or 7 at the unit's place. So, there are 2 ways of filling the unit's place.

Now, we cannot use 0 at the ten's place and the repetition of digits is allowed. So, we may fill up the ten's place by any of the digits 2, 5, 7. Thus, there are 3 ways of filling the ten's place.

Hence, the required type of 2-digit numbers = $(2 \times 3) = 6$.

Three-digit numbers: To have an odd 3-digit number, we may put 5 or 7 at the unit's place. So, there are 2 ways of filling the unit's place.

We may fill up the ten's place by any of the digits 0, 2, 5, 7. So, there are 4 ways of filling the ten's place.

We cannot put 0 at the hundred's place. So, the hundred's place can be filled by any of the digits 2, 5, 7 and so it can be done in 3 ways.

$\therefore$ the required number of 3-digit numbers = $(2 \times 4 \times 3) = 24$.

Hence, the total number of required type of numbers $= (2 + 6 + 24) = 32$.

EXAMPLE 14 *How many numbers are there between* 100 *and* 1000 *such that* 7 *is in the unit's place?*

SOLUTION Clearly, the numbers between 100 and 1000 are 3-digit numbers. So, we have to form 3-digit numbers with 7 at the unit's place.

Clearly, 0 cannot be there at the hundred's place.

So, the hundred's place can be filled with any of the digits from 1 to 9.

Thus, the hundred's place can be filled in 9 ways.

Now, the ten's place can be filled with any of the digits from 0 to 9.

So, there are 10 ways of filling the ten's place.

The unit's place can be filled with 7, i.e., in 1 way only.

Hence, the number of required type of numbers $= (9 \times 10 \times 1) = 90$.

EXAMPLE 15 *How many numbers are there between* 100 *and* 1000 *such that at least one of their digits is* 7*?*

SOLUTION Clearly, the numbers between 100 and 1000 are 3-digit numbers.

So, we have to form 3-digit numbers such that at least one of their digits is 7.

Case I *Three-digit numbers with* 7 *at unit's place:*

The number of ways to fill the hundred's place = 9 [by any digit from 1 to 9].

The number of ways to fill the ten's place = 10 [by any digit from 0 to 9].

The number of ways to fill the unit's place = 1 [by 7 only].

$\therefore$ the number of such numbers $= (9 \times 10 \times 1) = 90$.

Case II *Three-digit numbers with* 7 *at ten's place:*

The number of ways to fill the hundred's place = 9 [by any digit from 1 to 9].

The number of ways to fill the ten's place = 1 [by 7 only].

The number of ways to fill the unit place = 10 [by any digit from 0 to 9].

$\therefore$ the number of such numbers $= (9 \times 1 \times 10) = 90$.

Case III *Three-digit numbers with* 7 *at hundred's place:*

The number of ways to fill the hundred's place = 1 [by 7 only].

The number of ways to fill the ten's place = 10 [by any digit from 0 to 9].

The number of ways to fill the unit's place = 10 [by any digit from 0 to 9].

$\therefore$ the number of such numbers = $(1 \times 10 \times 10) = 100$.

Hence, the total number of required numbers = $(90 + 90 + 100) = 280$.

EXAMPLE 16 *How many numbers are there between* 100 *and* 1000, *which have exactly one of their digits as* 7?

SOLUTION Clearly, the numbers between 100 and 1000 are 3-digit numbers.

So, we have to form 3-digit numbers having exactly one of their digits as 7.

Case I 3-*digit numbers having* 7 *at unit's place only:*

Number of ways to fill the unit's place =1 [with 7 only].

Number of ways to fill the ten's place = 9
[any digit from 0 to 9 except 7].

Number of ways to fill the hundred's place = 8
[any digit from 1 to 9 except 7].

So, the number of such numbers = $(1 \times 9 \times 8) = 72$.

Case II 3-*digit numbers having* 7 *at ten's place only:*

Number of ways to fill the ten's place = 1 [with 7 only].

Number of ways to fill the unit's place = 9
[any digit from 0 to 9 except 7].

Number of ways to fill the hundred's place = 8
[any digit from 1 to 9 except 7].

So, the number of such numbers = $(1 \times 9 \times 8) = 72$.

Case III 3-*digit numbers having* 7 *at hundred's place only:*

Number of ways to fill the hundred's place = 1
[with 7 only].

Number of ways to fill the ten's place = 9
[any digit from 0 to 9 except 7].

Number of ways to fill the unit's place = 9
[any digit from 0 to 9 except 7].

So, the number of such numbers = $(1 \times 9 \times 9) = 81$.

Hence, the total number of required numbers = $(72 + 72 + 81) = 225$.

EXAMPLE 17 *How many numbers are there between* 100 *and* 1000 *such that every digit is either* 2 *or* 9?

SOLUTION Clearly, the numbers between 100 and 1000 are 3-digit numbers.

So, we have to form 3-digit numbers by using 2 and 9.

Clearly, the repetition of digits is allowed.

Each one of the unit's, ten's and hundred's places can be filled in 2 ways.

Hence, the required number of numbers = $(2 \times 2 \times 2) = 8$.

EXAMPLE 18 *How many 3-digit numbers can be formed without using the digits* 0, 2, 3, 4, 5 *and* 6?

SOLUTION Clearly, we have to form 3-digit numbers by using the digits 1, 7, 8 and 9 while the repetition of digits is allowed.

Clearly, each one of the unit's, ten's and hundred's places can be filled by any of the digits 1, 7, 8, 9, i.e., in four ways.

Hence, the number of required numbers = $(4 \times 4 \times 4) = 64$.

EXERCISE 8B

1. There are 10 buses running between Delhi and Agra. In how many ways can a man go from Delhi to Agra and return by a different bus?
2. A, B and C are three cities. There are 5 routes from A to B and 3 routes from B to C. How many different routes are there from A to C via B?
3. There are 12 steamers plying between A and B. In how many ways could the round trip from A be made if the return was made on (i) the same steamer? (ii) a different steamer?
4. In how many ways can 4 people be seated in a row containing 6 seats?
5. In how many ways can 5 ladies draw water from 5 taps, assuming that no tap remains unused?
6. In a textbook on mathematics there are three exercises A, B and C consisting of 12, 18 and 10 questions respectively. In how many ways can three questions be selected choosing one from each exercise?
7. In a school, there are four sections of 40 students each in XI standard. In how many ways can a set of 4 student representatives be chosen, one from each section?
8. In how many ways can a vowel, a consonant and a digit be chosen out of the 26 letters of the English alphabet and the 10 digits?
9. How many 8-digit telephone numbers can be constructed using the digits 0 to 9 if each number starts with 270 and no digit appears more than once?
10. (a) A coin is tossed three times and the outcomes are recorded. How many possible outcomes are there?

    (b) How many possible outcomes if the coin is tossed:

    (i) four times? (ii) five times? (iii) n times?
11. Find the number of different signals that can be generated by arranging at least 2 flags in order (one below the other) on a vertical staff, if five different flags are available.
12. How many 4-letter codes can be formed using the first 10 letters of the English alphabet, if no letter can be repeated?
13. Given, $A = \{2, 3, 5\}$ and $B = \{0, 1\}$. Find the number of different ordered pairs in which the first entry is an element of A and the second is an element of B.
14. How many arithmetic progressions with 10 terms are there whose first term is in the set $\{1, 2, 3\}$ and whose common difference is in the set $\{2, 3, 4\}$?

15. There are 6 items in column A and 6 items in column B. A student is asked to match each item in column A with an item in column B. How many possible (correct or incorrect) answers are there to this question?

16. A mint prepares metallic calendars specifying months, dates and days in the form of monthly sheets (one plate for each month). How many types of February calendars should it prepare to serve for all the possibilities in the future years?

17. From among the 36 teachers in a school, one principal and one vice-principal are to be appointed. In how many ways can this be done?

18. A sample of 3 bulbs is tested. A bulb is labelled as G if it is good and D if it is defective. Find the number of all possible outcomes.

19. For a set of five true or false questions, no student has written the all correct answer and no two students have given the same sequence of answers. What is the maximum number of students in the class for this to be possible?

20. In how many ways can the following prizes be given away to a class of 20 students:

first and second in mathematics; first and second in chemistry; first in physics and first in English?

21. Find the total number of ways of answering 5 objective-type questions, each question having 4 choices.

22. A gentleman has 6 friends to invite. In how many ways can he send invitation cards to them, if he has 3 servants to carry the cards?

23. In how many ways 6 rings of different types can be worn in 4 fingers?

24. In how many ways can 5 letters be posted in 4 letter boxes?

25. How many 3-letter words can be formed using a, b, c, d, e if

(i) repetition of letters is not allowed?

(ii) repetition of letters is allowed?

26. How many 4-digit numbers are there, when a digit may be repeated any number of times?

27. How many numbers can be formed from the digits 1, 3, 5, 9 if repetition of digits is not allowed?

28. How many 3-digit numbers are there with no digit repeated?

29. How many 3-digit numbers can be formed by using the digits 0, 1, 3, 5, 7 while each digit may be repeated any number of times?

30. How many 6-digit numbers can be formed from the digits 0, 1, 3, 5, 7, 9 when no digit is repeated? How many of them are divisible by 10?

31. How many natural numbers less than 1000 can be formed from the digits 0, 1, 2, 3, 4, 5 when a digit may be repeated any number of times?

32. How many 6-digit telephone numbers can be constructed using the digits 0 to 9, if each number starts with 67 and no digit appears more than once?

33. In how many ways can three jobs I, II and III be assigned to three persons A, B and C if one person is assigned only one job and all are capable of doing each job?

34. A number lock on a suitcase has three wheels each labelled with ten digits 0 to 9. If opening of the lock is a particular sequence of three digits with no repeats, how many such sequences will be possible? Also, find the number of unsuccessful attempts to open the lock.

35. A customer forgets a four-digit code for an automated teller machine (ATM) in a bank. However, he remembers that this code consists of digits 3, 5, 6, 9. Find the largest possible number of trials necessary to obtain the correct code.

36. In how many ways can 3 prizes be distributed among 4 girls, when
 (i) no girl gets more than one prize?
 (ii) a girl may get any number of prizes?
 (iii) no girl gets all the prizes?

ANSWERS (EXERCISE 8B)

1. 90	**2.** 15	**3.** (i) 12 (ii) 132	**4.** 360	**5.** 120	**6.** 2160
7. 2560000	**8.** 1050	**9.** 2520	**10.** (a) 8 (b) (i) 16 (ii) 32 (iii) 2^n		
11. 320	**12.** 5040	**13.** 6	**14.** 9	**15.** 720	**16.** 14
17. 1260	**18.** 8	**19.** 31	**20.** 57760000		**21.** 4^5
22. 3^6	**23.** 4^6	**24.** 4^5	**25.** (i) 60 (ii) 125		**26.** 9000
27. 64	**28.** 648	**29.** 100	**30.** 600, 120		**31.** 215
32. 1680	**33.** 6	**34.** 720, 719	**35.** 24	**36.** (i) 24 (ii) 64 (iii) 60	

HINTS TO SOME SELECTED QUESTIONS

3. (i) In this case the round trip from A is to be made by the same steamer. So, there are 12 ways of making the round trips.

(ii) In this case, number of ways to go from A to B = 12.
Number of ways to go from B to A = 11.
Hence, the required number of ways = $(12 \times 11) = 132$.

4. 1st person can be seated in 6 ways, 2nd person in 5 ways, 3rd person in 4 ways and 4th person in 3 ways.
Required number of ways = $(6 \times 5 \times 4 \times 3) = 360$.

5. 1st lady can draw water from any of the 5 taps.

So, first lady can draw water in 5 ways, 2nd in 4 ways, 3rd in 3 ways, 4th in 2 ways and 5th in 1 way.
Required number of ways = $(5 \times 4 \times 3 \times 2 \times 1) = 120$.

6. 1 question out of 12 can be chosen in 12 ways.
1 question out of 18 can be chosen in 18 ways.
1 question out of 10 can be chose in 10 ways.
Required number of ways = $(12 \times 18 \times 10) = 2160$.

7. From each of the 4 sections 1 representative out of 40 can be chosen in 40 ways.

So, by the fundamental principle of multiplication, the required number of ways $= (40 \times 40 \times 40 \times 40) = 2560000$.

8. There are 5 vowels, 21 consonants and 10 digits.

Number of ways of choosing 1 vowel out of 5 vowels = 5.

Number of ways of choosing 1 consonant out of 21 consonants = 21.

Number of ways of choosing 1 digit out of 10 digits = 10.

Required number of ways $= (5 \times 21 \times 10) = 1050$.

9. The remaining digits to be used are 1, 3, 4, 5, 6, 8, 9.

And, we have to select 5 digits out of these eight.

First digit can be anyone of these 7 digits.

So, there are 7 ways of choosing the 1st digit.

Similarly, we may choose 2nd digit in 6 ways, 3rd digit in 5 ways, 4th digit in 4 ways and 5th digit in 3 ways.

By the fundamental principle of multiplication, the required number of ways $= (7 \times 6 \times 5 \times 4 \times 3) = 2520$.

10. (a) In a single throw of a coin there are 2 possible outcomes namely Head (H) and Tail (T).

So, by the fundamental principle of multiplication, the total number of possible outcomes in 3 tosses $= (2 \times 2 \times 2) = 2^3 = 8$.

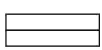
(b) Similarly, we have

(i) total number of possible outcomes in 4 tosses $= (2 \times 2 \times 2 \times 2) = 2^4 = 16$.

(ii) total number of possible outcomes in 5 tosses $= (2 \times 2 \times 2 \times 2 \times 2) = 2^5 = 32$.

(iii) total number of possible outcomes in n tosses $= 2^n$.

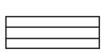
11. A signal can consist of either 2 flags, 3 flags, 4 flags or 5 flags.

2-flag signals: 

1st place can be filled by any of the 5 flags in 5 ways.

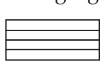
2nd place can be filled by any of the remaining 4 flags in 4 ways.

Total number of 2-flag signals $= (5 \times 4) = 20$.

3-flag signals: 

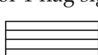
1st place can be filled in 5 ways, 2nd in 4 ways and 3rd in 3 ways.

Total number of 3-flag signals $= (5 \times 4 \times 3) = 60$.

4-flag signals: 

1st place can be filled in 5 ways, 2nd in 4 ways, 3rd in 3 ways and 4th in 2 ways.

Total number of 4-flag signals $= (5 \times 4 \times 3 \times 2) = 120$.

5-flag signals: 

1st place can be filled in 5 ways, 2nd in 4 ways, 3rd in 3 ways, 4th in 2 ways and 5th in 1 way only.

Total number of 5-flag signals $= (5 \times 4 \times 3 \times 2 \times 1) = 120$.

Hence, total number of signals $= (20 + 60 + 120 + 120) = 320$.

12. We have to fill up 4 places □□□□ by first 10 letters of the English alphabet. The first place can be filled by any of the 10 letters in 10 ways.

Similarly, 2nd place can be filled in 9 ways; 3rd place can be filled in 8 ways and 4th place can be filled in 7 ways.

Required number of 4-letter codes = $(10 \times 9 \times 8 \times 7) = 5040$.

13. If $n(A) = p$ and $n(B) = q$, then $n(A \times B) = pq$.

Here $n(A) = 3$ and $n(B) = 2$, then $n(A \times B) = (3 \times 2) = 6$.

14. Required number of AP's $= (3 \times 3) = 9$.

15. First item in column A can be matched with any of the 6 items in column B.

Thus, the first item of column A can be matched in 6 ways.

Similarly, 2nd item of column A can be matched in 5 ways.

Similarly, 3rd, 4th, 5th and 6th items of column A can be matched in 4, 3, 2 and 1 ways respectively.

By the fundamental principle of multiplication, the required number of ways $= (6 \times 5 \times 4 \times 3 \times 2 \times 1) = 720$.

16. There are two types of Februaries, one having 28 days and the other having 29 days. So, there are 2 ways of preparing calendars for February.

Also, the month may start from any day from Monday to Sunday. So, there are 7 ways of preparing it for various days.

Hence, the required number of plates $= (2 \times 7) = 14$.

17. Number of ways of choosing 1 principal out of 36 teachers = 36.

Number of ways of choosing 1 vice-principal out of 35 teachers = 35.

Hence, the required number of ways $= (36 \times 35) = 1260$.

18. All possible outcomes are

GGG, GGD, GDG, DGG, GDD, DGD, DDG, DDD.

Clearly, there are 8 possible outcomes.

19. Clearly there are 2 ways of answering each of the 5 questions, i.e., true or false.

So, the total number of different sequences of answers $= (2 \times 2 \times 2 \times 2 \times 2) = 32$.

There is only one all correct answer.

Maximum number of sequences of answers leaving aside the all correct answer $= (32 - 1) = 31$.

Hence, the maximum possible number of students = 31.

20. 1st and 2nd prizes in mathematics can be given in (20×19) ways;

1st and 2nd prizes in chemistry can be given in (20×19) ways;

1st prize in physics can be given in 20 ways;

1st prize in English can be given in 20 ways.

Required number of ways in all $= (20 \times 19) \times (20 \times 19) \times 20 \times 20 = 57760000$.

21. Since each question can be answered in 4 ways, total number of ways of answering 5 questions $= (4 \times 4 \times 4 \times 4 \times 4) = 4^5$.

22. Each friend can be invited by any of the 3 servants, i.e., in 3 ways. So, the required number of ways $= (3 \times 3 \times 3 \times 3 \times 3 \times 3) = 3^6$.

23. The first ring can be worn in any of the 4 fingers. So, there are 4 ways of wearing it. Similarly, each of the other rings can be worn in 4 ways.

So, the required number of ways $= (4 \times 4 \times 4 \times 4 \times 4 \times 4) = 4^6$.

24. Since each letter can be posted in anyone of the 4 letter boxes, so each letter can be posted in 4 ways.

So, the required number of ways = $(4 \times 4 \times 4 \times 4 \times 4) = 4^5$.

25. (i) Total number of 3-letter words is equal to the number of ways of filling 3 places. First place can be filled in 5 ways by any of the given five letters. Second place can be filled in 4 ways by any of the remaining 4 letters and the third place can be filled in 3 ways by any of the remaining 3 letters.

So, the required number of 3-letter words = $(5 \times 4 \times 3) = 60$.

(ii) When repetition of letters is allowed, each place can be filled by any of the 5 letters in 5 ways.

$\therefore$ the required number of ways = $(5 \times 5 \times 5) = 125$.

26. In a 4-digit number, 0 cannot be placed at the thousand's place. So, thousand's place can be filled by any digit from 1 to 9 in nine ways. Now, each of hundred's, ten's and one's places can be filled by any of the digits from 0 to 9 in ten ways.

So, the required number of numbers = $(9 \times 10 \times 10 \times 10) = 9000$.

27. One-digit numbers:

Clearly, there are four 1-digit numbers.

Two-digit numbers:

We may fill the unit's place by any of the four given digits.

Thus, there are 4 ways to fill the unit's place.

The ten's place may now be filled by any of the remaining three digits. So, there are 3 ways to fill the ten's place.

Number of 2-digit numbers = $(4 \times 3) = 12$.

Three-digit numbers:

Number of ways to fill the unit's, ten's and hundred's places are 4, 3 and 2 respectively.

Number of 3-digit numbers = $(4 \times 3 \times 2) = 24$.

Four-digit numbers:

Number of ways to fill the unit's, ten's, hundred's and thousand's places are 4, 3, 2 and 1 respectively.

Number of 4-digit numbers = $(4 \times 3 \times 2 \times 1) = 24$.

Hence, the number of required numbers = $(4 + 12 + 24 + 24) = 64$.

28. The hundred's place cannot contain 0. So, it can be filled by any digit from 1 to 9. Thus, there are 9 ways to fill this place. Now, the ten's digit can be filled by any of the remaining 9 digits out of 0 to 9. So, there are 9 ways to fill the ten's place. And, there are 8 ways to fill the unit's digit.

Required number of numbers = $(9 \times 9 \times 8) = 648$.

29. The hundred's place cannot contain 0. So, it can be filled by any of the digits 1, 3, 5, 7. So, there are 4 ways to fill it.

Now, each of the ten's and unit's digits can be filled by any of the given five digits. So, each can be filled in 5 ways.

$\therefore$ the required number of numbers = $(4 \times 5 \times 5) = 100$.

30. The lakh's place cannot be 0. So, it can be filled by any of the digits 1, 3, 5, 7, 9. So, there are 5 ways to fill it. The ten-thousand's place can be filled by 0 or the remaining four digits, i,e., in 5 ways. The thousand's, hundred's, ten's and unit's digits can be filled in 4, 3, 2 and 1 ways respectively.

Required number of numbers = $(5 \times 5 \times 4 \times 3 \times 2 \times 1) = 600$.

Particular case Let us put 0 at unit's place, i.e., it is filled in 1 way.

Now ten's, hundred's, thousand's, ten-thousand's and lakh's place can be filled in 5, 4, 3, 2 and 1 ways respectively.

So, the number of numbers having 0 as unit's digit $= (5 \times 4 \times 3 \times 2 \times 1) = 120$.

So, 120 numbers are divisible by 10.

31. There are 5 one-digit natural numbers out of the given digits.

For getting 2-digit natural numbers we cannot put 0 at the ten's place. So, this place can be filled by any of the given five nonzero digits in 5 ways.

The unit's digit can be filled by any of the given six digits in 6 ways.

So, the number of 2-digit natural numbers $= (5 \times 6) = 30$.

Similarly, to get a 3-digit number, we cannot put 0 at the hundred's place. So, this place can be filled in 5 ways, each of the ten's and unit's places can be filled in 6 ways.

$\therefore$ the number of 3-digit numbers $= (5 \times 6 \times 6) = 180$.

Required number of numbers $= (5 + 30 + 180) = 215$.

32. Out of 10 digits from 0 to 9, we have used 2 digits, namely 6 and 7.

So, the 3rd, 4th, 5th and 6th digit may be filled in $(10 - 2) = 8, 7, 6$ and 5 ways respectively.

Hence, the required number of numbers $= (8 \times 7 \times 6 \times 5) = 1680$.

33. A can be assigned any of the three jobs. So, there are 3 ways of assigning a job to A. B can now be assigned any of the two remaining jobs in 2 ways and C can now be assigned the job in 1 way.

So, the required number of ways $= (3 \times 2 \times 1) = 6$.

34. The first term of the sequence may be anyone of the digits from 0 to 9. So, there are 10 ways of getting the first term.

So, the 2nd and 3rd terms may be obtained in 9 ways and 8 ways respectively.

Required number of sequences $= (10 \times 9 \times 8) = 720$.

Number of unsuccessful attempts $= (720 - 1) = 719$.

35. 1st, 2nd, 3rd, 4th digits may be chosen in 4 ways, 3 ways, 2 ways and 1 way respectively.

Number of largest possible trials $= (4 \times 3 \times 2 \times 1) = 24$.

36. (i) The 1st prize can be given to anyone of the 4 girls in 4 ways.

Similarly, the second prize can be given in 3 ways and the 3rd prize can be given in 2 ways.

$\therefore$ the required number of ways $= (4 \times 3 \times 2) = 24$.

(ii) 1st prize can be given in 4 ways, 2nd prize can be given in 4 ways and the 3rd prize can be given in 4 ways.

$\therefore$ the required number of ways $= (4 \times 4 \times 4) = 64$.

(iii) Number of ways in which a girl does not get all the prizes

= (number of ways in which a girl may get any number of prizes)

– (number of ways in which a girl gets all the prizes)

$= (64 - 4) = 60$.

PERMUTATIONS

PERMUTATIONS *The different arrangements which can be made out of a given number of things by taking some or all at a time, are called permutations.*

REMARK In permutations, the order of arrangement of the objects is taken into consideration. When the order is changed, a different permutation is obtained.

ILLUSTRATIVE EXAMPLES

EXAMPLE 1 *All permutations of the letters A, B, C taking two at a time are:*

AB, BA, AC, CA, BC, CB.

Thus, we obtain 6 different permutations.

EXAMPLE 2 *All permutations of the letters A, B, C taking all at a time are:*

ABC, ACB, BAC, BCA, CAB, CBA.

Thus, the 3 letters A, B, C taking all at a time give 6 permutations.

EXAMPLE 3 *All permutations of the letters A, B, C, D taking* 3 *at a time are:*

ABC, ACB, BAC, BCA, CAB, CBA;

ABD, ADB, BAD, BDA, CAD, CDA;

BCD, BDC, CBD, CDB, DBC, DCB;

ACD, ADC, CAD, CDA, DAC, DCA.

Thus, the 4 letters A, B, C, D taking 3 at a time give 24 permutations.

PERMUTATIONS WHEN ALL THE OBJECTS ARE DISTINCT

THEOREM 1 *The number of permutations of n different objects taken r at a time, where $0 < r \leq n$ and the objects do not repeat, is given by*

$$P(n, r) \text{ or } {}^nP_r = n(n-1)(n-2)\ldots(n-r+1).$$

PROOF The number of permutations of n different objects taken r at a time is the same as the number of ways of filling in r vacant places by the n objects.

The first place can be filled up by anyone of these n objects.

Thus, there are n ways of filling up the first place.

Now, we are left with $(n-1)$ objects.

The second place can be filled up by anyone of these $(n-1)$ objects.

Thus, there are $(n-1)$ ways of filling up the second place.

Now, we are left with $(n-2)$ objects.

The third place can be filled up by any of these $(n-2)$ objects.

Thus, there are $(n-2)$ ways of filling up the third place.

Similarly, the 4th place can be filled in $(n-3)$ ways, the 5th place can be filled in $(n-4)$ ways and so on, and the rth place can be filled in $\{n-(r-1)\} = (n-r+1)$ ways.

Thus, the total number of ways of filling in r vacant places by n different objects in succession is $\{n(n-1)(n-2)\dots(n-r+1)\}$.

Hence, ${}^nP_r = n(n-1)(n-2)\dots(n-r+1)$.

REMARK We have

$$\begin{aligned} {}^nP_r &= n(n-1)(n-2)\dots(n-r+1) \\ &= \frac{n(n-1)(n-2)\dots(n-r+1)(n-r)(n-r-1)\dots 3\cdot 2\cdot 1}{(n-r)(n-r-1)\dots 3\cdot 2\cdot 1} \\ &= \frac{n!}{(n-r)!}, \text{ where } 0 < r \le n. \end{aligned}$$

THEOREM 2 *The number of all permutations of n different objects taken all at a time is given by* ${}^nP_n = n!$.

PROOF The number of all permutations of n different objects taken all at a time is the same as the number of ways of filling n vacant places by n different objects.

Proceeding as in Theorem 1, we have

$${}^nP_n = n(n-1)(n-2)(n-3)\dots\times 3\times 2\times 1 = n!.$$

THEOREM 3 *Prove that* $0! = 1$.

PROOF We have

$${}^nP_r = \frac{n!}{(n-r)!} \qquad \dots \text{(i)}$$

$$\Rightarrow \quad {}^nP_n = \frac{n!}{0!} \qquad [\text{putting } r = n \text{ in (i)}]$$

$$\Rightarrow \quad n! = \frac{n!}{0!} \qquad [\because\ {}^nP_n = n!]$$

$$\Rightarrow \quad 0! = \frac{n!}{n!} = 1.$$

Hence, $0! = 1$.

REMARK Thus, we have ${}^nP_r = \dfrac{n!}{(n-r)!}$, where $0 \le r \le n$.

SOLVED EXAMPLES

EXAMPLE 1 *Evaluate:*

(i) ${}^{12}P_4$ *(ii)* ${}^{75}P_2$ *(iii)* ${}^8P_8$

SOLUTION We have

(i) $${}^{12}P_4 = \frac{12!}{(12-4)!} = \frac{12!}{8!} = \frac{12\times 11\times 10\times 9\times (8!)}{8!} = 11880.$$

(ii) $${}^{75}P_2 = \frac{75!}{(75-2)!} = \frac{75!}{73!} = \frac{75\times 74\times (73!)}{73!} = (75\times 74) = 5550.$$

(iii) $${}^8P_8 = 8! = (8\times 7\times 6\times 5\times 4\times 3\times 2\times 1) = 40320.$$

EXAMPLE 2 *Find the value of n such that*

(i) $^nP_5 = 42 \times {}^nP_3,\ n > 4$ *(ii)* $\dfrac{^nP_4}{^{n-1}P_4} = \dfrac{5}{3},\ n > 4.$

SOLUTION We know that $^nP_r = n(n-1)(n-2)\ldots(n-r+1)$.

(i) We have

$$\frac{^nP_5}{^nP_3} = 42 \Rightarrow \frac{n(n-1)(n-2)(n-3)(n-4)}{n(n-1)(n-2)} = 42$$

$$\Rightarrow (n-3)(n-4) = 42 \qquad [\text{as } n(n-1)(n-2) \neq 0]$$

$$\Rightarrow n^2 - 7n - 30 = 0 \Rightarrow (n-10)(n+3) = 0$$

$$\Rightarrow n = 10 \qquad [\because\ n \neq -3, \text{ as } n \text{ cannot be negative}].$$

Hence, $n = 10$.

(ii) $$\frac{^nP_4}{^{n-1}P_4} = \frac{5}{3} \Rightarrow \frac{n(n-1)(n-2)(n-3)}{(n-1)(n-2)(n-3)(n-4)} = \frac{5}{3}$$

$$\Rightarrow 3n = 5(n-4) \qquad [\text{as } (n-1)(n-2)(n-3)(n-4) \neq 0]$$

$$\Rightarrow 2n = 20 \Rightarrow n = 10.$$

Hence, $n = 10$.

EXAMPLE 3 *If* $5 \times {}^4P_r = 6 \times {}^5P_{r-1}$, *find r.*

SOLUTION We have

$$5 \times {}^4P_r = 6 \times {}^5P_{r-1}$$

$$\Rightarrow \quad 5 \times \frac{4!}{(4-r)!} = 6 \times \frac{5!}{\{5-(r-1)\}!}$$

$$\Rightarrow \quad \frac{5!}{(4-r)!} = \frac{6 \times (5!)}{(6-r)!} \qquad [\because\ 5 \times (4!) = 5!]$$

$$\Rightarrow \quad \frac{1}{(4-r)!} = \frac{6}{(6-r)!} \Rightarrow \frac{1}{(4-r)!} = \frac{6}{(6-r)(5-r)\{(4-r)!\}}$$

$$\Rightarrow \quad \frac{6}{(6-r)(5-r)} = 1 \Rightarrow (6-r)(5-r) - 6 = 0$$

$$\Rightarrow \quad r^2 - 11r + 24 = 0 \Rightarrow (r-3)(r-8) = 0$$

$$\Rightarrow \quad r = 3 \qquad [\because\ r \neq 8, \text{ as } {}^4P_8 \text{ is not defined}].$$

Hence, $r = 3$.

EXAMPLE 4 *If* $^nP_4 = 2 \times {}^5P_3$, *find n.*

SOLUTION We have

$$^nP_4 = 2 \times {}^5P_3$$

$$\Rightarrow \quad n(n-1)(n-2)(n-3) = 2 \times 5 \times 4 \times 3$$

$$\Rightarrow \quad n(n-3)(n-1)(n-2) = 120$$

$$\Rightarrow \quad (n^2 - 3n)(n^2 - 3n + 2) = 120$$

$$\Rightarrow \quad m(m+2) - 120 = 0, \text{ where } m = n^2 - 3n$$

$$\Rightarrow \quad m^2 + 2m - 120 = 0 \Rightarrow (m+12)(m-10) = 0$$

$$\Rightarrow \quad m = -12 \text{ or } m = 10.$$

This gives

$$n^2 - 3n = -12 \quad \text{or} \quad n^2 - 3n = 10$$

$$\Rightarrow \quad n^2 - 3n + 12 = 0 \quad \text{or} \quad n^2 - 3n - 10 = 0$$

$$\Rightarrow \quad n = \frac{3 \pm \sqrt{9 - 48}}{2} \quad \text{or} \quad (n-5)(n+2) = 0$$

$$\Rightarrow \quad n = \frac{3 \pm i\sqrt{39}}{2} \quad \text{or} \quad n = 5 \quad \text{or} \quad n = -2$$

$\Rightarrow \quad n = 5$ [neglecting the negative and imaginary values of n].

Hence, $n = 5$.

EXAMPLE 5 *(i) If* ${}^{15}P_r = 2730$, *find the value of r.*

(ii) If ${}^{10}P_r = 5040$, *find the value of r.*

SOLUTION We have

(i) ${}^{15}P_r = 2730$

$\Rightarrow \quad {}^{15}P_r = 15 \times 182$

$\Rightarrow \quad {}^{15}P_r = 15 \times 14 \times 13$ (up to 3 factors)

$\Rightarrow \quad r = 3.$

Hence, $r = 3$.

(ii) ${}^{10}P_r = 5040$

$\Rightarrow \quad {}^{10}P_r = 10 \times 504$

$\Rightarrow \quad {}^{10}P_r = 10 \times 9 \times 56$

$\Rightarrow \quad {}^{10}P_r = 10 \times 9 \times 8 \times 7$ (up to 4 factors)

$\Rightarrow \quad r = 4.$

Hence, $r = 4$.

EXAMPLE 6 *If* ${}^{56}P_{r+6} : {}^{54}P_{r+3} = (30800 : 1)$, *find r.*

SOLUTION We have

$${}^{56}P_{r+6} = \frac{56\,!}{[56 - (r+6)]\,!} = \frac{56\,!}{(50 - r)\,!}.$$

And, $${}^{54}P_{r+3} = \frac{54\,!}{[54 - (r+3)]\,!} = \frac{54\,!}{(51 - r)\,!}.$$

$$\therefore \quad \frac{{}^{56}P_{r+6}}{{}^{54}P_{r+3}} = \frac{56\,!}{(50-r)\,!} \times \frac{(51-r)\,!}{54\,!}$$

$$= \frac{56 \times 55 \times (54\,!)}{(50-r)\,!} \times \frac{(51-r) \cdot [(50-r)\,!]}{54\,!}$$

$$= 56 \times 55 \times (51 - r).$$

Thus, $$\frac{{}^{56}P_{r+6}}{{}^{54}P_{r+3}} = \frac{30800}{1} \Rightarrow 56 \times 55 \times (51 - r) = 30800$$

$$\Rightarrow (51 - r) = 10 \Rightarrow r = 41.$$

Hence, $r = 41$.

EXAMPLE 7 *If* ${}^{2n+1}P_{n-1} : {}^{2n-1}P_n = 3:5$, *find n.*

SOLUTION We have

$${}^{2n+1}P_{n-1} : {}^{2n-1}P_n = 3:5$$

$$\Rightarrow \quad \frac{(2n+1)!}{\{(2n+1)-(n-1)\}!} : \frac{(2n-1)!}{\{(2n-1)-n\}!} = \frac{3}{5}$$

$$\Rightarrow \quad \frac{(2n+1)!}{(n+2)!} : \frac{(2n-1)!}{(n-1)!} = \frac{3}{5}$$

$$\Rightarrow \quad \frac{(2n+1)!}{(n+2)!} \times \frac{(n-1)!}{(2n-1)!} = \frac{3}{5}$$

$$\Rightarrow \quad \frac{(2n+1)\times 2n \times [(2n-1)!]}{(n+2)\times(n+1)\times n \times [(n-1)!]} \times \frac{(n-1)!}{(2n-1)!} = \frac{3}{5}$$

$$\Rightarrow \quad 10(2n+1) = 3(n+2)(n+1)$$

$$\Rightarrow \quad 20n + 10 = 3(n^2 + 3n + 2)$$

$$\Rightarrow \quad 3n^2 - 11n - 4 = 0$$

$$\Rightarrow \quad (n-4)(3n+1) = 0 \Rightarrow n = 4 \ \left[\because n \neq \frac{-1}{3}, \text{ as } n \text{ cannot be negative}\right].$$

Hence, $n = 4$.

EXAMPLE 8 *If* ${}^{22}P_{r+1} : {}^{20}P_{r+2} = 11:52$, *find r.*

SOLUTION We have

$${}^{22}P_{r+1} : {}^{20}P_{r+2} = 11:52$$

$$\Rightarrow \quad \frac{22!}{\{22-(r+1)\}!} : \frac{20!}{\{20-(r+2)\}!} = 11:52$$

$$\Rightarrow \quad \frac{22!}{(21-r)!} : \frac{20!}{(18-r)!} = \frac{11}{52}$$

$$\Rightarrow \quad \frac{22!}{(21-r)!} \times \frac{(18-r)!}{20!} = \frac{11}{52}$$

$$\Rightarrow \quad \frac{22\times 21\times(20!)}{(21-r)(20-r)(19-r)\times[(18-r)!]} \times \frac{(18-r)!}{20!} = \frac{11}{52}$$

$$\Rightarrow \quad \frac{22\times 21}{(21-r)(20-r)(19-r)} = \frac{11}{52}$$

$$\Rightarrow \quad (19-r)(20-r)(21-r) = 2\times 21\times 52 = 2\times 3\times 7\times 4\times 13$$

$$\Rightarrow \quad (19-r)(20-r)(21-r) = 12\times 13\times 14$$

$$\Rightarrow \quad 19 - r = 12 \Rightarrow r = (19-12) = 7.$$

Hence, $r = 7$.

EXAMPLE 9 *Prove that:*

(i) ${}^nP_n = {}^nP_{n-1}$ *(ii)* ${}^nP_r = n \cdot {}^{n-1}P_{r-1}$ *(iii)* ${}^{n-1}P_r + r \cdot {}^{n-1}P_{r-1} = {}^nP_r$

SOLUTION By using the formula for ${}^nP_r$, we have

(i) $${}^nP_{n-1} = \frac{n!}{\{n-(n-1)]!} = \frac{n!}{1!} = n! = {}^nP_n.$$

(ii) $^nP_r = \dfrac{n!}{(n-r)!} = \dfrac{n \cdot (n-1)!}{[(n-1)-(r-1)]!} = n \cdot {}^{n-1}P_{r-1}.$

(iii) $^{n-1}P_r + r \cdot {}^{n-1}P_{r-1}$

$$= \left\{\frac{(n-1)!}{(n-1-r)!} + r \cdot \frac{(n-1)!}{[(n-1)-(r-1)]!}\right\}$$

$$= \left\{\frac{(n-1)!}{(n-1-r)!} + r \cdot \frac{(n-1)!}{(n-r)!}\right\}$$

$$= \left\{\frac{(n-1)!}{(n-r-1)!} + r \cdot \frac{(n-1)!}{(n-r) \cdot [(n-r-1)!]}\right\}$$

$$= \frac{(n-1)!}{(n-r-1)!}\left\{1 + \frac{r}{(n-r)}\right\} = \frac{n \cdot [(n-1)!]}{(n-r) \cdot [(n-r-1)!]}$$

$$= \frac{n!}{(n-r)!} = {}^nP_r.$$

Hence, $^{n-1}P_r + r \cdot {}^{n-1}P_{r-1} = {}^nP_r$.

EXAMPLE 10 *If $r < s \le n$ then prove that ^nP_s is divisible by ^nP_r.*

SOLUTION We have

$$^nP_s = n(n-1)(n-2)\ldots(n-r+1)(n-r)\ldots(n-s+1)$$
$$= \{n(n-1)(n-2)\ldots(n-r+1)\} \times \{(n-r)(n-r-1)\ldots(n-s+1)\}$$
$$= {}^nP_r \times \{(n-r)(n-r-1)\ldots(n-s+1)\},$$

which is clearly divisible by ^nP_r.

Hence, ^nP_s is divisible by ^nP_r.

EXAMPLE 11 *Find the number of permutations of 7 objects, taken 3 at a time.*

SOLUTION Number of permutations of 7 objects, taken 3 at a time

$= {}^7P_3 = 7 \times 6 \times 5$ [up to three terms]

$= 210.$

EXERCISE 8C

1. Evaluate:
   (i) $^{10}P_4$ (ii) $^{62}P_3$ (iii) $^6P_6$ (iv) $^9P_0$
2. Prove that $^9P_3 + 3 \times {}^9P_2 = {}^{10}P_3$.
3. (i) If $^nP_5 = 20 \times {}^nP_3$, find n.
   (ii) If $16 \times {}^nP_3 = 13 \times {}^{n+1}P_3$, find n.
   (iii) If $^{2n}P_3 = 100 \times {}^nP_2$, find n.
4. (i) If $^5P_r = 2 \times {}^6P_{r-1}$, find r.
   (ii) If $^{20}P_r = 13 \times {}^{20}P_{r-1}$, find r.
   (iii) If $^{11}P_r = {}^{12}P_{r-1}$, find r.
5. (i) If $^nP_4 : {}^nP_5 = 1:2$, find n.
   (ii) If $^{n-1}P_3 : {}^{n+1}P_3 = 5:12$, find n.

6. If ${}^{15}P_{r-1} : {}^{16}P_{r-2} = 3:4$, find r.

7. If ${}^{2n-1}P_n : {}^{2n+1}P_{n-1} = 22:7$, find n.

8. Find n, if ${}^{n+5}P_{n+1} = \frac{11}{2}(n-1) \cdot {}^{n+3}P_n$.

9. Prove that $1 + 1 \cdot {}^1P_1 + 2 \cdot {}^2P_2 + 3 \cdot {}^3P_3 + \ldots + n \cdot {}^nP_n = {}^{n+1}P_{n+1}$.

10. Find the number of permutations of 10 objects, taken 4 at a time.

ANSWERS (EXERCISE 8C)

1. (i) 5040 (ii) 226920 (iii) 720 (iv) 1 **3.** (i) $n = 8$ (ii) $n = 15$ (iii) $n = 13$

4. (i) $r = 3$ (ii) $r = 8$ (iii) $r = 9$ **5.** (i) $n = 6$ (ii) $n = 8$

6. $r = 14$ **7.** $n = 10$ **8.** $n = 6$ or $n = 7$ **10.** 5040

HINTS TO SOME SELECTED QUESTIONS

5. (ii) $\frac{(n-1)(n-2)(n-3)}{(n+1)n(n-1)} = \frac{5}{12} \Rightarrow 12(n-2)(n-3) = 5(n+1)n$

$\Rightarrow 7n^2 - 65n + 72 = 0 \Rightarrow 7n^2 - 56n - 9n + 72 = 0$

$\Rightarrow 7n(n-8) - 9(n-8) = 0 \Rightarrow (n-8)(7n-9) = 0 \Rightarrow n = 8.$

6. $\frac{15!}{\{15-(r-1)\}!} : \frac{16!}{\{(16-(r-2)\}!} = 3:4 \Rightarrow \frac{15!}{(16-r)!} : \frac{16!}{(18-r)!} = 3:4.$

$\therefore \quad \frac{15!}{(16-r)!} \times \frac{(18-r)!}{16!} = \frac{3}{4} \Rightarrow \frac{15!}{(16-r)!} \times \frac{(18-r)(17-r)\cdot[(16-r)!]}{16 \times (15!)} = \frac{3}{4}.$

$\therefore \quad (18-r)(17-r) = 12 \Rightarrow r^2 - 35r + 294 = 0 \Rightarrow (r-14)(r-21) = 0 \Rightarrow r = 14$

$[\because r \neq 21].$

7. $\frac{(2n-1)!}{(2n-1-n)!} : \frac{(2n+1)!}{\{(2n+1)-(n-1)\}!} = \frac{22}{7}$

$\Rightarrow \quad \frac{(2n-1)!}{(n-1)!} \times \frac{(n+2)!}{(2n+1)!} = \frac{22}{7} \Rightarrow \frac{(2n-1)!}{(n-1)!} \times \frac{(n+2)(n+1)n\cdot[(n-1)!]}{(2n+1)(2n)\cdot[(2n-1)!]} = \frac{22}{7}$

$\Rightarrow \quad 7(n+2)(n+1) = 44(2n+1) \Rightarrow 7n^2 - 67n - 30 = 0 \Rightarrow (n-10)(7n+3) = 0.$

Hence, $n = 10$.

9. $1 + 1\cdot(1!) + 2\cdot(2!) + 3\cdot(3!) + \ldots n\cdot(n!)$

$= 1 + (2-1)(1!) + (3-1)\cdot(2!) + (4-1)\cdot(3!) + \ldots + \{(n+1)-1\}\cdot(n!)$

$= \{1 + 2\cdot(1!) + 3\cdot(2!) + 4\cdot(3!) + \ldots + (n+1)\cdot(n!)\} - \{1! + 2! + 3! + \ldots + n!\}$

$= \{1 + 2! + 3! + 4! + \ldots + n! + (n+1)!\} - \{1! + 2! + 3! + \ldots + n!\} = (n+1)!.$

WORD PROBLEMS ON PERMUTATIONS

EXAMPLE 1 *In how many ways can 6 persons stand in a queue?*

SOLUTION Required number of ways

= number of arrangements of 6 different things taking all at a time

$= {}^6P_6 = 6! = (6 \times 5 \times 4 \times 3 \times 2 \times 1) = 720.$

EXAMPLE 2 *How many different signals can be made by 4 flags from 6 flags of different colours?*

SOLUTION Required number of signals

= number of arrangements of 6 flags taking 4 at a time

$= {}^6P_4 = (6 \times 5 \times 4 \times 3) = 360.$

EXAMPLE 3 *Eight athletes are participating in a race. In how many ways can the first 3 prizes be won by them?*

SOLUTION Total number of ways of winning first 3 prizes by 8 athletes

= number of arrangements of 8 athletes taking 3 at a time

$= {}^8P_3 = (8 \times 7 \times 6) = 336.$

EXAMPLE 4 *In how many ways can 3 different rings be worn in 4 fingers with at most one in each finger?*

SOLUTION Required number of ways

= number of ways of arranging 4 fingers taking 3 at a time

$= {}^4P_3 = (4 \times 3 \times 2) = 24.$

EXAMPLE 5 *3 men have 4 coats, 6 waistcoats and 5 caps. In how many ways can they wear them?*

SOLUTION Total number of ways in which 3 men can wear 4 coats

= number of arrangements of 4 coats taking 3 at a time

$= {}^4P_3 = (4 \times 3 \times 2) = 24.$

Total number of ways in which 3 men can wear 6 waistcoats

= number of arrangements of 6 waistcoats taking 3 at a time

$= {}^6P_3 = (6 \times 5 \times 4) = 120.$

Total number of ways in which 3 men can wear 5 caps

$= {}^5P_3 = (5 \times 4 \times 3) = 60.$

Hence, by the fundamental principle of multiplication, the required number of ways $= (24 \times 120 \times 60) = 172800.$

EXAMPLE 6 *How many different signals can be given by using any number of flags from 4 flags of different colours?*

SOLUTION These signals can be made by using 1 or 2 or 3 or 4 flags at a time.

Number of signals made by 4 flags:

(i) taking 1 at a time $= {}^4P_1 = 4;$

(ii) taking 2 at a time $= {}^4P_2 = (4 \times 3) = 12;$

(iii) taking 3 at a time $= {}^4P_3 = (4 \times 3 \times 2) = 24;$

(iv) taking 4 at a time $= {}^4P_4 = (4 \times 3 \times 2 \times 1) = 24.$

Hence, by the fundamental principle of addition, the total number of signals $= (4 + 12 + 24 + 24) = 64.$

EXAMPLE 7 *A code is to consist of two distinct letters followed by two distinct numbers between* 1 *and* 9. *For example, CA* 35, *DQ* 72, *etc., are codes.*

(i) How many such codes are there?

(ii) How many of them end with an even integer?

SOLUTION (i) We know that there are 26 letters in the English alphabet.

Number of ways of choosing 2 letters out of 26

$$= {}^{26}P_2 = (26 \times 25) = 650.$$

Number of ways of choosing 2 numbers out of 9

$$= {}^{9}P_2 = (9 \times 8) = 72.$$

By the fundamental principle of multiplication, the total number of codes = $(650 \times 72) = 46800$.

(ii) **Particular case** *When each code ends with an even integer*

In this case, it can have any of the numbers 2, 4, 6, 8 at the extreme right position. This position can thus be filled in 4 ways.

The next left position can be filled with any of the remaining 8 numbers. So, there are 8 ways of filling this position.

Number of ways of choosing pairs of even numbers

$$= (4 \times 8) = 32.$$

Number of ways of choosing 2 letters out of 26

$$= {}^{26}P_2 = (26 \times 25) = 650.$$

Hence, the number of such codes which end with an even integer = $(32 \times 650) = 20800$.

EXAMPLE 8 *Find the number of ways in which* 5 *boys and* 3 *girls can be arranged in a row so that no two girls are together.*

SOLUTION In order that no two girls are together, we must arrange the 5 boys, each denoted by B, as under:

$$X\ B\ X\ B\ X\ B\ X\ B\ X\ B\ X$$

Here, B denotes the position of a boy and X that of a girl.

Number of ways in which 5 boys can be arranged at 5 places

$$= {}^{5}P_5 = 5\,! = (5 \times 4 \times 3 \times 2 \times 1) = 120.$$

Number of ways in which 3 girls can be arranged at 6 places (each marked X) = ${}^{6}P_3 = (6 \times 5 \times 4) = 120$.

Hence, by the fundamental principle of multiplication, the required number of ways = $(120 \times 120) = 14400$.

EXAMPLE 9 *There are* 2 *English,* 3 *Sanskrit and* 4 *Hindi books. In how many ways can they be arranged on a shelf so as to keep all the books of the same language together?*

SOLUTION Let us make 1 packet for each of the books on the same language.

Number of ways of arranging the 3 packets = ${}^{3}P_3 = 3\,! = 6$.

Number of ways of arranging 2 English books $= 2! = 2$.

Number of ways of arranging 3 Sanskrit books $= 3! = 6$.

Number of ways of arranging 4 Hindi books $= 4! = 24$.

Hence, the required number of ways $= (2 \times 6 \times 24) \times 6 = 1728$.

EXAMPLE 10 *In how many ways can 7 examination papers be arranged so that the best and the worst papers are never together?*

SOLUTION Let us tie the best paper (b) and the worst paper (w) together and consider (bw) as one paper.

Now, this (bw) and 5 other papers may be arranged in

$$^6P_6 = 6! = 720 \text{ ways.}$$

Also, these two papers may be arranged among themselves in $2! = 2$ ways.

Total number of arrangements with best and worst papers together

$$= (720 \times 2) = 1440.$$

Total number of ways of arranging 7 papers

$$= {}^7P_7 = 7! = 5040.$$

Number of arrangements with best and worst paper never together

$$= (5040 - 1440) = 3600.$$

EXAMPLE 11 *In how many ways can 6 balls of different colours, namely black, white, blue, red, green and yellow be arranged in a row in such a way that the black and white balls are never together?*

SOLUTION Let us tie the black ball (b) and white ball (w) together and consider (bw) as one ball.

Now, this (bw) and 4 other balls may be arranged in

$$^5P_5 = 5! = 120 \text{ ways.}$$

Also, these two balls may be arranged among themselves in $2! = 2$ ways.

Total number of arrangements with black and white balls together

$$= (120 \times 2) = 240.$$

Number of ways of arranging 6 balls among themselves

$$= {}^6P_6 = 6! = 720.$$

Number of ways of arranging 6 balls such that black and white balls are never together $= (720 - 240) = 480$.

EXAMPLE 12 *The principal wants to arrange 5 students on a platform such that the boy Sajal occupies the second position and the girl Tanvy is always adjacent to the girl Aditi. How many such arrangements are possible?*

SOLUTION Let the five seats be arranged as shown below.

I	II	III	IV	V
□	S	□	□	□

Keep Sajal fixed at the second position.

Since Tanvy and Aditi are to sit together, none of them can occupy the first seat.

The first seat can be occupied by any of the two remaining students in $2! = 2$ ways.

Number of ways in which 2 seats namely III, IV or IV, V may be occupied by Tanvy and Aditi $= (2!) + (2!) = (2 + 2) = 4$.

The remaining seat may now be occupied by the 5th student in 1 way only.

Hence, the required number of arrangements $= (2 \times 4 \times 1) = 8$.

EXAMPLE 13 *How many words (with or without meaning) can be formed by using all the letters of the word, 'DELHI' using each letter exactly once?*

SOLUTION The word, 'DELHI' contains 5 different letters.

Total number of words formed by using all the letters of the given word $= {}^5P_5 = 5! = (5 \times 4 \times 3 \times 2 \times 1) = 120$.

Hence, the required number of words = 120.

EXAMPLE 14 *How many 4-letter words (with or without meaning) can be formed out of the letters of the word, 'LOGARITHMS', if repetition of letters is not allowed?*

SOLUTION The word, 'LOGARITHMS' contains 10 different letters.

Number of 4-letter words formed out of 10 given letters

$$= {}^{10}P_4 = (10 \times 9 \times 8 \times 7) = 5040.$$

Hence, the required number of 4-letter words = 5040.

EXAMPLE 15 *How many words (with or without meaning) can be formed from the letters of the word, 'DAUGHTER', so that*

(i) all vowels occur together?

(ii) all vowels do not occur together?

SOLUTION The given word, 'DAUGHTER' contains 3 vowels A, U, E and 5 consonants D, G, H, T, R.

Case (i) *When all vowels occur together:*

Let us assume (AUE) as a single letter.

Then, this letter (AUE) along with 5 other letters can be arranged in ${}^6P_6 = (6!)$ ways $= (6 \times 5 \times 4 \times 3 \times 2 \times 1)$ ways

$$= 720 \text{ ways.}$$

These 3 vowels may be arranged among themselves in $3! = 6$ ways.

Hence, the required number of words with vowels together

$$= (6!) \times (3!) = (720 \times 6) = 4320.$$

Case (ii) *When all vowels do not occur together:*

Number of words formed by using all the 8 letters of the given word

$= {}^8P_8 = 8! = (8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1) = 40320.$

Number of words in which all vowels are never together

= (total number of words) – (number of words with all vowels together)

$= (40320 - 4320) = 36000.$

EXAMPLE 16 *How many words (with or without meaning) can be made from the letters of the word 'MONDAY', assuming that no letter is repeated, when:*

(i) 4 letters are used at a time?

(ii) all letters are used at a time?

(iii) all letters are used but the first letter is a vowel?

SOLUTION The given word 'MONDAY' contains 6 letters out of which two letters namely O and A are vowels.

Case (i) *When 4 letters are used at a time:*

Then, the required number of words

= number of arrangements of 6 letters taking 4 at a time

$= {}^6P_4 = (6 \times 5 \times 4 \times 3) = 360.$

Case (ii) *When all the letters are used at a time:*

Then, the required number of words

= number of arrangements of 6 letters taking all at a time

$= {}^6P_6 = 6! = (6 \times 5 \times 4 \times 3 \times 2 \times 1) = 720.$

Case (iii) *When all the letters are used and the first letter is a vowel:*

The given word contains 2 vowels, namely O and A.

So, the first letter can be chosen 1 out of 2 vowels.

Thus, there are 2 ways of choosing the first letter.

Number of ways of arranging the remaining 5 letters among themselves $= 5! = (5 \times 4 \times 3 \times 2 \times 1) = 120.$

Hence, the required number of words $= (2 \times 120) = 240.$

EXAMPLE 17 *(i) How many words can be formed from the letters of the word, 'TRIANGLE'?*

(ii) How many of them will begin with T and end with E?

SOLUTION (i) The given word 'TRIANGLE' contains 8 letters.

Total number of words formed with these 8 letters

$= {}^8P_8 = 8! = 40320.$

Hence, the required number of words = 40320.

(ii) By fixing T in the beginning and E at the end, the remaining 6 letters can be arranged among themselves in ${}^6P_6 = 6! = 720$ ways.

Hence, the total number of words, each beginning with T and ending with $E = 720$.

EXAMPLE 18 *How many words can be formed with the letters of the word 'ORDINATE' so that the vowels occupy odd places?*

SOLUTION The given word consists of 8 letters, out of which there are 4 vowels and 4 consonants.

Let us mark out the positions to be filled up, as shown below:

$$(^1)(^2)(^3)(^4)(^5)(^6)(^7)(^8).$$

Since vowels occupy odd places, they may be placed at 1, 3, 5, 7.

Number of ways of arranging 4 vowels at 4 odd places

$= {}^4P_4 = 4! = 24.$

The remaining 4 letters of the given word are consonants, which can be arranged at 4 even places marked 2, 4, 6, 8.

Number of ways of arranging 4 consonants at 4 even places

$= {}^4P_4 = 4! = 24.$

Hence, the total number of words in which the vowels occupy odd places $= (24 \times 24) = 576.$

EXAMPLE 19 *How many 3-digit numbers can be formed by using the digits 1 to 9, if no digit is repeated?*

SOLUTION Required number of numbers

= number of permutations of 9 digits taking 3 at a time

$= {}^9P_3 = (9 \times 8 \times 7) = 504.$

Hence, the required number of numbers = 504.

EXAMPLE 20 *Find the number of 4-digit numbers that can be formed using the digits 1, 2, 3, 4, 5 if no digit is repeated. How many of them will be even?*

SOLUTION Required number of 4-digit numbers

= number of arrangements of 5 digits taking 4 at a time

$= {}^5P_4 = (5 \times 4 \times 3 \times 2) = 120.$

Particular case To get an even number, the unit's digit is 2 or 4.

Thus, there are 2 ways of filling the unit's digit.

Number of ways of filling remaining 3 places with the remaining 4 digits $= {}^4P_3 = (4 \times 3 \times 2) = 24.$

Hence, the required number of even numbers $= (2 \times 24) = 48.$

EXAMPLE 21 *How many 6-digit telephone numbers can be constructed with the digits 0, 1, 2, 3, 4, 5, 6, 7, 8, 9 if each number starts with 35 and no digit appears more than once?*

SOLUTION Out of the given 10 digits, the two digits namely 3 and 5 are reserved for the two extreme left places of the desired numbers.

The remaining 8 digits can be arranged in the remaining four positions in ${}^8P_4 = (8 \times 7 \times 6 \times 5) = 1680$ ways.

Hence, 1680 telephone numbers can be constructed.

EXERCISE 8D

1. In how many ways can 5 persons occupy 3 vacant seats?
2. In how many ways can 7 people line up at a ticket window of a cinema hall?
3. In how many ways can 5 children stand in a queue?
4. In how many ways can 6 women draw water from 6 wells, if no well remains unused?
5. In how many ways can 4 different books, one each in chemistry, physics, biology and mathematics, be arranged on a shelf?
6. Six students are contesting the election for the presidentship of the students' union. In how many ways can their names be listed on the ballot papers?
7. It is required to seat 5 men and 3 women in a row so that the women occupy the even places. How many such arrangements are possible?
8. There are 6 items in column A and 6 items in column B. A student is asked to match each item in column A with an item in column B. How many possible, correct or incorrect answers are there to this question?
9. Five letters F, K, R, T and V one in each were purchased from a plastic warehouse. How many ordered pairs of letters, to be used as initials, can be formed from them?
10. Ten students are participating in a race. In how many ways can the first three prizes be won?
11. If there are 6 periods on each working day of a school, in how many ways can one arrange 5 subjects such that each subject is allowed at least one period?
12. In how many ways can 6 pictures be hung from 4 picture nails on a wall?
13. Find the number of words formed (may be meaningless) by using all the letters of the word 'EQUATION', using each letter exactly once.
14. Find the number of different 4-letter words (may be meaningless) that can be formed from the letters of the word 'NUMBERS'.
15. How many words can be formed from the letters of the word 'SUNDAY'? How many of these begin with D?
16. How many words beginning with C and ending with Y can be formed by using the letters of the word 'COURTESY'?
17. Find the number of permutations of the letters of the word 'ENGLISH'. How many of these begin with E and end with I?
18. In how many ways can the letters of the word 'HEXAGON' be permuted? In how many words will the vowels be together?
19. How many words can be formed out of the letters of the word 'ORIENTAL' so that the vowels always occupy the odd places?
20. In how many ways can the letters of the word 'FAILURE' be arranged so that the consonants may occupy only odd positions?
21. In how many arrangements of the word 'GOLDEN' will the vowels never occur together?

22. Find the number of ways in which the letters of the word 'MACHINE' can be arranged such that the vowels may occupy only odd positions.

23. How many permutations can be formed by the letters of the word 'VOWELS', when

(i) there is no restriction on letters;

(ii) each word begins with E;

(iii) each word begins with O and ends with L;

(iv) all vowels come together;

(v) all consonants come together?

24. How many numbers divisible by 5 and lying between 3000 and 4000 can be formed by using the digits 3, 4, 5, 6, 7, 8 when no digit is repeated in any such number?

25. In an examination, there are 8 candidates out of which 3 candidates have to appear in mathematics and the rest in different subjects. In how many ways can they be seated in a row, if candidates appearing in mathematics are not to sit together?

26. In how many ways can 5 children be arranged in a line such that

(i) two of them, Rajan and Tanvy, are always together?

(ii) two of them, Rajan and Tanvy, are never together?

27. When a group photograph is taken, all the seven teachers should be in the first row and all the twenty students should be in the second row. If the two corners of the second row are reserved for the two tallest students, interchangeable only between them, and if the middle seat of the front row is reserved for the principal, how many arrangements are possible?

28. Find the number of ways in which m boys and n girls may be arranged in a row so that no two of the girls are together, it being given that $m > n$.

ANSWERS (EXERCISE 8D)

1. 60	**2.** 5040	**3.** 120	**4.** 720	**5.** 24
6. 720	**7.** 7200	**8.** 720	**9.** 20	**10.** 720
11. 600	**12.** 360	**13.** 40320	**14.** 840	**15.** 720, 120
16. 720	**17.** 5040, 120	**18.** 5040, 720	**19.** 576	**20.** 576

21. 480 **22.** 576 **23.** (i) 720 (ii) 120 (iii) 24 (iv) 240 (v) 144

24. 12 **25.** 14400 **26.** (i) 48 (ii) 72 **27.** $(18\,!)\times(6\,!)\times 2$

28. $\dfrac{(m\,!)\times(m+1)\,!}{(m+1-n)\,!}$

HINTS TO SOME SELECTED QUESTIONS

7. 5 men can be arranged at 5 places in ${}^5P_5 = 5\,! = 120$ ways.

Starting from man, we leave alternate places for women, as shown below.

M □ M □ M □ M □ M □

Now, 3 women can be arranged at these 5 places, marked □, in

$^5P_3 = (5 \times 4 \times 3) = 60$ ways.

Required number of ways $= (120 \times 60) = 7200$.

9. Number of ordered pairs $= {^5P_2} = (5 \times 4) = 20$.

10. Required number of ways $= {^{10}P_3}$.

11. Out of 6 periods, 5 may be arranged for 5 subjects in $^5P_5$ ways.

Remaining 1 period may be arranged for anyone of the five subjects in $^5P_1$ ways.
Required number of ways $= ({^5P_5} \times {^5P_1})$.

18. The given word contains 7 letters, which may be arranged among themselves in $7! = 5040$ ways.

The given word contains 3 vowels and 4 consonants.
Taking the 3 vowels $\boxed{\text{EAO}}$ as one letter, this letter and 4 more letters can be arranged in $5! = 120$ ways.
The 3 vowels can be arranged among themselves in $3! = 6$ ways.
Required number of arrangements with vowels together $= (120 \times 6) = 720$.

19. The given word contains 4 vowels and 4 consonants.

Let us fix the 8 places as shown below.

(¹) (²) (³) (⁴) (⁵) (⁶) (⁷) (⁸)

The 4 vowels can be arranged at 4 places (namely 1, 3, 5, 7) in $^4P_4 = 4! = 24$ ways.

And, the 4 consonants can be arranged at 4 places (namely 2, 4, 6, 8) in $^4P_4 = 4! = 24$ ways.
Required number of ways $= (24 \times 24) = 576$.

20. The given word 'FAILURE' has 3 consonants and 4 vowels, which can be arranged at seven places, shown below.

(¹) (²) (³) (⁴) (⁵) (⁶) (⁷)

Now, 3 consonants may be placed at any of the 3 places out of the 4 marked 1, 3, 5, 7.
Number of ways of arranging the consonants $= {^4P_3} = 24$.
And, the 4 vowels can be arranged at the remaining 4 places in $^4P_4 = 24$ ways.
Required number of ways $= (24 \times 24) = 576$.

21. The given word 'GOLDEN' contains 6 letters which may be arranged among themselves in $6! = 720$ ways.

It has 2 vowels and 4 consonants.
Taking the 2 vowels $\boxed{\text{OE}}$ as a single letter, this letter and 4 consonants together may be arranged in $5! = 120$ ways.
Also, O and E may be arranged among themselves in $2!$ ways.
Number of arrangements with vowels together $= (120 \times 2) = 240$.
Number of arrangements with vowels never together $= (720 - 240) = 480$.

22. There are 7 letters in the given word, out of which there are 3 vowels and 4 consonants. Let us mark out the positions to be filled up as follows:

(¹) (²) (³) (⁴) (⁵) (⁶) (⁷)

Now, the 3 vowels can be placed at any of the three places out of the four, marked 1, 3, 5, 7.
So, the number of ways of arranging the vowels $= {^4P_3} = 4 \times 3 \times 2 = 24$.

Also, the 4 consonants at the remaining 4 positions may be arranged in $^4P_4 = 4!$ $= 24$ ways.

$\therefore$ the requisite number of ways $= (24 \times 24) = 576$.

24. Fix up 3 at the thousand's place and 5 at the unit's place. Now, the remaining 2 places can be filled by the remaining 4 digits in $^4P_2 = (4 \times 3) = 12$ ways.

25. Clearly, there are 5 candidates not appearing in mathematics.

Let us arrange these 5 in a row, each shown by X.

They can be arranged in $5! = 120$ ways.

On both sides of each X, we put an M, as shown below:

$$MXMXMXMXMXM$$

Now, 3 candidates in mathematics can be arranged at 6 places in $^6P_3$ ways $= (6 \times 5 \times 4)$ ways $= 120$ ways.

Total number of arrangements $= (120 \times 120) = 14400$.

26. Total number of ways of arranging 5 children in a row $= 5! = 120$.

(i) Considering $\boxed{\text{Rajan + Tanvy}}$ as one child, this one and 3 more can be arranged in $4! = 24$ ways.

Rajan and Tanvy can be arranged among themselves in $2! = 2$ ways.

Required number of ways $= (24 \times 2) = 48$.

(ii) When Rajan and Tanvy are never together, the number of arrangements $= (120 - 48) = 72$.

27. Keeping the middle seat of the front row fixed for the principal, the remaining 6 seats can be occupied by the remaining 6 teachers in $6!$ ways.

The two corner seats may be occupied by the two tallest students in 2 ways. The remaining 18 seats may be occupied by the remaining 18 students in $18!$ ways.

28. We may arrange m boys at m places in $m!$ ways, leaving a seat between each pair of boys as shown below:

$$\times B \times B \times \ldots \times B \times .$$

Now, n girls may be arranged in $(m+1)$ places, shown by cross marks.

The number of such arrangements is $^{m+1}P_n$.

PERMUTATIONS OF OBJECTS NOT ALL DIFFERENT

THEOREM 1 *Let there be n objects, of which m objects are alike of one kind, and the remaining $(n-m)$ objects are alike of another kind. Then, the total number of mutually distinguishable permutations that can be formed from these objects is*

$$\frac{n!}{(m!) \times (n-m)!}.$$

PROOF Let the required number of permutations be x.

Consider any of these x permutations.

Now, replace m like things in this permutation by m different things. These m different things may be arranged amongst themselves in $m!$ ways.

Similarly, if we replace $(n-m)$ like things by $(n-m)$ different things, then they can be arranged amongst themselves in $(n-m)!$ ways.

Thus, if both the replacements are done simultaneously then each one of the x permutations gives rise to $(m!) \times (n-m)!$ permutations.

x permutations give rise to $x \times (m!) \times (n-m)!$ permutations.

Now, each of these $x \times (m!) \times (n-m)!$ permutations is a permutation of n different things, taken all at a time.

$\therefore \quad x \times (m!) \times (n-m)! = n!.$

Hence, $x = \dfrac{n!}{(m!) \times (n-m)!}$.

REMARK 1 *Let there be n things of which p_1 are alike of one kind, p_2 are alike of another kind, p_3 are alike of 3rd kind, ..., p_r are alike of rth kind such that $p_1 + p_2 + \dots + p_r = n$. Then, the number of permutations of these n things is*

$$\frac{n!}{(p_1!) \times (p_2!) \times \dots (p_r!)}.$$

REMARK 2 *The number of permutations of n things of which p are alike of one kind, q are alike of another kind and remaining all are distinct is $\dfrac{n!}{(p!) \times (q!)}$.*

SOLVED EXAMPLES

EXAMPLE 1 *Find the number of different permutations of the letters of the word, 'BANANA'.*

SOLUTION The given word 'BANANA' contains 6 letters, out of which three are alike of one kind (3 A's), two are alike of second kind (2 N's) and one of its own kind (1 B).

Total number of their permutations $= \dfrac{6!}{(3!) \times (2!) \times (1!)} = 60.$

EXAMPLE 2 *How many words can be formed using the letter A thrice, the letter B twice and the letter C thrice?*

SOLUTION We are given 8 letters, namely AAABBCCC. Out of these letters three are alike of one kind (3 A's), two are alike of second kind (2 B's) and three are alike of third kind (3 C's).

Hence, the required number of permutations

$$= \frac{8!}{(3!) \times (2!) \times (3!)} = 560.$$

EXAMPLE 3 *(i) Find how many arrangements can be made with the letters of the word 'MATHEMATICS'.*

(ii) In how many of them are the vowels together?

SOLUTION (i) There are 11 letters in the word 'MATHEMATICS'. Out of these letters M occurs twice, A occurs twice, T occurs twice and the rest are all different.

Hence, the total number of arrangements of the given letters

$$= \frac{11!}{(2!)\times(2!)\times(2!)} = 4989600.$$

(ii) The given word contains 4 vowels, namely A, E, A, I. Treating these 4 vowels $\boxed{\text{AEAI}}$ as one letter, we have to arrange 8 letters MTHMTCS + $\boxed{\text{AEAI}}$, out of which M occurs twice, T occurs twice and the rest are all different.

So, the number of all such arrangements $= \frac{8!}{(2!)\times(2!)} = 10080.$

Now, out of 4 vowels, A occurs twice and the rest are all distinct.

So, the number of arrangements of these vowels $= \frac{4!}{2!} = 12.$

Hence, the number of arrangements in which 4 vowels are together $= (10080\times 12) = 120960.$

EXAMPLE 4 I. *Find the number of arrangements of the letters of the word, 'INDEPENDENCE'.*

II. *In how many of these arrangements:*

(i) do the words start with P?

(ii) do all the vowels occur together?

(iii) do the vowels never occur together?

(iv) do the words begin with I and end in P?

SOLUTION I. The given word contains 12 letters out of which N occurs 3 times, E occurs 4 times, D occurs 2 times and the rest are all different.

Hence, the required number of arrangements

$$= \frac{12!}{(3!)\times(4!)\times(2!)} = 1663200.$$

II. (i) After fixing P at the extreme left position, there are 11 letters consisting of 3 N's, 4 E's, 2 D's, 1 I and 1 C.

So, the number of words beginning with P

= number of arrangements of 11 letters out of which there are 3 N's, 4 E's, 2 D's, 1 I and 1 C

$$= \frac{11!}{(3!)\times(4!)\times(2!)} = 138600.$$

(ii) There are 5 vowels in the given word, namely 4 E's and 1 I.

Let us treat them as a single letter $\boxed{\text{EEEEI}}$.

This letter with 7 remaining letters will give us 8 letters in which there are 3 N's, 2 D's, 1 P, 1 C and 1 letter $\boxed{\text{EEEEI}}$.

Number of all such arrangements $= \frac{8!}{(3!)\times(2!)} = 3360.$

Number of arrangements of 5 vowels, namely 4 E's and 1 I

$$= \frac{5!}{4!} = 5.$$

Hence, the number of words in which vowels are together

$$= (3360 \times 5) = 16800.$$

(iii) Number of arrangements in which vowels do not occur together

= (total number of arrangements) – (number of arrangements in which vowels occur together)

$= (1663200 - 16800) = 1646400.$

(iv) Let us fix I at the left end and P at the right end of each arrangement.

Then, we are left with 10 letters, out of which there are 3 N's, 4 E's, 2 D's and 1 C.

Hence, the number of words which begin with I and end in P $= \frac{10!}{(3!) \times (4!) \times (2!)} = 12600.$

EXAMPLE 5 *(i) How many different words can be formed by using all the letters of the word, 'ALLAHABAD'?*

In how many of them:

(ii) both L's do not come together?

(iii) the vowels occupy the even positions?

SOLUTION (i) The given word, 'ALLAHABAD' contains 9 letters consisting of 4 A's, 2 L's, 1 H, 1 B and 1 D.

Hence, the number of different words formed by using all the letters of the given word $= \frac{9!}{(4!) \times (2!)} = 7560.$

(ii) Let us take both L together and we treat $\boxed{LL}$ as 1 letter.

Then, we will have to arrange 8 letters, namely $\boxed{LL}$, 4 A's, 1 H, 1 B and 1 D.

So, the number of words having both L together $= \frac{8!}{4!} = 1680.$

Hence, the number of words with both L not occurring together

$$= (7560 - 1680) = 5880.$$

(iii) There are 4 vowels and all are alike, i.e., 4 A's.

Also, there are 4 even places, namely 2nd, 4th, 6th and 8th.

So, we arrange A's at these 4 places, as shown below.

$$\square \ A \ \square \ A \ \square \ A \ \square \ A \ \square$$

Number of arrangements of 4 A's at 4 places $= \frac{4!}{4!} = 1.$

Now, we are left with 5 letters consisting of 2 L's, 1 H, 1 B and 1 D.

Number of arrangements of these 5 letters at 5 places $= \frac{5!}{2!} = 60$.

Hence, the number of words in which vowels occupy even places $= (1 \times 60) = 60$.

EXAMPLE 6 *In how many of the distinct permutations of the letters in 'MISSISSIPPI' do the 4 I's not come together?*

SOLUTION The given word 'MISSISSIPPI' contains 11 letters, out of which there are 4 I's, 4 S's, 2 P's and 1 M.

Total number of words formed by the letters of the given word

$$= \frac{11!}{(4!) \times (4!) \times (2!)} = 34650.$$

There are 4 I's in the given word and we treat $\boxed{\text{IIII}}$ as 1 letter.

This single letter with remaining 7 letters will give us 8 letters out of which there are 4 S's, 2 P's, 1 M and 1 $\boxed{\text{IIII}}$.

So, the number of all such arrangements in which 4 I's are together

$$= \frac{8!}{(4!) \times (2!)} = 840.$$

Thus, the number of words formed when 4 I's are together = 840.

Hence, the number of words formed when 4 I's are not together

$$= (34650 - 840) = 33810.$$

EXAMPLE 7 *(i) How many words can be formed with the letters of the word, 'HARYANA'?*

How many of these

(ii) have H and N together?

(iii) begin with H and end with N?

(iv) have 3 vowels together?

SOLUTION (i) The given word 'HARYANA' consists of 7 letters, out of which there are 1 H, 3 A's, 1 R, 1 Y and 1 N.

Total number of words formed by all the letters of the given word $= \frac{7!}{3!} = 840$.

(ii) Let us consider $\boxed{\text{HN}}$ as a single letter.

Now, $\boxed{\text{HN}}$ + ARYAA will give us 6 letters out of which there are 3 A's, 1 R, 1 Y and 1 $\boxed{\text{HN}}$.

Total number of all such arrangements $= \frac{6!}{3!} = 120$.

But, H and N can be arranged amongst themselves in 2 ! ways.

Hence, the number of words having H and N together

$$= (120 \times 2) = 240.$$

(iii) After fixing H in first place and N in last place, we have 5 letters, out of which there are 3 A's, 1 R and 1 Y.

Hence, the number of words beginning with H and ending with N $= \frac{5!}{3!} = 20$.

(iv) The given word contains 3 vowels AAA and let us treat $\boxed{\text{AAA}}$ as 1 letter.

Now, we have to arrange 5 letters HRYN + $\boxed{\text{AAA}}$ at 5 places.

Hence, total number of words formed having all vowels together $= 5! = (5 \times 4 \times 3 \times 2 \times 1) = 120$.

EXAMPLE 8 *In how many ways can the letters of the word 'PERMUTATIONS' be arranged, if*

(i) the words start with P and end with S?

(ii) the vowels are all together?

SOLUTION (i) Let us fix P at the left end and S at the right end of each arrangement.

Then, we are left with 10 letters, out of which T occurs 2 times and the rest are all different.

Hence, the required number of arrangements $= \frac{10!}{2!} = 1814400$.

(ii) The given word contains 5 vowels, namely E, U, A, I, O.

Treating these five vowels $\boxed{\text{EUAIO}}$ as one letter, we have to arrange 8 letters, namely $\boxed{\text{EUAIO}}$ + PRMTTNS, out of which T occurs 2 times and the rest are all different.

Number of all such arrangements $= \frac{8!}{2!} = 20160$.

Now, the 5 vowels are all different. They can be arranged among themselves in $5! = 120$ ways.

Hence, the number of arrangements in which 5 vowels are together $= (20160 \times 120) = 2419200$.

EXAMPLE 9 *If the different permutations of the word 'EXAMINATION' are listed as in a dictionary, how many items are there in the list before the first word starting with E?*

SOLUTION In a dictionary, the words at each stage are arranged in an alphabetical order.

When the letters of the word 'EXAMINATION' are arranged in the dictionary, the first word would be AAEIIMNNOTX.

Starting with A, we arrange the remaining 10 letters AEIIMNNOTX in which there are 2 I's, 2 N's and the rest are all different, in all possible ways.

Thus, the number of words formed beginning with A

$$= \frac{10!}{(2!)\times(2!)} = 907200.$$

In the dictionary, the next word would be EAAIIMNNOTX.

Hence, 907200 words are there in the dictionary before the first word starting with E.

EXAMPLE 10 *(i) Find the number of words which can be made using all the letters of the word, 'AGAIN'.*

(ii) If these words are written as in a dictionary, what will be the 50th word?

SOLUTION (i) The given word 'AGAIN' consists of 5 letters, out of which there are 2 A's and the rest are all distinct.

Hence, the number of words formed by using all the letters of the given word $= \frac{5!}{2!} = 60$.

(ii) When the letters of the word 'AGAIN' are listed in a dictionary, the first word would be AAGIN.

Starting with A and arranging the remaining 4 letters A, G, I, N, we obtain 4 ! words, i.e., 24 words.

The 25th word would be GAAIN, which starts with G.

Now, starting with G arrange the remaining 4 letters AAIN, (in which A occurs 2 times and the rest are different) in all possible ways.

Number of such arrangements $= \frac{4!}{2!} = 12$.

Thus, we obtain 12 new words.

The 37th word would be IAAGN, which starts with I.

Now, starting with I and arranging the remaining 4 letters, AAGN, we get $\frac{4!}{2!} = 12$ words.

The 49th word would be NAAGI and hence the 50th word is NAAIG.

EXAMPLE 11 *How many numbers greater than a million can be formed with the digits 2, 3, 0, 3, 4, 2, 3?*

SOLUTION Any number greater than one million will contain all the seven digits.

Now, we have to arrange seven digits, out of which 2 occurs twice, 3 occurs thrice and the rest are distinct.

Number of such arrangements $= \frac{7!}{(2!)\times(3!)} = 420$.

These arrangements will also include those which contain 0 at the million's place.

Keeping 0 fixed at the million's place, the remaining 6 digits out of which 2 occurs twice, 3 occurs thrice and the rest are distinct can be arranged in $= \frac{6!}{(2!) \times (3!)} = 60$ ways.

Hence, the number of required numbers $= (420 - 60) = 360$.

EXAMPLE 12 *How many numbers greater than* 400000 *can be formed by using the digits* 0, 2, 2, 4, 4, 5*?*

SOLUTION We are being given six digits and each number greater than 400000 is a six-digit number, having 4 or 5 in the extreme left position.

When 4 occupies the extreme left position, the remaining 5 places can be filled with the digits 0, 2, 2, 4, 5 (in which 2 occurs 2 times and rest are all different).

Number of such numbers $= \frac{5!}{2!} = 60$.

When 5 occupies the extreme left position, the remaining 5 places can be filled with the digits 0, 2, 2, 4, 4 (in which 2 occurs twice, 4 occurs twice and the rest are all distinct).

Number of such numbers $= \frac{5!}{(2!) \times (2!)} = 30$.

Hence, the required number of numbers $= (60 + 30) = 90$.

EXAMPLE 13 *How many numbers can be formed using the digits* 1, 2, 3, 4, 3, 2, 1 *so that the odd digits always occupy the odd places?*

SOLUTION We have been given seven digits, namely 1, 2, 3, 4, 3, 2, 1.

So, we have to form 7-digit numbers, so that odd digits occupy odd places.

Every 7-digit number has 4 odd places.

Given four odd digits are 1, 3, 3, 1 out of which 1 occurs 2 times and 3 occurs 2 times.

So, the number of ways to fill up 4 odd places $= \frac{4!}{(2!) \times (2!)} = 6$.

Given three even digits are 2, 4, 2 in which 2 occurs 2 times and 4 occurs 1 time.

So, the number of ways to fill up 3 even places $= \frac{3!}{2!} = 3$.

Hence, the required number of numbers $= (6 \times 3) = 18$.

EXERCISE 8E

1. Find the total number of permutations of the letters of each of the words given below:

(i) APPLE (ii) ARRANGE (iii) COMMERCE
(iv) INSTITUTE (v) ENGINEERING (vi) INTERMEDIATE

2. In how many ways can the letters of the expression $x^2y^3z^4$ be arranged when written without using exponents?

3. There are 3 blue balls, 4 red balls and 5 green balls. In how many ways can they be arranged in a row?

4. A child has three plastic toys bearing the digits 3, 3, 5 respectively. How many 3-digit numbers can he make using them?

5. How many different signals can be transmitted by arranging 2 red, 3 yellow and 2 green flags on a pole, if all the seven flags are used to transmit a signal?

6. How many words can be formed by arranging the letters of the word 'ARRANGEMENT', so that the vowels remain together?

7. How many words can be formed by arranging the letters of the word 'INDIA', so that the vowels are never together?

8. Find the number of arrangements of the letters of the word 'ALGEBRA' without altering the relative positions of the vowels and the consonants.

9. How many words can be formed from the letters of the word 'SERIES', which start with S and end with S?

10. In how many ways can the letters of the word 'PARALLEL' be arranged so that all L's do not come together?

11. How many different words can be formed with the letters of the word 'CAPTAIN'? In how many of these C and T are never together?

12. In how many ways can the letters of the word 'ASSASSINATION' be arranged so that all S's are together?

13. (i) How many arrangements can be made by using all the letters of the word 'MATHEMATICS'?

(ii) How many of them begin with C?

(iii) How many of them begin with T?

14. In how many ways can the letters of the word 'INTERMEDIATE' be arranged so that:

(i) the vowels always occupy even places?

(ii) the relative orders of vowels and consonants do not change?

15. (i) Find the number of different words formed by using all the letters of the word, 'INSTITUTION'.

In how many of them

(ii) are the three T's together?

(iii) are the first two letters the two N's?

16. How many five-digit numbers can be formed with the digits 5, 4, 3, 5, 3?

17. How many numbers can be formed with the digits 2, 3, 4, 5, 4, 3, 2 so that the odd digits occupy the odd places?

18. How many 7-digit numbers can be formed by using the digits 1, 2, 0, 2, 4, 2, 4?

19. How many 6-digit numbers can be formed by using the digits 4, 5, 0, 3, 4, 5?

20. The letters of the word 'INDIA' are arranged as in a dictionary. What are the 1st, 13th, 25th, 49th and 60th words?

ANSWERS (EXERCISE 8E)

1. (i) 60 (ii) 1260 (iii) 5040 (iv) 30240 (v) 277200 (vi) 19958400
2. 1260 **3.** 27720 **4.** 3 **5.** 210 **6.** 60480
7. 42 **8.** 72 **9.** 12 **10.** 3000 **11.** 2520, 1800
12. 151200 **13.** (i) 4989600 (ii) 453600 (iii) 907200
14. (i) 21600 (ii) 21600 **15.** (i) 554400 (ii) 30240 (iii) 10080
16. 30 **17.** 18 **18.** 360 **19.** 150
20. (i) ADIIN (ii) DAIIN (iii) IADIN (iv) NADII (v) NIIDA

HINTS TO SOME SELECTED QUESTIONS

2. There are 2 x's, 3 y's and 4 z's.

$\therefore$ total number of arrangements $= \dfrac{9!}{(2!) \times (3!) \times (4!)}$.

3. We have to arrange 12 balls, out of which 3 are of one kind, 4 are of second kind and 5 are of third kind.

Required number of arrangements $= \dfrac{12!}{(3!) \times (4!) \times (5!)}$.

4. We have to arrange 3 objects in a row, out of which 2 are alike of one kind and the third one of its own kind.

Total number of arrangements $= \dfrac{3!}{2!} = 3$.

5. Required number of signals $= \dfrac{7!}{(2!) \times (3!) \times (2!)}$.

6. Let us assume that the vowels $\boxed{\text{AAEE}}$ form a single letter.

Now, $\boxed{\text{AAEE}}$ + RRNGMNT gives us 8 letters, out of which there are 2 R's, 2 N's, 1 G, 1 M and 1 T.

Number of these arrangements $= \dfrac{8!}{(2!) \times (2!)} = 10080$.

Now, AAEE has 4 letters, out of which there are 2 A's and 2 E's.

Number of their arrangements $= \dfrac{4!}{(2!) \times (2!)} = 6$.

Required number of words formed $= (10080 \times 6) = 60480$.

7. The word 'INDIA' contains 2 I's, 1 A, 1 N and 1 D.

Number of permutations of the letters of the given word $= \dfrac{5!}{2!} = 60$.

Let us assume that the vowels $\boxed{\text{IIA}}$ form 1 letter.

Now, $\boxed{\text{IIA}}$ + ND gives 3 letters, all distinct.
Number of their arrangements = $3! = 6$.
Now, IIA has 3 letters, out of which there are 2 I's and 1 A.

Number of their arrangements = $\frac{3!}{2!} = 3$.

Number of words having all vowels together = $(6 \times 3) = 18$.
Number of words in which all vowels are not together = $(60 - 18) = 42$.

8. In the given word, there are 3 vowels, namely A, E, A out of which there are 2 A's and 1 E.

Number of arrangements of the vowels = $\frac{3!}{2!} = 3$.

The given word contains 4 consonants, namely L, G, B, R.
Number of arrangements of the consonants = $4! = 24$.
Required number of arrangements = $(3 \times 24) = 72$.

9. After fixing S at the beginning and S at the end of each word, we have to arrange 4 letters out of which there are 2 E's, 1 R and 1 I.

Number of their arrangements = $\frac{4!}{2!} = 12$.

10. The given word 'PARALLEL' has 8 letters, out of which there are 2 A's, 3 L's, 1 P, 1 R and 1 E.

Number of their arrangements = $\frac{8!}{(2!) \times (3!)} = 3360$.

Let us assume $\boxed{\text{LLL}}$ as 1 letter.

Then, $\boxed{\text{LLL}}$ + PARAE has 6 letters, out of which there are 2 A's and the rest are all distinct.

Number of their arrangements = $\frac{6!}{2!} = 360$.

Number of arrangements in which 3 L's are not together = $(3360 - 360) = 3000$.

11. The given word 'CAPTAIN' has 7 letters, out of which there are 2 A's and the rest are all different.

Number of their arrangements = $\frac{7!}{2!} = 2520$.

Let us assume $\boxed{\text{CT}}$ as 1 letter.

Then, $\boxed{\text{CT}}$ + APAIN has 6 letters, out of which there are 2 A's and the rest are all distinct.

Number of their arrangements = $\frac{6!}{2!} = 360$.

Also, CT can be arranged among themselves in $2! = 2$ ways.
Number of arrangements in which C and T are together = $(360 \times 2) = 720$.
Number of arrangements in which C and T are never together = $(2520 - 720) = 1800$.

12. The given word 'ASSASSINATION' has 3 A's, 2 I's, 2 N's, 4 S's, 1 T and 1 O.

We consider $\boxed{\text{SSSS}}$ as one letter.

Then, $\boxed{\text{SSSS}}$ + AAAIINNTO consists of 10 letters, out of which there are 3 A's, 2 I's, 2 N's, 1 T, 1 O and 1 $\boxed{\text{SSSS}}$.

Number of their arrangements = $\frac{10!}{(3!) \times (2!) \times (2!)} = 151200$.

13. (i) The given word 'MATHEMATICS' has 11 letters, out of which there are 2 M's, 2 A's, 2 T's and the rest are all distinct.

Number of their arrangements $= \dfrac{11!}{(2!)\times(2!)\times(2!)} = 4989600.$

(ii) Let each arrangement begin with C. Then, out of the remaining 10 letters, there are 2 M's, 2 A's, 2 T's and the rest are all distinct.

Number of all such arrangements $= \dfrac{10!}{(2!)\times(2!)\times(2!)} = 453600.$

(iii) Let each arrangement begin with T. Then, out of the remaining 10 letters, there are 2 M's, 2 A's and the rest are all distinct.

Number of all such arrangements $= \dfrac{10!}{(2!)\times(2!)} = 907200.$

14. (i) The given word 'INTERMEDIATE' has 12 letters, namely 6 vowels and 6 consonants. Since vowels occupy even places, we have

N □ T □ R □ M □ D □ T □

type of arrangements, where vowels occupy even places, as shown by boxes.

It has 6 consonants, namely 2 T's, 1 N, 1 R, 1 M and 1 D.

Number of arrangements of 6 consonants $= \dfrac{6!}{2!} = 360.$

The given word has 6 vowels, namely 2 I's, 3 E's and 1 A.

Number of arrangements of 6 vowels $= \dfrac{6!}{(2!)\times(3!)} = 60.$

$\therefore$ total number of words formed $= (360\times 60) = 21600.$

(ii) It is given that we may replace a vowel by another vowel and we can replace a consonant by a consonant, given herewith.

Number of arrangements of consonants $= \dfrac{6!}{2!} = 360.$

Number of arrangements of vowels $= \dfrac{6!}{(3!)\times(2!)} = 60.$

Total number of words formed $= (360\times 60) = 21600.$

15. (i) The given word 'INSTITUTION' has 11 letters, out of which there are 3 I's, 3 T's, 2 N's, 1 S, 1 U and 1 O.

Total number of words $= \dfrac{11!}{(3!)\times(3!)\times(2!)} = 554400.$

(ii) Treating $\boxed{\text{TTT}}$ as one letter, the given word is expressed as $\boxed{\text{TTT}}$ + INSIUION, which consists of 9 letters out of which there are 3 I's, 2 N's, 1 S, 1 U and 1 O.

Number of words in this case $= \dfrac{9!}{(3!)\times(2!)} = 30240.$

(iii) Keeping $\boxed{\text{NN}}$ as first letter, we arrange the remaining 9 letters, out of which there are 3 I's, 3 T's, 1 S, 1 U and 1 O.

Number of words formed $= \dfrac{9!}{(3!)\times(3!)} = 10080.$

16. We have to form 5-digit numbers in which 5 occurs 2 times; 3 occurs 2 times and 4 occurs 1 time.

Required number of numbers $= \dfrac{5!}{(2!)\times(2!)\times(1!)} = 30.$

17. There are three odd digits, namely 3, 5, 3 out of which 3 occurs twice and 5 occurs once.

Number of ways to fill the odd places $= \frac{3!}{2!} = 3$.

There are four even digits namely 2, 4, 4, 2 out of which 2 occurs twice and 4 occurs twice.

Number of ways to fill the even places $= \frac{4!}{(2!)\times(2!)} = 6$.

Required number of numbers $= (3 \times 6) = 18$.

18. Required number of numbers $= \left\{\frac{7!}{(3!)\times(2!)} - \frac{6!}{(3!)\times(2!)}\right\} = (420 - 60) = 360$.

19. Required number of numbers $= \left\{\frac{6!}{(2!)\times(2!)} - \frac{5!}{(2!)\times(2!)}\right\} = (180 - 30) = 150$.

20. The word 'INDIA' contains 5 letters out of which there are 2 I's, 1 N, 1 D and 1 A.

Number of arrangements of its letters $= \frac{5!}{2!} = 60$.

In the dictionary, the first word is ADIIN.

Starting with A, we arrange the letters D, I, I, N in $\frac{4!}{2!} = 12$ ways.

The 13th word is DAIIN.

Starting with D, we arrange the letters A, I, I, N in $\frac{4!}{2!} = 12$ ways.

The 25th word is IADIN.

Starting with I, we arrange the letters A, D, I, N in $4! = 24$ ways.

The 49th word is NADII.

Starting with N, we arrange the letters A, D, I, I in $\frac{4!}{2!} = 12$ ways.

The 60th word is NIIDA.

PERMUTATIONS WITH REPETITIONS

THEOREM 1 *The number of permutations of n different objects, taken r at a time when each may be repeated any number of times in each arrangement, is n^r.*

PROOF We know that the number of all permutations of n objects, taken r at a time is the same as the number of ways of filling up r places with n different objects.

Clearly, the first place can be filled in n ways. Since each object may be repeated, the second place can be filled in n ways.

Similarly, each of the 3rd, 4th, ..., rth places can be filled in n ways.

$\therefore$ the requisite number of permutations $= n \times n \times n \times \ldots r$ times $= n^r$.

Corollary When $r = n$,

i.e., the number of permutations of n different objects, taken all at a time when each may be repeated any number of times in each arrangement, is n^n.

SUMMARY

(i) Number of permutations of n different objects, taken r at a time when each may be repeated any number of times in each arrangement, is n^r.

(ii) Number of permutations of n different objects, taken all at a time when each may be repeated any number of times in each arrangement, is n^n.

SOLVED EXAMPLES

EXAMPLE 1 *In how many ways can 4 letters be posted in 3 letter boxes?*

SOLUTION Each letter can be posted in any of the 3 letter boxes.

So, each letter can be posted in 3 ways.

$\therefore$ 4 letters can be posted in $(3 \times 3 \times 3 \times 3) = 3^4 = 81$ ways.

EXAMPLE 2 *In how many ways can 6 different rings be worn in 4 fingers of the hand?*

SOLUTION Each ring may be worn in any of the 4 fingers.

So, each ring may be worn in 4 ways.

$\therefore$ 6 rings may be worn in $(4 \times 4 \times 4 \times 4 \times 4 \times 4) = 4^6 = 4096$ ways.

EXAMPLE 3 *In how many ways can 5 apples be distributed among 6 boys, there being no restriction to the number of apples each boy may get?*

SOLUTION Each apple may be given to any of the 6 boys.

So, each apple may be given in 6 ways.

$\therefore$ 5 apples may be given in $(6 \times 6 \times 6 \times 6 \times 6) = 6^5 = 7776$ ways.

EXAMPLE 4 *In how many ways can 3 prizes be distributed among 4 boys, when*

(i) no boy gets more than 1 prize;

(ii) a boy may get any number of prizes;

(iii) no boy gets all the prizes?

SOLUTION **Case (i)** *When no boy gets more than 1 prize.*

The first prize may be given to any of the 4 boys in 4 ways.

The 2nd prize may be given in 3 ways, since the boy who got the 1st prize cannot receive it.

The 3rd prize may be given to any of the remaining 2 boys in 2 ways.

Required number of ways = $(4 \times 3 \times 2) = 24$.

Case (ii) *When a boy may get any number of prizes.*

The 1st prize may be given to any of the 4 boys in 4 ways.

Since a boy may get any number of prizes, so 2nd prize may be given to any of the 4 boys in 4 ways.

Similarly, the 3rd prize may be given in 4 ways.

Required number of ways = $(4 \times 4 \times 4) = 64$.

Case (iii) *When no boy gets all the prizes.*

The number of ways in which a boy gets all the prizes is 4, as anyone of the 4 boys may get all the prizes.

Hence, the number of ways in which a boy does not get all the prizes = $(64-4) = 60$.

EXAMPLE 5 *How many four-digit numbers can be formed with the digits* 1, 2, 3, 4, 5, 6 *when a digit may be repeated any number of times in any arrangement?*

SOLUTION Clearly, the thousand's place can be filled in 6 ways as anyone of the 6 digits may be placed at it.

Since there is no restriction on repetition of digits, each one of the hundred's, ten's and unit's digits can be filled in 6 ways.

Hence, the required number of numbers = $(6 \times 6 \times 6) = 216$.

EXAMPLE 6 *How many* 4*-digit numbers are there when a digit may be repeated any number of times?*

SOLUTION Clearly, 0 cannot be placed at the thousand's place.

So, this place can be filled with any digit from 1 to 9.

Thus, there are 9 ways of filling the thousand's place.

Since repetition of digits is allowed, each one of the remaining 3 places can be filled in 10 ways, i.e., with any digit from 0 to 9.

So, the required number of numbers = $(9 \times 10^3) = 9000$.

EXERCISE 8F

1. A child has 6 pockets. In how many ways can he put 5 marbles in his pocket?
2. In how many ways can 5 bananas be distributed among 3 boys, there being no restriction to the number of bananas each boy may get?
3. In how many ways can 3 letters be posted in 2 letter boxes?
4. How many 3-digit numbers are there when a digit may be repeated any number of times?
5. How many 4-digit numbers can be formed with the digits 0, 2, 3, 4, 5 when a digit may be repeated any number of times in any arrangement?
6. In how many ways can 4 prizes be given to 3 boys when a boy is eligible for all prizes?
7. There are 4 candidates for the post of a chairman and one is to be elected by votes of 5 men. In how many ways can the votes be given?

ANSWERS (EXERCISE 8F)

1. 6^5 **2.** 3^5 **3.** 8 **4.** 900 **5.** 500 **6.** 81 **7.** 1024

HINTS TO SOME SELECTED QUESTIONS

2. Each banana may be given to any of the 3 boys.
So, each banana can be given in 3 ways.

3. Each letter may be posted in 2 ways.

4. We may fill hundred's digit by any digit from 1 to 9, i.e., in 9 ways. Each of the ten's and unit's digits may be filled in 10 ways.
Required number of numbers = $(9 \times 10 \times 10) = 900$.

5. Required number of numbers = $(4 \times 5 \times 5 \times 5) = 500$.

6. Each prize can be distributed in 3 ways.

7. Each man can vote in 4 ways.

CIRCULAR PERMUTATIONS

So far we have been considering the arrangements of objects in a line. Such permutations are known as linear permutations.

Instead of arranging the objects in a line, if we arrange them in the form of a circle, we call them *circular permutations*.

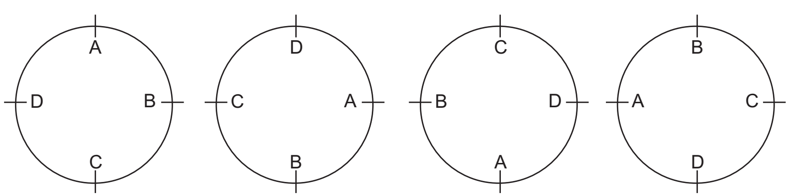
In circular permutations, what really matters is the position of an object relative to the others.

If we consider the linear permutations

$ABCD$, $BCDA$, $CDAB$ and $DABC$

then, clearly, they are distinct.

Now, we arrange A, B, C, D along the circumference of a circle as shown below:



If we consider the position of an object relative to others then we find that the above four arrangements are the same.

For example, if four persons A, B, C, D sit round a table as shown in the first arrangement and then each one shifts to the next chair to the left, we obtain the second arrangement.

In this arrangement each person has the same neighbour to the left, and the same neighbour to the right, as he had in the first arrangement.

Thus, the first and the second arrangements give rise to the same circular permutation.

Same is the case with the third and fourth arrangements given above.

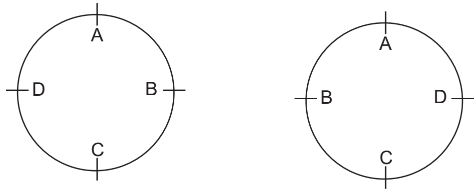
Thus, in circular permutations, we fix the position of one of the objects and then arrange the other objects in all possible ways.

There are two types of circular permutations:

(i) The circular permutations in which the clockwise and the anticlockwise arrangements give rise to different permutations, e.g., seating arrangements of persons round a table.

(ii) The circular permutations in which the clockwise and the anticlockwise arrangements give rise to same permutations, e.g., arranging some beads to form a necklace.

Look at the circular permutations, given below:



Suppose A, B, C, D are the four beads forming a necklace.

They have been arranged in clockwise and anticlockwise directions in the first and second arrangements respectively.

Now, if the necklace in the first arrangement be given a turn, from clockwise to anticlockwise, we obtain the second arrangement. Thus, there is no difference between the above two arrangements.

RESULTS ON CIRCULAR PERMUTATIONS

THEOREM 1 *The number of circular permutations of n different objects is* $(n-1)!$.

PROOF Fixing the position of an object can be done in n ways, as the position of anyone of them may be fixed.

Thus, each circular permutation corresponds to n linear permutations, depending upon where from we start.

Since there are $n!$ linear permutations, it follows that there are $\frac{n!}{n}$, i.e., $(n-1)!$ circular permutations.

THEOREM 2 *The number of ways in which n persons can be seated round a table is* $(n-1)!$.

PROOF Let us fix the position of one person and then arrange the remaining $(n-1)$ persons in all possible ways. Clearly, this can be done in $(n-1)!$ ways.

Hence, the required number of ways $= (n-1)!$.

THEOREM 3 *Show that the number of ways in which n different beads can be arranged to form a necklace is* $\frac{1}{2}(n-1)!$.

PROOF Fixing the position of one bead, the remaining $(n-1)$ beads can be arranged in $(n-1)!$ ways.

In case of arranging the beads, there is no distinction between the clockwise and the anticlockwise arrangements.

So, the required number of ways $= \frac{1}{2}(n-1)!$.

ILLUSTRATIVE EXAMPLES

EXAMPLE 1 *In how many ways can 8 students be arranged in*

(i) a line, *(ii) a circle?*

SOLUTION (i) The number of ways in which 8 students can be arranged in a line $= 8! = (8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1) = 40320$.

(ii) The number of ways in which 8 students can be arranged in a circle $= (8-1)! = 7! = (7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1) = 5040$.

EXAMPLE 2 *If 20 persons were invited for a party, in how many ways can they and the host be seated at a circular table?*

In how many of these ways will two particular persons be seated on either side of the host?

SOLUTION Clearly, there are 21 persons, to be seated round a circular table.

(i) Let us fix the seat of one person, say the host. The remaining 20 persons can now be arranged in 20 ! ways.

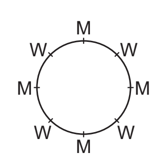
Hence, the number of ways in which these 21 persons can be seated round a circular table = 20 !.

(ii) The two particular persons can be seated on either side of the host in 2 ways and for each way of their taking seats, the remaining 18 persons can be seated at the circular table in 18 ! ways.

$\therefore$ the number of ways of seating 21 persons at a circular table with two particular persons on either side of the host $= (2 \times 18!)$.

EXAMPLE 3 *In how many ways can a party of 4 men and 4 women be seated at a circular table so that no two women are adjacent?*

SOLUTION The 4 men can be seated at the circular table such that there is a vacant seat between every pair of men.



The number of ways in which these 4 men can be seated at the circular table = 3 ! = 6.

Now, the 4 vacant seats may be occupied by 4 women in

$^4P_4 = 4! = 24$ ways.

$\therefore$ the required number of ways $= (6 \times 24) = 144$.

EXAMPLE 4 *A round table conference is to be held between delegates of 20 countries. In how many ways can they be seated if two particular delegates may wish to sit together?*

SOLUTION Regarding these two delegates as one, the 19 delegates can be arranged at a circular table in 18 ! ways.

These two delegates can be arranged among themselves in 2 !, i.e., 2 ways.

$\therefore$ the required number of ways $= 2 \times (18\,!)$.

EXAMPLE 5 *In how many ways can 7 persons sit around a table so that all shall not have the same neighbours in any two arrangements?*

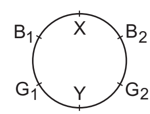
SOLUTION 7 persons sit at a round table in 6 ! ways.

But, in clockwise and anticlockwise arrangements, each person will have the same neighbours.

So, the required number of ways $= (1/2) \times (6\,!) = 360$.

EXAMPLE 6 *3 boys and 3 girls are to be seated around a table in a circle. Among them, the boy X does not want any girl neighbour and the girl Y does not want any boy neighbour. How many such arrangements are possible?*

SOLUTION Fix the positions of X and Y as shown in the figure. Now, the boys B_1, B_2 and the girls G_1, G_2 may have their neighbours X and Y respectively.



The boys B_1 and B_2 may be arranged among themselves in 2 ! ways. And, the girls G_1 and G_2 may be arranged among themselves in 2 ! ways.

Hence, the required number of arrangements $= (2 \times 2) = 4$.

EXAMPLE 7 *Find the number of ways in which 8 different beads can be arranged to form a necklace.*

SOLUTION Fixing the position of one bead, the remaining beads can be arranged in 7! ways. But, there is no distinction between the clockwise and anticlockwise arrangements.

So, the required number of arrangements $= (1/2) \times (7\,!) = 2520$.

EXERCISE 8G

1. In how many ways can 6 persons be arranged in

(i) a line, (ii) a circle?

2. There are 5 men and 5 ladies to dine at a round table. In how many ways can they seat themselves so that no two ladies are together?

3. In how many ways can 11 members of a committee sit at a round table so that the secretary and the joint secretary are always the neighbours of the president?

4. In how many ways can 8 persons be seated at a round table so that all shall not have the same neighbours in any two arrangements?

5. In how many different ways can 20 different pearls be arranged to form a necklace?

6. In how many different ways can a garland of 16 different flowers be made?

ANSWERS (EXERCISE 8G)

1. (i) $6! = 720$ (ii) 120 **2.** $(4!) \times (5!) = 2880$ **3.** 80640

4. 2520 **5.** $\frac{1}{2} \times (19!)$ **6.** $\frac{1}{2} \times (15!)$

EXERCISE 8H

Very-Short-Answer Questions

1. If $(n+1)! = 12 \times [(n-1)!]$, find the value of n.

2. If $\frac{1}{4!} + \frac{1}{5!} = \frac{x}{6!}$, find the value of x.

3. How many 3-digit numbers are there with no digit repeated?

4. How many 3-digit numbers above 600 can be formed by using the digits 2, 3, 4, 5, 6, if repetition of digits is allowed?

5. How many numbers divisible by 5 and lying between 4000 and 5000 can be formed from the digits 4, 5, 6, 7, 8, if repetition of digits is allowed?

6. In how many ways can the letters of the word 'CHEESE' be arranged?

7. In how many ways can the letters of the word 'PERMUTATIONS' be arranged if each word starts with P and ends with S?

8. How many different words can be formed by using all the letters of the word 'ALLAHABAD'?

9. How many permutations of the letters of the word 'APPLE' are there?

10. How many words can be formed by the letters of the word 'SUNDAY'?

11. In how many ways can 4 letters be posted in 5 letter boxes?

12. In how many ways can 4 women draw water from 4 taps, if no tap remains unused?

13. How many 5-digit numbers can be formed by using the digits 0, 1 and 2?

14. In how many ways can 5 boys and 3 girls be seated in a row so that each girl is between 2 boys?

15. A child has plastic toys bearing the digits 4, 4 and 5. How many 3-digit numbers can he make using them?

16. In how many ways can the letters of the word 'PENCIL' be arranged so that N is always next to E?

ANSWERS (EXERCISE 8H)

1. $n = 3$ **2.** $x = 36$ **3.** 648 **4.** 25 **5.** 25

6. 120 **7.** 1814400 **8.** 7560 **9.** 60 **10.** 720

11. 625 **12.** 24 **13.** 162 **14.** 24 **15.** 3
16. 120

HINTS TO SOME SELECTED QUESTIONS

1. $(n+1)\times n\times[(n-1)!]=12\times[(n-1)!] \Rightarrow (n+1)\times n=12=4\times 3 \Rightarrow n=3.$

2. $\frac{1}{4!}+\frac{1}{5\times(4!)}=\frac{x}{(6\times 5)\times(4!)} \Rightarrow \frac{x}{30}=\left(1+\frac{1}{5}\right)=\frac{6}{5} \Rightarrow x=\left(\frac{6}{5}\times 30\right)=36.$

3. The hundred's, ten's and unit's digits can be filled in 9, 9 and 8 ways respectively.
Hence, the required number of numbers $=(9\times 9\times 8)=648$.

4. Given digits are 2, 3, 4, 5, 6 and repetition of digits is allowed.
The hundred's place can be filled by 6 in 1 way only.
Each of the ten's and unit's digits can be filled by any of the given five digits in 5 ways.
Required number of numbers $=(1\times 5\times 5)=25$.

5. Given digits are 4, 5, 6, 7, 8 and repetition of digits is allowed.
Clearly, the required number must have 4 at the thousand's place.
So, this place can be filled in 1 way.
Since each number is divisible by 5, so we must have 5 at the unit's place. So, unit's place can be filled in 1 way.
Each of the hundred's and ten's places can be filled in 5 ways.
$\therefore$ the required number of numbers $=(1\times 1\times 5\times 5)=25$.

6. The given word contains 6 letters, out of which E is repeated 3 times and all others are different.
So, the required number of ways $=\frac{6!}{3!}=120$.

7. The given word contains 12 letters. Keeping P and S at first and last positions, 10 letters remain there out of which, there are 2 T's and the rest are different.
Required number of ways $=\frac{10!}{2!}=1814400$.

8. The given word contains 9 letters out of which there are 4 A's, 2 L's and others are all distinct.
Required number of words $=\frac{9!}{(4!)\times(2!)}=7560$.

9. The given word contains 5 letters, out of which there are 2 P's and the rest are all different.
Required number of permutations $=\frac{5!}{2!}=60$.

10. The given word contains 6 letters, all distinct.
Required number of words $=6!=720$.

11. Every letter can be posted in 5 ways.
Required number of ways $=5^4$.

12. 1st tap can be used by any of the 4 women in 4 ways.
2nd tap can be used by any of the remaining 3 women.

3rd tap can be used by any of the remaining 2 women.
4th tap can be used by the remaining 1 woman.
Required number of ways $= 4 \times 3 \times 2 \times 1 = 24$.

13. The ten-thousand's place can be filled by 1 or 2, i.e., in 2 ways.
Each of the other 4 digits can be filled in 3 ways.
Required number of numbers $= (2 \times 3 \times 3 \times 3 \times 3) = 162$.

14. Let us fix 5 boys, keeping a space between every 2 boys, as shown below:

B □ B □ B □ B □ B.

Required number of ways $= {}^4P_3 = (4 \times 3 \times 2) = 24$.

15. There are 3 digits out of which 4 is repeated 2 times and 5 occurs only once.

Required number of numbers $= \dfrac{3\,!}{2\,!} = \dfrac{3 \times 2 \times 1}{2} = 3$.

16. Keeping $\boxed{\text{EN}}$ as 1 letter, this letter and 4 other letters of the given word can be arranged among themselves in $5\,! = 120$ ways.

KEY FACTS AND FORMULAE

1. (i) $n\,! = n \times (n-1) \times (n-2) \times \ldots \times 3 \times 2 \times 1$

(ii) $n\,! = n \times [(n-1)\,!]$

(iii) $0\,! = 1$

2. Fundamental Principle of Multiplication

If there are two operations such that one of them can be performed in m ways and when it has been performed in anyone of these m ways, the second operation can be performed in n ways, then the two operations in succession can be performed in $(m \times n)$ ways.

3. Fundamental Principle of Addition

If there are two operations such that they can be performed independently in m and n ways respectively, then either of the two operations can be performed in $(m+n)$ ways.

4. Permutations The different arrangements which can be made out of a given number of things by taking some or all at a time, are called permutations.

5. (i) Number of permutations of n different objects taking r at a time, where $0 < r \le n$ and the objects do not repeat, is given by

$$ {}^nP_r = n(n-1)(n-2)\ldots(n-r+1) = \frac{n\,!}{(n-r)\,!}\cdot $$

(ii) ${}^nP_n = n\,!$.

(iii) Let there be n things of which m are alike of one kind and the remaining $(n-m)$ are alike of second kind, then

$$ \text{total number of permutations} = \frac{n\,!}{(m\,!) \times [(n-m)\,!]}\cdot $$

(iv) The number of permutations of n objects, where p_1 objects are of one kind, p_2 are of second kind, ..., p_k are of kth kind and the rest, if any, are distinct, is

$$\frac{n!}{(p_1!)\times(p_2!)\times\ldots\times(p_k!)}.$$

(v) Number of permutations of n different objects, taken r at a time when each may be repeated any number of times in each arrangement, is n^r.

6. The number of circular permutations of n different objects is $(n-1)!$.

7. The number of ways in which n persons can be seated round a table is $(n-1)!$.

8. The number of ways in which n different beads can be arranged to form a necklace is $\frac{1}{2}(n-1)!$.

❑

9
Combinations

COMBINATIONS *Each of the different groups or selections that can be made out of a given number of things by taking some or all of them at a time, irrespective of their arrangements, is called a combination.*

ILLUSTRATIVE EXAMPLES

EXAMPLE 1 *All combinations of three persons A, B, C taken two at a time are*

AB, AC, BC.

Clearly, *AB* and *BA* represent the same combination but they represent different permutations.

EXAMPLE 2 *All combinations of four persons A, B, C, D taken two at a time are*

AB, AC, AD, BC, BD, CD.

EXAMPLE 3 *All combinations of four persons A, B, C, D taken three at a time are*

ABC, ABD, ACD, BCD.

DIFFERENCE BETWEEN A PERMUTATION AND A COMBINATION

In a combination, only a group is made and the order in which the objects are arranged is immaterial.

On the other hand, in a permutation, not only the group is formed, but also an arrangement in a definite order is considered.

REMARK In general, we use the word *arrangements* for permutations and the word *selections* for combinations.

EXAMPLE 4 (i) *AB* and *BA* are two different permutations but each represents the same combination.

(ii) *ABC, ACB, BAC, BCA, CAB, CBA* are six different permutations but each one represents the same combination, namely *ABC*.

NOTATION *The number of all combinations of n things, taken r at a time, is denoted by* ${}^nC_r$, *or* $C(n, r)$, *where n and r are integers such that* $n > 0$, $r \geq 0$ *and* $n \geq r$.

Combinations of *n* Different Objects

THEOREM 1 *The number of all combinations of n distinct objects, taken r at a time, is given by*

$$ {}^nC_r = \frac{n!}{(r!) \times (n-r)!} . $$

PROOF Let the number of all combinations of n objects, taken r at a time, be x. Then, ${}^nC_r = x$.

Now, each combination contains r objects, which may be arranged among themselves in $r\,!$ ways.

Thus, each combination gives rise to $r\,!$ permutations.

$\therefore$ x combinations will give rise to $x \times (r\,!)$ permutations.

So, the number of permutations of n things, taken r at a time, is $x \times (r\,!)$.

Consequently, ${}^nP_r = x \times (r\,!) = {}^nC_r \times (r\,!)$.

$$\therefore \quad {}^nC_r = \frac{{}^nP_r}{r\,!} = \frac{n\,!}{(r\,!) \times (n-r)\,!} \quad \left[\because\ {}^nP_r = \frac{n\,!}{(n-r)\,!}\right].$$

Corollary 1 *Prove that* ${}^nC_r = \dfrac{n(n-1)(n-2)\ldots \text{to } r \text{ factors}}{r\,!}$.

PROOF We have

$$\begin{aligned} {}^nC_r &= \frac{n\,!}{(r\,!) \times (n-r)\,!} \\ &= \frac{n(n-1)(n-2)\ldots(n-r+1)(n-r)(n-r-1)\ldots 3\cdot 2\cdot 1}{(r\,!) \times [(n-r)(n-r-1)\ldots 3\cdot 2\cdot 1]} \\ &= \frac{n(n-1)(n-2)\ldots(n-r+1)}{r\,!} \\ &= \frac{n(n-1)(n-2)\ldots \text{to } r \text{ factors}}{r\,!}. \end{aligned}$$

Hence, ${}^nC_r = \dfrac{n(n-1)(n-2)\ldots \text{to } r \text{ factors}}{r\,!}$.

EXAMPLE 1 *Evaluate* ${}^{11}C_4$.

SOLUTION ${}^{11}C_4 = \dfrac{11 \times 10 \times 9 \times 8}{4\,!} = \dfrac{11 \times 10 \times 9 \times 8}{4 \times 3 \times 2 \times 1} = 330.$

Corollary 2 *Prove that* ${}^nC_n = {}^nC_0 = 1$.

PROOF We have

$${}^nC_r = \frac{n\,!}{(r\,!) \times (n-r)\,!}. \qquad \ldots \text{(i)}$$

Putting $r = n$ in (i), we get

$${}^nC_r = \frac{n\,!}{(n\,!) \times (n-n)\,!} = \frac{1}{0\,!} = 1 \qquad [\because\ 0\,! = 1].$$

Putting $r = 0$ in (i), we get

$${}^nC_0 = \frac{n\,!}{(0\,!) \times (n-0)\,!} = \frac{n\,!}{1 \times (n\,!)} = 1 \qquad [\because\ 0\,! = 1].$$

Hence, ${}^nC_n = {}^nC_0 = 1$.

Corollary 3 *Prove that* $^nC_r = \dfrac{^nP_r}{r\,!}\cdot$

PROOF We have

$$^nC_r = \frac{n\,!}{(r\,!)\times(n-r)\,!} = \frac{1}{(r\,!)}\cdot\left\{\frac{n\,!}{(n-r)\,!}\right\} = \frac{^nP_r}{r\,!}\cdot$$

EXAMPLE 2 *If* $^nP_r = 720$ *and* $^nC_r = 120$, *find* r.

SOLUTION We have

$$^nC_r = \frac{^nP_r}{r\,!} \Rightarrow \frac{720}{r\,!} = 120 \Rightarrow r\,! = \frac{720}{120} = 6 = (3\times 2\times 1) = 3\,!$$

$$\Rightarrow r = 3.$$

Hence, $r = 3$.

SUMMARY OF THE ABOVE RESULTS

(i) $^nC_r = \dfrac{n\,!}{(r\,!)\times(n-r)\,!} = \dfrac{n(n-1)(n-2)\ldots(n-r+1)}{r\,!}$

(ii) $^nC_r = \dfrac{n(n-1)(n-2)\ldots \text{ to } r \text{ factors}}{r\,!}$

(iii) $^nC_n = {}^nC_0 = 1$

(iv) $^nC_r = \dfrac{^nP_r}{r\,!}$

PROPERTIES OF ^nC_r

THEOREM 1 *For* $0 \le r \le n$, *prove that* $^nC_r = {}^nC_{n-r}$.

PROOF We have

$$^nC_{n-r} = \frac{n\,!}{(n-r)\,!\times\{n-(n-r)\}\,!} = \frac{n\,!}{(n-r)\,!\times(r\,!)} = {}^nC_r.$$

Hence, $^nC_r = {}^nC_{n-r}$.

EXAMPLE 1 *Evaluate* $^{100}C_{98}$.

SOLUTION Using the result $^nC_r = {}^nC_{n-r}$, we get

$$^{100}C_{98} = {}^{100}C_{(100-98)} = {}^{100}C_2 = \frac{100\times 99}{2} = 4950.$$

THEOREM 2 *If* $^nC_x = {}^nC_y$ *and* $x \ne y$ *then prove that* $x + y = n$.

PROOF We have

$$^nC_x = {}^nC_y = {}^nC_{n-y} \qquad [\because\ ^nC_r = {}^nC_{n-r}]$$

$$\Rightarrow \quad x = y \text{ or } x = n-y$$

$$\Rightarrow \quad x = n-y \qquad [\because\ x \ne y \text{ (given)}]$$

$$\Rightarrow \quad x + y = n.$$

Hence, $^nC_x = {}^nC_y$ and $x \ne y$ then $x + y = n$.

EXAMPLE 2 (i) *If* ${}^{n}C_{18} = {}^{n}C_{12}$ *then find the value of* ${}^{32}C_n$.

(ii) *If* ${}^{10}C_r = {}^{10}C_{r+4}$ *then find the value of* ${}^{5}C_r$.

SOLUTION (i) ${}^{n}C_{18} = {}^{n}C_{12} \Rightarrow n = (18+12) = 30.$

$$\therefore \quad {}^{32}C_n = {}^{32}C_{30} = {}^{32}C_{32-30} = {}^{32}C_2 = \frac{32 \times 31}{2\,!} = 496.$$

(ii) ${}^{10}C_r = {}^{10}C_{r+4} \Rightarrow r + (r+4) = 10 \Rightarrow r = 3.$

$$\therefore \quad {}^{5}C_r = {}^{5}C_3 = {}^{5}C_{5-3} = {}^{5}C_2 = \frac{5 \times 4}{2\,!} = 10.$$

THEOREM 3 *Let* $1 \le r \le n$. *Then, prove that*

$${}^{n}C_r + {}^{n}C_{r-1} = {}^{n+1}C_r. \quad [\textit{Pascal's rule}]$$

PROOF We have

$$\begin{aligned}
&{}^{n}C_r + {}^{n}C_{r-1}\\
&= \frac{n\,!}{r\,!\cdot(n-r)\,!} + \frac{n\,!}{(r-1)\,!\cdot\{n-(r-1)\}\,!}\\
&= \frac{n\,!}{r\,!\cdot(n-r)\,!} + \frac{n\,!}{(r-1)\,!\cdot(n-r+1)\,!}\\
&= \frac{(n\,!)\cdot(n-r+1)}{r\,!\cdot(n-r+1)\,!} + \frac{n\,!\cdot r}{r\,!\cdot(n-r+1)\,!}\\
&= \left\{\frac{n\,!}{r\,!\cdot(n-r+1)\,!}\right\}\cdot\{n-r+1+r\}\\
&= \frac{(n+1)\cdot(n\,!)}{(r\,!)\cdot(n-r+1)\,!}\\
&= \frac{(n+1)\,!}{(r\,!)(n+1-r)\,!}\\
&= {}^{n+1}C_r.
\end{aligned}$$

Hence, ${}^{n}C_r + {}^{n}C_{r-1} = {}^{n+1}C_r$.

THEOREM 4 *If* $1 \le r \le n$, *prove that* $n \times {}^{n-1}C_{r-1} = (n-r+1) \times {}^{n}C_{r-1}$.

PROOF We have

$$\begin{aligned}
n \times {}^{n-1}C_{r-1} &= n \times \frac{(n-1)\,!}{(r-1)\,!\times[(n-1)-(r-1)]\,!}\\
&= \frac{n \times (n-1)\,!}{(r-1)\,!\times(n-r)\,!}\\
&= \frac{(n\,!)\times(n-r+1)}{(r-1)\,!\times(n-r)\,!\times(n-r+1)}\\
&\qquad [\text{multiplying num. and denom. by } (n-r+1)]\\
&= (n-r+1) \times \frac{n\,!}{(r-1)\,!\times(n-r+1)\,!}\\
&= (n-r+1) \times {}^{n}C_{r-1}.
\end{aligned}$$

Hence, $n \times {}^{n-1}C_{r-1} = (n-r+1) \times {}^{n}C_{r-1}$.

REMARK The above result may be written as

$$\frac{{}^{n}C_{r-1}}{{}^{n-1}C_{r-1}} = \frac{n}{(n-r+1)}\cdot$$

THEOREM 5 *If n and r are positive integers such that* $1 \le r \le n$ *then prove that:*

(i) $\dfrac{{}^{n}C_{r}}{{}^{n}C_{r-1}} = \dfrac{n-r+1}{r}$ (ii) $\dfrac{{}^{n}C_{r}}{{}^{n-1}C_{r-1}} = \dfrac{n}{r}$

PROOF (i) We know that

$${}^{n}C_{r} = \frac{n!}{(r!)\times(n-r)!} \text{ and } {}^{n}C_{r-1} = \frac{n!}{(r-1)!\times(n-r+1)!}\cdot$$

$$\therefore \quad \frac{{}^{n}C_{r}}{{}^{n}C_{r-1}} = \frac{n!}{(r!)\times(n-r)!} \times \frac{(r-1)!\times(n-r+1)!}{n!}$$

$$= \frac{(r-1)!\times(n-r+1)\times(n-r)!}{r\times(r-1)!\times(n-r)!}$$

$$= \frac{n-r+1}{r}\cdot$$

Hence, $\dfrac{{}^{n}C_{r}}{{}^{n}C_{r-1}} = \dfrac{n-r+1}{r}\cdot$

(ii) We have

$${}^{n}C_{r} = \frac{n!}{(r!)\times(n-r)!}$$

and $${}^{n-1}C_{r-1} = \frac{(n-1)!}{(r-1)!\times\{(n-1)-(r-1)\}!} = \frac{(n-1)!}{(r-1)!\times(n-r)!}\cdot$$

$$\therefore \quad \frac{{}^{n}C_{r}}{{}^{n-1}C_{r-1}} = \frac{n!}{(r!)\times(n-r)!} \times \frac{(r-1)!\times(n-r)!}{(n-1)!} = \frac{n}{r}\cdot$$

Hence, $\dfrac{{}^{n}C_{r}}{{}^{n-1}C_{r-1}} = \dfrac{n}{r}\cdot$

SUMMARY OF THE ABOVE RESULTS

(i) ${}^{n}C_{r} = {}^{n}C_{n-r}$

(ii) ${}^{n}C_{x} = {}^{n}C_{y}$ and $x \ne y \Rightarrow x+y=n$

(iii) ${}^{n}C_{r} + {}^{n}C_{r-1} = {}^{n+1}C_{r}$ [Pascal's rule]

(iv) $\dfrac{{}^{n}C_{r-1}}{{}^{n-1}C_{r-1}} = \dfrac{n}{(n-r+1)}$ (v) $\dfrac{{}^{n}C_{r}}{{}^{n-1}C_{r-1}} = \dfrac{n}{r}$

(vi) $\dfrac{{}^{n}C_{r}}{{}^{n}C_{r+1}} = \dfrac{r+1}{n-r}$

SOLVED EXAMPLES

EXAMPLE 1 *Evaluate:*

(i) $^{10}C_3$ *(ii)* $^{11}C_8$ *(iii)* $^{50}C_{48}$ *(iv)* $^{63}C_{63}$

SOLUTION We have

(i) $^{10}C_3 = \dfrac{10\times 9\times 8}{3\,!} = \dfrac{10\times 9\times 8}{3\times 2\times 1} = 120.$

(ii) $^{11}C_8 = {}^{11}C_{(11-8)} = {}^{11}C_3 = \dfrac{11\times 10\times 9}{3\,!} = \dfrac{11\times 10\times 9}{3\times 2\times 1} = 165$

$[\because\ {}^nC_r = {}^nC_{n-r}].$

(iii) $^{50}C_{48} = {}^{50}C_{(50-48)} = {}^{50}C_2 = \dfrac{50\times 49}{2} = 1225.$

(iv) $^{63}C_{63} = 1$ $[\because\ {}^nC_n = 1].$

EXAMPLE 2 *If* $^nC_8 = {}^nC_6$, *find* $^nC_3$.

SOLUTION We know that $^nC_p = {}^nC_q$ and $p \neq q \Rightarrow n = p + q.$

$\therefore$ $^nC_8 = {}^nC_6 \Rightarrow n = (8+6) = 14.$

$\therefore$ $^nC_3 = {}^{14}C_3 = \dfrac{14\times 13\times 12}{3\,!} = \dfrac{14\times 13\times 12}{3\times 2\times 1} = 364.$

EXAMPLE 3 *If* $^nC_{r-1} : {}^nC_r : {}^nC_{r+1} = 3:4:5$, *find the values of n and r.*

SOLUTION We have, $\dfrac{^nC_r}{^nC_{r+1}} = \dfrac{r+1}{n-r}\cdot$ … (i)

Putting $\dfrac{^nC_r}{^nC_{r+1}} = \dfrac{4}{5}$ in (i), we get

$\dfrac{r+1}{n-r} = \dfrac{4}{5} \Rightarrow 5r + 5 = 4n - 4r \Rightarrow 4n - 9r = 5.$ … (ii)

Replacing r by $(r-1)$ in (i), we get

$\dfrac{^nC_{r-1}}{^nC_r} = \dfrac{r}{n-r+1} \Rightarrow \dfrac{r}{n-r+1} = \dfrac{3}{4}$

$\Rightarrow 3n - 3r + 3 = 4r \Rightarrow 3n - 7r = -3.$ … (iii)

On solving (ii) and (iii), we get $n = 62$ and $r = 27$.

EXAMPLE 4 *If* $^{2n}C_3 : {}^nC_3 = 11:1$, *find n.*

SOLUTION We have

$$^{2n}C_3 = \frac{2n(2n-1)(2n-2)}{3\,!} = \frac{2n(n-1)(2n-1)}{3}$$

and $^nC_3 = \dfrac{n(n-1)(n-2)}{3\,!} = \dfrac{n(n-1)(n-2)}{6}\cdot$

$$\therefore\quad \frac{^{2n}C_3}{^nC_3} = \frac{2n(n-1)(2n-1)}{3} \times \frac{6}{n(n-1)(n-2)} = \frac{4(2n-1)}{(n-2)}\cdot$$

Now, $\dfrac{^{2n}C_3}{^nC_3} = \dfrac{11}{1} \Rightarrow \dfrac{4(2n-1)}{(n-2)} = \dfrac{11}{1}$

$$\Rightarrow\ 8n-4=11n-22\ \Rightarrow\ 3n=18\ \Rightarrow\ n=6.$$

Hence, $n=6$.

EXAMPLE 5 *Prove that the product of r consecutive positive integers is divisible by $r\,!$.*

SOLUTION Let the r consecutive positive integers be $(n+1), (n+2), \ldots, (n+r)$.

Then, their product $=(n+1)(n+2)(n+3)\ldots(n+r)$

$$=\frac{(n\,!)(n+1)(n+2)(n+3)\ldots(n+r)}{n\,!}$$

$$=\frac{(n+r)\,!}{n\,!}=(r\,!)\cdot\frac{(n+r)\,!}{(r\,!)\times[(n+r)-r]\,!}$$

$=(r\,!)\times{}^{n+r}C_r$, which is divisible by $r\,!$, since ${}^{n+r}C_r$ is a positive integer.

EXAMPLE 6 *If ${}^nP_r={}^nP_{r+1}$ and ${}^nC_r={}^nC_{r-1}$, find n and r.*

SOLUTION ${}^nP_r={}^nP_{r+1}\ \Rightarrow\ \dfrac{n\,!}{(n-r)\,!}=\dfrac{n\,!}{(n-r-1)\,!}$

$$\Rightarrow\ (n-r)\,!=(n-r-1)\,!$$

$$\Rightarrow\ (n-r)\times(n-r-1)\,!=(n-r-1)\,!$$

$$\Rightarrow\ n-r=1. \qquad \ldots \text{(i)}$$

${}^nC_r={}^nC_{r-1}\ \Rightarrow\ \dfrac{{}^nC_r}{{}^nC_{r-1}}=1$

$$\Rightarrow\ \frac{n-r+1}{r}=1\quad\left[\because\ \frac{{}^nC_r}{{}^nC_{r-1}}=\frac{n-r+1}{r}\right]$$

$$\Rightarrow\ n-2r=-1. \qquad \ldots \text{(ii)}$$

Solving (i) and (ii), we get $n=3$ and $r=2$.

EXAMPLE 7 *Prove that* ${}^{2n}C_n=\dfrac{2^n\times[1\cdot3\cdot5\ldots(2n-1)]}{n\,!}$.

SOLUTION ${}^{2n}C_n=\dfrac{(2n)\,!}{(n\,!)(2n-n)\,!}=\dfrac{(2n)\,!}{(n\,!)^2}$

$$=\frac{(2n)(2n-1)(2n-2)\ldots4\cdot3\cdot2\cdot1}{(n\,!)^2}$$

$$=\frac{[(2n)(2n-2)(2n-4)\ldots4\cdot2]\times[(2n-1)(2n-3)\ldots5\cdot3\cdot1]}{(n\,!)^2}$$

$$=\frac{2^n[n(n-1)(n-2)\ldots2\cdot1]\times[(2n-1)(2n-3)\ldots5\cdot3\cdot1]}{(n\,!)^2}$$

$$=\frac{2^n\times(n\,!)\times[1\cdot3\cdot5\ldots(2n-3)(2n-1)]}{(n\,!)^2}$$

$$=\frac{2^n\times[1\cdot3\cdot5\ldots(2n-3)(2n-1)]}{(n\,!)}.$$

Hence, $^{2n}C_n = \dfrac{2^n \times \{1 \times 3 \times 5 \times \ldots \times (2n-1)\}}{n!}$.

EXAMPLE 8 *Prove that*

$$^{4n}C_{2n} : {}^{2n}C_n = \{1 \times 3 \times 5 \times \ldots \times (4n-1)\} : \{1 \times 3 \times 5 \times \ldots \times (2n-1)\}^2.$$

SOLUTION We have

$$\frac{^{4n}C_{2n}}{^{2n}C_n} = \frac{(4n)!}{(2n)! \times (2n)!} \times \frac{(n!) \times (n!)}{(2n)!}$$

$$= \frac{(4n)! \times (n!)^2}{(2n)! \times [(2n)!]^2}$$

$$= \frac{[1 \cdot 2 \cdot 3 \cdot 4 \ldots (4n-1)(4n)] \times (n!)^2}{(2n)! \times [1 \cdot 2 \cdot 3 \cdot 4 \ldots (2n-2)(2n-1)(2n)]^2}$$

$$= \frac{[1 \cdot 3 \cdot 5 \ldots (4n-1)] \times [2 \cdot 4 \cdot 6 \ldots (4n)] \times (n!)^2}{(2n)! \times [1 \cdot 3 \cdot 5 \ldots (2n-1)]^2 \times [2 \cdot 4 \cdot 6 \ldots (2n-2)(2n)]^2}$$

$$= \frac{[1 \cdot 3 \cdot 5 \ldots (4n-1)] \times 2^{2n} \times [1 \cdot 2 \cdot 3 \ldots (2n)] \times (n!)^2}{(2n)! \times [1 \cdot 3 \cdot 5 \ldots (2n-1)]^2 \times 2^{2n} \times (n!)^2}$$

$$= \frac{[1 \cdot 3 \cdot 5 \ldots (4n-1)]}{[1 \cdot 3 \cdot 5 \ldots (2n-1)]^2}.$$

$$\therefore \quad {}^{4n}C_{2n} : {}^{2n}C_n = [1 \cdot 3 \cdot 5 \ldots (4n-1)\} : \{1 \cdot 3 \cdot 5 \ldots (2n-1)\}^2.$$

EXAMPLE 9 *If* $^{n+2}C_8 : {}^{n-2}P_4 = 57 : 16$, *find n.*

SOLUTION We have

$$^{n+2}C_8 = \frac{(n+2)!}{(8!)(n+2-8)!} = \frac{(n+2)!}{(8!)(n-6)!} \qquad \left[\because \ {}^nC_r = \frac{n!}{(r!) \cdot (n-r)!}\right]$$

and $^{n-2}P_4 = \dfrac{(n-2)!}{(n-2-4)!} = \dfrac{(n-2)!}{(n-6)!}$ $\qquad \left[\because \ {}^nP_r = \dfrac{n!}{(n-r)!}\right]$.

Now, $\dfrac{^{n+2}C_8}{^{n-2}P_4} = \dfrac{57}{16}$

$$\Rightarrow \quad \frac{(n+2)!}{(8!)(n-6)!} \times \frac{(n-6)!}{(n-2)!} = \frac{57}{16}$$

$$\Rightarrow \quad \frac{(n+2)(n+1)n(n-1) \cdot (n-2)!}{(n-2)!} = \left(\frac{57}{16} \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1\right)$$

$$\Rightarrow \quad (n+2)(n+1)n(n-1) = 21 \times 20 \times 19 \times 18$$

$$\Rightarrow \quad n-1 = 18 \ \Rightarrow \ n = 19.$$

Hence, $n = 19$.

EXERCISE 9A

1. Evaluate:

(i) ${}^{20}C_4$ (ii) ${}^{16}C_{13}$ (iii) ${}^{90}C_{88}$

(iv) ${}^{71}C_{71}$ (v) ${}^{n+1}C_n$ (vi) $\sum_{r=1}^{6} {}^6C_r$

2. Verify that:

(i) ${}^{15}C_8 + {}^{15}C_9 - {}^{15}C_6 - {}^{15}C_7 = 0$ (ii) ${}^{10}C_4 + {}^{10}C_3 = {}^{11}C_4$

3. (i) If ${}^nC_7 = {}^nC_5$, find n.

(ii) If ${}^nC_{14} = {}^nC_{16}$, find ${}^nC_{28}$.

(iii) If ${}^nC_{16} = {}^nC_{14}$, find ${}^nC_{27}$.

4. (i) If ${}^{20}C_r = {}^{20}C_{r+6}$, find r.

(ii) If ${}^{18}C_r = {}^{18}C_{r+2}$, find ${}^rC_5$.

5. If ${}^nC_{r-1} = {}^nC_{3r}$, find r.

6. If ${}^{2n}C_3 : {}^nC_3 = 12:1$, find n.

7. If ${}^{15}C_r : {}^{15}C_{r-1} = 11:5$, find r.

8. If ${}^nP_r = 840$ and ${}^nC_r = 35$, find the value of r.

9. If ${}^nC_{r-1} = 36$, ${}^nC_r = 84$ and ${}^nC_{r+1} = 126$, find n and r.

10. If ${}^{n+1}C_{r+1} : {}^nC_r = 11:6$ and ${}^nC_r : {}^{n-1}C_{r-1} = 6:3$, find n and r.

11. How many different teams of 11 players can be chosen from 15 players?

12. If there are 12 persons in a party and if each two of them shake hands with each other, how many handshakes are possible?

13. How many chords can be drawn through 21 points on a circle?

14. From a class of 25 students, 4 are to be chosen for a competition. In how many ways can this be done?

ANSWERS (EXERCISE 9A)

1. (i) 4845 (ii) 560 (iii) 4005 (iv) 1 (v) $n+1$ (vi) 63

3. (i) $n = 12$ (ii) 435 (iii) 4060 **4.** (i) $r = 7$ (ii) 56

5. $r = \frac{1}{4}(n+1)$ **6.** $n = 5$ **7.** $r = 5$ **8.** $r = 4$ **9.** $n = 9, r = 3$

10. $n = 10, r = 5$ **11.** 1365 **12.** 66 **13.** 210 **14.** 12650

HINTS TO SOME SELECTED QUESTIONS

6. $\frac{2n(2n-1)(2n-2)}{3\times2\times1} \times \frac{3\times2\times1}{n(n-1)(n-2)} = \frac{12}{1}$

$\Rightarrow \quad 4(2n-1) = 12(n-2) \Rightarrow 2n-1 = 3n-6 \Rightarrow n = 5.$

7. $\frac{{}^nC_r}{{}^nC_{r-1}} = \frac{n-r+1}{r} \Rightarrow \frac{{}^{15}C_r}{{}^{15}C_{r-1}} = \frac{15-r+1}{r} = \frac{16-r}{r} \Rightarrow \frac{16-r}{r} = \frac{11}{5}.$

Find r.

8. Use ${}^nC_r = \dfrac{{}^nP_r}{r!}$.

9. $\dfrac{{}^nC_{r-1}}{{}^nC_r} = \dfrac{r}{n-r+1}$ and $\dfrac{{}^nC_r}{{}^nC_{r+1}} = \dfrac{r+1}{n-r}$.

$\therefore \quad \dfrac{r}{n-r+1} = \dfrac{36}{84} = \dfrac{3}{7} \Rightarrow 3n-3r+3 = 7r \Rightarrow 3n-10r = -3$ ... (i)

and $\dfrac{r+1}{n-r} = \dfrac{84}{126} = \dfrac{2}{3} \Rightarrow 2n-2r = 3r+3 \Rightarrow 2n-5r = 3.$ ... (ii)

10. Use $\dfrac{{}^nC_r}{{}^{n-1}C_{r-1}} = \dfrac{n}{r}$ and $\dfrac{{}^{n+1}C_{r+1}}{{}^nC_r} = \dfrac{n+1}{r+1}$.

$\therefore \quad \dfrac{n}{r} = \dfrac{6}{3} = \dfrac{2}{1}$ and $\dfrac{n+1}{r+1} = \dfrac{11}{6} \Rightarrow 6n+6 = 11r+11 \Rightarrow 6n-11r = 5.$

Solve $n-2r = 0$ and $6n-11r = 5$.

11. Required number of teams $= {}^{15}C_{11} = {}^{15}C_{(15-11)} = {}^{15}C_4$.

12. Number of handshakes $= {}^{12}C_2$.

13. Number of chords $= {}^{21}C_2$.

14. Required number of ways $= {}^{25}C_4$.

PRACTICAL PROBLEMS ON COMBINATIONS

SOLVED EXAMPLES

EXAMPLE 1 *In how many ways can a cricket eleven be chosen out of a batch of* 15 *players, if*

(i) there is no restriction on the selection?

(ii) a particular player is always chosen?

(iii) a particular player is never chosen?

SOLUTION (i) Total number of ways of selecting 11 players out of 15

$$= {}^{15}C_{11} = {}^{15}C_{15-11} = {}^{15}C_4 = \frac{15 \times 14 \times 13 \times 12}{4 \times 3 \times 2 \times 1} = 1365.$$

(ii) When a particular player is always chosen, then we will have to choose 10 players out of remaining 14.

$\therefore$ required number of ways

$$= {}^{14}C_{10} = {}^{14}C_{14-10} = {}^{14}C_4 = \frac{14 \times 13 \times 12 \times 11}{4 \times 3 \times 2 \times 1} = 1001.$$

(iii) When a particular player is never chosen, then we will have to choose 11 players out of remaining 14.

$\therefore$ required number of ways

$$= {}^{14}C_{11} = {}^{14}C_{14-11} = {}^{14}C_3 = \frac{14 \times 13 \times 12}{3 \times 2 \times 1} = 364.$$

EXAMPLE 2 *In how many ways can a committee of* 5 *members be selected from* 6 *men and* 5 *ladies, consisting of* 3 *men and* 2 *ladies?*

SOLUTION We have to select (3 men out of 6) and (2 ladies out of 5).

$\therefore$ required number of ways

$$= ({}^6C_3 \times {}^5C_2) = \left(\frac{6 \times 5 \times 4}{3 \times 2 \times 1} \times \frac{5 \times 4}{2 \times 1}\right) = (20 \times 10) = 200.$$

EXAMPLE 3 *Out of 5 men and 2 women, a committee of 3 is to be formed. In how many ways can it be formed if at least one woman is to be included?*

SOLUTION The committee can be formed by choosing

(i) 1 woman and 2 men;

or (ii) 2 women and 1 man.

Now, the number of ways of choosing (1 woman out of 2) and (2 men out of 5)

$$= ({}^2C_1 \times {}^5C_2) = \left(2 \times \frac{5 \times 4}{2 \times 1}\right) = (2 \times 10) = 20.$$

And, the number of ways of choosing (2 women out of 2) and (1 man out of 5)

$$= ({}^2C_2 \times {}^5C_1) = (1 \times 5) = 5.$$

Hence, the total number of ways of forming the committee

$$= (20 + 5) = 25.$$

EXAMPLE 4 *A committee of 5 is to be formed out of 6 men and 4 ladies. In how many ways can this be done, when*

(a) at least 2 ladies are included;

(b) at most 2 ladies are included?

SOLUTION (a) We have to make a selection of

(i) (2 ladies out of 4) and (3 men out of 6)

or (ii) (3 ladies out of 4) and (2 men out of 6)

or (iii) (4 ladies out of 4) and (1 man out of 6).

The number of ways of these selections are:

Case I ${}^4C_2 \times {}^6C_3 = 6 \times 20 = 120.$

Case II ${}^4C_3 \times {}^6C_2 = 4 \times 15 = 60.$

Case III ${}^4C_4 \times {}^6C_1 = 1 \times 6 = 6.$

Hence, the required number of ways $= (120 + 60 + 6) = 186.$

(b) We have to make a selection of

(i) (1 lady out of 4) and (4 men out of 6)

or (ii) (2 ladies out of 4) and (3 men out of 6).

The number of ways of these selections are:

Case I ${}^4C_1 \times {}^6C_4 = 4 \times 15 = 60.$

Case II ${}^4C_2 \times {}^6C_3 = 6 \times 20 = 120.$

Hence, the required number of ways $= (60 + 120) = 180.$

EXAMPLE 5 *An examination paper containing 12 questions consists of two parts, A and B. Part A contains 7 questions and part B contains 5 questions.*

A candidate is required to attempt 8 questions, selecting at least 3 from each part. In how many ways can the candidate select the questions?

SOLUTION Clearly, the candidate may select the questions as under:

(i) (3 out of 7 from A) and (5 out of 5 from B)

or (ii) (4 out of 7 from A) and (4 out of 5 from B)

or (iii) (5 out of 7 from A) and (3 out of 5 from B).

The number of ways of these selections are:

Case I ${}^7C_3 \times {}^5C_5 = 35 \times 1 = 35.$

Case II ${}^7C_4 \times {}^5C_4 = 35 \times 5 = 175.$

Case III ${}^7C_5 \times {}^5C_3 = 21 \times 10 = 210.$

Hence, the required number of ways $= (35 + 175 + 210) = 420.$

EXAMPLE 6 *For the post of 5 teachers, there are 23 applicants. 2 posts are reserved for SC candidates and there are 7 SC candidates among the applicants. In how many ways can the selection be made?*

SOLUTION Clearly, there are 7 SC candidates and 16 non-SC candidates.

So, we have to make a selection of (2 out of 7) and (3 out of 16).

Hence, the number of ways of making the selection

$$= {}^7C_2 \times {}^{16}C_3 = \left(\frac{7 \times 6}{2} \times \frac{16 \times 15 \times 14}{3 \times 2 \times 1}\right) = 11760.$$

EXAMPLE 7 *How many diagonals are there in a polygon of n sides?*

SOLUTION A polygon of n sides has n vertices.

By joining any two of these vertices, we obtain either a side or a diagonal of the polygon.

Number of all straight lines obtained by joining 2 vertices at a time $= {}^nC_2 = \frac{1}{2}n(n-1)$.

These straight lines include n sides of the polygon.

Hence, the number of diagonals of the polygon

$$= \left[\frac{1}{2}n(n-1) - n\right] = \frac{1}{2}n(n-3).$$

EXAMPLE 8 *There are 10 points in a plane, no three of which are in the same straight line, except 4 points, which are collinear.*

Find (i) the number of lines obtained from the pairs of these points;

(ii) the number of triangles that can be formed with vertices as these points.

SOLUTION (i) Number of lines formed by joining the 10 points, taking 2 at a time

$$= {}^{10}C_2 = \left(\frac{10 \times 9}{2 \times 1}\right) = 45.$$

Number of lines formed by joining the 4 points, taking 2 at a time

$$= {}^4C_2 = \left(\frac{4 \times 3}{2 \times 1}\right) = 6.$$

But, 4 collinear points, when joined pairwise, give 1 line.

Hence, the required number of straight lines

$$= (45 - 6 + 1) = 40.$$

(ii) Number of triangles formed by 10 points, taking 3 at a time

$$= {}^{10}C_3 = \frac{10 \times 9 \times 8}{3 \times 2 \times 1} = 120.$$

Number of triangles formed by 4 points, taking 3 at a time

$$= {}^4C_3 = {}^4C_{4-3} = {}^4C_1 = 4.$$

But, these 4 points being collinear, so no triangle is formed.

Hence, the required number of triangles = (120 − 4) = 116.

EXAMPLE 9 *A box contains 6 red and 5 white balls. In how many ways can 6 balls be selected so that there are at least two balls of each colour?*

SOLUTION We may select

(i) (2 red balls out of 6) and (4 white balls out of 5)

or (ii) (3 red balls out of 6) and (3 white balls out of 5)

or (iii) (4 red balls out of 6) and (2 white balls out of 5).

The number of ways of these selections are:

Case I $({}^6C_2 \times {}^5C_4) = ({}^6C_2 \times {}^5C_1) = \left(\frac{6 \times 5}{2} \times 5\right) = 75.$

Case II $({}^6C_3 \times {}^5C_3) = ({}^6C_3 \times {}^5C_2) = \left(\frac{6 \times 5 \times 4}{3 \times 2 \times 1} \times \frac{5 \times 4}{2 \times 1}\right) = 200.$

Case III $({}^6C_4 \times {}^5C_2) = ({}^6C_2 \times {}^5C_2) = \left(\frac{6 \times 5}{2 \times 1} \times \frac{5 \times 4}{2 \times 1}\right) = 150.$

Hence, the required number of ways = (75 + 200 + 150) = 425.

EXAMPLE 10 *How many committees of 5 persons with a chairperson can be selected from 12 persons?*

SOLUTION Number of ways of selecting a chairperson out of 12 persons

$$= {}^{12}C_1 = 12.$$

Number of ways of selecting 4 persons out of remaining 11

$$= {}^{11}C_4 = \frac{11 \times 10 \times 9 \times 8}{4 \times 3 \times 2 \times 1} = 330.$$

Hence, the required number of ways = (12 × 330) = 3960.

EXAMPLE 11 *A box contains 3 black, 2 white and 4 red balls. In how many ways can 3 balls be drawn from the box, if at least one black ball is to be included in the draw?*

SOLUTION In all, we have 3 black and 6 non-black balls.

We may select:

(i) (1 black ball out of 3) and (2 non-black balls out of 6)

or (ii) (2 black balls out of 3) and (1 non-black ball out of 6)

or (iii) (3 black balls out of 3).

The number of ways of these selections are:

Case I $({}^3C_1 \times {}^6C_2) = \left(3 \times \dfrac{6 \times 5}{2 \times 1}\right) = 45.$

Case II $({}^3C_2 \times {}^6C_1) = ({}^3C_1 \times {}^6C_1) = (3 \times 6) = 18.$

Case III ${}^3C_3 = 3\,! = (3 \times 2 \times 1) = 6.$

Hence, the required number of ways $= (45 + 18 + 6) = 69$.

EXAMPLE 12 *A group consists of* 7 *boys and* 4 *girls. In how many ways can a team of* 5 *members be selected, if the team has*

(i) no girls?
(ii) at least 1 *boy and* 1 *girl?*
(iii) at least 3 *girls?*

SOLUTION (i) *When no girl is taken*:

Then, we have to select 5 boys out of 7.

$\therefore$ required number of ways

$$= {}^7C_5 = {}^7C_{7-5} = {}^7C_2 = \frac{7 \times 6}{2 \times 1} = 21.$$

(ii) *When at least* 1 *boy and* 1 *girl are taken*:

In this case, we may choose:

(1 boy and 4 girls) or (2 boys and 3 girls)

or (3 boys and 2 girls) or (4 boys and 1 girl).

$\therefore$ required number of ways

$$= ({}^7C_1 \times {}^4C_4) + ({}^7C_2 \times {}^4C_3) + ({}^7C_3 \times {}^4C_2) + ({}^7C_4 \times {}^4C_1)$$
$$= (7 \times 1) + ({}^7C_2 \times {}^4C_1) + ({}^7C_3 \times {}^4C_2) + ({}^7C_3 \times {}^4C_1)$$
$$= 7 + \left(\frac{7 \times 6}{2 \times 1} \times 4\right) + \left(\frac{7 \times 6 \times 5}{3 \times 2 \times 1} \times \frac{4 \times 3}{2 \times 1}\right) + \left(\frac{7 \times 6 \times 5}{3 \times 2 \times 1} \times 4\right)$$
$$= (7 + 84 + 210 + 140) = 441.$$

(iii) *When at least* 3 *girls are taken*:

Then, we may choose:

(3 girls and 2 boys) or (4 girls and 1 boy).

$\therefore$ required number of ways

$$= ({}^4C_3 \times {}^7C_2) + ({}^4C_4 \times {}^7C_1)$$
$$= ({}^4C_1 \times {}^7C_2) + (1 \times {}^7C_1) = \left(4 \times \frac{7 \times 6}{2 \times 1}\right) + (1 \times 7)$$
$$= (84 + 7) = 91.$$

EXAMPLE 13 *A bag contains* 6 *white marbles and* 5 *red marbles. Find the number of ways in which* 4 *marbles can be drawn from the bag, if*

(a) they can be of any colour;
(b) 2 *must be white and* 2 *red;*
(c) they must all be of the same colour.

SOLUTION (a) *Drawing* 4 *marbles of any colour*:

Total number of marbles = $(6+5) = 11$.

In this case, we have to draw 4 marbles out of 11.

$\therefore$ required number of ways

$$= {}^{11}C_4 = \frac{11 \times 10 \times 9 \times 8}{4 \times 3 \times 2 \times 1} = 330.$$

(b) *Drawing* 2 *white and* 2 *red marbles*:

We may draw:

(2 white marbles out of 6) and (2 red marbles out of 5).

$\therefore$ required number of ways

$$= ({}^6C_2 \times {}^5C_2) = \left(\frac{6 \times 5}{2 \times 1} \times \frac{5 \times 4}{2 \times 1}\right) = (15 \times 10) = 150.$$

(c) *Drawing* 4 *marbles of the same colour*:

We may draw:

(4 white marbles out of 6) or (4 red marbles out of 5) [*Note it*]

$\therefore$ required number of ways

$$= ({}^6C_4 + {}^5C_4) = ({}^6C_2 + {}^5C_1) = \left(\frac{6 \times 5}{2 \times 1} + 5\right) = 20.$$

EXAMPLE 14 *We wish to select* 6 *persons from* 8, *but if the person A is chosen, then B must be chosen. In how many ways can the selection be made?*

SOLUTION We have two cases: (either A is chosen) or (A is not chosen).

Case I *When A is chosen:*

In this case A is chosen and B is chosen.

So, we must choose 4 persons out of remaining 6.

$\therefore$ required number of ways $= {}^6C_4 = {}^6C_2 == \frac{6 \times 5}{2 \times 1} = 15.$

Case II *When A is not chosen:*

In this case, we have to choose 6 persons out of 7.

$\therefore$ required number of ways $= {}^7C_6 = {}^7C_1 = 7.$

Combining both the cases, we have:

total number of ways = $(15+7) = 22$.

EXAMPLE 15 *Find the number of ways of choosing* 4 *cards from a pack of* 52 *playing cards. In how many of these*

(i) 4 *cards are of the same suit?*

(ii) 4 *cards belong to four different suits?*

(iii) 4 *cards are face cards?*

(iv) 2 *are red cards and* 2 *are black cards?*

(v) cards are of the same colour?

SOLUTION Number of ways of choosing 4 cards out of 52

$$= {}^{52}C_4 = \frac{52 \times 51 \times 50 \times 49}{4 \times 3 \times 2 \times 1} = 270725.$$

(i) *Choosing* 4 *cards of the same suit:*

We have to draw:

(4 cards from 13 cards of diamond)
or (4 cards from 13 cards of spade)
or (4 cards from 13 cards of club)
or (4 cards from 13 cards of heart).

$\therefore$ required number of ways

$$= ({}^{13}C_4 + {}^{13}C_4 + {}^{13}C_4 + {}^{13}C_4) = 4 \times {}^{13}C_4$$

$$= \frac{4 \times 13 \times 12 \times 11 \times 10}{4 \times 3 \times 2 \times 1} = 2860.$$

(ii) *Choosing* 4 *cards from* 4 *different suits:*

We have to draw:

(1 card from 13 cards of diamond)
and (1 card from 13 cards of spade)
and (1 card from 13 cards of club)
and (1 card from 13 cards of heart).

$\therefore$ required number of ways

$$= ({}^{13}C_1 \times {}^{13}C_1 \times {}^{13}C_1 \times {}^{13}C_1) = (13 \times 13 \times 13 \times 13) = (13)^4.$$

(iii) *Choosing* 4 *cards out of* 12 *face cards:*

We have to choose 4 cards out of 12 face cards.

$\therefore$ required number of ways

$$= {}^{12}C_4 = \frac{12 \times 11 \times 10 \times 9}{4 \times 3 \times 2 \times 1} = 495.$$

(iv) *Choosing* 2 *red cards and* 2 *black cards:*

We have to choose 2 red cards out of 26 and 2 black cards out of 26.

$\therefore$ required number of ways

$$= ({}^{26}C_2 \times {}^{26}C_2) = \left(\frac{26 \times 25}{2}\right)^2 = (325 \times 325) = 105625.$$

(v) *Choosing all the* 4 *cards of the same colour:*

We have to choose 4 cards out of 26 black cards or 4 cards out of 26 red cards.

$\therefore$ required number of ways

$$= ({}^{26}C_4 + {}^{26}C_4) = (2 \times {}^{26}C_4) = \left(2 \times \frac{26 \times 25 \times 24 \times 23}{4 \times 3 \times 2 \times 1}\right)$$

$$= 29900.$$

EXAMPLE 16 *From a class of* 25 *students,* 10 *are to be chosen for an excursion party. There are* 3 *students who decide that either all of them will join or none of them will join. In how many ways can they be chosen?*

SOLUTION Either 3 particular students join or none of them joins.

Case I *When 3 particular students join the party:*

In this case, we have to choose 7 students out of 22.

This can be done in ${}^{22}C_7$ ways.

Case II *When 3 particular students do not join the party:*

In this case, we have to choose 10 students out of 22.

This can be done in ${}^{22}C_{10}$ ways.

Hence, the required number of ways $= ({}^{22}C_7 + {}^{22}C_{10}) = 817190$.

EXAMPLE 17 *A boy has 3 library tickets and 8 books of his interest in the library. Of these 8, he does not want to borrow Mathematics Part II, unless Mathematics Part I is also borrowed. In how many ways can he choose the 3 books to be borrowed?*

SOLUTION Clearly, the boy may borrow Mathematics Part II or he may not borrow it.

Case I *When the boy borrows Mathematics Part II:*

In this case, he borrows Mathematics Part I also.

Since he has 3 library tickets in all,

he may now borrow 1 book out of the remaining 6 books.

This can be done in ${}^{6}C_1 = 6$ ways.

Case II *When the boy does not borrow Mathematics Part II:*

Then, he may borrow 3 books out of the remaining 7 books.

This can be done in ${}^{7}C_3 = \frac{7 \times 6 \times 5}{3 \times 2 \times 1} = 35$ ways.

Hence, the required number of ways $= (6 + 35) = 41$.

EXAMPLE 18 *A convex polygon has 44 diagonals. Find the number of its sides.*

SOLUTION Let the given convex polygon have n sides.

Then, the number of its diagonals $= ({}^{n}C_2 - n)$.

Now, ${}^{n}C_2 - n = 44 \Rightarrow \frac{n(n-1)}{2} - n = 44$

$$\Rightarrow n^2 - 3n - 88 = 0 \Rightarrow (n-11)(n+8) = 0$$

$$\Rightarrow n = 11 \quad [\because n \neq -8].$$

Hence, the given polygon has 11 sides.

EXAMPLE 19 *If m parallel lines in a plane are intersected by a family of n parallel lines, find the number of parallelograms formed.*

SOLUTION A parallelogram is formed by choosing two straight lines from the set of m parallel lines and two straight lines from the set of n parallel lines.

Number of ways of choosing 2 lines out of m parallel lines $= {}^{m}C_2$.

Number of ways of choosing 2 lines out of n parallel lines $= {}^{n}C_2$.

Hence, the required number of parallelograms formed

$$= ({}^mC_2 \times {}^nC_2) = \frac{m(m-1)}{2} \times \frac{n(n-1)}{2} = \frac{mn(m-1)(n-1)}{4}.$$

EXAMPLE 20 *In an examination, a candidate has to pass in each of the 5 subjects. In how many ways can he fail?*

SOLUTION The candidate can fail by failing in 1 or 2 or 3 or 4 or 5 subjects out of 5 in each case.

$\therefore$ the total number of ways in which he can fail

$$= {}^5C_1 + {}^5C_2 + {}^5C_3 + {}^5C_4 + {}^5C_5$$
$$= {}^5C_1 + {}^5C_2 + {}^5C_2 + {}^5C_1 + {}^5C_5 \qquad [\because\ {}^nC_r = {}^nC_{n-r}]$$
$$= (5 + 10 + 10 + 5 + 1) = 31.$$

EXAMPLE 21 *Determine the number of 5-card combinations out of a deck of 52 cards if there is exactly one ace in each combination.*

SOLUTION Number of ways of selecting 1 ace out of 4 aces $= {}^4C_1 = 4$.

Number of ways of selecting 4 cards out of remaining 48

$$= {}^{48}C_4 = \frac{48 \times 47 \times 46 \times 45}{4 \times 3 \times 2 \times 1} = 194580.$$

Hence, the required number of combinations

$$= (194580 \times 4) = 778320.$$

EXAMPLE 22 *In how many ways can one select a cricket team of eleven from 17 players in which only 5 players can bowl, if each cricket team of 11 must include exactly 4 bowlers?*

SOLUTION Number of ways of selecting 4 bowlers out of 5

$$= {}^5C_4 = {}^5C_{(5-4)} = {}^5C_1 = 5.$$

Number of ways of selecting 7 batsmen out of 12

$$= {}^{12}C_7 = {}^{12}C_{(12-7)} = {}^{12}C_5 = \frac{12 \times 11 \times 10 \times 9 \times 8}{5 \times 4 \times 3 \times 2 \times 1} = 792.$$

Hence, the number of ways of selecting the team $= (5 \times 792) = 3960$.

EXAMPLE 23 *A committee of 12 persons is to be formed from 9 women and 8 men. In how many ways can this be done if at least 5 women have to be included in a committee? In how many of these committees are (a) the women in majority? (b) the men in majority?*

SOLUTION There are 9 women and 8 men in all. In order to form a committee consisting of at least 5 women, we may choose

(i) 5 women and 7 men or (ii) 6 women and 6 men

or (iii) 7 women and 5 men or (iv) 8 women and 4 men

or (v) 9 women and 3 men.

Hence, the total number of ways of forming the committee

$$= ({}^9C_5 \times {}^8C_7) + ({}^9C_6 \times {}^8C_6) + ({}^9C_7 \times {}^8C_5) + ({}^9C_8 \times {}^8C_4) + ({}^9C_9 \times {}^8C_3)$$

$$= ({}^9C_4 \times {}^8C_1) + ({}^9C_3 \times {}^8C_2) + ({}^9C_2 \times {}^8C_3) + ({}^9C_1 \times {}^8C_4) + (1 \times {}^8C_3)$$

$$= \left\{\frac{9 \times 8 \times 7 \times 6}{4 \times 3 \times 2 \times 1} \times 8\right\} + \left\{\frac{9 \times 8 \times 7}{3 \times 2 \times 1} \times \frac{8 \times 7}{2 \times 1}\right\} + \left\{\frac{9 \times 8}{2 \times 1} \times \frac{8 \times 7 \times 6}{3 \times 2 \times 1}\right\}$$
$$+ \left\{9 \times \frac{8 \times 7 \times 6 \times 5}{4 \times 3 \times 2 \times 1}\right\} + \left(1 \times \frac{8 \times 7 \times 6}{3 \times 2 \times 1}\right)$$

$$= (1008 + 2352 + 2016 + 630 + 56) = 6062.$$

(a) Clearly, the women are in majority in (iii), (iv) and (v) cases.

So, the number of committees in which the women are in majority

$$= ({}^9C_7 \times {}^8C_5) + ({}^9C_8 \times {}^8C_4) + ({}^9C_9 \times {}^8C_3)$$
$$= ({}^9C_2 \times {}^8C_3) + ({}^9C_1 \times {}^8C_4) + (1 + {}^8C_3)$$
$$= (2016 + 630 + 56) = 2702.$$

(b) Clearly, the men are in majority in (i) only.

So, the number of committees in which the men are in majority

$$= ({}^9C_5 \times {}^8C_7) = ({}^9C_4 \times {}^8C_1) = \left(\frac{9 \times 8 \times 7 \times 6}{4 \times 3 \times 2 \times 1} \times 8\right) = 1008.$$

EXAMPLE 24 *In a village, there are* 87 *families of which* 52 *families have at most* 2 *children. In a rural development programme,* 20 *families are to be chosen for assistance, of which at least* 18 *families must have at most* 2 *children. In how many ways can the choice be made?*

SOLUTION It is given that 52 families have at most 2 children and $(87 - 52) = 35$ families have more than 2 children.

We have to select 20 families of which at least 18 must have at most 2 children.

The selection can be made as under:

(i) 18 families out of 52 and 2 families out of 35

or (ii) 19 families out of 52 and 1 family out of 35

or (iii) 20 families out of 52.

So, the number of ways in which these choices can be made

$$= ({}^{52}C_{18} \times {}^{35}C_2) + ({}^{52}C_{19} \times {}^{35}C_1) + ({}^{52}C_{20}).$$

EXAMPLE 25 *In how many ways can* 19 *identical books on English and* 17 *identical books on Hindi be placed in a row on a shelf so that two books on Hindi may not be together?*

SOLUTION In order that two books on Hindi are never together, we must place these books as under:

$$XEXEXEX \ldots XEX,$$

where E denotes the position of an English book and X that of a Hindi book.

Since there are 19 books on English, the number of cross marks are therefore, 20.

Since all books on English are identical, they can be placed in only 1 way at the places marked E.

Number of ways of placing 17 identical books on Hindi at 20 places, each marked $X = {}^{20}C_{17} = {}^{20}C_3 = \frac{20 \times 19 \times 18}{3 \times 2 \times 1} = 1140.$

Hence, the required number of ways $= (1 \times 1140) = 1140.$

EXERCISE 9B

1. In how many ways can 5 sportsmen be selected from a group of 10?
2. A bag contains 5 black and 6 red balls. Find the number of ways in which 2 black and 3 red balls can be selected.
3. Find the number of ways of selecting 9 balls from 6 red balls, 5 white balls and 4 blue balls if each selection consists of 3 balls of each colour.
4. How many different boat parties of 8 consisting of 5 boys and 3 girls can be made from 20 boys and 10 girls?
5. In how many ways can a student choose 5 courses out of 9 courses if 2 specific courses are compulsory for every student?
6. A sports team of 11 students is to be constituted, choosing at least 5 from class XI and at least 5 from class XII. If there are 20 students in each of these classes, in how many ways can the team be constituted?
7. From 4 officers and 8 clerks, in how many ways can 6 be chosen (i) to include exactly one officer, (ii) to include at least one officer?
8. A cricket team of 11 players is to be selected from 16 players including 5 bowlers and 2 wicketkeepers. In how many ways can a team be selected so as to consist of exactly 3 bowlers and 1 wicketkeeper?
9. In how many ways can a cricket team be selected from a group of 25 players containing 10 batsmen, 8 bowlers, 5 all-rounders and 2 wicketkeepers, assuming that the team of 11 players requires 5 batsmen, 3 all-rounders, 2 bowlers and 1 wicketkeeper?
10. A question paper has two parts, part A and part B, each containing 10 questions. If the student has to choose 8 from part A and 5 from part B, in how many ways can he choose the questions?
11. In an examination, a student has to answer 4 questions out of 5. Questions 1 and 2 are compulsory. Find the number of ways in which the student can make the choice.
12. In an examination, a student has to answer 10 questions, choosing at least 4 from each of part A and part B. If there are 6 questions in part A and 7 in part B, in how many ways can these questions be chosen?
13. In an examination, a candidate is required to answer 7 questions out of 12, which are divided into two groups, each containing 6 questions. One cannot attempt more than 5 questions from either group. In how many ways can he choose these questions?

14. Out of 6 teachers and 8 students, a committee of 11 is to be formed. In how many ways can this be done, if the committee contains

(i) exactly 4 teachers? (ii) at least 4 teachers?

15. A committee of 7 has to be formed from 9 boys and 4 girls. In how many ways can this be done when the committee consists of

(i) exactly 3 girls? (ii) at least 3 girls? (iii) at most 3 girls?

16. A committee of three persons is to be constituted from a group of 2 men and 3 women. In how many ways can this be done? How many of these committees would consist of 1 man and 2 women?

17. A committee of 5 is to be formed out of 6 gents and 4 ladies. In how many ways can this be done, when

(i) at least 2 ladies are included? (ii) at most 2 ladies are included?

18. From a class of 14 boys and 10 girls, 10 students are to be chosen for a competition, at least including 4 boys and 4 girls. The 2 girls who won the prizes last year should be included. In how many ways can the selection be made?

19. Find the number of 5-card combinations out of a deck of 52 cards if at least one of the five cards has to be a king.

20. Find the number of diagonals of

(i) a hexagon, (ii) a decagon, (iii) a polygon of 18 sides.

21. How many triangles can be obtained by joining 12 points, four of which are collinear?

22. How many triangles can be formed in a decagon?

23. How many different selections of 4 books can be made from 10 different books, if

(i) there is no restriction?

(ii) two particular books are always selected?

(iii) two particular books are never selected?

24. How many different products can be obtained by multiplying two or more of the numbers 3, 5, 7, 11 without repetition?

25. Find the number of ways in which a committee of 2 teachers and 3 students can be formed out of 10 teachers and 20 students. In how many of these committees

(i) a particular teacher is included?

(ii) a particular student is included?

(iii) a particular student is excluded?

26. There are 18 points in a plane of which 5 are collinear. How many straight lines can be formed by joining them?

ANSWERS (EXERCISE 9B)

1. 252 **2.** 200 **3.** 800 **4.** 1860480 **5.** 35 **6.** $2({}^{20}C_5 \times {}^{20}C_6)$

7. (i) 224 (ii) 896 **8.** 720 **9.** 141120 **10.** 11340 **11.** 3
12. 266 **13.** 780 **14.** (i) 120 (ii) 344 **15.** (i) 504 (ii) 588 (iii) 1632
16. 10, 6 **17.** (i) 186 (ii) 186 **18.** 266266 **19.** 886656
20. (i) 9 (ii) 35 (iii) 135 **21.** 216 **22.** 120 **23.** (i) 210 (ii) 28 (iii) 70
24. 11 **25.** 51300; (i) 10260 (ii) 7695 (iii) 43605 **26.** 144

HINTS TO SOME SELECTED QUESTIONS

5. We have to choose 3 courses out of 7.

6. We have to choose:

(5 from XI and 6 from XII) or (6 from XI and 5 from XII)

Required number of ways $= \{({}^{20}C_5 \times {}^{20}C_6) + ({}^{20}C_6 \times {}^{20}C_5)\}$.

7. (i) Number of ways of choosing (1 officer out of 4) and (5 clerks out of 8) in $({}^4C_1 \times {}^8C_5) = (4 \times {}^8C_3)$ ways.

(ii) We may choose:

(1 officer, 5 clerks) or (2 officers, 4 clerks) or (3 officers, 3 clerks)
or (4 officers, 2 clerks).

Required number of ways

$$= ({}^4C_1 \times {}^8C_5) + ({}^4C_2 \times {}^8C_4) + ({}^4C_3 \times {}^8C_3) + ({}^4C_4 \times {}^8C_2).$$

8. We have to choose 3 bowlers out of 5, 1 wicketkeeper out of 2 and 7 batsmen out of 9.

Required number of ways $= ({}^5C_3 \times {}^2C_1 \times {}^9C_7)$.

9. Required number of ways $= ({}^{10}C_5 \times {}^5C_3 \times {}^8C_2 \times {}^2C_1)$.

10. Required number of ways $= ({}^{10}C_8 \times {}^{10}C_5) = ({}^{10}C_2 \times {}^{10}C_5)$.

11. Number of ways of choosing 2 questions out of 3 $= {}^3C_2 = 3$.

12. He may choose:

(4 questions from A and 6 from B) or (5 from A and 5 from B)
or (6 from A and 4 from B).

Required number of ways $= ({}^6C_4 \times {}^7C_6) + ({}^6C_5 \times {}^7C_5) + ({}^6C_6 \times {}^7C_4)$
$= ({}^6C_2 \times {}^7C_1) + ({}^6C_1 \times {}^7C_2) + (1 \times {}^7C_3)$.

13. He may choose:

(5 from 1st group, 2 from 2nd group) or (4 from 1st group, 3 from 2nd group)
or (3 from 1st group, 4 from 2nd group) or (2 from 1st group, 5 from 2nd group).

Required number of ways

$$= ({}^6C_5 \times {}^6C_2) + ({}^6C_4 \times {}^6C_3) + ({}^6C_3 \times {}^6C_4) + ({}^6C_2 \times {}^6C_5)$$
$$= ({}^6C_1 \times {}^6C_2) + ({}^6C_2 \times {}^6C_3) + ({}^6C_3 \times {}^6C_2) + ({}^6C_2 \times {}^6C_1).$$

14. (i) $({}^6C_4 \times {}^8C_7) = ({}^6C_2 \times {}^8C_1)$.

(ii) $({}^6C_4 \times {}^8C_7) + ({}^6C_5 \times {}^8C_6) + ({}^6C_6 \times {}^8C_5) = ({}^6C_2 \times {}^8C_1) + ({}^6C_1 \times {}^8C_2) + (1 \times {}^8C_3)$.

15. (i) $({}^4C_3 \times {}^9C_4)$. (ii) $({}^4C_3 \times {}^9C_4) + ({}^4C_4 \times {}^9C_3)$.

(iii) $({}^9C_7) + ({}^9C_6 \times {}^4C_1) + ({}^9C_5 \times {}^4C_2) + ({}^9C_4 \times {}^4C_3)$
$= {}^9C_2 + ({}^9C_3 \times 4) + ({}^9C_4 \times {}^4C_2) + ({}^9C_4 \times {}^4C_1)$.

16. Total number of committees $= {}^5C_3 = {}^5C_2 = 10.$

Number of committees consisting of 1 man and 2 women

$= ({}^2C_1 \times {}^3C_2) = (2 \times {}^3C_1) = (2 \times 3) = 6.$

17. (i) $({}^4C_2 \times {}^6C_3) + ({}^4C_3 \times {}^6C_2) + ({}^4C_4 \times {}^6C_1).$

(ii) $({}^6C_5) + ({}^6C_4 \times {}^4C_1) + ({}^6C_3 \times {}^4C_2).$

18. 2 girls who won the prize last year are surely to be taken. So, we have to make a selection of 8 students out of 14 boys and 8 girls, choosing at least 4 boys and at least 2 girls.

Thus, we may choose:

(4 boys, 4 girls) or (5 boys, 3 girls) or (6 boys, 2 girls).

Required number of ways

$= ({}^{14}C_4 \times {}^8C_4) + ({}^{14}C_5 \times {}^8C_3) + ({}^{14}C_6 \times {}^8C_2).$

19. Required number of combinations

= (total number of 5-card combinations)

– (number of 5-card combinations having no king)

$= ({}^{52}C_5 - {}^{48}C_5).$

20. (i) Number of diagonals of a hexagon $= ({}^6C_2 - 6) = (15 - 6) = 9.$

(ii) Number of diagonals of a decagon $= ({}^{10}C_2 - 10) = (45 - 10) = 35.$

(iii) Number of diagonals of a polygon of 18 sides $= ({}^{18}C_2 - 18) = 135.$

21. Number of triangles $= ({}^{12}C_3 - {}^4C_3) = (220 - 4) = 216.$

22. Number of triangles in a decagon $= {}^{10}C_3 = 120.$

23. (i) Number of ways of selecting 4 books out of 10 $= {}^{10}C_4.$

(ii) After choosing 2 particular books, we have to choose 2 books out of remaining 8.

$\therefore$ required number of ways $= {}^8C_2.$

(iii) In this case, we have to choose 4 books out of remaining 8.

$\therefore$ required number of ways $= {}^8C_4.$

24. Required number of products

= number of ways of selecting 2 or 3 or 4 numbers out of 3, 5, 7, 11

$= ({}^4C_2 + {}^4C_3 + {}^4C_4) = (6 + 4 + 1) = 11.$

25. Number of ways of forming a committee of 2 teachers and 3 students $= ({}^{10}C_2 \times {}^{20}C_3).$

(i) Number of committees in which a particular teacher is included $= ({}^9C_1 \times {}^{20}C_3).$

(ii) Number of committees in which a particular student is included $= ({}^{10}C_2 \times {}^{19}C_2).$

(iii) Number of committees in which a particular student is excluded $= ({}^{10}C_2 \times {}^{19}C_3).$

26. Required number of straight lines $= ({}^{18}C_2 - {}^5C_2 + 1).$

MIXED PROBLEMS ON PERMUTATIONS AND COMBINATIONS

SOLVED EXAMPLES

EXAMPLE 1 *Out of 7 consonants and 4 vowels, how many words of 3 consonants and 2 vowels can be formed?*

SOLUTION Total number of ways of choosing (3 consonants out of 7) and (2 vowels out of 4)

$$= ({}^7C_3 \times {}^4C_2) = \left(\frac{7 \times 6 \times 5}{3 \times 2 \times 1} \times \frac{4 \times 3}{2 \times 1}\right) = 210.$$

Number of groups, each containing 3 consonants and 2 vowels = 210.

Each group contains 5 letters.

Number of ways of arranging 5 letters amongst themselves

$$= 5\,! = (5 \times 4 \times 3 \times 2 \times 1) = 120.$$

Hence, the required number of words = $(210 \times 120) = 25200$.

EXAMPLE 2 *The English alphabet has 21 consonants and 5 vowels. How many words with two different consonants and two different vowels can be formed from the alphabet?*

SOLUTION Total number of ways of choosing

(2 consonants out of 21) and (2 vowels out of 5)

$$= {}^{21}C_2 \times {}^5C_2 = \left(\frac{21 \times 20}{2 \times 1} \times \frac{5 \times 4}{2 \times 1}\right) = 2100.$$

Number of groups, each containing 2 consonants and 2 vowels

= 2100.

Each group contains 4 letters.

Number of ways of arranging 4 letters amongst themselves

$$= 4\,! = (4 \times 3 \times 2 \times 1) = 24.$$

Hence, the required number of words = $(2100 \times 24) = 50400$.

EXAMPLE 3 *How many words with or without meaning, each of 2 vowels and 3 consonants can be formed from the letters of the word 'DAUGHTER'?*

SOLUTION The given word 'DAUGHTER' contains 3 vowels and 5 consonants.

Total number of ways of choosing

(2 vowels out of 3) and (3 consonants out of 5)

$$= ({}^3C_2 \times {}^5C_3) = ({}^3C_1 \times {}^5C_2) = \left(3 \times \frac{5 \times 4}{2 \times 1}\right) = 30.$$

Number of groups, each containing 2 vowels and 3 consonants = 30.

Each group contains 5 letters.

Number of ways of arranging 5 letters amongst themselves

$$= 5\,! = (5 \times 4 \times 3 \times 2 \times 1) = 120.$$

Hence, the required number of words = $(30 \times 120) = 3600$.

EXAMPLE 4 *How many words with or without meaning, can be formed using all the letters of the word 'EQUATION' at a time so that vowels and consonants occur together?*

SOLUTION The given word 'EQUATION' contains 5 vowels and 3 consonants.

All vowels can be put together in 5 ! ways.

All consonants can be put together in 3 ! ways.

Considering all vowels as 1 letter and all consonants as 1 letter, vowels and consonants can be arranged in 2 ! ways.

$\therefore$ required number of words

$$= (5\,!) \times (3\,!) \times (2\,!)$$
$$= (5 \times 4 \times 3 \times 2 \times 1) \times (3 \times 2 \times 1) \times (2 \times 1) = 1440.$$

EXAMPLE 5 *In how many ways can 5 girls and 3 boys be seated in a row so that no two boys are together?*

SOLUTION Since no two boys are to be together, let us arrange 5 girls as under:

$$XGXGXGXGXGX,$$

where G denotes the position of a girl and X that of a boy.

These girls may be arranged at 5 places in 5 ! ways.

Now, 3 boys can be seated at 6 places, each marked X.

Number of ways of seating 3 boys at 6 places $= {}^6P_3$.

$\therefore$ required number of ways

$$= (5\,!) \times {}^6P_3 = (5\,!) \times \frac{6\,!}{3\,!}$$
$$= \left\{(5 \times 4 \times 3 \times 2 \times 1) \times \left(\frac{6 \times 5 \times 4 \times 3 \times 2 \times 1}{3 \times 2 \times 1}\right)\right\} = 14400.$$

Hence, the required number of ways is 14400.

EXERCISE 9C

1. Out of 12 consonants and 5 vowels, how many words, each containing 3 consonants and 2 vowels, can be formed?
2. How many words, each of 3 vowels and 2 consonants, can be formed from the letters of the word 'INVOLUTE'?
3. The English alphabet has 21 consonants and 5 vowels. How many words with two different consonants and three different vowels can be formed from the alphabet?
4. In how many ways can 4 girls and 3 boys be seated in a row so that no two boys are together?
5. How many words, with or without meaning, can be formed from the letters of the word, 'MONDAY', assuming that no letter is repeated, if
   (i) 4 ketters are used at a time?
   (ii) all letters are used at a time?
   (iii) all letters are used but first letter is a vowel?

ANSWERS (EXERCISE 9C)

1. 264000 **2.** 2880 **3.** 252000 **4.** 480

5. (i) 360 (ii) 720 (iii) 240

HINTS TO SOME SELECTED QUESTIONS

5. (i) Number of 4-letter words from the letters of 'MONDAY' = $^6P_4 = 360$.
(ii) Number of words formed from all the letters of 'MONDAY' = $^6P_6 = 6! = 720$.
(iii) The given word contains two vowels, namely A and O.
So, first letter of the word can be chosen in 2 ways.
Remaining 5 places can be filled in 5 ! ways.
$\therefore$ required number of words $= 2 \times (5!) = 240$.

EXERCISE 9D

Very-Short-Answer Questions

1. If $^{20}C_r = {}^{20}C_{r-10}$ then find the value of $^{17}C_r$.
2. If $^{20}C_{r+1} = {}^{20}C_{r-1}$ then find the value of $^{10}C_r$.
3. If $^nC_{12} = {}^nC_8$ then find the value of $^{22}C_n$.
4. If $^{35}C_{n+7} = {}^{35}C_{4n-2}$ then find the value of n.
5. Find the values of (i) $^{200}C_{198}$, (ii) $^{76}C_0$, (iii) $^{15}C_{15}$.
6. If $^mC_1 = {}^nC_2$ prove that $m = \frac{1}{2}n(n-1)$.
7. Write the value of $({}^5C_1 + {}^5C_2 + {}^5C_3 + {}^5C_4 + {}^5C_5)$.
8. If $^{n+1}C_3 = 2({}^nC_2)$, find the value of n.
9. If $^nP_r = 720$ and $^nC_r = 120$ then find the value of r.
10. If $^{(n^2-n)}C_2 = {}^{(n^2-n)}C_4$ then find the value of n.
11. How many words are formed by 2 vowels and 3 consonants, taken from 4 vowels and 5 consonants?
12. Find the number of diagonals in an n-sided polygon.
13. Three persons enter a railway compartment having 5 vacant seats. In how many ways can they seat themselves?
14. There are 12 points in a plane, out of which 3 points are collinear. How many straight lines can be drawn by joining any two of them?
15. In how many ways can a committee of 5 be made out of 6 men and 4 women, containing at least 2 women?
16. There are 13 cricket players, out of which 4 are bowlers. In how many ways can a team of 11 be selected from them so as to include at least 3 bowlers?
17. How many different committees of 5 can be formed from 6 men and 4 women, if each committee consists of 3 men and 2 women?
18. How many parallelograms can be formed from a set of 4 parallel lines intersecting another set of 3 parallel lines?

ANSWERS (EXERCISE 9D)

1. 136	**2.** 1	**3.** 231	**4.** $n = 6$	**5.** (i) 19900 (ii) 1 (iii) 1	
7. 31	**8.** $n = 5$	**9.** $r = 3$	**10.** $n = 3$	**11.** 60	**12.** $\dfrac{n(n-3)}{2}$
13. 60	**14.** 64	**15.** 186	**16.** 72	**17.** 120	**18.** 12

HINTS TO SOME SELECTED QUESTIONS

1. $r + r - 10 = 20 \Rightarrow r = 15$. So, ${}^{17}C_r = {}^{17}C_{15} = {}^{17}C_2 = \dfrac{17 \times 16}{2 \times 1} = 136$.

2. $r + 1 + r - 1 = 20 \Rightarrow r = 10$. So, ${}^{10}C_r = {}^{10}C_{10} = 1$.

3. $n = 12 + 8 = 20 \Rightarrow {}^{22}C_n = {}^{22}C_{20} = {}^{22}C_2 = \dfrac{22 \times 21}{2} = 231$.

4. $n + 7 + 4n - 2 = 35 \Rightarrow 5n = 30 \Rightarrow n = 6$.

5. (i) ${}^{200}C_{198} = {}^{200}C_2 = \dfrac{200 \times 199}{2} = 19900$.

(ii) ${}^{76}C_0 = 1$.

(iii) ${}^{15}C_{15} = 1$.

6. ${}^{m}C_1 = {}^{n}C_2 \Rightarrow m = \dfrac{n(n-1)}{2}$.

7. ${}^5C_1 + {}^5C_2 + {}^5C_3 + {}^5C_4 + {}^5C_5 = ({}^5C_1 + {}^5C_2 + {}^5C_2 + {}^5C_1 + {}^5C_5) = (5 + 10 + 10 + 5 + 1) = 31$.

8. ${}^{n+1}C_3 = 2({}^nC_2) \Rightarrow \dfrac{(n+1)n(n-1)}{6} = \dfrac{n(n-1)}{2} \times 2 \Rightarrow n + 1 = 6 \Rightarrow n = 5$.

9. ${}^nP_r = ({}^nC_r) \times (r\,!) \Rightarrow r\,! = \dfrac{720}{120} = 6 = 3 \times 2 \times 1 = 3\,! \Rightarrow r = 3$.

10. $n^2 - n = (2 + 4) \Rightarrow n^2 - n - 6 = 0 \Rightarrow (n-3)(n+2) = 0 \Rightarrow n = 3$.

11. Number of words $= ({}^4C_2 \times {}^5C_3) = ({}^4C_2 \times {}^5C_2) = (6 \times 10) = 60$.

12. Required number of diagonals $= ({}^nC_2 - n) = \dfrac{n(n-1)}{2} - n = \dfrac{n^2 - n - 2n}{2}$

$$= \frac{n^2 - 3n}{2} = \frac{n(n-3)}{2}.$$

13. First can occupy any of the 5 seats, 2nd may occupy any of 4 seats and third may occupy any of 3 seats.

Required number of ways $= (5 \times 4 \times 3) = 60$.

14. Required number of straight lines $= ({}^{12}C_2 - {}^3C_2 + 1) = 64$.

15. (2 women and 3 men) or (3 women and 2 men) or (4 women and 1 man)

$= ({}^4C_2 \times {}^6C_3) + ({}^4C_3 \times {}^6C_2) + ({}^4C_4 \times {}^6C_1) = (120 + 60 + 6) = 186$.

16. There are 4 bowlers and 9 batsmen.

We may choose (3 bowlers and 8 batsmen) or (4 bowlers and 7 batsmen).

Required number of ways $= ({}^4C_3 \times {}^9C_8) + ({}^4C_4 \times {}^9C_7) = 36 + 36 = 72$.

17. Number of committees $= ({}^6C_3 \times {}^4C_2) = (20 \times 6) = 120$.

18. Number of parallelograms $= (4 \times 3) = 12$.

KEY FACTS AND FORMULAE

1. COMBINATION Each of the different groups or selections that can be made out of a given number of things by taking some or all of them at a time, irrespective of their arrangements, is called a combination.

2. We use the word *arrangements* for permutations and *selections* for combinations.

3. The number of combinations of n things taken r at a time is denoted by ${}^nC_r$, or $C(n, r)$, where $n > 0$, $r \geq 0$ and $n \geq r$.

4. (i) ${}^nC_r = \dfrac{n!}{(r!) \times (n-r)!} = \dfrac{n(n-1)(n-2)\ldots \text{ to } r \text{ factors}}{r!}$.

(ii) ${}^nC_n = {}^nC_0 = 1$.

(iii) ${}^nC_r = \dfrac{{}^nP_r}{r!}$.

5. (i) ${}^nC_r = {}^nC_{n-r}$.

(ii) ${}^nC_x = {}^nC_y$ and $x \neq y \Rightarrow x + y = n$.

(iii) ${}^nC_r + {}^nC_{r-1} = {}^{n+1}C_r$ [Pascal's rule].

(iv) $\dfrac{{}^nC_r}{{}^{n-1}C_{r-1}} = \dfrac{n}{r}$ and $\dfrac{{}^nC_{r-1}}{{}^{n-1}C_{r-1}} = \dfrac{n}{(n-r+1)}$.

(v) $\dfrac{{}^nC_r}{{}^nC_{r+1}} = \dfrac{r+1}{n-r}$.

❑

10
Binomial Theorem

HISTORY The expression of the form $(a+b)^n$ is called a binomial. By direct multiplication it is easy to expand $(a+b)^2, (a+b)^3$, etc. The ancient mathematicians knew about the expansion of $(a+b)^n$ for $0 \le n \le 7$. Around 1660, B Pascal introduced Pascal's triangle for the coefficients in the expansion of $(a+b)^n$ and in the same year he gave the present form of binomial theorem.

BINOMIAL THEOREM (For Non-negative Integers)

$$(a+b)^n = {}^nC_0a^n + {}^nC_1a^{n-1}b + {}^nC_2a^{n-2}b^2 + \ldots + {}^nC_{n-1}ab^{n-1} + {}^nC_nb^n.$$

By direct multiplication, we have

(i) $(a+b)^0 = 1.$ $\quad [\because\ x^0 = 1$ for every real $x]$

(ii) $(a+b)^1 = a+b = {}^1C_0a + {}^1C_1b.$

(iii) $(a+b)^2 = a^2 + 2ab + b^2 = {}^2C_0a^2 + {}^2C_1ab + {}^2C_2b^2.$

(iv) $(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3 = {}^3C_0a^3 + {}^3C_1a^2b + {}^3C_2ab^2 + {}^3C_3b^3.$

(v) $(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$

$$= 4C_0a^4 + {}^4C_1a^3b + {}^4C_2a^2b^2 + {}^4C_3ab^3 + {}^4C_4b^4.$$

Given below is the ***Pascal's Triangle*** showing the coefficients of various terms in a binomial expansion.

Index	**Coefficients**
0	1
1	1 1
2	1 2 1
3	1 3 3 1
4	1 4 6 4 1
5	1 5 10 10 5 1
6	1 6 15 20 15 6 1
7	1 7 21 35 35 21 7 1
8	1 8 28 56 70 56 28 8 1

PASCAL'S TRIANGLE

Look at the pattern given below:

For $n = 4$, the various coefficients are ${}^4C_0, {}^4C_1, {}^4C_2, {}^4C_3, {}^4C_4$.

For $n = 5$, the various coefficients are ${}^5C_0, {}^5C_1, {}^5C_2, {}^5C_3, {}^5C_4, {}^5C_5$.

For $n = 6$, the various coefficients are ${}^6C_0, {}^6C_1, {}^6C_2, {}^6C_3, {}^6C_4, {}^6C_5, {}^6C_6$.

For $n = 7$, the various coefficients are ${}^7C_0, {}^7C_1, {}^7C_2, {}^7C_3, {}^7C_4, {}^7C_5, {}^7C_6, {}^7C_7$.

BINOMIAL THEOREM FOR POSITIVE INTEGRAL INDEX

THEOREM *If a and b are real numbers then for all* $n \in N$

$$(a+b)^n = {}^nC_0a^n + {}^nC_1a^{n-1}b + {}^nC_2a^{n-2}b^2 + \dots + {}^nC_{n-1}ab^{n-1} + {}^nC_nb^n.$$

PROOF We shall prove the theorem by using the principle of mathematical induction.

Let $P(n)$ be the statement:

$$(a+b)^n = {}^nC_0a^n + {}^nC_1a^{n-1}b + {}^nC_2a^{n-2}b^2 + \dots + {}^nC_{n-1}ab^{n-1} + {}^nC_nb^n.$$

Then, the statement $P(1)$ is

$$(a+b)^1 = {}^1C_0a^1 + {}^1C_1a^0b = (a+b), \text{ which is true.}$$

Thus, $P(1)$ is true.

Let $P(m)$ be true. Then,

$$(a+b)^m = {}^mC_0a^m + {}^mC_1a^{m-1}b + {}^mC_2a^{m-2}b^2 + \dots + {}^mC_{m-1}ab^{m-1} + {}^mC_mb^m. \quad \dots \text{(i)}$$

Multiplying both sides of (i) by $(a+b)$, we get

$$\begin{aligned}(a+b)^{m+1} &= {}^mC_0a^{m+1} + {}^mC_0a^mb + {}^mC_1a^mb + {}^mC_1a^{m-1}b^2 + {}^mC_2a^{m-1}b^2 \\ &\quad + {}^mC_2a^{m-2}b^3 + \dots + {}^mC_{m-1}a^2b^{m-1} + {}^mC_{m-1}ab^m + {}^mC_mab^m \\ &\quad + {}^mC_mb^{m+1} \\ &= {}^mC_0a^{m+1} + ({}^mC_0 + {}^mC_1)a^mb + ({}^mC_1 + {}^mC_2)a^{m-1}b^2 + \dots \\ &\quad + ({}^mC_{m-1} + {}^mC_m)ab^m + {}^mC_mb^{m+1} \\ &= {}^{m+1}C_0a^{m+1} + {}^{m+1}C_1a^mb + {}^{m+1}C_2a^{m-1}b^2 + \dots + {}^{m+1}C_mab^m \\ &\quad + {}^{m+1}C_{m+1}b^{m+1}\end{aligned}$$

$$[\because\ {}^mC_0 = 1 = {}^{m+1}C_0,\ {}^mC_m = 1 = {}^{m+1}C_{m+1} \text{ and } {}^mC_{r-1} + {}^mC_r = {}^{m+1}C_r]$$

This shows that $P(m+1)$ is true, whenever $P(m)$ is true.

Hence, by the principle of mathematical induction the theorem is true for all $n \in N$.

REMARK ${}^nC_0, {}^nC_1, {}^nC_2, \dots, {}^nC_n$ are known as binomial coefficients.

We may write the above result as:

$$\boldsymbol{(a+b)^n = \sum_{r=0}^{n} {}^nC_ra^{n-r}b^r.}$$

SOME OBSERVATIONS IN A BINOMIAL EXPANSION

(i) The expansion of $(a+b)^n$ has $(n+1)$ terms.

(ii) Since ${}^nC_r = {}^nC_{n-r}$, we have ${}^nC_0 = {}^nC_n$, ${}^nC_1 = {}^nC_{n-1}$, and so on.

Thus, the coefficients of the terms equidistant from the beginning and the end in a binomial expansion are equal.

(iii) General term in the expansion of $(a+b)^n$ is the $(r+1)$th term given by $T_{r+1} = {}^nC_r a^{n-r} b^r$.

(iv) In the expansion of $(a+b)^n$, there are $(n+1)$ terms.

I. When n is even, the middle term $= \left(\frac{n}{2}+1\right)$th term.

II. When n is odd, two middle terms are

$$\frac{1}{2}(n+1)\text{th term and } \left\{\frac{1}{2}(n+1)+1\right\}\text{th term.}$$

(v) pth term from the end in $(a+b)^n$

$= (n+1-p+1)$th term from the beginning

$= (n-p+2)$th term from the beginning.

(vi) **Results on binomial coefficients**

We have the binomial expansion:

$$(a+b)^n = {}^nC_0 a^n + {}^nC_1 a^{n-1} b + {}^nC_2 a^{n-2} b^2 + \ldots + {}^nC_{n-1} ab^{n-1} + {}^nC_n b^n \quad \ldots \text{(A)}$$

Putting $a = b = 1$ in (A), we get

$$ {}^nC_0 + {}^nC_1 + {}^nC_2 + {}^nC_3 + \ldots + {}^nC_n = 2^n. \quad \ldots \text{(B)}$$

Putting $a = 1$ and $b = -1$ in (A) and using (B), we get

$$ {}^nC_0 + {}^nC_2 + {}^nC_4 + \ldots = {}^nC_1 + {}^nC_3 + {}^nC_5 + \ldots = \frac{2^n}{2} = 2^{n-1}.$$

Hence, ${}^nC_0 + {}^nC_2 + {}^nC_4 + \ldots = {}^nC_1 + {}^nC_3 + {}^nC_5 + \ldots = 2^{n-1}$.

DEDUCTIONS

(i) Putting $a = 1$ and $b = x$ in the binomial expansion of $(a+b)^n$, we get

$$(1+x)^n = {}^nC_0 + {}^nC_1 x + {}^nC_2 x^2 + \ldots + {}^nC_n x^n.$$

Thus, $(1+x)^n = \sum\limits_{r=0}^{n} {}^nC_r x^r$.

And, $T_{r+1} = {}^nC_r x^r$.

(ii) Replacing b by $-b$ in the binomial expansion of $(a+b)^n$, we get

$$(a-b)^n = {}^nC_0 a^n - {}^nC_1 a^{n-1} b + {}^nC_2 a^{n-2} b^2 - {}^nC_3 a^{n-3} b^3 + \ldots$$

$\Rightarrow \quad (a-b)^n = \sum\limits_{r=0}^{n} (-1)^r \cdot {}^nC_r a^{n-r} b^r$.

$\therefore \quad T_{r+1}$ in the expansion of $(a-b)^n$ is given by

$$T_{r+1} = (-1)^r \cdot {}^nC_r a^{n-r} b^r.$$

(iii) Putting $a = 1$ and $b = x$ in the above expansion, we get

$$(1-x)^n = {}^nC_0 - {}^nC_1 x + {}^nC_2 x^2 - {}^nC_3 x^3 + \ldots + (-1)^n \cdot {}^nC_n x^n$$

$\Rightarrow \quad (1-x)^n = \sum\limits_{r=0}^{n} (-1)^r \cdot {}^nC_r x^r$.

$\therefore \quad T_{r+1}$ in the expansion of $(1-x)^n$ is given by

$$T_{r+1} = (-1)^r \cdot {}^nC_r x^r.$$

SUMMARY

1. We have

(i) $(a+b)^n = \sum_{r=0}^{n} {}^nC_r a^{n-r} b^r$ and $T_{r+1} = {}^nC_r a^{n-r} b^r$.

(ii) $(a-b)^n = \sum_{r=0}^{n} (-1)^r \cdot {}^nC_r a^{n-r} b^r$ and $T_{r+1} = (-1)^r \cdot {}^nC_r a^{n-r} b^r$.

(iii) When n is even, then

the middle term $= \left(\frac{n}{2}+1\right)$th term.

(iv) When n is odd, then

the middle terms are $\frac{1}{2}(n+1)$th and $\left\{\frac{1}{2}(n+1)+1\right\}$th.

(v) pth term from the end $= (n-p+2)$th term from the beginning.

2. (i) $(1+x)^n = \sum_{r=0}^{n} {}^nC_r x^r$ and $T_{r+1} = {}^nC_r x^r$.

(ii) $(1-x)^n = \sum_{r=0}^{n} (-1)^r \cdot {}^nC_r x^r$ and $T_{r+1} = (-1)^r \cdot {}^nC_r x^r$.

3. (i) ${}^nC_0 + {}^nC_1 + {}^nC_2 + \ldots + {}^nC_n = 2^n$.

(ii) ${}^nC_0 + {}^nC_2 + {}^nC_4 + \ldots = {}^nC_1 + {}^nC_3 + {}^nC_5 + \ldots = 2^{n-1}$.

SOLVED EXAMPLES

EXAMPLE 1 *Expand* $(x^2+2y)^5$ *using binomial expansion.*

SOLUTION Using the binomial expansion, we have

$$\begin{aligned}(x^2+2y)^5 &= {}^5C_0 (x^2)^5 + {}^5C_1 (x^2)^4 (2y) + {}^5C_2 (x^2)^3 (2y)^2 \\ &\quad + {}^5C_3 (x^2)^2 (2y)^3 + {}^5C_4 x^2 (2y)^4 + {}^5C_5 (2y)^5 \\ &= x^{10} + 10x^8 y + 40x^6 y^2 + 80x^4 y^3 + 80x^2 y^4 + 32y^5.\end{aligned}$$

EXAMPLE 2 *Expand* $\left(x^3 - \frac{2}{x^2}\right)^6$ *using binomial expansion.*

SOLUTION Using the binomial expansion, we get

$$\begin{aligned}\left(x^3 - \frac{2}{x^2}\right)^6 &= {}^6C_0 (x^3)^6 - {}^6C_1 (x^3)^5 \left(\frac{2}{x^2}\right) + {}^6C_2 (x^3)^4 \left(\frac{2}{x^2}\right)^2 - {}^6C_3 (x^3)^3 \left(\frac{2}{x^2}\right)^3 \\ &\quad + {}^6C_4 (x^3)^2 \left(\frac{2}{x^2}\right)^4 - {}^6C_5 x^3 \left(\frac{2}{x^2}\right)^5 + {}^6C_6 \left(\frac{2}{x^2}\right)^6 \\ &= x^{18} - 12x^{13} + 60x^8 - 160x^3 + \frac{240}{x^2} - \frac{192}{x^7} + \frac{64}{x^{12}}.\end{aligned}$$

EXAMPLE 3 *Expand* $(1+x+x^2)^3$ *using binomial expansion.*

SOLUTION Put $1+x=y$. Then,

$(1+x+x^2)^3 = (y+x^2)^3$

$$= {}^3C_0 y^3 + {}^3C_1 y^2 x^2 + {}^3C_2 y (x^2)^2 + {}^3C_3 (x^2)^3$$
$$= y^3 + 3y^2x^2 + 3yx^4 + x^6$$
$$= (1+x)^3 + 3x^2(1+x)^2 + 3x^4(1+x) + x^6 \qquad [\because\ y = (1+x)]$$
$$= (1 + {}^3C_1 x + {}^3C_2 x^2 + {}^3C_3 x^3) + 3x^2(1+2x+x^2) + 3x^4(1+x) + x^6$$
$$= (1+3x+3x^2+x^3) + (3x^2+6x^3+3x^4) + (3x^4+3x^5) + x^6$$
$$= x^6 + 3x^5 + 6x^4 + 7x^3 + 6x^2 + 3x + 1.$$

Hence, $(1+x+x^2)^3 = x^6 + 3x^5 + 6x^4 + 7x^3 + 6x^2 + 3x + 1$.

EXAMPLE 4 *Expand* $(1-x+x^2)^4$ *using binomial expansion.*

SOLUTION Put $1 - x = y$. Then,

$$(1-x+x^2)^4 = (y+x^2)^4$$
$$= {}^4C_0 y^4 + {}^4C_1 y^3 x^2 + {}^4C_2 y^2 (x^2)^2 + {}^4C_3 y (x^2)^3 + {}^4C_4 (x^2)^4$$
$$= y^4 + 4y^3x^2 + 6y^2x^4 + 4yx^6 + x^8$$
$$= (1-x)^4 + 4(1-x)^3x^2 + 6(1-x)^2x^4 + 4(1-x)x^6 + x^8 \qquad [\because\ y = 1-x]$$
$$= (1 - {}^4C_1 x + {}^4C_2 x^2 - {}^4C_3 x^3 + {}^4C_4 x^4) + 4x^2(1 - {}^3C_1 x + {}^3C_2 x^2 - {}^3C_3 x^3) + 6x^4(1-2x+x^2) + 4x^6(1-x) + x^8$$
$$= (1-4x+6x^2-4x^3+x^4) + 4x^2(1-3x+3x^2-x^3) + 6x^4(1-2x+x^2) + 4x^6(1-x) + x^8$$
$$= (1-4x+6x^2-4x^3+x^4) + (4x^2-12x^3+12x^4-4x^5) + (6x^4-12x^5+6x^6) + (4x^6-4x^7) + x^8$$
$$= x^8 - 4x^7 + 10x^6 - 16x^5 + 19x^4 - 16x^3 + 10x^2 - 4x + 1.$$

Hence,

$$(1-x+x^2)^4 = x^8 - 4x^7 + 10x^6 - 16x^5 + 19x^4 - 16x^3 + 10x^2 - 4x + 1.$$

EXAMPLE 5 *Expand* $\{(a+b)^4 + (a-b)^4\}$ *and use it to evaluate*

$$(x^2 + \sqrt{1-x^2})^4 + (x^2 - \sqrt{1-x^2})^4.$$

SOLUTION We have

$$(a+b)^4 + (a-b)^4 = [{}^4C_0 a^4 + {}^4C_1 a^3 b + {}^4C_2 a^2 b^2 + {}^4C_3 ab^3 + {}^4C_4 b^4] + [{}^4C_0 a^4 - {}^4C_1 a^3 b + {}^4C_2 a^2 b^2 - {}^4C_3 ab^3 + {}^4C_4 b^4]$$
$$= 2[{}^4C_0 a^4 + {}^4C_2 a^2 b^2 + {}^4C_4 b^4]. \qquad \ldots \text{(i)}$$

Putting $\sqrt{1-x^2} = y$, we get

$$(x^2 + \sqrt{1-x^2})^4 + (x^2 - \sqrt{1-x^2})^4$$
$$= (x^2+y)^4 + (x^2-y)^4$$
$$= 2[{}^4C_0 (x^2)^4 + {}^4C_2 (x^2)^2 y^2 + {}^4C_4 y^4] \qquad [\text{taking } a = x^2 \text{ and } b = y \text{ in (i)}]$$
$$= 2[x^8 + 6x^4y^2 + y^4]$$
$$= 2[x^8 + 6x^4(1-x^2) + (1-x^2)^2] \qquad [\because\ y^2 = (1-x^2)]$$

$$= 2[x^8 + (6x^4 - 6x^6) + (1 - 2x^2 + x^4)]$$
$$= 2[x^8 - 6x^6 + 7x^4 - 2x^2 + 1]$$
$$= 2x^8 - 12x^6 + 14x^4 - 4x^2 + 2.$$

EXAMPLE 6 *Find the* 10*th term in the expansion of* $\left(2x^2 + \frac{1}{x}\right)^{12}$.

SOLUTION The general term in the given expansion is given by

$$T_{r+1} = {}^{12}C_r \times (2x^2)^{(12-r)} \times \left(\frac{1}{x}\right)^r. \qquad \ldots \text{(i)}$$

$\therefore \quad T_{10} = T_{(9+1)}$

$$= {}^{12}C_9 \times (2x^2)^{(12-9)} \times \left(\frac{1}{x}\right)^9 \qquad [\text{putting } r = 9 \text{ in (i)}]$$

$$= {}^{12}C_3 \times (2x^2)^3 \times \left(\frac{1}{x^9}\right) \qquad [\because\ {}^{12}C_9 = {}^{12}C_{(12-9)} = {}^{12}C_3]$$

$$= \left(\frac{12 \times 11 \times 10}{3 \times 2 \times 1} \times 8x^6 \times \frac{1}{x^9}\right) = \frac{1760}{x^3}.$$

Hence, the 10th term in the given expansion is $\frac{1760}{x^3}$.

EXAMPLE 7 *Find the* 6*th term in the expansion of* $\left(\frac{4x}{5} - \frac{5}{2x}\right)^9$.

SOLUTION The general term in the given expansion is given by

$$T_{r+1} = (-1)^r \times {}^9C_r \times \left(\frac{4x}{5}\right)^{(9-r)} \times \left(\frac{5}{2x}\right)^r. \qquad \ldots \text{(i)}$$

$\therefore \quad T_6 = T_{(5+1)}$

$$= (-1)^5 \times {}^9C_5 \times \left(\frac{4x}{5}\right)^{(9-5)} \times \left(\frac{5}{2x}\right)^5 \qquad [\text{putting } r = 5 \text{ in (i)}]$$

$$= -{}^9C_4 \times \left(\frac{4x}{5}\right)^4 \times \left(\frac{5}{2x}\right)^5 \qquad [\because\ {}^9C_5 = {}^9C_{(9-5)} = {}^9C_4]$$

$$= -\left[\frac{9 \times 8 \times 7 \times 6}{4 \times 3 \times 2 \times 1} \times \frac{2^8 x^4}{5^4} \times \frac{5^5}{2^5 x^5}\right] = \frac{-5040}{x}.$$

Hence, the 6th term in the given expansion is $\frac{-5040}{x}$.

EXAMPLE 8 *Find the* 4*th term from the end in the expansion of* $\left(\frac{x^3}{2} - \frac{2}{x^2}\right)^9$.

SOLUTION The general term in the given expansion is given by

$$T_{r+1} = (-1)^r \times {}^9C_r \times \left(\frac{x^3}{2}\right)^{(9-r)} \times \left(\frac{2}{x^2}\right)^r. \qquad \ldots \text{(i)}$$

Now, pth term from the end

$= (n - p + 2)$th term from the beginning.

$\therefore$ 4th term from the end

$= (9 - 4 + 2)$th term from the beginning $\quad [\because\ n = 9, p = 4]$

$= 7$th term from the beginning

$$= T_7 = T_{(6+1)}$$

$$= (-1)^6 \times {}^9C_6 \times \left(\frac{x^3}{2}\right)^{(9-6)} \times \left(\frac{2}{x^2}\right)^6 \qquad [\text{putting } r = 6 \text{ in (i)}]$$

$$= {}^9C_3 \times \left(\frac{x^3}{2}\right)^3 \times \left(\frac{2}{x^2}\right)^6 \qquad [\because\ {}^9C_6 = {}^9C_{(9-6)} = {}^9C_3]$$

$$= \left(\frac{9 \times 8 \times 7}{3 \times 2 \times 1} \times \frac{x^9}{8} \times \frac{64}{x^{12}}\right) = \frac{672}{x^3}.$$

Hence, the 4th term from the end is $\frac{672}{x^3}$.

EXAMPLE 9 *If the ratio of the* 5*th term from the beginning to the* 5*th term from the end in the expansion of* $\left(\sqrt[4]{2} + \frac{1}{\sqrt[4]{3}}\right)^n$ *is* $\sqrt{6} : 1$ *then find the value of n.*

SOLUTION The given expression may be written as $(2^{\frac{1}{4}} + 3^{-\frac{1}{4}})^n$.

The general term in this expansion is given by

$$T_{r+1} = {}^nC_r \times (2^{\frac{1}{4}})^{(n-r)} \times (3^{-\frac{1}{4}})^r$$

$$\Rightarrow \quad T_{r+1} = {}^nC_r \times 2^{\frac{(n-r)}{4}} \times 3^{\left(\frac{-r}{4}\right)}. \qquad \dots \text{(i)}$$

Now, pth term from the end

$= (n-p+2)$th term from the beginning.

$\therefore$ 5th term from the end

$= (n-5+2)$th term from the beginning

$= (n-3)$th term from the beginning.

Now, $T_5 = T_{(4+1)}$

$$= {}^nC_4 \times 2^{\frac{(n-4)}{4}} \times 3^{\left(\frac{-4}{4}\right)} \qquad [\text{putting } r = 4 \text{ in (i)}]$$

$$= {}^nC_4 \times 2^{\frac{(n-4)}{4}} \times 3^{-1}. \qquad \dots \text{(ii)}$$

And, $T_{n-3} = T_{(n-4)+1}$

$$= {}^nC_{n-4} \times 2^1 \times 3^{\frac{-(n-4)}{4}} \qquad [\text{putting } r = (n-4) \text{ in (i)}]$$

$$= {}^nC_4 \times 2 \times 3^{\frac{-(n-4)}{4}}. \qquad \dots \text{(iii)} \quad [\because\ {}^nC_{n-r} = {}^nC_r]$$

$$\therefore \quad \frac{T_5}{T_{n-3}} = \frac{\sqrt{6}}{1}$$

$$\Rightarrow \quad \frac{{}^nC_4 \times 2^{\frac{(n-4)}{4}} \times 3^{-1}}{{}^nC_4 \times 2 \times 3^{\frac{-(n-4)}{4}}} = \frac{\sqrt{6}}{1}$$

$\Rightarrow \quad 2^{\left\{\frac{(n-4)}{4}-1\right\}} \times 3^{\left\{\frac{(n-4)}{4}-1\right\}} = \sqrt{6}$

$\Rightarrow \quad 2^{\frac{(n-8)}{4}} \times 3^{\frac{(n-8)}{4}} = 6^{1/2}$

$\Rightarrow \quad (2 \times 3)^{\frac{(n-8)}{4}} = 6^{1/2}$

$\Rightarrow \quad \frac{n-8}{4} = \frac{1}{2} \Rightarrow n-8=2 \Rightarrow n=10.$

Hence, $n = 10$.

EXAMPLE 10 *Find the term independent of x in the expansion of* $\left(\frac{3x^2}{2} - \frac{1}{3x}\right)^{15}$.

SOLUTION Let T_{r+1} be independent of x.

Here, $T_{r+1} = (-1)^r \times {}^{15}C_r \times \left(\frac{3x^2}{2}\right)^{(15-r)} \times \left(\frac{1}{3x}\right)^r$

$\Rightarrow \quad T_{r+1} = (-1)^r \times {}^{15}C_r \times 3^{(15-2r)} \times 2^{(r-15)} \times x^{(30-3r)}. \quad \ldots \text{(i)}$

Now, T_{r+1} will be independent of x only when the power of x in it is 0.

$\therefore \quad 30 - 3r = 0 \Rightarrow 3r = 30 \Rightarrow r = 10 \Rightarrow r+1 = 11.$

Thus, T_{11} is independent of x.

Now, $T_{11} = T_{(10+1)}$

$= (-1)^{10} \times {}^{15}C_{10} \times 3^{(15-20)} \times 2^{(10-15)} \times x^0$ [putting $r = 10$ in (i)]

$= {}^{15}C_5 \times 3^{-5} \times 2^{-5} = \frac{{}^{15}C_5}{6^5} \qquad \left[\because {}^{15}C_{10} = {}^{15}C_{(15-10)} = {}^{15}C_5\right]$

$= \left(\frac{15 \times 14 \times 13 \times 12 \times 11}{5 \times 4 \times 3 \times 2 \times 1} \times \frac{1}{36 \times 36 \times 6}\right) = \frac{1001}{2592}.$

Hence, the term independent of x in the given expansion is $\frac{1001}{2592}$.

EXAMPLE 11 *Find the term independent of x in the expansion of* $\left(\frac{\sqrt{x}}{\sqrt{3}} + \frac{\sqrt{3}}{2x^2}\right)^{10}$.

SOLUTION Let T_{r+1} be independent of x.

Here, $T_{r+1} = {}^{10}C_r \times \left(\sqrt{\frac{x}{3}}\right)^{(10-r)} \times \left(\frac{\sqrt{3}}{2x^2}\right)^r$

$\Rightarrow \quad T_{r+1} = {}^{10}C_r \times \left(\frac{x}{3}\right)^{\frac{(10-r)}{2}} \times 3^{\frac{r}{2}} \times \frac{1}{2^r x^{2r}}$

$= {}^{10}C_r \times x^{\left(5-\frac{r}{2}-2r\right)} \times \frac{1}{3^{\left(5-\frac{r}{2}\right)}} \times 3^{\frac{r}{2}} \times 2^{-r}$

$= {}^{10}C_r \times x^{\left(5-\frac{5r}{2}\right)} \times 3^{(r-5)} \times 2^{-r}. \quad \ldots \text{(i)}$

Now, T_{r+1} will be independent of x only when the power of x in it is 0.

$\therefore \quad 5-\frac{5r}{2}=0 \Rightarrow \frac{5r}{2}=5 \Rightarrow r=2 \Rightarrow r+1=3.$

Thus, T_3 is independent of x.

Putting $r=2$ in (i), we get

$$T_3 = T_{(2+1)}$$
$$= {}^{10}C_2 \times 3^{(2-5)} \times 2^{-2} \times x^0 = \left(\frac{10\times 9}{2\times 1} \times 3^{-3} \times 2^{-2}\right)$$
$$= \left(\frac{45}{3^3\times 2^2}\right) = \left(\frac{45}{27\times 4}\right) = \frac{5}{12}.$$

Hence, the term independent of x in the given expansion is $\frac{5}{12}$.

EXAMPLE 12 *If the term free from x in the expansion of* $\left(\sqrt{x}-\frac{m}{x^2}\right)^{10}$ *is* 405, *find the value of m.*

SOLUTION The general term in the given expansion is given by

$$T_{r+1} = (-1)^r \times {}^{10}C_r \times (\sqrt{x})^{(10-r)} \times \left(\frac{m}{x^2}\right)^r$$
$$= (-1)^r \times {}^{10}C_r \times x^{\left(5-\frac{r}{2}\right)} \times \frac{m^r}{x^{2r}}$$
$$= (-1)^r \times {}^{10}C_r \times x^{\left(5-\frac{r}{2}-2r\right)} \times m^r$$
$$= (-1)^r \times {}^{10}C_r \times x^{\left(5-\frac{5r}{2}\right)} \times m^r. \qquad \ldots \text{(i)}$$

Let T_{r+1} be free from x.

Then, the power of x in T_{r+1} must be 0.

$\therefore \quad 5-\frac{5r}{2}=0 \Rightarrow \frac{5r}{2}=5 \Rightarrow r=2 \Rightarrow r+1=3.$

So, T_3 will be free from x.

Now, $T_3 = T_{(2+1)}$

$$= (-1)^2 \times {}^{10}C_2 \times x^0 \times m^2 \qquad \text{[putting } r=2 \text{ in (i)]}$$
$$= \left(\frac{10\times 9}{2} \times m^2\right) = 45m^2.$$

But, it is given that the term free from x is 405.

$\therefore \quad 45m^2 = 405 \Rightarrow m^2 = 9 \Rightarrow m = \pm 3.$

Hence, $m = \pm 3$.

EXAMPLE 13 *Find the coefficients of* x^{32} *and* $\frac{1}{x^{17}}$ *in the expansion of* $\left(x^4-\frac{1}{x^3}\right)^{15}$.

SOLUTION The general term in the given expansion is given by

$$T_{r+1} = (-1)^r \times {}^{15}C_r \times (x^4)^{(15-r)} \times \left(\frac{1}{x^3}\right)^r$$
$$\Rightarrow \quad T_{r+1} = (-1)^r \times {}^{15}C_r \times x^{(60-7r)}. \qquad \ldots \text{(i)}$$

Putting $60-7r=32$, we get $7r=28 \Rightarrow r=4 \Rightarrow r+1=5$.

Now, $T_5 = T_{(4+1)} = (-1)^4 \times {}^{15}C_4 \times x^{(60-28)} = {}^{15}C_4 \times x^{32}$.

$\therefore$ coefficient of $x^{32} = {}^{15}C_4 = \left(\dfrac{15\times14\times13\times12}{4\times3\times2\times1}\right) = 1365.$

Let T_{s+1} be the term containing x^{-17} $\left[\because \dfrac{1}{x^{17}} = x^{-17}\right]$.

Then, $T_{s+1} = (-1)^s \times {}^{15}C_s \times x^{(60-7s)}$. ... (ii) [putting $r=s$ in (i)]

Putting $60-7s=-17$, we get $s=11$ and therefore, $s+1=12$.

Thus, the 12th term contains x^{-17}.

Now, $T_{12} = T_{(11+1)}$

$= (-1)^{11} \times {}^{15}C_{11} \times x^{(60-77)}$ [putting $s=11$ in (ii)]

$= -{}^{15}C_4 \times x^{-17}$.

$\therefore$ coefficient of $x^{-17} = -{}^{15}C_4 = -\left(\dfrac{15\times14\times13\times12}{4\times3\times2\times1}\right) = -1365.$

So, the coefficient of x^{-17} in the given expansion is –1365.

Hence, the coefficient of x^{32} is 1365 and that of x^{-17} is –1365.

EXAMPLE 14 *Prove that there is no term containing x^{10} in the expansion of* $\left(x^2 - \dfrac{2}{x}\right)^{18}\cdot$

SOLUTION The general term in the given expansion is given by

$$T_{r+1} = (-1)^r \times {}^{18}C_r \times (x^2)^{(18-r)} \times \left(\frac{2}{x}\right)^r$$

$$\Rightarrow \quad T_{r+1} = (-1)^r \times {}^{18}C_r \times 2^r \times x^{(36-3r)}.$$

Let T_{r+1} contain x^{10}. Then,

$$36-3r=10 \Rightarrow 3r=26 \Rightarrow r=\frac{26}{3}\cdot$$

Since the value of r cannot be a fraction, so there is no term containing x^{10}.

EXAMPLE 15 *Find the middle term in the expansion of* $\left(ax - \dfrac{b}{x^2}\right)^{12}\cdot$

SOLUTION The general term in the given expansion is given by

$$T_{r+1} = (-1)^r \times {}^{12}C_r \times (ax)^{(12-r)} \times \left(\frac{b}{x^2}\right)^r$$

$$\Rightarrow \quad T_{r+1} = (-1)^r \times {}^{12}C_r \times a^{(12-r)} \times b^r \times x^{(12-3r)}. \quad \text{... (i)}$$

Since the given binomial contains an even power, so it has only one middle term.

Middle term $= \left(\dfrac{12}{2}+1\right)$th term = 7th term.

Now, $T_7 = T_{(6+1)}$

$= (-1)^6 \times {}^{12}C_6 \times a^{(12-6)} \times b^6 \times x^{(12-3\times6)}$ [putting $r=6$ in (i)]

$= {}^{12}C_6 \times a^6 b^6 \times x^{-6}$

$$= \left(\frac{12 \times 11 \times 10 \times 9 \times 8 \times 7}{6 \times 5 \times 4 \times 3 \times 2 \times 1} \times \frac{a^6 b^6}{x^6} \right) = \frac{924 a^6 b^6}{x^6}.$$

Hence, the middle term in the given expansion is $\frac{924 a^6 b^6}{x^6}$.

EXAMPLE 16 *Find the middle terms in the expansion of* $\left(3x - \frac{x^3}{6}\right)^7$.

SOLUTION The general term in the given expansion is given by

$$T_{r+1} = (-1)^r \times {}^7C_r \times (3x)^{7-r} \times \left(\frac{x^3}{6}\right)^r$$

$$\Rightarrow \quad T_{r+1} = (-1)^r \times {}^7C_r \times 3^{(7-2r)} \times 2^{-r} \times x^{(7+2r)}. \qquad \text{... (i)}$$

Clearly, the given expansion has 8 terms.

So, it has two middle terms, namely $\left(\frac{8}{2}\right)$th, i.e., 4th and 5th.

Now, $T_4 = T_{(3+1)}$

$$= (-1)^3 \times {}^7C_3 \times 3^{(7-6)} \times 2^{-3} \times x^{13} \qquad \text{[putting } r = 3 \text{ in (i)]}$$

$$= -\left(\frac{7 \times 6 \times 5}{3 \times 2 \times 1} \times 3^1 \times \frac{1}{2^3} \times x^{13} \right)$$

$$= -\left(\frac{35 \times 3 \times x^{13}}{8} \right) = \frac{-105 x^{13}}{8}.$$

And, $T_5 = T_{(4+1)}$

$$= (-1)^4 \times {}^7C_4 \times 3^{(7-8)} \times 2^{-4} \times x^{15} \qquad \text{[putting } r = 4 \text{ in (i)]}$$

$$= {}^7C_3 \times 3^{-1} \times 2^{-4} \times x^{15} \qquad [\because {}^7C_4 = {}^7C_{(7-4)} = {}^7C_3]$$

$$= \left(\frac{7 \times 6 \times 5}{3 \times 2 \times 1} \times \frac{1}{3} \times \frac{1}{2^4} \times x^{15} \right) = \frac{35 x^{15}}{48}.$$

Hence, the required middle terms are $\frac{-105 x^{13}}{8}$ and $\frac{35 x^{15}}{48}$.

EXAMPLE 17 *Show that the middle term in the expansion of* $(1+x)^{2n}$ *is* $\frac{1 \cdot 3 \cdot 5 \ldots (2n-1)}{n!} \cdot 2^n \cdot x^n$, *where* $n \in N$.

SOLUTION Clearly, the number of terms in the expansion of $(1+x)^{2n}$ is $(2n+1)$.

$$\therefore \quad \text{middle term} = \left(\frac{2n}{2} + 1\right)\text{th term} = (n+1)\text{th term} = T_{n+1}.$$

Now, $T_{n+1} = {}^{2n}C_n \cdot x^n = \frac{(2n)!}{(n!) \times (n!)} \cdot x^n$

$$= \frac{1 \cdot 2 \cdot 3 \cdot 4 \ldots (2n-2)(2n-1)(2n)}{(n!) \times (n!)} \cdot x^n$$

$$= \frac{[1 \cdot 3 \cdot 5 \ldots (2n-3)(2n-1)] \times [2 \cdot 4 \cdot 6 \ldots (2n-2)(2n)]}{(n!)(n!)} \cdot x^n$$

$$= \frac{[1 \cdot 3 \cdot 5 \ldots (2n-1)] \times 2^n \times [1 \cdot 2 \cdot 3 \ldots (n-1) \cdot n]}{(n!) \times (n!)} \cdot x^n$$

$$= \frac{[1 \cdot 3 \cdot 5 \ldots (2n-1)] \times 2^n \times x^n}{(n!)}.$$

Hence, the middle term in the given expansion is

$$\frac{1 \cdot 3 \cdot 5 \ldots (2n-1) \times 2^n \times x^n}{(n!)}.$$

EXAMPLE 18 *If x^p occurs in the expansion of $\left(x^2 + \frac{1}{x}\right)^{2n}$, prove that its coefficient is*

$$\frac{(2n)!}{\left\{\left(\frac{4n-p}{3}\right)!\right\} \times \left\{\left(\frac{2n+p}{3}\right)!\right\}}.$$

SOLUTION The general term in the expansion of $\left(x^2 + \frac{1}{x}\right)^{2n}$ is given by

$$T_{r+1} = {}^{2n}C_r \times (x^2)^{(2n-r)} \times \left(\frac{1}{x}\right)^r$$

$$\Rightarrow \quad T_{r+1} = {}^{2n}C_r \times x^{(4n-3r)}. \qquad \ldots \text{(i)}$$

This term contains x^p only when $4n - 3r = p$.

And, $4n - 3r = p \Rightarrow r = \frac{(4n-p)}{3}$.

Putting $4n - 3r = p$ in (i), we get

coefficient of $x^p = {}^{2n}C_r$, where $r = \frac{(4n-p)}{3}$

$$= \frac{(2n)!}{(r!) \times (2n-r)!} = \frac{(2n)!}{\left\{\left(\frac{4n-p}{3}\right)!\right\} \times \left\{\left[2n - \frac{(4n-p)}{3}\right]!\right\}}$$

$$= \frac{(2n)!}{\left\{\left(\frac{4n-p}{3}\right)!\right\} \times \left\{\left(\frac{2n+p}{3}\right)!\right\}}.$$

Hence, the coefficient of x^p in the expansion of $\left(x^2 + \frac{1}{x}\right)^{2n}$ is

$$\frac{(2n)!}{\left\{\left(\frac{4n-p}{3}\right)!\right\} \times \left\{\left(\frac{2n+p}{3}\right)!\right\}}.$$

EXAMPLE 19 *Show that the coefficient of the middle term in the expansion of $(1+x)^{2n}$ is the sum of the coefficients of two middle terms in the expansion of $(1+x)^{2n-1}$.*

SOLUTION The expansion of $(1+x)^{2n}$ contains $(2n+1)$ terms and therefore, the $(n+1)$th term is the middle term.

$\therefore$ the middle term in the expansion of $(1+x)^{2n}$ is given by

$$t_{n+1} = {}^{2n}C_n x^n = \frac{(2n)!}{(n!)\cdot(n!)} x^n = \frac{(2n)!}{(n!)^2} x^n.$$

Thus, the coefficient of the middle term of $(1+x)^{2n}$ is $\dfrac{(2n)!}{(n!)^2}$.

Again, the expansion of $(1+x)^{2n-1}$ contains $2n$ terms.

So, the middle terms are the nth and the $(n+1)$th terms.

Now, $t_n = t_{(n-1)+1} = {}^{2n-1}C_{n-1} x^{n-1}$ and $t_{n+1} = {}^{2n-1}C_n x^n$.

$\therefore$ the sum of the coefficients of the middle terms of $(1+x)^{2n-1}$

$$= {}^{2n-1}C_{n-1} + {}^{2n-1}C_n$$

$$= \frac{(2n-1)!}{(n-1)! \times (n!)} + \frac{(2n-1)!}{(n!) \times (n-1)!}$$

$$= \frac{2 \times (2n-1)! \times n}{(n!) \times n \times (n-1)!} \quad \text{[multiplying num. and denom. by } n\text{]}$$

$$= \frac{(2n)\cdot(2n-1)!}{(n!)\cdot(n!)} = \frac{(2n)!}{(n!)^2}$$

$=$ the coefficient of the middle term of $(1+x)^{2n}$.

EXAMPLE 20 *If a_1, a_2, a_3, a_4 be the coefficients of four consecutive terms in the expansion of $(1+x)^n$ then prove that*

$$\frac{a_1}{(a_1+a_2)} + \frac{a_3}{(a_3+a_4)} = \frac{2a_2}{(a_2+a_3)}.$$

SOLUTION Let a_1, a_2, a_3 and a_4 be the coefficients of the rth, $(r+1)$th, $(r+2)$th and $(r+3)$th terms in the expansion of $(1+x)^n$. Then,

$$a_1 = {}^nC_{r-1};\ a_2 = {}^nC_r,\ a_3 = {}^nC_{r+1} \text{ and } a_4 = {}^nC_{r+2}.$$

Now, $a_1 + a_2 = ({}^nC_{r-1} + {}^nC_r) = {}^{n+1}C_r$;

$a_3 + a_4 = ({}^nC_{r+1} + {}^nC_{r+2}) = {}^{n+1}C_{r+2}$;

and $a_2 + a_3 = ({}^nC_r + {}^nC_{r+1}) = {}^{n+1}C_{r+1}$.

$$\therefore \quad \frac{a_1}{(a_1+a_2)} + \frac{a_3}{(a_3+a_4)} = \frac{{}^nC_{r-1}}{{}^{n+1}C_r} + \frac{{}^nC_{r+1}}{{}^{n+1}C_{r+2}}$$

$$= \frac{{}^nC_{r-1}}{\left(\frac{n+1}{r}\right)\cdot {}^nC_{r-1}} + \frac{{}^nC_{r+1}}{\left(\frac{n+1}{r+2}\right)\cdot {}^nC_{r+1}} \quad \left[\because\ {}^nC_r = \left(\frac{n}{r}\right)\cdot {}^{n-1}C_{r-1}\right]$$

$$= \frac{r}{(n+1)} + \frac{(r+2)}{(n+1)} = \frac{2(r+1)}{(n+1)}.$$

$$\text{And, } \frac{2a_2}{(a_2+a_3)} = \frac{2 \times {}^nC_r}{{}^{n+1}C_{r+1}} = \frac{2 \times {}^nC_r}{\left(\frac{n+1}{r+1}\right)\cdot {}^nC_r} = \frac{2(r+1)}{(n+1)}.$$

$$\text{Hence, } \frac{a_1}{(a_1+a_2)} + \frac{a_3}{(a_3+a_4)} = \frac{2a_2}{(a_2+a_3)}.$$

EXAMPLE 21 *In the binomial expansion of* $(1+x)^{m+n}$, *prove that the coefficients of* x^m *and* x^n *are equal.*

SOLUTION In the binomial expansion of $(1+x)^{m+n}$, the general term is given by

$$T_{r+1} = {}^{m+n}C_r \times x^r. \quad \ldots \text{(i)}$$

$\therefore$ coefficient of x^m in the expansion of $(1+x)^{m+n}$

$$= {}^{m+n}C_m = \frac{(m+n)!}{\{(m+n-m)!\} \times (m!)} = \frac{(m+n)!}{(n!) \times (m!)}.$$

coefficient of x^n in the expansion of $(1+x)^{m+n}$

$$= {}^{m+n}C_n = \frac{(m+n)!}{\{(m+n-n)!\} \times (n!)} = \frac{(m+n)!}{(m!) \times (n!)}.$$

Hence, the coefficients of x^m and x^n are equal.

EXAMPLE 22 *Find the value of r when it is being given that the coefficients of* $(2r+4)$*th and* $(r-2)$*th terms in the expansion of* $(1+x)^{18}$ *are equal.*

SOLUTION In the expansion of $(1+x)^{18}$, we have $T_{p+1} = {}^{18}C_p x^p$.

$$\therefore \quad T_{(2r+4)} = T_{(2r+3)+1} = {}^{18}C_{2r+3} \cdot x^{2r+3} \quad \ldots \text{(i)}$$

$$\text{and} \quad T_{(r-2)} = T_{(r-3)+1} = {}^{18}C_{(r-3)} \cdot x^{r-3}. \quad \ldots \text{(ii)}$$

It is being given that the coefficients of $T_{(2r+1)}$ and $T_{(r-2)}$ are equal.

So, from (i) and (ii), we get

$${}^{18}C_{2r+3} = {}^{18}C_{r-3} \Rightarrow (2r+3) = (r-3) \text{ or } (2r+3)+(r-3) = 18$$

$$[\because \; {}^nC_p = {}^nC_q \Rightarrow p = q \text{ or } p+q = n].$$

Now, $(2r+3) = (r-3)$ or $(2r+3)+(r-3) = 18$

$\Rightarrow \quad r = -6$ or $r = 6$

$\Rightarrow \quad r = 6 \quad$ [$\because$ the negative value of r is not permissible].

Hence, $r = 6$.

EXAMPLE 23 *Find the coefficient of* $x^6 y^3$ *in the expansion of* $(x+2y)^9$.

SOLUTION The general term in the expansion of $(x+2y)^9$ is given by

$$T_{r+1} = {}^9C_r \times x^{(9-r)} \times (2y)^r$$

$$\Rightarrow \quad T_{r+1} = {}^9C_r \times 2^r \times x^{(9-r)} \times y^r. \quad \ldots \text{(i)}$$

We have to find the coefficient of $x^6 y^3$.

Putting $r = 3$ in (i), we get

$$T_{(3+1)} = {}^9C_3 \times 2^3 \times (x^6 y^3).$$

$\therefore$ coefficient of $x^6 y^3$ in the given expansion

$$= ({}^9C_3 \times 2^3) = \left(\frac{9 \times 8 \times 7}{3 \times 2 \times 1} \times 8\right) = 672.$$

Hence, the coefficient of $x^6 y^3$ in the given expansion is 672.

EXAMPLE 24 *Write down the binomial expansion of* $(1+x)^{n+1}$ *when* $x = 8$. *Deduce that* $(9^{n+1} - 8n - 9)$ *is divisible by* 64, *where n is a positive integer.*

SOLUTION Using the binomial expansion, we get

$$(1+x)^{n+1} = {}^{n+1}C_0 + {}^{n+1}C_1 x + {}^{n+1}C_2 x^2 + {}^{n+1}C_3 x^3 + \ldots + {}^{n+1}C_{n+1} x^{n+1}.$$

Putting $x = 8$ in the above expansion, we get

$$9^{n+1} = 1 + (n+1)\times 8 + {}^{n+1}C_2 \times (8)^2 + {}^{n+1}C_3 \times (8)^3 + \ldots + {}^{n+1}C_{n+1} \times (8)^{n+1}$$

$$\Rightarrow \quad (9^{n+1} - 8n - 9) = (8)^2 \times \{{}^{n+1}C_2 + {}^{n+1}C_3 \times 8 + \ldots + (8)^{n-1}\}$$

$$\Rightarrow \quad (9^{n+1} - 8n - 9) = 64 \times (\text{an integer}).$$

Hence, $(9^{n+1} - 8n - 9)$ is exactly divisible by 64.

EXAMPLE 25 *Using the binomial theorem, prove that* $(6^n - 5n)$ *always leaves the remainder* 1 *when divided by* 25.

SOLUTION Using binomial expansion, we get

$$\begin{aligned}(6^n - 5n) &= (1+5)^n - 5n \\ &= \{{}^nC_0 + {}^nC_1 \times 5 + {}^nC_2 \times (5)^2 + {}^nC_3 \times (5)^3 + {}^nC_4 \times (5)^4 + \ldots + {}^nC_n \times 5^n\} - 5n \\ &= \{1 + 5n + {}^nC_2 \times (5)^2 + {}^nC_3 \times (5)^3 + {}^nC_4 \times (5)^4 + \ldots + {}^nC_n \times 5^n\} - 5n \\ &= \{1 + {}^nC_2 \times (5)^2 + {}^nC_3 \times (5)^3 + {}^nC_4 \times (5)^4 + \ldots + {}^nC_n \times 5^n\} \\ &= (5)^2 \times \{{}^nC_2 + {}^nC_3 \times 5 + {}^nC_4 \times (5)^2 + \ldots + {}^nC_n \times 5^{(n-2)}\} + 1 \\ &= 25 \times (\text{an integer}) + 1.\end{aligned}$$

Hence, $(6^n - 5n)$ when divided by 25 leaves the remainder 1.

EXAMPLE 26 *Show that* $2^{4n+4} - 15n - 16$, *where* $n \in N$ *is divisible by* 225.

SOLUTION We have

$$2^{4n+4} - 15n - 16 = 2^{4(n+1)} - 15n - 16 = 16^{(n+1)} - 15n - 16.$$

Now, $16^{(n+1)} - 15n - 16 = (1+15)^{n+1} - 15n - 16$

$$\begin{aligned}&= {}^{n+1}C_0 + {}^{n+1}C_1 \times 15 + {}^{n+1}C_2 \times (15)^2 + {}^{n+1}C_3 \times (15)^3 + \ldots + {}^{n+1}C_{n+1} \times (15)^{n+1} - 15n - 16 \\ &= 1 + (n+1)\times 15 + {}^{n+1}C_2 \times (15)^2 + {}^{n+1}C_3 \times (15)^3 + \ldots + (15)^{n+1} - 15n - 16 \\ &= {}^{n+1}C_2 \times (15)^2 + {}^{n+1}C_3 \times (15)^3 + \ldots + (15)^{n+1} \\ &= (15)^2 \times [{}^{n+1}C_2 + {}^{n+1}C_3 \times 15 + \ldots + (15)^{n-1}] \\ &= 225 \times (\text{an integer}).\end{aligned}$$

Hence, $(2^{4n+4} - 15n - 16)$ is divisible by 225.

EXAMPLE 27 *If a and b are distinct integers then prove that* $(a-b)$ *is a factor of* $(a^n - b^n)$, *whenever n is a positive integer.*

SOLUTION We may write

$$a^n = (a-b+b)^n = \{(a-b)+b\}^n$$
$$= {}^nC_0 \times (a-b)^n + {}^nC_1 \times (a-b)^{(n-1)} \times b + {}^nC_2 \times (a-b)^{(n-2)} \times b^2 + \dots + {}^nC_{n-1} \times (a-b) \times b^{n-1} + {}^nC_n \times b^n$$
$$= (a-b)^n + nb(a-b)^{n-1} + \frac{1}{2}n(n-1)b^2(a-b)^{n-2} + \dots + nb^{n-1}(a-b) + b^n$$
$$\Rightarrow \quad (a^n - b^n) = (a-b)^n + nb(a-b)^{n-1} + \frac{1}{2}n(n-1)b^2(a-b)^{n-2} + \dots + nb^n(a-b)$$
$$= (a-b) \times [(a-b)^{n-1} + nb(a-b)^{n-2} + \frac{1}{2}n(n-1)b^2(a-b)^{n-3} + nb^n]$$

[taking $(a-b)$ common throughout]

$$= (a-b) \times (\text{an integer}).$$

Hence, $(a-b)$ is a factor of $(a^n - b^n)$.

EXAMPLE 28 *The second, third and fourth terms in the binomial expansion of* $(x+a)^n$ *are* 240, 720 *and* 1080 *respectively. Find x, a and n.*

SOLUTION By the binomial expansion, we have

$$(x+a)^n = {}^nC_0 x^n + {}^nC_1 x^{n-1}a + {}^nC_2 x^{n-2}a^2 + {}^nC_3 x^{n-3}a^3 + \dots$$
$$\Rightarrow \quad (x+a)^n = x^n + nx^{n-1}a + \frac{1}{2}n(n-1)x^{n-2}a^2 + \frac{1}{6}n(n-1)(n-2)x^{n-3}a^3 + \dots.$$

$\therefore \quad T_2 = 240 \Rightarrow nx^{n-1}a = 240,$ ... (i)

$T_3 = 720 \Rightarrow n(n-1)x^{n-2}a^2 = 1440,$ ... (ii)

$T_4 = 1080 \Rightarrow n(n-1)(n-2)x^{n-3}a^3 = 6480.$ ... (iii)

On dividing (ii) by (i), we get

$$\frac{(n-1)a}{x} = \frac{1440}{240} \Rightarrow \frac{a}{x} = \frac{6}{(n-1)}. \qquad \text{... (iv)}$$

On dividing (iii) by (ii), we get

$$\frac{(n-2)a}{x} = \frac{6480}{1440} \Rightarrow \frac{a}{x} = \frac{9}{2(n-2)}. \qquad \text{... (v)}$$

Equating the values of $\frac{a}{x}$ from (iv) and (v), we get

$$\frac{6}{(n-1)} = \frac{9}{2(n-2)} \Rightarrow 12(n-2) = 9(n-1) \Rightarrow 3n = 15 \Rightarrow n = 5.$$

Putting $n = 5$ in (i), we get $x^4a = 48$. ... (vi)

Putting $n = 5$ in (iv), we get $\frac{a}{x} = \frac{6}{4} \Rightarrow \frac{a}{x} = \frac{3}{2} \Rightarrow a = \frac{3}{2}x$.

Putting $a=\frac{3}{2}x$ in (vi), we get

$$x^4\times\frac{3}{2}x=48 \Rightarrow x^5=48\times\frac{2}{3}=32=2^5 \Rightarrow x=2.$$

$$\therefore \quad a=\frac{3}{2}\times 2=3.$$

Hence, $x=2$, $a=3$, and $n=5$.

EXAMPLE 29 *If the coefficients of $(r-1)$th, rth and $(r+1)$th terms in the expansion of $(x+1)^n$ are in the ratio $1:3:5$, find the values of n and r.*

SOLUTION We have

$$(x+1)^n=(1+x)^n$$
$$={}^nC_0+{}^nC_1x+{}^nC_2x^2+\ldots+{}^nC_{r-2}x^{r-2}+{}^nC_{r-1}x^{r-1}+{}^nC_rx^r+\ldots.$$

$$\therefore \quad T_{(r-1)}=T_{(r-2)+1}={}^nC_{r-2}x^{r-2}, \qquad \ldots \text{(i)}$$

$$T_r=T_{(r-1)+1}={}^nC_{r-1}x^{r-1}, \qquad \ldots \text{(ii)}$$

$$T_{r+1}={}^nC_rx^r. \qquad \ldots \text{(iii)}$$

It is given that

$$\text{coeff. of } T_{(r-1)} : \text{coeff. of } T_r : \text{coeff. of } T_{(r+1)}=1:3:5$$

$$\Rightarrow \quad {}^nC_{(r-2)}:{}^nC_{(r-1)}:{}^nC_r=1:3:5$$

$$\Rightarrow \quad \frac{{}^nC_{r-2}}{{}^nC_{r-1}}=\frac{1}{3} \text{ and } \frac{{}^nC_{r-1}}{{}^nC_r}=\frac{3}{5}.$$

But, $\frac{{}^nC_{r-2}}{{}^nC_{r-1}}=\frac{1}{3} \Rightarrow \frac{n!}{(r-2)!\cdot(n-r+2)!}\times\frac{(r-1)!\cdot(n-r+1)!}{n!}=\frac{1}{3}$

$$\Rightarrow \frac{(r-1)}{(n-r+2)}=\frac{1}{3}$$

$$\Rightarrow n-4r=-5. \qquad \ldots \text{(i)}$$

And, $\frac{{}^nC_{r-1}}{{}^nC_r}=\frac{3}{5} \Rightarrow \frac{r}{(n-r+1)}=\frac{3}{5}$

$$\Rightarrow 3n-8r=-3. \qquad \ldots \text{(ii)}$$

Solving (i) and (ii), we get $n=7$ and $r=3$.

Hence, $n=7$ and $r=3$.

EXAMPLE 30 *If the coefficients of x^{r-1}, x^r and x^{r+1} in the binomial expansion of $(1+x)^n$ are in AP, prove that*

$$n^2-n(4r+1)+4r^2-2=0.$$

SOLUTION We know that

$$(1+x)^n={}^nC_0+{}^nC_1x+{}^nC_2x^2+\ldots+{}^nC_{r-1}x^{r-1}+{}^nC_rx^r+{}^nC_{r+1}x^{r+1}+\ldots.$$

So, the coefficients of x^{r-1}, x^r and x^{r+1} in the given expansion are ${}^nC_{r-1}$, ${}^nC_r$, and ${}^nC_{r+1}$ respectively.

It is given that ${}^nC_{r-1}$, ${}^nC_r$, and ${}^nC_{r+1}$ are in AP.

$$\therefore \quad 2 \times {}^nC_r = {}^nC_{r-1} + {}^nC_{r+1}$$

$$\Rightarrow \quad \frac{{}^nC_{r-1}}{{}^nC_r} + \frac{{}^nC_{r+1}}{{}^nC_r} = 2$$

$$\Rightarrow \quad \frac{n!}{(r-1)! \cdot (n-r+1)!} \times \frac{(r!) \cdot (n-r)!}{n!} + \frac{n!}{(r+1)! \cdot (n-r+1)!} \times \frac{(r!) \cdot (n-r)!}{n!} = 2$$

$$\Rightarrow \quad \frac{r}{(n-r+1)} + \frac{(n-r)}{(r+1)} = 2$$

$$[\because \ (n-r+1)! = (n-r+1) \cdot \{(n-r)!\} \text{ and } r! = r \cdot \{(r-1)!\}]$$

$$\Rightarrow \quad r(r+1) + (n-r)(n-r+1) = 2(r+1)(n-r+1)$$

$$\Rightarrow \quad n^2 - n(4r+1) + 4r^2 - 2 = 0.$$

Hence, $n^2 - n(4r+1) + 4r^2 - 2 = 0$.

EXAMPLE 31 *Simplify the expression*

$$(1+x)^{1000} + x(1+x)^{999} + x^2(1+x)^{998} + \ldots + x^{1000}$$

and find the coefficient of x^{50}.

SOLUTION Clearly, the given series is a geometric series in which

$$a = (1+x)^{1000}, \ r = \frac{x(1+x)^{999}}{(1+x)^{1000}} = \frac{x}{(1+x)} \text{ and } n = 1001.$$

$$\therefore \quad \text{given sum} = \frac{a(1-r^n)}{(1-r)}$$

$$= \frac{(1+x)^{1000} \times \left\{1 - \left(\frac{x}{1+x}\right)^{1001}\right\}}{\left(1 - \frac{x}{1+x}\right)} = (1+x)^{1001} - x^{1001}$$

$$= 1 + {}^{1001}C_1 x + {}^{1001}C_2 x^2 + \ldots {}^{1001}C_{1000} x^{1000} + x^{1001} - x^{1001}$$

$$= 1 + {}^{1001}C_1 x + {}^{1001}C_2 x^2 + \ldots + {}^{1001}C_1 x^{1000}.$$

$\therefore$ coefficient of x^{50} in the above expansion

$$= {}^{1001}C_{50} = \frac{(1001)!}{(50)! \cdot (1001-50)!} = \frac{(1001)!}{(50)! \cdot (951)!}.$$

EXAMPLE 32 *If n is a positive integer, find the coefficient of* x^{-1} *in the expansion of* $(1+x)^n\left(1+\frac{1}{x}\right)^n$.

SOLUTION We have

$$(1+x)^n\left(1+\frac{1}{x}\right)^n = (1+x)^n \times \frac{(1+x)^n}{x^n} = \frac{(1+x)^{2n}}{x^n}.$$

Coefficient of x^{-1} in the expansion of $(1+x)^n\left(1+\frac{1}{x}\right)^n$

$= \text{coefficient of } x^{-1} \text{ in the expansion of } \frac{(1+x)^{2n}}{x^n}$

$= \text{coefficient of } (x^{-1} \times x^n) \text{ in the expansion of } (1+x)^{2n}$

$= \text{coefficient of } x^{n-1} \text{ in the expansion of } (1+x)^{2n}$

$= {}^{2n}C_{n-1}$ $\quad \left[\because (1+x)^{2n} = \sum_{r=0}^{2n} {}^{2n}C_r x^r\right].$

EXAMPLE 33 *Find the coefficient of* x^4 *in the expansion of* $(1+x+x^2+x^3)^{11}$.

SOLUTION We have

$$(1+x+x^2+x^3)^{11} = \{(1+x)+x^2(1+x)\}^{11} = \{(1+x)(1+x^2)\}^{11}$$
$$= (1+x)^{11} \times (1+x^2)^{11}$$
$$= ({}^{11}C_0 x^0 + {}^{11}C_1 x + {}^{11}C_2 x^2 + {}^{11}C_3 x^3 + {}^{11}C_4 x^4 + \ldots)$$
$$\times [{}^{11}C_0 (x^2)^0 + {}^{11}C_1 (x^2) + {}^{11}C_2 (x^2)^2 + \ldots]$$
$$= (1+11x+55x^2+165x^3+330x^4+\ldots)$$
$$\times (1+11x^2+55x^4+\ldots).$$

$\therefore$ coefficient of x^4 in the given expansion

$$= (1 \times 55) + (55 \times 11) + (330 \times 1)$$
$$= (55+605+330) = 990.$$

Hence, the coefficient of x^4 in the given expansion is 990.

EXAMPLE 34 *Find the coefficient of* x^4 *in the product* $(1+2x)^4 \times (2-x)^5$.

SOLUTION Using the binomial expansion, we get

$$(1+2x)^4 = {}^4C_0 + {}^4C_1(2x) + {}^4C_2(2x)^2 + {}^4C_3(2x)^3 + {}^4C_4(2x)^4$$
$$= 1+8x+24x^2+32x^3+16x^4$$

and $(2-x)^5 = 2^5 - {}^5C_1(2^4)x + {}^5C_2(2^3)x^2 - {}^5C_3(2^2)x^3$
$$+ {}^5C_4(2)x^4 - {}^5C_5 x^5$$
$$= 32 - 80x + 80x^2 - 40x^3 + 10x^4 - x^5$$

$\therefore$ $(1+2x)^4 \times (2-x)^5$
$$= (1+8x+24x^2+32x^3+16x^4)$$
$$\times (32-80x+80x^2-40x^3+10x^4-x^5).$$

Sum of the terms containing x^4 in the given product

$$= (1 \times 10x^4) + 8x(-40x^3) + (24x^2)(80x^2)$$
$$+ (32x^3)(-80x) + (16x^4 \times 32)$$
$$= 10x^4 - 320x^4 + 1920x^4 - 2560x^4 + 512x^2$$
$$= (10-320+1920-2560+512)x^4 = -438x^4.$$

Hence, the coefficient of x^4 in the given product is -438.

EXAMPLE 35 *If P be the sum of all odd terms and Q that of all even terms in the expansion of* $(x+a)^n$, *prove that*

(i) $(P^2-Q^2)=(x^2-a^2)^n$,

(ii) $4PQ=\{(x+a)^{2n}-(x-a)^{2n}\}$,

(iii) $2(P^2+Q^2)=[(x+a)^{2n}+(x-a)^{2n}]$.

SOLUTION By the binomial expansion, we have

$$\begin{aligned}(x+a)^n &= {}^nC_0x^n+{}^nC_1x^{n-1}+{}^nC_2x^{n-2}+{}^nC_3x^{n-3}+{}^nC_4x^{n-4}+\\ &\qquad \ldots+{}^nC_na^n.\\ &=(T_1+T_2+T_3+T_4+T_5+\ldots+T_n+T_{n+1})\\ &=(T_1+T_3+T_5+\ldots)+(T_2+T_4+T_6+\ldots)\\ &=P+Q. \qquad \ldots \text{(I)}\end{aligned}$$

And, $$\begin{aligned}(x-a)^n &= {}^nC_0x^n-{}^nC_1x^{n-1}a+{}^nC_2x^{n-2}a^2-{}^nC_3x^{n-3}a^3+\\ &\qquad \ldots+(-1)^n\,{}^nC_na^n\\ &=(T_1-T_2+T_3-T_4+\ldots)\\ &=(T_1+T_3+T_5+\ldots)-(T_2+T_4+T_6+\ldots)\\ &=P-Q. \qquad \ldots \text{(II)}\end{aligned}$$

(i) On multiplying (I) and (II), we get

$$(x^2-a^2)^n=(P^2-Q^2).$$

(ii) On squaring (I) and (II) and subtracting, we get

$$\{(x+a)^{2n}-(x-a)^{2n}\}=\{(P+Q)^2-(P-Q)^2\}=4PQ.$$

(iii) On squaring (I) and (II) and adding, we get

$$\{(x+a)^{2n}+(x-a)^{2n}\}=\{(P+Q)^2+(P-Q)^2\}=2(P^2+Q^2).$$

EXAMPLE 36 *Using binomial theorem, find the value of* $(103)^4$.

SOLUTION By the binomial expansion, we have

$$\begin{aligned}(103)^4 &= (100+3)^4\\ &= {}^4C_0\cdot(100)^4+{}^4C_1\times(100)^3\times 3+{}^4C_2\times(100)^2\times 3^2\\ &\qquad +{}^4C_3\times 100\times 3^3+{}^4C_4\times 3^4\\ &=(100)^4+12\times(100)^3+54\times(100)^2+108\times 100+81\\ &=100000000+12\times 1000000+540000+10800+81\\ &=100000000+12000000+540000+10800+81\\ &=112550881.\end{aligned}$$

Hence, $(103)^4=112550881$.

EXAMPLE 37 *Using binomial theorem, find the value of* $(99)^4$.

SOLUTION By the binomial expansion, we have

$$\begin{aligned}(99)^4 &= (100-1)^4\\ &= {}^4C_0\times(100)^4-{}^4C_1\times(100)^3\times 1+{}^4C_2\times(100)^2\times 1^2\\ &\qquad -{}^4C_3\times 100\times 1^3+{}^4C_4\times 1^4\end{aligned}$$

$$= (100)^4 - 4 \times (100)^3 + 6 \times (100)^2 - 4 \times 100 + 1$$
$$= 100000000 - 4000000 + 60000 - 400 + 1$$
$$= 96059601.$$

Hence, $(99)^4 = 96059601$.

EXAMPLE 38 *Using binomial theorem, find the value of* $(0.99)^{15}$ *up to four places of decimal.*

SOLUTION We have

$$(0.99)^{15} = (1 - 0.01)^{15}$$
$$= 1 - {}^{15}C_1 \times (0.01) + {}^{15}C_2 \times (0.01)^2 - {}^{15}C_3 \times (0.01)^3 + \ldots$$
[neglecting higher powers of 0.01]
$$= 1 - 15 \times (0.01) + 105 \times (0.0001) - 455 \times (0.000001) + \ldots$$
$$= 1 - 0.15 + 0.0105 + 0.000455 = 0.860045.$$

Hence, $(0.99)^{15} = 0.860045$.

EXAMPLE 39 *Using binomial theorem, prove that* $(101)^{50} > (100^{50} + 99^{50})$.

SOLUTION Putting $(101)^{50} = a$ and $(100^{50} + 99^{50}) = b$, we get

$$(a - b) = (101)^{50} - (100)^{50} - (99)^{50}$$
$$= (101)^{50} - (99)^{50} - (100)^{50}$$
$$= (100 + 1)^{50} - (100 - 1)^{50} - (100)^{50}$$
$$= 2 \times [{}^{50}C_1 \times 100^{49} + {}^{50}C_3 \times 100^{47} + \ldots + {}^{50}C_{49} \times 100] - (100)^{50}$$
$$= [2 \times {}^{50}C_3 \times 100^{47} + 2 \times {}^{50}C_5 \times 100^{45} + \ldots + 2 \times {}^{50}C_{49} \times 100]$$
$$= \text{(a positive integer)}.$$

Thus, $a - b > 0 \Rightarrow a > b \Rightarrow (101)^{50} > (100^{50} + 99^{50})$.

EXAMPLE 40 *Which is larger,* $(1.01)^{1000000}$ *or* 10000?

SOLUTION We have

$$(1.01)^{1000000} - 10000$$
$$= (1.01)^p - 10000, \text{ where } 1000000 = p$$
$$= (1 + 0.01)^p - 10000$$
$$= {}^pC_0 + {}^pC_1 \times 0.01 + {}^pC_2 \times (0.01)^2 + \ldots + {}^pC_p \times (0.01)^p - 10000$$
$$= 1 + (p \times 0.01) + [{}^pC_2 \times (0.01)^2 + \ldots + (0.01)^p] - 10000$$
$$= 1 + (1000000 \times 0.01) + [{}^{1000000}C_2 \times (0.01)^2 + \ldots + (1.01)^{1000000}] - 10000$$
$$= 1 + 10000 + [\text{a positive real number}] - 10000$$
$$= 1 + (\text{a positive real number}) > 0$$

Hence, $(1.01)^{1000000} > 10000$.

EXERCISE 10A

Using binomial theorem, expand each of the following:

1. $(1-2x)^5$ **2.** $(2x-3)^6$ **3.** $(3x+2y)^5$

4. $(2x-3y)^4$ **5.** $\left(\frac{2x}{3}-\frac{3}{2x}\right)^6$ **6.** $\left(x^2-\frac{2}{x}\right)^7$

7. $\left(x-\frac{1}{y}\right)^5$ **8.** $(\sqrt{x}+\sqrt{y})^8$ **9.** $(\sqrt[3]{x}-\sqrt[3]{y})^6$

10. $(1+2x-3x^2)^4$ **11.** $\left(1+\frac{x}{2}-\frac{2}{x}\right)^4, x\neq 0$ **12.** $(3x^2-2ax+3a^2)^3$

Evaluate:

13. $(\sqrt{2}+1)^6+(\sqrt{2}-1)^6$ **14.** $(\sqrt{3}+1)^5-(\sqrt{3}-1)^5$

15. $(2+\sqrt{3})^7+(2-\sqrt{3})^7$ **16.** $(\sqrt{3}+\sqrt{2})^6-(\sqrt{3}-\sqrt{2})^6$

17. Prove that $\sum_{r=0}^{n} {}^nC_r \cdot 3^r = 4^n$.

18. Using binomial theorem, evaluate each of the following:

(i) $(101)^4$ (ii) $(98)^4$ (iii) $(1.2)^4$

19. Using binomial theorem, prove that $(2^{3n}-7n-1)$ is divisible by 49, where $n\in N$.

20. Prove that $(2+\sqrt{x})^4+(2-\sqrt{x})^4=2(16+24x+x^2)$.

21. Find the 7th term in the expansion of $\left(\frac{4x}{5}+\frac{5}{2x}\right)^8$.

22. Find the 9th term in the expansion of $\left(\frac{a}{b}-\frac{b}{2a^2}\right)^{12}$.

23. Find the 16th term in the expansion of $(\sqrt{x}-\sqrt{y})^{17}$.

24. Find the 13th term in the expansion of $\left(9x-\frac{1}{3\sqrt{x}}\right)^{18}, x\neq 0$.

25. If the coefficients of x^7 and x^8 in the expansion of $\left(2+\frac{x}{3}\right)^n$ are equal then find the value of n.

26. Find the ratio of the coefficient of x^{15} to the term independent of x in the expansion of $\left(x^2+\frac{2}{x}\right)^{15}$.

27. Show that the ratio of the coefficient of x^{10} in the expansion of $(1-x^2)^{10}$ and the term independent of x in the expansion of $\left(x-\frac{2}{x}\right)^{10}$ is 1 : 32.

28. Find the term independent of x in the expansion of $(1+x+2x^3)\left(\frac{3}{2}x^2-\frac{1}{3x}\right)^9$.

29. Find the coefficient of x in the expansion of $(1-3x+7x^2)(1-x)^{16}$.

30. Find the coefficient of

(i) x^5 in the expansion of $(x+3)^8$.

(ii) x^6 in the expansion of $\left(3x^2-\frac{1}{3x}\right)^9$.

(iii) x^{-15} in the expansion of $\left(3x^2-\frac{a}{3x^3}\right)^{10}$.

(iv) a^7b^5 in the expansion of $(a-2b)^{12}$.

31. Show that the term containing x^3 does not exist in the expansion of $\left(3x-\frac{1}{2x}\right)^8$.

32. Show that the expansion of $\left(2x^2-\frac{1}{x}\right)^{20}$ does not contain any term involving x^9.

33. Show that the expansion of $\left(x^2+\frac{1}{x}\right)^{12}$ does not contain any term involving x^{-1}.

34. Write the general term in the expansion of $(x^2-y)^6$.

35. Find the 5th term from the end in the expansion of $\left(x-\frac{1}{x}\right)^{12}$.

36. Find the 4th term from the end in the expansion of $\left(\frac{4x}{5}-\frac{5}{2x}\right)^9$.

37. If the 7th terms from the beginning and end in the expansion of $\left(\sqrt[3]{2}+\frac{1}{\sqrt[3]{3}}\right)^n$ are equal, find the value of n.

38. Find the middle term in the expansion of:

(i) $(3+x)^6$ (ii) $\left(\frac{x}{3}+3y\right)^8$

(iii) $\left(\frac{x}{a}-\frac{a}{x}\right)^{10}$ (iv) $\left(x^2-\frac{2}{x}\right)^{10}$

39. Find the two middle terms in the expansion of:

(i) $(x^2+a^2)^5$ (ii) $\left(x^4-\frac{1}{x^3}\right)^{11}$

(iii) $\left(\frac{p}{x}+\frac{x}{p}\right)^9$ (iv) $\left(3x-\frac{x^3}{6}\right)^9$

40. Find the term independent of x in the expansion of:

(i) $\left(2x+\frac{1}{3x^2}\right)^9$ (ii) $\left(\frac{3x^2}{2}-\frac{1}{3x}\right)^6$

(iii) $\left(x-\frac{1}{x^2}\right)^{3n}$ (iv) $\left(3x-\frac{2}{x^2}\right)^{15}$

41. Find the coefficient of x^5 in the expansion of $(1+x)^3(1-x)^6$.

42. Find numerically the greatest term in the expansion of $(2+3x)^9$, where

$x = \frac{3}{2}$.

43. If the coefficients of 2nd, 3rd and 4th terms in the expansion of $(1+x)^{2n}$ are in AP, show that $2n^2 - 9n + 7 = 0$.

44. Find the 6th term of the expansion $(y^{1/2} + x^{1/3})^n$, if the binomial coefficient of 3rd term from the end is 45.

45. If the 17th and 18th terms in the expansion of $(2+a)^{50}$ are equal, find the value of a.

46. Find the coefficient of x^4 in the expansion of $(1+x)^n(1-x)^n$. Deduce that $C_2 = C_0C_4 - C_1C_3 + C_2C_2 - C_3C_1 + C_4C_0$, where C_r stands for ${}^nC_r$.

47. Prove that the coefficient of x^n in the binomial expansion of $(1+x)^{2n}$ is twice the coefficient of x^n in the binomial expansion of $(1+x)^{2n-1}$.

48. If the middle term in the expansion of $\left(\frac{p}{2}+2\right)^8$ is 1120, find p.

ANSWERS (EXERCISE 10A)

1. $1 - 10x + 40x^2 - 80x^3 + 80x^4 - 32x^5$

2. $64x^6 - 576x^5 + 2160x^4 - 4320x^3 + 4860x^2 - 2916x + 729$

3. $243x^5 + 810x^4y + 1080x^3y^2 + 720x^2y^3 + 240xy^4 + 32y^5$

4. $16x^4 - 96x^3y + 216x^2y^2 - 216xy^3 + 81y^4$

5. $\frac{64x^6}{729} - \frac{32x^4}{27} + \frac{20x^2}{3} - 20 + \frac{135}{4x^2} - \frac{243}{8x^4} + \frac{729}{64x^6}$

6. $x^{14} - 14x^{11} + 84x^8 - 280x^5 + 560x^2 - \frac{672}{x} + \frac{448}{x^4} - \frac{128}{x^7}$

7. $x^5 - \frac{5x^4}{y} + \frac{10x^3}{y^2} - \frac{10x^2}{y^3} + \frac{5x}{y^4} - \frac{1}{y^5}$

8. $x^4 + 8x^{7/2}y^{1/2} + 28x^3y + 56x^{5/2}y^{3/2} + 70x^2y^2 + 56x^{3/2}y^{5/2} + 28xy^3 + 8x^{1/2}y^{7/2} + y^4$

9. $x^2 - 6x^{5/3}y^{1/3} + 15x^{4/3}y^{2/3} - 20xy + 15x^{2/3}y^{4/3} - 6x^{1/3}y^{5/3} + y^2$

10. $81x^8 - 216x^7 + 108x^6 + 120x^5 - 74x^4 - 40x^3 + 12x^2 + 8x + 1$

11. $\frac{x^4}{16} + \frac{x^3}{2} + \frac{x^2}{2} - 4x - 5 + \frac{16}{x} + \frac{8}{x^2} - \frac{32}{x^3} + \frac{16}{x^4}$

12. $27x^6 - 54ax^5 + 117a^2x^4 - 116a^3x^3 + 117a^4x^2 - 54a^5x + 27a^6$

13. 198 **14.** 152 **15.** 10084 **16.** $396\sqrt{6}$

18. (i) 104060401 (ii) 92236816 (iii) 2.0736

21. $\frac{4375}{x^4}$ **22.** $\frac{495b^4}{256a^{12}}$ **23.** $-136xy^{15/2}$ **24.** 18564

25. $n = 56$ **26.** 1 : 32 **28.** $\frac{17}{54}$ **29.** –19

30. (i) 1512 (ii) 378 (iii) $\frac{-40}{27}a^7$ (iv) -25344

34. $T_{r+1} = (-1)^r \times {}^6C_r \times x^{(12-2r)} \times y^r$ **35.** $495x^4$ **36.** $\frac{10500}{x^3}$

37. $n = 12$ **38.** (i) $540x^3$ (ii) $70x^4y^4$ (iii) -252 (iv) $-8064x^5$

39. (i) $10x^6a^4, 10x^4a^6$ (ii) $-462x^9, 462x^2$ (iii) $\frac{126p}{x}, \frac{126x}{p}$ (iv) $\frac{189x^{17}}{8}, \frac{-21x^{19}}{16}$

40. (i) $\frac{1792}{9}$ (ii) $\frac{5}{12}$ (iii) $(-1)^n \cdot {}^{3n}C_n$ (iv) $-3003 \times 3^{10} \times 2^5$

41. -6 **42.** $T_7 = \frac{7 \times 3^{13}}{2}$ **44.** $252y^{5/2}x^{5/3}$ **45.** $a = 1$

46. C_2 **48.** $p = \pm 2$

HINTS TO SOME SELECTED QUESTIONS

8. $$\begin{aligned}(x^{1/2}+y^{1/2})^8 &= {}^8C_0(x^{1/2})^8 + {}^8C_1(x^{1/2})^7(y^{1/2}) + {}^8C_2(x^{1/2})^6(y^{1/2})^2 + {}^8C_3(x^{1/2})^5(y^{1/2})^3 \\ &\quad + {}^8C_4(x^{1/2})^4(y^{1/2})^4 + {}^8C_5(x^{1/2})^3(y^{1/2})^5 + {}^8C_6(x^{1/2})^2(y^{1/2})^6 + {}^8C_7(x^{1/2})(y^{1/2})^7 \\ &\quad + {}^8C_8(y^{1/2})^8 \\ &= x^4 + 8x^{7/2}y^{1/2} + 28x^3y + 56x^{5/2}y^{3/2} + 70x^2y^2 + 56x^{3/2}y^{5/2} + 28xy^3 \\ &\quad + 8x^{1/2}y^{7/2} + y^4.\end{aligned}$$

9. $$\begin{aligned}(x^{1/3}-y^{1/3})^6 &= {}^6C_0(x^{1/3})^6 - {}^6C_1(x^{1/3})^5(y^{1/3}) + {}^6C_2(x^{1/3})^4(y^{1/3})^2 - {}^6C_3(x^{1/3})^3(y^{1/3})^3 \\ &\quad + {}^6C_4(x^{1/3})^2(y^{1/3})^4 - {}^6C_5(x^{1/3})(y^{1/3})^5 + {}^6C_6(y^{1/3})^6 \\ &= x^2 - 6x^{5/3}y^{1/3} + 15x^{4/3}y^{2/3} - 20xy + 15x^{2/3}y^{4/3} - 6x^{1/3}y^{5/3} + y^2.\end{aligned}$$

10. $$\begin{aligned}(1+2x-3x^2)^4 &= [1+(2x-3x^2)]^4 \\ &= (1+y)^4, \text{ where } 2x-3x^2 = y \\ &= {}^4C_0 + {}^4C_1y + {}^4C_2y^2 + {}^4C_3y^3 + {}^4C_4y^4 \\ &= 1 + 4y + 6y^2 + 4y^3 + y^4 \\ &= 1 + 4(2x-3x^2) + 6(2x-3x^2)^2 + 4(2x-3x^2)^3 + (2x-3x^2)^4 \\ &= 1 + (8x-12x^2) + 6[{}^2C_0(2x)^2 - {}^2C_1(2x)(3x^2) + {}^2C_2(3x^2)^2] \\ &\quad + 4[{}^3C_0(2x)^3 - {}^3C_1(2x)^2(3x^2) + {}^3C_2(2x)(3x^2)^2 - {}^3C_3(3x^2)^3] \\ &\quad + [{}^4C_0(2x)^4 - {}^4C_1(2x)^3(3x^2) + {}^4C_2(2x)^2(3x^2)^2 - {}^4C_3(2x)(3x^2)^3 + {}^4C_4(3x^2)^4] \\ &= 1 + (8x-12x^2) + 6(4x^2-12x^3+9x^4) + 4(8x^3-36x^4+54x^5-27x^6) \\ &\quad + (16x^4-96x^5+216x^6-216x^7+81x^8) \\ &= 1 + (8x-12x^2) + (24x^2-72x^3+54x^4) + (32x^3-144x^4+216x^5-108x^6) \\ &\quad + (16x^4-96x^5+216x^6-216x^7+81x^8) \\ &= 81x^8 - 216x^7 + 108x^6 + 120x^5 - 74x^4 - 40x^3 + 12x^2 + 8x + 1.\end{aligned}$$

11. $$\begin{aligned}\left(1+\frac{x}{2}-\frac{2}{x}\right)^4 &= \left\{1+\left(\frac{x}{2}-\frac{2}{x}\right)\right\}^4 \\ &= (1+y)^4, \text{ where } \left(\frac{x}{2}-\frac{2}{x}\right) = y\end{aligned}$$

$$= {}^4C_0 + {}^4C_1 y + {}^4C_2 y^2 + {}^4C_3 y^3 + {}^4C_4 y^4$$
$$= 1 + 4y + 6y^2 + 4y^3 + y^4$$
$$= 1 + 4\left(\frac{x}{2} - \frac{2}{x}\right) + 6\left(\frac{x}{2} - \frac{2}{x}\right)^2 + 4\left(\frac{x}{2} - \frac{2}{x}\right)^3 + \left(\frac{x}{2} - \frac{2}{x}\right)^4$$
$$= 1 + \left(2x - \frac{8}{x}\right) + 6\left(\frac{x^2}{4} + \frac{4}{x^2} - 2\right) + 4\left[{}^3C_0\left(\frac{x}{2}\right)^3 - {}^3C_1\left(\frac{x}{2}\right)^2\left(\frac{2}{x}\right) + {}^3C_2\left(\frac{x}{2}\right)\left(\frac{2}{x}\right)^2 - {}^3C_3\left(\frac{2}{x}\right)^3\right]$$
$$+\left[{}^4C_0\left(\frac{x}{2}\right)^4 - {}^4C_1\left(\frac{x}{2}\right)^3\left(\frac{2}{x}\right) + {}^4C_2\left(\frac{x}{2}\right)^2\left(\frac{2}{x}\right)^2 - {}^4C_3\left(\frac{x}{2}\right)\left(\frac{2}{x}\right)^3 + {}^4C_4\left(\frac{2}{x}\right)^4\right]$$
$$= 1 + \left(2x - \frac{8}{x}\right) + \left(\frac{3x^2}{2} + \frac{24}{x^2} - 12\right) + 4\left[\frac{x^3}{8} - \frac{3}{2}x + \frac{6}{x} - \frac{8}{x^3}\right] + \left[\frac{x^4}{16} - x^2 + 6 - \frac{16}{x^2} + \frac{16}{x^4}\right]$$
$$= 1 + \left(2x - \frac{8}{x}\right) + \left(\frac{3x^2}{2} + \frac{24}{x^2} - 12\right) + \left(\frac{x^3}{2} - 6x + \frac{24}{x} - \frac{32}{x^3}\right) + \left(\frac{x^4}{16} - x^2 + 6 - \frac{16}{x^2} + \frac{16}{x^4}\right)$$
$$= \frac{x^4}{16} + \frac{x^3}{2} + \frac{x^2}{2} - 4x - 5 + \frac{16}{x} + \frac{8}{x^2} - \frac{32}{x^3} + \frac{16}{x^4}.$$

12. $(3x^2 - 2ax + 3a^2)^3 = [(3x^2 - 2ax) + 3a^2]^3$
$$= {}^3C_0(3x^2 - 2ax)^3 + {}^3C_1(3x^2 - 2ax)^2(3a^2) + {}^3C_2(3x^2 - 2ax)(3a^2)^2 + {}^3C_3(3a^2)^3$$
$$= [{}^3C_0(3x^2)^3 - {}^3C_1(3x^2)^2 \times 2ax + {}^3C_2(3x^2) \times (2ax)^2 - {}^3C_3(2ax)^3]$$
$$+3(9x^4 + 4a^2x^2 - 12ax^3)(3a^2) + (27a^4)(3x^2 - 2ax) + 27a^6$$
$$= 27x^6 - 54ax^5 + 36a^2x^4 - 8a^3x^3 + 81a^2x^4 + 36a^4x^2 - 108a^3x^3 + 81a^4x^2 - 54a^5x + 27a^6$$
$$= 27x^6 - 54ax^5 + 117a^2x^4 - 116a^3x^3 + 117a^4x^2 - 54a^5x + 27a^6.$$

13. $(a+b)^6 + (a-b)^6 = 2[a^6 + {}^6C_2a^4b^2 + {}^6C_4a^2b^4 + {}^6C_6b^6]$.

$\therefore \quad (\sqrt{2}+1)^6 + (\sqrt{2}-1)^6 = 2[(\sqrt{2})^6 + {}^6C_2(\sqrt{2})^4 \times 1^2 + {}^6C_4(\sqrt{2})^2 \times 1^4 + {}^6C_6(1)^6]$
$$= 2(8 + 60 + 30 + 1) = 2(99) = 198.$$

16. $(a+b)^6 - (a-b)^6 = 2[{}^6C_1a^5b + {}^6C_3a^3b^3 + {}^6C_5ab^5]$.

$\therefore \quad (\sqrt{3}+\sqrt{2})^6 - (\sqrt{3}-\sqrt{2})^6 = 2[6(\sqrt{3})^5(\sqrt{2}) + 20(\sqrt{3})^3(\sqrt{2})^3 + 6(\sqrt{3})(\sqrt{2})^5]$
$$= 2[54\sqrt{6} + 120\sqrt{6} + 24\sqrt{6}] = 396\sqrt{6}.$$

17. LHS $= {}^nC_0 \times 3^0 + {}^nC_1 \times 3 + {}^nC_2 \times 3^2 + \ldots + {}^nC_n3^n$

$= (1+3)^n = 4^n =$ RHS [by binomial expansion].

18. (i) $(101)^4 = (100+1)^4$
$$= {}^4C_0(100)^4 + {}^4C_1(100)^3 \times 1 + {}^4C_2(100)^2 \times 1^2 + {}^4C_3(100) \times 1^3 + {}^4C_4 \times 1^4$$
$$= 100000000 + 4000000 + 60000 + 400 + 1 = 104060401.$$

19. $(1+x)^n = {}^nC_0 + {}^nC_1x + {}^nC_2x^2 + \ldots + {}^nC_nx^n$.

$(1+7)^n = {}^nC_0 + {}^nC_1 \times 7 + {}^nC_2 \times (7)^2 + \ldots + {}^nC_n \times (7)^n$
$$= 1 + 7n + (7)^2[{}^nC_2 + {}^nC_3 \times 7 + \ldots + (7)^{n-2}]$$

$\Rightarrow \quad (8^n - 7n - 1) = (7)^2 \times (\text{an integer}) = 49 \times (\text{an integer})$.

Hence, $(2^{3n} - 7n - 1)$ is divisible by 49.

20. $(a+b)^4 + (a-b)^4 = [{}^4C_0a^4 + {}^4C_1a^3b + {}^4C_2a^2b^2 + {}^4C_3ab^3 + {}^4C_4b^4]$
$$+[{}^4C_0a^4 - {}^4C_1a^3b + {}^4C_2a^2b^2 - {}^4C_3ab^3 + {}^4C_4b^4]$$
$$= 2[{}^4C_0a^4 + {}^4C_2a^2b^2 + {}^4C_4b^4] = 2[a^4 + 6a^2b^2 + b^4].$$

Putting $a = 2$ and $b = \sqrt{x}$, we get

$$(2+\sqrt{x})^4 + (2-\sqrt{x})^4 = 2[2^4 + 6 \times 2^2 \times (\sqrt{x})^2 + (\sqrt{x})^4] = 2(16 + 24x + x^2).$$

25. We have, $T_{r+1} = {}^nC_r 2^{n-r} \times \left(\frac{x}{3}\right)^r = {}^nC_r \times 2^{n-r} \times 3^{-r} \times x^r.$

Coeff. of x^7 = coeff. of $x^8 \Rightarrow {}^nC_7 \times 2^{n-7} \times 3^{-7} = {}^nC_8 \times 2^{n-8} \times 3^{-8}.$

$$\therefore \quad \frac{{}^nC_7}{{}^nC_8} = \frac{2^{n-8} \times 3^{-8}}{2^{n-7} \times 3^{-7}} = \frac{1}{2 \times 3} = \frac{1}{6} \Rightarrow \frac{n!}{(7!)(n-7)!} \times \frac{(8!) \times (n-8)!}{(n!)} = \frac{1}{6}.$$

$$\therefore \quad \frac{8}{n-8} = \frac{1}{6} \Rightarrow n - 8 = 48 \Rightarrow n = 56.$$

26. $T_{r+1} = {}^{15}C_r (x^2)^{15-r} \times \left(\frac{2}{x}\right)^r = {}^{15}C_r \times 2^r \times x^{30-3r}$

$[30 - 3r = 15 \Rightarrow 3r = 15 \Rightarrow r = 5]$ and $[30 - 3r = 0 \Rightarrow 3r = 30 \Rightarrow r = 10]$.

Find $(T_6 : T_{11}) = (T_{5+1} : T_{10+1})$.

28. In the expansion of $E = \left(\frac{3}{2}x^2 - \frac{1}{3x}\right)^9$, we have

$$T_{r+1} = (-1)^r \cdot {}^9C_r \cdot \left(\frac{3}{2}x^2\right)^{(9-r)} \left(\frac{1}{3x}\right)^r$$

$$\Rightarrow \quad T_{r+1} = (-1)^r \cdot {}^9C_r \cdot \frac{3^{(9-2r)}}{2^{(9-r)}} \cdot x^{(18-3r)}.$$

$$\therefore \quad (1 + x + 2x^3)\left[\left(a_0 \times \frac{1}{x^3} + a_1 \times \frac{1}{x} + a_2\right) \text{from } E\right]$$

$$= (1 + x + 2x^3)\left[\left\{(-1)^7 \cdot {}^9C_7 \cdot \frac{3^{-5}}{2^2} \times \frac{1}{x^3}\right\} + \left\{(-1)^6 \cdot {}^9C_6 \cdot \frac{3^{-3}}{2^3} \times x^0\right\}\right]$$

$$\begin{bmatrix} x = -1 \Rightarrow 18 - 3r = -1 \Rightarrow r \text{ is fraction} \\ 18 - 3r = 0 \Rightarrow r = 6 \text{ and } 18 - 3r = -3 \Rightarrow r = 7 \end{bmatrix}$$

$$= (1 + x + 2x^3)\left[\frac{-1}{27x^3} + \frac{7}{18}\right]$$

$$\therefore \quad \text{required term} = \left(\frac{-2}{27} + \frac{7}{18}\right) = \frac{17}{54}.$$

29. In the expansion of $E = (1 - x)^{16}$, we have

$$T_{r+1} = (-1)^r \cdot {}^{16}C_r x^r.$$

Product of given expressions $= (1 - 3x + 7x^2)(1 - 16x + \ldots)$.

Terms containing $x = [1 \times (-16x) + (-3x) \times 1] = (-19x)$.

$\therefore$ coefficient of $x = -19$.

30. (iii) $T_{r+1} = (-1)^r \times {}^{10}C_r (3x^2)^{(10-r)} \times \left(\frac{a}{3x^3}\right)^r$

$$\Rightarrow \quad T_{r+1} = (-1)^r \times {}^{10}C_r \times 3^{(10-2r)} \times x^{(20-5r)} \times a^r.$$

Now, $20 - 5r = -15 \Rightarrow 5r = 35 \Rightarrow r = 7$.

(iv) $T_{r+1} = (-1)^r \times {}^{12}C_r \times a^{(12-r)} \times (2b)^r$

$$\Rightarrow \quad T_{r+1} = (-1)^r \times {}^{12}C_r \times 2^r \times a^{(12-r)} b^r.$$

Put $r = 5$ and get T_6.

31. $T_{r+1} = (-1)^r \cdot {}^8C_r \cdot (3x)^{(8-r)} \cdot \left(\frac{1}{2x}\right)^r$

$$\Rightarrow \quad T_{r+1} = (-1)^r \cdot {}^8C_r \cdot 3^{(8-r)} \cdot \frac{1}{2^r} \cdot x^{(8-2r)}.$$

Now, $8-2r=3 \Rightarrow r=\frac{5}{2}$, which is a fraction.

34. $T_{r+1}=(-1)^r \times {}^6C_r \times x^{(12-2r)} \times y^r$.

35. pth term from the end = $(n-p+2)$th term from the beginning.

$\therefore$ 5th term from the end = $(12-5+2)$th = 9th term.

37. 7th term from the end = $(n-7+2)$th term = $(n-5)$th term.

$\therefore$ $T_7=T_{n-5} \Rightarrow n-5=7 \Rightarrow n=12$.

40. (iii) $T_{r+1}=(-1)^r \cdot {}^{3n}C_r x^{(3n-r)} \times \left(\frac{1}{x^2}\right)^r$

$\Rightarrow$ $T_{r+1}=(-1)^r \cdot {}^{3n}C_r x^{(3n-3r)}$.

Put $3n-3r=0 \Rightarrow r=n$.

So, T_{n+1} is free from x.

And, $T_{n+1}=(-1)^n \cdot {}^{3n}C_n$.

41. $$(1+x)^3(1-x)^6=({}^3C_0+{}^3C_1x+{}^3C_2x^2+{}^3C_3x^3) \times ({}^6C_0-{}^6C_1x+{}^6C_2x^2-{}^6C_3x^3+{}^6C_4x^4-{}^6C_5x^5+{}^6C_6x^6)$$
$$=(1+3x+3x^2+x^3)\times(1-6x+15x^2-20x^3+15x^4-6x^5+x^6).$$

Coeff. of x^5 in the given product $=(-6+45-60+15)=-6$.

42. We have, $(2+3x)^9=2^9\left(1+\frac{3x}{2}\right)^9$.

$$T_{r+1}=2^9\times {}^9C_r\left(\frac{3x}{2}\right)^r \text{ and } T_r=2^9\times {}^9C_{r-1}\left(\frac{3x}{2}\right)^{r-1}.$$

$$\therefore \quad \frac{T_{r+1}}{T_r}=2^9\times {}^9C_r\times\frac{3^r}{2^r}\times x^r\times\frac{2^{r-1}}{3^{r-1}}\times\frac{1}{x^{r-1}}\times\frac{1}{2^9\times {}^9C_{r-1}}$$
$$=\frac{3}{2}\times x\times\frac{{}^9C_r}{{}^9C_{r-1}}=\left(\frac{3}{2}x\right)\times\frac{(9\,!)}{(r\,!)\cdot(9-r)\,!}\times\frac{(r-1)\,!}{(9\,!)}\times(10-r)\,!$$
$$=\left(\frac{3}{2}\times\frac{3}{2}\right)\times\frac{(10-r)}{r}=\frac{9(10-r)}{4r} \qquad \left[\because x=\frac{3}{2}\right].$$

Let the greatest term be T_{r+1}.

Then, $T_{r+1}\ge T_r \Rightarrow \frac{T_{r+1}}{T_r}\ge 1 \Rightarrow \frac{9(10-r)}{4r}\ge 1$

$$\Rightarrow 4r\le 90-9r \Rightarrow 13r\le 90 \Rightarrow r\le 6\tfrac{12}{13}.$$

Thus, the maximum value of r is 6 for which T_{r+1} is greatest.

$\therefore$ greatest term $=T_{6+1}=2^9\times {}^9C_6\times\frac{3^6}{2^6}\times x^6$, where $x=\frac{3}{2}$.

$\therefore$ numerical value of greatest term at $x=\frac{3}{2}=2^9\times {}^9C_3\times\frac{3^6}{2^6}\times\frac{3^6}{2^6}$

$$=\frac{9\times8\times7}{3\times2\times1}\times\frac{3^{12}}{2^3}=\left(\frac{7\times3^{13}}{2}\right).$$

43. It is given that ${}^{2n}C_1$, ${}^{2n}C_2$ and ${}^{2n}C_3$ are in AP.

$\therefore$ $2\times {}^{2n}C_2={}^{2n}C_1+{}^{2n}C_3$

$\Rightarrow \quad 2 = \dfrac{{}^{2n}C_1}{{}^{2n}C_2} + \dfrac{{}^{2n}C_3}{{}^{2n}C_2}$

$\Rightarrow \quad 2 = \dfrac{4n}{2n(2n-1)} + \dfrac{2n-3+1}{3}$ $\quad \left[\because \dfrac{{}^{2n}C_r}{{}^{2n}C_{r-1}} = \dfrac{2n-r+1}{r} \text{ and } r=3\right]$

$\Rightarrow \quad \dfrac{2}{2n-1} + \dfrac{2n-2}{3} = 2$

$\Rightarrow \quad 6 + (2n-1)(2n-2) = 6(2n-1)$

$\Rightarrow \quad 4n^2 - 18n + 14 = 0 \Rightarrow 2n^2 - 9n + 7 = 0.$

44. Coefficient of 3rd term from the beginning is always equal to the coefficient of 3rd term from the end.

Now, $T_{r+1} = {}^nC_r (y^{1/2})^{n-r} \cdot (x^{1/3})^r$.

Coefficient of $T_3 = {}^nC_2 = \dfrac{n(n-1)}{2}$.

$\therefore \quad \dfrac{n(n-1)}{2} = 45 \Rightarrow n^2 - n - 90 = 0 \Rightarrow n^2 - 10n + 9n - 90 = 0$

$\Rightarrow (n-10)(n+9) = 0 \Rightarrow n = 10$ $\quad [\because n \neq -9].$

$\therefore \quad T_6 = T_{5+1} = {}^{10}C_5 (y^{1/2})^{(10-5)} \cdot (x^{1/3})^5$

$= \dfrac{10\times 9\times 8\times 7\times 6}{5\times 4\times 3\times 2\times 1} \times y^{5/2} x^{5/3} = 252 y^{5/2} x^{5/3}.$

45. $T_{r+1} = {}^{50}C_r \times 2^{(50-r)} \times a^r.$

Now, $T_{17} = T_{18} \Rightarrow T_{16+1} = T_{17+1}$

$\Rightarrow {}^{50}C_{16} \times 2^{34} \times a^{16} = {}^{50}C_{17} \times 2^{33} \times a^{17}$

$\Rightarrow a = \dfrac{2\times {}^{50}C_{16}}{{}^{50}C_{17}} = \dfrac{2\times 17}{(50-17+1)} = 1.$ $\quad \left[\because \dfrac{{}^nC_{r-1}}{{}^nC_r} = \dfrac{r}{n-r+1}\right]$

46. $(1+x)^n \times (1-x)^n = [C_0 + C_1 x + C_2 x^2 + \ldots + C_n x^n] \times [C_0 - C_1 x + C_2 x^2 - \ldots + (-1)^n C_n x^n]$

$(1+x)^n \times (1-x)^n = (1-x^2)^n = [C_0 - C_1 x^2 + C_2 x^4 - \ldots + (-1)^n C_n x^{2n}].$

$\therefore \quad C_0C_4 - C_1C_3 + C_2C_2 - C_3C_1 + C_4C_0 = C_2.$

47. In $(1+x)^{2n}$, we have $T_{r+1} = {}^{2n}C_r \cdot x^r$.

$\therefore \quad$ coefficient of x^n in $(1+x)^{2n} = {}^{2n}C_n = \dfrac{(2n)!}{(n!)(n!)} = \dfrac{(2n)(2n-1)!}{n(n-1)!\times(n!)}$

$= 2\times \dfrac{(2n-1)!}{(n-1)!\times(n!)}.$

In $(1+x)^{2n-1}$, we have $T_{r+1} = {}^{2n-1}C_r x^r$.

$\therefore \quad$ coeff. of x^n in $(1+x)^{2n-1} = {}^{2n-1}C_n = \dfrac{(2n-1)!}{(n-1)!\times n!}.$

Hence, coeff. of x^n in $(1+x)^{2n} = 2\times$ coeff. of x^n in $(1+x)^{2n-1}$.

48. T_{r+1} in $\left(\dfrac{p}{2}+2\right)^8 = {}^8C_r \left(\dfrac{p}{2}\right)^{8-r} \cdot 2^r.$

Middle term $= \left(\dfrac{8}{2}+1\right)$th $=$ 5th term $= {}^8C_4 \left(\dfrac{p}{2}\right)^4 \times 2^4.$

$\therefore \quad \dfrac{8\times 7\times 6\times 5}{4\times 3\times 2\times 1} \times p^4 = 1120 \Rightarrow p^4 = 16 \Rightarrow p = \pm 2.$

EXERCISE 10B

Very-Short-Answer Questions

1. Show that the term independent of x in the expansion of $\left(x-\frac{1}{x}\right)^{10}$ is -252.
2. If the coefficients of x^2 and x^3 in the expansion of $(3+px)^9$ are the same then prove that $p=\frac{9}{7}\cdot$
3. Show that the coefficient of x^{-3} in the expansion of $\left(x-\frac{1}{x}\right)^{11}$ is -330.
4. Show that the middle term in the expansion of $\left(\frac{2x^2}{3}+\frac{3}{2x^2}\right)^{10}$ is 252.
5. Show that the coefficient of x^4 in the expansion of $\left(\frac{x}{2}-\frac{3}{x^2}\right)^{10}$ is $\frac{405}{256}\cdot$
6. Prove that there is no term involving x^6 in the expansion of $\left(2x^2-\frac{3}{x}\right)^{11}\cdot$
7. Show that the coefficient of x^4 in the expansion of $(1+2x+x^2)^5$ is 212.
8. Write the number of terms in the expansion of $(\sqrt{2}+1)^5+(\sqrt{2}-1)^5$.
9. Which term is independent of x in the expansion of $\left(x-\frac{1}{3x^2}\right)^9$?
10. Write the coefficient of the middle term in the expansion of $(1+x)^{2n}$.
11. Write the coefficient of x^7y^2 in the expansion of $(x+2y)^9$.
12. If the coefficients of $(r-5)$th and $(2r-1)$th terms in the expansion of $(1+x)^{34}$ are equal, find the value of r.
13. Write the 4th term from the end in the expansion of $\left(\frac{3}{x^2}-\frac{x^3}{6}\right)^7\cdot$
14. Find the coefficient of x^n in the expansion of $(1+x)(1-x)^n$.
15. In the binomial expansion of $(a+b)^n$, the coefficients of the 4th and 13th terms are equal to each other. Find the value of n.
16. Find the positive value of m for which the coefficient of x^2 in the expansion of $(1+x)^m$ is 6.

ANSWERS (EXERCISE 10B)

8. Three **9.** 4th **10.** ${}^{2n}C_n$ **11.** 144 **12.** 14

13. $\frac{35x^6}{48}$ **14.** $(-1)^n\times(n-1)$ **15.** $n=15$ **16.** $m=4$

HINTS TO SOME SELECTED QUESTIONS

1. $T_{r+1} = (-1)^r \cdot {}^{10}C_r x^{(10-r)} \cdot \left(\frac{1}{x}\right)^r = (-1)^r \cdot {}^{10}C_r x^{(10-2r)}$.

Putting $10 - 2r = 0$, we get $r = 5$.

$\therefore \quad T_6 = (-1)^5 \cdot {}^{10}C_5 = -\frac{10 \times 9 \times 8 \times 7 \times 6}{5 \times 4 \times 3 \times 2 \times 1} = -252$.

2. $T_{r+1} = {}^9C_r 3^{(9-r)} \times (px)^r = {}^9C_r 3^{(9-r)} \cdot p^r x^r$.

Coeff. of x^2 = coeff. of $x^3 \Rightarrow {}^9C_2 \times 3^7 \times p^2 = {}^9C_3 \times 3^6 \times p^3$.

3. $T_{r+1} = (-1)^r \cdot {}^{11}C_r x^{(11-r)} \cdot \left(\frac{1}{x}\right)^r = (-1)^r \cdot {}^{11}C_r x^{(11-2r)}$.

Now, $11 - 2r = -3 \Rightarrow 2r = 14 \Rightarrow r = 7$.

$T_8 = (-1)^7 \cdot {}^{11}C_7 x^{-3} = -{}^{11}C_4 x^{-3} = -\frac{11 \times 10 \times 9 \times 8}{4 \times 3 \times 2 \times 1} x^{-3} = -330x^{-3}$.

4. $T_{r+1} = {}^{10}C_r \left(\frac{2x^2}{3}\right)^{10-r} \times \left(\frac{3}{2x^2}\right)^r = {}^{10}C_r \times 2^{10-2r} \times 3^{2r-10} \times x^{(20-4r)}$.

The expansion has 11 terms. So, middle term = 6th term.

$T_6 = T_{5+1} = {}^{10}C_5 \times 2^0 \times 3^0 \times x^0 = \left(\frac{10 \times 9 \times 8 \times 7 \times 6}{5 \times 4 \times 3 \times 2 \times 1}\right) = 252$.

5. $T_{r+1} = (-1)^r \cdot {}^{10}C_r \cdot \left(\frac{x}{2}\right)^{(10-r)} \times \left(\frac{3}{x^2}\right)^r = (-1)^r \cdot {}^{10}C_r \cdot 3^r \cdot 2^{r-10} \cdot x^{(10-3r)}$.

Putting $10 - 3r = 4$, we get $3r = 6 \Rightarrow r = 2$.

$T_3 = T_{(2+1)} = (-1)^2 \cdot {}^{10}C_2 \times 3^2 \times 2^{-8} \times x^4 = \frac{405}{256} x^4$.

6. $T_{r+1} = (-1)^r \cdot {}^{11}C_r (2x^2)^{(11-r)} \cdot \left(\frac{3}{x}\right)^r = (-1)^r \cdot {}^{11}C_r 2^{(11-r)} \times 3^r \cdot x^{22-3r}$

$22 - 3r = 6 \Rightarrow 3r = 16 \Rightarrow r = \frac{16}{3}$, which is a fraction.

7. $(1 + 2x + x^2)^5 = \{(1+x)^2\}^5 = (1+x)^{10}$.

$T_{r+1} = {}^{10}C_r x^r$. Coefficient of $x^4 = {}^{10}C_4 = \frac{10 \times 9 \times 8 \times 7}{4 \times 3 \times 2 \times 1} = 210$.

8. Each expansion gives 6 terms. Their sum will contain 3 terms, as 3 terms are cancelled.

9. $T_{r+1} = (-1)^r \cdot {}^9C_r \cdot x^{9-r} \cdot \left(\frac{1}{3x^2}\right)^r = (-1)^r \cdot {}^9C_r \cdot x^{(9-3r)} \cdot \frac{1}{3^r} \cdot$

Putting $9 - 3r = 0$, we get $r = 3$.

Hence, T_4, i.e., 4th term is independent of x.

10. Middle term $= \left(\frac{2n}{2} + 1\right)$th term $= (n+1)$th term $= T_{n+1}$.

Now, $T_{r+1} = {}^{2n}C_r x^r \Rightarrow T_{n+1} = {}^{2n}C_n x^n$. Its coefficient is ${}^{2n}C_n$.

11. $T_{r+1} = {}^9C_r x^{(9-r)} \cdot (2y)^r = {}^9C_r \times 2^r \times x^{(9-r)} \times y^r$.

Putting $r = 2$, we get the coefficient of x^7y^2 as ${}^9C_2 \times 2^2 = 144$.

12. $T_m = T_{(m-1)+1} = {}^{34}C_{m-1} \times x^{m-1}$.

$\therefore \quad T_{r-5} = {}^{34}C_{r-6} x^{r-6}$ and $T_{2r-1} = {}^{34}C_{2r-2} x^{2r-2}$.

Now, ${}^{34}C_{r-6} = {}^{34}C_{2r-2} \Rightarrow (r - 6 = 2r - 2)$ or $(r - 6 + 2r - 2 = 34)$

$\Rightarrow r = -4$ or $r = 14 \Rightarrow r = 14$.

13. rth term from the end $= (n - r + 2)$th term.

$\therefore \quad$ 4th term from the end $= (7 - 4 + 2)$th term = 5th term.

$T_{r+1} = (-1)^r \cdot {}^7C_r \left(\frac{3}{4x^2}\right)^{(7-r)} \cdot \left(\frac{x^3}{6}\right)^r = (-1)^r \times {}^7C_r \times 3^{(7-2r)} \times 2^{-r} \times x^{5r-14}.$

$T_5 = T_{(4+1)} = (-1)^4 \times {}^7C_4 \times 3^{-1} \times 2^{-4} \times x^6 = {}^7C_3 \times \frac{1}{(3 \times 2^4)} x^6 = \frac{35x^6}{48}.$

14. T_{r+1} in $(1-x)^n = (-1)^{r+1} \cdot {}^nC_r \cdot x^r$.

$(1+x)(1-x)^n = (1+x)[1 - {}^nC_1 x + \ldots + (-1)^n \times {}^nC_{n-1} x^{n-1} + (-1)^{n+1} \cdot {}^nC_n x^n]$.

Coeff. of x^n in the given product $(1+x)(1-x)^n = [(-1)^{n+1} \cdot {}^nC_n + (-1)^n \cdot {}^nC_{n-1}]$

$= (-1)^{n+1} + (-1)^n \times n = (-1)^n \times (n-1)$.

15. $T_{r+1} = {}^nC_r a^{n-r} \cdot b^r$.

Coeff. of T_4 = coeff. of $T_{13} \Rightarrow {}^nC_3 = {}^nC_{12} \Rightarrow (3+12) = n \Rightarrow n = 15$.

16. $(1+x)^m = {}^mC_0 + {}^mC_1 x + {}^mC_2 x^2 + \ldots + {}^mC_m x^m$.

$\therefore \quad {}^mC_2 = 6 \Rightarrow \frac{m(m-1)}{2!} = 6 \Rightarrow m^2 - m - 12 = 0 \Rightarrow (m-4)(m+3) = 0 \Rightarrow m = 4$.

REVIEW OF FACTS AND FORMULAE

1. Binomial Theorem

If x and a are real numbers then for all $n \in N$, we have

$$(x+a)^n = {}^nC_0 x^n a^0 + {}^nC_1 x^{n-1} a + {}^nC_2 x^{n-2} a^2 + \ldots + {}^nC_r x^{n-r} a^r + \ldots + {}^nC_n x^0 a^n$$

$$\Rightarrow \quad (x+a)^n = \sum_{r=0}^{n} {}^nC_r x^{n-r} a^r.$$

In this expansion, we have

(i) total number of terms $= (n+1)$.

(ii) sum of the powers of x and a in each term is n.

(iii) coefficients of terms equidistant from the beginning and the end are equal.

(iv) general term is given by $T_{r+1} = {}^nC_r x^{n-r} a^r$.

2. $(1+x)^n = {}^nC_0 + {}^nC_1 \times x + {}^nC_2 \times x^2 + \ldots + {}^nC_r \times x^r + \ldots + {}^nC_n x^n$

$$\Rightarrow \quad (1+x)^n = \sum_{r=0}^{n} {}^nC_r x^r.$$

In this expansion, we have

(i) total number of terms $= (n+1)$.

(ii) general term, $T_{r+1} = {}^nC_r x^r$.

3. $(x-a)^n = {}^nC_0 x^n - {}^nC_1 x^{n-1} a + {}^nC_2 x^{n-2} a^2 - \ldots + (-1)^r \cdot {}^nC_r x^{n-r} a^r + \ldots + (-1)^n \cdot {}^nC_n x^n$

$$\Rightarrow \quad (x-a)^n = \sum_{r=0}^{n} (-1)^r \cdot {}^nC_r x^{n-r} a^r.$$

In this expansion, we have

(i) total number of terms $= (n+1)$.

(ii) sum of the powers of x and a in each term is n.

(iii) general term is given by $T_{r+1} = (-1)^r \cdot {}^nC_r x^{n-r} a^r$.

4. $(1-x)^n = {}^nC_0 - {}^nC_1 x + {}^nC_2 x^2 - {}^nC_3 x^3 + \ldots + (-1)^r \cdot {}^nC_r x^r + \ldots + (-1)^n \cdot {}^nC_n x^n$

$\Rightarrow \quad (1-x)^n = \sum_{r=0}^{n} (-1)^r \cdot {}^nC_r x^r.$

In this expansion, we have

(i) total number of terms $= (n+1)$.

(ii) general term, $T_{r+1} = (-1)^r \cdot {}^nC_r x^r$.

5. (i) $(x+a)^n + (x-a)^n = 2[{}^nC_0 x^n a^0 + {}^nC_2 x^{n-2} a^2 + {}^nC_4 x^{n-r} a^4 + \ldots]$.

(ii) $(x+a)^n - (x-a)^n = 2[{}^nC_1 x^{n-1} a + {}^nC_3 x^{n-3} a^3 + {}^nC_5 x^{n-5} a^5 + \ldots]$.

6. Middle Terms

(i) If n is even then $\left(\frac{n}{2}+1\right)$th term is the middle term.

(ii) If n is odd then $\frac{1}{2}(n+1)$th and $\frac{1}{2}(n+3)$th terms are the middle terms.

❑

11
Arithmetic Progression

ARITHMETIC PROGRESSION

SEQUENCE *A succession of numbers arranged in a definite order according to a certain given rule is called a sequence.*

The number occurring at the nth place of a sequence is called its *nth term* or the *general term*, to be denoted by a_n.

A sequence is said to be finite or infinite according as the number of terms in it is finite or infinite respectively.

By adding the terms of a sequence, we get a series.

A series is said to be finite or infinite according as the number of terms in it is finite or infinite respectively.

EXAMPLE 1 *Write first five terms of the sequence given by the rule* $a_n = (2n+1)$ *and obtain the corresponding series.*

SOLUTION We have, $a_n = (2n+1)$. ... (i)

Putting $n = 1, 2, 3, 4, 5, \ldots$ successively in (i), we get

$$a_1 = (2\times 1+1) = 3;\ a_2 = (2\times 2+1) = 5;\ a_3 = (2\times 3+1) = 7;$$
$$a_4 = (2\times 4+1) = 9 \quad \text{and} \quad a_5 = (2\times 5+1) = 11.$$

Hence, the required sequence is 3, 5, 7, 9, 11, ...

The corresponding series is $3+5+7+9+11+\ldots$.

EXAMPLE 2 *Write first four terms of the sequence given by* $a_n = \frac{1}{6}(2n-3)$ *and obtain the corresponding series.*

SOLUTION We have, $a_n = \frac{1}{6}(2n-3)$. ... (i)

Putting $n = 1, 2, 3, 4, \ldots$ successively in (i), we get

$$a_1 = \frac{1}{6}(2\times 1-3) = \frac{-1}{6};\ a_2 = \frac{1}{6}(2\times 2-3) = \frac{1}{6};$$
$$a_3 = \frac{1}{6}(2\times 3-3) = \frac{3}{6} = \frac{1}{2};\ a_4 = \frac{1}{6}(2\times 4-3) = \frac{5}{6}.$$

Hence, the required sequence is $\frac{-1}{6}, \frac{1}{6}, \frac{1}{2}, \frac{5}{6}, \ldots$.

The corresponding series is $-\frac{1}{6}+\frac{1}{6}+\frac{1}{2}+\frac{5}{6}+\ldots$.

EXAMPLE 3 *The Fibonacci sequence is defined by*

$$1 = a_1 = a_2 \text{ and } a_n = a_{n-1} + a_{n-2},\ n > 2.$$

Find $\dfrac{a_{n+1}}{a_n}$ *for* $n = 1, 2, 3, 4, 5.$

SOLUTION We have, $a_1 = 1$, $a_2 = 1$ and $a_n = a_{n-1} + a_{n-2}$ for $n > 2$.

$$\therefore \quad a_3 = (a_2 + a_1) = (1+1) = 2,\ a_4 = (a_3 + a_2) = (2+1) = 3,$$

$$a_5 = (a_4 + a_3) = (3+2) = 5.$$

$$\therefore \quad \frac{a_2}{a_1} = 1,\ \frac{a_3}{a_2} = \frac{2}{1} = 2,\ \frac{a_4}{a_3} = \frac{3}{2} \text{ and } \frac{a_5}{a_4} = \frac{5}{3}.$$

PROGRESSIONS

Sequences following certain patterns are called progressions.

ARITHMETIC PROGRESSION (AP)

It is a sequence in which each term except the first one differs from its preceding term by a constant.

This constant difference is called the common difference of the AP.

In an AP we usually denote the *first term* by a, the *common difference* by d and the *nth term* by T_n.

SOME EXAMPLES OF AP

EXAMPLE 1 *Show that the sequence defined by* $T_n = 3n + 5$ *is an AP. Find its common difference.*

SOLUTION We have, $T_n = 3n + 5$. ... (i)

Replacing n by $(n-1)$ in (i), we get

$$T_{n-1} = 3(n-1) + 5 \Rightarrow T_{n-1} = 3n + 2. \quad \text{... (ii)}$$

Subtracting (ii) from (i), we get

$$(T_n - T_{n-1}) = (3n+5) - (3n+2) = 3, \text{ which is constant.}$$

Hence, the given sequence is an AP with common difference 3.

EXAMPLE 2 *Show that the sequence* $\log a, \log\left(\frac{a^2}{b}\right), \log\left(\frac{a^3}{b^2}\right), \log\left(\frac{a^4}{b^3}\right), \ldots$ *forms an AP. Find its common difference.*

SOLUTION By symmetry, we find that

$$T_n = \log\left(\frac{a^n}{b^{n-1}}\right) \text{ and } T_{n-1} = \log\left(\frac{a^{n-1}}{b^{n-2}}\right).$$

$$\therefore \quad (T_n - T_{n-1}) = \log\left(\frac{a^n}{b^{n-1}}\right) - \log\left(\frac{a^{n-1}}{b^{n-2}}\right)$$

$$= \log\left(\frac{a^n}{b^{n-1}} \times \frac{b^{n-2}}{a^{n-1}}\right) = \log\left(\frac{a}{b}\right) = \text{constant.}$$

Hence, the given sequence in an AP with common difference $\log\left(\frac{a}{b}\right)$.

EXAMPLE 3 *Show that the sequence defined by $T_n = 3n^2 + 2$ is not an AP.*

SOLUTION We have

$$T_n = 3n^2 + 2 \text{ and } T_{n-1} = 3(n-1)^2 + 2 \Rightarrow T_{n-1} = 3n^2 - 6n + 5.$$

$$\therefore \quad (T_n - T_{n-1}) = (3n^2 + 2) - (3n^2 - 6n + 5) \Rightarrow (T_n - T_{n-1}) = (6n - 3).$$

This shows that $(T_n - T_{n-1})$ is not independent of n and therefore, it is not constant.

Hence, the given sequence is not an AP.

GENERAL TERM OF AN AP

THEOREM 1 *Show that the nth term of an AP with first term a and common difference d is given by $T_n = a + (n-1)d$.*

PROOF Let us consider an AP with first term a and common difference d.

Then, the given AP is

$$a, (a+d), (a+2d), (a+3d), (a+4d), \ldots .$$

In this AP, we have

first term, $T_1 = a = (a + 0 \times d) = a + (1-1)d$;

second term, $T_2 = (a+d) = a + (2-1)d$;

third term, $T_3 = (a+2d) = a + (3-1)d$;

…… …… …… …… ……

…… …… …… …… ……

nth term, $T_n = a + (n-1)d$.

Hence, $T_n = a + (n-1)d$.

This is called the general term of the AP.

In an AP with first term = a and common difference = d, we have $T_n = a + (n-1)d$.

SOME SIMPLE PROPERTIES OF AN AP

(i) If a constant is added to each term of an AP then the resulting progression is an AP.

(ii) If a constant is subtracted from each term of an AP then the resulting progression is an AP.

(iii) If each term of an AP is multiplied by the same nonzero number then the resulting progression is an AP.

(iv) If each term of an AP is divided by the same nonzero number then the resulting progression is an AP.

SOLVED EXAMPLES

EXAMPLE 1 *Show that the progression 7, 12, 17, 22, 27, … is an AP. Find its general term and the 14th term.*

SOLUTION We have

$$(12-7)=(17-12)=(22-17)=(27-22)=5, \text{ which is constant.}$$

So, the given progression is an AP in which $a=7$ and $d=5$.

Its general term, $T_n=\{a+(n-1)d\}=7+(n-1)\times 5 \Rightarrow T_n=(5n+2)$.

$\therefore$ 14th term, $T_{14}=(5\times 14+2)=72$.

EXAMPLE 2 *Show that the progression* $\log a, \log(ab), \log(ab^2), \log(ab^3), \ldots$ *is an AP. Find its general term and the* 10*th term.*

SOLUTION We have

$$\log(ab)-\log a=\log a+\log b-\log a=\log b,$$

$$\log(ab^2)-\log(ab)=(\log a+2\log b)-(\log a+\log b)=\log b,$$

$$\log(ab^3)-\log(ab^2)=(\log a+3\log b)-(\log a+2\log b)=\log b.$$

$\therefore$ $(T_2-T_1)=(T_3-T_2)=(T_4-T_3)=\log b=\text{constant}$.

So, the given progression is an AP.

Let A be the first term and d be the common difference of this AP.

Then, $A=\log a$ and $d=\log b$.

$\therefore$ general term, $T_n=A+(n-1)d$

$\Rightarrow$ $T_n=\log a+(n-1)\log b$.

And, 10th term, $T_{10}=\log a+(10-1)\log b=\log a+9\log b$

$\Rightarrow$ $T_{10}=\log a+\log(b^9) \Rightarrow T_{10}=\log(ab^9)$.

EXAMPLE 3 *Show that the progression* 21, 16, 11, 6, 1, ... *is an AP. Write its first term and the common difference. Which term of this AP is* –54?

SOLUTION The given progression is 21, 16, 11, 6, 1, ...

We have

$$(16-21)=(11-16)=(6-11)=(1-6)=-5, \text{ which is constant.}$$

So, the given progression is an AP.

Its first term, $a=21$ and common difference, $d=-5$.

Let its nth term be –54. Then,

$$T_n=-54 \Rightarrow a+(n-1)d=-54$$
$$\Rightarrow 21+(n-1)\times(-5)=-54$$
$$\Rightarrow -5n+26=-54 \Rightarrow 5n=80 \Rightarrow n=16.$$

$\therefore$ $T_{16}=-54$.

Hence, the 16th term of the given AP is –54.

EXAMPLE 4 *Show that the progression* $-11, -7, -3, 1, 5, \ldots, 161$ *is an AP. How many terms does it have?*

SOLUTION We have

$$(-7)-(-11)=(-7+11)=4, (-3)-(-7)=(-3+7)=4,$$
$$1-(-3)=(1+3)=4 \text{ and } (5-1)=4.$$

$\therefore$ $\{(-7)-(-11)\}=\{(-3)-(-7)\}=\{1-(-3)\}=(5-1)=4$, which is constant.

So, the given progression is an AP in which $a = -11$ and $d = 4$.

Let it have n terms. Then,

$$T_n = 161 \Rightarrow a + (n-1)d = 161$$
$$\Rightarrow (-11) + (n-1) \times 4 = 161$$
$$\Rightarrow 4n = 176 \Rightarrow n = 44.$$

$\therefore$ $T_{44} = 161$ and hence there are 44 terms in the given AP.

EXAMPLE 5 *Is 319 a term of the AP* 11, 17, 23, 29, 35, …?

SOLUTION In the given AP, we have $a = 11$ and $d = (17 - 11) = 6$.

If possible, let the nth term of this AP be 319. Then,

$$T_n = 319 \Rightarrow a + (n-1)d = 319$$
$$\Rightarrow 11 + (n-1) \times 6 = 319$$
$$\Rightarrow 6n + 5 = 319 \Rightarrow 6n = 314$$
$$\Rightarrow 3n = 157 \Rightarrow n = \frac{157}{3} = 52\frac{1}{3}.$$

Since the number of terms cannot be a fraction, so it follows that 319 is not a term of the given AP.

EXAMPLE 6 *Which term of the AP* $30, 29\frac{1}{4}, 28\frac{1}{2}, 27\frac{3}{4}, \cdots$ *is the first negative term?*

SOLUTION We have

$$\left(\frac{117}{4} - 30\right) = \left(\frac{57}{2} - \frac{117}{4}\right) = \left(\frac{111}{4} - \frac{57}{2}\right) = \frac{-3}{4}.$$

$\therefore$ $a = 30$ and $d = \frac{-3}{4}$.

Let its nth term be the first negative term. Then,

$$T_n < 0 \Rightarrow a + (n-1)d < 0$$
$$\Rightarrow 30 + (n-1) \times \left(\frac{-3}{4}\right) < 0$$
$$\Rightarrow 123 - 3n < 0 \Rightarrow 123 < 3n$$
$$\Rightarrow 3n > 123 \Rightarrow n > 41.$$

Hence, the 42nd term of the given AP is the first negative term.

And, $T_{42} = a + (42-1)d = 30 + 41 \times \left(\frac{-3}{4}\right) = \frac{-3}{4}.$

Hence, the first negative term of the given AP is $\frac{-3}{4}$.

EXAMPLE 7 *Find the 10th common term between the arithmetic series* $3 + 7 + 11 + 15 + \ldots$ *and* $1 + 6 + 11 + 16 + \ldots$.

SOLUTION Clearly, the first common term in the two series is 11.

LCM of the common differences of the two series = LCM (4, 5) = 20.

Let us now consider an AP in which $a = 11$ and $d = 20$.

Every term of this AP is a common term of the two given series.

So, the required 10th common term

$$= \{11 + (10-1) \times 20\} = (11 + 180) = 191.$$

Hence, 191 is the 10th common term of the two given series.

EXAMPLE 8 *If* $a_1, a_2, a_3, \ldots, a_n$ *are in AP with common difference* d *(where* $d \neq 0$*) then prove that the sum of the series* $\sin d\,(\text{cosec}\, a_1 \,\text{cosec}\, a_2 + \text{cosec}\, a_2 \,\text{cosec}\, a_3 + \ldots + \text{cosec}\, a_{n-1} \,\text{cosec}\, a_n)$ *is equal to* $(\cot a_1 - \cot a_n)$.

SOLUTION Since $a_1, a_2, a_3, \ldots, a_n$ are in AP with common difference d, we have

$$(a_2 - a_1) = (a_3 - a_2) = (a_4 - a_3) = \ldots = (a_n - a_{n-1}) = d.$$

$$\therefore \quad \sin d\,(\text{cosec}\, a_1 \,\text{cosec}\, a_2 + \text{cosec}\, a_2 \,\text{cosec}\, a_3 + \ldots + \text{cosec}\, a_{n-1} \,\text{cosec}\, a_n)$$

$$= \frac{\sin d}{\sin a_1 \sin a_2} + \frac{\sin d}{\sin a_2 \sin a_3} + \frac{\sin d}{\sin a_3 \sin a_4} + \ldots + \frac{\sin d}{\sin a_{n-1} \sin a_n}$$

$$= \frac{\sin (a_2 - a_1)}{\sin a_1 \sin a_2} + \frac{\sin (a_3 - a_2)}{\sin a_2 \sin a_3} + \frac{\sin (a_4 - a_3)}{\sin a_3 \sin a_4} + \ldots + \frac{\sin (a_n - a_{n-1})}{\sin a_{n-1} \sin a_n}$$

$$[\because d = (a_2 - a_1) = (a_3 - a_2) = \ldots = (a_n - a_{n-1})]$$

$$= \frac{\sin a_2 \cos a_1 - \cos a_2 \sin a_1}{\sin a_1 \sin a_2} + \frac{\sin a_3 \cos a_2 - \cos a_3 \sin a_2}{\sin a_2 \sin a_3} + \ldots + \frac{\sin a_n \cos a_{n-1} - \cos a_n \sin a_{n-1}}{\sin a_{n-1} \sin a_n}$$

$$= \left(\frac{\cos a_1}{\sin a_1} - \frac{\cos a_2}{\sin a_2}\right) + \left(\frac{\cos a_2}{\sin a_2} - \frac{\cos a_3}{\sin a_3}\right) + \ldots + \left(\frac{\cos a_{n-1}}{\sin a_{n-1}} - \frac{\cos a_n}{\sin a_n}\right)$$

$$= (\cot a_1 - \cot a_2) + (\cot a_2 - \cot a_3) + (\cot a_3 - \cot a_4) + \ldots + (\cot a_{n-1} - \cot a_n)$$

$$= (\cot a_1 - \cot a_n).$$

Hence, $\sin d\,(\text{cosec}\, a_1 \,\text{cosec}\, a_2 + \text{cosec}\, a_2 \,\text{cosec}\, a_3 + \ldots + \text{cosec}\, a_{n-1} \,\text{cosec}\, a_n) = (\cot a_1 - \cot a_n)$.

EXAMPLE 9 *If* $a_1, a_2, a_3, \ldots, a_n$ *are in AP, where* $a_i > 0$ *for each* i*, show that*

$$\frac{1}{\sqrt{a_1} + \sqrt{a_2}} + \frac{1}{\sqrt{a_2} + \sqrt{a_3}} + \ldots + \frac{1}{\sqrt{a_{n-1}} + \sqrt{a_n}} = \frac{(n-1)}{\sqrt{a_1} + \sqrt{a_n}}.$$

SOLUTION Let d be the common difference of the given AP. Then,

$$(a_2 - a_1) = (a_3 - a_2) = \ldots = (a_n - a_{n-1}) = d.$$

$$\text{Now, LHS} = \frac{1}{(\sqrt{a_2} + \sqrt{a_1})} + \frac{1}{(\sqrt{a_3} + \sqrt{a_2})} + \ldots + \frac{1}{(\sqrt{a_n} + \sqrt{a_{n-1}})}$$

$$= \frac{(\sqrt{a_2} - \sqrt{a_1})}{(\sqrt{a_2} + \sqrt{a_1})(\sqrt{a_2} - \sqrt{a_1})} + \frac{(\sqrt{a_3} - \sqrt{a_2})}{(\sqrt{a_3} + \sqrt{a_2})(\sqrt{a_3} - \sqrt{a_2})} + \ldots + \frac{(\sqrt{a_n} - \sqrt{a_{n-1}})}{(\sqrt{a_n} + \sqrt{a_{n-1}})(\sqrt{a_n} - \sqrt{a_{n-1}})}$$

$$= \frac{(\sqrt{a_2} - \sqrt{a_1})}{(a_2 - a_1)} + \frac{(\sqrt{a_3} - \sqrt{a_2})}{(a_3 - a_2)} + \ldots + \frac{(\sqrt{a_n} - \sqrt{a_{n-1}})}{(a_n - a_{n-1})}$$

$$= \frac{(\sqrt{a_2} - \sqrt{a_1})}{d} + \frac{(\sqrt{a_3} - \sqrt{a_2})}{d} + \ldots + \frac{(\sqrt{a_n} - \sqrt{a_{n-1}})}{d}$$

$[\because\ (a_2 - a_1) = (a_3 - a_2) = (a_4 - a_3) = \ldots = (a_n - a_{n-1}) = d]$

$$= \frac{1}{d} \cdot \{(\sqrt{a_2} - \sqrt{a_1}) + (\sqrt{a_3} - \sqrt{a_2}) + (\sqrt{a_4} - \sqrt{a_3}) + \ldots + (\sqrt{a_n} - \sqrt{a_{n-1}})\}$$

$$= \frac{1}{d} \cdot (\sqrt{a_n} - \sqrt{a_1}) = \frac{1}{d} \cdot \left\{(\sqrt{a_n} - \sqrt{a_1}) \times \frac{(\sqrt{a_n} + \sqrt{a_1})}{(\sqrt{a_n} + \sqrt{a_1})}\right\}$$

$$= \frac{1}{d} \cdot \left\{\frac{(a_n - a_1)}{(\sqrt{a_n} + \sqrt{a_1})}\right\} = \frac{1}{d} \cdot \left\{\frac{a_1 + (n-1)d - a_1}{\sqrt{a_1} + \sqrt{a_n}}\right\}$$

$[\because\ a_n = a_1 + (n-1)d]$

$$= \frac{1}{d} \cdot \left\{\frac{(n-1)d}{\sqrt{a_1} + \sqrt{a_n}}\right\} = \frac{(n-1)}{(\sqrt{a_1} + \sqrt{a_n})} = \text{RHS.}$$

Thus, LHS = RHS and hence the result follows.

EXAMPLE 10 *If $a_1, a_2, a_3, \ldots, a_n$ be an AP of nonzero terms then prove that*

$$\frac{1}{a_1 a_2} + \frac{1}{a_2 a_3} + \frac{1}{a_3 a_4} + \ldots + \frac{1}{a_{n-1} a_n} = \frac{(n-1)}{a_1 a_n}.$$

SOLUTION Let d be the common difference of the given AP. Then,

$$(a_2 - a_1) = (a_3 - a_2) = (a_4 - a_3) = \ldots = (a_n - a_{n-1}) = d.$$

$$\therefore \quad \frac{1}{a_1 a_2} + \frac{1}{a_2 a_3} + \frac{1}{a_3 a_4} + \ldots + \frac{1}{a_{n-1} a_n}$$

$$= \frac{1}{d} \cdot \left\{\frac{d}{a_1 a_2} + \frac{d}{a_2 a_3} + \frac{d}{a_3 a_4} + \ldots + \frac{d}{a_{n-1} a_n}\right\}$$

$$= \frac{1}{d} \cdot \left\{\frac{(a_2 - a_1)}{a_1 a_2} + \frac{(a_3 - a_2)}{a_2 a_3} + \frac{(a_4 - a_3)}{a_3 a_4} + \ldots + \frac{(a_n - a_{n-1})}{a_{n-1} a_n}\right\}$$

$[\because\ (a_2 - a_1) = (a_3 - a_2) = \ldots = (a_n - a_{n-1}) = d]$

$$= \frac{1}{d} \cdot \left\{\left(\frac{1}{a_1} - \frac{1}{a_2}\right) + \left(\frac{1}{a_2} - \frac{1}{a_3}\right) + \left(\frac{1}{a_3} - \frac{1}{a_4}\right) + \ldots + \left(\frac{1}{a_{n-1}} - \frac{1}{a_n}\right)\right\}$$

$$= \frac{1}{d} \cdot \left(\frac{1}{a_1} - \frac{1}{a_n}\right) = \frac{1}{d} \cdot \frac{(a_n - a_1)}{a_1 a_n}$$

$$= \frac{1}{d} \cdot \left[\frac{\{a_1 + (n-1)d\} - a_1}{a_1 a_n}\right] \qquad [\because\ a_n = a_1 + (n-1)d]$$

$$= \frac{1}{d} \cdot \frac{(n-1)d}{a_1 a_n} = \frac{(n-1)}{a_1 a_n}.$$

Hence, the result follows.

EXAMPLE 11 *In an AP, the pth term is q and $(p+q)$th term is 0. Then, prove that its qth term is p.*

SOLUTION Let a be the first term and d be the common difference of the given AP. Then,

$$T_p = q \Rightarrow a + (p-1)d = q \qquad \ldots \text{(i)}$$

and $T_{p+q} = 0 \Rightarrow a + (p+q-1)d = 0.$ ... (ii)

On subtracting (i) from (ii), we get

$$qd = -q \Rightarrow d = -1.$$

Putting $d = -1$ in (i), we get $a = (q+p-1)$.

$\therefore \quad T_q = a + (q-1)d = (q+p-1) + (q-1) \times (-1) \Rightarrow T_q = p$

[$\because$ $a = (q+p-1)$ and $d = -1$].

Hence, the qth term is p.

TO FIND THE *n*TH TERM FROM THE END OF AN AP

THEOREM 2 *Show that the nth term from the end of an AP with the first term a, common difference d and the last term l is given by $\{l-(n-1)d\}$.*

PROOF Let a be the first term, d be the common difference and l be the last term of a given AP.

Then, the AP is given by

$$a, a+d, a+2d, \ldots, (l-2d), (l-d), l.$$

$\therefore$ last term $= l = l-(1-1)d$

2nd term from the end $= (l-d) = l-(2-1)d$

3rd term from the end $(l-2d) = l-(3-1)d$

... ...

... ...

$\therefore$ nth term from the end $= l-(n-1)d$.

Hence, nth term from the end $= l-(n-1)d$.

EXAMPLE 12 *Find the 17th term from the end of the AP* $-36, -31, -26, -21, \ldots, 79$.

SOLUTION Here $a = 36$, $d = \{-31-(-36)\} = 5$ and $l = 79$.

$\therefore$ 17th term from the end $= l-(n-1)d$

$= 79-(17-1)\times 5 = (79-80) = -1.$

Hence, the 17th term from the end is –1.

SOME RESULTS ON AP

THEOREM 1 *If the nth term of a progression is a linear expression in n then show that it is an AP.*

PROOF Let the nth term of a progression be a linear expression in n.

Then, $T_n = an + b$, ... (i)

where a and b are constants.

Replacing n by $(n-1)$ in (i), we get

$$T_{n-1} = a(n-1) + b. \qquad \ldots \text{(ii)}$$

On subtracting (ii) from (i), we get

$(T_n - T_{n-1}) = a$, which is constant.

This shows that the difference between any two consecutive terms of the given progression is constant.

Hence, the given progression is an AP.

THEOREM 2 *If the mth term of an AP be* $(1/n)$ *and its nth term be* $(1/m)$ *then show that its (mn)th term is* 1.

PROOF In the given AP, let the first term = a and the common difference = d.

Then, $T_m = \frac{1}{n}$ and $T_n = \frac{1}{m}$

$\Rightarrow \quad a + (m-1)d = \frac{1}{n} \quad \ldots$ (i) $\quad$ and $a + (n-1)d = \frac{1}{m}$. $\quad \ldots$ (ii)

On subtracting (ii) from (i), we get

$$(m-n)d = \left(\frac{1}{n} - \frac{1}{m}\right) \Rightarrow (m-n)d = \frac{(m-n)}{mn}$$

$$\Rightarrow d = \frac{1}{mn}.$$

Putting $d = \frac{1}{mn}$ in (i), we get

$$a + \frac{(m-1)}{mn} = \frac{1}{n} \Rightarrow a = \left\{\frac{1}{n} - \frac{(m-1)}{mn}\right\}$$

$$\Rightarrow a = \frac{m-(m-1)}{mn} \Rightarrow a = \frac{1}{mn}.$$

Thus, we get $a = \frac{1}{mn}$ and $d = \frac{1}{mn}$.

$\therefore \quad (mn)$th term $= a + (mn-1)d$

$$= \left\{\frac{1}{mn} + \frac{(mn-1)}{mn}\right\} = \frac{mn}{mn} = 1.$$

Hence, the (mn)th term of the given AP is 1.

THEOREM 3 *If the mth term of a given AP is n and its nth term is m then show that its pth term is* $(n+m-p)$.

PROOF In the given AP, let the first term = a and the common difference = d.

Then, $T_m = n$ and $T_n = m$

$\Rightarrow \quad a + (m-1)d = n \quad \ldots$ (i) $\quad$ and $a + (n-1)d = m$. $\quad \ldots$ (ii)

On subtracting (ii) from (i), we get

$(m-n)d = (n-m) \Rightarrow d = -1$.

Putting $d = -1$ in (i), we get $a = (n+m-1)$.

$\therefore \quad p$th term $= a + (p-1)d$

$= (n+m-1) + (p-1) \times (-1) \quad [\because a = (n+m-1)$ and $d = -1]$

$= (n+m-p)$.

Hence, the pth term of the given AP is $(n+m-p)$.

THEOREM 4 *If m times the mth term of an AP is equal to n times its nth term, show that the $(m+n)$th term of the AP is zero.*

PROOF In the given AP, let the first term = a and the common difference = d.

Now, $m \cdot T_m = n \cdot T_n$

$\Rightarrow \quad m \cdot [a+(m-1)d] = n \cdot [a+(n-1)d]$

$\Rightarrow \quad [(m^2-n^2)-(m-n)] \times d = (n-m) \times a$

$\Rightarrow \quad (m+n-1)d = -a$

$\Rightarrow \quad a+(m+n-1)d = 0$

$\Rightarrow \quad T_{m+n} = 0.$

Hence, the $(m+n)$th term of the given AP is zero.

THEOREM 5 *If the pth, qth and rth terms of an AP be a, b, c respectively, show that* $(q-r)a+(r-p)b+(p-q)c=0$.

PROOF Let x be the first term and d be the common difference of the given AP.

Then, $T_p = a$, $T_q = b$ and $T_r = c$

$$\Rightarrow \begin{cases} x+(p-1)d = a & \quad \ldots \text{(i)} \\ x+(q-1)d = b & \quad \ldots \text{(ii)} \\ x+(r-1)d = c. & \quad \ldots \text{(iii)} \end{cases}$$

On subtracting (ii) from (i), we get

$$(p-q)d = (a-b) \Rightarrow d = \frac{(a-b)}{(p-q)}.$$

Again, subtracting (iii) from (ii), we get

$$(q-r)d = (b-c) \Rightarrow d = \frac{(b-c)}{(q-r)}.$$

$$\therefore \quad \frac{(a-b)}{(p-q)} = \frac{(b-c)}{(q-r)} \qquad \text{[each equal to } d]$$

$\Rightarrow \quad (q-r)a-(q-r)b = (p-q)b-(p-q)c$

$\Rightarrow \quad (q-r)a+(r-p)b+(p-q)c = 0.$

Hence, $(q-r)a+(r-p)b+(p-q)c = 0$.

REMARK The above result may also be written as

$$(a-b)r+(b-c)p+(c-a)q = 0.$$

THEOREM 6 *Show that the sum of $(m+n)$th and $(m-n)$th terms of an AP is equal to twice the mth term.*

PROOF Let a be the first term and d be the common difference of the given AP. Then,

$$\begin{aligned} T_{m+n}+T_{m-n} &= \{a+(m+n-1)d\}+\{a+(m-n-1)d\} \\ &= 2a+2(m-1)d \\ &= 2\{a+(m-1)d\} = 2 \times T_m. \end{aligned}$$

Hence, $T_{m+n}+T_{m-n} = 2 \times T_m$.

THEOREM 7 *Show that in an AP, the sum of the terms equidistant from the beginning and end is always equal to the sum of the first and last terms.*

PROOF Let $a_1, a_2, a_3, \ldots, a_n$ be an AP with common difference d. Then,

(mth term from the beginning) + (mth term from the end)

$$= \{a_1 + (m-1)d\} + \{a_n - (m-1)d\} = (a_1 + a_n)$$

= sum of the first and last terms of the given AP.

PROBLEMS BASED ON AP

For solving problems based on AP it is always convenient to make following choices:

(i) 3 numbers in AP as $(a-d), a, (a+d)$;

(ii) 4 numbers in AP as $(a-3d), (a-d), (a+d), (a+3d)$;

(iii) 5 numbers in AP as $(a-2d), (a-d), a, (a+d), (a+2d)$;

(iv) 6 numbers in AP as $(a-5d), (a-3d), (a-d), (a+d), (a+3d), (a+5d)$.

SOLVED EXAMPLES

EXAMPLE 13 *The sum of three numbers in AP is* 24 *and their product is* 440. *Find the numbers.*

SOLUTION Let the required numbers be $(a-d)$, a and $(a+d)$. Then,

$$(a-d) + a + (a+d) = 24 \Rightarrow 3a = 24 \Rightarrow a = 8.$$

Thus, the numbers are $(8-d)$, 8 and $(8+d)$.

But their product is 440.

$$\therefore \quad (8-d) \times 8 \times (8+d) = 440$$

$$\Rightarrow \quad (8-d)(8+d) = 55$$

$$\Rightarrow \quad 64 - d^2 = 55 \Rightarrow d^2 = 64 - 55 = 9 \Rightarrow d = \pm 3.$$

Hence, the required numbers are 5, 8, 11 or 11, 8, 5.

EXAMPLE 14 *The product of three numbers in AP is* 224 *and the largest number is* 7 *times the smallest. Find the numbers.*

SOLUTION Let the required numbers be $(a-d)$, a and $(a+d)$.

Then, largest number $= (a+d)$ and smallest number $= (a-d)$.

$$\therefore \quad a + d = 7(a-d) \Rightarrow 6a = 8d \Rightarrow d = \frac{3a}{4}. \quad \ldots \text{(i)}$$

And, $(a-d) \times a \times (a+d) = 224$

$$\Rightarrow \quad a(a^2 - d^2) = 224$$

$$\Rightarrow \quad a\left(a^2 - \frac{9a^2}{16}\right) = 224 \Rightarrow a^3 = \left(224 \times \frac{16}{7}\right) = 512 = 8^3 \Rightarrow a = 8.$$

Putting $a = 8$ in (i), we get $d = \left(\frac{3}{4} \times 8\right) = 6$.

$\therefore$ the required numbers are $(8-6)$, 8 and $(8+6)$, i.e., 2, 8 and 14.

EXAMPLE 15 *Find four numbers in AP whose sum is* 20 *and the sum of whose squares is* 120.

SOLUTION Let the required numbers be

$$(a-3d), (a-d), (a+d) \text{ and } (a+3d).$$

Then, $(a-3d)+(a-d)+(a+d)+(a+3d)=20 \Rightarrow 4a=20 \Rightarrow a=5.$

And $(a-3d)^2+(a-d)^2+(a+d)^2+(a+3d)^2=120$

$\Rightarrow \quad 2(a^2+9d^2)+2(a^2+d^2)=120$

$\Rightarrow \quad 4a^2+20d^2=120 \Rightarrow a^2+5d^2=30$

$\Rightarrow \quad 25+5d^2=30 \Rightarrow 5d^2=5 \Rightarrow d^2=1 \Rightarrow d=\pm 1.$

Hence, the required numbers are 2, 4, 6, 8 or 8, 6, 4, 2.

EXAMPLE 16 *Divide* 32 *into four parts which are in AP such that the product of extremes is to the product of means as* $7:15$.

SOLUTION Let the required parts be $(a-3d)$, $(a-d)$, $(a+d)$ and $(a+3d)$.

Then, $(a-3d)+(a-d)+(a+d)+(a+3d)=32 \Rightarrow 4a=32 \Rightarrow a=8.$

So, the required parts are $(8-3d)$, $(8-d)$, $(8+d)$ and $(8+3d)$.

But product of extremes is to the product of means as 7 : 15.

$$\therefore \quad \frac{(8-3d)(8+3d)}{(8-d)(8+d)}=\frac{7}{15}$$

$$\Rightarrow \quad \frac{(64-9d^2)}{(64-d^2)}=\frac{7}{15}$$

$$\Rightarrow \quad 15(64-9d^2)=7(64-d^2)$$

$$\Rightarrow \quad 128d^2=512 \Rightarrow d^2=\frac{512}{128}=4 \Rightarrow d=\pm 2.$$

Thus, $(a=8 \text{ and } d=2)$ or $(a=8 \text{ and } d=-2)$.

Hence, the required parts are 2, 6, 10, 14 or 14, 10, 6, 2.

EXERCISE 11A

1. Write first 4 terms in each of the sequences:

(i) $a_n=(5n+2)$ (ii) $a_n=\dfrac{(2n-3)}{4}$ (iii) $a_n=(-1)^{n-1}\times 2^{n+1}$

2. Find first five terms of the sequence, defined by

$$a_1=1, a_n=a_{n-1}+3 \text{ for } n\geq 2.$$

3. Find first 5 terms of the sequence, defined by

$$a_1=-1, a_n=\frac{a_{n-1}}{n} \text{ for } n\geq 2.$$

4. Find the 23rd term of the AP 7, 5, 3, 1, –1, –3, … .

5. Find the 20th term of the AP $\sqrt{2}, 3\sqrt{2}, 5\sqrt{2}, 7\sqrt{2}, \ldots$.

6. Find the *n*th term of the AP 8, 3, –2, –7, –12, … .

7. Find the *n*th term of the AP $1, \frac{5}{6}, \frac{2}{3}, \frac{1}{2}, \ldots$.

8. Which term of the AP 9, 14, 19, 24, 29, … is 379?

9. Which term of the AP 64, 60, 56, 52, 48, … is 0?

10. How many terms are there in the AP 11, 18, 25, 32, 39, …, 207?

11. How many terms are there in the AP $1\frac{5}{6}, 1\frac{1}{6}, \frac{1}{2}, \frac{-1}{6}, \frac{-5}{6}, \ldots, -16\frac{1}{6}$?

12. Is –47 a term of the AP $5, 2, -1, -4, -7, \ldots$?

13. The 5th and 13th terms of an AP are 5 and –3 respectively. Find the AP and its 30th term.

14. The 2nd, 31st and the last terms of an AP are $7\frac{3}{4}, \frac{1}{2}$ and $-6\frac{1}{2}$ respectively. Find the first term and the number of terms.

15. If the 9th term of an AP is 0, prove that its 29th term is double the 19th term.

16. The 4th term of an AP is three times the first and the 7th term exceeds twice the third term by 1. Find the first term and the common difference.

17. If 7 times the 7th term of an AP is equal to 11 times its 11th term, show that the 18th term of the AP is zero.

18. Find the 28th term from the end of the AP 6, 9, 12, 15, 18, …, 102.

19. Find the 16th term from the end of the AP $7, 2, -3, -8, -13, \ldots, -113$.

20. How many 3-digit numbers are divisible by 7?

21. How many 2-digit numbers are divisible by 3?

22. If $\theta_1, \theta_2, \theta_3, \ldots, \theta_n$ are in AP whose common difference is d, show that

$$\sec\theta_1 \sec\theta_2 + \sec\theta_2 \sec\theta_3 + \ldots + \sec\theta_{n-1} \sec\theta_n = \frac{(\tan\theta_n - \tan\theta_1)}{\sin d}.$$

23. In an AP, it is being given that $\frac{T_4}{T_7} = \frac{2}{3}$. Find $\frac{T_7}{T_{10}}$.

24. Three numbers are in AP. If their sum is 27 and their product is 648, find the numbers.

25. The sum of three consecutive terms of an AP is 21 and the sum of the squares of these terms is 165. Find these terms.

26. The angles of a quadrilateral are in AP whose common difference is 10°. Find the angles.

27. The digits of a 3-digit number are in AP and their sum is 15. The number obtained by reversing the digits is 594 less than the original number. Find the number.

28. Find the number of terms common to the two arithmetic progressions 5, 9, 13, 17, …, 217 and 3, 9, 15, 21, …, 321.

29. We know that the sum of the interior angles of a triangle is 180°. Show that the sum of the interior angles of polygons with 3, 4, 5, 6, … sides form an arithmetic progression. Find the sum of the interior angles for a 21-sided polygon.

30. A side of an equilateral triangle is 24 cm long. A second equilateral triangle is inscribed in it by joining the midpoints of the sides of the first triangle. The process is continued. Find the perimeter of the sixth inscribed equilateral triangle.

31. A man starts repaying a loan as the first instalment of ₹ 10000. If he increases the instalments by ₹ 500 every month, what amount will he pay in 30th instalment?

ANSWERS (EXERCISE 11A)

1. (i) 7, 12, 17, 22 (ii) $\frac{-1}{4}, \frac{1}{4}, \frac{3}{4}, \frac{5}{4}$ (iii) 4, –8, 16, –32 **2.** 1, 4, 7, 10, 13

3. $-1, \frac{-1}{2}, \frac{-1}{6}, \frac{-1}{24}, \frac{-1}{120}$ **4.** –37 **5.** $39\sqrt{2}$ **6.** $T_n = (13 - 5n)$

7. $T_n = \frac{1}{6}(7 - n)$ **8.** 75th **9.** 17th **10.** 29 **11.** 28

12. No **13.** $(9, 8, 7, 6, \ldots), T_{30} = -20$ **14.** $a = 8, n = 59$

16. $a = 3, d = 2$ **18.** 21 **19.** –38 **20.** 128

21. 30 **23.** $\frac{3}{4}$ **24.** 6, 9, 12 **25.** 4, 7, 10

26. $75°, 85°, 95°, 105°$ **27.** 852 **28.** 18 **29.** 3420°

30. 2.25 cm **31.** ₹ 24500

HINTS TO SOME SELECTED QUESTIONS

14. Let a be the first term and d be the common difference of the given AP.

Then, $T_2 = \frac{31}{4} \Rightarrow a + d = \frac{31}{4} \Rightarrow 4a + 4d = 31$ … (i)

and $T_{31} = \frac{1}{2} \Rightarrow a + 30d = \frac{1}{2} \Rightarrow 2a + 60d = 1.$ … (ii)

On solving (i) and (ii), we get $a = 8$ and $d = \frac{-1}{4}\cdot$

Let $T_n = \frac{-13}{2}\cdot$ Then, $a + (n-1)d = \frac{-13}{2} \Rightarrow 8 + (n-1) \times \left(\frac{-1}{4}\right) = \frac{-13}{2} \Rightarrow n = 59.$

15. $T_9 = 0 \Rightarrow a + 8d = 0 \Rightarrow a = -8d.$

$T_{29} = a + 28d = -8d + 28d = 20d.$

$T_{19} = a + 18d = -8d + 18d = 10d.$

Hence, $T_{29} = 2 \times T_{19}.$

16. $T_4 = 3a$ and $T_7 - 2T_3 = 1$

$\Rightarrow a + 3d = 3a$ and $(a + 6d) - 2(a + 2d) = 1$

$\Rightarrow 2a - 3d = 0$ … (i) and $-a + 2d = 1.$ … (ii)

On solving (i) and (ii), we get $a = 3$ and $d = 2$.

20. The given numbers are 105, 112, 119, …, 994.

This is an AP for which $a = 105$, $d = 7$ and $l = 994$.

Let $T_n = 994$. Then, $105 + (n-1) \times 7 = 994 \Rightarrow n = 128.$

22. Let $\theta_2 - \theta_1 = \theta_3 - \theta_2 = \theta_4 - \theta_3 = \ldots = \theta_n - \theta_{n-1} = d.$

$\sec\theta_1 \sec\theta_2 + \sec\theta_2 \sec\theta_3 + \ldots + \sec\theta_{n-1} \sec\theta_n$

$$= \frac{1}{\sin d}\left\{\frac{\sin d}{\cos\theta_1\cos\theta_2}+\frac{\sin d}{\cos\theta_2\cos\theta_3}+\ldots+\frac{\sin d}{\cos\theta_{n-1}\cos\theta_n}\right\}$$

$$= \frac{1}{\sin d}\cdot\left\{\frac{\sin(\theta_2-\theta_1)}{\cos\theta_1\cos\theta_2}+\frac{\sin(\theta_3-\theta_2)}{\cos\theta_2\cos\theta_3}+\ldots+\frac{\sin(\theta_n-\theta_{n-1})}{\cos\theta_{n-1}\cos\theta_n}\right\}$$

$$= \frac{1}{\sin d}\cdot\left\{\frac{\sin\theta_2\cos\theta_1-\cos\theta_2\sin\theta_1}{\cos\theta_1\cos\theta_2}+\frac{\sin\theta_3\cos\theta_2-\cos\theta_3\sin\theta_2}{\cos\theta_2\cos\theta_3}+\ldots+\frac{\sin\theta_n\cos\theta_{n-1}-\cos\theta_n\sin\theta_{n-1}}{\cos\theta_{n-1}\cos\theta_n}\right\}$$

$$= \frac{1}{\sin d}\cdot\{(\tan\theta_2-\tan\theta_1)+(\tan\theta_3-\tan\theta_2)+\ldots+(\tan\theta_n-\tan\theta_{n-1})\}$$

$$= \frac{1}{\sin d}\cdot(\tan\theta_n-\tan\theta_1)=\frac{(\tan\theta_n-\tan\theta_1)}{\sin d}.$$

23. $\frac{a+3d}{a+6d}=\frac{2}{3} \Rightarrow 3a+9d=2a+12d \Rightarrow a=3d.$

$\therefore \quad \frac{T_7}{T_{10}}=\frac{a+6d}{a+9d}=\frac{3d+6d}{3d+9d}=\frac{9d}{12d}=\frac{3}{4}.$

26. Let the required angles be $a°, (a+10)°, (a+20)°, (a+30)°$. Then,

$a+a+10+a+20+a+30=360 \Rightarrow 4a=300 \Rightarrow a=75.$

27. Let the hundred's, ten's and unit digit's be $a+d, a, a-d$.

Then, $a+d+a+a-d=15 \Rightarrow 3a=15 \Rightarrow a=5.$

$[100(a+d)+10a+(a-d)]-[100(a-d)+10a+(a+d)]=594$

$\Rightarrow \quad 99d+99d=594 \Rightarrow 198d=594 \Rightarrow d=3.$

Hence, the required number is 852.

28. The first common term in two APs is 9.

LCM of their common differences = LCM (4, 6) = 12.

New AP of common elements is 9, 21, 33, 45, … .

Let it contain n terms. Then,

$T_n \le 216 \Rightarrow 9+(n-1)\times 12 \le 216 \Rightarrow (n-1)\times 12 \le 207$

$\Rightarrow (n-1)\times 12 = 204 \Rightarrow n=18.$

So, there are 18 common terms in two APs.

29. The sum of the interior angles of a triangle is 180°. A polygon of 4 sides can be divided into two triangles and therefore, the sum of its angles is 360°. A polygon of 5 sides can be divided into 3 triangles and therefore, the sum of its angles is $(3\times 180°)=540°$ and so on. Thus, a polygon of 21 sides is divided into $(21-2)=19$ triangles and therefore, the sum of its angles is $(19\times 180°)=3420°$.

30. We know that the line segment joining midpoints of the opposite sides of a triangle is parallel to the third side and equal to half of it.

Perimeters of 1st, 2nd, 3rd, 4th, 5th and 6th triangle are respectively 72 cm, 36 cm, 18 cm, 9 cm, $\frac{9}{2}$ cm, $\frac{9}{4}$ cm.

31. The instalments form the AP

10000 + 10500 + 11000 + 11500 + … up to 30 terms.

Here $a=10000$, $d=500$ and we have to find T_{30}.

SUM OF *n* TERMS OF AN AP

THEOREM 1 *Prove that the sum of n terms of an AP with first term a and common difference d is given by:*

I. $S_n = \frac{n}{2}(a+l)$, *where l is the last term.*

II. $S_n = \frac{n}{2} \cdot [2a+(n-1)d]$.

PROOF Let us consider an AP containing n terms with the first term a, the common difference d and the last term l.

Let the sum of these n terms be S_n. Then,

$$S_n = a + (a+d) + (a+2d) + \ldots + (l-2d) + (l-d) + l. \quad \text{... (i)}$$

Writing the above series in a reverse order, we get

$$S_n = l + (l-d) + (l-2d) + \ldots + (a+2d) + (a+d) + a. \quad \text{... (ii)}$$

Adding the corresponding terms of (i) and (ii), we get

$$2S_n = \{(a+l) + (a+l) + \ldots \text{ to } n \text{ times}\} = n(a+l).$$

$$\therefore \quad S_n = \frac{n}{2}(a+l).$$

But, $l = n$th term $= a + (n-1)d$.

$$\therefore \quad S_n = \frac{n}{2} \cdot \{a + a + (n-1)d\} = \frac{n}{2} \cdot \{2a + (n-1)d\}.$$

Hence, $S_n = \frac{n}{2} \cdot \{2a + (n-1)d\}$.

SUMMARY

(i) Sum of n terms of an AP with the first term $= a$ and the last term $= l$ is given by

$$S_n = \frac{n}{2}(a+l).$$

(ii) Sum of n terms of an AP with the first term $= a$ and the common difference $= d$ is given by

$$S_n = \frac{n}{2} \cdot \{2a + (n-1)d\}.$$

SOLVED EXAMPLES

EXAMPLE 1 *Find the sum of* 23 *terms of the AP* 5, 9, 13, 17, … .

SOLUTION Here, $a = 5$, $d = (9-5) = 4$ and $n = 23$.

Now, $S_n = \frac{n}{2} \times \{2a + (n-1) \times d\}$.

$$\therefore \quad S_{23} = \frac{23}{2} \times \{2 \times 5 + (23-1) \times 4\}$$

$$= \left(\frac{23}{2} \times 98\right) = 1127.$$

Hence, the sum of 23 terms of the given AP is 1127.

EXAMPLE 2 *The first, second and the last terms of an AP are a, b, c respectively. Prove that the sum of the AP is* $\dfrac{(a+c)(b+c-2a)}{2(b-a)}$.

SOLUTION Let d be the common difference of the given AP and let it contain n terms. Then, $d=(b-a)$.

$$\text{Now, } T_n = c \Rightarrow a+(n-1)d = c$$
$$\Rightarrow a+(n-1)(b-a)=c \qquad [\because d=(b-a)]$$
$$\Rightarrow (n-1)=\frac{(c-a)}{(b-a)}$$
$$\Rightarrow n=\left\{\frac{(c-a)}{(b-a)}+1\right\}=\frac{(b+c-2a)}{(b-a)}.$$

$\therefore$ sum of the given AP $=\dfrac{n}{2}(a+l)$, where l is the last term

$$=\frac{(b+c-2a)}{2(b-a)}\times(a+c)$$
$$\left[\because n=\frac{(b+c-2a)}{2(b-a)} \text{ and } l=c\right]$$
$$=\frac{(a+c)(b+c-2a)}{2(b-a)}.$$

EXAMPLE 3 *How many terms of the AP* $-6, \frac{-11}{2}, -5, \frac{-9}{2}, \ldots$ *are needed to give the sum* -25*? Explain the double answer.*

SOLUTION Let the required number of terms be n. Then,

$$a=-6,\ d=\left\{\frac{-11}{2}-(-6)\right\}=\left(\frac{-11}{2}+6\right)=\frac{1}{2} \text{ and } S_n=-25.$$

$$\therefore \quad S_n=-25 \Rightarrow \frac{n}{2}\cdot[2a+(n-1)d]=-25$$
$$\Rightarrow \frac{n}{2}\cdot\left[2\times(-6)+(n-1)\cdot\frac{1}{2}\right]=-25$$
$$\Rightarrow \frac{n}{2}\cdot\left(\frac{n}{2}-\frac{25}{2}\right)=-25 \Rightarrow \frac{n}{2}\cdot\frac{(n-25)}{2}=-25$$
$$\Rightarrow n^2-25n+100=0 \Rightarrow (n-5)(n-20)=0$$
$$\Rightarrow n=5 \text{ or } n=20.$$

This shows that the sum of first five terms is –25 and that the sum of first 20 terms is also –25.

It means that the sum of all the terms from 6th to 20th is zero.

EXAMPLE 4 *Find the value of x when* $1+6+11+16+\ldots+x=148$.

SOLUTION The given series is an arithmetic series in which $a=1$ and $d=(6-1)=5$.

Let it contain n terms. Then, we have to find the nth term.

Now, $S_n = 148$

$\Rightarrow \quad \frac{n}{2} \cdot [2a + (n-1)d] = 148$

$\Rightarrow \quad \frac{n}{2} \cdot [2 \times 1 + (n-1) \times 5] = 148$

$\Rightarrow \quad n(5n-3) = 296 \Rightarrow 5n^2 - 3n - 296 = 0$

$\Rightarrow \quad 5n^2 - 40n + 37n - 296 = 0$

$\Rightarrow \quad 5n(n-8) + 37(n-8) = 0 \Rightarrow (n-8)(5n+37) = 0$

$\Rightarrow \quad n = 8 \quad \left[\because n \neq \frac{-37}{5}\right].$

$\therefore \quad T_8 = 1 + (8-1) \times 5 = 36 \quad [\because T_n = a + (n-1)d].$

Hence, $x = 36$.

EXAMPLE 5 *Find the sum of all odd integers from* 1 *to* 1001.

SOLUTION The odd integers from 1 to 1001 are 1, 3, 5, 7, …, 999, 1001.

This is an AP in which $a = 1$, $d = (3-1) = 2$ and $l = 1001$.

Let the number of terms be n. Then,

$$T_n = 1001 \Rightarrow a + (n-1)d = 1001$$
$$\Rightarrow 1 + (n-1) \times 2 = 1001 \Rightarrow n = 501.$$

Now, $a = 1$, $l = 1001$ and $n = 501$.

$$\therefore \quad S_n = \frac{n}{2}(a+l) = \frac{501}{2} \cdot (1 + 1001) = (501 \times 501) = 251001.$$

Hence, the required sum is 251001.

EXAMPLE 6 *Find the sum of all natural numbers lying between* 100 *and* 1000, *which are multiples of* 5.

SOLUTION Required numbers are 105, 110, 115, …, 995.

This is an AP in which $a = 105$, $d = (110 - 105) = 5$ and $l = 995$.

Let the number of terms in this AP be n. Then,

$$T_n = 995 \Rightarrow a + (n-1)d = 995$$
$$\Rightarrow 105 + (n-1) \times 5 = 995$$
$$\Rightarrow (n-1) = \frac{890}{5} = 178 \Rightarrow n = 179.$$

$$\therefore \quad S_n = \frac{n}{2}(a+l) = \frac{179}{2} \times (105 + 995) = \left(\frac{179}{2} \times 1100\right)$$
$$= (179 \times 550) = 98450.$$

Hence, the required sum is 98450.

EXAMPLE 7 *Find the sum of integers from* 1 *to* 100 *that are divisible by* 2 *or* 5.

SOLUTION Required sum

= (sum of integers from 1 to 100, divisible by 2)

+ (sum of integers from 1 to 100, divisible by 5)

– (sum of integers from 1 to 100, divisible by 10)

$$= (2+4+6+\ldots+100)+(5+10+15+\ldots+100) \\ -(10+20+30+\ldots+100)$$

$$= \frac{50}{2}(2+100)+\frac{20}{2}(5+105)-\frac{10}{2}(10+100)$$

$$= (2550+1050-550) = 3050.$$

EXAMPLE 8 *Find the sum of all two-digit numbers which when divided by 4 yield 1 as remainder.*

SOLUTION Required sum

$$13+17+21+25+\ldots+97.$$

This is an AP in which $a = 13$, $d = (17-13) = 4$ and $l = 97$.

Let the number of terms be n. Then,

$$T_n = 97 \Rightarrow a+(n-1)d = 97$$

$$\Rightarrow 13+(n-1)\times 4 = 97 \Rightarrow n = 22.$$

$\therefore$ required sum $= \dfrac{n}{2}(a+l) = \dfrac{22}{2}\times(13+97) = (11\times 110) = 1210.$

Hence, the required sum is 1210.

EXAMPLE 9 *The sum of n terms of two arithmetic progressions are in the ratio* $(3n+8):(7n+15)$. *Find the ratio of their 12th terms.*

SOLUTION Let a_1, a_2 and d_1, d_2 be the first terms and common differences of the first and second AP respectively. Then, by the given condition, we have

$$\frac{\text{sum to } n \text{ terms of first AP}}{\text{sum to } n \text{ terms of second AP}} = \frac{3n+8}{7n+15}$$

$$\Rightarrow \quad \frac{\frac{n}{2}\cdot[2a_1+(n-1)d_1]}{\frac{n}{2}\cdot[2a_2+(n-1)d_2]} = \frac{3n+8}{7n+15}$$

$$\Rightarrow \quad \frac{2a_1+(n-1)d_1}{2a_2+(n-1)d_2} = \frac{3n+8}{7n+15}. \qquad \ldots \text{(i)}$$

Now, $\dfrac{\text{12th term of first AP}}{\text{12th term of second AP}} = \dfrac{a_1+11d_1}{a_2+11d_2}$

$$= \frac{2a_1+22d_1}{2a_2+22d_2} = \frac{2a_1+(23-1)d_1}{2a_2+(23-1)d_2}$$

$$= \frac{3\times 23+8}{7\times 23+15} \quad \text{[putting } n = 23 \text{ in (i)]}$$

$$= \frac{77}{176} = \frac{7}{16}.$$

Hence, the ratio of the 12th terms of given APs is 7 : 16.

EXAMPLE 10 *In an AP, the first term is 2 and the sum of first five terms is one-fourth of the sum of next five terms. Show that its 20th term is –112 and the sum of its first 20 terms is –1100.*

SOLUTION We have

$$(T_1+T_2+T_3+T_4+T_5)=\frac{1}{4}(T_6+T_7+T_8+T_9+T_{10})$$
$$=\frac{1}{4}\cdot[(T_1+T_2+\ldots+T_5+T_6+\ldots+T_{10})-(T_1+T_2+T_3+T_4+T_5)]$$

$\Rightarrow \quad S_5=\frac{1}{4}\cdot(S_{10}-S_5)$, where S_n denotes the sum of first n terms

$\Rightarrow \quad 4S_5=S_{10}-S_5$

$\Rightarrow \quad 5S_5=S_{10}$

$\Rightarrow \quad 5\times\frac{5}{2}\times[2\times2+(5-1)d]=\frac{10}{2}\times[2\times2+(10-1)d]$

$\Rightarrow \quad 50(1+d)=(20+45d)$

$\Rightarrow \quad 5d=-30 \Rightarrow d=-6.$

$\therefore \quad a=2$ and $d=-6$.

$\therefore \quad T_{20}=[2+(20-1)\times(-6)]=-112$

and $\quad S_{20}=\frac{20}{2}\times[2\times2+(20-1)\times(-6)]=-1100.$

EXAMPLE 11 *The difference between any two consecutive interior angles of a polygon is* $5°$. *If the smallest angle is* $120°$, *find the number of sides of the polygon.*

SOLUTION Let the number of sides of the polygon be n.

Then, the sum of its interior angles $=(2n-4)$ right angles

$$=\{(n-2)\times180\}°. \quad \ldots \text{(i)}$$

Since the difference between any two consecutive interior angles of the polygon is constant, its angles are in AP.

In this AP, we have $a=120$ and $d=5$.

Now, $S_n=(n-2)\times180$ [from (i)]

$\Rightarrow \quad \frac{n}{2}\cdot\{2\times120+(n-1)\times5\}=(n-2)\times180$

$\Rightarrow \quad \frac{235n}{2}+\frac{5n^2}{2}=180n-360 \Rightarrow 5n^2-125n+720=0$

$\Rightarrow \quad n^2-25n+144=0 \Rightarrow (n-9)(n-16)=0$

$\Rightarrow \quad n=9$ or $n=16$.

But, when $n=16$, we have

last angle $=\{120+(16-1)\times5\}°=195°$, which is not possible.

$\therefore \quad n=9.$

Hence, the number of sides of the given polygon = 9.

EXAMPLE 12 *The income of a man is ₹ 400000 in the first year and he receives an increase of ₹ 10000 to his income per year for 19 years. Find the total amount he received in 20 years.*

SOLUTION Here, we have an AP with $a=400000$, $d=10000$ and $n=20$.

Using the formula, $S_n = \frac{n}{2} \times [2a + (n-1)d]$, we have

$$S_{20} = \frac{20}{2} \times [2 \times 400000 + 19 \times 10000] = 10 \times 990000 = 9900000.$$

Hence, the man received ₹ 9900000 as the total amount in 20 years.

EXAMPLE 13 *A man saved* ₹ 660000 *in* 20 *years. In each succeeding year after the first year he saves* ₹ 2000 *more than what he saved in the previous year. How much did he save in the first year?*

SOLUTION Here, we have an AP in which $S_{20} = 660000$, $d = 2000$ and we have to find the value of a.

Using the formula, $S_n = \frac{n}{2} \times [2a + (n-1)d]$, we get

$$S_{20} = \frac{20}{2} \times [2a + (20-1) \times 2000]$$

$\Rightarrow \quad 10 \times [2a + 19 \times 2000] = 660000 \qquad [\because S_{20} = 660000]$

$\Rightarrow \quad 20a + 380000 = 660000 \Rightarrow 20a = 280000 \Rightarrow a = 14000.$

Hence, the man saved ₹ 14000 in the first year.

EXAMPLE 14 *In a potato race,* 20 *potatoes are placed in a line at intervals of* 4 *metres with the first potato* 24 *metres from the starting point. A contestant is required to bring the potatoes back to the starting place, one at a time. How far would he run in bringing back all the potatoes?*

SOLUTION Distances moved in bringing back the 1st, 2nd, 3rd, …, 20th potato to the starting point are (2×24) m, (2×28) m, (2×32) m, …, and so on.

Total distance moved $= \{48 + 56 + 64 + 72 + \ldots \text{ to } 20 \text{ terms}\}$

$$= \frac{20}{2} \times [2 \times 48 + (20-1) \times 8] \text{ m}$$

$[\because a = 48, d = 8 \text{ and } n = 20]$

$$= 10 \times (96 + 152) \text{ m} = 2480 \text{ m}.$$

Hence, total distance covered = 2480 m.

EXAMPLE 15 *In a cricket tournament* 16 *school teams participated. A sum of* ₹ 16000 *is to be awarded among themselves as prize money. If the team in the last place is awarded* ₹ 550 *in prize money and the award increases by the same amount for successive finishing places, how much amount will the team in the first place receive?*

SOLUTION Let the first place team receive ₹ a and let the award increase by ₹ d. Then, these prizes form an AP in which first term = a, common difference = d, $T_{16} = 550$ and $S_{16} = 16000$.

Now, $T_{16} = 550 \Rightarrow a + 15d = 550.$ … (i)

And, $S_{16} = 16000 \Rightarrow \frac{16}{2} \times [2a + 15d] = 16000$

$\Rightarrow 2a + 15d = 2000.$ … (ii)

On subtracting (i) from (ii), we get $a = 1450$.

Hence, the team in the first place receives ₹ 1450.

EXAMPLE 16 *The first term of an AP is a and the sum of the first p terms is zero. Show that the sum of the next q terms is* $\frac{-a(p+q)q}{(p-1)}$.

SOLUTION Let the common difference of the given AP be d. Then,

$$S_p = 0 \Rightarrow \frac{p}{2} \times [2a + (p-1)d] = 0$$

$$\Rightarrow 2a + (p-1)d = 0 \Rightarrow d = \frac{-2a}{(p-1)}. \quad \ldots \text{(i)}$$

$$\therefore \quad S_{p+q} = \frac{(p+q)}{2} \cdot [2a + (p+q-1)d]$$

$$= \frac{(p+q)}{2} \cdot \left[2a + (p+q-1) \times \frac{(-2a)}{(p-1)}\right] \quad \text{[using (i)]}$$

$$= a(p+q) \cdot \left[1 - \frac{(p+q-1)}{(p-1)}\right] = \frac{-a(p+q)q}{(p-1)}.$$

Hence, the required sum is $\frac{-a(p+q)q}{(p-1)}$.

EXAMPLE 17 *If the mth term of an AP is a and its nth term is b, show that the sum of its* $(m+n)$ *terms is* $\frac{(m+n)}{2} \cdot \left\{a + b + \frac{(a-b)}{(m-n)}\right\}$.

SOLUTION Let the first term and the common difference of the given AP be A and d respectively. Then,

$$T_m = a \Rightarrow A + (m-1)d = a \quad \ldots \text{(i)}$$

and $T_n = b \Rightarrow A + (n-1)d = b.$ … (ii)

On subtracting (ii) from (i), we get

$$(m-n)d = (a-b) \Rightarrow d = \frac{(a-b)}{(m-n)}. \quad \ldots \text{(iii)}$$

Adding (i) and (ii), we get

$$2A + (m+n-2)d = a + b$$

$$\Rightarrow \quad 2A + (m+n-1)d - d = a + b$$

$$\Rightarrow \quad 2A + (m+n-1)d = (a+b+d) = \left(a + b + \frac{a-b}{m-n}\right) \quad \ldots \text{(iv) [using (iii)].}$$

$$\therefore \quad T_{m+n} = \frac{(m+n)}{2} \cdot [2A + (m+n-1)d].$$

$$\Rightarrow \quad T_{m+n} = \frac{(m+n)}{2} \cdot \left[a + b + \frac{a-b}{m-n}\right] \quad \text{[using (iv)].}$$

EXAMPLE 18 *If the ratio of the sums of m and n terms of an AP be* $(m^2 : n^2)$, *show that the ratio of their mth and nth terms is* $(2m-1):(2n-1)$.

SOLUTION Let the first term and the common difference of the given AP be a and d respectively. Then,

$$\frac{S_m}{S_n} = \frac{m^2}{n^2} \Rightarrow \frac{\frac{m}{2} \cdot [2a + (m-1)d]}{\frac{n}{2} \cdot [2a + (n-1)d]} = \frac{m^2}{n^2}$$

$$\Rightarrow \frac{2a + (m-1)d}{2a + (n-1)d} = \frac{m}{n}$$

$$\Rightarrow n[2a + (m-1)d] = m[2a + (n-1)d]$$

$$\Rightarrow 2a(m-n) = d(m-n) \Rightarrow d = 2a.$$

$$\therefore \quad \frac{T_m}{T_n} = \frac{a + (m-1)d}{a + (n-1)d} = \frac{a + 2(m-1)a}{a + 2(n-1)a} = \frac{1 + 2m - 2}{1 + 2n - 2} = \frac{2m-1}{2n-1}.$$

Hence, the ratio of their mth and nth terms is $(2m-1):(2n-1)$.

SOME RESULTS ON SUMMATION OF AP

THEOREM 1 *If the sum of first n terms of a progression is a quadratic expression in n, show that it is an AP.*

PROOF Let $S_n = an^2 + bn + c$ ... (i), where a, b, c are constants and $a \neq 0$.

On replacing n by $(n-1)$ in (i), we get

$$S_{n-1} = a(n-1)^2 + b(n-1) + c.$$

$$\therefore \quad T_n = (S_n - S_{n-1})$$

$$= (an^2 + bn + c) - \{a(n-1)^2 + b(n-1) + c\}$$

$$= a\{n^2 - (n-1)^2\} + b\{n - (n-1)\}$$

$$= a(2n-1) + b = 2an + (b-a).$$

Thus, $T_n = 2an + (b-a)$. ... (ii)

Replacing n by $(n-1)$ in (ii), we get

$$T_{n-1} = 2a(n-1) + (b-a). \quad \text{... (iii)}$$

On subtracting (iii) from (ii), we get

$(T_n - T_{n-1}) = 2a$, which is constant.

Hence, the given progression is an AP.

THEOREM 2 *If there are $(2n+1)$ terms in an AP, prove that the sum of odd terms and the sum of even terms bear the ratio $(n+1):n$.*

PROOF Since the given AP contains $(2n+1)$ terms, so it has $(n+1)$ odd terms and n even terms.

Let a be the first term and d be the common difference of the given AP.

Let S_1 and S_2 be the sum of all odd terms and the sum of all even terms respectively. Then,

$$S_1 = T_1 + T_3 + T_5 + \ldots + T_{2n+1}$$

$$= \frac{(n+1)}{2} \cdot [T_1 + T_{2n+1}] \quad \text{[sum of } (n+1) \text{ odd terms]}$$

$$= \frac{(n+1)}{2} \cdot [a + \{a + (2n+1-1)d\}] \quad [\because T_{2n+1} = a + (2n+1-1)d]$$

$$= (n+1)(a + nd).$$

$$S_2 = T_2 + T_4 + T_6 + \ldots + T_{2n}$$
$$= \frac{n}{2} \cdot (T_2 + T_{2n}) \qquad \text{[sum of } n \text{ even terms]}$$
$$= \frac{n}{2} \cdot [(a+d) + \{a + (2n-1)d\}] \qquad [\because\ T_{2n} = a + (2n-1)d]$$
$$= n(a + nd).$$

$\therefore \quad \dfrac{S_1}{S_2} = \dfrac{(n+1)(a+nd)}{n(a+nd)} \Rightarrow \dfrac{S_1}{S_2} = \dfrac{n+1}{n}.$

Hence, the required ratio is $(n+1):n$.

THEOREM 3 *If the mth term of an AP is* $\left(\frac{1}{n}\right)$ *and its nth term is* $\left(\frac{1}{m}\right)$ *then show that the sum of its mn terms is* $\frac{1}{2}(mn+1)$.

PROOF Let a be the first term and d be the common difference of the given AP.

Then, $T_m = \dfrac{1}{n}$ and $T_n = \dfrac{1}{m}$.

$\therefore \quad a + (m-1)d = \dfrac{1}{n} \quad \ldots \text{(i)} \qquad \text{and} \quad a + (n-1)d = \dfrac{1}{m}. \quad \ldots \text{(ii)}$

On subtracting (ii) from (i), we get

$$(m-n)d = \left(\frac{1}{n} - \frac{1}{m}\right) = \frac{(m-n)}{mn} \Rightarrow d = \frac{1}{mn}.$$

Putting $d = \dfrac{1}{mn}$ in (i), we get

$$a + \frac{(m-1)}{mn} = \frac{1}{n} \Rightarrow a = \left\{\frac{1}{n} - \frac{(m-1)}{mn}\right\} = \frac{1}{mn}.$$

$\therefore \quad S_{mn} = \dfrac{mn}{2} \cdot [2a + (mn-1)d]$

$$= \frac{mn}{2} \cdot \left\{\frac{2}{mn} + \frac{(mn-1)}{mn}\right\} \qquad \left[\because\ a = \frac{1}{mn} \text{ and } d = \frac{1}{mn}\right]$$
$$= \frac{1}{2}(mn+1).$$

Hence, the sum of mn terms of the given AP is $\frac{1}{2}(mn+1)$.

THEOREM 4 *If the sum of first p terms of an AP is the same as the sum of its first q terms then show that the sum of its first* $(p+q)$ *terms is zero.*

PROOF Let a be the first term and d be the common difference of the given AP.

Then, $S_p = S_q$

$\Rightarrow \quad \dfrac{p}{2} \cdot [2a + (p-1)d] = \dfrac{q}{2} \cdot [2a + (q-1)d]$

$\Rightarrow \quad 2(p-q)a + \{p(p-1) - q(q-1)\}d = 0$

$\Rightarrow \quad 2(p-q)a + \{(p^2 - q^2) - (p-q)\}d = 0$

$\Rightarrow \quad 2(p-q)a + (p-q)(p+q-1)d = 0$

$\Rightarrow \quad (p-q)\{2a + (p+q-1)d\} = 0$

$\Rightarrow \quad 2a + (p+q-1)d = 0. \quad \ldots \text{(i)}$

$$\therefore \quad S_{p+q} = \frac{(p+q)}{2} \cdot [2a + (p+q-1)d] = 0 \qquad \text{[using (i)].}$$

Hence, the sum of the first $(p+q)$ term is zero.

THEOREM 5 *If the sum of m terms of an AP be n and the sum of its n terms be m, show that the sum of its* $(m+n)$ *terms is* $-(m+n)$.

PROOF Let a be the first term and d be the common difference of the given AP.

Then, $S_m = n \Rightarrow \frac{m}{2} \cdot \{2a + (m-1)d\} = n$

$$\Rightarrow 2am + m(m-1)d = 2n. \qquad \ldots \text{(i)}$$

And, $S_n = m \Rightarrow \frac{n}{2} \cdot \{2a + (n-1)d\} = m$

$$\Rightarrow 2an + n(n-1)d = 2m. \qquad \ldots \text{(ii)}$$

On subtracting (ii) from (i), we get

$$2a(m-n) + \{(m^2 - n^2) - (m-n)\} \cdot d = 2(n-m)$$

$$\Rightarrow \quad 2a(m-n) + (m-n)(m+n-1)d = 2(n-m)$$

$$\Rightarrow \quad (m-n)\{2a + (m+n-1)d\} = -2(m-n)$$

$$\Rightarrow \quad 2a + (m+n-1)d = -2. \qquad \ldots \text{(iii)}$$

$$\therefore \quad S_{m+n} = \frac{(m+n)}{2} \cdot \{2a + (m+n-1)d\}$$

$$= \frac{(m+n)}{2} \cdot (-2) = -(m+n) \qquad \text{[using (iii)].}$$

Hence, the sum of $(m+n)$ terms is $-(m+n)$.

THEOREM 6 *The sum of the first p, q and r terms of an AP are a, b, c respectively. Show that*

$$\frac{a}{p} \cdot (q-r) + \frac{b}{q} \cdot (r-p) + \frac{c}{r} \cdot (p-q) = 0.$$

PROOF Let x be the first term and d be the common difference of the given AP.

Then, $S_p = a$, $S_q = b$ and $S_r = c$.

Now, $S_p = a \Rightarrow \frac{p}{2} \cdot [2x + (p-1)d] = a$

$$\Rightarrow \frac{a}{p} = x + (p-1) \cdot \frac{d}{2}. \qquad \ldots \text{(i)}$$

$S_q = b \Rightarrow \frac{q}{2} \cdot [2x + (q-1)d] = b$

$$\Rightarrow \frac{b}{q} = x + (q-1) \cdot \frac{d}{2}. \qquad \ldots \text{(ii)}$$

And, $S_r = c \Rightarrow \frac{r}{2} \cdot [2x + (r-1)d] = c$

$$\Rightarrow \frac{c}{r} = x + (r-1) \cdot \frac{d}{2}. \qquad \ldots \text{(iii)}$$

On multiplying (i), (ii) and (iii) by $(q-r)$, $(r-p)$ and $(p-q)$ respectively and adding, we get

$$\frac{a}{p}\cdot(q-r)+\frac{b}{q}\cdot(r-p)+\frac{c}{r}\cdot(p-q)$$
$$=x\cdot[(q-r)+(r-p)+(p-q)]$$
$$+\frac{d}{2}\cdot[(p-1)(q-r)+(q-1)(r-p)+(r-1)(p-q)]$$
$$=(x\times 0)+\left(\frac{d}{2}\times 0\right)=0.$$

Hence, $\frac{a}{p}(q-r)+\frac{b}{q}(r-p)+\frac{c}{r}(p-q)=0.$

THEOREM 7 *If the sums of the first $n, 2n, 3n$ terms of an AP are S_1, S_2 and S_3 respectively, prove that $S_3=3(S_2-S_1)$.*

PROOF Let a be the first term and d be the common difference of the given AP.

Then, $S_1=\frac{n}{2}\cdot\{2a+(n-1)d\} \Rightarrow S_1=na+\frac{1}{2}n(n-1)d,$ ... (i)

$S_2=\frac{2n}{2}\cdot\{2a+(2n-1)d\} \Rightarrow S_2=2na+n(2n-1)d$ ... (ii)

and $S_3=\frac{3n}{2}\cdot\{2a+(3n-1)d\} \Rightarrow S_3=3na+\frac{3}{2}n(3n-1)d.$ ... (iii)

$$\therefore\quad (S_2-S_1)=na+nd\left\{(2n-1)-\frac{1}{2}(n-1)\right\}$$
$$=na+\frac{n}{2}\cdot(3n-1)d=\frac{n}{2}\cdot\{2a+(3n-1)d\}.$$
$$\therefore\quad 3(S_2-S_1)=\frac{3n}{2}\cdot\{2a+(3n-1)d\}=S_3.$$

Hence, $S_3=3(S_2-S_1)$.

THEOREM 8 *If $S_1, S_2, S_3, \ldots, S_m$ be the sums of n terms of m arithmetic progressions whose first terms are $1, 2, 3, \ldots, m$ respectively and the common differences are $1, 3, 5, \ldots, (2m-1)$ respectively then show that*

$$S_1+S_2+S_3+\ldots+S_m=\frac{1}{2}mn(mn+1).$$

PROOF Clearly, we have

$S_1=\frac{n}{2}\times\{2\times 1+(n-1)\times 1\}$ $[\because\ a=1, d=1]$

$S_2=\frac{n}{2}\times\{2\times 2+(n-1)\times 3\}$ $[\because\ a=2, d=3]$

$S_3=\frac{n}{2}\times\{2\times 3+(n-1)\times 5\}$ $[\because\ a=3, d=5]$

... ...

... ...

$S_m=\frac{n}{2}\times\{2\times m+(n-1)(2m-1)\}$ $[\because\ a=m, d=(2m-1)]$

Adding columnwise, we get

$$S_1+S_2+S_3+\ldots+S_m=\frac{n}{2}\times[2\times(1+2+3+\ldots+m)]$$
$$+\frac{n(n-1)}{2}\times[1+3+5+\ldots+(2m-1)]$$

$$= \left\{n \times \frac{m}{2}(1+m)\right\} + \frac{n(n-1)}{2} \times \frac{m}{2} \times [1 + (2m-1)]$$
$$= \left\{\frac{n}{2} \times m(m+1)\right\} + \left\{\frac{n}{2} \times m^2 (n-1)\right\}$$
$$= \frac{mn}{2} \times [(m+1) + m(n-1)] = \frac{1}{2} mn(mn+1).$$

Hence, $S_1 + S_2 + S_3 + \ldots + S_m = \frac{1}{2} mn(mn+1)$.

THEOREM 9 *If the sum of m terms of an AP is equal to the sum of either the next n terms or the next p terms then prove that*

$$(m+n)\left(\frac{1}{m} - \frac{1}{p}\right) = (m+p)\left(\frac{1}{m} - \frac{1}{n}\right).$$

PROOF Let us consider an AP with first term a and common difference d.

Then, $(T_1 + T_2 + T_3 + \ldots + T_m) = (T_{m+1} + T_{m+2} + T_{m+3} + \ldots + T_{m+n})$

$\Rightarrow \quad 2(T_1 + T_2 + T_3 + \ldots + T_m) = (T_1 + T_2 + \ldots + T_m + T_{m+1} + \ldots + T_{m+n})$

[adding $(T_1 + T_2 + T_3 + \ldots + T_m)$ on both sides]

$\Rightarrow \quad 2 \times \frac{m}{2} \times [2a + (m-1)d] = \frac{(m+n)}{2} \times [2a + (m+n-1)d]$

$\Rightarrow \quad mx = \frac{(m+n)}{2} \times (x + nd)$, where $[2a + (m-1)d] = x$

$\Rightarrow \quad 2mx = (m+n)x + (m+n)nd$

$\Rightarrow \quad (m-n)x = (m+n)nd.$ … (i)

Similarly, $(m-p)x = (m+p)pd.$ … (ii)

On dividing (i) by (ii), we get

$$\frac{(m-n)}{(m-p)} = \frac{(m+n)n}{(m+p)p}$$

$\Rightarrow \quad (m+n)(m-p)n = (m+p)(m-n)p$ [by cross multiplication]

$\Rightarrow \quad (m+n)\left(\frac{1}{p} - \frac{1}{m}\right) = (m+p)\left(\frac{1}{n} - \frac{1}{m}\right)$ [on dividing both sides by mnp]

$\Rightarrow \quad (m+n)\left(\frac{1}{m} - \frac{1}{p}\right) = (m+p)\left(\frac{1}{m} - \frac{1}{n}\right).$

Hence, $(m+n)\left(\frac{1}{m} - \frac{1}{p}\right) = (m+p)\left(\frac{1}{m} - \frac{1}{n}\right).$

THEOREM 10 *The first term of an AP is a and the sum of its first p terms is zero. Then, show that the sum of its next q terms is* $\frac{-a(p+q)q}{(p-1)}$.

PROOF Let d be the common difference of the given AP. Then,

$$S_p = 0 \Rightarrow \frac{p}{2}[2a + (p-1)d] = 0$$

$$\Rightarrow 2a + (p-1)d = 0 \Rightarrow d = \frac{2a}{(1-p)}. \quad \ldots \text{(i)} \quad [\because p \neq 0]$$

Let S be the sum of next q terms. Then,

$$S = T_{p+1} + T_{p+2} + \dots + T_{p+q}$$
$$= (T_1 + T_2 + \dots + T_p + T_{p+1} + T_{p+2} + \dots + T_{p+q}) - (T_1 + T_2 + \dots + T_p)$$
$$= S_{p+q} - S_p = S_{p+q} - 0 = S_{p+q} \qquad [\because\ S_p = 0]$$
$$= \frac{(p+q)}{2} \cdot [2a + (p+q-1)d]$$
$$= \frac{(p+q)}{2} \cdot \left[2a + (p+q-1) \times \frac{2a}{(1-p)}\right] \qquad \text{[using (i)]}$$
$$= (p+q)a \cdot \left[1 + \frac{(p+q-1)}{(1-p)}\right] = \frac{-a(p+q)q}{(p-1)}.$$

Hence, the sum of next q terms is $\dfrac{-a(p+q)q}{(p-1)}$.

THEOREM 11 *If the sum of n terms of an AP is $(cn + dn^2)$, where c and d are constants then show that the common difference of the AP is $2d$.*

PROOF Let $S_n = cn + dn^2$ ... (i), where c and d are constants.

Replacing n by $(n-1)$ in (i), we get

$$S_{n-1} = c(n-1) + d(n-1)^2$$
$$\Rightarrow \quad S_{n-1} = dn^2 + n(c-2d) + (d-c). \qquad \dots \text{(ii)}$$

On subtracting (ii) from (i), we get

$$(S_n - S_{n-1}) = 2dn + (c-d)$$
$$\Rightarrow \quad T_n = 2dn + (c-d). \qquad \dots \text{(iii)}$$

Replacing n by $(n-1)$ in (iii), we get

$$T_{n-1} = 2d(n-1) + (c-d). \qquad \dots \text{(iv)}$$

On subtracting (iv) from (iii), we get

$(T_n - T_{n-1}) = 2d$, which is constant.

Hence, the common difference of the given AP is $2d$.

EXERCISE 11B

1. Find the sum of 23 terms of the AP 17, 12, 7, 2, –3, ...
2. Find the sum of 16 terms of the AP $6, 5\frac{1}{3}, 4\frac{2}{3}, 4, \dots$.
3. Find the sum of 25 terms of the AP $\sqrt{2}, 2\sqrt{2}, 3\sqrt{2}, 4\sqrt{2}, \dots$.
4. Find the sum of 100 terms of the AP 0.6, 0.61, 0.62, 0.63, ...
5. Find the sum of 20 terms of the AP $(x+y), (x-y), (x-3y), \dots$.
6. Find the sum of n terms of the AP $\dfrac{x-y}{x+y}, \dfrac{3x-2y}{x+y}, \dfrac{5x-3y}{x+y}, \dots$.
7. Find the sum of the series $2 + 5 + 8 + 11 + \dots + 191$.
8. Find the sum of the series $101 + 99 + 97 + 95 + \dots + 43$.

9. Find the value of x such that $1+4+7+10+\ldots+x=715$.

10. Find the value of x such that $25+22+19+16+\ldots+x=112$.

11. Find the rth term of an AP, the sum of whose first n terms is $(3n^2+2n)$.

12. Find the sum of n terms of an AP whose rth term is $(5r+1)$.

13. If the sum of a certain number of terms of the AP 27, 24, 21, 18, … is –30, find the last term.

14. How many terms of the AP 26, 21, 16, 11, … are needed to give the sum 11?

15. How many terms of the AP 18, 16, 14, 12, … are needed to give the sum 78? Explain the double answer.

16. How many terms of the AP $20, 19\frac{1}{3}, 18\frac{2}{3}, \cdots$ must be taken to make the sum 300? Explain the double answer.

17. The sums of n terms of two arithmetic progressions are in the ratio $(7n-5):(5n+17)$. Show that their 6th terms are equal.

18. If the ratio between the sums of n terms of two arithmetic progressions is $(7n+1):(4n+27)$, find the ratio of their 11th terms.

19. Find the sum of all odd integers from 1 to 201.

20. Find the sum of all even integers between 101 and 199.

21. Find the sum of all integers between 101 and 500, which are divisible by 9.

22. Find the sum of all integers between 100 and 600, each of which when divided by 5 leaves 2 as remainder.

23. The sum of first 7 terms of an AP is 10 and that of next 7 terms is 17. Find the AP.

24. If the sum of n terms of an AP is $(3n^2+5n)$ and its mth term is 164, find the value of m.

25. Find the sum of all natural numbers from 1 and 100 which are divisible by 4 or 5.

26. If the sum of n terms of an AP is $\left\{nP+\frac{1}{2}n(n-1)Q\right\}$, where P and Q are constants then find the common difference.

27. If $S_m=m^2p$ and $S_n=n^2p$, where $m\neq n$ in an AP then prove that $S_p=p^3$.

28. A carpenter was hired to build 192 window frames. The first day he made 5 frames and each day, thereafter he made 2 more frames than he made the day before. How many days did he take to finish the job?

ANSWERS (EXERCISE 11B)

1. –874 **2.** 16 **3.** $325\sqrt{2}$ **4.** 109.5 **5.** $20(x-18y)$

6. $\frac{n}{2(x+y)}\cdot\{n(2x-y)-y\}$ **7.** 6176 **8.** 2160 **9.** $x=64$

10. $x=7$ **11.** $T_r=(6r-1)$ **12.** $S_n=\frac{1}{2}n(5n+7)$ **13.** –30

14. 11 **15.** 6 and 13, sum of all terms from 7th to 13th is zero

16. 25 and 36, sum of all terms from 26th to 36th is zero **18.** 4 : 3

19. 10201 **20.** 7350 **21.** 13266 **22.** 34950

23. $1+1\frac{1}{7}+1\frac{2}{7}+1\frac{3}{7}+\ldots$ **24.** $m=27$ **25.** 2050 **26.** Q

28. 12 days

HINTS TO SOME SELECTED QUESTIONS

5. $a=(x+y)$ and $d=(x-y)-(x+y)=-2y$.

$S_{20}=\frac{20}{2}\times[2a+(20-1)d]=10\times[2(x+y)+19\times(-2y)]=20(x-18y)$.

6. Required sum $=\frac{1}{(x+y)}\cdot\{(x-y)+(3x-2y)+(5x-3y)+\ldots$ to n terms]

$=\frac{1}{(x+y)}\cdot\frac{n}{2}\cdot[2(x-y)+(n-1)(2x-y)]$ $[\because\ a=(x-y)$ and $d=(2x-y)]$

$=\frac{n}{2(x+y)}\cdot\{n(2x-y)-y\}$.

9. Here, $a=1$ and $d=3$. Let it contain n terms. Then,

$\frac{n}{2}\cdot[2a+(n-1)d]=715 \Rightarrow \frac{n}{2}\cdot[2\times1+(n-1)\times3]=715$

$\Rightarrow\ n(3n-1)=1430 \Rightarrow 3n^2-n-1430=0 \Rightarrow 3n^2-66n+65n-1430=0$

$\Rightarrow\ 3n(n-22)+65(n-22)=0 \Rightarrow (n-22)(3n+65)=0 \Rightarrow n=22.$

$\therefore\ x=T_{22}=(a+21d)=(1+21\times3)=64.$

10. $3n^2-21n-32n+224=0.$

11. $S_n=3n^2+2n \Rightarrow S_{n-1}=3(n-1)^2+2(n-1) \Rightarrow S_{n-1}=3n^2-4n+1.$

$\therefore\ T_n=(S_n-S_{n-1})=(3n^2+2n)-(3n^2-4n+1)=(6n-1) \Rightarrow T_r=(6r-1).$

13. Here, $a=27$, $d=(24-27)=-3$ and $S_n=-30$. Then,

$\frac{n}{2}\cdot[2a+(n-1)d]=-30 \Rightarrow \frac{n}{2}\cdot[54+(n-1)\times(-3)]=-30$

$\therefore\ 3n^2-57n-60=0 \Rightarrow n^2-19n-20=0 \Rightarrow n=20.$

Now, $S_n=-30 \Rightarrow \frac{n}{2}(a+l)=-30 \Rightarrow \frac{20}{2}(27+l)=-30 \Rightarrow l=-30.$

16. Here, $a=20$ and $d=\left(\frac{58}{3}-20\right)=\frac{-2}{3}\cdot$ Let $S_n=300$. Then,

$\frac{n}{2}\times[2a+(n-1)d]=300 \Rightarrow \frac{n}{2}\times\left[40+(n-1)\times\left(\frac{-2}{3}\right)\right]=300$

$\therefore\ \frac{n}{2}\cdot\frac{(122-2n)}{3}=300 \Rightarrow n(61-n)=900 \Rightarrow n^2-61n+900=0$

$\Rightarrow\ n^2-36n-25n+900=0 \Rightarrow n(n-36)-25(n-36)=0 \Rightarrow (n-25)(n-36)=0.$

$\therefore\ S_{25}=300$ and $S_{36}=300.$

Sum of all terms from 26th to 36th is zero.

17. $\dfrac{\frac{n}{2}\cdot[2a_1+(n-1)d_1]}{\frac{n}{2}\cdot[2a_2+(n-1)d_2]}=\frac{7n-5}{5n+17} \Rightarrow \frac{2a_1+(n-1)d_1}{2a_2+(n-1)d_2}=\frac{7n-5}{5n+17}$

$$\Rightarrow \quad \frac{a_1+\left(\frac{n-1}{2}\right)d_1}{a_2+\frac{(n-1)}{2}d_2}=\frac{7n-5}{5n+17} \Rightarrow \frac{a_1+5d_1}{a_2+5d_2}=\frac{7\times 11-5}{5\times 11+17} \qquad \left[\frac{n-1}{2}=5 \Rightarrow n=11\right]$$

$$=\frac{72}{72}=1.$$

Hence, their 6th terms are equal.

18. Proceeding as above, we get

$$\frac{a_1+\frac{1}{2}(n-1)d_1}{a_2+\frac{1}{2}(n-1)d_2}=\frac{7n+1}{4n+27} \Rightarrow \frac{a_1+10d_1}{a_2+10d_2}=\frac{7\times 21+1}{4\times 21+27} \qquad \left[\frac{n-1}{2}=10 \Rightarrow n=21\right]$$

$$=\frac{148}{111}=\frac{4}{3}.$$

19. Required sum $= 1+3+5+7+\ldots+201$.

$T_n = 201 \Rightarrow 1+(n-1)\times 2 = 201 \Rightarrow n = 101$.

$\therefore \quad S_n = \frac{101}{2}\times(1+201) = (101\times 101) = 10201.$

20. Required sum $= 102+104+106+\ldots+198$.

21. Required sum $= 108+117+126+\ldots+495$.

22. Required sum $= 102+107+112+\ldots+597$.

23. $S_7 = 10$ and $S_{14} = (17+10) = 27$.

$\therefore \quad \frac{7}{2}\times[2a+6d] = 10 \Rightarrow 7a+21d = 10 \qquad \ldots \text{(i)}$

and $\frac{14}{2}\times[2a+13d] = 27 \Rightarrow 14a+91d = 27. \qquad \ldots \text{(ii)}$

Solve for a and d.

24. $T_n = (S_n - S_{n-1}) = (3n^2+5n)-(3n^2-n-2) = (6n+2)$.

$T_m = 164 \Rightarrow 6m+2 = 164 \Rightarrow 6m = 162 \Rightarrow m = 27$.

25. Required numbers = (numbers divisible by 4) + (numbers divisible by 5) – (numbers divisible by 20)

$$= (4+8+12+\ldots+100)+(5+10+15+\ldots+100) - (20+40+60+\ldots+100)$$

$$= \frac{25}{2}\times(4+100)+\frac{20}{2}\times(5+100)-\frac{5}{2}\times(20+100)$$

$$= (1300+1050-300) = 2050.$$

26. $S_n = nP+\frac{1}{2}n(n-1)Q \Rightarrow S_{n-1} = (n-1)P+\frac{1}{2}(n-1)(n-2)Q.$

$\therefore \quad T_n = (S_n - S_{n-1}) = P+(n-1)Q \Rightarrow T_{n-1} = P+(n-2)Q.$

$\therefore \quad d = (T_n - T_{n-1}) = Q.$

27. $S_m = m^2p$ and $S_{m-1} = (m-1)^2p \Rightarrow S_m - S_{m-1} = \{m^2-(m-1)^2\}p = (2m-1)p.$

$T_m = (2m-1)p \Rightarrow T_1 = p, T_2 = 3p, T_3 = 5p, \ldots \Rightarrow a = p$ and $d = 2p$.

$S_p = \frac{p}{2}\times[2p+(p-1)\times 2p] = \left(\frac{p}{2}\times 2p^2\right) = p^3.$

28. We have $5+7+9+\ldots = 192$

$$\frac{n}{2}\times[10+(n-1)\times 2] = 192 \Rightarrow n(n+4) = 192 \Rightarrow n^2+4n-192 = 0$$

$$\Rightarrow (n+16)(n-12) = 0 \Rightarrow n = 12.$$

WORD PROBLEMS ON AP

EXERCISE 11C

1. The interior angles of a polygon are in AP. The smallest angle is 52° and the common difference is 8°. Find the number of sides of the polygon.
2. A circle is completely divided into n sectors in such a way that the angles of the sectors are in AP. If the smallest of these angles is 8° and the largest is 72°, calculate n and the angle in the fifth sector.
3. There are 30 trees at equal distances of 5 metres in a line with a well, the distance of the well from the nearest tree being 10 metres. A gardner waters all the trees separately starting from the well and he returns to the well after watering each tree to get water for the next. Find the total distance the gardner will cover in order to water all the trees.
4. Two cars start together from the same place in same direction. The first goes with uniform speed of 60 km/hr. The second goes at a speed of 48 km/hr in the first hour and increases the speed by 1 km each succeeding hour. After how many hours will the second car overtake the first car if both cars go non-stop?
5. Arun buys a scooter for ₹ 44000. He pays ₹ 8000 in cash and agrees to pay the balance in annual instalments of ₹ 4000 each plus 10% interest on the unpaid amount. How much did he pay for it?
6. A man accepts a position with an initial salary of ₹ 26000 per month. It is understood that he will receive an automatic increase of ₹ 250 in the very next month and each month thereafter.

   Find his (i) salary for the 10th month, (ii) total earnings during the first year.
7. A man saved ₹ 660000 in 20 years. In each succeeding year after the first year he saved ₹ 2000 more than what he saved in the previous year. How much did he save in the first year?
8. 150 workers were engaged to finish a piece of work in a certain number of days. Four workers dropped the second day, four more workers dropped the third day, and so on. It takes 8 more days to finish the work now. Find the number of days in which the work was completed.
9. A man saves ₹ 4000 during first year, ₹ 5000 during second year and in this way he increases his savings by ₹ 1000 every year. Find in what time his savings will be ₹ 85000.
10. A man arranges to pay off a debt of ₹ 36000 by 40 annual instalments which form an AP. When 30 of the instalments are paid, he dies, leaving one-third of the debt unpaid. Find the value of first instalment.
11. A manufacturer of TV sets produced 6000 units in the third year and 7000 units in the seventh year. Assuming that the production increases uniformly by a fixed number every year, find the production (i) in the first year, (ii) in the 10th year, (iii) in 7 years.

12. A farmer buys a used tractor for ₹ 180000. He pays ₹ 90000 in cash and agrees to pay the balance in annual instalments of ₹ 9000 plus 12% interest on the unpaid amount. How much did the tractor cost him?

ANSWERS (EXERCISE 11C)

1. 3 **2.** $n = 9$, 5th angle = 40° **3.** 4795 m **4.** 25 hours **5.** ₹ 62000
6. (i) ₹ 28250 (ii) ₹ 328500 **7.** ₹ 14000 **8.** 25 days **9.** 10 years
10. ₹ 510 **11.** (i) 5500 (ii) 7750 (iii) 43750 **12.** ₹ 239400

HINTS TO SOME SELECTED QUESTIONS

1. Let the number of sides of the polygon be n. Then,

sum of its interior angles $= (2n - 4) \times 90^\circ = (180n - 360)^\circ$.

$\therefore \quad 52 + 60 + 68 + \ldots$ to n terms $= (180n - 360)$

$\Rightarrow \quad \frac{n}{2} \times [2 \times 52 + (n-1) \times 8] = 180n - 360$

$\Rightarrow \quad n^2 - 33n + 90 = 0 \Rightarrow (n-3)(n-30) = 0 \Rightarrow n = 3$ or $n = 30$.

When $n = 30$, then last angle $= [52 + 29 \times 8]^\circ = 284^\circ$, which is a reflex angle. So, $n \neq 30$.

Hence, $n = 3$.

2. Let the number of sectors be n. Then,

$\frac{n}{2}(a + l) = 360 \Rightarrow n(8 + 72) = 720 \Rightarrow n = \frac{720}{80} = 9.$

Now, $a = 8$, $T_n = 72$ and $n = 9$.

$\therefore \quad 8 + 8d = 72 \Rightarrow 8d = 64 \Rightarrow d = 8.$

So, the AP is 8, 16, 24, 32, … . So, 5th angle $= (a + 4d) = 40^\circ$.

3. Total distance covered in metres

$= 2 \times \{10 + 15 + 20 + \ldots$ up to 29 terms$\}$ + 30th term

$= 2 \times \frac{29}{2} \times \{20 + 28 \times 5\} + (10 + 29 \times 5)$

$= (4640 + 155) = 4795$ m.

4. Suppose the second car overtakes the first car in n hours.

Distance covered by the first car in n hours $= (60n)$ km.

Distance covered by the second car in n hours

$= [48 + 49 + 50 + \ldots$ to n terms$]$

$= \frac{n}{2} \times [2 \times 48 + (n-1) \times 1] = \frac{n}{2}(95 + n).$

$\therefore \quad \frac{n}{2}(95 + n) = 60n \Rightarrow 95 + n = 120 \Rightarrow n = 25$ hours.

5. Total cost = ₹ 44000. Paid in cash = ₹ 8000. Balance = ₹ 36000.

Amount of each instalment = ₹ 4000.

Number of instalments $= 36000 \div 4000 = 9$.

1st instalment = ₹ $\left(4000 + 36000 \times \frac{10}{100} \times 1\right)$ = ₹ 7600.

2nd instalment = ₹ $\left(4000 + 32000 \times \frac{10}{100} \times 1\right)$ = ₹ 7200.

3rd instalment = ₹ $\left(4000 + 28000 \times \frac{10}{100} \times 1\right)$ = ₹ 6800, and so on.

Total payment $= ₹\,\{8000 + (7600 + 7200 + 6800 + \ldots \text{ to 9 terms})\}$

$$= ₹\left\{8000 + \frac{9}{2}(15200 + 8 \times (-400))\right\} = ₹\,62000.$$

6. The amounts form an AP with $a = ₹\,26000$ and $d = ₹\,250$.

(i) $T_{10} = (a + 9d) = ₹\,(26000 + 9 \times 250) = ₹\,28250$.

(ii) Total earning during 1 year $= ₹\left\{\frac{12}{2} \times (2 \times 26000 + 11 \times 250)\right\} = ₹\,328500$.

7. Suppose that the man saved ₹ x in first year. Then,

$x + (x + 2000) + (x + 4000) + \ldots$ to 20 terms $= 660000$.

This is an AP in which $a = x$, $d = 2000$ and $n = 20$.

$\therefore \quad S_{20} = 660000 \Rightarrow \frac{20}{2} \times (2x + 19 \times 2000) = 660000 \Rightarrow x = 14000$.

8. Suppose that the work was completed in n days.

Total number of workers engaged for 1 day each

$= 150 + 146 + 142 + \ldots$ to n terms

$= \frac{n}{2} \times [2 \times 150 + (n-1) \times (-4)] = n(152 - 2n)$.

If 150 workers are engaged each day, time taken $= (n - 8)$ days.

Number of workers engaged for 1 day $= 150(n - 8)$.

$\therefore \quad n(152 - 2n) = 150(n - 8) \Rightarrow n^2 - n - 600 = 0$

$\Rightarrow (n - 25)(n + 24) = 0$

$\Rightarrow n = 25 \quad [\because n \neq -24]$.

9. Let the total savings be ₹ 85000 in n years. Then,

$\frac{n}{2} \times [2 \times 4000 + (n-1) \times 1000] = 85000$

$\Rightarrow \quad n^2 + 7n - 170 = 0 \Rightarrow (n + 17)(n - 10) = 0 \Rightarrow n = 10$ years.

10. Let the first instalment be ₹ a and let it increase by ₹ d each year.

Then, $S_{30} = \left(\frac{2}{3} \times 36000\right) = 24000$ and $S_{40} = 36000$

$\Rightarrow \quad \frac{30}{2} \times (2a + 29d) = 24000$ and $\frac{40}{2} \times (2a + 39d) = 36000$

$\Rightarrow \quad 2a + 29d = 1600 \quad \ldots \text{(i)} \quad$ and $\quad 2a + 39d = 1800. \quad \ldots \text{(ii)}$

On solving (i) and (ii), we get $a = 510$ and $d = 20$.

$\therefore \quad$ first instalment $= ₹\,510$.

11. Let the production in first year be a, increasing uniformly at the rate of d per year. Then,

$a + 2d = 6000$ and $a + 6d = 7000$.

On solving, we get $a = 5500$ and $d = 250$.

(i) Production in the 1st year $= a = 5500$.

(ii) Production in the 10th year $= (a + 9d) = (5500 + 9 \times 250)$.

(iii) Production in 7 years $= \frac{7}{2} \times [2a + (7-1)d] = \frac{7}{2} \times [2 \times 5500 + 6 \times 250]$.

12. Total cost $= ₹\,180000$. Paid in cash $= ₹\,90000$. Balance $= ₹\,90000$.

Amount of each instalment $= ₹\,9000$.

Number of instalments $= \frac{90000}{9000} = 10$.

1st instalment = ₹ $\left(9000 + 90000 \times \frac{12}{100} \times 1\right)$ = ₹ 19800.

2nd instalment = ₹ $\left(9000 + 81000 \times \frac{12}{100} \times 1\right)$ = ₹ 18720.

3rd instalment = ₹ $\left(9000 + 72000 \times \frac{12}{100} \times 1\right)$ = ₹ 17640, and so on.

Total payment = ₹ [90000 + (19800 + 18720 + 17640 + ... up to 10 terms]

= ₹ $\left[90000 + \frac{10}{2} \cdot \{39600 + 9 \times (-1080)\}\right]$

= ₹ (90000 + 149400) = ₹ 239400.

ARITHMETIC MEAN

ARITHMETIC MEAN

If a, A, b are in AP then we say that A is the arithmetic mean (AM) between a and b.

INSERTION OF A SINGLE ARITHMETIC MEAN BETWEEN *a* AND *b*

Let a and b be two given numbers and let A be the arithmetic mean between a and b. Then,

a, A, b are in AP

$\Rightarrow \quad A - a = b - A$

$\Rightarrow \quad 2A = a + b \Rightarrow A = \frac{a+b}{2}$.

Hence, the arithmetic mean between a and b is $\frac{a+b}{2}$.

EXAMPLE 1 *Find the arithmetic mean between*

(i) 14 *and* –6, *(ii)* $(a - b)$ *and* $(a + b)$.

SOLUTION (i) Arithmetic mean between 14 and –6

$$= \frac{14 + (-6)}{2} = \frac{8}{2} = 4.$$

(ii) Arithmetic mean between $(a - b)$ and $(a + b)$

$$= \frac{(a-b) + (a+b)}{2} = \frac{2a}{2} = a.$$

EXAMPLE 2 *If* $\left(\frac{a^n + b^n}{a^{n-1} + b^{n-1}}\right)$ *is the AM between a and b then find the value of n.*

SOLUTION We know that the AM between a and b is $\frac{(a+b)}{2}$.

$\therefore \quad \left(\frac{a^n + b^n}{a^{n-1} + b^{n-1}}\right)$ is the AM between a and b

$$\Rightarrow \quad \frac{a^n + b^n}{a^{n-1} + b^{n-1}} = \frac{a+b}{2}$$

$$\Rightarrow \quad 2a^n + 2b^n = a^n + a^{n-1}b + b^{n-1}a + b^n$$

$$\Rightarrow \quad a^n + b^n = a^{n-1}b + b^{n-1}a$$

$\Rightarrow \quad a^n - a^{n-1}b = b^{n-1}a - b^n$

$\Rightarrow \quad a^{n-1}(a-b) = b^{n-1}(a-b)$

$\Rightarrow \quad a^{n-1} = b^{n-1}$ [on dividing both sides by $(a-b)$]

$\Rightarrow \quad \left(\frac{a}{b}\right)^{n-1} = 1 = \left(\frac{a}{b}\right)^0 \Rightarrow n-1=0 \Rightarrow n=1.$

Hence, the required value of n is 1.

n ARITHMETIC MEANS BETWEEN *a* AND *b*

If $a, A_1, A_2, \ldots, A_n, b$ are in AP then we say that $A_1, A_2, \ldots, A_n$ are the n arithmetic means between a and b.

INSERTION OF *n* ARITHMETIC MEANS BETWEEN *a* AND *b*

Let a and b be two given numbers and let $A_1, A_2, \ldots, A_n$ be the n arithmetic means between a and b. Then,

$a, A_1, A_2, \ldots, A_n, b$ are in AP.

In this AP, we have first term $= a$, last term $= b$, number of terms $= (n+2)$.

Let the common difference be d. Then,

$b = T_{n+2} \Rightarrow b = a + (n+2-1)d$

$\Rightarrow d = \frac{b-a}{n+1}$. ... (i)

$\therefore \quad A_1 = (a+d), A_2 = (a+2d), A_3 = (a+3d), \ldots, A_n = (a+nd)$

$\Rightarrow \quad A_1 = \left\{a + \frac{(b-a)}{(n+1)}\right\}, A_2 = \left\{a + \frac{2(b-a)}{(n+1)}\right\}, \ldots, A_n = \left\{a + \frac{n(b-a)}{(n+1)}\right\}.$

These are the required n arithmetic means between a and b.

EXAMPLE 3 *Insert six arithmetic means between* 15 *and* –13.

SOLUTION Let $A_1, A_2, A_3, A_4, A_5, A_6$ be the six arithmetic means between 15 and –13. Then,

$15, A_1, A_2, A_3, A_4, A_5, A_6, -13$ are in AP.

Now, $d = \frac{(-13-15)}{(6+1)} = \frac{-28}{7} = -4$ $\left[\because d = \frac{(b-a)}{(n+1)} = \frac{(-13-15)}{(6+1)}\right]$.

$\therefore \quad A_1 = (15+d) = (15-4) = 11, A_2 = (15+2d) = (15-8) = 7,$

$A_3 = (15+3d) = (15-12) = 3, A_4 = (15+4d) = (15-16) = -1,$

$A_5 = (15+5d) = (15-20) = -5, A_6 = (15+6d) = (15-24) = -9.$

Hence, the required six AMs between 15 and –13 are

11, 7, 3, –1, –5 and –9.

EXAMPLE 4 *If m arithmetic means are inserted between* 1 *and* 31 *so that the ratio of the 7th and* $(m-1)$*th means is* 5 : 9 *then find the value of m.*

SOLUTION Let $A_1, A_2, \ldots, A_m$ be the m arithmetic means between 1 and 31.

Then, $1, A_1, A_2, \ldots, A_m, 31$ are in AP.

$\therefore \quad d = \frac{(31-1)}{(m+1)} = \frac{30}{m+1}$ $\left[\because d = \frac{(b-a)}{(m+1)}\right]$.

$$7\text{th mean} = A_7 = T_8 = (a+7d) = \left(1+\frac{7\times 30}{m+1}\right) = \frac{(m+211)}{(m+1)}.$$

$$(m-1)\text{th mean} = A_{m-1} = T_m = a+(m-1)d = \left\{1+\frac{(m-1)\times 30}{(m+1)}\right\}$$
$$= \frac{(31m-29)}{(m+1)}.$$

$$\text{Now, } \frac{A_7}{A_{m-1}} = \frac{5}{9} \Rightarrow \frac{(m+211)}{(m+1)} \times \frac{(m+1)}{(31m-29)} = \frac{5}{9}$$
$$\Rightarrow \frac{(m+211)}{(31m-29)} = \frac{5}{9} \Rightarrow 9m+1899 = 155m-145$$
$$\Rightarrow 146m = 2044 \Rightarrow m = \frac{2044}{146} = 14.$$

Hence, $m = 14$.

EXAMPLE 5 *If x, y, z are in AP and A_1 is the AM between x and y while A_2 is the AM between y and z then prove that the AM between A_1 and A_2 is y.*

SOLUTION Since x, y, z are in AP, we have $x+z=2y$. … (i)

Since A_1 is the AM between x and y, we have $A_1 = \frac{x+y}{2}$. … (ii)

Since A_2 is the AM between y and z, we have $A_2 = \frac{y+z}{2}$. … (iii)

$$\therefore \quad \text{AM between } A_1 \text{ and } A_2 = \frac{1}{2}(A_1+A_2) = \frac{1}{2}\left\{\frac{x+y}{2}+\frac{y+z}{2}\right\}$$
$$= \frac{x+2y+z}{4} = \frac{4y}{4} = y \qquad \text{[using (i)].}$$

Hence, the AM between A_1 and A_2 is y.

EXAMPLE 6 *If the AM between pth and qth terms of an AP be equal to the AM between its rth and sth terms then prove that $p+q=r+s$.*

SOLUTION Let a be the first term and d be the common difference of the given AP. Then,

$$\frac{1}{2}(T_p+T_q) = \frac{1}{2}(T_r+T_s)$$
$$\Rightarrow \quad T_p+T_q = T_r+T_s$$
$$\Rightarrow \quad \{a+(p-1)d\}+\{a+(q-1)d\} = \{a+(r-1)d\}+\{a+(s-1)d\}$$
$$\Rightarrow \quad 2a+(p+q-2)d = 2a+(r+s-2)d$$
$$\Rightarrow \quad (p+q-2)d = (r+s-2)d$$
$$\Rightarrow \quad p+q-2 = r+s-2 \Rightarrow p+q = r+s.$$

Hence, $p+q=r+s$.

EXAMPLE 7 *The n arithmetic means between 20 and 80 are such that the first mean : last mean $= 1:3$. Find the value of n.*

SOLUTION Let $A_1, A_2, \ldots, A_n$ be n AMs between 20 and 80.

Then, $20, A_1, A_2, \ldots, A_n, 80$ are in AP.

Let their common difference be d. Then,

$$T_{n+2} = 80 \Rightarrow 20 + (n+2-1)d = 80 \Rightarrow d = \frac{60}{(n+1)}. \quad \ldots \text{(i)}$$

First mean, $A_1 = T_2 = (20 + d) = \left(20 + \dfrac{60}{n+1}\right)$ [using (i)]

$$= \frac{(20n+80)}{(n+1)}.$$

Last mean, $A_n = T_{n+1} = [20 + (n+1-1)d]$

$$= \left\{20 + \frac{n \times 60}{(n+1)}\right\} \quad \text{[using (i)]}$$

$$= \frac{(80n+20)}{(n+1)}.$$

$$\therefore \quad \frac{A_1}{A_n} = \frac{1}{3} \Rightarrow \frac{(20n+80)}{(n+1)} \times \frac{(n+1)}{(80n+20)} = \frac{1}{3}$$

$$\Rightarrow 3(20n+80) = (80n+20) \Rightarrow 20n = 220 \Rightarrow n = 11.$$

Hence, $n = 11$.

EXAMPLE 8 *If n arithmetic means are inserted between two numbers, prove that the sum of the means equidistant from the beginning and the end is constant.*

SOLUTION Let $A_1, A_2, \ldots, A_n$ be n AMs between a and b.

Then, $a, A_1, A_2, \ldots, A_n, b$ are in AP.

$\therefore$ mth mean from the beginning + mth mean from the end

= $(m+1)$th term from the beginning + $(m+1)$th term from the end

$= \{a + (m+1-1)d\} + \{b - (m+1-1)d\} = (a+b) =$ constant.

Hence, the sum of the means equidistant from the beginning and the end is constant.

EXAMPLE 9 *Show that the sum of n arithmetic means between two numbers is n times the single arithmetic mean between them.*

SOLUTION Let a and b be two given numbers and let A be the single AM between them. Then, $A = \dfrac{a+b}{2}$.

Let $A_1, A_2, \ldots, A_n$ be the n arithmetic means between a and b.

Then $a, A_1, A_2, \ldots, A_n, b$ are in AP.

Now, $(A_1 + A_2 + \ldots + A_n) = \dfrac{n}{2}(A_1 + A_n)$ $\left[S_n = \dfrac{n}{2}(a+l)\right]$

$= \dfrac{n}{2}(a+b)$ [$\because$ $a, A_1, A_2, \ldots, A_n, b$ is an AP $\Rightarrow a + b = A_1 + A_n$]

$= n \times$ (AM between a and b).

Hence, the sum of n arithmetic means between a and b is n times the AM between a and b.

EXERCISE 11D

1. Find the arithmetic mean between

(i) 9 and 19, (ii) 15 and –7, (iii) –16 and –8.

2. Insert four arithmetic means between 4 and 29.

3. Insert three arithmetic means between 23 and 7.

4. Insert six arithmetic means between 11 and –10.

5. There are n arithmetic means between 9 and 27. If the ratio of the last mean to the first mean is 2 : 1, find the value of n.

6. Insert arithmetic means between 16 and 65 such that the 5th AM is 51. Find the number of arithmetic means.

7. Insert five numbers between 11 and 29 such that the resulting sequence is an AP.

8. Prove that the sum of m arithmetic means between the two numbers is to the sum of n arithmetic means between them is $m : n$.

ANSWERS (EXERCISE 11D)

1. (i) 14 (ii) 4 (iii) –12 **2.** 9, 14, 19, 24 **3.** 19, 15, 11 **4.** 8, 5, 2, –1, –4, –7

5. $n = 5$ **6.** six **7.** 14, 17, 20, 23, 26

HINTS TO SOME SELECTED QUESTIONS

5. Let $A_1, A_2, \ldots, A_n$ be the n arithmetic means between 9 and 27.

Then, 9, $A_1, A_2, \ldots, A_n$, 27 are in AP.

$$\therefore \quad 9 + (n + 2 - 1)d = 27 \Rightarrow d = \frac{18}{n+1}.$$

$$\frac{A_n}{A_1} = \frac{2}{1} \Rightarrow \frac{T_{n+1}}{T_2} = \frac{2}{1} \Rightarrow \frac{9 + (n + 1 - 1)d}{9 + (2 - 1)d} = \frac{2}{1}$$

$$\Rightarrow 9 + nd = 18 + 2d \Rightarrow d = \frac{9}{n-2}.$$

$$\therefore \quad \frac{18}{n+1} = \frac{9}{n-2} \Rightarrow 18(n - 2) = 9(n + 1) \Rightarrow n = 5.$$

6. 16, $A_1, A_2, A_3, A_4, A_5, \ldots, A_n$, 65 are in AP.

Then, $T_6 = 51 \Rightarrow a + 5d = 51 \Rightarrow 16 + 5d = 51 \Rightarrow d = 7$.

$T_{n+2} = 65 \Rightarrow 16 + (n + 2 - 1)d = 65 \Rightarrow 16 + (n + 1) \times 7 = 65 \Rightarrow n = 6$.

8. Sum of m AMs between a and b

$$= m \times (\text{AMs between } a \text{ and } b) = m \times \frac{(a+b)}{2}.$$

Sum of n AMs between a and b

$$= n \times (\text{AMs between } a \text{ and } b) = n \times \frac{(a+b)}{2}.$$

$\therefore$ required ratio $= m : n$.

PROPERTIES OF AP

THEOREM 1 *If to each term of an AP a fixed nonzero number is added then the resulting progression is an AP.*

PROOF Let $a, a+d, a+2d, a+3d, \ldots$ be a given AP and let k be a fixed nonzero number. Then,

$(a+k), \{(a+d)+k\}, \{(a+2d)+k\}, \ldots$ is an AP

in which first term $= (a+k)$ and common difference $= d$.

THEOREM 2 *If each term of a given AP is multiplied or divided by a given nonzero fixed number k then the resulting progression is an AP.*

PROOF (i) Let $a, a+d, a+2d, \ldots$ be the given AP and let k be a given nonzero number. Then,

$ak, (a+d)k, (a+2d)k, \ldots$ is clearly an AP

in which the first term $= ak$ and the common difference $= dk$.

(ii) Let $a, a+d, a+2d, \ldots$ be the given AP and let k be a given nonzero number. Then,

$\frac{a}{k}, \frac{(a+d)}{k}, \frac{(a+2d)}{k}, \ldots$ is clearly an AP

in which the first term $= \frac{a}{k}$ and the common difference $= \frac{d}{k}\cdot$

SUMMARY

If $a_1, a_2, a_3, a_4, \ldots$ are in AP and k is a nonzero number then

(i) $(a_1+k), (a_2+k), (a_3+k), (a_4+k), \ldots$ are in AP.

(ii) $(a_1-k), (a_2-k), (a_3-k), (a_4-k), \ldots$ are in AP.

(iii) $a_1k, a_2k, a_3k, a_4k, \ldots$ are in AP.

(iv) $\frac{a_1}{k}, \frac{a_2}{k}, \frac{a_3}{k}, \frac{a_4}{k}, \ldots$ are in AP.

SOLVED EXAMPLES

EXAMPLE 1 *If a, b, c, are in AP, show that*

(i) $(b+c), (c+a)$ *and* $(a+b)$ *are in AP.*

(ii) $a^2(b+c), b^2(c+a)$ *and* $c^2(a+b)$ *are in AP.*

SOLUTION Since a, b, c are in AP, we have

$$2b = (a+c). \qquad \ldots \text{(i)}$$

(i) $(b+c), (c+a), (a+b)$ will be in AP

if $(c+a)-(b+c) = (a+b)-(c+a)$

i.e., if $a-b = b-c$

i.e., if $2b = a+c$, which is true by (i).

$\therefore$ a, b, c are in AP $\Rightarrow$ $(b+c), (c+a), (a+b)$ are in AP.

(ii) $a^2(b+c), b^2(c+a), c^2(a+b)$ will be in AP

$$\text{if } b^2(c+a)-a^2(b+c)=c^2(a+b)-b^2(c+a)$$

i.e., if $c(b^2-a^2)+ab(b-a)=a(c^2-b^2)+bc(c-b)$

i.e., if $(b-a)(ab+bc+ca)=(c-b)(ab+bc+ca)$

i.e., if $(b-a)=(c-b)$

i.e., if $2b=(a+c)$, which is true by (i).

$\therefore$ a, b, c are in AP $\Rightarrow$ $a^2(b+c), b^2(c+a), c^2(a+b)$ are in AP.

EXAMPLE 2 *If a, b, c are in AP, show that*

$$\frac{1}{(\sqrt{b}+\sqrt{c})}, \frac{1}{(\sqrt{c}+\sqrt{a})}, \frac{1}{(\sqrt{a}+\sqrt{b})} \text{ are in AP.}$$

SOLUTION Since a, b, c are in AP, we have

$$2b=(a+c). \qquad \ldots \text{(i)}$$

Now, $\dfrac{1}{(\sqrt{b}+\sqrt{c})}, \dfrac{1}{(\sqrt{c}+\sqrt{a})}, \dfrac{1}{(\sqrt{a}+\sqrt{b})}$ will be in AP

$$\text{if } \frac{1}{(\sqrt{c}+\sqrt{a})}-\frac{1}{(\sqrt{b}+\sqrt{c})}=\frac{1}{(\sqrt{a}+\sqrt{b})}-\frac{1}{(\sqrt{c}+\sqrt{a})}$$

i.e., if $\dfrac{(\sqrt{b}+\sqrt{c})-(\sqrt{c}+\sqrt{a})}{(\sqrt{c}+\sqrt{a})(\sqrt{b}+\sqrt{c})}=\dfrac{(\sqrt{c}+\sqrt{a})-(\sqrt{a}+\sqrt{b})}{(\sqrt{a}+\sqrt{b})(\sqrt{c}+\sqrt{a})}$

i.e., if $\dfrac{(\sqrt{b}-\sqrt{a})}{(\sqrt{c}+\sqrt{a})(\sqrt{b}+\sqrt{c})}=\dfrac{(\sqrt{c}-\sqrt{b})}{(\sqrt{a}+\sqrt{b})(\sqrt{c}+\sqrt{a})}$

i.e., if $\dfrac{(\sqrt{b}-\sqrt{a})}{(\sqrt{b}+\sqrt{c})}=\dfrac{(\sqrt{c}-\sqrt{b})}{(\sqrt{a}+\sqrt{b})}$

i.e., if $b-a=c-b$

i.e., if $2b=a+c$, which is true by (i).

$\therefore$ a, b, c are in AP $\Rightarrow$ $\dfrac{1}{(\sqrt{b}+\sqrt{c})}, \dfrac{1}{(\sqrt{c}+\sqrt{a})}, \dfrac{1}{(\sqrt{a}+\sqrt{b})}$ are in AP.

EXAMPLE 3 *If a, b, c are in AP, show that*

$$[(b+c)^2-a^2], [(c+a)^2-b^2], [(a+b)^2-c^2] \text{ are in AP.}$$

SOLUTION Since a, b, c are in AP, we have

$$2b=(a+c). \qquad \ldots \text{(i)}$$

Now, $[(b+c)^2-a^2], [(c+a)^2-b^2], [(a+b)^2-c^2]$ will be in AP

$$\text{if } [(c+a)^2-b^2]-[(b+c)^2-a^2]=[(a+b)^2-c^2]-[(c+a)^2-b^2]$$

i.e., if $(a+b+c)(c+a-b)-(a+b+c)(b+c-a)$

$$=(a+b+c)(a+b-c)-(a+b+c)(c+a-b)$$

i.e., if $(a+b+c)[(c+a-b)-(b+c-a)]$

$$=(a+b+c)[(a+b-c)-(c+a-b)]$$

i.e., if $2(a-b)=2(b-c)$

i.e., if $(a-b)=(b-c)$

i.e., if $2b = (a + c)$, which is true by (i).

$\therefore$ a, b, c are in AP

$\Rightarrow$ $[(b+c)^2 - a^2], [(c+a)^2 - b^2], [(a+b)^2 - c^2]$ are in AP.

EXAMPLE 4 *If a, b, c are in AP, show that*

(i) $\dfrac{1}{bc}, \dfrac{1}{ca}, \dfrac{1}{ab}$ *are in AP.*

(ii) $a\left(\dfrac{1}{b}+\dfrac{1}{c}\right), b\left(\dfrac{1}{c}+\dfrac{1}{a}\right), c\left(\dfrac{1}{a}+\dfrac{1}{b}\right)$ *are in AP.*

SOLUTION (i) Since a, b, c are in AP, the terms obtained by dividing each term of this AP by abc are also in AP.

Consequently, $\dfrac{1}{bc}, \dfrac{1}{ca}, \dfrac{1}{ab}$ are in AP.

(ii) a, b, c are in AP

$\Rightarrow$ $\dfrac{1}{bc}, \dfrac{1}{ca}, \dfrac{1}{ab}$ are in AP [dividing each term by abc]

$\Rightarrow$ $\left(\dfrac{ab+bc+ca}{bc}\right), \left(\dfrac{(ab+bc+ca}{ca}\right), \left(\dfrac{ab+bc+ca}{ab}\right)$ are in AP

[multiplying each term by $(ab + bc + ca)$]

$\Rightarrow$ $\left(\dfrac{ab+bc+ca}{bc}-1\right), \left(\dfrac{ab+bc+ca}{ca}-1\right), \left(\dfrac{ab+bc+ca}{ab}-1\right)$ are in AP

[adding –1 to each term]

$\Rightarrow$ $\dfrac{(ac+ab)}{bc}, \dfrac{(ab+bc)}{ca}, \dfrac{(bc+ca)}{ab}$ are in AP

$\Rightarrow$ $a\left(\dfrac{c+b}{bc}\right), b\left(\dfrac{a+c}{ca}\right), c\left(\dfrac{b+a}{ab}\right)$ are in AP

$\Rightarrow$ $a\left(\dfrac{1}{b}+\dfrac{1}{c}\right), b\left(\dfrac{1}{c}+\dfrac{1}{a}\right), c\left(\dfrac{1}{a}+\dfrac{1}{b}\right)$ are in AP.

EXAMPLE 5 *If* a^2, b^2, c^2 *are in AP, prove that*

$$\frac{1}{(b+c)}, \frac{1}{(c+a)}, \frac{1}{(a+b)} \text{ are in AP.}$$

SOLUTION $\dfrac{1}{(b+c)}, \dfrac{1}{(c+a)}, \dfrac{1}{(a+b)}$ will be in AP

if $\dfrac{1}{(c+a)} - \dfrac{1}{(b+c)} = \dfrac{1}{(a+b)} - \dfrac{1}{(c+a)}$

i.e., if $\dfrac{(b+c)-(c+a)}{(c+a)(b+c)} = \dfrac{(c+a)-(a+b)}{(a+b)(c+a)}$

i.e., if $\dfrac{(b-a)}{(c+a)(b+c)} = \dfrac{(c-b)}{(a+b)(c+a)}$

i.e., if $\dfrac{(b-a)}{(b+c)} = \dfrac{(c-b)}{(a+b)}$

i.e., if $(b-a)(b+a)=(c+b)(c-b)$

i.e., if $b^2-a^2=c^2-b^2$

i.e., if a^2, b^2, c^2 are in AP.

Hence, a^2, b^2, c^2 are in AP $\Rightarrow \dfrac{1}{(b+c)}, \dfrac{1}{(c+a)}, \dfrac{1}{(a+b)}$ are in AP.

EXAMPLE 6 *If a^2, b^2, c^2 are in AP, prove that*

$$\frac{a}{b+c}, \frac{b}{c+a}, \frac{c}{a+b} \text{ are in AP.}$$

SOLUTION $\dfrac{a}{b+c}, \dfrac{b}{c+a}, \dfrac{c}{a+b}$ will be in AP

if $\dfrac{a}{b+c}+1, \dfrac{b}{c+a}+1, \dfrac{c}{a+b}+1$ are in AP

[on adding 1 to each term]

i.e., if $\dfrac{a+b+c}{b+c}, \dfrac{a+b+c}{c+a}, \dfrac{a+b+c}{a+b}$ are in AP

i.e., if $\dfrac{1}{b+c}, \dfrac{1}{c+a}, \dfrac{1}{a+b}$ are in AP

[on dividing each term by $(a+b+c)$]

i.e., if $\dfrac{1}{c+a}-\dfrac{1}{b+c}=\dfrac{1}{a+b}-\dfrac{1}{c+a}$

i.e., if $\dfrac{(b+c)-(c+a)}{(c+a)(b+c)}=\dfrac{(c+a)-(a+b)}{(a+b)(c+a)}$

i.e., if $\dfrac{(b-a)}{(b+c)}=\dfrac{(c-b)}{(a+b)}$

i.e., if $(b-a)(b+a)=(c-b)(c+b)$

i.e., if $b^2-a^2=c^2-b^2$

i.e., if a^2, b^2, c^2 are in AP.

Hence, a^2, b^2, c^2 are in AP $\Rightarrow \dfrac{a}{(b+c)}, \dfrac{b}{(c+a)}, \dfrac{c}{(a+b)}$ are in AP.

EXAMPLE 7 *Show that $(a^2+ab+b^2), (c^2+ac+a^2)$ and (b^2+bc+c^2) are in AP, if a, b, c are in AP.*

SOLUTION The terms $(a^2+ab+b^2), (c^2+ac+a^2), (b^2+bc+c^2)$ will be in AP

if $(c^2+ac+a^2)-(a^2+ab+b^2)=(b^2+bc+c^2)-(c^2+ac+a^2)$

i.e., if $c^2+ac-ab-b^2=b^2+bc-ac-a^2$

i.e., if $(a^2+c^2+2ac)-b^2=b^2+bc+ab$

i.e., if $(a+c)^2-b^2=b(a+b+c)$

i.e., if $(a+c+b)(a+c-b)=b(a+b+c)$

i.e., if $a+c-b=b$

i.e., if $a+c=2b$

i.e., if a, b, c are in AP.

Hence, a, b, c are in AP $\Rightarrow (a^2+ab+b^2), (c^2+ac+a^2)$ and (b^2+bc+c^2)

are in AP.

EXAMPLE 8 *If* $(b-c)^2, (c-a)^2, (a-b)^2$ *are in AP, prove that*

$$\frac{1}{(b-c)}, \frac{1}{(c-a)}, \frac{1}{(a-b)} \text{ are in AP.}$$

SOLUTION $(b-c)^2, (c-a)^2, (a-b)^2$ are in AP

$\Rightarrow \quad (c-a)^2 - (b-c)^2 = (a-b)^2 - (c-a)^2$

$\Rightarrow \quad (b-a)(2c-a-b) = (c-b)(2a-b-c)$. ... (i)

Now, $\frac{1}{(b-c)}, \frac{1}{(c-a)}, \frac{1}{(a-b)}$ will be in AP

if $\frac{1}{(c-a)} - \frac{1}{(b-c)} = \frac{1}{(a-b)} - \frac{1}{(c-a)}$

i.e., if $\frac{(b-c)-(c-a)}{(c-a)(b-c)} = \frac{(c-a)-(a-b)}{(a-b)(c-a)}$

i.e., if $\frac{(a+b-2c)}{(c-a)(b-c)} = \frac{(b+c-2a)}{(a-b)(c-a)}$

i.e., if $(a-b)(a+b-2c) = (b-c)(b+c-2a)$

i.e., if $(b-a)(2c-a-b) = (c-b)(2a-b-c)$, which is true by (i).

Hence, $(b-c)^2, (c-a)^2, (a-b)^2$ are in AP

$\Rightarrow \quad \frac{1}{(b-c)}, \frac{1}{(c-a)}, \frac{1}{(a-b)}$ are in AP.

EXAMPLE 9 *If* $(a^2+2bc), (b^2+2ac), (c^2+2ab)$ *are in AP, show that*

$$\frac{1}{(b-c)}, \frac{1}{(c-a)}, \frac{1}{(a-b)} \text{ are in AP.}$$

SOLUTION $a^2+2bc, b^2+2ac, c^2+2ab$ are in AP

$\Rightarrow \quad (a^2+2bc)-(ab+bc+ac), (b^2+2ac)-(ab+bc+ac), (c^2+2ab)$
$-(ab+bc+ac)$ are in AP

[by adding $(ab+bc+ac)$ to each term]

$\Rightarrow \quad (a^2+bc-ab-ac), (b^2+ac-ab-bc), (c^2+ab-bc-ac)$ are in AP

$\Rightarrow \quad (a-b)(a-c), (b-c)(b-a), (c-b)(c-a)$ are in AP

$\Rightarrow \quad -(a-b)(c-a), -(a-b)(b-c), -(b-c)(c-a)$ are in AP

$\Rightarrow \quad \frac{-1}{(b-c)}, \frac{-1}{(c-a)}, \frac{-1}{(a-b)}$ are in AP

[on dividing each term by $(a-b)(b-c)(c-a)$]

$\Rightarrow \quad \frac{1}{(b-c)}, \frac{1}{(c-a)}, \frac{1}{(a-b)}$ are in AP

[on multiplying each term by –1].

Hence, the result follows.

EXAMPLE 10 *If* $\frac{(b+c-a)}{a}, \frac{(c+a-b)}{b}, \frac{(a+b-c)}{c}$ *are in AP, prove that* $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ *are in AP.*

SOLUTION $\frac{(b+c-a)}{a}, \frac{(c+a-b)}{b}, \frac{(a+b-c)}{c}$ are in AP

$\Rightarrow \quad \left\{\frac{(b+c-a)}{a}+2\right\}, \left\{\frac{(c+a-b)}{b}+2\right\}, \left\{\frac{(a+b-c)}{c}+2\right\}$ are in AP

[adding 2 to each term]

$\Rightarrow \quad \frac{(a+b+c)}{a}, \frac{(a+b+c)}{b}, \frac{(a+b+c)}{c}$ are in AP

$\Rightarrow \quad \frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ are in AP [on dividing each term by $(a+b+c)$].

EXAMPLE 11 *If $a^2(b+c)$, $b^2(c+a)$, $c^2(a+b)$ are in AP then show that either a, b, c are in AP or $ab+bc+ca=0$.*

SOLUTION $a^2(b+c)$, $b^2(c+a)$, $c^2(a+b)$ are in AP

$$\begin{aligned}
&\Rightarrow \quad b^2(c+a)-a^2(b+c)=c^2(a+b)-b^2(c+a)\\
&\Rightarrow \quad (b^2a-a^2b)+(b^2c-a^2c)=(c^2b-b^2c)+(c^2a-b^2a)\\
&\Rightarrow \quad ab(b-a)+c(b^2-a^2)=bc(c-b)+a(c^2-b^2)\\
&\Rightarrow \quad (b-a)(ab+bc+ca)=(c-b)(ab+bc+ca)\\
&\Rightarrow \quad (ab+bc+ca)\{(b-a)-(c-b)\}=0\\
&\Rightarrow \quad (ab+bc+ca)(2b-a-c)=0\\
&\Rightarrow \quad ab+bc+ca=0 \text{ or } 2b-a-c=0\\
&\Rightarrow \quad ab+bc+ca=0 \text{ or } 2b=a+c\\
&\Rightarrow \quad ab+bc+ca=0 \text{ or } a, b, c \text{ are in AP.}
\end{aligned}$$

EXAMPLE 12 *If a, b, c are in AP, prove that $(a-c)^2=4(b^2-ac)$.*

SOLUTION Since a, b, c are in AP, we have $b=\frac{1}{2}(a+c)$.

$$\begin{aligned}
\therefore \quad \text{RHS} &= 4(b^2-ac)=4\times\left\{\frac{1}{4}(a+c)^2-ac\right\}\\
&=(a+c)^2-4ac=(a-c)^2=\text{LHS}.
\end{aligned}$$

Hence, $(a-c)^2=4(b^2-ac)$.

EXAMPLE 13 *If a, b, c are in AP, prove that $a^3+4b^3+c^3=3b(a^2+c^2)$.*

SOLUTION Since a, b, c are in AP, we have $b=\frac{1}{2}(a+c)$.

$\therefore \quad \text{RHS}=3\cdot\frac{1}{2}(a+c)(a^2+c^2)=\frac{3}{2}(a^3+c^3+ac^2+a^2c)$

and $\quad \text{LHS}=a^3+4\cdot\left(\frac{a+c}{2}\right)^3+c^3=\frac{3}{2}(a^3+c^3+ac^2+a^2c)$.

$\therefore \quad$ LHS = RHS.

EXERCISE 11E

1. If a, b, c are in AP, prove that
   (i) $(a-c)^2=4(a-b)(b-c)$ (ii) $a^2+c^2+4ac=2(ab+bc+ca)$
   (iii) $a^3+c^3+6abc=8b^3$

2. If a, b, c are in AP, show that

$$(a+2b-c)(2b+c-a)(c+a-b)=4abc.$$

3. If a, b, c are in AP, show that

(i) $(b+c-a), (c+a-b), (a+b-c)$ are in AP.

(ii) $(bc-a^2), (ca-b^2), (ab-c^2)$ are in AP.

4. If $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ are in AP, prove that

(i) $\frac{(b+c)}{a}, \frac{(c+a)}{b}, \frac{(a+b)}{c}$ are in AP.

(ii) $\frac{(b+c-a)}{a}, \frac{(c+a-b)}{b}, \frac{(a+b-c)}{c}$ are in AP.

5. If $a\left(\frac{1}{b}+\frac{1}{c}\right), b\left(\frac{1}{c}+\frac{1}{a}\right), c\left(\frac{1}{a}+\frac{1}{b}\right)$ are in AP, prove that $a^2(b+c), b^2(c+a), c^2(a+b)$ are in AP.

6. If a, b, c are in AP, show that $\frac{a(b+c)}{bc}, \frac{b(c+a)}{ca}, \frac{c(a+b)}{ab}$ are also in AP.

HINTS TO SOME SELECTED QUESTIONS

4. $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ are in AP

$\Rightarrow \quad \frac{a+b+c}{a}, \frac{a+b+c}{b}, \frac{a+b+c}{c}$ are in AP

$\Rightarrow \quad 1+\frac{b+c}{a}, 1+\frac{c+a}{b}, 1+\frac{a+b}{c}$ are in AP

$\Rightarrow \quad \frac{b+c}{a}, \frac{c+a}{b}, \frac{a+b}{c}$ are in AP. This proves (i).

$\therefore \quad \frac{b+c}{a}-1, \frac{c+a}{b}-1, \frac{a+b}{c}-1$ are in AP

$\Rightarrow \quad \frac{b+c-a}{a}, \frac{c+a-b}{b}, \frac{a+b-c}{c}$ are in AP. This proves (ii).

5. $a\left(\frac{1}{b}+\frac{1}{c}\right), b\left(\frac{1}{c}+\frac{1}{a}\right), c\left(\frac{1}{a}+\frac{1}{b}\right)$ are in AP

$\Rightarrow \quad \frac{a(b+c)}{bc}, \frac{b(c+a)}{ca}, \frac{c(a+b)}{ab}$ are in AP

$\Rightarrow \quad \frac{a^2(b+c)}{abc}, \frac{b^2(c+a)}{abc}, \frac{c^2(a+b)}{abc}$ are in AP

$\Rightarrow \quad a^2(b+c), b^2(c+a), c^2(a+b)$ are in AP.

6. $\frac{a(b+c)}{bc}, \frac{b(c+a)}{ca}, \frac{c(a+b)}{ab}$ will be in AP

if $\frac{a(b+c)}{bc}+1, \frac{b(c+a)}{ca}+1, \frac{c(a+b)}{ab}+1$ are in AP

i.e., if $\frac{ab+bc+ca}{bc}, \frac{ab+bc+ca}{ca}, \frac{ab+bc+ca}{ab}$ are in AP

i.e., if $\frac{1}{bc}, \frac{1}{ca}, \frac{1}{ab}$ are in AP i.e., if $\frac{abc}{bc}, \frac{abc}{ca}, \frac{abc}{ab}$ are in AP

i.e., if a, b, c are in AP.

EXERCISE 11F

Very-Short-Answer Questions

1. If the sum of n terms of an AP is given by $S_n = (2n^2 + 3n)$ then find its common difference.
2. If 9 times the 9th term of an AP is equal to 13 times the 13th term, show that its 22nd term is 0.
3. In an AP it is given that $S_n = qn^2$ and $S_m = qm^2$. Prove that $S_q = q^3$.
4. Find three arithmetic means between 6 and –6.
5. The 9th term of an AP is 0. Prove that its 29th term is double the 19th term.
6. How many terms are there in the AP 13, 16, 19, …, 43?
7. Find the 8th term from the end of the AP 7, 9, 11, …, 201.
8. How many 2-digit numbers are divisible by 7?
9. If 7th and 13th terms of an AP be 34 and 64 respectively then find its 18th term.
10. What is the 10th common term between the APs 3, 7, 11, 15, 19, … and 1, 6, 11, 16, …?
11. The first and last terms of an AP are 1 and 11 respectively. If the sum of its terms is 36, find the number of terms.
12. In an AP, the pth term is q and $(p+q)$th term is 0. Show that its qth term is p.
13. If $\dfrac{3+5+7+9+\ldots \text{ up to 35 terms}}{5+8+11+\ldots \text{ up to } n \text{ terms}} = 7$, find the value of n.
14. Write the sum of first n even natural numbers.
15. Write the sum of first n odd natural numbers.
16. The sum of n terms of an AP is $\frac{1}{2}an^2 + bn$. Find the common difference.
17. If the sums of n terms of two APs are in the ratio $(2n+3):(3n+2)$, find the ratio of their 10th terms.

ANSWERS (EXERCISE 11F)

1. 4 **4.** 3, 0, –3 **6.** 11 **7.** 197 **8.** 13

9. 89 **10.** 191 **11.** 6 **13.** $n = 10$ **14.** $n(n+1)$

15. n^2 **16.** a **17.** 41 : 59

HINTS TO SOME SELECTED QUESTIONS

1. $T_n = (S_n - S_{n-1}) = (2n^2 + 3n) - \{2(n-1)^2 + 3(n-1)\}$
$= (2n^2 + 3n) - (2n^2 - n - 1) = 4n + 1.$
$\therefore \quad d = (T_2 - T_1) = (4 \times 2 + 1) - (4 \times 1 + 1) = (9 - 5) = 4.$

2. $9(a + 8d) = 13(a + 12d) \Rightarrow 4a + 84d = 0 \Rightarrow a + 21d = 0 \Rightarrow T_{22} = 0.$

3. $S_n = qn^2 \Rightarrow S_1 = q$ and $S_2 = 4q$
$\Rightarrow T_1 = q$ and $T_1 + T_2 = 4q \Rightarrow T_1 = q$ and $T_2 = 3q$.
$\therefore \quad S_q = \frac{q}{2} \times [2q + (q-1) \times 2q] = \left(\frac{q}{2} \times 2q^2\right) = q^3.$

4. Let the required AM be x_1, x_2, x_3. Then,
$6, x_1, x_2, x_3, -6$ are in AP.
Now, $T_5 = -6 \Rightarrow a + 4d = -6 \Rightarrow 6 + 4d = -6 \Rightarrow 4d = -12 \Rightarrow d = -3.$
$\therefore \quad x_1 = 6 + d = (6 - 3) = 3,\ x_2 = 6 + 2d = (6 - 6) = 0,\ x_3 = 6 + 3d = (6 - 9) = -3.$

5. $T_9 = 0 \Rightarrow a + 8d = 0 \Rightarrow a = -8d.$
$T_{19} = a + 18d = -8d + 18d = 10d$ and $T_{29} = a + 28d = -8d + 28d = 20d.$
$\therefore \quad T_{29} = 2 \times T_{19}.$

6. Let it contain n terms. Then, $T_n = 43 \Rightarrow a + (n-1)d = 43.$
$\therefore \quad 13 + (n-1) \times 3 = 43 \Rightarrow n = 11.$

7. Here, $a = 7$, $d = (9 - 7) = 2$ and $l = 201$.
Let $T_n = 201$. Then, $a + (n-1)d = 201 \Rightarrow 7 + (n-1) \times 2 = 201 \Rightarrow n = 98.$
$\therefore$ 8th term from the end $= l - (n-1)d = (201 - 7 \times 2) = (201 - 14) = 197.$

8. 2-digit numbers divisible by 7 are 14, 21, 28, …, 98.
Let their number be n. Then, $T_n = 98$.
$\therefore \quad 14 + (n-1) \times 7 = 98 \Rightarrow n = 13.$

9. $a + 6d = 34$ … (i) and $a + 12d = 64.$ … (ii)
On solving (i) and (ii), we get $a = 4$ and $d = 5$.
$\therefore \quad T_{18} = a + 17d = (4 + 17 \times 5) = 89.$

10. The first common term = 11.
Common difference of new AP = LCM (4, 5) = 20.
So, the new AP is 11, 31, 51, … .
Its 10th term $= (11 + 9 \times 20) = 191.$

11. $\frac{n}{2}(a + l) = 36 \Rightarrow \frac{n}{2} \times (1 + 11) = 36 \Rightarrow n = 6.$

13. $\dfrac{\frac{35}{2} \cdot \{6 + 34 \times 2\}}{\frac{n}{2} \cdot \{10 + (n-1) \times 3\}} = 7 \Rightarrow 3n^2 + 7n - 370 = 0$
$\Rightarrow \quad (3n + 37)(n - 10) = 0 \Rightarrow n = 10.$

14. Given sum $= 2 + 4 + 6 + 8 + \ldots + 2n$
$= \frac{n}{2}(2 + 2n) = n(n+1).$

15. Given sum $= 1 + 3 + 5 + \ldots$ to n terms
$= \frac{n}{2} \times [2 \times 1 + (n-1) \times 2] = n^2.$

16. $T_n = (S_n - S_{n-1})$

$$= \left\{\frac{1}{2}an^2 + bn\right\} - \left\{\frac{1}{2}a(n-1)^2 + b(n-1)\right\}$$

$$= \left\{\frac{1}{2}an^2 + bn\right\} - \left\{\frac{1}{2}an^2 - an + \frac{1}{2}a + bn - b\right\} = \left(an + b - \frac{1}{2}a\right).$$

$$T_{n-1} = a(n-1) + b - \frac{1}{2}a = \left(an + b - \frac{1}{2}a\right) - a.$$

$\therefore \quad d = (T_n - T_{n-1}) = a.$

17. $$\frac{\frac{n}{2}\cdot[2a_1 + (n-1)d_1]}{\frac{n}{2}\cdot[2a_2 + (n-1)d_2]} = \frac{2n+3}{3n+2} \Rightarrow \frac{2a_1 + (n-1)d_1}{2a_2 + (n-1)d_2} = \frac{2n+3}{3n+2}.$$

$$\therefore \quad \frac{a_1 + \left(\frac{n-1}{2}\right)d_1}{a_2 + \left(\frac{n-1}{2}\right)d_2} = \frac{2n+3}{3n+2} \Rightarrow \frac{a_1 + 9d_1}{a_2 + 9d_2} = \frac{2\times 19 + 3}{3\times 19 + 2} \qquad \left[\text{putting } \frac{n-1}{2} = 9 \Rightarrow n = 19\right]$$

$$\Rightarrow \frac{a_1 + 9d_1}{a_2 + 9d_2} = \frac{41}{59}.$$

Hence, the ratio of their 10th terms is 41 : 59.

REVIEW OF FACTS AND FORMULAE

1. (i) **Sequence** A succession of numbers arranged in a definite order according to a certain given rule is called a sequence.

The number occurring at the nth place of a sequence is called its nth term, denoted by a_n.

(ii) By adding the terms of a sequence, we get a series.

(iii) A series is said to be finite or infinite according as the number of terms in it is finite or infinite respectively.

2. (i) **Progressions** Sequences following certain patterns are called progressions.

(ii) **Arithmetic Progression (AP)**

The progression of the form $a, (a+d), (a+2d), \ldots$ is called an AP with first term $= a$ and common difference $= d$.

We have, $T_n = a + (n-1)d$, called the general term.

Clearly, $T_n - T_{n-1} = d$.

(iii) If l is the last term of the AP then nth term from the end $= l - (n-1)d$.

(iv) If an AP contains n terms then

pth term from the end $= (n - p + 1)$th term from the beginning.

3. Sum to n terms of an AP is given by

(i) $S_n = \frac{n}{2}\cdot\{2a + (n-1)d\}$

(ii) $S_n = \frac{n}{2}(a + l)$

(iii) $S_1 = T_1, (S_2 - S_1) = T_2, S_n - S_{n-1} = T_n$

4. (i) 3 numbers in AP are taken as $(a-d), a, (a+d)$.

(ii) 4 numbers in AP are taken as $(a-3d), (a-d), (a+d), (a+3d)$.

(iii) 5 numbers in AP are taken as $(a-2d), (a-d), a, (a+d), (a+2d)$.

5. If $a_1, a_2, a_3, a_4, \ldots$ are in AP and k is a nonzero number then

(i) $(a_1+k), (a_2+k), (a_3+k), (a_4+k), \ldots$ are in AP.

(ii) $(a_1-k), (a_2-k), (a_3-k), (a_4-k), \ldots$ are in AP.

(iii) $a_1k, a_2k, a_3k, a_4k, \ldots$ are in AP.

(iv) $\frac{a_1}{k}, \frac{a_2}{k}, \frac{a_3}{k}, \frac{a_4}{k}, \ldots$ are in AP.

6. (i) **Arithmetic Mean (AM)** If a, A, b are in AP then A is the arithmetic mean between a and b and $A = \frac{1}{2}(a+b)$.

(ii) If $a, A_1, A_2, \ldots, A_n, b$ are in AP then $A_1, A_2, \ldots, A_n$ are n arithmetic means between a and b.

❑

12
Geometrical Progression

GEOMETRICAL PROGRESSION (GP)

A sequence $a_1, a_2, a_3, \ldots, a_n$ is called a geometrical progression, if each term is nonzero and $\dfrac{a_{k+1}}{a_k} = r$ (constant) for all $k \geq 1$.

The constant ratio is called its *common ratio*.

A geometrical progression is abbreviated as GP.

In a GP we usually denote the *first term* by a, the *common ratio* by r and the *nth term* by T_n.

The nth term of a GP is called its general term.

SOME EXAMPLES OF GP

EXAMPLE 1 *Show that the progression* 6, 18, 54, 162, ... *is a GP. Write down its first term and the common ratio.*

SOLUTION We have $\dfrac{18}{6} = \dfrac{54}{18} = \dfrac{162}{54} = 3$ (constant).

So, the given progression is a GP in which the first term = 6 and the common ratio = 3.

EXAMPLE 2 *Show that the progression* $-16, 4, -1, \dfrac{1}{4}, \cdots$ *is a GP. Write down its first term and the common ratio.*

SOLUTION We have $\dfrac{4}{-16} = \dfrac{-1}{4} = \dfrac{(1/4)}{-1} = \dfrac{-1}{4}$ (constant).

So, the given progression is a GP in which $a = -16$ and $r = \dfrac{-1}{4}\cdot$

EXAMPLE 3 *Show that the progression* $\dfrac{1}{2}, \dfrac{-1}{3}, \dfrac{2}{9}, \dfrac{-4}{27}, \cdots$ *is a GP. Write down its first term and find the common ratio.*

SOLUTION We have

$$\left(\frac{-1}{3}\right) \div \frac{1}{2} = \left(\frac{-1}{3} \times 2\right) = \frac{-2}{3},\ \frac{2}{9} \div \left(\frac{-1}{3}\right) = \left\{\frac{2}{9} \times \frac{3}{(-1)}\right\} = \frac{-2}{3},$$

$$\left(\frac{-4}{27}\right) \div \frac{2}{9} = \left(\frac{-4}{27} \times \frac{9}{2}\right) = \frac{-2}{3}.$$

So, the given progression is a GP in which $a = \dfrac{1}{2}$ and $r = \dfrac{-2}{3}\cdot$

EXAMPLE 4 *Show that the sequence given by* $T_n = (2 \times 3^n)$ *for all* $n \in N$ *is a GP. Find its first term and the common ratio.*

SOLUTION We have

$$T_n = (2 \times 3^n) \text{ and } T_{n+1} = (2 \times 3^{n+1}).$$

$$\therefore \quad \frac{T_{n+1}}{T_n} = \frac{2 \times 3^{n+1}}{2 \times 3^n} = 3 \text{ (constant) for all } n \in N.$$

Also, $T_1 = (2 \times 3^1) = (2 \times 3) = 6$.

$\therefore$ the first term = 6 and the common ratio = 3.

GEOMETRIC SERIES *If* $a_1, a_2, a_3, \ldots, a_n, \ldots$ *is a GP then the sum* $a_1 + a_2 + a_3 + \ldots + a_n + \ldots$ *is called a geometric series.*

A geometric series is finite or infinite according as the corresponding GP is finite or infinite.

GENERAL TERM OF A GP

THEOREM 1 *Prove that the nth term of a GP with first term a and common ratio r is given by* $T_n = ar^{n-1}$.

PROOF Let us consider a GP in which first term = a and common ratio = r. Then, the GP is $a, ar, ar^2, \ldots$.

Its first term, $T_1 = a = ar^{(1-1)}$;

second term, $T_2 = ar = ar^{(2-1)}$;

third term, $T_3 = ar^2 = ar^{(3-1)}$;

… … … … …

… … … … …

$\therefore$ nth term, $T_n = ar^{(n-1)}$.

Hence, $T_n = ar^{n-1}$ (known as the general term of the GP).

> **REMEMBER**
>
> In a GP with first term = a and common ratio = r, we have
>
> nth term, $T_n = ar^{n-1}$.

SOLVED EXAMPLES

EXAMPLE 1 *Find the* 10*th term and the general term of the progression*

$$\frac{1}{4}, \frac{-1}{2}, 1, -2, 4, \ldots .$$

SOLUTION In the given progression, we have

$$\left(\frac{-1}{2}\right) \div \frac{1}{4} = \left(\frac{-1}{2} \times 4\right) = -2, \; 1 \div \left(\frac{-1}{2}\right) = 1 \times (-2) = -2,$$

$$(-2) \div 1 = -2 \text{ and } 4 \div (-2) = -2.$$

So, the given progression is a GP in which $a = \frac{1}{4}$ and $r = -2$.

$\therefore$ the 10th term, $T_{10} = ar^{(10-1)} = ar^9 = \frac{1}{4} \times (-2)^9 = \frac{-512}{4} = -128.$

The general term, $T_n = ar^{(n-1)} = \frac{1}{4} \times (-2)^{(n-1)} = (-1)^{(n-1)} \times 2^{(n-3)}$.

EXAMPLE 2 *Show that the progression*

$$1, \frac{(\sqrt{2}-1)}{2\sqrt{3}}, \frac{(3-2\sqrt{2})}{12}, \frac{(5\sqrt{2}-7)}{24\sqrt{3}}, \ldots$$

is a GP. Find its 5th term.

SOLUTION In the given progression, we have

$$\frac{T_2}{T_1} = \left\{\frac{(\sqrt{2}-1)}{2\sqrt{3}} \times \frac{1}{1}\right\} = \frac{(\sqrt{2}-1)}{2\sqrt{3}}, \frac{T_3}{T_2} = \frac{(3-2\sqrt{2})}{12} \times \frac{2\sqrt{3}}{(\sqrt{2}-1)} = \frac{(\sqrt{2}-1)}{2\sqrt{3}},$$

$$\frac{T_4}{T_3} = \frac{(5\sqrt{2}-7)}{24\sqrt{3}} \times \frac{12}{(3-2\sqrt{2})} = \frac{(\sqrt{2}-1)}{2\sqrt{3}}.$$

$$\therefore \quad \frac{T_2}{T_1} = \frac{T_3}{T_2} = \frac{T_4}{T_3} = \ldots = \frac{(\sqrt{2}-1)}{2\sqrt{3}} \text{ (constant).}$$

So, the given progression is a GP in which $a = 1$ and $r = \frac{(\sqrt{2}-1)}{2\sqrt{3}}$.

$$\therefore \quad \text{the 5th term, } T_5 = ar^{(5-1)} = ar^4 = 1 \times \left(\frac{\sqrt{2}-1}{2\sqrt{3}}\right)^4$$

$$= \frac{(\sqrt{2}-1)^4}{144} = \frac{(3-2\sqrt{2})^2}{144}$$

$$= \frac{(17-12\sqrt{2})}{144}.$$

Hence, $T_5 = \frac{(17-12\sqrt{2})}{144}$.

EXAMPLE 3 *Which term of the GP* 2, 8, 32, 128, … *is* 131072?

SOLUTION In the given GP, we have $a = 2$ and $r = \frac{8}{2} = 4$.

Let its nth term be 131072. Then,

$$\begin{aligned} T_n = 131072 &\Rightarrow ar^{(n-1)} = 131072 \\ &\Rightarrow 2 \times 4^{(n-1)} = 131072 \\ &\Rightarrow 4^{(n-1)} = 65536 = 4^8 \\ &\Rightarrow n-1 = 8 \Rightarrow n = 9. \end{aligned}$$

Hence, the 9th term of the given GP is 131072.

EXAMPLE 4 *Which term of the GP* $\sqrt{3}, \frac{1}{\sqrt{3}}, \frac{1}{3\sqrt{3}}, \frac{1}{9\sqrt{3}}, \ldots$ *is* $\frac{1}{729\sqrt{3}}$?

SOLUTION In the given GP, we have $a = \sqrt{3}$ and $r = \left(\frac{1}{\sqrt{3}} \times \frac{1}{\sqrt{3}}\right) = \frac{1}{3}$.

Let its nth term be $\frac{1}{729\sqrt{3}}$. Then,

$$T_n = \frac{1}{729\sqrt{3}} \Rightarrow ar^{(n-1)} = \frac{1}{729\sqrt{3}}$$
$$\Rightarrow \sqrt{3} \times \left(\frac{1}{3}\right)^{(n-1)} = \frac{1}{729\sqrt{3}}$$
$$\Rightarrow \frac{1}{3^{(n-1)}} = \frac{1}{(729 \times 3)} = \frac{1}{2187} = \frac{1}{3^7}$$
$$\Rightarrow n-1=7 \Rightarrow n=8.$$

Hence, the 8th term of the given GP is $\frac{1}{729\sqrt{3}}$.

EXAMPLE 5 *If the* 4*th and* 9*th terms of a GP are* 54 *and* 13122 *respectively, find the GP. Also, find its general term.*

SOLUTION Let a be the first term and r be the common ratio of the given GP.

Then, $T_4 = 54 \Rightarrow ar^{(4-1)} = 54 \Rightarrow ar^3 = 54.$ ... (i)

And, $T_9 = 13122 \Rightarrow ar^{(9-1)} = 13122 \Rightarrow ar^8 = 13122.$ ... (ii)

On dividing (ii) by (i), we get

$$\frac{ar^8}{ar^3} = \frac{13122}{54} \Rightarrow r^5 = 243 = 3^5 \Rightarrow r = 3.$$

Putting $r = 3$ in (i), we get

$$a \times 3^3 = 54 \Rightarrow 27a = 54 \Rightarrow a = 2.$$

Thus, $a = 2$ and $r = 3$.

So, the required GP is 2, 6, 18, 54, ...

The general term of the GP is given by

$$T_n = ar^{(n-1)} = \{2 \times 3^{(n-1)}\}.$$

EXAMPLE 6 *The first term of a GP is* 1 *and the sum of its* 3*rd and* 5*th terms is* 90. *Find the common ratio of the GP.*

SOLUTION It is given that the first term of the GP is 1. Let r be the common ratio of this GP. Then,

$$T_3 = 1 \times r^{(3-1)} = r^2 \text{ and } T_5 = 1 \times r^{(5-1)} = r^4.$$

$$\therefore \quad (T_3 + T_5) = 90 \Rightarrow r^2 + r^4 = 90$$
$$\Rightarrow r^4 + r^2 - 90 = 0$$
$$\Rightarrow t^2 + t - 90 = 0, \text{ where } r^2 = t$$
$$\Rightarrow (t+10)(t-9) = 0$$
$$\Rightarrow t = -10 \text{ or } t = 9$$
$$\Rightarrow t = 9 \qquad [\because t = r^2 \neq -10]$$
$$\Rightarrow r^2 = 9$$
$$\Rightarrow r = 3 \text{ or } r = -3.$$

Hence, the common ratio of the given GP is 3 or –3.

EXAMPLE 7 *The 4th, 7th and 10th terms of a GP are a, b, c respectively. Prove that* $b^2 = ac$.

SOLUTION Let A be the first term and r be the common ratio of the given GP. Then,

$$a = Ar^{(4-1)} = Ar^3;\ b = Ar^{(7-1)} = Ar^6 \text{ and } c = Ar^{(10-1)} = Ar^9.$$

$$\therefore \quad ac = (Ar^3) \times (Ar^9) = A^2r^{12} = (Ar^6)^2 = b^2.$$

Hence, $b^2 = ac$.

EXAMPLE 8 *If a, b, c are three consecutive terms of an AP and x, y, z are three consecutive terms of a GP, then prove that*

$$x^{(b-c)} \cdot y^{(c-a)} \cdot z^{(a-b)} = 1.$$

SOLUTION It is given that a, b, c are in AP. Let d be the common difference of this AP. Then,

$$(b-c) = -(c-b) = -d,\ (c-a) = \{(c-b)+(b-a)\} = 2d$$

and $(a-b) = -(b-a) = -d$.

Also, x, y, z are in GP. So, $y = \sqrt{xz}$.

$$\therefore \quad x^{(b-c)} \cdot y^{(c-a)} \cdot z^{(a-b)}$$

$$= x^{-d} \times (\sqrt{xz})^{2d} \times z^{-d}$$

$$[\because\ (b-c) = -d, (c-a) = 2d, (a-b) = -d \text{ and } y = \sqrt{xz}]$$

$$= x^{-d} \times (xz)^d \times z^{-d}$$

$$= x^{-d} \times x^d \times z^d \times z^{-d} = x^{(-d+d)} \times z^{(-d+d)} = (x^0 \times z^0) = 1.$$

Hence, $x^{(b-c)} \cdot y^{(c-a)} \cdot z^{(a-b)} = 1$.

EXAMPLE 9 *If a, b, c, d are in GP, prove that*

$$(a^2+b^2+c^2)(b^2+c^2+d^2) = (ab+bc+cd)^2.$$

SOLUTION Let r be the common ratio of the GP a, b, c, d.

Then, $b = ar$, $c = ar^2$ and $d = ar^3$.

$$\therefore \quad \text{LHS} = (a^2+b^2+c^2)(b^2+c^2+d^2)$$

$$= (a^2+a^2r^2+a^2r^4)(a^2r^2+a^2r^4+a^2r^6)$$

$$= a^4r^2(1+r^2+r^4)^2.$$

And, $\text{RHS} = (ab+bc+cd)^2 = (a^2r+a^2r^3+a^2r^5)^2$

$$= a^4r^2(1+r^2+r^4)^2.$$

Hence, $(a^2+b^2+c^2)(b^2+c^2+d^2) = (ab+bc+cd)^2$.

EXAMPLE 10 *If a, b, c, d are in GP, prove that*

$$(b-c)^2+(c-a)^2+(d-b)^2 = (a-d)^2.$$

SOLUTION Let r be the common ratio of the GP a, b, c, d.

Then, $b = ar$, $c = ar^2$ and $d = ar^3$.

$$\therefore \quad \text{LHS} = (b-c)^2+(c-a)^2+(d-b)^2$$

$$= (ar-ar^2)^2+(ar^2-a)^2+(ar^3-ar)^2$$

$$= \{a(r-r^2)\}^2 + \{a(r^2-1)\}^2 + \{a(r^3-r)\}^2$$
$$= a^2(r-r^2)^2 + a^2(r^2-1)^2 + a^2(r^3-r)^2$$
$$= a^2 \cdot \{(r-r^2)^2 + (r^2-1)^2 + (r^3-r)^2\}$$
$$= a^2 \cdot \{(r^2+r^4-2r^3) + (r^4+1-2r^2) + (r^6+r^2-2r^4)\}$$
$$= a^2 \cdot (r^6-2r^3+1) = a^2(1-r^3)^2$$
$$= (a-ar^3)^2 = (a-d)^2 = \text{RHS}.$$

Hence, $(b-c)^2 + (c-a)^2 + (d-b)^2 = (a-d)^2$.

EXAMPLE 11 *If the pth, qth and rth terms of a GP be a, b, c respectively then prove that*
$$a^{(q-r)} \cdot b^{(r-p)} \cdot c^{(p-q)} = 1.$$

SOLUTION Let A be the first term and R be the common ratio of the given GP. Then,

$$T_p = a \Rightarrow a = AR^{(p-1)}$$
$$T_q = b \Rightarrow b = AR^{(q-1)}$$
$$T_r = c \Rightarrow c = AR^{(r-1)}.$$

$\therefore$ $a^{(q-r)} \cdot b^{(r-p)} \cdot c^{(p-q)}$

$$= \{AR^{(p-1)}\}^{(q-r)} \cdot \{AR^{(q-1)}\}^{(r-p)} \cdot \{AR^{(r-1)}\}^{(p-q)}$$
$$= A^{(q-r)} \cdot R^{(p-1)(q-r)} \cdot A^{(r-p)} \cdot R^{(q-1)(r-p)} \cdot A^{(p-q)} \cdot R^{(r-1)(p-q)}$$
$$= A^{\{(q-r)+(r-p)+(p-q)\}} \cdot R^{\{(p-1)(q-r)+(q-1)(r-p)+(r-1)(p-q)\}}$$
$$= A^0 \cdot R^{p(q-r)+q(r-p)+r(p-q)+(r-q)+(p-r)+(q-p)}$$
$$= (1 \times R^0) = (1 \times 1) = 1.$$

Hence, $a^{(q-r)} \cdot b^{(r-p)} \cdot c^{(p-q)} = 1$.

EXAMPLE 12 *If the first and the nth terms of a GP are a and b respectively and P is the product of its first n terms then prove that* $P^2 = (ab)^n$.

SOLUTION Let r be the comon ratio of the given GP. Then,

$$T_n = b \Rightarrow ar^{n-1} = b \Rightarrow r^{n-1} = \frac{b}{a} \Rightarrow r = \left(\frac{b}{a}\right)^{\frac{1}{(n-1)}}. \quad \ldots \text{(i)}$$

$\therefore$ P = product of first n terms of the GP

$$= a \cdot ar \cdot ar^2 \ldots \cdot ar^{n-1}$$
$$= a^n \cdot r^{\{1+2+3+\ldots+(n-1)\}} = a^n \cdot r^{\frac{1}{2}n(n-1)}$$

[taking the sum of the AP]

$$= a^n \cdot \left\{\left(\frac{b}{a}\right)^{\frac{1}{(n-1)}}\right\}^{\frac{1}{2}n(n-1)} \quad \text{[using (i)]}$$
$$= a^n \cdot \left(\frac{b}{a}\right)^{\left\{\frac{1}{(n-1)} \times \frac{1}{2}n(n-1)\right\}} = a^n \cdot \left(\frac{b}{a}\right)^{\frac{n}{2}}$$

$$\Rightarrow \quad P^2 = a^{2n} \cdot \left(\frac{b}{a}\right)^n = \left(a^{2n} \cdot \frac{b^n}{a^n}\right) = a^n b^n = (ab)^n.$$

Hence, $P^2 = (ab)^n$.

EXAMPLE 13 *The $(m+n)$th and the $(m-n)$th terms of a GP are p and q respectively. Show that the mth and the nth terms of the GP are $\sqrt{pq}$ and $p \cdot \left(\frac{q}{p}\right)^{(m/2n)}$ respectively.*

SOLUTION Let a be the first term and r be the common ratio of the given GP. Then,

$$T_{m+n} = p \text{ and } T_{m-n} = q$$

$$\Rightarrow \quad ar^{(m+n-1)} = p \quad \ldots \text{(i)} \qquad \text{and} \quad ar^{(m-n-1)} = q \quad \ldots \text{(ii)}$$

$$\Rightarrow \quad \frac{ar^{(m+n-1)}}{ar^{(m-n-1)}} = \frac{p}{q}$$

$$\Rightarrow \quad r^{\{(m+n-1)-(m-n-1)\}} = \frac{p}{q}$$

$$\Rightarrow \quad r^{2n} = \frac{p}{q} \Rightarrow r = \left(\frac{p}{q}\right)^{\frac{1}{2n}}$$

$$\Rightarrow \quad \frac{1}{r} = \left(\frac{q}{p}\right)^{\frac{1}{2n}}. \quad \ldots \text{(iii)}$$

$$\therefore \quad T_m = ar^{(m-1)}$$

$$= ar^{(m+n-1)} \times \left(\frac{1}{r}\right)^n$$

$$= p \times \left\{\left(\frac{q}{p}\right)^{\frac{1}{2n}}\right\}^n \quad \text{[using (i) and (iii)]}$$

$$= p \times \left(\frac{q}{p}\right)^{\left(\frac{1}{2n} \times n\right)} = p \times \left(\frac{q}{p}\right)^{\frac{1}{2}} = \sqrt{pq}.$$

And, $T_n = ar^{(n-1)}$

$$= ar^{(m+n-1)} \times \left(\frac{1}{r}\right)^m$$

$$= p \times \left\{\left(\frac{q}{p}\right)^{\frac{1}{2n}}\right\}^m = p \times \left(\frac{q}{p}\right)^{\frac{m}{2n}} \quad \text{[using (i) and (iii)].}$$

Hence, $T_m = \sqrt{pq}$ and $T_n = p \times \left(\frac{q}{p}\right)^{\frac{m}{2n}}$.

EXAMPLE 14 *The pth, qth and rth terms of an AP as well as those of a GP are a, b, c, respectively. Prove that*

$$a^{b-c} \cdot b^{c-a} \cdot c^{a-b} = 1.$$

SOLUTION Let x be the first term and d be the common difference of the AP. Then,

$$a = x + (p-1)d,\ b = x + (q-1)d \text{ and } c = x + (r-1)d.$$

$$\therefore \quad (b-c) = (q-r)d \quad \ldots \text{(i)}, \qquad (c-a) = (r-p)d \quad \ldots \text{(ii)},$$

$$\text{and} \quad (a-b) = (p-q)d \quad \ldots \text{(iii)}.$$

Now, let A be the first term and R be the common ratio of the GP. Then,

$$a = AR^{(p-1)} \quad \ldots \text{(iv)}, \qquad b = AR^{(q-1)} \quad \ldots \text{(v)},$$

and $c = AR^{(r-1)}$ ... (vi).

$\therefore \quad a^{(b-c)} \cdot b^{(c-a)} \cdot c^{(a-b)}$

$$= \{AR^{(p-1)}\}^{(q-r)d} \times \{AR^{(q-1)}\}^{(r-p)d} \times \{AR^{(r-1)}\}^{(p-q)d}$$

$$= A^{\{(q-r)d+(r-p)d+(p-q)d\}} \times R^{\{(p-1)(q-r)d+(q-1)(r-p)d+(r-1)(p-q)d\}}$$

$$= A^0 \times R^0 = (1 \times 1) = 1$$

$$\left[\begin{array}{l} \because \;\; (q-r)d+(r-p)d+(p-q)d = 0 \\ \text{and } p(q-r)d+q(r-p)d+r(p-q)d-(q-r)d-(r-p)d-(p-q)d=0 \end{array}\right].$$

Hence, $a^{(b-c)} \cdot b^{(c-a)} \cdot c^{(a-b)} = 1$.

EXAMPLE 15 *If x, $2y$, $3z$ are in AP, where the distinct numbers x, y, z are in GP then find the common ratio of the GP.*

SOLUTION It is given that the distinct numbers x, y, z are in GP.

Let r be the common ratio of this GP. Then,

$$y = xr \text{ and } z = xr^2.$$

Now, x, $2y$, $3z$ are in AP

$\Rightarrow \quad x, 2xr, 3xr^2$ are in AP

$\Rightarrow \quad 2xr - x = 3xr^2 - 2xr$

$\Rightarrow \quad 3xr^2 - 4xr + x = 0$

$\Rightarrow \quad 3r^2 - 4r + 1 = 0 \qquad [\because\; x \neq 0$, as it is a term of GP$]$

$\Rightarrow \quad (r-1)(3r-1) = 0 \Rightarrow r = \frac{1}{3} \qquad [\because\; x \neq y \neq z \Rightarrow r \neq 1]$.

Hence, the common ratio of the GP is $\frac{1}{3}$.

EXAMPLE 16 *In a GP of positive terms, if any term is equal to the sum of the next two terms then show that the common ratio of the GP is* $2\sin 18°$.

SOLUTION Let us consider a GP of positive terms with first term $= a$ and common ratio $= r$. Then, $a > 0$ and $r > 0$.

It is given that $T_n = T_{n+1} + T_{n+2}$.

Now, $T_n = T_{n+1} + T_{n+2}$

$\Rightarrow \quad ar^{n-1} = ar^n + ar^{n+1}$

$\Rightarrow \quad 1 = r + r^2 \qquad$ [dividing both sides by ar^{n-1}]

$\Rightarrow \quad r^2 + r - 1 = 0$

$$\Rightarrow \quad r = \frac{-1 \pm \sqrt{1+4}}{2} = \frac{-1 \pm \sqrt{5}}{2}$$

$$\Rightarrow \quad r = \frac{-1+\sqrt{5}}{2} = 2\left(\frac{\sqrt{5}-1}{4}\right) = 2\sin 18° \qquad [\because\; r > 0].$$

Hence, the common ratio of the GP is $2\sin 18°$.

EXAMPLE 17 *If p, q, r are in AP, prove that the pth, qth and rth terms of any GP are in GP.*

SOLUTION Since p, q, r are in AP, we have $(q-p)=(r-q)$. ... (i)

Let us consider a GP with first term = A and common ratio = R.

Then, $T_p = AR^{(p-1)}$, $T_q = AR^{(q-1)}$ and $T_r = AR^{(r-1)}$.

$$\therefore \quad \frac{T_q}{T_p} = \frac{AR^{(q-1)}}{AR^{(p-1)}} = R^{(q-p)} = R^{(r-q)} \qquad \text{[using (i)]}$$

and $$\frac{T_r}{T_q} = \frac{AR^{(r-1)}}{AR^{(q-1)}} = R^{(r-q)}.$$

Thus, $\dfrac{T_r}{T_p} = \dfrac{T_r}{T_q}$ and therefore, T_p, T_q, T_r are in GP.

Hence, the pth, qth and rth terms of any GP are in GP.

EXAMPLE 18 *Let S be the sum, P be the product and R be the sum of the reciprocals of three terms of a GP. Then, prove that* $P^2R^3 = S^3$.

SOLUTION Let the three terms of the given GP be $\frac{a}{r}$, a and ar. Then,

$$S = \left(\frac{a}{r} + a + ar\right) = a\left(\frac{1+r+r^2}{r}\right),\ P = \left(\frac{a}{r} \times a \times ar\right) = a^3$$

and $$R = \left(\frac{r}{a} + \frac{1}{a} + \frac{1}{ar}\right) = \frac{1}{a}\left(r + 1 + \frac{1}{r}\right) = \frac{1}{a}\left(\frac{r^2+r+1}{r}\right) = \frac{1}{a}\left(\frac{1+r+r^2}{r}\right).$$

$$\therefore \quad P^2R^3 = a^6 \times \frac{(1+r+r^2)^3}{a^3r^3} = a^3\left(\frac{1+r+r^2}{r}\right)^3 = S^3.$$

Hence, $P^2R^3 = S^3$.

EXAMPLE 19 *For a, b, c to be in GP, what is the value of* $\dfrac{a-b}{b-c}$*?*

SOLUTION For a, b, c to be in GP, we must have

$$\frac{b}{a} = \frac{c}{b} = r \quad \text{or} \quad \frac{b}{a} = \frac{c}{b} = \frac{1}{r}.$$

Case I *When* $\dfrac{b}{a} = \dfrac{c}{b} = r$.

In this case, $b = ar$ and $c = br$.

$$\therefore \quad \frac{a-b}{b-c} = \frac{a-ar}{b-br} = \frac{a(1-r)}{b(1-r)} = \frac{a}{b}.$$

Case II *When* $\dfrac{b}{a} = \dfrac{c}{b} = \dfrac{1}{r}$.

In this case, $a = br$ and $b = cr$.

$$\therefore \quad \frac{a-b}{b-c} = \frac{br-b}{cr-c} = \frac{b(r-1)}{c(r-1)} = \frac{b}{c}.$$

Hence, the value of $\left(\dfrac{a-b}{b-c}\right)$ is $\dfrac{a}{b}$ or $\dfrac{b}{c}$.

EXAMPLE 20 *If a, b, c are in GP and* $a^{1/x} = b^{1/y} = c^{1/z}$, *prove that x, y, z are in AP.*

SOLUTION Since a, b, c are in GP, we have

$$b^2 = ac. \quad \ldots \text{(i)}$$

Let $a^{1/x} = b^{1/y} = c^{1/z} = k$ (say).

Then, $a = k^x$, $b = k^y$ and $c = k^z$.

Putting these values in (i), we get

$$(k^y)^2 = (k^x) \times (k^z) \Rightarrow k^{2y} = k^{(x+z)}.$$

$\therefore \quad 2y = x + z.$

Hence, x, y, z are in AP.

EXAMPLE 21 *If a, b, c are in AP; b, c, d are in GP and* $\frac{1}{c}, \frac{1}{d}, \frac{1}{e}$ *are in AP, then show that a, c, e are in GP.*

SOLUTION a, b, c are in AP $\Rightarrow 2b = a + c$. $\quad \ldots$ (i)

b, c, d are in GP $\Rightarrow c^2 = bd$. $\quad \ldots$ (ii)

$\frac{1}{c}, \frac{1}{d}, \frac{1}{e}$ are in AP $\Rightarrow \frac{2}{d} = \left(\frac{1}{c} + \frac{1}{e}\right) = \frac{(c+e)}{ce}$

$$\Rightarrow d = \frac{2ce}{(c+e)}. \quad \ldots \text{(iii)}$$

Now, $c^2 = bd \Rightarrow c^2 = \frac{(a+c)}{2} \cdot \frac{2ce}{(c+e)}$ [from (i) and (iii)]

$$\Rightarrow c = \frac{(a+c)e}{(c+e)}$$

$$\Rightarrow c(c+e) = (a+c)e$$

$$\Rightarrow c^2 = ae$$

$\Rightarrow a, c, e$ are in GP.

Hence, a, c, e are in GP.

EXAMPLE 22 *If p, q, r are in GP and the equations*

$$px^2 + 2qx + r = 0 \text{ and } dx^2 + 2ex + f = 0$$

have a common root then show that $\frac{d}{p}, \frac{e}{q}, \frac{f}{r}$ *are in AP.*

SOLUTION Since p, q, r are in GP, we have

$$q^2 = pr. \quad \ldots \text{(i)}$$

On solving $px^2 + 2qx + r = 0$, we get

$$x = \frac{-2q \pm \sqrt{4q^2 - 4pr}}{2p} = \frac{-2q}{2p} = \frac{-q}{p} \quad \text{[using (i)].}$$

Thus, $x = \frac{-q}{p}$ is a repeated root of $px^2 + 2qr + r = 0$.

$\therefore \quad x = \frac{-q}{p}$ is also a root of $dx^2 + 2ex + f = 0$

$$\Rightarrow \quad d \cdot \left(\frac{-q}{p}\right)^2 + 2e\left(\frac{-q}{p}\right) + f = 0$$

$$\Rightarrow \quad dq^2 - 2eqp + fp^2 = 0 \qquad \ldots \text{(ii)}$$

$$\Rightarrow \quad \frac{d}{p} - \frac{2e}{q} + \frac{fp}{q^2} = 0 \qquad [\text{on dividing (ii) by } pq^2]$$

$$\Rightarrow \quad \frac{d}{p} - \frac{2e}{q} + \frac{f}{r} = 0 \qquad [\because q^2 = pr]$$

$$\Rightarrow \quad \frac{d}{p} + \frac{f}{r} = \frac{2e}{q}.$$

Hence, $\frac{d}{p}, \frac{e}{q}, \frac{f}{r}$ are in AP.

EXAMPLE 23 *Find all sequences which are simultaneously AP and GP.*

SOLUTION Let $a_1, a_2, a_3, \ldots, a_n, \ldots$ be a sequence which is both an AP as well as a GP.

Consider three consecutive terms a_n, a_{n+1}, a_{n+2} of this AP and GP.

Then, $2a_{n+1} = a_n + a_{n+2}$ … (i) [$\because$ a_n, a_{n+1}, a_{n+2} are in AP].

And, if r be the common ratio of the GP, then

$$a_n = a_1 r^{n-1}, a_{n+1} = a_1 r^n \text{ and } a_{n+2} = a_1 r^{n+1}.$$

Putting these values in (i), we get

$$2a_1 r^n = a_1 r^{n-1} + a_1 r^{n+1} \Rightarrow r^2 - 2r + 1 = 0 \Rightarrow (r-1)^2 = 0 \Rightarrow r = 1.$$

Putting $r = 1$, we get $a_n = a_1$, $a_{n+1} = a_1$ and $a_{n+2} = a_1$.

Thus, we get a constant sequence $a_1, a_1, a_1, a_1, \ldots$.

Hence, a constant sequence is the only sequence which is both AP and GP.

EXAMPLE 24 *If the pth, qth, rth and sth terms of an AP be in GP then prove that* $(p-q), (q-r), (r-s)$ *are in GP.*

SOLUTION Let a be the first term and d be the common difference of the given AP. Then,

$$T_p = a + (p-1)d, T_q = a + (q-1)d,$$
$$T_r = a + (r-1)d, T_s = a + (s-1)d.$$

It is given that T_p, T_q, T_r and T_s are in GP.

Let the first term of this GP be A and its common ratio be R. Then,

$$T_p = A \Rightarrow a + (p-1)d = A \qquad \ldots \text{(i)}$$

$$T_q = AR \Rightarrow a + (q-1)d = AR \qquad \ldots \text{(ii)}$$

$$T_r = AR^2 \Rightarrow a + (r-1)d = AR^2 \qquad \ldots \text{(iii)}$$

and $T_s = AR^3 \Rightarrow a + (s-1)d = AR^3.$ … (iv)

On subtracting (ii) from (i), we get

$$(p-q)d = A(1-R). \qquad \ldots \text{(v)}$$

On subtracting (iii) from (ii), we get

$$(q-r)d = AR(1-R). \qquad \ldots \text{(vi)}$$

On subtracting (iv) from (iii), we get

$$(r-s)d = AR^2(1-R). \qquad \dots \text{(vii)}$$

Now, we have

$$(q-r)^2 d^2 = A^2R^2(1-R)^2 \text{ and } (p-q)d \times (r-s)d = A^2R^2(1-R)^2.$$

$\therefore \quad (q-r)^2 d^2 = (p-q)d \times (r-s)d$

$\Rightarrow \quad (q-r)^2 = (p-q) \times (r-s)$

$\Rightarrow \quad (p-q), (q-r)$ and $(r-s)$ are in GP.

Hence, $(p-q), (q-r), (r-s)$ are in GP.

*n*th TERM FROM THE END OF A GP

THEOREM 2 *Show that the nth term from the end of a GP with the first term a, the common ratio r and the last term l is given by* $\dfrac{l}{r^{(n-1)}}$.

PROOF Let a be the first term, r be the common ratio and l be the last term of a given GP. Then,

2nd term from the end $= \dfrac{l}{r} = \dfrac{l}{r^{(2-1)}}$;

3rd term from the end $= \dfrac{l}{r^2} = \dfrac{l}{r^{(3-1)}}$;

... ...

... ...

nth term from the end $= \dfrac{l}{r^{(n-1)}}$.

Hence, the nth term from the end $= \dfrac{l}{r^{(n-1)}}$.

EXAMPLE 25 *Find the 8th term from the end of the GP* 3, 6, 12, 24, …, 12288.

SOLUTION Here $r = 2$ and $l = 12288$.

$\therefore \quad$ 8th term from the end $= \dfrac{l}{r^{(8-1)}} = \dfrac{l}{r^7} = \dfrac{12288}{2^7} = \dfrac{12288}{128} = 96.$

Hence, the 8th term from the end = 96.

EXERCISE 12A

1. Find the 6th and nth terms of the GP 2, 6, 18, 54, … .
2. Find the 17th and nth terms of the GP $2, 2\sqrt{2}, 4, 8\sqrt{2}, \dots$.
3. Find the 7th and nth terms of the GP 0.4, 0.8, 1.6, … .
4. Find the 10th and nth terms of the GP $\dfrac{-3}{4}, \dfrac{1}{2}, \dfrac{-1}{3}, \dfrac{2}{9}, \dots$.
5. Which term of the GP 3, 6, 12, 24, … is 3072?
6. Which term of the GP $\dfrac{1}{4}, \dfrac{-1}{2}, 1, \dots$ is –128?
7. Which term of the GP $\sqrt{3}, 3, 3\sqrt{3}, \dots$ is 729?

8. Find the geometric series whose 5th and 8th terms are 80 and 640 respectively.

9. Find the GP whose 4th and 7th terms are $\frac{1}{18}$ and $\frac{-1}{486}$ respectively.

10. The 5th, 8th and 11th terms of a GP are a, b, c respectively. Show that $b^2 = ac$.

11. The first term of a GP is –3 and the square of the second term is equal to its 4th term. Find its 7th term.

12. Find the 6th term from the end of the GP $8, 4, 2, \ldots, \frac{1}{1024}$.

13. Find the 4th term from the end of the GP $\frac{2}{27}, \frac{2}{9}, \frac{2}{3}, \ldots, 162$.

14. If a, b, c are the pth, qth and rth terms of a GP, show that $(q-r)\log a + (r-p)\log b + (p-q)\log c = 0$.

15. The third term of a GP is 4. Find the product of its first five terms.

16. In a finite GP, prove that the product of the terms equidistant from the beginning and end is the product of first and last terms.

17. If $\frac{a+bx}{a-bx} = \frac{b+cx}{b-cx} = \frac{c+dx}{c-dx}$ $(x \neq 0)$ then show that a, b, c, d are in GP.

18. If a and b are the roots of $x^2 - 3x + p = 0$ and c and d are the roots of $x^2 - 12x + q = 0$, where a, b, c, d form a GP, prove that $(q+p):(q-p) = 17:15$.

ANSWERS (EXERCISE 12A)

1. $486, 2 \times 3^{(n-1)}$ **2.** $512, (\sqrt{2})^{(n+1)}$ **3.** $25.6, \frac{2^n}{5}$ **4.** $\frac{128}{6561}, \frac{-3}{4} \times \left(\frac{-2}{3}\right)^{(n-1)}$

5. 11th **6.** 10th **7.** 12th **8.** $5 + 10 + 20 + 40 + 80 + \ldots$

9. $\frac{-3}{2}, \frac{1}{2}, \frac{-1}{6}, \frac{1}{18}, \ldots$ **11.** –2187 **12.** $\frac{1}{32}$ **13.** 6

15. 1024

HINTS TO SOME SELECTED QUESTIONS

14. Let A be the first term and R be the common ratio of the given GP. Then,

$$a = AR^{(p-1)} \Rightarrow \log a = \log A + (p-1)\log R \quad \ldots \text{(i)}$$

$$b = AR^{(q-1)} \Rightarrow \log b = \log A + (q-1)\log R \quad \ldots \text{(ii)}$$

$$c = AR^{(r-1)} \Rightarrow \log c = \log A + (r-1)\log R. \quad \ldots \text{(iii)}$$

$\therefore$ $(q-r)\log a + (r-p)\log b + (p-q)\log c$

$$= (q-r)\log [AR^{(p-1)}] + (r-p)\log \{AR^{(q-1)}] + (p-q)\log [AR^{(r-1)}]$$

$$= (\log A)\{(q-r)+(r-p)+(p-q)\} + (\log R)\{(p-1)(q-r)+(q-1)(r-p) + (r-1)(p-q)\}$$

$$= (\log A) \times 0 + (\log R) \times 0 = 0.$$

15. Let a be the first term and r be the common ratio of the GP. Then, the first five terms are a, ar, ar^2, ar^3, ar^4.

Also, $T_3 = 4 \Rightarrow ar^2 = 4$.

Product of first five terms $= (a \times ar \times ar^2 \times ar^3 \times ar^4) = a^5 r^{10} = (ar^2)^5 = (4)^5 = 1024$.

16. Let a be the first term, r be the common ratio and l be the last term of a GP containing n terms. Then,

$$\begin{aligned}&(p\text{th term from the beginning}) \times (p\text{th term from the end})\\ &= (p\text{th term from the beginning}) \times \{(n-p+1)\text{th term from the beginning}\}\\ &= T_p \times T_{(n-p+1)} = ar^{p-1} \times ar^{(n-p+1)-1} = ar^{(p-1)} \times ar^{(n-p)}\\ &= a \times (ar^{n-1}) = (T_1 \times T_n) = (\text{first term} \times \text{last term}).\end{aligned}$$

17. $\dfrac{a+bx}{a-bx} = \dfrac{b+cx}{b-cx} \Rightarrow \dfrac{(a+bx)+(a-bx)}{(a+bx)-(a-bx)} = \dfrac{(b+cx)+(b-cx)}{(b+cx)-(b-cx)} \Rightarrow \dfrac{b}{a} = \dfrac{c}{b}$.

Similarly, $\dfrac{b+cx}{b-cx} = \dfrac{c+dx}{c-dx} \Rightarrow \dfrac{c}{b} = \dfrac{d}{c}$.

$\therefore \quad \dfrac{b}{a} = \dfrac{c}{b} = \dfrac{d}{c} \Rightarrow a, b, c, d$ are in GP.

18. Let r be the common ratio of the given GP.

Then, $b = ar$, $c = ar^2$ and $d = ar^3$.

Also, $a + b = 3$, $ab = p$, $c + d = 12$ and $cd = q$.

Now, $a + b = 3, c + d = 12 \Rightarrow a(1+r) = 3$ and $ar^2(1+r) = 12$

$$\Rightarrow \frac{ar^2(1+r)}{a(1+r)} = 4 \Rightarrow r^2 = 4 \Rightarrow r = 2$$

$$\Rightarrow a(1+2) = 3 \Rightarrow a = 1.$$

$\therefore \quad p = ab = a \times ar = a^2 r = (1^2 \times 2) = 2$ and $q = cd = ar^2 \times ar^3 = a^2 r^5 = (1)^2 \times 2^5 = 32$.

Hence, $\dfrac{q+p}{q-p} = \dfrac{32+2}{32-2} = \dfrac{34}{30} = \dfrac{17}{15}$.

PROBLEMS ON GP

For solving problems on GP, it is always convenient to take:

(i) 3 numbers in GP as $\dfrac{a}{r}, a\ ar$;

(ii) 4 numbers in GP $\dfrac{a}{r^3}, \dfrac{a}{r}, ar, ar^3$;

(iii) 5 numbers in GP as $\dfrac{a}{r^2}, \dfrac{a}{r}, a, ar, ar^2$;

(iv) the terms as $a, ar, ar^2, \ldots$ when their product is not given.

SOLVED EXAMPLES

EXAMPLE 1 *The sum of first three terms of a GP is* $\dfrac{13}{12}$ *and their product is* -1. *Find these terms.*

SOLUTION Let the first three terms of the given GP be $\frac{a}{r}$, a and ar.

Then, $\frac{a}{r} \times a \times ar = -1 \Rightarrow a^3 = -1 \Rightarrow a = -1$.

And, $\frac{a}{r} + a + ar = \frac{13}{12} \Rightarrow \frac{-1}{r} - 1 - r = \frac{13}{12}$ $[\because a = -1]$

$$\Rightarrow \frac{-1}{r} - r = \frac{13}{12} + 1$$

$$\Rightarrow \frac{-1 - r^2}{r} = \frac{25}{12}$$

$$\Rightarrow 25r = -12 - 12r^2$$

$$\Rightarrow 12r^2 + 25r + 12 = 0$$

$$\Rightarrow 12r^2 + 16r + 9r + 12 = 0$$

$$\Rightarrow 4r(3r + 4) + 3(3r + 4) = 0$$

$$\Rightarrow (3r + 4)(4r + 3) = 0 \Rightarrow r = \frac{-4}{3} \text{ or } r = \frac{-3}{4}.$$

So, the required terms are

$$\left\{\frac{-1}{(-4/3)}, -1, \frac{4}{3}\right\} \text{ or } \left\{\frac{-1}{(-3/4)}, -1, \frac{3}{4}\right\}$$

i.e., $\frac{3}{4}, -1, \frac{4}{3}$ or $\frac{4}{3}, -1, \frac{3}{4}$.

EXAMPLE 2 *Find three numbers in GP whose sum is* 13 *and the sum of whose squares is* 91.

SOLUTION Let the required numbers be $\frac{a}{r}$, a and ar. Then,

$$\frac{a}{r} + a + ar = 13 \quad \ldots \text{(i)}$$

and $$\frac{a^2}{r^2} + a^2 + a^2r^2 = 91. \quad \ldots \text{(ii)}$$

On squaring both sides of (i), we get

$$\left(\frac{a}{r} + a + ar\right)^2 = 169$$

$$\Rightarrow \left(\frac{a^2}{r^2} + a^2 + a^2r^2\right) + 2\left(\frac{a^2}{r} + a^2 + a^2r\right) = 169$$

$$\Rightarrow 91 + 2a\left(\frac{a}{r} + a + ar\right) = 169 \quad \text{[using (ii)]}$$

$$\Rightarrow 91 + 26a = 169 \quad \text{[using (i)]}$$

$$\Rightarrow 26a = 78 \Rightarrow a = 3.$$

Putting $a = 3$ in (i), we get

$$\frac{3}{r} + 3 + 3r = 13 \Rightarrow \frac{3}{r} + 3r = 10 \Rightarrow 3r^2 - 10r + 3 = 0$$

$$\Rightarrow (r - 3)(3r - 1) = 0 \Rightarrow r = 3 \text{ or } r = \frac{1}{3}.$$

Hence, the required numbers are 1, 3, 9 or 9, 3, 1.

EXAMPLE 3 *Find three numbers in GP whose sum is* 52 *and the sum of whose products in pairs is* 624.

SOLUTION Let the required numbers be a, ar, ar^2. Then,

$$(a + ar + ar^2) = 52 \Rightarrow a(1 + r + r^2) = 52 \qquad \text{... (i)}$$

and $a \cdot ar + ar \cdot ar^2 + a \cdot ar^2 = 624 \Rightarrow a^2 r(1 + r + r^2) = 624. \qquad \text{... (ii)}$

On dividing (ii) by (i), we get

$$ar = 12 \text{ and therefore, } a = \frac{12}{r}.$$

Putting $a = \frac{12}{r}$ in (i), we get

$$\frac{12}{r} \cdot (1 + r + r^2) = 52 \Rightarrow 3(1 + r + r^2) = 13r$$
$$\Rightarrow 3r^2 - 10r + 3 = 0$$
$$\Rightarrow (3r - 1)(r - 3) = 0$$
$$\Rightarrow r = \frac{1}{3} \text{ or } r = 3.$$

$\therefore \quad a = 36$ or $a = 4$.

Hence, the required numbers are 36, 12, 4 or 4, 12, 36.

EXAMPLE 4 *If the product of three numbers in GP is* 216 *and the sum of their products in pairs is* 156, *find the numbers.*

SOLUTION Let the required numbers be $\frac{a}{r}$, a and ar. Then,

$$\frac{a}{r} \times a \times ar = 216 \Rightarrow a^3 = 216 = 6^3 \Rightarrow a = 6.$$

And, $\frac{a}{r} \times a + a \times ar + \frac{a}{r} \times ar = 156$

$$\Rightarrow \quad a^2\left(\frac{1}{r} + r + 1\right) = 156 \Rightarrow (6^2)(1 + r + r^2) = 156r \qquad [\because a = 6]$$

$$\Rightarrow \quad 36(r^2 + r + 1) = 156r \Rightarrow 3(r^2 + r + 1) = 13r$$

$$\Rightarrow \quad 3r^2 - 10r + 3 = 0 \Rightarrow (3r - 1)(r - 3) = 0 \Rightarrow r = \frac{1}{3} \text{ or } r = 3.$$

So, the required numbers are 18, 6, 2 or 2, 6, 18.

EXAMPLE 5 *Find four numbers in GP such that the third term is greater than the first by* 9 *and the second term is greater than the fourth by* 18.

SOLUTION Let the required numbers be a, ar, ar^2 and ar^3. Then,

$$T_3 - T_1 = 9 \Rightarrow ar^2 - a = 9 \Rightarrow a(r^2 - 1) = 9 \qquad \text{... (i)}$$

and $T_2 - T_4 = 18 \Rightarrow ar - ar^3 = 18 \Rightarrow ar(1 - r^2) = 18. \qquad \text{... (ii)}$

On dividing (ii) by (i), we get $r = -2$.

Putting $r = -2$ in (i), we get $a(4 - 1) = 9 \Rightarrow a = 3$.

Hence, the required numbers are 3, –6, 12 and –24.

EXAMPLE 6 *The sum of three numbers in GP is 56. If we subtract 1, 7, 21 from these numbers in that order, we get an AP. Find the numbers.*

SOLUTION Let the required numbers be a, ar and ar^2. Then,

$$a + ar + ar^2 = 56. \qquad \ldots \text{(i)}$$

Also, $(a-1)$, $(ar-7)$ and (ar^2-21) are in AP.

$$\therefore \quad 2(ar-7) = (a-1) + (ar^2-21) \Rightarrow a + ar^2 = 2ar + 8. \qquad \ldots \text{(ii)}$$

Using (ii) in (i), we get

$$ar + (2ar+8) = 56 \Rightarrow 3ar = 48 \Rightarrow ar = 16 \Rightarrow r = \frac{16}{a}\cdot$$

Putting $r = \frac{16}{a}$ in (i), we get

$$\begin{aligned} a + 16 + \frac{256}{a} = 56 &\Rightarrow a^2 + 16a + 256 = 56a \\ &\Rightarrow a^2 - 40a + 256 = 0 \\ &\Rightarrow a^2 - 8a - 32a + 256 = 0 \\ &\Rightarrow a(a-8) - 32(a-8) = 0 \\ &\Rightarrow (a-8)(a-32) = 0 \\ &\Rightarrow a = 8 \text{ or } a = 32. \end{aligned}$$

Now, $a = 8 \Rightarrow r = \frac{16}{8} \Rightarrow r = 2.$

And, $a = 32 \Rightarrow r = \frac{16}{32} \Rightarrow r = \frac{1}{2}\cdot$

Hence, the required numbers are (8, 16, 32) or (32, 16, 8).

EXERCISE 12B

1. For what values of x are the numbers $\frac{-2}{7}, x, \frac{-7}{2}$ in GP?
2. For what values of x are the numbers $(x+9)$, $(x-6)$ and 4 in GP?
3. The sum of three numbers in GP is $\frac{39}{10}$ and their product is 1. Find the numbers.
4. The sum of first three terms of a GP is $\frac{13}{12}$ and their product is –1. Find the GP.
5. Find three numbers in GP whose sum is 38 and whose product is 1728.
6. Find three numbers in GP whose sum is 65 and whose product is 3375.
7. The sum of three numbers in GP is 21 and the sum of their squares is 189. Find the numbers.
8. The product of three numbers in GP is 216. If 2, 8, 6 be added to them in that order, we get an AP. Find the numbers.
9. The product of three numbers in GP is 1000. If 6 is added to the second number and 7 is added to the third number, we get an AP. Find the numbers.

ANSWERS (EXERCISE 12B)

1. $x = 1$ or $x = -1$ **2.** $x = 0$ or $x = 16$ **3.** $\frac{5}{2}, 1, \frac{2}{5}$ or $\frac{2}{5}, 1, \frac{5}{2}$

4. $\frac{4}{3}, -1, \frac{3}{4}$ or $\frac{3}{4}, -1, \frac{4}{3}$ **5.** 18, 12, 8 or 8, 12, 18 **6.** 45, 15, 5 or 5, 15, 45

7. 12, 6, 3 or 3, 6, 12 **8.** 18, 6, 2 or 2, 6, 18 **9.** 5, 10, 20 or 20, 10, 5

SUM OF *n* TERMS OF A GP

THEOREM *Prove that the sum of n terms of a GP with the first term a and the common ratio r is given by*

$$S_n = \begin{cases} na, \text{ when } r = 1; \\ \dfrac{a(1-r^n)}{(1-r)}, \text{ when } r < 1; \\ \dfrac{a(r^n-1)}{(r-1)}, \text{ when } r > 1. \end{cases}$$

PROOF Let us consider a GP with the first term a and the common ratio r. Then,

$$S_n = a + ar + ar^2 + \ldots + ar^{n-1}. \qquad \ldots \text{(i)}$$

Case 1 If $r = 1$, we have

$$S_n = a + a + \ldots \text{ to } n \text{ terms} = na.$$

Case 2 If $r \neq 1$, we have

$$rS_n = ar + ar^2 + \ldots + ar^{n-1} + ar^n. \qquad \ldots \text{(ii)}$$

On subtracting (ii) from (i), we get

$$(1-r)S_n = (a - ar^n) = a(1-r^n)$$

$$\Rightarrow \quad S_n = \frac{a(1-r^n)}{(1-r)} \text{ or } S_n = \frac{a(r^n-1)}{(r-1)}.$$

$$\therefore \quad S_n = \begin{cases} \dfrac{a(1-r^n)}{(1-r)}, \text{ when } r < 1 \\ \dfrac{a(r^n-1)}{(r-1)}, \text{ when } r > 1 \end{cases}$$

REMARK If a GP contains n terms with first term $= a$, common ratio $= r$ and last term $= l$ then

$$l = ar^{n-1}. \qquad \ldots \text{(i)}$$

Case 1 When $r < 1$, we have

$$S_n = \frac{a(1-r^n)}{(1-r)} = \frac{(a - ar^n)}{(1-r)} = \frac{(a - lr)}{(1-r)} \qquad \text{[using (i)]}.$$

Case 2 When $r > 1$, we have

$$S_n = \frac{a(r^n - 1)}{(r-1)} = \frac{(ar^n - a)}{(r-1)} = \frac{(lr-a)}{(r-1)} \qquad \text{[using (i)].}$$

SUMMARY

In a GP with first term = a and common ratio = r, we have

$$S_n = \begin{cases} na, \text{ when } r = 1; \\ \dfrac{a(1-r^n)}{(1-r)} = \dfrac{(a-lr)}{(1-r)}, \text{ when } r < 1; \\ \dfrac{a(r^n-1)}{(r-1)} = \dfrac{(lr-a)}{(r-1)}, \text{ when } r > 1. \end{cases}$$

SOLVED EXAMPLES

EXAMPLE 1 *Find the sum of 8 terms of the GP 3, 6, 12, 24, … .*

SOLUTION Here $a = 3$, $r = 2 > 1$ and $n = 8$.

Using the formula, $S_n = \dfrac{a(r^n-1)}{(r-1)}$, we get

$$S_8 = \frac{3 \times (2^8 - 1)}{(2-1)} = 3 \times (256 - 1) = 3 \times 255 = 765.$$

EXAMPLE 2 *Find the sum of the geometric series* $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots$ *to 12 terms.*

SOLUTION Here $a = 1$, $r = \frac{1}{2} < 1$ and $n = 12$.

Using the formula, $S_n = \dfrac{a(1-r^n)}{(1-r)}$, we get

$$S_{12} = \frac{1 \times \left\{1 - \left(\frac{1}{2}\right)^n\right\}}{\left(1 - \frac{1}{2}\right)} = \frac{\left(1 - \frac{1}{2^{12}}\right)}{\left(\frac{1}{2}\right)} = \frac{(2^{12}-1)}{2^{11}} = \frac{4095}{2048}.$$

EXAMPLE 3 *How many terms of the geometric series* $1 + 4 + 16 + 64 + \ldots$ *will make the sum 5461?*

SOLUTION Let the required number of terms be n.

Here, $a = 1$, $r = 4 > 1$ and $S_n = 5461$.

$\therefore \quad S_n = \dfrac{a(r^n-1)}{(r-1)} \Rightarrow \dfrac{1 \times (4^n - 1)}{(4-1)} = 5461$

$\Rightarrow \quad (4^n - 1) = 16383$

$\Rightarrow \quad 4^n = 16384 = 4^7$

$\Rightarrow \quad n = 7.$

Hence, the required number of terms is 7.

EXAMPLE 4 *Find the sum of the series* $2+6+18+54+\ldots+4374$.

SOLUTION Clearly, the given series is a geometric series in which $a=2$, $r=3>1$ and $l=4374$.

$\therefore$ the required sum $=\dfrac{(lr-a)}{(r-1)}=\dfrac{(4374\times 3-2)}{(3-1)}=\dfrac{13120}{2}=6560$.

Hence, the sum of the given series is 6560.

EAMPLE 5 *In a GP, it is being given that* $T_1=3$, $T_n=96$ *and* $S_n=189$. *Find the value of* n.

SOLUTION Here, $a=3$, $l=96$ and $S_n=189$.

Let the common ratio of the given GP be r.

Then, $S_n=\dfrac{(lr-a)}{(r-1)} \Rightarrow \dfrac{(96r-3)}{(r-1)}=189$

$\Rightarrow (96r-3)=(189r-189)$

$\Rightarrow 93r=186 \Rightarrow r=2$.

Now, $l=ar^{n-1} \Rightarrow 3\times 2^{n-1}=96$

$\Rightarrow 2^{n-1}=32=2^5 \Rightarrow n-1=5 \Rightarrow n=6$.

Hence, $n=6$.

EXAMPLE 6 *Sum the series* $5+55+555+\ldots$ *to* n *terms.*

SOLUTION We have

$5+55+555+\ldots$ to n terms

$=5\times\{1+11+111+\ldots \text{ to } n \text{ terms}\}$

$=\dfrac{5}{9}\times\{9+99+999+\ldots \text{ to } n \text{ terms}\}$

$=\dfrac{5}{9}\times\{(10-1)+(10^2-1)+(10^3-1)+\ldots \text{ to } n \text{ terms}\}$

$=\dfrac{5}{9}\times\{(10+10^2+10^3+\ldots \text{ to } n \text{ terms})-n\}$

$=\dfrac{5}{9}\times\left\{\dfrac{10\times(10^n-1)}{(10-1)}-n\right\}=\dfrac{5}{81}\times(10^{n+1}-9n-10)$.

Hence, the required sum is $\dfrac{5}{81}\times(10^{n+1}-9n-10)$.

EXAMPLE 7 *Sum the series* $.4+.44+.444+\ldots$ *to* n *terms.*

SOLUTION We have

$.4+.44+.444+\ldots$ to n terms

$=4\times\{.1+.11+.111+\ldots \text{ to } n \text{ terms}\}$

$=\dfrac{4}{9}\times\{.9+.99+.999+\ldots \text{ to } n \text{ terms}\}$

$=\dfrac{4}{9}\times\{(1-.1)+(1-.01)+(1-.001)+\ldots \text{ to } n \text{ terms}\}$

$$= \frac{4}{9} \times \{(1 + 1 + \ldots \text{ to } n \text{ terms}) - (.1 + .01 + .001 + \ldots \text{ to } n \text{ terms})\}$$

$$= \frac{4}{9} \times \left[n - \frac{.1 \times \{1 - (.1)^n\}}{(1 - .1)} \right] \qquad \left\{ \because \; S_n = \frac{a(1 - r^n)}{(1 - r)} \right\}$$

$$= \frac{4}{9} \times \left[n - \frac{\frac{1}{10} \cdot \left\{ 1 - \frac{1}{(10)^n} \right\}}{\left(1 - \frac{1}{10}\right)} \right]$$

$$= \frac{4}{9} \times \left\{ n - \frac{(10^n - 1)}{9 \cdot 10^n} \right\} = \frac{4}{9} \times \left[n - \frac{1}{9} \left\{ 1 - \frac{1}{10^n} \right\} \right]$$

$$= \frac{4}{81} \times \left[9n - \left\{ 1 - \frac{1}{10^n} \right\} \right] = \frac{4}{81} \times \left\{ 9n - 1 + \frac{1}{10^n} \right\}.$$

Hence, the required sum is $\frac{4}{81} \times \left(9n - 1 + \frac{1}{10^n}\right)$.

EXAMPLE 8 *The sum of some terms of a GP is* 315. *Its first term is* 5 *and the common ratio is* 2. *Find the number of its terms and the last term.*

SOLUTION Let the given GP contain n terms. Then,

$$a = 5, r = 2 > 1 \text{ and } S_n = 315.$$

$$\therefore \quad S_n = 315 \Rightarrow \frac{a(r^n - 1)}{(r - 1)} = 315$$

$$\Rightarrow \frac{5 \times (2^n - 1)}{(2 - 1)} = 315$$

$$\Rightarrow (2^n - 1) = 63 \Rightarrow 2^n = 64 = 2^6 \Rightarrow n = 6.$$

Last term $= ar^{(n-1)} = 5 \times 2^{(6-1)} = (5 \times 2^5) = (5 \times 32) = 160$.

Hence, the given GP contains 6 terms and its last term is 160.

EXAMPLE 9 *The sum of first three terms of a GP is* 16 *and the sum of its next three terms is* 128. *Find the sum of n terms of the GP.*

SOLUTION Let a be the first term and r be the common ratio of the given GP.

Then, $a + ar + ar^2 = 16$ and $ar^3 + ar^4 + ar^5 = 128$

$\Rightarrow \quad a(1 + r + r^2) = 16 \quad \ldots$ (i) and $ar^3(1 + r + r^2) = 128 \quad \ldots$ (ii).

On dividing (ii) by (i), we get $r^3 = 8 = 2^3 \Rightarrow r = 2$.

Putting $r = 2$ in (i), we get

$$a(1 + 2 + 4) = 16 \Rightarrow a = \frac{16}{7}.$$

Thus, in the given GP, we have $a = \frac{16}{7}$ and $r = 2 > 1$.

$$\therefore \quad S_n = \frac{a(r^n - 1)}{(r - 1)} = \frac{\frac{16}{7} \times (2^n - 1)}{(2 - 1)} = \frac{16}{7}(2^n - 1).$$

EXAMPLE 10 *In a GP, the sum of first two terms is –4 and the 5th term is 4 times the 3rd term. Find the GP.*

SOLUTION Let a be the first term and r be the common ratio of the given GP. Then,

$$T_1 + T_2 = -4 \text{ and } T_5 = 4 \times T_3$$

$$\Rightarrow \quad a + ar = -4 \text{ and } ar^4 = 4 \times ar^2$$

$$\Rightarrow \quad a(1+r) = -4 \text{ and } r^2 = 4$$

$$\Rightarrow \quad a(1+r) = -4 \quad \ldots \text{(i)} \quad \text{and} \quad r = \pm 2.$$

Putting $r = 2$ in (i), we get $a = \frac{-4}{3}\cdot$

Putting $r = -2$ in (i), we get $a = 4$.

$\therefore \quad a = \frac{-4}{3}$ and $r = 2$ gives the GP $\frac{-4}{3}, \frac{-8}{3}, \frac{-16}{3}, \ldots$

and, $a = 4$ and $r = -2$ gives the GP $4, -8, 16, \ldots$.

Hence, the required GP is $\left\{\frac{-4}{3}, \frac{-8}{3}, \frac{-16}{3}, \ldots\right\}$ or $\{4, -8, 16, \ldots\}$.

EXAMPLE 11 *In an increasing GP, the sum of the first and last terms is 66, the product of the second and the last but one is 128 and the sum of the terms is 126. How many terms are there in this GP?*

SOLUTION Let the given GP contain n terms. Let a be the first term and r be the common ratio of this GP.

Since the given GP is increasing, we have $r > 1$.

Now, $T_1 + T_n = 66 \Rightarrow a + ar^{(n-1)} = 66.$ … (i)

And, $T_2 \times T_{n-1} = 128 \Rightarrow ar \times ar^{(n-2)} = 128$

$$\Rightarrow a^2 r^{(n-1)} = 128 \Rightarrow ar^{(n-1)} = \frac{128}{a}\cdot \quad \ldots \text{(ii)}$$

Using (ii) in (i), we get

$$a + \frac{128}{a} = 66 \Rightarrow a^2 - 66a + 128 = 0$$

$$\Rightarrow a^2 - 2a - 64a + 128 = 0$$

$$\Rightarrow a(a-2) - 64(a-2) = 0$$

$$\Rightarrow (a-2)(a-64) = 0$$

$$\Rightarrow a = 2 \text{ or } a = 64.$$

Putting $a = 2$ in (ii), we get

$$r^{(n-1)} = \frac{128}{a^2} = \frac{128}{4} = 32. \quad \ldots \text{(iii)}$$

Putting $a = 64$ in (ii), we get

$$r^{(n-1)} = \frac{128}{a^2} = \frac{128}{64 \times 64} = \frac{1}{32}, \text{ which is rejected, since } r > 1.$$

Thus, $a = 2$ and $r^{(n-1)} = 32$.

Now, $S_n = 126 \Rightarrow \dfrac{a(r^n - 1)}{(r-1)} = 126$

$$\Rightarrow 2\left(\frac{r^n - 1}{r-1}\right) = 126 \Rightarrow \frac{r^n - 1}{r-1} = 63$$

$$\Rightarrow \frac{r^{(n-1)} \times r - 1}{r-1} = 63 \Rightarrow \frac{32r - 1}{r-1} = 63$$

$$\Rightarrow 32r - 1 = 63r - 63 \Rightarrow 31r = 62 \Rightarrow r = 2.$$

$\therefore \quad r^{(n-1)} = 32 = 2^5 \Rightarrow n - 1 = 5 \Rightarrow n = 6.$

Hence, there are 6 terms in the given GP.

REMARK Consider two GPs $a, ar, ar^2, ar^3, \ldots$ *and* $A, AR, AR^2, AR^3, \ldots$.

Taking the products of their corresponding elements, we get

$$aA, aA(rR), aA(rR)^2, aA(rR)^3, \ldots$$

which is clearly a GP with first term $= aA$ and common ratio $= rR$.

Based upon this result, we take the following example.

EXAMPLE 12 *Find the sum of the products of the corresponding terms of finite geometrical progressions*

$$2, 4, 8, 16, 32 \textit{ and } 128, 32, 8, 2, \frac{1}{2}.$$

SOLUTION Taking the products of the corresponding terms of two given GPs, we get a new progression

$$(2 \times 128), (4 \times 32), (8 \times 8), (16 \times 2), \left(32 \times \frac{1}{2}\right),$$

i.e., 256, 128, 64, 32, 16, which is clearly a GP in which $a = 256$ and $r = \dfrac{16}{32} = \dfrac{1}{2} < 1.$

$\therefore$ the required sum $= \dfrac{a(1 - r^5)}{(1-r)} = \dfrac{256 \times \left\{1 - \left(\frac{1}{2}\right)^5\right\}}{\left(1 - \frac{1}{2}\right)}$

$$= \frac{256 \times \left(1 - \frac{1}{2^5}\right)}{\left(\frac{1}{2}\right)} = 256 \times 2 \times \left(1 - \frac{1}{32}\right)$$

$$= \left(256 \times 2 \times \frac{31}{32}\right) = 496.$$

EXAMPLE 13 *Find the sum of n terms of the sequence given by* $a_n = (3^n + 5n), n \in N,$

SOLUTION Let the sum of n terms of the given sequence be S_n. Then,

$$S_n = a_1 + a_2 + a_3 + \ldots + a_n$$

$$= (3^1 + 5 \times 1) + (3^2 + 5 \times 2) + (3^3 + 5 \times 3) + \ldots + (3^n + 5 \times n)$$

$$= (3 + 3^2 + 3^3 + \ldots + 3^n) + (5 \times 1 + 5 \times 2 + 5 \times 3 + \ldots + 5 \times n)$$

$$= (3 + 3^2 + 3^3 + \ldots + 3^n) + 5 \times (1 + 2 + 3 + \ldots + n)$$

$$= \frac{3(3^n - 1)}{(3 - 1)} + 5 \times \frac{n}{2}(1 + n)$$

$$= \frac{3}{2}(3^n - 1) + \frac{5}{2} \times n(n + 1).$$

Hence, the required sum is $\frac{3}{2}(3^n - 1) + \frac{5}{2}n(n + 1)$.

EXAMPLE 14 *If S_1, S_2 and S_3 be respectively the sum of n, $2n$ and $3n$ terms of a GP then prove that $S_1(S_3 - S_2) = (S_2 - S_1)^2$.*

SOLUTION Let a be the first term and r be the common ratio of the given GP. Then,

$$S_1(S_3 - S_2) = \frac{a(1 - r^n)}{(1 - r)} \times \left\{\frac{a(1 - r^{3n})}{(1 - r)} - \frac{a(1 - r^{2n})}{(1 - r)}\right\}$$

$$= \frac{a(1 - r^n)}{(1 - r)} \times \frac{(a - ar^{3n} - a + ar^{2n})}{(1 - r)}$$

$$= \frac{a(1 - r^n)}{(1 - r)} \times \frac{ar^{2n}(1 - r^n)}{(1 - r)}$$

$$= \frac{a^2 r^{2n}(1 - r^n)^2}{(1 - r)^2}.$$

And, $$(S_2 - S_1)^2 = \left\{\frac{a(1 - r^{2n})}{(1 - r)} - \frac{a(1 - r^n)}{(1 - r)}\right\}^2$$

$$= \frac{(a - ar^{2n} - a + ar^n)^2}{(1 - r)^2}$$

$$= \frac{\{ar^n(1 - r^n)\}^2}{(1 - r)^2}$$

$$= \frac{a^2 r^{2n}(1 - r^n)^2}{(1 - r)^2}.$$

Hence, $S_1(S_3 - S_2) = (S_2 - S_1)^2$.

EXAMPLE 15 *If S be the sum, P be the product and R be the sum of the reciprocals of n terms in a GP, prove that $P^2 = \left(\frac{S}{R}\right)^n$.*

SOLUTION Let the given GP be $a, ar, ar^2, \ldots, ar^{(n-1)}$. Then,

$$S = \frac{a(1 - r^n)}{(1 - r)} \qquad \ldots \text{(i)}$$

and $P = [a \times ar \times ar^2 \times \ldots \times ar^{(n-1)}]$

$\Rightarrow \quad P = a^n r^{[1+2+3+\ldots+(n-1)]} = a^n r^{\frac{1}{2}(n-1)n}$

$\Rightarrow \quad P = a^n r^{\frac{1}{2}(n-1)n}$

$\Rightarrow \quad P^2 = a^{2n} r^{(n-1)n}$. … (ii)

And, $R = \left\{\frac{1}{a} + \frac{1}{ar} + \ldots + \frac{1}{ar^{n-1}}\right\} = \left(\frac{1}{a}\right) \cdot \frac{\left\{\frac{1}{r^n} - 1\right\}}{\left\{\frac{1}{r} - 1\right\}}$

$\Rightarrow \quad R = \left(\frac{r}{a}\right) \cdot \frac{(1 - r^n)}{(1 - r) \cdot r^n}$

$\Rightarrow \quad R = \frac{(1 - r^n)}{a(1 - r) \cdot r^{(n-1)}}$. … (iii)

On dividing (i) by (iii), we get

$$\frac{S}{R} = \frac{a(1 - r^n)}{(1 - r)} \cdot \frac{a(1 - r) \cdot r^{(n-1)}}{(1 - r^n)} = a^2 r^{(n-1)}$$

$\therefore \quad \left(\frac{S}{R}\right)^n = \{a^2 r^{(n-1)}\}^n = a^{2n} \cdot r^{(n-1)n} = P^2$ [using (ii)].

Hence, $P^2 = \left(\frac{S}{R}\right)^n$.

EXERCISE 12C

1. Find the sum of the GP:

(i) $1 + 3 + 9 + 27 + \ldots$ to 7 terms

(ii) $1 + \sqrt{3} + 3 + 3\sqrt{3} + \ldots$ to 10 terms

(iii) $0.15 + 0.015 + 0.0015 + \ldots$ to 6 terms

(iv) $1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \ldots$ to 9 terms

(v) $\sqrt{2} + \frac{1}{\sqrt{2}} + \frac{1}{2\sqrt{2}} + \ldots$ to 8 terms

(vi) $\frac{2}{9} - \frac{1}{3} + \frac{1}{2} - \frac{3}{4} + \ldots$ to 6 terms

2. Find the sum of the GP:

(i) $\sqrt{7} + \sqrt{21} + 3\sqrt{7} + \ldots$ to n terms

(ii) $1 - \frac{1}{3} + \frac{1}{3^2} - \frac{1}{3^3} + \ldots$ to n terms

(iii) $1 - a + a^2 - a^3 + \ldots$ to n terms $(a \neq 1)$

(iv) $x^3 + x^5 + x^7 + \ldots$ to n terms

(v) $x(x + y) + x^2(x^2 + y^2) + x^3(x^3 + y^3) + \ldots$ to n terms

3. Find the sum to n terms of the sequence:

(i) $\left(x+\frac{1}{x}\right)^2, \left(x^2+\frac{1}{x^2}\right)^2, \left(x^3+\frac{1}{x^3}\right)^2, \ldots$ to n terms

(ii) $(x+y), (x^2+xy+y^2), (x^3+x^2y+xy^2+y^3), \ldots$ to n terms

4. Find the sum:

$$\frac{3}{5}+\frac{4}{5^2}+\frac{3}{5^3}+\frac{4}{5^4}+\ldots \text{ to } 2n \text{ terms.}$$

5. Evaluate:

(i) $\sum_{n=1}^{10}(2+3^n)$ (ii) $\sum_{k=1}^{n}[2^k+3^{(k-1)}]$ (iii) $\sum_{n=1}^{8}5^n$

6. Find the sum of the series:

(i) $8+88+888+\ldots$ to n terms

(ii) $3+33+333+\ldots$ to n terms

(iii) $0.7+0.77+0.777+\ldots$ to n terms

7. The sum of n terms of a progression is (2^n-1). Show that it is a GP and find its common ratio.

8. In a GP, the ratio of the sum of first three terms is to that of first six terms is 125 : 152. Find the common ratio.

9. Find the sum of the geometric series $3+6+12+\ldots+1536$.

10. How many terms of the series $2+6+18+\ldots$ must be taken to make the sum equal to 728?

11. The common ratio of a finite GP is 3 and its last term is 486. If the sum of these terms is 728, find the first term.

12. The first term of a GP is 27 and its 8th term is $\frac{1}{81}\cdot$ Find the sum of its first 10 terms.

13. The 2nd and 5th terms of a GP are $\frac{-1}{2}$ and $\frac{1}{16}$ respectively. Find the sum of the GP up to 8 terms.

14. The 4th and 7th terms of a GP are $\frac{1}{27}$ and $\frac{1}{729}$ respectively. Find the sum of n terms of the GP.

15. A GP consists of an even number of terms. If the sum of all the terms is 5 times the sum of the terms occupying the odd places, find the common ratio of the GP.

16. Show that the ratio of the sum of first n terms of a GP to the sum of the terms from $(n+1)$th to $(2n)$th term is $\frac{1}{r^n}\cdot$

ANSWERS (EXERCISE 12C)

1. (i) 1093 (ii) $121(\sqrt{3}+1)$ (iii) $\frac{333333}{2000000}$ (iv) $\frac{171}{256}$

(v) $\frac{255\sqrt{2}}{128}$ (vi) $\frac{-133}{144}$

2. (i) $\frac{\sqrt{7}}{2}(\sqrt{3}+1)(3^{n/2}-1)$ (ii) $\frac{\{3^n-(-1)^n\}}{4\times 3^{(n-1)}}$ (iii) $\frac{\{1-(-a)^n\}}{(1+a)}$

(iv) $\frac{x^3\times(x^{2n}-1)}{(x^2-1)}$ (v) $\frac{x^2(x^{2n}-1)}{(x^2-1)}+\frac{xy(x^ny^n-1)}{(xy-1)}$

3. (i) $\left(\frac{x^{2n}-1}{x^2-1}\right)\left(x^2+\frac{1}{x^{2n}}\right)+2n$ (ii) $\frac{1}{(x-y)}\cdot\left\{x^2\left(\frac{x^n-1}{x-1}\right)-y^2\left(\frac{y^n-1}{y-1}\right)\right\}$

4. $\frac{19}{24}\cdot\left(\frac{5^{2n}-1}{5^{2n}}\right)$ **5.** (i) 29544 (ii) $\frac{1}{2}[2^{(n+2)}+3^n-5]$ (iii) 488280

6. (i) $\frac{8}{81}[10^{(n+1)}-10-9n]$ (ii) $\frac{1}{27}[10^{(n+1)}-10-9n]$ (iii) $\frac{7}{81}\left(9n-1+\frac{1}{10^n}\right)$

7. $r=2$ **8.** $r=\frac{3}{5}$ **9.** 3069 **10.** 6 **11.** $a=2$

12. $\frac{81}{2}\left(1-\frac{1}{3^{10}}\right)$ **13.** $\frac{85}{128}$ **14.** $\frac{3}{2}\left(1-\frac{1}{3^n}\right)$ **15.** $r=4$

HINTS TO SOME SELECTED QUESTIONS

1. (i) $S_7=\frac{1\times(3^7-1)}{(3-1)}=\frac{(2187-1)}{2}=\frac{2186}{2}=1093.$

(ii) $S_{10}=\frac{1\times\{(\sqrt{3})^{10}-1\}}{(\sqrt{3}-1)}=\frac{(3^5-1)}{(\sqrt{3}-1)}\times\frac{(\sqrt{3}+1)}{(\sqrt{3}+1)}=\frac{242(\sqrt{3}+1)}{(3-1)}=121(\sqrt{3}+1).$

(iii) $a=\frac{15}{100}=\frac{3}{20}$ and $r=\frac{0.015}{0.15}=\frac{1}{10}$.

$$\therefore\quad S_6=\frac{\frac{3}{20}\times\left(1-\frac{1}{10^6}\right)}{\left(1-\frac{1}{10}\right)}=\left(\frac{3}{20}\times\frac{10}{9}\right)\cdot\frac{(1000000-1)}{1000000}=\frac{1}{6}\times\frac{999999}{1000000}=\frac{333333}{2000000}.$$

(iv) $a=1$, $r=\frac{-1}{2}$ and $n=9$.

$$\therefore\quad S_9=\frac{1\times\left[1-\left(\frac{-1}{2}\right)^9\right]}{\left(1+\frac{1}{2}\right)}=\frac{2}{3}\times\left(1+\frac{1}{2^9}\right)=\left(\frac{2}{3}\times\frac{513}{512}\right)=\frac{171}{256}.$$

(v) $a=\sqrt{2}$, $r=\frac{1}{2}$ and $n=8$.

$$\therefore\quad S_8=\frac{\sqrt{2}\left(1-\frac{1}{2^8}\right)}{\left(1-\frac{1}{2}\right)}=\left(\sqrt{2}\times\frac{255}{256}\times 2\right)=\frac{255\sqrt{2}}{128}.$$

(vi) $a=\frac{2}{9}$, $r=\left(\frac{-1}{3}\times\frac{9}{2}\right)=\frac{-3}{2}$ and $n=6$.

$$\therefore \quad S_6 = \frac{\frac{2}{9} \times \left\{1 - \left(\frac{-3}{2}\right)^6\right\}}{\left(1 + \frac{3}{2}\right)} = \frac{\frac{2}{9} \times \left\{1 - \frac{729}{64}\right\}}{\left(\frac{5}{2}\right)} = \left(\frac{2}{9} \times \frac{2}{5}\right) \times \left(\frac{-665}{64}\right) = \frac{-133}{144}.$$

2. (i) $a = \sqrt{7}$ and $r = \frac{\sqrt{21}}{\sqrt{7}} = \sqrt{3}$.

$$\therefore \quad S_n = \frac{\sqrt{7} \times \{(\sqrt{3})^n - 1\}}{(\sqrt{3} - 1)} \times \frac{(\sqrt{3} + 1)}{(\sqrt{3} + 1)} = \frac{\sqrt{7}}{2} \cdot (\sqrt{3} + 1)(3^{n/2} - 1).$$

(ii) $a = 1$ and $r = \frac{-1}{3}$.

$$\therefore \quad S_n = \frac{1 \times \left\{1 - \left(\frac{-1}{3}\right)^n\right\}}{\left(1 + \frac{1}{3}\right)} = \frac{3}{4} \times \left\{1 - \frac{(-1)^n}{3^n}\right\} = \frac{\{3^n - (-1)^n\}}{4 \times 3^{(n-1)}}.$$

(iii) $A = 1$ and $R = -a$.

$$\therefore \quad S_n = \frac{A \times \{1 - R^n\}}{(1 - R)} = \frac{1 \times \{1 - (-a)^n\}}{(1 + a)} = \frac{\{1 - (-a)^n\}}{(1 + a)}.$$

(iv) $a = x^3$ and $r = \frac{x^5}{x^3} = x^2$.

$$\therefore \quad S_n = \frac{x^3 \times \{(x^2)^n - 1\}}{(x^2 - 1)} = \frac{x^3 \times (x^{2n} - 1)}{(x^2 - 1)}.$$

(v) Given expression

$$= \{x^2 + x^4 + x^6 + \ldots \text{ to } n \text{ terms}\} + \{xy + x^2y^2 + x^3y^3 + \ldots \text{ to } n \text{ terms}\}$$

$$= x^2(1 + x^2 + x^4 + \ldots \text{ to } n \text{ terms}\} + xy(1 + xy + x^2y^2 + \ldots \text{ to } n \text{ terms}\}$$

$$= \frac{x^2 \times 1 \times \{(x^2)^n - 1\}}{(x^2 - 1)} + \frac{xy \times 1 \times \{(xy)^n - 1\}}{(xy - 1)} = \frac{x^2(x^{2n} - 1)}{(x^2 - 1)} + \frac{xy(x^n y^n - 1)}{(xy - 1)}.$$

3. (i) Given sum

$$= \left(x + \frac{1}{x}\right)^2 + \left(x^2 + \frac{1}{x^2}\right)^2 + \left(x^3 + \frac{1}{x^3}\right)^2 + \ldots + \left(x^n + \frac{1}{x^n}\right)^2$$

$$= \left(x^2 + \frac{1}{x^2} + 2\right) + \left(x^4 + \frac{1}{x^4} + 2\right) + \left(x^6 + \frac{1}{x^6} + 2\right) + \ldots + \left(x^{2n} + \frac{1}{x^{2n}} + 2\right)$$

$$= (x^2 + x^4 + x^6 + \ldots + x^{2n}) + \left(\frac{1}{x^2} + \frac{1}{x^4} + \frac{1}{x^6} + \ldots + \frac{1}{x^{2n}}\right) + (2 + 2 + \ldots n \text{ times})$$

$$= \frac{x^2[(x^2)^n - 1]}{(x^2 - 1)} + \frac{\frac{1}{x^2} \cdot \left\{1 - \left(\frac{1}{x^2}\right)^n\right\}}{\left(1 - \frac{1}{x^2}\right)} + 2n = \frac{x^2(x^{2n} - 1)}{(x^2 - 1)} + \frac{(x^{2n} - 1)}{x^{2n}(x^2 - 1)} + 2n$$

$$= x^2\left(\frac{x^{2n} - 1}{x^2 - 1}\right) + \frac{1}{x^{2n}}\left(\frac{x^{2n} - 1}{x^2 - 1}\right) + 2n = \left(\frac{x^{2n} - 1}{x^2 - 1}\right)\left(x^2 + \frac{1}{x^{2n}}\right) + 2n.$$

(ii) Multiplying and dividing each term by $(x-y)$, we get the given sum as

$$\frac{1}{(x-y)}\cdot\{(x^2-y^2)+(x^3-y^3)+(x^4-y^4)+\ldots\text{ to } n \text{ terms}\}$$

$$=\frac{1}{(x-y)}\cdot\{(x^2+x^3+x^4+\ldots\text{ to } n\text{ terms})-(y^2+y^3+y^4+\ldots\text{ to } n\text{ terms})\}$$

$$=\frac{1}{(x-y)}\cdot\left\{x^2\left(\frac{x^n-1}{x-1}\right)-y^2\left(\frac{y^n-1}{y-1}\right)\right\}.$$

4. Given sum $=\left\{\frac{3}{5}+\frac{3}{5^3}+\frac{3}{5^5}+\ldots\text{ to } n\text{ terms}\right\}+\left\{\frac{4}{5^2}+\frac{4}{5^4}+\ldots\text{ to } n\text{ terms}\right\}$

$$=\frac{\frac{3}{5}\times\left\{1-\left(\frac{1}{5^2}\right)^n\right\}}{\left(1-\frac{1}{5^2}\right)}+\frac{\frac{4}{5^2}\times\left\{1-\left(\frac{1}{5^2}\right)^n\right\}}{\left(1-\frac{1}{5^2}\right)}$$

$$=\left(\frac{3}{5}\times\frac{25}{24}\right)\frac{(5^{2n}-1)}{5^{2n}}+\left(\frac{4}{25}\times\frac{25}{24}\right)\frac{(5^{2n}-1)}{5^{2n}}$$

$$=\left(\frac{5}{8}+\frac{1}{6}\right)\left(\frac{5^{2n}-1}{5^{2n}}\right)=\frac{19}{24}\cdot\left(\frac{5^{2n}-1}{5^{2n}}\right).$$

5. (i) $\sum\limits_{n=1}^{10}(2+3^n)=(2+3^1)+(2+3^2)+(2+3^3)+\ldots+(2+3^{10})$

$$=(2+2+2+\ldots\text{ taken 10 times})+(3+3^2+3^3+\ldots+3^{10})$$

$$=(2\times 10)+\left\{3\times\left(\frac{3^{10}-1}{3-1}\right)\right\}$$

$$=20+\left(\frac{59049-1}{2}\right)=(20+29524)=29544.$$

(ii) $\sum\limits_{k=1}^{n}[2^k+3^{(k-1)}]=(2^1+3^0)+(2^2+3^1)+(2^3+3^2)+\ldots\text{ up to } n\text{ terms}$

$$=(2^1+2^2+2^3+\ldots\text{ to } n\text{ terms})+(1+3+3^2+\ldots\text{ to } n\text{ terms})$$

$$=2\left(\frac{2^n-1}{2-1}\right)+1\cdot\left(\frac{3^n-1}{3-1}\right)$$

$$=2(2^n-1)+\frac{1}{2}(3^n-1)=2^{(n+1)}-2+\frac{1}{2}\cdot 3^n-\frac{1}{2}$$

$$=2^{(n+1)}+\frac{1}{2}\cdot 3^n-\frac{5}{2}=2^{(n+1)}+\frac{1}{2}(3^n-5)=\frac{1}{2}[2^{(n+2)}+3^n-5].$$

(iii) $\sum\limits_{n=1}^{8}5^n=[5+5^2+5^3+\ldots\text{ to 8 terms}]$

$$=5\left(\frac{5^8-1}{5-1}\right)=\frac{5(390625-1)}{4}=\frac{5\times 390624}{4}$$

$$=(5\times 97656)=488280.$$

6. (i) Given sum $=8\times[1+11+111+\ldots\text{ to } n\text{ terms}]$

$$=\frac{8}{9}\times[9+99+999+\ldots\text{ to } n\text{ terms}]$$

$$=\frac{8}{9}\times[(10-1)+(10^2-1)+(10^3-1)+\ldots\text{ to } n\text{ terms}]$$

$$= \frac{8}{9} \times \{(10 + 10^2 + 10^3 + \ldots \text{ to } n \text{ terms}) - n\}$$

$$= \frac{8}{9} \times \left\{10 \times \left(\frac{10^n - 1}{10 - 1}\right) - n\right\} = \frac{8}{81}[10^{(n-1)} - 10 - 9n].$$

(iii) Given series $= \frac{7}{10} + \frac{77}{100} + \frac{777}{1000} + \ldots$ to n terms

$$= 7 \times \left\{\frac{1}{10} + \frac{11}{100} + \frac{111}{1000} + \ldots \text{ to } n \text{ terms}\right\}$$

$$= \frac{7}{9} \times \left\{\frac{9}{10} + \frac{99}{100} + \frac{999}{1000} + \ldots \text{ to } n \text{ terms}\right\}$$

$$= \frac{7}{9} \times \left\{\left(1 - \frac{1}{10}\right) + \left(1 - \frac{1}{10^2}\right) + \left(1 - \frac{1}{10^3}\right) + \ldots \text{ to } n \text{ terms}\right\}$$

$$= \frac{7}{9} \times \left\{n - \left(\frac{1}{10} + \frac{1}{10^2} + \frac{1}{10^3} + \ldots \text{ to } n \text{ terms}\right)\right\}$$

$$= \frac{7}{9} \times \left\{n - \frac{\frac{1}{10}\left(1 - \frac{1}{10^n}\right)}{\left(1 - \frac{1}{10}\right)}\right\} = \frac{7}{81} \times \left(9n - 1 + \frac{1}{10^n}\right).$$

7. $S_n = (2^n - 1)$ and $S_{n-1} = [2^{(n-1)} - 1]$.

$\therefore \quad T_n = (S_n - S_{n-1}) = [2^n - 2^{(n-1)}] \Rightarrow T_1 = 1, T_2 = 2, T_3 = 4, T_4 = 8, \ldots$.

Clearly, 1, 2, 4, 8, … is a GP with common ratio = 2.

15. Let us consider a GP with first term a and common ratio r and having $2n$ terms. Then,

$$T_1 + T_2 + T_3 + T_4 + \ldots + T_{2n} = 5(T_1 + T_3 + T_5 + \ldots + T_{2n-1})$$

$$\Rightarrow \quad a + ar + ar^2 + \ldots + ar^{(2n-1)} = 5[a + ar^2 + ar^4 + \ldots + ar^{(2n-2)}]$$

$$\Rightarrow \quad a\left(\frac{1 - r^{2n}}{1 - r}\right) = 5a\left[\frac{1 - (r^2)^n}{1 - r^2}\right] \Rightarrow 1 + r = 5 \Rightarrow r = 4.$$

16. Required ratio $= S_n : (S_{2n} - S_n)$.

Now, $S_n = \frac{a(1 - r^n)}{(1 - r)}$ and $S_{2n} = \frac{a(1 - r^{2n})}{(1 - r)} = \frac{a(1 - r^n)(1 + r^n)}{(1 - r)}$

$$\therefore \quad (S_{2n} - S_n) = \frac{a(1 - r^n)(1 + r^n)}{(1 - r)} - \frac{a(1 - r^n)}{(1 - r)} = \frac{a(1 - r^n)(1 + r^n - 1)}{(1 - r)} = \frac{a(1 - r^n)r^n}{(1 - r)}.$$

Hence, the required ratio is $\frac{1}{r^n}$.

WORD PROBLEMS ON GP

EXAMPLE 1 *A man has 2 parents, 4 grandparents, 8 great-grandparents, and so on. Find the number of his ancestors during the ten generations preceding his own.*

SOLUTION Required number of ancestors

$$= 2 + 4 + 6 + 8 + \ldots \text{ up to 10 terms}$$

$$= 2 \times \frac{(2^{10} - 1)}{(2 - 1)} = 2(2^{10} - 1) \qquad [\text{GP with } a = 2, r = 2 \text{ and } n = 10]$$

$$= 2(1024 - 1) = (2 \times 1023) = 2046.$$

Hence, the required number of the man's ancestors is 2046.

EXAMPLE 2 *A man writes a letter to four of his friends. He asks each one of them to copy the letter and mail to four different persons with the instruction that they move the chain similarly. Assuming that the chain is not broken and it costs* ₹ 4 *to mail one letter, find the amount spent on postage when 6th set of letters is mailed.*

SOLUTION Successive number of letters are 4, 16, 64, … .

This is a GP with $a = 4$ and $r = 4$.

Number of letters in the 6th set $= ar^{(6-1)} = (4 \times 4^5) = 4^6 = 4096$.

Cost of postage on these leters = ₹ (4096 × 4) = ₹ 16384.

EXAMPLE 3 *What will* ₹ 10000 *amount to in* 4 *years after its deposit in a bank which pays annual interest at the rate of* 10% *per annum compounded annually?*

SOLUTION Original sum = ₹ 10000.

Amount after 1 year $= ₹\ 10000 + ₹\left(10000 \times \frac{10}{100} \times 1\right) = ₹\ 11000$,

amount after 2 years $= ₹\ 11000 + ₹\left(11000 \times \frac{10}{100} \times 1\right) = ₹\ 12100$,

amount after 3 years $= ₹\ 12100 + ₹\left(12100 \times \frac{10}{100} \times 1\right) = ₹\ 13310$,

and so on.

Thus, these amounts form a GP

10000, 11000, 12100, 13310, …

such that $\frac{11000}{10000} = \frac{12100}{11000} = \frac{13310}{12100} = \frac{11}{10}$.

In this GP, we have $a = 10000$, $r = \frac{11}{10}$.

Amount after 4 years $= T_5 = ar^{(5-1)} = ar^4$

$$= ₹\left[10000 \times \left(\frac{11}{10}\right)^4\right] = ₹\ 14641.$$

Hence, the amount after 4 years is ₹ 14641.

Alternative method

Here, $P = ₹\ 10000$, $R = 10\%$ p.a. and $T = 4$ years.

$\therefore$ amount after 4 years $= ₹\left\{P \times \left(1 + \frac{R}{100}\right)^T\right\}$

$$= ₹\left\{10000 \times \left(1 + \frac{10}{100}\right)^4\right\}$$

$$= ₹\left\{10000 \times \left(\frac{11}{10}\right)^4\right\} = ₹\ 14641.$$

Hence, the amount after 4 years is ₹ 14641.

EXAMPLE 4 *A manufacturer reckons that the value of a machine which costs him* ₹ 125000 *will depreciate each year by* 20%. *Find the estimated value of the machine at the end of* 5 *years.*

SOLUTION **Short-cut method**

Value of the machine after 5 years $= ₹ \left\{125000 \times \left(1 - \frac{20}{100}\right)^5\right\}$

$= ₹ \left\{125000 \times \left(\frac{4}{5}\right)^5\right\}$

$= ₹ \left(\frac{125000 \times 1024}{3125}\right) = ₹\ 40960.$

Hence, the value of the machine after 5 years is ₹ 40960.

Alternative method

Initial value of the machine = ₹ 125000.

Value of the machine after 1 year = 80% of ₹ 125000

$= ₹ \left(125000 \times \frac{80}{100}\right) = ₹\ 100000.$

Value of the machine after 2 years = 80% of ₹ 100000

$= ₹ \left(100000 \times \frac{80}{100}\right) = ₹\ 80000.$

Thus, the yearwise values of the machine are

₹ 125000, ₹ 100000, ₹ 80000, … .

Here, $\frac{100000}{125000} = \frac{4}{5}, \frac{80000}{100000} = \frac{4}{5}$, etc.

Thus, these amounts form a GP with $a = 125000$ and $r = \frac{4}{5}\cdot$

Value after 5 years = 6th term $= T_6 = ar^{(6-1)} = ar^5$

$= ₹ \left\{125000 \times \left(\frac{4}{5}\right)^5\right\}$

$= ₹ \left(\frac{125000 \times 1024}{3125}\right) = ₹\ 40960.$

EXAMPLE 5 *The number of bacteria in a certain culture doubles every hour. If there were* 30 *bacteria present in the culture originally, how many bacteria would be present at the end of* 2*nd hour,* 4*th hour and nth hour?*

SOLUTION The bacteria present in the culture originally, at the end of 1st hour, at the end of 2nd hour, at the end of 3rd hour and so on, are

30, 60, 120, 240, … .

This is a GP with $a = 30$ and $r = \frac{60}{30} = 2$.

Number of bacteria at the end of 2nd hour

$= T_3 = ar^2 = (30 \times 2^2) = 120.$

Number of bacteria at the end of 4th hour

$= T_5 = ar^4 = (30 \times 2^4) = 480.$

Number of bacteria at the end of nth hour

$$= T_{n+1} = ar^{(n+1-1)} = 30 \times 2^n.$$

EXAMPLE 6 *The inventor of the chessboard suggested a reward of one grain of wheat for the first square; 2 grains for the second; 4 grains for the third; and so on, doubling the number of grains for subsequent squares. How many grains would have to be given to the inventor? (Note that there are 64 squares in the chessboard.)*

SOLUTION Required number of grains

$$= 1 + 2^1 + 2^2 + 2^3 + \dots \text{ to 64 terms}$$
$$= 1 + (2^1 + 2^2 + 2^3 + \dots 2^{63})$$
$$= 1 + \frac{2(2^{63} - 1)}{(2-1)} = (1 + 2^{64} - 2) = (2^{64} - 1).$$

EXAMPLE 7 *The lengths of three unequal edges of a rectangular solid block are in GP. The volume of the block is 216 cm^3 and the total surface area is 252 cm^2. Find the length of its longest edge.*

SOLUTION Let the lengths of its edges be $\frac{a}{r}$ cm, a cm and ar cm.

Then, its volume $= \left(\frac{a}{r} \times a \times ar\right) cm^3 = a^3\ cm^3$.

$\therefore \quad a^3 = 216 = 6^3 \Rightarrow a = 6.$

So, the edges are $6r$ cm, 6 cm and $\frac{6}{r}$ cm.

$\therefore \quad$ surface area $= 2[lb + bh + lh]$

$$= 2\left[6r \times 6 + 6 \times \frac{6}{r} + 6r \times \frac{6}{r}\right] cm^2$$
$$= 2\left[36r + \frac{36}{r} + 36\right] cm^2 = 72 \times \left[r + \frac{1}{r} + 1\right] cm^2.$$

$\therefore \quad 72 \times \left(r + \frac{1}{r} + 1\right) = 252$

$\Rightarrow \quad 2(r^2 + 1 + r) = 7r \Rightarrow 2r^2 - 5r + 2 = 0$

$\Rightarrow \quad (r-2)(2r-1) = 0 \Rightarrow r = 2 \text{ or } r = \frac{1}{2}.$

Each value of r gives the longest edge = 12 cm.

EXERCISE 12D

1. What will ₹ 15625 amount to in 3 years after its deposit in a bank which pays annual interest at the rate of 8% per annum, compounded annually?
2. The value of a machine costing ₹ 80000 depreciates at the rate of 15% per annum. What will be the worth of this machine after 3 years?
3. Three years before the population of a village was 10000. If at the end of each year, 20% of the people migrated to a nearby town, what is its present population?

4. What will ₹ 5000 amount to in 10 years, compounded annually at 10% per annum? [Given $(1.1)^{10} = 2.594$]

5. A manufacturer reckons that the value of a machine which costs him ₹ 156250, will depreciate each year by 20%. Find the estimated value at the end of 5 years.

6. The number of bacteria in a certain culture doubles every hour. If there were 50 bacteria present in the culture originally, how many bacteria would be present at the end of (i) 2nd hour, (ii) 5th hour and (iii) nth hour?

Hint At the end of 2nd hour means 3rd term $= ar^2 = (50 \times 2^2)$.

ANSWERS (EXERCISE 12D)

1. ₹ 19683 **2.** ₹ 49130 **3.** 5120 **4.** ₹ 12970

5. ₹ 51200 **6.** (i) 200 (ii) 1600 (iii) 50×2^n

PROPERTIES OF GP

Let $a, ar, ar^2, ar^3, \ldots$ be a GP. Then,

(i) $\frac{1}{a}, \frac{1}{ar}, \frac{1}{ar^2}, \frac{1}{ar^3}, \ldots$ is a GP.

(ii) $ka, k(ar), k(ar^2), k(ar^3), \ldots$ is a GP, where $k \neq 0$.

(iii) $\frac{k}{a}, \frac{k}{ar}, \frac{k}{ar^2}, \frac{k}{ar^3}, \ldots$ is a GP, where $k \neq 0$.

(iv) $a^k, (ar)^k, (ar^2)^k, (ar^3)^k, \ldots$ is a GP, where $k \neq 0$.

SOLVED EXAMPLES

EXAMPLE 1 *If a, b, c are in GP, prove that* $\log a, \log b, \log c$ *are in AP.*

SOLUTION Let a, b, c be in GP. Then,

$$b^2 = ac \Rightarrow \log b^2 = \log(ac)$$
$$\Rightarrow 2\log b = \log a + \log c$$
$$\Rightarrow \log a, \log b, \log c \text{ are in AP.}$$

EXAMPLE 2 *If a, b, c, d are in GP then prove that* $a+b, b+c, c+d$ *are also in GP.*

SOLUTION Let a, b, c, d be in GP with common ratio r. Then,

$$b = ar, c = ar^2 \text{ and } d = ar^3.$$

$$\therefore \quad (a+b) = (a+ar) = a(1+r),$$
$$(b+c) = (ar+ar^2) = ar(1+r),$$
$$(c+d) = (ar^2+ar^3) = ar^2(1+r).$$

$$\therefore \quad (b+c)^2 = a^2r^2(1+r)^2 \text{ and } (a+b)(c+d) = a^2r^2(1+r)^2.$$

Consequently, $(b+c)^2 = (a+b)(c+d)$.

Hence, $(a+b)$, $(b+c)$ and $(c+d)$ are in GP.

EXAMPLE 3 *If a, b, c, d are in GP then prove that $a^n+b^n, b^n+c^n, c^n+d^n$ are also in GP.*

SOLUTION Let a, b, c, d be in GP with common ratio $= r$. Then,

$$b = ar, c = ar^2 \text{ and } d = ar^3.$$

$\therefore$ $(a^n+b^n) = (a^n+a^nr^n) = a^n(1+r^n)$,

$(b^n+c^n) = (a^nr^n+a^nr^{2n}) = a^nr^n(1+r^n)$

and $(c^n+d^n) = (a^nr^{2n}+a^nr^{3n}) = a^nr^{2n}(1+r^n)$.

$\therefore$ $(b^n+c^n)^2 = a^{2n}r^{2n}(1+r^n)^2$ and $(a^n+b^n)(c^n+d^n) = a^{2n}r^{2n}(1+r^n)^2$.

Consequently, $(b^n+c^n)^2 = (a^n+b^n)(c^n+d^n)$.

Hence, (a^n+b^n), (b^n+c^n) and (c^n+d^n) are in GP.

EXAMPLE 4 *If (a^2+b^2), $(ab+bc)$, (b^2+c^2) are in GP then prove that a, b, c are also in GP.*

SOLUTION Let (a^2+b^2), $(ab+bc)$ and (b^2+c^2) be in GP. Then,

$$(ab+bc)^2 = (a^2+b^2)(b^2+c^2)$$

$\Rightarrow$ $a^2b^2+b^2c^2+2ab^2c = a^2b^2+a^2c^2+b^4+b^2c^2$

$\Rightarrow$ $b^4+a^2c^2-2ab^2c = 0$

$\Rightarrow$ $(b^2-ac)^2 = 0 \Rightarrow b^2-ac = 0 \Rightarrow b^2 = ac$.

Hence, a, b, c are in GP.

EXAMPLE 5 *If mth, nth and pth terms of a GP form three consecutive terms of a GP then prove that m, n, p are three consecutive terms of a AP.*

SOLUTION Let a be the first term and r be the common ratio of the given GP. Then,

$$T_m = ar^{(m-1)}, T_n = ar^{(n-1)} \text{ and } T_p = ar^{(p-1)}.$$

Since T_m, T_n, T_p are in GP, we have

$$T_n^2 = T_m \times T_p$$

$\Rightarrow$ $[ar^{(n-1)}]^2 = [ar^{(m-1)} \times ar^{(p-1)}]$

$\Rightarrow$ $a^2r^{(2n-2)} = a^2r^{(m+p-2)}$

$\Rightarrow$ $r^{(2n-2)} = r^{(m+p-2)}$

$\Rightarrow$ $2n-2 = m+p-2$

$\Rightarrow$ $2n = m+p \Rightarrow m, n, p$ are in AP.

Hence, m, n, p are three consecutive terms of an AP.

EXAMPLE 6 *If the 4th, 10th and 16th terms of a GP are x, y, z respectively, prove that x, y, z are in GP.*

SOLUTION Let a be the first term and r be the common ratio of the given GP.

Then, $T_4 = x$, $T_{10} = y$ and $T_{16} = z$.

$\Rightarrow \quad ar^3 = x, ar^9 = y$ and $ar^{15} = z$.

$\therefore \quad y^2 = (ar^9)^2 = a^2r^{18}$ *and* $xz = (ar^3)(ar^{15}) = a^2r^{18}$.

Consequently, we have $y^2 = xz$.

Hence, x, y, z are in GP.

EXAMPLE 7 *Three numbers are in AP and their sum is* 15. *If* 1, 3, 9 *be added to them respectively, they form a GP. Find the numbers.*

SOLUTION Let the required numbers be $(a-d)$, a and $(a+d)$. Then,

$$(a-d)+a+(a+d)=15 \Rightarrow 3a=15 \Rightarrow a=5.$$

So, the numbers are $(5-d)$, 5 and $(5+d)$.

Adding 1, 3, 9 respectively to these numbers, we get the numbers

$$(6-d), 8 \text{ and } (14+d).$$

These numbers are in GP.

$$\begin{aligned}\therefore \quad 8^2 = (6-d)(14+d) &\Rightarrow 84-8d-d^2=64\\ &\Rightarrow d^2+8d-20=0\\ &\Rightarrow d^2+10d-2d-20=0\\ &\Rightarrow d(d+10)-2(d+10)=0\\ &\Rightarrow (d+10)(d-2)=0\\ &\Rightarrow d=-10 \text{ or } d=2.\end{aligned}$$

So, the required numbers are (3, 5, 7) or (15, 5, –5).

EXERCISE 12E

1. If p, q, r are in AP then prove that pth, qth and rth terms of any GP are in GP.
2. If a, b, c are in GP then show that $\log a^n, \log b^n, \log c^n$ are in AP.
3. If a, b, c are in GP then show that $\frac{1}{\log_a m}, \frac{1}{\log_b m}, \frac{1}{\log_c m}$ are in AP.
4. Find the values of k for which $k+12$, $k-6$ and 3 are in GP.
5. Three numbers are in AP and their sum is 15. If 1, 4, 19 be added to them respectively then they are in GP. Find the numbers.
6. Three numbers are in AP and their sum is 21. If the second number is reduced by 1 and the third is increased by 1, we obtain three numbers in GP. Find the numbers.
7. The sum of three numbers in GP is 56. If 1, 7, 21 be subtracted from them respectively, we obtain the numbers in AP. Find the numbers.
8. If a, b, c are in GP, prove that $\frac{a^2+ab+b^2}{ab+bc+ca} = \frac{b+a}{c+b}$.
9. If $(a-b), (b-c), (c-a)$ are in GP then prove that $(a+b+c)^2 = 3(ab+bc+ca)$.
10. If a, b, c are in GP, prove that
    (i) $a(b^2+c^2) = c(a^2+b^2)$
    (ii) $\frac{1}{(a^2-b^2)} + \frac{1}{b^2} = \frac{1}{(b^2-c^2)}$

(iii) $(a+2b+2c)(a-2b+2c)=a^2+4c^2$

(iv) $a^2b^2c^2\left(\frac{1}{a^3}+\frac{1}{b^3}+\frac{1}{c^3}\right)=a^3+b^3+c^3.$

11. If a, b, c, d are in GP, prove that

(i) $(b+c)(b+d)=(c+a)(c+d)$

(ii) $\frac{ab-cd}{b^2-c^2}=\frac{a+c}{b}$

(iii) $(a+b+c+d)^2=(a+b)^2+2(b+c)^2+(c+d)^2.$

12. If a, b, c are in GP, prove that $\frac{1}{(a+b)}, \frac{1}{2b}, \frac{1}{b+c}$ are in AP.

13. If a, b, c are in GP, prove that a^2, b^2, c^2 are in GP.

14. If a, b, c are in GP, prove that a^3, b^3, c^3 are in GP.

15. If a, b, c are in GP, prove that $(a^2+b^2), (ab+bc), (b^2+c^2)$ are in GP.

16. If a, b, c, d are in GP, prove that $(a^2-b^2), (b^2-c^2), (c^2-d^2)$ are in GP.

17. If a, b, c, d are in GP then prove that

$$\frac{1}{(a^2+b^2)}, \frac{1}{(b^2+c^2)}, \frac{1}{(c^2+d^2)} \text{ are in GP.}$$

18. If $(p^2+q^2), (pq+qr), (q^2+r^2)$ are in GP then prove that p, q, r are in GP.

19. If a, b, c are in AP and a, b, d are in GP, show that $a, (a-b)$ and $(d-c)$ are in GP.

20. If a, b, c are in AP, and a, x, b and b, y, c are in GP then show that x^2, b^2, y^2 are in AP.

ANSWERS (EXERCISE 12E)

4. $(k=0)$ or $(k=15)$ **5.** $(2, 5, 8)$ or $(26, 5, -16)$ **6.** $(12, 7, 2)$ or $(3, 7, 11)$

7. 8, 16, 32

HINTS TO SOME SELECTED QUESTIONS

1. We have, $2q=p+r$.

Let T_p, T_q, T_r be the given terms of a GP with first term $= A$ and common ratio $= R$. Then,

$$(T_q)^2=\{AR^{(q-1)}\}^2=A^2R^{(2q-2)}$$

and $(T_p \times T_r)=\{AR^{(p-1)} \times AR^{(r-1)}\}=A^2R^{(p+r-2)}=A^2R^{(2q-2)}.$

$\therefore \quad (T_q)^2=(T_p \times T_r)$ and hence T_p, T_q, T_r are in GP.

2. a, b, c are in GP $\Rightarrow b^2=ac \Rightarrow (b^2)^n=(ac)^n \Rightarrow b^{2n}=a^nc^n.$

$\therefore \quad 2\log b^n=\log a^n+\log c^n \Rightarrow \log a^n, \log b^n, \log c^n$ are in AP.

3. a, b, c are in GP $\Rightarrow b^2=ac$

$\therefore \quad 2\log_m b=\log_m a+\log_m c \Rightarrow 2\times\frac{\log b}{\log m}=\frac{\log a}{\log m}+\frac{\log c}{\log m}$

$\therefore \quad 2 \times \dfrac{1}{\log_b m} = \dfrac{1}{\log_a m} + \dfrac{1}{\log_c m} \Rightarrow \dfrac{1}{\log_a m}, \dfrac{1}{\log_b m}, \dfrac{1}{\log_c m}$ are in AP.

4. $\dfrac{k-6}{k+12} = \dfrac{3}{k-6} \Rightarrow (k-6)^2 = 3k+36$

$\Rightarrow k^2 - 12k = 3k \Rightarrow k^2 - 15k = 0 \Rightarrow k(k-15) = 0$

$\therefore \quad k = 0$ or $k = 15$.

6. Let the required numbers be $(a-d), a, (a+d)$. Then, $3a = 21 \Rightarrow a = 7$.

So, these numbers are $(7-d), 7, (7+d)$.

$\therefore \quad (7-d), 6$ and $(8+d)$ are in GP.

So, $\dfrac{6}{(7-d)} = \dfrac{(8+d)}{6} \Rightarrow (7-d)(8+d) = 36 \Rightarrow d^2 + d - 20 = 0$

$\Rightarrow (d+5)(d-4) = 0 \Rightarrow d = -5$ or $d = 4$.

Hence, the required numbers are (12, 7, 2) or (3, 7, 11).

8. Let r be the common ratio of the given GP. Then,

$b = ar$ and $c = ar^2$.

$\therefore \quad \text{LHS} = \dfrac{a^2 + a^2r + a^2r^2}{a^2r + a^2r^3 + a^2r^2} = \dfrac{a^2(1+r+r^2)}{a^2r(1+r+r^2)} = \dfrac{1}{r}$

$\text{RHS} = \dfrac{ar+a}{ar^2+ar} = \dfrac{a(1+r)}{ar(1+r)} = \dfrac{1}{r}$.

Hence, LHS = RHS.

9. $\dfrac{(b-c)}{(a-b)} = \dfrac{(c-a)}{(b-c)} \Rightarrow (b-c)^2 = (a-b)(c-a)$

$\Rightarrow b^2 + c^2 - 2bc = ac - a^2 - bc + ab$

$\Rightarrow a^2 + b^2 + c^2 = ab + bc + ac$

$\Rightarrow (a+b+c)^2 = 3(ab+bc+ac)$ [adding $2(ab+bc+ac)$ on both sides].

12. Let $b = ar$ and $c = ar^2$. Then,

$\dfrac{1}{a+b} = \dfrac{1}{a+ar} = \dfrac{1}{a(1+r)}, \ \dfrac{1}{2b} = \dfrac{1}{2ar}$, and $\dfrac{1}{(b+c)} = \dfrac{1}{ar(1+r)}$.

$\therefore \quad \dfrac{1}{(a+b)} + \dfrac{1}{(b+c)} = \dfrac{1}{a(1+r)} + \dfrac{1}{ar(1+r)} = \dfrac{(1+r)}{ar(1+r)} = \dfrac{1}{ar} = 2 \times \left(\dfrac{1}{2b}\right)$.

Hence, $\dfrac{1}{(a+b)}, \dfrac{1}{2b}, \dfrac{1}{(b+c)}$ are in AP.

13. Let $b = ar$ and $c = ar^2$. Then

$a^2c^2 = a^2(ar^2)^2 = a^2 \times a^2r^4 = a^4r^4 = (b^2)^2$.

19. $(a+c) = 2b$ and $b^2 = ad$.

We have to prove that $(a-b)^2 = a(d-c)$.

Now, $(a-b)^2 = a^2 + b^2 - 2ab = a^2 + ad - a(a+c)$

$\Rightarrow \quad (a-b)^2 = ad - ac = a(d-c)$.

Hence, $a, (a-b), (d-c)$ are in GP.

20. $x^2 = ab, y^2 = bc$ and $a + c = 2b$.

$\therefore \quad x^2 + y^2 = (ab+bc) = b(a+c) = 2b^2$.

Hence, x^2, b^2, y^2 are in AP.

GEOMETRIC MEAN

GEOMETRIC MEAN *Let a and b be two given numbers. We say that G is the geometric mean (GM) between a and b if a, G, b are in GP.*

$$G \text{ is the GM between } a \text{ and } b \Leftrightarrow a, G, b \text{ are in GP}$$
$$\Leftrightarrow \frac{G}{a} = \frac{b}{G}$$
$$\Leftrightarrow G^2 = ab \Leftrightarrow G = \sqrt{ab}.$$

So, *GM between a and b* $= \sqrt{ab}$.

REMARKS (i) The GM between two positive numbers is positive.
(ii) The GM between two negative numbers is negative.
(iii) The GM between two numbers of opposite signs does not exist.

SOLVED EXAMPLES

EXAMPLE 1 *Find the geometric mean between*

(i) 6 *and* 24 *(ii)* –9 *and* –25 *(iii)* –6 *and* 9

SOLUTION (i) The GM between 6 and 24 $= \sqrt{6 \times 24} = \sqrt{144} = 12$.

(ii) The GM between –9 and $-25 = \sqrt{(-9) \times (-25)} = \sqrt{225} = -15$.

(iii) Since the GM between two numbers of opposite signs does not exist, so the GM between –6 and 9 does not exist.

EXAMPLE 2 *Find two positive numbers a and b whose AM and GM are* 34 *and* 16 *respectively.*

SOLUTION We have

$$\frac{a+b}{2} = 34 \text{ and } \sqrt{ab} = 16$$
$$\Rightarrow a + b = 68 \text{ and } ab = 256$$
$$\Rightarrow (a-b) = \sqrt{(a+b)^2 - 4ab} = \sqrt{(68)^2 - 4 \times 256} = \sqrt{3600} = \pm 60$$
$$\Rightarrow a + b = 68 \text{ and } a - b = \pm 60$$
$$\Rightarrow (a+b = 68, a-b = 60) \text{ or } (a+b = 68, a-b = -60)$$
$$\Rightarrow (a = 64, b = 4) \text{ or } (a = 4, b = 64)$$

Hence, the required numbers are $(a = 64, b = 4)$ or $(a = 4, b = 64)$.

EXAMPLE 3 *Find the value of n so that* $\dfrac{a^{(n+1)} + b^{(n+1)}}{a^n + b^n}$ *may be the geometric mean between a and b, where* $a \neq b$.

SOLUTION $\dfrac{a^{(n+1)} + b^{(n+1)}}{a^n + b^n}$ is GM between a and b

$$\Rightarrow \frac{a^{(n+1)} + b^{(n+1)}}{a^n + b^n} = a^{\frac{1}{2}} b^{\frac{1}{2}}$$
$$\Rightarrow [a^{(n+1)} + b^{(n+1)}] = (a^n + b^n)\left(a^{\frac{1}{2}} b^{\frac{1}{2}}\right)$$

$\Rightarrow \quad [a^{(n+1)} + b^{(n+1)}] = a^{\left(n+\frac{1}{2}\right)} b^{\frac{1}{2}} + a^{\frac{1}{2}} b^{\left(n+\frac{1}{2}\right)}$

$\Rightarrow \quad \left\{a^{(n+1)} - a^{\left(n+\frac{1}{2}\right)} b^{\frac{1}{2}}\right\} = a^{\frac{1}{2}} b^{\left(n+\frac{1}{2}\right)} - b^{(n+1)}$

$\Rightarrow \quad a^{\left(n+\frac{1}{2}\right)}\left(a^{\frac{1}{2}} - b^{\frac{1}{2}}\right) = b^{\left(n+\frac{1}{2}\right)}\left(a^{\frac{1}{2}} - b^{\frac{1}{2}}\right)$

$\Rightarrow \quad a^{\left(n+\frac{1}{2}\right)} = b^{\left(n+\frac{1}{2}\right)} \qquad \left[\because \left(a^{\frac{1}{2}} - b^{\frac{1}{2}}\right) \neq 0, \text{ since } a \neq b\right]$

$\Rightarrow \quad \left(\frac{a}{b}\right)^{\left(n+\frac{1}{2}\right)} = 1 = \left(\frac{a}{b}\right)^0$

$\Rightarrow \quad n + \frac{1}{2} = 0 \Rightarrow n = -\frac{1}{2}.$

Hence, $n = -\frac{1}{2}$.

EXAMPLE 4 *If A and G are respectively the arithmetic and geometric means between two distinct positive numbers a and b then prove that $A > G$.*

SOLUTION We have

$$A = \frac{a+b}{2} \text{ and } G = \sqrt{ab}$$

$\Rightarrow \quad A - G = \frac{a+b}{2} - \sqrt{ab} = \frac{a+b-2\sqrt{ab}}{2} = \frac{1}{2}(\sqrt{a} - \sqrt{b})^2 > 0$

$\Rightarrow \quad A > G.$

Hence, AM > GM.

EXAMPLE 5 *If A and G are respectively the arithmetic and geometric means between two positive numbers a and b then prove that the quadratic equation having a, b as its roots is given by $x^2 - 2Ax + G^2 = 0$.*

SOLUTION We have

$$A = \frac{a+b}{2} \text{ and } G = \sqrt{ab}.$$

The equation having a, b as its roots is given by

$x^2 - (a+b)x + ab = 0$

$\Rightarrow \quad x^2 - 2Ax + G^2 = 0 \qquad \left[\because A = \frac{a+b}{2} \text{ and } G = \sqrt{ab}\right].$

Hence, the quadratic equation having a, b as its roots is $x^2 - 2Ax + G^2 = 0$.

EXAMPLE 6 *If A and G be the AM and GM between two positive numbers then prove that the numbers are $A \pm \sqrt{A^2 - G^2}$.*

SOLUTION Let the required numbers be a and b. Then,

$$A = \frac{a+b}{2} \text{ and } G = \sqrt{ab}.$$

The quadratic equation having a, b as its roots is given by

$x^2 - (a+b)x + ab = 0$

$\Rightarrow \quad x^2 - 2Ax + G^2 = 0$

$$\Rightarrow \quad x = \frac{2A \pm \sqrt{4A^2 - 4G^2}}{2} \Rightarrow x = A \pm \sqrt{A^2 - G^2}.$$

Hence, the required numbers are $A \pm \sqrt{A^2 - G^2}$.

EXAMPLE 7 *If x, y, z are distinct positive numbers then prove that* $(x+y)(y+z)(z+x) > 8xyz$.

SOLUTION Using AM > GM, we have

$$\frac{x+y}{2} > \sqrt{xy}, \ \frac{y+z}{2} > \sqrt{yz} \text{ and } \frac{z+x}{2} > \sqrt{zx}$$

$$\Rightarrow \quad x+y > 2\sqrt{xy}, \ y+z > 2\sqrt{yz} \text{ and } z+x > 2\sqrt{zx}$$

$$\Rightarrow \quad (x+y)(y+z)(z+x) > 2\sqrt{xy} \times 2\sqrt{yz} \times 2\sqrt{zx}$$

$$\Rightarrow \quad (x+y)(y+z)(z+x) > 8xyz.$$

Hence, $(x+y)(y+z)(z+x) > 8xyz$.

EXAMPLE 8 *If a, b, c, d are four distinct positive numbers in GP then prove that* $a+d > b+c$.

SOLUTION a, b, c, d are in GP

$\Rightarrow$ a, b, c are in GP

$\Rightarrow$ b is the GM between a and c.

Also, we know that $\frac{a+c}{2}$ is the AM between a and c.

But, AM > GM.

$$\therefore \quad \frac{a+c}{2} > b \Rightarrow a+c > 2b. \qquad \ldots \text{(i)}$$

Again, a, b, c, d are in GP

$\Rightarrow$ b, c, d are in GP

$\Rightarrow$ c is the GM between b and d.

Also, we know that $\frac{b+d}{2}$ is the AM between b and d.

But, AM > GM.

$$\therefore \quad \frac{b+d}{2} > c \Rightarrow b+d > 2c. \qquad \ldots \text{(ii)}$$

From (i) and (ii), we get

$$a+c+b+d > 2b+2c.$$

Hence, $a+d > b+c$.

EXAMPLE 9 *The sum of two numbers is 6 times their geometric mean. Show that the numbers are in the ratio* $(3+2\sqrt{2}):(3-2\sqrt{2})$.

SOLUTION Let the required numbers be a and b. Then,

$$a+b = 6\sqrt{ab}$$

$$\Rightarrow \quad \frac{a+b}{2\sqrt{ab}} = \frac{3}{1}$$

$$\Rightarrow \quad \frac{a+b+2\sqrt{ab}}{a+b-2\sqrt{ab}} = \frac{3+1}{3-1} = \frac{2}{1} \qquad \text{[by componendo and dividendo]}$$

$$\Rightarrow \quad \frac{(\sqrt{a}+\sqrt{b})^2}{(\sqrt{a}-\sqrt{b})^2} = \frac{(\sqrt{2})^2}{1^2} \Rightarrow \frac{\sqrt{a}+\sqrt{b}}{\sqrt{a}-\sqrt{b}} = \frac{\sqrt{2}}{1}$$

$$\Rightarrow \quad \frac{(\sqrt{a}+\sqrt{b})+(\sqrt{a}-\sqrt{b})}{(\sqrt{a}+\sqrt{b})-(\sqrt{a}-\sqrt{b})} = \frac{\sqrt{2}+1}{\sqrt{2}-1}$$

[by componendo and dividendo]

$$\Rightarrow \quad \frac{\sqrt{a}}{\sqrt{b}} = \frac{\sqrt{2}+1}{\sqrt{2}-1} \Rightarrow \left(\frac{\sqrt{a}}{\sqrt{b}}\right)^2 = \frac{(\sqrt{2}+1)^2}{(\sqrt{2}-1)^2}$$

$$\Rightarrow \quad \frac{a}{b} = \frac{3+2\sqrt{2}}{3-2\sqrt{2}}.$$

Hence, $a:b = (3+2\sqrt{2}):(3-2\sqrt{2})$.

EXAMPLE 10 *If the AM and GM of the roots of a quadratic equation be* 8 *and* 5 *respectively then obtain the equation.*

SOLUTION Let the required roots be a and b.

Then, AM $= \frac{a+b}{2} \Rightarrow \frac{a+b}{2} = 8 \Rightarrow a+b = 16$ … (i)

and GM $= \sqrt{ab} \Rightarrow \sqrt{ab} = 5 \Rightarrow ab = 25.$ … (ii)

$\therefore$ sum of roots = 16 and product of roots = 25.

Hence, the required equation is $x^2 - 16x + 25 = 0$.

EXAMPLE 11 *If the AM and GM of two positive numbers a and b are in the ratio $m:n$, show that*

$$a:b = (m+\sqrt{m^2-n^2}):(m-\sqrt{m^2-n^2}).$$

SOLUTION Let A and G be respectively the AM and GM of a and b. Then,

$$A = \frac{a+b}{2} \text{ and } G = \sqrt{ab}$$

$$\Rightarrow \quad a+b = 2A \text{ and } G^2 = ab.$$

Now, the equation having roots a and b is

$$x^2 - (a+b)x + ab = 0$$

$$\Rightarrow \quad x^2 - 2Ax + G^2 = 0 \qquad [\because a+b = 2A \text{ and } ab = G^2]$$

$$\Rightarrow \quad x = \frac{2A \pm \sqrt{4A^2 - 4G^2}}{2} = A \pm \sqrt{A^2 - G^2}$$

$$\Rightarrow \quad a = A + \sqrt{A^2-G^2} \text{ and } b = A - \sqrt{A^2-G^2}.$$

Now, $A:G = m:n$ (given).

Let $A = km$ and $G = kn$ for some constant k. Then,

$$\frac{a}{b} = \frac{A+\sqrt{A^2-G^2}}{A-\sqrt{A^2-G^2}} = \frac{km+\sqrt{k^2m^2-k^2n^2}}{km-\sqrt{k^2m^2-k^2n^2}} = \frac{m+\sqrt{m^2-n^2}}{m-\sqrt{m^2-n^2}}.$$

Hence, $a:b = (m+\sqrt{m^2-n^2}):(m-\sqrt{m^2-n^2})$.

EXAMPLE 12 *Find two positive numbers whose difference is* 12 *and whose AM exceeds the GM by* 2.

SOLUTION Let the required positive numbers be a and b, where $a > b$.

Then, $a - b = 12$. ... (i)

Also, AM − GM = 2

$$\Rightarrow \quad \frac{a+b}{2} - \sqrt{ab} = 2$$

$$\Rightarrow \quad a + b - 2\sqrt{ab} = 4$$

$$\Rightarrow \quad (\sqrt{a} - \sqrt{b})^2 = 4$$

$$\Rightarrow \quad \sqrt{a} - \sqrt{b} = 2. \quad \text{... (ii)}$$

Now, $a - b = 12$ [from (i)]

$$\Rightarrow \quad (\sqrt{a} + \sqrt{b})(\sqrt{a} - \sqrt{b}) = 12$$

$$\Rightarrow \quad (\sqrt{a} + \sqrt{b}) \times 2 = 12 \Rightarrow \sqrt{a} + \sqrt{b} = 6. \quad \text{... (iii)} \quad \text{[using (ii)]}$$

On solving (ii) and (iii), we get $\sqrt{a} = 4$ and $\sqrt{b} = 2$.

$\therefore \quad a = 16$ and $b = 4$.

Hence, the required numbers are 16 and 4.

EXAMPLE 13 *If $x \in R$, find the minimum value of* $3^x + 3^{(1-x)}$.

SOLUTION We know that AM > GM.

$\therefore$ for all $x \in R$, we have

$$\frac{3^x + 3^{(1-x)}}{2} > \sqrt{3^x \times 3^{(1-x)}}$$

$$\Rightarrow \quad \frac{3^x + 3^{(1-x)}}{2} > \sqrt{3}$$

$$\Rightarrow \quad \{3^x + 3^{(1-x)}\} > 2\sqrt{3}.$$

Hence, the minimum value of $3^x + 3^{(1-x)}$ is $2\sqrt{3}$ for any $x \in R$.

n GEOMETRIC MEANS BETWEEN TWO GIVEN NUMBERS

Let a and b be two given numbers. We say that $G_1, G_2, G_3, \ldots, G_n$ are n geometric means between a and b, if $a, G_1, G_2, G_3, \ldots, G_n, b$ are in GP.

EXAMPLE 14 *Insert n geometric means between a and b.*

SOLUTION Let $G_1, G_2, G_3, \ldots, G_n$ be the n geometric means between a and b.

Then $a, G_1, G_2, G_3, \ldots, G_n, b$ are in GP.

This GP contains $(n + 2)$ terms.

Let the common ratio of this GP be r. Then,

$$T_{n+2} = b \Rightarrow ar^{(n+1)} = b \Rightarrow r = \left(\frac{b}{a}\right)^{\frac{1}{(n+1)}}.$$

$$\therefore \quad G_1 = ar = a \cdot \left(\frac{b}{a}\right)^{\frac{1}{(n+1)}};$$

$$G_2 = ar^2 = a \cdot \left(\frac{b}{a}\right)^{\frac{2}{(n+1)}};$$

$$G_3 = ar^3 = a \cdot \left(\frac{b}{a}\right)^{\frac{3}{(n+1)}};$$

$$\cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots$$

$$\cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots$$

$$G_n = ar^n = a \cdot \left(\frac{b}{a}\right)^{\frac{n}{(n+1)}}.$$

EXAMPLE 15 *Insert two numbers between* 3 *and* 81 *so that the resulting sequence is a GP.*

SOLUTION Let the required numbers be G_1 and G_2. Then,

$3, G_1, G_2, 81$ are in GP.

Let the common ratio of this GP be r. Then,

$$T_4 = 81 \Rightarrow ar^{(4-1)} = 81$$

$$\Rightarrow 3 \times r^3 = 81 \Rightarrow r^3 = 27 = (3)^3 \Rightarrow r = 3.$$

$\therefore \quad G_1 = (3 \times r) = (3 \times 3) = 9$ and $G_2 = (3 \times r^2) = (3 \times 9) = 27$.

Hence, the required numbers are 3 and 27.

EXAMPLE 16 *Insert three numbers between* 1 *and* 256 *so that the resulting sequence is a GP.*

SOLUTION Let the required numbers be G_1, G_2, G_3. Then,

$1, G_1, G_2, G_3, 256$ are in GP.

Let r be the common ratio of this GP. Then,

$$T_5 = 256 \Rightarrow ar^{(5-1)} = 256$$

$$\Rightarrow 1 \times r^4 = 256 \Rightarrow r^4 = 256 = (4)^4 \Rightarrow r = 4.$$

$\therefore \quad G_1 = (1 \times 4) = 4, G_2 = (1 \times 4^2) = 16$ and $G_3 = (1 \times 4^3) = 64$.

Hence, the required numbers are 4, 16, 64.

EXAMPLE 17 *If G be the GM between two given numbers and A_1, A_2 be the two AMs between them, prove that*

$$G^2 = (2A_1 - A_2)(2A_2 - A_1).$$

SOLUTION Let a, b be the two given numbers. Then,

$$G^2 = ab. \qquad \ldots \text{(i)}$$

Also, a, A_1, A_2, b are in AP

$\Rightarrow \quad 2A_1 = a + A_2$ and $2A_2 = A_1 + b$

$\Rightarrow \quad (2A_1 - A_2) = a$ and $(2A_2 - A_1) = b$

$\Rightarrow \quad (2A_1 - A_2)(2A_2 - A_1) = ab$

$\Rightarrow \quad (2A_1 - A_2)(2A_2 - A_1) = G^2 \quad$ [using (i)].

Hence, $G^2 = (2A_1 - A_2)(2A_2 - A_1)$.

EXAMPLE 18 *If A is the arithmetic mean and G_1, G_2 be the two geometric means between any two numbers then prove that*

$$2A = \frac{G_1^2}{G_2} + \frac{G_2^2}{G_1}.$$

SOLUTION Let the given numbers be a and b. Then,

$$A = \frac{a+b}{2} \Rightarrow a + b = 2A. \qquad \ldots \text{(i)}$$

And, a, G_1, G_2, b are in GP.

$$\therefore \quad \frac{G_1}{a} = \frac{G_2}{G_1} = \frac{b}{G_2}$$

$$\Rightarrow \quad a = \frac{G_1^2}{G_2} \text{ and } b = \frac{G_2^2}{G_1}$$

$$\Rightarrow \quad a + b = \frac{G_1^2}{G_2} + \frac{G_2^2}{G_1}$$

$$\Rightarrow \quad 2A = \frac{G_1^2}{G_2} + \frac{G_2^2}{G_1} \qquad \text{[using (i)].}$$

Hence, $2A = \dfrac{G_1^2}{G_2} + \dfrac{G_2^2}{G_1}$.

EXERCISE 12F

1. Find two positive numbers a and b, whose
(i) AM = 25 and GM = 20 (ii) AM = 10 and GM = 8.

2. Find the GM between the numbers
(i) 5 and 125 (ii) 1 and $\frac{9}{16}$ (iii) 0.15 and 0.0015
(iv) –8 and –2 (v) –6.3 and –2.8 (vi) a^3b and ab^3

3. Insert two geometric means between 9 and 243.

4. Insert three geometric means between $\frac{1}{3}$ and 432.

5. Insert four geometric means between 6 and 192.

6. The AM between two positive numbers a and $b (a > b)$ is twice their GM. Prove that $a : b = (2+\sqrt{3}) : (2-\sqrt{3})$.

7. If a, b, c are in AP; x is the GM between a and b; y is the GM between b and c; then show that b^2 is the AM between x^2 and y^2.

8. Show that the product of n geometric means between a and b is equal to the nth power of the single GM between a and b.

9. If AM and GM of the roots of a quadratic equation are 10 and 8 respectively then obtain the quadratic equation.

ANSWERS (EXERCISE 12F)

1. (i) (a = 40 and b = 10) or (a = 10 and b = 40)
(ii) (a = 16 and b = 4) or (a = 4 and b = 16)

2. (i) 25 (ii) $\frac{3}{4}$ (iii) 0.015 (iv) –4 (v) –0.42 (vi) a^2b^2

3. 27, 81 **4.** 2, 12, 72 **5.** 12, 24, 48, 96 **9.** $x^2 - 20x + 64 = 0$

HINTS TO SOME SELECTED QUESTIONS

6. AM $= 2(\text{GM}) \Leftrightarrow \frac{1}{2}(a+b) = 2\sqrt{ab} \Leftrightarrow \frac{a+b}{2\sqrt{ab}} = \frac{2}{1}$

$$\Leftrightarrow \frac{a+b+2\sqrt{ab}}{a+b-2\sqrt{ab}} = \frac{2+1}{2-1} \Leftrightarrow \frac{(\sqrt{a}+\sqrt{b})^2}{(\sqrt{a}-\sqrt{b})^2} = \frac{(\sqrt{3})^2}{(1)^2}$$

$$\Leftrightarrow \frac{\sqrt{a}+\sqrt{b}}{\sqrt{a}-\sqrt{b}} = \frac{\sqrt{3}}{1} \Leftrightarrow \frac{(\sqrt{a}+\sqrt{b})+(\sqrt{a}-\sqrt{b})}{(\sqrt{a}+\sqrt{b})-(\sqrt{a}-\sqrt{b})} = \frac{\sqrt{3}+1}{\sqrt{3}-1}$$

$$\Leftrightarrow \frac{\sqrt{a}}{\sqrt{b}} = \frac{\sqrt{3}+1}{\sqrt{3}-1} \Leftrightarrow \frac{a}{b} = \frac{(\sqrt{3}+1)^2}{(\sqrt{3}-1)^2} = \frac{2+\sqrt{3}}{2-\sqrt{3}}.$$

7. $(a+c) = 2b$, $x = \sqrt{ab}$ and $y = \sqrt{bc}$

$\Rightarrow \quad x^2 + y^2 = (ab+bc) = b(a+c) = 2b^2.$

8. Let $G_1, G_2, \ldots, G_n$ be n GMs between a and b, and let r be its common ratio. Then, $a, G_1, G_2, \ldots, G_n, b$ are in GP.

$$\therefore \quad T_{n+2} = b \Rightarrow ar^{(n+2-1)} = b \Rightarrow r^{(n+1)} = \left(\frac{b}{a}\right) \Rightarrow r = \left(\frac{b}{a}\right)^{\frac{1}{(n+1)}}. \qquad \ldots \text{(i)}$$

$$\therefore \quad G_1 \times G_2 \times G_3 \times \ldots \times G_n = (ar) \times (ar^2) \times \ldots \times (ar^n)$$

$$= a^n \times r^{(1+2+\ldots+n)} = a^n \times r^{\frac{1}{2}n(n+1)}$$

$$= a^n \times \left(\frac{b}{a}\right)^{n/2} = a^{n/2} \times b^{n/2} \qquad \text{[using (i)]}$$

$$= (\sqrt{ab})^n = G^n, \text{ where } G = \sqrt{ab}.$$

9. Let a and b be the roots of the quadratic equation.

Then, $x^2 - (a+b)x + ab = 0$.

Now, $\frac{a+b}{2} = 10$ and $\sqrt{ab} = 8 \Rightarrow a+b = 20$ and $ab = 64$.

Hence, the required equation is $x^2 - 20x + 64 = 0$.

INFINITE GEOMETRIC SERIES

THEOREM *Prove that the sum of an infinite GP with first term a and common ratio r, where* $|r| < 1$, *is given by*

$$S = \frac{a}{(1-r)}.$$

PROOF Let us consider an infinite GP with first term a and common ratio r, where $-1 < r < 1$, i.e., $|r| < 1$.

The sum of n terms of a GP is given by

$$S_n = \frac{a(1-r^n)}{(1-r)} \Rightarrow S_n = \left\{\frac{a}{(1-r)} - \frac{ar^n}{(1-r)}\right\}. \qquad \ldots \text{(i)}$$

Since $|r| < 1$, so when n increases then r^n decreases.

Thus, when $n \to \infty$ then $r^n \to 0$.

$$\therefore \qquad \lim_{n \to \infty} \frac{r^n}{(1-r)} = 0.$$

Hence, the sum of an infinite GP is given by

$$S = \lim_{n \to \infty} S_n = \lim_{n \to \infty} \left\{ \frac{a}{(1-r)} - \frac{ar^n}{(1-r)} \right\}$$

$$= \frac{a}{(1-r)} - \lim_{n \to \infty} \frac{ar^n}{(1-r)} = \left\{ \frac{a}{(1-r)} - 0 \right\} = \frac{a}{(1-r)}$$

[using (i)].

This gives, $S = \dfrac{a}{(1-r)}$, when $|r| < 1$.

REMARKS 1. We shall denote the sum of an infinite GP by S.

2. If $r \geq 1$ then sum of an infinite GP is $S = \infty$.

SUMMARY

Sum of an infinite GP with the first term a and the common ratio r, where $|r| < 1$, is given by $S = \dfrac{a}{(1-r)}$.

SOLVED EXAMPLES

EXAMPLE 1 *Find the sum of the infinite geometric series* $\left(1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \ldots \infty\right)$.

SOLUTION In the given infinite geometric series, we have

$$a = 1 \text{ and } r = \frac{1}{3} \text{ such that } |r| = \frac{1}{3} < 1.$$

Hence, the sum of the given infinite series is

$$S = \frac{a}{(1-r)} = \frac{1}{\left(1 - \frac{1}{3}\right)} = \frac{1}{\left(\frac{2}{3}\right)} = \frac{3}{2}.$$

Hence, the required sum is $\frac{3}{2}$.

EXAMPLE 2 *Find the sum of the infinite geometric series* $\left(1 - \frac{1}{3} + \frac{1}{3^2} - \frac{1}{3^3} + \ldots \infty\right)$.

SOLUTION The given series is an infinite geometric series in which

$$a = 1, r = -\frac{1}{3} \text{ and } |r| = \frac{1}{3} < 1.$$

Hence, the sum of the given infinite geometric series is

$$S = \frac{a}{(1-r)} = \frac{1}{\left(1 + \frac{1}{3}\right)} = \frac{1}{\left(\frac{4}{3}\right)} = \frac{3}{4}.$$

EXAMPLE 3 *Find the sum of the infinite geometric series* $\left(\frac{-5}{4} + \frac{5}{16} - \frac{5}{64} + \ldots \infty\right)$.

SOLUTION In the given infinite geometric series, we have

$$a = \frac{-5}{4},\ r = \left(\frac{5}{16} \times \frac{4}{-5}\right) = -\frac{1}{4} \text{ and } |r| = \frac{1}{4} < 1.$$

Hence, the sum of the given infinite geometric series is

$$S = \frac{a}{(1-r)} = \frac{(-5/4)}{\left(1 + \frac{1}{4}\right)} = \left(\frac{-5}{4} \times \frac{4}{5}\right) = -1.$$

EXAMPLE 4 *Find the sum of the infinite geometric series*

$$(\sqrt{2}+1) + 1 + (\sqrt{2}-1) + \ldots \infty.$$

SOLUTION We have

$$\frac{1}{(\sqrt{2}+1)} = \frac{1}{(\sqrt{2}+1)} \times \frac{(\sqrt{2}-1)}{(\sqrt{2}-1)} = \frac{(\sqrt{2}-1)}{1}.$$

So, the given series is an infinite geometric series in which $a = (\sqrt{2}+1)$ and $r = (\sqrt{2}-1) < 1$.

Hence, the sum of the given infinite geometric series is

$$S = \frac{a}{(1-r)} = \frac{(\sqrt{2}+1)}{\{1-(\sqrt{2}-1)\}} = \frac{(\sqrt{2}+1)}{(2-\sqrt{2})} \times \frac{(2+\sqrt{2})}{(2+\sqrt{2})}$$

$$= \frac{4+3\sqrt{2}}{(4-2)} = \frac{(4+3\sqrt{2})}{2}.$$

EXAMPLE 5 *Find the sum of the infinite series:*

$$\left\{\frac{1}{2} + \frac{1}{3^2} + \frac{1}{2^3} + \frac{1}{3^4} + \frac{1}{2^5} + \frac{1}{3^6} + \ldots \infty\right\}.$$

SOLUTION The given series can be expressed as the sum of two infinite geometric series, shown below.

$$\left\{\frac{1}{2} + \frac{1}{3^2} + \frac{1}{2^3} + \frac{1}{3^4} + \frac{1}{2^5} + \frac{1}{3^6} + \ldots \infty\right\}$$

$$= \left\{\frac{1}{2} + \frac{1}{2^3} + \frac{1}{2^5} + \ldots \infty\right\} + \left\{\frac{1}{3^2} + \frac{1}{3^4} + \frac{1}{3^6} + \ldots \infty\right\}$$

$$\left\{\text{an infinite GP with } a = \frac{1}{2} \text{ and } r = \frac{1}{4}\right\}$$

$$\left\{\text{an infinite GP with } a = \frac{1}{9} \text{ and } r = \frac{1}{9}\right\}$$

$$= \frac{(1/2)}{\left(1-\frac{1}{4}\right)} + \frac{(1/9)}{\left(1-\frac{1}{9}\right)} = \frac{(1/2)}{(3/4)} + \frac{(1/9)}{(8/9)}$$

$$= \left(\frac{1}{2} \times \frac{4}{3}\right) + \left(\frac{1}{9} \times \frac{9}{8}\right) = \left(\frac{2}{3} + \frac{1}{8}\right)$$

$$= \frac{16+3}{24} = \frac{19}{24}.$$

EXAMPLE 6 *Prove that* $6^{1/2} \cdot 6^{1/4} \cdot 6^{1/8} \cdot \ldots \infty = 6.$

SOLUTION We observe here that $\left(\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\ldots\infty\right)$ is an infinite geometric series in which $a=\frac{1}{2}$ and $r=\left(\frac{1}{4}\times\frac{2}{1}\right)=\frac{1}{2}$ such that $|r|<1$.

So, this sum is given by

$$S=\frac{\left(\frac{1}{2}\right)}{\left(1-\frac{1}{2}\right)}=\frac{\left(\frac{1}{2}\right)}{\left(\frac{1}{2}\right)}=1. \qquad \ldots \text{(i)}$$

$$\therefore \quad 6^{\frac{1}{2}} \cdot 6^{\frac{1}{4}} \cdot 6^{\frac{1}{8}} \cdot \ldots \infty = 6^{\left(\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\ldots\infty\right)} = 6^1 = 6 \qquad \text{[using (i)]}.$$

Hence, $6^{\frac{1}{2}} \cdot 6^{\frac{1}{4}} \cdot 6^{\frac{1}{8}} \cdot \ldots \infty = 6.$

EXAMPLE 7 *If* $|r|<1$, $x=\left(a+\frac{a}{r}+\frac{a}{r^2}+\ldots\infty\right)$, $y=\left(b-\frac{b}{r}+\frac{b}{r^2}-\ldots\infty\right)$ *and* $z=\left(c+\frac{c}{r^2}+\frac{c}{r^4}+\ldots\infty\right)$, *prove that* $\frac{xy}{z}=\frac{ab}{c}\cdot$

SOLUTION Clearly, each one of the given series is an infinite geometric series. Taking the sum of each series, we get

$$x=\frac{a}{\left(1-\frac{1}{r}\right)}=\frac{ar}{(r-1)} \qquad \left[\because \text{ 1st term} = a \text{ and common ratio} = \frac{1}{r}\right],$$

$$y=\frac{b}{\left(1+\frac{1}{r}\right)}=\frac{br}{(r+1)} \qquad \left[\because \text{ 1st term} = b \text{ and common ratio} = \frac{-1}{r}\right]$$

and
$$z=\frac{c}{\left(1-\frac{1}{r^2}\right)}=\frac{cr^2}{(r^2-1)} \qquad \left[\because \text{ 1st term} = c \text{ and common ratio} = \frac{1}{r^2}\right]$$

$$\therefore \quad \frac{xy}{z}=\frac{ar}{(r-1)}\times\frac{br}{(r+1)}\times\frac{(r^2-1)}{cr^2}=\frac{ab}{c}\cdot$$

Hence, $\frac{xy}{z}=\frac{ab}{c}\cdot$

EXAMPLE 8 *If* $y=x+x^2+x^3+\ldots\infty$, *where* $|x|<1$, *prove that* $x=\frac{y}{(1+y)}\cdot$

SOLUTION By summing the given infinite GS, we get

$$y=\frac{x}{(1-x)} \Rightarrow y(1-x)=x$$
$$\Rightarrow y-xy=x \Rightarrow x+xy=y$$
$$x(1+y)=y \Rightarrow x=\frac{y}{(1+y)}\cdot$$

Hence, $x=\frac{y}{(1+y)}\cdot$

EXAMPLE 9 *If* $x = 1 + a + a^2 + \dots \infty$, *where* $|a| < 1$ *and* $y = 1 + b + b^2 + \dots \infty$, *where* $|b| < 1$, *prove that* $(1 + ab + a^2b^2 + \dots \infty) = \dfrac{xy}{(x + y - 1)}$.

SOLUTION By summing the given infinite GS, we get

$$x = \frac{1}{(1-a)} \text{ and } y = \frac{1}{(1-b)}.$$

$$\therefore \quad \frac{xy}{(x+y-1)} = \frac{\left\{\dfrac{1}{(1-a)} \cdot \dfrac{1}{(1-b)}\right\}}{\left\{\dfrac{1}{(1-a)} + \dfrac{1}{(1-b)} - 1\right\}}$$

$$= \left\{\frac{1}{(1-a)(1-b)} \times \frac{(1-a)(1-b)}{(1-b)+(1-a)-(1-a)(1-b)}\right\}$$

$$= \frac{1}{(1-ab)} = (1-ab)^{-1}$$

$$= (1 + ab + a^2b^2 + \dots \infty).$$

Hence, $(1 + ab + a^2b^2 + \dots \infty) = \dfrac{xy}{(x+y-1)}$.

EXAMPLE 10 *If* $x = (1 + r^a + r^{2a} + \dots \infty)$ *and* $y = (1 + r^b + r^{2b} + \dots \infty)$, *prove that* $r = \left(\dfrac{x-1}{x}\right)^{\frac{1}{a}} = \left(\dfrac{y-1}{y}\right)^{\frac{1}{b}}$.

SOLUTION By summing the given infinite geometric series, we get

$$x = \frac{1}{(1-r^a)} \text{ and } y = \frac{1}{(1-r^b)}$$

$$\Rightarrow \quad (1-r^a) = \frac{1}{x} \text{ and } (1-r^b) = \frac{1}{y}$$

$$\Rightarrow \quad r^a = \left(1 - \frac{1}{x}\right) \text{ and } r^b = \left(1 - \frac{1}{y}\right)$$

$$\Rightarrow \quad r = \left(\frac{x-1}{x}\right)^{\frac{1}{a}} \text{ and } r = \left(\frac{y-1}{y}\right)^{\frac{1}{b}}.$$

Hence, $r = \left(\dfrac{x-1}{x}\right)^{\frac{1}{a}} = \left(\dfrac{y-1}{y}\right)^{\frac{1}{b}}$.

EXAMPLE 11 *Use geometric series to express* $0.555\dots = 0.\overline{5}$ *as a rational number.*

SOLUTION We have

$$0.\overline{5} = 0.5555\dots$$
$$= 0.5 + 0.05 + 0.005 + 0.0005 + \dots \infty$$
$$= \frac{5}{10} + \frac{5}{10^2} + \frac{5}{10^3} + \frac{5}{10^4} + \dots \infty$$

$$= \frac{\left(\frac{5}{10}\right)}{\left(1 - \frac{1}{10}\right)} = \frac{5}{9} \qquad \left[\because S = \frac{a}{(1-r)} \text{ in an infinite GS}\right].$$

Hence, $0.\overline{5} = \frac{5}{9}$.

EXAMPLE 12 *Find the rational number whose decimal form is* $0.1\overline{42}$.

SOLUTION We have

$$0.1\overline{42} = 0.142424242\ldots.$$
$$= 0.1 + 0.042 + 0.00042 + 0.0000042 + \ldots\infty$$
$$= \frac{1}{10} + \left\{\frac{42}{10^3} + \frac{42}{10^5} + \frac{42}{10^7} + \ldots\infty\right\}$$
$$= \frac{1}{10} + \frac{\left(\frac{42}{10^3}\right)}{\left(1 - \frac{1}{10^2}\right)} \quad \left[\begin{array}{l}\text{this being a GS in which} \\ a = \frac{42}{10^3} \text{ and } r = \left(\frac{42}{10^5} \times \frac{10^3}{42}\right) = \frac{1}{10^2} < 1\end{array}\right]$$
$$= \frac{1}{10} + \left(\frac{42}{1000} \times \frac{100}{99}\right) = \left(\frac{1}{10} + \frac{42}{990}\right) = \frac{141}{990}.$$

Hence, $0.1\overline{42} = \frac{141}{990}$.

EXAMPLE 13 *Find the rational number whose decimal form is* $1.3\overline{45}$.

SOLUTION We have

$$1.3\overline{45} = 1.3454545\ldots\infty$$
$$= 1.3 + 0.045 + 0.00045 + \ldots\infty$$
$$= 1.3 + \left\{\frac{45}{10^3} + \frac{45}{10^5} + \ldots\infty\right\}$$
$$= 1.3 + \frac{\left(\frac{45}{10^3}\right)}{\left(1 - \frac{1}{10^2}\right)} \quad \left[\begin{array}{l}\text{this being a GS in which} \\ a = \frac{45}{10^3} \text{ and } r = \frac{1}{10^2} < 1\end{array}\right]$$
$$= 1.3 + \frac{45}{1000} \times \frac{100}{99} = \frac{13}{10} + \frac{45}{990} = \left(\frac{13}{10} + \frac{1}{22}\right)$$
$$= \frac{143 + 5}{110} = \frac{148}{110} = \frac{74}{55}.$$

Hence, $1.3\overline{45} = \frac{74}{55}$.

EXAMPLE 14 *The sum of an infinite GP is* $\frac{80}{9}$ *and its common ratio is* $\frac{-4}{5}$. *Find its first term.*

SOLUTION Let a be the first term of the given infinite GP.

Here, $r = \frac{-4}{5}$ and $|r| = \left|\frac{-4}{5}\right| = \frac{4}{5} < 1$.

The sum of this infinite GP is $S = \frac{80}{9}\cdot$

$$\therefore \quad S = \frac{a}{(1-r)} \Rightarrow \frac{a}{\left(1+\frac{4}{5}\right)} = \frac{80}{9}$$

$$\Rightarrow \frac{5a}{9} = \frac{80}{9} \Rightarrow 5a = 80 \Rightarrow a = 16.$$

Hence, the first term is 16.

EXAMPLE 15 *The sum of first two terms of an infinite geometric series is* 15 *and each term of the series is equal to the sum of all the terms following it. Find the series.*

SOLUTION Let the series be $a + ar + ar^2 + \ldots + ar^n + ar^{(n+1)} + \ldots \infty$.

Then, first term $= a$ and common ratio $= r$.

Clearly, $a + ar = 15 \Rightarrow a(1+r) = 15.$ ... (i)

Also, $ar^{(n-1)} = ar^n + ar^{(n+1)} + \ldots \infty$ $\quad [\because a_n = a_{n+1} + a_{n+2} + \ldots \infty]$

$\Rightarrow \quad ar^{(n-1)} = \frac{ar^n}{(1-r)}$ $\quad \left[\because \text{ RHS is a GS in which first term} = ar^n \text{ and common ratio} = r\right]$

$\Rightarrow \quad 1 - r = r \Rightarrow 2r = 1 \Rightarrow r = \frac{1}{2}\cdot$

Putting $r = \frac{1}{2}$ in (i), we get $\frac{3}{2}a = 15 \Rightarrow a = \left(15 \times \frac{2}{3}\right) = 10.$

Thus, $a = 10$ and $r = \frac{1}{2}\cdot$

Hence, the required series is $\left(10 + 5 + \frac{5}{2} + \frac{5}{4} + \ldots \infty\right)\cdot$

EXAMPLE 16 *The sum of an infinite geometric series is* 8. *If its second term is* 2, *find its common ratio.*

SOLUTION Let the infinite geometric series be $a, ar, ar^2, \ldots, \infty$. Then,

$$ar = 2 \qquad \ldots \text{(i)}$$

and $\quad \frac{a}{(1-r)} = 8.$ ... (ii)

Putting $r = \frac{2}{a}$ from (i) in (ii), we get

$$\frac{a}{\left(1-\frac{2}{a}\right)} = 8 \Rightarrow \frac{a^2}{(a-2)} = 8 \Rightarrow a^2 - 8a + 16 = 0$$

$$\Rightarrow (a-4)^2 = 0 \Rightarrow a - 4 = 0 \Rightarrow a = 4.$$

Putting $a = 4$ in (i), we get $r = \frac{2}{4} \Rightarrow r = \frac{1}{2}\cdot$

Hence, the common ratio of the given geometric series is $\frac{1}{2}\cdot$

EXAMPLE 17 *The sum of an infinite geometric series is* 15 *and the sum of the squares of these terms is* 45. *Find the series.*

SOLUTION Let a be the first term and r be the common ratio.

Then, the series is $a + ar + ar^2 + \ldots \infty$.

Now, $(a + ar + ar^2 + \ldots \infty) = 15 \Rightarrow \dfrac{a}{(1-r)} = 15$ … (i)

and $(a^2 + a^2r^2 + a^2r^4 + \ldots \infty) \Rightarrow \dfrac{a^2}{(1-r^2)} = 45.$ … (ii)

On squaring both sides of (i), we get $\dfrac{a^2}{(1-r)^2} = 225.$ … (iii)

On dividing (iii) by (ii), we get

$$\frac{a^2}{(1-r)^2} \times \frac{(1-r^2)}{a^2} = \frac{225}{45} \Rightarrow \frac{1+r}{1-r} = 5$$

$$\Rightarrow 1 + r = 5 - 5r$$

$$\Rightarrow 6r = 4 \Rightarrow r = \frac{2}{3}\cdot$$

Putting $r = \dfrac{2}{3}$ in (i), we get

$$a = 15 \times \left(1 - \frac{2}{3}\right) = \left(15 \times \frac{1}{3}\right) = 5.$$

Thus, $a = 5$ and $r = \dfrac{2}{3}\cdot$

Hence, the required series is $\left(5 + \dfrac{10}{3} + \dfrac{20}{9} + \dfrac{40}{27} + \ldots \infty\right)\cdot$

EXAMPLE 18 *If $S_1, S_2, S_3, \ldots, S_p$ denote the sums of infinite geometric series whose first terms are $1, 2, 3, \ldots, p$ respectively and whose common ratios are $\dfrac{1}{2}, \dfrac{1}{3}, \dfrac{1}{4}, \ldots, \dfrac{1}{(p+1)}$ respectively, then show that*

$$S_1 + S_2 + S_3 + \ldots + S_p = \frac{1}{2}p(p+3).$$

SOLUTION By the summation of an infinite geometric series, we get

$$S_1 = \frac{1}{\left(1 - \frac{1}{2}\right)} = 2;\ S_2 = \frac{2}{\left(1 - \frac{1}{3}\right)} = 3;\ S_3 = \frac{3}{\left(1 - \frac{1}{4}\right)} = 4, \ldots,$$

$$S_p = \frac{p}{\left(1 - \frac{1}{p+1}\right)} = (p+1).$$

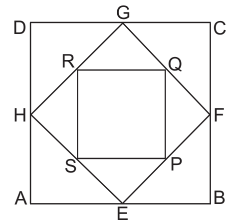
$$\therefore \quad S_1 + S_2 + S_3 + \ldots + S_p = [2 + 3 + 4 + \ldots + (p+1)]$$

$$= \frac{p}{2} \cdot \{2 + (p+1)\} = \frac{1}{2}p\,(p+3).$$

Hence, $S_1 + S_2 + S_3 + \ldots + S_p = \dfrac{1}{2}p\,(p+3)$.

EXAMPLE 19 *The side of a given square is* 10 cm. *The midpoints of its sides are joined to form a new square. Again, the midpoints of the sides of this new square are joined to form another square. This process is continued indefinitely. Find (i) the sum of the areas and (ii) the sum of the perimeters of the squares.*

SOLUTION Let $ABCD$ be the given square with each side equal to 10 cm. Let E, F, G, H be the midpoints of the sides AB, BC, CD and DA respectively. Let P, Q, R, S be the midpoints of the sides EF, FG, GH and HE respectively.



$\therefore \quad BE = BF = 5$ cm

$\Rightarrow \quad EF = \sqrt{BE^2 + BF^2} = \sqrt{25+25}$ cm $= \sqrt{50}$ cm $= 5\sqrt{2}$ cm.

$\therefore \quad FQ = FP = \frac{1}{2}EF = \frac{5\sqrt{2}}{2}$ cm $= \frac{5}{\sqrt{2}}$ cm

$\Rightarrow \quad PQ = \sqrt{FP^2 + FQ^2} = \sqrt{\frac{25}{2} + \frac{25}{2}}$ cm $= \sqrt{25}$ cm $= 5$ cm.

Thus, the sides of the squares are 10 cm, $5\sqrt{2}$ cm, 5 cm, ...

(i) Sum of the areas of the squares formed

$$= \{(10)^2 + (5\sqrt{2})^2 + 5^2 + \ldots \infty\} \text{ cm}^2$$

$$= (100 + 50 + 25 + \ldots \infty) \text{ cm}^2$$

$$= \frac{100}{\left(1 - \frac{1}{2}\right)} \text{ cm}^2 = 200 \text{ cm}^2 \quad \left[\begin{array}{c}\text{taking the sum of infinite GP} \\ \text{with } a = 100 \text{ and } r = \frac{1}{2}\end{array}\right].$$

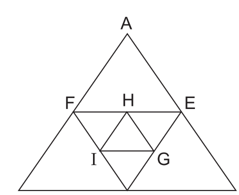
(ii) Sum of perimeters of the squares formed

$$= (40 + 20\sqrt{2} + 20 + \ldots) \text{ cm}$$

$$= \frac{40}{\left(1 - \frac{1}{\sqrt{2}}\right)} \text{ cm} = \frac{40\sqrt{2}}{(\sqrt{2}-1)} \times \frac{(\sqrt{2}+1)}{(\sqrt{2}+1)} \text{ cm} = (80 + 40\sqrt{2}) \text{ cm}.$$

EXAMPLE 20 *Each side of an equilateral triangle is* 18 cm. *The midpoints of its sides are joined to form another triangle whose midpoints, in turn, are joined to form still another triangle. The process is continued indefinitely. Find the sum of the (i) perimeters of all the triangles and (ii) areas of all the triangles.*

SOLUTION Let $\triangle ABC$ be the given triangle having each side 18 cm. Let D, E, F be the midpoints of BC, CA, AB respectively to form $\triangle DEF$. Let G, H, I be the midpoints of DE, EF and FD respectively to form $\triangle GHI$.
We continue this process indefinitely.



The sides of these triangle are 18 cm, 9 cm, $\frac{9}{2}$ cm, ..., and so on.

(i) Sum of the perimeters of all triangles so formed

$$= 3\left\{18 + 9 + \frac{9}{2} + \frac{9}{4} + \ldots \infty\right\} \text{ cm}$$

$$= \left\{3 \times \frac{a}{(1-r)}\right\} \text{ cm, where } a = 18 \text{ and } r = \frac{9}{18} = \frac{1}{2}$$

$$= \left\{ 3 \times \frac{18}{\left(1 - \frac{1}{2}\right)} \right\} \text{ cm} = (3 \times 36) \text{ cm} = 108 \text{ cm}.$$

(ii) Sum of the areas of all the triangles so formed

$$= \frac{\sqrt{3}}{4} \cdot \left\{ (18)^2 + (9)^2 + \left(\frac{9}{2}\right)^2 + \left(\frac{9}{4}\right)^2 + \ldots \infty \right\} \text{ cm}^2$$

$$= \frac{\sqrt{3}}{4} \cdot \left\{ 324 + 81 + \frac{81}{4} + \frac{81}{16} + \ldots \infty \right\} \text{ cm}^2$$

$$= \frac{\sqrt{3}}{4} \cdot \frac{a}{(1-r)} \text{ cm}^2, \text{ where } a = 324 \text{ and } r = \frac{81}{324} = \frac{1}{4}$$

$$= \left\{ \frac{\sqrt{3}}{4} \times \frac{324}{\left(1 - \frac{1}{4}\right)} \right\} \text{ cm}^2 = 108\sqrt{3} \text{ cm}^2.$$

EXAMPLE 21 *After striking a floor a certain ball rebounds* $\frac{4}{5}$ *of the height from which it has fallen. Find the total distance that it travels before coming to rest, if it is gently dropped from a height of* 120 *metres.*

SOLUTION Initially, the ball falls from a height of 120 m. After striking the floor, it rebounds and goes to a height of $\left(\frac{4}{5} \times 120\right)$ m.

Now, it falls from this height and after striking the floor, it rebounds and goes to a height of $\frac{4}{5}$ of $\left(\frac{4}{5} \times 120\right)$ m $= \left\{ \left(\frac{4}{5}\right)^2 \times 120 \right\}$ m.

This process continues indefinitely till the ball comes to rest.

Total distance covered by the ball in metres

$$= 120 + 2 \times \left[\left\{ \frac{4}{5} \times 120 \right\} + \left\{ \left(\frac{4}{5}\right)^2 \times 120 \right\} + \left\{ \left(\frac{4}{5}\right)^3 \times 120 \right\} + \ldots \infty \right]$$

$$= 120 + \left\{ 2 \times \frac{a}{(1-r)} \right\}, \text{ where } a = \left(\frac{4}{5} \times 120\right) = 96 \text{ and } r = \frac{4}{5}$$

$$= \left\{ 120 + \frac{2 \times 96}{\left(1 - \frac{4}{5}\right)} \right\} = \{120 + (192 \times 5)\} = (120 + 960) = 1080.$$

Hence, the total distance covered by the ball is 1080 m.

EXAMPLE 22 *If* $x = \sum_{n=0}^{\infty} \cos^{2n}\theta$, $y = \sum_{n=0}^{\infty} \sin^{2n}\phi$ *and* $z = \sum_{n=0}^{\infty} \cos^{2n}\theta \sin^{2n}\phi$, *where* $0 < \theta < \phi < \frac{\pi}{2}$ *then prove that* $xz + yz - z = xy$.

SOLUTION We have

$$x = \sum_{n=0}^{\infty} \cos^{2n}\theta = 1 + \cos^2\theta + \cos^4\theta + \ldots \infty$$

$$= \frac{a}{(1-r)}, \text{ where } a = 1 \text{ and } r = \cos^2\theta \quad \text{[sum of an infinite GP]}$$

$$= \frac{1}{(1-\cos^2\theta)} = \frac{1}{\sin^2\theta}.$$

$$\therefore \quad \sin^2\theta = \frac{1}{x}. \quad \ldots \text{(i)}$$

$$y = \sum_{n=0}^{\infty} \sin^{2n}\phi = 1 + \sin^2\phi + \sin^4\phi + \ldots \infty$$

$$= \frac{a}{(1-r)}, \text{ where } a = 1 \text{ and } r = \sin^2\phi \quad \text{[sum of an infinite GP]}$$

$$= \frac{1}{(1-\sin^2\phi)} = \frac{1}{\cos^2\phi}.$$

$$\therefore \quad \cos^2\phi = \frac{1}{y}. \quad \ldots \text{(ii)}$$

$$z = \sum_{n=0}^{\infty} \cos^{2n}\theta \sin^{2n}\phi = 1 + \cos^2\theta\sin^2\phi + \cos^4\theta\sin^4\phi + \ldots \infty$$

$$= \frac{a}{(1-r)}, \text{ where } a = 1 \text{ and } r = \cos^2\theta\sin^2\phi$$

$$= \frac{1}{(1-\cos^2\theta\sin^2\phi)} = \frac{1}{1-(1-\sin^2\theta)(1-\cos^2\phi)}$$

$$= \frac{1}{1-\left(1-\frac{1}{x}\right)\left(1-\frac{1}{y}\right)} = \frac{1}{1-\left(1-\frac{1}{x}-\frac{1}{y}+\frac{1}{xy}\right)} \quad \text{[using (i) and (ii)]}$$

$$= \frac{1}{\left(\frac{1}{x}+\frac{1}{y}-\frac{1}{xy}\right)} = \frac{xy}{(x+y-1)}.$$

$$\therefore \quad z = \frac{xy}{x+y-1} \Rightarrow xz + yz - z = xy.$$

Hence, $xz + yz - z = xy$.

EXERCISE 12G

Find the sum of each of the following infinite series:

1. $8 + 4\sqrt{2} + 4 + 2\sqrt{2} + \ldots \infty$

2. $6 + 1.2 + 0.24 + \ldots \infty$

3. $\sqrt{2} - \frac{1}{\sqrt{2}} + \frac{1}{2\sqrt{2}} - \frac{1}{4\sqrt{2}} + \ldots \infty$

4. $10 - 9 + 8.1 - \ldots \infty$

5. $\frac{2}{5} + \frac{3}{5^2} + \frac{2}{5^3} + \frac{3}{5^4} + \ldots \infty$

6. Prove that $9^{1/3} \times 9^{1/9} \times 9^{1/27} \times \ldots \infty = 3$.

7. Find the rational number whose decimal expansion is given below:

(i) $0.\overline{3}$ (ii) $0.\overline{231}$ (iii) $3.5\overline{2}$

8. Express the recurring decimal $0.125125125\ldots = 0.\overline{125}$ as a rational number.

9. Write the value of $0.4\overline{23}$ in the form of a simple fraction.

10. Write the value of $2.1\overline{34}$ in the form of a simple fraction.

11. The sum of an infinite geometric series is 6. If its first term is 2, find its common ratio.

12. The sum of an infinite geometric series is 20 and the sum of the squares of these terms is 100. Find the series.

13. The sum of an infinite GP is 57 and the sum of their cubes is 9747. Find the GP.

ANSWERS (EXERCISE 12G)

1. $8(2+\sqrt{2})$ **2.** 7.5 **3.** $\frac{2\sqrt{2}}{3}$ **4.** $\frac{100}{19}$ **5.** $\frac{13}{24}$

7. (i) $\frac{1}{3}$ (ii) $\frac{231}{999}$ (iii) $\frac{317}{90}$ **8.** $\frac{125}{999}$ **9.** $\frac{419}{990}$ **10.** $\frac{2113}{990}$

11. $\frac{2}{3}$ **12.** $\left(8+\frac{24}{5}+\frac{72}{25}+\ldots\infty\right)$ **13.** $\left(19, \frac{38}{3}, \frac{76}{9}, \frac{152}{27}, \ldots\infty\right)$

HINTS TO SOME SELECTED QUESTIONS

1. $a = 8$ and $r = \frac{4\sqrt{2}}{8} = \frac{1}{\sqrt{2}}$.

$$\therefore \quad S = \frac{a}{(1-r)} = \frac{8}{\left(1-\frac{1}{\sqrt{2}}\right)} = \frac{8\sqrt{2}}{(\sqrt{2}-1)} \times \frac{(\sqrt{2}+1)}{(\sqrt{2}+1)} = 8(2+\sqrt{2}).$$

2. $a = 6$ and $r = \frac{1.2}{6} = 0.2$.

$$\therefore \quad S = \frac{a}{(1-r)} = \frac{6}{0.8} = \frac{60}{8} = \frac{15}{2} = 7.5.$$

3. $a = \sqrt{2}$ and $r = \frac{-1}{\sqrt{2}} \times \frac{1}{\sqrt{2}} = \frac{-1}{2}$.

$$\therefore \quad S = \frac{a}{(1-r)} = \frac{\sqrt{2}}{\left(1+\frac{1}{2}\right)} = \frac{2\sqrt{2}}{3}.$$

4. $a = 10$ and $r = \frac{-9}{10} = -0.9$.

$$\therefore \quad S = \frac{a}{(1-r)} = \frac{10}{(1+0.9)} = \frac{10}{1.9} = \frac{100}{19}.$$

5. $S = \left(\frac{2}{5}+\frac{2}{5^3}+\frac{2}{5^5}+\ldots\infty\right)+\left(\frac{3}{5^2}+\frac{3}{5^4}+\frac{3}{5^6}+\ldots\infty\right)$

$$= \frac{a_1}{(1-r_1)} + \frac{a_2}{(1-r_2)}, \text{ where } \left(a_1 = \frac{2}{5}, r_1 = \frac{1}{25}\right) \text{ and } \left(a_2 = \frac{3}{25}, r_2 = \frac{1}{25}\right).$$

6. $9^{1/3} \times 9^{1/9} \times 9^{1/27} \times \ldots \infty$

$= 9^{\left(\frac{1}{3}+\frac{1}{9}+\frac{1}{27}+\ldots\infty\right)} = 9^{\frac{a}{(1-r)}}$, where $a = \dfrac{1}{3}$ and $r = \dfrac{1}{3}$

$= 9^{\frac{(1/3)}{\left(1-\frac{1}{3}\right)}} = 9^{\frac{1}{2}} = \sqrt{9} = 3.$

7. (ii) $x = 0.231231231\ldots$ and $1000x = 231.231231231\ldots$.

On subraction, we get $999x = 231 \Rightarrow x = \dfrac{231}{999}$.

Hence, $0.\overline{231} = \dfrac{231}{999}$.

(iii) $x = 3.5222\ldots \Rightarrow 10x = 35.222\ldots$ and $100x = 352.222\ldots$.

$\therefore \quad 90x = 317 \Rightarrow x = \dfrac{317}{90}$.

8. $x = 0.125125\ldots \Rightarrow 1000x = 125.125125\ldots$.

$\therefore \quad 999x = 125 \Rightarrow x = \dfrac{125}{999}$.

Hence, $0.125125\ldots = \dfrac{125}{999}$.

9. $x = 0.42323\ldots \Rightarrow 10x = 4.232323\ldots$ and $1000x = 423.2323\ldots$.

$\therefore \quad 990x = (423 - 4) = 419 \Rightarrow x = \dfrac{419}{990}$.

Hence, $0.4\overline{23} = \dfrac{419}{990}$.

10. $x = 2.1343434\ldots \Rightarrow 10x = 21.343434\ldots$ and $1000x = 2134.3434\ldots$.

$\therefore \quad 990x = (2134 - 21) = 2113 \Rightarrow x = \dfrac{2113}{990}$.

Hence, $2.1\overline{34} = \dfrac{2113}{990}$.

11. $\dfrac{a}{1-r} = 6 \Rightarrow \dfrac{2}{1-r} = 6 \Rightarrow 1 - r = \dfrac{1}{3} \Rightarrow r = \left(1 - \dfrac{1}{3}\right) = \dfrac{2}{3}$.

12. Let a be the first term and r be the common ratio of the given series.

Then, $\dfrac{a}{(1-r)} = 20 \Rightarrow \dfrac{a^2}{(1-r)^2} = 400.$ … (i)

And, $\dfrac{a^2}{(1-r^2)} = 100.$ … (ii)

$\therefore \quad \dfrac{a^2}{(1-r)^2} \times \dfrac{(1-r^2)}{a^2} = \dfrac{400}{100} \Rightarrow \dfrac{1+r}{1-r} = 4 \Rightarrow r = \dfrac{3}{5}$.

$\therefore \quad \dfrac{a}{\left(1-\dfrac{3}{5}\right)} = 20 \Rightarrow a = 8.$

Hence, the required series is $\left(8 + \dfrac{24}{5} + \dfrac{72}{25} + \ldots\infty\right)$.

13. Let a be the first term and r be the common ratio of the given GP.

Then, $\dfrac{a}{1-r} = 57 \Rightarrow \dfrac{a^3}{(1-r)^3} = (57)^3.$ … (i)

And, $a^3 + a^3r^3 + a^3r^6 + \ldots \infty = 9747$

$$\Rightarrow \quad \frac{a^3}{(1-r^3)} = 9747. \qquad \ldots \text{(ii)}$$

$$\therefore \quad \frac{a^3}{(1-r)^3} \times \frac{(1-r^3)}{a^3} = \frac{(57)^3}{9747} \Rightarrow \frac{(1-r^3)}{(1-r)^3} = 19 \Rightarrow \frac{1+r+r^2}{(1-r)^2} = 19$$

$$\Rightarrow 1 + r + r^2 = 19r^2 - 38r + 19$$

$$\Rightarrow 18r^2 - 39r + 18 = 0$$

$$\Rightarrow 18r^2 - 27r - 12r + 18 = 0$$

$$\Rightarrow (3r-2)(6r-9) = 0 \Rightarrow r = \frac{2}{3}$$

$$\left[\because r \neq \frac{3}{2}, \text{ as } -1 < r < 1 \text{ for infinite GP}\right].$$

$$\therefore \quad \frac{a}{\left(1-\frac{2}{3}\right)} = 57 \Rightarrow a = \left(57 \times \frac{1}{3}\right) = 19.$$

Hence, the required GP is $19, \frac{38}{3}, \frac{76}{9}, \frac{152}{27}, \ldots \infty$.

EXERCISE 12H

Very-Short-Answer Questions

1. If the 5th term of a GP is 2, find the product of its first nine terms.
2. If the $(p+q)$th and $(p-q)$th terms of a GP are m and n respectively, find its pth term.
3. If 2nd, 3rd and 6th terms of an AP are the three consecutive terms of a GP then find the common ratio of the GP.
4. Write the quadratic equation, the arithmetic and geometric means of whose roots are A and G respectively.
5. If a, b, c are in GP and $a^{1/x} = b^{1/y} = c^{1/z}$ then prove that x, y, z are in AP.
6. If a, b, c are in AP and x, y, z are in GP then prove that the value of $x^{b-c} \cdot y^{c-a} \cdot z^{a-b}$ is 1.
7. Prove that $\left(1 - \frac{1}{3} + \frac{1}{3^2} - \frac{1}{3^3} + \frac{1}{3^4} - \ldots \infty\right) = \frac{3}{4}$.
8. Express $0.\overline{123}$ as a rational number.
9. Express $0.\overline{6}$ as a rational number.
10. Express $0.6\overline{8}$ as a rational number.
11. The second term of a GP is 24 and its fifth term is 81. Find the sum of its first five terms.
12. The ratio of the sum of first three terms is to that of first six terms of a GP is 125 : 152. Find the common ratio.
13. The sum of first three terms of a GP is $\frac{39}{10}$ and their product is 1. Find the common ratio and these three terms.

ANSWERS (EXERCISE 12H)

1. 512 **2.** $\sqrt{mn}$ **3.** 3 **4.** $x^2 - 2Ax + G^2 = 0$

8. $\frac{123}{999}$ **9.** $\frac{2}{3}$ **10.** $\frac{31}{45}$ **11.** 211 **12.** $\frac{3}{5}$

13. $r = \frac{5}{2}$ or $\frac{2}{5}$; $\frac{2}{5}, 1, \frac{5}{2}$ or $\frac{5}{2}, 1, \frac{2}{5}$

HINTS TO SOME SELECTED QUESTIONS

1. Let first term $= a$ and common ratio $= r$. Then,

$$T_5 = 2 \Rightarrow ar^4 = 2.$$

Required product

$$= (a \times ar \times ar^2 \times \ldots \times ar^8)$$

$$= a^9 \times r^{(1+2+3+\ldots+8)} = a^9 r^{36} = (ar^4)^9 = 2^9 = 512.$$

2. Let first term $= a$ and common ratio $= r$. Then,

$$T_{p+q} = m \Rightarrow ar^{(p+q-1)} = m$$

$$T_{p-q} = n \Rightarrow ar^{(p-q-1)} = n$$

$$\therefore \quad mn = a^2 r^{2(p-1)} = \{ar^{(p-1)}\}^2 \Rightarrow ar^{(p-1)} = \sqrt{mn} \Rightarrow T_p = \sqrt{mn}.$$

3. $(a+d), (a+2d), (a+5d)$ are in GP.

$$\therefore \quad \frac{a+2d}{a+d} = \frac{a+5d}{a+2d} \Rightarrow (a+2d)^2 = (a+d)(a+5d)$$

$$\Rightarrow d^2 + 2ad = 0 \Rightarrow d(d+2a) = 0 \Rightarrow d = -2a \qquad [\because d \neq 0].$$

So, these terms are $-a, -3a, -9a$. Its common ratio $= \frac{-3a}{-a} = 3$.

4. Let the roots of the quadratic equation be α and β.

Then, the equation is $x^2 - (\alpha+\beta)x + \alpha\beta = 0$.

$$\therefore \quad A = \frac{\alpha+\beta}{2} \Rightarrow \alpha+\beta = 2A.$$

$$G = \sqrt{\alpha\beta} \Rightarrow \alpha\beta = G^2.$$

Hence, the required equation is $x^2 - 2Ax + G^2 = 0$.

5. Let $a^{1/x} = b^{1/y} = c^{1/z} = m$. Then,

$$a = m^x, b = m^y \text{ and } c = m^z.$$

Since a, b, c are in GP, we have

$$b^2 = ac \Rightarrow (m^y)^2 = (m^x \times m^z)$$

$$\Rightarrow m^{2y} = m^{x+z} \Rightarrow 2y = x+z.$$

$\therefore$ x, y, z are in AP.

6. a, b, c are in AP $\Rightarrow 2b = a+c$.

x, y, z are in GP $\Rightarrow y^2 = xz$.

$$\therefore \quad x^{(b-c)} \cdot y^{(c-a)} \cdot z^{(a-b)}$$

$$= x^{(b-c)} \cdot (\sqrt{xz})^{(c-a)} \cdot z^{(a-b)} \qquad [\because y = \sqrt{xz}]$$

$$= x^{(b-c)} \cdot x^{\frac{1}{2}(c-a)} \cdot z^{\frac{1}{2}(c-a)} \cdot z^{(a-b)}$$

$$= x^{(b-c)+\frac{1}{2}(c-a)} \cdot z^{\frac{1}{2}(c-a)+(a-b)}$$

$$= x^{\frac{1}{2}\{2b-(a+c)\}} \cdot z^{\frac{1}{2}\{c+a-2b\}} = x^0 \times z^0 = 1.$$

7. This is an infinite GP in which $a = 1$ and $r = \frac{-1}{3}\cdot$

$$S_\infty = \frac{a}{(1-r)} = \frac{1}{\left(1+\frac{1}{3}\right)} = \frac{3}{4}\cdot$$

8. $x = 0.123123\ldots$ and $1000x = 123.123123\ldots$.

$\therefore \quad 999x = 123 \Rightarrow x = \frac{123}{999}\cdot$

10. $x = 0.688\ldots \Rightarrow 10x = 6.888\ldots$ and $100x = 68.888\ldots$.

$\therefore \quad 90x = 62 \Rightarrow x = \frac{62}{90} = \frac{31}{45}\cdot$

11. $ar = 24$ and $ar^4 = 81$.

$\therefore \quad \frac{ar^4}{ar} = \frac{81}{24} = \frac{27}{8} \Rightarrow r^3 = \frac{27}{8} = \left(\frac{3}{2}\right)^3 \Rightarrow r = \frac{3}{2}\cdot$

$\therefore \quad a \times \frac{3}{2} = 24 \Rightarrow a = \left(24 \times \frac{2}{3}\right) = 16.$

$$S_5 = \frac{a(r^5-1)}{(r-1)} = \frac{16 \times \left\{\left(\frac{3}{2}\right)^5 - 1\right\}}{\left(\frac{3}{2}-1\right)} = \frac{32 \times (3^5 - 2^5)}{2^5} = (243-32) = 211.$$

12. $\frac{a(r^3-1)}{(r-1)} \times \frac{(r-1)}{a(r^6-1)} = \frac{125}{152} \Rightarrow \frac{1}{(r^3+1)} = \frac{125}{152} \Rightarrow (r^3+1) = \frac{152}{125}\cdot$

$\therefore \quad r^3 = \left(\frac{152}{125} - 1\right) = \frac{27}{125} = \left(\frac{3}{5}\right)^3 \Rightarrow r = \frac{3}{5}\cdot$

13. Let the three terms of the GP be $\frac{a}{r}$, a and ar. Then,

$$\frac{a}{r} \times a \times ar = 1 \Rightarrow a^3 = 1 \Rightarrow a = 1.$$

So, the given terms are $\frac{1}{r}$, 1 and r.

$\therefore \quad \frac{1}{r} + 1 + r = \frac{39}{10} \Rightarrow r + \frac{1}{r} = \left(\frac{39}{10} - 1\right) = \frac{29}{10}$

$$\Rightarrow 10r^2 + 10 = 29r$$

$$\Rightarrow 10r^2 - 29r + 10 = 0$$

$$\Rightarrow (2r-5)(5r-2) = 0$$

$$\Rightarrow r = \frac{5}{2} \text{ or } r = \frac{2}{5}\cdot$$

Hence, these terms are $\frac{2}{5}, 1, \frac{5}{2}$ or $\frac{5}{2}, 1, \frac{2}{5}\cdot$

SUMMARY OF FACTS AND FORMULAE

1. (i) A sequence $a_1, a_2, a_3, \ldots, a_n, \ldots$ is called a geometrical progression (abbreviated as GP), if each term is nonzero and $\frac{a_{k+1}}{a_k} = r$ (constant) for all $k \geq 1$.

The constant ratio $= r$ is called the common ratio.

(ii) In a GP, we denote the first term by a, common ratio by r and nth term by T_n.

(iii) We have, $T_n = ar^{n-1}$.

(iv) nth term from the end of a GP with first term a, common ratio r and the last term l is $\frac{l}{r^{(n-1)}}\cdot$

2. For solving problems on GP, it is convenient to take

(i) 3 numbers in GP as $\frac{a}{r}, a, ar$.

(ii) 4 numbers in GP as $\frac{a}{r^3}, \frac{a}{r}, ar, ar^3$.

(iii) 5 numbers in GP as $\frac{a}{r^2}, \frac{a}{r}, a, ar, ar^2$.

(iv) $a, ar, ar^2, ar^3, \ldots$ when their product is not given.

3. Sum of n terms of a GP with first term $= a$ and common ratio $= r$ is given by

$$S_n = \begin{cases} na, \text{ when } r = 1 \\ \dfrac{a(1-r^n)}{(1-r)}, \text{ when } r < 1 \\ \dfrac{a(r^n - 1)}{(r-1)}, \text{ when } r > 1. \end{cases}$$

4. The sum of an infinite GP with first term $= a$ and common ratio $= r$, where $|r| < 1$ is given by $S = \frac{a}{(1-r)}\cdot$

5. Properties of GP

Let $a, ar, ar^2, ar^3, \ldots$ be a GP. Then,

(i) $\frac{1}{a}, \frac{1}{ar}, \frac{1}{ar^2}, \frac{1}{ar^3}, \ldots$ is a GP with first term $= \frac{1}{a}$ and common ratio $= \frac{1}{r}\cdot$

(ii) $ka, k(ar), k(ar^2), k(ar^3), \ldots$ is a GP (where $k \neq 0$) with first term $= ka$ and common ratio $= r$.

(iii) $\frac{k}{a}, \frac{k}{ar}, \frac{k}{ar^2}, \frac{k}{ar^3}, \ldots$ is a GP (where $k \neq 0$) with first term $= \frac{k}{a}$ and common ratio $= \frac{1}{r}\cdot$

(iv) $a^k, (ar)^k, (ar^2)^k, \ldots$ is a GP (where $k \neq 0$) with first term $= a^k$ and common ratio $= r^k$.

❑

13
Some Special Series

SUM OF FIRST *n* NATURAL NUMBERS

THEOREM 1 *Prove that* $(1+2+3+\ldots+n)=\frac{1}{2}n(n+1)$

i.e., $\sum_{k=1}^{n} k=\frac{1}{2}n(n+1)$.

PROOF Consider the series $(1+2+3+\ldots+n)$.

This is an arithmetic series in which $a=1$, $d=1$ and $l=n$.

$\therefore \quad S_n=\frac{n}{2}(1+n) \Rightarrow S_n=\frac{1}{2}n(n+1) \qquad \left[\because S_n=\frac{1}{2}(a+l)\right]$.

Hence, $\sum_{k=1}^{n} k=\frac{1}{2}n(n+1)$.

SUM OF THE SQUARES OF FIRST *n* NATURAL NUMBERS

THEOREM 2 *Prove that* $(1^2+2^2+3^2+\ldots+n^2)=\frac{1}{6}n(n+1)(2n+1)$

i.e., $\sum_{k=1}^{n} k^2=\frac{1}{6}n(n+1)(2n+1)$.

PROOF Let $S_n=1^2+2^2+3^2+\ldots+n^2$.

Consider the identity: $k^3-(k-1)^3=3k^2-3k+1$. ... (i)

Putting $k=1, 2, 3, \ldots, (n-1), n$ successively in (i), we get

$$1^3-0^3=3\cdot1^2-3\cdot1+1;$$
$$2^3-1^3=3\cdot2^2-3\cdot2+1;$$
$$3^3-2^3=3\cdot3^2-3\cdot3+1;$$
$$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$$
$$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$$
$$(n-1)^3-(n-2)^3=3\cdot(n-1)^2-3\cdot(n-1)+1;$$
$$n^3-(n-1)^3=3\cdot n^2-3\cdot n+1.$$

Adding columnwise, we get

$$(n^3-0^3)=3(1^2+2^2+3^2+\ldots+n^2)-3(1+2+3+\ldots+n)+n$$

$$\Rightarrow \quad n^3=3\cdot\sum_{k=1}^{n} k^2-3\cdot\frac{1}{2}n(n+1)+n \qquad \left[\because (1+2+3+\ldots+n)=\frac{1}{2}n(n+1)\right]$$

$$\Rightarrow \quad 3\left(\sum_{k=1}^{n} k^2\right)=n^3+\frac{3}{2}n(n+1)-n=\frac{1}{2}(2n^3+3n^2+n)$$

$$\Rightarrow \quad \left[\sum_{k=1}^{n} k^2\right] = \frac{1}{6}(2n^3 + 3n^2 + n) = \frac{1}{6}n(2n^2 + 3n + 1) = \frac{1}{6}n(n+1)(2n+1).$$

Hence, $\sum_{k=1}^{n} k^2 = \frac{1}{6}n(n+1)(2n+1)$.

SUM OF THE CUBES OF FIRST *n* NATURAL NUMBERS

THEOREM 3 *Prove that* $(1^3 + 2^3 + 3^3 + \ldots + n^3) = \left\{\frac{1}{2}n(n+1)\right\}^2$

i.e., $\sum_{k=1}^{n} k^3 = \left\{\frac{1}{2}n(n+1)\right\}^2$.

PROOF Let $S_n = 1^3 + 2^3 + 3^3 + \ldots + n^3$.

Consider the identity: $k^4 - (k-1)^4 = 4k^3 - 6k^2 + 4k - 1$. … (i)

Putting $k = 1, 2, 3, \ldots, (n-1), n$ successively in (i), we get

$$\begin{aligned}
1^4 - 0^4 &= 4 \cdot 1^3 - 6 \cdot 1^2 + 4 \cdot 1 - 1 \\
2^4 - 1^4 &= 4 \cdot 2^3 - 6 \cdot 2^2 + 4 \cdot 2 - 1 \\
3^4 - 2^4 &= 4 \cdot 3^3 - 6 \cdot 3^2 + 4 \cdot 3 - 1 \\
\cdots \quad &\cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \\
\cdots \quad &\cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \\
(n-1)^4 - (n-2)^4 &= 4 \cdot (n-1)^3 - 6 \cdot (n-1)^2 + 4 \cdot (n-1) - 1 \\
n^4 - (n-1)^4 &= 4 \cdot n^3 - 6 \cdot n^2 + 4 \cdot n - 1.
\end{aligned}$$

Adding columnwise, we get

$$(n^4 - 0^4) = 4 \cdot \{1^3 + 2^3 + 3^3 + \ldots + n^3\} - 6 \cdot \{1^2 + 2^2 + 3^2 + \ldots + n^2\}$$
$$+ 4 \cdot \{1 + 2 + 3 + \ldots + n\} - (1 + 1 + 1 + \ldots \text{ to } n \text{ terms})$$

$$\Rightarrow \quad n^4 = 4 \cdot \left(\sum_{k=1}^{n} k^3\right) - 6 \cdot \left(\sum_{k=1}^{n} k^2\right) + 4 \cdot \left(\sum_{k=1}^{n} k\right) - n$$

$$\Rightarrow \quad n^4 = 4 \cdot \left(\sum_{k=1}^{n} k^3\right) - 6 \cdot \frac{1}{6}n(n+1)(2n+1) + 4 \cdot \frac{1}{2}n(n+1) - n$$

$$\left\{\because \sum_{k=1}^{n} k^2 = \frac{1}{6}n(n+1)(2n+1) \text{ and } \sum_{k=1}^{n} k = \frac{1}{2}n(n+1)\right\}$$

$$\Rightarrow \quad 4 \cdot \left(\sum_{k=1}^{n} k^3\right) = n^4 + n(n+1)(2n+1) - 2n(n+1) + n$$

$$\Rightarrow \quad \left(\sum_{k=1}^{n} k^3\right) = \frac{1}{4} \cdot (n^4 + 2n^3 + n^2) = \frac{1}{4}n^2(n^2 + 2n + 1) = \left\{\frac{1}{2}n(n+1)\right\}^2$$

$$\therefore \quad \left(\sum_{k=1}^{n} k^3\right) = \left\{\frac{1}{2}n(n+1)\right\}^2$$

i.e., $(1^3 + 2^3 + 3^3 + \ldots + n^3) = \left\{\frac{1}{2}n(n+1)\right\}^2$.

SUM OF THE CUBES OF FIRST *n* ODD NATURAL NUMBERS

THEOREM 4 *Prove that* $[1^3 + 3^3 + 5^3 + \ldots (2n-1)^3] = n^2(2n^2 - 1)$

i.e., $\sum_{k=1}^{n} (2k-1)^3 = n^2(2n^2 - 1)$.

PROOF Let $S_n = 1^3 + 3^3 + 5^3 + \ldots + (2n-1)^3$.

Then, $S_n = \sum_{k=1}^{n} (2k-1)^3$

$$= \sum_{k=1}^{n} (8k^3 - 12k^2 + 6k - 1)$$

$$= 8\left(\sum_{k=1}^{n} k^3\right) - 12\left(\sum_{k=1}^{n} k^2\right) + 6\left(\sum_{k=1}^{n} k\right) - n \quad [\because\ 1 + 1 + \ldots \text{ to } n \text{ terms} = n]$$

$$= 8 \cdot \frac{1}{4} n^2 (n+1)^2 - 12 \cdot \frac{1}{6} n(n+1)(2n+1) + 6 \cdot \frac{1}{2} n(n+1) - n$$

$$= 2n^2 (n+1)^2 - 2n(n+1)(2n+1) + 3n(n+1) - n$$

$$= n(n+1) \cdot \{2n(n+1) - 2(2n+1) + 3\} - n$$

$$= n(n+1)(2n^2 - 2n + 1) - n = n \cdot \{(n+1)(2n^2 - 2n + 1) - 1\}$$

$$= n(2n^3 - n) = n^2 (2n^2 - 1).$$

$\therefore \quad \{1^3 + 3^3 + 5^3 + \ldots + (2n-1)^3\} = n^2 (2n^2 - 1)$.

SUMMARY

(i) $(1 + 2 + 3 + \ldots + n) = \frac{1}{2} n(n+1)$

i.e., $\left(\sum_{k=1}^{n} k\right) = \frac{1}{2} n(n+1)$.

(ii) $(1^2 + 2^2 + 3^2 + \ldots + n^2) = \frac{1}{6} n(n+1)(2n+1)$

i.e., $\left(\sum_{k=1}^{n} k^2\right) = \frac{1}{6} n(n+1)(2n+1)$.

(iii) $(1^3 + 2^3 + 3^3 + \ldots + n^3) = \left\{\frac{1}{2} n(n+1)\right\}^2$

i.e., $\left(\sum_{k=1}^{n} k^3\right) = \left\{\frac{1}{2} n(n+1)\right\}^2$.

(iv) $\{1^3 + 3^3 + 5^3 + \ldots + (2n-1)^3\} = n^2 (2n^2 - 1)$

i.e., $\sum_{k=1}^{n} (2k-1)^3 = n^2 (2n^2 - 1)$.

SOLVED EXAMLES

EXAMPLE 1 *If S_1, S_2 and S_3 are the sums of first n natural numbers, their squares and their cubes respectively then show that $9S_2^2 = S_3(1 + 8S_1)$.*

SOLUTION We have

$$S_1 = (1 + 2 + 3 + \ldots + n) \Rightarrow S_1 = \frac{1}{2} n(n+1);$$

$$S_2 = (1^2 + 2^2 + 3^2 + \ldots + n^2) \Rightarrow S_2 = \frac{1}{6} n(n+1)(2n+1);$$

and $\quad S_3 = (1^2 + 2^3 + 3^3 + \ldots + n^3) \Rightarrow S_3 = \frac{1}{4} n^2 (n+1)^2$.

$\therefore \quad 9S_2^2 = 9 \times \frac{1}{36} \cdot n^2(n+1)^2(2n+1)^2 = \frac{1}{4}n^2(n+1)^2(2n+1)^2.$

And, $S_3(1+8S_1) = \frac{1}{4}n^2(n+1)^2 \cdot \left\{1 + 8 \cdot \frac{1}{2}n(n+1)\right\}$

$= \frac{1}{4}n^2(n+1)^2(4n^2+4n+1) = \frac{1}{4}n^2(n+1)^2(2n+1)^2.$

Hence, $9S_2^2 = S_3(1+8S_1)$.

EXAMPLE 2 *Find the sum to n terms of the series whose nth term is* $n(n+3)$.

SOLUTION We have, $T_k = k(k+3) = (k^2+3k)$.

$\therefore$ sum to n terms is given by

$$S_n = \sum_{k=1}^{n} T_k$$

$$= \sum_{k=1}^{n}(k^2+3k) = \sum_{k=1}^{n} k^2 + 3\sum_{k=1}^{n} k$$

$$= \frac{1}{6}n(n+1)(2n+1) + 3 \cdot \frac{1}{2}n(n+1) = \frac{1}{6}\{n(n+1)(2n+1) + 9n(n+1)\}$$

$$= \frac{1}{6}n(n+1)(2n+1+9) = \frac{1}{6}n(n+1) \cdot 2(n+5) = \frac{1}{3}n(n+1)(n+5).$$

Hence, the required sum is $\frac{1}{3}n(n+1)(n+5)$.

EXAMPLE 3 *Find the sum to n terms of the series whose nth term is* $(2n-1)^2$.

SOLUTION We have, $T_k = (2k-1)^2 = (4k^2-4k+1)$.

$\therefore$ sum to n terms is given by

$$S_n = \sum_{k=1}^{n} T_k$$

$$= \sum_{k=1}^{n}(4k^2-4k+1) = 4 \cdot \sum_{k=1}^{n} k^2 - 4 \cdot \sum_{k=1}^{n} k + n$$

$[\because 1 + 1 + \ldots n \text{ times} = n]$

$$= 4 \cdot \frac{1}{6}n(n+1)(2n+1) - 4 \cdot \frac{1}{2}n(n+1) + n$$

$$= \frac{2}{3}n(n+1)(2n+1) - 2n(n+1) + n$$

$$= \frac{1}{3} \cdot \{2n(n+1)(2n+1) - 6n(n+1) + 3n\}$$

$$= \frac{1}{3}[n(n+1)\{2(2n+1) - 6\} + 3n] = \frac{1}{3} \cdot [4n(n+1)(n-1) + 3n]$$

$$= \frac{1}{3}(4n^3 - n) = \frac{1}{3}n(4n^2-1).$$

Hence, the required sum is $\frac{1}{3}n(4n^2-1)$.

EXAMPLE 4 *Find the sum to n terms of the series whose nth term is* (n^2+2^n).

SOLUTION We have, $T_k = (k^2+2^k)$.

$$\therefore \quad S_n = \sum_{k=1}^{n} T_k$$

$$= \sum_{k=1}^{n}(k^2+2^k) = \sum_{k=1}^{n} k^2 + \sum_{k=1}^{n} 2^k$$

$$= \frac{1}{6}n(n+1)(2n+1) + (2 + 2^2 + 2^3 + \ldots + 2^n)$$

$$\left[\because \sum_{k=1}^{n} k^2 = \frac{1}{6}n(n+1)(2n+1)\right]$$

$$= \frac{1}{6}n(n+1)(2n+1) + \frac{2(2^n - 1)}{(2-1)} \quad [\because\ 2 + 2^2 + \ldots + 2^n \text{ is a GP}]$$

$$= \frac{1}{6}n(n+1)(2n+1) + 2(2^n - 1).$$

Hence, the required sum is $\frac{1}{6}n(n+1)(2n+1) + 2(2^n - 1)$.

EXAMPLE 5 *Sum the series* $3 \cdot 8 + 6 \cdot 11 + 9 \cdot 14 + \ldots$ *to n terms.*

SOLUTION We have

$$T_k = (k\text{th term of } 3, 6, 9, \ldots) \times (k\text{th term of } 8, 11, 14, \ldots)$$

$$= \{3 + (k-1) \times 3\} \times \{8 + (k-1) \times 3\} = 3k(3k+5)$$

$$= (9k^2 + 15k).$$

$$\therefore \quad S_n = \sum_{k=1}^{n} T_k$$

$$= \sum_{k=1}^{n} (9k^2 + 15k) = 9\left(\sum_{k=1}^{n} k^2\right) + 15\left(\sum_{k=1}^{n} k\right)$$

$$= 9 \cdot \left\{\frac{1}{6}n(n+1)(2n+1)\right\} + 15 \cdot \left\{\frac{1}{2}n(n+1)\right\}$$

$$= \frac{3}{2}n(n+1)\{(2n+1) + 5\} = 3n(n+1)(n+3).$$

Hence, the required sum is $3n(n+1)(n+3)$.

EXAMPLE 6 *Sum the series* $1 \cdot 2 \cdot 3 + 2 \cdot 3 \cdot 4 + 3 \cdot 4 \cdot 5 + \ldots$ *to n terms.*

SOLUTION We have

$$T_k = (k\text{th term of } 1, 2, 3, \ldots) \times (k\text{th term of } 2, 3, 4, \ldots) \times (k\text{th term of } 3, 4, 5, \ldots)$$

$$= \{1 + (k-1) \times 1\} + \{2 + (k-1) \times 1\} \times \{3 + (k-1) \times 1\}$$

$$= k(k+1)(k+2) = (k^3 + 3k^2 + 2k).$$

$$\therefore \quad S_n = \sum_{k=1}^{n} T_k$$

$$= \sum_{k=1}^{n} (k^3 + 3k^2 + 2k) = \sum_{k=1}^{n} k^3 + 3\sum_{k=1}^{n} k^2 + 2\sum_{k=1}^{n} k$$

$$= \frac{1}{4}n^2(n+1)^2 + 3 \cdot \frac{1}{6}n(n+1)(2n+1) + 2 \cdot \frac{1}{2}n(n+1)$$

$$\left\{\because \sum_{k=1}^{n} k^3 = \left\{\frac{1}{2}n(n+1)\right\}^2, \sum_{k=1}^{n} k^2 = \frac{1}{6}n(n+1)(2n+1), \sum_{k=1}^{n} k = \frac{1}{2}n(n+1)\right\}$$

$$= \frac{1}{4}n(n+1)\{n(n+1) + 2(2n+1) + 4\}$$

$$= \frac{1}{4}n(n+1)(n^2+5n+6) = \frac{1}{4}n(n+1)(n+2)(n+3).$$

Hence, the required sum is $\frac{1}{4}n(n+1)(n+2)(n+3)$.

EXAMPLE 7 *Find the sum of the series* $1^2+3^2+5^2+\ldots$ *to n terms.*

SOLUTION We have

$$T_k = k\text{th term of } (1^2+3^2+5^2+\ldots)$$
$$= \{1+(k-1)\times 2\}^2 = (2k-1)^2 = (4k^2-4k+1).$$

$$\therefore \quad S_n = \sum_{k=1}^{n} T_k$$
$$= \sum_{k=1}^{n}(4k^2-4k+1)$$
$$= 4\sum_{k=1}^{n}k^2 - 4\sum_{k=1}^{n}k + (1+1+\ldots n \text{ times})$$
$$= 4\cdot\frac{1}{6}n(n+1)(2n+1) - 4\cdot\frac{1}{2}n(n+1)+n$$
$$= \frac{n}{3}\cdot\{2(n+1)(2n+1)-6(n+1)+3\} = \frac{n}{3}(4n^2-1).$$

Hence, the required sum is $\frac{n}{3}(4n^2-1)$.

EXAMPLE 8 *Find the sum* $1^2+(1^2+2^2)+(1^2+2^2+3^2)+\ldots$ *to n terms.*

SOLUTION We have

$$T_k = (1^2+2^2+3^2+\ldots+k^2) = \left\{\frac{1}{6}k(k+1)(2k+1)\right\} = \frac{1}{6}(2k^3+3k^2+k).$$

$$\therefore \quad S_n = \sum_{k=1}^{n} T_k$$
$$= \frac{1}{6}\cdot\sum_{k=1}^{n}(2k^3+3k^2+k) = \frac{1}{6}\cdot 2\cdot\sum_{k=1}^{n}k^3 + \frac{1}{6}\cdot 3\cdot\sum_{k=1}^{n}k^2 + \frac{1}{6}\cdot\sum_{k=1}^{n}k$$
$$= \frac{1}{3}\cdot\sum_{k=1}^{n}k^3 + \frac{1}{2}\cdot\sum_{k=1}^{n}k^2 + \frac{1}{6}\cdot\sum_{k=1}^{n}k$$
$$= \frac{1}{3}\cdot\left\{\frac{1}{2}n(n+1)\right\}^2 + \frac{1}{2}\cdot\frac{1}{6}n(n+1)(2n+1) + \frac{1}{6}\cdot\frac{1}{2}n(n+1)$$
$$= \frac{1}{12}n^2(n+1)^2 + \frac{1}{12}n(n+1)(2n+1) + \frac{1}{12}n(n+1)$$
$$= \frac{1}{12}n(n+1)\cdot\{n(n+1)+(2n+1)+1\} = \frac{1}{12}n(n+1)^2(n+2).$$

Hence, the required sum is $\frac{1}{12}n(n+1)^2(n+2)$.

EXAMPLE 9 *Prove that* $\dfrac{1\times 2^2+2\times 3^2+\ldots+n\times(n+1)^2}{1^2\times 2+2^2\times 3+\ldots+n^2\times(n+1)} = \dfrac{3n+5}{3n+1}$.

SOLUTION We have

kth term of the numerator $= k(k+1)^2 = (k^3+2k^2+k)$.

kth term of the denominator $= k^2(k+1) = (k^3+k^2)$.

$$\therefore \quad \text{LHS} = \frac{\sum\limits_{k=1}^{n}(k^3+2k^2+k)}{\sum\limits_{k=1}^{n}(k^3+k^2)} = \frac{\left(\sum\limits_{k=1}^{n}k^3\right)+2\cdot\left(\sum\limits_{k=1}^{n}k^2\right)+\left(\sum\limits_{k=1}^{n}k\right)}{\left(\sum\limits_{k=1}^{n}k^3\right)+\left(\sum\limits_{k=1}^{n}k^2\right)}$$

$$= \frac{\frac{1}{4}n^2(n+1)^2+2\cdot\frac{1}{6}n(n+1)(2n+1)+\frac{1}{2}n(n+1)}{\frac{1}{4}n^2(n+1)^2+\frac{1}{6}n(n+1)(2n+1)}$$

$$\left\{\because \left(\sum_{k=1}^{n}k\right)=\frac{1}{2}n(n+1);\left(\sum_{k=1}^{n}k^2\right)=\frac{1}{6}n(n+1)(2n+1) \text{ and } \left(\sum_{k=1}^{n}k^3\right)=\frac{1}{4}n^2(n+1)^2\right\}$$

$$= \frac{n(n+1)(n+2)(3n+5)}{12}\times\frac{12}{n(n+1)(n+2)(3n+1)}$$

[on simplifying]

$$= \frac{(3n+5)}{(3n+1)} = \text{RHS}.$$

EXAMPLE 10 *Find the sum of n terms of the series*

$$\frac{1}{(2\times 5)}+\frac{1}{(5\times 8)}+\frac{1}{(8\times 11)}+\dots .$$

SOLUTION We have

$$T_k = \frac{1}{(k\text{th term of } 2, 5, 8, \dots)\times(k\text{th term of } 5, 8, 11, \dots)}$$

$$= \frac{1}{\{2+(k-1)\times 3\}\times\{5+(k-1)\times 3\}}$$

$$= \frac{1}{(3k-1)(3k+2)} = \frac{1}{3}\left\{\frac{1}{(3k-1)}-\frac{1}{(3k+2)}\right\}.$$

$$\therefore \quad T_k = \frac{1}{3}\left\{\frac{1}{(3k-1)}-\frac{1}{(3k+2)}\right\}. \quad \dots \text{(i)}$$

Putting $k = 1, 2, 3, \dots, n$ successively in (i), we get

$$T_1 = \frac{1}{3}\left(\frac{1}{2}-\frac{1}{5}\right)$$

$$T_2 = \frac{1}{3}\left(\frac{1}{5}-\frac{1}{8}\right)$$

$$T_3 = \frac{1}{3}\left(\frac{1}{8}-\frac{1}{11}\right)$$

… … … …

… … … …

$$T_n = \frac{1}{3}\left\{\frac{1}{(3n-1)}-\frac{1}{(3n+2)}\right\}.$$

Adding columnwise, we get

$$S_n = (T_1+T_2+T_3+\dots+T_n)$$

$$= \frac{1}{3}\left(\frac{1}{2}-\frac{1}{3n+2}\right) = \frac{n}{2(3n+2)}.$$

EXAMPLE 11 *Find the sum of n terms of the series* $5+11+19+29+41+\dots$.

SOLUTION Let $S_n = 5+11+19+29+\dots+T_{n-1}+T_n$

$S_n = \quad 5+11+19+\dots+T_{n-2}+T_{n-1}+T_n$

On subtraction, we get

$$0 = 5+[6+8+10+12+\dots \text{ to } (n-1) \text{ terms}]-T_n$$

$$\Rightarrow \quad T_n = 5+\frac{(n-1)\times[2\times 6+(n-2)\times 2]}{2}$$

$$\Rightarrow \quad T_n = 5+(n-1)(n+4)$$

$$\Rightarrow \quad T_n = n^2+3n+1.$$

$$\therefore \quad S_n = \sum_{k=1}^{n} T_k = \sum_{k=1}^{n}(k^2+3k+1) = \sum_{k=1}^{n} k^2 + 3\sum_{k=1}^{n} k + n$$

$$= \frac{1}{6}n(n+1)(2n+1)+3\times\frac{1}{2}n(n+1)+n$$

$$= \frac{1}{3}n(n+2)(n+4).$$

EXAMPLE 12 *Find the sum of the first n terms of the series* $3+7+13+21+31+\dots$.

SOLUTION Let $S_n = 3+7+13+21+31+\dots+T_{n-1}+T_n$

$S_n = \quad 3+\ 7+13+21+\dots+T_{n-2}+T_{n-1}+T_n$

On subtraction, we get

$$0 = 3+[4+6+8+10+\dots \text{ to } (n-1) \text{ terms}]-T_n$$

$$\Rightarrow \quad T_n = 3+[4+6+8+\dots \text{ to } (n-1) \text{ terms}]$$

$$= 3+\frac{(n-1)}{2}\times[2\times 4+(n-2)\times 2]$$

$$= 3+(n-1)(n+2) = (n^2+n+1).$$

$$\therefore \quad S_n = \sum_{k=1}^{n} T_k = \sum_{k=1}^{n}(k^2+k+1)$$

$$= \sum_{k=1}^{n} k^2 + \sum_{k=1}^{n} k + n \qquad [\because\ 1+1+1+\dots \text{ to } n \text{ terms} = n]$$

$$= \frac{1}{6}n(n+1)(2n+1)+\frac{1}{2}n(n+1)+n = \frac{1}{3}n(n^2+3n+5).$$

EXAMPLE 13 *Find the* 50*th term of the series* $2+3+6+11+18+\dots$.

SOLUTION Let $S_n = 2+3+6+11+18+\dots+T_{n-1}+T_n$.

Again $S_n = \quad 2+3+\ 6+11+\dots+T_{n-1}+T_n$.

On subtracting, we get

$$0 = 2+[1+3+5+7+\dots \text{ to } (n-1) \text{ terms}]-T_n$$

$$\Rightarrow \quad T_n = 2+[1+3+5+7+\dots \text{ to } (n-1) \text{ terms}]$$

$$= 2+\frac{(n-1)}{2}\cdot[2\times 1+(n-2)\times 2] = (n-1)^2+2.$$

$$\Rightarrow \quad T_{50} = \{(50-1)^2+2\} = \{(49)^2+2\} = (2401+2) = 2403.$$

EXAMPLE 14 *Find the sum of the series*

$$(3^3 - 2^3) + (5^3 - 4^3) + (7^3 - 6^3) + \ldots \textit{ to (i) n terms} \quad \textit{(ii) 10 terms.}$$

SOLUTION We have

$$T_k = \{(2k+1)^3 - (2k)^3\}$$
$$= \{8k^3 + 1 + 6k(2k+1) - 8k^3\} = (12k^2 + 6k + 1).$$

(i) sum to n terms is given by

$$S_n = 12 \sum_{k=1}^{n} k^2 + 6 \sum_{k=1}^{n} k + n$$
$$= 12 \times \frac{1}{6} n(n+1)(2n+1) + 6 \times \frac{1}{2} n(n+1) + n$$
$$= 2n(n+1)(2n+1) + 3n(n+1) + n$$
$$= n(n+1)[4n+2+3] + n = n[(n+1)(4n+5)+1]$$
$$= n(4n^2 + 9n + 6) = (4n^3 + 9n^2 + 6n).$$

(ii) Sum to 10 terms is given by

$$S_{10} = 10 \times [4 \times 10^2 + 9 \times 10 + 6] = 4960.$$

EXERCISE 13A

Find the sum of the series whose nth term is given by:

1. $(3n^2 + 2n)$
2. $n(n+1)(n+4)$
3. $(4n^3 + 6n^2 + 2n)$
4. $(3n^2 - 3n + 2)$
5. $(2n^2 - 3n + 5)$
6. $(n^3 - 3^n)$

Find the sum of the series:

7. $(2^2 + 4^2 + 6^2 + 8^2 + \ldots$ to n terms)
8. $(2^3 + 4^3 + 6^3 + 8^3 + \ldots$ to n terms)
9. $(5^2 + 6^2 + 7^2 + \ldots + 20^2)$
10. $1 \times 2 + 2 \times 3 + 3 \times 4 + 4 \times 5 + \ldots$ to n terms
11. $3 \times 8 + 6 \times 11 + 9 \times 14 + \ldots$ to n terms
12. $1 \times 2^2 + 2 \times 3^2 + 3 \times 4^2 + \ldots$ to n terms
13. $1 \times 2^2 + 3 \times 3^2 + 5 \times 4^2 + \ldots$ to n terms
14. $3 \times 1^2 + 5 \times 2^2 + 7 \times 3^2 + \ldots$ to n terms
15. $(1 \times 2 \times 3) + (2 \times 3 \times 4) + (3 \times 4 \times 5) + \ldots$ to n terms
16. $(1 \times 2 \times 4) + (2 \times 3 \times 7) + (3 \times 4 \times 10) + \ldots$ to n terms
17. $\frac{1}{(1 \times 2)} + \frac{1}{(2 \times 3)} + \frac{1}{(3 \times 4)} + \ldots$ to n terms
18. $\frac{1}{(1 \times 3)} + \frac{1}{(3 \times 5)} + \frac{1}{(5 \times 7)} + \ldots + \frac{1}{(2n-1)(2n+1)}$
19. $\frac{1}{(1 \times 6)} + \frac{1}{(6 \times 11)} + \frac{1}{(11 \times 16)} + \ldots + \frac{1}{(5n-4)(5n+1)}$

20. $\frac{1^3}{1}+\frac{1^3+2^3}{1+3}+\frac{1^3+2^3+3^3}{1+3+5}+\ldots$ to n terms

21. $3+15+35+63+\ldots$ to n terms

22. $1+5+12+22+35+\ldots$ to n terms

23. $5+7+13+31+85+\ldots$ to n terms

24. If $S_k=\frac{(1+2+3+\ldots+k)}{k}$, prove that

$$(S_1^2+S_2^2+S_3^2+\ldots+S_n^2)=\frac{1}{24}n(2n^2+9n+13).$$

25. If S_n denotes the sum of the cubes of the first n natural numbers and s_n denotes the sum of the first n natural numbers then find the value of $\sum_{k=1}^{n}\frac{S_k}{s_k}$.

ANSWERS (EXERCISE 13A)

1. $\frac{1}{2}n(n+1)(2n+3)$ **2.** $\frac{1}{12}n(n+1)(n+2)(3n+17)$ **3.** $n(n+1)^2(n+2)$

4. $2n(n^2+1)$ **5.** $\frac{n}{6}(4n^2-3n+23)$ **6.** $\frac{1}{4}n^2(n+1)^2-\frac{3}{2}(3^n-1)$

7. $\frac{2}{3}n(n+1)(2n+1)$ **8.** $2n^2(n+1)^2$ **9.** 2840

10. $\frac{1}{3}n(n+1)(n+2)$ **11.** $3n(n+1)(n+3)$ **12.** $\frac{n(n+1)(n+2)(3n+5)}{12}$

13. $\frac{n}{2}(n^3+4n^2+4n-1)$ **14.** $\frac{n(n+1)(3n^2+5n+1)}{6}$ **15.** $\frac{n(n+1)(n+2)(n+3)}{4}$

16. $\frac{n(n+1)(3n^2+19n+14)}{12}$ **17.** $\frac{n}{(n+1)}$ **18.** $\frac{n}{(2n+1)}$ **19.** $\frac{n}{(5n+1)}$

20. $\frac{1}{24}n(2n^2+9n+13)$ **21.** $\frac{1}{3}n(4n^2+6n-1)$ **22.** $\frac{1}{2}n^2(n+1)$

23. $\frac{1}{2}(3^n+8n-1)$ **25.** $\frac{1}{6}n(n+1)(n+2)$

HINTS TO SOME SELECTED QUESTIONS

4. $T_n=(3n^2-3n+2)$.

$\therefore\quad S_n=3\left(\sum_{k=1}^{n}k^2\right)-3\left(\sum_{k=1}^{n}k\right)+2n$ $\quad[\because\ 2+2+2+\ldots\ n \text{ times}=2n]$.

6. $T_n=(n^3-3^n)$.

$\therefore\quad S_n=\left(\sum_{k=1}^{n}k^3\right)-\left(\sum_{k=1}^{n}3^k\right)=\frac{1}{4}n^2(n+1)^2-\{3+3^2+\ldots+3^n\}$

$\Rightarrow\quad S_n=\left\{\frac{1}{4}n^2(n+1)^2-\frac{3(3^n-1)}{(3-1)}\right\}$.

7. $T_n=\{2+(n-1)\times 2\}^2=(2n)^2=4n^2$.

$\therefore\quad S_n=\sum_{k=1}^{n}T_k=4\left(\sum_{k=1}^{n}k^2\right)=4\times\frac{1}{6}n(n+1)(2n+1)$.

8. $T_n = \{2 + (n-1) \times 2\}^3 = (2n)^3 = 8n^3$.

$\therefore \quad S_n = 8\left\{\sum_{k=1}^{n} k^3\right\} = 8 \times \frac{1}{4} n^2 (n+1)^2 = 2n^2 (n+1)^2$.

9. Given series $= (1^2 + 2^2 + 3^2 + 4^2 + 5^2 + 6^2 + \ldots + 20^2) - (1^2 + 2^2 + 3^2 + 4^2)$

$= \left\{\frac{1}{6} n(n+1)(2n+1), \text{ when } n = 20\right\} - 30$

$= \left(\frac{1}{6} \times 20 \times 21 \times 41\right) - 30 = 2840$.

10. $T_n = n(n+1) = n^2 + n$.

$\therefore \quad S_n = \left(\sum_{k=1}^{n} k^2\right) + \left(\sum_{k=1}^{n} k\right)\cdot$

11. $T_n = (n\text{th term of } 3, 6, 9, \ldots) \times (n\text{th term of } 8, 11, 14, \ldots)$

$= \{3 + (n-1) \times 3\} \times \{8 + (n-1) \times 3\} = 3n(3n+5) = 9n^2 + 15n$.

$\therefore \quad S_n = 9\left(\sum_{k=1}^{n} k^2\right) + 15\left(\sum_{k=1}^{n} k\right)\cdot$

12. $T_n = (n\text{th term of } 1, 2, 3, \ldots) \times (n\text{th term of } 2^2, 3^2, 4^2, \ldots)$

$= n \times (n+1)^2 = n \times (n^2 + 2n + 1) = n^3 + 2n^2 + n$.

$\therefore \quad S_n = \left(\sum_{k=1}^{n} k^3\right) + 2\left(\sum_{k=1}^{n} k^2\right) + \left(\sum_{k=1}^{n} k\right)\cdot$

13. $T_n = (n\text{th term of } 1, 3, 5, \ldots) \times (n\text{th term of } 2^2, 3^2, 4^2, \ldots)$

$= \{1 + (n-1) \times 2\} \times (n+1)^2 = (2n-1)(n^2 + 2n + 1) = (2n^3 + 3n^2 - 1)$.

$\therefore \quad S_n = 2\left(\sum_{k=1}^{n} k^3\right) + 3\left(\sum_{k=1}^{n} k^2\right) - n$.

14. $T_n = (n\text{th term of } 3, 5, 7, \ldots) \times (n\text{th term of } 1^2, 2^2, 3^2, \ldots)$

$= \{3 + (n-1) \times 2\} \times n^2 = (2n+1)n^2 = (2n^3 + n^2)$.

$\therefore \quad S_n = 2\left(\sum_{k=1}^{n} k^3\right) + \left(\sum_{k=1}^{n} k^2\right)\cdot$

15. $T_n = n(n+1)(n+2) = (n^3 + 3n^2 + 2n)$.

$\therefore \quad S_n = \left(\sum_{k=1}^{n} k^3\right) + 3\left(\sum_{k=1}^{n} k^2\right) + 2\left(\sum_{k=1}^{n} k\right)\cdot$

16. $T_n = (n\text{th term of } 1, 2, 3, \ldots) \times (n\text{th term of } 2, 3, 4, \ldots) \times (n\text{th term of } 4, 7, 10, \ldots)$

$= n(n+1) \times [4 + (n-1) \times 3] = n(n+1)(3n+1) = (3n^3 + 4n^2 + n)$.

17. $T_n = \frac{1}{n(n+1)} = \frac{A}{n} + \frac{B}{n+1} = \frac{A(n+1) + Bn}{n(n+1)} = \frac{(A+B)n + A}{n(n+1)}\cdot$

$\therefore \quad A + B = 0$ and $A = 1 \Rightarrow A = 1$ and $B = -1$

$\therefore \quad T_n = \left(\frac{1}{n} - \frac{1}{n+1}\right)$

$\therefore \quad T_1 = \left(\frac{1}{1} - \frac{1}{2}\right)$

$T_2 = \left(\frac{1}{2} - \frac{1}{3}\right)$

$T_3 = \left(\frac{1}{3} - \frac{1}{4}\right)$

… … … …

… … … …

$$T_{n-1} = \left(\frac{1}{n-1} - \frac{1}{n}\right)$$
$$T_n = \left(\frac{1}{n} - \frac{1}{n+1}\right).$$

Adding columnwise, we get

$$S_n = (T_1 + T_2 + T_3 + \dots + T_n) = \left(1 - \frac{1}{n+1}\right) = \frac{(n+1)-1}{n+1} = \frac{n}{n+1}.$$

18. $T_n = \dfrac{1}{(2n-1)(2n+1)} = \dfrac{A}{2n-1} + \dfrac{B}{2n+1} = \dfrac{A(2n+1)+B(2n-1)}{(2n-1)(2n+1)} = \dfrac{2(A+B)n+(A-B)}{(2n-1)(2n+1)}.$

$\therefore\quad 2(A+B) = 0$ and $A - B = 1 \Rightarrow A + B = 0, A - B = 1 \Rightarrow A = \frac{1}{2}, B = \frac{-1}{2}.$

$$\therefore\quad T_n = \frac{1}{2} \times \left\{\frac{1}{2n-1} - \frac{1}{2n+1}\right\}.$$

$$\therefore\quad T_1 = \frac{1}{2} \times \left(\frac{1}{1} - \frac{1}{3}\right)$$
$$T_2 = \frac{1}{2} \times \left(\frac{1}{3} - \frac{1}{5}\right)$$
$$T_3 = \frac{1}{2} \times \left(\frac{1}{5} - \frac{1}{7}\right)$$

... ...
... ...

$$T_{n-1} = \frac{1}{2} \times \left\{\frac{1}{2n-3} - \frac{1}{2n-1}\right\}$$
$$T_n = \frac{1}{2} \times \left\{\frac{1}{2n-1} - \frac{1}{2n+1}\right\}.$$

Adding columnwise, we get

$$S_n = (T_1 + T_2 + T_3 + \dots + T_n) = \frac{1}{2}\left(1 - \frac{1}{2n+1}\right) = \frac{2n+1-1}{2(2n+1)} = \frac{n}{(2n+1)}.$$

19. $T_n = \dfrac{1}{5} \times \left\{\dfrac{1}{5n-4} - \dfrac{1}{5n+1}\right\}.$

$$\therefore\quad T_1 = \frac{1}{5} \times \left\{\frac{1}{1} - \frac{1}{6}\right\}$$
$$T_2 = \frac{1}{5} \times \left\{\frac{1}{6} - \frac{1}{11}\right\}$$
$$T_3 = \frac{1}{5} \times \left\{\frac{1}{11} - \frac{1}{16}\right\}$$

... ...
... ...

$$T_{n-1} = \frac{1}{5} \times \left\{\frac{1}{5n-9} - \frac{1}{5n-4}\right\}$$
$$T_n = \frac{1}{5} \times \left\{\frac{1}{5n-4} - \frac{1}{5n+1}\right\}.$$

Adding columnwise, we get

$$S_n = (T_1 + T_2 + \dots + T_n) = \frac{1}{5}\left(1 - \frac{1}{5n+1}\right) = \frac{(5n+1-1)}{5(5n+1)} = \frac{n}{(5n+1)}.$$

20. $T_n = \dfrac{1^3 + 2^3 + 3^3 + \dots + n^3}{1 + 3 + 5 + \dots + (2n-1)} = \dfrac{\left\{\frac{1}{2}n(n+1)\right\}^2}{\frac{n}{2}\{1 + (2n-1)\}} = \dfrac{(n+1)^2}{4} = \dfrac{1}{4}n^2 + \dfrac{1}{2}n + \dfrac{1}{4}.$

$$\therefore\quad S_n = \sum_{k=1}^{n} T_k$$
$$= \frac{1}{4}\left(\sum_{k=1}^{n} k^2\right) + \frac{1}{2}\left(\sum_{k=1}^{n} k\right) + \frac{1}{4}n.$$

21. $S_n = 3 + 15 + 35 + 63 + \ldots + T_{n-1} + T_n$ … (i)

$S_n = \quad 3 + 15 + 35 + \ldots + T_{n-2} + T_{n-1} + T_n.$ … (ii)

On subtracting (ii) from (i), we get

$$0 = 3 + \{12 + 20 + 28 + \ldots \text{ to } (n-1) \text{ terms}\} - T_n$$

$$\Rightarrow \quad T_n = 3 + \frac{1}{2}(n-1)\{2 \times 12 + (n-2) \times 8\} = (4n^2 - 1)$$

$$\Rightarrow \quad S_n = 4\left(\sum_{k=1}^{n} k^2\right) - n = \left\{4 \times \frac{1}{6} n(n+1)(2n+1)\right\} - n = \frac{1}{3} n(4n^2 + 6n - 1).$$

24. $S_k = \dfrac{(1 + 2 + 3 + \ldots + k)}{k} = \dfrac{k(k+1)}{2k} = \dfrac{k+1}{2}.$

$$\therefore \quad \sum_{k=1}^{n} S_k^2 = \sum_{k=1}^{n} \frac{1}{4}(k+1)^2 = \frac{1}{4}\left(\sum_{k=1}^{n} k^2\right) + \frac{1}{2}\left(\sum_{k=1}^{n} k\right) + \frac{1}{4}(1 + 1 + \ldots n \text{ times})$$

$$= \frac{1}{4} \times \frac{1}{6} n(n+1)(2n+1) + \frac{1}{4} n(n+1) + \frac{1}{4} n$$

$$= \frac{n}{24}(2n^2 + 9n + 13) \qquad \text{[on simplification].}$$

25. $S_k = \left\{\sum_{r=1}^{k} r^3\right\} = \left\{\frac{1}{2} k(k+1)\right\}^2$ and $s_k = \left\{\sum_{r=1}^{k} r\right\} = \frac{1}{2} k(k+1).$

$$\therefore \quad \frac{S_k}{s_k} = \frac{\left\{\frac{1}{2} k(k+1)\right\}^2}{\frac{1}{2} k(k+1)} = \frac{1}{2} k(k+1) = \frac{1}{2} k^2 + \frac{1}{2} k$$

$$\Rightarrow \quad \sum_{k=1}^{n} \frac{S_k}{s_k} = \frac{1}{2}\left\{\sum_{k=1}^{n} k^2\right\} + \frac{1}{2}\left\{\sum_{k=1}^{n} k\right\}$$

$$= \frac{1}{2} \times \frac{1}{6} \times n(n+1)(2n+1) + \frac{1}{2} \times \frac{1}{2} n(n+1)$$

$$= \frac{1}{12}\{n(n+1)(2n+1) + 3n(n+1)\} = \frac{1}{6} n(n+1)(n+2).$$

EXERCISE 13B

Very-Short-Answer Questions

1. Find the sum $(2 + 4 + 6 + 8 + \ldots + 100)$.
2. Find the sum $(41 + 42 + 43 + \ldots + 100)$.
3. Find the sum $\{(11)^2 + (12)^2 + (13)^2 + \ldots + (20)^2\}$.
4. Find the sum $\{(6)^3 + (7)^3 + (8)^3 + (9)^3 + (10)^3\}$.
5. If $\sum_{k=1}^{n} k = 210$, find the value of $\sum_{k=1}^{n} k^2$.
6. If $\sum_{k=1}^{n} k = 45$, find the value of $\sum_{k=1}^{n} k^3$.
7. Find the sum of the series $\{2^2 + 4^2 + 6^2 + \ldots + (2n)^2\}$.
8. Find the sum of 10 terms of the geomeric series $\sqrt{2} + \sqrt{6} + \sqrt{18} + \ldots$.
9. Find the sum of n terms of the series whose rth term is $(r + 2^r)$.

ANSWERS (EXERCISE 13B)

1. 2550 **2.** 4230 **3.** 2485 **4.** 2800 **5.** 2870 **6.** 2025

7. $\frac{2}{3}n(n+1)(2n+1)$ **8.** $121(\sqrt{6}+\sqrt{2})$ **9.** $\frac{1}{2}n(n+1)+2^{n+1}-2$

HINTS TO SOME SELECTED QUESTIONS

1. We know that $(1+2+3+\ldots+n)=\frac{1}{2}n(n+1)$.

$\therefore\quad (2+4+6+8+\ldots+100)=2\times(1+2+3+\ldots+50)=\left(2\times\frac{1}{2}\times 50\times 51\right)=2550.$

2. Given sum $=(1+2+3+\ldots+40+41+\ldots+100)-(1+2+3+\ldots+40)$

$$=\left(\frac{1}{2}\times 100\times 101\right)-\left(\frac{1}{2}\times 40\times 41\right)=(5050-820)=4230.$$

3. We know that $(1^2+2^2+3^2+\ldots+n^2)=\frac{1}{6}n(n+1)(2n+1)$.

$\therefore\quad$ given sum $=\{1^2+2^2+\ldots+(10)^2+(11)^2+\ldots+(20)^2\}-\{1^2+2^2+\ldots+(10)^2\}$

$$=\left(\frac{1}{6}\times 20\times 21\times 41\right)-\left(\frac{1}{6}\times 10\times 11\times 21\right)=(2870-385)=2485.$$

4. We know that $(1^3+2^3+3^3+\ldots+n^3)=\left\{\frac{1}{2}n(n+1)\right\}^2$.

$\therefore\quad$ given sum $=\{1^3+2^3+3^3+4^3+5^3+6^3+\ldots+(10)^3\}-\{1^3+2^3+3^3+4^3+5^3\}$

$$=\left(\frac{1}{2}\times 10\times 11\right)^2-\left(\frac{1}{2}\times 5\times 6\right)^2=(55)^2-(15)^2=(3025-225)=2800.$$

5. $(1+2+3+\ldots+n)=210 \Rightarrow \frac{1}{2}n(n+1)=210 \Rightarrow n^2+n-420=0$

$$\Rightarrow (n+21)(n-20)=0 \Rightarrow n=20.$$

$\therefore\quad (1^2+2^2+3^2+\ldots+n^2)=\frac{1}{6}n(n+1)(2n+1)$, where $n=20$

$$=\left(\frac{1}{6}\times 20\times 21\times 41\right)=2870.$$

6. Given $(1+2+3+\ldots+n)=45 \Rightarrow \frac{1}{2}n(n+1)=45$.

$\therefore\quad \left(\sum_{k=1}^{n} k^3\right)=\left\{\frac{1}{2}n(n+1)\right\}^2=(45)^2=2025.$

7. $T_k=(2k)^2=4k^2$.

$\therefore\quad S_n=\sum_{k=1}^{n} T_k=4\left(\sum_{k=1}^{n} k^2\right)=\left\{4\times\frac{1}{6}n(n+1)(2n+1)\right\}=\frac{2}{3}n(n+1)(2n+1).$

8. Given series is a geometric series in which $a=\sqrt{2}$, $r=\sqrt{3}$ and $n=10$.

$\therefore\quad S=\frac{a(r^n-1)}{(r-1)}=\frac{\sqrt{2}\{(\sqrt{3})^{10}-1\}}{(\sqrt{3}-1)}\times\frac{(\sqrt{3}+1)}{(\sqrt{3}+1)}=\frac{\sqrt{2}(243-1)(\sqrt{3}+1)}{2}$

$\Rightarrow\quad S=121(\sqrt{6}+\sqrt{2}).$

9. Required sum $=\sum_{r=1}^{n} T_r=\sum_{r=1}^{n}(r+2^r)$

$$=\sum_{r=1}^{n} r+\sum_{r=1}^{n} 2^r=\frac{1}{2}n(n+1)+(2+2^2+2^3+\ldots+2^n)$$

$$=\frac{1}{2}n(n+1)+\frac{2(2^n-1)}{(2-1)}=\frac{1}{2}n(n+1)+2^{n+1}-2.$$

❑

14

Measurement of Angles

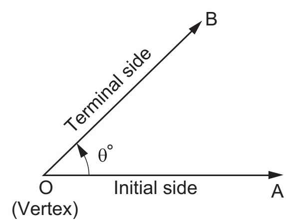
TRIGONOMETRY

The literal meaning of the word *trigonometry* is the science of measuring the sides and the angles of triangles. However, in modern times it is used in a wider context.

We define it *as the branch of mathematics which deals with the measurements of angles of triangles and the problems related to these angles.*

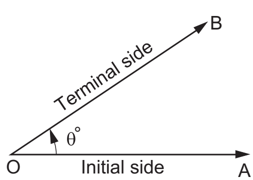
ANGLE When a ray $\overrightarrow{OA}$ starting from its *initial position OA* rotates about its end point O and takes the *final position OB*, we say that angle AOB (written as $\angle AOB$) has been formed.

Here, OA and OB are respectively known as the *initial side* and the *terminal side* of $\angle AOB$, and the point O is called its *vertex*.

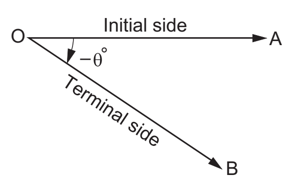


The amount of rotation from the initial side to the terminal side is called the *measure of the angle.*

POSITIVE AND NEGATIVE ANGLES An angle formed by a rotating ray is said to be positive or negative depending on whether it moves in an anticlockwise or a clockwise direction respectively.



(Positive angle)



(Negative angle)

MEASURING ANGLES Mainly, there are two systems of measuring angles.

(i) Sexagesimal System (Degree Measure)

The angle traced by a moving line about a point from its initial position to the terminating position in making $\frac{1}{360}$ *of the complete revolution of a circle is said to have a measure of* 1 *degree, written as* $1°$.

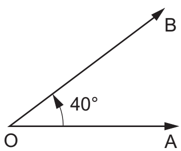
The one-sixtieth part of a degree, i.e., $\frac{1^\circ}{60}$, is called a *minute,* written as $1'$.

The one-sixtieth part of a minute, i.e., $\frac{1'}{60}$, is called a *second,* written as $1''$.

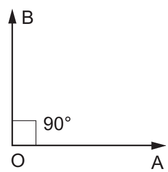
Also, we define 1 right angle = 90°.

An angle measuring 180° is called a *straight angle.*

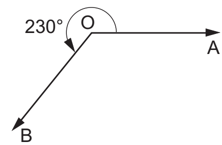
The measures of the angles given below in (i), (ii), (iii), (iv), (v), (vi) are respectively 40°, 90°, 230°, 420°, –30° and –230°.



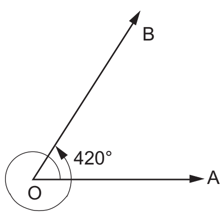
(i)



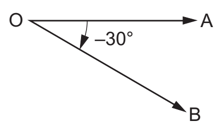
(ii)



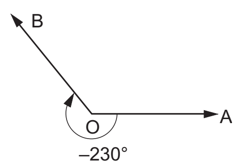
(iii)



(iv)

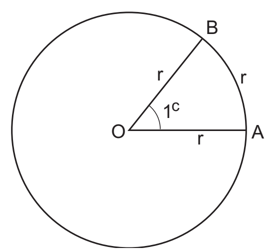


(v)



(vi)

(ii) Circular System (Radian Measure)



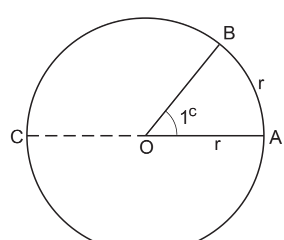
In this system, an angle is measured in *radians.*

A radian is an angle subtended at the centre of a circle by an arc whose length is equal to the radius of the circle.

We denote 1 radian by 1^c.

THEOREM 1 *A radian is a constant angle.*

PROOF Let us consider a circle with centre O and radius r.



Let AB be an arc of length r. Join OA and OB.

Then, $\angle AOB = 1^c$.

Produce AO to meet the circle at C.

Then, $\angle AOC$ = a straight angle = 2 rt. $\angle$s.

Since the angles at the centre of a circle are proportional to the lengths of the arcs subtending them, we have

$$\frac{\angle AOB}{\angle AOC} = \frac{\text{arc } AB}{\text{arc } ABC} = \frac{r}{\frac{1}{2}\cdot(2\pi r)} = \frac{1}{\pi}$$

$$\Rightarrow \quad \angle AOB = \frac{\angle AOC}{\pi} = \frac{2\text{rt}\angle s}{\pi} = \text{constant}$$

$$\Rightarrow \quad 1^c = \text{constant.}$$

Hence, a radian is a constant angle.

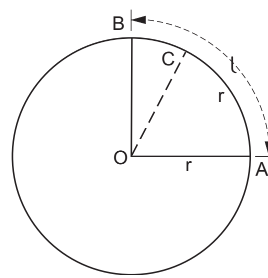
THEOREM 2 *In a circle of radius r, if an arc of length l subtends an angle* θ^c *at the centre then* $\theta = \frac{l}{r}\cdot$

PROOF Let us consider a circle with centre O and radius r.

Let $\angle AOB = \theta^c$ and let arc $AB = l$.

Let C be a point on AB such that arc $AC = r$.

Then, $\angle AOC = 1^c$.



We know that the angles at the centre of a circle are proportional to the lengths of the arcs subtending them.

$$\therefore \quad \frac{\angle AOB}{\angle AOC} = \frac{\text{arc } AB}{\text{arc } AC} = \frac{l}{r}$$

$$\Rightarrow \quad \frac{\theta}{1} = \frac{l}{r}$$

$$\Rightarrow \theta = \frac{l}{r}\cdot$$

RELATION BETWEEN RADIAN AND DEGREE We know that a complete circle subtends at its centre an angle whose measure is 2π radians as well as $360°$.

$\therefore \quad (2\pi)^c = 360°$.

Hence, $\pi^c = 180°$.

TWO IMPORTANT CONVERSIONS

(i) $\pi^c = 180° \Rightarrow 1^c = \left(\frac{180}{\pi}\right)^{\circ} = \left(\frac{180\times 7}{22}\right)^{\circ} = 57°16'\,21''$ (approx.)

(ii) $180° = \pi^c \Rightarrow 1° = \left(\frac{\pi}{180}\right)^c = \left(\frac{22}{7}\times\frac{1}{180}\right)^c = (0.01746)^c$.

Degrees	30°	45°	60°	90°	180°	270°	360°
Radians	$\pi/6$	$\pi/4$	$\pi/3$	$\pi/2$	π	$3\pi/2$	2π

REMARK We express θ^c by the symbol θ only.

SUMMARY

(i) In the sexagesimal system, we measure angles in degrees, minutes and seconds.
1 right angle = 90°, 1° = 60′ and 1′ = 60″.

(ii) In the circular measure, we measure angles in radians. Thus,

$$\pi^c = 180°.$$

(iii) If an arc of length l makes an angle θ^c at the centre of a circle of radius r, we have

$$\theta = \frac{l}{r}.$$

SOLVED EXAMPLES

EXAMPLE 1 *Find the degree measure corresponding to each of the following radian measures:*

(i) $\left(\frac{7\pi}{12}\right)^c$ *(ii)* $\left(\frac{3}{4}\right)^c$ *(iii)* $(-2)^c$

SOLUTION (i) $\pi^c = 180° \Rightarrow 1^c = \left(\frac{180}{\pi}\right)^\circ$

$$\Rightarrow \left(\frac{7\pi}{12}\right)^c = \left(\frac{180}{\pi} \times \frac{7\pi}{12}\right)^\circ = 105°.$$

Hence, $\left(\frac{7\pi}{12}\right)^c = 105°$.

(ii) $\pi^c = 180° \Rightarrow 1^c = \left(\frac{180}{\pi}\right)^\circ$

$$\Rightarrow \left(\frac{3}{4}\right)^c = \left(\frac{180}{\pi} \times \frac{3}{4}\right)^\circ = \left(180 \times \frac{7}{22} \times \frac{3}{4}\right)^\circ$$
$$= 42° \, 57' \, 16''.$$

$\therefore \left(\frac{3}{4}\right)^c = 42° \, 57' \, 16''$.

(iii) $\pi^c = 180° \Rightarrow 1^c = \left(\frac{180}{\pi}\right)^\circ$

$$\Rightarrow (-2)^c = \left\{\frac{180}{\pi} \times (-2)\right\}^\circ = \left\{180 \times \frac{7}{22} \times (-2)\right\}^\circ$$
$$= -(114° \, 32' \, 44'').$$

$\therefore (-2)^c = -(114° \, 32' \, 44'')$.

EXAMPLE 2 *Find the radian measure corresponding to each of the following degree measures:*

(i) 15° *(ii)* 240° *(iii)* −37°30′

SOLUTION (i) $180° = \pi^c \Rightarrow 1° = \left(\frac{\pi}{180}\right)^c$

$$\Rightarrow 15° = \left(\frac{\pi}{180} \times 15\right)^c = \left(\frac{\pi}{12}\right)^c.$$

Hence, $15° = \left(\frac{\pi}{12}\right)^c$.

(ii) $180° = \pi^c \Rightarrow 1° = \left(\frac{\pi}{180}\right)^c$

$$\Rightarrow 240° = \left(\frac{\pi}{180} \times 240\right)^c = \left(\frac{4\pi}{3}\right)^c.$$

Hence, $240° = \left(\frac{4\pi}{3}\right)^c$.

(iii) $180° = \pi^c \Rightarrow 1° = \left(\frac{\pi}{180}\right)^c$

$$\Rightarrow -37°\,30' = \left\{\frac{\pi}{180} \times \left(\frac{-75}{2}\right)\right\}^c$$

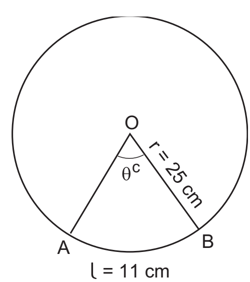
$$\left[-37°\,30' = -\left(37\frac{1}{2}\right)^\circ = -\left(\frac{75}{2}\right)^\circ\right]$$

$$= \left(\frac{-5\pi}{24}\right)^c.$$

Hence, $(-37°30') = \left(\frac{-5\pi}{24}\right)^c$.

EXAMPLE 3 *Find in degrees the angle subtended at the centre of a circle of diameter* 50 cm *by an arc of length* 11 cm.

SOLUTION Here, $r = 25$ cm and $l = 11$ cm.
Let the measure of the required angle be θ^c.



Then, $\theta^c = \left(\frac{l}{r}\right)^c = \left(\frac{11}{25}\right)^c$

$$= \left(\frac{11}{25} \times \frac{180}{\pi}\right)^\circ \quad [\because \pi^c = 180°]$$

$$= \left(\frac{11}{25} \times \frac{7}{22} \times 180\right)^\circ$$

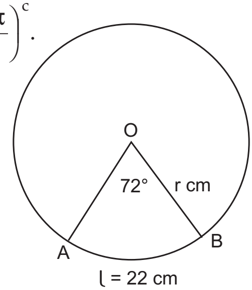
$$= \left(\frac{126}{5}\right)^\circ = 25°\,12'.$$

Hence, the required angle is $25°\,12'$.

EXAMPLE 4 *Find the radius of a circle in which a central angle of* 72° *intercepts an arc of length* 22 cm. (Use $\pi = 22/7$.)

SOLUTION Let the required radius be r cm. Then,

$$l = 22 \text{ cm and } \theta = 72° = \left(72 \times \frac{\pi}{180}\right)^c = \left(\frac{2\pi}{5}\right)^c.$$



$$\text{Now, } \theta = \frac{l}{r} \Rightarrow r = \frac{l}{\theta} = \left(22 \times \frac{5}{2\pi}\right) \text{ cm}$$

$$= \left(22 \times \frac{5}{2} \times \frac{7}{22}\right) \text{ cm}$$

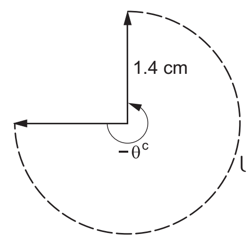
$$= \frac{35}{2} \text{ cm} = 17.5 \text{ cm}.$$

Hence, the radius of the circle is 17.5 cm.

EXAMPLE 5 *The minute hand of a watch is* 1.4 cm *long. How far does its tip move in* 45 *minutes?* (Use $\pi = 22/7$.)

SOLUTION In 60 minutes, the minute hand moves through $(2\pi)^c$.

In 45 minutes, the minute hand moves through $\left(\frac{2\pi}{60} \times 45\right)^c = \left(\frac{3\pi}{2}\right)^c.$



$\therefore \quad r = 1.4 \text{ cm and } \theta = \left(\frac{3\pi}{2}\right)^c.$

Distance moved by the tip of the minute hand in 45 minutes is given by

$$l = r\theta = \left(1.4 \times \frac{3\pi}{2}\right) \text{ cm} = \left(1.4 \times \frac{3}{2} \times \frac{22}{7}\right) \text{ cm} = 6.6 \text{ cm}.$$

EXAMPLE 6 *If the arcs of the same length in two circles subtend angles of* 60° *and* 75° *at their respective centres, find the ratio of their radii.*

SOLUTION Let r_1 and r_2 be the radii of the two circles. Then,

$$\theta_1 = 60° = \left(60 \times \frac{\pi}{180}\right)^c = \left(\frac{\pi}{3}\right)^c,$$

$$\text{and } \theta_2 = 75° = \left(75 \times \frac{\pi}{180}\right)^c = \left(\frac{5\pi}{12}\right)^c.$$

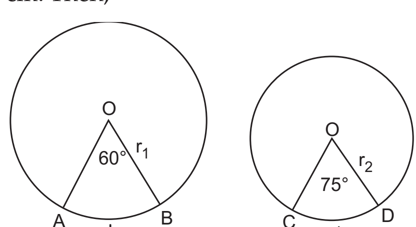
Let the length of each arc be l cm. Then,

$$l = r_1\theta_1 = r_2\theta_2$$

$$\Rightarrow \left(r_1 \times \frac{\pi}{3}\right) = \left(r_2 \times \frac{5\pi}{12}\right)$$

$$\Rightarrow \frac{r_1}{r_2} = \frac{5}{4}.$$

Hence, $r_1 : r_2 = 5 : 4$.



EXAMPLE 7 *The angles of a triangle are in AP and the ratio of the number of degrees in the least to the number of radians in the greatest is* $60 : \pi$. *Find the angles in degrees and radians.*

SOLUTION Let the angles of the triangle be $(a-d)^\circ$, a° and $(a+d)^\circ$. Then,

$$(a-d) + a + (a+d) = 180 \Rightarrow 3a = 180 \Rightarrow a = 60.$$

Thus, the angles are $(60-d)^\circ$, 60° and $(60+d)^\circ$.

Number of degrees in the least angle $= (60-d)$.

Number of radians in the greatest angle $= \left\{(60+d) \times \frac{\pi}{180}\right\}$.

$$\frac{(60-d)}{(60+d) \times \frac{\pi}{180}} = \frac{60}{\pi} \Leftrightarrow (60-d) \times \pi = 60 \times (60+d) \times \frac{\pi}{180}$$

$$\Leftrightarrow 3(60-d) = (60+d)$$

$$\Leftrightarrow 4d = 120 \Leftrightarrow d = 30.$$

$\therefore$ the required angles are $(60-30)^\circ$, 60° and $(60+30)^\circ$, i.e., 30°, 60° and 90°.

These angles in radians are

$$\left(30 \times \frac{\pi}{180}\right)^c, \left(60 \times \frac{\pi}{180}\right)^c, \left(90 \times \frac{\pi}{180}\right)^c, \text{i.e.,} \left(\frac{\pi}{6}\right)^c, \left(\frac{\pi}{3}\right)^c, \left(\frac{\pi}{2}\right)^c.$$

EXAMPLE 8 *In a right-angled triangle, the difference between the two acute angles is* $(\pi/15)^c$. *Find the angles in degrees.*

SOLUTION Clearly, the sum of the two acute angles of a right triangle is 90°.

Difference between the acute angles $= \left(\frac{\pi}{15}\right)^c = \left(\frac{\pi}{15} \times \frac{180}{\pi}\right)^\circ = 12^\circ$.

Let the two acute angles be x° and y°. Then,

$$x + y = 90 \quad \text{... (i)}$$

$$x - y = 12 \quad \text{... (ii)}$$

Solving (i) and (ii), we get $x = 51$ and $y = 39$.

Hence, the angles of the triangle are 51°, 39° and 90°.

EXAMPLE 9 *A wheel makes* 360 *revolutions in* 1 *minute. Through how many radians does it turn in* 1 *second?*

SOLUTION Number of revolutions made in 60 seconds = 360.

Number of revolutions made in 1 second $= \frac{360}{60} = 6$.

Angle moved in 1 revolution $= (2\pi)^c$.

Angle moved in 6 revolutions $= (2\pi \times 6)^c = (12\pi)^c$.

EXAMPLE 10 *Find the angle between the minute hand and the hour hand of a clock when the time is* 7.20.

SOLUTION Angle traced by the hour hand in 12 hours = 360°.

Angle traced by it in 7 h 20 min, i.e., in $\frac{22}{3}$ hours $= \left(\frac{360}{12} \times \frac{22}{3}\right)^\circ = 220°$.

Angle traced by the minute hand in 60 min = 360°.

Angle traced by the minute hand in 20 min $\left(\frac{360}{60} \times 20\right)^\circ = 120°$.

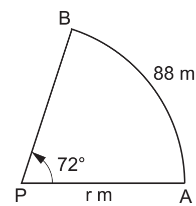
Hence, the required angle between the two hands

$$= (220° - 120°) = 100°.$$

EXAMPLE 11 *A horse is tied to a post by a rope. If the horse moves along a circular path, always keeping the rope tight, and describes* 88 *metres when it traces* 72° *at the centre, find the length of the rope.*

SOLUTION Let us denote the post by P and let PA be the length of the rope in the tightest position. Suppose that the horse moves along the arc AB so that $\angle APB = 72°$ and arc $AB = 88$ m.

Let the length of the rope PA be r metres.



Then, $\theta = 72° = \left(72 \times \frac{\pi}{180}\right)^c = \left(\frac{2\pi}{5}\right)^c$ and $l = 88$ m.

$\therefore \quad r = \frac{l}{\theta} = \frac{88}{(2\pi/5)}$ m $= \left(88 \times \frac{5}{2} \times \frac{7}{22}\right)$ m $= 70$ m.

Hence, the length of the rope is 70 m.

EXERCISE 14

1. Using a protractor, draw each of the following angles.

(i) 60° (ii) 130° (iii) 300° (iv) 430°

(v) –40° (vi) –220° (vii) –310° (viii) –400°

2. Express each of the following angles in radians.

(i) 36° (ii) 120° (iii) 225° (iv) 330°

(v) 400° (vi) 7° 30′ (vii) –270° (viii) –(22° 30′)

3. Express each of the following angles in degrees.

(i) $\left(\frac{5\pi}{12}\right)^c$ (ii) $-\left(\frac{18\pi}{5}\right)^c$ (iii) $\left(\frac{5}{6}\right)^c$ (iv) -4^c

4. The angles of a triangle are in AP and the greatest angle is double the least. Find all the angles in degrees and radians.

5. The difference between the two acute angles of a right triangle is $(\pi/5)^c$. Find these angles in radians and degrees.

6. Find the radius of a circle in which a central angle of 45° intercepts an arc of length 33 cm. (Take $\pi = 22/7$.)

7. Find the length of an arc of a circle of radius 14 cm which subtends an angle of 36° at the centre.

8. If the arcs of the same length in two circles subtend angles $75°$ and $120°$ at the centre, find the ratio of their radii.

9. Find the degree measure of the angle subtended at the centre of a circle of diameter 60 cm by an arc of length 16.5 cm.

10. In a circle of diameter 30 cm, the length of a chord is 15 cm. Find the length of the minor arc of the chord.

11. Find the angle in radians as well as in degrees through which a pendulum swings if its length is 45 cm and its tip describes an arc of length 11 cm.

12. The large hand of a clock is 42 cm long. How many centimetres does its extremity move in 20 minutes?

13. A wheel makes 180 revolutions in 1 minute. Through how many radians does it turn in 1 second?

14. A train is moving on a circular curve of radius 1500 m at the rate of 66 km per hour. Through what angle has it turned in 10 seconds?

15. A wire of length 121 cm is bent so as to lie along the arc of a circle of radius 180 cm. Find in degrees, the angle subtended at the centre by the arc.

16. The angles of a quadrilateral are in AP, and the greatest angle is double the least. Express the least angle in radians.

ANSWERS (EXERCISE 14)

2. (i) $\left(\frac{\pi}{5}\right)^c$ (ii) $\left(\frac{2\pi}{3}\right)^c$ (iii) $\left(\frac{5\pi}{4}\right)^c$ (iv) $\left(\frac{11\pi}{6}\right)^c$ (v) $\left(\frac{20\pi}{9}\right)^c$ (vi) $\left(\frac{\pi}{24}\right)^c$ (vii) $-\left(\frac{3\pi}{2}\right)^c$ (viii) $-\left(\frac{\pi}{8}\right)^c$

3. (i) $75°$ (ii) $-648°$ (iii) $47°\,43'\,38''$ (iv) $-(229°\,5'\,27'')$

4. $(40°, 60°, 80°)$, $\left\{\left(\frac{2\pi}{9}\right)^c, \left(\frac{\pi}{3}\right)^c, \left(\frac{4\pi}{9}\right)^c\right\}$ **5.** $(63°, 27°)$, $\left\{\left(\frac{7\pi}{20}\right)^c, \left(\frac{3\pi}{20}\right)^c\right\}$

6. 42 cm **7.** 8.8 cm **8.** 8 : 5 **9.** $31°\,30'$ **10.** 15.71 cm

11. $\left(\frac{11}{45}\right)^c$, $14°$ **12.** 88 cm **13.** $(6\pi)^c$ **14.** $7°$ **15.** $38°\,30'$ **16.** $\left(\frac{\pi}{3}\right)^c$

HINTS TO SOME SELECTED QUESTIONS

4. Let the required angles be $(a-d)°$, $a°$ and $(a+d)°$.

Then, the sum of the angles of a triangle being $180°$, we have

$$(a-d)+a+(a+d)=180 \Leftrightarrow 3a=180 \Leftrightarrow a=60.$$

Also, $(a+d)=2(a-d) \Leftrightarrow a=3d$

$$\Leftrightarrow 3d=60 \quad [\because a=60]$$

$$\Leftrightarrow d=20.$$

Hence, the angles in degrees are 40°, 60°, 80°.

These angles in radians are $\left(40\times\frac{\pi}{180}\right)^c$, $\left(60\times\frac{\pi}{180}\right)^c$ and $\left(80\times\frac{\pi}{180}\right)^c$, i.e., $\left(\frac{2\pi}{9}\right)^c$, $\left(\frac{\pi}{3}\right)^c$ and $\left(\frac{4\pi}{9}\right)^c$.

5. Let the acute angles of the given triangle be x^c and y^c. Then,

$$x+y=\frac{\pi}{2} \qquad \text{... (i)}$$

$$x-y=\frac{\pi}{5} \qquad \text{... (ii)}$$

Solving (i) and (ii), we get $x=\frac{7\pi}{20}$ and $y=\frac{3\pi}{20}$.

Thus, the required angles are $\left(\frac{7\pi}{20}\right)^c$ and $\left(\frac{3\pi}{20}\right)^c$.

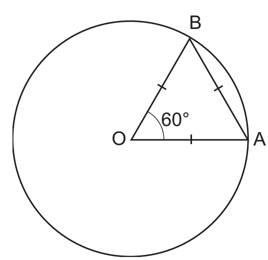
These angles in degrees are $\left(\frac{7\pi}{20}\times\frac{180}{\pi}\right)^\circ$ and $\left(\frac{3\pi}{20}\times\frac{180}{\pi}\right)^\circ$, i.e., 63° and 27°.

10. Let O be the centre of the given circle of radius 15 cm and let AB be the chord of length 15 cm. Then, $OA=OB=AB=15$ cm.

$\therefore$ $\triangle AOB$ is an equilateral triangle.

And, therefore, $\theta=\angle AOB=60^\circ$

$$=\left(60\times\frac{\pi}{180}\right)^c=\left(\frac{\pi}{3}\right)^c.$$



Let the minor arc $AB=l$ cm. Then,

$\theta=\left(\frac{\pi}{3}\right)^c$ and $r=15$ cm.

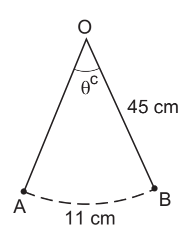
$\therefore$ $\theta=\frac{l}{r} \Rightarrow l=r\theta=\left(15\times\frac{\pi}{3}\right)$ cm

$$=\left(5\times\frac{22}{7}\right)\text{ cm}=\frac{110}{7}\text{ cm}=15.71\text{ cm}.$$

11. Here, $r=45$ cm and $l=11$ cm.

$\therefore$ Let the angle be θ^c. Then,

$$\theta=\frac{l}{r}=\left(\frac{11}{45}\right)^c$$



Now, $\pi^c=180^\circ$

$$\Rightarrow \left(\frac{11}{45}\right)^c=\left(\frac{180}{\pi}\times\frac{11}{45}\right)^\circ$$

$$=\left(180\times\frac{7}{22}\times\frac{11}{45}\right)^\circ=14^\circ.$$

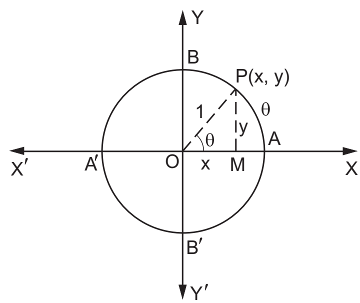
❑

15

Trigonometric, or Circular, Functions

TRIGONOMETRIC (OR CIRCULAR) FUNCTIONS

Let $X'OX$ and YOY' be the coordinate axes. Taking O as the centre and a unit radius, draw a circle, cutting the coordinate axes at A, B, A' and B', as shown in the figure.



Suppose that a moving point starts from A and moves along the circumference of the circle in an anticlockwise direction. Let it cover an arc length θ and take the final position $P(x, y)$. Join OP. Then,

$\angle AOP = \theta$

$$\left[\because \ \angle AOP = \frac{\text{arc } AP}{\text{radius } OP} = \frac{\theta}{1} = \theta^c, \text{ using } \theta = \frac{l}{r}\right].$$

Now, the six circular functions may be defined as under:

(i) $\cos\theta = x$ (ii) $\sin\theta = y$

(iii) $\sec\theta = \frac{1}{x}$, $x \neq 0$ (iv) $\operatorname{cosec}\theta = \frac{1}{y}$, $y \neq 0$

(v) $\tan\theta = \frac{y}{x}$, $x \neq 0$ (vi) $\cot\theta = \frac{x}{y}$, $y \neq 0$

REMARK Clearly, we have

(i) $\sec\theta = \dfrac{1}{\cos\theta}$ (ii) $\operatorname{cosec}\theta = \dfrac{1}{\sin\theta}$

(iii) $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$ (iv) $\cot\theta = \dfrac{1}{\tan\theta} = \dfrac{\cos\theta}{\sin\theta}$

THEOREM 1 *For any acute angle, say* θ, *we have*

(i) $\cos^2\theta + \sin^2\theta = 1$ *(ii)* $1 + \tan^2\theta = \sec^2\theta$

(iii) $1 + \cot^2\theta = \operatorname{cosec}^2\theta$

PROOF Let $X'OX$ and YOY' be the coordinate axes. Taking O as the centre and a unit radius, draw a circle, meeting OX at A.

Let $P(x, y)$ be a point on the circle with $\angle AOP = \theta$. Join OP.

Draw $PM \perp OA$.

Then, $\cos\theta = x$ and $\sin\theta = y$.

(i) From right $\triangle OMP$, we have

$$OM^2 + PM^2 = OP^2$$

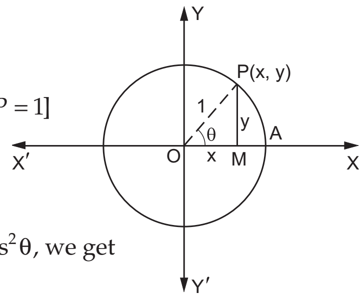
$\Rightarrow \quad x^2 + y^2 = 1$

$[\because OM = x,\ PM = y \text{ and } OP = 1]$

$\Rightarrow \quad \cos^2\theta + \sin^2\theta = 1.\ \ldots \text{(i)}$

$[\because x = \cos\theta,\ y = \sin\theta]$

This proves (i).



(ii) Dividing both sides of (i) by $\cos^2\theta$, we get

$$1 + \frac{\sin^2\theta}{\cos^2\theta} = \frac{1}{\cos^2\theta}$$

$$\Rightarrow \quad 1 + \tan^2\theta = \sec^2\theta \qquad \left[\because \frac{\sin\theta}{\cos\theta} = \tan\theta \text{ and } \frac{1}{\cos\theta} = \sec\theta\right].$$

(iii) Dividing both sides of (i) by $\sin^2\theta$, we get

$$\frac{\cos^2\theta}{\sin^2\theta} + 1 = \frac{1}{\sin^2\theta}$$

$$\Rightarrow \quad \cot^2\theta + 1 = \operatorname{cosec}^2\theta \qquad \left[\because \frac{\cos\theta}{\sin\theta} = \cot\theta \text{ and } \frac{1}{\sin\theta} = \operatorname{cosec}\theta\right]$$

$$\Rightarrow \quad 1 + \cot^2\theta = \operatorname{cosec}^2\theta.$$

NEGATIVE ARC LENGTH *If a point moves in a circle then the arc length covered by it is said to be positive or negative depending on whether the point moves in the anticlockwise or clockwise direction respectively.*

THEOREM 2 *For any acute angle, say* θ, *we have*

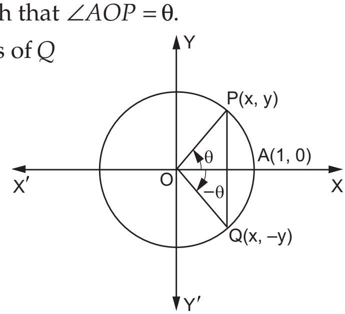
(i) $\cos(-\theta) = \cos\theta$ *(ii)* $\sin(-\theta) = -\sin\theta$ *(iii)* $\tan(-\theta) = -\tan\theta$

(iv) $\cot(-\theta) = -\cot\theta$ *(v)* $\sec(-\theta) = \sec\theta$ *(vi)* $\operatorname{cosec}(-\theta) = -\operatorname{cosec}\theta$

PROOF Let $X'OX$ and YOY' be the coordinate axes. With O as the centre, draw a circle of unit radius, meeting OX at $A(1, 0)$.

Let $P(x, y)$ be a point on the circle such that $\angle AOP = \theta$.

Let $\angle AOQ = -\theta$. Then, the coordinates of Q are $Q(x, -y)$.



$\therefore$ (i) $\cos(-\theta) = x = \cos\theta$

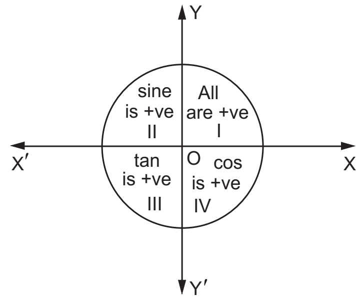
(ii) $\sin(-\theta) = -y = -\sin\theta$

(iii) $$\tan(-\theta) = \frac{\sin(-\theta)}{\cos(-\theta)} = \frac{-\sin\theta}{\cos\theta} = -\tan\theta$$

Working in this order, you can prove the remaining identities by yourself.

Signs of Trigonometric Functions in Various Quadrants

Quadrant	Signs of various T-functions
I	All T-functions are positive.
II	sin θ and cosec θ are positive. All others are negative.
III	tan θ and cot θ are positive. All others are negative.
IV	cos θ and sec θ are positive. All others are negative.



Remember

I	II	III	IV
All are +ve	sin is +ve	tan is +ve	cos is +ve

Values of T-functions of Some Particular Angles

$\theta \rightarrow$	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
$\sin\theta$	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1	0	–1	0
$\cos\theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0	–1	0	1
$\tan\theta$	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	not defined	0	not defined	0

Domain and Range of Trigonometric Functions

The domain and range of each one of the six trigonometric functions is given below.

T-function	Domain	Range
$\sin x$	R	$[-1, 1]$
$\cos x$	R	$[-1, 1]$
$\tan x$	$\{x \in R : x \neq (2n+1)\,\pi/2,\ n \in I\}$	R
$\cot x$	$\{x \in R : x \neq n\pi,\ n \in I\}$	R
$\operatorname{cosec} x$	$\{x \in R : x \neq n\pi,\ n \in I\}$	$R -]-1, 1[$
$\sec x$	$\{x \in R : x \neq (2n+1)\,\pi/2,\ n \in I\}$	$R -]-1, 1[$

PERIODIC FUNCTIONS *A function $f(x)$ is said to be periodic if there exists a constant real number p such that $f(x+p) = f(x)$ for all x.*

The least positive value of p for which $f(x+p) = f(x)$ is called the *period* of $f(x)$.

AN IMPORTANT RESULT If $f(x)$ is a periodic function with period p then $f(ax+b)$ with $a > 0$ is a periodic function with period (p/a).

THEOREM 3 $\sin(\theta + 2\pi) = \sin\theta$ *and* $\cos(\theta + 2\pi) = \cos\theta$.

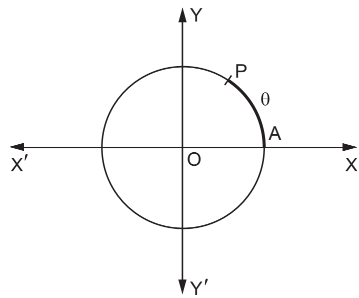
PROOF Let $X'OX$ and YOY' be the coordinate axes. With O as the centre, draw a circle of unit radius, meeting OX at A.

The circumference of this circle of unit radius is 2π.

Let P be a point on this circle such that arc $AP = \theta$.

If we start from the point A, move along the circle and after making a complete revolution, reach P then the length of the arc covered is $(\theta + 2\pi)$.

Since the trigonometric functions are defined in terms of the coordinates of P, we have



$$\cos(\theta + 2\pi) = \cos\theta \text{ and } \sin(\theta + 2\pi) = \sin\theta.$$

REMARKS (i) Since 2π is the least angle for which $\cos(\theta + 2\pi) = \cos\theta$ and $\sin(\theta + 2\pi) = \sin\theta$, it follows that

$\cos\theta$ *and* $\sin\theta$ *are perodic functions, each with period* 2π.

Similarly, $\sec\theta$ *and* $\operatorname{cosec}\theta$ *are periodic, each with period* 2π*; and* $\tan\theta$ *and* $\cot\theta$ *are periodic, each with period* π.

A constant function is periodic, having no period.

(ii) For any real number θ and any integer n, we have:

$$\boxed{\begin{aligned} &\sin(\theta + 2n\pi) = \sin\theta,\ \cos(\theta + 2n\pi) = \cos\theta \\ &\tan(\theta + n\pi) = \tan\theta,\ \cot(\theta + n\pi) = \cot\theta \\ &\sec(\theta + 2n\pi) = \sec\theta,\ \operatorname{cosec}(\theta + 2n\pi) = \operatorname{cosec}\theta \end{aligned}}$$

SOLVED EXAMPLES

EXAMPLE 1 *If* $\sec\theta = \frac{-13}{12}$ *and* θ *lies in the second quadrant, find the values of all the other five trigonometric functions.*

SOLUTION We know that in the second quadrant, $\sin\theta$ and $\operatorname{cosec}\theta$ are positive and all the other trigonometric functions are negative.

Now, $\sec\theta = \frac{-13}{12} \Rightarrow \cos\theta = \frac{1}{\sec\theta} = \frac{-12}{13}$.

$\therefore \quad \tan^2\theta = (\sec^2\theta - 1) = \left\{\left(\frac{-13}{12}\right)^2 - 1\right\} = \left(\frac{169}{144} - 1\right) = \frac{25}{144}$

$$\Rightarrow \quad \tan\theta = -\sqrt{\frac{25}{144}} = \frac{-5}{12}$$

$$\Rightarrow \quad \cot\theta = \frac{1}{\tan\theta} = \frac{-12}{5}.$$

Also, $\sin\theta = \tan\theta \cdot \cos\theta = \left(\frac{-5}{12}\right) \times \left(\frac{-12}{13}\right) = \frac{5}{13}$.

And, $\operatorname{cosec}\theta = \frac{1}{\sin\theta} = \frac{13}{5}$.

Hence, $\cos\theta = \frac{-12}{13}$; $\tan\theta = \frac{-5}{12}$; $\cot\theta = \frac{-12}{5}$;

$\sin\theta = \frac{5}{13}$ and $\operatorname{cosec}\theta = \frac{13}{5}$.

EXAMPLE 2 *If* $\sin\theta = \frac{-4}{5}$ *and* $\pi < \theta < \frac{3\pi}{2}$, *find the values of all the other five trigonometric functions.*

SOLUTION Clearly, θ lies in the third quadrant in which $\tan\theta$ and $\cot\theta$ are positive and all the other trigonometric functions are negative.

Now, $\sin\theta = \frac{-4}{5} \Rightarrow \operatorname{cosec}\theta = \frac{1}{\sin\theta} = \frac{-5}{4}$.

$$\therefore \quad \cot^2\theta = (\operatorname{cosec}^2\theta - 1) = \left(\frac{25}{16} - 1\right) = \frac{9}{16}$$

$$\Rightarrow \quad \cot\theta = +\sqrt{\frac{9}{16}} = \frac{3}{4}$$

$$\Rightarrow \quad \tan\theta = \frac{1}{\cot\theta} = \frac{4}{3}.$$

Also, $\cos\theta = \cot\theta \sin\theta = \left(\frac{3}{4}\right) \times \left(\frac{-4}{5}\right) = \frac{-3}{5}$.

$$\therefore \quad \sec\theta = \frac{1}{\cos\theta} = \frac{-5}{3}.$$

Hence, $\cos\theta = \frac{-3}{5}$; $\tan\theta = \frac{4}{3}$; $\cot\theta = \frac{3}{4}$;

$\sec\theta = \frac{-5}{3}$ and $\operatorname{cosec}\theta = \frac{-5}{4}$.

EXAMPLE 3 *Find the value of*

(i) $\sin\left(\frac{25\pi}{3}\right)$ *(ii)* $\cos\left(\frac{41\pi}{4}\right)$ *(iii)* $\tan\left(\frac{-16\pi}{3}\right)$

(iv) $\cot\left(\frac{29\pi}{4}\right)$ *(v)* $\sec\left(-\frac{19\pi}{3}\right)$ *(vi)* $\operatorname{cosec}\left(-\frac{33\pi}{4}\right)$

SOLUTION (i) $\sin\left(\frac{25\pi}{3}\right) = \sin\left(8\pi + \frac{\pi}{3}\right)$

$= \sin\frac{\pi}{3}$ $\quad [\because \sin(2n\pi + \theta) = \sin\theta]$

$= \frac{\sqrt{3}}{2}$.

(ii) $\cos\left(\frac{41\pi}{4}\right) = \cos\left(10\pi + \frac{\pi}{4}\right)$

$= \cos\frac{\pi}{4}$ $\quad [\because \cos(2n\pi + \theta) = \cos\theta]$

$= \frac{1}{\sqrt{2}}$.

(iii) $\tan\left(-\frac{16\pi}{3}\right) = -\tan\frac{16\pi}{3}$ $\quad [\because \tan(-\theta) = -\tan\theta]$

$= -\tan\left(5\pi + \frac{\pi}{3}\right)$

$= -\tan\frac{\pi}{3}$ $\quad [\because \tan(n\pi + \theta) = \tan\theta]$

$= -\sqrt{3}$.

(iv) $\cot\left(\frac{29\pi}{4}\right) = \cot\left(7\pi + \frac{\pi}{4}\right)$

$= \cot\frac{\pi}{4}$ $\quad [\because \cot(n\pi + \theta) = \cot\theta]$

$= 1$.

(v) $\sec\left(-\frac{19\pi}{3}\right) = \sec\left(\frac{19\pi}{3}\right)$ $\quad [\because \sec(-\theta) = \sec\theta]$

$= \sec\left(6\pi + \frac{\pi}{3}\right) = \sec\left(\frac{\pi}{3}\right)$ $\quad [\because \sec(2n\pi + \theta) = \sec\theta]$

$= 2$.

(vi) $\operatorname{cosec}\left(\frac{-33\pi}{4}\right) = -\operatorname{cosec}\frac{33\pi}{4}$ $\quad [\because \operatorname{cosec}(-\theta) = -\operatorname{cosec}\theta]$

$= -\operatorname{cosec}\left(8\pi + \frac{\pi}{4}\right)$

$= -\operatorname{cosec}\frac{\pi}{4}$ $\quad [\because \operatorname{cosec}(2n\pi + \theta) = \operatorname{cosec}\theta]$

$= -\sqrt{2}$.

EXAMPLE 4 *Find the value of*

(i) $\sin(765^\circ)$ *(ii)* $\operatorname{cosec}(-1110^\circ)$ *(iii)* $\cot(-600^\circ)$

SOLUTION (i) $180° = \pi^c$

$$\Rightarrow \quad 765° = \left(\frac{\pi}{180} \times 765\right)^c = \left(\frac{17\pi}{4}\right)^c.$$

$$\therefore \quad \sin(765°) = \sin\left(\frac{17\pi}{4}\right)$$

$$= \sin\left(4\pi + \frac{\pi}{4}\right)$$

$$= \sin\frac{\pi}{4} \quad [\because \sin(2n\pi + \theta) = \sin\theta]$$

$$= \frac{1}{\sqrt{2}}.$$

(ii) $180° = \pi^c$

$$\Rightarrow \quad (1110)° = \left(\frac{\pi}{180} \times 1110\right)^c = \left(\frac{37\pi}{6}\right)^c.$$

$$\therefore \quad \operatorname{cosec}(-1110°) = -\operatorname{cosec}(1110°) \quad [\because \operatorname{cosec}(-\theta) = -\operatorname{cosec}\theta]$$

$$= -\operatorname{cosec}\left(\frac{37\pi}{6}\right)$$

$$= -\operatorname{cosec}\left(6\pi + \frac{\pi}{6}\right)$$

$$= -\operatorname{cosec}\frac{\pi}{6} \quad [\because \operatorname{cosec}(2n\pi + \theta) = \operatorname{cosec}\theta]$$

$$= -2.$$

(iii) $180° = \pi^c$

$$\Rightarrow \quad 600° = \left(\frac{\pi}{180} \times 600\right)^c = \left(\frac{10\pi}{3}\right)^c.$$

$$\therefore \quad \cot(-600°) = -\cot 600° \quad [\because \cot(-\theta) = -\cot\theta]$$

$$= -\cot\left(\frac{10\pi}{3}\right)$$

$$= -\cot\left(3\pi + \frac{\pi}{3}\right)$$

$$= -\cot\frac{\pi}{3} \quad [\because \cot(n\pi + \theta) = \cot\theta]$$

$$= -\frac{1}{\sqrt{3}}.$$

EXAMPLE 5 *Find the value of*

(i) $\cos 15\pi$ *(ii)* $\sin 16\pi$ *(iii)* $\cos(-\pi)$

(iv) $\sin 5\pi$ *(v)* $\tan\left(\frac{5\pi}{4}\right)$ *(vi)* $\sec 6\pi$

SOLUTION (i) $\cos 15\pi = \cos(14\pi + \pi)$

$= \cos \pi$ $\quad [\because \cos(2n\pi + \theta) = \cos\theta]$

$= -1.$

(ii) $\sin 16\pi = \sin(16\pi + 0)$

$= \sin 0°$ $\quad [\because \sin(2n\pi + \theta) = \sin\theta]$

$= 0.$

(iii) $\cos(-\pi) = \cos\pi$ $\quad [\because \cos(-\theta) = \cos\theta]$

$= -1.$

(iv) $\sin 5\pi = \sin(4\pi + \pi)$

$= \sin\pi$ $\quad [\because \sin(2n\pi + \theta) = \sin\theta]$

$= 0.$

(v) $\tan\frac{5\pi}{4} = \tan\left(\pi + \frac{\pi}{4}\right)$

$= \tan\frac{\pi}{4}$ $\quad [\because \tan(n\pi + \theta) = \tan\theta]$

$= 1.$

(vi) $\sec 6\pi = \sec(6\pi + 0)$

$= \sec 0°$ $\quad [\because \sec(2n\pi + \theta) = \sec\theta]$

$= 1.$

EXAMPLE 6 *Prove that* $\sin^2\frac{\pi}{6} + \cos^2\frac{\pi}{3} - \tan^2\frac{\pi}{4} = -\frac{1}{2}\cdot$

SOLUTION LHS $= \sin^2\frac{\pi}{6} + \cos^2\frac{\pi}{3} - \tan^2\frac{\pi}{4}$

$= \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2 - 1^2$ $\quad \left[\because \sin\frac{\pi}{6} = \frac{1}{2}, \cos\frac{\pi}{3} = \frac{1}{2} \text{ and } \tan\frac{\pi}{4} = 1\right]$

$= \left(\frac{1}{4} + \frac{1}{4} - 1\right) = -\frac{1}{2} = \text{RHS}.$

EXERCISE 15A

1. If $\cos\theta = \frac{-\sqrt{3}}{2}$ and θ lies in Quadrant III, find the values of all the other five trigonometric functions.
2. If $\sin\theta = \frac{-1}{2}$ and θ lies in Quadrant IV, find the values of all the other five trigonometric functions.
3. If $\operatorname{cosec}\theta = \frac{5}{3}$ and θ lies in Quadrant II, find the values of all the other five trigonometric functions.

4. If $\sec\theta = \sqrt{2}$ and θ lies in Quadrant IV, find the values of all the other trigonometric functions.

5. If $\sin x = \dfrac{-2\sqrt{6}}{5}$ and x lies in Quadrant III, find the values of $\cos x$ and $\cot x$.

6. If $\cos x = \dfrac{-\sqrt{15}}{4}$ and $\dfrac{\pi}{2} < x < \pi$, find the value of $\sin x$.

7. If $\sec x = -2$ and $\pi < x < \dfrac{3\pi}{2}$, find the values of all the other five trigonometric functions.

8. Find the value of

(i) $\sin\left(\dfrac{31\pi}{3}\right)$ (ii) $\cos\left(\dfrac{17\pi}{2}\right)$ (iii) $\tan\left(\dfrac{-25\pi}{3}\right)$

(iv) $\cot\left(\dfrac{13\pi}{4}\right)$ (v) $\sec\left(\dfrac{-25\pi}{3}\right)$ (vi) $\operatorname{cosec}\left(\dfrac{-41\pi}{4}\right)$

9. Find the value of

(i) $\sin 405°$ (ii) $\sec(-1470°)$ (iii) $\tan(-300°)$

(iv) $\cot(585°)$ (v) $\operatorname{cosec}(-750°)$ (vi) $\cos(-2220°)$

10. Prove that

(i) $\tan^2\dfrac{\pi}{3} + 2\cos^2\dfrac{\pi}{4} + 3\sec^2\dfrac{\pi}{6} + 4\cos^2\dfrac{\pi}{2} = 8$

(ii) $\sin\dfrac{\pi}{6}\cos 0 + \sin\dfrac{\pi}{4}\cos\dfrac{\pi}{4} + \sin\dfrac{\pi}{3}\cos\dfrac{\pi}{6} = \dfrac{7}{4}$

(iii) $4\sin\dfrac{\pi}{6}\sin^2\dfrac{\pi}{3} + 3\cos\dfrac{\pi}{3}\tan\dfrac{\pi}{4} + \operatorname{cosec}^2\dfrac{\pi}{2} = 4$

ANSWERS (EXERCISE 15A)

1. $\sin\theta = -\dfrac{1}{2}$, $\tan\theta = \dfrac{1}{\sqrt{3}}$, $\cot\theta = \sqrt{3}$, $\operatorname{cosec}\theta = -2$, $\sec\theta = \dfrac{-2}{\sqrt{3}}$

2. $\cos\theta = \dfrac{\sqrt{3}}{2}$, $\tan\theta = \dfrac{-1}{\sqrt{3}}$, $\cot\theta = -\sqrt{3}$, $\sec\theta = \dfrac{2}{\sqrt{3}}$, $\operatorname{cosec}\theta = -2$

3. $\sin\theta = \dfrac{3}{5}$, $\cos\theta = \dfrac{-4}{5}$, $\tan\theta = \dfrac{-3}{4}$, $\cot\theta = \dfrac{-4}{3}$, $\sec\theta = \dfrac{-5}{4}$

4. $\sin\theta = \dfrac{-1}{\sqrt{2}}$, $\cos\theta = \dfrac{1}{\sqrt{2}}$, $\tan\theta = -1$, $\cot\theta = -1$, $\operatorname{cosec}\theta = -\sqrt{2}$

5. $\cos x = -\frac{1}{5}$, $\cot x = \frac{1}{2\sqrt{6}}$ **6.** $\frac{1}{4}$

7. $\sin x = \frac{-\sqrt{3}}{2}$, $\cos x = -\frac{1}{2}$, $\tan x = \sqrt{3}$, $\cot x = \frac{1}{\sqrt{3}}$, $\operatorname{cosec} x = \frac{-2}{\sqrt{3}}$

8. (i) $\frac{\sqrt{3}}{2}$ (ii) 0 (iii) $-\sqrt{3}$ (iv) 1 (v) 2 (vi) $-\sqrt{2}$

9. (i) $\frac{1}{\sqrt{2}}$ (ii) $\frac{2}{\sqrt{3}}$ (iii) $\sqrt{3}$ (iv) 1 (v) –2 (vi) $\frac{1}{2}$

TRIGONOMETRIC FUNCTIONS OF SUM AND DIFFERENCE OF NUMBERS

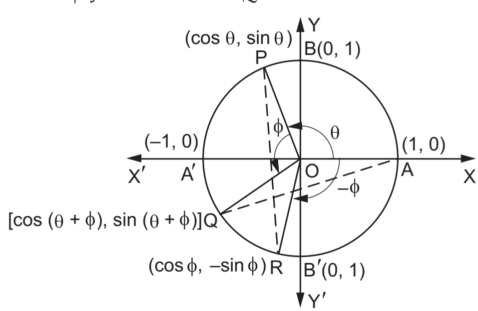
THEOREM 1 *(i)* $\cos(\theta + \phi) = \cos\theta\cos\phi - \sin\theta\sin\phi$

(ii) $\cos(\theta - \phi) = \cos\theta\cos\phi + \sin\theta\sin\phi$

PROOF Let $X'OX$ and YOY' be the coordinate axes. With O as the centre and taking a unit radius, draw a circle, cutting the axes at A, A', B and B' as shown in the figure given below.

Let $\angle AOP = \theta$ and $\angle POQ = \phi$. Then, $\angle AOQ = (\theta + \phi)$.

Let $\angle AOR = -\phi$. Join PR and AQ.



(i) The points on the circle are given by

$P(\cos\theta, \sin\theta)$, $Q[\cos(\theta+\phi), \sin(\theta+\phi)]$, $R[\cos(-\phi), \sin(-\phi)]$,

i.e., $R(\cos\phi, -\sin\phi)$, $A(1, 0)$, $B(0, 1)$, $A'(-1, 0)$ and $B'(0, 1)$.

In $\triangle POR$ and $\triangle QOA$, we have

$PO = QO = 1$ unit, $OR = OA = 1$ unit and

$\angle POR = \angle QOA$ [each equal to $(\phi + \angle QOR)$].

$\therefore$ $\triangle POR \cong \triangle QOA$.

Hence, $PR = QA$.

$\therefore$ $PR^2 = QA^2$

$$\Rightarrow (\cos\theta - \cos\phi)^2 + (\sin\theta + \sin\phi)^2$$
$$= [\cos(\theta + \phi) - 1]^2 + [\sin(\theta + \phi) - 0]^2$$

$\Rightarrow \quad (\cos^2\theta + \cos^2\phi - 2\cos\theta\cos\phi) + (\sin^2\theta + \sin^2\phi + 2\sin\theta\sin\phi)$
$= [\cos^2(\theta+\phi) + 1 - 2\cos(\theta+\phi)] + \sin^2(\theta+\phi)$

$\Rightarrow \quad (\cos^2\theta + \sin^2\theta) + (\cos^2\phi + \sin^2\phi) - 2(\cos\theta\cos\phi - \sin\theta\sin\phi)$
$= [\cos^2(\theta+\phi) + \sin^2(\theta+\phi)] + 1 - 2\cos(\theta+\phi)$

$\Rightarrow \quad 2 - 2(\cos\theta\cos\phi - \sin\theta\sin\phi) = 2 - 2\cos(\theta+\phi)$

$\Rightarrow \quad (\cos\theta\cos\phi - \sin\theta\sin\phi) = \cos(\theta+\phi)$.

Hence, $\cos(\theta+\phi) = \cos\theta\cos\phi - \sin\theta\sin\phi$. ... (1)

(ii) Replacing ϕ by $-\phi$ in identity (1), we get

$$\cos(\theta-\phi) = \cos\theta\cos(-\phi) - \sin\theta\sin(-\phi).$$

Hence, $\cos(\theta-\phi) = \cos\theta\cos\phi + \sin\theta\sin\phi$

[$\because$ $\cos(-\phi) = \cos\phi$ and $\sin(-\phi) = -\sin\phi$].

SUMMARY

(i) $\cos(\theta+\phi) = \cos\theta\cos\phi - \sin\theta\sin\phi$

(ii) $\cos(\theta-\phi) = \cos\theta\cos\phi + \sin\theta\sin\phi$

THEOREM 2 *(i)* $\cos\left(\frac{\pi}{2} - x\right) = \sin x$ *(ii)* $\left(\frac{\pi}{2} - x\right) = \cos x$

(iii) $\tan\left(\frac{\pi}{2} - x\right) = \cot x$ *(iv)* $\cot\left(\frac{\pi}{2} - x\right) = \tan x$

(v) $\sec\left(\frac{\pi}{2} - x\right) = \operatorname{cosec} x$ *(vi)* $\operatorname{cosec}\left(\frac{\pi}{2} - x\right) = \sec x$

PROOF (i) $\cos\left(\frac{\pi}{2} - x\right) = \cos\frac{\pi}{2}\cos x + \sin\frac{\pi}{2}\sin x$

$= (0 \times \cos x) + (1 \times \sin x)$ $\left[\because \cos\frac{\pi}{2} = 0 \text{ and } \sin\frac{\pi}{2} = 1\right]$

$= \sin x$.

(ii) $\sin\left(\frac{\pi}{2} - x\right) = \sin\theta$, where $\theta = \left(\frac{\pi}{2} - x\right)$

$= \cos\left(\frac{\pi}{2} - \theta\right)$ $\left[\because \cos\left(\frac{\pi}{2} - \theta\right) = \sin\theta\right]$

$= \cos\left\{\frac{\pi}{2} - \left(\frac{\pi}{2} - x\right)\right\}$

$= \cos x$.

(iii) $\tan\left(\frac{\pi}{2} - x\right) = \dfrac{\sin\left(\frac{\pi}{2} - x\right)}{\cos\left(\frac{\pi}{2} - x\right)}$

$= \dfrac{\cos x}{\sin x} = \cot x$ [using (ii) and (i)].

(iv) $\cot\left(\dfrac{\pi}{2}-x\right)=\dfrac{\cos\left(\dfrac{\pi}{2}-x\right)}{\sin\left(\dfrac{\pi}{2}-x\right)}=\dfrac{\sin x}{\cos x}=\tan x$ [using (i) and (ii)].

(v) $\sec\left(\dfrac{\pi}{2}-x\right)=\dfrac{1}{\cos\left(\dfrac{\pi}{2}-x\right)}=\dfrac{1}{\sin x}=\operatorname{cosec} x\left[\because \cos\left(\dfrac{\pi}{2}-x\right)=\sin x\right]$

$= \operatorname{cosec} x.$

(vi) $\operatorname{cosec}\left(\dfrac{\pi}{2}-x\right)=\dfrac{1}{\sin\left(\dfrac{\pi}{2}-x\right)}=\dfrac{1}{\cos x}$ $\left[\because \sin\left(\dfrac{\pi}{2}-x\right)=\cos x\right]$

$= \sec x.$

SUMMARY

(i) $\cos\left(\dfrac{\pi}{2}-x\right)=\sin x$ **(ii)** $\sin\left(\dfrac{\pi}{2}-x\right)=\cos x$

(iii) $\tan\left(\dfrac{\pi}{2}-x\right)=\cot x$ **(iv)** $\cot\left(\dfrac{\pi}{2}-x\right)=\tan x$

(v) $\sec\left(\dfrac{\pi}{2}-x\right)=\operatorname{cosec} x$ **(vi)** $\operatorname{cosec}\left(\dfrac{\pi}{2}-x\right)=\sec x$

THEOREM 3 *(i)* $\cos\left(\dfrac{\pi}{2}+x\right)=-\sin x$ *(ii)* $\sin\left(\dfrac{\pi}{2}+x\right)=\cos x$

(iii) $\tan\left(\dfrac{\pi}{2}+x\right)=-\cot x$ *(iv)* $\cot\left(\dfrac{\pi}{2}+x\right)=-\tan x$

(v) $\sec\left(\dfrac{\pi}{2}+x\right)=-\operatorname{cosec} x$ *(vi)* $\operatorname{cosec}\left(\dfrac{\pi}{2}+x\right)=\sec x$

PROOF (i) $\cos\left(\dfrac{\pi}{2}+x\right)=\cos\dfrac{\pi}{2}\cos x-\sin\dfrac{\pi}{2}\sin x$

$=(0\times\cos x)-(1\times\sin x)$ $\left[\because \cos\dfrac{\pi}{2}=0 \text{ and } \sin\dfrac{\pi}{2}=1\right]$

$=-\sin x.$

(ii) $\sin\left(\dfrac{\pi}{2}+x\right)=\sin\left[\dfrac{\pi}{2}-(-x)\right]$

$=\cos(-x)$ $\left[\because \sin\left(\dfrac{\pi}{2}-\theta\right)=\cos\theta\right]$

$=\cos x$ $[\because \cos(-x)=\cos x].$

(iii) $\tan\left(\frac{\pi}{2}+x\right) = \frac{\sin\left(\frac{\pi}{2}+x\right)}{\cos\left(\frac{\pi}{2}+x\right)}$

$= \frac{\cos x}{-\sin x}$

$\left[\because \sin\left(\frac{\pi}{2}+x\right) = \cos x \text{ and } \cos\left(\frac{\pi}{2}+x\right) = -\sin x\right]$

$= -\cot x.$

(iv) $\cot\left(\frac{\pi}{2}+x\right) = \frac{1}{\tan\left(\frac{\pi}{2}+x\right)} = \frac{1}{-\cot x}$ $\left[\because \tan\left(\frac{\pi}{2}+x\right) = -\cot x\right]$

$= -\tan x.$

(v) $\sec\left(\frac{\pi}{2}+x\right) = \frac{1}{\cos\left(\frac{\pi}{2}+x\right)} = \frac{1}{-\sin x}$ $\left[\because \cos\left(\frac{\pi}{2}+x\right) = -\sin x\right]$

$= -\operatorname{cosec} x.$

(vi) $\operatorname{cosec}\left(\frac{\pi}{2}+x\right) = \frac{1}{\sin\left(\frac{\pi}{2}+x\right)} = \frac{1}{\cos x}$ $\left[\because \sin\left(\frac{\pi}{2}+x\right) = \cos x\right]$

$= \sec x.$

SUMMARY

(i) $\cos\left(\frac{\pi}{2}+x\right) = -\sin x$ **(ii)** $\sin\left(\frac{\pi}{2}+x\right) = \cos x$

(iii) $\tan\left(\frac{\pi}{2}+x\right) = -\cot x$ **(iv)** $\cot\left(\frac{\pi}{2}+x\right) = -\tan x$

(v) $\sec\left(\frac{\pi}{2}+x\right) = -\operatorname{cosec} x$ **(vi)** $\operatorname{cosec}\left(\frac{\pi}{2}+x\right) = \sec x$

THEOREM 4 *(i)* $\cos(\pi - x) = -\cos x$ *(ii)* $\sin(\pi - x) = \sin x$

(iii) $\tan(\pi - x) = -\tan x$ *(iv)* $\cot(\pi - x) = -\cot x$

(v) $\sec(\pi - x) = -\sec x$ *(vi)* $\operatorname{cosec}(\pi - x) = \operatorname{cosec} x$

PROOF (i) $\cos(\pi - x) = \cos\pi\cos x + \sin\pi\sin x$

$= (-1\times\cos x) + (0\times\sin x)$ $[\because \cos\pi = -1 \text{ and } \sin\pi = 0]$

$= -\cos x.$

(ii) $\sin(\pi - x) = \sin\left[\frac{\pi}{2} + \left(\frac{\pi}{2} - x\right)\right]$

$= \cos\left(\frac{\pi}{2} - x\right)$ $\left[\because \sin\left(\frac{\pi}{2} + \theta\right) = \cos\theta\right]$

$= \sin x$ $\left[\because \cos\left(\frac{\pi}{2} - x\right) = \sin x\right]$.

(iii) $\tan(\pi - x) = \frac{\sin(\pi - x)}{\cos(\pi - x)}$

$= \frac{\sin x}{-\cos x}$ $[\because \sin(\pi - x) = \sin x, \cos(\pi - x) = -\cos x]$

$= -\tan x$.

(iv) $\cot(\pi - x) = \frac{1}{\tan(\pi - x)} = \frac{1}{-\tan x}$ $[\because \tan(\pi - x) = -\tan x]$

$= -\cot x$.

(v) $\sec(\pi - x) = \frac{1}{\cos(\pi - x)} = \frac{1}{-\cos x}$ $[\because \cos(\pi - x) = -\cos x]$

$= -\sec x$.

(vi) $\operatorname{cosec}(\pi - x) = \frac{1}{\sin(\pi - x)} = \frac{1}{\sin x}$ $[\because \sin(\pi - x) = \sin x]$

$= \operatorname{cosec} x$.

SUMMARY

(i) $\cos(\pi - x) = -\cos x$ **(ii) $\sin(\pi - x) = \sin x$**

(iii) $\tan(\pi - x) = -\tan x$ **(iv) $\cot(\pi - x) = -\cot x$**

(v) $\sec(\pi - x) = -\sec x$ **(vi) $\operatorname{cosec}(\pi - x) = \operatorname{cosec} x$**

THEOREM 5 *(i)* $\cos(\pi + x) = -\cos x$ *(ii)* $\sin(\pi + x) = -\sin x$
(iii) $\tan(\pi + x) = \tan x$ *(iv)* $\cot(\pi + x) = \cot x$
(v) $\sec(\pi + x) = -\sec x$ *(vi)* $\operatorname{cosec}(\pi + x) = -\operatorname{cosec} x$

PROOF (i) $\cos(\pi + x) = \cos\pi\cos x - \sin\pi\sin x$

$= (-1)\cdot\cos x - (0 \times \sin x)$ $[\because \cos\pi = -1 \text{ and } \sin\pi = 0]$

$= -\cos x$.

(ii) $\sin(\pi + x) = \sin\left[\frac{\pi}{2} + \left(\frac{\pi}{2} + x\right)\right]$

$= \cos\left(\frac{\pi}{2} + x\right)$ $\left[\because \sin\left(\frac{\pi}{2} + \theta\right) = \cos\theta\right]$

$= -\sin x$.

(iii) $\tan(\pi + x) = \dfrac{\sin(\pi + x)}{\cos(\pi + x)} = \dfrac{-\sin x}{-\cos x}$

$= \tan x.$ $[\because \sin(\pi + x) = -\sin x, \cos(\pi + x) = -\cos x]$

(iv) $\cot(\pi + x) = \dfrac{1}{\tan(\pi + x)} = \dfrac{1}{\tan x}$ $[\because \tan(\pi + x) = \tan x]$

$= \cot x.$

(v) $\sec(\pi + x) = \dfrac{1}{\cos(\pi + x)} = \dfrac{1}{-\cos x}$ $[\because \cos(\pi + x) = -\cos x]$

$= -\sec x.$

(vi) $\operatorname{cosec}(\pi + x) = \dfrac{1}{\sin(\pi + x)} = \dfrac{1}{-\sin x}$ $[\because \sin(\pi + x) = -\sin x]$

$= -\operatorname{cosec} x.$

SUMMARY

(i) $\cos(\pi + x) = -\cos x$ **(ii)** $\sin(\pi + x) = -\sin x$

(iii) $\tan(\pi + x) = \tan x$ **(iv)** $\cot(\pi + x) = \cot x$

(v) $\sec(\pi + x) = -\sec x$ **(vi)** $\operatorname{cosec}(\pi + x) = -\operatorname{cosec} x$

THEOREM 6 *(i)* $\cos(2\pi - x) = \cos x$ *(ii)* $\sin(2\pi - x) = -\sin x$
(iii) $\tan(2\pi - x) = -\tan x$ *(iv)* $\cot(2\pi - x) = -\cot x$
(v) $\sec(2\pi - x) = \sec x$ *(vi)* $\operatorname{cosec}(2\pi - x) = -\operatorname{cosec} x$

PROOF (i) $\cos(2\pi - x) = \cos 2\pi \cos x + \sin 2\pi \sin x$

$= (1 \times \cos x) + (0 \times \sin x)$ $[\because \cos 2\pi = 1 \text{ and } \sin 2\pi = 0]$

$= \cos x.$

(ii) $\sin(2\pi - x) = \sin[\pi + (\pi + x)]$

$= -\sin(\pi + x)$ $[\because \sin(\pi + \theta) = -\sin\theta]$

$= -[-\sin x] = \sin x.$

The remaining parts are left to the student as an exercise.

SUMMARY

(i) $\cos(2\pi - x) = \cos x$ **(ii)** $\sin(2\pi - x) = -\sin x$

(iii) $\tan(2\pi - x) = -\tan x$ **(iv)** $\cot(2\pi - x) = -\cot x$

(v) $\sec(2\pi - x) = \sec x$ **(vi)** $\operatorname{cosec}(2\pi - x) = -\operatorname{cosec} x$

Further, it is easy to verify that

(i) $\cos(2\pi + x) = \cos x$ **(ii)** $\sin(2\pi + x) = \sin x$

(iii) $\tan(2\pi + x) = \tan x$ **(iv)** $\cot(2\pi + x) = \cot x$

(v) $\sec(2\pi + x) = \sec x$ **(vi)** $\operatorname{cosec}(2\pi + x) = \operatorname{cosec} x$

Also, we have

(i) $\cos\left(\frac{3\pi}{2} - x\right) = -\sin x$	**(ii)** $\sin\left(\frac{3\pi}{2} - x\right) = -\cos x$
(iii) $\cos\left(\frac{3\pi}{2} + x\right) = \sin x$	**(iv)** $\sin\left(\frac{3\pi}{2} + x\right) = -\cos x$

Summarised Table

	$(-x)$	$\left(\frac{\pi}{2} - x\right)$	$\left(\frac{\pi}{2} + x\right)$	$(\pi - x)$	$(\pi + x)$	$\left(\frac{3\pi}{2} - x\right)$	$\left(\frac{3\pi}{2} + x\right)$
sin	$-\sin x$	$\cos x$	$\cos x$	$\sin x$	$-\sin x$	$-\cos x$	$-\cos x$
cos	$\cos x$	$\sin x$	$-\sin x$	$-\cos x$	$-\cos x$	$-\sin x$	$\sin x$
tan	$-\tan x$	$\cot x$	$-\cot x$	$-\tan x$	$\tan x$	$\cot x$	$-\cot x$

REMARKS (i) The T-ratios for $(2\pi - x)$ and $(2n\pi - x)$ are the same as those for $(-x)$.
(ii) The T-ratios for $(2\pi + x)$ and $(2n\pi + x)$ are the same as those for x.

THEOREM 7 *(i)* $\sin(\theta + \phi) = \sin\theta\cos\phi + \cos\theta\sin\phi$
(ii) $\sin(\theta - \phi) = \sin\theta\cos\phi - \cos\theta\sin\phi$

PROOF (i) $\sin(\theta + \phi) = \cos\left[\frac{\pi}{2} - (\theta + \phi)\right]$

$$= \cos\left[\left(\frac{\pi}{2} - \theta\right) - \phi\right]$$

$$= \cos\left(\frac{\pi}{2} - \theta\right)\cos\phi + \sin\left(\frac{\pi}{2} - \theta\right)\sin\phi$$

$$= \sin\theta\cos\phi + \cos\theta\sin\phi.$$

Hence, $\sin(\theta + \phi) = \sin\theta\cos\phi + \cos\theta\sin\phi$.

(ii) Replacing ϕ by $-\phi$ in (i), we get

$$\sin[\theta + (-\phi)] = \sin\theta\cos(-\phi) + \cos\theta\sin(-\phi)$$

$\Rightarrow \sin(\theta - \phi) = \sin\theta\cos\phi - \cos\theta\sin\phi$

[$\because$ $\cos(-\phi) = \cos\phi$, $\sin(-\phi) = -\sin\phi$].

Hence, $\sin(\theta - \phi) = \sin\theta\cos\phi - \cos\theta\sin\phi$.

SUMMARY

(i) $\sin(\theta + \phi) = \sin\theta\cos\phi + \cos\theta\sin\phi$

(ii) $\sin(\theta - \phi) = \sin\theta\cos\phi - \cos\theta\sin\phi$

THEOREM 8 *If none of the angles* θ, ϕ *and* $(\theta+\phi)$ *is an odd multiple of* $\pi/2$ *then*

(i) $\tan(\theta+\phi)=\dfrac{\tan\theta+\tan\phi}{1-\tan\theta\tan\phi}$

(ii) $\tan(\theta-\phi)=\dfrac{\tan\theta-\tan\phi}{1+\tan\theta\tan\phi}$

PROOF (i) $$\tan(\theta+\phi)=\frac{\sin(\theta+\phi)}{\cos(\theta+\phi)}$$

$$=\frac{(\sin\theta\cos\phi+\cos\theta\sin\phi)}{(\cos\theta\cos\phi-\sin\theta\sin\phi)}$$

$$=\frac{\left(\dfrac{\sin\theta\cos\phi}{\cos\theta\cos\phi}+\dfrac{\cos\theta\sin\phi}{\cos\theta\cos\phi}\right)}{\left(\dfrac{\cos\theta\cos\phi}{\cos\theta\cos\phi}-\dfrac{\sin\theta\sin\phi}{\cos\theta\cos\phi}\right)}$$

[dividing num. and denom. by $\cos\theta\cos\phi$]

$$=\frac{\tan\theta+\tan\phi}{1-\tan\theta\tan\phi}.$$

Hence, $\tan(\theta+\phi)=\dfrac{\tan\theta+\tan\phi}{1-\tan\theta\tan\phi}$.

(ii) Replacing ϕ by $-\phi$ in (i), we get

$$\tan[\theta+(-\phi)]=\frac{\tan\theta+\tan(-\phi)}{1-\tan\theta\tan(-\phi)}$$

$$\Rightarrow\quad \tan(\theta-\phi)=\frac{\tan\theta-\tan\phi}{1+\tan\theta\tan\phi}.$$

Hence, $\tan(\theta-\phi)=\dfrac{\tan\theta-\tan\phi}{1+\tan\theta\tan\phi}$.

THEOREM 9 *If none of the angles* θ, ϕ *and* $(\theta+\phi)$ *is a multiple of* π *then*

(i) $\cot(\theta+\phi)=\dfrac{\cot\theta\cot\phi-1}{\cot\phi+\cot\theta}$

(ii) $\cot(\theta-\phi)=\dfrac{\cot\theta\cot\phi+1}{\cot\phi-\cot\theta}$

PROOF (i) $$\cot(\theta+\phi)=\frac{\cos(\theta+\phi)}{\sin(\theta+\phi)}$$

$$=\frac{(\cos\theta\cos\phi-\sin\theta\sin\phi)}{(\sin\theta\cos\phi+\cos\theta\sin\phi)}$$

$$= \frac{\left(\frac{\cos\theta\cos\phi}{\sin\theta\sin\phi} - \frac{\sin\theta\sin\phi}{\sin\theta\sin\phi}\right)}{\left(\frac{\sin\theta\cos\phi}{\sin\theta\sin\phi} + \frac{\cos\theta\sin\phi}{\sin\theta\sin\phi}\right)}$$

[dividing num. and denom. by $\sin\theta\sin\phi$]

$$= \frac{\cot\theta\cot\phi - 1}{\cot\phi + \cot\theta}.$$

Hence, $\cot(\theta+\phi) = \dfrac{\cot\theta\cot\phi - 1}{\cot\phi + \cot\theta}$.

(ii) Replacing ϕ by $-\phi$ in (i), we get

$$\cot[\theta + (-\phi)] = \frac{\cot\theta\cot(-\phi) - 1}{\cot(-\phi) + \cot\theta}$$

$$\Rightarrow \cot(\theta-\phi) = \frac{-\cot\theta\cot\phi - 1}{-\cot\phi + \cot\theta} \quad [\because \cot(-\phi) = -\cot\phi]$$

$$\Rightarrow \cot(\theta-\phi) = \frac{\cot\theta\cot\phi + 1}{\cot\phi - \cot\theta}.$$

Hence, $\cot(\theta-\phi) = \dfrac{\cot\theta\cot\phi + 1}{\cot\phi - \cot\theta}$.

SUMMARY

(i) $\tan(\theta+\phi) = \dfrac{\tan\theta + \tan\phi}{1 - \tan\theta\tan\phi}$

(ii) $\tan(\theta-\phi) = \dfrac{\tan\theta - \tan\phi}{1 + \tan\theta\tan\phi}$

(iii) $\cot(\theta+\phi) = \dfrac{\cot\theta\cot\phi - 1}{\cot\phi + \cot\theta}$

(iv) $\cot(\theta-\phi) = \dfrac{\cot\theta\cot\phi + 1}{\cot\phi - \cot\theta}$

THEOREM 10 *For all $x, y \in R$, we have*

(i) $\sin(x+y)\sin(x-y) = \sin^2 x - \sin^2 y$

(ii) $\cos(x+y)\cos(x-y) = \cos^2 x - \sin^2 y$

PROOF (i) $\sin(x+y)\sin(x-y)$

$$= (\sin x\cos y + \cos x\sin y)(\sin x\cos y - \cos x\sin y)$$

$$= \sin^2 x\cos^2 y - \cos^2 x\sin^2 y$$

$$= \sin^2 x(1 - \sin^2 y) - (1 - \sin^2 x)\sin^2 y$$

$$= \sin^2 x - \sin^2 x\sin^2 y - \sin^2 y + \sin^2 x\sin^2 y$$

$$= \sin^2 x - \sin^2 y.$$

(ii) $\cos(x + y)\ \cos(x - y)$

$= (\cos x \cos y - \sin x \sin y)(\cos x \cos y + \sin x \sin y)$

$= \cos^2 x \cos^2 y - \sin^2 x \sin^2 y$

$= \cos^2 x(1 - \sin^2 y) - (1 - \cos^2 x)\sin^2 y$

$= \cos^2 x - \cos^2 x \sin^2 y - \sin^2 y + \cos^2 x \sin^2 y$

$= \cos^2 x - \sin^2 y.$

SOLVED EXAMPLES

EXAMPLE 1 *Find the value of*

(i) $\cos 480°$ *(ii)* $\sin 1230°$ *(iii)* $\cot(-135°)$

(iv) $\operatorname{cosec}(-1410°)$ *(v)* $\cos(-870°)$ *(vi)* $\tan 330°$

SOLUTION (i) $\cos 480° = \cos(360° + 120°)$

$= \cos 120°$ $[\because \cos(360° + \theta) = \cos\theta]$

$= \cos(180° - 60°) = -\cos 60°$ $[\because \cos(180° - \theta) = -\cos\theta]$

$= -\frac{1}{2}\cdot$

(ii) $\sin 1230° = \sin(3 \times 360° + 150°)$

$= \sin 150°$ $[\because \sin(2n\pi + \theta) = \sin\theta]$

$= \sin(180° - 30°) = \sin 30°$ $[\because \sin(180° - \theta) = \sin\theta]$

$= \frac{1}{2}\cdot$

(iii) $\cot(-135°) = -\cot 135°$ $[\because \cot(-\theta) = -\cot\theta]$

$= -\cot(90° + 45°) = \tan 45°$

$[\because \cot(90° + \theta) = -\tan\theta]$

$= 1.$

(iv) $\operatorname{cosec}(-1410°) = -\operatorname{cosec} 1410° = -\operatorname{cosec}(4 \times 360° - 30°)$

$= \operatorname{cosec} 30°$ $[\because \operatorname{cosec}(2n\pi - \theta) = -\operatorname{cosec}\theta]$

$= 2.$

(v) $\cos(-870°) = \cos 870°$ $[\because \cos(-\theta) = \cos\theta]$

$= \cos(2 \times 360° + 150°)$

$= \cos 150°$ $[\because \cos(2n\pi + \theta) = \cos\theta]$

$= \cos(180° - 30°) = -\cos 30°$

$[\because \cos(180° - \theta) = -\cos\theta]$

$= \frac{-\sqrt{3}}{2}\cdot$

(vi) $\tan 330° = \tan(360° - 30°) = -\tan 30°$

$[\because \tan(360° - \theta) = -\tan\theta]$

$= -\frac{1}{\sqrt{3}}\cdot$

EXAMPLE 2 *Find the values of all trigonometric functions of (i)* 120° *(ii)* 150°.

SOLUTION (i) $\sin 120° = \sin (180° - 60°) = \sin 60° = \dfrac{\sqrt{3}}{2}$;

$\cos 120° = \cos (180° - 60°) = -\cos 60° = \dfrac{-1}{2}$;

$\tan 120° = \dfrac{\sin 120°}{\cos 120°} = \left\{\dfrac{\sqrt{3}}{2} \times (-2)\right\} = -\sqrt{3}$;

$\operatorname{cosec} 120° = \dfrac{1}{\sin 120°} = \dfrac{2}{\sqrt{3}}$;

$\sec 120° = \dfrac{1}{\cos 120°} = -2$;

$\cot 120° = \dfrac{1}{\tan 120°} = \dfrac{-1}{\sqrt{3}}$.

(ii) $\sin 150° = \sin (180° - 30°) = \sin 30° = \dfrac{1}{2}$;

$\cos 150° = \cos (180° - 30°) = -\cos 30° = \dfrac{-\sqrt{3}}{2}$;

$\tan 150° = \dfrac{\sin 150°}{\cos 150°} = \left(\dfrac{1}{2} \times \dfrac{2}{-\sqrt{3}}\right) = \dfrac{-1}{\sqrt{3}}$;

$\cot 150° = \dfrac{1}{\tan 150°} = -\sqrt{3}$;

$\sec 150° = \dfrac{1}{\cos 150°} = \dfrac{-2}{\sqrt{3}}$;

$\operatorname{cosec} 150° = \dfrac{1}{\sin 150°} = 2$.

EXAMPLE 3 *Show that* $\sin 105° + \cos 105° = \dfrac{1}{\sqrt{2}}$.

SOLUTION We have

$$\begin{aligned}
\text{LHS} &= \sin 105° + \cos 105° \\
&= \sin (60° + 45°) + \cos (60° + 45°) \\
&= (\sin 60° \cos 45° + \cos 60° \sin 45°) \\
&\qquad + (\cos 60° \cos 45° - \sin 60° \sin 45°) \\
&= \left\{\left(\frac{\sqrt{3}}{2} \times \frac{1}{\sqrt{2}}\right) + \left(\frac{1}{2} \times \frac{1}{\sqrt{2}}\right)\right\} + \left\{\left(\frac{1}{2} \times \frac{1}{\sqrt{2}}\right) - \left(\frac{\sqrt{3}}{2} \times \frac{1}{\sqrt{2}}\right)\right\} \\
&= \left(\frac{\sqrt{3}}{2\sqrt{2}} + \frac{1}{2\sqrt{2}} + \frac{1}{2\sqrt{2}} - \frac{\sqrt{3}}{2\sqrt{2}}\right) = \frac{1}{\sqrt{2}} = \text{RHS}.
\end{aligned}$$

EXAMPLE 4 *Calculate the value of*

(i) sin 15° *(ii)* cos 15° *(iii)* tan 15°

(iv) sin 75° *(v)* cos 75° *(vi)* tan 75°

SOLUTION (i) $\sin 15° = \sin (45° - 30°)$

$$= \sin 45° \cos 30° - \cos 45° \sin 30°$$

$$= \left(\frac{1}{\sqrt{2}} \times \frac{\sqrt{3}}{2}\right) - \left(\frac{1}{\sqrt{2}} \times \frac{1}{2}\right) = \frac{(\sqrt{3}-1)}{2\sqrt{2}}.$$

(ii) $\cos 15° = \cos (45° - 30°)$

$$= \cos 45° \cos 30° + \sin 45° \sin 30°$$

$$= \left(\frac{1}{\sqrt{2}} \times \frac{\sqrt{3}}{2}\right) + \left(\frac{1}{\sqrt{2}} \times \frac{1}{2}\right) = \frac{(\sqrt{3}+1)}{2\sqrt{2}}.$$

(iii) $\tan 15° = \tan (45° - 30°)$

$$= \frac{\tan 45° - \tan 30°}{1 + \tan 45° \tan 30°} = \frac{\left(1 - \frac{1}{\sqrt{3}}\right)}{1 + \left(1 \times \frac{1}{\sqrt{3}}\right)} = \frac{(\sqrt{3}-1)}{(\sqrt{3}+1)}.$$

(iv) $\sin 75° = \sin (45° + 30°)$

$$= \sin 45° \cos 30° + \cos 45° \sin 30°$$

$$= \left(\frac{1}{\sqrt{2}} \times \frac{\sqrt{3}}{2}\right) + \left(\frac{1}{\sqrt{2}} \times \frac{1}{2}\right) = \frac{(\sqrt{3}+1)}{2\sqrt{2}}.$$

(v) $\cos 75° = \cos (45° + 30°)$

$$= \cos 45° \cos 30° - \sin 45° \sin 30°$$

$$= \left(\frac{1}{\sqrt{2}} \times \frac{\sqrt{3}}{2}\right) - \left(\frac{1}{\sqrt{2}} \times \frac{1}{2}\right)$$

$$= \frac{(\sqrt{3}-1)}{2\sqrt{2}}.$$

(vi) $\tan 75° = \tan (45° + 30°)$

$$= \frac{\tan 45° + \tan 30°}{1 - \tan 45° \tan 30°} = \frac{\left(1 + \frac{1}{\sqrt{3}}\right)}{\left(1 - \frac{1}{\sqrt{3}}\right)}$$

$$= \frac{(\sqrt{3}+1)}{(\sqrt{3}-1)}.$$

EXAMPLE 5 *Evaluate* $\tan \frac{13\pi}{12}$.

SOLUTION We have

$$\tan \frac{13\pi}{12} = \tan \left(\pi + \frac{\pi}{12}\right)$$

$$= \tan \frac{\pi}{12} \qquad [\because \tan (\pi + \theta) = \tan \theta]$$

$$= \tan \left(\frac{\pi}{3} - \frac{\pi}{4}\right)$$

$$= \frac{\tan \frac{\pi}{3} - \tan \frac{\pi}{4}}{1 + \tan \frac{\pi}{3} \tan \frac{\pi}{4}} = \frac{\sqrt{3} - 1}{1 + \sqrt{3}}$$

$$= \frac{(\sqrt{3} - 1)}{(\sqrt{3} + 1)} \times \frac{(\sqrt{3} - 1)}{(\sqrt{3} - 1)} = \frac{(\sqrt{3} - 1)^2}{(3 - 1)}$$

$$= \frac{(3 + 1 - 2\sqrt{3})}{2} = \frac{(4 - 2\sqrt{3})}{2} = (2 - \sqrt{3}).$$

EXAMPLE 6 *Prove that*

(i) $\sin 70° \cos 10° - \cos 70° \sin 10° = \frac{\sqrt{3}}{2}$

(ii) $\cos 50° \cos 10° - \sin 50° \sin 10° = \frac{1}{2}$

(iii) $\cos 80° \cos 20° + \sin 80° \sin 20° = \frac{1}{2}$

(iv) $\sin 36° \cos 9° + \cos 36° \sin 9° = \frac{1}{\sqrt{2}}$

SOLUTION (i) We know that $\sin \theta \cos \phi - \cos \theta \sin \phi = \sin (\theta - \phi)$.

Putting $\theta = 70°$ and $\phi = 10°$ in the this identity, we get

$$\sin 70° \cos 10° - \cos 70° \sin 10° = \sin (70° - 10°) = \sin 60° = \frac{\sqrt{3}}{2}.$$

(ii) We know that $\cos \theta \cos \phi - \sin \theta \sin \phi = \cos (\theta + \phi)$.

Putting $\theta = 50°$ and $\phi = 10°$ in the this identity, we get

$$\cos 50° \cos 10° - \sin 50° \sin 10° = \cos (50° + 10°) = \cos 60° = \frac{1}{2}.$$

(iii) We know that $\cos \theta \cos \phi + \sin \theta \sin \phi = \cos (\theta - \phi)$.

Putting $\theta = 80°$ and $\phi = 20°$ in this identity, we get

$$\cos 80° \cos 20° + \sin 80° \sin 20° = \cos (80° - 20°) = \cos 60° = \frac{1}{2}.$$

(iv) We know that $\sin \theta \cos \phi + \cos \theta \sin \phi = \sin (\theta + \phi)$.

Putting $\theta = 36°$ and $\phi = 9°$ in this identity, we get

$$\sin 36° \cos 9° + \cos 36° \sin 9° = \sin (36° + 9°) = \sin 45° = \frac{1}{\sqrt{2}}.$$

EXAMPLE 7 *Prove that*

$$\sin (40° + \theta) \cos (10° + \theta) - \cos (40° + \theta) \sin (10° + \theta) = \frac{1}{2}.$$

SOLUTION We know that $\sin x \cos y - \cos x \sin y = \sin (x - y)$.

Putting $x = (40° + \theta)$ and $y = (10° + \theta)$ in this identity, we get

$$\sin (40° + \theta) \cos (10° + \theta) - \cos (40° + \theta) \sin (10° + \theta)$$
$$= \sin [(40° + \theta) - (10° + \theta)]$$
$$= \sin 30° = \frac{1}{2}.$$

EXAMPLE 8 *Prove that*

$$\cos\theta + \sin(270^\circ + \theta) - \sin(270^\circ - \theta) + \cos(180^\circ + \theta) = 0.$$

SOLUTION We have

$$\cos\theta + \sin(270^\circ + \theta) - \sin(270^\circ - \theta) + \cos(180^\circ + \theta)$$
$$= \cos\theta + \sin\{180^\circ + (90^\circ + \theta)\} - \sin\{180^\circ + (90^\circ - \theta)\} + \cos(180^\circ + \theta)$$
$$= \cos\theta - \sin(90^\circ + \theta) + \sin(90^\circ - \theta) - \cos\theta$$
$$[\because \sin(180^\circ + x) = -\sin x,\ \cos(180^\circ + x) = -\cos x]$$
$$= \cos\theta - \cos\theta + \cos\theta - \cos\theta = 0$$
$$[\because \sin(90^\circ + \theta) = \cos\theta,\ \sin(90^\circ - \theta) = \cos\theta].$$

EXAMPLE 9 *Prove that*

$$\frac{\cos(90^\circ + \theta)\sec(270^\circ + \theta)\sin(180^\circ + \theta)}{\operatorname{cosec}(-\theta)\cos(270^\circ - \theta)\tan(180^\circ + \theta)} = \cos\theta.$$

SOLUTION We have

$$\text{LHS} = \frac{\cos(90^\circ + \theta)\sec(270^\circ + \theta)\sin(180^\circ + \theta)}{\operatorname{cosec}(-\theta)\cos(270^\circ - \theta)\tan(180^\circ + \theta)}$$
$$= \frac{\cos(90^\circ + \theta)\sec\{180^\circ + (90^\circ + \theta)\}\sin(180^\circ + \theta)}{-\operatorname{cosec}\theta\cos\{180^\circ + (90^\circ - \theta)\}\tan(180^\circ + \theta)}$$
$$= \frac{\cos(90^\circ + \theta)\{-\sec(90^\circ + \theta)\}\{-\sin\theta\}}{-\operatorname{cosec}\theta\{-\cos(90^\circ - \theta)\}\tan\theta}$$
$$\left[\begin{array}{l}\because \sin(180^\circ + x) = -\sin x,\ \cos(180^\circ + x) = -\cos x \\ \quad \sec(180^\circ + x) = -\sec x,\ \tan(180^\circ + x) = \tan x\end{array}\right]$$
$$= \frac{(-\sin\theta)(\operatorname{cosec}\theta)(-\sin\theta)}{(-\operatorname{cosec}\theta)(-\sin\theta)\tan\theta}$$
$$\left[\because \cos(90^\circ + \theta) = -\sin\theta,\ \sec(90^\circ + \theta) = -\operatorname{cosec}\theta,\ \cos(90^\circ - \theta) = \sin\theta\right]$$
$$= \cos\theta = \text{RHS}.$$

EXAMPLE 10 *Prove that*

$$\cos\left(\frac{3\pi}{2} + \theta\right)\cos(2\pi + \theta)\left[\cot\left(\frac{3\pi}{2} - \theta\right) + \cot(2\pi + \theta)\right] = 1.$$

SOLUTION We have

$$\cos\left(\frac{3\pi}{2} + \theta\right)\cos(2\pi + \theta)\left[\cot\left(\frac{3\pi}{2} - \theta\right) + \cot(2\pi + \theta)\right]$$
$$= \cos\left[\pi + \left(\frac{\pi}{2} + \theta\right)\right]\cos(2\pi + \theta)\left[\cot\left\{\pi + \left(\frac{\pi}{2} - \theta\right)\right\} + \cot(2\pi + \theta)\right]$$
$$= -\cos\left(\frac{\pi}{2} + \theta\right)\cos\theta\left[\cot\left(\frac{\pi}{2} - \theta\right) + \cot\theta\right]$$
$$[\because \cos(\pi + x) = -\cos x,\ \cot(\pi + x) = \cot x \text{ and } \cot(2\pi + x) = \cot x]$$

$$= \sin\theta\cos\theta[\tan\theta + \cot\theta] \quad \left[\because \cos\left(\frac{\pi}{2}+\theta\right) = -\sin\theta\right]$$

$$= (\sin\theta\cos\theta)\left[\frac{\sin\theta}{\cos\theta} + \frac{\cos\theta}{\sin\theta}\right] = (\sin\theta\cos\theta)\cdot\frac{(\sin^2\theta+\cos^2\theta)}{(\sin\theta\cos\theta)} = 1.$$

EXAMPLE 11 *Prove that*

(i) $\sin\frac{7\pi}{12}\cos\frac{\pi}{4} - \cos\frac{7\pi}{12}\sin\frac{\pi}{4} = \frac{\sqrt{3}}{2}$

(ii) $\sin\frac{\pi}{4}\cos\frac{\pi}{12} + \cos\frac{\pi}{4}\sin\frac{\pi}{12} = \frac{\sqrt{3}}{2}$

(iii) $\cos\frac{2\pi}{3}\cos\frac{\pi}{4} - \sin\frac{2\pi}{3}\sin\frac{\pi}{4} = \frac{-(\sqrt{3}+1)}{2\sqrt{2}}$

SOLUTION (i) $\sin\frac{7\pi}{12}\cos\frac{\pi}{4} - \cos\frac{7\pi}{12}\sin\frac{\pi}{4}$

$$= \sin\left(\frac{7\pi}{12} - \frac{\pi}{4}\right) \quad [\because \sin x\cos y - \cos x\sin y = \sin(x-y)]$$

$$= \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}.$$

(ii) $\sin\frac{\pi}{4}\cos\frac{\pi}{12} + \cos\frac{\pi}{4}\sin\frac{\pi}{12}$

$$= \sin\left(\frac{\pi}{4} + \frac{\pi}{12}\right) \quad [\because \sin x\cos y + \cos x\sin y = \sin(x+y)]$$

$$= \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}.$$

(iii) $\sin\frac{2\pi}{3}\cos\frac{\pi}{4} - \sin\frac{2\pi}{3}\sin\frac{\pi}{4}$

$$= \cos\left(\pi - \frac{\pi}{3}\right)\cos\frac{\pi}{4} - \sin\left(\pi - \frac{\pi}{3}\right)\sin\frac{\pi}{4}$$

$$= -\cos\frac{\pi}{3}\cos\frac{\pi}{4} - \sin\frac{\pi}{3}\cos\frac{\pi}{4}$$

$$[\because \sin(\pi - x) = \sin x,\ \cos(\pi - x) = -\cos x]$$

$$= \left(-\frac{1}{2}\times\frac{1}{\sqrt{2}}\right) - \left(\frac{\sqrt{3}}{2}\times\frac{1}{\sqrt{2}}\right) = -\left\{\frac{(\sqrt{3}+1)}{2\sqrt{2}}\right\}.$$

EXAMPLE 12 *Evaluate: (i)* $\sin\frac{\pi}{12}$ *(ii)* $\sin\frac{5\pi}{12}$

SOLUTION (i) $\sin\frac{\pi}{12} = \sin\left(\frac{\pi}{4} - \frac{\pi}{6}\right)$

$$= \sin\frac{\pi}{4}\cos\frac{\pi}{6} - \cos\frac{\pi}{4}\sin\frac{\pi}{6}$$

$$= \left(\frac{1}{\sqrt{2}}\times\frac{\sqrt{3}}{2}\right) - \left(\frac{1}{\sqrt{2}}\times\frac{1}{2}\right) = \frac{(\sqrt{3}-1)}{2\sqrt{2}}.$$

(ii) $\sin \dfrac{5\pi}{12} = \sin\left(\dfrac{\pi}{4} + \dfrac{\pi}{6}\right)$

$$= \sin \frac{\pi}{4} \cos \frac{\pi}{6} + \cos \frac{\pi}{4} \sin \frac{\pi}{6}$$

$$= \left(\frac{1}{\sqrt{2}} \times \frac{\sqrt{3}}{2}\right) + \left(\frac{1}{\sqrt{2}} \times \frac{1}{2}\right) = \frac{(\sqrt{3}+1)}{2\sqrt{2}}\cdot$$

EXAMPLE 13 *Prove that* $\dfrac{\sin(x+y)}{\sin(x-y)} = \dfrac{\tan x + \tan y}{\tan x - \tan y}\cdot$

SOLUTION We have

$$\text{LHS} = \frac{\sin(x+y)}{\sin(x-y)}$$

$$= \frac{\sin x \cos y + \cos x \sin y}{\sin x \cos y - \cos x \sin y}$$

$$= \frac{\tan x + \tan y}{\tan x - \tan y} \quad [\text{dividing num. and denom. by } \cos x \cos y]$$

$= \text{RHS}.$

EXAMPLE 14 *If* $\sin\theta = \dfrac{15}{17}$ *and* $\cos\phi = \dfrac{12}{13}$, *where* θ *and* ϕ *both lie in the first quadrant, find the values of*

(i) $\sin(\theta+\phi)$, (ii) $\cos(\theta-\phi)$, (iii) $\tan(\theta+\phi)$.

SOLUTION Given: $\sin\theta = \dfrac{15}{17}$ and $\cos\phi = \dfrac{12}{13}\cdot$

Since θ lies in the first quadrant, we have

$\sin\theta > 0$, $\cos\theta > 0$ and $\tan\theta > 0$.

Again, since ϕ lies in the first quadrant, we have

$\sin\phi > 0$, $\cos\phi > 0$ and $\tan\phi > 0$.

Now, $\cos^2\theta = (1 - \sin^2\theta) = \left(1 - \dfrac{225}{289}\right) = \dfrac{64}{289}$

$\Rightarrow \quad \cos\theta = +\sqrt{\dfrac{64}{289}} = \dfrac{8}{17}\cdot$

And, $\sin^2\phi = (1 - \cos^2\phi) = \left(1 - \dfrac{144}{169}\right) = \dfrac{25}{169}$

$\Rightarrow \quad \sin\phi = +\sqrt{\dfrac{25}{169}} = \dfrac{5}{13}\cdot$

$\therefore \quad \tan\theta = \dfrac{\sin\theta}{\cos\theta} = \left(\dfrac{15}{17} \times \dfrac{17}{8}\right) = \dfrac{15}{8}$ and

$\tan\phi = \dfrac{\sin\phi}{\cos\phi} = \left(\dfrac{5}{13} \times \dfrac{13}{12}\right) = \dfrac{5}{12}\cdot$

Now, (i) $\sin(\theta+\phi) = \sin\theta\cos\phi + \cos\theta\sin\phi$

$$= \left(\frac{15}{17}\times\frac{12}{13}\right) + \left(\frac{8}{17}\times\frac{5}{13}\right) = \left(\frac{180}{221}+\frac{40}{221}\right) = \frac{220}{221}.$$

(ii) $\cos(\theta-\phi) = \cos\theta\cos\phi + \sin\theta\sin\phi$

$$= \left(\frac{8}{17}\times\frac{12}{13}\right) + \left(\frac{15}{17}\times\frac{5}{13}\right) = \left(\frac{96}{221}+\frac{75}{221}\right) = \frac{171}{221}.$$

(iii) $\tan(\theta+\phi) = \dfrac{\tan\theta+\tan\phi}{1-\tan\theta\tan\phi}$

$$= \frac{\left(\frac{15}{8}+\frac{5}{12}\right)}{\left\{1-\left(\frac{15}{8}\times\frac{5}{12}\right)\right\}} = \frac{\left(\frac{55}{24}\right)}{\left(1-\frac{25}{32}\right)} = \left(\frac{55}{24}\times\frac{32}{7}\right) = \frac{220}{21}.$$

<u>EXAMPLE 15</u> *If* $\sin\theta = \dfrac{3}{5}$ *and* $\cos\phi = \dfrac{-12}{13}$, *where* θ *and* ϕ *both lie in the second quadrant, find the values of*

(i) $\sin(\theta-\phi)$, *(ii)* $\cos(\theta+\phi)$, *(iii)* $\tan(\theta-\phi)$.

SOLUTION Given: $\sin\theta = \dfrac{3}{5}$ and $\cos\phi = \dfrac{-12}{13}$.

Since θ lies in the second quadrant, we have

$\sin\theta > 0$, $\cos\theta < 0$ and $\tan\theta < 0$.

Again, since ϕ lies in the second quadrant, we have

$\sin\phi > 0$, $\cos\phi < 0$ and $\tan\phi < 0$.

Now, $\cos\theta = -\sqrt{1-\sin^2\theta} = -\sqrt{1-\dfrac{9}{25}} = -\sqrt{\dfrac{16}{25}} = \dfrac{-4}{5}$

and $\tan\theta = \dfrac{\sin\theta}{\cos\theta} = \dfrac{3}{5}\times\left(-\dfrac{5}{4}\right) = \dfrac{-3}{4}$.

Also, $\sin\phi = +\sqrt{1-\cos^2\phi} = +\sqrt{1-\dfrac{144}{169}} = +\sqrt{\dfrac{25}{169}} = +\dfrac{5}{13}$

and $\tan\phi = \dfrac{\sin\phi}{\cos\phi} = \dfrac{5}{13}\times\left(\dfrac{-13}{12}\right) = \dfrac{-5}{12}$.

$\therefore$ (i) $\sin(\theta-\phi) = \sin\theta\cos\phi - \cos\theta\sin\phi$

$$= \left\{\frac{3}{5}\times\left(\frac{-12}{13}\right)\right\} - \left\{\left(\frac{-4}{5}\right)\times\frac{5}{13}\right\} = \left(\frac{-36}{65}+\frac{20}{65}\right) = \frac{-16}{65}.$$

(ii) $\cos(\theta+\phi) = \cos\theta\cos\phi - \sin\theta\sin\phi$

$$= \left\{\left(\frac{-4}{5}\right)\times\left(\frac{-12}{13}\right)\right\} - \left\{\frac{3}{5}\times\frac{5}{13}\right\} = \left(\frac{48}{65}-\frac{3}{13}\right) = \frac{33}{65}.$$

(iii) $\tan(\theta - \phi) = \dfrac{\tan\theta - \tan\phi}{1 + \tan\theta\tan\phi}$

$$= \frac{\left(\frac{-3}{4}\right) - \left(\frac{-5}{12}\right)}{1 + \left\{\left(\frac{-3}{4}\right) \times \left(\frac{-5}{12}\right)\right\}} = \frac{\left(\frac{-3}{4} + \frac{5}{12}\right)}{\left(1 + \frac{5}{16}\right)} = \left(\frac{-1}{3}\right) \times \frac{16}{21} = \frac{-16}{63}.$$

EXAMPLE 16 *If* $\cos\theta = \dfrac{4}{5}$ *and* $\cos\phi = \dfrac{12}{13}$, *where* θ *and* ϕ *both lie in the fourth quadrant, find the values of*

(i) $\cos(\theta + \phi)$, *(ii)* $\sin(\theta - \phi)$, *(iii)* $\tan(\theta + \phi)$.

SOLUTION Given: $\cos\theta = \dfrac{4}{5}$ and $\cos\phi = \dfrac{12}{13}$.

Since θ lies in the fourth quadrant, we have

$\cos\theta > 0$, $\sin\theta < 0$ and $\tan\theta < 0$.

Again, since ϕ lies in the fourth quadrant, we have

$\cos\phi > 0$, $\sin\phi < 0$ and $\tan\phi < 0$.

Now, $\sin\theta = -\sqrt{1 - \cos^2\theta} = -\sqrt{1 - \dfrac{16}{25}} = -\sqrt{\dfrac{9}{25}} = \dfrac{-3}{5}$

and $\tan\theta = \dfrac{\sin\theta}{\cos\theta} = \left(\dfrac{-3}{5}\right) \times \dfrac{5}{4} = \dfrac{-3}{4}$.

Also, $\sin\phi = -\sqrt{1 - \cos^2\phi} = -\sqrt{1 - \dfrac{144}{169}} = -\sqrt{\dfrac{25}{169}} = \dfrac{-5}{13}$

and $\tan\phi = \dfrac{\sin\phi}{\cos\phi} = \left(\dfrac{-5}{13}\right) \times \dfrac{13}{12} = \dfrac{-5}{12}$.

$\therefore$ (i) $\cos(\theta + \phi) = \cos\theta\cos\phi - \sin\theta\sin\phi$

$$= \left(\frac{4}{5} \times \frac{12}{13}\right) - \left\{\left(\frac{-3}{5}\right) \times \left(\frac{-5}{13}\right)\right\} = \left(\frac{48}{65} - \frac{15}{65}\right) = \frac{33}{65}.$$

(ii) $\sin(\theta - \phi) = \sin\theta\cos\phi - \cos\theta\sin\phi$

$$= \left\{\left(\frac{-3}{5}\right) \times \left(\frac{12}{13}\right)\right\} - \left\{\frac{4}{5} \times \left(\frac{-5}{13}\right)\right\} = \left(\frac{-36}{65} + \frac{20}{65}\right) = \frac{-16}{65}.$$

(iii) $\tan(\theta + \phi) = \dfrac{\tan\theta + \tan\phi}{1 - \tan\theta\tan\phi}$

$$= \frac{\left(\frac{-3}{4}\right) + \left(\frac{-5}{12}\right)}{1 - \left\{\left(\frac{-3}{4}\right) \times \left(\frac{-5}{12}\right)\right\}} = \frac{\left(\frac{-7}{6}\right)}{\left(1 - \frac{5}{16}\right)} = \left(\frac{-7}{6}\right) \times \frac{16}{11} = \frac{-56}{33}.$$

EXAMPLE 17 *If* $\cot\alpha = \dfrac{1}{2}$ *and* $\sec\beta = \dfrac{-5}{3}$, *where* $\alpha \in \left]\pi, \dfrac{3\pi}{2}\right[$ *and* $\beta \in \left]\dfrac{\pi}{2}, \pi\right[$, *find the value of* $\tan(\alpha + \beta)$.

SOLUTION Here, α lies in Quadrant III and, therefore, $\tan\alpha$ is positive.

Now, $\cot\alpha = \frac{1}{2} \Rightarrow \tan\alpha = 2$.

Again, β lies in Quadrant II and, therefore, $\sin\beta$ is positive and $\cos\beta$ is negative.

Now, $\sec\beta = \frac{-5}{3} \Rightarrow \cos\beta = \frac{-3}{5}$.

And, $\sin\beta = +\sqrt{1-\cos^2\beta} = +\sqrt{1-\frac{9}{25}} = +\sqrt{\frac{16}{25}} = \frac{4}{5}$.

$$\therefore \quad \tan\beta = \frac{\sin\beta}{\cos\beta} = \frac{4}{5} \times \left(\frac{-5}{3}\right) = \frac{-4}{3}.$$

$$\therefore \quad \tan(\alpha+\beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$$

$$= \frac{2 - \frac{4}{3}}{1 - 2 \times \left(\frac{-4}{3}\right)} = \frac{\left(\frac{2}{3}\right)}{\left(\frac{11}{3}\right)} = \left(\frac{2}{3} \times \frac{3}{11}\right) = \frac{2}{11}.$$

EXAMPLE 18 *Prove that* $\tan 56° = \dfrac{\cos 11° + \sin 11°}{\cos 11° - \sin 11°}$.

SOLUTION We have

$$\text{LHS} = \tan 56° = \tan(45° + 11°)$$

$$= \frac{\tan 45° + \tan 11°}{1 - \tan 45° \tan 11°}$$

$$= \frac{1 + \tan 11°}{1 - \tan 11°}$$

$$= \frac{1 + \frac{\sin 11°}{\cos 11°}}{1 - \frac{\sin 11°}{\cos 11°}} = \frac{\cos 11° + \sin 11°}{\cos 11° - \sin 11°} = \text{RHS}.$$

EXAMPLE 19 *Prove that* $\tan 70° = \tan 20° + 2\tan 50°$.

SOLUTION We have

$$\tan 70° = \tan(20° + 50°)$$

$$\Rightarrow \quad \tan 70° = \frac{\tan 20° + \tan 50°}{1 - \tan 20° \tan 50°}$$

$$\Rightarrow \quad \tan 70° - \tan 70° \tan 20° \tan 50° = \tan 20° + \tan 50°$$

$$\Rightarrow \quad \tan 70° - \tan(90° - 20°) \tan 20° \tan 50° = \tan 20° + \tan 50°$$

$$\Rightarrow \quad \tan 70° - \cot 20° \tan 20° \tan 50° = \tan 20° + \tan 50°$$

$$\Rightarrow \quad \tan 70° - \tan 50° = \tan 20° + \tan 50°$$

$$\Rightarrow \quad \tan 70° = \tan 20° + 2\tan 50°.$$

EXAMPLE 20 *Show that* $\cot 2x \cot x - \cot 3x \cot 2x - \cot 3x \cot x = 1$.

SOLUTION We have

$$\cot 3x = \cot(2x + x)$$

$$\Leftrightarrow \quad \cot 3x = \frac{\cot 2x \cot x - 1}{\cot 2x + \cot x}$$

$$\Leftrightarrow \quad \cot 3x \cot 2x + \cot 3x \cot x = \cot 2x \cot x - 1$$

$$\Leftrightarrow \quad \cot 2x \cot x - \cot 3x \cot 2x - \cot 3x \cot x = 1.$$

EXERCISE 15B

1. Find the value of

(i) $\cos 840°$ (ii) $\sin 870°$ (iii) $\tan(-120°)$

(iv) $\sec(-420°)$ (v) $\operatorname{cosec}(-690°)$ (vi) $\tan(225°)$

(vii) $\cot(-315°)$ (viii) $\sin(-1230°)$ (ix) $\cos(495°)$

2. Find the values of all trigonometric functions of 135°.

3. Prove that

(i) $\sin 80° \cos 20° - \cos 80° \sin 20° = \frac{\sqrt{3}}{2}$

(ii) $\cos 45° \cos 15° - \sin 45° \sin 15° = \frac{1}{2}$

(iii) $\cos 75° \cos 15° + \sin 75° \sin 15° = \frac{1}{2}$

(iv) $\sin 40° \cos 20° + \cos 40° \sin 20° = \frac{\sqrt{3}}{2}$

(v) $\cos 130° \cos 40° + \sin 130° \sin 40° = 0$

4. Prove that

(i) $\sin(50° + \theta)\cos(20° + \theta) - \cos(50° + \theta)\sin(20° + \theta) = \frac{1}{2}$

(ii) $\cos(70° + \theta)\cos(10° + \theta) + \sin(70° + \theta)\sin(10° + \theta) = \frac{1}{2}$

5. Prove that

(i) $\cos(n + 2)x \cos(n + 1)x + \sin(n + 2)x \sin(n + 1)x = \cos x$

(ii) $\cos\left(\frac{\pi}{4} - x\right)\cos\left(\frac{\pi}{4} - y\right) - \sin\left(\frac{\pi}{4} - x\right)\sin\left(\frac{\pi}{4} - y\right) = \sin(x + y)$

6. Prove that $\dfrac{\tan\left(\dfrac{\pi}{4}+x\right)}{\tan\left(\dfrac{\pi}{4}-x\right)} = \left(\dfrac{1+\tan x}{1-\tan x}\right)^2.$

7. Prove that

(i) $\sin 75° = \dfrac{(\sqrt{6}+\sqrt{2})}{4}$

(ii) $\dfrac{\cos 135° - \cos 120°}{\cos 135° + \cos 120°} = (3 - 2\sqrt{2})$

(iii) $\tan 15° + \cot 15° = 4$

8. Prove that

(i) $\cos 15° - \sin 15° = \dfrac{1}{\sqrt{2}}$

(ii) $\cot 105° - \tan 105° = 2\sqrt{3}$

(iii) $\dfrac{\tan 69° + \tan 66°}{1 - \tan 69° \tan 66°} = -1$

9. Prove that $\dfrac{\cos 9° + \sin 9°}{\cos 9° - \sin 9°} = \tan 54°.$

10. Prove that $\dfrac{\cos 8° - \sin 8°}{\cos 8° + \sin 8°} = \tan 37°.$

11. Prove that $\dfrac{\cos(\pi+\theta)\cos(-\theta)}{\cos(\pi-\theta)\cos\left(\dfrac{\pi}{2}+\theta\right)} = -\cot\theta.$

12. Prove that $\dfrac{\cos\theta}{\sin(90°+\theta)} + \dfrac{\sin(-\theta)}{\sin(180°+\theta)} - \dfrac{\tan(90°+\theta)}{\cot\theta} = 3.$

13. Prove that $\dfrac{\sin(180°+\theta)\cos(90°+\theta)\tan(270°-\theta)\cot(360°-\theta)}{\sin(360°-\theta)\cos(360°+\theta)\operatorname{cosec}(-\theta)\sin(270°+\theta)} = 1.$

14. If θ and ϕ lie in the first quadrant such that $\sin\theta = \dfrac{8}{17}$ and $\cos\phi = \dfrac{12}{13}$, find the values of

(i) $\sin(\theta-\phi)$, (ii) $\cos(\theta+\phi)$, (iii) $\tan(\theta-\phi)$.

15. If x and y are acute angles such that $\sin x = \dfrac{1}{\sqrt{5}}$ and $\sin y = \dfrac{1}{\sqrt{10}}$, prove that $(x+y) = \dfrac{\pi}{4}\cdot$

16. If x and y are acute angles such that $\cos x = \dfrac{13}{14}$ and $\cos y = \dfrac{1}{7}$, prove that $(x-y) = -\dfrac{\pi}{3}\cdot$

17. If $\sin x = \frac{12}{13}$ and $\sin y = \frac{4}{5}$, where $\frac{\pi}{2} < x < \pi$ and $0 < y < \frac{\pi}{2}$, find the values of

(i) $\sin(x + y)$, (ii) $\cos(x + y)$, (iii) $\tan(x - y)$.

18. If $\cos x = \frac{3}{5}$ and $\cos y = \frac{-24}{25}$, where $\frac{3\pi}{2} < x < 2\pi$ and $\pi < y < \frac{3\pi}{2}$, find the values of

(i) $\sin(x + y)$, (ii) $\cos(x - y)$, (iii) $\tan(x + y)$.

19. Prove that

(i) $\cos\left(\frac{\pi}{3} + x\right) = \frac{1}{2}(\cos x - \sqrt{3}\sin x)$

(ii) $\sin\left(\frac{\pi}{4} + x\right) + \sin\left(\frac{\pi}{4} - x\right) = \sqrt{2}\cos x$

(iii) $\frac{1}{\sqrt{2}}\cos\left(\frac{\pi}{4} + x\right) = \frac{1}{2}(\cos x - \sin x)$

(iv) $\cos x + \cos\left(\frac{2\pi}{3} + x\right) + \cos\left(\frac{2\pi}{3} - x\right) = 0$

20. Prove that

(i) $2\sin\frac{5\pi}{12}\sin\frac{\pi}{12} = \frac{1}{2}$

(ii) $2\cos\frac{5\pi}{12}\cos\frac{\pi}{12} = \frac{1}{2}$

(iii) $2\sin\frac{5\pi}{12}\cos\frac{\pi}{12} = \frac{(2+\sqrt{3})}{2}$

ANSWERS (EXERCISE 15B)

1. (i) $-\frac{1}{2}$ (ii) $\frac{1}{2}$ (iii) $\sqrt{3}$ (iv) 2 (v) 2 (vi) 1 (vii) 1 (viii) $-\frac{1}{2}$ (ix) $-\frac{1}{\sqrt{2}}$

2. $\sin 135° = \frac{1}{\sqrt{2}}$, $\cos 135° = \frac{-1}{\sqrt{2}}$, $\tan 135° = -1$, $\cot 135° = -1$,

$\sec 135° = -\sqrt{2}$, $\text{cosec}\, 135° = \sqrt{2}$

14. (i) $\frac{21}{221}$ (ii) $\frac{140}{221}$ (iii) $\frac{21}{220}$

17. (i) $\frac{16}{65}$ (ii) $\frac{-63}{65}$ (iii) $\frac{56}{33}$

18. (i) $\frac{3}{5}$ (ii) $\frac{-44}{125}$ (iii) $\frac{-3}{4}$

HINTS TO SOME SELECTED QUESTIONS

4. (i) LHS $= \sin x \cos y - \cos x \sin y$

$= \sin(x-y) = \sin[(50° + \theta) - (20° + \theta)] = \sin 30°.$

(ii) LHS $= \cos x \cos y + \sin x \sin y$

$= \cos(x-y) = \cos[(70° + \theta) - (10° + \theta)] = \cos 60°.$

5. (i) LHS $= \cos A \cos B + \sin A \sin B$

$= \cos(A-B) = \cos[(n+2)x - (n+1)x] = \cos x.$

(ii) LHS $= \cos A \cos B - \sin A \sin B$

$= \cos(A+B) = \cos\left[\left(\frac{\pi}{4} - x\right) + \left(\frac{\pi}{4} - y\right)\right]$

$= \cos\left[\frac{\pi}{2} - (x+y)\right] = \sin(x+y).$

7. (i) $\sin 75° = \sin(45° + 30°)$

(ii) $\cos 135° = \cos(180° - 45°) = -\cos 45°.$
$\cos 120° = \cos(180° - 60°) = -\cos 60°.$

(iii) $\tan 15° = \tan(45° - 30°) = \dfrac{\tan 45° - \tan 30°}{1 + \tan 45° \tan 30°}\cdot$

8. (i) $\cos 15° = \cos(45° - 30°)$ and $\sin 15° = \sin(45° - 30°).$

(ii) $\tan 105° = \tan(60° + 45°).$

(iii) LHS $= \tan(69° + 66°) = \tan 135°$
$= \tan(180° - 45°) = -\tan 45° = -1.$

9. Dividing num. and denom. by $\cos 9°$, we get

$$\text{LHS} = \frac{1 + \tan 9°}{1 - \tan 9°} = \frac{\tan 45° + \tan 9°}{1 - \tan 45° \tan 9°}$$

$= \tan(45° + 9°) = \tan 54°.$

17. Clearly, $\sin x > 0$, $\cos x < 0$ and $\tan x < 0$.

Also, $\sin y > 0$, $\cos y > 0$ and $\tan y > 0$.

18. In the 4th quadrant, $\cos x > 0$, $\sin x < 0$ and $\tan x < 0$.

In the 3rd quadrant, $\tan y > 0$, $\sin y < 0$ and $\cos y < 0$.

$\therefore\quad \sin x = -\sqrt{1 - \dfrac{9}{25}} = -\sqrt{\dfrac{16}{25}} = \dfrac{-4}{5},$

$\tan x = \left(\dfrac{-4}{5} \times \dfrac{5}{3}\right) = \dfrac{-4}{3}$

$\cos y = \dfrac{-24}{25},\ \sin y = -\sqrt{1 - \dfrac{576}{625}} = -\sqrt{\dfrac{49}{625}} = \dfrac{-7}{25}$ and $\tan y = \dfrac{7}{24}\cdot$

Now, find the required values.

SOME MORE TRIGONOMETRIC FUNCTIONS

THEOREM 1 *For all $x, y \in R$, we have*

(a) $2\sin x \cos y = \sin(x + y) + \sin(x - y)$

(b) $2\cos x \sin y = \sin(x + y) - \sin(x - y)$

(c) $2\cos x \cos y = \cos(x + y) + \cos(x - y)$

(d) $2\sin x \sin y = \cos(x - y) - \cos(x + y)$

PROOF We know that

$$\sin(x + y) = \sin x \cos y + \cos x \sin y \qquad \ldots \text{(i)}$$
$$\sin(x - y) = \sin x \cos y - \cos x \sin y \qquad \ldots \text{(ii)}$$

(a) Adding (i) and (ii), we get

$$2\sin x \cos y = \sin(x + y) + \sin(x - y).$$

(b) Subtracting (ii) from (i), we get

$$2\cos x \sin y = \sin(x + y) - \sin(x - y).$$

Further, we know that

$$\cos(x + y) = \cos x \cos y - \sin x \sin y \qquad \ldots \text{(iii)}$$
$$\cos(x - y) = \cos x \cos y + \sin x \sin y \qquad \ldots \text{(iv)}$$

(c) Adding (iii) and (iv), we get

$$2\cos x \cos y = \cos(x + y) + \cos(x - y).$$

(d) Subtracting (iii) from (iv), we get

$$2\sin x \sin y = \cos(x - y) - \cos(x + y).$$

THEOREM 2 *For all $x, y \in R$, we have*

(i) $\sin x + \sin y = 2\sin\left(\frac{x + y}{2}\right)\cos\left(\frac{x - y}{2}\right)$

(ii) $\sin x - \sin y = 2\cos\left(\frac{x + y}{2}\right)\sin\left(\frac{x - y}{2}\right)$

(iii) $\cos x + \cos y = 2\cos\left(\frac{x + y}{2}\right)\cos\left(\frac{x - y}{2}\right)$

(iv) $\cos x - \cos y = -2\sin\left(\frac{x + y}{2}\right)\sin\left(\frac{x - y}{2}\right)$

PROOF Putting $x = (a + b)$ and $y = (a - b)$, we get $a = \left(\frac{x + y}{2}\right)$ and $b = \left(\frac{x - y}{2}\right)$.

(i)
$$\begin{aligned}\sin x + \sin y &= \sin(a + b) + \sin(a - b)\\ &= 2\sin a \cos b\\ &= 2\sin\left(\frac{x + y}{2}\right)\cos\left(\frac{x - y}{2}\right).\end{aligned}$$

(ii)
$$\begin{aligned}\sin x - \sin y &= \sin(a + b) - \sin(a - b)\\ &= 2\cos a \sin b\\ &= 2\cos\left(\frac{x + y}{2}\right)\sin\left(\frac{x - y}{2}\right).\end{aligned}$$

(iii) $\cos x + \cos y = \cos(a+b) + \cos(a-b)$

$$= 2\cos a \cos b$$

$$= 2\cos\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right).$$

(iv) $\cos x - \cos y = \cos(a+b) - \cos(a-b)$

$$= -2\sin a \sin b$$

$$= -2\sin\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right)$$

These formulae may be listed as under:

I. $\sin C + \sin D = 2\sin\left(\frac{C+D}{2}\right)\cos\left(\frac{C-D}{2}\right)$

II. $\sin C - \sin D = 2\cos\left(\frac{C+D}{2}\right)\sin\left(\frac{C-D}{2}\right)$

III. $\cos C + \cos D = 2\cos\left(\frac{C+D}{2}\right)\cos\left(\frac{C-D}{2}\right)$

IV. $\cos C - \cos D = -2\sin\left(\frac{C+D}{2}\right)\sin\left(\frac{C-D}{2}\right)$

SOLVED EXAMPLES

EXAMPLE 1 *Prove that*

(i) $\cos\left(\frac{\pi}{4}+x\right) + \cos\left(\frac{\pi}{4}-x\right) = \sqrt{2}\cos x$

(ii) $\cos\left(\frac{3\pi}{4}+x\right) - \cos\left(\frac{3\pi}{4}-x\right) = -\sqrt{2}\sin x$

SOLUTION (i) LHS $= \cos\left(\frac{\pi}{4}+x\right) + \cos\left(\frac{\pi}{4}-x\right)$

$$= 2\cos\frac{\pi}{4}\cos x \quad [\because \cos(A+B)+\cos(A-B) = 2\cos A\cos B]$$

$$= \left(2\times\frac{1}{\sqrt{2}}\cos x\right) = \sqrt{2}\cos x = \text{RHS.}$$

(ii) LHS $= \cos\left(\frac{3\pi}{4}+x\right) - \cos\left(\frac{3\pi}{4}-x\right)$

$$= -2\sin\frac{3\pi}{4}\sin x$$

$$[\because \cos(A+B) - \cos(A-B) = -2\sin A\sin B]$$

$$= -2\sin\left(\pi-\frac{\pi}{4}\right)\sin x$$

$$= -2\sin\frac{\pi}{4}\sin x \quad \left[\because \sin\left(\pi - \frac{\pi}{4}\right) = \sin\frac{\pi}{4}\right]$$

$$= \left(-2 \times \frac{1}{\sqrt{2}}\right)\sin x = -\sqrt{2}\sin x = \text{RHS}.$$

EXAMPLE 2 *Express each of the following as an algebraic sum of sines or cosines:*

(i) $2\sin 3x \cos 2x$ *(ii)* $2\cos 4x \sin 2x$

(iii) $2\cos 6x \cos 4x$ *(iv)* $2\sin 3x \sin 5x$

SOLUTION (i) We know that $2\sin A \cos B = \sin(A+B) + \sin(A-B)$.

$$\therefore \quad 2\sin 3x \cos 2x = \sin(3x+2x) + \sin(3x-2x) = \sin 5x + \sin x.$$

(ii) We know that $2\cos A \sin B = \sin(A+B) - \sin(A-B)$.

$$\therefore \quad 2\cos 4x \sin 2x = \sin(4x+2x) - \sin(4x-2x) = \sin 6x - \sin 2x.$$

(iii) We know that $2\cos A \cos B = \cos(A+B) + \cos(A-B)$.

$$\therefore \quad 2\cos 6x \cos 4x = \cos(6x+4x) + \cos(6x-4x) = \cos 10x + \cos 2x.$$

(iv) We know that $2\sin A \sin B = \cos(A-B) - \cos(A+B)$.

$$\therefore \quad 2\sin 3x \sin 5x = 2\sin 5x \sin 3x = \cos(5x-3x) - \cos(5x+3x) = \cos 2x - \cos 8x.$$

EXAMPLE 3 *Express each of the following as a product of sines or cosines or sine and cosine:*

(i) $\cos 5x + \cos 3x$ *(ii)* $\cos 5x - \cos 7x$

(iii) $\sin 7x + \sin 3x$ *(iv)* $\sin 5x - \sin 3x$

SOLUTION (i) We know that $\cos C + \cos D = 2\cos\left(\frac{C+D}{2}\right)\cos\left(\frac{C-D}{2}\right)$.

$$\therefore \quad \cos 5x + \cos 3x = 2\cos\left(\frac{5x+3x}{2}\right)\cos\left(\frac{5x-3x}{2}\right) = 2\cos 4x \cos x.$$

(ii) We know that $\cos C - \cos D = -2\sin\left(\frac{C+D}{2}\right)\sin\left(\frac{C-D}{2}\right)$.

$$\therefore \quad \cos 5x - \cos 7x = -2\sin\left(\frac{5x+7x}{2}\right)\sin\left(\frac{5x-7x}{2}\right) = -2\sin 6x \sin(-x) = 2\sin 6x \sin x.$$

(iii) We know that $\sin C + \sin D = 2\sin\left(\frac{C+D}{2}\right)\cos\left(\frac{C-D}{2}\right)$.

$$\therefore \quad \sin 7x + \sin 3x = 2\sin\left(\frac{7x+3x}{2}\right)\cos\left(\frac{7x-3x}{2}\right) = 2\sin 5x \cos 2x.$$

(iv) We know that $\sin C - \sin D = 2\cos\left(\frac{C+D}{2}\right)\sin\left(\frac{C-D}{2}\right)$.

$$\therefore \quad \sin 5x - \sin 3x = 2\cos\left(\frac{5x+3x}{2}\right)\sin\left(\frac{5x-3x}{2}\right)$$

$$= 2\cos 4x \sin x.$$

EXAMPLE 4 *Prove that* $\dfrac{\cos 6x + \cos 4x}{\sin 6x - \sin 4x} = \cot x$.

SOLUTION We have

$$\text{LHS} = \frac{\cos 6x + \cos 4x}{\sin 6x - \sin 4x}$$

$$= \frac{2\cos\left(\frac{6x+4x}{2}\right)\cos\left(\frac{6x-4x}{2}\right)}{2\cos\left(\frac{6x+4x}{2}\right)\sin\left(\frac{6x-4x}{2}\right)}$$

$$\left\{\begin{array}{l} \because \cos C + \cos D = 2\cos\left(\frac{C+D}{2}\right)\cos\left(\frac{C-D}{2}\right) \\ \sin C - \sin D = 2\cos\left(\frac{C+D}{2}\right)\sin\left(\frac{C-D}{2}\right) \end{array}\right\}$$

$$= \frac{2\cos 5x \cos x}{2\cos 5x \sin x} = \frac{\cos x}{\sin x}$$

$$= \cot x = \text{RHS}.$$

EXAMPLE 5 *Prove that* $\dfrac{\sin 3x - \sin x}{\cos x - \cos 3x} = \cot 2x$.

SOLUTION We have

$$\text{LHS} = \frac{\sin 3x - \sin x}{\cos x - \cos 3x}$$

$$= \frac{2\cos\left(\frac{3x+x}{2}\right)\sin\left(\frac{3x-x}{2}\right)}{-2\sin\left(\frac{x+3x}{2}\right)\sin\left(\frac{x-3x}{2}\right)}$$

$$\left\{\begin{array}{l} \because \sin C - \sin D = 2\cos\left(\frac{C+D}{2}\right)\sin\left(\frac{C-D}{2}\right) \\ \cos C - \cos D = -2\sin\left(\frac{C+D}{2}\right)\sin\left(\frac{C-D}{2}\right) \end{array}\right\}$$

$$= \frac{\cos 2x \sin x}{-\sin 2x \sin(-x)} = \frac{\cos 2x \sin x}{\sin 2x \sin x}$$

$$= \cot 2x = \text{RHS}.$$

EXAMPLE 6 *Prove that* $\dfrac{\sin 3x - \sin x}{\cos 2x} = 2\sin x.$

SOLUTION We have

$$\text{LHS} = \frac{\sin 3x - \sin x}{\cos 2x}$$

$$= \frac{2\cos\left(\frac{3x+x}{2}\right)\sin\left(\frac{3x-x}{2}\right)}{\cos 2x}$$

$$\left[\because \sin C - \sin D = 2\cos\left(\frac{C+D}{2}\right)\sin\left(\frac{C-D}{2}\right)\right]$$

$$= \frac{2\cos 2x \sin x}{\cos 2x} = 2\sin x = \text{RHS}.$$

EXAMPLE 7 *Prove that* $\dfrac{\sin 5x - 2\sin 3x + \sin x}{\cos 5x - \cos x} = \operatorname{cosec} 2x - \cot 2x.$

SOLUTION We have

$$\text{LHS} = \frac{(\sin 5x + \sin x) - 2\sin 3x}{\cos 5x - \cos x}$$

$$= \frac{2\sin\left(\frac{5x+x}{2}\right)\cos\left(\frac{5x-x}{2}\right) - 2\sin 3x}{-2\sin\left(\frac{5x+x}{2}\right)\sin\left(\frac{5x-x}{2}\right)}$$

$$\left[\because \sin C + \sin D = 2\sin\left(\frac{C+D}{2}\right)\cos\left(\frac{C-D}{2}\right) \text{ and } \cos C - \cos D = -2\sin\left(\frac{C+D}{2}\right)\sin\left(\frac{C-D}{2}\right)\right]$$

$$= \frac{2\sin 3x \cos 2x - 2\sin 3x}{-2\sin 3x \sin 2x} = \frac{2\sin 3x(\cos 2x - 1)}{-2\sin 3x \sin 2x}$$

$$= \frac{(1-\cos 2x)}{\sin 2x} = \frac{1}{\sin 2x} - \frac{\cos 2x}{\sin 2x}$$

$$= \operatorname{cosec} 2x - \cot 2x = \text{RHS}.$$

EXAMPLE 8 *If* $\dfrac{\cos(A+B)}{\cos(A-B)} = \dfrac{\sin(C+D)}{\sin(C-D)}$, *prove that* $\tan A \tan B \tan C + \tan D = 0.$

SOLUTION We have

$$\frac{\cos(A+B)}{\cos(A-B)} = \frac{\sin(C+D)}{\sin(C-D)}$$

$$\Rightarrow \frac{\cos(A+B) + \cos(A-B)}{\cos(A+B) - \cos(A-B)} = \frac{\sin(C+D) + \sin(C-D)}{\sin(C+D) - \sin(C-D)}$$

$$\Rightarrow \quad \frac{2\cos A \cos B}{-2\sin A \sin B} = \frac{2\sin C \cos D}{2\cos C \sin D}$$

$\Rightarrow \quad -\cot A \cot B = \tan C \cot D$

$\Rightarrow \quad \tan A \tan B \tan C = -\tan D$

$\Rightarrow \quad \tan A \tan B \tan C + \tan D = 0.$

Hence, $\tan A \tan B \tan C + \tan D = 0.$

EXAMPLE 9 *Prove that* $\dfrac{\cos 4x + \cos 3x + \cos 2x}{\sin 4x + \sin 3x + \sin 2x} = \cot 3x.$

SOLUTION We have

$$\text{LHS} = \frac{(\cos 4x + \cos 2x) + \cos 3x}{(\sin 4x + \sin 2x) + \sin 3x}$$

$$= \frac{2\cos\left(\frac{4x+2x}{2}\right)\cos\left(\frac{4x-2x}{2}\right) + \cos 3x}{2\sin\left(\frac{4x+2x}{2}\right)\cos\left(\frac{4x-2x}{2}\right) + \sin 3x}$$

[using formulae for $(\cos C + \cos D)$ and $(\sin C + \sin D)$]

$$= \frac{2\cos 3x \cos x + \cos 3x}{2\sin 3x \cos x + \sin 3x} = \frac{\cos 3x(2\cos x + 1)}{\sin 3x(2\cos x + 1)}$$

$$= \frac{\cos 3x}{\sin 3x} = \cot 3x = \text{RHS}.$$

EXAMPLE 10 *Prove that* $\sin x + \sin 3x + \sin 5x + \sin 7x = 4\sin 4x \cos 2x \cos x.$

SOLUTION We have

$$\text{LHS} = (\sin 7x + \sin x) + (\sin 5x + \sin 3x)$$

$$= 2\sin\left(\frac{7x+x}{2}\right)\cos\left(\frac{7x-x}{2}\right) + 2\sin\left(\frac{5x+3x}{2}\right)\cos\left(\frac{5x-3x}{2}\right)$$

$$= 2\sin 4x \cos 3x + 2\sin 4x \cos x$$

$$= 2\sin 4x\,(\cos 3x + \cos x)$$

$$= (2\sin 4x) \times 2\cos\left(\frac{3x+x}{2}\right)\cos\left(\frac{3x-x}{2}\right)$$

$$= (2\sin 4x) \times 2\cos 2x \cos x$$

$$= 4\sin 4x \cos 2x \cos x = \text{RHS}.$$

EXAMPLE 11 *Prove that* $\cos 2x \cos \dfrac{x}{2} - \cos 3x \cos \dfrac{9x}{2} = -\sin 5x \sin\left(\dfrac{5x}{2}\right)\cdot$

SOLUTION We have

$$\text{LHS} = \frac{1}{2}\left[2\cos 2x \cos\frac{x}{2} - 2\cos 3x \cos\frac{9x}{2}\right]$$

$$= \frac{1}{2}\left[\cos\left(2x + \frac{x}{2}\right) + \cos\left(2x - \frac{x}{2}\right)\right] - \frac{1}{2}\left[\cos\left(\frac{9x}{2} + 3x\right) + \cos\left(\frac{9x}{2} - 3x\right)\right]$$

$$= \frac{1}{2}\left(\cos\frac{5x}{2} + \cos\frac{3x}{2}\right) - \frac{1}{2}\left(\cos\frac{15x}{2} + \cos\frac{3x}{2}\right)$$

$$= \frac{1}{2}\left(\cos\frac{5x}{2} - \cos\frac{15x}{2}\right)$$

$$= \frac{1}{2}\left[-2\sin\frac{\left(\frac{5x}{2}+\frac{15x}{2}\right)}{2}\sin\frac{\left(\frac{15x}{2}-\frac{5x}{2}\right)}{2}\right]$$

$$= -\sin 5x \sin\left(\frac{5x}{2}\right) = \text{RHS.}$$

EXAMPLE 12 *Prove that* $\dfrac{\sin 8x\cos x - \sin 6x\cos 3x}{\cos 2x\cos x - \sin 4x\sin 3x} = \tan 2x.$

SOLUTION We have

$$\text{LHS} = \frac{2\sin 8x\cos x - 2\sin 6x\cos 3x}{2\cos 2x\cos x - 2\sin 4x\sin 3x}$$

$$= \frac{[\sin(8x+x) + \sin(8x-x)] - [\sin(6x+3x) + \sin(6x-3x)]}{[\cos(2x+x) + \cos(2x-x)] - [\cos(4x-3x) - \cos(4x+3x)]}$$

$$[\because 2\sin A\cos B = \sin(A+B) + \sin(A-B), \text{ etc.}]$$

$$= \frac{(\sin 9x + \sin 7x) - (\sin 9x + \sin 3x)}{(\cos 3x + \cos x) - (\cos x - \cos 7x)}$$

$$= \frac{(\sin 7x - \sin 3x)}{(\cos 3x + \cos 7x)}$$

$$= \frac{2\cos\left(\frac{7x+3x}{2}\right)\sin\left(\frac{7x-3x}{2}\right)}{2\cos\left(\frac{7x+3x}{2}\right)\cos\left(\frac{7x-3x}{2}\right)}$$

$$\left[\because \sin C - \sin D = 2\cos\frac{(C+D)}{2}\sin\frac{(C-D)}{2}\right]$$

$$= \frac{\cos 5x\sin 2x}{\cos 5x\cos 2x} = \frac{\sin 2x}{\cos 2x}$$

$$= \tan 2x = \text{RHS.}$$

EXAMPLE 13 *Prove that* $\dfrac{(\sin x - \sin y)}{(\cos x + \cos y)} = \tan\left(\dfrac{x-y}{2}\right).$

SOLUTION We have

$$\text{LHS} = \frac{\sin x - \sin y}{\cos x + \cos y}$$

$$= \frac{2\cos\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right)}{2\cos\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)} = \tan\left(\frac{x-y}{2}\right) = \text{RHS.}$$

EXAMPLE 14 *Prove that* $(\cos x + \cos y)^2 + (\sin x + \sin y)^2 = 4\cos^2\left(\dfrac{x-y}{2}\right)$.

SOLUTION We have

$$\begin{aligned}
\text{LHS} &= (\cos x + \cos y)^2 + (\sin x + \sin y)^2 \\
&= \left\{2\cos\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)\right\}^2 + \left\{2\sin\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)\right\}^2 \\
&= 4\cos^2\left(\frac{x+y}{2}\right)\cos^2\left(\frac{x-y}{2}\right) + 4\sin^2\left(\frac{x+y}{2}\right)\cos^2\left(\frac{x-y}{2}\right) \\
&= 4\cos^2\left(\frac{x-y}{2}\right)\cdot\left\{\cos^2\left(\frac{x+y}{2}\right) + \sin^2\left(\frac{x+y}{2}\right)\right\} \\
&= 4\cos^2\left(\frac{x-y}{2}\right) = \text{RHS.}
\end{aligned}$$

EXAMPLE 15 *Prove that*

$$\cos\alpha + \cos\beta + \cos\gamma + \cos(\alpha+\beta+\gamma) = 4\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\beta+\gamma}{2}\right)\cos\left(\frac{\gamma+\alpha}{2}\right).$$

SOLUTION We have

$$\begin{aligned}
\text{LHS} &= (\cos\alpha + \cos\beta) + \{\cos(\alpha+\beta+\gamma) + \cos\gamma\} \\
&= 2\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right) + 2\cos\left(\frac{\alpha+\beta+2\gamma}{2}\right)\cos\left(\frac{\alpha+\beta}{2}\right) \\
&= 2\cos\left(\frac{\alpha+\beta}{2}\right)\cdot\left\{\cos\left(\frac{\alpha-\beta}{2}\right) + \cos\left(\frac{\alpha+\beta+2\gamma}{2}\right)\right\} \\
&= 2\cos\left(\frac{\alpha+\beta}{2}\right)\cdot\left\{2\cos\left(\frac{\gamma+\alpha}{2}\right)\cos\left(\frac{\beta+\gamma}{2}\right)\right\} \\
&\qquad \left[\text{using } \cos C + \cos D = 2\cos\frac{(C+D)}{2}\cos\frac{(C-D)}{2}\right] \\
&= 4\cos\left(\frac{\alpha+\beta}{2}\right)2\cos\left(\frac{\beta+\gamma}{2}\right)\cos\left(\frac{\gamma+\alpha}{2}\right) = \text{RHS.}
\end{aligned}$$

EXAMPLE 16 *Prove that*

$$\sin^2 6x - \sin^2 4x = \sin 10x \sin 2x.$$

SOLUTION

$$\begin{aligned}
\text{LHS} &= \sin^2 6x - \sin^2 4x \\
&= \sin(6x+4x)\sin(6x-4x) \\
&\qquad [\because \sin^2 x - \sin^2 y = \sin(x+y)\sin(x-y)] \\
&= \sin 10x \sin 2x = \text{RHS.}
\end{aligned}$$

EXAMPLE 17 *Prove that* $\cos 20^\circ \cos 40^\circ \cos 60^\circ \cos 80^\circ = \frac{1}{16}\cdot$

SOLUTION We have

$$\begin{aligned}
&\cos 20^\circ \cos 40^\circ \cos 60^\circ \cos 80^\circ\\
&= \frac{1}{2} \cos 60^\circ \cos 40^\circ (2\cos 80^\circ \cos 20^\circ)\\
&= \frac{1}{2} \times \frac{1}{2} \cos 40^\circ \cdot \{\cos (80^\circ + 20^\circ) + \cos (80^\circ - 20^\circ)\}\\
&= \frac{1}{4} \cos 40^\circ (\cos 100^\circ + \cos 60^\circ)\\
&= \frac{1}{4} \cos 40^\circ \left(\cos 100^\circ + \frac{1}{2}\right)\\
&= \frac{1}{8} (2\cos 100^\circ \cos 40^\circ) + \frac{1}{8} \cos 40^\circ\\
&= \frac{1}{8} [\cos (100^\circ + 40^\circ) + \cos (100^\circ - 40^\circ)] + \frac{1}{8} \cos 40^\circ\\
&= \frac{1}{8} (\cos 140^\circ + \cos 60^\circ) + \frac{1}{8} \cos 40^\circ\\
&= \frac{1}{8} \cos 140^\circ + \frac{1}{16} + \frac{1}{8} \cos 40^\circ\\
&= \frac{1}{8} \cos (180^\circ - 40^\circ) + \frac{1}{16} + \frac{1}{8} \cos 40^\circ\\
&= -\frac{1}{8} \cos 40^\circ + \frac{1}{16} + \frac{1}{8} \cos 40^\circ\\
&= \frac{1}{16} = \text{RHS.}
\end{aligned}$$

EXAMPLE 18 *Prove that* $\sin 10^\circ \sin 50^\circ \sin 60^\circ \sin 70^\circ = \frac{\sqrt{3}}{16}\cdot$

SOLUTION We have

$$\begin{aligned}
&\sin 10^\circ \sin 50^\circ \sin 60^\circ \sin 70^\circ\\
&= \frac{1}{2} \sin 60^\circ \sin 10^\circ (2\sin 70^\circ \sin 50^\circ)\\
&= \frac{1}{2} \times \frac{\sqrt{3}}{2} \times \sin 10^\circ \cdot \{\cos (70^\circ - 50^\circ) - \cos (70^\circ + 50^\circ)\}\\
&= \frac{\sqrt{3}}{4} \sin 10^\circ \cdot \{\cos 20^\circ - \cos 120^\circ\}\\
&= \frac{\sqrt{3}}{4} \sin 10^\circ \left(\cos 20^\circ + \frac{1}{2}\right) \qquad \left[\because \cos 120^\circ = -\frac{1}{2}\right]\\
&= \frac{\sqrt{3}}{8} (2\cos 20^\circ \sin 10^\circ) + \frac{\sqrt{3}}{8} \sin 10^\circ
\end{aligned}$$

$$= \frac{\sqrt{3}}{8}[\sin(20° + 10°) - \sin(20° - 10°)] + \frac{\sqrt{3}}{8}\sin 10°$$
$$= \frac{\sqrt{3}}{8}\sin 30° = \left(\frac{\sqrt{3}}{8} \times \frac{1}{2}\right) = \frac{\sqrt{3}}{16}.$$

EXERCISE 15C

Prove that

1. $\sin(150° + x) + \sin(150° - x) = \cos x$
2. $\cos x + \cos(120° - x) + \cos(120° + x) = 0$
3. $\sin\left(x - \frac{\pi}{6}\right) + \cos\left(x - \frac{\pi}{3}\right) = \sqrt{3}\sin x$
4. $\tan\left(\frac{\pi}{4} + x\right) = \frac{1 + \tan x}{1 - \tan x}$
5. $\tan\left(\frac{\pi}{4} - x\right) = \frac{1 - \tan x}{1 + \tan x}$
6. Express each of the following as a product:
   (i) $\sin 10x + \sin 6x$ (ii) $\sin 7x - \sin 3x$
   (iii) $\cos 7x + \cos 5x$ (iv) $\cos 2x - \cos 4x$
7. Express each of the following as an algebraic sum of sines or cosines:
   (i) $2\sin 6x \cos 4x$ (ii) $2\cos 5x \sin 3x$
   (iii) $2\cos 7x \cos 3x$ (iv) $2\sin 8x \sin 2x$

Prove that

8. $\dfrac{\sin x + \sin 3x}{\cos x - \cos 3x} = \cot x$
9. $\dfrac{\sin 7x - \sin 5x}{\cos 7x + \cos 5x} = \tan x$
10. $\dfrac{\sin 5x + \sin 3x}{\cos 5x + \cos 3x} = \tan 4x$
11. $\dfrac{\cos 9x - \cos 5x}{\sin 17x - \sin 3x} = \dfrac{-\sin 2x}{\cos 10x}$
12. $\dfrac{\sin x + \sin 3x + \sin 5x}{\cos x + \cos 3x + \cos 5x} = \tan 3x$
13. $\dfrac{(\sin 7x + \sin 5x) + (\sin 9x + \sin 3x)}{(\cos 7x + \cos 5x) + (\cos 9x + \cos 3x)} = \tan 6x$
14. $\cot 4x(\sin 5x + \sin 3x) = \cot x(\sin 5x - \sin 3x)$
15. $(\sin 3x + \sin x)\sin x + (\cos 3x - \cos x)\cos x = 0$
16. $(\cos x - \cos y)^2 + (\sin x - \sin y)^2 = 4\sin^2\left(\frac{x - y}{2}\right)$

17. $\dfrac{\sin 2x - \sin 2y}{\cos 2y - \cos 2x} = \cot(x + y)$

18. $\dfrac{\cos x + \cos y}{\cos y - \cos x} = \cot\left(\dfrac{x+y}{2}\right)\cot\left(\dfrac{x-y}{2}\right)$

19. $\dfrac{\sin x + \sin y}{\sin x - \sin y} = \tan\left(\dfrac{x+y}{2}\right)\cot\left(\dfrac{x-y}{2}\right)$

20. $\sin 3x + \sin 2x - \sin x = 4\sin x \cos\dfrac{x}{2}\cos\dfrac{3x}{2}$

21. $\dfrac{\cos 4x \sin 3x - \cos 2x \sin x}{\sin 4x \sin x + \cos 6x \cos x} = \tan 2x$

22. $\dfrac{\cos 2x \sin x + \cos 6x \sin 3x}{\sin 2x \sin x + \sin 6x \sin 3x} = \cot 5x$

23. $\sin 10° \sin 30° \sin 50° \sin 70° = \dfrac{1}{16}$

24. $\sin 20° \sin 40° \sin 60° \sin 80° = \dfrac{3}{16}$

25. $\cos 10° \cos 30° \cos 50° \cos 70° = \dfrac{3}{16}$

26. If $\cos x + \cos y = \dfrac{1}{3}$ and $\sin x + \sin y = \dfrac{1}{4}$, prove that $\tan\left(\dfrac{x+y}{2}\right) = \dfrac{3}{4}\cdot$

27. Prove that

(i) $2\cos 45° \cos 15° = \dfrac{(\sqrt{3}+1)}{2}$ (ii) $2\sin 75° \sin 15° = \dfrac{1}{2}$

(iii) $\cos 15° - \sin 15 = \dfrac{1}{\sqrt{2}}$

ANSWERS (EXERCISE 15C)

6. (i) $2\sin 8x \cos 2x$ (ii) $2\cos 5x \sin 2x$ (iii) $2\cos 6x \cos x$ (iv) $2\sin 3x \sin x$

7. (i) $\sin 10x + \sin 2x$ (ii) $\sin 8x - \sin 2x$ (iii) $\cos 10x + \cos 4x$ (iv) $\cos 6x - \cos 10x$

HINTS TO SOME SELECTED QUESTIONS

16. LHS $= \left\{-2\sin\left(\dfrac{x+y}{2}\right)\sin\left(\dfrac{x-y}{2}\right)\right\}^2 + \left\{\cos\left(\dfrac{x+y}{2}\right)\sin\left(\dfrac{x-y}{2}\right)\right\}^2$

$= 4\sin^2\left(\dfrac{x-y}{2}\right)\cdot\left\{\sin^2\left(\dfrac{x+y}{2}\right) + \cos^2\left(\dfrac{x+y}{2}\right)\right\} = 4\sin^2\left(\dfrac{x-y}{2}\right)$

21. LHS $= \dfrac{2\cos 4x \sin 3x - 2\cos 2x \sin x}{2\sin 4x \sin x + 2\cos\ 6x \cos x}$

$= \dfrac{\{\sin(4x+3x) - \sin(4x-3x)\} - \{\sin(2x+x) - \sin(2x-x)\}}{\{\cos(4x-x) - \cos(4x+x)\} + \{\cos(6x+x) + \cos(6x-x)\}}$

$= \dfrac{(\sin 7x - \sin x) - (\sin 3x - \sin x)}{(\cos 3x - \cos 5x) + (\cos 7x + \cos 5x)} = \dfrac{(\sin 7x - \sin 3x)}{(\cos 7x + \cos 3x)}$

$= \dfrac{2\cos\left(\dfrac{7x+3x}{2}\right)\sin\left(\dfrac{7x-3x}{2}\right)}{2\cos\left(\dfrac{7x+3x}{2}\right)\cos\left(\dfrac{7x-3x}{2}\right)} = \dfrac{\sin 2x}{\cos 2x} = \tan 2x =$ RHS.

26. $\cos x + \cos y = \dfrac{1}{3} \Rightarrow 2\cos\left(\dfrac{x+y}{2}\right)\cos\left(\dfrac{x-y}{2}\right) = \dfrac{1}{3}$ … (i)

$\sin x + \sin y = \dfrac{1}{4} \Rightarrow 2\sin\left(\dfrac{x+y}{2}\right)\cos\left(\dfrac{x-y}{2}\right) = \dfrac{1}{4}$ … (ii)

Dividing (ii) by (i), we get $\tan\left(\dfrac{x+y}{2}\right) = \dfrac{3}{4}$.

27. (i) Use $2\cos x \cos y = \cos\ (x+y) + \cos\ (x-y)$.

(ii) Use $2\sin x \sin y = \cos\ (x-y) - \cos\ (x+y)$.

(iii) LHS $= \cos 15° - \cos\ (90° - 15°) = \cos 15° - \cos 75°$.

Now, use $\cos C - \cos D = 2\sin\left(\dfrac{C+D}{2}\right)\sin\left(\dfrac{D-C}{2}\right)$

TRIGONOMETRIC FUNCTIONS OF MULTIPLES OF ANGLES

THEOREM 1 *(i)* $\sin 2x = 2\sin x \cos x$

(ii) $\cos 2x = (\cos^2 x - \sin^2 x) = (2\cos^2 x - 1) = (1 - 2\sin^2 x)$

(iii) $\tan 2x = \dfrac{2\tan x}{1 - \tan^2 x}$

PROOF (i) $\sin 2x = \sin(x + x)$

$= \sin x \cos x + \cos x \sin x$

$= 2\sin x \cos x.$

(ii) $\cos 2x = \cos(x + x)$

$= \cos x \cos x - \sin x \sin x$

$= \cos^2 x - \sin^2 x.$

Now, $\cos 2x = \cos^2 x - \sin^2 x$

$= (1 - \sin^2 x - \sin^2 x) = (1 - 2\sin^2 x)$

$[\because\ \cos^2 x = (1 - \sin^2 x)].$

Also, $\cos 2x = \cos^2 x - \sin^2 x$

$= \cos^2 x - (1 - \cos^2 x)$ $[\because\ \sin^2 x = 1 - \cos^2 x]$

$= (2\cos^2 x - 1).$

$\therefore\quad \cos 2x = (\cos^2 x - \sin^2 x) = (1 - 2\sin^2 x) = (2\cos^2 x - 1).$

(iii) $\tan 2x = \tan(x + x)$

$$= \frac{\tan x + \tan x}{1 - (\tan x \times \tan x)}$$

$$= \frac{2\tan x}{1 - \tan^2 x}.$$

$$\therefore \quad \tan 2x = \frac{2\tan x}{1 - \tan^2 x}.$$

REMARK The above results may be expressed as

(i) $(1 - \cos 2x) = 2\sin^2 x$

(ii) $(1 + \cos 2x) = 2\cos^2 x$

(iii) $\dfrac{(1 - \cos 2x)}{(1 + \cos 2x)} = \tan^2 x$

THEOREM 2 *If x is not an odd multiple of* $\frac{\pi}{2}$ *then prove that*

(i) $\sin 2x = \dfrac{2\tan x}{1 + \tan^2 x}$ *(ii)* $\cos 2x = \dfrac{1 - \tan^2 x}{1 + \tan^2 x}$

PROOF Since x is not an odd multiple of $\frac{\pi}{2}$, we have $\cos x \neq 0$.

(i) $\sin 2x = 2\sin x \cos x$

$$= \frac{2\sin x \cos x}{\cos^2 x + \sin^2 x} \qquad [\because\ 1 = \cos^2 x + \sin^2 x]$$

$$= \frac{\left(\dfrac{2\sin x}{\cos x}\right)}{1 + \tan^2 x} \qquad [\text{dividing num. and denom. by } \cos^2 x]$$

$$= \frac{2\tan x}{1 + \tan^2 x}.$$

$$\therefore \quad \sin 2x = \frac{2\tan x}{1 + \tan^2 x}.$$

(ii) $\cos 2x = \cos^2 x - \sin^2 x$

$$= \frac{\cos^2 x - \sin^2 x}{\cos^2 x + \sin^2 x} \qquad [\because\ 1 = \cos^2 x + \sin^2 x]$$

$$= \frac{1 - \dfrac{\sin^2 x}{\cos^2 x}}{1 + \dfrac{\sin^2 x}{\cos^2 x}} \qquad [\text{dividing num. and denom. by } \cos^2 x]$$

$$= \frac{1 - \tan^2 x}{1 + \tan^2 x}.$$

$$\therefore \quad \cos 2x = \frac{1 - \tan^2 x}{1 + \tan^2 x}.$$

SUMMARY

(i) $\sin 2x = 2\sin x \cos x$

(ii) $\cos 2x = \cos^2 x - \sin^2 x = 2\cos^2 x - 1 = 1 - 2\sin^2 x$

(iii) $\tan 2x = \dfrac{2\tan x}{1 - \tan^2 x}$

(iv) $\sin 2x = \dfrac{2\tan x}{1 + \tan^2 x}$

(v) $\cos 2x = \dfrac{1 - \tan^2 x}{1 + \tan^2 x}$

THEOREM 3 *(i)* $\sin 3x = 3\sin x - 4\sin^3 x$ *(ii)* $\cos 3x = 4\cos^3 x - 3\cos x$

(iii) $\tan 3x = \dfrac{3\tan x - \tan^3 x}{1 - 3\tan^2 x}$

PROOF (i) $\sin 3x = \sin (2x + x)$

$= \sin 2x \cos x + \cos 2x \sin x$

$= 2\sin x \cos^2 x + \cos 2x \sin x$ $[\because \sin 2x = 2\sin x \cos x]$

$= 2\sin x(1 - \sin^2 x) + (1 - 2\sin^2 x)\sin x$

$[\because \cos^2 x = (1 - \sin^2 x)$ and $\cos 2x = (1 - 2\sin^2 x)]$

$= 3\sin x - 4\sin^3 x.$

(ii) $\cos 3x = \cos (2x + x)$

$= \cos 2x \cos x - \sin 2x \sin x$

$= (2\cos^2 x - 1)\cos x - 2\sin^2 x \cos x$

$[\because \cos 2x = (2\cos^2 x - 1)$ and $\sin 2x = 2\sin x \cos x]$

$= (2\cos^2 x - 1)\cos x + 2(\cos^2 x - 1)\cos x$

$[\because \sin^2 x = (1 - \cos^2 x)]$

$= 4\cos^3 x - 3\cos x.$

(iii) $\tan 3x = \tan (2x + x) = \dfrac{\tan 2x + \tan x}{1 - \tan 2x \tan x}$

$$= \frac{\dfrac{2\tan x}{(1 - \tan^2 x)} + \tan x}{1 - \left(\dfrac{2\tan x}{1 - \tan^2 x}\right)\cdot \tan x} = \frac{3\tan x - \tan^3 x}{1 - 3\tan^2 x}.$$

SUMMARY

(i) $\sin 3x = 3\sin x - 4\sin^3 x$

(ii) $\cos 3x = 4\cos^3 x - 3\cos x$

(iii) $\tan 3x = \dfrac{3\tan x - \tan^3 x}{1 - 3\tan^2 x}$

SOLVED EXAMPLES

EXAMPLE 1 *If* $\sin x = -\frac{1}{2}$ *and* $\pi < x < \frac{3\pi}{2}$, *find the values of*

(i) $\sin 2x$, *(ii)* $\cos 2x$, *(iii)* $\tan 2x$.

SOLUTION Since x lies in Quadrant III, we have $\sin x < 0$, $\cos x < 0$ and $\tan x > 0$.

Now, $\sin x = -\frac{1}{2}$ (given).

$$\therefore \quad \cos x = -\sqrt{1-\sin^2 x} = -\sqrt{\left(1-\frac{1}{4}\right)} = -\sqrt{\frac{3}{4}} = \frac{-\sqrt{3}}{2}$$

and $\tan x = \frac{\sin x}{\cos x} = \left(-\frac{1}{2} \times \frac{2}{-\sqrt{3}}\right) = \frac{1}{\sqrt{3}}$.

(i) $\sin 2x = 2\sin x \cos x$

$$= 2 \times \left(-\frac{1}{2}\right) \times \left(-\frac{\sqrt{3}}{2}\right) = \frac{\sqrt{3}}{2}.$$

(ii) $\cos 2x = (2\cos^2 x - 1)$

$$= \left(2 \times \frac{3}{4}\right) - 1 = \left(\frac{3}{2} - 1\right) = \frac{1}{2}.$$

(iii) $\tan 2x = \frac{\sin 2x}{\cos 2x} = \left(\frac{\sqrt{3}}{2} \times \frac{2}{1}\right) = \sqrt{3}$.

EXAMPLE 2 *If* $\sec x = -\frac{13}{12}$ *and* $\frac{\pi}{2} < x < \pi$, *find the values of*

(i) $\sin 2x$, *(ii)* $\cos 2x$, *(iii)* $\tan 2x$.

SOLUTION Since x lies in Quadrant II, we have $\sin x > 0$, $\cos x < 0$ and $\tan x < 0$.

Now, $\sec x = \frac{-13}{12} \Rightarrow \cos x = \frac{1}{\sec x} = \frac{-12}{13}$.

$$\therefore \quad \sin x = +\sqrt{1-\cos^2 x} = +\sqrt{1-\frac{144}{169}} = +\sqrt{\frac{25}{169}} = \frac{5}{13}.$$

And, $\tan x = \frac{\sin x}{\cos x} = \frac{5}{13} \times \left(\frac{-13}{12}\right) = \frac{-5}{12}$.

(i) $\sin 2x = 2\sin x \cos x$

$$= \left\{2 \times \frac{5}{13} \times \frac{(-12)}{13}\right\} = \frac{-120}{169}.$$

(ii) $\cos 2x = 2\cos^2 x - 1$

$$= \left(2 \times \frac{144}{169} - 1\right) = \frac{119}{169}.$$

(iii) $\tan 2x = \frac{\sin 2x}{\cos 2x} = \left(\frac{-120}{169} \times \frac{169}{119}\right) = \frac{-120}{119}$.

EXAMPLE 3 *If* $\tan x = \frac{-3}{4}$ *and* $\frac{3\pi}{2} < x < 2\pi$, *find the values of*

(i) $\sin 2x$, *(ii)* $\cos 2x$, *(iii)* $\tan 2x$.

SOLUTION Since x lies in Quadrant IV, we have

$$\cos x > 0,\ \sin x < 0 \text{ and } \tan x < 0.$$

$$\therefore \quad \sec x = \sqrt{1 + \tan^2 x} = \sqrt{\left(1 + \frac{9}{16}\right)} = \sqrt{\frac{25}{16}} = \frac{5}{4} \Rightarrow \cos x = \frac{4}{5}.$$

And, $\sin x = -\sqrt{1 - \cos^2 x} = -\sqrt{\left(1 - \frac{16}{25}\right)} = -\sqrt{\frac{9}{25}} = \frac{-3}{5}.$

$$\therefore \quad \tan x = \frac{\sin x}{\cos x} = \left(\frac{-3}{5} \times \frac{5}{4}\right) = \frac{-3}{4}.$$

(i) $\sin 2x = 2\sin x \cos x$

$$= \left\{2 \times \frac{(-3)}{5} \times \frac{4}{5}\right\} = \frac{-24}{25}.$$

(ii) $\cos 2x = (2\cos^2 x - 1)$

$$= \left(2 \times \frac{16}{25} - 1\right) = \frac{7}{25}.$$

(iii) $\tan 2x = \frac{\sin 2x}{\cos 2x} = \left(\frac{-24}{25} \times \frac{25}{7}\right) = \frac{-24}{7}.$

EXAMPLE 4 *(i) If* $\sin x = \frac{1}{3}$, *find the value of* $\sin 3x$.

(ii) If $\cos x = \frac{1}{2}$, *find the value of* $\cos 3x$.

SOLUTION (i) $\sin 3x = (3\sin x - 4\sin^3 x)$

$$= \left\{\left(3 \times \frac{1}{3}\right) - 4 \times \left(\frac{1}{3}\right)^3\right\} = \left(1 - \frac{4}{27}\right) = \frac{23}{27}.$$

(ii) $\cos 3x = (4\cos^3 x - 3\cos x)$

$$= \left\{\left(4 \times \frac{1}{8}\right) - \left(3 \times \frac{1}{2}\right)\right\} = \left(\frac{1}{2} - \frac{3}{2}\right) = -1.$$

EXAMPLE 5 *If* $\cos x = \frac{4}{5}$ *and x is acute, find the value of* $\tan 2x$.

SOLUTION $\cos x = \frac{4}{5} \Rightarrow \sec x = \frac{5}{4}$

$$\Rightarrow \tan x = \sqrt{\sec^2 x - 1} = \sqrt{\frac{25}{16} - 1} = \sqrt{\frac{9}{16}} = \frac{3}{4}.$$

$$\therefore \quad \tan 2x = \frac{2\tan x}{1 - \tan^2 x} = \frac{\left(2 \times \frac{3}{4}\right)}{\left(1 - \frac{9}{16}\right)} = \left(\frac{3}{2} \times \frac{16}{7}\right) = \frac{24}{7}.$$

EXAMPLE 6 *If* $\tan x = \frac{1}{7}$ *and* $\tan y = \frac{1}{3}$, *show that* $\cos 2x = \sin 4y$.

SOLUTION We have

$$\cos 2x = \frac{(1-\tan^2 x)}{(1+\tan^2 x)} = \frac{\left(1-\frac{1}{49}\right)}{\left(1+\frac{1}{49}\right)} = \left(\frac{48}{49}\times\frac{49}{50}\right) = \frac{24}{25}.$$

$$\sin 4y = 2\sin 2y \cos 2y$$

$$= 2\times\frac{(2\tan y)}{(1+\tan^2 y)}\times\frac{(1-\tan^2 y)}{(1+\tan^2 y)}$$

$$= \frac{2\times\left(2\times\frac{1}{3}\right)}{\left(1+\frac{1}{9}\right)}\times\frac{\left(1-\frac{1}{9}\right)}{\left(1+\frac{1}{9}\right)} = \left(\frac{4}{3}\times\frac{9}{10}\times\frac{8}{9}\times\frac{9}{10}\right) = \frac{24}{25}.$$

$\therefore \quad \cos 2x = \sin 4y$.

EXAMPLE 7 *Prove that*

(i) $\sin\frac{\pi}{6}\cos\frac{\pi}{6} = \frac{\sqrt{3}}{4}$ (ii) $\cos^2\frac{\pi}{12} - \sin^2\frac{\pi}{12} = \frac{\sqrt{3}}{2}$

SOLUTION (i) $\sin\frac{\pi}{6}\cos\frac{\pi}{6} = \frac{1}{2}\left(2\sin\frac{\pi}{6}\cos\frac{\pi}{6}\right)$

$$= \frac{1}{2}\times\sin\left(2\times\frac{\pi}{6}\right) \qquad [\because 2\sin x\cos x = \sin 2x]$$

$$= \frac{1}{2}\sin\frac{\pi}{3} = \left(\frac{1}{2}\times\frac{\sqrt{3}}{2}\right) = \frac{\sqrt{3}}{4}.$$

(ii) $\cos^2\frac{\pi}{12} - \sin^2\frac{\pi}{12} = \cos\left(2\times\frac{\pi}{12}\right) \qquad [\because \cos^2 x - \sin^2 x = \cos 2x]$

$$= \cos\frac{\pi}{6} = \frac{3}{2}.$$

EXAMPLE 8 *Prove that*

(i) $\frac{\sin 2x}{1-\cos 2x} = \cot x$ (ii) $\frac{1-\cos 2x}{1+\cos 2x} = \tan^2 x$

SOLUTION (i) LHS $= \frac{\sin 2x}{1-\cos 2x}$

$$= \frac{2\sin x\cos x}{2\sin^2 x}$$

$[\because \sin 2x = 2\sin x\cos x, (1-\cos 2x) = 2\sin^2 x]$

$= \cot x =$ RHS.

Hence, $\frac{\sin 2x}{1-\cos 2x} = \cot x$.

(ii) LHS $= \dfrac{1-\cos 2x}{1+\cos 2x} = \dfrac{2\sin^2 x}{2\cos^2 x}$ $\left[\begin{array}{l}\because\ (1-\cos 2x) = 2\sin^2 x, \\ (1+\cos 2x) = 2\cos^2 x\end{array}\right]$

$= \tan^2 x = \text{RHS}.$

Hence, $\dfrac{1-\cos 2x}{1+\cos 2x} = \tan^2 x.$

EXAMPLE 9 *Prove that* $\cos 4x = 1 - 8\sin^2 x \cos^2 x.$

SOLUTION We have

$$\begin{aligned}\text{LHS} &= \cos 4x \\ &= \cos 2(2x) = \cos 2\theta, \text{ where } 2x = \theta \\ &= \cos^2\theta - \sin^2\theta \qquad [\because \cos 2\theta = \cos^2\theta - \sin^2\theta] \\ &= (1 - 2\sin^2\theta) = \{1 - 2(\sin 2x)^2\} \\ &= 1 - 2(2\sin x \cos x)^2 \qquad [\because \sin 2x = 2\sin x \cos x] \\ &= 1 - 8\sin^2 x \cos^2 x = \text{RHS}.\end{aligned}$$

Hence, $\cos 4x = 1 - 8\sin^2 x \cos^2 x.$

EXAMPLE 10 *Prove that* $\dfrac{1+\sin 2x - \cos 2x}{1+\sin 2x + \cos 2x} = \tan x.$

SOLUTION We have

$$\begin{aligned}\text{LHS} &= \frac{1+\sin 2x - \cos 2x}{1+\sin 2x + \cos 2x} \\ &= \frac{(1-\cos 2x) + \sin 2x}{(1+\cos 2x) + \sin 2x} \\ &= \frac{2\sin^2 x + 2\sin x \cos x}{2\cos^2 x + 2\sin x \cos x}\end{aligned}$$

$$\left[\begin{array}{l}\because\ (1-\cos 2x) = 2\sin^2 x, (1+\cos 2x) = 2\cos^2 x \\ \quad \text{and } \sin 2x = 2\sin x \cos x\end{array}\right]$$

$$= \frac{2\sin x(\sin x + \cos x)}{2\cos x(\sin x + \cos x)} = \tan x = \text{RHS}.$$

EXAMPLE 11 *Prove that* $\dfrac{1-\sin 2x}{1+\sin 2x} = \tan^2\left(\dfrac{\pi}{4} - x\right).$

SOLUTION We have

$$\begin{aligned}\text{LHS} &= \frac{1-\sin 2x}{1+\sin 2x} \\ &= \frac{1-\cos\left(\frac{\pi}{2} - 2x\right)}{1+\cos\left(\frac{\pi}{2} - 2x\right)} \qquad \left[\because \sin 2x = \cos\left(\frac{\pi}{2} - 2x\right)\right]\end{aligned}$$

$$= \frac{2\sin^2\left(\frac{\pi}{4} - x\right)}{2\cos^2\left(\frac{\pi}{4} - x\right)} = \tan^2\left(\frac{\pi}{4} - x\right) = \text{RHS.}$$

$[\because\ (1 - \cos 2\theta) = 2\sin^2\theta \text{ and } (1 + \cos 2\theta) = 2\cos^2\theta]$

$$\therefore \quad \frac{1 - \sin 2x}{1 + \sin 2x} = \tan^2\left(\frac{\pi}{4} - x\right).$$

EXAMPLE 12 *Prove that* $\dfrac{\cos x + \sin x}{\cos x - \sin x} - \dfrac{\cos x - \sin x}{\cos x + \sin x} = 2\tan 2x.$

SOLUTION We have

$$\begin{aligned}
\text{LHS} &= \frac{\cos x + \sin x}{\cos x - \sin x} - \frac{\cos x - \sin x}{\cos x + \sin x} \\
&= \frac{(\cos x + \sin x)^2 - (\cos x - \sin x)^2}{\cos^2 x - \sin^2 x} \\
&= \frac{4\sin x \cos x}{\cos^2 x - \sin^2 x} = \frac{2\sin 2x}{\cos 2x} \\
&= 2\tan 2x = \text{RHS.}
\end{aligned}$$

EXAMPLE 13 *Prove that* $\tan 4\theta = \dfrac{4\tan\theta(1 - \tan^2\theta)}{1 - 6\tan^2\theta + \tan^4\theta}$.

SOLUTION We have

$$\begin{aligned}
\text{LHS} &= \tan 4\theta = \tan 2(2\theta) \\
&= \tan 2x, \text{ where } x = 2\theta \\
&= \frac{2\tan x}{1 - \tan^2 x} = \frac{2\tan 2\theta}{1 - \tan^2 2\theta} \\
&= \frac{2 \times \dfrac{2\tan\theta}{(1 - \tan^2\theta)}}{1 - \left(\dfrac{2\tan\theta}{1 - \tan^2\theta}\right)^2} = \frac{4\tan\theta(1 - \tan^2\theta)}{(1 - \tan^2\theta)^2 - 4\tan^2\theta} \\
&= \frac{4\tan\theta(1 - \tan^2\theta)}{1 - 6\tan^2\theta + \tan^4\theta} = \text{RHS.}
\end{aligned}$$

EXAMPLE 14 *Prove that* $\dfrac{\tan 5\theta + \tan 3\theta}{\tan 5\theta - \tan 3\theta} = 4\cos 2\theta \cos 4\theta.$

SOLUTION We have

$$\text{LHS} = \frac{\tan 5\theta + \tan 3\theta}{\tan 5\theta - \tan 3\theta}$$

$$= \frac{\left(\dfrac{\sin 5\theta}{\cos 5\theta} + \dfrac{\sin 3\theta}{\cos 3\theta}\right)}{\left(\dfrac{\sin 5\theta}{\cos 5\theta} - \dfrac{\sin 3\theta}{\cos 3\theta}\right)} = \frac{(\sin 5\theta \cos 3\theta + \cos 5\theta \sin 3\theta)}{(\sin 5\theta \cos 3\theta - \cos 5\theta \sin 3\theta)}$$

$$= \frac{\sin (5\theta + 3\theta)}{\sin (5\theta - 3\theta)} = \frac{\sin 8\theta}{\sin 2\theta} = \frac{2\sin 4\theta \cos 4\theta}{\sin 2\theta}$$

$$= \frac{4\sin 2\theta \cos 2\theta \cos 4\theta}{\sin 2\theta} = 4\cos 2\theta \cos 4\theta = \text{RHS.}$$

EXAMPLE 15 *Prove that* $\dfrac{\sec 8\theta - 1}{\sec 4\theta - 1} = \dfrac{\tan 8\theta}{\tan 2\theta}$.

SOLUTION We have

$$\text{LHS} = \frac{\sec 8\theta - 1}{\sec 4\theta - 1}$$

$$= \frac{\left(\dfrac{1}{\cos 8\theta} - 1\right)}{\left(\dfrac{1}{\cos 4\theta} - 1\right)} = \frac{(1 - \cos 8\theta)}{(1 - \cos 4\theta)} \cdot \frac{\cos 4\theta}{\cos 8\theta}$$

$$= \frac{(2\sin^2 4\theta)(\cos 4\theta)}{(2\sin^2 2\theta)(\cos 8\theta)}$$

$$= \frac{(2\sin 4\theta \cos 4\theta)(\sin 4\theta)}{(2\sin^2 2\theta)(\cos 8\theta)}$$

$$= \frac{(\sin 8\theta)(2\sin 2\theta \cos 2\theta)}{(\cos 8\theta)(2\sin^2 2\theta)}$$

$$= \tan 8\theta \cot 2\theta = \frac{\tan 8\theta}{\tan 2\theta} = \text{RHS.}$$

EXAMPLE 16 *Show that* $\sqrt{2 + \sqrt{2 + 2\cos 4\theta}} = 2\cos\theta$.

SOLUTION $\text{LHS} = \sqrt{2 + \sqrt{2 + 2\cos 4\theta}}$

$$= \sqrt{2 + \sqrt{2(1 + \cos 4\theta)}} = \sqrt{2 + \sqrt{4\cos^2 2\theta}}$$

$$= \sqrt{2 + 2\cos 2\theta} = \sqrt{2(1 + \cos 2\theta)}$$

$$= \sqrt{4\cos^2\theta} = 2\cos\theta = \text{RHS.}$$

EXAMPLE 17 *Prove that* $\cos 5x = 16\cos^5 x - 20\cos^3 x + 5\cos x$.

SOLUTION We have

$$\cos 5x = \cos(3x + 2x)$$

$$= \cos 3x \cos 2x - \sin 3x \sin 2x$$

$$= (4\cos^3 x - 3\cos x)(2\cos^2 x - 1) - (3\sin x - 4\sin^3 x)(2\sin x \cos x)$$

$$= (8\cos^5 x - 10\cos^3 x + 3\cos x) - 6\sin^2 x \cos x + 8\sin^4 x \cos x$$

$$= (8\cos^5 x - 10\cos^3 x + 3\cos x) - 6(1-\cos^2 x)\cos x + 8(1-\cos^2 x)^2 \cdot \cos x$$

$$= 16\cos^5 x - 20\cos^3 x + 5\cos x.$$

Hence, $\cos 5x = 16\cos^5 x - 20\cos^3 x + 5\cos x$.

EXAMPLE 18 *Prove that* $\cos 6x = 32\cos^6 x - 48\cos^4 x + 18\cos^2 x - 1$.

SOLUTION We have

$$\begin{aligned}\cos 6x &= \cos 2(3x)\\ &= \cos 2\theta, \text{ where } 3x = \theta\\ &= 2\cos^2\theta - 1 = 2\cos^2 3x - 1\\ &= 2(4\cos^3 x - 3\cos x)^2 - 1 \quad [\because \cos 3x = (4\cos^3 x - 3\cos x)]\\ &= 2(9\cos^2 x + 16\cos^6 x - 24\cos^4 x) - 1\\ &= 32\cos^6 x - 48\cos^4 x + 18\cos^2 x - 1.\end{aligned}$$

Hence, $\cos 6x = 32\cos^6 x - 48\cos^4 x + 18\cos^2 x - 1$.

EXAMPLE 19 *Prove that* $\cot 2x \cot x - \cot 3x \cot 2x - \cot 3x \cot x = 1$.

SOLUTION We have

$$\cot 3x = \cot(2x + x)$$

$$\Leftrightarrow \quad \cot 3x = \frac{\cot 2x \cot x - 1}{\cot 2x + \cot x}$$

$$\Leftrightarrow \quad \cot 3x \cot 2x + \cot 3x \cot x = \cot 2x \cot x - 1$$

$$\Leftrightarrow \quad \cot 2x \cot x - \cot 3x \cot 2x - \cot 3x \cot x = 1.$$

Hence, $\cot 2x \cot x - \cot 3x \cot 2x - \cot 3x \cot x = 1$.

EXAMPLE 20 *Find the value of*

(i) $\sin 18°$ *(ii)* $\cos 18°$ *(iii)* $\cos 36°$ *(iv)* $\sin 36°$
(v) $\sin 72°$ *(vi)* $\cos 72°$ *(vii)* $\sin 54°$ *(viii)* $\cos 54°$

SOLUTION (i) Let $\theta = 18°$. Then,

$$\begin{aligned}\theta = 18° &\Rightarrow 5\theta = 90°\\ &\Rightarrow 2\theta = (90° - 3\theta)\\ &\Rightarrow \sin 2\theta = \sin(90° - 3\theta) = \cos 3\theta\\ &\Rightarrow 2\sin\theta\cos\theta = 4\cos^3\theta - 3\cos\theta\\ &\Rightarrow 2\sin\theta\cos\theta - 4\cos^3\theta + 3\cos\theta = 0\\ &\Rightarrow \cos\theta(2\sin\theta - 4\cos^2\theta + 3) = 0\\ &\Rightarrow 2\sin\theta - 4\cos^2\theta + 3 = 0 \quad [\because \cos\theta = \cos 18° \neq 0]\\ &\Rightarrow 2\sin\theta - 4(1 - \sin^2\theta) + 3 = 0\\ &\Rightarrow 4\sin^2\theta + 2\sin\theta - 1 = 0\\ &\Rightarrow \sin\theta = \frac{-2 \pm \sqrt{4+16}}{8} = \frac{(-1 \pm \sqrt{5})}{4}\\ &\Rightarrow \sin\theta = \frac{(\sqrt{5} - 1)}{4} \qquad [\because \sin\theta = \sin 18° > 0].\end{aligned}$$

$$\therefore \quad \sin 18° = \frac{(\sqrt{5} - 1)}{4}.$$

(ii) $\cos^2 18° = (1 - \sin^2 18°)$

$$= \left\{1 - \frac{(\sqrt{5}-1)^2}{16}\right\} = \left\{1 - \frac{(6-2\sqrt{5})}{16}\right\} = \frac{10+2\sqrt{5}}{16}$$

$$\Leftrightarrow \quad \cos 18° = \frac{\sqrt{10+2\sqrt{5}}}{4} \qquad [\because \cos 18° > 0].$$

(iii) $\cos 36° = (1 - 2\sin^2 18°)$

$$= \left\{1 - 2\cdot\frac{(\sqrt{5}-1)^2}{16}\right\} = \left\{1 - \frac{(6-2\sqrt{5})}{8}\right\}$$

$$= \frac{\sqrt{5}+1}{4}.$$

(iv) $\sin 36° = \sqrt{1-\cos^2 36°} = \left\{1 - \frac{(\sqrt{5}+1)^2}{16}\right\}^{1/2}$

$$= \left\{\frac{10-2\sqrt{5})}{16}\right\}^{1/2} = \frac{\sqrt{10-2\sqrt{5}}}{4}.$$

(v) $\sin 72° = \sin\ (90° - 18°) = \cos 18° = \dfrac{\sqrt{10+2\sqrt{5}}}{4}.$

(vi) $\cos 72° = \cos\ (90° - 18°) = \sin 18° = \dfrac{(\sqrt{5}-1)}{4}.$

(vii) $\sin 54° = \sin\ (90° - 36°) = \cos 36° = \dfrac{(\sqrt{5}+1)}{4}.$

(viii) $\cos 54° = \cos\ (90° - 36°) = \sin 36° = \dfrac{\sqrt{10-2\sqrt{5}}}{4}.$

SUMMARY

(i) $\boldsymbol{\sin 18° = \dfrac{(\sqrt{5}-1)}{4} = \cos 72°}$

(ii) $\boldsymbol{\cos 18° = \dfrac{\sqrt{(10+2\sqrt{5})}}{4} = \sin 72°}$

(iii) $\boldsymbol{\cos 36° = \dfrac{(\sqrt{5}+1)}{4} = \sin 54°}$

(iv) $\boldsymbol{\sin 36° = \dfrac{\sqrt{(10-2\sqrt{5})}}{4} = \cos 54°}$

EXAMPLE 21 *Prove that* $\sin\dfrac{\pi}{10} + \sin\dfrac{13\pi}{10} = \dfrac{-1}{2}.$

SOLUTION We have

$$\text{LHS} = \sin\frac{\pi}{10} + \sin\frac{13\pi}{10}$$

$$= \sin\frac{\pi}{10} + \sin\left(\pi + \frac{3\pi}{10}\right) = \sin\frac{\pi}{10} - \sin\frac{3\pi}{10} \quad [\because \sin(\pi + \theta) = -\sin\theta]$$

$$= \sin 18° - \sin 54° = \sin 18° - \sin(90° - 36°)$$

$$= (\sin 18° - \cos 36°) = \left\{\frac{(\sqrt{5}-1)}{4} - \frac{(\sqrt{5}+1)}{4}\right\} = \frac{-1}{2}$$

$$\left[\because \sin 18° = \frac{(\sqrt{5}-1)}{4} \text{ and } \cos 36° = \frac{(\sqrt{5}+1)}{4}\right]$$

EXAMPLE 22 *Prove that* $(\cos^2 48° - \sin^2 12°) = \frac{(\sqrt{5}+1)}{8}$.

SOLUTION We have

$$\text{LHS} = (\cos^2 48° - \sin^2 12°)$$

$$= \frac{1}{2}(2\cos^2 48° - 2\sin^2 12°)$$

$$= \frac{1}{2}\{(1 + \cos 96°) - (1 - \cos 24°)\}$$

$$[\because 2\cos^2\theta = (1 + \cos 2\theta) \text{ and } 2\sin^2\theta = (1 - \cos 2\theta)]$$

$$= \frac{1}{2}(\cos 96° + \cos 24°)$$

$$= \frac{1}{2}\left[2\cos\left(\frac{96° + 24°}{2}\right)\cos\left(\frac{96° - 24°}{2}\right)\right]$$

$$= \frac{1}{2} \times 2\cos 60° \cos 36° = \frac{1}{2} \times \frac{(\sqrt{5}+1)}{4} \quad \left[\because \cos 36° = \frac{(\sqrt{5}+1)}{4}\right]$$

$$= \frac{(\sqrt{5}+1)}{8} = \text{RHS.}$$

EXAMPLE 23 *Prove that* $(\sin^2 72° - \sin^2 60°) = \frac{(\sqrt{5}-1)}{8}$.

SOLUTION We have

$$\text{LHS} = (\sin^2 72° - \sin^2 60°)$$

$$= \frac{1}{2}(2\sin^2 72° - 2\sin^2 60°)$$

$$= \frac{1}{2}\{(1 - \cos 144°) - (1 - \cos 120°)\} \quad [\because 2\sin^2\theta = (1 - \cos 2\theta)]$$

$$= \frac{1}{2}(\cos 120° - \cos 144°)$$

$$= -\frac{1}{4} - \frac{1}{2}\cos 144° = -\frac{1}{4} - \frac{1}{2}\cos(180° - 36°) \left[\because \cos 120° = -\frac{1}{2}\right]$$

$$= -\frac{1}{4} + \frac{1}{2}\cos 36° = -\frac{1}{4} + \frac{(\sqrt{5}+1)}{8} \quad \left[\because \cos 36° = \frac{(\sqrt{5}+1)}{4}\right]$$

$$= \frac{(\sqrt{5}-1)}{8} = \text{RHS.}$$

EXAMPLE 24 *Prove that* $\cos 6^\circ \cos 42^\circ \cos 66^\circ \cos 78^\circ = \frac{1}{16}$.

SOLUTION We have

$$\begin{aligned}
\text{LHS} &= \cos 6^\circ \cos 42^\circ \cos 66^\circ \cos 78^\circ \\
&= \frac{1}{4}(2\cos 66^\circ \cos 6^\circ)(2\cos 78^\circ \cos 42^\circ) \\
&= \frac{1}{4}[\cos(66^\circ + 6^\circ) + \cos(66^\circ - 6^\circ)] \times \\
&\qquad\qquad [\cos(78^\circ + 42^\circ) + \cos(78^\circ - 42^\circ)] \\
&= \frac{1}{4}(\cos 72^\circ + \cos 60^\circ)(\cos 120^\circ + \cos 36^\circ) \\
&= \frac{1}{4}\left(\sin 18^\circ + \frac{1}{2}\right)\left(-\frac{1}{2} + \cos 36^\circ\right) \\
&\qquad [\because \cos 72^\circ = \cos(90^\circ - 18^\circ) = \sin 18^\circ] \\
&= \frac{1}{4}\left[\frac{(\sqrt{5}-1)}{4} + \frac{1}{2}\right]\left[-\frac{1}{2} + \frac{(\sqrt{5}+1)}{4}\right] \\
&\qquad \left[\because \sin 18^\circ = \frac{(\sqrt{5}-1)}{4} \text{ and } \cos 36^\circ = \frac{(\sqrt{5}+1)}{4}\right] \\
&= \frac{1}{4}\cdot\frac{(\sqrt{5}+1)}{4}\cdot\frac{(\sqrt{5}-1)}{4} = \frac{(5-1)}{64} \\
&= \frac{4}{64} = \frac{1}{16} = \text{RHS.}
\end{aligned}$$

EXAMPLE 25 *Prove that* $\sin\frac{\pi}{5}\sin\frac{2\pi}{5}\sin\frac{3\pi}{5}\sin\frac{4\pi}{5} = \frac{5}{16}$.

SOLUTION We have

$$\begin{aligned}
\text{LHS} &= \sin\frac{\pi}{5}\sin\frac{2\pi}{5}\sin\frac{3\pi}{5}\sin\frac{4\pi}{5} \\
&= \sin\frac{\pi}{5}\sin\frac{2\pi}{5}\sin\left(\pi - \frac{2\pi}{5}\right)\sin\left(\pi - \frac{\pi}{5}\right) \\
&= \sin^2\frac{\pi}{5}\sin^2\frac{2\pi}{5} \qquad [\because \sin(\pi - \theta) = \sin\theta] \\
&= (\sin 36^\circ)^2 \times (\sin 72^\circ)^2 \\
&= (\sin 36^\circ)^2 \times (\cos 18^\circ)^2 \\
&\qquad [\because \sin 72^\circ = \sin(90^\circ - 18^\circ) = \cos 18^\circ] \\
&= \frac{(10-2\sqrt{5})}{16} \times \frac{(10+2\sqrt{5})}{16} = \frac{(100-20)}{(16\times 16)} \\
&\qquad \left[\because \sin 36^\circ = \frac{\sqrt{10-2\sqrt{5}}}{4} \text{ and } \cos 18^\circ = \frac{\sqrt{10+2\sqrt{5}}}{4}\right] \\
&= \frac{80}{(16\times 16)} = \frac{5}{16} = \text{RHS.}
\end{aligned}$$

EXAMPLE 26 *Find the value of*

(i) $\sin 22° 30'$ *(ii)* $\cos 22° 30'$ *(iii)* $\tan 22° 30'$

SOLUTION (i) $\sin^2\theta = \frac{(1-\cos 2\theta)}{2}$

$$\Rightarrow \sin^2(22° 30') = \frac{(1-\cos 45°)}{2} = \frac{\left(1-\frac{1}{\sqrt{2}}\right)}{2} = \frac{\sqrt{2}-1}{2\sqrt{2}}$$

$$\Rightarrow \sin(22° 30') = \sqrt{\frac{(\sqrt{2}-1)}{2\sqrt{2}}}.$$

(ii) $\cos^2\theta = \frac{(1+\cos 2\theta)}{2}$

$$\Rightarrow \cos^2(22° 30') = \frac{(1+\cos 45°)}{2} = \frac{\left(1+\frac{1}{\sqrt{2}}\right)}{2} = \frac{(\sqrt{2}+1)}{2\sqrt{2}}$$

$$\Rightarrow \cos(22° 30') = \sqrt{\frac{(\sqrt{2}+1)}{2\sqrt{2}}}.$$

(iii) $$\tan^2(22° 30') = \frac{\sin^2(22° 30')}{\cos^2(22° 30')} = \frac{(\sqrt{2}-1)}{(2\sqrt{2})} \times \frac{(2\sqrt{2})}{(\sqrt{2}+1)}$$

$$= \frac{\sqrt{2}-1}{\sqrt{2}+1} = \frac{(\sqrt{2}-1)}{(\sqrt{2}+1)} \times \frac{(\sqrt{2}-1)}{(\sqrt{2}-1)}$$

$$= (\sqrt{2}-1)^2$$

$$\Rightarrow \tan(22° 30') = (\sqrt{2}-1).$$

EXERCISE 15D

1. If $\sin x = \frac{\sqrt{5}}{3}$ and $0 < x < \frac{\pi}{2}$, find the values of

(i) $\sin 2x$, (ii) $\cos 2x$, (iii) $\tan 2x$.

2. If $\cos x = \frac{-3}{5}$ and $\pi < x < \frac{3\pi}{2}$, find the values of

(i) $\sin 2x$, (ii) $\cos 2x$, (iii) $\tan 2x$.

3. If $\tan x = \frac{-5}{12}$ and $\frac{\pi}{2} < x < \pi$, find the values of

(i) $\sin 2x$, (ii) $\cos 2x$, (iii) $\tan 2x$.

4. (i) If $\sin x = \frac{1}{6}$, find the value of $\sin 3x$.

(ii) If $\cos x = \frac{-1}{2}$, find the value of $\cos 3x$.

Prove that

5. $\dfrac{\cos 2x}{\cos x - \sin x} = \cos x + \sin x$

6. $\dfrac{\sin 2x}{1 + \cos 2x} = \tan x$

7. $\dfrac{\sin 2x}{1 - \cos 2x} = \cot x$

8. $\dfrac{\tan 2x}{1 + \sec 2x} = \tan x$

9. $\sin 2x(\tan x + \cot x) = 2$

10. $\operatorname{cosec} 2x + \cot 2x = \cot x$

11. $\cos 2x + 2\sin^2 x = 1$

12. $(\sin x - \cos x)^2 = 1 - \sin 2x$

13. $\cot x - 2\cot 2x = \tan x$

14. $(\cos^4 x + \sin^4 x) = \dfrac{1}{2}(2 - \sin^2 2x)$

15. $\dfrac{\cos^3 x - \sin^3 x}{\cos x - \sin x} = \dfrac{1}{2}(2 + \sin 2x)$

16. $\dfrac{1 - \cos 2x + \sin x}{\sin 2x + \cos x} = \tan x$

17. $\cos x \cos 2x \cos 4x \cos 8x = \dfrac{\sin 16x}{16\sin x}$

18. Prove that

(i) $2\sin 22\dfrac{1^\circ}{2}\cos 22\dfrac{1^\circ}{2} = \dfrac{1}{\sqrt{2}}$

(ii) $2\cos^2 15^\circ - 1 = \dfrac{\sqrt{3}}{2}$

(iii) $8\cos^3 20^\circ - 6\cos 20^\circ = 1$

(iv) $3\sin 40^\circ - 4\sin^3 40^\circ = \dfrac{\sqrt{3}}{2}$

19. Prove that

(i) $\sin^2 24^\circ - \sin^2 6^\circ = \dfrac{(\sqrt{5}-1)}{8}$

(ii) $\sin^2 72^\circ - \cos^2 30^\circ = \dfrac{(\sqrt{5}-1)}{8}$

20. Prove that $\tan 6^\circ \tan 42^\circ \tan 66^\circ \tan 78^\circ = 1.$

21. If $\tan\theta = \dfrac{a}{b}$, prove that $a\sin 2\theta + b\cos 2\theta = b.$

ANSWERS (EXERCISE 15D)

1. (i) $\dfrac{4\sqrt{5}}{9}$ (ii) $\dfrac{-1}{9}$ (iii) $-4\sqrt{5}$

2. (i) $\dfrac{24}{25}$ (ii) $\dfrac{-7}{25}$ (iii) $\dfrac{-24}{7}$

3. (i) $\dfrac{-120}{169}$ (ii) $\dfrac{119}{169}$ (iii) $\dfrac{-120}{119}$

4. (i) $\dfrac{13}{27}$ (ii) 1

HINTS TO SOME SELECTED QUESTIONS

3. Use the relations $\sin 2x = \dfrac{2\tan x}{1+\tan^2 x}$ and $\cos 2x = \dfrac{1-\tan^2 x}{1+\tan^2 x}$.

5. LHS $= \dfrac{\cos^2 x - \sin^2 x}{\cos x - \sin x} = \cos x + \sin x =$ RHS.

6. LHS $= \dfrac{2\sin x\cos x}{2\cos^2 x} = \tan x =$ RHS.

7. LHS $= \dfrac{2\sin x\cos x}{2\sin^2 x} = \cot x =$ RHS.

8. LHS $= \dfrac{\sin 2x}{(1+\cos 2x)} = \dfrac{2\sin x\cos x}{2\cos^2 x} = \tan x =$ RHS.

10. LHS $= \dfrac{1}{\sin 2x} + \dfrac{\cos 2x}{\sin 2x} = \dfrac{1+\cos 2x}{\sin 2x} = \dfrac{2\cos^2 x}{2\sin x\cos x} = \cot x =$ RHS.

13. LHS $= \left(\dfrac{1}{\tan x} - \dfrac{2}{\tan 2x}\right) = \left\{\dfrac{1}{\tan x} - \dfrac{2(1-\tan^2 x)}{2\tan x}\right\} = \dfrac{\tan^2 x}{\tan x} = \tan x =$ RHS.

14. LHS $= (\cos^2 x + \sin^2 x)^2 - 2\cos^2 x\sin^2 x$

$= 1 - 2\cos^2 x\sin^2 x = 1 - \dfrac{1}{2}(\sin^2 2x) = \dfrac{1}{2}(2 - \sin^2 2x) =$ RHS.

16. LHS $= \dfrac{2\sin^2 x + \sin x}{2\sin x\cos x + \cos x} = \dfrac{\sin x(2\sin x + 1)}{\cos x(2\sin x + 1)} = \dfrac{\sin x}{\cos x} = \tan x =$ RHS.

17. LHS $= \dfrac{1}{2\sin x}(2\sin x\cos x)\cos 2x\cos 4x\cos 8x$

$= \dfrac{1}{2\sin x}(\sin 2x\cos 2x)\cos 4x\cos 8x$

$= \dfrac{1}{4\sin x}(2\sin 2x\cos 2x)\cos 4x\cos 8x$

$= \dfrac{1}{4\sin x}(\sin 4x\cos 4x)\cos 8x$

$= \dfrac{1}{8\sin x}(2\sin 4x\cos 4x)\cos 8x$

$= \dfrac{1}{8\sin x}(\sin 8x\cos 8x)$

$= \dfrac{1}{16\sin x}(2\sin 8x\cos 8x)$

$= \dfrac{1}{16\sin x}(\sin 16x) = \dfrac{\sin 16x}{16\sin x} =$ RHS.

18. (i) LHS $= \sin\left(2\times 22\dfrac{1^\circ}{2}\right) = \sin 45^\circ = \dfrac{1}{\sqrt{2}}\cdot$

(ii) LHS $= (2\cos^2\theta - 1) = \cos 2\theta$, where $\theta = 15^\circ$

$= \cos 30^\circ = \dfrac{\sqrt{3}}{2}\cdot$

(iii) LHS $= 2(4\cos^3\theta - 3\cos\theta)$, where $\theta = 20^\circ$

$= 2\times\cos 3\theta = 2\times\cos 60^\circ = \left(2\times\dfrac{1}{2}\right) = 1.$

(iv) LHS $= (3\sin\theta - 4\sin^3\theta)$, where $\theta = 40^\circ$

$= \sin 3\theta = \sin 120^\circ = \sin 60^\circ = \dfrac{\sqrt{3}}{2}\cdot$

20. LHS $= \dfrac{\sin 6^\circ\sin 42^\circ\sin 66^\circ\sin 78^\circ}{\cos 6^\circ\cos 42^\circ\cos 66^\circ\cos 78^\circ}\cdot$

21. Use the relations

$$\sin 2\theta = \frac{2\tan\theta}{(1+\tan^2\theta)} \text{ and } \cos 2\theta = \frac{1-\tan^2\theta}{1+\tan^2\theta}\cdot$$

TRIGONOMETRICAL FUNCTIONS OF HALF ANGLES

THEOREM 1 *For all $x \in R$, we have*

(i) $\sin x = 2\sin\frac{x}{2}\cos\frac{x}{2}$ (ii) $1 - \cos x = 2\sin^2\frac{x}{2}$

(iii) $1 + \cos x = 2\cos^2\frac{x}{2}$

PROOF (i) $\sin x = \sin\left(\frac{x}{2} + \frac{x}{2}\right)$

$$= \sin\frac{x}{2}\cos\frac{x}{2} + \cos\frac{x}{2}\sin\frac{x}{2}$$

$$= 2\sin\frac{x}{2}\cos\frac{x}{2}$$

$\therefore \sin x = 2\sin\frac{x}{2}\cos\frac{x}{2}\cdot$

(ii) $\cos x = \cos\left(\frac{x}{2} + \frac{x}{2}\right)$

$$= \cos\frac{x}{2}\cos\frac{x}{2} - \sin\frac{x}{2}\sin\frac{x}{2}$$

$$= \cos^2\frac{x}{2} - \sin^2\frac{x}{2}$$

$$= \left(1 - \sin^2\frac{x}{2}\right) - \sin^2\frac{x}{2}$$

$$= \left(1 - 2\sin^2\frac{x}{2}\right)\cdot$$

$\therefore (1 - \cos x) = 2\sin^2\frac{x}{2}\cdot$

(iii) $\cos x = \cos\left(\frac{x}{2} + \frac{x}{2}\right)$

$$= \cos\frac{x}{2}\cos\frac{x}{2} - \sin\frac{x}{2}\sin\frac{x}{2}$$

$$= \cos^2\frac{x}{2} - \sin^2\frac{x}{2}$$

$$= \cos^2\frac{x}{2} - \left(1 - \cos^2\frac{x}{2}\right)$$

$$= \left(2\cos^2\frac{x}{2} - 1\right)\cdot$$

$\therefore (1 + \cos x) = 2\cos^2\frac{x}{2}\cdot$

THEOREM 2 *For all $x \in R$, we have*

(i) $\sin x = \dfrac{2\tan(x/2)}{1 + \tan^2(x/2)}$ (ii) $\cos x = \dfrac{1 - \tan^2(x/2)}{1 + \tan^2(x/2)}$

PROOF (i) $\sin x = 2\sin\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right)$

$$= \frac{2\sin\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right)}{\cos^2\left(\frac{x}{2}\right)+\sin^2\left(\frac{x}{2}\right)}$$

$$= \frac{\left\{\frac{2\sin\left(\frac{x}{2}\right)}{\cos\left(\frac{x}{2}\right)}\right\}}{1+\left\{\frac{\sin^2\left(\frac{x}{2}\right)}{\cos^2\left(\frac{x}{2}\right)}\right\}} \quad \left[\text{dividing num. and denom. by } \cos^2\left(\frac{x}{2}\right)\right]$$

$$= \frac{2\tan\left(\frac{x}{2}\right)}{\left\{1+\tan^2\left(\frac{x}{2}\right)\right\}}.$$

$$\therefore \quad \sin x = \frac{2\tan\left(\frac{x}{2}\right)}{\left\{1+\tan^2\left(\frac{x}{2}\right)\right\}}.$$

(ii) $\cos x = \cos^2\left(\frac{x}{2}\right) - \sin^2\left(\frac{x}{2}\right)$

$$= \frac{\cos^2\left(\frac{x}{2}\right)-\sin^2\left(\frac{x}{2}\right)}{\cos^2\left(\frac{x}{2}\right)+\sin^2\left(\frac{x}{2}\right)} \quad \left[\because\ 1 = \cos^2\left(\frac{x}{2}\right)+\sin^2\left(\frac{x}{2}\right)\right]$$

$$= \frac{1-\tan^2\left(\frac{x}{2}\right)}{1+\tan^2\left(\frac{x}{2}\right)} \quad \left[\text{dividing num. and denom. by } \cos^2\left(\frac{x}{2}\right)\right].$$

$$\therefore \quad \cos x = \frac{1-\tan^2\left(\frac{x}{2}\right)}{1+\tan^2\left(\frac{x}{2}\right)}.$$

SUMMARY

(i) $\sin x = 2\sin\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right)$

(ii) $(1 - \cos x) = 2\sin^2\left(\frac{x}{2}\right)$

(iii) $(1 + \cos x) = 2\cos^2\left(\frac{x}{2}\right)$

(iv) $\sin x = \dfrac{2\tan\left(\frac{x}{2}\right)}{1 + \tan^2\left(\frac{x}{2}\right)}$

(v) $\cos x = \dfrac{1 - \tan^2\left(\frac{x}{2}\right)}{1 + \tan^2\left(\frac{x}{2}\right)}$

SOLVED EXAMPLES

EXAMPLE 1 *If* $\tan x = \frac{-4}{3}$ *and* $\frac{\pi}{2} < x < \pi$, *find the values of*

(i) $\sin\frac{x}{2}$, *(ii)* $\cos\frac{x}{2}$, *(iii)* $\tan\frac{x}{2}$.

SOLUTION Since x lies in Quadrant II, we have $\cos x < 0$.

$$\because \quad \tan x = \frac{-4}{3} \Rightarrow \sec^2 x = (1 + \tan^2 x) = \left(1 + \frac{16}{9}\right) = \frac{25}{9}$$

$$\Rightarrow \cos^2 x = \frac{1}{\sec^2 x} = \frac{9}{25}$$

$$\Rightarrow \cos x = -\sqrt{\frac{9}{25}} = -\frac{3}{5}.$$

Also, $\frac{\pi}{2} < x < \pi \Rightarrow \frac{\pi}{4} < \frac{x}{2} < \frac{\pi}{2}$

$\Rightarrow \frac{x}{2}$ lies in Quadrant I

$\Rightarrow \sin\frac{x}{2} > 0$ and $\cos\frac{x}{2} > 0$.

(i) $2\sin^2\frac{x}{2} = (1 - \cos x) = \left(1 + \frac{3}{5}\right) = \frac{8}{5}$

$$\Rightarrow \sin^2\frac{x}{2} = \frac{8}{10} = \frac{4}{5}$$

$$\Rightarrow \sin\frac{x}{2} = +\sqrt{\frac{4}{5}} = \frac{2}{\sqrt{5}} \quad \left[\because \sin\frac{x}{2} > 0\right].$$

(ii) $2\cos^2\frac{x}{2} = (1 + \cos x) = \left(1 - \frac{3}{5}\right) = \frac{2}{5}$

$\Rightarrow \cos^2\frac{x}{2} = \left(\frac{2}{5 \times 2}\right) = \frac{1}{5}$

$\Rightarrow \cos\frac{x}{2} = +\sqrt{\frac{1}{5}} = \frac{1}{\sqrt{5}} \qquad \left[\because \cos\frac{x}{2} > 0\right].$

(iii) $\tan\frac{x}{2} = \dfrac{\sin\left(\frac{x}{2}\right)}{\cos\left(\frac{x}{2}\right)} = \left(\frac{2}{\sqrt{5}} \times \sqrt{5}\right) = 2.$

EXAMPLE 2 *If* $\tan x = \frac{3}{4}$ *and* $\pi < x < \frac{3\pi}{2}$, *find the values of*

(i) $\sin\frac{x}{2}$, *(ii)* $\cos\frac{x}{2}$, *(iii)* $\tan\frac{x}{2}\cdot$

SOLUTION Since x lies in Quadrant III, we have $\cos x < 0$.

Now, $\tan x = \frac{3}{4} \Rightarrow \sec^2 x = (1 + \tan^2 x) = \left(1 + \frac{9}{16}\right) = \frac{25}{16}$

$\Rightarrow \cos^2 x = \frac{1}{\sec^2 x} = \frac{16}{25}$

$\Rightarrow \cos x = -\sqrt{\frac{16}{25}} = \frac{-4}{5}\cdot$

Also, $\pi < x < \frac{3\pi}{2} \Rightarrow \frac{\pi}{2} < \frac{x}{2} < \frac{3\pi}{4}$

$\Rightarrow \frac{x}{2}$ lies in Quadrant II

$\Rightarrow \sin\frac{x}{2} > 0$ and $\cos\frac{x}{2} < 0.$

(i) $2\sin^2\frac{x}{2} = (1 - \cos x) = \left(1 + \frac{4}{5}\right) = \frac{9}{5}$

$\Rightarrow \sin^2\frac{x}{2} = \frac{9}{10}$

$\Rightarrow \sin\frac{x}{2} = +\sqrt{\frac{9}{10}} = \frac{3}{\sqrt{10}} \qquad \left[\because \sin\frac{x}{2} > 0\right].$

(ii) $2\cos^2\frac{x}{2} = (1 + \cos x) = \left(1 - \frac{4}{5}\right) = \frac{1}{5}$

$\Rightarrow \cos^2\frac{x}{2} = \frac{1}{10}$

$\Rightarrow \cos\frac{x}{2} = -\sqrt{\frac{1}{10}} = \frac{-1}{\sqrt{10}} \qquad \left[\because \cos\frac{x}{2} < 0\right].$

$$\text{(iii)}\ \tan\frac{x}{2} = \frac{\sin\left(\frac{x}{2}\right)}{\cos\left(\frac{x}{2}\right)} = \left(\frac{3}{\sqrt{10}} \times \frac{\sqrt{10}}{-1}\right) = -3.$$

EXAMPLE 3 *If* $\cos x = -\frac{1}{3}$ *and x lies in Quadrant III, find the values of*

(i) $\sin\frac{x}{2}$, *(ii)* $\cos\frac{x}{2}$, *(iii)* $\tan\frac{x}{2}\cdot$

SOLUTION Since x lies in Quadrant III, we have

$$\pi < x < \frac{3\pi}{2} \Rightarrow \frac{\pi}{2} < \frac{x}{2} < \frac{3\pi}{4}$$

$$\Rightarrow \frac{x}{2} \text{ lies in Quadrant II}$$

$$\Rightarrow \sin\frac{x}{2} > 0 \text{ and } \cos\frac{x}{2} < 0.$$

$$\text{(i)}\ 2\sin^2\frac{x}{2} = (1 - \cos x) = \left(1 + \frac{1}{3}\right) = \frac{4}{3}$$

$$\Rightarrow \sin^2\frac{x}{2} = \frac{2}{3}$$

$$\Rightarrow \sin\frac{x}{2} = +\sqrt{\frac{2}{3}} = \frac{\sqrt{2}}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}} = \frac{\sqrt{6}}{3} \quad \left[\because \sin\frac{x}{2} > 0\right]\cdot$$

$$\text{(ii)}\ 2\cos^2\frac{x}{2} = (1 + \cos x) = \left(1 - \frac{1}{3}\right) = \frac{2}{3}$$

$$\Rightarrow \cos^2\frac{x}{2} = \frac{1}{3}$$

$$\Rightarrow \cos\frac{x}{2} = -\frac{1}{\sqrt{3}} = \frac{-1}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}} = -\frac{\sqrt{3}}{3}\cdot$$

$$\text{(iii)}\ \tan\frac{x}{2} = \frac{\sin\left(\frac{x}{2}\right)}{\cos\left(\frac{x}{2}\right)} = \left(\frac{\sqrt{6}}{3} \times \frac{3}{-\sqrt{3}}\right) = -\sqrt{2}.$$

EXAMPLE 4 *Prove that* $\frac{1 + \cos x}{1 - \cos x} = (\operatorname{cosec} x + \cot x)^2$.

SOLUTION We have

$$\text{LHS} = \frac{1 + \cos x}{1 - \cos x}$$

$$= \frac{2\cos^2\left(\frac{x}{2}\right)}{2\sin^2\left(\frac{x}{2}\right)} = \cot^2\frac{x}{2} \quad \left[\begin{array}{l}\because 1 + \cos x = 2\cos^2\left(\frac{x}{2}\right), \\ 1 - \cos x = 2\sin^2\left(\frac{x}{2}\right)\end{array}\right]$$

$$\text{RHS} = (\operatorname{cosec} x + \cot x)^2$$

$$= \left(\frac{1}{\sin x} + \frac{\cos x}{\sin x}\right)^2 = \left(\frac{1 + \cos x}{\sin x}\right)^2$$

$$= \left\{\frac{2\cos^2\left(\frac{x}{2}\right)}{2\sin\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right)}\right\}^2 = \left\{\frac{\cos\left(\frac{x}{2}\right)}{\sin\left(\frac{x}{2}\right)}\right\}^2 = \cot^2\frac{x}{2}.$$

$\therefore$ LHS = RHS.

EXAMPLE 5 *Prove that* $\dfrac{\cos x}{(1 - \sin x)} = \tan\left(\dfrac{\pi}{4} + \dfrac{x}{2}\right)$.

SOLUTION We have

$$\text{LHS} = \frac{\cos x}{(1 - \sin x)}$$

$$= \frac{\left(\cos^2\frac{x}{2} - \sin^2\frac{x}{2}\right)}{\left(\cos^2\frac{x}{2} + \sin^2\frac{x}{2} - 2\sin\frac{x}{2}\cos\frac{x}{2}\right)}$$

$$\left[\because \cos x = \left(\cos^2\frac{x}{2} - \sin^2\frac{x}{2}\right),\ \cos^2\frac{x}{2} + \sin^2\frac{x}{2} = 1 \text{ and } \sin x = 2\sin\frac{x}{2}\cos\frac{x}{2}\right]$$

$$= \frac{\left(\cos\frac{x}{2} - \sin\frac{x}{2}\right)\left(\cos\frac{x}{2} + \sin\frac{x}{2}\right)}{\left(\cos\frac{x}{2} - \sin\frac{x}{2}\right)^2}$$

$$= \frac{\left(\cos\frac{x}{2} + \sin\frac{x}{2}\right)}{\left(\cos\frac{x}{2} - \sin\frac{x}{2}\right)} = \frac{\left(1 + \tan\frac{x}{2}\right)}{\left(1 - \tan\frac{x}{2}\right)} = \frac{\left(\tan\frac{\pi}{4} + \tan\frac{x}{2}\right)}{\left(1 - \tan\frac{\pi}{4}\cdot\tan\frac{x}{2}\right)}$$

$$\left[\text{dividing num. and denom. by } \cos\frac{x}{2}\right]$$

$$= \tan\left(\frac{\pi}{4} + \frac{x}{2}\right) = \text{RHS}.$$

EXAMPLE 6 *If* α *and* β *be two distinct real numbers such that* $(\alpha - \beta) \neq 2n\pi$ *for any intetger* n*, satisfying the equation* $a\cos\theta + b\sin\theta = c$ *then prove that*

(i) $\cos(\alpha + \beta) = \dfrac{a^2 - b^2}{a^2 + b^2}$ *(ii)* $\sin(\alpha + \beta) = \dfrac{2ab}{a^2 + b^2}$.

SOLUTION Since α and β satisfy the equation $a\cos\theta + b\sin\theta = c$, we have

$$a\cos\alpha + b\sin\alpha = c, \quad \ldots \text{(i)}$$

$$a\cos\beta + b\sin\beta = c. \quad \ldots \text{(ii)}$$

Subtracting (ii) from (i), we get

$$a(\cos\alpha - \cos\beta) + b(\sin\alpha - \sin\beta) = 0$$

$$\Rightarrow \quad -2a\sin\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right) + 2b\cos\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right) = 0$$

$$\Rightarrow \quad -2\sin\left(\frac{\alpha-\beta}{2}\right)\left[a\sin\left(\frac{\alpha+\beta}{2}\right) - b\cos\left(\frac{\alpha+\beta}{2}\right)\right] = 0$$

$$\Rightarrow \quad a\sin\left(\frac{\alpha+\beta}{2}\right) - b\cos\left(\frac{\alpha+\beta}{2}\right) = 0$$

$$\left[\because (\alpha-\beta) \neq 2n\pi \Rightarrow \left(\frac{\alpha-\beta}{2}\right) \neq n\pi \quad \therefore \sin\left(\frac{\alpha-\beta}{2}\right) \neq \sin n\pi \neq 0\right]$$

$$\Rightarrow \quad \tan\left(\frac{\alpha+\beta}{2}\right) = \frac{b}{a}. \qquad \ldots \text{(iii)}$$

(i) $$\cos(\alpha+\beta) = \frac{1-\tan^2\left(\frac{\alpha+\beta}{2}\right)}{1+\tan^2\left(\frac{\alpha+\beta}{2}\right)} \qquad \left[\because \cos x = \frac{1-\tan^2\left(\frac{x}{2}\right)}{1+\tan^2\left(\frac{x}{2}\right)}\right]$$

$$= \frac{\left(1-\frac{b^2}{a^2}\right)}{\left(1+\frac{b^2}{a^2}\right)} = \frac{a^2-b^2}{a^2+b^2} \quad \text{[using (iii)].}$$

(ii) $$\sin(\alpha+\beta) = \frac{2\tan\left(\frac{\alpha+\beta}{2}\right)}{1+\tan^2\left(\frac{\alpha+\beta}{2}\right)} \qquad \left[\because \sin x = \frac{2\tan\left(\frac{x}{2}\right)}{1+\tan^2\left(\frac{x}{2}\right)}\right]$$

$$= \frac{2\left(\frac{b}{a}\right)}{\left(1+\frac{b^2}{a^2}\right)} = \frac{2ab}{a^2+b^2} \quad \text{[using (iii)].}$$

EXERCISE 15E

1. If $\sin x = \frac{\sqrt{5}}{3}$ and $\frac{\pi}{2} < x < \pi$, find the values of

(i) $\sin\frac{x}{2}$, (ii) $\cos\frac{x}{2}$, (iii) $\tan\frac{x}{2}$.

2. If $\cos x = \frac{-3}{5}$ and $\frac{\pi}{2} < x < \pi$, find the values of

(i) $\sin\frac{x}{2}$, (ii) $\cos\frac{x}{2}$, (iii) $\tan\frac{x}{2}$.

3. If $\sin x = \frac{-1}{2}$ and x lies in Quadrant IV, find the values of

(i) $\sin \frac{x}{2}$, (ii) $\cos \frac{x}{2}$, (iii) $\tan \frac{x}{2}$.

4. If $\cos \frac{x}{2} = \frac{12}{13}$ and x lies in Quadrant I, find the values of

(i) $\sin x$, (ii) $\cos x$, (iii) $\cot x$.

5. If $\sin x = \frac{3}{5}$ and $0 < x < \frac{\pi}{2}$, find the value of $\tan \frac{x}{2}$.

Prove that

6. $\cot \frac{x}{2} - \tan \frac{x}{2} = 2\cot x$

7. $\tan\left(\frac{\pi}{4} + \frac{x}{2}\right) = \tan x + \sec x$

8. $\sqrt{\frac{1 + \sin x}{1 - \sin x}} = \tan\left(\frac{\pi}{4} + \frac{x}{2}\right)$

9. $\tan\left(\frac{\pi}{4} + \frac{x}{2}\right) + \tan\left(\frac{\pi}{4} - \frac{x}{2}\right) = 2\sec x$

10. $\frac{\sin x}{1 + \cos x} = \tan \frac{x}{2}$

ANSWERS (EXERCISE 15E)

1. (i) $\frac{\sqrt{30}}{6}$ (ii) $\frac{\sqrt{6}}{6}$ (iii) $\sqrt{5}$ **2.** (i) $\frac{2\sqrt{5}}{5}$ (ii) $\frac{1}{\sqrt{5}}$ (iii) 2

3. (i) $\sin \frac{x}{2} = \frac{\sqrt{(2 - \sqrt{3})}}{2}$ (ii) $\cos \frac{x}{2} = -\frac{\sqrt{(2 + \sqrt{3})}}{2}$ (iii) $\tan \frac{x}{2} = -(2 - \sqrt{3})$

4. (i) $\frac{120}{169}$ (ii) $\frac{119}{169}$ (iii) $\frac{119}{120}$ **5.** $\frac{1}{3}$

HINTS TO SOME SELECTED QUESTIONS

5. $\tan \frac{x}{2} = \frac{\sqrt{1 - \cos x}}{\sqrt{1 + \cos x}}$, where $\cos x = \sqrt{1 - \frac{9}{25}} = \frac{4}{5}$.

6. LHS $= \frac{\cos\left(\frac{x}{2}\right)}{\sin\left(\frac{x}{2}\right)} - \frac{\sin\left(\frac{x}{2}\right)}{\cos\left(\frac{x}{2}\right)} = \frac{\left\{\cos^2\left(\frac{x}{2}\right) - \sin^2\left(\frac{x}{2}\right)\right\}}{\sin \frac{x}{2} \cos \frac{x}{2}}$

$$= \frac{2\cos x}{\sin x} = 2\cot x = \text{RHS.}$$

7. RHS $= \left(\dfrac{1}{\cos x} + \dfrac{\sin x}{\cos x}\right) = \dfrac{(1+\sin x)}{\cos x} = \dfrac{1-\cos\left(\dfrac{\pi}{2}+x\right)}{\sin\left(\dfrac{\pi}{2}+x\right)}$

$$= \frac{2\sin^2\left(\dfrac{\pi}{4}+\dfrac{x}{2}\right)}{2\sin\left(\dfrac{\pi}{4}+\dfrac{x}{2}\right)\cos\left(\dfrac{\pi}{4}+\dfrac{x}{2}\right)} = \tan\left(\frac{\pi}{4}+\frac{x}{2}\right) = \text{LHS.}$$

8. LHS $= \dfrac{\sqrt{1+\sin x}}{\sqrt{1-\sin x}} \times \dfrac{\sqrt{1+\sin x}}{\sqrt{1+\sin x}} = \dfrac{(1+\sin x)}{\cos x} = \dfrac{1-\cos\left(\dfrac{\pi}{2}+x\right)}{\sin\left(\dfrac{\pi}{2}+x\right)}$

$$= \frac{2\sin^2\left(\dfrac{\pi}{4}+\dfrac{x}{2}\right)}{2\sin\left(\dfrac{\pi}{4}+\dfrac{x}{2}\right)\cos\left(\dfrac{\pi}{4}+\dfrac{x}{2}\right)} = \tan\left(\frac{\pi}{4}+\frac{x}{2}\right) = \text{RHS.}$$

10. LHS $= \dfrac{\sin x}{(1+\cos x)} = \dfrac{2\sin\left(\dfrac{x}{2}\right)\cos\left(\dfrac{x}{2}\right)}{2\cos^2\left(\dfrac{x}{2}\right)} = \tan\dfrac{x}{2} = \text{RHS.}$

❑

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Conditional Identities Involving the Angles of a Triangle

Before we begin, let us first recall the following formulae:

$$\textbf{(i)}\ \sin x + \sin y = 2\sin\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)$$

$$\textbf{(ii)}\ \sin x - \sin y = 2\cos\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right)$$

$$\textbf{(iii)}\ \cos x + \cos y = 2\cos\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)$$

$$\textbf{(iv)}\ \cos x - \cos y = -2\sin\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right)$$

TYPE 1 IDENTITIES INVOLVING SINES AND COSINES

METHOD

1. Express the sum of the first two terms as a product.
2. In this product, express the sum of two angles in terms of the third angle, using $A + B + C = \pi$.
3. Expand the third term by using one of the following relations:
   $\sin 2\theta = 2\sin\theta\cos\theta$ and $\cos 2\theta = (2\cos^2\theta - 1) = (1 - 2\sin^2\theta)$.
4. Take the common factor outside.
5. Express the T-ratio of a single angle into a sum of two angles, and use the necessary formula from the ones given in the box above.

SOLVED EXAMPLES

EXAMPLE 1 *If $A + B + C = \pi$, prove that*

$$\sin 2A + \sin 2B + \sin 2C = 4\sin A\sin B\sin C.$$

SOLUTION We have

$$\begin{aligned}\text{LHS} &= (\sin 2A + \sin 2B) + \sin 2C\\ &= 2\sin(A+B)\cos(A-B) + 2\sin C\cos C \qquad \text{[by formula]}\\ &= 2\sin(\pi - C)\cos(A-B) + 2\sin C\cos C\\ &= 2\sin C\cos(A-B) + 2\sin C\cos C \quad [\because \sin(\pi - C) = \sin C]\end{aligned}$$

$$= 2\sin C[\cos(A-B) + \cos C]$$
$$= 2\sin C[\cos(A-B) + \cos\{\pi - (A+B)\}] \quad [\because C = \pi - (A+B)]$$
$$= 2\sin C[\cos(A-B) - \cos(A+B)] \quad [\because \cos(\pi - \theta) = \cos\theta]$$
$$= 2\sin C[2\sin A \sin B]$$
$$= 4\sin A \sin B \sin C = \text{RHS}.$$

$\therefore \quad \sin 2A + \sin 2B + \sin 2C = 4\sin A \sin B \sin C.$

EXAMPLE 2 *If* $A + B + C = \pi$, *prove that*
$$\sin 2A - \sin 2B + \sin 2C = 4\cos A \sin B \cos C.$$

SOLUTION We have

$$\text{LHS} = (\sin 2A - \sin 2B) + \sin 2C$$
$$= 2\cos(A+B)\sin(A-B) + 2\sin C\cos C \quad \text{[by formula]}$$
$$= 2\cos(\pi - C)\sin(A-B) + 2\sin C\cos C$$
$$= -2\cos C\sin(A-B) + 2\sin C\cos C \; [\because \cos(\pi - C) = -\cos C]$$
$$= -2\cos C[\sin(A-B) - \sin C]$$
$$= -2\cos C[\sin(A-B) - \sin\{\pi - (A+B)\}] \; [\because C = \pi - (A+B)]$$
$$= -2\cos C[\sin(A-B) - \sin(A+B)]$$
$$= -2\cos C[-2\cos A \sin B]$$
$$= 4\cos A \sin B \cos C = \text{RHS}.$$

$\therefore \quad \sin 2A - \sin 2B + \sin 2C = 4\cos A \sin B \cos C.$

EXAMPLE 3 *If* $A + B + C = \pi$, *prove that*
$$\cos 2A + \cos 2B + \cos 2C = -1 - 4\cos A \cos B \cos C.$$

SOLUTION We have

$$\text{LHS} = \cos 2A + \cos 2B + \cos 2C$$
$$= 2\cos(A+B)\cos(A-B) + 2\cos^2 C - 1$$
$$= 2\cos(\pi - C)\cos(A-B) + 2\cos^2 C - 1$$
$$= -2\cos C\cos(A-B) + 2\cos^2 C - 1$$
$$= -2\cos C[\cos(A-B) - \cos C] - 1$$
$$= -1 - 2\cos C[\cos(A-B) - \cos\{\pi - (A+B)\}]$$
$$= -1 - 2\cos C[\cos(A-B) + \cos(A+B)]$$
$$= -1 - 2\cos C[2\cos A \cos B]$$
$$= -1 - 4\cos A \cos B \cos C = \text{RHS}.$$

$\therefore \quad \cos 2A + \cos 2B + \cos 2C = -1 - 4\cos A \cos B \cos C.$

EXAMPLE 4 *If* $A + B + C = \pi$, *prove that*
$$\cos 2A + \cos 2B - \cos 2C = 1 - 4\sin A \sin B \cos C.$$

SOLUTION We have

$$\text{LHS} = (\cos 2A + \cos 2B) - \cos 2C$$
$$= 2\cos(A+B)\cos(A-B) - 2\cos^2 C + 1$$

$$= 2\cos(\pi - C)\cos(A - B) - 2\cos^2 C + 1$$
$$= -2\cos C\cos(A - B) - 2\cos^2 C + 1$$
$$= 1 - 2\cos C[\cos(A - B) + \cos C]$$
$$= 1 - 2\cos C[\cos(A - B) + \cos\{\pi - (A + B)\}]$$
$$= 1 - 2\cos C[\cos(A - B) - \cos(A + B)]$$
$$= 1 - 2\cos C[2\sin A\sin B]$$
$$= 1 - 4\sin A\sin B\cos C = \text{RHS}.$$
$$\therefore \quad \cos 2A + \cos 2B - \cos 2C = 1 - 4\sin A\sin B\cos C.$$

EXAMPLE 5 *If $A + B + C = \pi$, prove that*
$$\cos 4A + \cos 4B + \cos 4C = -1 + 4\cos 2A\cos 2B\cos 2C.$$

SOLUTION We have

$$\text{LHS} = \cos 4A + \cos 4B + \cos 4C$$
$$= 2\cos(2A + 2B)\cos(2A - 2B) + \cos 4C$$
$$= 2\cos(2\pi - 2C)\cos(2A - 2B) + (2\cos^2 2C - 1)$$
$$[\because A + B + C = \pi \Rightarrow (2A + 2B) = (2\pi - 2C)]$$
$$= 2\cos 2C\cos(2A - 2B) + 2\cos^2 2C - 1$$
$$= 2\cos 2C[\cos(2A - 2B) + \cos 2C] - 1$$
$$= 2\cos 2C[\cos(2A - 2B) + \cos\{2\pi - (2A + 2B)\}] - 1$$
$$[\because 2A + 2B + 2C = 2\pi \Rightarrow 2C = 2\pi - (2A + 2B)]$$
$$= 2\cos 2C[\cos(2A - 2B) + \cos(2A + 2B)] - 1$$
$$= 2\cos 2C[2\cos 2A\cos 2B] - 1$$
$$= -1 + 4\cos 2A\cos 2B\cos 2C = \text{RHS}.$$
$$\therefore \quad \cos 4A + \cos 4B + \cos 4C = -1 + 4\cos 2A\cos 2B\cos 2C.$$

EXAMPLE 6 *If $A + B + C = \pi$, prove that*
$$\sin A + \sin B - \sin C = 4\sin\frac{A}{2}\sin\frac{B}{2}\cos\frac{C}{2}.$$

SOLUTION We have

$$\text{LHS} = (\sin A + \sin B) - \sin C$$
$$= 2\sin\left(\frac{A + B}{2}\right)\cos\left(\frac{A - B}{2}\right) - 2\sin\frac{C}{2}\cos\frac{C}{2}$$
$$= 2\sin\left(\frac{\pi}{2} - \frac{C}{2}\right)\cos\left(\frac{A - B}{2}\right) - 2\sin\frac{C}{2}\cos\frac{C}{2}\left[\because \frac{A + B}{2} = \frac{\pi}{2} - \frac{C}{2}\right]$$
$$= 2\cos\frac{C}{2}\cos\left(\frac{A - B}{2}\right) - 2\sin\frac{C}{2}\cos\frac{C}{2}$$
$$= 2\cos\frac{C}{2}\left[\cos\left(\frac{A - B}{2}\right) - \sin\frac{C}{2}\right]$$

$$= 2\cos\frac{C}{2}\left[\cos\left(\frac{A-B}{2}\right) - \sin\left\{\frac{\pi}{2} - \left(\frac{A+B}{2}\right)\right\}\right]$$

$$= 2\cos\frac{C}{2}\left[\cos\left(\frac{A-B}{2}\right) - \cos\left(\frac{A+B}{2}\right)\right]$$

$$= 2\cos\frac{C}{2}\left[2\sin\frac{A}{2}\sin\frac{B}{2}\right]$$

$$= 4\sin\frac{A}{2}\sin\frac{B}{2}\cos\frac{C}{2} = \text{RHS}.$$

$$\therefore \quad \sin A + \sin B - \sin C = 4\sin\frac{A}{2}\sin\frac{B}{2}\cos\frac{C}{2}.$$

EXAMPLE 7 *If $A + B + C = \pi$, prove that*

$$\cos A + \cos B - \cos C = \left(4\cos\frac{A}{2}\cos\frac{B}{2}\sin\frac{C}{2}\right) - 1.$$

SOLUTION We have

$$\text{LHS} = (\cos A + \cos B) - \cos C$$

$$= 2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right) - \left(1 - 2\sin^2\frac{C}{2}\right)$$

$$= 2\cos\left(\frac{\pi}{2} - \frac{C}{2}\right)\cos\left(\frac{A-B}{2}\right) - 1 + 2\sin^2\frac{C}{2}$$

$$= 2\sin\frac{C}{2}\left[\cos\left(\frac{A-B}{2}\right) + \sin\frac{C}{2}\right] - 1$$

$$= 2\sin\frac{C}{2}\left[\cos\left(\frac{A-B}{2}\right) + \sin\left\{\frac{\pi}{2} - \frac{(A+B)}{2}\right\}\right] - 1$$

$$= 2\sin\frac{C}{2}\left[\cos\left(\frac{A-B}{2}\right) + \cos\left(\frac{A+B}{2}\right)\right] - 1$$

$$= 2\sin\frac{C}{2}\left\{2\cos\frac{A}{2}\cos\frac{B}{2}\right\} - 1$$

$$= \left\{4\cos\frac{A}{2}\cos\frac{B}{2}\sin\frac{C}{2}\right\} - 1 = \text{RHS}.$$

$$\therefore \quad \cos A + \cos B - \cos C = \left(4\cos\frac{A}{2}\cos\frac{B}{2}\sin\frac{C}{2}\right) - 1.$$

TYPE 2 IDENTITIES INVOLVING SQUARES OF SINES AND COSINES

METHOD Change the squares of sines and cosines into cosines of double the angles by using the formulae

$$\cos^2\theta = \frac{1}{2}(1 + \cos 2\theta) \text{ and } \sin^2\theta = \frac{1}{2}(1 - \cos 2\theta).$$

Now, proceed as above.

EXAMPLE 8 *If $A + B + C = \pi$, prove that*

$$\sin^2 A + \sin^2 B + \sin^2 C = 2(1 + \cos A\cos B\cos C).$$

SOLUTION We have

$$\begin{aligned}\text{LHS} &= \sin^2 A + \sin^2 B + \sin^2 C\\ &= \frac{1}{2}(1 - \cos 2A) + \frac{1}{2}(1 - \cos 2B) + \frac{1}{2}(1 - \cos 2C)\\ &= \frac{3}{2} - \frac{1}{2}(\cos 2A + \cos 2B + \cos 2C)\\ &= \frac{3}{2} - \frac{1}{2}(-1 - 4\cos A \cos B \cos C) \quad \text{[see Example 3]}\\ &= 2 + 2\cos A \cos B \cos C = 2(1 + \cos A \cos B \cos C) = \text{RHS}.\end{aligned}$$

$\therefore \quad \sin^2 A + \sin^2 B + \sin^2 C = 2(1 + \cos A \cos B \cos C).$

EXAMPLE 9 *If $A + B + C = \pi$, prove that*

$$\cos^2 A + \cos^2 B - \cos^2 C = 1 - 2 \sin A \sin B \cos C.$$

SOLUTION We have

$$\begin{aligned}\text{LHS} &= \cos^2 A + \cos^2 B - \cos^2 C\\ &= \frac{1}{2}(1 + \cos 2A) + \frac{1}{2}(1 + \cos 2B) - \frac{1}{2}(1 + \cos 2C)\\ &= \frac{1}{2} + \frac{1}{2}(\cos 2A + \cos 2B - \cos 2C)\\ &= \frac{1}{2} + \frac{1}{2}(1 - 4\sin A \sin B \cos C) \quad \text{[see Example 4]}\\ &= 1 - 2\sin A \sin B \cos C = \text{RHS}.\end{aligned}$$

$\therefore \quad \cos^2 A + \cos^2 B - \cos^2 C = 1 - 2\sin A \sin B \cos C.$

EXAMPLE 10 *In a $\triangle ABC$, prove that*

$$\cos^2 A + \cos^2\left(A + \frac{\pi}{3}\right) + \cos^2\left(A - \frac{\pi}{3}\right) = \frac{3}{2}\cdot$$

SOLUTION We have

$$\begin{aligned}\text{LHS} &= \cos^2 A + \cos^2\left(A + \frac{\pi}{3}\right) + \cos^2\left(A - \frac{\pi}{3}\right)\\ &= \frac{1}{2}(1 + \cos 2A) + \frac{1}{2}\left\{1 + \cos\left(2A + \frac{2\pi}{3}\right)\right\} + \frac{1}{2}\left\{1 + \cos\left(2A - \frac{2\pi}{3}\right)\right\}\\ &= \frac{3}{2} + \frac{1}{2}\cos 2A + \frac{1}{2}\left\{\cos\left(2A + \frac{2\pi}{3}\right) + \cos\left(2A - \frac{2\pi}{3}\right)\right\}\\ &= \frac{3}{2} + \frac{1}{2}\cos 2A + \cos 2A \cos\frac{2\pi}{3}\\ &\qquad [\because \cos(A + B) + \cos(A - B) = 2\cos A \cos B]\\ &= \frac{3}{2} + \frac{1}{2}\cos 2A - \frac{1}{2}\cos 2A \quad \left[\because \cos\frac{2\pi}{3} = -\frac{1}{2}\right]\\ &= \frac{3}{2} = \text{RHS}.\end{aligned}$$

EXAMPLE 11 *If $A+B+C=\pi$, prove that*

$$\sin^2\frac{A}{2}+\sin^2\frac{B}{2}-\sin^2\frac{C}{2}=1-2\cos\frac{A}{2}\cos\frac{B}{2}\sin\frac{C}{2}.$$

SOLUTION We have

$$\begin{aligned}\text{LHS} &= \sin^2\frac{A}{2}+\sin^2\frac{B}{2}-\sin^2\frac{C}{2}\\ &=\frac{1}{2}(1-\cos A)+\frac{1}{2}(1-\cos B)-\frac{1}{2}(1-\cos C)\\ &=\frac{1}{2}-\frac{1}{2}(\cos A+\cos B-\cos C)\\ &=\frac{1}{2}-\frac{1}{2}\left[\left(4\cos\frac{A}{2}\cos\frac{B}{2}\sin\frac{C}{2}\right)-1\right] \quad \text{[see Example 7]}\\ &=1-2\cos\frac{A}{2}\cos\frac{B}{2}\sin\frac{C}{2}=\text{RHS.}\end{aligned}$$

$$\therefore\quad \sin^2\frac{A}{2}+\sin^2\frac{B}{2}-\sin^2\frac{C}{2}=1-2\cos\frac{A}{2}\cos\frac{B}{2}\sin\frac{C}{2}.$$

EXAMPLE 12 *If $A+B+C=\pi$, prove that*

$$\cos^2\frac{A}{2}+\cos^2\frac{B}{2}-\cos^2\frac{C}{2}=2\cos\frac{A}{2}\cos\frac{B}{2}\sin\frac{C}{2}.$$

SOLUTION We have

$$\begin{aligned}\text{LHS} &= \cos^2\frac{A}{2}+\cos^2\frac{B}{2}-\cos^2\frac{C}{2}\\ &=\frac{1}{2}(1+\cos A)+\frac{1}{2}(1+\cos B)-\frac{1}{2}(1+\cos C)\\ &=\frac{1}{2}+\frac{1}{2}(\cos A+\cos B-\cos C)\\ &=\frac{1}{2}+\frac{1}{2}\left[\left(4\cos\frac{A}{2}\cos\frac{B}{2}\sin\frac{C}{2}\right)-1\right]\\ &=2\cos\frac{A}{2}\cos\frac{B}{2}\sin\frac{C}{2}=\text{RHS.}\end{aligned}$$

$$\therefore\quad \cos^2\frac{A}{2}+\cos^2\frac{B}{2}-\cos^2\frac{C}{2}=2\cos\frac{A}{2}\cos\frac{B}{2}\sin\frac{C}{2}.$$

EXAMPLE 13 *If $A+B+C=\pi$, prove that*

$$\cos^2\frac{A}{2}+\cos^2\frac{B}{2}+\cos^2\frac{C}{2}=2\left(1+\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\right).$$

SOLUTION We have

$$\begin{aligned}\text{LHS} &= \cos^2\frac{A}{2}+\cos^2\frac{B}{2}+\cos^2\frac{C}{2}\\ &=\frac{1}{2}(1+\cos A)+\frac{1}{2}(1+\cos B)+\frac{1}{2}(1+\cos C)\\ &=\frac{3}{2}+\frac{1}{2}(\cos A+\cos B+\cos C)\end{aligned}$$

$$= \frac{3}{2} + \frac{1}{2}\left[2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right) + \cos C\right]$$

$$= \frac{3}{2} + \cos\left(\frac{\pi}{2} - \frac{C}{2}\right)\cos\left(\frac{A-B}{2}\right) + \frac{1}{2}\left(1 - 2\sin^2\frac{C}{2}\right)$$

$$\left[\because \left(\frac{A+B}{2}\right) = \left(\frac{\pi}{2} - \frac{C}{2}\right) \text{ and } \cos C = 1 - 2\sin^2\frac{C}{2}\right]$$

$$= 2 + \sin\frac{C}{2}\cos\left(\frac{A-B}{2}\right) - \sin^2\frac{C}{2}$$

$$= 2 + \sin\frac{C}{2}\left[\cos\left(\frac{A-B}{2}\right) - \sin\frac{C}{2}\right]$$

$$= 2 + \sin\frac{C}{2}\left[\cos\left(\frac{A-B}{2}\right) - \sin\left\{\frac{\pi}{2} - \left(\frac{A+B}{2}\right)\right\}\right]$$

$$= 2 + \sin\frac{C}{2}\left[\cos\left(\frac{A-B}{2}\right) - \cos\left(\frac{A+B}{2}\right)\right]$$

$$= 2 + \sin\frac{C}{2}\left(2\sin\frac{A}{2}\sin\frac{B}{2}\right)$$

$$= 2\left(1 + \sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\right) = \text{RHS}.$$

$$\therefore \quad \cos^2\frac{A}{2} + \cos^2\frac{B}{2} + \cos^2\frac{C}{2} = 2\left(1 + \sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\right).$$

TYPE 3 IDENTITIES INVOLVING TANGENTS

METHOD

1. Using $A + B + C = \pi$, express the sum of two angles in terms of the third.
2. Take tangents on both sides and expand the LHS.
3. Cross multiply and transpose.

Similarly, we proceed for identities involving cotangents.

EXAMPLE 14 *If* $A + B + C = \pi$, *prove that*

$$\tan A + \tan B + \tan C = \tan A \tan B \tan C.$$

SOLUTION Here, $A + B + C = \pi$

$$\Rightarrow \quad (A + B) = (\pi - C)$$

$$\Rightarrow \quad \tan(A + B) = \tan(\pi - C)$$

$$\Rightarrow \quad \frac{\tan A + \tan B}{1 - \tan A \tan B} = -\tan C$$

$$\Rightarrow \quad \tan A + \tan B = -\tan C + \tan A \tan B \tan C$$

$$\Rightarrow \quad \tan A + \tan B + \tan C = \tan A \tan B \tan C.$$

EXAMPLE 15 *If* $A + B + C = \pi$, *prove that*

$$\tan\frac{A}{2}\tan\frac{B}{2} + \tan\frac{B}{2}\tan\frac{C}{2} + \tan\frac{C}{2}\tan\frac{A}{2} = 1.$$

SOLUTION Here, $A + B + C = \pi$

$$\Rightarrow \left(\frac{A}{2} + \frac{B}{2}\right) = \left(\frac{\pi}{2} - \frac{C}{2}\right)$$

$$\Rightarrow \tan\left(\frac{A}{2} + \frac{B}{2}\right) = \tan\left(\frac{\pi}{2} - \frac{C}{2}\right) = \cot\frac{C}{2}$$

$$\Rightarrow \frac{\tan\frac{A}{2} + \tan\frac{B}{2}}{1 - \tan\frac{A}{2}\tan\frac{B}{2}} = \frac{1}{\tan\frac{C}{2}}$$

$$\Rightarrow \tan\frac{C}{2}\tan\frac{A}{2} + \tan\frac{B}{2}\tan\frac{C}{2} = 1 - \tan\frac{A}{2}\tan\frac{B}{2}$$

$$\Rightarrow \tan\frac{A}{2}\tan\frac{B}{2} + \tan\frac{B}{2}\tan\frac{C}{2} + \tan\frac{C}{2}\tan\frac{A}{2} = 1.$$

EXAMPLE 16 *If* $A + B + C = \pi$, *prove that*

$$\cot B \cot C + \cot C \cot A + \cot A \cot B = 1.$$

SOLUTION Here, $A + B + C = \pi$

$$\Rightarrow (A + B) = (\pi - C)$$

$$\Rightarrow \cot(A + B) = \cot(\pi - C) = -\cot C$$

$$\Rightarrow \frac{\cot A \cot B - 1}{\cot A + \cot B} = -\cot C$$

$$\Rightarrow \cot A \cot B - 1 = -\cot C \cot A - \cot B \cot C$$

$$\Rightarrow \cot B \cot C + \cot C \cot A + \cot A \cot B = 1.$$

EXAMPLE 17 *If* $A + B + C = \pi$, *prove that*

$$\cot\frac{A}{2} + \cot\frac{B}{2} + \cot\frac{C}{2} = \cot\frac{A}{2}\cot\frac{B}{2}\cot\frac{C}{2}\cdot$$

SOLUTION Here, $A + B + C = \pi$

$$\Rightarrow \left(\frac{A}{2} + \frac{B}{2}\right) = \left(\frac{\pi}{2} - \frac{C}{2}\right)$$

$$\Rightarrow \cot\left(\frac{A}{2} + \frac{B}{2}\right) = \cot\left(\frac{\pi}{2} - \frac{C}{2}\right) = \tan\frac{C}{2}$$

$$\Rightarrow \frac{\cot\frac{A}{2}\cot\frac{B}{2} - 1}{\cot\frac{A}{2} + \cot\frac{B}{2}} = \frac{1}{\cot\frac{C}{2}}$$

$$\Rightarrow \cot\frac{A}{2} + \cot\frac{B}{2} = \cot\frac{A}{2}\cot\frac{B}{2}\cot\frac{C}{2} - \cot\frac{C}{2}$$

$$\Rightarrow \cot\frac{A}{2} + \cot\frac{B}{2} + \cot\frac{C}{2} = \cot\frac{A}{2}\cot\frac{B}{2}\cot\frac{C}{2}\cdot$$

EXERCISE 16

If A + B + C = π, prove that

1. $\sin 2A + \sin 2B - \sin 2C = 4\cos A \cos B \sin C$

2. $\cos 2A - \cos 2B - \cos 2C = -1 + 4\cos A \sin B \sin C$

3. $\cos 2A - \cos 2B + \cos 2C = 1 - 4\sin A \cos B \sin C$

4. $\sin A + \sin B + \sin C = 4\cos \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2}$

5. $\cos A + \cos B + \cos C = 1 + 4\sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}$

6. $\dfrac{\sin 2A + \sin 2B + \sin 2C}{\sin A + \sin B + \sin C} = 8\sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}$

7. $\sin (B + C - A) + \sin (C + A - B) - \sin (A + B - C) = 4\cos A \cos B \sin C$

8. $\dfrac{\cos A}{\sin B \sin C} + \dfrac{\cos B}{\sin C \sin A} + \dfrac{\cos C}{\sin A \sin B} = 2$

9. $\cos^2 A + \cos^2 B + \cos^2 C = 1 - 2\cos A \cos B \cos C$

10. $\sin^2 A - \sin^2 B + \sin^2 C = 2\sin A \cos B \sin C$

11. $\sin^2 \frac{A}{2} + \sin^2 \frac{B}{2} + \sin^2 \frac{C}{2} = 1 - 2\sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}$

12. $\tan 2A + \tan 2B + \tan 2C = \tan 2A \tan 2B \tan 2C$

HINTS TO SOME SELECTED QUESTIONS

7. LHS $= \sin (A + B + C - 2A) + \sin (A + B + C - 2B) - \sin (A + B + C - 2C)$
$= \sin (\pi - 2A) + \sin (\pi - 2B) - \sin (\pi - 2C)$
$= \sin 2A + \sin 2B - \sin 2C.$

8. Multiply both sides by $2 \sin A \sin B \sin C$.

12. $A + B + C = \pi \Rightarrow 2A + 2B = 2\pi - 2C$
$\Rightarrow \tan (2A + 2B) = \tan (2\pi - 2C) = -\tan 2C.$

❑

17
Trigonometric Equations

TRIGONOMETRIC EQUATIONS

Equations which involve trigonometric functions of unknown angles are known as trigonometric equations.

For example, $\sin x = \frac{1}{2}$, $\tan x = -\sqrt{3}$, $\cos x = \sin 2x$, etc.

SOLUTION OF A TRIGONOMETRIC EQUATION *A solution of a trigonometric equation is the value of the unknown angle that satisfies the equation.*

A trigonometric equation may have an infinite number of solutions.

PRINCIPAL SOLUTIONS *The solutions of a trigonometric equation for which* $0 \le x < 2\pi$ *are called its principal solutions.*

GENERAL SOLUTION *A solution of a trigonometric equation, generalised by means of periodicity, is known as the general solution.*

REMARK We shall use 'I' to denote the set of all integers.

GENERAL SOLUTIONS OF TRIGONOMETRIC EQUATIONS

THEOREM 1 *(i)* $\sin\theta = 0 \Leftrightarrow \theta = n\pi,\ n \in I$

(ii) $\cos\theta = 0 \Leftrightarrow \theta = (2n+1)\frac{\pi}{2},\ n \in I$

(iii) $\tan\theta = 0 \Leftrightarrow \theta = n\pi,\ n \in I$

PROOF (i) $\sin\theta = 0 \Leftrightarrow \theta = 0, \pm\pi, \pm 2\pi, \pm 3\pi, \ldots$

$\Leftrightarrow \theta = n\pi,\ n \in I.$

(ii) $\cos\theta = 0 \Leftrightarrow \theta = 0, \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}, \ldots$

$\Leftrightarrow \theta = (2n+1)\frac{\pi}{2},\ n \in I.$

(iii) $\tan\theta = 0 \Leftrightarrow \frac{\sin\theta}{\cos\theta} = 0$

$\Leftrightarrow \sin\theta = 0$

$\Leftrightarrow \theta = 0, \pm\pi, \pm 2\pi, \pm 3\pi, \ldots$

$\Leftrightarrow \theta = n\pi,\ n \in I.$

EXAMPLE *Find the general solution of each of the following equations:*

(i) $\sin 2x = 0$ (ii) $\sin 3x = 0$ (iii) $\sin \dfrac{x}{2} = 0$

(iv) $\sin\left(x - \dfrac{\pi}{4}\right) = 0$ (v) $\cos 2x = 0$ (vi) $\cos\left(x + \dfrac{\pi}{10}\right) = 0$

(vii) $\tan 3x = 0$ (viii) $\tan\left(2x + \dfrac{\pi}{8}\right) = 0$ (ix) $\tan\left(4x + \dfrac{\pi}{6}\right) = 0$

SOLUTION (i) $\sin 2x = 0 \Rightarrow 2x = n\pi \Rightarrow x = \dfrac{n\pi}{2}$, where $n \in I$.

(ii) $\sin 3x = 0 \Rightarrow 3x = n\pi \Rightarrow x = \dfrac{n\pi}{3}$, where $n \in I$.

(iii) $\sin \dfrac{x}{2} = 0 \Rightarrow \dfrac{x}{2} = n\pi \Rightarrow x = 2n\pi$, where $n \in I$.

(iv) $\sin\left(x - \dfrac{\pi}{4}\right) = 0 \Rightarrow \left(x - \dfrac{\pi}{4}\right) = n\pi \Rightarrow x = \left(n\pi + \dfrac{\pi}{4}\right)$, where $n \in I$.

(v) $\cos 2x = 0 \Rightarrow 2x = (2n+1)\dfrac{\pi}{2} \Rightarrow x = (2n+1)\dfrac{\pi}{4}$, where $n \in I$.

(vi) $\cos\left(x + \dfrac{\pi}{10}\right) = 0 \Rightarrow \left(x + \dfrac{\pi}{10}\right) = (2n+1)\dfrac{\pi}{2} \Rightarrow x = \left\{(2n+1)\dfrac{\pi}{2} - \dfrac{\pi}{10}\right\}$,
where $n \in I$.

(vii) $\tan 3x = 0 \Rightarrow 3x = n\pi \Rightarrow x = \dfrac{n\pi}{3}$, where $n \in I$.

(viii) $\tan\left(2x + \dfrac{\pi}{8}\right) = 0 \Rightarrow \left(2x + \dfrac{\pi}{8}\right) = n\pi \Rightarrow x = \dfrac{1}{2}\left(n\pi - \dfrac{\pi}{8}\right)$,
where $n \in I$.

(ix) $\tan\left(4x + \dfrac{\pi}{6}\right) = 0 \Rightarrow \left(4x + \dfrac{\pi}{6}\right) = n\pi \Rightarrow x = \dfrac{1}{4}\left(n\pi - \dfrac{\pi}{6}\right)$,
where $n \in I$.

THEOREM 2 $\sin\theta = \sin\alpha \Rightarrow \theta = n\pi + (-1)^n\alpha$, *where* $n \in I$.

PROOF We have

$$\begin{aligned}
\sin\theta = \sin\alpha &\Rightarrow \sin\theta - \sin\alpha = 0 \\
&\Rightarrow 2\cos\frac{(\theta+\alpha)}{2}\sin\frac{(\theta-\alpha)}{2} = 0 \\
&\Rightarrow \cos\frac{(\theta+\alpha)}{2} = 0 \text{ or } \sin\frac{(\theta-\alpha)}{2} = 0 \\
&\Rightarrow \frac{(\theta+\alpha)}{2} = (2n+1)\frac{\pi}{2} \text{ or } \frac{(\theta-\alpha)}{2} = n\pi \\
&\Rightarrow (\theta+\alpha) = (2n+1)\pi \text{ or } (\theta-\alpha) = 2n\pi \\
&\Rightarrow \theta = [(2n+1)\pi - \alpha] \text{ or } \theta = (2n\pi + \alpha)
\end{aligned}$$

$$\Rightarrow\ \theta = [(\text{an odd multiple of } \pi) - \alpha]$$
$$\text{or } \theta = [(\text{an even multiple of } \pi) + \alpha]$$
$$\Rightarrow\ \theta = n\pi + (-1)^n\alpha, \text{ where } n \in I.$$

$\therefore\quad \sin\theta = \sin\alpha \ \Rightarrow\ \theta = n\pi + (-1)^n\alpha$, where $n \in I$.

THEOREM 3 $\cos\theta = \cos\alpha \ \Rightarrow\ \theta = 2n\pi \pm \alpha$, *where* $n \in I$.

PROOF We have

$$\cos\theta = \cos\alpha \ \Rightarrow\ \cos\theta - \cos\alpha = 0$$
$$\Rightarrow\ -2\sin\frac{(\theta+\alpha)}{2}\sin\frac{(\theta-\alpha)}{2} = 0$$
$$\Rightarrow\ \sin\frac{(\theta+\alpha)}{2} = 0 \text{ or } \sin\frac{(\theta-\alpha)}{2} = 0$$
$$\Rightarrow\ \frac{(\theta+\alpha)}{2} = n\pi \text{ or } \frac{(\theta-\alpha)}{2} = n\pi$$
$$\Rightarrow\ (\theta+\alpha) = 2n\pi \text{ or } (\theta-\alpha) = 2n\pi$$
$$\Rightarrow\ \theta = (2n\pi - \alpha) \text{ or } \theta = (2n\pi + \alpha)$$
$$\Rightarrow\ \theta = (2n\pi \pm \alpha).$$

$\therefore\quad \cos\theta = \cos\alpha \ \Rightarrow\ \theta = (2n\pi \pm \alpha)$, where $n \in I$.

THEOREM 4 *If* θ *and* α *are not odd multiples of* $\frac{\pi}{2}$ *then*

$$\tan\theta = \tan\alpha \ \Rightarrow\ \theta = n\pi + \alpha, \textit{ where } n \in I.$$

PROOF We have

$$\tan\theta = \tan\alpha \ \Rightarrow\ \frac{\sin\theta}{\cos\theta} = \frac{\sin\alpha}{\cos\alpha}$$
$$\Rightarrow\ \sin\theta\cos\alpha - \cos\theta\sin\alpha = 0$$
$$\Rightarrow\ \sin(\theta-\alpha) = 0$$
$$\Rightarrow\ (\theta-\alpha) = n\pi, \text{ where } n \in I$$
$$\Rightarrow\ \theta = (n\pi + \alpha), \text{ where } n \in I$$

$\therefore\quad \tan\theta = \tan\alpha \ \Rightarrow\ \theta = (n\pi + \alpha)$, where $n \in I$.

THEOREM 5 *(i)* $\sin^2\theta = \sin^2\alpha \ \Leftrightarrow\ \theta = n\pi \pm \alpha$

(ii) $\cos^2\theta = \cos^2\alpha \ \Leftrightarrow\ \theta = n\pi \pm \alpha$

(iii) $\tan^2\theta = \tan^2\alpha \ \Leftrightarrow\ \theta = n\pi \pm \alpha$

PROOF (i) $\sin^2\theta = \sin^2\alpha$

$$\Rightarrow\ \frac{(1-\cos 2\theta)}{2} = \frac{(1-\cos 2\alpha)}{2}$$
$$\Rightarrow\ \cos 2\theta = \cos 2\alpha$$
$$\Rightarrow\ 2\theta = 2n\pi \pm 2\alpha$$
$$\Rightarrow\ \theta = (n\pi \pm \alpha),\ n \in I.$$

(ii) $\cos^2\theta = \cos^2\alpha$

$$\Rightarrow \quad \frac{(1+\cos 2\theta)}{2} = \frac{(1+\cos 2\alpha)}{2}$$

$$\Rightarrow \quad \cos 2\theta = \cos 2\alpha$$

$$\Rightarrow \quad 2\theta = 2n\pi \pm 2\alpha$$

$$\Rightarrow \quad \theta = (n\pi \pm \alpha), \text{ where } n \in I.$$

(iii) $\tan^2\theta = \tan^2\alpha$

$$\Rightarrow \quad \frac{1-\tan^2\theta}{1+\tan^2\theta} = \frac{1-\tan^2\alpha}{1+\tan^2\alpha}$$

$$\Rightarrow \quad \cos 2\theta = \cos 2\alpha$$

$$\Rightarrow \quad 2\theta = 2n\pi \pm 2\alpha, \text{ where } n \in I$$

$$\Rightarrow \quad \theta = (n\pi \pm \alpha), \text{ where } n \in I.$$

THEOREM 6 $a\cos\theta + b\sin\theta = c \iff \theta = 2n\pi + \alpha \pm \beta$,

where $\cos\alpha = \dfrac{a}{\sqrt{a^2+b^2}}$, $\cos\beta = \dfrac{c}{\sqrt{a^2+b^2}}$ *and* $\sin\alpha = \dfrac{b}{\sqrt{a^2+b^2}}$.

PROOF The given equation is $a\cos\theta + b\sin\theta = c$. … (i)

Dividing (i) throughout by $\sqrt{a^2+b^2}$, we get

$$\frac{a}{\sqrt{a^2+b^2}}\cdot\cos\theta + \frac{b}{\sqrt{a^2+b^2}}\cdot\sin\theta = \frac{c}{\sqrt{a^2+b^2}}$$

$\Rightarrow \quad \cos\theta\cos\alpha + \sin\theta\sin\alpha = \cos\beta$,

where $\cos\alpha = \dfrac{a}{\sqrt{a^2+b^2}}$, $\sin\alpha = \dfrac{b}{\sqrt{a^2+b^2}}$ and $\cos\beta = \dfrac{c}{\sqrt{a^2+b^2}}$

$$\Rightarrow \quad \cos(\theta-\alpha) = \cos\beta$$

$$\Rightarrow \quad (\theta-\alpha) = 2n\pi \pm \beta$$

$$\Rightarrow \quad \theta = (2n\pi + \alpha \pm \beta).$$

SUMMARY

1. $\left.\begin{array}{l} \textbf{(i)}\ \sin\theta = 0 \Rightarrow \theta = n\pi \\ \textbf{(ii)}\ \cos\theta = 0 \Rightarrow \theta = (2n+1)\dfrac{\pi}{2} \\ \textbf{(iii)}\ \tan\theta = 0 \Rightarrow \theta = n\pi \end{array}\right\}$ **where $n \in I$**

2. $\left.\begin{array}{l} \textbf{(i)}\ \sin\theta = \sin\alpha \Rightarrow \theta = n\pi + (-1)^n\alpha \\ \textbf{(ii)}\ \cos\theta = \cos\alpha \Rightarrow \theta = 2n\pi \pm \alpha \\ \textbf{(iii)}\ \tan\theta = \tan\alpha \Rightarrow \theta = n\pi + \alpha \end{array}\right\}$ **where $n \in I$**

3. $\left.\begin{array}{l} \textbf{(i)}\ \sin^2\theta = \sin^2\alpha \Rightarrow \theta = n\pi \pm \alpha \\ \textbf{(ii)}\ \cos^2\theta = \cos^2\alpha \Rightarrow \theta = n\pi \pm \alpha \\ \textbf{(iii)}\ \tan^2\theta = \tan^2\alpha \Rightarrow \theta = n\pi \pm \alpha \end{array}\right\}$ **where $n \in I$**

SOLVED EXAMPLES

EXAMPLE 1 *Find the principal solutions of each of the following equations:*

(i) $\sin x = \frac{1}{2}$ *(ii)* $\cos x = \frac{1}{\sqrt{2}}$ *(iii)* $\tan x = \frac{1}{\sqrt{3}}$

SOLUTION (i) The given equation is $\sin x = \frac{1}{2}$.

We know that $\sin \frac{\pi}{6} = \frac{1}{2}$ and $\sin\left(\pi - \frac{\pi}{6}\right) = \frac{1}{2}$.

$\therefore \sin \frac{\pi}{6} = \frac{1}{2}$ and $\sin \frac{5\pi}{6} = \frac{1}{2}$.

Hence, the principal solutions are $x = \frac{\pi}{6}$ and $x = \frac{5\pi}{6}$.

(ii) The given equation is $\cos x = \frac{1}{\sqrt{2}}$.

We know that $\cos \frac{\pi}{4} = \frac{1}{\sqrt{2}}$ and $\cos\left(2\pi - \frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$.

$\therefore \cos \frac{\pi}{4} = \frac{1}{\sqrt{2}}$ and $\cos \frac{7\pi}{4} = \frac{1}{\sqrt{2}}$.

Hence, the principal solutions are $x = \frac{\pi}{4}$ and $x = \frac{7\pi}{4}$.

(iii) The given equation is $\tan x = \frac{1}{\sqrt{3}}$.

We know that $\tan \frac{\pi}{6} = \frac{1}{\sqrt{3}}$ and $\tan\left(\pi + \frac{\pi}{6}\right) = \frac{1}{\sqrt{3}}$.

$\therefore \tan \frac{\pi}{6} = \frac{1}{\sqrt{3}}$ and $\tan \frac{7\pi}{6} = \frac{1}{\sqrt{3}}$.

Hence, the principal solutions are $x = \frac{\pi}{6}$ and $x = \frac{7\pi}{6}$.

EXAMPLE 2 *Find the principal solutions of each of the following equations:*

(i) $\sin x = \frac{-\sqrt{3}}{2}$ *(ii)* $\cos x = \frac{-1}{2}$ *(iii)* $\cot x = -\sqrt{3}$

SOLUTION (i) The given equation is $\sin x = \frac{-\sqrt{3}}{2}$.

We know that $\sin \frac{\pi}{3} = \frac{\sqrt{3}}{2}$.

$\therefore \sin\left(\pi + \frac{\pi}{3}\right) = -\sin \frac{\pi}{3} = \frac{-\sqrt{3}}{2}$, and $\left(2\pi - \frac{\pi}{3}\right) = -\sin \frac{\pi}{3} = \frac{-\sqrt{3}}{2}$.

$\therefore \sin \frac{4\pi}{3} = \frac{-\sqrt{3}}{2}$ and $\sin \frac{5\pi}{3} = \frac{-\sqrt{3}}{2}$.

Hence, the principal solutions are $x = \frac{4\pi}{3}$ and $x = \frac{5\pi}{3}$.

(ii) The given equation is $\cos x = \frac{-1}{2}\cdot$

We know that $\cos\frac{\pi}{3} = \frac{1}{2}\cdot$

$\therefore\ \cos\left(\pi - \frac{\pi}{3}\right) = -\cos\frac{\pi}{3} = -\frac{1}{2}$, and $\cos\left(\pi + \frac{\pi}{3}\right) = -\cos\frac{\pi}{3} = -\frac{1}{2}\cdot$

$\therefore\ \cos\frac{2\pi}{3} = -\frac{1}{2}$ and $\cos\frac{4\pi}{3} = -\frac{1}{2}\cdot$

Hence, the principal solutions are $x = \frac{2\pi}{3}$ and $x = \frac{4\pi}{3}\cdot$

(iii) The given equation is $\cot x = -\sqrt{3} \Leftrightarrow \tan x = -\frac{1}{\sqrt{3}}\cdot$

We know that $\tan\frac{\pi}{6} = \frac{1}{\sqrt{3}}\cdot$

$\therefore\ \tan\left(\pi - \frac{\pi}{6}\right) = -\tan\frac{\pi}{6} = \frac{-1}{\sqrt{3}}$, and $\tan\left(2\pi - \frac{\pi}{6}\right) = -\tan\frac{\pi}{6} = \frac{-1}{\sqrt{3}}\cdot$

$\therefore\ \tan\frac{5\pi}{6} = \frac{-1}{\sqrt{3}}$ and $\tan\frac{11\pi}{6} = \frac{-1}{\sqrt{3}}\cdot$

Hence, the principal solutions are $x = \frac{5\pi}{6}$ and $x = \frac{11\pi}{6}\cdot$

EXAMPLE 3 *In each of the following, find the general value of x satisfying the equation:*

(i) $\sin x = \frac{1}{\sqrt{2}}$ *(ii)* $\cos x = \frac{1}{2}$ *(iii)* $\tan x = \frac{1}{\sqrt{3}}$

SOLUTION (i) Given: $\sin x = \frac{1}{\sqrt{2}}\cdot$

The least value of x in $[0, 2\pi[$ for which $\sin x = \frac{1}{\sqrt{2}}$ is $x = \frac{\pi}{4}\cdot$

$\therefore\ \sin x = \sin\frac{\pi}{4} \Rightarrow x = n\pi + (-1)^n\cdot\frac{\pi}{4}$, where $n \in I$.

Hence, the general solution is $x = n\pi + (-1)^n\cdot\frac{\pi}{4}$, where $n \in I$.

(ii) Given: $\cos x = \frac{1}{2}\cdot$

The least value of x in $[0, 2\pi[$ for which $\cos x = \frac{1}{2}$ is $x = \frac{\pi}{3}\cdot$

$\therefore\ \cos x = \cos\frac{\pi}{3} \Rightarrow x = \left(2n\pi \pm \frac{\pi}{3}\right)$, where $n \in I$.

Hence, the general solution is $x = \left(2n\pi \pm \frac{\pi}{3}\right)$, where $n \in I$.

(iii) Given: $\tan x = \frac{1}{\sqrt{3}}$.

The least value of x in $[0, 2\pi[$ for which $\tan x = \frac{1}{\sqrt{3}}$ is $\frac{\pi}{6}$.

$\therefore \tan x = \tan \frac{\pi}{6} \Rightarrow x = \left(n\pi + \frac{\pi}{6}\right)$, where $n \in I$.

Hence, the general solution is $x = \left(n\pi + \frac{\pi}{6}\right)$, where $n \in I$.

EXAMPLE 4 *Find the general value of x for which* $\sqrt{3}$ cosec $x = 2$.

SOLUTION Given: $\sqrt{3} \operatorname{cosec} x = 2 \Rightarrow \operatorname{cosec} x = \frac{2}{\sqrt{3}} \Rightarrow \sin x = \frac{\sqrt{3}}{2}$.

The least value of x in $[0, 2\pi[$ for which $\sin x = \frac{\sqrt{3}}{2}$ is $\frac{\pi}{3}$.

$\therefore \quad \sin x = \sin \frac{\pi}{3} \Rightarrow x = \left\{n\pi + (-1)^n \cdot \frac{\pi}{3}\right\}$, where $n \in I$.

Hence, the general solution is $x = \left\{n\pi + (-1)^n \cdot \frac{\pi}{3}\right\}$, where $n \in I$.

EXAMPLE 5 *In each of the following, find the general value of x satisfying the equation:*

(i) $\sin x = \frac{-\sqrt{3}}{2}$ *(ii)* $\cos x = \frac{-1}{2}$ *(iii)* $\cot x = -\sqrt{3}$

SOLUTION (i) $\sin x = \frac{-\sqrt{3}}{2} = -\sin \frac{\pi}{3} = \sin\left(\pi + \frac{\pi}{3}\right) = \sin \frac{4\pi}{3}$.

$\therefore \sin x = \sin \frac{4\pi}{3} \Rightarrow x = \left\{n\pi + (-1)^n \cdot \frac{4\pi}{3}\right)$, where $n \in I$.

Hence, the general solution is $x = \left\{n\pi + (-1)^n \cdot \frac{4\pi}{3}\right)$, where $n \in I$.

(ii) $\cos x = \frac{-1}{2} = -\cos \frac{\pi}{3} = \cos\left(\pi - \frac{\pi}{3}\right) = \cos \frac{2\pi}{3}$.

$\therefore \cos x = \cos \frac{2\pi}{3} \Rightarrow x = \left(2n\pi \pm \frac{2\pi}{3}\right)$, where $n \in I$.

Hence, the general solution is $x = \left(2n\pi \pm \frac{2\pi}{3}\right)$, where $n \in I$.

(iii) $\cot x = -\sqrt{3}$

$\Rightarrow \tan x = \frac{-1}{\sqrt{3}} = -\tan \frac{\pi}{6} = \tan\left(\pi - \frac{\pi}{6}\right) = \tan \frac{5\pi}{6}$

$\Rightarrow \tan x = \tan \frac{5\pi}{6}$

$\Rightarrow x = \left(n\pi + \frac{5\pi}{6}\right)$, where $n \in I$.

Hence, the general solution is $x = \left(n\pi + \dfrac{5\pi}{6}\right)$, where $n \in I$.

EXAMPLE 6 *Find the general solution of each equation:*

(i) $\sqrt{3} \cot x + 1 = 0$ *(ii)* $\operatorname{cosec} x + \sqrt{2} = 0$

SOLUTION (i) $\sqrt{3} \cot x + 1 = 0$

$$\Rightarrow \cot x = \frac{-1}{\sqrt{3}}$$

$$\Rightarrow \tan x = -\sqrt{3} = -\tan\frac{\pi}{3} = \tan\left(\pi - \frac{\pi}{3}\right) = \tan\frac{2\pi}{3}$$

$$\Rightarrow \tan x = \tan\frac{2\pi}{3}$$

$$\Rightarrow x = \left(n\pi + \frac{2\pi}{3}\right), \text{ where } n \in I.$$

Hence, the general solution is $x = \left(n\pi + \dfrac{2\pi}{3}\right)$, where $n \in I$.

(ii) $\operatorname{cosec} x + \sqrt{2} = 0$

$$\Rightarrow \sin x = -\frac{1}{\sqrt{2}} = -\sin\frac{\pi}{4} = \sin\left(\pi + \frac{\pi}{4}\right) = \sin\frac{5\pi}{4}$$

$$\Rightarrow \sin x = \sin\frac{5\pi}{4}$$

$$\Rightarrow x = \left\{n\pi + (-1)^n \cdot \frac{5\pi}{4}\right\}, \text{ where } n \in I.$$

Hence, the general solution is $x = \left\{n\pi + (-1)^n \cdot \dfrac{5\pi}{4}\right\}$, where $n \in I$.

EXAMPLE 7 *Find the general solution of each of the equations:*

(i) $\sin 2x = -\dfrac{1}{2}$ *(ii)* $\tan 3x = -1$

SOLUTION (i) $\sin 2x = -\dfrac{1}{2} = -\sin\dfrac{\pi}{6} = \sin\left(\pi + \dfrac{\pi}{6}\right) = \sin\dfrac{7\pi}{6}$

$$\Rightarrow \sin 2x = \sin\frac{7\pi}{6}$$

$$\Rightarrow 2x = \left\{n\pi + (-1)^n \cdot \frac{7\pi}{6}\right\}, \text{ where } n \in I$$

$$\Rightarrow x = \left\{\frac{n\pi}{2} + (-1)^n \cdot \frac{7\pi}{12}\right\}, \text{ where } n \in I.$$

Hence, the general solution is $x = \left\{\dfrac{n\pi}{2} + (-1)^n \cdot \dfrac{7\pi}{12}\right\}$, where $n \in I$.

(ii) $\tan 3x = -1 = -\tan\dfrac{\pi}{4} = \tan\left(\pi - \dfrac{\pi}{4}\right) = \tan\dfrac{3\pi}{4}$

$$\Rightarrow \tan 3x = \tan\frac{3\pi}{4}$$

$\Rightarrow\ 3x = \left(n\pi + \dfrac{3\pi}{4}\right)$, where $n \in I$.

$\Rightarrow\ x = \left(\dfrac{n\pi}{3} + \dfrac{\pi}{4}\right)$, where $n \in I$.

Hence, the general solution is $x = \left(\dfrac{n\pi}{3} + \dfrac{\pi}{4}\right)$, where $n \in I$.

EXAMPLE 8 *Find the general solution of each of the equations:*
(i) $4\sin^2 x = 1$ *(ii)* $2\cos^2 x = 1$ *(iii)* $\cot^2 x = 3$

SOLUTION (i) $4\sin^2 x = 1 \Rightarrow \sin^2 x = \dfrac{1}{4} = \left(\dfrac{1}{2}\right)^2 = \sin^2 \dfrac{\pi}{6}$

$\Rightarrow\ \sin^2 x = \sin^2 \dfrac{\pi}{6}$

$\Rightarrow\ x = \left\{n\pi \pm \dfrac{\pi}{6}\right\}$, where $n \in I$.

Hence, the general solution is $x = \left(n\pi \pm \dfrac{\pi}{6}\right)$, $n \in I$.

(ii) $2\cos^2 x = 1 \Rightarrow \cos^2 x = \dfrac{1}{2} = \left(\dfrac{1}{\sqrt{2}}\right)^2 = \cos^2 \dfrac{\pi}{4}$

$\Rightarrow\ \cos^2 x = \cos^2 \dfrac{\pi}{4}$

$\Rightarrow\ x = \left(n\pi \pm \dfrac{\pi}{4}\right)$, where $n \in I$.

Hence, the general solution is $x = \left(n\pi \pm \dfrac{\pi}{4}\right)$, $n \in I$.

(iii) $\cot^2 x = 3 \Rightarrow \tan^2 x = \dfrac{1}{3} = \left(\dfrac{1}{\sqrt{3}}\right)^2 = \tan^2 \dfrac{\pi}{6}$

$\Rightarrow\ \tan^2 x = \tan^2 \dfrac{\pi}{6}$

$\Rightarrow\ x = \left(n\pi \pm \dfrac{\pi}{6}\right)$, where $n \in I$.

Hence, the general solution is $\left(n\pi \pm \dfrac{\pi}{6}\right)$, where $n \in I$.

EXAMPLE 9 *Find the general solution of the equation* $\sin 2x + \sin 4x + \sin 6x = 0$.

SOLUTION The given equation may be written as $(\sin 6x + \sin 2x) + \sin 4x = 0$

$$\Rightarrow\ 2\sin \frac{(6x + 2x)}{2} \cos \frac{(6x - 2x)}{2} + \sin 4x = 0$$

$$\left[\because\ \sin C + \sin D = 2\sin \frac{(C + D)}{2} \cos \frac{(C - D)}{2}\right]$$

$\Rightarrow \quad 2\sin 4x \cos 2x + \sin 4x = 0$

$\Rightarrow \quad \sin 4x(2\cos 2x + 1) = 0$

$\Rightarrow \quad \sin 4x = 0 \text{ or } 2\cos 2x + 1 = 0$

$\Rightarrow \quad \sin 4x = 0 \text{ or } \cos 2x = -\frac{1}{2} = -\cos\frac{\pi}{3} = \cos\left(\pi - \frac{\pi}{3}\right) = \cos\frac{2\pi}{3}$

$\Rightarrow \quad \sin 4x = 0 \text{ or } \cos 2x = \cos\frac{2\pi}{3}$

$\Rightarrow \quad 4x = n\pi \text{ or } 2x = \left(2m\pi \pm \frac{2\pi}{3}\right)$, where $m, n \in I$

$\Rightarrow \quad x = \frac{n\pi}{4} \text{ or } x = \left(m\pi \pm \frac{\pi}{3}\right)$, where $m, n \in I$.

Hence, the general solution is given by $x = \frac{n\pi}{4}$ or $x = \left(m\pi \pm \frac{\pi}{3}\right)$, where $m, n \in I$.

EXAMPLE 10 *Solve:* $2\cos^2 x + 3\sin x = 0$.

SOLUTION We have

$2\cos^2 x + 3\sin x = 0$

$\Rightarrow \quad 2(1 - \sin^2 x) + 3\sin x = 0$

$\Rightarrow \quad 2\sin^2 x - 3\sin x - 2 = 0$

$\Rightarrow \quad 2\sin^2 x - 4\sin x + \sin x - 2 = 0$

$\Rightarrow \quad 2\sin x(\sin x - 2) + (\sin x - 2) = 0$

$\Rightarrow \quad (\sin x - 2)(2\sin x + 1) = 0$

$\Rightarrow \quad (\sin x - 2) = 0 \text{ or } (2\sin x + 1) = 0$

$\Rightarrow \quad \sin x = 2 \text{ or } (2\sin x + 1) = 0$

$\Rightarrow \quad 2\sin x + 1 = 0 \quad [\because \sin x = 2 \text{ is not possible}]$

$\Rightarrow \quad \sin x = -\frac{1}{2} = -\sin\frac{\pi}{6} = \sin\left(\pi + \frac{\pi}{6}\right) = \sin\frac{7\pi}{6}$

$\Rightarrow \quad \sin x = \sin\frac{7\pi}{6}$

$\Rightarrow \quad x = \left\{n\pi + (-1)^n \cdot \frac{7\pi}{6}\right\}$, where $n \in I$.

Hence, the general solution is given by $x = \left\{n\pi + (-1)^n \cdot \frac{7\pi}{6}\right\}$, where $n \in I$.

EXAMPLE 11 *Find the general solution for each of the following equations:*

(i) $\cos 4x = \cos 2x$ *(ii)* $\cos 3x = \sin 2x$

(iii) $\sin 3x + \cos 2x = 0$ *(iv)* $\sin mx + \sin nx = 0$

SOLUTION (i) $\cos 4x = \cos 2x$

$\Rightarrow \cos 4x - \cos 2x = 0$

$$\Rightarrow -2\sin\frac{(4x+2x)}{2}\sin\frac{(4x-2x)}{2}=0$$

$$\left[\because \cos C-\cos D=-2\sin\frac{(C+D)}{2}\sin\frac{(C-D)}{2}\right]$$

$\Rightarrow -2\sin 3x \sin x = 0$

$\Rightarrow \sin 3x = 0$ or $\sin x = 0$

$\Rightarrow 3x = n\pi$ or $x = m\pi$, where $m, n \in I$

$\Rightarrow x = \frac{n\pi}{3}$ or $x = m\pi$, where $m, n \in I$.

Hence, the general solution is $x = \frac{n\pi}{3}$ or $x = m\pi$, where $m, n \in I$.

(ii) $\cos 3x = \sin 2x$

$\Rightarrow \cos 3x = \cos\left(\frac{\pi}{2}-2x\right)$

$\Rightarrow 3x = 2n\pi \pm \left(\frac{\pi}{2}-2x\right)$ $[\cos\theta=\cos\alpha \Rightarrow \theta = 2n\pi \pm \alpha]$

$\Rightarrow 3x = 2n\pi + \left(\frac{\pi}{2}-2x\right)$ or $3x = 2n\pi - \left(\frac{\pi}{2}-2x\right)$, where $n \in I$

$\Rightarrow 5x = 2n\pi + \frac{\pi}{2}$ or $x = \left(2n\pi - \frac{\pi}{2}\right)$, where $n \in I$

$\Rightarrow x = \left(\frac{2n\pi}{5}+\frac{\pi}{10}\right)$ or $x = \left(2n\pi - \frac{\pi}{2}\right)$, where $n \in I$.

Hence, the general solution is $x = \left(\frac{2n\pi}{5}+\frac{\pi}{10}\right)$ or $x = \left(2n\pi - \frac{\pi}{2}\right)$, where $n \in I$.

(iii) $\sin 3x + \cos 2x = 0$

$\Rightarrow \cos 2x = -\sin 3x = \cos\left(\frac{\pi}{2}+3x\right)$

$\Rightarrow \cos 2x = \cos\left(\frac{\pi}{2}+3x\right)$

$\Rightarrow 2x = 2n\pi \pm \left(\frac{\pi}{2}+3x\right)$, where $n \in I$

$\Rightarrow 2x = 2n\pi + \left(\frac{\pi}{2}+3x\right)$ or $2x = 2n\pi - \left(\frac{\pi}{2}+3x\right)$, where $n \in I$

$\Rightarrow x = \left(-2n\pi - \frac{\pi}{2}\right)$ or $x = \left(\frac{2n\pi}{5}-\frac{\pi}{10}\right)$, where $n \in I$

$\Rightarrow x = \left(2n\pi - \frac{\pi}{2}\right)$ or $x = \left(\frac{2n\pi}{5}-\frac{\pi}{10}\right)$, where $n \in I$.

Hence, the general solution is $x = \left(2n\pi - \frac{\pi}{2}\right)$ or $x = \left(\frac{2n\pi}{5} - \frac{\pi}{10}\right)$, where $n \in I$.

NOTE $\left(-2n\pi - \frac{\pi}{2}\right)$ and $\left(2n\pi - \frac{\pi}{2}\right)$ give the same result as $n \in I$.

(iv) $\sin mx + \sin nx = 0$

$$\Rightarrow \quad 2\sin\frac{(m+n)x}{2}\cos\frac{(m-n)x}{2} = 0$$

$$\text{or } \sin\frac{(m+n)x}{2} = 0 \quad \text{or} \quad \cos\frac{(m-n)x}{2} = 0$$

$$\Rightarrow \quad \frac{(m+n)x}{2} = p\pi \text{ or } \frac{(m-n)x}{2} = (2q+1)\frac{\pi}{2}, \text{ where } p, q \in I$$

$$\Rightarrow \quad x = \frac{2p\pi}{(m+n)} \text{ or } x = \frac{(2q+1)\pi}{(m-n)}, \text{ where } p, q \in I.$$

Hence, the general solution is $x = \frac{2p\pi}{(m+n)}$ or $x = \frac{(2q+1)\pi}{(m-n)}$, where $p, q \in I$.

EXAMPLE 12 *Solve:* $\sqrt{3}\cos x - \sin x = 1$.

SOLUTION Given: $\sqrt{3}\cos x - \sin x = 1$. … (i)

Dividing both sides of (i) by $\sqrt{(\sqrt{3})^2 + (-1)^2}$, i.e., by 2, we get

$$\frac{\sqrt{3}}{2}\cos x - \frac{1}{2}\sin x = \frac{1}{2}$$

$$\Rightarrow \quad \cos x \cos\frac{\pi}{6} - \sin x \sin\frac{\pi}{6} = \frac{1}{2}$$

$$\Rightarrow \quad \cos\left(x + \frac{\pi}{6}\right) = \cos\frac{\pi}{3} \quad \left[\because \frac{1}{2} = \cos\frac{\pi}{3}\right]$$

$$\Rightarrow \quad \left(x + \frac{\pi}{6}\right) = 2n\pi \pm \frac{\pi}{3}, \text{ where } n \in I \; [\because \cos\theta = \cos\alpha \Rightarrow \theta = 2n\pi \pm \alpha]$$

$$\Rightarrow \quad \left(x + \frac{\pi}{6}\right) = \left(2n\pi + \frac{\pi}{3}\right) \text{ or } \left(x + \frac{\pi}{6}\right) = \left(2n\pi - \frac{\pi}{3}\right),$$

$$x = \left(2n\pi + \frac{\pi}{6}\right) \text{ or } x = \left(2n\pi - \frac{\pi}{2}\right), \text{ where } n \in I.$$

Hence, the general solution is $x = \left(2n\pi + \frac{\pi}{6}\right)$ or $x = \left(2n\pi - \frac{\pi}{2}\right)$, where $n \in I$.

EXAMPLE 13 *Solve:* $\sec x - \tan x = \sqrt{3}$.

SOLUTION We have

$$\sec x - \tan x = \sqrt{3} \Rightarrow \frac{1}{\cos x} - \frac{\sin x}{\cos x} = \sqrt{3} \Rightarrow (1 - \sin x) = \sqrt{3}\cos x$$

$$\Rightarrow \sqrt{3}\cos x + \sin x = 1.$$

Thus, the given equation becomes

$$\sqrt{3}\cos x + \sin x = 1. \qquad \dots \text{(i)}$$

Dividing both sides of (i) by $\sqrt{(\sqrt{3})^2 + 1^2}$, i.e., by 2, we get

$$\frac{\sqrt{3}}{2}\cos x + \frac{1}{2}\sin x = \frac{1}{2}$$

$$\Rightarrow \cos x \cos\frac{\pi}{6} + \sin x \sin\frac{\pi}{6} = \frac{1}{2}$$

$$\Rightarrow \cos\left(x - \frac{\pi}{6}\right) = \cos\frac{\pi}{3}$$

$$\Rightarrow \left(x - \frac{\pi}{6}\right) = 2n\pi \pm \frac{\pi}{3}, \text{ where } n \in I \; [\because \cos\theta = \cos\alpha \Rightarrow \theta = 2n\pi \pm \alpha]$$

$$\Rightarrow \left(x - \frac{\pi}{6}\right) = \left(2n\pi + \frac{\pi}{3}\right) \text{ or } \left(x - \frac{\pi}{6}\right) = \left(2n\pi - \frac{\pi}{3}\right)$$

$$\Rightarrow x = 2n\pi + \left(+\frac{\pi}{3} + \frac{\pi}{6}\right) \text{ or } x = 2n\pi + \left(-\frac{\pi}{3} + \frac{\pi}{6}\right), \text{ where } n \in I$$

$$\Rightarrow x = \left(2n\pi + \frac{\pi}{2}\right) \text{ or } x = \left(2n\pi - \frac{\pi}{6}\right), \text{ where } n \in I$$

$$\Rightarrow x = \left(2n\pi - \frac{\pi}{6}\right), \text{ where } n \in I$$

$$\left[\because \sec x \text{ is not defined when } x = \left(2n\pi + \frac{\pi}{2}\right)\right].$$

Hence, the general solution is $x = \left(2n\pi - \frac{\pi}{6}\right)$, where $n \in I$.

EXERCISE 17

1. Find the principal solutions of each of the following equations:

(i) $\sin x = \frac{\sqrt{3}}{2}$ (ii) $\cos x = \frac{1}{2}$ (iii) $\tan x = \sqrt{3}$

(iv) $\cot x = \sqrt{3}$ (v) $\operatorname{cosec} x = 2$ (vi) $\sec x = \frac{2}{\sqrt{3}}$

2. Find the principal solutions of each of the following equations:

(i) $\sin x = \frac{-1}{2}$ (ii) $\sqrt{2}\cos x + 1 = 0$

(iii) $\tan x = -1$ (iv) $\sqrt{3}\operatorname{cosec} x + 2 = 0$

(v) $\tan x = -\sqrt{3}$ (vi) $\sqrt{3}\sec x + 2 = 0$

Find the general solution of each of the following equations:

3. (i) $\sin 3x = 0$ (ii) $\sin\frac{3x}{2} = 0$ (iii) $\sin\left(x + \frac{\pi}{5}\right) = 0$

(iv) $\cos 2x = 0$ (v) $\cos\frac{5x}{2} = 0$ (vi) $\cos\left(x + \frac{\pi}{10}\right) = 0$

(vii) $\tan 2x = 0$ (viii) $\tan\left(3x + \frac{\pi}{6}\right) = 0$ (ix) $\tan\left(2x - \frac{\pi}{4}\right) = 0$

4. (i) $\sin x = \frac{\sqrt{3}}{2}$ (ii) $\cos x = 1$ (iii) $\sec x = \sqrt{2}$

5. (i) $\cos x = \frac{-1}{2}$ (ii) $\operatorname{cosec} x = -\sqrt{2}$ (iii) $\tan x = -1$

6. (i) $\sin 2x = \frac{1}{2}$ (ii) $\cos 3x = \frac{1}{\sqrt{2}}$ (iii) $\tan \frac{2x}{3} = \sqrt{3}$

7. (i) $\sec 3x = -2$ (ii) $\cot 4x = -1$ (iii) $\operatorname{cosec} 3x = \frac{-2}{\sqrt{3}}$

8. (i) $4\cos^2 x = 1$ (ii) $4\sin^2 x - 3 = 0$ (iii) $\tan^2 x = 1$

9. (i) $\cos 3x = \cos 2x$ (ii) $\cos 5x = \sin 3x$ (iii) $\cos mx = \sin nx$

10. $\sin x = \tan x$

11. $4\sin x \cos x + 2\sin x + 2\cos x + 1 = 0$

12. $\sec^2 2x = 1 - \tan 2x$

13. $\tan^3 x - 3\tan x = 0$

14. $\sin x + \sin 3x + \sin 5x = 0$

15. $\sin x \tan x - 1 = \tan x - \sin x$

16. $\cos x + \sin x = 1$

17. $\cos x - \sin x = -1$

18. $\sqrt{3}\cos x + \sin x = 1$

19. $2\tan x - \cot x + 1 = 0$

20. $\sin x \tan x - 1 = \tan x - \sin x$

21. $\cot x + \tan x = 2\operatorname{cosec} x$

ANSWERS (EXERCISE 17)

1. (i) $\frac{\pi}{3}, \frac{2\pi}{3}$ (ii) $\frac{\pi}{3}, \frac{5\pi}{3}$ (iii) $\frac{\pi}{3}, \frac{4\pi}{3}$

(iv) $\frac{\pi}{6}, \frac{7\pi}{6}$ (v) $\frac{\pi}{6}, \frac{5\pi}{6}$ (vi) $\frac{\pi}{6}, \frac{11\pi}{6}$

2. (i) $\frac{7\pi}{6}, \frac{11\pi}{6}$ (ii) $\frac{3\pi}{4}, \frac{5\pi}{4}$ (iii) $\frac{3\pi}{4}, \frac{7\pi}{4}$

(iv) $\frac{4\pi}{3}, \frac{5\pi}{3}$ (v) $\frac{2\pi}{3}, \frac{5\pi}{3}$ (vi) $\frac{5\pi}{6}, \frac{7\pi}{6}$

3. (i) $\frac{n\pi}{3}, n \in I$ (ii) $\frac{2n\pi}{3}, n \in I$ (iii) $\left(n\pi - \frac{\pi}{5}\right), n \in I$

(iv) $(2n+1)\frac{\pi}{4}, n \in I$ (v) $(2n+1)\frac{\pi}{5}, n \in I$ (vi) $\left(n\pi + \frac{2\pi}{5}\right), n \in I$

(vii) $\frac{n\pi}{2}, n \in I$ (viii) $\left(\frac{n\pi}{3} - \frac{\pi}{18}\right), n \in I$ (ix) $\left(\frac{n\pi}{2} + \frac{\pi}{8}\right), n \in I$

4. (i) $x = n\pi + (-1)^n \cdot \frac{\pi}{3}, n \in I$ (ii) $x = 2n\pi, n \in I$

(iii) $x = \left(2n\pi \pm \frac{\pi}{4}\right), n \in I$

5. (i) $x = \left(2\pi \pm \frac{2\pi}{3}\right), n \in I$ (ii) $x = n\pi + (-1)^n \cdot \frac{5\pi}{4}, n \in I$

(iii) $x = \left(n\pi + \frac{3\pi}{4}\right), n \in I$

6. (i) $x = \frac{n\pi}{2} + (-1)^n \cdot \frac{\pi}{12}, n \in I$ (ii) $x = \left(\frac{2n\pi}{3} \pm \frac{\pi}{12}\right), n \in I$

(iii) $x = (3n + 1)\frac{\pi}{2}, n \in I$

7. (i) $x = \left(\frac{2n\pi}{3} \pm \frac{2\pi}{9}\right), n \in I$ (ii) $x = (4n + 3)\frac{\pi}{16}, n \in I$

(iii) $x = \frac{n\pi}{3} + (-1)^n \cdot \frac{4\pi}{9}, n \in I$

8. (i) $x = \left(n\pi \pm \frac{\pi}{3}\right), n \in I$ (ii) $x = \left(n\pi \pm \frac{\pi}{3}\right), n \in I$

(iii) $x = \left(n\pi \pm \frac{\pi}{4}\right), n \in I$

9. (i) $x = 2n\pi$ or $x = \frac{2n\pi}{5}$, where $n \in I$

(ii) $x = \left(\frac{n\pi}{4} + \frac{\pi}{16}\right)$ or $x = \left(n\pi - \frac{\pi}{4}\right)$, where $n \in I$

(iii) $x = \frac{(4k+1)\pi}{2(m+n)}$ or $x = \frac{(4k-1)\pi}{2(m-n)}$, where $k \in I$

10. $x = n\pi$ or $x = 2m\pi$, where $m, n \in I$

11. $x = \left(2n\pi \pm \frac{2\pi}{3}\right)$ or $x = m\pi + (-1)^m \cdot \frac{7\pi}{6}$, where $m, n \in I$

12. $x = \frac{n\pi}{2}$ or $x = \left(\frac{m\pi}{2} + \frac{3\pi}{8}\right)$, where $m, n \in I$

13. $x = n\pi$ or $x = \left(m\pi + \frac{\pi}{3}\right)$ or $x = \left(p\pi + \frac{2\pi}{3}\right)$, where $m, n, p \in I$

14. $x = \frac{n\pi}{3}$ or $x = \left(m\pi \pm \frac{\pi}{3}\right)$, where $m, n \in I$

15. $x = n\pi + (-1)^n \cdot \frac{\pi}{2}$ or $x = \left(m\pi + \frac{3\pi}{4}\right)$, where $m, n \in I$

16. $x = \left(2n\pi + \frac{\pi}{2}\right)$ or $x = 2n\pi$, where $n \in I$

17. $x = \left(2n\pi + \frac{\pi}{2}\right)$ or $x = (2n-1)\pi$, where $n \in I$

18. $x = \left(2n\pi + \frac{\pi}{2}\right)$ or $x = \left(2n\pi - \frac{\pi}{6}\right)$, where $n \in I$

19. $x = n\pi + \frac{3\pi}{4}$ or $x = m\pi + \tan^{-1}\frac{1}{2}$, where $m, n \in I$

20. $x = \left(m\pi - \frac{\pi}{4}\right)$ or $x = n\pi + (-1)^n \frac{\pi}{2}$, where $m, n \in I$

21. $x = 2n\pi \pm \frac{\pi}{3}$, where $n \in I$

HINTS TO SOME SELECTED QUESTIONS

10. $\sin x = \tan x \Rightarrow \sin x = \frac{\sin x}{\cos x}$

$\Rightarrow \sin x \cos x - \sin x = 0$

$\Rightarrow \sin x(\cos x - 1) = 0$

$\Rightarrow \sin x = 0$ or $\cos x = 1$

$\Rightarrow x = n\pi$ or $x = 2m\pi$, where $m, n \in I$.

11. $4\sin x\cos x + 2\sin x + 2\cos x + 1 = 0$

$\Rightarrow 2\sin x(2\cos x + 1) + (2\cos x + 1) = 0$

$\Rightarrow (2\cos x + 1)(2\sin x + 1) = 0$

$\Rightarrow \cos x = -\frac{1}{2}$ or $\sin x = -\frac{1}{2}$

$\Rightarrow \cos x = -\frac{1}{2} = -\cos\frac{\pi}{3} = \cos\left(\pi - \frac{\pi}{3}\right) = \cos\frac{2\pi}{3}$

or $\sin x = -\frac{1}{2} = -\sin\frac{\pi}{6} = \sin\left(\pi + \frac{\pi}{6}\right) = \sin\frac{7\pi}{6}$

$\Rightarrow x = \left(2n\pi \pm \frac{2\pi}{3}\right)$ or $x = m\pi + (-1)^m \cdot \frac{7\pi}{6}$, where $m, n \in I$.

12. $\sec^2 2x = 1 - \tan 2x$

$\Rightarrow 1 + \tan^2 2x = 1 - \tan 2x$

$\Rightarrow \tan^2 2x + \tan 2x = 0$

$\Rightarrow \tan 2x(\tan 2x + 1) = 0$

$\Rightarrow \tan 2x = 0$ or $\tan 2x = -1$

$\Rightarrow 2x = n\pi$ or $\tan 2x = -\tan\frac{\pi}{4} = \tan\left(\pi - \frac{\pi}{4}\right) = \tan\frac{3\pi}{4}$

$\Rightarrow x = \frac{n\pi}{2}$ or $2x = m\pi + \frac{3\pi}{4}$, where $m, n \in I$

$$\Rightarrow \quad x = \frac{n\pi}{2} \text{ or } x = \frac{m\pi}{2} + \frac{3\pi}{8}, \text{where } m, n \in I.$$

13. $\tan^3 x - 3\tan x = 0 \Rightarrow \tan x(\tan^2 x - 3) = 0$

$$\Rightarrow \tan x = 0 \text{ or } \tan x = \sqrt{3} \text{ or } \tan x = -\sqrt{3}$$

$$\Rightarrow x = n\pi \text{ or } \tan x = \tan\frac{\pi}{3} \text{ or } \tan x = -\tan\frac{\pi}{3} = \tan\left(\pi - \frac{\pi}{3}\right)$$

$$\Rightarrow x = n\pi \text{ or } x = m\pi + \frac{\pi}{3} \text{ or } x = p\pi + \frac{2\pi}{3}, \text{where } m, n, p \in I.$$

15. The given equation becomes

$$(\sin x - 1)(\sin x + \cos x) = 0$$

$$\Rightarrow \quad \sin x = 1 \text{ or } \tan x = -1$$

$$\Rightarrow \quad \sin x = \sin\frac{\pi}{2} \text{ or } \tan x = \tan\frac{3\pi}{4}$$

$$\Rightarrow \quad x = n\pi + (-1)^n\frac{\pi}{2} \text{ or } x = m\pi + \frac{3\pi}{4}, \text{where } m, n \in I.$$

16. $\cos x + \sin x = 1 \Rightarrow \frac{1}{\sqrt{2}}\cos x + \frac{1}{\sqrt{2}}\sin x = \frac{1}{\sqrt{2}}$

$$\Rightarrow \cos x\cos\frac{\pi}{4} + \sin x\sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}$$

$$\Rightarrow \cos\left(x - \frac{\pi}{4}\right) = \cos\frac{\pi}{4}$$

$$\Rightarrow x - \frac{\pi}{4} = 2n\pi \pm \frac{\pi}{4}$$

$$\Rightarrow x - \frac{\pi}{4} = 2n\pi + \frac{\pi}{4} \text{ or } x - \frac{\pi}{4} = 2n\pi - \frac{\pi}{4}$$

$$\Rightarrow x = 2n\pi + \frac{\pi}{2} \text{ or } x = 2n\pi, \text{where } n \in I.$$

17. $\cos x - \sin x = -1$

$$\Rightarrow \quad \frac{1}{\sqrt{2}}\cos x - \frac{1}{\sqrt{2}}\sin x = \frac{-1}{\sqrt{2}}$$

$$\Rightarrow \quad \cos\frac{\pi}{4}\cos x - \sin\frac{\pi}{4}\sin x = -\cos\frac{\pi}{4} = \cos\left(\pi - \frac{\pi}{4}\right)$$

$$\Rightarrow \quad \cos\left(x + \frac{\pi}{4}\right) = \cos\frac{3\pi}{4}$$

$$\Rightarrow \quad x + \frac{\pi}{4} = 2n\pi \pm \frac{3\pi}{4}$$

$$\Rightarrow \quad x + \frac{\pi}{4} = 2n\pi + \frac{3\pi}{4} \text{ or } x + \frac{\pi}{4} = 2n\pi - \frac{3\pi}{4}$$

$$\Rightarrow \quad x = 2n\pi + \frac{\pi}{2} \text{ or } x = (2n-1)\pi, \text{where } n \in I.$$

19. The given equation is

$$2\tan^2 x + \tan x - 1 = 0$$

$$\Rightarrow \quad (\tan x + 1)(2\tan x - 1) = 0$$

$$\Rightarrow \quad \tan x = -1 \text{ or } \tan x = \frac{1}{2}$$

$$\Rightarrow \quad \tan x = \tan\frac{3\pi}{4} \text{ or } \tan x = \tan\left(\tan^{-1}\frac{1}{2}\right)$$

$$\Rightarrow \quad x = n\pi + \frac{3\pi}{4} \text{ or } x = m\pi + \tan^{-1}\frac{1}{2}, \text{ where } m, n \in I.$$

20. The given equation is

$$\sin x \tan x + \sin x - 1 - \tan x = 0$$

$$\Rightarrow \quad \sin x(\tan x + 1) - (\tan x + 1) = 0$$

$$\Rightarrow \quad (\tan x + 1)(\sin x - 1) = 0$$

$$\Rightarrow \quad \tan x = -1 = \tan\left(-\frac{\pi}{4}\right) \text{ or } \sin x = 1 = \sin\frac{\pi}{2}$$

$$\Rightarrow \quad x = \left(m\pi - \frac{\pi}{4}\right) \text{ or } x = n\pi + (-1)^n \frac{\pi}{2}.$$

21. The given equation is

$$\frac{\cos x}{\sin x} + \frac{\sin x}{\cos x} = \frac{2}{\sin x} \Rightarrow \cos^2 x + \sin^2 x = 2\cos x$$

$$\Rightarrow \quad \cos x = \frac{1}{2} = \cos\frac{\pi}{3}$$

$$\Rightarrow \quad x = 2n\pi \pm \frac{\pi}{3}, \text{ where } n \in I.$$

❑

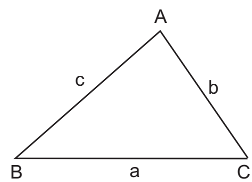
18

Solution of Triangles

INTRODUCTION The three sides and three angles of a triangle are called the *parts of the triangle.*

When at least three parts of a triangle are known (i.e., *any* three parts), we can compute the measures of the remaining three parts.

The process of computing the measures of unknown parts of a triangle from those of the given parts is known as solving the triangle.



THE OBLIQUE TRIANGLE *A triangle which does not contain a right angle is called an oblique triangle.*

In a $\triangle ABC$, we denote the lengths of the sides as $BC = a$, $CA = b$ and $AB = c$.

THEOREM 1 (Sine Formula or Sine Rule) *In any triangle, the sides are proportional to the sines of the opposite angles.*
That is, in a $\triangle ABC$,

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}.$$

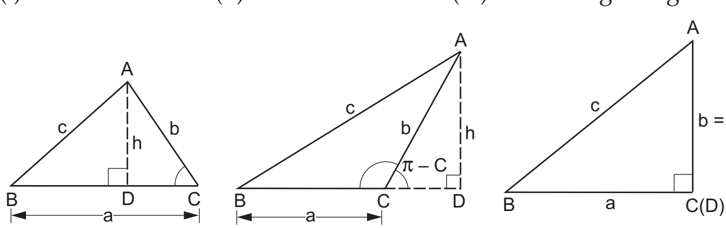
PROOF Let $\triangle ABC$ be any triangle.

We know that the sum of all the angles of a triangle is 180°. So, each angle of a triangle cannot be obtuse.

Let us assume here that $\angle B$ is acute.

Now, we consider the following cases:

(i) $\angle C$ is acute (ii) $\angle C$ is obtuse (iii) $\angle C$ is a right angle



(i) (ii) (iii)

Draw $AD \perp BC$ or BC produced.

Let $AD = h$.

Then, in each figure, we have:

$$\frac{AD}{AB} = \sin B \Rightarrow \frac{h}{c} = \sin B \Rightarrow h = c\sin B \quad \ldots (1)$$

Also, we have

In Fig. (i), $\frac{AD}{AC} = \sin C \Rightarrow \frac{h}{b} = \sin C \Rightarrow h = b\sin C.$

In Fig. (ii), $\frac{AD}{AC} = \sin(\pi - C) = \sin C \Rightarrow \frac{h}{b} = \sin C \Rightarrow h = b\sin C.$

In Fig. (iii), $\frac{AD}{AC} = 1 = \sin\frac{\pi}{2} = \sin C \Rightarrow \frac{h}{b} = \sin C \Rightarrow b\sin C.$

Thus, in each case, we have: $h = b\sin C$ $\quad \ldots (2)$

From (1) and (2), we get:

$$c\sin B = b\sin C \Rightarrow \frac{b}{\sin B} = \frac{c}{\sin C}.$$

Similarly, $\frac{b}{\sin B} = \frac{a}{\sin A}.$

Hence, $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}.$

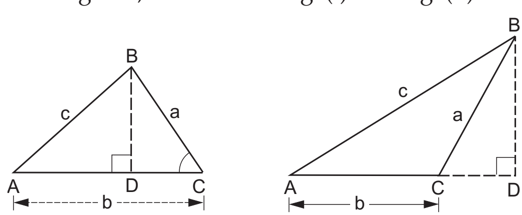
THEOREM 2 (Cosine Formulae) *Let a, b and c be the lengths of sides of $\triangle ABC$, opposite to $\angle A$, $\angle B$ and $\angle C$ respectively. Then,*

(i) $a^2 = b^2 + c^2 - 2bc\cos A$

(ii) $b^2 = c^2 + a^2 - 2ca\cos B$

(iii) $c^2 = a^2 + b^2 - 2ab\cos C$

PROOF Let $\triangle ABC$ be given, as shown in Fig. (i) and Fig. (ii).



(i) (ii)

From Fig. (i), we get

$$\begin{aligned} BC^2 &= BD^2 + DC^2 \\ &= BD^2 + (AC - AD)^2 \\ &= BD^2 + AD^2 + AC^2 - 2AC \cdot AD \\ &= AB^2 + AC^2 - 2AC \cdot AB\cos A \\ &\qquad [\because BD^2 + AD^2 = AB^2 \text{ and } AD = AB\cos A] \\ &= AC^2 + AB^2 - 2AC \cdot AB\cos A \end{aligned}$$

$$\Rightarrow \quad a^2 = b^2 + c^2 - 2bc\cos A.$$

From Fig. (ii), we get

$$BC^2 = BD^2 + CD^2$$
$$= BD^2 + (AD - AC)^2$$
$$= BD^2 + AD^2 + AC^2 - 2AC \cdot AD$$
$$= AB^2 + AC^2 - 2AC \cdot AB \cos A$$
$$[\because BD^2 + AD^2 = AB^2 \text{ and } AD = AB \cos A]$$

$\Rightarrow \quad a^2 = b^2 + c^2 - 2bc \cos A.$

Hence, from both the cases, we get

$$a^2 = b^2 + c^2 - 2bc \cos A.$$

Similarly, $b^2 = c^2 + a^2 - 2ca \cos B$ and $c^2 = a^2 + b^2 - 2ab \cos C$.

REMARKS *A convenient method writing cosine formulae is given below:*

$$(i)\ \cos A = \frac{(b^2 + c^2 - a^2)}{2bc}$$

$$(ii)\ \cos B = \frac{(c^2 + a^2 - b^2)}{2ca}$$

$$(iii)\ \cos C = \frac{(a^2 + b^2 - c^2)}{2ab}$$

SUMMARY

1. Sine Formula

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

2. Cosine Formulae

(i) $a^2 = b^2 + c^2 - 2bc \cos A$

(ii) $b^2 = c^2 + a^2 - 2ca \cos B$

(iii) $c^2 = a^2 + b^2 - 2ab \cos C$

THEOREM 3 **(Napier's Analogies)** *In any* $\triangle ABC$, *prove that*

$$(i)\ \tan \frac{(B - C)}{2} = \frac{(b - c)}{(b + c)} \cot \frac{A}{2}$$

$$(ii)\ \tan \frac{(C - A)}{2} = \frac{(c - a)}{(c + a)} \cot \frac{B}{2}$$

$$(iii)\ \tan \frac{(A - B)}{2} = \frac{(a - b)}{(a + b)} \cot \frac{C}{2}$$

PROOF By the sine formula, we have

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = k \text{ (say)}.$$

Then, $a = k \sin A$, $b = k \sin B$ and $c = k \sin C$.

(i) $\therefore \quad \dfrac{(b - c)}{(b + c)} = \dfrac{k \sin B - k \sin C}{k \sin B + k \sin C}$

$$= \frac{(\sin B - \sin C)}{(\sin B + \sin C)} = \frac{2\cos \frac{(B+C)}{2} \sin \frac{(B-C)}{2}}{2\sin \frac{(B+C)}{2} \cos \frac{(B-C)}{2}}$$

$$= \cot \frac{(B+C)}{2} \tan \frac{(B-C)}{2}$$

$$= \cot \left(\frac{\pi}{2} - \frac{A}{2}\right) \tan \frac{(B-C)}{2} \qquad \left[\because \frac{A}{2} + \frac{(B+C)}{2} = \frac{\pi}{2}\right]$$

$$= \tan \frac{A}{2} \tan \frac{(B-C)}{2}$$

$$\Rightarrow \quad \tan \frac{(B-C)}{2} = \frac{(b-c)}{(b+c)} \cot \frac{A}{2}.$$

Similarly, (ii) and (iii) may be proved.

SOLVED EXAMPLES

EXAMPLE 1 *In any $\triangle ABC$, prove that*

$$a(\sin B - \sin C) + b(\sin C - \sin A) + c(\sin A - \sin B) = 0.$$

SOLUTION By the sine rule, we have

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = k \text{ (say)}$$

$\Rightarrow \quad a = k \sin A$, $b = k \sin B$ and $c = k \sin C$.

$$\begin{aligned}
\therefore \quad \text{LHS} &= a(\sin B - \sin C) + b(\sin C - \sin A) + c(\sin A - \sin B) \\
&= k\sin A(\sin B - \sin C) + k\sin B(\sin C - \sin A) \\
&\qquad + k\sin C(\sin A - \sin B) \\
&= k[(\sin A \sin B - \sin A \sin C) + (\sin B \sin C - \sin A \sin B) \\
&\qquad + (\sin A \sin C - \sin B \sin C)] \\
&= (k \times 0) = 0 = \text{RHS}.
\end{aligned}$$

EXAMPLE 2 *In any $\triangle ABC$, prove that*

$$a\sin (B - C) + b\sin (C - A) + c\sin (A - B) = 0.$$

SOLUTION By the sine rule, we have

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

$$\Rightarrow \quad \frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c} = k \text{ (say)}$$

$\Rightarrow \quad \sin A = ka$, $\sin B = kb$ and $\sin C = kc$.

Substituting the values of sin B and sin C and using cosine formulae, we have

$$a\sin\ (B-C) = a(\sin B\cos C - \cos B\sin C)$$
$$= a\left[kb\cdot\frac{(a^2+b^2-c^2)}{2ab} - kc\cdot\frac{(c^2+a^2-b^2)}{2ac}\right]$$
$$= \frac{k}{2}\cdot[(a^2+b^2-c^2)-(c^2+a^2-b^2)] = k(b^2-c^2)].$$

Similarly, $b\sin(C-A) = b(\sin C\cos A - \cos C\sin A) = k(c^2-a^2)$.

And, $c\sin(A-B) = c(\sin A\cos B - \cos A\sin B) = k(a^2-b^2)$.

$\therefore\quad a\sin(B-C) + b\sin(C-A) + c\sin(A-B)$
$$= k(b^2-c^2) + k(c^2-a^2) + k(a^2-b^2)$$
$$= k(b^2-c^2+c^2-a^2+a^2-b^2) = k\times 0 = 0.$$

EXAMPLE 3 *In any* $\triangle ABC$, *prove that*
$$\frac{\sin\ (B-C)}{\sin\ (B+C)} = \frac{(b^2-c^2)}{a^2}.$$

SOLUTION By the sine rule, we have
$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = k\ (\text{say})$$
$\Rightarrow\quad a = k\sin A$, $b = k\sin B$ and $c = k\sin C$.

$$\therefore\quad \text{RHS} = \frac{(b^2-c^2)}{a^2} = \frac{k^2\sin^2 B - k^2\sin^2 C}{k^2\sin^2 A}$$
$$= \frac{(\sin^2 B - \sin^2 C)}{\sin^2 A} = \frac{\sin\ (B+C)\cdot\sin\ (B-C)}{\sin^2(B+C)}$$
$$\left[\begin{array}{l}\because\ (A+B+C)=\pi\ \Rightarrow\ A=\pi-(B+C)\\ \therefore\ \sin A = \sin\{\pi-(B+C)\} = \sin(B+C)\end{array}\right]$$
$$= \frac{\sin\ (B-C)}{\sin\ (B+C)} = \text{LHS}.$$

$\therefore\quad$ RHS = LHS.

Hence, $\dfrac{\sin\ (B-C)}{\sin\ (B+C)} = \dfrac{(b^2-c^2)}{a^2}$.

EXAMPLE 4 *In any* $\triangle ABC$, *prove that*
$$\frac{(a-b)}{c}\cos\frac{C}{2} = \sin\frac{(A-B)}{2}.$$

SOLUTION By the sine rule, we have
$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = k\ (\text{say})$$
$\Rightarrow\quad a = k\sin A$, $b = k\sin B$ and $c = k\sin C$.

$$\therefore \quad \text{LHS} = \frac{(a-b)}{c}\cos\frac{C}{2}$$

$$= \frac{(k\sin A - k\sin B)}{k\sin C}\cos\frac{C}{2} = \frac{k(\sin A - \sin B)}{k\sin C}\cdot\cos\frac{C}{2}$$

$$= \frac{(\sin A - \sin B)}{\sin C}\cdot\cos\frac{C}{2} = \frac{2\cos\frac{(A+B)}{2}\sin\frac{(A-B)}{2}}{2\sin\frac{C}{2}\cos\frac{C}{2}}\cdot\cos\frac{C}{2}$$

$$\left[\because (\sin C - \sin D) = 2\cos\frac{(C+D)}{2}\sin\frac{(C-D)}{2}\right]$$

$$= \frac{\cos\left(\frac{\pi}{2}-\frac{C}{2}\right)\cdot\sin\frac{(A-B)}{2}}{\sin\frac{C}{2}} \qquad \left[\because \frac{A+B}{2} = \left(\frac{\pi}{2}-\frac{C}{2}\right)\right]$$

$$= \frac{\sin\frac{C}{2}\cdot\sin\frac{(A-B)}{2}}{\sin\frac{C}{2}} = \sin\frac{(A-B)}{2} = \text{RHS}.$$

Hence, $\dfrac{(a-b)}{c}\cos\dfrac{C}{2} = \sin\dfrac{(A-B)}{2}$.

EXAMPLE 5 *In any* $\triangle ABC$, *prove that*

$$\frac{(b^2-c^2)}{a^2}\sin 2A + \frac{(c^2-a^2)}{b^2}\sin 2B + \frac{(a^2-b^2)}{c^2}\sin 2C = 0.$$

SOLUTION By the sine rule, we have

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = k\,(\text{say})$$

$\Rightarrow \quad a = k\sin A$, $b = k\sin B$ and $c = k\sin C$.

Applying sine rule and cosine formula, we get:

$$\frac{(b^2-c^2)}{a^2}\sin 2A = \frac{(b^2-c^2)}{a^2}\cdot(2\sin A\cos A)$$

$$= \frac{(b^2-c^2)}{a^2}\left(\frac{2a}{k}\right)\cdot\frac{(b^2+c^2-a^2)}{2bc}$$

$$\left[\because \sin A = \frac{a}{k} \text{ and } \cos A = \frac{(b^2+c^2-a^2)}{2bc}\right]$$

$$= \frac{1}{(kabc)}\cdot(b^2-c^2)(b^2+c^2-a^2) \qquad \ldots\text{(i)}$$

Similarly, $\dfrac{(c^2-a^2)}{b^2}\sin 2B = \dfrac{1}{(kabc)}\cdot(c^2-a^2)(c^2+a^2-b^2) \qquad \ldots\text{(ii)}$

And, $\dfrac{(a^2-b^2)}{c^2}\sin 2C = \dfrac{1}{(kabc)}\cdot(a^2-b^2)(a^2+b^2-c^2)$. … (iii)

From (i), (ii) and (iii), we get

$$\text{LHS} = \frac{(b^2-c^2)}{a^2}\sin 2A + \frac{(c^2-a^2)}{b^2}\sin 2B + \frac{(a^2-b^2)}{c^2}\sin 2C$$

$$= \frac{1}{(kabc)}\cdot[(b^2-c^2)(b^2+c^2-a^2)+(c^2-a^2)(c^2+a^2-b^2)$$
$$+(a^2-b^2)(a^2+b^2-c^2)]$$

$$= \frac{1}{(kabc)}\times 0 = 0 = \text{RHS}.$$

Hence, $\dfrac{(b^2-c^2)}{a^2}\sin 2A + \dfrac{(c^2-a^2)}{b^2}\sin 2B + \dfrac{(a^2-b^2)}{c^2}\sin 2C = 0.$

EXAMPLE 6 *In any $\triangle ABC$, prove that*

$$\frac{(b-c)}{(b+c)} = \frac{\tan\frac{1}{2}(B-C)}{\tan\frac{1}{2}(B+C)}.$$

SOLUTION By the sine rule, we have

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = k \text{ (say)}$$

$\Rightarrow$ $a = k\sin A$, $b = k\sin B$ and $c = k\sin C$.

$$\therefore \quad \text{LHS} = \frac{(b-c)}{(b+c)} = \frac{k\sin B - k\sin C}{k\sin B + k\sin C} = \frac{k(\sin B - \sin C)}{k(\sin B + \sin C)}$$

$$= \frac{(\sin B - \sin C)}{(\sin B + \sin C)} = \frac{2\cos\frac{(B+C)}{2}\sin\frac{(B-C)}{2}}{2\sin\frac{(B+C)}{2}\cos\frac{(B-C)}{2}}$$

$$= \frac{\tan\frac{1}{2}(B-C)}{\tan\frac{1}{2}(B+C)} = \text{RHS}.$$

Hence, $\dfrac{(b-c)}{(b+c)} = \dfrac{\tan\frac{1}{2}(B-C)}{\tan\frac{1}{2}(B+C)}$.

EXAMPLE 7 *In any $\triangle ABC$, prove that*

$$a(\cos C - \cos B) = 2(b-c)\cos^2\frac{A}{2}.$$

SOLUTION Applying cosine formulae, we have

$\text{LHS} = a(\cos C - \cos B)$

$$= a\left[\frac{(a^2+b^2-c^2)}{2ab} - \frac{(a^2+c^2-b^2)}{2ac}\right] = \frac{(a^2c+b^2c-c^3-a^2b-bc^2+b^3)}{2bc}$$

$$= \frac{(b^3 - c^3) + (b^2c - bc^2) - (a^2b - a^2c)}{2bc} = \frac{(b^3 - c^3) + bc(b - c) - a^2(b - c)}{2bc}$$

$$= (b - c)\,\frac{[(b^2 + c^2 + bc) + bc - a^2]}{2bc} = (b - c)\cdot\left\{\frac{(b^2 + c^2 - a^2)}{2bc} + \frac{2bc}{2bc}\right\}$$

$$= (b - c)\left\{\frac{(b^2 + c^2 - a^2)}{2bc} + 1\right\} = (b - c)(1 + \cos A)$$

$$= 2(b - c)\cos^2\frac{A}{2} = \text{RHS}.$$

$$\therefore \quad a(\cos C - \cos B) = 2(b - c)\cos^2\frac{A}{2}\cdot$$

EXAMPLE 8 *In any $\triangle ABC$, prove that*

$$a\cos A + b\cos B + c\cos C = 2a\sin B\sin C.$$

SOLUTION By the sine rule, we have

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = k\text{ (say)}$$

$\Rightarrow \quad a = k\sin A$, $b = k\sin B$ and $c = k\sin C$.

$$\begin{aligned}
\therefore \quad \text{LHS} &= a\cos A + b\cos B + c\cos C\\
&= k\sin A\cos A + k\sin B\cos B + k\sin C\cos C\\
&= \frac{1}{2}k(\sin 2A + \sin 2B + \sin 2C)\\
&= \frac{1}{2}k[2\sin(A + B)\cos(A - B) + 2\sin C\cos C]\\
&= k[\sin(A + B)\cos(A - B) + \sin C\cos C]\\
&= k[\sin(\pi - C)\cos(A - B) + \sin C\cos C]\\
&= k\sin C[\cos(A - B) + \cos C]\\
&= k\sin C[\cos(A - B) + \cos\{\pi - (A + B)\}]\\
&= k\sin C[\cos(A - B) - \cos(A + B)]\\
&= k\sin C \times 2\sin A\sin B\\
&= 2(k\sin A)\sin B\sin C\\
&= 2a\sin B\sin C = \text{RHS}.
\end{aligned}$$

Hence, $a\cos A + b\cos B + c\cos C = 2a\sin B\sin C$.

EXAMPLE 9 *In any $\triangle ABC$, prove that*

$$(b^2 - c^2)\cot A + (c^2 - a^2)\cot B + (a^2 - b^2)\cot C = 0.$$

SOLUTION By the sine rule, we have

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

$$\Rightarrow \quad \frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c} = k\text{ (say)}$$

$\Rightarrow \quad \sin A = ka$, $\sin B = kb$ and $\sin C = kc$.

$$\therefore \quad (b^2 - c^2)\cot A = \frac{(b^2 - c^2)\cdot\cos A}{\sin A} = \frac{(b^2 - c^2)}{ka}\cdot\frac{(b^2 + c^2 - a^2)}{2bc}$$

$$= \frac{1}{(2kabc)} \cdot (b^2 - c^2)(b^2 + c^2 - a^2) = \frac{(b^4 - c^4 - a^2b^2 + a^2c^2)}{2kabc}.$$

Similarly, $(c^2 - a^2) \cot B = \dfrac{(c^4 - a^4 - b^2c^2 + a^2b^2)}{2kabc}$.

And, $(a^2 - b^2) \cot C = \dfrac{(a^4 - b^4 - a^2c^2 + b^2c^2)}{2kabc}$.

$\therefore$ LHS $= (b^2 - c^2) \cot A + (c^2 - a^2) \cot B + (a^2 - b^2) \cot C$

$$= \frac{1}{2kabc} \cdot [(b^4 - c^4 - a^2b^2 + a^2c^2) + (c^4 - a^4 - b^2c^2 + a^2b^2) + (a^4 - b^4 - a^2c^2 + b^2c^2)]$$

$$= \frac{1}{2kabc} \times 0 = 0 = \text{RHS}.$$

Hence, $(b^2 - c^2) \cot A + (c^2 - a^2) \cot B + (a^2 - b^2) \cot C = 0$.

EXAMPLE 10 *In any $\triangle ABC$, prove that*

$$a^3 \sin(B - C) + b^3 \sin(C - A) + c^3 \sin(A - B) = 0.$$

SOLUTION By the sine rule, we have

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = k \text{ (say)}$$

$\Rightarrow$ $a = k \sin A$, $b = k \sin B$ and $c = k \sin C$.

$\therefore$ $a^3 \sin(B - C) = k^3 \sin^3 A \cdot \sin(B - C)$

$= k^3 \sin^2 A \cdot \sin A \cdot \sin(B - C)$

$= k^3 \sin^2 A[\sin\{\pi - (B + C)\} \sin(B - C)]$

$[\because A + (B + C) = \pi]$

$= k^3 \sin^2 A[\sin(B + C) \sin(B - C)]$

$= k^3 \sin^2 A \cdot (\sin^2 B - \sin^2 C)$.

Similarly, $b^3 \sin(C - A) = k^3 \sin^2 B(\sin^2 C - \sin^2 A)$.

And, $c^3 \sin(A - B) = k^3 \sin^2 C(\sin^2 A - \sin^2 B)$.

$\therefore$ $a^3 \sin(B - C) + b^3 \sin(C - A) + c^3 \sin(A - B)$

$$= k^3 \sin^2 A(\sin^2 B - \sin^2 C) + k^3 \sin^2 B(\sin^2 C - \sin^2 A) + k^3 \sin^2 C(\sin^2 A - \sin^2 B)$$

$= 0$.

EXAMPLE 11 *In any $\triangle ABC$, prove that*

$$(a - b)^2 \cos^2 \frac{C}{2} + (a + b)^2 \sin^2 \frac{C}{2} = c^2.$$

SOLUTION We have

$$\text{LHS} = (a - b)^2 \cos^2 \frac{C}{2} + (a + b)^2 \sin^2 \frac{C}{2}$$

$$= a^2\left(\cos^2\frac{C}{2} + \sin^2\frac{C}{2}\right) + b^2\left(\cos^2\frac{C}{2} + \sin^2\frac{C}{2}\right) - 2ab\cos^2\frac{C}{2} + 2ab\sin^2\frac{C}{2}$$

$$= a^2 + b^2 - 2ab\left(\cos^2\frac{C}{2} - \sin^2\frac{C}{2}\right)$$

$$= a^2 + b^2 - 2ab\cos C = c^2 = \text{RHS}. \qquad \text{[by cosine formula]}$$

$$\therefore \quad (a-b)^2\cos^2\frac{C}{2} + (a+b)^2\sin^2\frac{C}{2} = c^2.$$

EXAMPLE 12 *In a $\triangle ABC$, prove that*

$$(b-c)\cot\frac{A}{2} + (c-a)\cot\frac{B}{2} + (a-b)\cot\frac{C}{2} = 0.$$

SOLUTION By the sine rule, we have

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = k \text{ (say)}$$

$$\Rightarrow \quad a = k\sin A,\ b = k\sin B \text{ and } c = k\sin C.$$

$$\therefore \quad (b-c)\cot\frac{A}{2} = k(\sin B - \sin C)\cot\frac{A}{2}$$

$$= 2k\cos\frac{(B+C)}{2}\sin\frac{(B-C)}{2}\cdot\frac{\cos\left(\frac{A}{2}\right)}{\sin\left(\frac{A}{2}\right)}$$

$$= 2k\cos\left(\frac{\pi}{2} - \frac{A}{2}\right)\sin\frac{(B-C)}{2}\cdot\frac{\cos\left(\frac{A}{2}\right)}{\sin\left(\frac{A}{2}\right)}$$

$$= 2k\sin\frac{A}{2}\sin\frac{(B-C)}{2}\cdot\frac{\cos\left(\frac{A}{2}\right)}{\sin\left(\frac{A}{2}\right)}$$

$$= 2k\sin\frac{(B-C)}{2}\cdot\cos\frac{A}{2}$$

$$= 2k\sin\frac{(B-C)}{2}\cdot\cos\left\{\frac{\pi}{2} - \frac{(B+C)}{2}\right\}$$

$$= 2k\sin\frac{(B-C)}{2}\cdot\sin\frac{(B+C)}{2}$$

$$= k(\cos C - \cos B).$$

Similarly, we have

$$(c-a)\cot\frac{B}{2} = k(\cos A - \cos C) \text{ and } (a-b)\cot\frac{C}{2} = k(\cos B - \cos A).$$

$$\therefore \quad (b-c)\cot\frac{A}{2} + (c-a)\cot\frac{B}{2} + (a-b)\cot\frac{C}{2}$$

$$= k\cdot[(\cos C - \cos B) + (\cos A - \cos C) + (\cos B - \cos A)] = 0.$$

EXAMPLE 13 *In a* $\triangle ABC$, *if* $\cos A = \dfrac{\sin B}{2\sin C}$, *show that the triangle is isosceles.*

SOLUTION By the sine rule, we have

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

$$\Rightarrow \quad \frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c} = k \text{ (say)}$$

$$\Rightarrow \quad \sin A = ka, \ \sin B = kb \text{ and } \sin C = kc.$$

$$\therefore \quad \cos A = \frac{\sin B}{2\sin C} \Rightarrow \frac{(b^2 + c^2 - a^2)}{2bc} = \frac{kb}{2kc}$$

$$\Rightarrow (b^2 + c^2 - a^2) = b^2$$

$$\Rightarrow c^2 = a^2 \Rightarrow |c| = |a|$$

$$\Rightarrow AB = BC.$$

Hence, $\triangle ABC$ is isosceles.

EXAMPLE 14 *In a* $\triangle ABC$, *if* $a\cos A = b\cos B$, *show that the triangle is either isosceles or right-angled.*

SOLUTION By the sine rule, we have

$$\frac{a}{\sin A} = \frac{b}{\sin B} = k \text{ (say)}$$

$$\Rightarrow \quad a = k\sin A \text{ and } b = k\sin B.$$

$$\therefore \quad a\cos A = b\cos B$$

$$\Rightarrow \quad k\sin A \cos A = k\sin B \cos B$$

$$\Rightarrow \quad \frac{1}{2}(\sin 2A) = \frac{1}{2}(\sin 2B)$$

$$\Rightarrow \quad \sin 2A = \sin 2B$$

$$\Rightarrow \quad \sin 2A - \sin 2B = 0$$

$$\Rightarrow \quad 2\cos(A + B)\sin(A - B) = 0$$

$$\Rightarrow \quad \cos(A + B) = 0 \quad \text{or} \quad \sin(A - B) = 0$$

$$\Rightarrow \quad (A + B) = \frac{\pi}{2} \quad \text{or} \quad (A - B) = 0$$

$$\Rightarrow \quad \angle C = \frac{\pi}{2} \quad \text{or} \quad A = B.$$

Hence, $\triangle ABC$ is right-angled or isosceles.

EXAMPLE 15 *In a* $\triangle ABC$, *if* $a = 18$, $b = 24$, $c = 30$; *find*
(i) $\sin A$, $\sin B$, $\sin C$ *(ii)* $\cos A$, $\cos B$, $\cos C$

SOLUTION Here, $a^2 + b^2 = (18)^2 + (24)^2 = (324 + 576) = 900 = (30)^2 = c^2$.

$\therefore$ $\triangle ABC$ is right-angled at C. Thus, $\angle C = 90°$.

(i) By sine rule, we have

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

$$\Rightarrow \quad \frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c} = k \text{ (say)}$$

$$\Rightarrow \quad \frac{\sin A}{18} = \frac{\sin B}{24} = \frac{\sin C}{30} = k$$

$\therefore \quad \sin A = 18k$, $\sin B = 24k$ and $\sin C = 30k$.

But, $\angle C = 90° \Rightarrow \sin C = \sin 90° = 1 \Rightarrow 30k = 1 \Rightarrow k = \frac{1}{30}$.

$$\therefore \quad \sin A = 18k = \left(18 \times \frac{1}{30}\right) = \frac{3}{5}; \ \sin B = 24k = \left(24 \times \frac{1}{30}\right) = \frac{4}{5}$$

and $\sin C = 30k = \left(30 \times \frac{1}{30}\right) = 1.$

Hence, $\sin A = \frac{3}{5}$, $\sin B = \frac{4}{5}$ and $\sin C = 1$.

(ii) Using cosine formulae, we get

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc} = \frac{(24)^2 + (30)^2 - (18)^2}{2 \times 24 \times 30}$$

$$= \frac{576 + 900 - 324}{1440} = \frac{1152}{1440} = \frac{4}{5}.$$

$$\cos B = \frac{c^2 + a^2 - b^2}{2ca} = \frac{(30)^2 + (18)^2 - (24)^2}{2 \times 30 \times 18}$$

$$= \frac{(900 + 324 - 576)}{1080} = \frac{648}{1080} = \frac{3}{5}.$$

$\cos C = \cos 90° = 0.$

Hence, $\cos A = \frac{4}{5}$, $\cos B = \frac{3}{5}$ and $\cos C = 0$.

EXAMPLE 16 *In a $\triangle ABC$, if $\angle A = 45°$, $\angle B = 60°$ and $\angle C = 75°$, find the ratio of its sides.*

SOLUTION By the sine formula, we have

$$a : b : c = \sin A : \sin B : \sin C$$

$$= \sin 45° : \sin 60° : \sin 75°$$

$$= \frac{1}{\sqrt{2}} : \frac{\sqrt{3}}{2} : \frac{(\sqrt{3} + 1)}{2\sqrt{2}} = 2 : \sqrt{6} : (\sqrt{3} + 1).$$

EXAMPLE 17 *In a $\triangle ABC$, if $a = 2$, $b = 3$ and $\sin A = \left(\frac{2}{3}\right)$, find $\angle B$.*

SOLUTION $\frac{a}{\sin A} = \frac{b}{\sin B} \Rightarrow \frac{2}{(2/3)} = \frac{3}{\sin B}$

$$\Rightarrow \sin B = \left(3 \times \frac{2}{3} \times \frac{1}{2}\right) = 1$$

$$\Rightarrow \angle B = 90°.$$

EXAMPLE 18 *The angles of a triangle ABC are in AP and if it is being given that* $b : c = \sqrt{3} : \sqrt{2}$, *find* $\angle A$, $\angle B$ *and* $\angle C$.

SOLUTION $\angle A$, $\angle B$, $\angle C$ are in AP

$\Rightarrow \quad 2\angle B = \angle A + \angle C$

$\Rightarrow \quad 3\angle B = \angle A + \angle B + \angle C = 180°$

$\Rightarrow \quad \angle B = 60°.$

Now, $\dfrac{b}{\sin B} = \dfrac{c}{\sin C} \Rightarrow \dfrac{b}{c} = \dfrac{\sin B}{\sin C}$

$$\Rightarrow \frac{\sqrt{3}}{\sqrt{2}} = \frac{\sin 60°}{\sin C}$$

$$\Rightarrow \sin C = \left(\frac{\sqrt{3}}{2} \times \sqrt{2} \times \frac{1}{\sqrt{3}}\right) = \frac{1}{\sqrt{2}}$$

$$\Rightarrow \angle C = 45°.$$

$\therefore \quad \angle A = 180° - (\angle B + \angle C) = [180° - (60° + 45°)] = 75°.$

Hence, $\angle A = 75°$, $\angle B = 60°$ and $\angle C = 45°$.

EXAMPLE 19 *In a* $\triangle ABC$, *if* $\angle A = 30°$ *and* $b : c = 2 : \sqrt{3}$, *find* $\angle B$.

SOLUTION Let $b = 2k$ and $c = \sqrt{3}k$. Then,

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

$$\Rightarrow \cos 30° = \frac{4k^2 + 3k^2 - a^2}{4\sqrt{3}\,k^2}$$

$$\Rightarrow \frac{\sqrt{3}}{2} \times 4\sqrt{3}\,k^2 = 7k^2 - a^2$$

$\Rightarrow \quad k^2 = a^2 \Rightarrow k = a.$

Thus, $b = 2a$ and $c = \sqrt{3}\,a$.

Now, $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} \Rightarrow \dfrac{a}{\sin 30°} = \dfrac{2a}{\sin B}$

$$\Rightarrow \sin B = \left(2a \times \frac{1}{2} \times \frac{1}{a}\right) = 1$$

$\Rightarrow \quad B = 90°.$

EXAMPLE 20 *Solve the triangle in which* $a = (\sqrt{3} + 1)$, $b = (\sqrt{3} - 1)$ *and* $\angle C = 60°$.

SOLUTION Using $\tan \dfrac{(A - B)}{2} = \dfrac{(a - c)}{(a + b)} \cot \dfrac{C}{2}$, we get

$$\tan \frac{(A - B)}{2} = \frac{2}{2\sqrt{3}} \cot 30° = \left(\frac{1}{\sqrt{3}} \times \sqrt{3}\right) = 1$$

$\Rightarrow \quad \dfrac{A - B}{2} = 45° \Rightarrow A - B = 90° \qquad \ldots \text{(i)}$

Also, $A + B + C = 180° \Rightarrow A + B = 120° \qquad \ldots \text{(ii)}$

On solving (i) and (ii), we get $A = 105°$ and $B = 15°$.

Also, $\cos C = \dfrac{a^2 + b^2 - c^2}{2ab}$

$\Rightarrow \quad \cos 60° = \dfrac{(\sqrt{3}+1)^2 + (\sqrt{3}-1)^2 - c^2}{2(\sqrt{3}+1)(\sqrt{3}-1)} = \dfrac{8 - c^2}{4}$

$\therefore \quad \dfrac{8 - c^2}{4} = \dfrac{1}{2} \Rightarrow 8 - c^2 = 2 \Rightarrow c^2 = 6 \Rightarrow c = \sqrt{6}.$

Hence, $c = \sqrt{6}$ cm, $\angle A = 105°$ and $\angle B = 15°$.

EXERCISE 18A

In any $\triangle ABC$, prove that

1. $a(b\cos C - c\cos B) = (b^2 - c^2)$

2. $ac\cos B - bc\cos A = (a^2 - b^2)$

3. $\dfrac{\cos A}{a} + \dfrac{\cos B}{b} + \dfrac{\cos C}{c} = \dfrac{(a^2 + b^2 + c^2)}{2abc}$

4. $\dfrac{c - b\cos A}{b - c\cos A} = \dfrac{\cos B}{\cos C}$

5. $2(bc\cos A + ca\cos B + ab\cos C) = (a^2 + b^2 + c^2)$

6. $4\left(bc\cos^2\dfrac{A}{2} + ca\cos^2\dfrac{B}{2} + ab\cos^2\dfrac{C}{2}\right) = (a + b + c)^2$

7. $a\sin A - b\sin B = c\sin(A - B)$

8. $a^2\sin(B - C) = (b^2 - c^2)\sin A$

9. $\dfrac{\sin(A - B)}{\sin(A + B)} = \dfrac{(a^2 - b^2)}{c^2}$

10. $\dfrac{(b - c)}{a}\cos\dfrac{A}{2} = \sin\dfrac{(B - C)}{2}$

11. $\dfrac{(a + b)}{c}\sin\dfrac{C}{2} = \cos\dfrac{(A - B)}{2}$

12. $\dfrac{(b + c)}{a}\cdot\cos\dfrac{(B + C)}{2} = \cos\dfrac{(B - C)}{2}$

13. $a^2(\cos^2 B - \cos^2 C) + b^2(\cos^2 C - \cos^2 A) + c^2(\cos^2 A - \cos^2 B) = 0$

14. $\dfrac{(\cos^2 B - \cos^2 C)}{b + c} + \dfrac{(\cos^2 C - \cos^2 A)}{c + a} + \dfrac{(\cos^2 A - \cos^2 B)}{a + b} = 0$

15. $\dfrac{\cos 2A}{a^2} - \dfrac{\cos 2B}{b^2} = \left(\dfrac{1}{a^2} - \dfrac{1}{b^2}\right)$

16. $(c^2 - a^2 + b^2)\tan A = (a^2 - b^2 + c^2)\tan B = (b^2 - c^2 + a^2)\tan C$

17. If in a $\triangle ABC$, $\angle C = 90°$, then prove that $\sin(A - B) = \dfrac{(a^2 - b^2)}{(a^2 + b^2)}$.

18. In a $\triangle ABC$, if $\dfrac{\cos A}{a} = \dfrac{\cos B}{b}$, show that the triangle is isosceles.

19. In a $\triangle ABC$, if $\sin^2 A + \sin^2 B = \sin^2 C$, show that the triangle is right-angled.

20. Solve the triangle in which $a = 2$ cm, $b = 1$ cm and $c = \sqrt{3}$ cm.

21. In a $\triangle ABC$, if $a = 3$ cm, $b = 5$ cm and $c = 7$ cm, find $\cos A$, $\cos B$, $\cos C$

22. If the angles of a triangle are in the ratio $1:2:3$, prove that its corresponding sides are in the ratio $1:\sqrt{3}:2$.

ANSWERS (EXERCISE 18A)

20. $\angle A = 90°$, $\angle B = 30°$ and $\angle C = 60°$

21. $\cos A = \frac{13}{14}$, $\cos B = \frac{11}{14}$ and $\cos C = \frac{-1}{2}$

HINTS TO SOME SELECTED QUESTIONS

6. LHS $= 2[bc(1+\cos A) + ca(1+\cos B) + ab(1+\cos C)]$

$= 2[(bc + ca + ab) + bc \cdot \cos A + ca \cdot \cos B + ab \cdot \cos C]$

$= 2\left[(ab + bc + ca) + bc \cdot \frac{(b^2 + c^2 - a^2)}{2bc} + ca \cdot \frac{(c^2 + a^2 - b^2)}{2ca} + ab \cdot \frac{(a^2 + b^2 - c^2)}{2ab}\right]$

$= a^2 + b^2 + c^2 + 2(ab + bc + ca) = (a + b + c)^2 =$ RHS.

7. $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = k$ (say)

$\Rightarrow$ $a = k \sin A$, $b = k \sin B$ and $c = k \sin C$.

$\therefore$ LHS $= k \sin^2 A - k \sin^2 B$

$= k(\sin^2 A - \sin^2 B) = k \sin(A + B) \cdot \sin(A - B)$

$= k \sin(\pi - C) \sin(A - B) = k \sin C \sin(A - B)$

$= c \sin(A - B) =$ RHS.

8. LHS $= k^2 \sin^2 A[\sin(B - C)]$

$= k^2 \sin A \sin[\pi - (B + C)] \sin(B - C)$

$= k^2 \sin A \sin(B + C) \sin(B - C)$

$= k^2 \sin A \cdot (\sin^2 B - \sin^2 C) =$ RHS.

9. See Example 3.

For questions **10, 11, 12** see Example 4.

13. $a^2(\cos^2 B - \cos^2 C) = a^2[(1 - \sin^2 B) - (1 - \sin^2 C)] = a^2(\sin^2 C - \sin^2 B)$

$= a^2\left(\frac{c^2}{k^2} - \frac{b^2}{k^2}\right) = \frac{1}{k^2} \cdot a^2(c^2 - b^2).$

14. $\frac{\cos^2 B - \cos^2 C}{(b + c)} = \frac{(1 - \sin^2 B) - (1 - \sin^2 C)}{(b + c)}$

$= \frac{(\sin^2 C - \sin^2 B)}{(b + c)} = \frac{(c^2 - b^2)}{k^2(b + c)} = \frac{1}{k^2}(c - b).$

15. LHS $= \frac{(1 - 2\sin^2 A)}{a^2} - \frac{(1 - 2\sin^2 B)}{b^2} = \left(\frac{1}{a^2} - \frac{1}{b^2}\right) - \frac{2\sin^2 A}{a^2} + \frac{2\sin^2 B}{b^2}$

$= \left(\frac{1}{a^2} - \frac{1}{b^2}\right) - \frac{2\sin^2 A}{k^2 \sin^2 A} + \frac{2\sin^2 B}{k^2 \sin^2 B}$

$= \left(\frac{1}{a^2} - \frac{1}{b^2}\right) - \frac{2}{k^2} + \frac{2}{k^2} = \left(\frac{1}{a^2} - \frac{1}{b^2}\right).$

17. RHS $= \dfrac{(a^2 - b^2)}{(a^2 + b^2)}$

$$= \frac{k^2 \sin^2 A - k^2 \sin^2 B}{k^2 \sin^2 A + k^2 \sin^2 B} = \frac{(\sin^2 A - \sin^2 B)}{\sin^2 A + \sin^2 (90° - A)} = (\sin^2 A - \sin^2 B)$$

$$= \sin (A + B) \sin (A - B) = (\sin 90°) \sin (A - B) = \sin (A - B).$$

18. $\dfrac{\cos A}{a} = \dfrac{\cos B}{b} \Rightarrow \dfrac{(b^2 + c^2 - a^2)}{2abc} = \dfrac{a^2 + c^2 - b^2}{2abc}$

$$\Rightarrow b^2 + c^2 - a^2 = a^2 + c^2 - b^2 \Rightarrow 2a^2 = 2b^2$$

$$\Rightarrow a^2 = b^2 \Rightarrow a = b.$$

19. $\sin^2 A + \sin^2 B = \sin^2 C \Rightarrow \dfrac{a^2}{k^2} + \dfrac{b^2}{k^2} = \dfrac{c^2}{k^2}$ $\left[\because \dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C} = k \text{ (say)}\right]$

$$\Rightarrow a^2 + b^2 = c^2 \Rightarrow \triangle ABC \text{ is right-angled.}$$

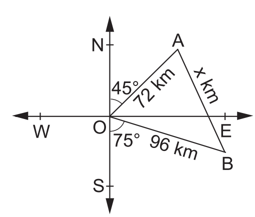
22. Let $\angle A = x°$, $\angle B = (2x)°$ and $\angle C = (3x)°$. Then,
$A + B + C = 180° \Rightarrow x + 2x + 3x = 180 \Rightarrow 6x = 180 \Rightarrow x = 30.$
$\therefore \angle A = 30°, \angle B = 60°$ and $\angle C = 90°$.
Ratio of sides $= \sin A : \sin B : \sin C = \sin 30° : \sin 60° : \sin 90°$

$$= \frac{1}{2} : \frac{\sqrt{3}}{2} : 1 = 1 : \sqrt{3} : 2.$$

PROBLEMS BASED ON SINE AND COSINE FORMULAE

EXAMPLE 1 *Two ships leave a port at the same time. One goes* 24 km *per hour in the direction* N 45° E *and the other travels* 32 km *per hour in the direction* S 75° E. *Find the distance between the ships at the end of* 3 *hours.*

SOLUTION Let A and B be the positions of the two ships at the end of 3 hours.
Then, $OA = (24 \times 3)$ km $= 72$ km.
And, $OB = (32 \times 3)$ km $= 96$ km.
Let, $AB = x$ km.
We have: $\angle NOA = 45°$ and $\angle SOB = 75°$
$\therefore \angle AOB = 180° - (45° + 75°) = 60°.$
Applying cosine formula on $\triangle AOB$, we get



$$\cos 60° = \frac{(72)^2 + (96)^2 - x^2}{2 \times 72 \times 96}$$

$$\Rightarrow (72)^2 + (96)^2 - x^2 = \frac{1}{2} \times 2 \times 72 \times 96$$

$$\Rightarrow 5184 + 9216 - x^2 = 6912$$

$$\Rightarrow x^2 = 14400 - 6912 = 7488$$

$$\Rightarrow x = \sqrt{7488} = 86.53 \text{ km.}$$

Hence, the distance between the ships at the end of 3 hours is 86.53 km.

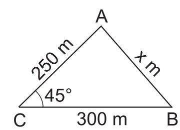
EXAMPLE 2 *Two trees A and B are on the same side of a river. From a point C in the river, the distances of trees A and B are* 250 m *and* 300 m *respectively. If* $\angle C = 45°$, *find the distance between the trees.* (*Use* $\sqrt{2} = 1.44$.)

SOLUTION Let A and B be the trees and C be a point in the river.

Then, $CA = 250$ m, $CB = 300$ m and $\angle ACB = 45°$.

Let $AB = x$ metres.

Applying cosine formula on $\triangle ACB$, we get



$$\cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

$$\Rightarrow \quad \cos 45° = \frac{(300)^2 + (250)^2 - x^2}{2 \times 300 \times 250}$$

$$\Rightarrow \quad \frac{1}{\sqrt{2}} = \frac{90000 + 62500 - x^2}{150000}$$

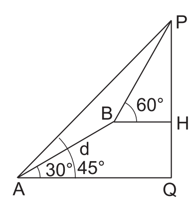
$$\Rightarrow \quad \frac{152500 - x^2}{150000} = \frac{1}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$

$$\Rightarrow \quad 152500 - x^2 = 75000 \times \sqrt{2} = 75000 \times 1.44 = 108000$$

$$\Rightarrow \quad x^2 = 152500 - 108000 = 44500$$

$$\Rightarrow \quad x = \sqrt{44500} = 210.95 \text{ m.}$$

EXAMPLE 3 *The angle of elevation of the top point P of the vertical tower PQ of height h from a point A on the ground is* 45° *and from a point B, the angle of elevation is* 60°, *where B is a point at a distance d from the point A, measured along the line AB, which makes an angle* 30° *with AQ. Prove that* $d = h(\sqrt{3} - 1)$.



SOLUTION In $\triangle PAQ$, we have

$$\frac{AQ}{PQ} = \cot 45° = 1 \quad \Rightarrow \quad \frac{AQ}{h} = 1 \quad \Rightarrow \quad AQ = h.$$

From right $\triangle AQP$, we get

$$AP^2 = AQ^2 + PQ^2 = (h^2 + h^2) = 2h^2 \quad \Rightarrow \quad AP = \sqrt{2}h.$$

In $\triangle PBH$, $\angle BPH = 180° - (60° + 90°) = 30°$.

In $\triangle PAQ$, $\angle APQ = 180° - (45° + 90°) = 45°$.

$\therefore \quad \angle APB = (\angle APQ - \angle BPH) = (45° - 30°) = 15°$.

In $\triangle APB$, we have

$$\angle PAB = 15°, \ \angle APB = 15° \text{ and } \angle ABP = 180° - (15° + 15°) = 150°.$$

Using sine formula on $\triangle ABP$, we have

$$\frac{AB}{\sin 15°} = \frac{AP}{\sin 15°}$$

$$\Rightarrow \quad \frac{d}{\sin 15°} = \frac{\sqrt{2}h}{\sin 30°}$$

$\Rightarrow \quad d = 2\sqrt{2}h \cdot \sin 15°$

$\Rightarrow \quad d = 2\sqrt{2}h \cdot \frac{(\sqrt{3}-1)}{2\sqrt{2}}$

$= (\sqrt{3}-1)h.$

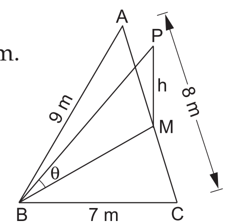
Hence, $d = (\sqrt{3}-1)h.$

$\left[\because \sin 15° = (\sin 45° - 30°) = \sin 45° \cos 30° - \cos 45° \sin 30° = \frac{1}{\sqrt{2}} \times \frac{\sqrt{3}}{2} - \frac{1}{\sqrt{2}} \times \frac{1}{2} = \frac{(\sqrt{3}-1)}{2\sqrt{2}}.\right]$

EXAMPLE 4 *A lamp post is situated at the middle point M of the side AC of a triangular plot ABC with* $BC = 7$ m, $CA = 8$ m *and* $AB = 9$ m. *The lamp post subtends an angle* $\tan^{-1} 3$ *at the point B. Determine the height of the lamp post.*

SOLUTION We have:

$a = BC = 7$ m, $b = AC = 8$ m and $C = AB = 9$ m.



Let M be the midpoint of AC and let MP be the lamp post of height h metres.

Then, $\angle MBP = \theta = \tan^{-1} \frac{1}{3}.$

Applying cosine formula on $\triangle ABC$, we get

$$\cos C = \frac{(a^2 + b^2 - c^2)}{2ab} = \frac{(49 + 64 - 81)}{2 \times 7 \times 8} = \frac{32}{112} = \frac{2}{7} \qquad \ldots \text{(i)}$$

In $\triangle BMC$, we have

$$BC = 7 \text{ m}, CM = \frac{1}{2} CA = 4 \text{ m}.$$

$$\therefore \quad BM^2 = BC^2 + CM^2 - 2BC \times CM \times \cos C$$

$$= \left(49 + 16 - 2 \times 7 \times 4 \times \frac{2}{7}\right) = 49 \qquad \text{[Using (i)]}$$

$$\Rightarrow \quad BM = 7 \text{ m}.$$

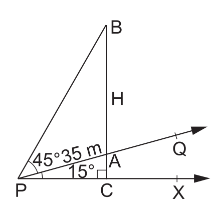
From right $\triangle BMP$, we have

$$\tan\theta = \frac{PM}{BM} = \frac{h}{7} \Rightarrow \frac{h}{7} = \tan(\tan^{-1} 3) = 3 \Rightarrow h = 21 \text{ m}.$$

Hence, the height of the lamp post is 21 m.

EXAMPLE 5 *A tree stands vertically on a hill side which makes an angle of* 15° *with the horizontal. From a point on the ground* 35 m *down the hill from the base of the tree, the angle of elevation of the top of tree is* 60°. *Find the height of the tree.*

SOLUTION Let PAQ be the hill, AB be the tree and PCX be the horizontal. Let P be the point of observation. Produce BA to meet PX at C. Let $AB = H$ metres.



Then, $\angle XPA = 15°$, $PA = 35$ m,

$\angle CPB = 60°$ and $\angle PCA = 90°$.

$\therefore \quad \angle APB = (60° - 15°) = 45°.$

In $\triangle PAC$, $\angle PAC = 180° - (15° + 90°) = 75°$

$\therefore \quad \angle PAB = (180° - 75) = 105°.$

And, $\angle PBA = 180° - (45° + 105°) = 30°.$

Applying sine rule on $\triangle\, PAB$, we get

$$\frac{PA}{\sin \angle PBA} = \frac{AB}{\sin \angle APB} \Rightarrow \frac{35}{\sin 30°} = \frac{H}{\sin 45°}$$

$\therefore \quad 35 \times 2 = H \times \sqrt{2} \Rightarrow H = 35\sqrt{2}$ m.

Hence, the height of the tree is $35\sqrt{2}$ m.

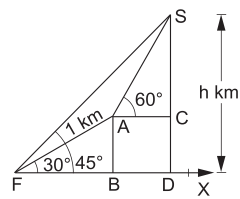
EXERCISE 18B

1. Two boats leave a port at the same time. One travels 60 km in the direction N 50° E while the other travels 50 km in the direction S 70° E. What is the distance between the boats?

2. A town B is 12 km south and 18 km west of a town A. Show that the bearing of B from A is S $56°20'$ W. Also, find the distance of B from A.

[Given: $\tan 56°20' = 1.5$.]

3. At the foot of a mountain, the angle of elevation of its summit is 45°. After ascending 1 km towards the mountain up an incline of 30°, the elevation changes to 60° (as shown in the given figure). Find the height of the mountain. [Given: $\sqrt{3} = 1.73$.]



ANSWERS (EXERCISE 18B)

1. 55.68 km **2.** 21.63 km **3.** 1.365 km

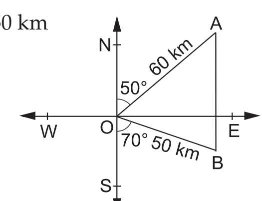
HINTS TO SOME SELECTED QUESTIONS

1. Let the original position of each boat be at O. Let A and B be their respective final positions, as shown in the figure.

Then, $\angle NOA = 50°$, $\angle SOB = 70°$, $OA = 60$ km and $OB = 50$ km

$\therefore \quad \angle AOB = 180° - (50° + 70°) = 60°$. Let $AB = x$ km.

By cosine formula on $\triangle AOB$, we have



$$\cos \angle AOB = \frac{OA^2 + OB^2 - AB^2}{2 \times OA \times OB}$$

$$\Rightarrow \quad \cos 60° = \frac{(60)^2 + (50)^2 - x^2}{2 \times 60 \times 50}$$

$$\Rightarrow \quad 3600 + 2500 - x^2 = \frac{1}{2} \times 2 \times 60 \times 50 \Rightarrow x^2 = 3100 \Rightarrow x = \sqrt{3100} = 55.68 \text{ km.}$$

Hence, the required distance between the boats is 55.68 km.

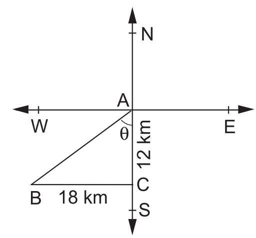
2. Let B be the position of the town relative to A, as shown in the figure.

From B, draw $BC \perp AS$.

Then, $AC = 12$ km and $BC = 18$ km.

Let $\angle BAC = \theta$. Then,

$$\tan\theta = \frac{BC}{AC} = \frac{18}{12} = \frac{3}{2} \Rightarrow \theta = 56°20'.$$



$\therefore$ bearing of B from A is S 56°20′ W.

Also, $AB = \sqrt{BC^2 + AC^2} = \sqrt{(18)^2 + (12)^2} = \sqrt{324 + 144}$

$= \sqrt{468} = 21.63$ km.

Hence, the distance of B from A is 21.63 km.

3. $\frac{SD}{FD} = \tan 45° = 1 \Rightarrow SD = FD = h$ km

$\Rightarrow \angle FSD = \angle SFD = 45°.$

$\angle ASC = 180° - (60° + 90°) = 30°$

$\Rightarrow$ $\angle FSA = (45° - 30°) = 15°$ and $\angle SFA = 15°.$

$\therefore$ $\angle SAF = 180° - (15° + 15°) = 150°.$

By the sine formula on $\triangle SAF$, we have

$$\frac{AF}{\sin\angle FSA} = \frac{FS}{\sin\angle SAF} \Rightarrow \frac{1}{\sin 15°} = \frac{\sqrt{2}\,h}{\sin 150°} = \frac{\sqrt{2}\,h}{\sin 30°} \Rightarrow \frac{2\sqrt{2}}{(\sqrt{3}-1)} = 2\sqrt{2}h$$

$$\left[\because FS = \sqrt{h^2 + h^2} = \sqrt{2}h \text{ and } \sin 15° = \frac{(\sqrt{3}-1)}{2\sqrt{2}}.\right]$$

$$\Rightarrow h = \frac{1}{(\sqrt{3}-1)} \times \frac{(\sqrt{3}+1)}{(\sqrt{3}+1)} = \frac{(\sqrt{3}+1)}{2} = \frac{(1.73+1)}{2} = \frac{2.73}{2} = 1.365 \text{ km}.$$

❑

19

Graphs of Trigonometric Functions

GRAPH OF $y = \sin x$

We know that $\sin x$ is periodic with period 2π.

So, we draw the graph of $\sin x$ in the interval $[0, 2\pi]$.

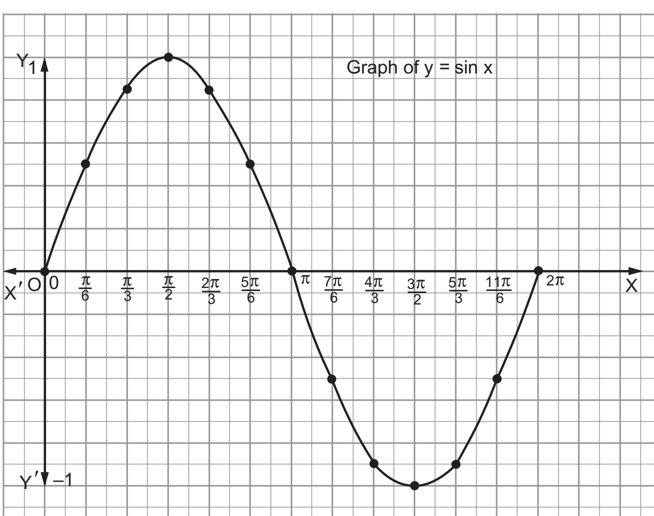
We can then extend it easily by repeating it over intervals of length 2π.

For this, we first construct the table given below.

x	0	$\pi/6$	$\pi/3$	$\pi/2$	$2\pi/3$	$5\pi/6$	π	$7\pi/6$	$4\pi/3$	$3\pi/2$	$5\pi/3$	$11\pi/6$	2π
$\sin x$	0	$1/2$ $= 0.5$	$\sqrt{3}/2$ $= 0.87$	1	$\sqrt{3}/2$ $= 0.87$	$1/2$ $= 0.5$	0	$-1/2$ $= -0.5$	$-\sqrt{3}/2$ $= -0.87$	-1	$-\sqrt{3}/2$ $= -0.87$	$-1/2$ $= -0.5$	0

On a graph paper we take 1 cm as $\pi/6$ units on the x-axis, and 5 cm as 1 unit on the y-axis. Taking the values of x along the x-axis and the values of $\sin x$ along the y-axis, we plot the points $(0, 0)$, $(\pi/6, 0.5)$, $(\pi/3, 0.87)$, $(\pi/2, 1)$, $(2\pi/3, 0.87)$, $(5\pi/6, 0.5)$, $(\pi, 0)$, $(7\pi/6, -0.5)$, $(4\pi/3, -0.87)$, $(3\pi/2, -1)$, $(5\pi/3, -0.87)$, $(11\pi/6, -0.5)$ and $(2\pi, 0)$.

We join these points by a freehand curve and obtain the graph of $y = \sin x$, as shown below.



GRAPH OF y = cos x

We know that cos x is periodic with period 2π.

So, we draw the graph of cos x in the interval $[0, 2\pi]$.

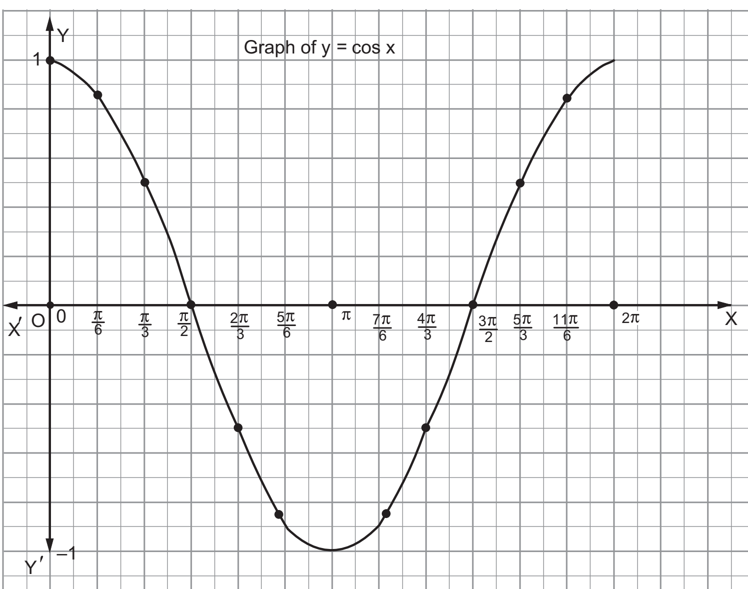
We can then extend it easily by repeating it over intervals of length 2π.

For this, we first construct the table given below.

x	0	$\pi/6$	$\pi/3$	$\pi/2$	$2\pi/3$	$5\pi/6$	π	$7\pi/6$	$4\pi/3$	$3\pi/2$	$5\pi/3$	$11\pi/6$	2π
cos x	1	$\sqrt{3}/2$ = 0.87	$1/2$ = 0.5	0	$-1/2$ = −0.5	$-\sqrt{3}/2$ = −0.87	−1	$-\sqrt{3}/2$ = −0.87	$-1/2$ = −0.5	0	$1/2$ = 0.5	$\sqrt{3}/2$ = 0.87	1

On a graph paper, we take 1 cm as $\pi/6$ units on the x-axis, and 5 cm as 1 unit on the y-axis.

Taking the values of x along the x-axis and the values of cos x along the y-axis, we plot the points $(0, 1)$, $(\pi/6, 0.87)$, $(\pi/3, 0.5)$, $(\pi/2, 0)$, $(2\pi/3, -0.5)$, $(5\pi/6, -0.87)$, $(\pi, -1)$, $(7\pi/6, -0.87)$, $(4\pi/3, -0.5)$, $(3\pi/2, 0)$, $(5\pi/3, 0.5)$, $(11\pi/6, 0.87)$ and $(2\pi, 1)$ on the graph paper and, joining these points successively by a freehand curve, we obtain the graph of cos x, as shown below.



GRAPH OF y = tan x

We know that $y = \tan x$ is defined for all real values of x, except when x is an odd multiple of $\pi/2$.

Also, tan x is periodic with period π.

So, we draw the graph of $y = \tan x$ in $[0, \pi]$ and then extend it over other intervals.

As x increases from 0 to $\pi/2$, $\tan x$ increases from 0 to ∞.

As x crosses the value of $\pi/2$, $\tan x$ becomes negative and is arbitrarily large in magnitude.

As x approaches $\pi/2$ from π, $\tan x$ comes to 0.

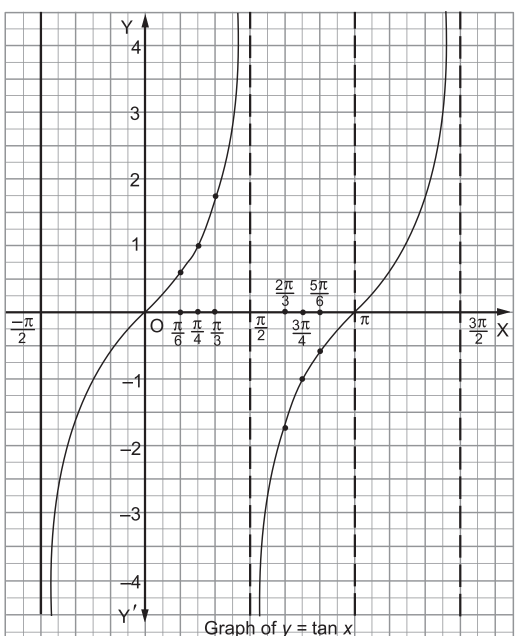
Now, we can construct the table, given below.

x	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π
$\tan x$	0	$\frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3} = \frac{1.73}{3} = 0.58$	1	$\sqrt{3} = 1.73$	∞	$-\sqrt{3} = -1.73$	-1	$-\frac{1}{\sqrt{3}} = \frac{-\sqrt{3}}{3} = \frac{-1.73}{3} = -0.58$	0

On a graph paper, we take 1 cm as $\pi/6$ units on the x-axis, and 2 cm as 1 unit on the y-axis.

Taking the values of x on the x-axis and the values of $\tan x$ along the y-axis, we plot the points $(0, 0)$, $(\pi/6, 0.58)$, $(\pi/4, 1)$, $(\pi/3, 1.73)$, $(\pi/2, \infty)$, $(2\pi/3, -1.73)$, $(3\pi/4, -1)$, $(5\pi/6, -0.58)$ and $(\pi, 0)$.

We join these points by a freehand curve and obtain the graph of $\tan x$ as shown below.



Graph of $y = \tan x$

GRAPH OF $y = a \sin bx$

We know that $-1 \le \sin bx \le 1$.

$\therefore \quad -a \le a \sin bx \le a.$

Also, $\sin x$ is periodic with period 2π.

$\therefore$ $\sin bx$ is periodic with period $\dfrac{2\pi}{b}$.

Hence, $a \sin bx$ is periodic with period $\dfrac{2\pi}{b}$.

EXAMPLE *Draw the graph of* $y = 3 \sin 2x$.

SOLUTION $-1 \le \sin 2x \le 1 \;\Rightarrow\; -3 \le 3 \sin 2x \le 3$.

Also, $3 \sin 2x$ is periodic with period $\dfrac{2\pi}{2}$, i.e., π.

So, we draw the graph of $y = 3 \sin 2x$ in the interval $[0, \pi]$.

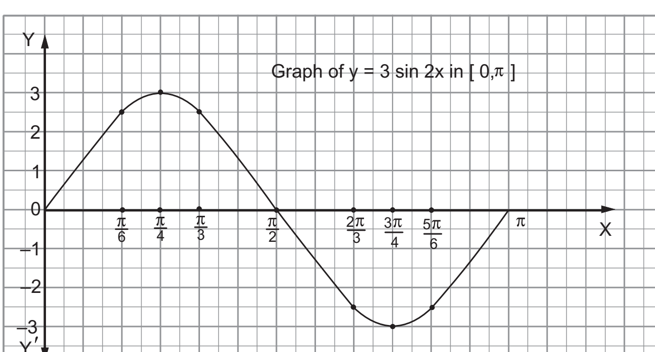
We thus prepare the table given below.

x	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π
$2x$	0	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	π	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	2π
$\sin 2x$	0	$\frac{\sqrt{3}}{2} = 0.87$	1	$\frac{\sqrt{3}}{2} = 0.87$	0	$\frac{-\sqrt{3}}{2} = -0.87$	–1	–0.87	0
$3 \sin 2x$	0	2.61	3	2.61	0	–2.61	–3	–2.61	0

On a graph paper, we take 4 cm = $\pi/3$ units on the x-axis and 1 cm = 1 unit on the y-axis.

Now, we plot the points $(0, 0)$, $(\pi/6, 2.61)$, $(\pi/4, 3)$, $(\pi/3, 2.61)$, $(\pi/2, 0)$, $(2\pi/3, -2.61)$, $(3\pi/4, -3)$, $(5\pi/6, -2.61)$, and $(\pi, 0)$.

Join these points with a freehand curve to obtain the graph of $y = 3 \sin 2x$, given below.



GRAPH OF $y = a \cos bx$

We know that $-1 \le \cos bx \le 1$.

$\therefore \quad -a \le a \cos bx \le a$.

Also, $\cos x$ is periodic with period 2π.

$\therefore$ $\cos bx$ is periodic with period $\frac{2\pi}{b}$.

Hence, $a \cos bx$ is periodic with period $\frac{2\pi}{b}$.

EXAMPLE 1 *Draw the graph of $y = 3\cos 2x$.*

SOLUTION $-1 \le \cos 2x \le 1 \Rightarrow -3 \le 3\cos 2x \le 3$.

Also, $3\cos 2x$ is periodic with period $\frac{2\pi}{2}$, i.e., π.

So, we draw the graph of $y = 3\cos 2x$ in the interval $[0, \pi]$.

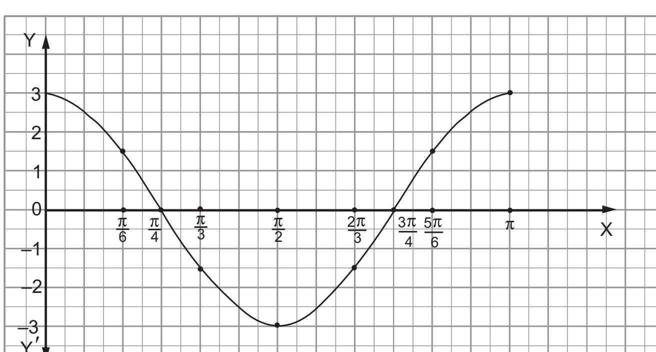
We thus prepare the table given below.

x	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π
$2x$	0	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	π	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	2π
$\cos 2x$	1	$\frac{1}{2} = 0.5$	0	$-\frac{1}{2} = -0.5$	-1	$-\frac{1}{2} = -0.5$	0	$\frac{1}{2} = 0.5$	1
$3\cos 2x$	3	1.5	0	-1.5	-3	-1.5	0	1.5	3

On a graph paper, we take 4 cm as $\pi/3$ units on the x-axis, and 1 cm = 1 unit on the y-axis.

Now, plot the points $(0, 3)$, $(\pi/6, 1.5)$, $(\pi/4, 0)$, $(\pi/3, -1.5)$, $(\pi/2, -3)$, $(2\pi/3, -1.5)$, $(3\pi/4, 0)$, $(5\pi/6, 1.5)$ and $(\pi, 3)$.

Join them with a freehand curve to obtain the required graph, given below.



<u>EXAMPLE 2</u> *Draw the graphs of* $y = \cos x$ *and* $y = \cos \frac{1}{2} x$ *in* $[0, 2\pi]$ *on the same scale.*

SOLUTION We know that

$y = \cos x$ is a periodic function with period 2π and amplitude 1

$y = \cos \frac{1}{2} x$ is a periodic function with period 4π and amplitude 1.

We may prepare the table as under:

x	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{11\pi}{6}$	2π
$\cos x$	1	0.87	0.5	0	–0.5	–0.87	–1	–0.87	–0.5	0	0.5	0.87	1
$\frac{1}{2} x$	0	$\frac{\pi}{12}$	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{5\pi}{12}$	$\frac{\pi}{2}$	$\frac{7\pi}{12}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	$\frac{11\pi}{12}$	π
$\cos \frac{1}{2} x$	1	0.97	0.87	0.7	0.5	0.26	0	–0.26	–0.5	–0.7	–0.87	–0.97	–1

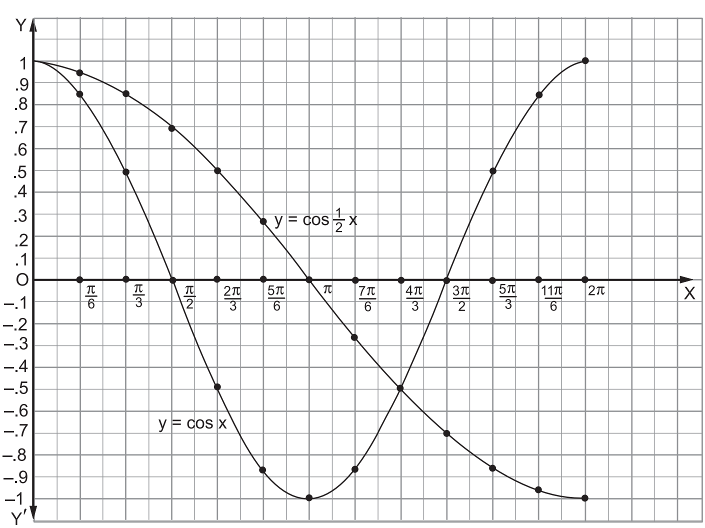
On a graph paper, we take 1 cm = $\pi/6$ units on the x-axis, and 0.5 cm = 0.1 unit on the y-axis.

For $y = \cos x$, we plot the points $(0, 1)$, $(\pi/6, 0.87)$, $(\pi/3, 0.5)$, $(\pi/2, 0)$, $(2\pi/3, -0.5)$, $(5\pi/6, -0.87)$, $(\pi, -1)$, $(7\pi/6, -0.87)$, $(4\pi/3, -0.5)$, $(3\pi/2, 0)$, $(5\pi/3, 0.5)$, $(11\pi/6, 0.87)$ and $(2\pi, 1)$.

We join these points by a freehand curve to obtain the curve $y = \cos x$.

For $y = \cos \frac{1}{2} x$, we plot the points $(0, 1)$, $(\pi/6, 0.97)$, $(\pi/3, 0.87)$, $(\pi/2, 0.7)$, $(2\pi/3, 0.5)$, $(5\pi/6, 0.26)$, $(\pi, 0)$, $(7\pi/6, -0.26)$, $(4\pi/3, -0.5)$, $(3\pi/2, -0.7)$, $(5\pi/3, -0.87)$, $(11\pi/6, -0.97)$ and $(2\pi, -1)$.

We join these points by a freehand curve to obtain the curve $y = \cos \frac{1}{2} x$.



EXERCISE 19

Draw the graph of each of the following functions:

1. $\sin 3x$ **2.** $3\sin x$ **3.** $2\sin 3x$

4. $2\cos 3x$ **5.** $\sin \frac{x}{2}$

6. Draw the graphs of $y = \sin x$ and $y = \cos x$ in $[0, 2\pi]$ on the same axes.

7. Draw the graphs of $y = \cos x$ and $y = \cos 2x$ in $[0, 2\pi]$ on the same axes.

❑

20
Straight Lines

INTRODUCTION

A systematic study of geometry by using algebra was first carried out by French mathematician *R Descartes* in 1637. This subject is now known as *Coordinate Geometry*.

In earlier classes, we have studied about coordinate axis, coordinate plane, points in a plane, distance between two points and section formulae.

Here, we give, in brief, the results derived so far.

1. Distance between the points $A(x_1, y_1)$ and $B(x_2, y_2)$ is given by
$$AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$
2. Area of $\triangle ABC$ whose vertices are $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ is given by
$$\triangle = \frac{1}{2} \cdot |x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)| \text{ sq units.}$$
REMARK The points A, B, C are collinear $\Leftrightarrow$ area of $\triangle ABC = 0$.
3. (i) If the point $P(x, y)$ divides the join of $A(x_1, y_1)$ and $B(x_2, y_2)$ in the ratio $m : n$ then
$$x = \frac{mx_2 + nx_1}{m + n} \quad \text{and} \quad y = \frac{my_2 + ny_1}{m + n}.$$
(ii) If $P(x, y)$ is the midpoint of the line segment joining $A(x_1, y_1)$ and $B(x_2, y_2)$ then
$$x = \frac{1}{2}(x_1 + x_2) \quad \text{and} \quad y = \frac{1}{2}(y_1 + y_2).$$

SOLVED EXAMPLES

EXAMPLE 1 *Find the distance between the points* $(2, -3)$ *and* $(-6, 3)$.

SOLUTION Let $A(2, -3)$ and $B(-6, 3)$ be the given points. Then,
$$AB = \sqrt{(-6-2)^2 + \{3-(-3)\}^2} = \sqrt{(-8)^2 + (3+3)^2} = \sqrt{(-8)^2 + 6^2}$$
$$= \sqrt{64 + 36} = \sqrt{100} = 10 \text{ units.}$$

EXAMPLE 2 *Using the distance formula, prove that the points $A(-2, 3)$, $B(1, 2)$ and $C(7, 0)$ are collinear.*

SOLUTION We have

$$AB = \sqrt{(1+2)^2 + (2-3)^2} = \sqrt{3^2 + (-1)^2} = \sqrt{10} \text{ units;}$$

$$BC = \sqrt{(7-1)^2 + (0-2)^2} = \sqrt{6^2 + (-2)^2} = \sqrt{40} = 2\sqrt{10} \text{ units;}$$

$$AC = \sqrt{(7+2)^2 + (0-3)^2} = \sqrt{9^2 + (-3)^2} = \sqrt{90} = 3\sqrt{10} \text{ units.}$$

$$\therefore \quad AB + BC = (\sqrt{10} + 2\sqrt{10}) \text{ units} = 3\sqrt{10} \text{ units} = AC.$$

Thus, $AB + BC = AC$, showing that the points A, B, C are collinear.

SOME IMPORTANT RESULTS

A quadrilateral is a

(i) rectangle, if its opposite sides are equal and the diagonals are equal;
(ii) square, if its all sides are equal and the diagonals are equal;
(iii) a parallelogram but not a rectangle, if its opposite sides are equal but the diagonals are not equal;
(iv) a rhombus but not a square, if its all sides are equal but the diagonals are not equal.

EXAMPLE 3 *Prove that the points $(0, 5)$, $(-2, -2)$, $(5, 0)$ and $(7, 7)$ are the vertices of a rhombus.*

SOLUTION Let the given points be $A(0, 5)$, $B(-2, -2)$, $C(5, 0)$ and $D(7, 7)$. Then, $ABCD$ is a quadrilateral in which:

$$AB = \sqrt{(-2-0)^2 + (-2-5)^2} = \sqrt{(-2)^2 + (-7)^2} = \sqrt{4+49} = \sqrt{53} \text{ units;}$$

$$BC = \sqrt{(5+2)^2 + (0+2)^2} = \sqrt{7^2 + 2^2} = \sqrt{49+4} = \sqrt{53} \text{ units;}$$

$$CD = \sqrt{(7-5)^2 + (7-0)^2} = \sqrt{2^2 + 7^2} = \sqrt{4+49} = \sqrt{53} \text{ units;}$$

$$DA = \sqrt{(7-0)^2 + (7-5)^2} = \sqrt{7^2 + 2^2} = \sqrt{49+4} = \sqrt{53} \text{ units.}$$

Thus, $AB = BC = CD = DA = \sqrt{53}$ units.

Also, $AC = \sqrt{(5-0)^2 + (0-5)^2} = \sqrt{5^2 + (-5)^2} = \sqrt{50} = 5\sqrt{2}$ units

And, $BD = \sqrt{(7+2)^2 + (7+2)^2} = \sqrt{9^2 + 9^2} = \sqrt{81+81}$

$= \sqrt{162} = 9\sqrt{2}$ units.

$\therefore \quad AC \neq BD.$

Thus, $ABCD$ is a quadrilateral all of whose sides are equal but its diagonals are not equal.

Hence, $ABCD$ is a rhombus.

EXAMPLE 4 *Find the area of the triangle whose vertices are $A(4, 4)$, $B(3, -16)$ and $C(3, -2)$.*

SOLUTION Let $(x_1 = 4, y_1 = 4)$; $(x_2 = 3, y_2 = -16)$ and $(x_3 = 3, y_3 = -2)$.

Then, area of $\triangle ABC$ is given by

$$\Delta = \frac{1}{2} \cdot |x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)|$$
$$= \frac{1}{2} \cdot |4(-16+2) + 3(-2-4) + 3(4+16)|$$
$$= \frac{1}{2} \cdot |-56 - 18 + 60| = \frac{1}{2} \cdot |-14| = \left(\frac{1}{2} \times 14\right) = 7 \text{ sq units.}$$

Hence, the area of the given triangle is 7 sq units.

EXAMPLE 5 *Find the coordinates of the point which divides the line segment joining the points $A(5, -2)$ and $B(9, 6)$ in the ratio $3:1$.*

SOLUTION Here $(x_1 = 5, y_1 = -2)$ and $(x_2 = 9, y_2 = 6)$. Also, $m = 3$ and $n = 1$.

Let the required point be $P(x, y)$. Then,

$$x = \frac{(mx_2 + nx_1)}{(m+n)} = \frac{(3 \times 9) + (1 \times 5)}{(3+1)} = \frac{32}{4} = 8,$$
$$y = \frac{(my_2 + ny_1)}{(m+n)} = \frac{(3 \times 6) + \{1 \times (-2)\}}{(3+1)} = \frac{(18-2)}{4} = \frac{16}{4} = 4.$$

Hence, the required point is (8, 4).

EXAMPLE 6 *Find the coordinates of the midpoint of the line segment joining the points $A(-2, -5)$ and $B(3, -1)$.*

SOLUTION Let the required point be $P(x, y)$. Then,

$$x = \frac{(-2+3)}{2} = \frac{1}{2}, \; y = \frac{-5+(-1)}{2} = -3.$$

Hence, the required point is $\left(\frac{1}{2}, -3\right)$.

EXAMPLE 7 *In what ratio is the line joining $A(-1, 1)$ and $B(5, 7)$ divided by the line $x + y = 4$?*

SOLUTION Let the given line divide AB in the ratio $m:1$.

Then, the point of division is $C\left(\frac{5m-1}{m+1}, \frac{7m+1}{m+1}\right)$.

This point C must lie on the line $x + y = 4$.

$$\therefore \quad \frac{5m-1}{m+1} + \frac{7m+1}{m+1} = 4 \Leftrightarrow (5m-1) + (7m+1) = 4(m+1)$$
$$\Leftrightarrow 8m = 4 \Leftrightarrow m = \frac{1}{2}.$$

So, the required ratio is $\frac{1}{2}:1$, i.e., $1:2$.

EXERCISE 20A

1. Find the distance between the points:

(i) $A(2, -3)$ and $B(-6, 3)$ (ii) $C(-1, -1)$ and $D(8, 11)$

(iii) $P(-8, -3)$ and $Q(-2, -5)$ (iv) $R(a+b, a-b)$ and $S(a-b, a+b)$

2. Find the distance of the point $P(6, -6)$ from the origin.

3. If a point $P(x, y)$ is equidistant from the points $A(6, -1)$ and $B(2, 3)$, find the relation between x and y.

4. Find a point on the x-axis which is equidistant from the points $A(7, 6)$ and $B(-3, 4)$.

5. Find the distance between the points $A(x_1, y_1)$ and $B(x_2, y_2)$, when (i) AB is parallel to the x-axis (ii) AB is parallel to the y-axis.

6. A is a point on the x-axis with abscissa -8 and B is a point on the y-axis with ordinate 15. Find the distance AB.

7. Find a point on the y-axis which is equidistant from $A(-4, 3)$ and $B(5, 2)$.

8. Using the distance formula, show that the points $A(3, -2)$, $B(5, 2)$ and $C(8, 8)$ are collinear.

9. Show that the points $A(7, 10)$, $B(-2, 5)$ and $C(3, -4)$ are the vertices of an isosceles right-angled triangle.

10. Show that the points $A(1, 1)$, $B(-1, -1)$ and $C(-\sqrt{3}, \sqrt{3})$ are the vertices of an equilateral triangle each of whose sides is $2\sqrt{2}$ units.

11. Show that the points $A(2, -2)$, $B(8, 4)$, $C(5, 7)$ and $D(-1, 1)$ are the angular points of a rectangle.

12. Show that $A(3, 2)$, $B(0, 5)$, $C(-3, 2)$ and $D(0, -1)$ are the vertices of a square.

13. Show that $A(1, -2)$, $B(3, 6)$, $C(5, 10)$ and $D(3, 2)$ are the vertices of a parallelogram.

14. Show that the points $A(2, -1)$, $B(3, 4)$, $C(-2, 3)$ and $D(-3, -2)$ are the vertices of a rhombus.

15. If the points $A(-2, -1)$, $B(1, 0)$, $C(x, 3)$ and $D(1, y)$ are the vertices of a parallelogram, find the values of x and y.

16. Find the area of $\triangle ABC$ whose vertices are $A(-3, -5)$, $B(5, 2)$ and $C(-9, -3)$.

17. Show that the points $A(-5, 1)$, $B(5, 5)$ and $C(10, 7)$ are collinear.

18. Find the value of k for which the points $A(-2, 3)$, $B(1, 2)$ and $C(k, 0)$ are collinear.

19. Find the area of the quadrilateral whose vertices are $A(-4, 5)$, $B(0, 7)$, $C(5, -5)$ and $D(-4, -2)$.

20. Find the area of $\triangle ABC$, the midpoints of whose sides AB, BC and CA are $D(3, -1)$, $E(5, 3)$ and $F(1, -3)$ respectively.

21. Find the coordinates of the point which divides the join of $A(-5, 11)$ and $B(4, -7)$ in the ratio $2 : 7$.

22. Find the ratio in which the x-axis cuts the join of the points $A(4, 5)$ and $B(-10, -2)$. Also, find the point of intersection.

23. In what ratio is the line segment joining the points $A(-4, 2)$ and $B(8, 3)$ divided by the y-axis? Also, find the point of intersection.

ANSWERS (EXERCISE 20A)

1. (i) 10 units (ii) 15 units (iii) $2\sqrt{10}$ units (iv) $(2\sqrt{2}b)$ units **2.** $6\sqrt{2}$ units

3. $x - y = 3$ **4.** $P(3, 0)$ **5.** (i) $|x_2 - x_1|$ (ii) $|y_2 - y_1|$

6. 17 units **7.** $P(0, -2)$ **15.** $x = 4$, $y = 2$ **16.** 29 sq units

18. $k = 7$ **19.** 60.5 sq units **20.** 8 sq units **21.** $P(-3, 7)$

22. $(5 : 2)$, $P(-6, 0)$ **23.** $(1 : 2)$, $P\left(0, \frac{7}{3}\right)$

HINTS TO SOME SELECTED QUESTIONS

4. Let the required point be $P(x, 0)$.
Then, $AP = BP \Leftrightarrow AP^2 = BP^2$.

5. (i) When AB is parallel to the x-axis, then A and B have the same ordinate.
Then, $A(x_1, y)$ and $B(x_2, y)$ are the given points. So, $AB = |x_2 - x_1|$.

(ii) When AB is parallel to the y-axis, then A and B have the same abscissa.
Then, $A(x, y_1)$ and $B(x, y_2)$ are the given points. So, $AB = |y_2 - y_1|$.

6. Given points are $A(-8, 0)$ and $B(0, 15)$.

7. Let the required point be $P(0, y)$.

8. Show that $AB + BC = AC$.

11. Show that $AB = DC$, $BC = AD$ and diag. $AC =$ diag. BD.

12. Show that $AB = BC = CD = DA$ and diag. $AC =$ diag. BD.

13. Show that $AB = CD$ and $BC = AD$.

15. Midpoint of $AC =$ Midpoint of BD.

17. Show that area of $\triangle ABC$ is zero.

19. Area of quad. $ABCD =$ area $(\triangle ABC) +$ area $(\triangle ACD)$.

20. Let $\triangle ABC$ be given and let D, E, F be the midpoints of AB, BC and CA respectively. Then, ar $(\triangle ABC) = 4 \times$ ar $(\triangle DEF)$.

22. Let the required ratio be $k : 1$ and let $P(x, 0)$ be the point of division. Then, $\frac{k \times (-2) + 1 \times 5}{k + 1} = 0 \Leftrightarrow k = \frac{5}{2}$.

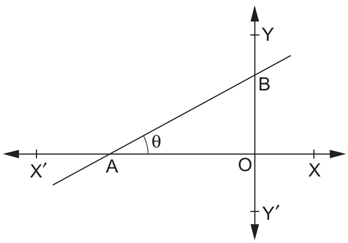
So, the required ratio $= 5 : 2$.

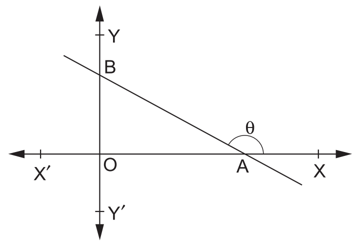
23. Let the ratio be $k : 1$ and let $P(0, y)$ be the point of division.

SLOPE OF A LINE

INCLINATION OF A LINE

The angle of inclination or simply the inclination of a line is the angle θ *which the part of the line above the x-axis makes with the positive direction of the x-axis, measured in anticlockwise direction.*





Clearly, $0° \le \theta < 180°$.

REMARKS (i) The inclination of a line parallel to the x-axis or the x-axis itself is 0°.

(ii) The inclination of a line parallel to the y-axis or the y-axis itself is 90°.

HORIZONTAL LINE *Any line parallel to the x-axis or the x-axis itself is called a horizontal line.*

VERTICAL LINE *Any line parallel to the y-axis or the y-axis itself is called a vertical line.*

OBLIQUE LINE *A line which is neither horizontal nor vertical is called a oblique line.*

SLOPE OR GRADIENT OF A LINE

If θ *is the inclination of a non-vertical line, then* $m = \tan\theta$ *is called the slope of the line.*

EXAMPLE 1 *Find the slope of a line whose inclination is*

(i) 45° *(ii)* 60° *(iii)* 150°

SOLUTION Let m be the slope of the line. Then,

(i) $m = \tan 45° = 1.$

(ii) $m = \tan 60° = \sqrt{3}.$

(iii) $m = \tan 150° = \tan(180° - 30°) = -\tan 30° = \dfrac{-1}{\sqrt{3}}\cdot$

REMARK 1 *The slope of a horizontal line is 0.*

We know that the inclination of a horizontal line is 0°.

So, slope of a horizontal line is $m = \tan 0° = 0.$

REMARK 2 *The slope of a vertical line is not defined.*

We know that the inclination of vertical line is 90°.

So, slope of a vertical line is $m = \tan 90°$, which is not defined.

EXAMPLE 2 *What is the inclination of a line whose slope is*

(i) zero? *(ii) positive?* *(iii) negative?* *(iv) not defined?*

SOLUTION Let θ be the inclination of the given line. Then, $m = \tan\theta$.

(i) $m = 0 \Rightarrow \tan\theta = 0$
$\Rightarrow \theta = 0°$ [$\because$ $0° \le \theta < 180°$]

(ii) $m > 0 \Rightarrow \tan\theta > 0$
$\Rightarrow \theta$ lies between $0°$ and $90°$
$\Rightarrow \theta$ is acute.

(iii) $m < 0 \Rightarrow \tan\theta < 0$
$\Rightarrow \theta$ lies between $90°$ and $180°$
$\Rightarrow \theta$ is obtuse.

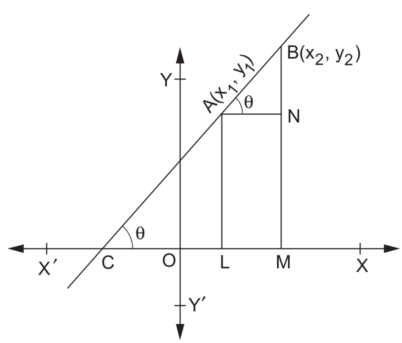
(iv) We know that a vertical line is the only line whose slope is not defined. And, the inclination of a vertical line is $90°$.

Hence, the inclination of a line whose slope is not defined, is $90°$.

SLOPE OF A LINE PASSING THROUGH TWO GIVEN POINTS

THEOREM 1 *Prove that the slope of a non-vertical line passing through the points $A(x_1, y_1)$ and $B(x_2, y_2)$ is given by* $m = \dfrac{(y_2 - y_1)}{(x_2 - x_1)}$.

PROOF Let $X'OX$ and YOY' be the coordinate axes. Let us consider a non-vertical line CAB, passing through the points $A(x_1, y_1)$ and $B(x_2, y_2)$. Let θ be its inclination.



Draw $AL \perp OX$, $BM \perp OX$ and $AN \perp BM$.

Then, $\angle BAN = \angle ACX = \theta$ (corr. ∠s).

From right triangle ANB, we have:

$$\tan\theta = \frac{BN}{AN} = \frac{(BM - NM)}{LM} = \frac{(BM - AL)}{(OM - OL)} = \frac{(y_2 - y_1)}{(x_2 - x_1)}$$

$$\Rightarrow m = \frac{(y_2 - y_1)}{(x_2 - x_1)} \qquad [\because \tan\theta = m].$$

Hence, the slope of the line joining the points $A(x_1, y_1)$ and $B(x_2, y_2)$ is given by, $m = \frac{(y_2 - y_1)}{(x_2 - x_1)}$.

EXAMPLE 3 *Find the slope of the line passing through the points*
(i) $(-2, 3)$ *and* $(8, -5)$ *(ii)* $(4, -3)$ *and* $(6, -3)$ *(iii)* $(3, -1)$ *and* $(3, 2)$

SOLUTION (i) Let $A(-2, 3)$ and $B(8, -5)$ be the given points. Then,

$$\text{slope of } AB = \frac{-5-3}{8-(-2)} = \frac{-8}{10} = \frac{-4}{5}. \qquad \left[\because\ m = \frac{(y_2 - y_1)}{(x_2 - x_1)}\right]$$

(ii) Let $C(4, -3)$ and $D(6, -3)$ be the given points. Then,

$$\text{slope of } CD = \frac{-3-(-3)}{6-4} = \frac{-3+3}{2} = \frac{0}{2} = 0.$$

ALITER The points $C(4, -3)$ and $D(6, -3)$ have the same y-coordinate. So, CD is a line parallel to the x-axis. Hence, its slope is 0.

(iii) Let $P(3, -1)$ and $Q(3, 2)$ be the given points. Then,

$$\text{slope of } PQ = \frac{2-(-1)}{3-3} = \frac{3}{0}, \text{ which is not defined.}$$

ALITER The points $P(3, -1)$ and $Q(3, 2)$ have the same x-coordinate. So, PQ is a vertical line.
Hence, its slope is not defined.

EXAMPLE 4 *If the slope of the line passing through the points* $(2, 5)$ *and* $(x, 3)$ *is* 2, *find the value of* x.

SOLUTION Let $A(2, 5)$ and $B(x, 3)$ be the given points. Then,

$$\text{slope of } AB = \frac{3-5}{x-2} = \frac{-2}{(x-2)}.$$

$$\therefore \quad \frac{-2}{(x-2)} = 2 \Leftrightarrow 2x - 4 = -2$$

$$\Leftrightarrow 2x = 2 \Leftrightarrow x = 1.$$

Hence, $x = 1$.

EXAMPLE 5 *Find the value of* x *so that the inclination of the line joining the points* $(x, -3)$ *and* $(2, 5)$ *is* $135°$.

SOLUTION Let $A(x, -3)$ and $B(2, 5)$ be the given points. Then,

$$\text{slope of } AB = \frac{5-(-3)}{2-x} = \frac{8}{(2-x)}.$$

But, slope of $AB = \tan 135° = \tan(180° - 45°) = -\tan 45° = -1$.

$$\therefore \quad \frac{8}{(2-x)} = -1 \Leftrightarrow x - 2 = 8 \Leftrightarrow x = 10.$$

Hence, $x = 10$.

EXAMPLE 6 *Find the angle between the x-axis and the line joining the points* $(3, -1)$ *and* $(4, -2)$.

SOLUTION Let $A(3, -1)$ and $B(4, -2)$ be the given points and let m be the slope of the line AB. Then,

$$m = \frac{-2-(-1)}{4-3} = \frac{-2+1}{1} = -1. \qquad \left[\because\ m = \frac{(y_2 - y_1)}{(x_2 - x_1)}\right]$$

Let θ be the angle between the x-axis and the line AB. Then,

$$\tan\theta = m = -1 = -\tan 45° = \tan(180° - 45°) = \tan 135°.$$

$\therefore \quad \theta = 135°.$

Hence, the required angle is $135°$.

SLOPES OF PARALLEL LINES

THEOREM 2 *Prove that two non-vertical lines are parallel, if and only if their slopes are equal.*

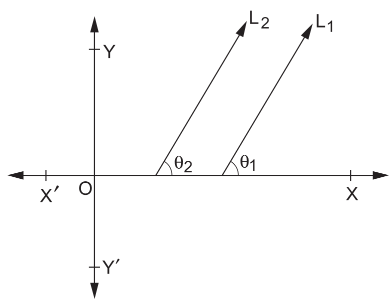
PROOF Let L_1 and L_2 be two non-vertical lines with slopes m_1 and m_2 respectively. Let their inclinations be θ_1 and θ_2 respectively.

First we assume that L_1 and L_2 are parallel.

Then, L_1 and L_2 are parallel $\Rightarrow \theta_1 = \theta_2$

$\Rightarrow \tan\theta_1 = \tan\theta_2$

$\Rightarrow m_1 - m_2.$



Thus, non-vertical parallel lines have equal slopes.

Conversely, let $m_1 = m_2$. Then,

$m_1 = m_2 \Rightarrow \tan\theta_1 = \tan\theta_2$

$\Rightarrow \theta_1 = \theta_2 \qquad [\because\ 0° \le \theta < 180°]$

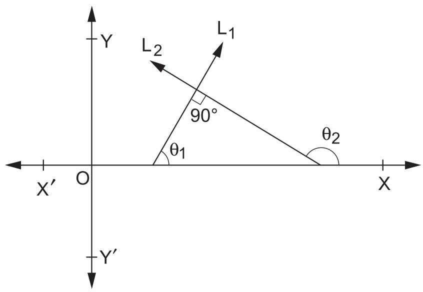
$\Rightarrow L_1$ and L_2 are parallel.

Thus, non-vertical lines having equal slopes, are parallel. Hence, two non-vertical lines are parallel if and only if their slopes are equal.

SLOPES OF PERPENDICULAR LINES

THEOREM 3 *Prove that two non-vertical lines with slopes m_1 and m_2 are perpendicular if and only if $m_1 m_2 = -1$.*

PROOF Let L_1 and L_2 be two non-vertical lines with slopes m_1 and m_2 respectively. Let their inclinations be θ_1 and θ_2 respectively. Then, $m_1 = \tan\theta_1$ and $m_2 = \tan\theta_2$.



First we assume that L_1 and L_2 are perpendicular to each other. Then,

$$\theta_2 = (90° + \theta_1)$$

$$\Rightarrow \tan\theta_2 = \tan(90° + \theta_1) = -\cot\theta_1$$

$$\Rightarrow \tan\theta_2 = -\frac{1}{\tan\theta_1}$$

$$\Rightarrow m_2 = -\frac{1}{m_1} \Rightarrow m_1 m_2 = 1.$$

Thus, when $L_1 \perp L_2$, then $m_1 m_2 = -1$.

Conversely, let L_1 and L_2 be two lines with slopes m_1 and m_2 respectively such that $m_1 m_2 = -1$.

Let $m_1 = \tan\theta_1$ and $m_2 = \tan\theta_2$. Then,

$$m_1 m_2 = -1 \Rightarrow \tan\theta_1 \cdot \tan\theta_2 = -1$$

$$\Rightarrow \tan\theta_2 = \frac{-1}{\tan\theta_1} = -\cot\theta_1 = \tan(90° + \theta_1)$$

$$\Rightarrow \theta_2 = 90° + \theta_1 \Rightarrow \theta_2 \text{ and } \theta_1 \text{ differ by } 90°$$

$\Rightarrow$ L_1 and L_2 are perpendicular to each other.

Thus, when $m_1 m_2 = -1$, then $L_1 \perp L_2$.

Hence, L_1 and L_2 are perpendicular to each other, if and only if $m_1 m_2 = -1$.

SUMMARY

Let L_1 and L_2 be two lines whose slopes are m_1 and m_2 respectively. Then,

(i) $L_1 \parallel L_2 \Leftrightarrow m_1 = m_2$

(ii) $L_1 \perp L_2 \Rightarrow m_1 m_2 = -1$

EXAMPLE 7 *Show that the line joining the points* $(2, -3)$ *and* $(-5, 1)$ *is parallel to the line joining the points* $(7, -1)$ *and* $(0, 3)$.

SOLUTION Let $A(2, -3)$, $B(-5, 1)$, $C(7, -1)$ and $D(0, 3)$ be the given points.

Then, slope of $AB = \frac{1-(-3)}{-5-2} = \frac{4}{-7} = -\frac{4}{7}$

And, slope of $CD = \frac{3-(-1)}{0-7} = \frac{4}{-7} = -\frac{4}{7}$.

$\therefore$ slope of AB = slope of CD.

Hence, $AB \parallel CD$.

EXAMPLE 8 *Show that the line joining the points* $(2, -5)$ *and* $(-2, 5)$ *is perpendicular to the line joining the points* $(6, 3)$ *and* $(1, 1)$.

SOLUTION Let $A(2, -5)$, $B(-2, 5)$, $C(6, 3)$ and $D(1, 1)$ be the given points.

Let m_1 and m_2 be the slopes of AB and CD respectively. Then,

$$m_1 = \text{slope of } AB = \frac{5-(-5)}{-2-2} = \frac{10}{-4} = -\frac{5}{2}.$$

$$m_2 = \text{slope of } CD = \frac{1-3}{1-6} = \frac{-2}{-5} = \frac{2}{5}.$$

$$\therefore \quad m_1 m_2 = \left(\frac{-5}{2}\right) \times \frac{2}{5} = -1.$$

Hence, $AB \perp CD$.

EXAMPLE 9 *If the line through the points* $(-2, 6)$ *and* $(4, 8)$ *is perpendicular to the line through the points* $(8, 12)$ *and* $(x, 24)$, *find the value of* x.

SOLUTION Let $A(-2, 6)$, $B(4, 8)$, $C(8, 12)$ and $D(x, 24)$ be the given points.

Let m_1 and m_2 be the slopes of AB and CD respectively. Then,

$$m_1 = \text{slope of } AB = \frac{(8-6)}{4-(-2)} = \frac{2}{6} = \frac{1}{3}.$$

$$m_2 = \text{slope of } CD = \frac{(24-12)}{(x-8)} = \frac{12}{(x-8)}.$$

Now, $AB \perp CD \iff m_1 m_2 = -1$

$$\iff \frac{1}{3} \times \frac{12}{(x-8)} = -1$$

$$\iff -x + 8 = 4 \iff x = 4.$$

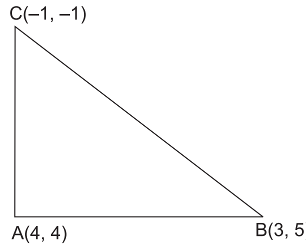
Hence, the required value of x is 4.

EXAMPLE 10 *Without using Pythagoras's theorem show that the points* $(4, 4)$, $(3, 5)$ *and* $(-1, -1)$ *are the vertices of a right-angled triangle.*

SOLUTION Let $A(4, 4)$, $B(3, 5)$ and $C(-1, -1)$ be the vertices of $\triangle ABC$.

Let m_1 and m_2 be the slopes of AB and AC respectively. Then,

$$m_1 = \text{slope of } AB = \frac{(5-4)}{(3-4)} = -1.$$



$m_2 =$ slope of $AC = \dfrac{(-1-4)}{(-1-4)} = \dfrac{-5}{-5} = 1.$

$\therefore \quad m_1 m_2 = -1.$

So, $AB \perp AC$ and therefore, $\angle CAB = 90°$.

Hence, the given points are the vertices of a right triangle.

EXAMPLE 11 *Using slopes, show that the points* $(5, 1)$, $(1, -1)$ *and* $(11, 4)$ *are collinear.*

SOLUTION Let $A(5, 1)$, $B(1, -1)$ and $C(11, 4)$ be the given points. Then,

$$\text{slope of } AB = \frac{(-1-1)}{(1-5)} = \frac{-2}{-4} = \frac{1}{2}$$

$$\text{and slope of } BC = \frac{4-(-1)}{11-1} = \frac{5}{10} = \frac{1}{2}.$$

$\therefore$ slope of AB = slope of BC

$\Rightarrow$ $AB \parallel BC$ and have a point B in common

$\Rightarrow$ A, B, C are collinear.

Hence, the given points are collinear.

EXAMPLE 12 *Find the value of* x *for which the points* $(x, -1)$, $(2, 1)$ *and* $(4, 5)$ *are collinear.*

SOLUTION Let $A(x, -1)$, $B(2, 1)$ and $C(4, 5)$ be the given collinear points.

Then, by collinearity of A, B, C we have: slope of AB = slope of BC

$$\frac{1-(-1)}{2-x} = \frac{(5-1)}{(4-2)} \Rightarrow \frac{2}{2-x} = 2 \Rightarrow 2-x = 1 \Rightarrow x = 1.$$

Hence, $x = 1$.

EXAMPLE 13 *If the points* $(h, 0)$, (a, b) *and* $(0, k)$ *lie on a line, show that* $\dfrac{a}{h} + \dfrac{b}{k} = 1.$

SOLUTION Let $A(h, 0)$, $B(a, b)$ and $C(0, k)$ be the given collinear points.

Since the given points A, B, C are collinear, we have

slope of AB = slope of BC

$$\therefore \quad \frac{b-0}{a-h} = \frac{k-b}{0-a} \Leftrightarrow \frac{b}{(a-h)} = \frac{(b-k)}{a}$$

$$\Leftrightarrow ab = (a-h)(b-k)$$

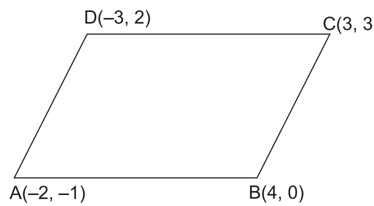
$$\Leftrightarrow \; ak + hb = hk$$

$$\Leftrightarrow \; \frac{a}{h} + \frac{b}{k} = 1 \qquad \text{[on dividing both sides by } hk\text{]}$$

Hence, $\frac{a}{h} + \frac{b}{k} = 1$.

EXAMPLE 14 *Using slopes, show that the vertices* $(-2, -1)$, $(4, 0)$, $(3, 3)$ *and* $(-3, 2)$ *are the vertices of a parallelogram.*

SOLUTION Let $A(-2, -1)$, $B(4, 0)$, $C(3, 3)$ and $D(-3, 2)$ be the vertices of the given quadrilateral $ABCD$. Then,



slope of $AB = \frac{0-(-1)}{4-(-2)} = \frac{1}{6}$;

slope of $DC = \frac{3-2}{3-(-3)} = \frac{1}{6}$;

slope of $BC = \frac{3-0}{3-4} = \frac{3}{-1} = -3$;

slope of $AD = \frac{2-(-1)}{-3-(-2)} = \frac{3}{-1} = -3$;

$\therefore$ slope of AB = slope of DC $\Rightarrow$ $AB \parallel DC$;

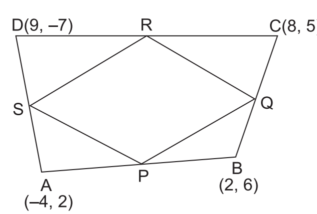
slope of BC = slope of AD $\Rightarrow$ $BC \parallel AD$.

Hence, $ABCD$ is a parallelogram.

EXAMPLE 15 $A(-4, 2)$, $B(2, 6)$, $C(8, 5)$ *and* $D(9, -7)$ *are the vertices of a quadrilateral ABCD. If P, Q, R, S are the midpoints of AB, BC, CD and DA respectively, using slopes, show that PQRS is a parallelogram.*

SOLUTION Clearly, the points P, Q, R, S are given by

$$P(-1, 4),\; Q\left(5, \frac{11}{2}\right),\; R\left(\frac{17}{2}, -1\right) \text{ and } S\left(\frac{5}{2}, \frac{-5}{2}\right).$$



$\therefore$ slope of $PQ = \frac{\left(\frac{11}{2} - 4\right)}{(5+1)} = \frac{3}{12} = \frac{1}{4}$;

slope of $SR = \frac{\left(-1 + \frac{5}{2}\right)}{\left(\frac{17}{2} - \frac{5}{2}\right)} = \frac{3}{12} = \frac{1}{4}$;

slope of $QR = \frac{\left(-1 - \frac{11}{2}\right)}{\left(\frac{17}{2} - 5\right)} = \left(\frac{-13}{2} \times \frac{2}{7}\right) = \frac{-13}{7}$;

slope of $PS = \frac{\left(\frac{5}{2} - 4\right)}{\left(\frac{5}{2} + 1\right)} = \left(\frac{-13}{2} \times \frac{2}{7}\right) = \frac{-13}{7}$.

$\therefore$ slope of PQ = slope of $SR \Rightarrow PQ \parallel SR$;
slope of QR = slope of $PS \Rightarrow QR \parallel PS$.

Hence, $PQRS$ is a parallelogram.

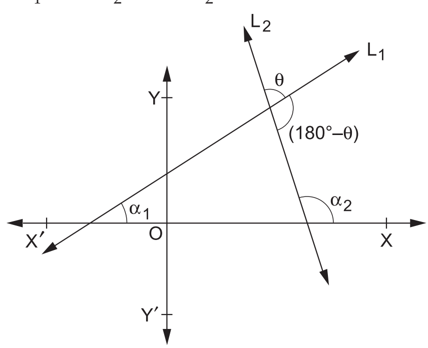
ANGLE BETWEEN TWO NON-VERTICAL AND NON-PERPENDICULAR LINES

THEOREM 4 *If α is the acute angle between two non-vertical and non-perpendicular lines L_1 and L_2 with slopes m_1 and m_2 then prove that*

$$\tan\alpha = \left|\frac{m_2 - m_1}{1 + m_1 m_2}\right|, \text{ where } 1 + m_1 m_2 \neq 0.$$

PROOF Let L_1 and L_2 be the two given non-vertical and non-perpendicular lines with slopes m_1 and m_2 respectively. Let α_1 and α_2 be their respective inclinations. Then,

$$m_1 = \tan\alpha_1 \text{ and } m_2 = \tan\alpha_2.$$



Let θ and $(180° - \theta)$ be the angles between L_1 and L_2.

From the given figure, we have

$$\theta = \alpha_2 - \alpha_1, \text{ where } \alpha_1, \alpha_2 \neq 90°$$

$$\Rightarrow \tan\theta = \tan(\alpha_2 - \alpha_1) = \frac{\tan\alpha_2 - \tan\alpha_1}{1 + \tan\alpha_1 \tan\alpha_2}$$

$$\Rightarrow \tan\theta = \frac{(m_2 - m_1)}{(1 + m_1 m_2)} \quad \ldots \text{(i)}, \qquad \text{where } 1 + m_1 m_2 \neq 0.$$

$$\text{Also, } \tan(180° - \theta) = -\tan\theta = -\frac{(m_2 - m_1)}{(1 + m_1 m_2)} \qquad \ldots \text{(ii)}$$

Thus, tangent of the angle between given lines $= \pm\dfrac{(m_2 - m_1)}{(1 + m_1 m_2)}$

[from (i) and (ii)].

Let $\alpha = \min.\{\theta, (180° - \theta)\}$, then α is acute and so $\tan\alpha > 0$.

$$\therefore \quad \tan\alpha = \left|\frac{m_2 - m_1}{1 + m_1 m_2}\right|.$$

EXAMPLE 16 *Find the angle between the lines whose slopes are* $\frac{1}{2}$ *and* 3.

SOLUTION Let θ be the angle between the given lines.

Let $m_1 = \frac{1}{2}$ and $m_2 = 3$. Then,

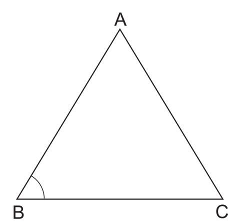
$$\tan\theta = \left|\frac{m_2 - m_1}{1 + m_1 m_2}\right| = \left|\frac{\left(3 - \frac{1}{2}\right)}{\left(1 + 3\cdot\frac{1}{2}\right)}\right| = \left|\frac{(5/2}{(5/2)}\right| = |1| = 1$$

$\Rightarrow \quad \theta = 45°$.

Hence, the angle between the given lines is $45°$.

EXAMPLE 17 *If* $A(-2, 1)$, $B(2, 3)$ *and* $C(-2, -4)$ *be the vertices of a* $\triangle ABC$, *show that* $\tan B = \frac{2}{3}\cdot$

SOLUTION Clearly, B is the angle between BA and BC.



Now, $m_1 =$ slope of $BA = \frac{3-1}{2+2} = \frac{1}{2}$

and $m_2 =$ slope of $BC = \frac{-4-3}{-2-2} = \frac{7}{4}\cdot$

$$\therefore \quad \tan B = \left|\frac{m_2 - m_1}{1 + m_1 m_2}\right| = \left|\frac{\frac{7}{4} - \frac{1}{2}}{1 + \frac{7}{4}\cdot\frac{1}{2}}\right| = \frac{2}{3}\cdot$$

Hence, $\tan B = \frac{2}{3}\cdot$

EXAMPLE 18 *If the angle between two lines is* $\frac{\pi}{4}$ *and the slope of one of the lines is* $\frac{1}{2}$, *find the slope of the other line.*

SOLUTION We know that the acute angle θ between two lines having slopes m_1 and m_2 is given by

$$\tan\theta = \left|\frac{m_2 - m_1}{1 + m_1 m_2}\right|\cdot \qquad \text{... (i)}$$

Let $m_1 = \frac{1}{2}$, $m_2 = m$ and $\theta = \frac{\pi}{4}\cdot$

Putting these values in (i), we get

$$\left|\frac{m - \frac{1}{2}}{1 + \frac{1}{2}m}\right| = \tan\frac{\pi}{4} \Leftrightarrow \left|\frac{2m-1}{2+m}\right| = 1$$

$$\Leftrightarrow \left(\frac{2m-1}{2+m}\right) = \pm 1$$

$$\Leftrightarrow \frac{2m-1}{2+m} = 1 \quad \text{or} \quad \frac{2m-1}{2+m} = -1$$

$$\Leftrightarrow (2m-1) = (2+m) \quad \text{or} \quad (2m-1) = (-2-m)$$

$$\Leftrightarrow m = 3 \quad \text{or} \quad m = \frac{-1}{3}.$$

Hence, the slope of the other line is 3 or $\frac{-1}{3}$.

EXERCISE 20B

1. Find the slope of a line whose inclination is

(i) 30° (ii) 120° (iii) 135° (iv) 90°

2. Find the inclination of a line whose slope is

(i) $\sqrt{3}$ (ii) $\frac{1}{\sqrt{3}}$ (iii) 1 (iv) -1

(v) $-\sqrt{3}$

3. Find the slope of a line which passes through the points

(i) (0, 0) and (4, − 2) (ii) (0, − 3) and (2, 1)

(iii) (2, 5) and (−4, − 4) (iv) (−2, 3) and (4, − 6)

4. If the slope of the line joining the points $A(x, 2)$ and $B(6, -8)$ is $\frac{-5}{4}$, find the value of x.

5. Show that the line through the points (5, 6) and (2, 3) is parallel to the line through the points (9, −2) and (6, −5).

6. Find the value of x so that the line through $(3, x)$ and (2, 7) is parallel to the line through (−1, 4) and (0, 6).

7. Show that the line through the points (−2, 6) and (4, 8) is perpendicular to the line through the points (3, −3) and (5, −9).

8. If $A(2, -5)$, $B(-2, 5)$, $C(x, 3)$ and $D(1, 1)$ be four points such that AB and CD are perpendicular to each other, find the value of x.

9. Without using Pythagoras's theorem, show that the points $A(1, 2)$, $B(4, 5)$ and $C(6, 3)$ are the vertices of a right-angled triangle.

10. Using slopes, show that the points $A(6, -1)$, $B(5, 0)$ and $C(2, 3)$ are collinear.

11. Using slopes, find the value of x for which the points $A(5, 1)$, $B(1, -1)$ and $C(x, 4)$ are collinear.

12. Using slopes, show that the points $A(-4, -1)$, $B(-2, -4)$, $C(4, 0)$ and $D(2, 3)$ taken in order, are the vertices of a rectangle.

13. Using slopes, prove that the points $A(-2, -1)$, $B(1, 0)$, $C(4, 3)$ and $D(1, 2)$ are the vertices of a parallelogram.

14. If the three points $A(h, k)$, $B(x_1, y_1)$ and $C(x_2, y_2)$ lie on a line then show that
$$(h - x_1)(y_2 - y_1) = (k - y_1)(x_2 - x).$$

15. If the points $A(a, 0)$, $B(0, b)$ and $P(x, y)$ are collinear, using slopes, prove that
$$\frac{x}{a} + \frac{y}{b} = 1.$$

16. A line passes through the points $A(4, -6)$ and $B(-2, -5)$. Show that the line AB makes an obtuse angle with the x-axis.

17. The vertices of a quadrilateral are $A(-4, 2)$, $B(2, 6)$, $C(8, 5)$ and $D(9, -7)$. Using slopes, show that the midpoints of the sides of the quad. $ABCD$ form a parallelogram.

18. Find the slope of the line which makes an angle of 30° with the positive direction of the y-axis, measured anticlockwise.

19. Find the angle between the lines whose slopes are $\sqrt{3}$ and $\frac{1}{\sqrt{3}}$.

20. Find the angle between the lines whose slopes are $(2 - \sqrt{3})$ and $(2 + \sqrt{3})$.

21. If $A(1, 2)$, $B(-3, 2)$ and $C(3, -2)$ be the vertices of a $\triangle ABC$, show that

(i) $\tan A = 2$ (ii) $\tan B = \frac{2}{3}$ (iii) $\tan C = \frac{4}{7}$.

22. If θ is the angle between the lines joining the points $A(0, 0)$ and $B(2, 3)$, and the points $C(2, -2)$ and $D(3, 5)$, show that $\tan \theta = \frac{11}{23}$.

23. If θ is the angle between the diagonals of a parallelogram $ABCD$ whose vertices are $A(0, 2)$, $B(2, -1)$, $C(4, 0)$ and $D(2, 3)$. Show that $\tan \theta = 2$.

24. Show that the points $A(0, 6)$, $B(2, 1)$ and $C(7, 3)$ are three corners of a square $ABCD$. Find (i) the slope of the diagonal BD and (ii) the coordinates of the fourth vertex D.

25. $A(1, 1)$, $B(7, 3)$ and $C(3, 6)$ are the vertices of a $\triangle ABC$. If D is the midpoint of BC and $AL \perp BC$, find the slopes of (i) AD and (ii) AL.

ANSWERS (EXERCISE 20B)

1. (i) $\frac{1}{\sqrt{3}}$ (ii) $-\sqrt{3}$ (iii) -1 (iv) not defined

2. (i) 60° (ii) 30° (iii) 45° (iv) 135° (v) 120°

3. (i) $-\frac{1}{2}$ (ii) 2 (iii) $\frac{3}{2}$ (iv) $-\frac{3}{2}$ **4.** $x = -2$ **6.** $x = 9$

8. $x = 6$ **11.** $x = 11$ **18.** $-\sqrt{3}$ **19.** 30°

20. 60° **24.** (i) $\frac{7}{3}$ (ii) (5, 8) **25.** (i) $\frac{7}{8}$ (ii) $\frac{4}{3}$

HINTS TO SOME SELECTED QUESTIONS

11. For collinearity, we must have, slope of AB = slope of BC.

12. Let m_1, m_2, m_3, m_4 be the slopes of AB, BC, CD and DA respectively. Then, show that $m_1 = m_3, m_2 = m_4$ and $m_1 m_2 = -1$.

14. Slope of AB = slope of BC

$$\Rightarrow \frac{(y_1 - k)}{(x_1 - h)} = \frac{(y_2 - y_1)}{(x_2 - x_1)} \Rightarrow \frac{(k - y_1)}{(h - x_1)} = \frac{(y_2 - y_1)}{(x_2 - x_1)}$$

$$\Rightarrow (h - x_1)(y_2 - y_1) = (k - y_1)(x_2 - x_1).$$

15. Since the points A, B, P are collinear, we have

(slope of AB = slope of BP) $\Rightarrow \frac{b - 0}{0 - a} = \frac{y - b}{x - 0}$

$$\Rightarrow bx + ay = ab \Rightarrow \frac{x}{a} + \frac{y}{b} = 1.$$

16. Slope of AB is negative.

17. Midpoints of AB, BC, CD and DA are $P(-1, 4)$, $Q\left(5, \frac{11}{2}\right)$, $R\left(\frac{17}{2}, -1\right)$ and $S\left(\frac{5}{2}, \frac{-5}{2}\right)$.

Now, show that (slope of PQ = slope of $RS = \frac{1}{4}$) and (slope of QR = slope of $PS = \frac{-13}{7}$).

18. The given line makes an angle of $(90° + 30°) = 120°$ with the positive direction of the x-axis. Hence, $m = \tan 120° = -\sqrt{3}$.

24. $BD \perp AC$.

VARIOUS FORMS OF EQUATIONS OF A LINE

EQUATION OF A LINE

The equation of a straight line is the linear relation between two variables x and y, which is satisfied by the coordinates of each and every point on the line and not by those of any other point in the cartesian plane.

SOME ELEMENTARY RESULTS

(i) EQUATION OF X-AXIS:

We know that the ordinate of each point on the x-axis is 0. If $P(x, y)$ is any point on the x-axis, then $y = 0$.

Hence, *the equation of x-axis is $y = 0$.*

(ii) EQUATION OF Y-AXIS:

We know that the abscissa of each point on the y-axis is 0. If $P(x, y)$ is any point on the y-axis, then $x = 0$.

Hence, *the equation of y-axis is $x = 0$.*

(iii) EQUATION OF A LINE PARALLEL TO Y-AXIS:

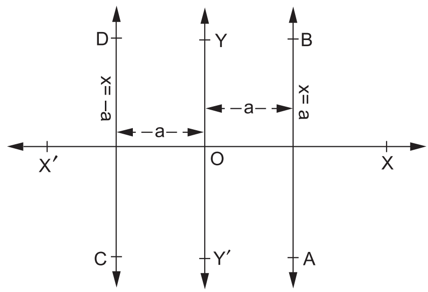
Let AB be a straight line parallel to the y-axis lying on its right-hand side at a distance a from it.

Then, the abscissa of each point on AB is a.

If $P(x, y)$ is any point on AB then $x = a$.

Thus, *the equation of a vertical line at a distance a from the y-axis, lying on its right-hand side is* $x = a$.

Similarly, *the equation of a vertical line at a distance a from the y-axis, lying on its left-hand side is* $x = -a$.



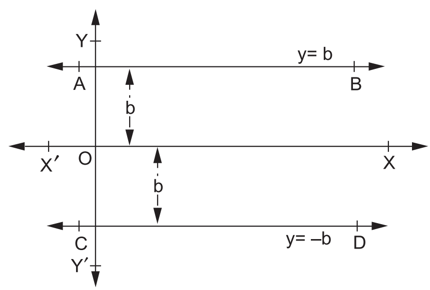
(iv) EQUATION OF A LINE PARALLEL TO X-AXIS:

Let AB be a straight line parallel to the x-axis, lying above it at a distance b from it.

Then, the ordinate of each point on AB is b.

If $P(x, y)$ is any point on AB, then $y = b$.

Thus, *the equation of a horizontal line at a distance b from the x-axis, lying above the x-axis is* $y = b$.



Similarly, *the equation of a horizontal line at a distance b from the x-axis, lying below the x-axis is* $y = -b$.

SUMMARY

(i) *Equation of x-axis is* $\boldsymbol{y = 0}$.

(ii) *Equation of y-axis is* $\boldsymbol{x = 0}$.

(iii) *Equation of a vertical line on RHS of the y-axis at a distance a from it is* $\boldsymbol{x = a}$..

(iv) *Equation of a vertical line on LHS of the x-axis at a distance a from it is* $\boldsymbol{x = -a}$..

(v) *Equation of a horizontal line lying above the x-axis at a distance b from it is* $\boldsymbol{y = b}$.

(vi) *Equation of a horizontal line lying below the x-axis at a distance b from it is* $\boldsymbol{y = -b}$.

EXAMPLE 1 *Write down the equation of (i) x-axis (ii) y-axis.*

SOLUTION We know that

(i) the equation of x-axis is $y = 0$.

(ii) the equation of y-axis is $x = 0$.

EXAMPLE 2 *Write down the equation of a line parallel to the x-axis*

(i) at a distance of 5 *units above the x-axis,*

(ii) at a distance of 4 *units below the x-axis.*

SOLUTION (i) The equation of a line parallel to the x-axis at a distance of 5 units above x-axis is $y = 5$.

(ii) The equation of a line parallel to the x-axis at a distance of 4 units below the x-axis is $y = -4$.

EXAMPLE 3 *Write down the equation of a line parallel to the y-axis*

(i) at a distance of 7 *units on left-hand side of the y-axis,*

(ii) at a distance of 3 *units on right-hand side of the y-axis.*

SOLUTION (i) The equation of a line parallel to the y-axis at a distance of 7 units on its left is $x = -7$.

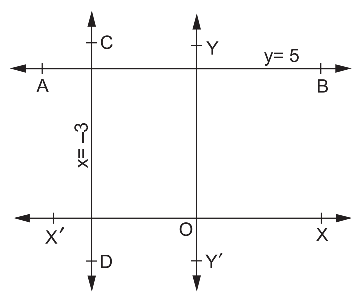
(ii) The equation of a line parallel to the y-axis at a distance of 3 units on its right is $x = 3$.

EXAMPLE 4 *Find the equations of the lines parallel to the axes and passing through the point* $(-3, 5)$.

SOLUTION We know that

(i) the equation of a line parallel to the x-axis and passing through $(-3, 5)$ is $y = 5$.

(ii) the equation of a line parallel to the y-axis and passing through $(-3, 5)$ is $x = -3$.



EXAMPLE 5 *Find the values of k for which the line*

$$(k - 3)x + (k^2 - 4)y + (k - 1)(k - 6) = 0$$

(i) is parallel to the x-axis,

(ii) is parallel to the y-axis,

(iii) passes through the origin.

SOLUTION Given equation of the line is $(k - 3)x + (k^2 - 4)y + (k - 1)(k - 6) = 0$.

(i) The given line will be parallel to the x-axis, when the coefficient of y is zero,

i.e., when $k^2 - 4 = 0$.

This happens when $k = \pm 2$.

(ii) The given line will be parallel to the y-axis, when the coefficient of x is zero,

i.e., when $k - 3 = 0$ or $k = 3$.

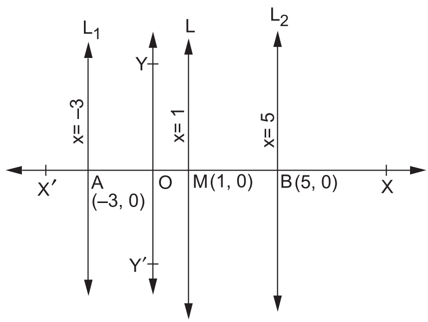
(iii) The given line will pass through the origin, if (0, 0) satisfies the given equation,

i.e., when $(k-1)(k-6)=0$.

This happens when $k=1$ or $k=6$.

EXAMPLE 6 *Find the equation of a line which is equidistant from the lines $x=-3$ and $x=5$.*

SOLUTION Let $X'OX$ and YOY' be the coordinate axes



Let L_1 be the line $x=-3$. Then, it is parallel to the y-axis.

It meets the x-axis at $A(-3, 0)$.

Let L_2 be the line $x=5$. Then, it is parallel to the y-axis.

It meets the x-axis at $B(5, 0)$.

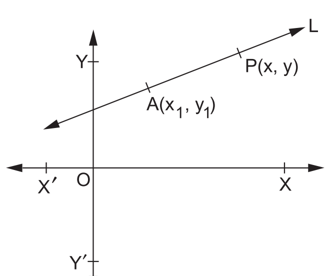
Midpoint of AB is $M\left(\frac{-3+5}{2}, \frac{0+0}{2}\right)$, i.e., $M(1, 0)$.

So, the line $x=1$ is equidistant from L_1 and L_2.

EQUATION OF A LINE IN POINT–SLOPE FORM

THEOREM 1 *Prove that the equation of a non-vertical line with slope m and passing through the point (x_1, y_1) is $(y-y_1)=m(x-x_1)$.*

SOLUTION Let L be a non-vertical line with slope m and passing through a point $A(x_1, y_1)$.



Let $P(x, y)$ be an arbitrary point on L.

Then, slope of line $L = \dfrac{(y-y_1)}{(x-x_1)}$.

$\therefore \quad \dfrac{(y-y_1)}{(x-x_1)} = m \iff (y-y_1) = m(x-x_1)$.

Hence, the required equation is, $(y-y_1) = m(x-x_1)$.

EXAMPLE 1 *Find the equation of a line passing through the point* (4, 3) *and having slope* 2.

SOLUTION We know that the equation of a line with slope m and passing through the point (x_1, y_1) is given by

$$(y-y_1) = m(x-x_1).$$

Here, $m = 2$, $x_1 = 4$ and $y_1 = 3$.

Hence, the required equation is

$$(y-3) = 2(x-4), \quad \text{i.e.,} \quad 2x - y - 5 = 0.$$

EXAMPLE 2 *Find the equation of a line which makes an angle of* $135°$ *with the x-axis and passes through the point* (3, 5).

SOLUTION Here, $m = \tan 135° = \tan(180° - 45°) = -\tan 45° = -1$.

Hence, the required equation is

$$\frac{y-5}{x-3} = -1 \iff (y-5) = 3 - x \iff x + y - 8 = 0.$$

EXAMPLE 3 *Find the equation of a line passing through the point* $(3, -4)$ *and parallel to the x-axis.*

SOLUTION Since the given line is parallel to the x-axis, we have $m = 0$.

So, the required equation is

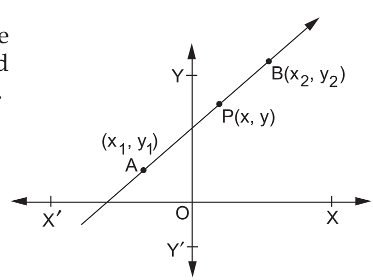
$$\frac{y+4}{y-3} = 0 \iff y + 4 = 0.$$

EQUATION OF A LINE IN TWO-POINT FORM

THEOREM 2 *Prove that the equation of a non-vertical line passing through two given points* $A(x_1, y_1)$ *and* $B(x_2, y_2)$ *is given by*

$$\frac{(y-y_1)}{(x-x_1)} = \frac{(y_2-y_1)}{(x_2-x_1)}.$$

PROOF Let L be the line passing through the points $A(x_1, y_1)$ and $B(x_2, y_2)$ and let $P(x, y)$ be an arbitrary point on L.



Then, clearly we have

slope of AP = slope of AB

$$\Rightarrow \frac{(y-y_1)}{(x-x_1)} = \frac{(y_2-y_1)}{(x_2-x_1)},$$

which is the required equation.

EXAMPLE 4 *Find the equation of a line passing through the points* $(-1, 1)$ *and* $(2, -4)$.

SOLUTION Let the given points be $A(-1, 1)$ and $B(2, -4)$.

We know that the equation of a line passing through the points (x_1, y_1) and (x_2, y_2) is given by $\frac{(y-y_1)}{(x-x_1)} = \frac{(y_2-y_1)}{(x_2-x_1)}$.

Here, $x_1 = -1$, $y_1 = 1$, $x_2 = 2$ and $y_2 = -4$.

So, the equation of a line passing through the given points is

$$\frac{(y-1)}{\{x-(-1)\}} = \frac{(-4-1)}{2-(-1)} \Leftrightarrow \frac{(y-1)}{(x+1)} = \frac{-5}{3}$$

$$\Leftrightarrow 3(y-1) = -5(x+1)$$

$$\Leftrightarrow 5x + 3y + 2 = 0.$$

Hence, the required equation is $5x + 3y + 2 = 0$.

EXAMPLE 5 *Show that the three points* $(3, 0)$, $(-2, -2)$ *and* $(8, 2)$ *are collinear. Also, find the equation of the straight line on which these points lie.*

SOLUTION Let $A(3, 0)$, $B(-2, -2)$ and $C(8, 2)$ be the given points.

Then, the equation of line AB is given by

$$\frac{y-0}{x-3} = \frac{-2-0}{-2-3} \qquad \left[\text{using } \frac{y-y_1}{x-x_1} = \frac{y_2-y_1}{x_2-x_1}\right]$$

$$\Rightarrow \frac{y}{x-3} = \frac{2}{5}$$

$$\Rightarrow 5y = 2x - 6 \qquad \text{... (i)}$$

$\Rightarrow$ $2x - 5y - 6 = 0$, which is the required equation.

Putting $x = 8$ and $y = 2$ in (i), we get

$$\text{LHS} = (2 \times 8) - (5 \times 2) - 6 = 0 = \text{RHS}.$$

Thus, the point $C(8, 2)$ also lies on (i).

Hence, the given points lie on the same straight line, whose equation is $2x - 5y - 6 = 0$.

EXAMPLE 6 *Show that the points* $(a, 0)$, $(0, b)$ *and* $(3a, -2b)$ *are collinear. Also find the equation of the line containing them.*

SOLUTION Let $A(a, 0)$, $B(0, b)$ and $C(3a, -2b)$ be the given points.

Then, the equation of line AB is given by

$$\frac{y-0}{x-a} = \frac{b-0}{0-a} \Rightarrow -ay = bx - ab$$

$$\Rightarrow bx + ay - ab = 0.$$

Thus, the equation of line AB is $bx + ay - ab = 0$. ... (i)

Putting $x = 3a$ and $y = -2b$ in (i), we get

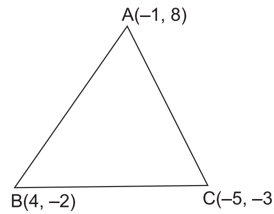
$$\text{LHS} = b \cdot 3a + a(-2b) - ab = 3ab - 2ab - ab = 0 = \text{RHS}.$$

Thus, the point $C(3a, -2b)$ also lies on AB.

Hence, the given points are collinear and the equation of the line containing them is $bx + ay - ab = 0$.

EXAMPLE 7 *Find the equations of the sides of a triangle whose vertices are* $A(-1, 8)$, $B(4, -2)$ *and* $C(-5, -3)$.

SOLUTION Here, we make use of the equation of a line in two-point form.



Equation of AB is $\dfrac{y-8}{x+1} = \dfrac{-2-8}{4+1}$

$\Rightarrow 5(y-8) + 10(x+1) = 0$

$\Rightarrow 10x + 5y - 30 = 0$

$\Rightarrow 2x + y - 6 = 0;$

equation of BC is $\dfrac{y+2}{x-4} = \dfrac{-3+2}{-5-4} \Rightarrow -9(y+2) = -(x-4)$

$\Rightarrow x - 9y - 22 = 0;$

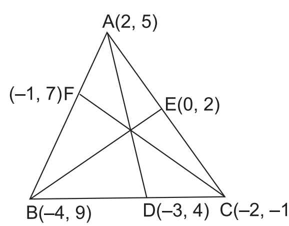
equation of AC is $\dfrac{y-8}{x+1} = \dfrac{-3-8}{-5+1} \Rightarrow -4(y-8) = -11(x+1)$

$\Rightarrow 11x - 4y + 43 = 0.$

EXAMPLE 8 *Find the equations of the medians of a* $\triangle ABC$ *whose vertices are* $A(2, 5)$, $B(-4, 9)$ *and* $C(-2, -1)$.

SOLUTION The vertices of $\triangle ABC$ are $A(2, 5)$, $B(-4, 9)$ and $C(-2, -1)$.

Let D, E, F be the midpoints of BC, CA and AB respectively.



Then, these points are

$$D\left(\frac{-4-2}{2}, \frac{9-1}{2}\right), E\left(\frac{-2+2}{2}, \frac{-1+5}{2}\right)$$

and $F\left(\dfrac{2-4}{2}, \dfrac{5+9}{2}\right)$

i.e., $D(-3, 4)$, $E(0, 2)$ and $F(-1, 7)$.

Thus, AD, BE and CF are the medians of $\triangle ABC$.

Equation of median AD is $\dfrac{y-5}{x-2} = \dfrac{4-5}{-3-2} \Rightarrow \dfrac{y-5}{x-2} = \dfrac{1}{5}$

$\Rightarrow 5(y-5) = x - 2$

$\Rightarrow x - 5y + 23 = 0.$

Equation of median BE is $\dfrac{y-9}{x+4} = \dfrac{2-9}{0+4} \Rightarrow \dfrac{y-9}{x+4} = \dfrac{-7}{4}$

$\Rightarrow 4(y-9) + 7(x+4) = 0$

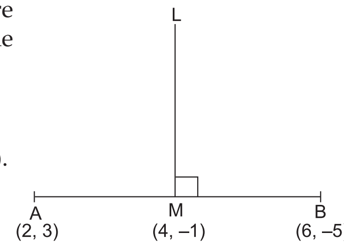
$\Rightarrow 7x + 4y - 8 = 0.$

Equation of median CF is $\dfrac{y+1}{x+2} = \dfrac{7+1}{-1+2} \Rightarrow (y+1) = 8(x+2)$

$$\Rightarrow 8x - y + 15 = 0.$$

EXAMPLE 9 *Find the equation of the perpendicular bisector of the line joining the points $A(2, 3)$ and $B(6, -5)$.*

SOLUTION Here, $A(2, 3)$ and $B(6, -5)$ are the end points of the given line segment.



The midpoint of AB is

$$M\left(\frac{2+6}{2}, \frac{3-5}{2}\right), \text{ i.e., } M(4, -1).$$

Slope of $AB = \dfrac{-5-3}{6-2} = -2.$

Let $LM \perp AB$ and let the slope of LM be m.

Then, $m \times (-2) = -1 \Rightarrow m = \dfrac{1}{2}$ $\quad [\because LM \perp AB].$

Clearly, LM is the perpendicular bisector of AB.

Now, LM is a line with slope $= \dfrac{1}{2}$ and passing through $M(4, -1)$.

So, the required equation is

$$\frac{y-(-1)}{x-4} = \frac{1}{2} \Rightarrow 2(y+1) = (x-4)$$

$$\Rightarrow x - 2y - 6 = 0.$$

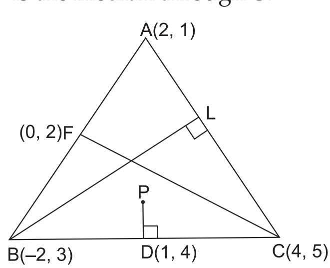
EXAMPLE 10 *If $A(2, 1)$, $B(-2, 3)$ and $C(4, 5)$ are the vertices of a $\triangle ABC$ then find the equation of*

(i) the median through C *(ii) the altitude through B*

(iii) the right bisector of side BC.

SOLUTION (i) Let F be the midpoint of side AB.

Then, CF is the median through C.



Coordinates of F are $F\left(\dfrac{2+(-2)}{2}, \dfrac{1+3}{2}\right)$, i.e., $F(0, 2)$.

So, the equation of median CF is given by

$$\frac{y-5}{x-4} = \frac{2-5}{0-4} \Leftrightarrow \frac{y-5}{x-4} = \frac{3}{4}$$

$$\Leftrightarrow 4(y-5) = 3(x-4) \Leftrightarrow 3x - 4y + 8 = 0.$$

Hence, the equation of median CF is $3x - 4y + 8 = 0$.

(ii) Draw $BL \perp AC$. Then, BL is the altitude through B.

Slope of $AC = \dfrac{5-1}{4-2} = 2.$

Let the slope BL be m.

Since $BL \perp AC$, we have $2m = -1$ and therefore, $m = \dfrac{-1}{2}\cdot$

Thus, the slope of BL is $\dfrac{-1}{2}\cdot$

So, the equation of BL is given by

$$\frac{y-3}{x+2} = \frac{-1}{2} \Leftrightarrow 2(y-3) = -(x+2) \Leftrightarrow x + 2y - 4 = 0.$$

Hence, the equation of altitude BL is $x + 2y - 4 = 0$.

(iii) Let D be the midpoint of BC.

Then, the coordinates of D are $D\left(\dfrac{-2+4}{2}, \dfrac{3+5}{2}\right)$, i.e., $D(1, 4)$.

Through D, draw $DP \perp BC$.

Slope of $BC = \dfrac{5-3}{4+2} = \dfrac{2}{6} = \dfrac{1}{3}\cdot$

Let the slope of PD be m.

Since $PD \perp BC$, we have $m \times \dfrac{1}{3} = -1 \Rightarrow m = -3.$

So, the slope of PD is -3.

So, the equation of PD is given by

$$\frac{y-4}{x-1} = -3 \Leftrightarrow -3(x-1) = (y-4) \Leftrightarrow 3x + y - 7 = 0.$$

Hence, the equation of the right bisector of BC is $3x + y - 7 = 0$.

EXAMPLE 11 *Find the equation of the bisector of $\angle A$ of $\triangle ABC$, whose vertices are $A(-2, 4)$, $B(5, 5)$ and $C(4, -2)$.*

SOLUTION Let AD be the bisector of $\angle A$.

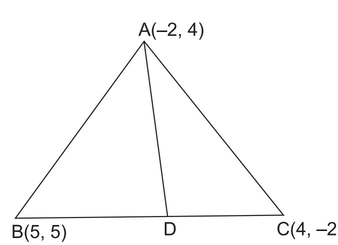
Then, $BD : DC = AB : AC$.

Now, $|AB| = \sqrt{(5-4)^2 + (5+2)^2}$

$= \sqrt{50} = 5\sqrt{2}.$

and $|AC| = \sqrt{(-2-4)^2 + (4+2)^2}$

$= \sqrt{72} = 6\sqrt{2}.$



$\therefore \quad \dfrac{BD}{DC} = \dfrac{AB}{AC} = \dfrac{5\sqrt{2}}{6\sqrt{2}} = \dfrac{5}{6} \Rightarrow BD : DC = 5 : 6.$

Thus, D divides BC in the ratio $5 : 6$.

So, the coordinates of D are $\left(\dfrac{5 \times 4 + 6 \times 5}{5+6}, \dfrac{5 \times (-2) + 6 \times 5}{5+6}\right)$,

i.e., $D\left(\dfrac{50}{11}, \dfrac{20}{11}\right)$

$\therefore$ equation of line AD is given by

$$\frac{y-4}{x+2} = \frac{\frac{20}{11} - 4}{\frac{50}{11} + 2} \Leftrightarrow \frac{y-4}{x+2} = \frac{-24}{72} = \frac{-1}{3}$$

$$\Leftrightarrow 3(y-4) = -(x+2) \Leftrightarrow x + 3y - 10 = 0.$$

Hence, the equation of the bisector of $\angle A$ is $x + 3y - 10 = 0$.

EXERCISE 20C

1. Find the equation of a line parallel to the x-axis at a distance of
   (i) 4 units above it (ii) 5 units below it.
2. Find the equation of a line parallel to the y-axis at a distance of
   (i) 6 units to its right (ii) 3 units to its left.
3. Find the equation of a line parallel to the x-axis and having intercept -3 on the y-axis.
4. Find the equation of a horizontal line passing through the point $(4, -2)$.
5. Find the equation of a vertical line passing through the point $(-5, 6)$.
6. Find the equation of a line which is equidistant from the lines $x = -2$ and $x = 6$.
7. Find the equation of a line which is equidistant from the lines $y = 8$ and $y = -2$.
8. Find the equation of a line
   (i) whose slope is 4 and which passes through the point $(5, -7)$;
   (ii) whose slope is -3 and which passes through the point $(-2, 3)$;
   (iii) which makes an angle of $\dfrac{2\pi}{3}$ with the positive direction of the x-axis and passes through the point $(0, 2)$.
9. Find the equation of a line whose inclination with the x-axis is $30°$ and which passes through the point $(0, 5)$.
10. Find the equation of a line whose inclination with the x-axis is $150°$ and which passes through the point $(3, -5)$.
11. Find the equation of a line passing through the origin and making an angle of $120°$ with the positive direction of the x-axis.

12. Find the equation of a line which cuts off intercept 5 on the x-axis and makes an angle of $60°$ with the positive direction of the x-axis.

13. Find the equation of the line passing through the point $P(4, -5)$ and parallel to the line joining the points $A(3, 7)$ and $B(-2, 4)$.

14. Find the equation of the line passing through the point $P(-3, 5)$ and perpendicular to the line passing through the points $A(2, 5)$ and $B(-3, 6)$.

15. Find the slope and the equation of the line passing through the points:

(i) $(3, -2)$ and $(-5, -7)$ (ii) $(-1, 1)$ and $(2, -4)$

(iii) $(5, 3)$ and $(-5, -3)$ (iv) (a, b) and $(-a, b)$

16. Find the angle which the line joining the points $(1, \sqrt{3})$ and $(\sqrt{2}, \sqrt{6})$ makes with the x-axis.

17. Prove that the points $A(1, 4)$, $B(3, -2)$ and $C(4, -5)$ are collinear. Also find the equation of the line on which these points lie.

18. If $A(0, 0)$, $B(2, 4)$ and $C(6, 4)$ are the vertices of a $\triangle ABC$, find the equations of its sides.

19. If $A(-1, 6)$, $B(-3, -9)$ and $C(5, -8)$ are the vertices of a $\triangle ABC$, find the equations of its medians.

20. Find the equation of the perpendicular bisector of the line segment whose end points are $A(10, 4)$ and $B(-4, 9)$.

21. Find the equations of the altitudes of a $\triangle ABC$, whose vertices are $A(2, -2)$, $B(1, 1)$ and $C(-1, 0)$.

22. If $A(4, 3)$, $B(0, 0)$ and $C(2, 3)$ are the vertices of a $\triangle ABC$, find the equation of the bisector of $\angle A$.

23. The midpoints of the sides BC, CA and AB of a $\triangle ABC$ are $D(2, 1)$, $E(-5, 7)$ and $F(-5, -5)$ respectively. Find the equations of the sides of $\triangle ABC$.

24. If $A(1, 4)$, $B(2, -3)$ and $C(-1, -2)$ are the vertices of a $\triangle ABC$, find the equation of

(i) the median through $\boldsymbol{A}$ (ii) the altitude through $\boldsymbol{A}$

(iii) the perpendicular bisector of $\boldsymbol{BC}$.

ANSWERS (EXERCISE 20C)

1. (i) $y - 4 = 0$ (ii) $y + 5 = 0$ **2.** (i) $x - 6 = 0$ (ii) $x + 3 = 0$

3. $y + 3 = 0$ **4.** $y + 2 = 0$ **5.** $x + 5 = 0$ **6.** $x = 2$

7. $y = 3$ **8.** (i) $4x - y - 27 = 0$ (ii) $3x + y + 3 = 0$ (iii) $\sqrt{3}x + y - 2 = 0$

9. $x - \sqrt{3}y + 5\sqrt{3} = 0$ **10.** $x + \sqrt{3}y + (-3 + 5\sqrt{3}) = 0$

11. $\sqrt{3}x + y = 0$ **12.** $\sqrt{3}x - y - 5\sqrt{3} = 0$ **13.** $3x - 5y - 37 = 0$ **14.** $5x - y + 20 = 0$

15. (i) $\frac{5}{8}$, $5x - 8y - 31 = 0$ (ii) $\frac{-5}{3}$, $5x + 3y + 2 = 0$ (iii) $\frac{3}{5}$, $3x - 5y = 0$ (iv) 0, $y = b$

16. $60°$ **17.** $3x + y - 7 = 0$ **18.** $y = 4$, $2x - 3y = 0$, $2x - y = 0$

19. $29x + 4y + 5 = 0,\ 8x - 5y - 21 = 0,\ 13x + 14y + 47 = 0$ **20.** $28x - 10y - 19 = 0$

21. $2x + y - 2 = 0,\ 3x - 2y - 1 = 0,\ x - 3y + 1 = 0$ **22.** $x - 3y + 5 = 0$

23. $x - 2 = 0,\ 6x - 7y + 79 = 0,\ 6x + 7y + 65 = 0$

24. (i) $13x - y - 9 = 0$ (ii) $3x - y + 1 = 0$ (iii) $3x - y - 4 = 0$

HINTS TO SOME SELECTED QUESTIONS

6. Midpoint of $A(-2, 0)$ and $B(6, 0)$ is $M\left(\frac{-2+6}{2}, 0\right)$, i.e., $M(2, 0)$.

So, the required line is $x = 2$.

7. Midpoint of $A(0, 8)$ and $B(0, -2)$ is $M\left(0, \frac{8-2}{2}\right)$, i.e., $M(0, 3)$.

So, the required line is $y = 3$.

12. The given line passes through the point (5, 0) and $m = \tan 60° = \sqrt{3}$. So, the required equation is $\frac{y-0}{x-5} = \sqrt{3} \Leftrightarrow y = \sqrt{3}(x-5)$.

13. Slope of the given line = slope of $AB = \frac{4-7}{-2-3} = \frac{3}{5} \cdot$ Thus, $m = \frac{3}{5} \cdot$

$\therefore$ required equation is $\frac{y+5}{x-4} = \frac{3}{5} \Leftrightarrow 3x - 5y - 37 = 0$.

14. Slope of $AB = m_1 = \frac{6-5}{-3-2} = \frac{-1}{5} \cdot$

Now, $m_1 m_2 = -1 \Leftrightarrow \frac{-1}{5} \times m_2 = -1 \Leftrightarrow m_2 = 5$.

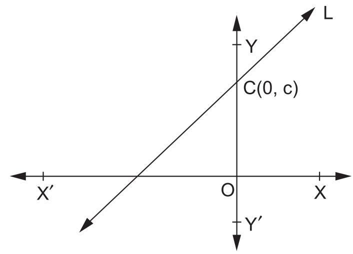
$\therefore$ required equation is $\frac{y-5}{x+3} = 5 \Leftrightarrow 5x - y + 20 = 0$.

16. $m = \tan\theta = \left(\frac{\sqrt{6}-\sqrt{3}}{\sqrt{2}-1}\right) = \sqrt{3} \Leftrightarrow \theta = 60°$.

EQUATION OF A LINE IN SLOPE–INTERCEPT FORM

THEOREM 1 *Prove that the equation of a line with slope m and making an intercept c on the y-axis, is given by $y = mx + c$.*

PROOF Let L be a given line with slope m, making an intercept c on the y-axis. Then, L cuts the y-axis at a point C such that $OC = c$.



Clearly, the coordinates of C are $(0, c)$.

Thus, the line L has slope m and passes through the point $C(0, c)$.

So, by the point–slope form, the equation of L is

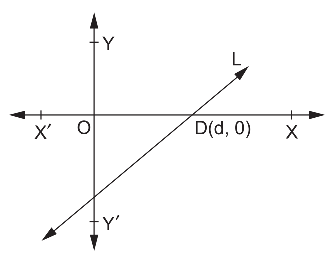
$$(y - c) = m(x - 0) \Rightarrow y = mx + c.$$

Hence, $y = mx + c$ is the required equation.

REMARK The equation of a line with slope m and y-intercept c is given by $y = mx + c$.

THEOREM 2 *Prove that the equation of a line with slope m and making an intercept d on the x-axis, is given by $y = m(x - d)$.*

PROOF Let L be a given line with slope m, making an intercept d on the x-axis. Then, L cuts the x-axis at a point D such that $OD = d$.



Clearly, the coordinates of D are $(d, 0)$.

Thus, the line L has slope m and passes through the point $D(d, 0)$.

So, by the point–slope form, the equation of L is

$$\frac{(y - 0)}{(x - d)} = m \Leftrightarrow y = m(x - d).$$

Hence, $y = m(x - d)$ is the required equation.

REMARK The equation of a line with slope m and x-intercept d is given by $y = m(x - d)$.

SUMMARY

(i) The equation of a line with slope m and y-intercept c is given by

$$y = mx + c.$$

(ii) The equation of a line with slope m and x-intercept d is given by

$$y = m(x - d).$$

SOLVED EXAMPLES

EXAMPLE 1 *Find the equation of a line whose slope is* $\frac{1}{2}$ *and y-intercept equal to* $\frac{-5}{4}$.

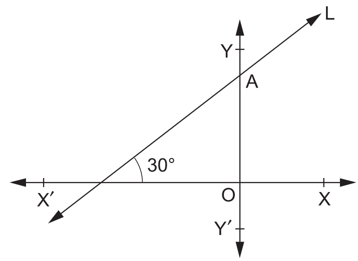
SOLUTION We know that the equation of a line with slope m and y-intercept c is given by $y = mx + c$.

Here, $m = \frac{1}{2}$ and $c = \frac{-5}{4}$.

Hence, the required equation of the line is

$$y = \frac{1}{2}x - \frac{5}{4} \Rightarrow 4y = 2x - 5$$
$$\Rightarrow 2x - 4y - 5 = 0.$$

EXAMPLE 2 *Find the equation of a line which intersects the y-axis at a distance of 3 units above the origin and makes an angle of 30° with the positive direction of the x-axis.*



SOLUTION Here, $m = \tan 30° = \frac{1}{\sqrt{3}}$ and $c = 3$.

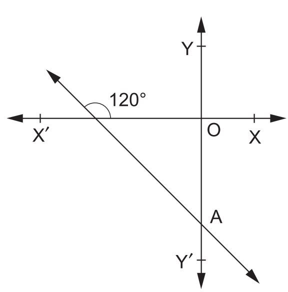
Hence, the required equation of the line is

$$y = \frac{1}{\sqrt{3}}x + 3 \Rightarrow \sqrt{3}y = x + 3\sqrt{3} \qquad [\because y = mx + c]$$
$$\Rightarrow x - \sqrt{3}y + 3\sqrt{3} = 0.$$

EXAMPLE 3 *Find the equation of a line which cuts off an intercept of 4 units on negative direction of the y-axis and makes an angle of 120° with the positive direction of the x-axis.*

SOLUTION We know that the equation of a line with slope m and y-intercept c is given by

$$y = mx + c.$$



Here, $m = \tan 120° = \tan (180° - 60°) = -\tan 60° = -\sqrt{3}$ and $c = -4$.

Hence, the required equation of the line is

$$y = -\sqrt{3}x - 4 \Rightarrow \sqrt{3}x + y + 4 = 0 \qquad [\because y = mx + c].$$

EXAMPLE 4 *Find the equation of a line for which* $\tan\theta = \frac{1}{3}$ *and x-intercept equal to* 5 *units.*

SOLUTION We know that the equation of a line with slope m and x-intercept d is given by $y = m(x - d)$.

Here, $m = \tan\theta = \frac{1}{3}$ and $d = 5$.

Hence, the required equation of the line is

$$y = \frac{1}{3}(x - 5) \Rightarrow 3y = x - 5$$
$$\Rightarrow x - 3y - 5 = 0.$$

EXAMPLE 5 *Find the equation of a line which cuts the x-axis at a distance of* 3 *units to the left of the origin and has a slope equal to* -2.

SOLUTION We know that the equation of a line with slope m and x-intercept d is given by $y = m(x - d)$.

Here, $m = -2$ and $d = -3$.

Hence, the required equation of the line is

$$y = -2(x + 3) \Rightarrow 2x + y + 6 = 0.$$

EXAMPLE 6 *Reduce the equation* $6x + 3y - 5 = 0$ *to the slope–intercept form and find its slope and y-intercept.*

SOLUTION We know that the equation of a line with slope m and y-intercept c is given by $y = mx + c$.

Now, $6x + 3y - 5 = 0 \Rightarrow 3y = -6x + 5$

$$\Rightarrow y = -2x + \frac{5}{3}.$$

$\therefore \quad m = -2$ and $c = \frac{5}{3}$.

Hence, slope $= -2$ and y-intercept $= \frac{5}{3}$.

EXAMPLE 7 *Prove that the lines* $x + 2y - 9 = 0$ *and* $2x + 4y + 5 = 0$ *are parallel.*

SOLUTION Let the slopes of the given lines be m_1 and m_2 respectively.

Then, $x + 2y - 9 = 0 \Rightarrow 2y = -x + 9$

$$\Rightarrow y = -\frac{1}{2}x + \frac{9}{2}.$$

And, $2x + 4y + 5 = 0 \Rightarrow 4y = -2x - 5$

$$\Rightarrow y = -\frac{1}{2}x - \frac{5}{4}.$$

$\therefore \quad m_1 = -\frac{1}{2}$ and $m_2 = -\frac{1}{2}$.

Thus, $m_1 = m_2$.

Hence, the given lines are parallel.

EXAMPLE 8 *Show that the lines* $27x - 18y + 25 = 0$ *and* $2x + 3y + 7 = 0$ *are perpendicular to each other.*

SOLUTION Let the slopes of the given lines be m_1 and m_2 respectively.

Then, $27x - 18y + 25 = 0 \Rightarrow 18y = 27x + 25$

$$\Rightarrow y = \frac{3}{2}x + \frac{25}{18}$$

And, $2x + 3y + 7 = 0 \Rightarrow 3y = -2x - 7$

$$\Rightarrow y = \frac{-2}{3}x - \frac{7}{3}.$$

$\therefore \quad m_1 = \frac{3}{2}$ and $m_2 = \frac{-2}{3}$

$$\Rightarrow \quad m_1 m_2 = \left(\frac{3}{2}\right) \times \left(\frac{-2}{3}\right) = -1$$

Hence, the given lines are perpendicular to each other.

EXAMPLE 9 *Find the angle made by the line* $x + \sqrt{3}y - 6 = 0$ *with the positive direction of the x-axis.*

SOLUTION Let the required angle be θ.

Now, $x + \sqrt{3}y - 6 = 0 \Rightarrow y = -\frac{1}{\sqrt{3}}x + \frac{6}{\sqrt{3}}$.

$$\therefore \quad m = -\frac{1}{\sqrt{3}} \Leftrightarrow \tan\theta = -\frac{1}{\sqrt{3}} = -\tan 30° = \tan(180° - 30°) = \tan 150°.$$

$\therefore \quad \theta = 150°$.

Hence, the given line makes an angle of 150° with the positive direction of the x-axis.

EXAMPLE 10 *Find the angle made by the line* $x\cos 30° + y\sin 30° + \sin 120° = 0$ *with the positive direction of the x-axis.*

SOLUTION Let the required angle be θ.

Now, $x\cos 30° + y\sin 30° + \sin 120° = 0$

$$\Rightarrow \quad \left(\frac{\sqrt{3}}{2}\right)x + \left(\frac{1}{2}\right)y + \frac{\sqrt{3}}{2} = 0$$

$$\Rightarrow \quad \sqrt{3}x + y + \sqrt{3} = 0$$

$$\Rightarrow \quad y = (-\sqrt{3})x + (-\sqrt{3})$$

$\Rightarrow \quad y = mx + c$, where $m = -\sqrt{3}$ and $c = -\sqrt{3}$.

Now, $m = -\sqrt{3}$

$$\Leftrightarrow \quad \tan\theta = -\sqrt{3} = -\tan 60° = \tan(180° - 60°) = \tan 120°.$$

$\therefore \quad \theta = 120°$.

Hence, the given line makes an angle of 120° with the positive direction of the x-axis.

EXAMPLE 11 *Find the angles between the lines* $\sqrt{3}\,x + y = 1$ *and* $x + \sqrt{3}\,y = 1$.

SOLUTION Let the slopes of the given lines be m_1 and m_2 respectively.

Now, $\sqrt{3}x + y = 1 \Rightarrow y = -\sqrt{3}x + 1$

and $x + \sqrt{3}y = 1 \Rightarrow y = -\dfrac{1}{\sqrt{3}}x + \dfrac{1}{\sqrt{3}}$.

$\therefore \quad m_1 = -\sqrt{3}$ and $m_2 = \dfrac{-1}{\sqrt{3}}$.

Let θ be the angle between the given lines. Then,

$$\tan\theta = \left|\frac{m_2 - m_1}{1 + m_1 m_2}\right|$$

$$= \left|\frac{\dfrac{-1}{\sqrt{3}} + \sqrt{3}}{\left\{1 + (-\sqrt{3}) \times \left(\dfrac{-1}{\sqrt{3}}\right)\right\}}\right| = \left|\frac{2}{\sqrt{3}} \times \frac{1}{2}\right| = \left|\frac{1}{\sqrt{3}}\right| = \frac{1}{\sqrt{3}}$$

$\Rightarrow \quad \theta = 30°$ and $(180° - \theta) = (180° - 30°) = 150°$.

Hence, the angles between the given lines are 30° and 150°.

EXAMPLE 12 *Show that the lines* $a_1x + b_1y + c_1 = 0$ *and* $a_2x + b_2y + c_2 = 0$, *where* $b_1, b_2 \neq 0$ *are (i) parallel, if* $\dfrac{a_1}{b_1} = \dfrac{a_2}{b_2}$ *(ii) perpendicular, if* $a_1a_2 + b_1b_2 = 0$.

SOLUTION Let the slopes of the given lines be m_1 and m_2 respectively.

Now, $a_1x + b_1y + c_1 = 0 \Rightarrow y = \dfrac{-a_1}{b_1} \cdot x - \dfrac{c_1}{b_1}$

and, $a_2x + b_2y + c_2 = 0 \Rightarrow y = \dfrac{-a_2}{b_2} \cdot x - \dfrac{c_2}{b_2}$.

$\therefore \quad m_1 = \dfrac{-a_1}{b_1}$ and $m_2 = \dfrac{-a_2}{b_2}$.

(i) Given lines are parallel, if $m_1 = m_2$ which gives

$$\frac{-a_1}{b_1} = \frac{-a_2}{b_2} \quad \text{or} \quad \frac{a_1}{a_2} = \frac{b_1}{b_2}.$$

(ii) Given lines are perpendicular, if $m_1m_2 = -1$ which gives

$$\left(\frac{-a_1}{b_1}\right) \cdot \left(\frac{-a_2}{b_2}\right) = -1 \quad \text{or} \quad a_1a_2 + b_1b_2 = 0.$$

EXAMPLE 13 *Find the equation of the line passing through the point* $(2, -5)$ *and parallel to the line* $2x - 3y = 7$.

SOLUTION We have

$$2x - 3y = 7 \Rightarrow y = \frac{2}{3}x - \frac{7}{3}.$$

$\therefore$ slope of the given line $= \frac{2}{3}$

and slope of a line parallel to the given line $= \frac{2}{3}$.

Now, the equation of a line with slope $\frac{2}{3}$ and passing through the point $(2, -5)$ is given by

$$\frac{y-(-5)}{x-2} = \frac{2}{3} \Rightarrow \frac{y+5}{x-2} = \frac{2}{3} \qquad \left[\because \frac{(y-y_1)}{(x-x_1)} = m\right]$$

$$\Rightarrow 3(y+5) = 2(x-2)$$

$\Rightarrow 2x - 3y - 19 = 0$, which is the required equation.

EXAMPLE 14 *Find the equation of the line passing through the point $(-2, -4)$ and perpendicular to the line $3x - y + 5 = 0$.*

SOLUTION We have

$$3x - y + 5 = 0 \Rightarrow y = 3x + 5$$

$\therefore$ slope of the given line $= 3$ $\qquad [\because y = mx + c]$

and slope of a line perpendicular to the given line $= -\frac{1}{3}$

$$\left[\because m_2 = -\frac{1}{m_1}\right].$$

Now, the equation of a line with slope $-\frac{1}{3}$ and passing through the point $(-2, -4)$ is given by

$$\frac{y-(-4)}{x-(-2)} = -\frac{1}{3} \Rightarrow \frac{y+4}{x+2} = \frac{-1}{3} \qquad \left[\because \frac{(y-y_1)}{(x-x_1)} = m\right]$$

$$\Rightarrow 3(y+4) = -(x+2)$$

$\Rightarrow x + 3y + 14 = 0$, which is the required equation.

EXAMPLE 15 *Find the equation of the line whose y-intercept is -3 and which is perpendicular to the line $3x - 2y + 5 = 0$.*

SOLUTION $3x - 2y + 5 = 0 \Rightarrow y = \frac{3}{2}x + \frac{5}{2}$

$\therefore$ slope of the given line $= \frac{3}{2}$.

$\therefore$ slope of a line perpendicular to the given line $= \frac{-2}{3}$ $\left[\because m_2 = \frac{-1}{m_1}\right]$.

Now, the equation of a line with slope $\frac{-2}{3}$ and y-intercept -3 is given by

$$y = \frac{-2}{3}x - 3 \qquad [\because\ y = mx + c]$$

$\Rightarrow\quad 2x + 3y + 9 = 0$, which is the required equation.

EXAMPLE 16 *Find the equation of the line perpendicular to the line $x - 7y + 5 = 0$ and having x-intercept 3.*

SOLUTION $x - 7y + 5 = 0 \;\Rightarrow\; y = \frac{1}{7}x + \frac{5}{7}\cdot$

$\therefore\quad$ slope of the given line $= \frac{1}{7}\cdot$

$\therefore\quad$ slope of a line perpendicular to the given line $= -7 \left[\because\ m_2 = \frac{-1}{m_1}\right]\cdot$

Thus, $m = -7$ and $d = 3$, where $d = x$-intercept.

Now, the equation of a line with slope -7 and x-intercept 3 is given by

$$y = -7(x - 3) \qquad [\because\ y = m(x - d)]$$

$\Rightarrow\quad 7x + y - 21 = 0$, which is the required equation.

EXAMPLE 17 *Find the equations of the lines through the point* (3, 2), *which make an angle of* 45° *with the line $x - 2y = 3$.*

SOLUTION Let the slope of the required line be m.

Then, its equation is $\frac{y-2}{x-3} = m$. ... (i)

Given line is $x - 2y = 3 \;\Rightarrow\; y = \frac{1}{2}x - \frac{3}{2}\cdot$ ... (ii)

Clearly, the slope of this line is $\frac{1}{2}\cdot$

It is given that the angle between (i) and (ii) is 45°.

$$\therefore\quad \left|\frac{m - \frac{1}{2}}{1 + \frac{1}{2}m}\right| = \tan 45^\circ \qquad \left[\text{using } \tan\theta = \left|\frac{m_2 - m_1}{1 + m_1 m_2}\right|\right]$$

$$\Leftrightarrow\quad \frac{2m-1}{2+m} = 1 \quad\text{or}\quad \frac{2m-1}{2+m} = -1 \;\Leftrightarrow\; m = 3 \quad\text{or}\quad m = \frac{-1}{3}\cdot$$

$\therefore\quad$ the required equation is $\frac{y-2}{x-3} = 3$ or $\frac{y-2}{x-3} = \frac{-1}{3}$

i.e., $3x - y - 7 = 0$ or $x + 3y - 9 = 0$.

EXERCISE 20D

1. Find the equation of the line whose
   (i) slope = 3 and y-intercept = 5
   (ii) slope = −1 and y-intercept = 4
   (iii) slope = $-\frac{2}{5}$ and y-intercept = −3
2. Find the equation of the line which makes an angle of 30° with the positive direction of the x-axis and cuts off an intercept of 4 units with the negative direction of the y-axis.
3. Find the equation of the line whose inclination is $\frac{5\pi}{6}$ and which makes an intercept of 6 units on the negative direction of the y-axis.
4. Find the equation of the line cutting off an intercept −2 from the y-axis and equally inclined to the axes.
5. Find the equation of the bisectors of the angles between the coordinate axes.
6. Find the equation of the line through the point (−1, 5) and making an intercept of −2 on the y-axis.
7. Find the equation of the line which is parallel to the line $2x - 3y = 8$ and whose y-intercept is 5 units.
8. Find the equation of the line passing through the point (0, 3) and perpendicular to the line $x - 2y + 5 = 0$.
9. Find the equation of the line passing through the point (2, 3) and perpendicular to the line $4x + 3y = 10$.
10. Find the equation of the line passing through the point (2, 4) and perpendicular to the x-axis.
11. Find the equation of the line that has x-intercept −3 and which is perpendicular to the line $3x + 5y = 4$.
12. Find the equation of the line which is perpendicular to the line $3x + 2y = 8$ and passes through the midpoint of the line joining the points (6, 4) and (4, −2).
13. Find the equation of the line whose y-intercept is −3 and which is perpendicular to the line joining the points (−2, 3) and (4, −5).
14. Find the equation of the line passing through (−3, 5) and perpendicular to the line through the points (2, 5) and (−3, 6).
15. A line perpendicular to the line segment joining the points (1, 0) and (2, 3) divides it in the ratio 1 : 2. Find the equation of the line.

ANSWERS (EXERCISE 20D)

1. (i) $3x - y + 5 = 0$ (ii) $x + y - 4 = 0$ (iii) $2x + 5y + 15 = 0$

2. $x - \sqrt{3}y - 4\sqrt{3} = 0$ **3.** $x + \sqrt{3}y + 6\sqrt{3} = 0$

4. $x - y - 2 = 0$ or $x + y + 2 = 0$ **5.** $x - y = 0$ or $x + y = 0$

6. $7x+y+2=0$ **7.** $2x-3y+15=0$ **8.** $2x+y-3=0$ **9.** $3x-4y+6=0$
10. $x=2$ **11.** $5x-3y+15=0$ **12.** $2x-3y-7=0$ **13.** $3x-4y-12=0$
14. $5x-y+20=0$ **15.** $3x+9y-13=0$

HINTS TO SOME SELECTED QUESTIONS

2. $m=\tan 30°=\dfrac{1}{\sqrt{3}}$ and $c=-4$.

3. $m=\tan\dfrac{5\pi}{6}=\tan\left(\pi-\dfrac{\pi}{6}\right)=-\tan\dfrac{\pi}{6}=\dfrac{-1}{\sqrt{3}}\cdot$ And, $c=-6$.

4. $m=\tan 45°=1$ or $m=\tan 135°=\tan(180°-45°)=-\tan 45°=-1$. Also, $c=-2$.

5. $m=\tan 45°=1$ or $m=\tan 135°=-1$. And, $c=0$.

6. Let the required equation be $y=mx-2$.
Since it passes through $(-1,5)$, we have $m\times(-1)-2=5 \Rightarrow m=-7$.

7. Given line is $y=\dfrac{2}{3}x-\dfrac{8}{3}\cdot$ Its slope $=\dfrac{2}{3}\cdot$

$\therefore$ slope of the required line $=\dfrac{2}{3}$ and $c=5$.

8. Given line is $y=\dfrac{1}{2}x+\dfrac{5}{2}\cdot$ Its slope is $\dfrac{1}{2}\cdot$

$$m\times\frac{1}{2}=-1 \Rightarrow m=-2.$$

Let $y=-2x+c$ be the required line.
Since it passes through (0, 3), we have $3=(-2)\times 0+c \Rightarrow c=3$.
$\therefore$ $m=-2$ and $c=3$.

10. Required line is parallel to the y-axis. So, its equation is $x=2$.

11. Required line passes through $(-3,0)$.

12. Required line passes through the point $\left(\dfrac{5+3}{2},0\right)$, i.e., (4, 0).

13. Let $A(-2,3)$ and $B(4,-5)$ be the given points.

Slope of $AB=\dfrac{-5-3}{4+2}=\dfrac{-4}{3}\cdot$ So, $m\times\left(\dfrac{-4}{3}\right)=-1 \Rightarrow m=\dfrac{3}{4}\cdot$

$\therefore$ $y=\dfrac{3}{4}x-3$ $[\because c=-3]$.

14. Given points are $A(2,5)$ and $B(-3,6)$.

Slope of $AB=\dfrac{6-5}{-3-2}=\dfrac{-1}{5}\cdot$ So, $m\times\left(\dfrac{-1}{5}\right)=-1 \Rightarrow m=5$.

$\therefore$ required equation is $\dfrac{y-5}{x+3}=5 \Leftrightarrow 5x-y+20=0$.

15. Let M divide the join of $A(1, 0)$ and $B(2, 3)$ in the ratio 1 : 2.

Then, the point M is $\left(\frac{1\times 2+2\times 1}{1+2}, \frac{1\times 3+2\times 0}{1+2}\right)$, i.e., $M\left(\frac{4}{3}, 1\right)$.

Slope of $AB = \frac{3-0}{2-1} = 3$.

Let L be the required line. Then, slope of $L = -\frac{1}{3}$ and it passes through $M\left(\frac{4}{3,} 1\right)$.

Let L be the required line. Then, $L \perp AB$.

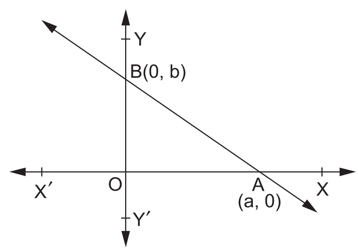
$\therefore$ slope of $L = -\frac{1}{3}$ and it passes through $M\left(\frac{4}{3}, 1\right)$.

EQUATION OF A LINE IN INTERCEPTS FORM

THEOREM 1 *Prove that the equation of a line making intercepts a and b on the x-axis and the y-axis respectively, is*

$$\frac{x}{a}+\frac{y}{b}=1.$$

PROOF Let $X'OX$ and YOY' be the coordinate axes.



Let AB be the line cutting the x-axis and the y-axis at A and B respectively such that $OA = a$ and $OB = b$.

Then, these points are $A(a, 0)$ and $B(0, b)$.

So, the equation of line AB is given by

$$\frac{y-0}{x-a} = \frac{b-0}{0-a} \qquad \text{[two-point form]}$$

$$\Leftrightarrow \frac{y}{x-a} = \frac{b}{-a} \Leftrightarrow b(x-a) = -ay$$

$$\Leftrightarrow bx + ay = ab$$

$$\Leftrightarrow \frac{x}{a}+\frac{y}{b}=1 \qquad \text{[on dividing both sides by } ab].$$

Thus, the line making intercepts a and b on the x- and y-axes respectively is $\frac{x}{a}+\frac{y}{b}=1$.

SOLVED EXAMPLES

EXAMPLE 1 *Find the equation of the line which makes intercepts* 2 *and* –3 *on the x-axis and the y-axis respectively.*

SOLUTION We know that the equation of a line making intercepts a and b on the x-axis and y-axis respectively, is $\frac{x}{a}+\frac{y}{b}=1$.

Here $a = 2$ and $b = -3$.

Hence, the required equation is

$$\frac{x}{2} + \frac{y}{-3} = 1 \iff \frac{x}{2} - \frac{y}{3} = 1 \iff 3x - 2y - 6 = 0.$$

EXAMPLE 2 *Find the equation of the line which passes through the point* (3, 4) *and the sum of whose intercepts on the axes is* 14.

SOLUTION Let the intercepts made by the line on the x-axis and y-axis be a and $(14 - a)$ respectively.

Then, its equation is $\dfrac{x}{a} + \dfrac{y}{(14-a)} = 1$ [intercept form].

Since it passes through the point (3, 4), we have

$$\frac{3}{a} + \frac{4}{(14-a)} = 1 \iff 3(14-a) + 4a = a(14-a)$$

$$\iff a^2 - 13a + 42 = 0 \iff (a-6)(a-7) = 0$$

$$\iff a = 6 \quad \text{or} \quad a = 7.$$

Now, $a = b \iff b = 14 - 6 = 8.$

And, $a = 7 \iff b = 14 - 7 = 7.$

So, the required equation is

$$\frac{x}{6} + \frac{y}{8} = 1 \quad \text{or} \quad \frac{x}{7} + \frac{y}{7} = 1$$

i.e., $4x + 3y - 24 = 0$ or $x + y - 7 = 0.$

EXAMPLE 3 *Find the equations of the lines which cut off intercepts on the axes whose sum and product are 1 and –6 respectively.*

SOLUTION Let the required equation be $\dfrac{x}{a} + \dfrac{y}{b} = 1.$

Then x-intercept $= a$ and y-intercept $= b$.

$\therefore \quad a + b = 1$ ... (i)

and $ab = -6$ ... (ii)

Putting $b = \dfrac{-6}{a}$ from (ii) in (i), we get

$$a - \frac{6}{a} = 1 \iff a^2 - a - 6 = 0 \iff (a-3)(a+2) = 0$$

$$\iff a = 3 \quad \text{or} \quad a = -2.$$

Now, $a = 3 \iff b = (1-a) = (1-3) = -2.$

And, $a = -2 \iff b = (1-a) = (1+2) = 3.$

$\therefore$ the required equation is

$$\frac{x}{3} + \frac{y}{-2} = 1 \quad \text{or} \quad \frac{x}{-2} + \frac{y}{3} = 1$$

i.e., $2x - 3y - 6 = 0$ or $3x - 2y + 6 = 0.$

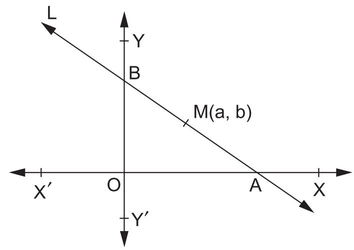
EXAMPLE 4 *If $M(a, b)$ is the midpoint of a line segment intercepted between the axes, show that the equation of the line is* $\dfrac{x}{a} + \dfrac{y}{b} = 2.$

SOLUTION Let the required equation of the line be

$$\frac{x}{c} + \frac{y}{d} = 1. \qquad \text{... (i)}$$

Then, it cuts the x-axis and y-axis at the points $A(c, 0)$ and $B(0, d)$.

Let $M(a, b)$ be the midpoint of AB.



Then, $\dfrac{c+0}{2} = a$ and $\dfrac{0+d}{2} = b$

$\Leftrightarrow$ $c = 2a$ and $d = 2b$.

So, the required equation is

$$\frac{x}{2a} + \frac{y}{2b} = 1 \;\Leftrightarrow\; \frac{x}{a} + \frac{y}{b} = 2.$$

Hence, the required equation is $\dfrac{x}{a} + \dfrac{y}{b} = 2$.

EXAMPLE 5 *Find the equation of a line which passes through the point* $(-3, 7)$ *and makes intercepts on the axes, equal in magnitude but opposite in sign.*

SOLUTION Let the required line make intercepts a and $-a$ on the x-axis and y-axis respectively.

Then, its equation is $\dfrac{x}{a} + \dfrac{y}{-a} = 1 \;\Leftrightarrow\; x - y = a.$ ... (i)

Since this line passes through the point $(-3, 7)$, we have

$$a = (-3 - 7) = -10.$$

So, the required equation of the line is

$$\frac{x}{-10} + \frac{y}{10} = 1 \;\Leftrightarrow\; -x + y = 10 \;\Leftrightarrow\; x - y + 10 = 0.$$

Hence, the required equation is $x - y + 10 = 0$.

EXAMPLE 6 *Find the intercepts cut off by the line* $2x - y + 16 = 0$ *on the coordinate axes.*

SOLUTION We have

$$2x - y + 16 = 0 \;\Leftrightarrow\; 2x - y = -16 \;\Leftrightarrow\; \frac{x}{-8} + \frac{y}{16} = 1$$

[on dividing both sides by -16]

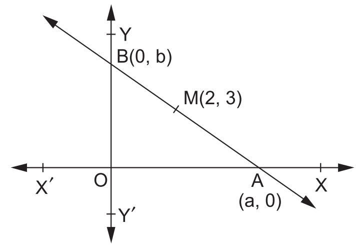
$\therefore$ the x-intercept $= -8$ and the y-intercept $= 16$.

EXAMPLE 7 *Find the equation of the line so that the line segment intercepted between the axes is bisected at the point* $(2, 3)$.

SOLUTION Let AB be the given line segment making intercepts a and b on the x-axis and y-axis respectively.

Then, the equation of line AB is $\dfrac{x}{a} + \dfrac{y}{b} = 1$.

Moreover, these points are $A(a, 0)$ and $B(0, b)$. Let $M(2, 3)$ be the midpoint of AB. Then,



$$\frac{a+0}{2}=2 \quad \text{and} \quad \frac{0+b}{2}=3$$

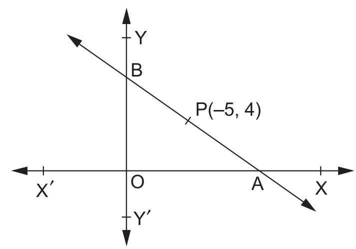
$\Rightarrow \quad a = 4$ and $b = 6$.

Hence, the required equation of the given line is

$$\frac{x}{4}+\frac{y}{6}=1 \Leftrightarrow 3x+2y-12=0.$$

Hence, the required equation is $3x + 2y - 12 = 0$.

EXAMPLE 8 *Find the equation of the line so that the line segment intercepted between the axes is divided by the point $P(-5, 4)$ in the ratio $1:2$.*



SOLUTION Let AB be the given line segment making intercepts a and b on the x-axis and y-axis respectively.

Then, the equation of line AB is

$$\frac{x}{a}+\frac{y}{b}=1.$$

So, these points are $A(a, 0)$ and $B(0, b)$.

Now, $P(-5, 4)$ divides AB in the ratio $1:2$.

$$\therefore \quad \frac{(1\times 0+2\times a)}{(1+2)}=-5 \text{ and } \frac{(1\times b+2\times 0)}{(1+2)}=4 \Leftrightarrow a=\frac{-15}{2} \text{ and } b=12.$$

So, the required equation of the line is

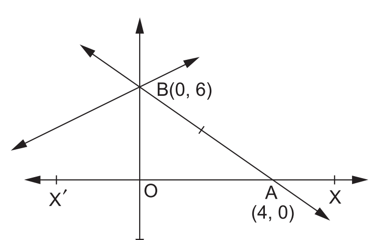
$$\frac{x}{\left(\frac{-15}{2}\right)}+\frac{y}{12}=1 \Leftrightarrow \frac{-2x}{15}+\frac{y}{12}=1$$

$$\Leftrightarrow 8x-5y+60=0.$$

Hence, the required equation of the line is $8x - 5y + 60 = 0$.

EXAMPLE 9 *Find the equation of a line drawn perpendicular to the line $\frac{x}{4}+\frac{y}{6}=1$ through the point, where it meets the y-axis.*

SOLUTION The given line is $\frac{x}{4}+\frac{y}{6}=1$.



It meets the y-axis at $B(0, 6)$.

Also, $\frac{x}{4}+\frac{y}{6}=1 \Leftrightarrow 3x+2y=12$

$$\Leftrightarrow y=\frac{-3}{2}x+6.$$

Slope of the given line is $\frac{-3}{2}$.

Let the slope of the line perpendicular to the given line be m. Then,

$$m \times \left(-\frac{3}{2}\right) = -1 \Rightarrow m = \frac{2}{3} \qquad [\because\ m_1 m_2 = -1].$$

Now, the required line has slope $\frac{2}{3}$ and passes through the point $B(0, 6)$. So, its equation is

$$\frac{y-6}{x-0} = \frac{2}{3} \Leftrightarrow 2x = 3y - 18 \Leftrightarrow 2x - 3y + 18 = 0.$$

Hence, the required equation is $2x - 3y + 18 = 0$.

EXAMPLE 10 *Find the equation of the line passing through the point of intersection of the lines $4x + 7y - 3 = 0$ and $2x - 3y + 1 = 0$, which has equal intercepts on the axes.*

AN IMPORTANT NOTE The point of intersection of two lines is obtained by solving their equations simultaneously.

SOLUTION Let the equation of the required line be $\frac{x}{a} + \frac{y}{a} = 1$,

i.e., $x + y = a$ ... (i)

The equations of the given lines are:

$$4x + 7y = 3 \quad \text{... (ii)}$$

$$2x - 3y = -1 \quad \text{... (iii)}$$

On solving (ii) and (iii), we get $x = \frac{1}{13}$, $y = \frac{5}{13}$.

So, the point of intersection of the given lines is $P\left(\frac{1}{13}, \frac{5}{13}\right)$.

Now, line (ii) passes through the point $P\left(\frac{1}{13}, \frac{5}{13}\right)$.

Putting $x = \frac{1}{13}$ and $y = \frac{5}{13}$ in (i), we get

$$a = \left(\frac{1}{13} + \frac{5}{13}\right) = \frac{6}{13}.$$

Hence, the required equation is $x + y = \frac{6}{13}$,

i.e., $13x + 13y - 6 = 0$.

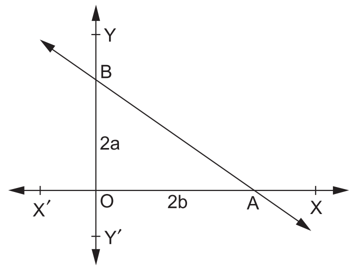
EXAMPLE 11 *Find the area of the triangle formed by the coordinate axes and the line $ax + by = 2ab$.*

SOLUTION Let $X'OX$ and YOY' be the coordinate axes.

Now, $ax + by = 2ab \Rightarrow \frac{x}{2b} + \frac{y}{2a} = 1$.

Suppose that this line meets the x-axis and y-axis at A and B respectively.

Then, $OA = 2b$ and $OB = 2a$.

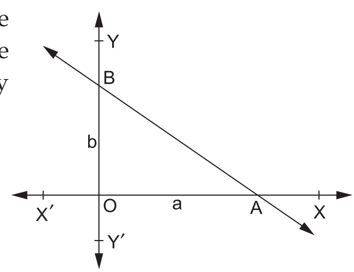


$\therefore$ area of $\triangle BOA = \left(\frac{1}{2} \times OA \times OB\right)$ sq units

$= \left(\frac{1}{2} \times 2b \times 2a\right) = 2ab$ sq units.

EXAMPLE 12 *The area of the triangle formed by the coordinate axes and a line is* 6 *square units and the length of its hypotenuse is* 5 *units. Find the equation of the line.*

SOLUTION Let $X'OX$ and YOY' be the coordinate axes, and let AB be the part of the given line intercepted by the axes.



Let $OA = a$ and $OB = b$.

Then, $\frac{1}{2}ab = 6 \Leftrightarrow ab = 12$ ... (i)

and, $a^2 + b^2 = 25$ ... (ii)

Putting $b = \frac{12}{a}$ from (i) in (ii), we get

$$a^2 + \frac{144}{a^2} = 25 \Rightarrow a^4 - 25a^2 + 144 = 0$$

$$\Rightarrow (a^2 - 16)(a^2 - 9) = 0 \Rightarrow a^2 = 16 \quad \text{or} \quad a^2 = 9$$

$$\Rightarrow a = 4 \quad \text{or} \quad a = 3 \quad [\because \text{ length is always positive}].$$

Now, $(a = 4 \Rightarrow b = 3)$ and $(a = 3 \Rightarrow b = 4)$.

Hence, the required equation of the line is

$$\frac{x}{4} + \frac{y}{3} = 1 \quad \text{or} \quad \frac{x}{3} + \frac{y}{4} = 1$$

i.e., $3x + 4y - 12 = 0$ or $4x + 3y - 12 = 0$.

EXERCISE 20E

1. Find the equation of the line which cuts off intercepts -3 and 5 on the x-axis and y-axis respectively.
2. Find the equation of the line which cuts off intercepts 4 and -6 on the x-axis and y-axis respectively.

3. Find the equation of the line that cuts off equal intercepts on the coordinate axes and passes through the point (4, 7).

4. Find the equation of the line which passes through the point $(3, -5)$ and cuts off intercepts on the axes which are equal in magnitude but opposite in sign.

5. Find the equation of the line passing through the point (2, 2) and cutting off intercepts on the axes, whose sum is 9.

6. Find the equation of the line which passes through the point $(22, -6)$ and whose intercept on the x-axis exceeds the intercept on the y-axis by 5.

7. Find the equation of the line whose portion intercepted between the axes is bisected at the point $(3, -2)$.

8. Find the equation of the line whose portion intercepted between the coordinate axes is divided at the point (5, 6) in the ratio 3 : 1.

9. A straight line passes through the point $(-5, 2)$ and the portion of the line intercepted between the axes is divided at this point in the ratio 2 : 3. Find the equation of the line.

10. If the straight line $\frac{x}{a} + \frac{y}{b} = 1$ passes through the points $(8, -9)$ and $(12, -15)$, find the values of a and b.

ANSWERS (EXERCISE 20E)

1. $5x - 3y + 15 = 0$ **2.** $3x - 2y - 12 = 0$ **3.** $x + y - 11 = 0$ **4.** $x - y - 8 = 0$
5. $x + 2y - 6 = 0$ or $2x + y - 6 = 0$ **6.** $6x + 11y - 66 = 0$ or $x + 2y - 10 = 0$
7. $2x - 3y - 12 = 0$ **8.** $2x + 5y - 40 = 0$ **9.** $3x - 5y + 25 = 0$ **10.** $a = 2$, $b = 3$

HINTS TO SOME SELECTED QUESTIONS

6. Let the required equation be $\frac{x}{a+5} + \frac{y}{a} = 1$.

Since it passes through $(22, -6)$, we have $\frac{22}{a+b} - \frac{6}{a} = 1$.

This gives $a^2 - 11a + 30 = 0 \Leftrightarrow (a-6)(a-5) = 0 \Leftrightarrow a = 6$ or $a = 5$.

So, the required equations are $\frac{x}{11} + \frac{y}{6} = 1$ or $\frac{x}{10} + \frac{y}{5} = 1$.

7. Let the required equation be $\frac{x}{a} + \frac{y}{b} = 1$.

Then, it cuts the axes at $A(a, 0)$ and $B(0, b)$.

$\therefore \left(\frac{a+0}{2} = 3 \text{ and } \frac{0+b}{2} = -2\right) \Leftrightarrow a = 6, b = -4.$

8. Let the required equation be $\frac{x}{a} + \frac{y}{b} = 1$.

Then, it cuts the axes at $A(a, 0)$ and $B(0, b)$.

Let $P(5, 6)$ divide AB in the ratio 3 : 1.

Then, $\dfrac{3\times 0 + 1\times a}{3+1} = 5$ and $\dfrac{3\times b + 1\times 0}{3+1} = 6 \Leftrightarrow a = 20$ and $b = 8$.

10. Since $\dfrac{x}{a} + \dfrac{y}{b} = 1$ passes through the points $A(8, -9)$ and $B(12, -15)$,

we have $\dfrac{8}{a} - \dfrac{9}{b} = 1$ and $\dfrac{12}{a} - \dfrac{15}{b} = 1$.

On solving, we get $\dfrac{3}{b} = 1$ and $\dfrac{8}{a} = 4$. This gives $a = 2$, $b = 3$.

EQUATION OF A LINE IN NORMAL FORM

THEOREM 1 *Let p be the length of perpendicular (or normal) from the origin to a given non-vertical line L, and let α be the angle between the normal and the positive direction of the x-axis. Then, prove that the equation of the line L is given by*

$$x\cos\alpha + y\sin\alpha = p.$$

PROOF Let $X'OX$ and YOY' be the coordinate axes and let L be the given line.

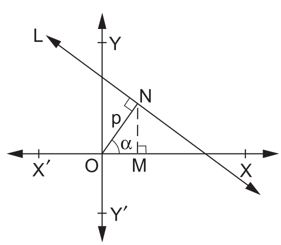
Draw ON perpendicular to L.

Let $ON = p$, and let the angle between the positive direction of the x-axis and ON be α.

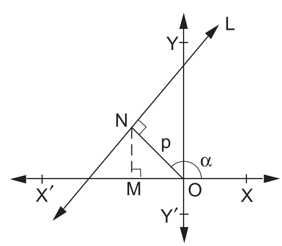
Draw perpendicular NM on the x-axis.

All possible positions of the line L are shown below.

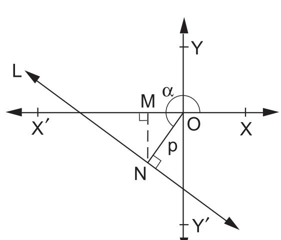
In each case, we have



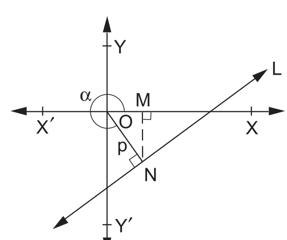
(i)



(ii)



(iii)



(iv)

$$\frac{OM}{ON} = \cos\alpha \quad \text{and} \quad \frac{NM}{ON} = \sin\alpha$$

$\Leftrightarrow$ $OM = p\cos\alpha$ and $NM = p\sin\alpha$.

Thus, the coordinates of N are $(p\cos\alpha,\ p\sin\alpha)$.

Moreover, line L is perpendicular to ON.

$\therefore$ slope of the line $L = \dfrac{-1}{\text{slope of } ON} = \dfrac{-1}{\tan\alpha} = \dfrac{-\cos\alpha}{\sin\alpha}$.

Thus, the line L passes through the point $N(p\cos\alpha,\ p\sin\alpha)$ and has a slope $m = \dfrac{-\cos\alpha}{\sin\alpha}$.

So, the equation of the line L is given by

$$\frac{y - p\sin\alpha}{x - p\cos\alpha} = \frac{-\cos\alpha}{\sin\alpha} \qquad \text{[point–slope form]}$$

$\Leftrightarrow$ $(y - p\sin\alpha)\sin\alpha = -\cos\alpha\,(x - p\cos\alpha)$

$\Leftrightarrow$ $x\cos\alpha + y\sin\alpha = p(\cos^2\alpha + \sin^2\alpha)$

$\Leftrightarrow$ $x\cos\alpha + y\sin\alpha = p$, which is the required equation.

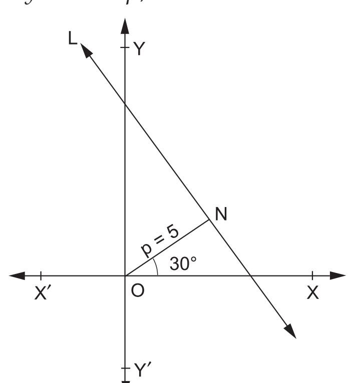
SOLVED EXAMPLES

EXAMPLE 1 *Find the equation of a line whose perpendicular distance from the origin is 5 units and the angle between the positive direction of the x-axis and the perpendicular is 30°.*

SOLUTION Here $p = 5$ units and $\alpha = 30°$.

So, the equation of the given line in normal form is

$$x\cos\alpha + y\sin\alpha = p, \text{ where } \alpha = 30° \text{ and } p = 5 \text{ units}$$



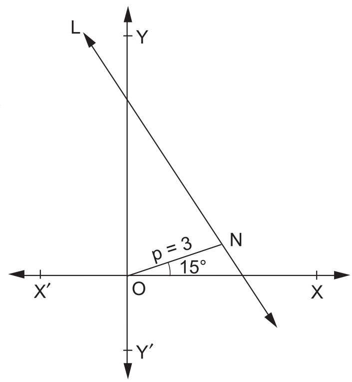
$\Leftrightarrow$ $x\cos 30° + y\sin 30° = 5$

$$\Leftrightarrow \quad x\left(\frac{\sqrt{3}}{2}\right) + y\left(\frac{1}{2}\right) = 5$$

$$\Leftrightarrow \quad \sqrt{3}x + y - 10 = 0,$$

which is the required equation.

EXAMPLE 2 *Find the equation of the line whose perpendicular distance from the origin is 3 units and the angle between the positive direction of x-axis and the perpendicular is* $15°$.



SOLUTION Here $p = 3$ units and $\alpha = 15°$.

So, the equation of the given line in normal form is

$x \cos \alpha + y \sin \alpha = p$, where $\alpha = 15°$ and $p = 3$ units

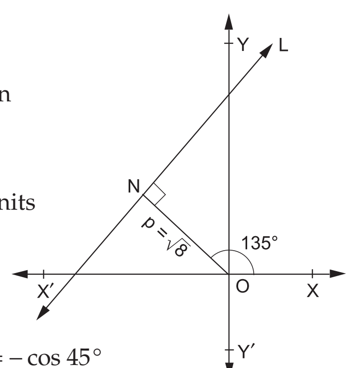
$$\Leftrightarrow \quad x \cos 15° + y \sin 15° = 3$$

$$\Leftrightarrow \quad \frac{x(\sqrt{3}+1)}{2\sqrt{2}} + \frac{y(\sqrt{3}-1)}{2\sqrt{2}} = 3$$

[$\because$ $\cos 15° = \cos(45° - 30°) = \cos 45° \cos 30° + \sin 45° \sin 30°$
$\sin 15° = \sin(45° - 30°) = \sin 45° \cos 30° - \cos 45° \sin 30°$]

$\Leftrightarrow \quad (\sqrt{3}+1)x + (\sqrt{3}-1)y - 6\sqrt{2} = 0$, which is the required equation.

EXAMPLE 3 *Find the equation of a line whose perpendicular distance from the origin is* $\sqrt{8}$ *units and the angle between the positive direction of the x-axis and the perpendicular is* $135°$.



SOLUTION Here $p = \sqrt{8}$ units and $\alpha = 135°$.

So, the equation of the given line in normal form is

$x \cos \alpha + y \sin \alpha = p$,
where $\alpha = 135°$ and $p = \sqrt{8}$ units

$$\Leftrightarrow \quad x \cos 135° + y \sin 135° = \sqrt{8}$$

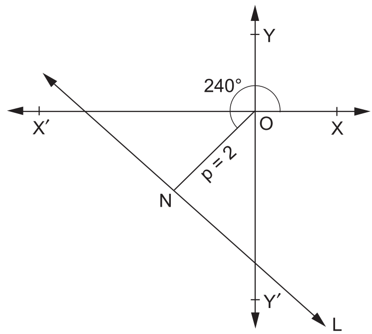
$$\Leftrightarrow \quad x\left(\frac{-1}{\sqrt{2}}\right) + y\left(\frac{1}{\sqrt{2}}\right) = \sqrt{18}$$

[$\because$ $\cos 135° = \cos(180° - 45°) = -\cos 45°$
$\sin 135° = \sin(180° - 45°) = \sin 45°$]

$\Leftrightarrow \quad -x + y = 4 \Leftrightarrow x - y + 4 = 0$, which is the required equation.

EXAMPLE 4 *Find the equation of a line whose perpendicular distance from the origin is 2 units and the angle between the perpendicular segment and the positive direction of the x-axis is* $240°$.

SOLUTION Here, $p = 2$ units and $\alpha = 240°$.



So, the equation of the given line in normal form is

$x\cos\alpha + y\sin\alpha = p$, where $\alpha = 240°$ and $p = 2$ units

$\Leftrightarrow \quad x\cos 240° + y\sin 240° = 2$

$\Leftrightarrow \quad x\left(\frac{-1}{2}\right) + y\left(\frac{-\sqrt{3}}{2}\right) = 2$ [$\because$ $\cos 240° = \cos(180° + 60°) = -\cos 60°$
$\sin 240° = \sin(180° + 60°) = -\sin 60°$]

$\Leftrightarrow \quad -x - \sqrt{3}y = 4 \Leftrightarrow x + \sqrt{3}y + 4 = 0$, which is the required equation.

EXERCISE 20F

1. Find the equation of the line for which

(i) $p = 3$ and $\alpha = 45°$ (ii) $p = 5$ and $\alpha = 135°$

(iii) $p = 8$ and $\alpha = 150°$ (iv) $p = 3$ and $\alpha = 225°$

(v) $p = 2$ and $\alpha = 300°$ (vi) $p = 4$ and $\alpha = 180°$

2. The length of the perpendicular segment from the origin to a line is 2 units and the inclination of this perpendicular is α such that $\sin\alpha = \frac{1}{3}$ and α is acute. Find the equation of the line.

3. Find the equation of the line which is at a distance of 3 units from the origin such that $\tan\alpha = \frac{5}{12}$, where α is the acute angle which this perpendicular makes with the positive direction of the x-axis.

ANSWERS (EXERCISE 20F)

1. (i) $x + y - 3\sqrt{2} = 0$ (ii) $x - y + 5\sqrt{2} = 0$ (iii) $\sqrt{3}x - y + 16 = 0$

(iv) $x + y + 3\sqrt{2} = 0$ (v) $x - \sqrt{3}y - 4 = 0$ (vi) $x + 4 = 0$

2. $2\sqrt{2}x + y - 6 = 0$ **3.** $12x + 5y - 39 = 0$

GENERAL EQUATION OF A LINE

THEOREM 1 *Prove that the equation $Ax + By + C = 0$ always represents a line, provided A and B are not simultaneously zero.*

PROOF Let $Ax + By + C = 0$, where $A \neq 0$ or $B \neq 0$.

CASE 1 *When $A = 0$ and $B \neq 0$*

In this case, the given equation becomes

$$By + C = 0 \Rightarrow y = \left(\frac{-C}{B}\right),$$

which represents a line parallel to the x-axis.

CASE 2 *When $A \neq 0$ and $B = 0$*

In this case, the given equation becomes

$$Ax + C = 0 \Rightarrow x = \left(\frac{-C}{A}\right),$$

which represents a line parallel to the y-axis.

CASE 3 *When $A \neq 0$ and $B \neq 0$*

In this case, we have

$$Ax + By + C = 0 \Rightarrow By = -Ax + (-C)$$

$$\Rightarrow y = \left(\frac{-A}{B}\right)x + \left(\frac{-C}{B}\right).$$

This is clearly an equation of a line in slope–intercept form,

$$y = mx + c, \text{ where } m = \left(\frac{-A}{B}\right) \text{ and } c = \frac{-C}{B}.$$

Thus, $Ax + By + C = 0$, when $A \neq 0$ or $B \neq 0$, always represents a line.

THEOREM 2 *Prove that every line has an equation of the form $Ax + By + c = 0$, where A, B, C are constants.*

PROOF For a given line, there are two possibilities.

CASE 1 *The given line is parallel to the y-axis.*

In this case, the equation is of the form, $x = c$.

Now, $x = c \Leftrightarrow 1 \cdot x + 0 \cdot y - c = 0$

$\Leftrightarrow Ax + By + C = 0$, where $C = -c$.

Thus, it is of the form $Ax + By + C = 0$.

CASE 2 *The given line is not parallel to the y-axis.*

In this case, the equation is of the form

$$y = mx + c \Rightarrow mx - y + c = 0$$

$$\Rightarrow Ax + By + C = 0,$$

where $A = m$, $B = -1$ and $C = c$.

Thus, it takes the form $Ax + By + C = 0$.

REDUCTION OF GENERAL FORM TO STANDARD FORM

Let the given equation of the line be $Ax + By + C = 0$.

I. SLOPE–INTERCEPT FORM:

$$Ax + By + C = 0 \Rightarrow By = -Ax - C$$

$$\Rightarrow y = \left(\frac{-A}{B}\right)x + \left(\frac{-C}{B}\right).$$

$$\Rightarrow y = mx + c, \text{ where } m = \frac{-A}{B} \text{ and } c = \frac{-C}{B}.$$

This is clearly the equation of a line in *slope–intercept* form, $y = mx + c$,

where $m = \frac{-A}{B}$ and $c = \frac{-C}{B}$.

II. INTERCEPTS FORM:

$$Ax + By + C = 0 \Rightarrow Ax + By = -C$$

$$\Rightarrow \left(\frac{A}{-C}\right)x + \left(\frac{B}{-C}\right)y = 1 \quad \text{[on dividing both sides by } -C\text{]}$$

$$\Rightarrow \frac{x}{\left(\frac{-C}{A}\right)} + \frac{y}{\left(\frac{-C}{B}\right)} = 1$$

$$\Rightarrow \frac{x}{a} + \frac{y}{b} = 1, \text{ where } a = \frac{-C}{A} \text{ and } b = \frac{-C}{B}.$$

This is clearly the equation of a line in intercept form, $\frac{x}{a} + \frac{y}{b} = 1$,

where $a = \frac{-C}{A}$ and $b = \frac{-C}{B}$.

III. NORMAL FORM:

$$Ax + By + C = 0 \Rightarrow \frac{A}{\sqrt{A^2 + B^2}}x + \frac{B}{\sqrt{A^2 + B^2}}\cdot y + \frac{C}{\sqrt{A^2 + B^2}} = 0$$

[on dividing throughout by $\sqrt{A^2 + B^2}$]

$$\Rightarrow \frac{-A}{\sqrt{A^2 + B^2}}x + \frac{-B}{\sqrt{A^2 + B^2}}y = \frac{C}{\sqrt{A^2 + B^2}}$$

$$\Rightarrow x\cos\alpha + y\sin\alpha = p,$$

where $\cos\alpha = \frac{-A}{\sqrt{A^2 + B^2}}$, $\sin\alpha = \frac{-B}{\sqrt{A^2 + B^2}}$ and $p = \frac{C}{\sqrt{A^2 + B^2}}$.

This is the equation of a line in normal form, $x\cos\alpha + y\sin\alpha = p$,

where $\cos\alpha = \frac{-A}{\sqrt{A^2 + B^2}}$, $\sin\alpha = \frac{-B}{\sqrt{A^2 + B^2}}$ and $p = \frac{C}{\sqrt{A^2 + B^2}}$.

SOLVED EXAMPLES

EXAMPLE 1 *Reduce the equation* $\sqrt{3}x + y + 2 = 0$ *to*

(i) slope–intercept form and find the slope and y-intercept.

(ii) intercepts form and find the intercepts on the axes.

SOLUTION We have

(i) $\sqrt{3}x + y + 2 = 0 \Rightarrow y = -\sqrt{3}x - 2.$

This is of the form $y = mx + c$, where $m = -\sqrt{3}$ and $c = -2$.

$\therefore \quad y = -\sqrt{3}x - 2$ is in slope–intercept form.

Here, slope $= -\sqrt{3}$ and y-intercept $= -2$.

(ii) $\sqrt{3}x + y + 2 = 0 \Rightarrow \sqrt{3}x + y = -2$

$$\Rightarrow \left(\frac{\sqrt{3}}{-2}\right)x + \left(\frac{1}{-2}\right)y = 1$$

$$\Rightarrow \frac{x}{\left(\frac{-2}{\sqrt{3}}\right)} + \frac{y}{-2} = 1.$$

This is of the form $\frac{x}{a} + \frac{y}{b} = 1$, where $a = \frac{-2}{\sqrt{3}}$ and $b = -2$.

Thus, $\frac{x}{\left(\frac{-2}{\sqrt{3}}\right)} + \frac{y}{-2} = 1$ is in intercepts form.

Here, x-intercept $= \frac{-2}{\sqrt{3}}$ and y-intercept $= -2$.

EXAMPLE 2 *Reduce the equation* $3x - 2y + 4 = 0$ *to intercepts form and find the length of the segment intercepted between the axes.*

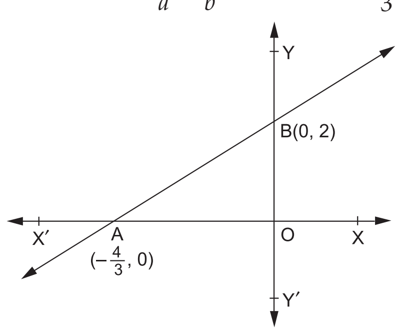
SOLUTION We have

$$3x - 2y + 4 = 0 \Rightarrow 3x - 2y = -4$$

$$\Rightarrow \frac{3x}{-4} + \frac{y}{2} = 1 \quad \text{[on dividing both sides by } -4]$$

$$\Rightarrow \frac{x}{\left(\frac{-4}{3}\right)} + \frac{y}{2} = 1$$

$$\Rightarrow \frac{x}{a} + \frac{y}{b} = 1, \text{ where } a = \frac{-4}{3} \text{ and } b = 1.$$



$\therefore \quad \dfrac{x}{\left(\dfrac{-4}{3}\right)} + \dfrac{y}{2} = 1$ is the required equation in intercepts form.

x-intercept $= \dfrac{-4}{3}$ and y-intercept $= 2$.

If AB is the part of the line intercepted between the axes then the end points of this line segment are $A\left(\dfrac{-4}{3}, 0\right)$ and $B(0, 2)$.

Length $AB = \sqrt{\left(\dfrac{-4}{3} - 0\right)^2 + (0-2)^2} = \sqrt{\dfrac{16}{9} + 4} = \dfrac{2}{3}\sqrt{13}$ units.

EXAMPLE 3 *Reduce the equation* $\sqrt{3}x + y + 2 = 0$ *to the normal form* $x\cos\alpha + y\sin\alpha = p$*, and hence find the values of* α *and* p.

SOLUTION We have

$$\sqrt{3}x + y + 2 = 0 \Rightarrow -\sqrt{3}x - y = 2 \qquad \text{[keeping constant +ve]}$$

$$\Rightarrow \left(\frac{-\sqrt{3}}{2}\right)x + \left(\frac{-1}{2}\right)y = 1$$

[on dividing throughout by $\sqrt{(\sqrt{3})^2 + 1^2}$]

$$\Rightarrow x\cos\alpha + y\sin\alpha = p,$$

where $\cos\alpha = \dfrac{-\sqrt{3}}{2}$, $\sin\alpha = \dfrac{-1}{2}$ and $p = 1$.

Since $\cos\alpha < 0$ and $\sin\alpha < 0$, α lies in the third quadrant.

Now, $\tan\alpha = \dfrac{\sin\alpha}{\cos\alpha} = \left(\dfrac{-1}{2}\right) \times \left(\dfrac{-2}{\sqrt{3}}\right) = \dfrac{1}{\sqrt{3}} = \tan(180° + 30°) = \tan 210°$

$\Rightarrow \alpha = 210°$.

Thus, $\alpha = 210°$ and $p = 1$.

Hence, the given equation in normal form is given by

$$x\cos 210° + y\sin 210° = 1.$$

EXAMPLE 4 *Reduce the equation* $x + \sqrt{3}y + 5 = 0$ *to the normal form* $x\cos\alpha + y\sin\alpha = p$*, and hence find the values of* α *and* p.

SOLUTION We have

$$x + \sqrt{3}y + 5 = 0 \Rightarrow -x - \sqrt{3}y = 5 \qquad \text{[keeping constant +ve]}$$

$$\Rightarrow \left(\frac{-1}{2}\right)x + \left(\frac{-\sqrt{3}}{2}\right)y = \frac{5}{2}$$

[on dividing throughout by $\sqrt{(-1)^2 + (-\sqrt{3})^2}$]

$$\Rightarrow x\cos\alpha + y\sin\alpha = p,$$

where $\cos\alpha = \dfrac{-1}{2}$, $\sin\alpha = \dfrac{-\sqrt{3}}{2}$ and $p = \dfrac{5}{2}\cdot$

Since $\cos\alpha < 0$ and $\sin\alpha < 0$, α lies in third quadrant.

Now, $\tan\alpha = \dfrac{\sin\alpha}{\cos\alpha} = \left(\dfrac{-\sqrt{3}}{2}\right) \times (-2) = \sqrt{3} = \tan(180° + 60°) = \tan 240°$

$\therefore \quad \alpha = 240°$.

Thus, $\alpha = 240°$ and $p = \dfrac{5}{2}\cdot$

Hence, the given equation in normal form is

$$x\cos 240° + y\sin 240° = \frac{5}{2}\cdot$$

EXAMPLE 5 *Reduce the equation $y + 4 = 0$ to the normal form $x\cos\alpha + y\sin\alpha = p$, and hence find the values of α and p.*

SOLUTION We have

$y + 4 = 0 \Rightarrow -y = 4$ [keeping constant positive]

$\Rightarrow x\cos\alpha + y\sin\alpha = p$,

where $\cos\alpha = 0$, $\sin\alpha = -1$ and $p = 4$.

Now, $(\cos\alpha = 0$ and $\sin\alpha = -1) \Rightarrow \alpha = 270°$.

Hence, the required normal form of the given equation is

$$x\cos 270° + y\sin 270° = 4.$$

Clearly, $\alpha = 270°$ and $p = 4$.

EXERCISE 20G

1. Reduce the equation $2x - 3y - 5 = 0$ to slope–intercept form, and find from it the slope and y-intercept.
2. Reduce the equation $5x + 7y - 35 = 0$ to slope–intercept form, and hence find the slope and the y-intercept of the line.
3. Reduce the equation $y + 5 = 0$ to slope–intercept form, and hence find the slope and the y-intercept of the line.
4. Reduce the equation $3x - 4y + 12 = 0$ to intercepts form. Hence, find the length of the portion of the line intercepted between the axes.
5. Reduce the equation $5x - 12y = 60$ to intercepts form. Hence, find the length of the portion of the line intercepted between the axes.
6. Find the inclination of the line:
   (i) $x + \sqrt{3}y + 6 = 0$ (ii) $3x + 3y + 8 = 0$ (iii) $\sqrt{3}x - y - 4 = 0$
7. Reduce the equation $x + y - \sqrt{2} = 0$ to the normal form $x\cos\alpha + y\sin\alpha = p$, and hence find the values of α and p.
8. Reduce the equation $x + \sqrt{3}y - 4 = 0$ to the normal form $x\cos\alpha + y\sin\alpha = p$, and hence find the values of α and p.
9. Reduce each of the following equations to normal form:
   (i) $x + y - 2 = 0$ (ii) $x + y + \sqrt{2} = 0$ (iii) $x + 5 = 0$
   (iv) $2y - 3 = 0$ (v) $4x + 3y - 9 = 0$

ANSWERS (EXERCISE 20G)

1. $y = \frac{2}{3}x - \frac{5}{3}$, $m = \frac{2}{3}$ and $c = \frac{-5}{3}$

2. $y = \frac{-5}{7}x + 5$, $m = \frac{-5}{7}$ and $c = 5$

3. $y = 0 \cdot x - 5$, $m = 0$ and $c = -5$

4. $\frac{x}{-4} + \frac{y}{3} = 1$, 5 units

5. $\frac{x}{12} + \frac{y}{-5} = 1$, 13 units

6. (i) $150°$ (ii) $135°$ (iii) $60°$

7. $x\cos 45° + y\sin 45° = 1$; $\alpha = 45°$, $p = 1$

8. $x\cos 60° + y\sin 60° = 2$; $\alpha = 60°$, $p = 2$

9. (i) $x\cos 45° + y\sin 45° = \sqrt{2}$ (ii) $x\cos 225° + y\sin 225° = 1$

   (iii) $x\cos 180° + y\sin 180° = 5$ (iv) $x\cos 90° + y\sin 90° = \frac{3}{2}$

   (v) $x\cos\alpha + y\sin\alpha = p$, where $\cos\alpha = \frac{4}{5}$, $\sin\alpha = \frac{3}{5}$ and $p = \frac{9}{5}$

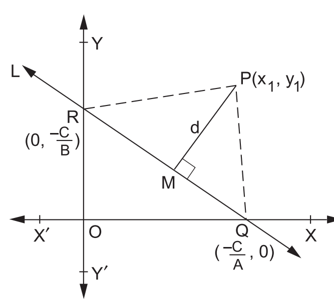
DISTANCE OF A POINT FROM A LINE

THEOREM 3 *Prove that the length of perpendicular from a given point $P(x_1, y_1)$ on a line $Ax + By + C = 0$, is given by*

$$d = \frac{|Ax_1 + By_1 + C|}{\sqrt{A^2 + B^2}}.$$

PROOF Let L be the line represented by the equation $Ax + By + C = 0$ and let $P(x, y)$ be a given point outside it.

Let this line L intersect the x-axis and y-axis at the points Q and R respectively, and let $PM \perp QR$.



Now,

$$Ax + By + C = 0 \Rightarrow Ax + By = -C$$

$$\Rightarrow \frac{Ax}{-C} + \frac{By}{-C} = 1$$

$$\Rightarrow \frac{x}{\left(\frac{-C}{A}\right)} + \frac{y}{\left(\frac{-C}{B}\right)} = 1.$$

Thus, the given line intersects the x-axis and the y-axis at the points

$$Q\left(\frac{-C}{A}, 0\right) \text{ and } R\left(0, \frac{-C}{B}\right).$$

Draw $PM \perp QR$.

$$\text{Area of } \triangle PQR = \frac{1}{2}\cdot\left|x_1\left(\frac{-C}{B}-0\right)+0\cdot(0-y_1)+\left(\frac{-C}{A}\right)\cdot\left(y_1+\frac{C}{B}\right)\right|$$

$$= \frac{1}{2}\cdot\left|-C\left(\frac{x_1}{B}+\frac{y_1}{A}+\frac{C}{AB}\right)\right| = \frac{1}{2}\cdot\left|\frac{C}{AB}(Ax_1+By_1+C)\right|$$

$$= \left|\frac{C}{2AB}(Ax_1+By_1+C)\right| \qquad \text{... (i)}$$

$$\text{Also, area of } \triangle PQR = \frac{1}{2}\times QR\times PM = \frac{1}{2}\times PM\times\sqrt{(OQ)^2+(OR)^2}$$

$$= \frac{1}{2}\times PM\times\sqrt{\left(\frac{-C}{A}\right)^2+\left(\frac{-C}{B}\right)^2}$$

$$\left[\because\ OQ = \frac{-C}{A} \text{ and } OR = \frac{-C}{B}\right]$$

$$= \frac{1}{2}\times PM\times\sqrt{\frac{C^2}{A^2}+\frac{C^2}{B^2}} \qquad \text{... (ii)}$$

From (i) and (ii), we get

$$\frac{1}{2}\times PM\times\sqrt{\frac{C^2}{A^2}+\frac{C^2}{B^2}} = \left|\frac{C}{2AB}\cdot(Ax_1+By_1+C)\right|$$

$$\Rightarrow \frac{1}{2}\times\frac{C}{AB}\times\sqrt{A^2+B^2}\times PM = \frac{C}{2AB}\cdot|Ax_1+By_1+C|$$

$$\Rightarrow PM = \frac{|Ax_1+By_1+C|}{\sqrt{A^2+B^2}}.$$

Hence, $d = \dfrac{|Ax_1+By_1+C|}{\sqrt{A^2+B^2}}$, where $PM = d$.

REMARK Clearly, the length of perpendicular from the origin on the line $Ax+By+C=0$ is $\dfrac{|C|}{\sqrt{A^2+B^2}}$.

DISTANCE BETWEEN TWO PARALLEL LINES

THEOREM 4 *Prove that the distance between two parallel lines $Ax+By+C_1=0$ and $Ax+By+C_2=0$ is given by*

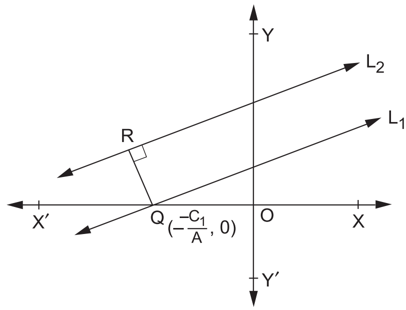
$$d = \frac{|C_2-C_1|}{\sqrt{A^2+B^2}}.$$

SOLUTION Let L_1 and L_2 be two given parallel lines represented by

$$Ax+By+C_1=0 \qquad \text{... (i)}$$

and $Ax+By+C_2=0$. ... (ii)

Putting $y=0$ in (i), we get $x = \left(-\dfrac{C_1}{A}\right)$.



Thus, L_1 intersects the x-axis at $Q = \left(-\frac{C_1}{A}, 0\right)$.

Let $QR \perp L_2$ and let $d = |QR|$. Then,

$d =$ length of perpendicular from $Q\left(-\frac{C_1}{A}, 0\right)$ on $Ax + By + C_2 = 0$

$$\Rightarrow \quad d = \frac{\left|A \times \frac{(-C_1)}{A} + B \times 0 + C_2\right|}{\sqrt{A^2 + B^2}} = \frac{|C_2 - C_1|}{\sqrt{A^2 + B^2}}.$$

REMARK Distance between two parallel lines $y = mx + C_1$ and $y = mx + C_2$ is given by

$$d = \frac{|C_2 - C_1|}{\sqrt{1 + m^2}}.$$

SUMMARY

(i) Length of perpendicular from a point $P(x_1, y_1)$ on the line $Ax + By + C = 0$ is given by, $d = \frac{|Ax_1 + By_1 + C|}{\sqrt{A^2 + B^2}}$.

(ii) Distance between two parallel lines $Ax + By + C_1 = 0$ and $Ax + By + C_2 = 0$ is given by, $d = \frac{|C_2 - C_1|}{\sqrt{A^2 + B^2}}$.

(iii) Distance between $y = mx + C_1$ and $y = mx + C_2$ is given by, $d = \frac{|C_2 - C_1|}{\sqrt{1 + m^2}}$.

SOLVED EXAMPLES

EXAMPLE 1 *Find the distance of the point* (4, 1) *from the line* $3x - 4y + 12 = 0$.

SOLUTION Clearly, the required distance is the length of perpendicular from the point $P(4, 1)$ on the line $3x - 4y + 12 = 0$.

Let the required distance be d. Then,

$$d = \frac{|3 \times 4 - 4 \times 1 + 12|}{\sqrt{3^2 + (-4)^2}} = \frac{20}{5} \text{ units} = 4 \text{ units}.$$

EXAMPLE 2 *Find the distance of the point* $(-1, 1)$ *from the line* $12(x + 6) = 5(y - 2)$.

SOLUTION The given point is $P(-1, 1)$ and the given line is $12x - 5y + 82 = 0$.

Let the required distance be d. Then,

$d =$ length of perpendicular from $P(-1, 1)$ on the line $12x - 5y + 82 = 0$

$$= \frac{|12 \times (-1) - 5 \times 1 + 82|}{\sqrt{(12)^2 + (-5)^2}} = \frac{65}{\sqrt{169}} = \frac{65}{13} = 5 \text{ units}.$$

EXAMPLE 3 *Find the length of perpendicular from the point* (a, b) *to the line* $\frac{x}{a} + \frac{y}{b} = 1$.

SOLUTION The given point is $P(a, b)$ and the given line is $bx + ay - ab = 0$.

Let d be the length of perpendicular from $P(a, b)$ to the line $bx + ay - ab = 0$.

Then, $d = \dfrac{|b \times a + a \times b - ab|}{\sqrt{b^2 + a^2}} = \dfrac{|ab|}{\sqrt{a^2 + b^2}}$ units.

EXAMPLE 4 *Find the length of perpendicular from the origin to the line* $4x + 3y - 2 = 0$.

SOLUTION The given point is $P(0, 0)$ and the given line is $4x + 3y - 2 = 0$.

Led d be the length of perpendicular from $P(0, 0)$ to the line $4x + 3y - 2 = 0$.

Then, $d = \dfrac{|4 \times 0 + 3 \times 0 - 2|}{\sqrt{4^2 + 3^2}} = \dfrac{2}{5}$ unit.

EXAMPLE 5 *If* p *is the length of perpendicular from the origin to the line whose intercepts on the axes are* a *and* b *then show that*

$$\frac{1}{p^2} = \frac{1}{a^2} + \frac{1}{b^2}.$$

SOLUTION The equation of the line making intercepts a and b on the axes is given by

$$\frac{x}{a} + \frac{y}{b} = 1, \text{ i.e., } \frac{x}{a} + \frac{y}{b} - 1 = 0. \qquad \text{... (i)}$$

Since p is the length of perpendicular from $O(0, 0)$ to line (i), we have

$$p = \frac{\left|\frac{1}{a} \times 0 + \frac{1}{b} \times 0 - 1\right|}{\sqrt{\frac{1}{a^2} + \frac{1}{b^2}}} = \frac{|-1|}{\sqrt{\frac{1}{a^2} + \frac{1}{b^2}}} = \frac{1}{\sqrt{\frac{1}{a^2} + \frac{1}{b^2}}}$$

$$\Rightarrow \quad p^2 = \frac{1}{\left(\frac{1}{a^2} + \frac{1}{b^2}\right)} = \frac{a^2b^2}{(b^2 + a^2)}$$

$$\Rightarrow \quad \frac{1}{p^2} = \frac{(b^2 + a^2)}{a^2b^2} = \left(\frac{b^2}{a^2b^2} + \frac{a^2}{a^2b^2}\right) = \left(\frac{1}{a^2} + \frac{1}{b^2}\right)$$

Hence, $\frac{1}{p^2} = \frac{1}{a^2} + \frac{1}{b^2}$.

EXAMPLE 6 *Find the perpendicular distance of the line joining the points $A(\cos\theta, \sin\theta)$ and $B(\cos\phi, \sin\phi)$ from the origin.*

SOLUTION We know that the equation of the line AB is given by

$$\frac{y - \sin\theta}{x - \cos\theta} = \frac{\sin\phi - \sin\theta}{\cos\phi - \cos\theta}$$

$$\Rightarrow \quad \frac{y - \sin\theta}{x - \cos\theta} = \frac{2\cos\left(\frac{\phi + \theta}{2}\right)\sin\left(\frac{\phi - \theta}{2}\right)}{-2\sin\left(\frac{\phi + \theta}{2}\right)\sin\left(\frac{\phi - \theta}{2}\right)}$$

$$\Rightarrow \quad \frac{y - \sin\theta}{x - \cos\theta} = \frac{\cos\left(\frac{\phi + \theta}{2}\right)}{-\sin\left(\frac{\phi + \theta}{2}\right)}$$

$$\Rightarrow \quad x\cos\left(\frac{\phi + \theta}{2}\right) + y\sin\left(\frac{\phi + \theta}{2}\right) - \left\{\cos\theta \cdot \cos\left(\frac{\phi + \theta}{2}\right) + \sin\theta \cdot \sin\left(\frac{\phi + \theta}{2}\right)\right\} = 0$$

$$\Rightarrow \quad x\cos\left(\frac{\phi + \theta}{2}\right) + y\sin\left(\frac{\phi + \theta}{2}\right) - \cos\left(\frac{\phi + \theta}{2} - \theta\right) = 0$$

$$\Rightarrow \quad x\cos\left(\frac{\phi + \theta}{2}\right) + y\sin\left(\frac{\phi + \theta}{2}\right) - \cos\left(\frac{\phi - \theta}{2}\right) = 0. \qquad \text{... (i)}$$

Let d be the perpendicular distance from the origin to line (i). Then,

$$d = \frac{\left|0 \times \cos\left(\frac{\phi + \theta}{2}\right) + 0 \times \sin\left(\frac{\phi + \theta}{2}\right) - \cos\left(\frac{\phi - \theta}{2}\right)\right|}{\sqrt{\cos^2\left(\frac{\phi + \theta}{2}\right) + \sin^2\left(\frac{\phi + \theta}{2}\right)}} = \left|\cos\left(\frac{\phi - \theta}{2}\right)\right|.$$

Hence, the required distance is $\left|\cos\left(\frac{\phi - \theta}{2}\right)\right|$.

EXAMPLE 7 *If p_1 and p_2 are the lengths of perpendiculars from the origin to the line $x\sec\theta + y\operatorname{cosec}\theta = a$ and $x\cos\theta - y\sin\theta = a\cos 2\theta$ respectively then prove that $4p_1^2 + p_2^2 = a^2$.*

SOLUTION We have

$$x\sec\theta + y\operatorname{cosec}\theta = a \Rightarrow \frac{x}{\cos\theta} + \frac{y}{\sin\theta} - a = 0. \quad \text{... (i)}$$

Now, p_1 is the length of perpendicular from the origin to line (i), so we have

$$p_1 = \frac{\left|\frac{1}{\cos\theta} \times 0 + \frac{1}{\sin\theta} \times 0 - a\right|}{\sqrt{\frac{1}{\cos^2\theta} + \frac{1}{\sin^2\theta}}} = |a| \sin\theta\cos\theta = \frac{|a|}{2}\sin 2\theta$$

$\Rightarrow \quad 2p_1 = |a| \sin 2\theta$

$\Rightarrow \quad 4p_1^2 = a^2 \sin^2 2\theta.$ ... (ii)

The other line is $x\cos\theta - y\sin\theta - a\cos 2\theta = 0.$ ... (iii)

Now, p_2 is the length of perpendicular from the origin to line (ii), so we have

$$p_2 = \frac{|0 \times \cos\theta - 0 \times \sin\theta - a\cos 2\theta|}{\sqrt{\cos^2\theta + \sin^2\theta}} = |a\cos 2\theta|$$

$\Rightarrow \quad p_2^2 = a^2 \cos^2 2\theta.$ ... (iv)

Adding (ii) and (iv), we get

$$4p_1^2 + p_2^2 = a^2(\sin^2 2\theta + \cos^2 2\theta)$$

$\Rightarrow \quad 4p_1^2 + p_2^2 = a^2.$

EXAMPLE 8 *What are the points on the y-axis whose perpendicular distance from the line $\frac{x}{3} - \frac{y}{4} = 1$ is 3 units?*

SOLUTION $\frac{x}{3} - \frac{y}{4} = 1 \Rightarrow 4x - 3y - 12 = 0.$ ... (i)

Let $P(0, y)$ be a point on the y-axis whose perpendicular distance from the line $4x - 3y - 12 = 0$ is 3 units. Then,

$$\frac{|4 \times 0 - 3y - 12|}{\sqrt{4^2 + (-3)^2}} = 3 \Rightarrow |-3y - 12| = 15 \Rightarrow |-3(y + 4)| = 15$$

$$\Rightarrow |-(y + 4)| = 5 \Rightarrow |y + 4| = 5$$

$$\Rightarrow (y + 4 = 5) \text{ or } (y + 4 = -5)$$

$$\Rightarrow y = 1 \quad \text{or} \quad y = -9.$$

Hence, the required points are $(0, 1)$ and $(0, -9)$.

EXAMPLE 9 *Find the distance between the parallel lines $15x + 8y - 34 = 0$ and $15x + 8y + 31 = 0$.*

SOLUTION Converting each of the given equations to the form $y = mx + C$, we get

$$15x + 8y - 34 = 0 \Rightarrow y = \frac{-15}{8}x + \frac{17}{4} \qquad \text{... (i)}$$

$$15x + 8y + 31 = 0 \Rightarrow y = \frac{-15}{8}x - \frac{31}{8} \qquad \text{... (ii)}$$

Clearly, the slopes of the given lines are equal and so they are parallel.

The given lines are of the form $y = mx + C_1$ and $y = mx + C_2$, where $m = \frac{-15}{8}$, $C_1 = \frac{17}{4}$ and $C_2 = \frac{-31}{8}$.

$\therefore$ distance between the given lines

$$= \frac{|C_2 - C_1|}{\sqrt{1 + m^2}}, \text{ where } m = \frac{-15}{8}, C_1 = \frac{17}{4} \text{ and } C_2 = \frac{-31}{8}$$

$$= \frac{\left|\frac{-31}{8} - \frac{17}{4}\right|}{\sqrt{1 + \left(\frac{-15}{8}\right)^2}} = \frac{\left|-\frac{65}{8}\right|}{\sqrt{1 + \frac{225}{64}}} = \frac{\left(\frac{65}{8}\right)}{\sqrt{\frac{289}{64}}} = \left(\frac{65}{8} \times \frac{8}{17}\right) = \frac{65}{17} \text{ units.}$$

Hence, the distance between the given lines is $\frac{65}{17}$ units.

EXAMPLE 10 *Find the distance between the lines* $3x - 4y + 9 = 0$ *and* $6x - 8y - 17 = 0$.

SOLUTION Converting each of the given equations to the form $y = mx + C$, we get

$$3x - 4y + 9 = 0 \Rightarrow y = \frac{3}{4}x + \frac{9}{4} \qquad \text{... (i)}$$

$$6x - 8y - 17 = 0 \Rightarrow y = \frac{3}{4}x - \frac{17}{8} \qquad \text{... (ii)}$$

Clearly, the slopes of the given lines are equal and so they are parallel.

Let the given lines be $y = mx + C_1$ and $y = mx + C_2$, where $m = \frac{3}{4}$, $C_1 = \frac{9}{4}$ and $C_2 = \frac{-17}{8}$.

$\therefore$ distance between given lines $= \frac{|C_2 - C_1|}{\sqrt{1 + m^2}}$

$$= \frac{\left|\frac{-17}{8} - \frac{9}{4}\right|}{\sqrt{1 + \frac{9}{16}}} = \frac{\left|\frac{-35}{8}\right|}{\left(\frac{5}{4}\right)} = \left(\frac{35}{8} \times \frac{4}{5}\right)$$

$$= \frac{7}{2} \text{ units.}$$

Hence, the distance between the given lines is $\frac{7}{2}$ units.

EXAMPLE 11 *Prove that the line* $5x - 2y - 1 = 0$ *is mid-parallel to the lines* $5x - 2y - 9 = 0$ *and* $5x - 2y + 7 = 0$.

SOLUTION Converting each of the given equations to the form $y = mx + C$, we get

$$5x - 2y - 1 = 0 \Rightarrow y = \frac{5}{2}x - \frac{1}{2} \quad \text{... (i)}$$

$$5x - 2y - 9 = 0 \Rightarrow y = \frac{5}{2}x - \frac{9}{2} \quad \text{... (ii)}$$

$$5x - 2y + 7 = 0 \Rightarrow y = \frac{5}{2}x + \frac{7}{2} \quad \text{... (iii)}$$

Clearly, the slope of (i) is equal to the slope of each of (ii) and (iii).

So, the line (i) is parallel to each of the given lines (ii) and (iii).

Let the given lines be $y = mx + C$, $y = mx + C_1$ and $y = mx + C_2$ respectively. Then, $m = \frac{5}{2}$, $C = -\frac{1}{2}$, $C_1 = -\frac{9}{2}$ and $C_2 = \frac{7}{2}$.

Let d_1 and d_2 be the distances of (i) from (ii) and (iii) respectively.

Then, $$d_1 = \frac{|C_1 - C|}{\sqrt{1 + m^2}} = \frac{\left|-\frac{9}{2} + \frac{1}{2}\right|}{\sqrt{1 + \frac{25}{4}}} = |-4| \cdot \frac{2}{\sqrt{29}} = \frac{(4 \times 2)}{\sqrt{29}} = \frac{8}{\sqrt{29}} \text{ units}$$

and $$d_2 = \frac{|C_2 - C|}{\sqrt{1 + m^2}} = \frac{\left|\frac{7}{2} + \frac{1}{2}\right|}{\sqrt{1 + \frac{25}{4}}} = \left(4 \times \frac{2}{\sqrt{29}}\right) = \frac{8}{\sqrt{29}} \text{ units.}$$

Thus, $d_1 = d_2$.

This shows that (i) is equidistant from (ii) and (iii).

Hence, $5x - 2y - 1 = 0$ is mid-parallel to the lines $5x - 2y - 9 = 0$ and $5x - 2y + 7 = 0$.

EXAMPLE 12 *Find the equation of the line midway between the parallel lines* $9x + 6y - 7 = 0$ *and* $3x + 2y + 6 = 0$.

SOLUTION Converting each of the given equations to the form $y = mx + C$, we get

$$9x + 6y - 7 = 0 \Rightarrow y = \frac{-3}{2}x + \frac{7}{6} \quad \text{... (i)}$$

$$3x + 2y + 6 = 0 \Rightarrow y = \frac{-3}{2}x - 3 \quad \text{... (ii)}$$

Clearly, the slope of each one of the given lines is $\frac{-3}{2}$.

Let the given lines be $y = mx + C_1$ and $y = mx + C_2$.

Then, $m = \frac{-3}{2}$, $C_1 = \frac{7}{6}$ and $C_2 = -3$.

Let L be the required line. Then, L is parallel to each one of (i) and (ii), and equidistant from each one of them.

$\therefore$ slope of $L = \frac{-3}{2}$.

Let the equation of L be $y = \frac{-3}{2}x + C$ ... (iii)

Then, distance between (i) and (iii) must be equal to the distance between (ii) and (iii).

$$\therefore \quad \frac{|C_1 - C|}{\sqrt{1+m^2}} = \frac{|C_2 - C|}{\sqrt{1+m^2}} \Rightarrow |C_1 - C| = |C_2 - C|$$

$$\Rightarrow \left|\frac{7}{6} - C\right| = |-3 - C| \Rightarrow \left|\frac{7}{6} - C\right| = |3 + C|$$

$$\Rightarrow \frac{7}{6} - C = 3 + C \Rightarrow 2C = \frac{-11}{6} \Rightarrow C = \frac{-11}{12}.$$

$\therefore$ equation of L is $y = \frac{-3}{2}x - \frac{11}{12}$, i.e., $18x + 12y + 11 = 0$.

Hence, the line $18x + 12y + 11 = 0$ is midway between the parallel lines $9x + 6y - 7 = 0$ and $3x + 2y + 6 = 0$.

EXAMPLE 13 *Find the coordinates of a point on the line $x + y + 3 = 0$, whose distance from the line $x + 2y + 2 = 0$ is $\sqrt{5}$.*

SOLUTION Let the required point be $P(a, b)$. Then,

$a + b + 3 = 0$. ... (i)

Also, $\frac{|a + 2b + 2|}{\sqrt{1^2 + 2^2}} = \sqrt{5} \Rightarrow |a + 2b + 2| = 5$

$$\Rightarrow (a + 2b + 2 = 5) \quad \text{or} \quad (a + 2b + 2 = -5)$$

$$\Rightarrow a + 2b = 3 \quad \text{or} \quad a + 2b = -7.$$

Thus, $a + 2b = 3$ ... (ii) and $a + 2b = -7$ ... (iii)

On solving (i) and (ii), we get $a = -9$ and $b = 6$.

On solving (i) and (iii), we get $a = 1$ and $b = -4$.

Hence, the required points are $A(-9, 6)$ and $B(1, -4)$.

EXERCISE 20H

1. Find the distance of the point $(3, -5)$ from the line $3x - 4y = 27$.
2. Find the distance of the point $(-2, 3)$ from the line $12x = 5y + 13$.
3. Find the distance of the point $(-4, 3)$ from the line $4(x + 5) = 3(y - 6)$.
4. Find the distance of the point $(2, 3)$ from the line $y = 4$.
5. Find the distance of the point $(4, 2)$ from the line joining the points $(4, 1)$ and $(2, 3)$.

6. Find the length of perpendicular from the origin to each of the following lines:

(i) $7x + 24y = 50$ (ii) $4x + 3y = 9$ (iii) $x = 4$

7. Prove that the product of the lengths of perpendiculars drawn from the points $A(\sqrt{a^2 - b^2}, 0)$ and $B(-\sqrt{a^2 - b^2}, 0)$ to the line $\frac{x}{a}\cos\theta + \frac{y}{b}\sin\theta = 1$, is b^2.

8. Find the values of k for which the length of perpendicular from the point $(4, 1)$ on the line $3x - 4y + k = 0$ is 2 units.

9. Show that the length of perpendicular from the point $(7, 0)$ to the line $5x + 12y - 9 = 0$ is double the length of perpendicular to it from the point $(2, 1)$.

10. The points $A(2, 3)$, $B(4, -1)$ and $C(-1, 2)$ are the vertices of $\triangle ABC$. Find the length of perpendicular from C on AB and hence find the area of $\triangle ABC$.

11. What are the points on the x-axis whose perpendicular distance from the line $\frac{x}{3} + \frac{y}{4} = 1$ is 4 units?

12. Find all the points on the line $x + y = 4$ that lie at a unit distance from the line $4x + 3y = 10$.

13. A vertex of a square is at the origin and its one side lies along the line $3x - 4y - 10 = 0$. Find the area of the square.

14. Find the distance between the parallel lines $4x - 3y + 5 = 0$ and $4x - 3y + 7 = 0$.

15. Find the distance between the parallel lines $8x + 15y - 36 = 0$ and $8x + 15y + 32 = 0$.

16. Find the distance between the parallel lines $y = mx + c$ and $y = mx + d$.

17. Find the distance between the parallel lines $p(x + y) + q = 0$ and $p(x + y) - r = 0$.

18. Prove that the line $12x - 5y - 3 = 0$ is mid-parallel to the lines $12x - 5y + 7 = 0$ and $12x - 5y - 13 = 0$.

19. The perpendicular distance of a line from the origin is 5 units and its slope is -1. Find the equation of the line.

ANSWERS (EXERCISE 20H)

1. $\frac{2}{5}$ unit **2.** 4 units **3.** $\frac{13}{5}$ units **4.** 1 unit

5. $\frac{\sqrt{2}}{2}$ units **6.** (i) 2 units (ii) $\frac{9}{5}$ units (iii) 4 units **8.** $k = 2$ or $k = -18$

10. $\frac{7}{\sqrt{5}}$ units, 7 sq units **11.** $(8, 0)$ and $(-2, 0)$

12. (3, 1) and (–7, 11) **13.** 4 sq units **14.** $\frac{2}{5}$ units

15. 4 units **16.** $\frac{|d-c|}{\sqrt{1+m^2}}$ **17.** $\frac{|q+r|}{\sqrt{2}p}$

19. $x+y+5\sqrt{2}=0$ or $x+y-5\sqrt{2}=0$

HINTS TO SOME SELECTED QUESTIONS

1. The given line is $0\cdot x+1\cdot y-4=0$.

5. Equation of the line joining the points $A(4,1)$ and $B(2,3)$ is

$$\frac{y-1}{x-4}=\frac{3-1}{2-4}, \quad \text{i.e., } x+y-5=0.$$

6. (iii) The given line is $x+0\cdot y-4=0$.

7. Put $\sqrt{a^2-b^2}=c$. Then,

$$(d_1\times d_2)=\frac{\left|\frac{c\cos\theta}{a}+0-1\right|}{\sqrt{\frac{\cos^2\theta}{a^2}+\frac{\sin^2\theta}{b^2}}}\times\frac{\left|\frac{-c\cos\theta}{a}+0-1\right|}{\sqrt{\frac{\cos^2\theta}{a^2}+\frac{\sin^2\theta}{b^2}}}=\frac{\left|\frac{c\cos\theta}{a}-1\right|\left|\frac{c\cos\theta}{a}+1\right|}{\left(\frac{\cos^2\theta}{a^2}+\frac{\sin^2\theta}{b^2}\right)}$$

$$=\frac{\left|\frac{c^2\cos^2\theta}{a^2}-1\right|}{(b^2\cos^2\theta+a^2\sin^2\theta)}\times a^2b^2=\frac{|(a^2-b^2)\cos^2\theta-a^2|}{(b^2\cos^2\theta+a^2\sin^2\theta)}\times b^2$$

$$=\frac{|a^2(1-\sin^2\theta)-b^2\cos^2\theta-a^2|}{(b^2\cos^2+a^2\sin^2\theta)}\times b^2=\frac{|-(a^2\sin^2\theta+b^2\cos^2\theta)|}{(b^2\cos^2\theta+a^2\sin^2\theta)}\times b^2$$

$$=b^2.$$

8. $\frac{|3\times 4-4\times 1+k|}{\sqrt{3^2+(-4)^2}}=2 \Leftrightarrow |8+k|=10 \Leftrightarrow (8+k)=10 \quad \text{or} \quad (8+k)=-10.$

10. Equation of AB is $\frac{y-3}{x-2}=\frac{-4}{2} \Rightarrow 2x+y-7=0.$

Length of perpendicular from $C(-1,2)$ to $2x+y-7=0$ is equal to $\frac{|2\times(-1)+2-7|}{\sqrt{2^2+1^2}}=\frac{7}{\sqrt{5}}$ units.

$$\text{ar}(\triangle ABC)=\frac{1}{2}\times AB\times\frac{7}{\sqrt{5}}=\left(\frac{1}{2}\times\sqrt{20}\times\frac{7}{\sqrt{5}}\right)\text{sq units}=7\text{ sq units.}$$

11. Let the required point be $P(x,0)$.

Given line is $4x+3y-12=0$.

Now $\frac{|4x+3\times 0-12|}{\sqrt{4^2+3^2}}=4 \Leftrightarrow |x-3|=5 \Rightarrow x-3=5 \quad \text{or} \quad x-3=-5$

$$\Rightarrow x=8 \quad \text{or} \quad x=-2.$$

12. Let $x = a$. Then, $x + y = 4 \Rightarrow y = (4 - a)$.

Let the required point be $P(a, 4 - a)$. Then,

$$\frac{|4a + 3(4 - a) - 10|}{\sqrt{4^2 + 3^2}} = 1 \Rightarrow |a + 2| = 5 \Rightarrow a + 2 = 5 \quad \text{or} \quad a + 2 = -5$$

$$\Rightarrow a = 3 \quad \text{or} \quad a = -7.$$

$\therefore$ required points are (3, 1) and $(-7, 11)$.

13. Side of the square = length of perp. from (0, 0) on $3x - 4y - 10 = 0$

$$= \frac{|3 \times 0 - 4 \times 0 - 10|}{\sqrt{3^2 + (-4)^2}} = 2 \text{ units.}$$

Area of the square $= (2 \times 2)$ sq units = 4 sq units.

16. Putting $x = 0$ in $y = mx + c$, we get $y = c$.

Thus, $P(0, c)$ is a point on $y = mx + c$.

Required distance = length of perp. from $P(0, c)$ on $y = mx + d$

$$= \frac{|m \times 0 - c + d|}{\sqrt{1 + m^2}} = \frac{|d - c|}{\sqrt{1 + m^2}}.$$

17. Putting $x = 0$ in $px + py + q = 0$, we get $y = (-q/p)$.

Required distance = length of perp. from $P\left(0, \frac{-q}{p}\right)$ on $px + py - r = 0$

$$= \frac{\left|p \times 0 + p \times \left(\frac{-q}{p}\right) - r\right|}{\sqrt{p^2 + p^2}} = \frac{|-q - r|}{\sqrt{2}p} = \frac{|q + r|}{\sqrt{2}p}.$$

19. Let the required equation be $y = -x + c$.

$$\therefore \quad \frac{|0 + 0 - c|}{\sqrt{1^2 + 1^2}} = 5 \Rightarrow |c| = 5\sqrt{2} \Rightarrow c = 5\sqrt{2} \quad \text{or} \quad c = -5\sqrt{2}.$$

MISCELLANEOUS PROBLEMS

It may be mentioned here that when the equations of two lines are given, then on solving them simultaneously, we get the point of intersection of the given lines.

EXAMPLE 1 *Find the point of intersection of the lines* $5x + 7y = 3$ *and* $2x - 3y = 7$.

SOLUTION The given equations are:

$$5x + 7y - 3 = 0 \qquad \text{... (i)}$$

$$2x - 3y - 7 = 0 \qquad \text{... (ii)}$$

On solving (i) and (ii), by cross multiplication, we get

$$\frac{x}{(-49 - 9)} = \frac{y}{(-6 + 35)} = \frac{1}{(-15 - 14)}$$

$$\Rightarrow \quad x = \left(\frac{-58}{-29}\right) = 2 \quad \text{and} \quad y = \left(\frac{29}{-29}\right) = -1.$$

Hence, the point of intersection of the given lines is $P(2, -1)$.

EXAMPLE 2 *Find the equation of the line parallel to the y-axis and drawn through the point of intersection of the lines* $x - 7y + 15 = 0$ *and* $2x + y = 0$.

SOLUTION The given equations are:

$$x - 7y + 15 = 0 \qquad \text{... (i)}$$
$$2x + y = 0 \qquad \text{... (ii)}$$

On solving (i) and (ii), we get $x = -1$ and $y = 2$.

Thus, the given lines intersect at the point $P(-1, 2)$.

The line parallel to the y-axis and drawn through $P(-1, 2)$ is given by $x = -1$, i.e., $x + 1 = 0$.

Hence, the required equation is $x + 1 = 0$.

EXAMPLE 3 *Find the equation of the line passing through the intersection of the lines* $x + 2y + 3 = 0$ *and* $3x + 4y + 7 = 0$, *and parallel to the line* $y - x = 8$.

SOLUTION The given intersecting lines are:

$$x + 2y + 3 = 0 \qquad \text{... (i)}$$
$$3x + 4y + 7 = 0 \qquad \text{... (ii)}$$

By cross multiplication, we get

$$\frac{x}{(14 - 12)} = \frac{y}{(9 - 7)} = \frac{1}{(4 - 6)} \Rightarrow \frac{x}{2} = \frac{y}{2} = \frac{1}{-2}$$

$$\Rightarrow \quad x = \frac{2}{-2} = -1 \quad \text{and} \quad y = \frac{2}{-2} = -1.$$

Thus, the lines (i) and (ii) intersect at the point $P(-1, -1)$.

Now, $y - x = 8 \Rightarrow y = x + 8$. ... (iii)

Slope of line (iii) is 1.

Slope of the line parallel to this line is 1.

So, the equation of the required line is $\dfrac{y + 1}{x + 1} = 1 \Rightarrow x - y = 0$.

EXAMPLE 4 *Find the value of k for which the lines* $3x + y = 2$, $kx + 2y = 3$ *and* $2x - y = 3$ *may intersect at a point.*

SOLUTION The given lines are:

$$3x + y - 2 = 0 \qquad \text{... (i)}$$
$$kx + 2y - 3 = 0 \qquad \text{... (ii)}$$
$$2x - y - 3 = 0 \qquad \text{... (iii)}$$

On solving (i) and (iii) by cross multiplication, we get

$$\frac{x}{(-3 - 2)} = \frac{y}{(-4 + 9)} = \frac{1}{(-3 - 2)} \Rightarrow \frac{x}{-5} = \frac{y}{5} = \frac{1}{-5}$$

$$\Rightarrow \quad x = \left(\frac{-5}{-5}\right) = 1 \quad \text{and} \quad y = \left(\frac{5}{-5}\right) = -1.$$

Thus, the point of intersection of (i) and (iii) is $P(1, -1)$.

For the given lines to intersect at a point, $x = 1$ and $y = -1$ must satisfy (ii) also.

$\therefore \quad (k \times 1) + 2 \times (-1) - 3 = 0 \;\Rightarrow\; k - 5 = 0 \;\Rightarrow\; k = 5.$

Hence, $k = 5$.

EXAMPLE 5 *Show that the lines $x - y = 6$, $4x - 3y = 20$ and $6x + 5y + 8 = 0$ are concurrent. Also find their point of intersection.*

SOLUTION The given equations are:

$$x - y - 6 = 0 \quad \text{... (i)}$$
$$4x - 3y - 20 = 0 \quad \text{... (ii)}$$
$$6x + 5y + 8 = 0 \quad \text{... (iii)}$$

On solving (i) and (ii) by cross multiplication, we get

$$\frac{x}{(20-18)} = \frac{y}{(-24+20)} = \frac{1}{(-3+4)} \;\Rightarrow\; \frac{x}{2} = \frac{y}{-4} = \frac{1}{1} \;\Rightarrow\; x = 2,\ y = -4.$$

Thus, the lines (i) and (ii) intersect at the point $P(2, -4)$.

Putting $x = 2$ and $y = -4$ in (iii), we get

$$\text{LHS} = 6 \times 2 + 5 \times (-4) + 8 = 0 = \text{RHS}.$$

This shows that the point $P(2, -4)$ also lies on (iii).

Thus, all the given three lines intersect at the same point.

Hence, the given lines are concurrent and their point of intersection is $P(2, -4)$.

EXAMPLE 6 *If the three lines $y = m_1x + c_1$, $y = m_2x + c_2$ and $y = m_3x + c_3$ are concurrent then show that*

$$m_1(c_2 - c_3) + m_2(c_3 - c_1) + m_3(c_1 - c_2) = 0.$$

SOLUTION The given lines are:

$$m_1x - y + c_1 = 0 \quad \text{... (i)}$$
$$m_2x - y + c_2 = 0 \quad \text{... (ii)}$$
$$m_3x - y + c_3 = 0 \quad \text{... (iii)}$$

On solving (i) and (ii) by cross multiplication, we get

$$\frac{x}{(-c_2 + c_1)} = \frac{y}{(m_2c_1 - m_1c_2)} = \frac{1}{(-m_1 + m_2)}$$

$$\Rightarrow \quad x = \frac{(c_1 - c_2)}{(m_2 - m_1)} \quad \text{and} \quad y = \frac{(m_2c_1 - m_1c_2)}{(m_2 - m_1)}.$$

Thus, the point of intersection of (i) and (ii) is $P\left(\dfrac{c_1 - c_2}{m_2 - m_1}, \dfrac{m_2c_1 - m_1c_2}{m_2 - m_1}\right)$.

Since the given three lines meet at a point, the point P must therefore lie on (iii) also.

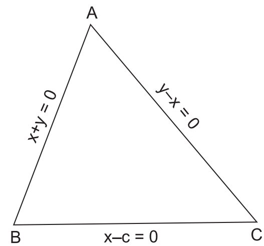
$$\therefore \quad m_3 \cdot \frac{(c_1 - c_2)}{(m_2 - m_1)} - \frac{(m_2c_1 - m_1c_2)}{(m_2 - m_1)} + c_3 = 0$$

$$\Rightarrow \quad m_3(c_1 - c_2) - (m_2c_1 - m_1c_2) + c_3(m_2 - m_1) = 0$$

$$\Rightarrow \quad m_1(c_2 - c_3) + m_2(c_3 - c_1) + m_3(c_1 - c_2) = 0.$$

EXAMPLE 7 *Find the area of the triangle formed by the lines* $y - x = 0$, $x + y = 0$ *and* $x - c = 0$.

SOLUTION Let the sides AC, AB and BC of $\triangle ABC$ be represented by $y - x = 0$, $x + y = 0$ and $x - c = 0$ respectively.



On solving these equations pairwise, we get the points $A(0, 0)$, $B(c, -c)$ and $C(c, c)$.

$\therefore$ area of $\triangle ABC$ is given by

$$\Delta = \frac{1}{2}|x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)|$$

$$= \frac{1}{2} \cdot |0 \cdot (-c - c) + c(c - 0) + c(0 + c)| \text{ sq units}$$

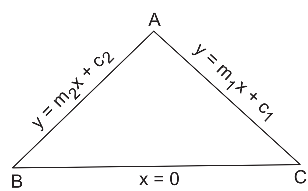
$$= \left(\frac{1}{2} \times 2c^2\right) \text{sq units} = c^2 \text{ sq units.}$$

Hence, the required area is c^2 sq units.

EXAMPLE 8 *Find the area of the triangle formed by the lines* $y = m_1 x + c_1$, $y = m_2 x + c_2$ *and* $x = 0$.

SOLUTION Let the sides AC, AB and BC of $\triangle ABC$ be represented by the equations:

$$m_1 x - y + c_1 = 0 \quad \text{... (i)}$$
$$m_2 x - y + c_2 = 0 \quad \text{... (ii)}$$
$$x = 0 \quad \text{... (iii)}$$



On solving (i) and (ii) by cross multiplication, we have

$$\frac{x}{(-c_2 + c_1)} = \frac{y}{(m_2 c_1 - m_1 c_2)} = \frac{1}{(-m_1 + m_2)}$$

$$\Rightarrow \quad x = \frac{(c_1 - c_2)}{(m_2 - m_1)} \quad \text{and} \quad y = \frac{(m_2 c_1 - m_1 c_2)}{(m_2 - m_1)}.$$

Thus, the lines AC and AB intersect at $A\left(\frac{c_1 - c_2}{m_2 - m_1}, \frac{m_2 c_1 - m_1 c_2}{m_2 - m_1}\right)$.

On solving (ii) and (iii), we get $B(0, c_2)$.

On solving (i) and (iii), we get $C(0, c_1)$.

$\therefore$ area of $\triangle ABC$ is given by

$$\Delta = \frac{1}{2} \cdot |x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)|$$

$$= \frac{1}{2} \cdot \left|\frac{(c_1 - c_2)}{(m_2 - m_1)} \cdot (c_2 - c_1) + 0 + 0\right| = \frac{1}{2} \cdot \frac{(c_1 - c_2)^2}{|m_1 - m_2|}.$$

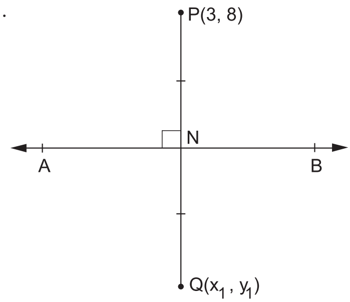
Hence, the area of the triangle formed by the given lines is $\frac{1}{2} \cdot \frac{(c_1 - c_2)^2}{|m_1 - m_2|}$.

EXAMPLE 9 *Find the image of the point P(3, 8) with respect to the line $x + 3y = 7$, assuming the given line to be a plane mirror.*

SOLUTION Let AB be the given line whose equation is $x + 3y - 7 = 0$ .... (i)

Now, $x + 3y = 7 \Rightarrow y = -\frac{1}{3}x + \frac{7}{3}$.

Slope of $AB = -\frac{1}{3}$.



Let $P(3, 8)$ be the given point and let $Q(x_1, y_1)$ be its image in AB. Join PQ, intersecting AB at N.

Then, $PN \perp AB$ and $PN = QN$.

Let the slope of PQ be m. Then,

$$m \times \left(-\frac{1}{3}\right) = -1 \Rightarrow m = 3 \quad [\because AB \perp PQ].$$

Equation of line PQ is given by

$$\frac{y-8}{x-3} = 3 \Rightarrow y - 8 = 3x - 9 \Rightarrow 3x - y - 1 = 0. \quad \text{... (ii)}$$

On solving (i) and (ii), we get

$$\frac{x}{(-3-7)} = \frac{y}{(-21+1)} = \frac{1}{(-1-9)} \Rightarrow \frac{x}{-10} = \frac{y}{-20} = \frac{1}{-10}$$

$$\Rightarrow x = 1 \quad \text{and} \quad y = 2.$$

Thus, AB and PQ intersect at the point $N(1, 2)$.

Also, N bisects PQ.

$$\therefore \left(\frac{3 + x_1}{2} = 1, \frac{8 + y_1}{2} = 2\right) \Rightarrow x_1 = -1 \quad \text{and} \quad y_1 = -4.$$

Hence, the image of the given point $P(3, 8)$ is $Q(-1, -4)$.

EXERCISE 20I

1. Find the points of intersection of the lines
   $4x + 3y = 5$ and $x = 2y - 7$.
2. Show that the lines $x + 7y = 23$ and $5x + 2y = 16$ intersect at the point (2, 3).
3. Show that the lines $3x - 4y + 5 = 0$, $7x - 8y + 5 = 0$ and $4x + 5y = 45$ are concurrent. Also find their point of intersection.
4. Find the value of k so that the lines $3x - y - 2 = 0$, $5x + ky - 3 = 0$ and $2x + y - 3 = 0$ are concurrent.

5. Find the image of the point $P(1, 2)$ in the line $x - 3y + 4 = 0$.
6. Find the area of the triangle formed by the lines $x + y = 6$, $x - 3y = 2$ and $5x - 3y + 2 = 0$.
7. Find the area of the triangle formed by the lines $x = 0$, $y = 1$ and $2x + y = 2$.
8. Find the area of the triangle, the equations of whose sides are $y = x$, $y = 2x$ and $y - 3x = 4$.
9. Find the equation of the perpendicular drawn from the origin to the line $4x - 3y + 5 = 0$. Also, find the coordinates of the foot of the perpendicular.
10. Find the equation of the perpendicular drawn from the point $P(-2, 3)$ to the line $x - 4y + 7 = 0$. Also, find the coordinates of the foot of the perpendicular.
11. Find the equations of the medians of a triangle whose sides are given by the equations $3x + 2y + 6 = 0$, $2x - 5y + 4 = 0$ and $x - 3y - 6 = 0$.

ANSWERS (EXERCISE 20I)

1. $(-1, 3)$ **3.** $(5, 5)$ **4.** $k = -2$ **5.** $\left(\frac{6}{5}, \frac{7}{5}\right)$

6. 12 sq units **7.** 1 sq unit **8.** 4 sq units

9. $3x + 4y = 0$, $\left(-\frac{4}{5}, \frac{3}{5}\right)$ **10.** $4x + y + 5 = 0$, $\left(\frac{-27}{17}, \frac{23}{17}\right)$

11. $41x - 112y - 70 = 0$, $16x - 59y - 120 = 0$ and $25x - 53y + 50 = 0$

SHIFTING OF ORIGIN

TRANSLATION OF AXES

Sometimes it is desirable to shift the origin to a new point and the new axes are transformed parallel to the original axes.

A transformation of this kind is known as translation of axes.

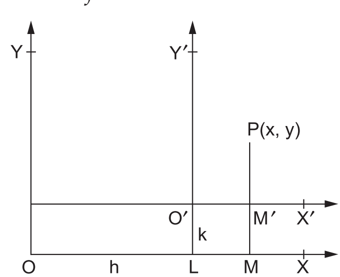
Let $P(x, y)$ be a point with reference to O as origin and OX and OY as the axes. Draw $PM \perp OX$, so that $OM = x$ and $PM = y$.

Let the new origin be $O'(h, k)$ referred to the old axes, i.e., $OL = h$ and $O'L = k$.

Draw $O'X' \parallel OX$, meeting PM at M'.

Also, draw $O'Y' \parallel OY$, meeting OM at L.

Now, we have the new origin at O' and the new axes as $O'X'$ and $O'Y'$.



Let $O'M' = x'$ and $M'P = y'$ with reference to the new axes. Then, $P(x', y')$ are the new coordinates of P.

Now, $OM = OL + LM \Rightarrow x = h + x' \Rightarrow x = x' + h \Rightarrow x' = x - h$.

And, $MP = MM' + PM' \Rightarrow y = k + y' \Rightarrow y = y' + k \Rightarrow y' = y - k$.

Hence, $x' = x - h$ and $y' = y - k$ are the relations between the new and old coordinates.

SOLVED EXAMPLES

EXAMPLE 1 *If the origin be shifted to the point $(2, -3)$ by a translation of coordinate axes, find the new coordiantes of the point $(4, 7)$.*

SOLUTION Let the given point be $P(x, y)$. Then, $x = 4$ and $y = 7$.

Let the origin O be shifted to the point $O'(h, k)$, where $h = 2$, $k = -3$.

Then, the new coordinates of P are (x', y'), where

$$x' = x - h = (4 - 2) = 2$$

and $y' = y - k = (7 + 3) = 10$.

Hence, the coordinates of (4, 7) in the new system are (2, 10).

EXAMPLE 2 *If the origin is shifted to the point $(2, 3)$, the coordinates of a point become $(5, -4)$. Find the original coordinates, when the axes are parallel.*

SOLUTION Let the original coordinates be $P(x, y)$.

Let the origin O be shifted to the new point $O'(h, k)$, where $h = 2$ and $k = 3$.

Then, the new coordinates of P are (x', y'), where $x' = 5$ and $y' = -4$.

Now, $x' = x - h \Rightarrow 5 = x - 2 \Rightarrow x = 7$.

And, $y' = y - k \Rightarrow -4 = y - 3 \Rightarrow y = -1$.

Hence, the original coordinates are (7, –1).

EXAMPLE 3 *At what point the origin be shifted, if the coordinates of a point $(4, 5)$ become $(3, 7)$?*

SOLUTION Let $O'(h, k)$ be the point to which the origin O be shifted.

Then, $(x = 4, y = 5)$ and $(x' = 3, y' = 7)$.

Now, $x' = x - h \Rightarrow h = x - x' = (4 - 3) = 1$.

And, $y' = y - k \Rightarrow k = y - y' = (5 - 7) = -2$.

Hence, the origin O must be shifted to the point $O'(1, -2)$.

EXAMPLE 4 *If the origin be shifted to the point $(3, -1)$, find the new equation of the line $2x - 3y + 5 = 0$.*

SOLUTION Let the origin O be shifted to the point $O'(h, k)$, where $h = 3$ and $k = -1$.

Let the new coordinates of $P(x, y)$ be $P(x', y')$.

Then, $x' = x - h \Rightarrow x' = x - 3 \Rightarrow x = x' + 3$.

And, $y' = y - k \Rightarrow y' = y + 1 \Rightarrow y = y' - 1$.

So, the new equation becomes:

$$2(x' + 3) - 3(y' - 1) + 5 = 0 \Rightarrow 2x' - 3y' + 14 = 0.$$

Hence, the equation of the straight line in the new system is $2x - 3y + 14 = 0$.

EXAMPLE 5 *Find the point to which the origin be shifted after a translation, so that the equation* $x^2 + y^2 - 4x - 8y + 3 = 0$ *will have no first degree terms.*

SOLUTION Let the origin O be shifted to a point $O'(h, k)$.

Let the new coordinates of $P(x, y)$ be $P(x', y')$.

Then, $x' = x - h \Rightarrow x = x' + h$.

And, $y' = y - k \Rightarrow y = y' + k$.

So, the new equation becomes:

$$(x' + h)^2 + (y' + k)^2 - 4(x' + h) - 8(y' + k) + 3 = 0$$

$$\Rightarrow (x'^2 + h^2 + 2x'h) + (y'^2 + k^2 + 2y'k) - 4(x' + h) - 8(y' + k) + 3 = 0$$

$$\Rightarrow x'^2 + y'^2 + (2h - 4)x' + (2k - 8)y' + (h^2 + k^2 - 4h - 8k + 3) = 0.$$

Since we are required to get an equation free from first degree terms, so we have:

$$(2h - 4 = 0 \text{ and } 2k - 8 = 0)$$

$$\Rightarrow (2h = 4 \text{ and } 2k = 8) \Rightarrow (h = 2 \text{ and } k = 4).$$

Hence, the origin O should be shifted to the point $O'(2, 4)$.

EXERCISE 20J

1. If the origin is shifted to the point (1, 2) by a translation of the axes, find the new coordinates of the point (3, –4).
2. If the origin is shifted to the point (–3, –2) by a translation of the axes, find the new coordinates of the point (3, –5).
3. If the origin is shifted to the point (0, –2) by a translation of the axes, the coordinates of a point become (3, 2). Find the original coordinates of the point.
4. If the origin is shifted to the point (2, –1) by a translation of the axes, the coordinates of a point become (–3, 5). Find the original coordinates of the point.
5. At what point must the origin be shifted, if the coordinates of a point (–4, 2) become (3, –2)?

***Find what the given equation becomes when the origin is shifted to the point* (1, 1).**

6. $x^2 + xy - 3x - y + 2 = 0$
7. $xy - y^2 - x + y = 0$
8. $x^2 - y^2 - 2x + 2y = 0$
9. $xy - x - y + 1 = 0$
10. Transform the equation $2x^2 + y^2 - 4x + 4y = 0$ to parallel axes when the origin is shifted to the point (1, –2).

ANSWERS (EXERCISE 20J)

1. (2, – 6) **2.** (6, – 3) **3.** (3, 0) **4.** (–1, 4)
5. (–7, 4) **6.** $x^2 + xy = 0$ **7.** $y^2 - xy = 0$ **8.** $x^2 - y^2 = 0$

9. $xy = 0$ **10.** $2x^2 + y^2 - 6 = 0$

EQUATION OF FAMILY OF LINES PASSING THROUGH THE POINT OF INTERSECTION OF TWO LINES

Let the two intersecting lines L_1 and L_2 be given by

$$A_1x + B_1y + C_1 = 0 \qquad \text{... (i)}$$

and $$A_2x + B_2y + C_2 = 0 \qquad \text{... (ii)}$$

From (i) and (ii), we can form an equation

$$(A_1x + B_1y + C_1) + k(A_2x + B_2y + C_2) = 0, \qquad \text{... (iii)}$$

where k is an arbitrary constant, called *parameter*.

Clearly, for any value of k, the equation (iii) is of first degree in x and y. Hence, it represents a family of lines.

Putting some value of k, we get a particular member of this family.

SOLVED EXAMPLES

EXAMPLE 1 *Find the equation of the line drawn through the point of intersection of the lines* $4x - 3y + 7 = 0$ *and* $2x + 3y + 5 = 0$ *and passing through the point* $(-4, 5)$.

SOLUTION The equation of any line through the point of intersection of the given lines is of the form

$$(4x - 3y + 7) + k(2x + 3y + 5) = 0$$

$$\Rightarrow \quad (4 + 2k)x + (3k - 3)y + (5k + 7) = 0 \qquad \text{... (i)}$$

If it passes through the point $(-4, 5)$, we have

$$(4 + 2k)(-4) + (3k - 3) \cdot 5 + (5k + 7) = 0$$

$$\Rightarrow \quad -16 - 8k + 15k - 15 + 5k + 7 = 0 \;\Rightarrow\; 12k = 24 \;\Rightarrow\; k = 2.$$

Substituting $k = 2$ in (i), we get

$8x + 3y + 17 = 0$, which is the required equation.

EXAMPLE 2 *Find the equation of the line through the intersection of lines* $3x + 4y = 7$ *and* $x - y + 2 = 0$ *and whose slope is* 5.

SOLUTION The given lines are $3x + 4y - 7 = 0$ and $x - y + 2 = 0$.

The equation of any line through the point of intersection of the given lines is of the form.

$$(3x + 4y - 7) + k(x - y + 2) = 0 \qquad \text{... (i)}$$

$$\Rightarrow \quad (3 + k)x + (4 - k)y + (2k - 7) = 0$$

$$\Rightarrow \quad (4 - k)y = -(3 + k)x + (7 - 2k)$$

$$\Rightarrow \quad y = \frac{-(3 + k)}{4 - k}x + \frac{(7 - 2k)}{(4 - k)}$$

$$\Rightarrow \quad y = \frac{(k + 3)}{(k - 4)}x + \frac{(7 - 2k)}{(4 - 2k)}\cdot$$

Slope of this line is $\dfrac{(k+3)}{(k-4)}\cdot$

$\therefore \quad \dfrac{k+3}{k-4} = 5 \Rightarrow k + 3 = 5k - 20 \Rightarrow 4k = 23 \Rightarrow \dfrac{23}{4}\cdot$

Substituting $k = \dfrac{23}{4}$ in (i), we get

$$(3x + 4y - 7) + \frac{23}{4}(x - y + 2) = 0$$

$\Rightarrow \quad 4(3x + 4y - 7) + 23(x - y + 2) = 0$

$\Rightarrow \quad 35x - 7y + 18 = 0$, which is the required equation.

EXAMPLE 3 *Find the equation of the line through the intersection of lines* $x + 2y - 3 = 0$ *and* $4x - y + 7 = 0$ *and which is parallel to the line* $5x + 4y - 20 = 0$.

SOLUTION $5x + 4y - 20 = 0 \Rightarrow y = \dfrac{-5}{4}x + 5$

$\therefore \quad$ slope of the given line $= \dfrac{-5}{4}$

and slope of the required line $= \dfrac{-5}{4}\cdot$

Now, the equation of any line through the intersection of the given lines is of the form

$$(x + 2y - 3) + k(4x - y + 7) = 0 \qquad \dots \text{(i)}$$

$\Rightarrow \quad (1 + 4k)x + (2 - k)y + (7k - 3) = 0$

$\Rightarrow \quad (2 - k)y = -(1 + 4k)x + (3 - 7k)$

$\Rightarrow \quad y = \dfrac{-(1+4k)}{(2-k)}x + \dfrac{(3-7k)}{(2-k)}$

$\Rightarrow \quad y = \dfrac{(1+4k)}{(k-2)}x + \dfrac{(3-7k)}{(2-k)}\cdot$

Slope of this line $= \dfrac{(1+4k)}{(k-2)}$

$\therefore \quad \dfrac{(1+4k)}{(k-2)} = \dfrac{-5}{4} \Rightarrow 4 + 16k = -5k + 10$

$\Rightarrow 21k = 6 \Rightarrow k = \dfrac{6}{21} = \dfrac{2}{7}\cdot$

Substituting $k = \dfrac{2}{7}$ in (i), we get

$$(x + 2y - 3) + \frac{2}{7}(4x - y + 7) = 0$$

$\Rightarrow \quad (7x + 14y - 21) + (8x - 2y + 14) = 0$

$\Rightarrow \quad 15x + 12y - 7 = 0$, which is the required equation.

EXAMPLE 4 *Find the equation of the line through the intersection of the lines* $3x + y - 9 = 0$ *and* $4x + 3y - 7 = 0$ *and which is perpendicular to the line* $5x - 4y + 1 = 0$.

SOLUTION $5x - 4y + 1 = 0 \Rightarrow y = \frac{5}{4}x + \frac{1}{4}$.

Slope of the given line is $\frac{5}{4}$.

Now, the equation of any line through the intersection of the given lines is of the form

$$(3x + y - 9) + k(4x + 3y - 7) = 0 \quad \ldots \text{(i)}$$

$$\Rightarrow \quad (3 + 4k)x + (1 + 3k)y - (9 + 7k) = 0$$

$$\Rightarrow \quad (1 + 3k)y = -(3 + 4k)x + (9 + 7k)$$

$$\Rightarrow \quad y = -\frac{(3 + 4k)}{(1 + 3k)}x + \frac{(9 + 7k)}{(1 + 3k)} \quad \ldots \text{(ii)}$$

Let m be the slope of the line perpendicular to the required line.

Then, $m \times \frac{5}{4} = -1 \Rightarrow m = \frac{-4}{5}$.

$\therefore$ we must have $\frac{-(3 + 4k)}{(1 + 3k)} = \frac{-4}{5}$

$$\Rightarrow \quad 15 + 20k = 4 + 12k \Rightarrow 8k = -11 \Rightarrow k = \frac{-11}{8}.$$

Substituting $k = \frac{-11}{8}$ in (i), we get

$$(3x + y - 9) - \frac{11}{8}(4 + 3y - 7) = 0$$

$$\Rightarrow \quad (24x + 8y - 72) - 44x - 33y + 77 = 0$$

$$\Rightarrow \quad 20x + 25y - 5 = 0 \Rightarrow 4x + 5y - 1 = 0.$$

EXAMPLE 5 *Find the equation of the x line parallel to the y-axis and drawn through the point of intersection of the lines* $x - 7y + 5 = 0$ *and* $3x + y - 7 = 0$.

SOLUTION The equation of any line through the point of intersection of the given lines is of the form

$$(x - 7y + 5) + k(3x + y - 7) = 0$$

$$\Rightarrow \quad (1 + 3k)x + (k - 7)y + (5 - 7k) = 0 \quad \ldots \text{(i)}$$

If the line is parallel to y-axis then coefficient of y should be 0, i.e., $k - 7 = 0$, which gives $k = 7$.

Substituting $k = 7$ in (i), we get: $22x - 44 = 0$

$\therefore$ $x - 2 = 0$, which is the required equation.

EXAMPLE 6 *Find the equation of the line through the intersection of the lines* $2x + 3y = 4$ *and* $x - 5y + 7 = 0$ *that has its x-intercept equal to* -4.

SOLUTION The given lines are $2x + 3y - 4 = 0$ and $x - 5y + 7 = 0$.

The equation of any line through the intersection of the given lines is of the form

$$(2x + 3y - 4) + k(x - 5y + 7) = 0$$

$$\Rightarrow \quad (2 + k)x + (3 - 5k)y + (7k - 4) = 0 \quad \ldots \text{(i)}$$

If this line has x-intercept –4, then the point (–4, 0) lies on (i).

$\therefore \quad (2+k)(-4)+(7k-4)=0 \Rightarrow -8-4k+7k-4=0$

$\Rightarrow \quad 3k=12 \Rightarrow k=4.$

Substituting $k=4$ in (i), we get

$6x-17y+24=0$, which is the required equation.

EXERCISE 20K

1. Find the equation of the line drawn through the point of intersection of the lines $x-2y+3=0$ and $2x-3y+4=0$ and passing through the point (4, –5).

2. Find the equation of the line drawn through the point of intersection of the lines $x-y=7$ and $2x+y=2$ and passing through the origin.

3. Find the equation of the line drawn through the point of intersection of the lines $x+y=9$ and $2x-3y+7=0$ and whose slope is $\frac{-2}{3}\cdot$

4. Find the equation of the line drawn through the point of intersection of the lines $x-y=1$ and $2x-3y+1=0$ and which is parallel to the line $3x+4y=12$.

5. Find the equation of the line through the intersection of the lines $5x-3y=1$ and $2x+3y=23$ and which is perpendicular to the line $5x-3y=1$.

6. Find the equation of the line through the intersection of the lines $2x-3y=0$ and $4x-5y=2$ and which is perpendicular to the line $x+2y+1=0$.

7. Find the equation of the line through the intersection of the lines $x-7y+5=0$ and $3x+y-7=0$ and which is parallel to x-axis.

8. Find the equation of the line through the intersection of the lines $2x-3y+1=0$ and $x+y-2=0$ and drawn parallel to y-axis.

9. Find the equation of the line through the intersection of the lines $2x+3y-2=0$ and $x-2y+1=0$ and having x-intercept equal to 3.

10. Find the equation of the line passing through the intersection of the lines $3x-4y+1=0$ and $5x+y-1=0$ and which cuts off equal intercepts from the axes.

ANSWERS (EXERCISE 20K)

1. $7x+3y-13=0$ **2.** $4x+3y=0$

3. $2x+3y-23=0$ **4.** $3x-4y+24=0$

5. $63x+105y-781=0$ **6.** $2x-y-4=0$

7. $y=1$ **8.** $x=1$

9. $x+5y-3=0$ **10.** $23x+23y=11$

HINTS TO SOME SELECTED QUESTIONS

7. When a line is parallel to x-axis, then coefficient of x should be 0.

10. Let the required line be

$$(3x - 4y + 1) + k(5x + y - 1) = 0$$

$$\Rightarrow \quad (3 + 5k)\,x + (k - 4)\,y + (1 - k) = 0$$

$$\Rightarrow \quad (3 + 5k)\,x + (k - 4)\,y = (k - 1)$$

$$\Rightarrow \quad \frac{(3 + 5k)}{(k - 1)}x + \frac{(k - 4)}{(k - 1)}y = 0 \Rightarrow \frac{x}{\frac{(k - 1)}{(3 + 5k)}} + \frac{y}{\frac{(k - 1)}{(k - 4)}} = 1$$

$$\therefore \quad \frac{k - 1}{3 + 5k} = \frac{k - 1}{k - 4} \Rightarrow k - 4 = 3 + 5k \Rightarrow 4k = -7 \Rightarrow k = \frac{-7}{4}\cdot$$

❑

21
Circle

CIRCLE *A circle is the set of all points in a plane which are at a constant distance from a fixed point in the plane.*

The fixed point is called the *centre* and the constant distance is called the *radius* of the circle.

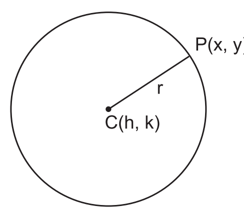
Equation of a Circle in Standard Form

THEOREM 1 *The equation of a circle with centre (h, k) and radius r is given by*

$$(x-h)^2+(y-k)^2=r^2.$$

PROOF Let us consider a circle with centre $C(h, k)$ and radius r.

Let $P(x, y)$ be an arbitrary point on the circle.



Then, $|CP|=r \Rightarrow \sqrt{(x-h)^2+(y-k)^2}=r$

$\Rightarrow (x-h)^2+(y-k)^2=r^2$,

which is the required equation of the given circle.

THEOREM 2 *The equation of a circle with centre at the origin and radius r is given by*

$$x^2+y^2=r^2.$$

PROOF The equation of a circle with centre $C(0, 0)$ and radius r is given by

$$(x-0)^2+(y-0)^2=r^2 \Rightarrow x^2+y^2=r^2.$$

Equation of a Circle, the End Points of whose Diameter are given

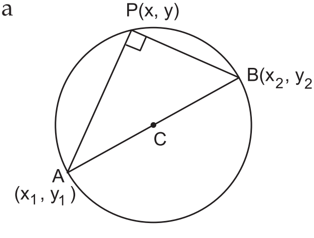
THEOREM 3 *The equation of a circle described on the line joining $A(x_1, y_1)$ and $B(x_2, y_2)$ as a diameter is $(x-x_1)(x-x_2)+(y-y_1)(y-y_2)=0$.*

PROOF Let $A(x_1, y_1)$ and $B(x_2, y_2)$ be the end points of a diameter of the given circle, and let $P(x, y)$ be any point on the circle.

Join AP and BP.

Then, the angle in a semicircle being a right angle,

we have $\angle APB = 90°$.



Now, slope of $AP=\dfrac{(y-y_1)}{(x-x_1)}$.

And, slope of $BP=\dfrac{(y-y_2)}{(x-x_2)}$.

Since $AP \perp BP$, we have

$$\frac{(y-y_1)}{(x-x_1)} \cdot \frac{(y-y_2)}{(x-x_2)} = -1 \Rightarrow (x-x_1)(x-x_2)+(y-y_1)(y-y_2)=0,$$

which is the required equation of the circle.

SUMMARY

(i) The equation of a circle with centre $C(h, k)$ and radius r is given by $\mathbf{(x-h)^2+(y-k)^2=r^2}$.

(ii) The equation of a circle with centre $C(0, 0)$ and radius r is given by $\mathbf{x^2+y^2=r^2}$.

(iii) The equation of a circle with $A(x_1, y_1)$ and $B(x_2, y_2)$ as the end points of a diameter is given by

$$\mathbf{(x-x_1)(x-x_2)+(y-y_1)(y-y_2)=0.}$$

SOLVED EXAMPLES

EXAMPLE 1 *Find the equation of a circle with centre* $(3, -2)$ *and radius* 5.

SOLUTION We know that the equation of a circle with centre $C(h, k)$ and radius r is given by

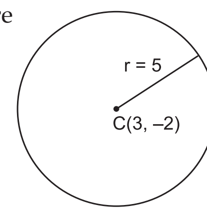
$$(x-h)^2+(y-k)^2=r^2.$$

Here, $h=3$, $k=-2$ and $r=5$.

$\therefore$ the required equation of the circle is

$$(x-3)^2+(y+2)^2=5^2$$

$\Rightarrow$ $x^2+y^2-6x+4y-12=0$.



EXAMPLE 2 *Find the equation of a circle whose centre is* $(2, -1)$ *and which passes through the point* $(3, 6)$.

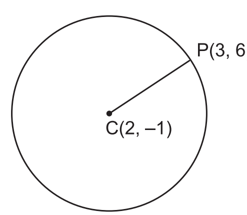
SOLUTION Let $C(2, -1)$ be the centre of the given circle and let it pass through the point $P(3, 6)$. Then, radius of the circle

$$= |CP| = \sqrt{(3-2)^2+(6+1)^2} = \sqrt{50}.$$

$\therefore$ the required equation of the circle is

$$(x-2)^2+(y+1)^2=(\sqrt{50})^2$$

$\Rightarrow$ $x^2+y^2-4x+2y-45=0$.



EXAMPLE 3 *Find the equation of the circle passing through the point* $(2, 4)$ *and having its centre at the intersection of the lines* $x-y=4$ *and* $2x+3y+7=0$.

SOLUTION The equations of the given lines are:

$$x-y=4 \qquad \ldots \text{(i)}$$

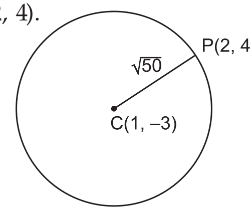
$$2x+3y=-7 \qquad \ldots \text{(ii)}$$

Solving (i) and (ii) simultaneously, we get $x = 1$ and $y = -3$.

So, the point of intersection of the given lines is $C(1, -3)$.

$\therefore$ centre of the given circle is $C(1, -3)$.

Also, the circle passes through the point $P(2, 4)$.



$\therefore$ radius of the circle

$= |CP| = \sqrt{(1-2)^2 + (-3-4)^2} = \sqrt{50}.$

$\therefore$ the required equation of the circle is

$(x-1)^2 + (y+3)^2 = (\sqrt{50})^2$

$\Rightarrow x^2 + y^2 - 2x + 6y - 40 = 0.$

EXAMPLE 4 *Find the equation of a circle of radius 5 units, whose centre lies on the x-axis and which passes through the point (2, 3).*

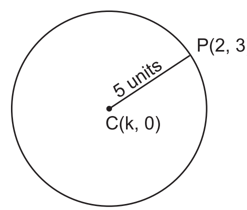
SOLUTION It is given that the centre of the circle lies on the x-axis.

So, let $C(k, 0)$ be the centre of the circle.

Also, it is given that it passes through the point $P(2, 3)$.

$\therefore$ radius of the circle $= |CP|$

$$= \sqrt{(k-2)^2 + (0-3)^2}$$
$$= \sqrt{k^2 - 4k + 13}$$



$\Rightarrow \sqrt{k^2 - 4k + 13} = 5$ [$\because$ radius = 5 units]

$\Rightarrow k^2 - 4k + 13 = 25 \Rightarrow k^2 - 4k - 12 = 0$

$\Rightarrow (k-6)(k+2) = 0 \Rightarrow k = 6$ or $k = -2$.

$\therefore$ centre of the circle is $(6, 0)$ or $(-2, 0)$.

Hence, the required equation of the circle is

$(x-6)^2 + (y-0)^2 = 5^2$ or $(x+2)^2 + (y-0)^2 = 5^2$

$\Rightarrow x^2 + y^2 - 12x + 11 = 0$ or $x^2 + y^2 + 4x - 21 = 0.$

Thus, there are two circles satisfying the given conditions.

EXAMPLE 5 *Find the equation of a circle, the end points of one of whose diameters are $A(2, -3)$ and $B(-3, 5)$.*

SOLUTION We know that the equation of a circle, the end points of one of whose diameters are (x_1, y_1) and (x_2, y_2), is given by

$$(x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0.$$

Here, $x_1 = 2$, $y_1 = -3$ and $x_2 = -3$, $y_2 = 5$.

$\therefore$ the required equation of the circle is

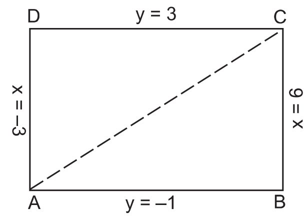
$(x-2)(x+3) + (y+3)(y-5) = 0.$

$\Rightarrow x^2 + y^2 + x - 2y - 21 = 0.$

EXAMPLE 6 *Find the equation of the circle drawn on the diagonal of the rectangle as its diameter, whose sides are $x = 6$, $x = -3$, $y = 3$ and $y = -1$.*

SOLUTION Let $ABCD$ be the given rectangle and let its sides AD, BC, AB and CD be represented by the equations $x = -3$, $x = 6$, $y = -1$ and $y = 3$ respectively.

Then, the coordinates of A and C are $A(-3, -1)$ and $C(6, 3)$.



So, the equation of the circle with AC as diameter is given by

$$(x+3)(x-6)+(y+1)(y-3)=0$$

$$\Rightarrow \quad x^2+y^2-3x-2y-21=0.$$

EXAMPLE 7 *If $y = 2x$ is a chord of the circle $x^2 + y^2 - 10x = 0$, find the equation of the circle with this chord as a diameter.*

SOLUTION The points of intersection of the given chord and the given circle are obtained by simultaneously solving

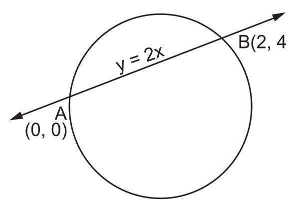
$$y = 2x \text{ and } x^2 + y^2 - 10x = 0.$$

Putting $y = 2x$ in $x^2 + y^2 - 10x = 0$, we get

$$5x^2 - 10x = 0 \Leftrightarrow 5x(x-2) = 0 \Leftrightarrow x = 0 \text{ or } x = 2.$$

Now, $x = 0 \Rightarrow y = 0$, and $x = 2 \Rightarrow y = 4$.

$\therefore$ the points of intersection of the given chord and the given circle are $A(0, 0)$ and $B(2, 4)$.



$\therefore$ the required equation of the circle with AB as diameter is

$$(x-0)(x-2)+(y-0)(y-4)=0$$

$$\Rightarrow \quad x^2+y^2-2x-4y=0.$$

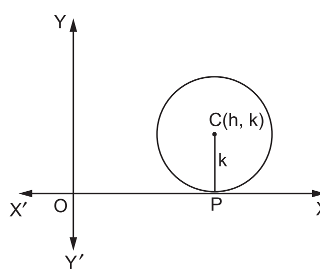
EXAMPLE 8 *Find the equation of a circle with centre (h, k) and touching the x-axis.*

SOLUTION Clearly, the radius of the circle $= k$.

So, the required equation is

$$(x-h)^2+(y-k)^2=k^2$$

$$\Rightarrow \quad x^2+y^2-2hx-2ky+h^2=0.$$



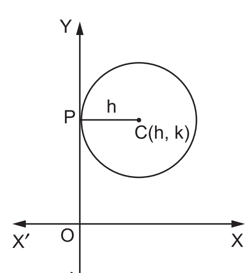
EXAMPLE 9 *Find the equation of a circle with centre (h, k) and touching the y-axis.*

SOLUTION Clearly, the radius of the circle $= h$.

So, the required equation is

$$(x-h)^2+(y-k)^2=h^2$$

$$\Rightarrow \quad x^2+y^2-2hx-2ky+k^2=0.$$



EXAMPLE 10 *Find the equation of a circle with centre (h, k) and touching both the axes.*

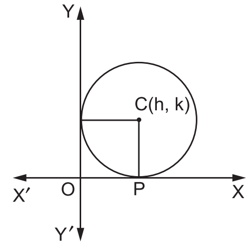
SOLUTION Clearly, radius $= h = k = c$ (say).

$\therefore$ the equation of the circle is

$(x - c)^2 + (y - c)^2 = c^2$

$\Rightarrow \quad x^2 + y^2 - 2c(x + y) + c^2 = 0,$

where $c = h = k$.



EXERCISE 21A

Find the equation of a circle with

1. centre (2, 4) and radius 5
2. centre (–3, –2) and radius 6
3. centre (a, a) and radius $\sqrt{2}$
4. centre $(a\cos\alpha, a\sin\alpha)$ and radius a
5. centre $(-a, -b)$ and radius $\sqrt{a^2 - b^2}$
6. centre at the origin and radius 4
7. Find the centre and radius of each of the following circles:

   (i) $(x - 3)^2 + (y - 1)^2 = 9$ (ii) $\left(x - \frac{1}{2}\right)^2 + \left(y + \frac{1}{3}\right)^2 = \frac{1}{16}$

   (iii) $(x + 5)^2 + (y - 3)^2 = 20$ (iv) $x^2 + (y - 1)^2 = 2$
8. Find the equation of the circle whose centre is (2, −5) and which passes through the point (3, 2).
9. Find the equation of the circle of radius 5 cm, whose centre lies on the y-axis and which passes through the point (3, 2).
10. Find the equation of the circle whose centre is (2, –3) and which passes through the intersection of the lines $3x + 2y = 11$ and $2x + 3y = 4$.
11. Find the equation of the circle passing through the point (–1, 3) and having its centre at the point of intersection of the lines $x - 2y = 4$ and $2x + 5y + 1 = 0$.
12. If two diameters of a circle lie along the lines $x - y = 9$ and $x - 2y = 7$, and the area of the circle is 38.5 sq cm, find the equation of the circle.
13. Find the equation of the circle, the coordinates of the end points of one of whose diameters are

    (i) $A(3, 2)$ and $B(2, 5)$ (ii) $A(5, -3)$ and $B(2, -4)$

    (iii) $A(-2, -3)$ and $B(-3, 5)$ (iv) $A(p, q)$ and $B(r, s)$
14. The sides of a rectangle are given by the equations $x = -2$, $x = 4$, $y = -2$ and $y = 5$. Find the equation of the circle drawn on the diagonal of this rectangle as its diameter.

ANSWERS (EXERCISE 21A)

1. $x^2 + y^2 - 4x - 8y - 5 = 0$
2. $x^2 + y^2 + 6x + 4y - 23 = 0$
3. $x^2 + y^2 - 2ax - 2ay = 0$
4. $x^2 + y^2 - 2ax\cos\alpha - 2ay\sin\alpha = 0$
5. $x^2 + y^2 + 2ax + 2by + 2b^2 = 0$
6. $x^2 + y^2 - 16 = 0$
7. (i) Centre (3, 1), radius = 3 (ii) Centre $\left(\frac{1}{2}, \frac{-1}{3}\right)$, radius $= \frac{1}{4}$
   (iii) Centre (–5, 3), radius $= 2\sqrt{5}$ (iv) Centre (0, 1), radius $= \sqrt{2}$
8. $x^2 + y^2 - 4x + 10y - 21 = 0$
9. $(x^2 + y^2 - 12y + 11 = 0)$ or $(x^2 + y^2 + 4y - 21 = 0)$
10. $x^2 + y^2 - 4x + 6y + 3 = 0$
11. $x^2 + y^2 - 4x + 2y - 20 = 0$
12. $4x^2 + 4y^2 - 88x - 16y + 451 = 0$
13. (i) $x^2 + y^2 - 5x - 7y + 16 = 0$ (ii) $x^2 + y^2 - 7x + 7y + 22 = 0$
    (iii) $x^2 + y^2 + 5x - 2y - 9 = 0$ (iv) $(x - p)(x - r) + (y - q)(y - s) = 0$
14. $x^2 + y^2 - 2x - 3y - 18 = 0$

HINTS TO SOME SELECTED QUESTIONS

9. Let the centre of the circle be $C(0, k)$.

It is given that the circle passes through $P(3, 2)$.

Then, $|CP|^2 = 5^2 \Leftrightarrow (0-3)^2 + (2-k)^2 = 5^2$

$\Leftrightarrow (2-k) = \pm 4 \Leftrightarrow k = 6$ or $k = -2$.

$\therefore$ the centre of the circle is (0, 6) or (0, –2).

Hence, the equation of the circle is

$$[(x-0)^2 + (y-6)^2 = 5^2] \text{ or } [(x-0)^2 + (y+2)^2 = 5^2].$$

12. We know that the point of intersection of two diameters of a circle is the centre of the circle.

On solving $x - y = 9$ and $x - 2y = 7$, we get the centre as $C(11, 2)$.

Now, $\pi r^2 = \frac{77}{2} \Leftrightarrow |r^2| = \left(\frac{77}{2} \times \frac{7}{22}\right) = \frac{49}{4} \Leftrightarrow r = \frac{7}{2}$.

$\therefore$ the required equation is $(x-11)^2 + (y-2)^2 = \frac{49}{4}$.

GENERAL EQUATION OF A CIRCLE

THEOREM *The general equation of a circle is of the form $x^2 + y^2 + 2gx + 2fy + c = 0$. Also, every equation of the form $x^2 + y^2 + 2gx + 2fy + c = 0$ represents a circle if $(g^2 + f^2 - c) > 0$.*

PROOF We know that the equation of a circle with centre (h, k) and radius r is given by

$$(x-h)^2 + (y-k)^2 = r^2$$

$\Rightarrow \quad x^2 + y^2 - 2hx - 2ky + (h^2 + k^2 - r^2) = 0$

$\Rightarrow \quad x^2 + y^2 + 2gx + 2fy + c = 0$, where $-h = g$, $-k = f$ and $(h^2 + k^2 - r^2) = c$.

Hence, the general equation of a circle is of the form

$$x^2 + y^2 + 2gx + 2fy + c = 0.$$

Conversely, let $x^2 + y^2 + 2gx + 2fy + c = 0$ be the given equation. Then,

$x^2 + y^2 + 2gx + 2fy + c = 0$

$\Rightarrow \quad (x^2 + 2gx + g^2) + (y^2 + 2fy + f^2) = (g^2 + f^2 - c)$

$\Rightarrow \quad (x + g)^2 + (y + f)^2 = (\sqrt{g^2 + f^2 - c})^2$

$\Rightarrow \quad \{x - (-g)\}^2 + \{y - (-f)\}^2 = (\sqrt{g^2 + f^2 - c})^2$

$\Rightarrow \quad (x - h)^2 + (y - k)^2 = r^2$, where $h = -g$, $k = -f$ and $r = \sqrt{g^2 + f^2 - c}$.

Clearly, this equation will represent a circle if $(g^2 + f^2 - c) > 0$.

The centre of this circle is $(-g, -f)$ and its radius is $\sqrt{g^2 + f^2 - c}$.

NOTE (i) If $(g^2 + f^2 - c) = 0$ then the above equation represents a *point circle.*

(ii) If $(g^2 + f^2 - c) < 0$ then the above equation does not represent a circle.

REMARK The equation of a circle has the following properties:

(i) It is a second-degree equation in x and y.

(ii) It contains no term of xy.

(iii) Coefficient of x^2 = coefficient of y^2.

Thus, the equation $ax^2 + by^2 + 2hxy + 2gx + 2fy + c = 0$ will represent a circle only when $a = b$ and $h = 0$.

SUMMARY

The general equation of a circle is

$$\mathbf{x^2 + y^2 + 2gx + 2fy + c = 0.}$$

Its centre is $(-g, -f)$ and radius $= \sqrt{g^2 + f^2 - c}$.

SOLVED EXAMPLES

EXAMPLE 1 *Show that the equation $x^2 + y^2 - 6x + 4y - 36 = 0$ represents a circle. Also, find its centre and radius.*

SOLUTION The given equation is $x^2 + y^2 - 6x + 4y - 36 = 0$.

This is of the form $x^2 + y^2 + 2gx + 2fy + c = 0$,

where $2g = -6$, $2f = 4$ and $c = -36$.

$\therefore \quad g = -3$, $f = 2$ and $c = -36$.

Hence, the given equation represents a circle.

Centre of the circle $= (-g, -f) = (3, -2)$.

Radius of the circle $= \sqrt{g^2 + f^2 - c} = \sqrt{3^2 + (-2)^2 + 36} = \sqrt{49} = 7$ units.

EXAMPLE 2 *Show that the equation* $3x^2 + 3y^2 + 12x - 18y - 11 = 0$ *represents a circle. Also, find its centre and radius.*

SOLUTION $3x^2 + 3y^2 + 12x - 18y - 11 = 0$

$$\Rightarrow \quad x^2 + y^2 + 4x - 6y - \frac{11}{3} = 0. \qquad \ldots \text{(i)}$$

This is of the form $x^2 + y^2 + 2gx + 2fy + c = 0$,

where $2g = 4$, $2f = -6$ and $c = -\frac{11}{3}$.

$\therefore \quad g = 2$, $f = -3$ and $c = -\frac{11}{3}$.

Hence, the given equation represents a circle.

Centre of the circle $= (-g, -f) = (-2, 3)$.

Radius of the circle $= \sqrt{g^2 + f^2 - c}$

$$= \sqrt{(-2)^2 + 3^2 + \frac{11}{3}} = \sqrt{\frac{50}{3}} \text{ units.}$$

EXAMPLE 3 *Find the equation of a circle passing through the points* $(5, 7)$, $(6, 6)$ *and* $(2, -2)$. *Find its centre and radius.*

SOLUTION Let the required equation of the circle be

$$x^2 + y^2 + 2gx + 2fy + c = 0. \qquad \ldots \text{(i)}$$

Since it passes through each of the points $(5, 7)$, $(6, 6)$ and $(2, -2)$, each one of these points must satisfy (i).

$$\therefore \quad 25 + 49 + 10g + 14f + c = 0 \Rightarrow 10g + 14f + c + 74 = 0 \qquad \ldots \text{(ii)}$$

$$36 + 36 + 12g + 12f + c = 0 \Rightarrow 12g + 12f + c + 72 = 0 \qquad \ldots \text{(iii)}$$

$$4 + 4 + 4g - 4f + c = 0 \Rightarrow 4g - 4f + c + 8 = 0 \qquad \ldots \text{(iv)}$$

Subtracting (ii) from (iii), we get

$$2g - 2f - 2 = 0 \Rightarrow g - f = 1. \qquad \ldots \text{(v)}$$

Subtracting (iv) from (iii), we get

$$8g + 16f + 64 = 0 \Rightarrow g + 2f = -8. \qquad \ldots \text{(vi)}$$

Solving (v) and (vi), we get $g = -2$ and $f = -3$.

Putting $g = -2$ and $f = -3$ in (ii), we get $c = -12$.

Putting $g = -2$, $f = -3$ and $c = -12$ in (i), we get

$$x^2 + y^2 - 4x - 6y - 12 = 0,$$

which is the required equation of the circle.

Centre of this circle $= (-g, -f) = (2, 3)$.

And, its radius $= \sqrt{g^2 + f^2 - c} = \sqrt{4 + 9 + 12} = \sqrt{25} = 5$ units.

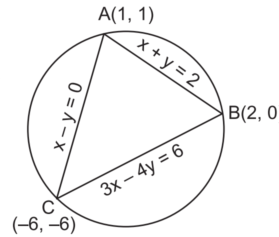
EXAMPLE 4 *Find the equation of the circle passing through the vertices of a triangle whose sides are represented by the equations* $x + y = 2$, $3x - 4y = 6$ *and* $x - y = 0$.

SOLUTION Let the sides AB, BC and CA of $\triangle ABC$ be represented by the equations $x + y = 2$, $3x - 4y = 6$ and $x - y = 0$ respectively.

Solving $x + y = 2$ and $3x - 4y = 6$, we get $B(2, 0)$.

Solving $3x - 4y = 6$ and $x - y = 0$, we get $C(-6, -6)$.

Solving $x + y = 2$ and $x - y = 0$, we get $A(1, 1)$.



Let the required equation of the circle be

$$x^2 + y^2 + 2gx + 2fy + c = 0. \quad \ldots \text{(i)}$$

Since it passes through $A(1, 1)$, $B(2, 0)$ and $C(-6, -6)$, each of these points must satisfy (i).

$$\therefore \quad 1^2 + 1^2 + 2g + 2f + c = 0 \Rightarrow 2g + 2f + c + 2 = 0 \quad \ldots \text{(ii)}$$

$$2^2 + 0^2 + 4g + c = 0 \Rightarrow 4g + c + 4 = 0 \quad \ldots \text{(iii)}$$

$$(-6)^2 + (-6)^2 - 12g - 12f + c = 0 \Rightarrow -12g - 12f + c + 72 = 0 \quad \ldots \text{(iv)}$$

Subtracting (ii) from (iii), we get

$$2g - 2f + 2 = 0 \Leftrightarrow g - f = -1. \quad \ldots \text{(v)}$$

Subtracting (iv) from (iii), we get

$$16g + 12f - 68 = 0 \Leftrightarrow 4g + 3f = 17. \quad \ldots \text{(vi)}$$

Solving (v) and (vi), we get $g = 2$ and $f = 3$.

Putting $g = 2$ in (iii), we get $c = -12$.

Hence, the required equation is

$$x^2 + y^2 + 4x + 6y - 12 = 0.$$

EXAMPLE 5 *Show that the points* $(5, 5)$, $(6, 4)$, $(-2, 4)$ *and* $(7, 1)$ *are concyclic, i.e., all lie on the same circle. Find the equation, centre and radius of this circle.*

SOLUTION Let the equation of the circle passing through the points $(5, 5)$, $(6, 4)$ and $(7, 1)$ be given by

$$x^2 + y^2 + 2gx + 2fy + c = 0. \quad \ldots \text{(i)}$$

Then, each of these points must satisfy (i).

$$25 + 25 + 10g + 10f + c = 0 \Rightarrow 10g + 10f + c = -50. \quad \ldots \text{(ii)}$$

$$36 + 16 + 12g + 8f + c = 0 \Rightarrow 12g + 8f + c = -52. \quad \ldots \text{(iii)}$$

$$49 + 1 + 14g + 2f + c = 0 \Rightarrow 14g + 2f + c = -50. \quad \ldots \text{(iv)}$$

Subtracting (ii) from (iv), we get

$$4g - 8f = 0 \Rightarrow g - 2f = 0. \quad \ldots \text{(v)}$$

Subtracting (iii) from (iv), we get

$$2g - 6f = 2 \Rightarrow g - 3f = 1. \quad \ldots \text{(vi)}$$

Solving (v) and (vi), we get $g = -2$ and $f = -1$.

Putting these values in (ii), we get $c = -20$.

Putting $g = -2$, $f = -1$ and $c = -20$ in (i), we get

$$x^2 + y^2 - 4x - 2y - 20 = 0. \quad \ldots \text{(vii)}$$

This is the equation of the circle passing through the points $(5, 5)$, $(6, 4)$ and $(7, 1)$.

Putting $x = -2$ and $y = 4$ in (vii), we get

$$\text{LHS} = 4 + 16 + 8 - 8 - 20 = 0 = \text{RHS}.$$

This shows that the point $(-2, 4)$ also lies on (vii).

Hence, the points $(5, 5)$, $(6, 4)$, $(-2, 4)$ and $(7, 1)$ all lie on the same circle, given by (vii).

Centre of this circle $= (-g, -f) = (2, 1)$.

Radius of this circle $= \sqrt{g^2 + f^2 - c} = \sqrt{(-2)^2 + (-1)^2 + 20}$

$= \sqrt{25} = 5$ units.

EXAMPLE 6 *Find the equation of the circle which passes through the centre of the circle $x^2 + y^2 + 8x + 10y - 7 = 0$ and is concentric with the circle $2x^2 + 2y^2 - 8x - 12y - 9 = 0$.*

SOLUTION We have $2x^2 + 2y^2 - 8x - 12y - 9 = 0$

$\Rightarrow \quad x^2 + y^2 - 4x - 6y - \dfrac{9}{2} = 0 \Rightarrow x^2 + y^2 + 2gx + 2fy + c = 0,$

where $g = -2$, $f = -3$ and $c = \dfrac{-9}{2}\cdot$

Centre of this circle $= (-g, -f) = (2, 3)$.

$\therefore$ the centre of the required circle $= C(2, 3)$.

Again, $x^2 + y^2 + 8x + 10y - 7 = 0$

$\Rightarrow \quad x^2 + y^2 + 2gx + 2fy + c = 0$, where $g = 4$, $f = 5$ and $c = -7$.

Centre of this circle $= (-g, -f) = (-4, -5)$.

So, the required circle passes through the point $P(-4, -5)$.

Radius of the required circle $= CP = \sqrt{(2 + 4)^2 + (3 + 5)^2}$

$= \sqrt{36 + 64} = \sqrt{100} = 10.$

Hence, the required equation of the circle is

$$(x - 2)^2 + (y - 3)^2 = (10)^2 \Rightarrow x^2 + y^2 - 4x - 6y - 87 = 0.$$

EXAMPLE 7 *Find the equation of the circle whose centre lies on the line $x - 4y = 1$ and which passes through the points $(3, 7)$ and $(5, 5)$.*

SOLUTION Let the required equation of the circle be

$$x^2 + y^2 + 2gx + 2fy + c = 0. \quad \ldots \text{(i)}$$

Its centre is $(-g, -f)$, which lies on the line $x - 4y = 1$.

$$\therefore \quad -g + 4f = 1. \quad \ldots \text{(ii)}$$

Also, it is given that (i) passes through the points $(3, 7)$ and $(5, 5)$.

$$\therefore \quad 3^2 + 7^2 + 6g + 14f + c = 0 \Rightarrow 6g + 14f + c = -58. \quad \ldots \text{(iii)}$$

$$5^2 + 5^2 + 10g + 10f + c = 0 \Rightarrow 10g + 10f + c = -50. \quad \ldots \text{(iv)}$$

Subtracting (iii) from (iv), we get

$$4g - 4f = 8 \Leftrightarrow g - f = 2. \quad \ldots \text{(v)}$$

Solving (ii) and (v), we get $g = 3$ and $f = 1$.

Putting $g = 3$ and $f = 1$ in (iii), we get $c = -90$.

$\therefore \quad g = 3, f = 1$ and $c = -90$.

Putting these values in (i), we get the required equation as

$$x^2 + y^2 + 6x + 2y - 90 = 0.$$

EXAMPLE 8 *Find the equation of a circle concentric with the circle*

$$2x^2 + 2y^2 - 6x + 8y + 1 = 0$$

and of double its area.

SOLUTION The given equation of the circle is

$$2x^2 + 2y^2 - 6x + 8y + 1 = 0$$

$$\Rightarrow \quad x^2 + y^2 - 3x + 4y + \frac{1}{2} = 0 \Rightarrow x^2 + y^2 + 2gx + 2fy + c = 0,$$

where $g = \frac{-3}{2}$, $f = 2$ and $c = \frac{1}{2}$.

Centre of this circle $= (-g, -f) = \left(\frac{3}{2}, -2\right)$.

Radius of this circle $= \sqrt{g^2 + f^2 - c}$

$$= \sqrt{\frac{9}{4} + 4 - \frac{1}{2}} = \frac{\sqrt{23}}{2}.$$

$\therefore$ the centre of the required circle $= \left(\frac{3}{2}, -2\right)$.

Let r_1 be the radius of the required circle. Then,

$$\pi r_1^2 = 2\left\{\pi \times \left(\frac{\sqrt{23}}{2}\right)^2\right\} \Rightarrow r_1^2 = \frac{23}{2}.$$

Hence, the equation of the required circle is

$$\left(x - \frac{3}{2}\right)^2 + (y + 2)^2 = \frac{23}{2}$$

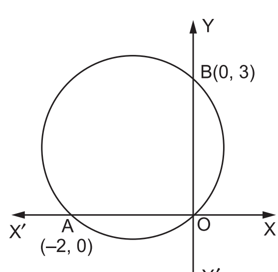
$$\Rightarrow \quad x^2 + y^2 - 3x + 4y - \frac{21}{4} = 0$$

$$\Rightarrow \quad 4x^2 + 4y^2 - 12x + 16y - 21 = 0.$$

EXAMPLE 9 *Find the equation of a circle which passes through the origin and cuts off intercepts -2 and 3 from the x-axis and the y-axis respectively.*

SOLUTION Let the required equation of the circle be

$$x^2 + y^2 + 2gx + 2fy + c = 0. \quad \dots \text{(i)}$$



Clearly, the circle passes through the points $O(0, 0)$, $A(-2, 0)$ and $B(0, 3)$.

Putting $x = 0$ and $y = 0$ in (i), we get $c = 0$.

Thus, (i) becomes

$$x^2 + y^2 + 2gx + 2fy = 0. \quad \dots \text{(ii)}$$

Putting $x = -2$ and $y = 0$ in (ii), we get

$$4g = 4 \Leftrightarrow g = 1.$$

Putting $x = 0$ and $y = 3$ in (ii), we get

$$6f = -9 \Leftrightarrow f = \frac{-3}{2}.$$

Putting $g = 1$ and $f = \frac{-3}{2}$ in (ii), we get $x^2 + y^2 + 2x - 3y = 0$,

which is the required equation of the circle.

EXERCISE 21B

1. Show that the equation $x^2 + y^2 - 4x + 6y - 5 = 0$ represents a circle. Find its centre and radius.

2. Show that the equation $x^2 + y^2 + x - y = 0$ represents a circle. Find its centre and radius.

3. Show that the equation $3x^2 + 3y^2 + 6x - 4y - 1 = 0$ represents a circle. Find its centre and radius.

4. Show that the equation $x^2 + y^2 + 2x + 10y + 26 = 0$ represents a point circle. Also, find its centre.

5. Show that the equation $x^2 + y^2 - 3x + 3y + 10 = 0$ does not represent a circle.

6. Find the equation of the circle passing through the points

(i) (0, 0), (5, 0) and (3, 3)

(ii) (1, 2), (3, –4) and (5, –6)

(iii) (20, 3), (19, 8) and (2, –9)

Also, find the centre and radius in each case.

7. Find the equation of the circle which is circumscribed about the triangle whose vertices are $A(-2, 3)$, $B(5, 2)$ and $C(6, -1)$. Find the centre and radius of this circle.

8. Find the equation of the circle concentric with the circle $x^2 + y^2 + 4x + 6y + 11 = 0$ and passing through the point $P(5, 4)$.

9. Show that the points $A(1, 0)$, $B(2, -7)$, $C(8, 1)$ and $D(9, -6)$ all lie on the same circle. Find the equation of this circle, its centre and radius.

10. Find the equation of the circle which passes through the points (1, 3) and (2, –1), and has its centre on the line $2x + y - 4 = 0$.

11. Find the equation of the circle concentric with the circle $x^2 + y^2 - 4x - 6y - 3 = 0$ and which touches the y-axis.

12. Find the equation of the circle concentric with the circle $x^2 + y^2 - 6x + 12y + 15 = 0$ and of double its area.

13. Prove that the centres of the three circles $x^2 + y^2 - 4x - 6y - 12 = 0$, $x^2 + y^2 + 2x + 4y - 5 = 0$ and $x^2 + y^2 - 10x - 16y + 7 = 0$ are collinear.

14. Find the equation of the circle which passes through the points $A(1, 1)$ and $B(2, 2)$ and whose radius is 1. Show that there are two such circles.

15. Find the equation of a circle passing through the origin and intercepting lengths a and b on the axes.

16. Find the equation of the circle circumscribing the triangle formed by the lines $x + y = 6$, $2x + y = 4$ and $x + 2y = 5$.

17. Show that the quadrilateral formed by the straight lines $x - y = 0$, $3x + 2y = 5$, $x - y = 10$ and $2x + 3y = 0$ is cyclic and hence find the equation of the circle.

18. If (–1, 3) and (α, β) are the extremities of the diameter of the circle $x^2 + y^2 - 6x + 5y - 7 = 0$, find the coordinates (α, β).

ANSWERS (EXERCISE 21B)

1. Centre (2, –3), radius $= 3\sqrt{2}$

2. Centre $\left(-\frac{1}{2}, \frac{1}{2}\right)$, radius $= \frac{1}{\sqrt{2}}$

3. Centre $\left(-1, \frac{2}{3}\right)$, radius $= \frac{4}{3}$

4. Centre $(-1, -5)$

6. (i) $x^2 + y^2 - 5x - y = 0$, centre $\left(\frac{5}{2}, \frac{1}{2}\right)$ and radius $= \frac{\sqrt{26}}{2}$

(ii) $x^2 + y^2 - 22x - 4y + 25 = 0$, centre (11, 2) and radius = 10

(iii) $x^2 + y^2 - 14x - 6y - 111 = 0$, centre (7, 3) and radius = 13

7. $x^2 + y^2 - 2x + 2y - 23 = 0$, centre (1, –1) and radius = 5

8. $x^2 + y^2 + 4x + 6y - 85 = 0$

9. $x^2 + y^2 - 10x + 6y + 9 = 0$, centre $C(5, -3)$ and radius = 5

10. $x^2 + y^2 - 3x - 2y - 1 = 0$

11. $x^2 + y^2 - 4x - 6y + 9 = 0$

12. $x^2 + y^2 - 6x + 12y - 15 = 0$

14. $x^2 + y^2 - 4x - 2y + 4 = 0$, $x^2 + y^2 - 2x - 4y + 4 = 0$

15. $x^2 + y^2 - ax - by = 0$

16. $x^2 + y^2 - 17x - 19y + 50 = 0$

17. $x^2 + y^2 - 6x + 4y = 0$

18. $B(7, -8)$

HINTS TO SOME SELECTED QUESTIONS

4. Show that the radius of the circle is 0.

5. Show that $(g^2 + f^2 - c) < 0$, where $g = \frac{-3}{2}$, $f = \frac{3}{2}$ and $c = 10$.

7. Find the equation of the circle passing through the points $A(-2, 3)$, $B(5, 2)$ and $C(6, -1)$.

8. Centre of the required circle is $C(-2, -3)$.

The circle passes through the point $P(5, 4)$.

Radius $= |CP| = \sqrt{(5+2)^2 + (4+3)^2} = 7\sqrt{2}$.

$\therefore$ the required equation is $(x+2)^2 + (y+3)^2 = 98$.

11. Centre of the given circle is (2, 3).

Centre of the required circle is (2, 3).

The required circle touches the y-axis, so its radius = 2.

$\therefore$ the required equation is $(x-2)^2 + (y-3)^2 = 2^2$.

13. Centres of the given circles are $A(2, 3)$, $B(-1, -2)$ and $C(5, 8)$ respectively. Now, show that $BA + AC = BC$.

Hence B, A, C are collinear.

14. Let the centre of the circle be $C(h, k)$.

Then, $|CA| = 1$ and $|CB| = 1$

$\Rightarrow \quad (h-1)^2 + (k-1)^2 = 1^2$ and $(h-2)^2 + (k-2)^2 = 1^2$

$\Rightarrow \quad h^2 + k^2 - 2h - 2k + 1 = 0$ ... (i)

and $h^2 + k^2 - 4h - 4k + 7 = 0.$ ... (ii)

Subtracting (ii) from (i), we get

$$2h + 2k - 6 = 0 \Rightarrow h + k = 3 \Rightarrow k = (3 - h).$$

Putting this value in (i), we get

$$h^2 + (3-h)^2 - 2h - 2(3-h) + 1 = 0$$

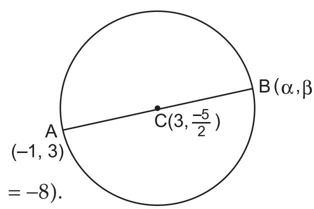
$\Rightarrow \quad (h-2)(h-1) = 0 \Rightarrow h = 2$ or $h = 1$.

$\therefore \quad k = 1$ or $k = 2$.

Hence, the centres are (2, 1) and (1, 2).

15. The circle passes through the points $O(0, 0)$, $A(a, 0)$ and $B(0, b)$.

18. Centre of the circle is $C\left(3, -\frac{5}{2}\right)$.



One end of the diameter AB is $A(-1, 3)$.

Let the other end of the diameter be $B(\alpha, \beta)$.

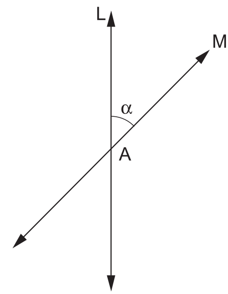
Then, C is the midpoint of AB.

$\therefore \quad \left(\frac{-1+\alpha}{2} = 3 \text{ and } \frac{3+\beta}{2} = \frac{-5}{2}\right) \Rightarrow (\alpha = 7 \text{ and } \beta = -8).$

❑

22
Parabola

CONIC SECTIONS Under this heading, we shall study about the curves, namely *parabola, ellipse* and *hyperbola*. These curves can be obtained as intersections of a plane with a double-napped right circular cone.

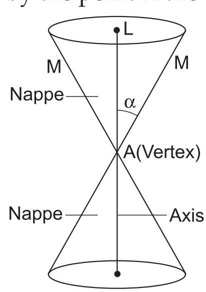


NAPPES Let L be a fixed vertical line and let M be another line intersecting it at a fixed point A and inclined to it at an angle α.

When M is rotated around the line L such that α remains constant, then the surface generated is a double-napped right circular hollow cone.

The line L is called the *axis* of the cone, A is called its *vertex* and the line M is called the *generator*.

The two parts, separated by the point A are called *nappes*.



CONIC SECTIONS *When a right circular cone is intersected by a plane, the curves obtained are known as conic sections.*

INTERSECTION OF NAPPES OF A CONE AND A PLANE

Suppose that a plane cuts double-napped cones each with semi-vertical angle α.

Let β be the angle made by the plane with the axis of the cone.

CASE 1 *When the plane cuts the nappes, but not at the vertex*

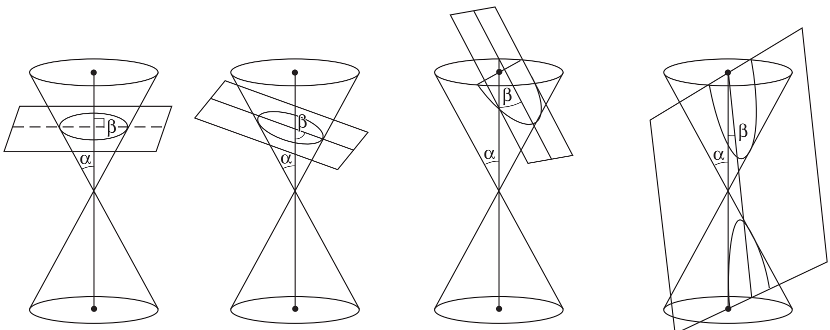
In this case, the following situations arise.

(i) When $\alpha < \beta = 90°$, the section is a ***circle***. [See Fig. (i)]

(ii) When $\alpha < \beta < 90°$, the section is an ***ellipse***. [See Fig. (ii)]

(iii) When $\beta = \alpha$, the section is a ***parabola***. [See Fig. (iii)]

(iv) When $0 \le \beta < \alpha$, the plane cuts both the nappes and the curves of intersection is a ***hyperbola***. [See Fig. (iv)]



($\alpha < \beta = 90°$) (i)   ($\alpha < \beta < 90°$) (ii)   ($\beta = \alpha$) (iii)   ($0 \leqslant \beta < \alpha$) (iv)

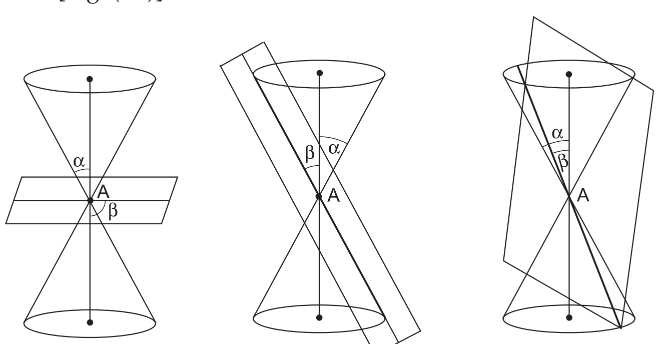
CASE 2 *When the plane cuts the nappe at the vertex*

In this case, the following situations arise.

(i) When $\alpha < \beta \le 90°$, then the section is a ***point.*** [Fig. (v)]

(ii) When $\beta = \alpha$, then the section is a ***line.*** [Fig. (vi)]

(iii) When $0 \le \beta < \alpha$, then the section is a ***pair of straight lines.*** [Fig. (vii)]



(v)   (vi)   (vii)

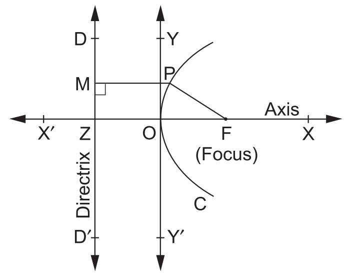
PARABOLA

PARABOLA *It is the path traced by a point which moves in a plane in such a way that its distance from a fixed point is always equal to its distance from a fixed line, both lying in the same plane, whereas the given fixed point does not lie on the given line.*

The fixed point is called the *focus* of the parabola and the fixed line is called its *directrix*.

A line through the focus and perpendicular to the directrix is called the *axis* of the parabola.

The point of intersection of the parabola with its axis is called the *vertex* of the parabola.



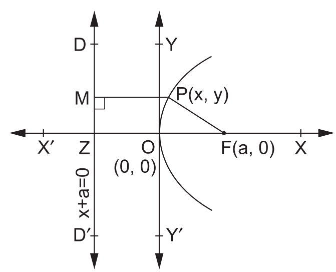
In the adjoining figure, C is a parabola with focus F and the line DD' as its directrix.

Clearly, x-axis is the axis of the parabola and $O(0, 0)$ is its vertex.

If we take an arbitrary point P on the parabola and draw $PM \perp DD'$ then by the definition of a parabola, we have $PF = PM$.

RIGHT-HANDED PARABOLA (or First Standard Form): $y^2 = 4ax,\ a > 0$

Let $X'OX$ and YOY' be the coordinate axes and let $a > 0$ be given.



Let us consider a parabola whose focus is $F(a, 0)$ and the directrix is the line DD', whose equation is $x + a = 0$.

Let $P(x, y)$ be an arbitrary point on the parabola. Let $PM \perp DD'$.

Then, by the definition of a parabola, we have $PF = PM$.

Now, $PF = PM \Rightarrow PF^2 = PM^2$

$\Rightarrow (x - a)^2 + y^2 = (x + a)^2$ [$\because$ length of perp. from $P(x, y)$ on the line $(x + a) = 0$ is $(x + a)$]

$\Rightarrow y^2 = (x + a)^2 - (x - a)^2$

$\Rightarrow y^2 = 4ax\ (a > 0)$.

Thus, every point on the parabola satisfies the equation, $y^2 = 4ax$.

Conversely, let $P(x, y)$ be a point satisfying the equation, $y^2 = 4ax$.

Let $F(a, 0)$ be a given point and $(x + a) = 0$ be a given line DD', and let $PM \perp DD'$. Then,

$$PF = \sqrt{(x - a)^2 + y^2}$$

$$= \sqrt{(x - a)^2 + 4ax} \qquad [\because\ y^2 = 4ax]$$

$$= \sqrt{(x + a)^2}$$

$$= (x + a) = PM$$

[$\because$ PM = length of perp. from $P(x, y)$ on the line $(x + a) = 0$].

This shows that $P(x, y)$ lies on the parabola, $y^2 = 4ax$.

Hence, the equation of a parabola with vertex at the origin, focus at the point $F(a, 0)$ and directrix $x = -a$ is $y^2 = 4ax$.

This equation is known as the *first standard form* of a parabola or a *right-handed parabola*.

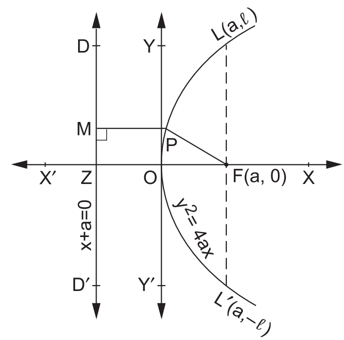
FOCAL CHORD *Any chord of a parabola passing through its focus is called a focal chord.*

FOCAL DISTANCE OF A POINT *The distance of any point of a parabola from its focus is called the focal distance of that point.*

LATUS RECTUM OF A PARABOLA *A chord of a parabola, passing through its focus and perpendicular to its axis, is called the latus rectum of the parabola.*

In the figure given below, LFL' is the latus rectum of the given parabola with focus F.

LENGTH OF LATUS RECTUM OF A PARABOLA



Let LFL' be the latus rectum of a parabola $y^2 = 4ax$ and let $|FL| = l$.

Then, clearly the coordinates of L and L' are $L(a, l)$ and $L'(a, -l)$.

Since the point $L(a, l)$ lies on the parabola $y^2 = 4ax$, we have

$$l^2 = 4a^2 \Rightarrow l = 2a.$$

$\therefore \quad |LL'| = 2l = 4a.$

Hence, the length of the latus rectum is $4a$.

The end points of the latus rectum are $L(a, 2a)$ and $L'(a, -2a)$.

The equation of the latus rectum is $x = a$.

SUMMARY

$y^2 = 4ax$, $a > 0$ is a parabola whose

(i) focus is $F(a, 0)$

(ii) vertex is $O(0, 0)$

(iii) directrix is the line $x + a = 0$

(iv) axis is the line $y = 0$

(v) length of the latus rectum is $4a$

(vi) latus rectum is the line $x = a$.

LEFT-HANDED PARABOLA (or 2nd Standard Form): $y^2 = -4ax$, $a > 0$

Proceeding as above, we can easily show that $y^2 = -4ax$ is the equation of a parabola whose focus is $F(-a, 0)$ and directrix is $x - a = 0$.

We summarise the properties of the parabola, $y^2 = -4ax$ as given below.

SUMMARY

$y^2 = -4ax,\ a > 0$ is a parabola whose

(i) focus is $F(-a, 0)$

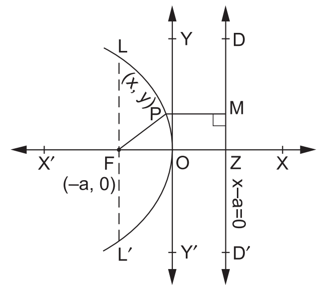
(ii) vertex is $O(0, 0)$

(iii) directrix is the line $x - a = 0$

(iv) axis is the line $y = 0$

(v) length of the latus rectum is $4a$

(vi) latus rectum is the line $x = -a$.



UPWARD PARABOLA (or 3rd Standard Form): $x^2 = 4ay, a > 0$

This is easy to show that $x^2 = 4ay$ is the equation of a parabola whose focus is $F(0, a)$ and directrix is $y + a = 0$.

We summarise the properties of the parabola, $x^2 = 4ay$ as given below.

SUMMARY

$x^2 = 4ay,\ a > 0$ is a parabola whose

(i) focus is $F(0, a)$

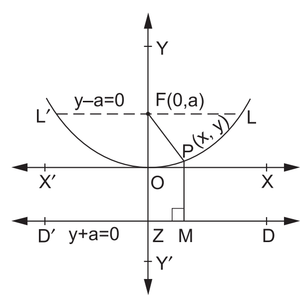
(ii) vertex is $O(0, 0)$

(iii) directrix is the line $y + a = 0$

(iv) axis is the line $x = 0$

(v) length of latus rectum is $4a$

(vi) latus rectum is the line $y - a = 0$.



DOWNWARD PARABOLA (or 4th Standard Form): $x^2 = -4ay, a > 0$

We can easily prove that $x^2 = -4ay$ is the equation of a parabola whose focus is $F(0, -a)$ and directrix is $y - a = 0$.

We summarise the properties of the parabola, $x^2 = -4ay$ as given below.

SUMMARY

$x^2 = -4ay,\ a > 0$ is a parabola whose

(i) focus is $F(0, -a)$

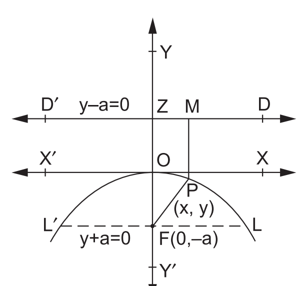
(ii) vertex is $O(0, 0)$

(iii) directrix is the line $y - a = 0$

(iv) axis is the line $x = 0$

(v) length of latus rectum is $4a$

(vi) latus rectum is the line $y + a = 0$.



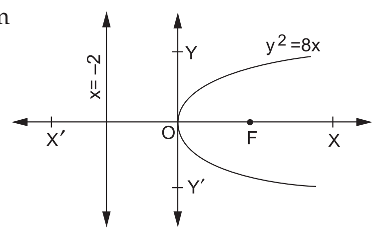
MAIN FACTS ABOUT ALL TYPES OF PARABOLAS

Parabola (Equation)	Focus	Vertex	Equation of directrix	Equation of axis	Length of latus rectum	Equation of latus rectum
(i) $y^2 = 4ax, a > 0$ (Right-handed)	$(a, 0)$	$(0, 0)$	$x + a = 0$	$y = 0$	$4a$	$x - a = 0$
(ii) $y^2 = -4ax, a > 0$ (Left-handed)	$(-a, 0)$	$(0, 0)$	$x - a = 0$	$y = 0$	$4a$	$x + a = 0$
(iii) $x^2 = 4ay, a > 0$ (Upward)	$(0, a)$	$(0, 0)$	$y + a = 0$	$x = 0$	$4a$	$y - a = 0$
(iv) $x^2 = -4ay, a > 0$ (Downward)	$(0, -a)$	$(0, 0)$	$y - a = 0$	$x = 0$	$4a$	$y + a = 0$

SOLVED EXAMPLES

EXAMPLE 1 *Find the coordinates of the focus and the vertex, the equations of the directrix and the axis, and length of the latus rectum of the parabola $y^2 = 8x$.*

SOLUTION The given equation is of the form $y^2 = 4ax$, where $4a = 8$, i.e., $a = 2$.



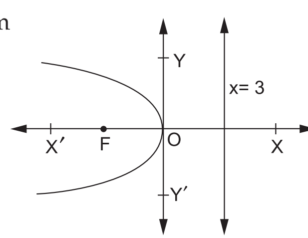
This is a *right-handed parabola*.
Its focus is $F(a, 0)$, i.e., $F(2, 0)$.
Its vertex is $O(0, 0)$.
The equation of the directrix is

$x = -a$, i.e., $x = -2$.

Its axis is x-axis, whose equation is $y = 0$.
Length of latus rectum $= 4a = (4 \times 2)$ units $= 8$ units.

EXAMPLE 2 *Find the coordinates of the focus and the vertex, the equations of the directrix and the axis, and length of the latus rectum of the parabola $y^2 = -12x$.*

SOLUTION The given equation is of the form $y^2 = -4ax$, where $4a = 12$, i.e., $a = 3$.



This is a *left-handed parabola*.
Its focus is $F(-a, 0)$, i.e., $F(-3, 0)$.
Its vertex is $O(0, 0)$.
The equation of the directrix is $x = a$, i.e., $x = 3$.
Its axis is x-axis, whose equation is $y = 0$.
Length of latus rectum $= 4a = (4 \times 3)$ units $= 12$ units.

EXAMPLE 3 *Find the coordinates of the focus and the vertex, the equations of the directrix and the axis, and length of the latus rectum of the parabola $x^2 = 6y$.*

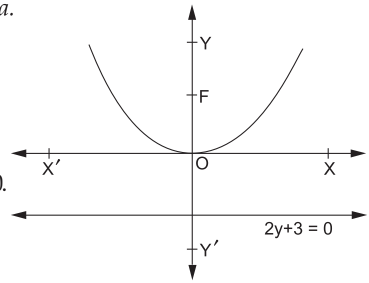
SOLUTION The given equation is of the form $x^2 = 4ay$, where $4a = 6$, i.e., $a = \frac{3}{2}\cdot$

So, this is a case of *upward parabola.*

Its focus is $F(0, a)$, i.e., $F\left(0, \frac{3}{2}\right)\cdot$

Its vertex is $O(0, 0)$.

The equation of the directrix is $y = -a$, i.e., $y = \frac{-3}{2}$, i.e., $2y + 3 = 0$.



Its axis is y-axis, whose equation is $x = 0$.

Length of latus rectum $= 4a = \left(4 \times \frac{3}{2}\right)$ units $= 6$ units.

EXAMPLE 4 *Find the coordinates of the focus and the vertex, the equations of the directrix and the axis, and length of latus rectum of the parabola $x^2 = -16y$.*

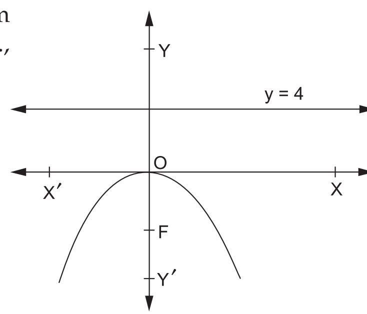
SOLUTION The given equation is of the form $x^2 = -4ay$, where $4a = 16$, i.e., $a = 4$.

So, it is a case of *downward parabola*.

Its focus is $F(0, -a)$, i.e., $F(0, -4)$.

Its vertex is $O(0, 0)$.

The equation of the directrix is $y = a$, i.e., $y = 4$.



Its axis is y-axis, whose equation is $x = 0$.

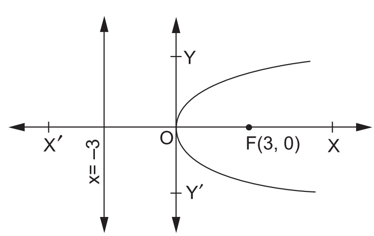
Length of latus rectum $= 4a = (4 \times 4)$ units $= 16$ units.

EXAMPLE 5 *Find the equation of the parabola with focus at $F(3, 0)$ and directrix $x = -3$.*

SOLUTION Since the focus lies on the x-axis, so x-axis is the axis of the parabola.

Focus $F(3, 0)$ lies to the right-hand side of the origin.

So, it is a *right-handed parabola.*



Let the required equation be $y^2 = 4ax$.

Then, focus is $F(a, 0)$. So, $a = 3$.

Hence, the required equation is $y^2 = 12x$.

EXAMPLE 6 *Find the equation of the parabola with vertex at the origin and $y + 3 = 0$ as its directrix. Also, find its focus.*

SOLUTION Let the vertex of the parabola be $O(0, 0)$.

Now, $y + 3 = 0 \Rightarrow y = -3$.

Thus, the directrix is a line parallel to the x-axis at a distance of 3 units below the x-axis.

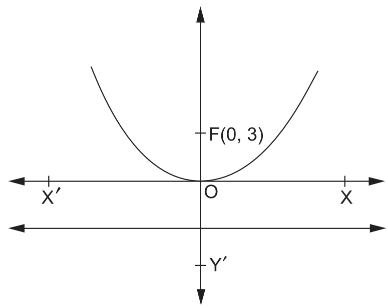
So, the focus of the parabola lies on the y-axis.

Consequently, the focus is $F(0, 3)$.

Hence, the equation of the parabola is

$$x^2 = 4ay, \quad \text{where } a = 3,$$

i.e., $x^2 = 12y$.



EXAMPLE 7 *Find the equation of the parabola with vertex at* $(0, 0)$ *and focus at* $(0, 2)$. *Also, find the equation of its directrix.*

SOLUTION Let $O(0, 0)$ be the vertex and $F(0, 2)$ be the focus of the given parabola.

Since focus lies on the y-axis, so y-axis is the axis of the parabola.

Also, the focus lies above the x-axis.

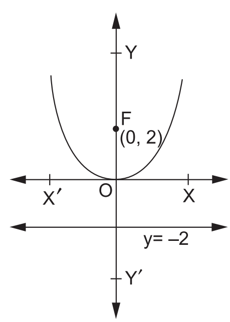
So, it is a case of *upward parabola.*

Let the equation of the parabola be $x^2 = 4ay$.

Then, its focus is $F(0, a)$ and so $a = 2$.

Hence, the required equation is $x^2 = 8y$.

Also, the equation of the directrix is $y = -2$.



EXAMPLE 8 *Find the equation of the parabola with vertex at the origin, passing through the point $P(3, -4)$ and symmetric about the y-axis.*

SOLUTION It is given that the vertex of the parabola is $O(0, 0)$ and it is symmetric about the y-axis.

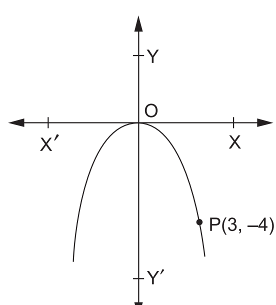
So, its equation is $x^2 = 4ay$ or $x^2 = -4ay$.

Since the parabola passes through the point $P(3, -4)$, so it lies in the 4th quadrant.

$\therefore$ it is a *downward parabola.*

Let its equation be $x^2 = -4ay$.

Since it passes through the point $P(3, -4)$, we have



$$3^2 = -4 \times a \times (-4) \Rightarrow a = \frac{9}{16}.$$

So, the required equation is

$$x^2 = -4 \times \frac{9}{16} y \Rightarrow x^2 = \frac{-9}{4} y \Rightarrow 4x^2 + 9y = 0.$$

EXAMPLE 9 *Find the equation of the parabola with vertex at the origin, the axis along the x-axis and passing through the point P(2, 3).*

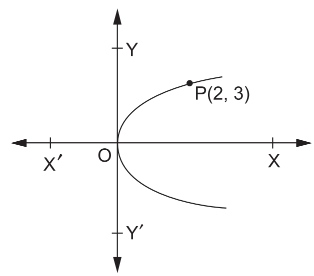
SOLUTION It is given that the vertex of the parabola is $O(0, 0)$ and its axis lies along the x-axis.

So, its equation is $y^2 = 4ax$ or $y^2 = -4ax$.

Since it passes through the point $P(2, 3)$, so it lies in the first quadrant.

$\therefore$ it is a *right-handed parabola.*

So, its equation is $y^2 = 4ax$.



Now, $P(2, 3)$ lies on it, so

$$3^2 = 4a \times 2 \Rightarrow a = \frac{9}{8}.$$

Hence, the required equation is $y^2 = 4 \times \frac{9}{8} x \Rightarrow y^2 = \frac{9}{2} x$.

EXERCISE 22

1. Find the coordinates of the focus and the vertex, the equations of the directrix and the axis, and length of the latus rectum of the parabola:
   (i) $y^2 = 12x$ (ii) $y^2 = 10x$ (iii) $3y^2 = 8x$
2. Find the coordinates of the focus and the vertex, the equations of the directrix and the axis, and length of the latus rectum of the parabola:
   (i) $y^2 = -8x$ (ii) $y^2 = -6x$ (iii) $5y^2 = -16x$
3. Find the coordinates of the focus and the vertex, the equations of the directrix and the axis, and length of the latus rectum of the parabola:
   (i) $x^2 = 16y$ (ii) $x^2 = 10y$ (iii) $3x^2 = 8y$
4. Find the coordinates of the focus and the vertex, the equations of the directrix and the axis, and length of the latus rectum of the parabola:
   (i) $x^2 = -8y$ (ii) $x^2 = -18y$ (iii) $3x^2 = -16y$
5. Find the equation of the parabola with vertex at the origin and focus at $F(-2, 0)$.
6. Find the equation of the parabola with focus $F(4, 0)$ and directrix $x = -4$.
7. Find the equation of the parabola with focus $F(0, -3)$ and directrix $y = 3$.
8. Find the equation of the parabola with vertex at the origin and focus $F(0, 5)$.

9. Find the equation of the parabola with vertex at the origin, passing through the point $P(5, 2)$ and symmetric with respect to the y-axis.

10. Find the equation of the parabola which is symmetric about the y-axis and passes through the point $P(2, -3)$.

ANSWERS (EXERCISE 22)

1. (i) $F(3, 0)$, $O(0, 0)$, $x + 3 = 0$, $y = 0$, 12 units

(ii) $F\left(\frac{5}{2}, 0\right)$, $O(0, 0)$, $2x + 5 = 0$, $y = 0$, 10 units

(iii) $F\left(\frac{2}{3}, 0\right)$, $O(0, 0)$, $3x + 2 = 0$, $y = 0$, $\frac{8}{3}$ units

2. (i) $F(-2, 0)$, $O(0, 0)$, $x = 2$, $y = 0$, 8 units

(ii) $F\left(\frac{-3}{2}, 0\right)$, $O(0, 0)$, $2x - 3 = 0$, $y = 0$, 6 units

(iii) $F\left(\frac{-4}{5}, 0\right)$, $O(0, 0)$, $5x - 4 = 0$, $y = 0$, $\frac{16}{5}$ units

3. (i) $F(0, 4)$, $O(0, 0)$, $y + 4 = 0$, $x = 0$, 16 units

(ii) $F\left(0, \frac{5}{2}\right)$, $O(0, 0)$, $2y + 5 = 0$, $x = 0$, 10 units

(iii) $F\left(0, \frac{2}{3}\right)$, $O(0, 0)$, $3y + 2 = 0$, $x = 0$, $\frac{8}{3}$ units

4. (i) $F(0, -2)$, $O(0, 0)$, $y = 2$, $x = 0$, 8 units

(ii) $F\left(0, \frac{-9}{2}\right)$, $O(0, 0)$, $2y - 9 = 0$, $x = 0$, 18 units

(iii) $F\left(0, \frac{-4}{3}\right)$, $O(0, 0)$, $3y - 4 = 0$, $x = 0$, $\frac{16}{3}$ units

5. $y^2 = -8x$ **6.** $y^2 = 16x$

7. $x^2 = -12y$ **8.** $x^2 = 20y$

9. $2x^2 = 25y$ **10.** $3x^2 = -4y$

❑

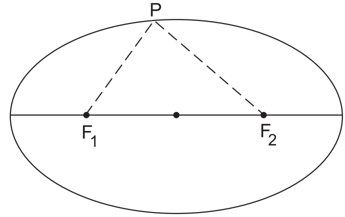
23
Ellipse

ELLIPSE *It is the path traced by a point which moves in a plane in such a way that the sum of its distance from two fixed points in the plane is a constant.*

The two fixed points are called the *foci* of the ellipse.

NOTE The plural of focus is foci.

In the given figure, F_1 and F_2 are two fixed points and P is a point which moves in such a way that $PF_1 + PF_2 = constant$.



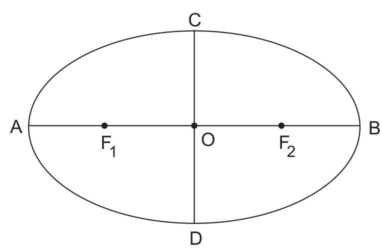
The path traced by the point P is called an ellipse, and the points F_1 and F_2 are called its *foci.*

SOME MORE TERMS RELATED TO AN ELLIPSE

(I) CENTRE OF THE ELLIPSE

The midpoint of the line segment joining the foci, is called the centre of the ellipse.

In the given figure, F_1 and F_2 are the foci of the ellipse and O is its *centre*, where $OF_1 = OF_2$.



(II) AXES OF THE ELLIPSE

MAJOR AXIS: *The line segment through the foci of the ellipse with its end points on the ellipse, is called its major axis.*

In the given figure, AB is the major axis of the ellipse.

MINOR AXIS: *The line segment through the centre and perpendicular to the major axis with its end points on the ellipse, is called its minor axis.*

In the given figure, CD is the minor axis of the ellipse.

(III) VERTICES OF AN ELLIPSE

The end points of the major axis of an ellipse are called its vertices.

In the given figure, A and B are the vertices of the ellipse.

AN IMPORTANT NOTE In an ellipse, we take:

Length of the major axis $= AB = 2a$.

Length of the minor axis $= CD = 2b$.

Distance between the foci $= F_1F_2 = 2c$.
Length of the semi-major axis $= a$.
Length of the semi-minor axis $= b$.

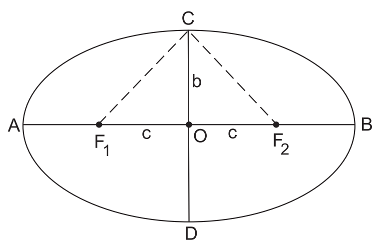
(IV) ECCENTRICITY OF AN ELLIPSE

The ratio $\frac{c}{a}$ is always constant and it is denoted by e, called the *eccentricity* of the ellipse.

For an ellipse, we have $0 < e < 1$ $\quad \left[\because\ c < a \iff e = \frac{c}{a} < 1\right]$.

AN IMPORTANT RESULT

In the given ellipse, it is being given that $AB = 2a$, $CD = 2b$ and $F_1F_2 = 2c$.
Prove that: $(a^2 - c^2) = b^2$.



PROOF Clearly, COD is the perpendicular bisector of F_1F_2.

Join F_1C and F_2C.

Now, $F_1O = c$, $OC = b \Rightarrow F_1C = \sqrt{c^2 + b^2}$.

And, $OF_2 = c$, $OC = b \Rightarrow F_2C = \sqrt{c^2 + b^2}$.

Since B and C both lie on the given ellipse, we have

$$F_1B + F_2B = F_1C + F_2C = \text{constant} \quad \text{[by the definition of an ellipse]}$$

$$\Rightarrow (F_1O + OB) + (OB - OF_2) = F_1C + F_2C$$

$$\Rightarrow (c + a) + (a - c) = 2\sqrt{c^2 + b^2}$$

$$[\because\ F_1O = c,\ OF_2 = c,\ OB = a \text{ and } F_1C = F_2C = \sqrt{c^2 + b^2}]$$

$$\Rightarrow a = \sqrt{c^2 + b^2} \Rightarrow a^2 = (c^2 + b^2) \Rightarrow (a^2 - c^2) = b^2.$$

Hence, $(a^2 - c^2) = b^2$.

STANDARD EQUATION OF AN ELLIPSE

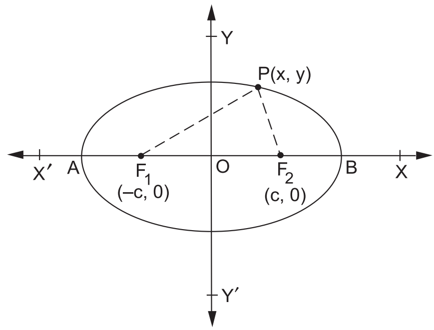
THEOREM *Prove that the standard equation of an ellipse is*

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

where a and b are the lengths of the semi-major axis and the semi-minor axis respectively and $a > b$.

PROOF Let $X'OX$ and YOY' be the coordinate axes.

Let us consider an ellipse in which a and b are the lengths of the semi-major axis and semi-minor axis respectively.



Choose a real number c such that $c^2 = (a^2 - b^2)$.

Let $F_1(-c, 0)$ and $F_2(c, 0)$ be the two fixed points and let $P(x, y)$ be an arbitrary point on the given ellipse, moving on it in such a way that $PF_1 + PF_2 = 2a$. Then,

$$PF_1 + PF_2 = 2a$$

$$\Rightarrow \sqrt{(x+c)^2 + y^2} + \sqrt{(x-c)^2 + y^2} = 2a$$

$$\Rightarrow \sqrt{(x+c)^2 + y^2} = 2a - \sqrt{(x-c)^2 + y^2}$$

$$\Rightarrow (x+c)^2 + y^2 = 4a^2 + (x-c)^2 + y^2 - 4a\sqrt{(x-c)^2 + y^2}$$

[on squaring both sides]

$$\Rightarrow (x+c)^2 - (x-c)^2 - 4a^2 = -4a\sqrt{(x-c)^2 + y^2}$$

$$\Rightarrow 4cx - 4a^2 = -4a\sqrt{(x-c)^2 + y^2}$$

$$\Rightarrow \sqrt{(x-c)^2 + y^2} = a - \frac{cx}{a} \qquad \text{[on dividing both sides by } -4a\text{]}$$

$$\Rightarrow (x-c)^2 + y^2 = a^2 - 2cx + \frac{c^2x^2}{a^2} \qquad \text{[on squaring both sides]}$$

$$\Rightarrow x^2 - \frac{c^2x^2}{a^2} + y^2 = (a^2 - c^2)$$

$$\Rightarrow x^2\left(1 - \frac{c^2}{a^2}\right) + y^2 = (a^2 - c^2)$$

$$\Rightarrow \frac{x^2(a^2 - c^2)}{a^2} + y^2 = (a^2 - c^2)$$

$$\Rightarrow \frac{x^2}{a^2} + \frac{y^2}{(a^2 - c^2)} = 1 \qquad \text{[on dividing both sides by } (a^2 - c^2)\text{]}$$

$$\Rightarrow \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \qquad [\because\ (a^2 - c^2) = b^2].$$

Thus, every point on the ellipse satisfies the equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Conversely, let the equation of the given curve be $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ and let $P(x, y)$ be an arbitrary point on this curve.

Then, $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \Rightarrow y^2=b^2\left(1-\frac{x^2}{a^2}\right)$

$$\Rightarrow y^2=\frac{b^2(a^2-x^2)}{a^2}. \qquad \ldots \text{(i)}$$

Also, let $(a^2-b^2)=c^2$. $\qquad \ldots$ (ii)

Let $F_1(-c, 0)$ and $F_2(c, 0)$ be two fixed points on the x-axis. Then,

$$PF_1=\sqrt{(x+c)^2+y^2}$$

$$=\sqrt{(x+c)^2+\frac{b^2(a^2-x^2)}{a^2}} \qquad \text{[using (i)]}$$

$$=\sqrt{(x+c)^2+\frac{(a^2-c^2)(a^2-x^2)}{a^2}} \qquad \text{[using (ii)]}$$

$$=\sqrt{a^2+2cx+\frac{c^2x^2}{a^2}}=\sqrt{\left(a+\frac{cx}{a}\right)^2}=\left(a+\frac{cx}{a}\right)$$

Similarly, $PF_2=\left(a-\frac{cx}{a}\right)$.

$\therefore \quad PF_1+PF_2=\left(a+\frac{cx}{a}+a-\frac{cx}{a}\right)$

$\Leftrightarrow \quad PF_1+PF_2=2a$.

This shows that the given curve is an ellipse.

Hence, the equation of the ellipse is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

AN IMPORTANT NOTE The foci of an ellipse always lie on the major axis.

HORIZONTAL ELLIPSE

In the given equation of an ellipse, if the coefficient of x^2 has the larger denominator then its major axis lies along the x-axis.

Such an ellipse is called a *horizontal ellipse.*

Thus, $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is an horizontal ellipse, if $a^2>b^2$.

EXAMPLE $\frac{x^2}{16}+\frac{y^2}{9}=1$ is an horizontal ellipse.

LATUS RECTUM OF A HORIZONTAL ELLIPSE

The latus rectum of an ellipse is a line segment perpendicular to the major axis, passing through any of the foci with end points lying on the ellipse.

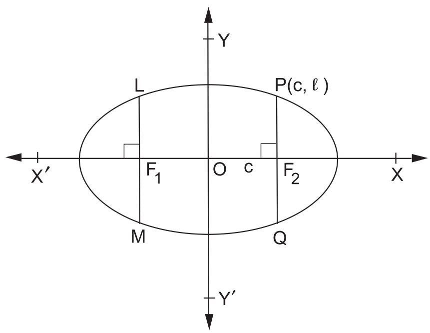
To Find the Length of Latus Rectum of the Ellipse $\boldsymbol{\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,\ a>b.}$

Let $X'OX$ and YOY' be the coordinate axes.

Let $F_1(-c, 0)$ and $F_2(c, 0)$ be the foci of the given ellipse, where $c^2=(a^2-b^2)$.

Let LF_1M and PF_2Q be the latus rectums.

Let $PF_2 = l$. Then, the coordinates of P are $P(c, l)$.



Since $P(c, l)$ lies on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, we have

$$\frac{c^2}{a^2}+\frac{l^2}{b^2}=1 \Rightarrow \frac{l^2}{b^2}=\left(1-\frac{c^2}{a^2}\right)$$

$$\Rightarrow l^2=b^2\left(1-\frac{c^2}{a^2}\right)=b^2\left[1-\frac{(a^2-b^2)}{a^2}\right] \qquad [\because c^2=(a^2-b^2)]$$

$$\Rightarrow l^2=\frac{b^4}{a^2} \Rightarrow l=\frac{b^2}{a}.$$

$$\therefore \quad PF_2=l=\frac{b^2}{a} \Leftrightarrow PF_2Q=2l=\frac{2b^2}{a}.$$

Similarly, $LF_1M=2l=\frac{2b^2}{a}$.

Hence, the length of the latus rectum is $\frac{2b^2}{a}$.

RESULTS ON HORIZONTAL ELLIPSE

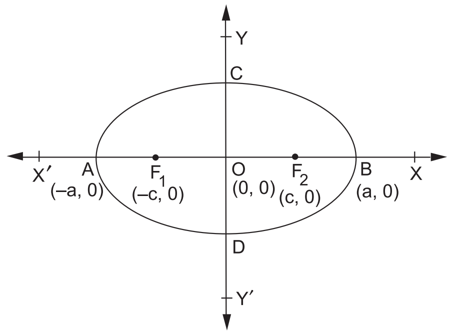
(i) The standard form of equation of a horizontal ellipse is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1, 0<b<a.$$

(ii) Its centre is $O(0, 0)$.

(iii) Its vertices are $A(-a, 0)$ and $B(a, 0)$.

(iv) Its foci are $F_1(-c, 0)$ and $F_2(c, 0)$, where $c^2=(a^2-b^2)$.



(v) Length of the major axis, $AB = 2a$ and length of the minor axis, $CD = 2b$.

(vi) Equation of the major axis is $y = 0$ and that of the minor axis is $x = 0$.

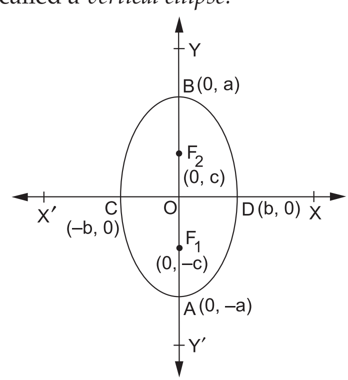
(vii) Length of the latus rectum $= \dfrac{2b^2}{a}\cdot$

(viii) Eccentricity, $e = \dfrac{c}{a} = \dfrac{\sqrt{a^2 - b^2}}{a}\cdot$

VERTICAL ELLIPSE

In the given equation of an ellipse, if the coefficient of x^2 has the smaller denominator, then its major axis lies along the y-axis.

Such an ellipse is called a *vertical ellipse.*



RESULTS ON VERTICAL ELLIPSE

(i) The standard form of the equation of a vertical ellipse is $\dfrac{x^2}{b^2} + \dfrac{y^2}{a^2} = 1$, where $0 < b < a$.

(ii) Its centre is $O(0, 0)$.

(iii) Its vertices are $A(0, -a)$ and $B(0, a)$.

(iv) Its foci are $F_1(0, -c)$ and $F_2(0, c)$, where $a^2 - b^2 = c^2$,

i.e., $F_1(0, -ae)$ and $F_2(0, ae)$.

(v) Length of the major axis $= 2a$ and length of the minor axis $= 2b$.

(vi) Equation of the major axis is $x = 0$ and that of the minor axis is $y = 0$.

(vii) Length of latus rectum $= \frac{2b^2}{a}$.

(viii) Eccentricity, $e = \frac{c}{a} = \frac{\sqrt{a^2 - b^2}}{a}$.

SUMMARY

Properties	**Horizontal Ellipse** $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ $0 < b < a$ **and** $c^2 = (a^2 - b^2)$	**Vertical Ellipse** $\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$ $0 < b < a$ **and** $c^2 = (a^2 - b^2)$
(i) Centre	(0, 0)	(0, 0)
(ii) Vertices	$A(-a, 0)$, $B(a, 0)$	$A(0, -a)$ and $B(0, a)$
(iii) Foci	$F_1(-c, 0)$ and $F_2(c, 0)$ or $(-ae, 0)$ and $(ae, 0)$	$F_1(0, -c)$ and $F_2(0, c)$ or $(0, -ae)$ and $(0, ae)$
(iv) Length of the major axis	$2a$	$2a$
(v) Length of the minor axis	$2b$	$2b$
(vi) Equation of the major axis	$y = 0$	$x = 0$
(vii) Equation of the minor axis	$x = 0$	$y = 0$
(viii) Length of the latus rectum	$\frac{2b^2}{a}$	$\frac{2b^2}{a}$
(ix) Eccentricity	$e = \frac{c}{a} = \frac{\sqrt{a^2 - b^2}}{a}$	$e = \frac{c}{a} = \frac{\sqrt{a^2 - b^2}}{a}$

SOLVED EXAMPLES

EXAMPLE 1 *Find the lengths of the major and minor axes; coordinates of the vertices and the foci, the eccentricity and length of the latus rectum of the ellipse:*

$$\frac{x^2}{16} + \frac{y^2}{9} = 1.$$

SOLUTION Given equation is $\frac{x^2}{16} + \frac{y^2}{9} = 1$.

This is of the form $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, where $a^2 > b^2$.

So, it is an equation of a *horizontal ellipse.*

Now, $(a^2 = 16$ and $b^2 = 9 \Rightarrow (a = 4$ and $b = 3)$.

$\therefore \quad c = \sqrt{a^2 - b^2} = \sqrt{16 - 9} = \sqrt{7}$.

Thus, $a = 4$, $b = 3$ and $c = \sqrt{7}$.

(i) Length of the major axis $= 2a = (2 \times 4)$ units $= 8$ units.
Length of the minor axis $= 2b = (2 \times 3)$ units $= 6$ units.

(ii) Coordinates of the vertices are $A(-a, 0)$ and $B(a, 0)$, i.e., $A(-4, 0)$ and $B(4, 0)$.

(iii) Coordinates of the foci are $F_1(-c, 0)$ and $F_2(c, 0)$, i.e., $F_1(-\sqrt{7}, 0)$ and $F_2(\sqrt{7}, 0)$.

(iv) Eccentricity, $e = \dfrac{c}{a} = \dfrac{\sqrt{7}}{4}$.

(v) Length of the latus rectum $= \dfrac{2b^2}{a} = \dfrac{(2 \times 9)}{3}$ units $= \dfrac{9}{2}$ units.

EXAMPLE 2 *Find the lengths of the major and minor axes; coordinates of the vertices and the foci; the eccentricity and length of the latus rectum of the ellipse:*

$$4x^2 + 9y^2 = 144.$$

SOLUTION The given equation may be written as

$$\frac{x^2}{36} + \frac{y^2}{16} = 1.$$

This is of the form $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$, where $a^2 > b^2$.

So, it is an equation of a *horizontal ellipse.*

Now, $(a^2 = 36$ and $b^2 = 16) \Rightarrow (a = 6$ and $b = 4)$.

$\therefore \quad c = \sqrt{a^2 - b^2} = \sqrt{36 - 16} = \sqrt{20} = 2\sqrt{5}$.

Thus, $a = 6$, $b = 4$ and $c = 2\sqrt{5}$.

(i) Length of the major axis $= 2a = (2 \times 6)$ units $= 12$ units.
Length of the minor axis $= 2b = (2 \times 4)$ units $= 8$ units.

(ii) Coordinates of the vertices are $A(-a, 0)$ and $B(a, 0)$, i.e., $A(-6, 0)$ and $B(6, 0)$.

(iii) Coordinates of the foci are $F_1(-c, 0)$ and $F_2(c, 0)$, i.e., $F_1(-2\sqrt{5}, 0)$ and $F_2(2\sqrt{5}, 0)$.

(iv) Eccentricity, $e = \dfrac{c}{a} = \dfrac{2\sqrt{5}}{6} = \dfrac{\sqrt{5}}{3}$.

(v) Length of the latus rectum $= \dfrac{2b^2}{a} = \dfrac{(2 \times 16)}{6}$ units $= \dfrac{16}{3}$ units.

EXAMPLE 3 *Find the lengths of the major and minor axes; coordinates of the vertices and the foci; the eccentricity and length of the latus rectum of the ellipse:*

$$\frac{x^2}{4} + \frac{y^2}{36} = 1.$$

SOLUTION Given equation is $\dfrac{x^2}{4} + \dfrac{y^2}{36} = 1$.

This is of the form $\dfrac{x^2}{b^2} + \dfrac{y^2}{a^2} = 1$, where $a^2 > b^2$.

So, it is an equation of a *vertical ellipse.*

Now, $(b^2 = 4$ and $a^2 = 36) \Rightarrow (b = 2$ and $a = 6)$.

$\therefore \quad c = \sqrt{a^2 - b^2} = \sqrt{36 - 4} = \sqrt{32} = 4\sqrt{2}$.

Thus, $a = 6$, $b = 2$ and $c = 4\sqrt{2}$.

(i) Length of the major axis $= 2a = (2 \times 6)$ units $= 12$ units.
Length of the minor axis $= 2b = (2 \times 2)$ units $= 4$ units.

(ii) Coordinates of its vertices are $A(0, -a)$ and $B(0, a)$, i.e., $A(0, -6)$ and $B(0, 6)$.

(iii) Coordinates of its foci are $F_1(0, -c)$ and $F_2(0, c)$, i.e., $F_1(0, -4\sqrt{2})$ and $F_2(0, 4\sqrt{2})$.

(iv) Eccentricity, $e = \frac{c}{a} = \frac{4\sqrt{2}}{6} = \frac{2\sqrt{2}}{3}$.

(v) Length of the latus rectum $= \frac{2b^2}{a} = \frac{(2 \times 4)}{6}$ units $= \frac{4}{3}$ units.

EXAMPLE 4 *Find the lengths of the major and minor axes; coordinates of the vertices and the foci; the eccentricity and length of the latus rectum of the ellipse:*

$$4x^2 + y^2 = 100.$$

SOLUTION The given equation of the ellipse may be written as

$$\frac{x^2}{25} + \frac{y^2}{100} = 1.$$

This is of the form $\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$, where $a^2 > b^2$.

So, it is an equation of a *vertical ellipse.*

Now, $(b^2 = 25$ and $a^2 = 100) \Rightarrow (b = 5$ and $a = 10)$.

$\therefore \quad c = \sqrt{a^2 - b^2} = \sqrt{100 - 25} = \sqrt{75} = 5\sqrt{3}$.

Thus, $a = 10$, $b = 5$ and $c = 5\sqrt{3}$.

(i) Length of the major axis $= 2a = (2 \times 10)$ units $= 20$ units.
Length of the minor axis $= 2b = (2 \times 5)$ units $= 10$ units.

(ii) Coordinates of the vertices are $A(0, -a)$ and $B(0, a)$, i.e., $A(0, -10)$ and $B(0, 10)$.

(iii) Coordinates of the foci are $F_1(0, -c)$ and $F_2(0, c)$, i.e., $F_1(0, -5\sqrt{3})$ and $F_2(0, 5\sqrt{3})$.

(iv) Eccentricity, $e = \frac{c}{a} = \frac{5\sqrt{3}}{10} = \frac{\sqrt{3}}{2}$.

(v) Length of the latus rectum $= \frac{2b^2}{a} = \frac{(2 \times 25)}{10}$ units $= 5$ units.

EXAMPLE 5 *Find the equation of an ellipse whose vertices are at* $(\pm 5, 0)$ *and foci at* $(\pm 4, 0)$.

SOLUTION Since the vertices of the given ellipse are on the x-axis, so it is a *horizontal ellipse.*

Let the equation of the ellipse be $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, where $a^2 > b^2$.

Its vertices are $(\pm a, 0)$ and therefore, $a = 5$.

Its foci are $(\pm c, 0)$ and therefore, $c = 4$.

$\therefore \quad c^2 = (a^2 - b^2) \Rightarrow b^2 = (a^2 - c^2) = (25 - 16) = 9.$

Thus, $a^2 = 5^2 = 25$ and $b^2 = 9$.

Hence, the required equation is $\frac{x^2}{25} + \frac{y^2}{9} = 1$.

EXAMPLE 6 *Find the equation of an ellipse whose foci are* $(\pm 4, 0)$ *and the eccentricity is* $\frac{1}{3}\cdot$

SOLUTION Since the foci of the given ellipse are on the x-axis, so it is a *horizontal ellipse.*

Let the required equation of the ellipse be

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \quad \text{where } a^2 > b^2.$$

Let its foci be $(\pm c, 0)$. Then, $c = 4$.

Also, $e = \frac{c}{a} \Leftrightarrow a = \frac{c}{e} = \frac{4}{(1/3)} = 12.$

Now, $c^2 = (a^2 - b^2) \Rightarrow b^2 = (a^2 - c^2) = (144 - 16) = 128.$

$\therefore \quad a^2 = (12)^2 = 144$ and $b^2 = 128$.

Hence, the required equation is $\frac{x^2}{144} + \frac{y^2}{128} = 1$.

EXAMPLE 7 *Find the equation of an ellipse whose major axis lies on the x-axis and which passes through the points* (4, 3) *and* (6, 2).

SOLUTION Since the major axis of the ellipse lies on the x-axis, so it is a *horizontal ellipse.*

Let the required equation of the ellipse be

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad (\text{where } a^2 > b^2). \qquad \ldots \text{(i)}$$

Since (4, 3) lies on (i), we have $\frac{16}{a^2} + \frac{9}{b^2} = 1$. ... (ii)

Also, since (6, 2) lies on (i), we have $\frac{36}{a^2} + \frac{4}{b^2} = 1$. ... (iii)

Putting $\frac{1}{a^2} = u$ and $\frac{1}{b^2} = v$ in (ii) and (iii), we get

$$16u + 9v = 1 \qquad \ldots \text{(iv)}$$
$$36u + 4v = 1 \qquad \ldots \text{(v)}$$

On multiplying (iv) by 9 and (v) by 4, and subtracting, we get

$$65v = 5 \Leftrightarrow v = \frac{1}{13} \Leftrightarrow \frac{1}{b^2} = \frac{1}{13} \Leftrightarrow b^2 = 13.$$

Putting $v = \frac{1}{13}$ in (iv), we get

$$16u = \left(1 - \frac{9}{13}\right) \Leftrightarrow 16u = \frac{4}{13} \Leftrightarrow u = \left(\frac{4}{13} \times \frac{1}{16}\right) = \frac{1}{52}$$
$$\Leftrightarrow \frac{1}{a^2} = \frac{1}{52} \Leftrightarrow a^2 = 52.$$

Thus, $a^2 = 52$ and $b^2 = 13$.

Hence, the required equation is $\dfrac{x^2}{52} + \dfrac{y^2}{13} = 1.$

EXAMPLE 8 *Find the equation of the ellipse, the ends of whose major axis are* $(\pm 3, 0)$ *and the ends of whose minor axis are* $(0, \pm 2)$.

SOLUTION Clearly, the major axis of the given ellipse lies on the x-axis. So, it is a *horizontal ellipse.*

Let the required equation be $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$, where $a^2 > b^2$.

Its vertices are $(\pm a, 0)$ and, therefore, $a = 3$

[$\because$ ends of the major axis are the vertices].

Ends of the minor axis are $C(0, -2)$ and $D(0, 2)$.

$\therefore \quad CD = 4$, i.e., length of the minor axis = 4 units.

$\therefore \quad 2b = 4 \Leftrightarrow b = 2.$

Now, $(a = 3$ and $b = 2) \Rightarrow (a^2 = 9$ and $b^2 = 4)$.

Hence, the required equation is $\dfrac{x^2}{9} + \dfrac{y^2}{4} = 1.$

EXAMPLE 9 *Find the equation of the ellipse whose centre lies at the origin, major axis lies on the x-axis, the eccentricity is* $\dfrac{2}{3}$ *and the length of the latus rectum is* 5 *units.*

SOLUTION Since the major axis of the ellipse lies on the x-axis, it is a horizontal ellipse.

Let the required equation be $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$, where $a^2 > b^2$.

Length of its latus rectum $= \dfrac{2b^2}{a}$.

$$\therefore \quad \frac{2b^2}{a} = 5 \Leftrightarrow \frac{2a^2(1-e^2)}{a} = 5 \qquad [\because b^2 = a^2(1-e^2)]$$

$$\Leftrightarrow \quad a = \frac{5}{2(1-e^2)} = \frac{5}{2\left(1 - \frac{4}{9}\right)} = \left(\frac{5}{2} \times \frac{9}{5}\right) = \frac{9}{2}.$$

Also, $b^2 = a^2(1-e^2) = \dfrac{81}{4} \times \left(1 - \dfrac{4}{9}\right) = \left(\dfrac{81}{4} \times \dfrac{5}{9}\right) = \dfrac{45}{4}$.

$$\therefore \quad a^2 = \left(\frac{9}{2}\right)^2 = \frac{81}{4} \text{ and } b^2 = \frac{45}{4}.$$

Hence, the required equation is

$$\frac{x^2}{(81/4)} + \frac{y^2}{(45/4)} = 1 \Leftrightarrow \frac{4x^2}{81} + \frac{4y^2}{45} = 1 \Leftrightarrow 20x^2 + 36y^2 = 405.$$

EXAMPLE 10 *Find the equation of an ellipse whose vertices are* $(0, \pm 13)$ *and the foci are* $(0, \pm 5)$.

SOLUTION Since the vertices of the ellipse lie on the y-axis, it is a *vertical ellipse.*

Let the required equation be $\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$, where $a^2 > b^2$.

Its vertices are $(0, \pm a)$ and therefore, $a = 13$.

Let its foci be $(0, \pm c)$. Then, $c = 5$.

$\therefore \quad b^2 = (a^2 - c^2) = (169 - 25) = 144.$

Thus, $b^2 = 144$ and $a^2 = (13)^2 = 169$.

Hence, the required equation is $\frac{x^2}{144} + \frac{y^2}{169} = 1$.

EXAMPLE 11 *Find the equation of the ellipse whose foci are* $(0, \pm 6)$ *and the length of whose minor axis is* 16.

SOLUTION Since the foci of the given ellipse lie on the y-axis, it is a *vertical ellipse.*

Let the required equation be $\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$, where $a^2 > b^2$.

Let $c^2 = (a^2 - b^2)$.

Then, its foci are $(0, \pm c)$ and therefore, $c = 6$.

Also, b = length of the semi-minor axis $= \left(\frac{1}{2} \times 16\right) = 8$.

$\therefore \quad a^2 = (c^2 + b^2) = (36 + 64) = 100.$

Thus, $b^2 = 64$ and $a^2 = 100$.

Hence, the required equation is $\frac{x^2}{64} + \frac{y^2}{100} = 1$.

EXAMPLE 12 *Find the equation of the ellipse whose foci are* $(0, \pm 5)$ *and the length of whose major axis is* 20.

SOLUTION Since the foci of the ellipse lie on the y-axis, it is a *vertical ellipse.*

Let the required equation be $\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$, where $a^2 > b^2$.

Let $c^2 = (a^2 - b^2)$.

Its foci are $(0, \pm c)$ and therefore, $c = 5$.

Also, a = length of the semi-major axis $= \left(\frac{1}{2} \times 20\right) = 10$.

Now, $c^2 = (a^2 - b^2) \Leftrightarrow b^2 = (a^2 - c^2) = (100 - 25) = 75$.

Thus, $a^2 = (10)^2 = 100$ and $b^2 = 75$.

Hence, the required equation is $\frac{x^2}{75} + \frac{y^2}{100} = 1$.

EXAMPLE 13 *Find the equation of the ellipse for which* $e = \frac{4}{5}$ *and whose vertices are* $(0, \pm 10)$.

SOLUTION Since the vertices of the ellipse lie on the y-axis, it is a *vertical ellipse.*

Let the required equation be $\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$, where $a^2 > b^2$.

Its vertices are $(0, \pm a)$ and therefore, $a = 10$.

Let $c^2 = (a^2 - b^2)$.

Then, $e = \frac{c}{a} \Rightarrow c = ae = \left(10 \times \frac{4}{5}\right) = 8$.

Now, $c^2 = (a^2 - b^2) \Leftrightarrow b^2 = (a^2 - c^2) = (100 - 64) = 36$.

$\therefore \quad a^2 = (10)^2 = 100$ and $b^2 = 36$.

Hence, the required equation is $\frac{x^2}{36} + \frac{y^2}{100} = 1$.

EXAMPLE 14 *Find the equation of the ellipse with centre at the origin, major-axis on the y-axis and passing through the points* $(3, 2)$ *and* $(1, 6)$.

SOLUTION Since the major axis of the ellipse lies on the y-axis, it is a *vertical ellipse.*

Let the required equation be $\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$ (where $a^2 > b^2$). ... (i)

Since $(3, 2)$ lies on (i), we have $\frac{9}{b^2} + \frac{4}{a^2} = 1$. ... (ii)

Also, since $(1, 6)$ lies on (i), we have $\frac{1}{b^2} + \frac{36}{a^2} = 1$. ... (iii)

Putting $\frac{1}{b^2} = u$ and $\frac{1}{a^2} = v$, these equations become:

$9u + 4v = 1$ ... (iv)

and $u + 36v = 1$ ... (v)

On multiplying (v) by 9 and subtracting (iv) from it, we get

$$320v = 8 \Leftrightarrow v = \frac{8}{320} = \frac{1}{40} \Leftrightarrow \frac{1}{a^2} = \frac{1}{40} \Leftrightarrow a^2 = 40.$$

Putting $v = \frac{1}{40}$ in (v), we get

$$u + \left(36 \times \frac{1}{40}\right) = 1 \Leftrightarrow u = \left(1 - \frac{9}{10}\right) = \frac{1}{10} \Leftrightarrow \frac{1}{b^2} = \frac{1}{10} \Leftrightarrow b^2 = 10.$$

Thus, $b^2 = 10$ and $a^2 = 40$.

Hence, the required equation is $\frac{x^2}{10} + \frac{y^2}{40} = 1$.

EXERCISE 23

Find the (i) lengths of major and minor axes, (ii) coordinates of the vertices, (iii) coordinates of the foci, (iv) eccentricity, and (v) length of the latus rectum of each of the following ellipses.

1. $\frac{x^2}{25}+\frac{y^2}{9}=1$ **2.** $\frac{x^2}{49}+\frac{y^2}{36}=1$ **3.** $16x^2+25y^2=400$

4. $x^2+4y^2=100$ **5.** $9x^2+16y^2=144$ **6.** $4x^2+9y^2=1$

7. $\frac{x^2}{4}+\frac{y^2}{25}=1$ **8.** $\frac{x^2}{9}+\frac{y^2}{16}=1$ **9.** $3x^2+2y^2=18$

10. $9x^2+y^2=36$ **11.** $16x^2+y^2=16$ **12.** $25x^2+4y^2=100$

13. Find the equation of the ellipse whose vertices are at $(\pm 6, 0)$ and foci at $(\pm 4, 0)$.

14. Find the equation of the ellipse whose vertices are at $(0, \pm 4)$ and foci at $(0, \pm\sqrt{7})$.

15. Find the equation of the ellipse the ends of whose major and minor axes are $(\pm 4, 0)$ and $(0, \pm 3)$ respectively.

16. The length of the major axis of an ellipse is 20 units and its foci are $(\pm 5\sqrt{3}, 0)$. Find the equation of the ellipse.

17. Find the equation of the ellipse whose foci are $(\pm 2, 0)$ and the eccentricity is $\frac{1}{2}\cdot$

18. Find the equation of the ellipse whose foci are at $(\pm 1, 0)$ and $e=\frac{1}{2}\cdot$

19. Find the equation of the ellipse whose foci are at $(0, \pm 4)$ and $e=\frac{4}{5}\cdot$

20. Find the equation of the ellipse with centre at the origin, major axis on the x-axis and passing through the points (4, 3) and (–1, 4).

21. Find the equation of the ellipse with eccentricity $\frac{3}{4}$, foci on the y-axis, centre at the origin and passing through the point (6, 4).

22. Find the equation of the ellipse which passes through the point (4, 1) and having its foci at $(\pm 3, 0)$.

23. Find the equation of an ellipse, the lengths of whose major and minor axes are 10 and 8 units respectively.

24. Find the equation of an ellipse whose eccentricity is $\frac{2}{3}$, the latus rectum is 5 and the centre is at the origin.

25. Find the eccentricity of an ellipse whose latus rectum is one half of its minor axis.

26. Find the eccentricity of an ellipse whose latus rectum is one half of its major axis.

ANSWERS (EXERCISE 23)

1. (i) 10 units, 6 units (ii) $A(-5, 0)$ and $B(5, 0)$ (iii) $F_1(-4, 0)$ and $F_2(4, 0)$
(iv) $e = \frac{4}{5}$ (v) 3.6 units

2. (i) 14 units, 12 units (ii) $A(-7, 0)$ and $B(7, 0)$
(iii) $F_1(-\sqrt{13}, 0)$ and $F_2(\sqrt{13}, 0)$ (iv) $e = \frac{\sqrt{13}}{7}$ (v) $\frac{72}{7}$ units

3. (i) 10 units, 8 units (ii) $A(-5, 0)$ and $B(5, 0)$ (iii) $F_1(-3, 0)$ and $F_2(3, 0)$
(iv) $e = \frac{3}{5}$ (v) $\frac{32}{5}$ units

4. (i) 20 units, 10 units (ii) $A(-10, 0)$ and $B(10, 0)$
(iii) $F_1(-5\sqrt{3}, 0)$ and $F_2(5\sqrt{3}, 0)$ (iv) $e = \frac{\sqrt{3}}{2}$ (v) 5 units

5. (i) 8 units, 6 units (ii) $A(-4, 0)$ and $B(4, 0)$ (iii) $F_1(-\sqrt{7}, 0)$, $F_2(\sqrt{7}, 0)$
(iv) $e = \frac{\sqrt{7}}{4}$ (v) $\frac{9}{2}$ units

6. (i) 1 unit, $\frac{2}{3}$ unit (ii) $A\left(-\frac{1}{2}, 0\right)$ and $B\left(\frac{1}{2}, 0\right)$
(iii) $F_1\left(\frac{-\sqrt{5}}{6}, 0\right)$ and $F_2\left(\frac{\sqrt{5}}{6}, 0\right)$ (iv) $e = \frac{\sqrt{5}}{3}$ (v) $\frac{4}{9}$ unit

7. (i) 10 units, 4 units (ii) $A(0, -5)$ and $B(0, 5)$
(iii) $F_1(0, -\sqrt{21})$ and $F_2(0, \sqrt{21})$ (iv) $e = \frac{\sqrt{21}}{5}$ (v) $\frac{8}{5}$ units

8. (i) 8 units, 6 units (ii) $A(0, -4)$ and $B(0, 4)$
(iii) $F_1(0, -\sqrt{7})$ and $F_2(0, \sqrt{7})$ (iv) $e = \frac{\sqrt{7}}{4}$ (v) $4\frac{1}{2}$ units

9. (i) 6 units, $2\sqrt{6}$ units (ii) $A(0, -3)$ and $B(0, 3)$
(iii) $F_1(0, -\sqrt{3})$ and $F_2(0, \sqrt{3})$ (iv) $e = \frac{1}{\sqrt{3}}$ (v) 4 units

10. (i) 12 units, 4 units (ii) $A(0, -6)$ and $B(0, 6)$

(iii) $F_1(0, -4\sqrt{2})$ and $F_2(0, 4\sqrt{2})$ (iv) $e = \frac{2\sqrt{2}}{3}$ (v) $1\frac{1}{3}$ units

11. (i) 8 units, 2 units (ii) $A(0, -4)$ and $B(0, 4)$

(iii) $F_1(0, -\sqrt{15})$ and $F_2(0, \sqrt{15})$ (iv) $e = \frac{\sqrt{15}}{4}$ (v) $\frac{1}{2}$ unit

12. (i) 10 units, 4 units (ii) $A(0, -5)$ and $B(0, 5)$

(iii) $F_1(0, -\sqrt{21})$ and $F_2(0, \sqrt{21})$ (iv) $e = \frac{\sqrt{21}}{5}$ (v) $1\frac{3}{5}$ units

13. $\frac{x^2}{36} + \frac{y^2}{20} = 1$ **14.** $\frac{x^2}{9} + \frac{y^2}{16} = 1$ **15.** $\frac{x^2}{16} + \frac{y^2}{9} = 1$

16. $\frac{x^2}{100} + \frac{y^2}{25} = 1$ **17.** $\frac{x^2}{16} + \frac{y^2}{12} = 1$ **18.** $\frac{x^2}{4} + \frac{y^2}{3} = 1$

19. $\frac{x^2}{9} + \frac{y^2}{25} = 1$ **20.** $7x^2 + 15y^2 = 247$ **21.** $16x^2 + 7y^2 = 688$

22. $\frac{x^2}{18} + \frac{y^2}{9} = 1$ **23.** $\frac{x^2}{25} + \frac{y^2}{16} = 1$ **24.** $\frac{4x^2}{81} + \frac{4y^2}{45} = 1$

25. $e = \frac{\sqrt{3}}{2}$ **26.** $e = \frac{1}{\sqrt{2}}$

HINTS TO SOME SELECTED QUESTIONS

6. Given equation is $\frac{x^2}{(1/4)} + \frac{y^2}{(1/9)} = 1$.

$\therefore \quad a^2 = \frac{1}{4}, b^2 = \frac{1}{9}$ and $c^2 = (a^2 - b^2) = \left(\frac{1}{4} - \frac{1}{9}\right) = \frac{5}{36}$.

$\therefore \quad a = \frac{1}{2}, b = \frac{1}{3}$ and $c = \frac{\sqrt{5}}{6}$.

15. The ends of the major axis are $(\pm a, 0)$ and the ends of the minor axis are $(0, \pm b)$.

$\therefore \quad a = 4$ and $b = 3$.

16. Let the equation of the ellipse be $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Length of the major axis = 20 $\Leftrightarrow 2a = 20 \Leftrightarrow a = 10$.

Foci are $(\pm c, 0) \Rightarrow c = 5\sqrt{3}$.

Now, $c^2 = (a^2 - b^2) \Leftrightarrow b^2 = (a^2 - c^2) = (100 - 75) = 25$.

$\therefore$ the required equation is $\frac{x^2}{100} + \frac{y^2}{25} = 1$.

19. Clearly, the major axis lies along the y-axis.

Let the required equation be $\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$.

Here, $c = 4$ and $e = \frac{4}{5} \cdot$ So, $a = \frac{c}{e} = 5.$

$\therefore \quad b^2 = (a^2 - c^2) = (25 - 16) = 9.$

21. Since the centre of the ellipse lies at the origin and its foci lie on the y-axis, it is a vertical ellipse.

Let its equation be $\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1.$ … (i)

Let $c^2 = a^2 - b^2.$

Now, $e = \frac{3}{4} \Leftrightarrow \frac{c}{a} = \frac{3}{4} \Leftrightarrow c = \frac{3}{4}a.$

$\therefore \quad b^2 = (a^2 - c^2) = \left(a^2 - \frac{9}{16}a^2\right) = \frac{7a^2}{16} \cdot$

So, the equation of the ellipse is

$$\frac{x^2}{\frac{7a^2}{16}} + \frac{y^2}{a^2} = 1 \Leftrightarrow 16x^2 + 7y^2 = 7a^2. \qquad \text{... (ii)}$$

Since it passes through (6, 4), putting $x = 6$ and $y = 4$ in (ii), we get

$$7a^2 = 688.$$

Hence, the required equation is $16x^2 + 7y^2 = 688.$

23. $2a = 10$ and $2b = 8.$

24. $\frac{2b^2}{a} = 5 \Rightarrow \frac{2a^2(1 - e^2)}{a} = 5 \Rightarrow a = \frac{9}{2} \cdot$

Also, $b^2 = \frac{5a}{2} = \left(\frac{5}{2} \times \frac{9}{2}\right) = \frac{45}{4} \cdot$

25. $\left\{\frac{2b^2}{a} = \frac{1}{2} \times 2b \Rightarrow b = \frac{a}{2}\right\} \cdot$

So, $b^2 = a^2(1 - e^2) \Rightarrow \frac{a^2}{4} = a^2(1 - e^2).$

26. $\frac{2b^2}{a} = \frac{1}{2} \times 2a \Rightarrow a^2 = 2b^2 \Rightarrow a = \sqrt{2}\, b.$ So, $e = \frac{b}{a} = \frac{1}{\sqrt{2}} \cdot$

❑

24
Hyperbola

HYPERBOLA *It is the set of all points in a plane, the difference of whose distances from two fixed points in the plane is a constant.*

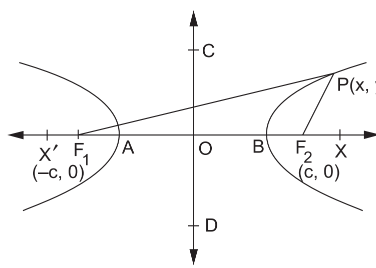
The two fixed points are called the *foci* of the hyperbola.

SOME MORE TERMS RELATED TO A HYPERBOLA

(I) CENTRE OF THE HYPERBOLA

The midpoint of the line segment joining the foci is called the centre of the hyperbola.

In the given figure, F_1 and F_2 are the foci of the hyperbola and O is its centre, where $OF_1 = OF_2$.



(II) AXES OF THE HYPERBOLA

TRANSVERSE AXIS: *The line through the foci of the hyperbola is called its transverse axis.*

In the given figure, $X'OX$ is the transverse axis of the hyperbola.

CONJUGATE AXIS: *The line through the centre and perpendicular to the transverse axis of the hyperbola is called its conjugate axis.*

Thus, COD is the conjugate axis of the hyperbola in the given figure.

(III) VERTICES OF THE HYPERBOLA

The points at which the hyperbola intersects the transverse axis are called its vertices.

In the given figure, A and B are the vertices of the hyperbola.

(IV) LENGTH OF TRANSVERSE AXIS

The distance between the two vertices of a hyperbola is called the length of its transverse axis.

In the given hyperbola, length of the transverse axis = AB.

AN IMPORTANT NOTE In a hyperbola, we take:

Length of the transverse axis $= AB = 2a$.

Distance between the foci $= F_1F_2 = 2c$, where $c > a$.

Length of the conjugate axis $= 2b$, where $b^2 = (c^2 - a^2)$.

(V) ECCENTRICITY OF A HYPERBOLA

The ratio $\frac{c}{a}$ *is always constant, called the eccentricity of the hyperbola and it is denoted by e.*

Here, $c > a \Leftrightarrow \frac{c}{a} > 1 \Leftrightarrow e > 1$.

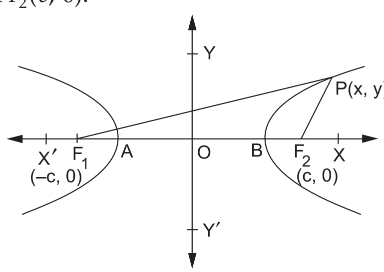
STANDARD EQUATION OF A HYPERBOLA

THEOREM *Prove that the standard equation of a hyperbola is*

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.$$

PROOF Let $X'OX$ and YOY' be the coordinate axes.

Let us consider a hyperbola with centre at $O(0, 0)$ and its foci at $F_1(-c, 0)$ and $F_2(c, 0)$.



Let $P(x, y)$ be an arbitrary point on the hyperbola such that $PF_1 - PF_2 = 2a$. Then,

$$PF_1 - PF_2 = 2a$$

$$\Rightarrow \sqrt{(x+c)^2 + y^2} - \sqrt{(x-c)^2 + y^2} = 2a$$

$$\Rightarrow \sqrt{(x+c)^2 + y^2} = 2a + \sqrt{(x-c)^2 + y^2}$$

$$\Rightarrow (x+c)^2 + y^2 = 4a^2 + (x-c)^2 + y^2 + 4a\sqrt{(x-c)^2 + y^2}$$

$$\Rightarrow (x+c)^2 - (x-c)^2 - 4a^2 = 4a\sqrt{(x-c)^2 + y^2}$$

$$\Rightarrow 4cx - 4a^2 = 4a\sqrt{(x-c)^2 + y^2}$$

$$\Rightarrow \left(\frac{cx}{a} - a\right) = \sqrt{(x-c)^2 + y^2}$$

$$\Rightarrow \left(\frac{cx}{a} - a\right)^2 = (x-c)^2 + y^2$$

$\Rightarrow \quad \dfrac{c^2x^2}{a^2} + a^2 = x^2 + c^2 + y^2$

$\Rightarrow \quad x^2\left(\dfrac{c^2}{a^2} - 1\right) - y^2 = (c^2 - a^2)$

$\Rightarrow \quad x^2(c^2 - a^2) - a^2y^2 = a^2(c^2 - a^2)$

$\Rightarrow \quad \dfrac{x^2}{a^2} - \dfrac{y^2}{(c^2 - a^2)} = 1$

$\Rightarrow \quad \dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$, where $(c^2 - a^2) = b^2$.

Thus, every point on the given hyperbola satisfies the equation

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.$$

Conversely, let the equation of the given curve be $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ and let $P(x, y)$ be an arbitrary point on it. Then,

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \Rightarrow y^2 = b^2\left(\frac{x^2}{a^2} - 1\right)$$

$$\Rightarrow y^2 = \frac{b^2(x^2 - a^2)}{a^2}. \qquad \ldots \text{(i)}$$

Let $c^2 = (a^2 + b^2)$. ... (ii)

Let $F_1(-c, 0)$ and $F_2(c, 0)$ be two fixed points on the x-axis. Then,

$$PF_1 = \sqrt{(x + c)^2 + y^2}$$

$$= \sqrt{(x + c^2) + \frac{b^2(x^2 - a^2)}{a^2}} \qquad \text{[using (i)]}$$

$$= \sqrt{(x + c)^2 + \frac{(c^2 - a^2)(x^2 - a^2)}{a^2}} \qquad \text{[using (ii)}$$

$$= \sqrt{(x^2 + c^2 + 2cx) + \frac{(c^2x^2 - c^2a^2 - a^2x^2 + a^4)}{a^2}}$$

$$= \sqrt{(x^2 + c^2 + 2cx) + \frac{c^2x^2}{a^2} - c^2 - x^2 + a^2}$$

$$= \sqrt{\frac{c^2x^2}{a^2} + a^2 + 2cx} = \sqrt{\left(\frac{cx}{a} + a\right)^2} = \left(\frac{cx}{a} + a\right).$$

Similarly, $PF_2 = \left(\dfrac{cx}{a} - a\right)\left[x \geq a \text{ and } c \geq a \Rightarrow \dfrac{x}{a} \geq 1 \text{ and } c \geq a \Rightarrow \dfrac{cx}{a} \geq a\right]$

$\therefore \quad PF_1 - PF_2 = \left(\dfrac{cx}{a} + a\right) - \left(\dfrac{cx}{a} - a\right) = 2a.$

This shows that P lies on the hyperbola.

Thus, every point that satisfies $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, lies on the hyperbola.

Hence, the equation of a hyperbola with origin (0, 0) and transverse axis along the x-axis is given by

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1.$$

REMARK A hyperbola in which $a=b$ is called an equilateral hyperbola.

LATUS RECTUM OF A HYPERBOLA

The latus rectum of a hyperbola is a line segment perpendicular to the transverse axis, through any of the foci with its end points lying on the hyperbola.

TO FIND THE LENGTH OF LATUS RECTUM OF THE HYPERBOLA $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$

Let $X'OX$ and YOY' be the coordinate axes.

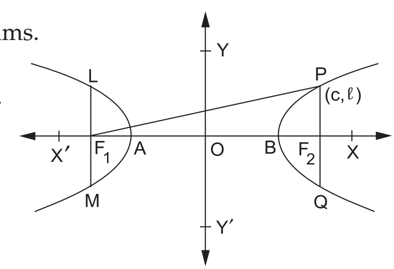
Let $F_1(-c, 0)$ and $F_2(c, 0)$ be the foci of the given hyperbola, where $c^2=(a^2+b^2)$.

Let LF_1M and PF_2Q be the latus rectums.

Let $PF_2=l$.

Then, the coordinates of P are $P(c, l)$.

Since $P(c, l)$ lies on the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, we have



$$\frac{c^2}{a^2}-\frac{l^2}{b^2}=1 \Rightarrow \frac{l^2}{b^2}=\left(\frac{c^2}{a^2}-1\right)$$

$$\Rightarrow l^2=b^2\left(\frac{c^2}{a^2}-1\right)=b^2\cdot\left\{\frac{(a^2+b^2)}{a^2}-1\right\} \quad [\because c^2=(a^2+b^2)]$$

$$\Rightarrow l^2=\frac{b^4}{a^2}$$

$$\Rightarrow l=\frac{b^2}{a}\cdot$$

$\therefore \quad PF_2=l=\frac{b^2}{a} \Leftrightarrow PF_2Q=2l=\frac{2b^2}{a}\cdot$

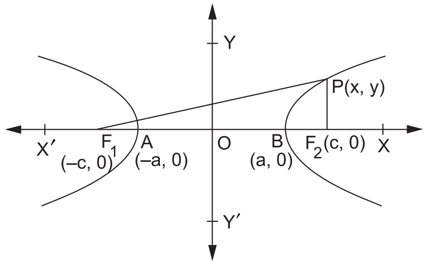
Similarly, $LF_1M=2l=\frac{2b^2}{a}\cdot$

Hence, the length of the latus rectum is $\frac{2b^2}{a}\cdot$

SUMMARY OF THE RESULTS ON HORIZONTAL HYPERBOLA

(i) The standard equation of a horizontal hyperbola is $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$.

(ii) Its *centre* is $O(0, 0)$.



(iii) $X'OX$ is the *transverse axis* and YOY' is the *conjugate axis.*

(iv) Its *foci* are $F_1(-c, 0)$ and $F_2(c, 0)$, i.e., $F_1(-ae, 0)$ and $F_2(ae, 0)$.

(v) Its *vertices* are $A(-a, 0)$ and $B(a, 0)$.

(vi) Its *eccentricity* is, $e = \frac{c}{a} = \frac{\sqrt{a^2 + b^2}}{a}$.

(vii) Length of the transverse axis = $2a$ and its equation is $y = 0$.

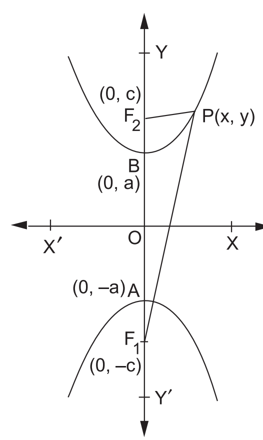
(viii) Length of the conjugate axis = $2b$ and its equation is $x = 0$.

(ix) Length of its *latus rectum* = $\frac{2b^2}{a}$.

We have similar results for *vertical hyperbola*, given below.

SUMMARY OF THE RESULTS ON VERTICAL HYPERBOLA

(i) The standard equation of a vertical hyperbola is $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$.



(ii) Its *centre* is $O(0, 0)$.

(iii) YOY' is the *transverse axis* and $X'OX$ is the *conjugate axis.*

(iv) Its *foci* are $F_1(0, -c)$ and $F_2(0, c)$, i.e., $F_1(0, -ae)$ and $F_2(0, ae)$.

(v) Its *vertices* are $A(0, -a)$ and $B(0, a)$.

(vi) Its *eccentricity* is, $e = \frac{c}{a} = \frac{\sqrt{a^2 + b^2}}{a}$.

(vii) Length of the transverse axis = $2a$ and its equation is $x = 0$.

(viii) Length of the conjugate axis = $2b$ and its equation is $y = 0$.

(ix) Length of its *latus rectum* = $\frac{2b^2}{a}$.

SOLVED EXAMPLES

EXAMPLE 1 *Find the lengths of the axes; the coordinates of the vertices and the foci; the eccentricity and length of the latus rectum of the hyperbola*

$$\frac{x^2}{36} - \frac{y^2}{64} = 1.$$

SOLUTION The equation of the given hyperbola is $\frac{x^2}{36} - \frac{y^2}{64} = 1$.

Comparing the given equation with $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, we get

$a^2 = 36$ and $b^2 = 64$.

$\therefore$ $a = 6$, $b = 8$ and $c = \sqrt{a^2 + b^2} = \sqrt{36 + 64} = \sqrt{100} = 10$.

(i) Length of the transverse axis $= 2a = (2 \times 6)$ units $= 12$ units.
Length of the conjugate axis $= 2b = (2 \times 8)$ units $= 16$ units.

(ii) The coordinates of the vertices are $A(-a, 0)$ and $B(a, 0)$, i.e., $A(-6, 0)$ and $B(6, 0)$.

(iii) The coordinates of the foci are $F_1(-c, 0)$ and $F_2(c, 0)$, i.e., $F_1(-10, 0)$ and $F_2(10, 0)$.

(iv) Eccentricity, $e = \frac{c}{a} = \frac{10}{6} = \frac{5}{3}\cdot$

(v) Length of the latus rectum $= \frac{2b^2}{a} = \left(\frac{2 \times 64}{6}\right)$ units $= \frac{64}{3}$ units.

EXAMPLE 2 *Find the lengths of the axes; the coordinates of the vertices and the foci; the eccentricity and length of the latus rectum of the hyperbola*

$$9x^2 - 16y^2 = 144.$$

SOLUTION $9x^2 - 16y^2 = 144 \Rightarrow \frac{x^2}{16} - \frac{y^2}{9} = 1$.

Thus, the equation of the given hyperbola is $\frac{x^2}{16} - \frac{y^2}{9} = 1$.

Comparing the given equation with $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, we get

$a^2 = 16$ and $b^2 = 9$.

$\therefore$ $a = 4$, $b = 3$ and $c = \sqrt{a^2 + b^2} = \sqrt{16 + 9} = \sqrt{25} = 5$.

(i) Length of the transverse axis $= 2a = (2 \times 4)$ units $= 8$ units.
Length of the conjugate axis $= 2b = (2 \times 3)$ units $= 6$ units.

(ii) The coordinates of the vertices are $A(-a, 0)$ and $B(a, 0)$, i.e., $A(-4, 0)$ and $B(4, 0)$.

(iii) The coordinates of the foci are $F_1(-c, 0)$ and $F_2(c, 0)$, i.e., $F_1(-5, 0)$ and $F_2(5, 0)$.

(iv) Eccentricity, $e = \frac{c}{a} = \frac{5}{4}\cdot$

(v) Length of the latus rectum $= \frac{2b^2}{a} = \left(\frac{2 \times 9}{4}\right)$ units $= \frac{9}{2}$ units.

EXAMPLE 3 *Find the equation of the hyperbola whose vertices are* $(\pm 2,\ 0)$ *and the foci are* $(\pm 3,\ 0)$.

SOLUTION Since the vertices of the given hyperbola are of the form $(\pm a,\ 0)$, it is a horizontal hyperbola.

Let the required equation be $\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}=1$.

Then, its vertices are $(\pm a,\ 0)$.

But, it is given that the vertices are $(\pm 2,\ 0)$.

$\therefore\quad a=2$.

Let its foci be $(\pm c,\ 0)$.

But, it is given that the foci are $(\pm 3,\ 0)$.

$\therefore\quad c=3$.

Now, $b^2=(c^2-a^2)=(3^2-2^2)=(9-4)=5$.

Thus, $a^2=2^2=4$ and $b^2=5$.

Hence, the required equation is $\dfrac{x^2}{4}-\dfrac{y^2}{5}=1$.

EXAMPLE 4 *Find the equation of the hyperbola whose foci are* $(\pm 5,\ 0)$ *and the transverse axis is of length* 8.

SOLUTION Since the foci of the given hyperbola are of the form $(\pm c,\ 0)$, it is a horizontal hyperbola.

Let the required equation be $\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}=1$.

Length of its transverse axis $=2a$.

$\therefore\quad 2a=8 \Leftrightarrow a=4 \Leftrightarrow a^2=16$.

Let its foci be $(\pm c,\ 0)$.

Then, $c=5$ $\quad$ [$\because$ foci are $(\pm 5,\ 0)$].

$\therefore\quad b^2=(c^2-a^2)=(5^2-4^2)=(25-16)=9$.

Thus, $a^2=16$ and $b^2=9$.

Hence, the required equation is $\dfrac{x^2}{16}-\dfrac{y^2}{9}=1$.

EXAMPLE 5 *Find the equation of the hyperbola whose foci are* $(\pm 4,\ 0)$ *and the length of the latus rectum is* 12 *units.*

SOLUTION Since the foci of the given hyperbola are of the form $(\pm c,\ 0)$, it is a horizontal hyperbola.

Let the required equation be $\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}=1$.

Let its foci be $(\pm c,\ 0)$.

Then, $c=4$ $\quad$ [$\because$ foci are $(\pm 4,\ 0)$].

Now, $c = 4 \Leftrightarrow c^2 = 16 \Leftrightarrow a^2 + b^2 = 16.$ ... (i)

Length of the latus rectum $= 12 \Leftrightarrow \dfrac{2b^2}{a} = 12 \Leftrightarrow b^2 = 6a.$ ... (ii)

From (i) and (ii), we get

$$\begin{aligned} a^2 + 6a = 16 &\Leftrightarrow a^2 + 6a - 16 = 0 \\ &\Leftrightarrow (a+8)(a-2) = 0 \\ &\Leftrightarrow a = 2 \qquad [\because a \text{ cannot be negative}]. \end{aligned}$$

Putting $a = 2$ in (ii), we get $b^2 = (6 \times 2) = 12.$

Thus, $a^2 = 2^2 = 4$ and $b^2 = 12.$

Hence, the required equation is $\dfrac{x^2}{4} - \dfrac{y^2}{12} = 1.$

EXAMPLE 6 *Find the equation of the hyperbola whose vertices are* $(\pm 7, 0)$ *and the eccentricity is* $\dfrac{4}{3}\cdot$

SOLUTION Since the vertices of the given hyperbola are of the form $(\pm a, 0)$, it is a horizontal hyperbola.

Let the required equation of the hyperbola be $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1.$

Then, its vertices are $(\pm a, 0)$.

But, the vertices are $(\pm 7, 0)$.

$\therefore \quad a = 7 \Leftrightarrow a^2 = 49.$

Also, $e = \dfrac{c}{a} \Leftrightarrow c = ae = \left(7 \times \dfrac{4}{3}\right) = \dfrac{28}{3}\cdot$

Now, $c^2 = (a^2 + b^2) \Leftrightarrow b^2 = (c^2 - a^2) = \left[\left(\dfrac{28}{3}\right)^2 - 49\right] = \dfrac{343}{9}\cdot$

Thus, $a^2 = 49$ and $b^2 = \dfrac{343}{9}\cdot$

$\therefore$ the required equation is

$$\frac{x^2}{49} - \frac{y^2}{(343/9)} = 1 \Leftrightarrow \frac{x^2}{49} - \frac{9y^2}{343} = 1 \Leftrightarrow 7x^2 - 9y^2 = 343.$$

EXAMPLE 7 *Find the equation of the hyperbola whose eccentricity is* $\dfrac{3}{2}$ *and whose foci are* $(\pm 2, 0)$.

SOLUTION Since the foci of the given hyperbola are of the form $(\pm c, 0)$, it is a horizontal hyperbola.

Let the required equation be $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1.$

Let the foci be $(\pm c, 0)$.

Then, $c = 2 \qquad [\because$ foci are $(\pm 2, 0)]$.

$\therefore \quad e = \dfrac{c}{a} \Leftrightarrow a = \dfrac{c}{e} = \dfrac{2}{(3/2)} = \dfrac{4}{3}.$

And, $c^2 = (a^2 + b^2) \Leftrightarrow b^2 = (c^2 - a^2) = \left(4 - \dfrac{16}{9}\right) = \dfrac{20}{9}.$

Thus, $a^2 = \dfrac{16}{9}$ and $b^2 = \dfrac{20}{9}.$

Hence, the required equation is

$$\frac{x^2}{\left(\frac{16}{9}\right)} - \frac{y^2}{\left(\frac{20}{9}\right)} = 1 \Leftrightarrow \frac{9x^2}{16} - \frac{9y^2}{20} = 1 \Leftrightarrow 45x^2 - 36y^2 = 80.$$

EXAMPLE 8 *Find the lengths of the axes; the coordinates of the vertices and the foci; the eccentricity and length of the latus rectum of the hyperbola*

$$\frac{y^2}{4} - \frac{x^2}{9} = 1.$$

SOLUTION The given equation is $\dfrac{y^2}{4} - \dfrac{x^2}{9} = 1.$

Clearly, the given equation represents a vertical hyperbola.

Comparing the given equation with $\dfrac{y^2}{a^2} - \dfrac{x^2}{b^2} = 1$, we get

$(a^2 = 4 \text{ and } b^2 = 9) \Rightarrow (a = 2 \text{ and } b = 3).$

$\therefore \quad c = \sqrt{a^2 + b^2} = \sqrt{4 + 9} = \sqrt{13}.$

(i) Length of the transverse axis $= 2a = (2 \times 2)$ units $= 4$ units.
Length of the conjugate axis $= 2b = (2 \times 3)$ units $= 6$ units.

(ii) The coordinates of the vertices are $A(0, -a)$ and $B(0, a)$, i.e., $A(0, -2)$ and $B(0, 2)$.

(iii) The coordinates of the foci are $F_1(0, -c)$ and $F_2(0, c)$, i.e., $F_1(0, -\sqrt{13})$ and $F_2(0, \sqrt{13})$.

(iv) Eccentricity, $e = \dfrac{c}{a} = \dfrac{\sqrt{13}}{2}.$

(v) Length of the latus rectum $= \dfrac{2b^2}{a} = \left(\dfrac{2 \times 9}{2}\right)$ units $= 9$ units.

EXAMPLE 9 *Find the lengths of the axes; the coordinates of the vertices and the foci; the eccentricity and length of the latus rectum of the hyperbola*

$$y^2 - 16x^2 = 16.$$

SOLUTION $y^2 - 16x^2 = 16 \Rightarrow \dfrac{y^2}{16} - \dfrac{x^2}{1} = 1.$

Clearly, the given equation represents a vertical hyperbola.

Comparing the given equation with $\dfrac{y^2}{a^2} - \dfrac{x^2}{b^2} = 1$, we get

$(a^2 = 16 \text{ and } b^2 = 1) \Rightarrow (a = 4 \text{ and } b = 1).$

$\therefore \quad c = \sqrt{a^2 + b^2} = \sqrt{16 + 1} = \sqrt{17}.$

(i) Length of the transverse axis = $2a = (2 \times 4)$ units = 8 units.
Length of the conjugate axis = $2b = (2 \times 1)$ units = 2 units.

(ii) The coordinates of the vertices are $A(0, -a)$ and $B(0, a)$, i.e., $A(0, -4)$ and $B(0, 4)$.

(iii) The coordinates of the foci are $F_1(0, -c)$ and $F_2(0, c)$, i.e., $F_1(0, -\sqrt{17})$ and $F_2(0, \sqrt{17})$.

(iv) Eccentricity, $e = \frac{c}{a} = \frac{\sqrt{17}}{4}$.

(v) Length of the latus rectum $= \frac{2b^2}{a} = \left(\frac{2 \times 1}{4}\right)$ unit $= \frac{1}{2}$ unit.

EXAMPLE 10 *Find the equation of the hyperbola whose vertices are* $(0, \pm 3)$ *and the foci are* $(0, \pm 5)$.

SOLUTION Since the vertices of the given hyperbola are of the form $(0, \pm a)$, it is a vertical hyperbola.

Let the required equation be $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$.

Then, its vertices are $(0, \pm a)$.

But, it is given that the vertices are $(0, \pm 3)$.

$\therefore \quad a = 3$.

Let its foci be $(0, \pm c)$.

But, it is given that the foci are $(0, \pm 5)$.

$\therefore \quad c = 5$.

Now, $b^2 = (c^2 - a^2) = (5^2 - 3^2) = (25 - 9) = 16$.

Thus, $a^2 = 3^2 = 9$ and $b^2 = 16$.

Hence, the required equation is $\frac{y^2}{9} - \frac{x^2}{16} = 1$.

EXAMPLE 11 *Find the equation of the hyperbola whose eccentricity is* $\frac{5}{3}$ *and whose vertices are* $(0, \pm 6)$. *Also, find the coordinates of its foci.*

SOLUTION Since the vertices of the given hyperbola are of the form $(0, \pm a)$, it is a vertical hyperbola.

Let the required equation be $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$.

Then, its vertices are $(0, \pm a)$.

But, it is given that the vertices are $(0, \pm 6)$.

$\therefore \quad a = 6$. Also, $e = \frac{5}{3}$.

Now, $\frac{c}{a} = e \Rightarrow c = ae = \left(6 \times \frac{5}{3}\right) = 10$.

And, $c^2 = (a^2 + b^2) \Leftrightarrow b^2 = (c^2 - a^2) = \{(10)^2 - 6^2\} = (100 - 36) = 64$.

Thus, $a^2 = 6^2 = 36$ and $b^2 = 64$.

Hence, the required equation is $\frac{y^2}{36} - \frac{x^2}{64} = 1$.

Also, the coordinates of its foci are $(0, \pm c)$, i.e., $(0, \pm 10)$.

EXAMPLE 12 *Find the equation of the hyperbola whose foci are* $(0, \pm 12)$ *and the length of whose latus rectum is* 36.

SOLUTION Since the coordinates of the foci of the given hyperbola are of the form $(0, \pm c)$, it is a case of vertical hyperbola.

Let the required equation be $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$.

Clearly, $c = 12$.

Length of the latus rectum $= 36 \Leftrightarrow \frac{2b^2}{a} = 36 \Leftrightarrow b^2 = 18a$.

Now, $c^2 = (a^2 + b^2) \Leftrightarrow a^2 = (c^2 - b^2) = (144 - 18a)$

$\Leftrightarrow a^2 + 18a - 144 = 0 \Leftrightarrow (a + 24)(a - 6) = 0$

$\Leftrightarrow a = 6 \quad [\because\ a \text{ is non-negative}]$.

Thus, $a^2 = 6^2 = 36$ and $b^2 = 18a = (18 \times 6) = 108$.

Hence, the required equation is $\frac{x^2}{36} - \frac{y^2}{108} = 1$.

EXAMPLE 13 *Find the equation of the hyperbola with centre at the origin, length of the transverse axis 6 and one focus at* $(0, 4)$.

SOLUTION Since the coordinates of the foci are of the form $(0, \pm c)$, it is a case of vertical hyperbola.

Let its equation be $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$.

Clearly, $c = 4 \quad [\because\ (0, c) = (0, 4)]$.

Length of the transverse axis $= 6 \Leftrightarrow 2a = 6 \Leftrightarrow a = 3$.

Also, $c^2 = (a^2 + b^2) \Leftrightarrow b^2 = (c^2 - a^2) = (4^2 - 3^2) = (16 - 9) = 7$.

Thus, $a^2 = 3^2 = 9$ and $b^2 = 7$.

Hence, the required equation is $\frac{y^2}{9} - \frac{x^2}{7} = 1$.

EXAMPLE 14 *Find the equation of the hyperbola whose foci are at* $(0, \pm 6)$ *and the length of whose conjugate axis is* $2\sqrt{11}$.

SOLUTION Since the foci of the given hyperbola are of the form $(0, \pm c)$, it is a case of vertical hyperbola.

Let its equation be $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$.

Let its foci be $(0, \pm c)$.

But, the foci are at $(0, \pm 6)$.

$\therefore \quad c = 6.$

Length of the conjugate axis $= 2\sqrt{11}$

$2b = 2\sqrt{11} \Leftrightarrow b = \sqrt{11} \Leftrightarrow b^2 = 11.$

Also, $c^2 = (a^2 + b^2) \Leftrightarrow a^2 = (c^2 - b^2) = (36 - 11) = 25.$

Thus, $a^2 = 25$ and $b^2 = 11$.

Hence, the required equation is $\frac{y^2}{25} - \frac{x^2}{11} = 1.$

EXAMPLE 15 *Find the equation of the hyperbola whose foci are at* $(0, \pm\sqrt{10})$ *and which passes through the point* $(2, 3)$.

SOLUTION Since the foci of the given hyperbola are of the form $(0, \pm c)$, it is a case of vertical hyperbola.

Let its equation be $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1.$ ... (i)

Let its foci be $(0, \pm c)$.

But, the foci are $(0, \pm\sqrt{10})$.

$\therefore \quad c = \sqrt{10} \Leftrightarrow c^2 = 10 \Leftrightarrow (a^2 + b^2) = 10$ ... (ii)

$[\because c^2 = (a^2 + b^2)]$.

Since (i) passes through (2, 3), we have $\frac{9}{a^2} - \frac{4}{b^2} = 1.$

Now, $\frac{9}{a^2} - \frac{4}{b^2} = 1 \Leftrightarrow \frac{9}{a^2} - \frac{4}{(10 - a^2)} = 1$ [using (ii)]

$\Leftrightarrow 9(10 - a^2) - 4a^2 = a^2(10 - a^2)$

$\Leftrightarrow a^4 - 23a^2 + 90 = 0$

$\Leftrightarrow (a^2 - 18)(a^2 - 5) = 0$

$\Leftrightarrow a^2 = 5$

$[\because a^2 = 18 \Rightarrow b^2 = -8$, which is not possible].

Thus, $a^2 = 5$ and $b^2 = 5$.

Hence, the required equation is $\frac{y^2}{5} - \frac{x^2}{5} = 1$, i.e., $y^2 - x^2 = 5$.

EXERCISE 24

Find the (i) lengths of the axes, (ii) coordinates of the vertices, (iii) coordinates of the foci, (iv) eccentricity and (iv) length of the latus rectum of each of the following the hyperbola:

1. $\frac{x^2}{9} - \frac{y^2}{16} = 1$

2. $\frac{x^2}{25} - \frac{y^2}{4} = 1$

3. $x^2 - y^2 = 1$

4. $3x^2 - 2y^2 = 6$

5. $25x^2 - 9y^2 = 225$ **6.** $24x^2 - 25y^2 = 600$ **7.** $\frac{y^2}{16} - \frac{x^2}{49} = 1$

8. $\frac{y^2}{9} - \frac{x^2}{27} = 1$ **9.** $3y^2 - x^2 = 108$ **10.** $5y^2 - 9x^2 = 36$

11. Find the equation of the hyperbola with vertices at $(\pm 6, 0)$ and foci at $(\pm 8, 0)$.

12. Find the equation of the hyperbola with vertices at $(0, \pm 5)$ and foci at $(0, \pm 8)$.

13. Find the equation of the hyperbola whose foci are $(\pm\sqrt{29}, 0)$ and the transverse axis is of the length 10.

14. Find the equation of the hyperbola whose foci are $(\pm 5, 0)$ and the conjugate axis is of the length 8. Also, find its eccentricity.

15. Find the equation of the hyperbola whose foci are $(\pm 3\sqrt{5}, 0)$ and the length of the latus rectum is 8 units.

16. Find the equation of the hyperbola whose vertices are $(\pm 2, 0)$ and the eccentricity is 2.

17. Find the equation of the hyperbola whose foci are $(\pm\sqrt{5}, 0)$ and the eccentricity is $\sqrt{\frac{5}{3}}\cdot$

18. Find the equation of the hyperbola, the length of whose latus rectum is 4 and the eccentricity is 3.

19. Find the equation of the hyperbola with eccentricity $\sqrt{2}$ and the distance between whose foci is 16.

20. Find the equation of the hyperbola whose vertices are $(0, \pm 3)$ and the eccentricity is $\frac{4}{3}\cdot$ Also find the coordinates of its foci.

21. Find the equation of the hyperbola whose foci are $(0, \pm 13)$ and the length of whose conjugate axis is 24.

22. Find the equation of the hyperbola whose foci are $(0, \pm 10)$ and the length of whose latus rectum is 9 units.

23. Find the equation of the hyperbola having its foci at $(0, \pm\sqrt{14})$ and passing through the point $P(3, 4)$.

ANSWERS (EXERCISE 24)

1. (i) 6 units, 8 units (ii) $A(-3, 0)$, $B(3, 0)$ (iii) $F_1(-5, 0)$, $F_2(5, 0)$

(iv) $e = \frac{5}{3}$ (v) $10\frac{2}{3}$ units

2. (i) 10 units, 4 units (ii) $A(-5, 0)$, $B(5, 0)$ (iii) $F_1(-\sqrt{29}, 0)$, $F_2(\sqrt{29}, 0)$

(iv) $e = \frac{\sqrt{29}}{5}$ (v) $1\frac{3}{5}$ units

3. (i) 2 units, 2 units (ii) $A(-1, 0)$, $B(1, 0)$ (iii) $F_1(-\sqrt{2}, 0)$, $F_2(\sqrt{2}, 0)$

(iv) $e = \sqrt{2}$ (v) 2 units

4. (i) $2\sqrt{2}$ units, $2\sqrt{3}$ units (ii) $A(-\sqrt{2}, 0)$, $B(\sqrt{2}, 0)$ (iii) $F_1(-\sqrt{5}, 0)$, $F_2(\sqrt{5}, 0)$

(iv) $e = \sqrt{\frac{5}{2}}$ (v) $3\sqrt{2}$ units

5. (i) 6 units, 10 units (ii) $A(-3, 0)$, $B(3, 0)$ (iii) $F_1(-\sqrt{34}, 0)$, $F_2(\sqrt{34}, 0)$

(iv) $e = \frac{\sqrt{34}}{3}$ (v) $16\frac{2}{3}$ units

6. (i) 10 units, $4\sqrt{6}$ units (ii) $A(-5, 0)$, $B(5, 0)$ (iii) $F_1(-7, 0)$, $F_2(7, 0)$

(iv) $e = 1\frac{2}{5}$ (v) $9\frac{3}{5}$ units

7. (i) 8 units, 14 units (ii) $A(0, -4)$, $B(0, 4)$ (iii) $F_1(0, -\sqrt{65})$, $F_2(0, \sqrt{65})$

(iv) $e = \frac{\sqrt{65}}{4}$ (v) $24\frac{1}{2}$ units

8. (i) 6 units, $6\sqrt{3}$ units (ii) $A(0, -3)$, $B(0, 3)$ (iii) $F_1(0, -6)$, $F_2(0, 6)$

(iv) $e = 2$ (v) 18 units

9. (i) 12 units, $12\sqrt{3}$ units (ii) $A(0, -6)$, $B(0, 6)$ (iii) $F_1(0, -12)$, $F_2(0, 12)$

(iv) $e = 2$ (v) 36 units

10. (i) $\frac{12}{\sqrt{5}}$ units, 4 units (ii) $A\left(0, -\frac{6}{\sqrt{5}}\right)$, $B\left(0, \frac{6}{\sqrt{5}}\right)$

(iii) $F_1\left(0, -2\sqrt{\frac{14}{5}}\right)$, $F_2\left(0, 2\sqrt{\frac{14}{5}}\right)$ (iv) $e = \frac{\sqrt{14}}{3}$ (v) $\frac{4\sqrt{5}}{3}$ units

11. $\frac{x^2}{36} - \frac{y^2}{28} = 1$ **12.** $\frac{y^2}{25} - \frac{x^2}{39} = 1$ **13.** $\frac{x^2}{25} - \frac{y^2}{4} = 1$

14. $\frac{x^2}{9} - \frac{y^2}{16} = 1$, $e = \frac{5}{3}$ **15.** $\frac{x^2}{25} - \frac{y^2}{20} = 1$ **16.** $\frac{x^2}{4} - \frac{y^2}{12} = 1$

17. $\frac{x^2}{3} - \frac{y^2}{2} = 1$ **18.** $16x^2 - 2y^2 = 1$ **19.** $x^2 - y^2 = 32$

20. $\frac{y^2}{9} - \frac{x^2}{7} = 1$, $(0, \pm 4)$ **21.** $\frac{y^2}{25} - \frac{x^2}{144} = 1$

22. $\frac{y^2}{64} - \frac{x^2}{36} = 1$ **23.** $y^2 - x^2 = 7$

10. Given equation is $\dfrac{y^2}{(36/5)} - \dfrac{x^2}{4} = 1.$

$\therefore \quad a^2 = \dfrac{36}{5}, b^2 = 4$ and $c^2 = (a^2 + b^2) = \left(\dfrac{36}{5} + 4\right) = \dfrac{56}{5}\cdot$

Also, $e = \dfrac{c}{a} = \left(\dfrac{\sqrt{56}}{\sqrt{5}} \times \dfrac{\sqrt{5}}{6}\right) = \dfrac{2\sqrt{14}}{6} = \dfrac{\sqrt{14}}{3}\cdot$

18. $e = 3 \Leftrightarrow \dfrac{c}{a} = 3 \Leftrightarrow c = 3a \Leftrightarrow c^2 = 9a^2 \Leftrightarrow a^2 + b^2 = 9a^2 \Leftrightarrow 8a^2 = b^2.$ … (i)

$\dfrac{2b^2}{a} = 4 \Leftrightarrow b^2 = 2a.$ … (ii)

$\therefore \quad 8a^2 = 2a \Leftrightarrow 2a(4a - 1) = 0 \Leftrightarrow a = \dfrac{1}{4}\cdot$

$\therefore \quad a^2 = \dfrac{1}{16}$ and $b^2 = \left(2 \times \dfrac{1}{4}\right) = \dfrac{1}{2}\cdot$

19. $2c = 16 \Leftrightarrow c = 8.$ Also, $e = \sqrt{2}.$

Now, $e = \dfrac{c}{a} \Rightarrow a = \dfrac{c}{e} = \dfrac{8}{\sqrt{2}} = 4\sqrt{2}.$

$\therefore \quad b^2 = (c^2 - a^2) = (64 - 32) = 32.$

Thus, $a^2 = 32$ and $b^2 = 32.$

❑

25

Applications of Conic Sections

Conic sections have their applications in various fields. Some of them are listed below.

1. We know that when an object is thrown in space then the path traced by the object (called a projectile) is a parabola. By using various properties of a parabola, we can find the equation of the path traced by the projectile.

Also, we can obtain other useful information such as the greatest height attained by the projectile, its horizontal range, and so on.

2. In a parabolic reflector, light rays or sound waves directed parallel to its axis converge at the focus and are then reflected by it, parallel to the axis.

Because of this property, parabolic reflectors are used in cars, automobiles, loudspeakers, solar cookers, telescopes, etc.

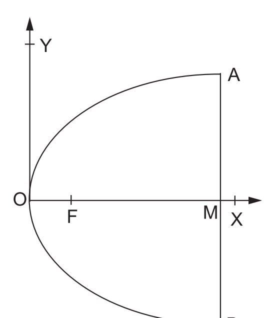
3. Cables shaped as parabolic arcs are used in the construction of suspension bridges.

4. The planets of the solar system move in elliptical paths with the sun at one of the foci. Artificial satellites are made to move in elliptical paths around the earth.

5. Hyperbolas have their applications in the field of ballistics. Suppose a gun is fired and its sound reaches two posts, situated at the two foci of a hyperbola, at different times. From the time difference, the distance between the two listening posts can be calculated.

SOLVED EXAMPLES

EXAMPLE 1 *The focus of a parabolic mirror is at a distance of* 5 cm *from its vertex and the mirror is* 15 cm *deep. Find the diameter of the mirror.*



SOLUTION Let us take the vertex of the mirror at the origin with its axis lying along the positive direction of the x-axis.

Let the equation of the parabolic shape of the mirror be $y^2 = 4ax$.

Since the distance between the focus and the vertex is 5 cm, we have $a = 5$.

$\therefore$ the equation of the parabola is

$$y^2 = 4 \times 5 \times x, \text{ i.e., } y^2 = 20x. \qquad \dots \text{(i)}$$

The depth of the mirror, $OM = 15$ cm.

$\therefore$ the coordinates of one end of the diameter AB are $(15, p)$, where $AM = p$ cm.

Clearly, $A(15, p)$ lies on the parabola $y^2 = 20x$.

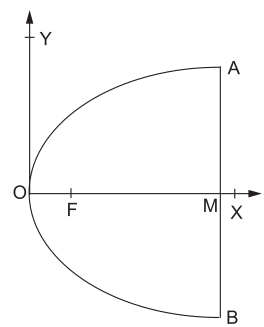
$\therefore$ $p^2 = (20 \times 15) \Rightarrow p = \sqrt{300} = 10\sqrt{3}$

$\Rightarrow AM = 10\sqrt{3}$ cm $\Rightarrow AB = 2 \times AM = 20\sqrt{3}$ cm.

Hence, the diameter of the mirror is $20\sqrt{3}$ cm.

EXAMPLE 2 *A parabolic reflector is* 9 cm *deep and its diameter is* 24 cm. *How far is its focus from the vertex?*

SOLUTION Let us take the vertex of the reflector at the origin, and let its axis lie along the positive direction of the x-axis.



Let the equation of the parabolic shape of the reflector be

$$y^2 = 4ax. \qquad \dots \text{(i)}$$

As given, we have

$OM = 9$ cm and $AB = 24$ cm

$\Rightarrow OM = 9$ cm and $AM = \frac{1}{2} AB = 12$ cm.

$\therefore$ the coordinates of A are $(9, 12)$.

Since A lies on (i), we have

$12 \times 12 = 4 \times a \times 9 \Rightarrow a = 4.$

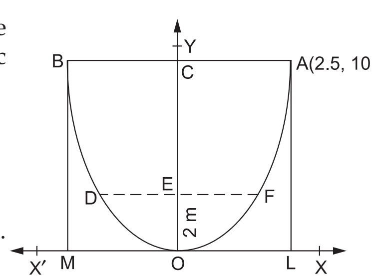
$\therefore$ the equation of the parabola is $y^2 = 16x$.

Its focus is $F(a, 0)$, i.e., $F(4, 0)$.

Hence, the focus is at a distance of 4 cm from the vertex.

EXAMPLE 3 *An arch is in the form of a parabola with its axis vertical. The arch is* 10 m *high and* 5 m *wide at the base. Find its width at a distance of* 2 m *from its vertex.*

SOLUTION Let O be the origin and OY be the vertical axis of the parabolic arch.



Then, the equation of the arch is

$$x^2 = 4ay. \qquad \dots \text{(i)}$$

$\therefore BA = 5$ m and $OC = 10$ m.

Now, $OL = \frac{1}{2} ML = \frac{1}{2} BA = 2.5$ m.

And, $AL = OC = 10$ m.

$\therefore$ the coordinates of A are (2.5, 10). Since $A(2.5, 10)$ lies on (i), we have

$$(2.5)^2 = 4a \times 10 \quad \Rightarrow \quad a = \left(\frac{2.5 \times 2.5}{4 \times 10}\right) = \frac{5}{32}.$$

$$\therefore \quad x^2 = \left(4 \times \frac{5}{32} \times y\right) \quad \Rightarrow \quad x^2 = \frac{5}{8}y. \qquad \ldots \text{(ii)}$$

Draw $DEF \parallel ML$ such that $OE = 2$ m.

Let the coordinates of F be $F(x_1, 2)$.

Since $F(x_1, 2)$ lies on (ii), we have

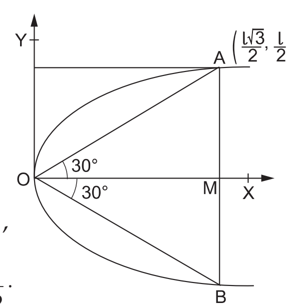
$$x_1^2 = \left(\frac{5}{8} \times 2\right) \quad \Rightarrow \quad x_1^2 = \frac{5}{4} \quad \Rightarrow \quad x_1 = \frac{\sqrt{5}}{2}.$$

$$\therefore \quad EF = \frac{\sqrt{5}}{2}\text{ m} \quad \Rightarrow \quad DF = 2 \times EF = \left(2 \times \frac{\sqrt{5}}{2}\right)\text{ m} = \sqrt{5}\text{ m}.$$

Hence, the required width of the arch at a distance of 2 m from its vertex is $\sqrt{5}$ metres.

EXAMPLE 4 *An equilateral triangle is inscribed in the parabola $y^2 = 4ax$ so that one angular point of the triangle is at the vertex of the parabola. Find the length of each side of the triangle.*

SOLUTION Let $\triangle OAB$ be an equilateral triangle inscribed in the parabola $y^2 = 4ax$.



Let $OA = OB = AB = l$.

Let AB cut the x-axis at M.

Then, $\angle AOM = \angle BOM = 30°$.

$$\therefore \frac{OM}{OA} = \cos 30° \Rightarrow OM = l \cos 30° = \frac{l\sqrt{3}}{2},$$

and $\dfrac{AM}{OA} = \sin 30° \Rightarrow AM = l \sin 30° = \dfrac{l}{2}.$

$\therefore$ the coordinates of A are $\left(\dfrac{l\sqrt{3}}{2}, \dfrac{l}{2}\right)$.

Since A lies on the parabola $y^2 = 4ax$, we have

$$\frac{l^2}{4} = 4a \times \frac{l\sqrt{3}}{2} \quad \Rightarrow \quad l = 8a\sqrt{3}.$$

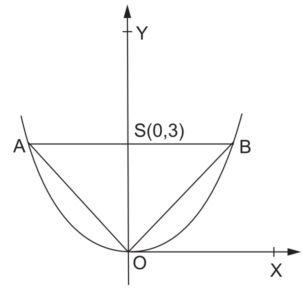
Hence, the length of each side of the triangle is $8a\sqrt{3}$ units.

EXAMPLE 5 *Find the area of the triangle formed by the lines joining the vertex of the parabola $x^2 = 12y$ to the ends of its latus rectum.*

SOLUTION The vertex of the parabola $x^2 = 12y$ is $O(0, 0)$.

Comparing $x^2 = 12y$ with $x^2 = 4ay$, we get $a = 3$.

The coordinates of its focus S are $(0, 3)$.



Clearly, the ends of its latus rectum are

$A(-2a, a)$ and $B(2a, a)$

i.e., $A(-6, 3)$ and $B(6, 3)$.

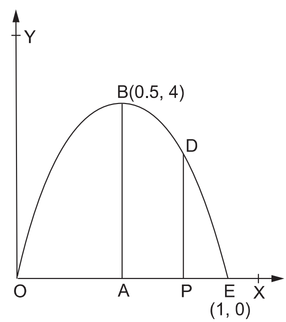
$$\therefore \quad \text{area of } \triangle OBA = \frac{1}{2} \cdot \begin{vmatrix} 0 & 0 & 1 \\ 6 & 3 & 1 \\ -6 & 3 & 1 \end{vmatrix}$$

$$= \frac{1}{2}\{1 \times (18 + 18)\}$$

$$= 18 \text{ sq units.}$$

EXAMPLE 6 *A water jet coming out of the small opening O of a fountain reaches its maximum height of* 4 *metres at a distance of* 0.5 *metre from the vertical. Find the height of the jet above the horizontal OX at a distance of* 0.75 *metre from the point O.*

SOLUTION The path of a water jet is a parabola.



Let the equation of this parabolic path be

$$y = ax^2 + bx + c. \qquad \ldots \text{(i)}$$

Let AB be the maximum height reached.

The path is symmetrical about AB.

Let the jet strike the x-axis at E.

Then, $AE = OA = 0.5$ m

$\Rightarrow \quad OE = 2(OA) = (2 \times 0.5)$ m $= 1$ m.

Clearly, $AB = 4$ m.

Thus, the points O, B and E are

$O(0, 0)$, $B(0.5, 4)$ and $E(1, 0)$.

Since these points lie on (i), we have

$$c = 0, \ \frac{1}{4}a + \frac{1}{2}b = 4 \quad \text{and} \quad a + b = 0$$

$$\Rightarrow \quad c = 0, \ a + 2b = 16 \quad \text{and} \quad a + b = 0$$

$$\Rightarrow \quad a = -16, \ b = 16 \quad \text{and} \quad c = 0.$$

$\therefore$ the equation of the parabola is $y = -16x^2 + 16x$. $\qquad \ldots$ (ii)

Let P be a point on OE such that $OP = 0.75$ m.

Draw $PD \perp OX$, meeting the parabola at D.

Let $PD = h$ metres.

Then, the coordinates of D are $(0.75, h)$.

Since $D(0.75, h)$ lies on (ii), we have

$$h = -16 \times \left(\frac{3}{4}\right)^2 + \left(16 \times \frac{3}{4}\right) \Rightarrow h = (-9 + 12) = 3.$$

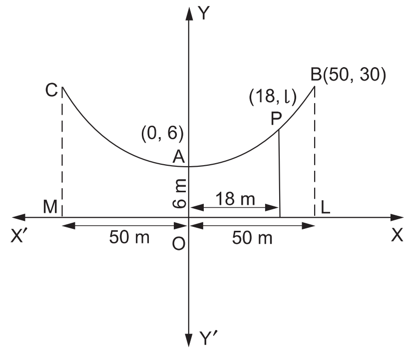
$\therefore$ the required height, $PD = h$ m $= 3$ m.

EXAMPLE 7 *The cable of a uniformly loaded suspension bridge hangs in the form of a parabola. The roadway which is horizontal and* 100 m *long is supported by vertical wires attached to the cable, the longest wire being* 30 m *and the shortest being* 6 m. *Find the length of a supporting wire attached to the roadway* 18 m *from the middle.*

SOLUTION Let $X'OX$ be the x-axis, taken along the roadway, and let YOY' be the y-axis.

Let CAB be the bridge in the form of a parabola having its vertex at $A(0, 6)$ such that A is the centre of the bridge.

Let BL and CM be the longest vertical wires, each of length 30 m, and let OA be the shortest vertical wire, 6 m in length.



The equation of the parabolic form of the bridge is

$$x^2 = 4a(y - 6). \qquad \dots \text{(i)}$$

It passes through $B(50, 30)$.

Putting $x = 50$ and $y = 30$ in (i), we get

$$2500 = 4a(30 - 6) \quad \Rightarrow \quad a = \frac{2500}{96}.$$

Putting $a = \dfrac{2500}{96}$ in (i), we get

$$x^2 = 4 \times \frac{2500}{96}(y - 6) \quad \Rightarrow \quad 6x^2 = 625y - 3750. \qquad \dots \text{(ii)}$$

Let l m be the length of the supporting wire 18 m from the middle.

Then, $P(18, l)$ must satisfy (ii).

$$\therefore \quad 6 \times 18 \times 18 = 625l - 3750 \quad \Rightarrow \quad 625l = 5694$$
$$\Rightarrow \quad l = 9.11.$$

Hence, the length of the supporting wire 18 m from the middle is 9.11 m.

EXAMPLE 8 *A rod of length* 12 cm *moves with its ends always touching the coordinate axes. Determine the equation of the path of a moving point P on the rod which is* 3 cm *from the end in contact with the x-axis.*

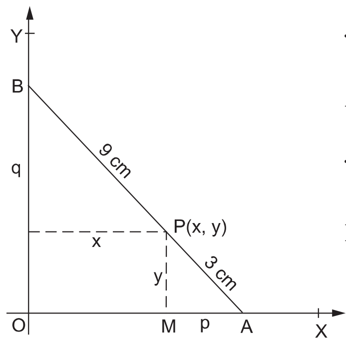
SOLUTION Let AB be the rod and $P(x, y)$ be a point on it such that

$$AP = 3 \text{ cm} \quad \text{and} \quad PB = 9 \text{ cm}.$$

From P, draw $PM \perp OX$ and $PN \perp OY$.

Let $AM = p$ and $BN = q$.

Then $\triangle BNP$ and $\triangle PMA$ are similar.



$\therefore \quad \dfrac{BN}{PM} = \dfrac{BP}{PA} \Rightarrow \dfrac{q}{y} = \dfrac{9}{3} \Rightarrow q = 3y.$

And, $\dfrac{MA}{NP} = \dfrac{PA}{BP} \Rightarrow \dfrac{p}{x} = \dfrac{3}{9} \Rightarrow p = \dfrac{1}{3}x.$

$\therefore \quad OA = OM + MA = x + p = x + \dfrac{1}{3}x = \dfrac{4x}{3},$

and $OB = ON + BN = y + q = y + 3y = 4y.$

In $\triangle BOA$, we have

$$OA^2 + OB^2 = AB^2$$

$$\Rightarrow \left(\frac{4x}{3}\right)^2 + (4y)^2 = (12)^2$$

$$\Rightarrow \frac{16x^2}{9} + 16y^2 = 144$$

$$\Rightarrow \frac{x^2}{81} + \frac{y^2}{9} = 1.$$

Hence, the path of P is an ellipse whose equation is $\dfrac{x^2}{81} + \dfrac{y^2}{9} = 1.$

EXAMPLE 9 *A man running in a racecourse notes that the sum of the distances of the two flag posts from him is always* 10 m, *and the distance between the flag posts is* 8 m. *Find the equation of the path traced by the man.*

SOLUTION We know that an ellipse is the locus of a point that moves in such a way that the sum of its distances from two fixed points (called foci) is constant.

So, the path traced by the man is an ellipse.

Let the equation of the ellipse be

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \text{ where } b^2 = a^2(1 - e^2).$$

Clearly, $\quad 2a = 10 \quad \text{and} \quad 2ae = 8$

$\Rightarrow \quad a = 5 \quad \text{and} \quad e = \dfrac{4}{5}$

$\Rightarrow \quad b^2 = a^2(1 - e^2) = 25\left(1 - \dfrac{16}{25}\right) = 9$

$\Rightarrow \quad b = 3.$

Hence, the equation of the path is $\dfrac{x^2}{25} + \dfrac{y^2}{9} = 1.$

EXAMPLE 10 *An arch is in the form of a semi-ellipse. It is* 8 m *wide and* 2 m *high at the centre. Find the height of the arch at a point* 1.5 m *from one end.*

SOLUTION Let the equation of the ellipse of which the given arch is a part be

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1. \qquad \text{... (i)}$$

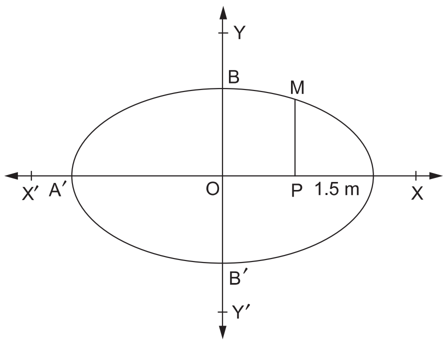
It is being given that

$$A'A = 8\,\text{m} \quad \text{and} \quad OB = 2\,\text{m}$$
$$\Rightarrow \quad 2a = 8 \quad \text{and} \quad b = 2$$
$$\Rightarrow \quad a = 4 \quad \text{and} \quad b = 2.$$

So, the equation of the arch is

$$\frac{x^2}{16} + \frac{y^2}{4} = 1.$$

Let P be a point on $A'OA$ such that $AP = 1.5$ m.



Draw $PM \perp A'A$, meeting the ellipse at M.

Then, $AP = 1.5$ m (given).

Now, $OP = (OA - AP) = (4 - 1.5)$ m $= 2.5$ m. Let $PM = p$ m.

So, the coordinates of M are $(2.5, p)$.

Since M lies on the ellipse, we have

$$\frac{(2.5)^2}{16} + \frac{p^2}{4} = 1 \;\Rightarrow\; \frac{p^2}{4} = \left(1 - \frac{625}{1600}\right) \;\Rightarrow\; p^2 = \frac{39}{16}$$

$$\Rightarrow \; p = \frac{\sqrt{39}}{4} = \frac{6.24}{4} = 1.56.$$

Hence, the height of the arch at a point 1.5 m from one end is 1.56 m.

EXERCISE 25

1. The focus of a parabolic mirror is at a distance of 6 cm from its vertex. If the mirror is 20 cm deep, find its diameter.

2. A parabolic reflector is 5 cm deep and its diameter is 20 cm. How far is its focus from the vertex?

3. The towers of a bridge, hung in the form of a parabola, have their tops 30 m above the roadway, and are 200 m apart. If the cable is 5 m above the roadway at the centre of the bridge, find the length of the vertical supporting cable, 30 m from the centre.

4. A rod of length 15 cm moves with its ends always touching the coordinate axes. Find the equation of the locus of a point P on the rod, which is at a distance of 3 cm from the end in contact with the x-axis.

5. A beam is supported at its ends by supports which are 12 m apart. Since the load is concentrated at its centre, there is a deflection of 3 cm at the centre, and the deflected beam is in the shape of a parabola. How far from the centre is the deflection 1 cm?

ANSWERS (EXERCISE 25)

1. $8\sqrt{30}$ cm **2.** 5 cm **3.** 7.25 m **4.** $5x^2 + 9y^2 = 81$ **5.** $2\sqrt{6}$ m

HINTS TO SOME SELECTED QUESTIONS

5. The beam takes the shape of a parabola, whose equation is of the form $x^2 = 4ay$. … (i)

Since the point $P\left(6, \frac{3}{100}\right)$ lies on it, we have

$$\left(4 \times a \times \frac{3}{100}\right) = 36 \Leftrightarrow a = 300 \text{ m}.$$

Let AB be the deflection. Then,

$$AB = \frac{1}{100} \text{ m and } B \text{ is } B\left(x, \frac{2}{100}\right).$$

$$\therefore \quad x^2 = 4 \times 300 \times \frac{2}{100} \Leftrightarrow x = 2\sqrt{6} \text{ m}.$$

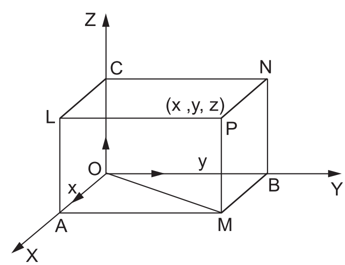
❑

26

Three-Dimensional Geometry

COORDINATES OF A POINT IN SPACE Let O be the origin, and let OX, OY and OZ be three mutually perpendicular lines, taken as the x-axis, the y-axis and the z-axis respectively in such a way that they form a right-handed system.

The planes, YOZ, ZOX and XOY are respectively known as *yz-plane, zx-plane* and *xy-plane.*



These planes, known as coordinate planes, divide the space into eight parts, called *octants.*

Let P be a point in space. Through P, draw planes parallel to coordinate planes, and meeting the axes OX, OY and OZ in points A, B and C respectively. Complete the parallelepiped whose coterminous edges are OA, OB and OC.

Let $OA = x$, $OB = y$ and $OC = z$.

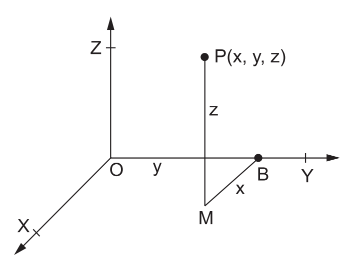
Then, we say that the *coordinates of P are (x, y, z).*

REMARK In fact, we have,

(i) $x =$ distance of P from yz-plane;

(ii) $y =$ distance of P from xz-plane;

(iii) $z =$ distance of P from xy-plane.



As a consequence of the above facts, we find that

(i) Every point in yz-plane has x-coordinate zero.

(ii) Every point in xz-plane has y-coordinate zero.

(iii) Every point in xy-plane has z-coordinate zero.

(iv) Any point on the x-axis is of the form $(x, 0, 0)$.

(v) Any point on the y-axis is of the form $(0, y, 0)$.

(vi) Any point on the z-axis is of the form $(0, 0, z)$.

The following table shows the signs of coordinates in eight octants.

	I	II	III	IV	V	VI	VII	VIII
x	+	–	–	+	+	–	–	+
y	+	+	–	–	+	+	–	–
z	+	+	+	+	–	–	–	–

EXAMPLE 1 *In which octant does the given point lie?*

(i) $(-2, 4, 3)$ *(ii)* $(3, -2, -5)$ *(iii)* $(-6, 3, -4)$

(iv) $(-3, -1, 4)$ *(v)* $(1, -3, 6)$ *(vi)* $(4, 7, -2)$

SOLUTION It is clear from the above table that the point

(i) $(-2, 4, 3)$ lies in octant II;

(ii) $(3, -2, -5)$ lies in octant VIII;

(iii) $(-6, 3, -4)$ lies in octant VI;

(iv) $(-3, -1, 4)$ lies in octant III;

(v) $(1, -3, 6)$ lies in octant IV;

(vi) $(4, 7, -2)$ lies in octant V.

EXAMPLE 2 *If a point lies on the y-axis then what are its x-coordinate and z-coordinate?*

SOLUTION If a point lies on the y-axis then its x-coordinate is 0 and its z-coordinate is 0.

EXAMPLE 3 *If a point lies in xy-plane then what is its z-coordinate?*

SOLUTION If a point lies in xy-plane then its z-coordinate is 0.

EXAMPLE 4 *In which plane does the point* $(0, 5, -4)$ *lie?*

SOLUTION Clearly, the point $(0, 5, -4)$ lies in the yz-plane.

EXERCISE 26A

1. If a point lies on the z-axis then find its x-coordinate and y-coordinate.
2. If a point lies on yz-plane then what is its x-coordinate?
3. In which plane does the point $(4, -3, 0)$ lie?
4. In which octant does each of the given points lie?

(i) $(-4, -1, -6)$ (ii) $(2, 3, -4)$ (iii) $(-6, 5, -1)$

(iv) $(4, -3, -2)$ (v) $(-1, -6, 5)$ (vi) $(4, 6, 8)$

ANSWERS (EXERCISE 26A)

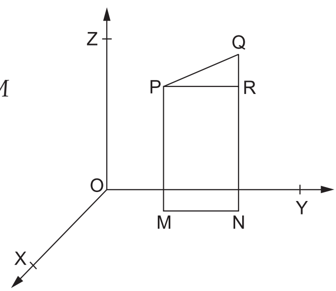
1. $x = 0, y = 0$ **2.** $x = 0$ **3.** xy-plane

4. (i) VII (ii) V (iii) VI (iv) VIII (v) III (vi) I

THEOREM 1 **(Distance Formula)** *Prove that the distance between the points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ is given by*

$$PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}.$$

PROOF Let O be the origin, and let $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ be the given point. From P and Q, draw perpendiculars PM and QN respectively on the xy-plane.



Then, the coordinates of M and N are

$M(x_1, y_1, 0)$ and $N(x_2, y_2, 0)$.

The two-dimensional coordinates of M and N referred to OX and OY are

$M(x_1, y_1)$ and $N(x_2 y_2)$.

$\therefore \quad MN^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2.$

Now, from P draw $PR \perp QN$.

Then, PR is parallel and equal to MN.

Now, in right triangle PRQ, we have

$$\begin{aligned} PQ^2 &= PR^2 + RQ^2 \\ &= MN^2 + (QN - RN)^2 \\ &= MN^2 + (QN - PM)^2 \\ &= (x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2 \quad [\because \; PM = z_1 \text{ and } QN = z_2]. \end{aligned}$$

$$\therefore \quad PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}.$$

COROLLARY *The distance of the point $P(x, y, z)$ from the origin $O(0, 0, 0)$ is*

$$OP = \sqrt{(x - 0)^2 + (y - 0)^2 + (z - 0)^2} = \sqrt{x^2 + y^2 + z^2}.$$

EXAMPLE 1 *Find the distance between the points $A(-2, 1, -3)$ and $B(4, 3, -6)$.*

SOLUTION We have

$$AB = \sqrt{\{[4 - (-2)]^2 + (3 - 1)^2 + [-6 - (-3)]^2\}}$$

$$= \sqrt{\{6^2 + 2^2 + (-3)^2\}} = \sqrt{49} = 7 \text{ units}.$$

EXAMPLE 2 *Show that the points $A(0, 7, 10)$, $B(-1, 6, 6)$ and $C(-4, 9, 6)$ form an isosceles right-angled triangle.*

SOLUTION We have

$$AB = \sqrt{(-1 - 0)^2 + (6 - 7)^2 + (6 - 10)^2} = \sqrt{18} = 3\sqrt{2},$$

$$BC = \sqrt{(-4 + 1)^2 + (9 - 6)^2 + (6 - 6)^2} = \sqrt{18} = 3\sqrt{2},$$

and $AC = \sqrt{(-4 - 0)^2 + (9 - 7)^2 + (6 - 7)^2} = \sqrt{36} = 6.$

Clearly, $AB = BC$ and $AB^2 + BC^2 = (18 + 18) = 36 = AC^2$.

Hence, triangle ABC is an isosceles right-angled triangle.

EXAMPLE 3 *Prove that the points* $A(3, -2, 4)$, $B(1, 1, 1)$ *and* $C(-1, 4, -2)$ *are collinear.*

SOLUTION We have

$$AB = \sqrt{(1-3)^2 + (1+2)^2 + (1-4)^2} = \sqrt{4+9+9} = \sqrt{22},$$

$$BC = \sqrt{(-1-1)^2 + (4-1)^2 + (-2-1)^2} = \sqrt{4+9+9} = \sqrt{22},$$

and $AC = \sqrt{(-1-3)^2 + (4+2)^2 + (-2-4)^2} = \sqrt{16+36+36}$

$$= \sqrt{88} = 2\sqrt{22}.$$

$\therefore \quad AB + BC = AC.$

This shows that the given points are collinear.

EXAMPLE 4 *Find the equation of the curve formed by the set of all points whose distances from the points* $(3, 4, -5)$ *and* $(-2, 1, 4)$ *are equal.*

SOLUTION Let $P(x, y, z)$ be any point on the given curve, and let $A(3, 4, -5)$ and $B(-2, 1, 4)$ be the given points.

Then, $PA = PB$

$\Rightarrow \quad PA^2 = PB^2$

$\Rightarrow \quad (x-3)^2 + (y-4)^2 + (z+5)^2 = (x+2)^2 + (y-1)^2 + (z-4)^2$

$\Rightarrow \quad 10x + 6y - 18z - 29 = 0.$

Hence, the required curve is $10x + 6y - 18z - 29 = 0$.

EXAMPLE 5 *Find the equation of the curve formed by the set of all those points the sum of whose distances from the points* $A(4, 0, 0)$ *and* $B(-4, 0, 0)$ *is* 10 *units.*

SOLUTION Let $P(x, y, z)$ be an arbitrary point on the given curve. Then,

$PA + PB = 10$

$\Rightarrow \quad \sqrt{(x-4)^2 + y^2 + z^2} + \sqrt{(x+4)^2 + y^2 + z^2} = 10$

$\Rightarrow \quad \sqrt{(x+4)^2 + y^2 + z^2} = 10 - \sqrt{(x-4)^2 + y^2 + z^2}$ ... (i)

$\Rightarrow \quad (x+4)^2 + y^2 + z^2 = 100 + (x-4)^2 + y^2 + z^2 - 20\sqrt{(x-4)^2 + y^2 + z^2}$

[on squaring both sides of (i)]

$\Rightarrow \quad 16x = 100 - 20\sqrt{(x-4)^2 + y^2 + z^2}$

$\Rightarrow \quad 5\sqrt{(x-4)^2 + y^2 + z^2} = (25 - 4x)$

$\Rightarrow \quad 25[(x-4)^2 + y^2 + z^2] = 625 + 16x^2 - 200x$

$\Rightarrow \quad 9x^2 + 25y^2 + 25z^2 - 225 = 0.$

Hence, the required equation of the curve is

$$9x^2 + 25y^2 + 25z^2 - 225 = 0.$$

SOME USEFUL RESULTS

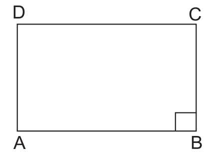
A quadrilateral $ABCD$ is a

(i) parallelogram, if $AB = CD$ and $BC = DA$;

(ii) rectangle, if $AB = CD$, $BC = DA$ and $AC = BD$;

(iii) rhombus, if $AB = BC = CD = DA$ but $AC \neq BD$;

(iv) square, if $AB = BC = CD = DA$ and $AC = BD$.



EXERCISE 26B

1. Find the distance between the points:

(i) $A(5, 1, 2)$ and $B(4, 6, -1)$ (ii) $P(1, -1, 3)$ and $Q(2, 3, -5)$

(iii) $R(1, -3, 4)$ and $S(4, -2, -3)$ (iv) $C(9, -12, -8)$ and the origin

2. Show that the points $A(1, -1, -5)$, $B(3, 1, 3)$ and $C(9, 1, -3)$ are the vertices of an equilateral triangle.

3. Show that the points $A(4, 6, -5)$, $B(0, 2, 3)$ and $C(-4, -4, -1)$ form the vertices of an isosceles triangle.

4. Show that the points $A(0, 1, 2)$, $B(2, -1, 3)$ and $C(1, -3, 1)$ are the vertices of an isosceles right-angled triangle.

5. Show that the points $A(1, 1, 1)$, $B(-2, 4, 1)$, $C(1, -5, 5)$ and $D(2, 2, 5)$ are the vertices of a square.

6. Show that the points $A(1, 2, 3)$, $B(-1, -2, -1)$, $C(2, 3, 2)$ and $D(4, 7, 6)$ are the vertices of a parallelogram. Show that $ABCD$ is not a rectangle.

7. Show that the points $P(2, 3, 5)$, $Q(-4, 7, -7)$, $R(-2, 1, -10)$ and $S(4, -3, 2)$ are the vertices of a rectangle.

8. Show that the points $P(1, 3, 4)$, $Q(-1, 6, 10)$, $R(-7, 4, 7)$ and $S(-5, 1, 1)$ are the vertices of a rhombus.

9. Show that $D(-1, 4, -3)$ is the circumcentre of triangle ABC with vertices $A(3, 2, -5)$, $B(-3, 8, -5)$ and $C(-3, 2, 1)$.

10. Show that the following points are collinear:

(i) $A(-2, 3, 5)$, $B(1, 2, 3)$ and $C(7, 0, -1)$

(ii) $A(3, -5, 1)$, $B(-1, 0, 8)$ and $C(7, -10, -6)$

(iii) $P(3, -2, 4)$, $Q(1, 1, 1)$ and $R(-1, 4, 2)$

11. Find the equation of the curve formed by the set of all points which are equidistant from the points $A(-1, 2, 3)$ and $B(3, 2, 1)$.

12. Find the point on the y-axis which is equidistant from the points $A(3, 1, 2)$ and $B(5, 5, 2)$.

13. Find the point on the z-axis which is equidistant from the points $A(1, 5, 7)$ and $B(5, 1, -4)$.

14. Find the coordinates of the point which is equidistant from the points $A(a, 0, 0)$, $B(0, b, 0)$, $C(0, 0, c)$ and $O(0, 0, 0)$.

15. Find the point in yz-plane which is equidistant from the points $A(3, 2, -1)$, $B(1, -1, 0)$ and $C(2, 1, 2)$.

16. Find the point in xy-plane which is equidistant from the points $A(2, 0, 3)$, $B(0, 3, 2)$ and $C(0, 0, 1)$.

ANSWERS (EXERCISE 26B)

1. (i) $\sqrt{35}$ units (ii) 9 units (iii) $\sqrt{59}$ units (iv) 17 units

11. $2x - z = 0$ **12.** $(0, 3, 0)$ **13.** $\left(0, 0, \frac{3}{2}\right)$ **14.** $\left(\frac{a}{2}, \frac{b}{2}, \frac{c}{2}\right)$

15. $\left(0, \frac{31}{16}, \frac{-3}{16}\right)$ **16.** $(3, 2, 0)$

HINTS TO SOME SELECTED QUESTIONS

6. Show that $AB = CD$, $BC = DA$ and $AC \neq BD$.

7. Show that $AB = CD$, $BC = DA$ and $AC = BD$.

8. Show that $PQ = QR = RS = SP$ and $PR \neq QS$.

9. Show that $DA = DB = DC$.

11. Let $P(x, y, z)$ be an arbitrary point on the given curve. Then,

$$PA = PB \Rightarrow PA^2 = PB^2.$$

12. Any point on the y-axis is of the form $P(0, y, 0)$.

14. Let the required point be $P(x, y, z)$. Then,

$$PA^2 = PB^2 = PC^2 = PO^2 \Leftrightarrow PA^2 - PO^2 = 0, PB^2 - PO^2 = 0 \text{ and } PC^2 - PO^2 = 0.$$

15. Any point in the yz-plane is of the form $P(0, y, z)$.

$$|AP| = |BP| = |CP| \Leftrightarrow AP^2 = BP^2 \text{ and } BP^2 = CP^2.$$

$$(0-3)^2 + (y-2)^2 + (z+1)^2 = (0-1)^2 + (y+1)^2 + (z-0)^2$$

and $(0-1)^2 + (y+1)^2 + (z-0)^2 = (0-2)^2 + (y-1)^2 + (z-2)^2$

$\Rightarrow \quad 3y - z - 6 = 0 \quad$ and $\quad 4y + 4z - 7 = 0.$

On solving, we get $y = \frac{31}{16}$ and $z = \frac{-3}{16}$.

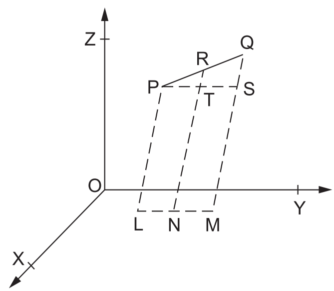
Hence, the required point is $\left(0, \frac{31}{16}, \frac{-3}{16}\right)$.

SECTION FORMULAE

THEOREM 2 **(Section Formula)** *Let $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ be two points and let $R(x, y, z)$ be a point on PQ dividing it in the ratio m : n. Prove that*

$$x = \frac{mx_2 + nx_1}{m+n},\ y = \frac{my_2 + ny_1}{m+n} \text{ and } z = \frac{mz_2 + nz_1}{m+n}.$$

PROOF From P, Q and R, draw perpendiculars PL, QM and RN on the xy-plane. Also, draw $PS \perp QM$, meeting QM and RN at S and T respectively. From similar triangles PRT and PQS, we have



$$\frac{RT}{QS} = \frac{PR}{PQ}$$

$$\Rightarrow \frac{RN - TN}{QM - SM} = \frac{m}{m+n}$$

$$\Rightarrow \frac{RN - PL}{QM - PL} = \frac{m}{m+n}$$

$$\Rightarrow \frac{z - z_1}{z_2 - z_1} = \frac{m}{m+n}$$

$$\Rightarrow z = \frac{mz_2 + nz_1}{m+n}.$$

Similarly, $x = \dfrac{mx_2 + nx_1}{m+n}$ and $y = \dfrac{my_2 + ny_1}{m+n}$.

Hence, $x = \dfrac{mx_2 + nx_1}{m+n}$, $y = \dfrac{my_2 + ny_1}{m+n}$, $z = \dfrac{mz_2 + nz_1}{m+n}$.

COROLLARY 1 *The coordinates of the midpoint of the join of $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ are*

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2}\right).$$

PROOF Let R be the midpoint of PQ.

Putting $m = n = 1$ in the above result, we find that the coordinates of R are

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2}\right).$$

COROLLARY 2 *The coordinates of a point R which divides the join of $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ externally in the ratio m : n are*

$$\left(\frac{mx_2 - nx_1}{m-n}, \frac{my_2 - ny_1}{m-n}, \frac{mz_2 - nz_1}{m-n}\right).$$

PROOF Replacing n by $-n$ in Theorem 2, we find that the coordinates of R are

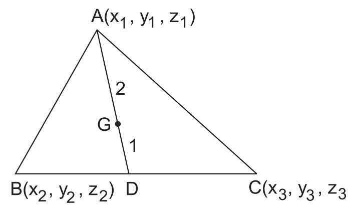
$$\left(\frac{mx_2 - nx_1}{m - n}, \frac{my_2 - ny_1}{m - n}, \frac{mz_2 - nz_1}{m - n}\right).$$

CENTROID OF A TRIANGLE

THEOREM 3 *Show that the centroid of the triangle with vertices $A(x_1, y_1, z_1)$, $B(x_2, y_2, z_2)$ and $C(x_3, y_3, z_3)$ is*

$$\left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}, \frac{z_1 + z_2 + z_3}{3}\right).$$

PROOF Let D be the midpoint of BC. Join A and D.



Then, the coordinates of D are

$$\left(\frac{x_2 + x_3}{2}, \frac{y_2 + y_3}{2}, \frac{z_2 + z_3}{2}\right).$$

Let G be the centroid of triangle ABC. Then, G lies on AD and divides it in the ratio 2 : 1.

$\therefore$ the coordinates of G are

$$\left[\frac{2\cdot\left(\frac{x_2 + x_3}{2}\right) + 1\cdot x_1}{2+1}, \frac{2\cdot\left(\frac{y_2 + y_3}{2}\right) + 1\cdot y_1}{2+1}, \frac{2\cdot\left(\frac{z_2 + z_3}{2}\right) + 1\cdot z_1}{2+1}\right],$$

i.e., $\left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}, \frac{z_1 + z_2 + z_3}{3}\right).$

SOLVED EXAMPLES

EXAMPLE 1 *Find the coordinates of the point which divides the join of the points $P(5, 4, 2)$ and $Q(-1, -2, 4)$ in the ratio $2 : 3$.*

SOLUTION Let $R(x, y, z)$ be the required point. Then,

$$x = \frac{2(-1) + 3\times 5}{2+3}, \; y = \frac{2(-2) + 3\times 4}{2+3}, \; z = \frac{2\times 4 + 3\times 2}{2+3}$$

or $x = \frac{13}{5}$, $y = \frac{8}{5}$ and $z = \frac{14}{5}$.

So, the required point is $R\left(\frac{13}{5}, \frac{8}{5}, \frac{14}{5}\right)$.

EXAMPLE 2 *Find the coordinates of the point which divides the join of the points $A(2, -1, 3)$ and $B(4, 3, 1)$ externally in the ratio $3 : 4$.*

SOLUTION Let $C(x, y, z)$ be the required point. Then,

$$x = \frac{3 \times 4 - 4 \times 2}{3 - 4}, \; y = \frac{3 \times 3 - 4(-1)}{3 - 4}, \; \frac{3 \times 1 - 4 \times 3}{3 - 4}$$

or $x = -4$, $y = -13$ and $z = 9$.

$\therefore$ the required point is $C(-4, -13, 9)$.

EXAMPLE 3 *Find the ratio in which the join of the points $P(2, -1, 3)$ and $Q(4, 3, 1)$ is divided by the point* $\left(\frac{20}{7}, \frac{5}{7}, \frac{15}{7}\right)$.

SOLUTION Let the required ratio be $\lambda : 1$. Then, the coordinates of R are

$$\left(\frac{4\lambda + 2}{\lambda + 1}, \frac{3\lambda - 1}{\lambda + 1}, \frac{\lambda + 3}{\lambda + 1}\right).$$

But, the coordinates of R are $\left(\frac{20}{7}, \frac{5}{7}, \frac{15}{7}\right)$

$$\therefore \quad \frac{4\lambda + 2}{\lambda + 1} = \frac{20}{7} \quad \text{or} \quad \lambda = \frac{3}{4}.$$

So, the required ratio is $\frac{3}{4} : 1$, i.e., $3 : 4$.

EXAMPLE 4 *Find the ratio in which the line segment, joining the points $P(2, 3, 4)$ and $Q(-3, 5, -4)$ is divided by the yz-plane. Also, find the point of intersection.*

SOLUTION Let PQ be the divided by the yz-plane at a point R in the ratio $\lambda : 1$. Then, the coordinates of R are

$$\left(\frac{-3\lambda + 2}{\lambda + 1}, \frac{5\lambda + 3}{\lambda + 1}, \frac{-4\lambda + 4}{\lambda + 1}\right). \qquad \text{... (i)}$$

Since R lies on the yz-plane, the x-coordinate or R is therefore 0.

$$\therefore \quad \frac{-3\lambda + 2}{\lambda + 1} = 0, \quad \text{or} \quad \lambda = \frac{2}{3}.$$

So, the required ratio is $\frac{2}{3} : 1$, i.e., $2 : 3$.

Putting $\lambda = \frac{2}{3}$ in (i), the point of intersection of the line segment PQ and the yz-plane is $\left(0, \frac{19}{5}, \frac{4}{5}\right)$.

EXAMPLE 5 *Find the ratio in which the join of $A(2, 1, 5)$ and $B(3, 4, 3)$ is divided by the plane $2x + 2y - 2z = 1$. Also, find the coordinates of the point of division.*

SOLUTION Suppose the given plane intersects AB at a point C and let the required ratio be $\lambda : 1$.

Then, the coordinates of C are

$$\left(\frac{3\lambda+2}{\lambda+1}, \frac{4\lambda+1}{\lambda+1}, \frac{3\lambda+5}{\lambda+1}\right). \quad \ldots (1)$$

Since C lies on the plane $2x + 2y - 2z = 1$, this point must satisfy the equation of the plane.

$$\therefore \quad 2\left(\frac{3\lambda+2}{\lambda+1}\right) + 2\left(\frac{4\lambda+1}{\lambda+1}\right) - 2\left(\frac{3\lambda+5}{\lambda+1}\right) = 1 \quad \text{or} \quad \lambda = \frac{5}{7}.$$

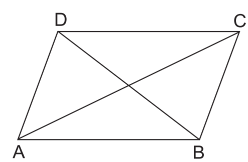
So, the required ratio is $\frac{5}{7} : 1$, i.e., $5 : 7$.

Putting $\lambda = \frac{5}{7}$ in (i), the required point of division is $C\left(\frac{29}{12}, \frac{9}{4}, \frac{25}{6}\right)$.

EXAMPLE 6 *Three vertices of a parallelogram ABCD are $A(3, -1, 2)$, $B(1, 2, -4)$ and $C(-1, 1, 2)$. Find the coordinates of the fourth vertex D.*

SOLUTION Let $D(x, y, z)$ be the required point. Then, the midpoint of diagonal BD is

$$\left(\frac{x+1}{2}, \frac{y+2}{2}, \frac{z-4}{2}\right).$$



And, the midpoint of diagonal AC is

$$\left(\frac{3-1}{2}, \frac{-1+1}{2}, \frac{2+2}{2}\right), \text{ i.e., } (1, 0, 2).$$

But, the midpoints of the diagonals of a parallelogram always coincide.

$$\therefore \quad \frac{x+1}{2} = 1, \frac{y+2}{2} = 0 \text{ and } \frac{z-4}{2} = 2.$$

So, $x = 1$, $y = -2$ and $z = 8$.

Hence, the required point is $D(1, -2, 8)$.

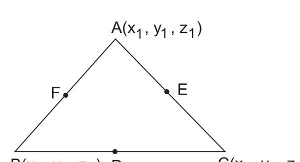
EXAMPLE 7 *The midpoints of the sides of a triangle are $(1, 5, -1)$, $(0, 4, -2)$ and $(2, 3, 4)$. Find its vertices.*

SOLUTION Let $A(x_1, y_1, z_1)$, $B(x_2, y_2, z_2)$ and $C(x_3, y_3, z_3)$ be the vertices of the given triangle, and let $D(1, 5, -1)$, $E(0, 4, -2)$ and $F(2, 3, 4)$ be the midpoints of the sides BC, CA and AB respectively. Then,

$$\frac{x_2+x_3}{2} = 1; \frac{y_2+y_3}{2} = 5; \frac{z_2+z_3}{2} = 1;$$

$$\frac{x_3+x_1}{2} = 0; \frac{y_3+y_1}{2} = 4; \frac{z_3+z_1}{2} = -2;$$

$$\frac{x_1+x_2}{2} = 2; \frac{y_1+y_2}{2} = 3 \text{ and } \frac{z_1+z_2}{2} = 4.$$



Thus, $x_2 + x_3 = 2; x_3 + x_1 = 0; x_1 + x_2 = 4;$

$y_2 + y_3 = 10; y_3 + y_1 = 8; y_1 + y_2 = 6;$

$z_2 + z_3 = -2; z_3 + z_1 = -4; z_1 + z_2 = 8.$

Adding first three equations, we get

$$2(x_1 + x_2 + x_3) = 6 \quad \text{or} \quad x_1 + x_2 + x_3 = 3.$$

Thus, $x_1 = 1, x_2 = 3$ and $x_3 = -1.$

Adding next three equations, we get

$$2(y_1 + y_2 + y_3) = 24 \quad \text{or} \quad y_1 + y_2 + y_3 = 12.$$

$\therefore \quad y_1 = 2; y_2 = 4$ and $y_3 = 6.$

Adding last three equations, we get

$$2(z_1 + z_2 + z_3) = 2 \quad \text{or} \quad z_1 + z_2 + z_3 = 1.$$

$\therefore \quad z_1 = 3, z_2 = 5$ and $z_3 = -7.$

Hence, the vertices of the given triangle are

$A(1, 2, 3); B(3, 4, 5)$ and $C(-1, 6, -7).$

EXAMPLE 8 *Using the section formula, prove that the three points $A(-2, 3, 5)$, $B(1, 2, 3)$ and $C(7, 0, -1)$ are collinear.*

SOLUTION Suppose C divides AB in the ratio $\lambda : 1$.

Then, $\dfrac{\lambda - 2}{\lambda + 1} = 7, \dfrac{2\lambda + 3}{\lambda + 1} = 0$ and $\dfrac{3\lambda + 5}{\lambda + 1} = -1.$

From each of these equations, we get $\lambda = \dfrac{-3}{2}\cdot$

This shows that C divides AB externally in the ratio $3 : 2$.

So, C lies on the line joining A and B.

Hence, the given points A, B, C are collinear.

EXERCISE 26C

1. Find the coordinates of the point which divides the join of $A(3, 2, 5)$ and $B(-4, 2, -2)$ in the ratio $4 : 3$.

2. Let $A(2, 1, -3)$ and $B(5, -8, 3)$ be two given points. Find the coordinates of the points of trisection of the line segment AB.

3. Find the coordinates of the point that divides the join of $A(-2, 4, 7)$ and $B(3, -5, 8)$ extremally in the ratio $2 : 1$.

4. Find the ratio in which the point $R(5, 4, -6)$ divides the join of $P(3, 2, -4)$ and $Q(9, 8, -10)$.

5. Find the ratio in which the point $C(5, 9, -14)$ divides the join of $A(2, -3, 4)$ and $B(3, 1, -2)$.

6. Find the ratio in which the line segment having the end points $A(-1, -3, 4)$ and $B(4, 2, -1)$ is divided by the xz-plane. Also, find the coordinates of the point of division.

7. Find the coordinates of the point where the line joining $A(3, 4, 1)$ and $B(5, 1, 6)$ crosses the xy-plane.

8. Find the ratio in which the plane $x - 2y + 3z = 5$ divides the join of $A(3, -5, 4)$ and $B(2, 3, -7)$. Find the coordinates of the point of intersection of the line and the plane.

9. The vertices of a triangle ABC are $A(3, 2, 0)$, $B(5, 3, 2)$ and $C(-9, 6, -3)$. The bisector AD of $\angle A$ meets BC at D. Find the coordinates of D.

10. If the three consecutive vertices of a parallelogram be $A(3, 4, -1)$, $B(7, 10, -3)$ and $C(5, -2, 7)$, find the fourth vertex D.

11. Two vertices of a triangle ABC are $A(2, -4, 3)$ and $B(3, -1, -2)$, and its centroid is $(1, 0, 3)$. Find its third vertex C.

12. If origin is the centroid of traingle ABC with vertices $A(a, 1, 3)$, $B(-2, b, -5)$ and $C(4, 7, c)$, find the values of a, b, c.

13. The midpoints of the sides of a triangle are $(1, 5, -1)$, $(0, 4, -2)$ and $(2, 3, 4)$. Find its vertices.

ANSWERS (EXERCISE 26C)

1. $(-1, 2, 1)$ **2.** $(3, -2, -1), (4, -5, 1)$ **3.** $(8, -14, 9)$

4. $1:2$ **5.** $3:2$ (externally) **6.** $3:2, (2, 0, 1)$

7. $\left(\frac{13}{5}, \frac{23}{5}, 0\right)$ **8.** $2:3, \left(\frac{13}{5}, \frac{-9}{5}, \frac{-2}{5}\right)$ **9.** $\left(\frac{19}{8}, \frac{57}{16}, \frac{17}{16}\right)$

10. $(1, -8, 9)$ **11.** $(-2, 5, 8)$ **12.** $a = -2, b = -8, c = 2$

13. $(1, 2, 3); (3, 4, 5); (-1, 6, -7)$

HINTS TO SOME SELECTED QUESTIONS

2. Let P and Q be the points of trisection of AB. Then, P divides AB in the ratio $1:2$ and Q divides AB in the ratio $2:1$.

7. Suppose the xy-plane divides AB in the ratio $\lambda : 1$.

Then, $\frac{6\lambda + 1}{\lambda + 1} = 0 \Rightarrow \lambda = -\frac{1}{6}$.

Divide AB in the ratio $1:(-6)$.

8. D divides BC in the ratio $AB : AC$.

❑

27
Limits

CONCEPT OF LIMIT

Consider the function $f(x) = \dfrac{(x^2 - 1)}{(x - 1)}$.

Clearly, $f(1) = \dfrac{0}{0}$, which is meaningless.

Thus, $f(x)$ is not defined at $x = 1$.

Now, $f(x) = \dfrac{(x^2 - 1)}{(x - 1)} = \dfrac{(x - 1)(x + 1)}{(x - 1)} = (x + 1)$, only when $x \neq 1$.

If we give to x, a value not exactly 1 but slightly more than 1 then clearly, the value of $f(x)$ is slightly more than 2. Now, if we go on decreasing this value and take it nearer to 1 then clearly the value of $f(x)$ will come nearer to 2, as shown below.

If $x = 1.1$	then $f(x) = 2.1$.
If $x = 1.01$	then $f(x) = 2.01$.
If $x = 1.001$	then $f(x) = 2.001$.
… … …	… … …
If $x = 1.0000001$	then $f(x) = 2.0000001$.
… … …	… … …

Thus, as the value of x approaches 1, the value of $f(x)$ approaches 2, and we write:

as $x \to 1, f(x) \to 2$.

(The symbol '$\to$' stands for 'approaches to'.)

Similarly, if we give to x, a value slightly less than 1 then the value of $f(x)$ is slightly less than 2. Now, if we go on increasing this value and take it nearer to 1, the value of $f(x)$ will come nearer to 2, as shown below.

If $x = 0.9$	then $f(x) = 1.9$.
If $x = 0.99$	then $f(x) = 1.99$.

If $x = 0.999$ then $f(x) = 1.999$.

... ...

If $x = 0.999999$ then $f(x) = 1.999999$.

... ...

Thus, in this case also

$$\text{as } x \to 1, f(x) \to 2.$$

We express this fact as $\lim\limits_{x \to 1} \left(\dfrac{x^2 - 1}{x - 1} \right) = 2.$

LIMIT *We say that* $\lim\limits_{x \to a} f(x) = l$ *if whenever* $x \to a$, $f(x) \to l$.

Working rules for finding $\lim\limits_{x \to a} f(x)$

RULE I Put $x = a$ in the given function. If $f(a)$ is a definite value then

$$\lim_{x \to a} f(x) = f(a).$$

EXAMPLES 1. $\lim\limits_{x \to 0} \sin x = \sin 0 = 0$.

2. $\lim\limits_{x \to 1} (x^2 + 5x - 2) = (1^2 + 5 \times 1 - 2) = 4$.

If $f(a)$ is indeterminate, we adopt the rules given below.

RULE II If $f(x)$ is a rational function then factorize the numerator and the denominator. Cancel out the common factors and then put $x = a$.

EXAMPLE *Evaluate* *(i)* $\lim\limits_{x \to 3} \left(\dfrac{x^2 - 9}{x - 3} \right)$ *(ii)* $\lim\limits_{x \to 1} \dfrac{(x^2 - 4x + 3)}{(x - 1)}$.

SOLUTION (i) $\lim\limits_{x \to 3} \left(\dfrac{x^2 - 9}{x - 3} \right) = \lim\limits_{x \to 3} \dfrac{(x + 3)(x - 3)}{(x - 3)} = \lim\limits_{x \to 3} (x + 3) = 6$.

(ii) $\lim\limits_{x \to 1} \dfrac{(x^2 - 4x + 3)}{(x - 1)} = \lim\limits_{x \to 1} \dfrac{(x - 1)(x - 3)}{(x - 1)} = \lim\limits_{x \to 1} (x - 3) = -2$.

RULE III If the given function contains a surd then simplify it by using conjugate surds. After simplification, put $x = a$.

EXAMPLE *Evaluate* $\lim\limits_{x \to 0} \left\{ \dfrac{\sqrt{1 + x} - \sqrt{1 - x}}{x} \right\}$.

SOLUTION We have

$$\lim_{x \to 0} \left\{ \frac{\sqrt{1 + x} - \sqrt{1 - x}}{x} \right\} = \lim_{x \to 0} \left\{ \frac{(\sqrt{1 + x} - \sqrt{1 - x}}{x} \cdot \frac{(\sqrt{1 + x} + \sqrt{1 - x})}{(\sqrt{1 + x} + \sqrt{1 - x})} \right\}$$

$$= \lim_{x\to 0} \frac{\{(1+x)-(1-x)\}}{x(\sqrt{1+x}+\sqrt{1-x})}$$

$$= \lim_{x\to 0} \frac{2x}{x(\sqrt{1+x}+\sqrt{1-x})}$$

$$= \lim_{x\to 0} \frac{2}{(\sqrt{1+x}+\sqrt{1-x})} = 1 \qquad \text{[putting } x = 0\text{].}$$

RULE IV If the given function contains a series which is capable of being expanded then after making proper expansion and simplifying, cancel the common factors in the numerator and denominator, if any. Then, put $x = a$.

Some important expansions are given below:

(i) For $|x| < 1$, we have the binomial expansion

$$(1+x)^n = \left\{1 + nx + \frac{n(n-1)}{2\,!}x^2 + \frac{n(n-1)(n-2)}{3\,!}x^3 + \ldots\right\}$$

(ii) $\left(\dfrac{x^n - a^n}{x-a}\right) = (x^{n-1} + ax^{n-2} + a^2x^{n-3} + \ldots + a^{n-1})$

(iii) $e^x = \left\{1 + x + \dfrac{x^2}{2\,!} + \dfrac{x^3}{3\,!} + \ldots + \dfrac{x^n}{n\,!} + \ldots\right\}$

(iv) $a^x = \left\{1 + x(\log a) + \dfrac{x^2}{2\,!}(\log a)^2 + \ldots\right\}$

(v) $\log(1+x) = \left\{x - \dfrac{x^2}{2} + \dfrac{x^3}{3} - \dfrac{x^4}{4} + \ldots\right\}$

(vi) $\sin x = \left\{x - \dfrac{x^3}{3\,!} + \dfrac{x^5}{5\,!} - \dfrac{x^7}{7\,!} + \ldots\right\}$

(vii) $\cos x = \left\{1 - \dfrac{x^2}{2\,!} + \dfrac{x^4}{4\,!} - \dfrac{x^6}{6\,!} + \ldots\right\}$ (viii) $\tan x = \left\{x + \dfrac{x^3}{3} + \dfrac{2}{15}x^5 + \ldots\right\}$

We shall also make use of the following theorems on limits. The proof of these theorems is beyond the scope of this book.

Fundamental Theorems on Limits (without proof)

(i) $\lim\limits_{x\to a} \{f(x) + g(x)\} = \left\{\lim\limits_{x\to a} f(x)\right\} + \left\{\lim\limits_{x\to a} g(x)\right\}$

(ii) $\lim\limits_{x\to a} \{f(x) - g(x)\} = \left\{\lim\limits_{x\to a} f(x)\right\} - \left\{\lim\limits_{x\to a} g(x)\right\}$

(iii) $\lim\limits_{x\to a} \{c \cdot f(x)\} = c \cdot \left\{\lim\limits_{x\to a} f(x)\right\}$, where c is a constant

(iv) $\lim\limits_{x\to a} \{f(x) \cdot g(x)\} = \left\{\lim\limits_{x\to a} f(x)\right\} \cdot \left\{\lim\limits_{x\to a} g(x)\right\}$

(v) $\lim_{x \to a} \left\{\dfrac{f(x)}{g(x)}\right\} = \dfrac{\left\{\lim_{x \to a} f(x)\right\}}{\left\{\lim_{x \to a} g(x)\right\}}$, provided $\lim_{x \to a} g(x) \neq 0$

(vi) If $f(x) \leq g(x)$ for all x then $\lim_{x \to a} f(x) \leq \lim_{x \to a} g(x)$.

Things to Remember

(i) $\sin C + \sin D = 2\sin\left(\dfrac{C+D}{2}\right)\cos\left(\dfrac{C-D}{2}\right)$

(ii) $\sin C - \sin D = 2\cos\left(\dfrac{C+D}{2}\right)\sin\left(\dfrac{C-D}{2}\right)$

(iii) $\cos C + \cos D = 2\cos\left(\dfrac{C+D}{2}\right)\cos\left(\dfrac{C-D}{2}\right)$

(iv) $\cos C - \cos D = -2\sin\left(\dfrac{C+D}{2}\right)\sin\left(\dfrac{C-D}{2}\right)$

Some more Important Theorems on Limits

THEOREM 1 $\lim_{x \to a}\left(\dfrac{x^n - a^n}{x - a}\right) = na^{n-1}$, *where* $a > 0$.

PROOF We know that $\left(\dfrac{x^n - a^n}{x - a}\right) = (x^{n-1} + ax^{n-2} + a^2x^{n-3} + \ldots + a^{n-1})$.

$$\therefore \quad \lim_{x \to a}\left(\frac{x^n - a^n}{x - a}\right) = \lim_{x \to a}(x^{n-1} + ax^{n-2} + a^2x^{n-3} + \ldots + a^{n-1})$$

$$= na^{n-1} \qquad \text{[putting } x = a\text{]}.$$

THEOREM 2 $\lim_{x \to 0}\left(\dfrac{e^x - 1}{x}\right) = 1$.

PROOF We have

$$\lim_{x \to 0}\left(\frac{e^x-1}{x}\right) = \lim_{x \to 0}\left\{\frac{\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots\right) - 1}{x}\right\} \left\{\because\ e^x = \left(1 + x + \frac{x^2}{2!} + \ldots\right)\right\}$$

$$= \lim_{x \to 0}\left\{\frac{\left(x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots\right)}{x}\right\} = \lim_{x \to 0}\left\{\frac{x\left(1 + \frac{x}{2!} + \frac{x^2}{3!} + \ldots\right)}{x}\right\}$$

$$= \lim_{x \to 0}\left(1 + \frac{x}{2!} + \frac{x^2}{3!} + \ldots\right)$$

$$= 1 \qquad \text{[putting } x = 0\text{]}.$$

THEOREM 3 $\lim_{x \to 0} \left(\frac{a^x - 1}{x} \right) = \log a.$

PROOF $\lim_{x \to 0} \left(\frac{a^x - 1}{x} \right) = \lim_{x \to 0} \left\{ \frac{\left\{ 1 + x(\log a) + \frac{x^2}{2!}(\log a)^2 + \dots \right\} - 1}{x} \right\}$ [on expanding a^x]

$$= \lim_{x \to 0} \frac{x \left\{ (\log a) + \frac{x}{2!}(\log a)^2 + \dots \right\}}{x}$$

$$= \lim_{x \to 0} \left\{ (\log a) + \frac{x}{2!}(\log a)^2 + \frac{x^2}{3!}(\log a)^3 + \dots \right\}$$

$$= \log a \qquad \text{[putting } x = 0\text{]}.$$

THEOREM 4 $\lim_{x \to 0} (1 + x)^{1/x} = e.$

PROOF $\lim_{x \to 0} (1 + x)^{1/x} = \lim_{x \to 0} \left\{ 1 + \frac{1}{x} \cdot x + \frac{\frac{1}{x}\left(\frac{1}{x} - 1\right)}{2!} \cdot x^2 + \frac{\frac{1}{x}\left(\frac{1}{x} - 1\right)\left(\frac{1}{x} - 2\right)}{3!} \cdot x^3 + \dots \infty \right\}$

[using the binomial expansion of $(1 + x)^{1/x}$]

$$= \lim_{x \to 0} \left\{ 1 + 1 + \frac{(1 - x)}{2!} + \frac{(1 - x)(1 - 2x)}{3!} + \dots \right\}$$

$$= \left(1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \dots \right) \qquad \text{[putting } x = 0\text{]}$$

$$= e.$$

THEOREM 5 $\lim_{x \to 0} \frac{\log (1 + x)}{x} = 1.$

PROOF $\lim_{x \to 0} \frac{\log (1 + x)}{x} = \lim_{x \to 0} \frac{\left(x - \frac{x^2}{2} + \frac{x^3}{3} - \dots \right)}{x}$ [expanding $\log (1 + x)$]

$$= \lim_{x \to 0} \left(1 - \frac{x}{2} + \frac{x^2}{3} - \dots \right) = 1 \qquad \text{[putting } x = 0\text{]}.$$

SUMMARY

1. $\lim_{x \to a} \left(\frac{x^n - a^n}{x - a} \right) = na^{n-1}$, where $a > 0$.

2. $\lim_{x \to 0} \left(\frac{e^x - 1}{x} \right) = 1.$

3. $\lim_{x \to 0} \left(\frac{a^x - 1}{x} \right) = \log_e a.$

4. $\lim_{x \to 0} (1 + x)^{1/x} = e.$

5. $\lim_{x \to 0} \frac{\log (1 + x)}{x} = 1.$

SOLVED EXAMPLES

EXAMPLE 1 *Evaluate* $\lim\limits_{x \to a} \left\{ \dfrac{x^{12} - a^{12}}{x - a} \right\}$.

SOLUTION $\lim\limits_{x \to a} \left\{ \dfrac{x^{12} - a^{12}}{x - a} \right\} = 12a^{(12-1)} = 12a^{11}$ $\left[\because \lim\limits_{x \to a} \left(\dfrac{x^n - a^n}{x - a} \right) = na^{n-1} \right]$.

EXAMPLE 2 *Evaluate* $\lim\limits_{x \to 2} \left(\dfrac{x^5 - 32}{x^3 - 8} \right)$.

SOLUTION
$$\lim_{x \to 2} \left(\frac{x^5 - 32}{x^3 - 8} \right) = \lim_{x \to 2} \left(\frac{x^5 - 2^5}{x^3 - 2^3} \right)$$
$$= \lim_{x \to 2} \left\{ \left(\frac{x^5 - 2^5}{x - 2} \right) \div \left(\frac{x^3 - 2^3}{x - 2} \right) \right\}$$
$$= \lim_{x \to 2} \left(\frac{x^5 - 2^5}{x - 2} \right) \div \lim_{x \to 2} \left(\frac{x^3 - 2^3}{x - 2} \right)$$
$$= [(5 \times 2^4) \div (3 \times 2^2)] \qquad \left[\because \lim_{x \to a} \left(\frac{x^n - a^n}{x - a} \right) = na^{n-1} \right]$$
$$= \left(\frac{80}{12} \right) = \frac{20}{3}.$$

EXAMPLE 3 *Evaluate* $\lim\limits_{x \to 1} \dfrac{x^{15} - 1}{x^{10} - 1}$.

SOLUTION
$$\lim_{x \to 1} \frac{x^{15} - 1}{x^{10} - 1} = \lim_{x \to 1} \left\{ \left(\frac{x^{15} - 1}{x - 1} \right) \div \left(\frac{x^{10} - 1}{x - 1} \right) \right\}$$
$$= \lim_{x \to 1} \left(\frac{x^{15} - 1}{x - 1} \right) \div \lim_{x \to 1} \left(\frac{x^{10} - 1}{x - 1} \right)$$
$$= (15 \times 1^{14}) \div (10 \times 1^9) = \frac{15}{10} = \frac{3}{2}.$$

EXAMPLE 4 *Evaluate* $\lim\limits_{x \to a} \left(\dfrac{x^m - a^m}{x^n - a^n} \right)$.

SOLUTION
$$\lim_{x \to a} \left(\frac{x^m - a^m}{x^n - a^n} \right) = \lim_{x \to a} \left\{ \left(\frac{x^m - a^m}{x - a} \right) \div \left(\frac{x^n - a^n}{x - a} \right) \right\}$$
$$= \lim_{x \to a} \left(\frac{x^m - a^m}{x - a} \right) \div \lim_{x \to a} \left(\frac{x^n - a^n}{x - a} \right)$$
$$= (ma^{m-1}) \quad (na^{n-1})$$
$$= \frac{ma^{m-1}}{na^{n-1}} = \frac{m}{n} a^{m-n}.$$

EXAMPLE 5 *Evaluate* $\lim_{x \to a} \dfrac{(x+2)^{3/2} - (a+2)^{3/2}}{x-a}$.

SOLUTION $\lim_{x \to a} \dfrac{(x+2)^{3/2} - (a+2)^{3/2}}{x-a}$

$$= \lim_{(x+2) \to (a+2)} \frac{(x+2)^{3/2} - (a+2)^{3/2}}{(x+2)-(a+2)}$$

$$= \frac{3}{2} \cdot (a+2)^{\left(\frac{3}{2}-1\right)} = \frac{3}{2}(a+2)^{1/2} \qquad \left[\because \lim_{x \to a} \left(\frac{x^n - a^n}{x-a}\right) = na^{n-1}\right].$$

EXAMPLE 6 *Evaluate* $\lim_{x \to 0} \dfrac{(1+x)^n - 1}{x}$.

SOLUTION Put $(1+x) = y$, so that when $x \to 0$ then $y \to 1$.

$$\therefore \quad \lim_{x \to 0} \frac{(1+x)^n - 1}{x} = \lim_{y \to 1} \left(\frac{y^n - 1}{y-1}\right)$$

$$= \lim_{y \to 1} \left(\frac{y^n - 1^n}{y-1}\right) = n \times 1^{(n-1)} = (n \times 1) = n.$$

EXAMPLE 7 *Evaluate* $\lim_{x \to 1} \left(\dfrac{1 - x^{-1/3}}{1 - x^{-2/3}}\right)$.

SOLUTION $\lim_{x \to 1} \left(\dfrac{1 - x^{-1/3}}{1 - x^{-2/3}}\right) = \lim_{x \to 1} \left[\dfrac{(1 - x^{-1/3})}{(1)^2 - (x^{-1/3})^2}\right]$

$$= \lim_{x \to 1} \frac{(1 - x^{-1/3})}{(1 - x^{-1/3})(1 + x^{-1/3})}$$

$$= \lim_{x \to 1} \frac{1}{(1 + x^{-1/3})} = \frac{1}{\{1 + (1)^{-1/3}\}} = \frac{1}{2}.$$

EXAMPLE 8 *Evaluate* $\lim_{x \to 0} \dfrac{(\sqrt{1+3x} - \sqrt{1-3x})}{x}$. **[CBSE 2000C]**

SOLUTION $\lim_{x \to 0} \dfrac{(\sqrt{1+3x} - \sqrt{1-3x})}{x}$

$$= \lim_{x \to 0} \left\{\frac{(\sqrt{1+3x} - \sqrt{1-3x})}{x} \times \frac{(\sqrt{1+3x} + \sqrt{1-3x})}{(\sqrt{1+3x} + \sqrt{1-3x})}\right\}$$

$$= \lim_{x \to 0} \frac{\{(1+3x) - (1-3x)}{x(\sqrt{1+3x} + \sqrt{1-3x})} = \lim_{x \to 0} \frac{6x}{x(\sqrt{1+3x} + \sqrt{1-3x})}$$

$$= \lim_{x \to 0} \frac{6}{(\sqrt{1+3x} + \sqrt{1-3x})} = \frac{6}{(\sqrt{1} + \sqrt{1})} = \frac{6}{2} = 3.$$

EXAMPLE 9 *Evaluate* $\lim_{x \to 2} \left\{ \frac{(x^2 - 4)}{\sqrt{3x - 2} - \sqrt{x + 2}} \right\}$.

SOLUTION $\lim_{x \to 2} \left\{ \frac{(x^2 - 4)}{\sqrt{3x - 2} - \sqrt{x + 2}} \right\}$

$$= \lim_{x \to 2} \left\{ \frac{(x^2 - 4)}{(\sqrt{3x - 2} - \sqrt{x + 2})} \times \frac{(\sqrt{3x - 2} + \sqrt{x - 2})}{(\sqrt{3x - 2} + \sqrt{x - 2})} \right\}$$

$$= \lim_{x \to 2} \left\{ \frac{(x^2 - 4)(\sqrt{3x - 2} + \sqrt{x - 2})}{(3x - 2) - (x + 2)} \right\}$$

$$= \lim_{x \to 2} \left\{ \frac{(x^2 - 4)(\sqrt{3x - 2} + \sqrt{x - 2})}{2(x - 2)} \right\}$$

$$= \lim_{x \to 2} \frac{(x + 2)(\sqrt{3x - 2} + \sqrt{x - 2})}{2} \qquad [\text{cancelling } (x - 2)]$$

$$= 8 \qquad [\text{putting } x = 2].$$

EXAMPLE 10 *Evaluate* $\lim_{x \to 0} \left(\frac{e^{3x} - 1}{x} \right)$.

SOLUTION $\lim_{x \to 0} \left(\frac{e^{3x} - 1}{x} \right) = \lim_{3x \to 0} \left\{ \left(\frac{e^{3x} - 1}{3x} \right) \times 3 \right\} \qquad [\because (x \to 0) \Rightarrow (3x \to 0)]$

$$= 3 \times \lim_{y \to 0} \left(\frac{e^y - 1}{y} \right), \quad \text{where } y = 3x$$

$$= (3 \times 1) = 3 \qquad \left[\because \lim_{y \to 0} \left(\frac{e^y - 1}{y} \right) = 1 \right].$$

EXAMPLE 11 *Evaluate* $\lim_{x \to 2} \left(\frac{e^x - e^2}{x - 2} \right)$. **[CBSE 2003]**

SOLUTION Put $(x - 2) = y$, so that when $x \to 2$ then $y \to 0$.

$$\therefore \lim_{x \to 2} \left(\frac{e^x - e^2}{x - 2} \right) = \lim_{y \to 0} \left(\frac{e^{y+2} - e^2}{y} \right)$$

$$= \lim_{y \to 0} \left\{ e^2 \cdot \left(\frac{e^y - 1}{y} \right) \right\}$$

$$= e^2 \cdot \lim_{y \to 0} \left(\frac{e^y - 1}{y} \right) = (e^2 \times 1) = e^2 \left[\because \lim_{y \to 0} \left(\frac{e^y - 1}{y} \right) = 1 \right].$$

EXAMPLE 12 *Evaluate:*

(i) $\lim_{x \to 0} \left(\frac{e^{-x} - 1}{x} \right)$ *(ii)* $\lim_{x \to 0} \left(\frac{e^x - e^{-x}}{x} \right)$ *(iii)* $\lim_{x \to 0} \left(\frac{e^x + e^{-x} - 2}{x^2} \right)$ **[CBSE 2004]**

SOLUTION (i) $\lim\limits_{x \to 0} \left(\dfrac{e^{-x} - 1}{x} \right) = \lim\limits_{y \to 0} \left(\dfrac{e^{y} - 1}{-y} \right)$, where $-x = y$

$$= -\lim_{y \to 0} \left(\frac{e^{y} - 1}{y} \right) = -1 \qquad \left[\because \lim_{y \to 0} \left(\frac{e^{y} - 1}{y} \right) = 1 \right].$$

(ii) $$\lim_{x \to 0} \left(\frac{e^{x} - e^{-x}}{x} \right) = \lim_{x \to 0} \left\{ \frac{(e^{x} - 1) - (e^{-x} - 1)}{x} \right\}$$

$$= \lim_{x \to 0} \left\{ \left(\frac{e^{x} - 1}{x} \right) - \left(\frac{e^{-x} - 1}{x} \right) \right\}$$

$$= \lim_{x \to 0} \left(\frac{e^{x} - 1}{x} \right) - \lim_{x \to 0} \left(\frac{e^{-x} - 1}{x} \right)$$

$$= 1 - \lim_{y \to 0} \left(\frac{e^{y} - 1}{-y} \right),$$

where $-x = y$ $\left[\because \lim\limits_{x \to 0} \left(\dfrac{e^{x} - 1}{x} \right) = 1 \right]$

$$= 1 + \lim_{y \to 0} \left(\frac{e^{y} - 1}{y} \right) = (1 + 1) = 2 \left[\because \lim_{y \to 0} \left(\frac{e^{y} - 1}{y} \right) = 1 \right].$$

(iii) $$\lim_{x \to 0} \left(\frac{e^{x} + e^{-x} - 2}{x^2} \right) = \lim_{x \to 0} \left(\frac{e^{2x} + 1 - 2e^{x}}{x^2 e^{x}} \right)$$

$$= \lim_{x \to 0} \left\{ \left(\frac{e^{x} - 1}{x} \right)^2 \cdot \frac{1}{e^{x}} \right\}$$

$$= \lim_{x \to 0} \left(\frac{e^{x} - 1}{x} \right)^2 \cdot \lim_{x \to 0} \frac{1}{e^{x}}$$

$$= \left(1^2 \times \frac{1}{e^0} \right) = (1 \times 1) = 1 \qquad \left[\because \lim_{x \to 0} \left(\frac{e^{x} - 1}{x} \right) = 1 \right].$$

EXAMPLE 13 *Evaluate* $\lim\limits_{x \to 0} \left(\dfrac{3^x - 2^x}{x} \right)$.

SOLUTION $\lim\limits_{x \to 0} \left(\dfrac{3^x - 2^x}{x} \right) = \lim\limits_{x \to 0} \left\{ \dfrac{(3^x - 1) - (2^x - 1)}{x} \right\}$

$$= \lim_{x \to 0} \left(\frac{3^x - 1}{x} \right) - \lim_{x \to 0} \left(\frac{2^x - 1}{x} \right)$$

$$= (\log 3 - \log 2) = \log \frac{3}{2} \qquad \left[\because \lim_{x \to 0} \left(\frac{a^x - 1}{x} \right) = \log a \right].$$

EXAMPLE 14 *Evaluate* $\lim_{x \to 0} \left(\frac{3^{2x} - 2^{3x}}{x} \right)$.

SOLUTION $\lim_{x \to 0} \left(\frac{3^{2x} - 2^{3x}}{x} \right) = \lim_{x \to 0} \left\{ \frac{(3^{2x} - 1) - (2^{3x} - 1)}{x} \right\}$

$$= \lim_{x \to 0} \left(\frac{3^{2x} - 1}{x} \right) - \lim_{x \to 0} \left(\frac{2^{3x} - 1}{x} \right)$$

$$= \lim_{x \to 0} \left\{ \left(\frac{3^{2x} - 1}{2x} \right) \cdot 2 \right\} - \lim_{x \to 0} \left\{ \left(\frac{2^{3x} - 1}{3x} \right) \cdot 3 \right\}$$

$$= 2 \cdot \lim_{2x \to 0} \left(\frac{3^{2x} - 1}{2x} \right) - 3 \cdot \lim_{3x \to 0} \left(\frac{2^{3x} - 1}{3x} \right)$$

$$= (2 \log 3 - 3 \log 2) \qquad \left[\because \lim_{y \to 0} \left(\frac{a^y - 1}{y} \right) = \log a \right]$$

$$= [\log (3)^2 - \log (2)^3] = (\log 9 - \log 8) = \log \left(\frac{9}{8} \right).$$

EXAMPLE 15 *Evaluate* $\lim_{x \to 0} \left(\frac{3^{2x} - 1}{2^{3x} - 1} \right)$.

SOLUTION $\lim_{x \to 0} \left(\frac{3^{2x} - 1}{2^{3x} - 1} \right) = \lim_{x \to 0} \frac{\left\{ \left(\frac{3^{2x} - 1}{2x} \right) \cdot 2x \right\}}{\left\{ \left(\frac{2^{3x} - 1}{3x} \right) \cdot 3x \right\}}$

$$= \frac{2}{3} \cdot \frac{\lim_{2x \to 0} \left(\frac{3^{2x} - 1}{2x} \right)}{\lim_{3x \to 0} \left(\frac{2^{3x} - 1}{3x} \right)}$$

$$= \frac{2}{3} \cdot \frac{\log 3}{\log 2} \qquad \left[\because \lim_{y \to 0} \left(\frac{a^y - 1}{y} \right) = \log a \right]$$

$$= \frac{2 \log 3}{3 \log 2} = \frac{\log (3^2)}{\log (2^3)} = \frac{\log 9}{\log 8}.$$

EXAMPLE 16 *Evaluate* $\lim_{x \to 1} \frac{x^2 - \sqrt{x}}{\sqrt{x} - 1}$.

SOLUTION $\lim_{x \to 1} \frac{(x^2 - \sqrt{x})}{(\sqrt{x} - 1)} = \lim_{x \to 1} \left\{ \frac{(x^2 - \sqrt{x})}{(\sqrt{x} - 1)} \times \frac{(\sqrt{x} + 1)}{(\sqrt{x} + 1)} \times \frac{(x^2 + \sqrt{x})}{(x^2 + \sqrt{x})} \right\}$

$$= \lim_{x \to 1} \frac{(x^4 - x)(\sqrt{x} + 1)}{(x - 1)(x^2 + \sqrt{x})} = \lim_{x \to 1} \frac{x(x^3 - 1)(\sqrt{x} + 1)}{(x - 1)(x^2 + \sqrt{x})}$$

$$= \lim_{x \to 1} \frac{x(x-1)(x^2+x+1)(\sqrt{x}+1)}{(x-1)(x^2+\sqrt{x})}$$

$$= \lim_{x \to 1} \frac{x(x^2+x+1)(\sqrt{x}+1)}{(x^2+\sqrt{x})}$$

$$= \frac{1 \times (1^2+1+1)(\sqrt{1}+1)}{(1^2+\sqrt{1})} = \left(\frac{1 \times 3 \times 2}{2}\right) = 3 \quad [\text{putting } x = 1].$$

EXAMPLE 17 *Evaluate* $\lim_{x \to a} \frac{\sqrt{a+2x} - \sqrt{3x}}{\sqrt{3a+x} - 2\sqrt{x}}$.

SOLUTION $\lim_{x \to a} \frac{(\sqrt{a+2x} - \sqrt{3x})}{(\sqrt{3a+x} - 2\sqrt{x})}$

$$= \lim_{x \to a} \frac{(\sqrt{a+2x} - \sqrt{3x})}{(\sqrt{3a+x} - 2\sqrt{x})} \times \frac{(\sqrt{3a+x} + 2\sqrt{x})}{(\sqrt{3a+x} + 2\sqrt{x})} \times \frac{(\sqrt{a+2x} + \sqrt{3x})}{(\sqrt{a+2x} + \sqrt{3x})}$$

$$\lim_{x \to a} \frac{[(a+2x) - 3x] \times (\sqrt{3a+x} + 2\sqrt{x})}{[(3a+x) - 4x] \times (\sqrt{a+2x} + \sqrt{3x})}$$

$$= \lim_{x \to a} \frac{(a-x) \times (\sqrt{3a+x} + 2\sqrt{x})}{3(a-x) \times (\sqrt{a+2x} + \sqrt{3x})} = \lim_{x \to a} \frac{(\sqrt{3a+x} + 2\sqrt{x})}{3(\sqrt{a+2x} + \sqrt{3x})}$$

$$= \frac{(\sqrt{4a} + 2\sqrt{a})}{3(\sqrt{3a} + \sqrt{3a})} = \frac{4\sqrt{a}}{6\sqrt{3}\,\sqrt{a}} = \frac{2}{3\sqrt{3}}.$$

EXAMPLE 18 *Evaluate* $\lim_{x \to 1} \frac{(2x-3)(\sqrt{x}-1)}{(2x^2+x-3)}$. **[CBSE 2005C]**

SOLUTION $\lim_{x \to 1} \frac{(2x-3)(\sqrt{x}-1)}{(2x^2+x-3)} = \lim_{x \to 1} \frac{(2x-3)(\sqrt{x}-1)}{(2x+3)(x-1)}$

$$= \lim_{x \to 1} \frac{(2x-3)(\sqrt{x}-1)(\sqrt{x}+1)}{(2x+3)(x-1)(\sqrt{x}+1)}$$

$$= \lim_{x \to 1} \frac{(2x-3)(x-1)}{(2x+3)(x-1)(\sqrt{x}+1)}$$

$$= \lim_{x \to 1} \frac{(2x-3)}{(2x+3)(\sqrt{x}+1)} = \frac{(2 \times 1 - 3)}{(2 \times 1 + 3)(\sqrt{1}+1)} = \frac{-1}{10}.$$

EXAMPLE 19 *Evaluate* $\lim_{x \to 0} \frac{(1+x)^4 - 1}{x}$.

SOLUTION Put $(1+x) = y$, so that when $x \to 0$ then $y \to 1$.

$$\therefore \quad \lim_{x \to 0} \frac{(1+x)^4 - 1}{x} = \lim_{y \to 1} \left(\frac{y^4 - 1}{y - 1} \right) = \lim_{y \to 1} \left(\frac{y^4 - 1^4}{y - 1} \right)$$

$$= 4 \times 1^{(4-1)} \qquad \left[\because \lim_{x \to a} \left(\frac{x^n - a^n}{x - a} \right) = na^{n-1} \right]$$

$$= (4 \times 1^3) = 4.$$

EXAMPLE 20 *Evaluate* $\lim_{x \to 0} \frac{(1+x)^6 - 1}{(1+x)^5 - 1}$.

SOLUTION Put $(1 + x) = y$, so that when $x \to 0$ then $y \to 1$.

$$\therefore \quad \lim_{x \to 0} \frac{(1+x)^6 - 1}{(1+x)^5 - 1}$$

$$= \lim_{y \to 1} \left(\frac{y^6 - 1}{y^5 - 1} \right) = \frac{\lim_{y \to 1} \left(\frac{y^6 - 1}{y - 1} \right)}{\lim_{y \to 1} \left(\frac{y^5 - 1}{y - 1} \right)} = \frac{\lim_{y \to 1} \left(\frac{y^6 - 1^6}{y - 1} \right)}{\lim_{y \to 1} \left(\frac{y^5 - 1^5}{y - 1} \right)} = \frac{6 \times 1^{(6-1)}}{5 \times 1^{(5-1)}}$$

$$= \frac{6 \times 1^5}{5 \times 1^4} = \frac{6}{5} \qquad \left[\because \lim_{x \to a} \left(\frac{x^n - a^n}{x - a} \right) = na^{n-1} \right].$$

EXERCISE 27A

Evaluate the following limits:

1. $\lim_{x \to 2} (5 - x)$

2. $\lim_{x \to 1} (6x^2 - 4x + 3)$

3. $\lim_{x \to 3} \left(\frac{x^2 + 9}{x + 3} \right)$

4. $\lim_{x \to 3} \left(\frac{x^2 - 4x}{x - 2} \right)$

5. $\lim_{x \to 5} \left(\frac{x^2 - 25}{x - 5} \right)$

6. $\lim_{x \to 1} \left(\frac{x^3 - 1}{x - 1} \right)$

7. $\lim_{x \to -2} \left(\frac{x^3 + 8}{x + 2} \right)$

8. $\lim_{x \to 3} \left(\frac{x^4 - 81}{x - 3} \right)$

9. $\lim_{x \to 3} \left(\frac{x^2 - 4x + 3}{x^2 - 2x - 3} \right)$

10. $\lim_{x \to \frac{1}{2}} \left(\frac{4x^2 - 1}{2x - 1} \right)$

11. $\lim_{x \to 4} \left(\frac{x^3 - 64}{x^2 - 16} \right)$

12. $\lim_{x \to 2} \left(\frac{x^5 - 32}{x^3 - 8} \right)$

13. $\lim_{x \to a} \left(\frac{x^{5/2} - a^{5/2}}{x - a} \right)$

14. $\lim_{x \to a} \left\{ \frac{(x+2)^{5/3} - (a+2)^{5/3}}{x - a} \right\}$

15. $\lim\limits_{x \to 1} \left(\dfrac{x^n - 1}{x - 1} \right)$

16. $\lim\limits_{x \to a} \left(\dfrac{\sqrt{x} - \sqrt{a}}{x - a} \right).$

17. $\lim\limits_{h \to 0} \left(\dfrac{\sqrt{x + h} - \sqrt{x}}{h} \right)$

18. $\lim\limits_{h \to 0} \dfrac{1}{h} \left\{ \dfrac{1}{\sqrt{x + h}} - \dfrac{1}{\sqrt{x}} \right\}$

19. $\lim\limits_{x \to 0} \left(\dfrac{\sqrt{1 + x} - 1}{x} \right)$

20. $\lim\limits_{x \to 0} \left(\dfrac{\sqrt{2 - x} - \sqrt{2 + x}}{x} \right)$

21. $\lim\limits_{x \to 0} \left(\dfrac{\sqrt{1 + x + x^2} - 1}{x} \right)$

22. $\lim\limits_{x \to 0} \left(\dfrac{\sqrt{3 - x} - 1}{2 - x} \right)$

23. $\lim\limits_{x \to 0} \left(\dfrac{2x}{\sqrt{a + x} - \sqrt{a - x}} \right)$

24. $\lim\limits_{x \to 1} \left(\dfrac{\sqrt{3 + x} - \sqrt{5 - x}}{x^2 - 1} \right)$

25. $\lim\limits_{x \to 2} \left(\dfrac{x^2 - 4}{\sqrt{x + 2} - \sqrt{3x - 2}} \right)$

26. $\lim\limits_{x \to 4} \left(\dfrac{3 - \sqrt{5 + x}}{1 - \sqrt{5 - x}} \right)$

27. $\lim\limits_{x \to 0} \left(\dfrac{\sqrt{a + x} - \sqrt{a}}{x\sqrt{a(a + x)}} \right)$

28. $\lim\limits_{x \to 0} \left(\dfrac{\sqrt{1 + x^2} - \sqrt{1 + x}}{\sqrt{1 + x^3} - \sqrt{1 + x}} \right)$

29. $\lim\limits_{x \to 1} \left(\dfrac{x^4 - 3x^2 + 2}{x^3 - 5x^2 + 3x + 1} \right)$

30. $\lim\limits_{x \to 2} \left(\dfrac{3^x + 3^{3-x} - 12}{3^{3-x} - 3^{x/2}} \right)$

31. $\lim\limits_{x \to 0} \left(\dfrac{e^{4x} - 1}{x} \right)$

32. $\lim\limits_{x \to 0} \left(\dfrac{e^{2+x} - e^2}{x} \right)$

33. $\lim\limits_{x \to 4} \left(\dfrac{e^x - e^4}{x - 4} \right)$

34. $\lim\limits_{x \to 0} \left(\dfrac{e^{3x} - e^{2x}}{x} \right)$

35. $\lim\limits_{x \to 0} \left(\dfrac{e^x - x - 1}{x} \right)$

36. $\lim\limits_{x \to 0} \left(\dfrac{e^{bx} - e^{ax}}{x} \right), 0 < a < b$

37. $\lim\limits_{x \to 0} \left(\dfrac{a^x - b^x}{x} \right)$

38. $\lim\limits_{x \to 0} \left(\dfrac{a^x - a^{-x}}{x} \right)$

39. $\lim\limits_{x \to 0} \left(\dfrac{2^x - 1}{x} \right)$

40. $\lim\limits_{x \to 0} \left(\dfrac{3^{2+x} - 9}{x} \right)$

ANSWERS (EXERCISE 27B)

1. 3 **2.** 5 **3.** 3 **4.** –3 **5.** 10 **6.** 3 **7.** 12 **8.** 108

9. $\dfrac{1}{2}$ **10.** 2 **11.** 6 **12.** $\dfrac{20}{3}$ **13.** $\dfrac{5}{2}a^{3/2}$ **14.** $\dfrac{5}{3}(a + 2)^{2/3}$ **15.** n

16. $\dfrac{1}{2\sqrt{a}}$ **17.** $\dfrac{1}{2\sqrt{x}}$ **18.** $\dfrac{-1}{2x^{3/2}}$ **19.** $\dfrac{1}{2}$ **20.** $\dfrac{-1}{\sqrt{2}}$ **21.** $\dfrac{1}{2}$ **22.** $\dfrac{1}{(\sqrt{3} + 1)}$ **23.** $2\sqrt{a}$

24. $\frac{1}{4}$ **25.** -8 **26.** $\frac{-1}{3}$ **27.** $\frac{1}{2a^{3/2}}$ **28.** 1 **29.** $\frac{1}{2}$ **30.** $\frac{-4}{3}$ **31.** 4 **32.** e^2 **33.** e^4

34. 1 **35.** 0 **36.** $(b-a)$ **37.** $(\log a - \log b)$ **38.** $2\log a$ **39.** $\log 2$ **40.** $9\log 3$

HINTS TO SOME SELECTED QUESTIONS

18. Given limit $= \lim\limits_{h \to 0} \left\{ \frac{(\sqrt{x} - \sqrt{x+h})}{h(\sqrt{x+h})\sqrt{x}} \times \frac{(\sqrt{x} + \sqrt{x+h})}{(\sqrt{x} + \sqrt{x+h})} \right\}$

$$= \lim_{h \to 0} \frac{-h}{h(\sqrt{x+h})\sqrt{x}(\sqrt{x} + \sqrt{x+h})}$$

$$= \lim_{h \to 0} \frac{-1}{(\sqrt{x+h})\sqrt{x}(\sqrt{x} + \sqrt{x+h})} = \frac{-1}{2x^{3/2}}.$$

26. Given limit $= \lim\limits_{x \to 4} \left\{ \frac{(3 - \sqrt{5+x})}{(1 - \sqrt{5-x})} \times \frac{(1 + \sqrt{5-x})}{(1 + \sqrt{5-x})} \times \frac{(3 + \sqrt{5+x})}{(3 + \sqrt{5+x})} \right\}$

$$= \lim_{x \to 4} \frac{\{9 - (5+x)\}(1 + \sqrt{5-x})}{\{1 - (5-x)\}(3 + \sqrt{5+x})} = \lim_{x \to 4} \frac{(4-x)(1+\sqrt{5-x})}{-(4-x)(3+\sqrt{5+x})}$$

$$= \lim_{x \to 4} \frac{(1+\sqrt{5-x})}{-(3+\sqrt{5+x})} = \frac{(1+\sqrt{1})}{-(3+\sqrt{9})} = \frac{2}{-6} = -\frac{1}{3}.$$

28. Given limit $= \lim\limits_{x \to 0} \left\{ \frac{(\sqrt{1+x^2} - \sqrt{1+x})}{(\sqrt{1+x^3} - \sqrt{1+x})} \times \frac{(\sqrt{1+x^3} + \sqrt{1+x})}{(\sqrt{1+x^3} + \sqrt{1+x})} \times \frac{(\sqrt{1+x^2} + \sqrt{1+x})}{(\sqrt{1+x^2} + \sqrt{1+x})} \right\}$

$$= \lim_{x \to 0} \frac{\{(1+x^2) - (1+x)\} \times (\sqrt{1+x^3} + \sqrt{1+x})}{\{(1+x^3) - (1+x)\} \times (\sqrt{1+x^2} + \sqrt{1+x})}$$

$$= \lim_{x \to 0} \frac{x(x-1)(\sqrt{1+x^3} + \sqrt{1+x})}{x(x-1)(x+1)(\sqrt{1+x^2} + \sqrt{1+x})}$$

$$= \lim_{x \to 0} \frac{(\sqrt{1+x^3} + \sqrt{1+x})}{(x+1)(\sqrt{1+x^2} + \sqrt{1+x})} = \frac{(\sqrt{1} + \sqrt{1})}{1 \cdot (\sqrt{1} + \sqrt{1})} = 1.$$

29. Given limit $= \lim\limits_{x \to 1} \frac{(x-1)(x+1)(x^2-2)}{(x-1)(x^2-4x-1)} = \lim\limits_{x \to 1} \frac{(x+1)(x^2-2)}{(x^2-4x-1)} = \frac{1}{2}.$

30. Given limit $= \lim\limits_{x \to 2} \left\{ \dfrac{3^x + \dfrac{3^3}{3^x} - 12}{\dfrac{3^3}{3^x} - 3^{x/2}} \right\} = \lim\limits_{t \to 3} \left(\dfrac{t^2 + \dfrac{27}{t^2} - 12}{\dfrac{27}{t^2} - t} \right)$ [put $3^{x/2} = t$]

$$= \lim_{t \to 3} \left(\frac{t^4 - 12t^2 + 27}{27 - t^3} \right) = \lim_{t \to 3} \frac{(t^2-3)(t+3)(t-3)}{(3-t)(9+3t+t^2)}$$

$$= \lim_{t \to 3} \frac{(t^2-3)(t+3)}{-(9+3t+t^2)} = \frac{(9-3)(3+3)}{-(9+9+9)} = \frac{-36}{27} = \frac{-4}{3}.$$

33. Put $(x-4)=y$, so that when $x \to 4$ then $y \to 0$.

$\therefore$ given limit $= \lim_{y \to 0} \left(\dfrac{e^{y+4} - e^4}{y} \right) = e^4 \cdot \lim_{y \to 0} \left(\dfrac{e^y - 1}{y} \right) = (e^4 \times 1) = e^4.$

34. Given limit $= \lim_{x \to 0} \dfrac{(e^{3x}-1)-(e^{2x}-1)}{x}$

$$= \lim_{x \to 0} \frac{(e^{3x}-1)}{x} - \lim_{x \to 0} \left(\frac{e^{2x}-1}{x} \right)$$

$$= \lim_{3x \to 0} \left\{ 3\left(\frac{e^{3x}-1}{3x} \right) \right\} - \lim_{2x \to 0} \left\{ 2\left(\frac{e^{2x}-1}{2x} \right) \right\}$$

$$= (3 \times 1) - (2 \times 1) = (3-2) = 1.$$

35. Given limit $= \lim_{x \to 0} \left\{ \left(\dfrac{e^x - 1}{x} \right) - 1 \right\} = \lim_{x \to 0} \left(\dfrac{e^x - 1}{x} \right) - 1 = (1-1) = 0.$

36. Given limit $= \lim_{x \to 0} \left\{ \dfrac{(e^{bx}-1)-(e^{ax}-1)}{x} \right\} = \lim_{bx \to 0} \left\{ \left(\dfrac{e^{bx}-1}{bx} \right) \cdot b \right\} - \lim_{ax \to 0} \left\{ \left(\dfrac{e^{ax}-1}{ax} \right) \cdot a \right\}$

$$= (1 \times b) - (1 \times a) = (b-a).$$

37. Given limit $= \lim_{x \to 0} \left\{ \dfrac{(a^x-1)-(b^x-1)}{x} \right\} = \lim_{x \to 0} \left(\dfrac{a^x-1}{x} \right) - \lim_{x \to 0} \left(\dfrac{b^x-1}{x} \right) = (\log a - \log b).$

38. Given limit $= \lim_{x \to 0} \left\{ \left(\dfrac{a^{2x}-1}{2x} \right) \times \dfrac{2}{a^x} \right\} = \left[(\log a) \times \dfrac{2}{1} \right] = 2 \log a.$

TRIGONOMETRIC LIMITS

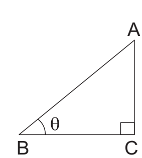
For finding the limits of trigonometric functions, we use trigonometric transformations and simplify.

The following theorems are quite important.

THEOREM 1 *(i)* $\lim_{\theta \to 0} \sin\theta = 0$ *(ii)* $\lim_{\theta \to 0} \cos\theta = 1$

PROOF Consider a triangle ABC in which $\angle C$ is a right angle and $\angle ABC = \theta$ radians.

Then, $\sin\theta = \dfrac{CA}{BA}$ and $\cos\theta = \dfrac{BC}{BA}$.



If we keep BC fixed and go on decreasing θ, we find that A keeps on coming nearer and nearer to C.

$\therefore$ when $\theta \to 0$ then $A \to C$.

$\therefore$ $\theta \to 0 \Rightarrow CA \to 0$ and $BA \to BC$

$$\Rightarrow \frac{CA}{BA} \to 0 \text{ and } \frac{BC}{BA} \to 1$$
$$\Rightarrow \sin\theta \to 0 \text{ and } \cos\theta \to 1.$$

Thus, we have

(i) $\lim\limits_{\theta \to 0} \sin\theta = 0$, and (ii) $\lim\limits_{\theta \to 0} \cos\theta = 1$.

THEOREM 2 *(i)* $\lim\limits_{x \to 0} \frac{\sin x}{x} = 1$ *(ii)* $\lim\limits_{x \to 0} \frac{\tan x}{x} = 1$

PROOF (i) $\lim\limits_{x \to 0} \frac{\sin x}{x} = \lim\limits_{x \to 0} \frac{\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots\right)}{x}$ [by sine series]

$$= \lim_{x \to 0} \frac{x\left(1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \dots\right)}{x} = \lim_{x \to 0} \left(1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \dots\right) = 1.$$

$$\therefore \quad \lim_{x \to 0} \frac{\sin x}{x} = 1.$$

(ii) $\lim\limits_{x \to 0} \frac{\tan x}{x} = \lim\limits_{x \to 0} \left[\frac{\sin x}{x} \cdot \frac{1}{\cos x}\right] = \lim\limits_{x \to 0} \frac{\sin x}{x} \cdot \frac{1}{\lim\limits_{x \to 0} \cos x} = (1 \times 1) = 1.$

SOLVED EXAMPLES

EXAMPLE 1 *Evaluate:*

(i) $\lim\limits_{x \to 0} \frac{\sin 2x}{x}$ *(ii)* $\lim\limits_{x \to 0} \frac{\sin 3x}{5x}$ *(iii)* $\lim\limits_{x \to 0} \left(\frac{\sin ax}{\sin bx}\right)$

SOLUTION (i) $\lim\limits_{x \to 0} \frac{\sin 2x}{x} = \lim\limits_{x \to 0} \left(\frac{2 \times \sin 2x}{2x}\right)$

$$= 2 \cdot \lim_{2x \to 0} \frac{\sin 2x}{2x} \qquad [\because \text{ when } x \to 0 \text{ then } 2x \to 0]$$

$$= (2 \times 1) = 2 \qquad \left[\because \lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1\right].$$

(ii) $\lim\limits_{x \to 0} \frac{\sin 3x}{5x} = \lim\limits_{x \to 0} \left(\frac{3}{5} \cdot \frac{\sin 3x}{3x}\right)$

$$= \frac{3}{5} \cdot \lim_{3x \to 0} \frac{\sin 3x}{3x} \qquad [\because \text{ when } x \to 0 \text{ then } 3x \to 0]$$

$$= \left(\frac{3}{5} \times 1\right) = \frac{3}{5} \qquad \left[\because \lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1\right].$$

(iii) $\lim\limits_{x\to 0}\left(\dfrac{\sin ax}{\sin bx}\right)=\lim\limits_{x\to 0}\dfrac{ax\cdot\left(\dfrac{\sin ax}{ax}\right)}{bx\cdot\left(\dfrac{\sin bx}{bx}\right)}$

$$=\frac{a}{b}\cdot\lim_{x\to 0}\frac{\left(\dfrac{\sin ax}{ax}\right)}{\left(\dfrac{\sin bx}{bx}\right)}$$

$$=\frac{a}{b}\cdot\frac{\lim\limits_{ax\to 0}\dfrac{\sin ax}{ax}}{\lim\limits_{bx\to 0}\dfrac{\sin bx}{bx}}\qquad [\because\ x\to 0\ \Rightarrow\ ax\to 0 \text{ and } bx\to 0]$$

$$=\left(\frac{a}{b}\times\frac{1}{1}\right)=\frac{a}{b}\qquad\left[\because\ \lim_{\theta\to 0}\frac{\sin\theta}{\theta}=1\right].$$

EXAMPLE 2 *Evaluate* $\lim\limits_{x\to 0}\dfrac{\sin 5x}{\tan 3x}$.

SOLUTION $\lim\limits_{x\to 0}\dfrac{\sin 5x}{\tan 3x}=\lim\limits_{x\to 0}\dfrac{5x\cdot\left(\dfrac{\sin 5x}{5x}\right)}{3x\cdot\left(\dfrac{\tan 3x}{3x}\right)}$

$$=\frac{5}{3}\cdot\frac{\lim\limits_{5x\to 0}\left(\dfrac{\sin 5x}{5x}\right)}{\lim\limits_{3x\to 0}\left(\dfrac{\tan 3x}{3x}\right)}=\left(\frac{5}{3}\times\frac{1}{1}\right)=\frac{5}{3}.$$

EXAMPLE 3 *Evaluate* $\lim\limits_{x\to 0}\dfrac{(1-\cos x)}{x^2}$. **[CBSE 2000]**

SOLUTION $\lim\limits_{x\to 0}\dfrac{(1-\cos x)}{x^2}=\lim\limits_{x\to 0}\dfrac{2\sin^2(x/2)}{\left(\dfrac{x}{2}\right)^2\times 4}$

$$=\frac{1}{2}\lim_{x\to 0}\left\{\frac{\sin(x/2)}{(x/2)}\right\}^2=\left(\frac{1}{2}\times 1^2\right)=\frac{1}{2}.$$

EXAMPLE 4 *Evaluate (i)* $\lim\limits_{x\to 0}\left(\dfrac{1-\cos 4x}{1-\cos 5x}\right)$ **[CBSE 2002C]**

(ii) $\lim\limits_{x\to 0}\left(\dfrac{1-\cos mx}{1-\cos nx}\right)$.

SOLUTION (i) $\lim_{x\to 0}\left(\frac{1-\cos 4x}{1-\cos 5x}\right)=\lim_{x\to 0}\frac{2\sin^2 2x}{2\sin^2(5x/2)}$

$$=\lim_{x\to 0}\frac{\sin^2 2x}{\sin^2(5x/2)}=\lim_{x\to 0}\left\{\frac{\left\{\frac{\sin 2x}{2x}\right\}^2\cdot 4x^2}{\left\{\frac{\sin(5x/2)}{(5x/2)}\right\}^2\cdot\frac{25x^2}{4}}\right\}$$

$$=\frac{16}{25}\cdot\frac{\lim_{2x\to 0}\left(\frac{\sin 2x}{2x}\right)^2}{\lim_{\frac{5x}{2}\to 0}\left\{\frac{\sin(5x/2)}{(5x/2)}\right\}^2}=\left(\frac{16}{25}\times\frac{1^2}{1^2}\right)=\frac{16}{25}.$$

(ii) $\lim_{x\to 0}\left(\frac{1-\cos mx}{1-\cos nx}\right)$

$$=\lim_{x\to 0}\left\{\frac{2\sin^2(mx/2)}{2\sin^2(nx/2)}\right\}=\lim_{x\to 0}\left\{\frac{\sin^2(mx/2)}{\sin^2(nx/2)}\right\}$$

$$=\lim_{x\to 0}\left\{\frac{\frac{\sin^2(mx/2)}{(mx/2)^2}\times\frac{m^2x^2}{4}}{\frac{\sin^2(nx/2)}{(nx/2)^2}\times\frac{n^2x^2}{4}}\right\}=\frac{m^2}{n^2}\cdot\frac{\lim_{\frac{mx}{2}\to 0}\left\{\frac{\sin(mx/2)}{(mx/2)}\right\}^2}{\lim_{\frac{nx}{2}\to 0}\left\{\frac{\sin(nx/2)}{(nx/2)}\right\}^2}$$

$$=\left(\frac{m^2}{n^2}\times\frac{1^2}{1^2}\right)=\frac{m^2}{n^2}.$$

EXAMPLE 5 *Evaluate* $\lim_{x\to 0}\left(\frac{\sin 2x+\sin 6x}{\sin 5x-\sin 3x}\right)$.

SOLUTION $\lim_{x\to 0}\left(\frac{\sin 2x+\sin 6x}{\sin 5x-\sin 3x}\right)$

$$=\lim_{x\to 0}\left(\frac{2\sin 4x\cos 2x}{2\cos 4x\sin x}\right)$$

$$=\lim_{x\to 0}\left(\frac{\sin 4x\cos 2x}{\cos 4x\sin x}\right\}$$

$$=\lim_{x\to 0}\left\{\frac{\sin 4x}{4x}\times\frac{x}{\sin x}\times\cos 2x\times\frac{1}{\cos 4x}\times 4\right\}$$

$$=4\times\lim_{4x\to 0}\frac{\sin 4x}{4x}\times\lim_{x\to 0}\left(\frac{x}{\sin x}\right)\times\lim_{2x\to 0}\cos 2x\times\frac{1}{\lim_{4x\to 0}\cos 4x}$$

$$=\left(4\times 1\times 1\times 1\times\frac{1}{1}\right)=4.$$

EXAMPLE 6 *Evaluate* $\lim_{x \to \pi} \left(\frac{1 + \cos x}{\tan^2 x} \right)$.

SOLUTION $\lim_{x \to \pi} \left(\frac{1 + \cos x}{\tan^2 x} \right) = \lim_{x \to \pi} \frac{\cos^2 x(1 + \cos x)}{\sin^2 x}$

$$= \lim_{x \to \pi} \frac{\cos^2 x(1 + \cos x)}{(1 - \cos^2 x)}$$

$$= \lim_{x \to \pi} \frac{\cos^2 x}{(1 - \cos x)} = \frac{1}{2} \qquad \text{[putting } x = \pi\text{]}.$$

EXAMPLE 7 *Evaluate* $\lim_{x \to 0} \frac{x \tan x}{(1 - \cos x)}$.

SOLUTION $\lim_{x \to 0} \frac{x \tan x}{(1 - \cos x)} = \lim_{x \to 0} \frac{x \sin x}{\cos x(1 - \cos x)}$

$$= \lim_{x \to 0} \frac{x \sin^2 x}{\sin x \cos x(1 - \cos x)}$$

$$= \lim_{x \to 0} \frac{x\,(1 - \cos^2 x)}{\sin x \cos x\,(1 - \cos x)} = \lim_{x \to 0} \frac{x\,(1 + \cos x)}{\sin x \cos x}$$

$$= \lim_{x \to 0} \frac{x}{\sin x} \cdot \lim_{x \to 0} \frac{(1 + \cos x)}{\cos x} = \left(1 \times \frac{2}{1}\right) = 2.$$

EXAMPLE 8 *Evaluate:*

(i) $\lim_{x \to \frac{\pi}{2}} (\sec x - \tan x)$ *(ii)* $\lim_{x \to 0} \frac{(\operatorname{cosec} x - \cot x)}{x}$

SOLUTION (i) $\lim_{x \to \frac{\pi}{2}} (\sec x - \tan x) = \lim_{x \to \frac{\pi}{2}} \left(\frac{1}{\cos x} - \frac{\sin x}{\cos x} \right)$

$$= \lim_{x \to \frac{\pi}{2}} \frac{(1 - \sin x)}{\cos x} = \lim_{x \to \frac{\pi}{2}} \frac{(1 - \sin x) \cos x}{\cos^2 x}$$

$$= \lim_{x \to \frac{\pi}{2}} \frac{(1 - \sin x) \cos x}{(1 - \sin^2 x)}$$

$$= \lim_{x \to \frac{\pi}{2}} \frac{\cos x}{(1 + \sin x)} = 0 \qquad \left[\text{putting } x = \frac{\pi}{2}\right].$$

(ii) $\lim\limits_{x\to 0} \dfrac{(\operatorname{cosec} x - \cot x)}{x} = \lim\limits_{x\to 0} \dfrac{\left(\dfrac{1}{\sin x} - \dfrac{\cos x}{\sin x}\right)}{x}$

$= \lim\limits_{x\to 0} \dfrac{(1-\cos x)}{x \sin x} = \lim\limits_{x\to 0} \dfrac{(1-\cos x)\sin x}{x \sin^2 x}$

$= \lim\limits_{x\to 0} \dfrac{(1-\cos x)\sin x}{x(1-\cos^2 x)} = \lim\limits_{x\to 0} \dfrac{\sin x}{x(1+\cos x)}$

$= \lim\limits_{x\to 0} \dfrac{\sin x}{x} \cdot \lim\limits_{x\to 0} \dfrac{1}{(1+\cos x)} = \left(1 \times \dfrac{1}{2}\right) = \dfrac{1}{2}.$

EXAMPLE 9 *Evaluate:* *(i)* $\lim\limits_{x\to 0} \left(\dfrac{\tan x - \sin x}{\sin^3 x}\right)$ *(ii)* $\lim\limits_{x\to 0} \dfrac{(\tan x - \sin x)}{x^3}$

SOLUTION (i) $\lim\limits_{x\to 0} \dfrac{(\tan x - \sin x)}{\sin^3 x} = \lim\limits_{x\to 0} \left(\dfrac{\sin x - \sin x \cos x}{\cos x \sin^3 x}\right)$

$= \lim\limits_{x\to 0} \dfrac{\sin x(1-\cos x)}{\cos x \sin^3 x} = \lim\limits_{x\to 0} \dfrac{(1-\cos x)}{\cos x \sin^2 x}$

$= \lim\limits_{x\to 0} \dfrac{(1-\cos x)}{\cos x (1-\cos^2 x)}$

$= \lim\limits_{x\to 0} \dfrac{1}{\cos x(1+\cos x)} = \dfrac{1}{2}$ [putting $x = 0$].

(ii) $\lim\limits_{x\to 0} \dfrac{(\tan x - \sin x)}{x^3} = \lim\limits_{x\to 0} \left(\dfrac{\sin x - \sin x \cos x}{x^3 \cos x}\right)$

$= \lim\limits_{x\to 0} \dfrac{\sin x(1-\cos x)}{x^3 \cos x} = \lim\limits_{x\to 0} \left\{\dfrac{\sin x}{x} \cdot \dfrac{2\sin^2(x/2)}{x^2} \cdot \dfrac{1}{\cos x}\right\}$

$= \lim\limits_{x\to 0} \left\{\dfrac{\sin x}{x} \cdot \dfrac{2\sin^2(x/2)}{(x/2)^2} \cdot \dfrac{1}{4} \cdot \dfrac{1}{\cos x}\right\}$

$= \dfrac{1}{2} \cdot \lim\limits_{x\to 0} \dfrac{\sin x}{x} \cdot \lim\limits_{x\to 0} \left\{\dfrac{\sin(x/2)}{(x/2)}\right\}^2 \cdot \lim\limits_{x\to 0} \dfrac{1}{\cos x}$

$= \left(\dfrac{1}{2} \times 1 \times 1^2 \times 1\right) = \dfrac{1}{2}.$

EXAMPLE 10 *Evaluate* $\lim\limits_{x\to 0} \left(\dfrac{x^3 \cot x}{1-\cos x}\right).$

SOLUTION $\lim\limits_{x\to 0} \left(\dfrac{x^3 \cot x}{1-\cos x}\right)$

$= \lim\limits_{x\to 0} \left\{\dfrac{x^3 \cot x}{(1-\cos x)} \times \dfrac{(1+\cos x)}{(1+\cos x)}\right\}$

$= \lim\limits_{x\to 0} \dfrac{x^3 \cos x(1+\cos x)}{\sin x(1-\cos^2 x)} = \lim\limits_{x\to 0} \dfrac{x^3 \cos x(1+\cos x)}{\sin^3 x}$

$$= \lim_{x \to 0} \left(\frac{x}{\sin x} \right)^3 \times \lim_{x \to 0} \cos x \times \lim_{x \to 0} (1 + \cos x)$$

$$= (1^3 \times 1 \times 2) = 2 \cdot$$

EXAMPLE 11 *Evaluate* $\lim_{x \to 0} \left(\frac{\sin 3x + 7x}{4x + \sin 2x} \right)$.

SOLUTION $\lim_{x \to 0} \left(\frac{\sin 3x + 7x}{4x + \sin 2x} \right) = \lim_{x \to 0} \left(\frac{\frac{\sin 3x}{x} + 7}{4 + \frac{\sin 2x}{x}} \right)$

[dividing num. and denom. by x]

$$= \lim_{x \to 0} \left[\frac{\left(\frac{\sin 3x}{3x} \right) \cdot 3 + 7}{4 + \left(\frac{\sin 2x}{2x} \right) \cdot 2} \right] = \frac{\left[3 \times \lim_{3x \to 0} \frac{\sin 3x}{3x} \right] + 7}{4 + \left[2 \times \lim_{2x \to 0} \left(\frac{\sin 2x}{2x} \right) \right]}$$

$$= \frac{(3 \times 1) + 7}{4 + (2 \times 1)} = \frac{10}{6} = \frac{5}{3} \cdot$$

EXAMPLE 12 *Evaluate* $\lim_{x \to 0} \left(\frac{\tan 3x - 2x}{3x - \sin^2 x} \right)$.

SOLUTION $\lim_{x \to 0} \left(\frac{\tan 3x - 2x}{3x - \sin^2 x} \right)$

$$= \lim_{x \to 0} \frac{\left(\frac{\tan 3x}{3x} - \frac{2}{3} \right)}{\left(1 - \frac{1}{3} \cdot \frac{\sin x}{x} \cdot \sin x \right)} \quad \text{[dividing num. and denom. by } 3x\text{]}$$

$$= \frac{\left(\lim_{3x \to 0} \frac{\tan 3x}{3x} - \frac{2}{3} \right)}{\left\{ 1 - \frac{1}{3} \lim_{x \to 0} \frac{\sin x}{x} \cdot \lim_{x \to 0} \sin x \right\}} = \frac{\left(1 - \frac{2}{3} \right)}{1 - \left(\frac{1}{3} \times 1 \times 0 \right)} = \frac{\left(\frac{1}{3} \right)}{1} = \frac{1}{3} \cdot$$

EXAMPLE 13 *Evaluate* $\lim_{x \to 0} \frac{x \tan 4x}{1 - \cos 4x} \cdot$ **[CBSE 2004]**

SOLUTION $\lim_{x \to 0} \frac{x \tan 4x}{1 - \cos 4x} = \lim_{x \to 0} \frac{x \sin 4x}{\cos 4x \, (2 \sin^2 2x)}$

$$= \lim_{x \to 0} \frac{2x \sin 2x \cos 2x}{\cos 4x \, (2 \sin^2 2x)} = \lim_{x \to 0} \left(\frac{\cos 2x}{\cos 4x} \cdot \frac{2x}{\sin 2x} \times \frac{1}{2} \right)$$

$$= \frac{1}{2} \times \frac{\lim_{2x \to 0} \cos 2x}{\lim_{4x \to 0} \cos 4x} \times \lim_{2x \to 0} \left(\frac{2x}{\sin 2x} \right) = \left(\frac{1}{2} \times \frac{1}{1} \times 1 \right) = \frac{1}{2} \cdot$$

EXAMPLE 14 *Evaluate* $\lim_{x \to 0} \frac{(1 - \cos x \sqrt{\cos 2x})}{x^2}$. **[CBSE 2004C, '05]**

SOLUTION $\lim_{x \to 0} \frac{(1 - \cos x \sqrt{\cos 2x})}{x^2}$

$$= \lim_{x \to 0} \left\{ \frac{(1 - \cos x \sqrt{\cos 2x})}{x^2} \times \frac{(1 + \cos x \sqrt{\cos 2x})}{(1 + \cos x \sqrt{\cos 2x})} \right\}$$

$$= \lim_{x \to 0} \left\{ \frac{(1 - \cos^2 x \cos 2x)}{x^2} \times \frac{1}{(1 + \cos x \sqrt{\cos 2x})} \right\}$$

$$= \lim_{x \to 0} \frac{(1 - \cos^2 x \cos 2x)}{x^2} \times \lim_{x \to 0} \frac{1}{(1 + \cos x \sqrt{\cos 2x})}$$

$$= \lim_{x \to 0} \left\{ \frac{1 - \cos^2 x (1 - 2\sin^2 x)}{x^2} \right\} \times \frac{1}{(1 + 1\sqrt{1})}$$

$$= \lim_{x \to 0} \frac{(1 - \cos^2 x + 2\sin^2 x \cos^2 x)}{x^2} \times \frac{1}{2}$$

$$= \frac{1}{2} \times \lim_{x \to 0} \frac{(\sin^2 x + 2\sin^2 x \cos^2 x)}{x^2} = \frac{1}{2} \times \lim_{x \to 0} \frac{\sin^2 x (1 + 2\cos^2 x)}{x^2}$$

$$= \frac{1}{2} \times \lim_{x \to 0} \left(\frac{\sin x}{x} \right)^2 \times \lim_{x \to 0} (1 + 2\cos^2 x)$$

$$= \frac{1}{2} \times 1^2 \times (1 + 2 \times 1^2) = \frac{3}{2}.$$

EXAMPLE 15 *Evaluate* $\lim_{x \to \frac{\pi}{4}} \frac{(\sin x - \cos x)}{\left(x - \frac{\pi}{4} \right)}$. **[CBSE 2000, '03]**

SOLUTION Put $\left(x - \frac{\pi}{4} \right) = y$ so that when $x \to \frac{\pi}{4}$ then $y \to 0$.

$$\therefore \quad \lim_{x \to \frac{\pi}{4}} \frac{(\sin x - \cos x)}{\left(x - \frac{\pi}{4} \right)}$$

$$= \lim_{y \to 0} \frac{\left\{ \sin\left(\frac{\pi}{4} + y \right) - \cos\left(\frac{\pi}{4} + y \right) \right\}}{y} \qquad \left[\text{putting } (x - \tfrac{\pi}{4}) = y\right]$$

$$= \lim_{y \to 0} \frac{\left\{ \left(\sin\frac{\pi}{4} \cos y + \cos\frac{\pi}{4} \sin y \right) - \left(\cos\frac{\pi}{4} \cos y - \sin\frac{\pi}{4} \sin y \right) \right\}}{y}$$

$$= \frac{2}{\sqrt{2}} \times \lim_{y \to 0} \left(\frac{\sin y}{y} \right) = (\sqrt{2} \times 1) = \sqrt{2}.$$

EXAMPLE 16 *Evaluate* $\lim_{x \to \frac{\pi}{2}} \dfrac{\cos x}{\left(\frac{\pi}{2} - x\right)}$.

SOLUTION Put $\left(x - \frac{\pi}{2}\right) = y$, so that when $x \to \frac{\pi}{2}$ then $y \to 0$.

$$\therefore \lim_{x \to \frac{\pi}{2}} \frac{\cos x}{\left(\frac{\pi}{2} - x\right)} = \lim_{y \to 0} \frac{\cos\left(\frac{\pi}{2} + y\right)}{-y} = \lim_{y \to 0} \left(\frac{-\sin y}{-y}\right) = \lim_{y \to 0} \frac{\sin y}{y} = 1.$$

EXAMPLE 17 *Evaluate* $\lim_{x \to \frac{\pi}{6}} \dfrac{(\sqrt{3} \sin x - \cos x)}{\left(x - \frac{\pi}{6}\right)}$.

SOLUTION $\lim_{x \to \frac{\pi}{6}} \dfrac{(\sqrt{3} \sin x - \cos x)}{\left(x - \frac{\pi}{6}\right)}$

$$= \lim_{x \to \frac{\pi}{6}} \frac{2\left(\frac{\sqrt{3}}{2} \sin x - \frac{1}{2} \cos x\right)}{\left(x - \frac{\pi}{6}\right)} = \lim_{x \to \frac{\pi}{6}} \frac{2\left(\sin x \cos \frac{\pi}{6} - \cos x \sin \frac{\pi}{6}\right)}{\left(x - \frac{\pi}{6}\right)}$$

$$= 2 \lim_{x \to \frac{\pi}{6}} \frac{\sin\left(x - \frac{\pi}{6}\right)}{\left(x - \frac{\pi}{6}\right)} = 2 \lim_{y \to 0} \frac{\sin y}{y} = 2 \qquad \left[\text{putting } \left(x - \frac{\pi}{6}\right) = y\right].$$

EXERCISE 27B

Evaluate the following limits:

1. $\lim_{x \to 0} \dfrac{\sin 4x}{6x}$

2. $\lim_{x \to 0} \dfrac{\sin 5x}{\sin 8x}$

3. $\lim_{x \to 0} \dfrac{\tan 3x}{\tan 5x}$

4. $\lim_{x \to 0} \dfrac{\tan \alpha x}{\tan \beta x}$

5. $\lim_{x \to 0} \dfrac{\sin 4x}{\tan 7x}$

6. $\lim_{x \to 0} \dfrac{\tan 3x}{\sin 4x}$

7. $\lim_{x \to 0} \dfrac{\sin mx}{\tan nx}$

8. $\lim_{x \to 0} \dfrac{(\sin x - 2\sin 3x + \sin 5x)}{x}$

9. $\lim_{x \to \pi/6} \dfrac{(2\sin^2 x + \sin x - 1)}{(2\sin^2 x - 3\sin x + 1)}$

10. $\lim_{x \to 0} \dfrac{(\sin 2x + 3x)}{(2x + \sin 3x)}$

11. $\lim_{x \to 0} \frac{(\tan 2x - x)}{(3x - \tan x)}$

12. $\lim_{x \to 0} \frac{(x^2 - \tan 2x)}{\tan x}$ **[CBSE 1998]**

13. $\lim_{x \to 0} \frac{(x \cos x + \sin x)}{(x^2 + \tan x)}$

14. $\lim_{x \to 0} \frac{(\tan x - \sin x)}{\sin^3 x}$

15. $\lim_{x \to 0} (x \operatorname{cosec} x)$

16. $\lim_{x \to 0} (x \cot 2x)$

17. $\lim_{x \to 0} \frac{\sin x \cos x}{3x}$

18. $\lim_{x \to 0} \frac{\sin (x/4)}{x}$

19. $\lim_{x \to 0} \frac{\tan (x/2)}{3x}$

20. $\lim_{x \to 0} \frac{(1 - \cos x)}{\sin^2 x}$ **[CBSE 2000]**

21. $\lim_{x \to 0} \frac{(1 - \cos 3x)}{x^2}$ **[CBSE 2000]**

22. $\lim_{x \to 0} \frac{(1 - \cos 2x)}{\sin^2 2x}$

23. $\lim_{x \to 0} \frac{(1 - \cos 2x)}{3 \tan^2 x}$

24. $\lim_{x \to 0} \frac{(1 - \cos 4x)}{(1 - \cos 6x)}$ **[CBSE 2002C]**

25. $\lim_{x \to 0} \frac{(1 - \cos mx)}{(1 - \cos nx)}$

26. $\lim_{x \to 0} \frac{(2 \sin x - \sin 2x)}{x^3}$

27. $\lim_{x \to 0} \frac{(\tan x - \sin x)}{x^3}$

28. $\lim_{x \to 0} \frac{(\tan 2x - \sin 2x)}{x^3}$

29. $\lim_{x \to 0} \frac{(\operatorname{cosec} x - \cot x)}{x}$

30. $\lim_{x \to 0} \frac{(\cot 2x - \operatorname{cosec} 2x)}{x}$

31. $\lim_{x \to 0} \frac{\sin 2x(1 - \cos 2x)}{x^3}$

32. $\lim_{x \to \frac{\pi}{4}} \frac{(\sec^2 x - 2)}{(\tan x - 1)}$ **[CBSE 2001, '2C]**

33. $\lim_{x \to \frac{\pi}{4}} \frac{(\operatorname{cosec}^2 x - 2)}{(\cot x - 1)}$ **[CBSE 2001]**

34. $\lim_{x \to \frac{\pi}{4}} \frac{(1 - \tan x)}{\left(x - \frac{\pi}{4}\right)}$ **[CBSE 2003C]**

35. $\lim_{x \to \pi} \frac{(\sin 3x - 3 \sin x)}{(\pi - x)^3}$

36. $\lim_{x \to \frac{\pi}{2}} \frac{(1 + \cos 2x)}{(\pi - 2x)^2}$ **[CBSE 2003C]**

37. $\lim_{x \to a} \frac{(\cos x - \cos a)}{(x - a)}$

38. $\lim_{x \to a} \frac{(\sin x - \sin a)}{(x - a)}$

39. $\lim_{x \to a} \frac{(\sin x - \sin a)}{(\sqrt{x} - \sqrt{a})}$

40. $\lim_{x \to 0} \frac{(\sin 5x - \sin 3x)}{\sin x}$

41. $\lim_{x \to 0} \frac{(\cos 3x - \cos 5x)}{x^2}$

42. $\lim_{x \to 0} \frac{(\sin 3x + \sin 5x)}{(\sin 6x - \sin 4x)}$

43. $\lim_{x \to 0} \frac{[\sin (2 + x) - \sin (2 - x)]}{x}$

44. $\lim_{x \to 0} \frac{(1 - \cos 2x)}{(\cos 2x - \cos 8x)}$

45. $\lim_{x \to \frac{\pi}{2}} \left(\frac{\pi}{2} - x\right) \tan x$

46. $\lim_{x \to 0} \frac{(\sqrt{1+2x} - \sqrt{1-2x})}{\sin x}$ **[CBSE 1998C]**

47. $\lim_{h \to 0} \frac{(a+h)^2 \sin(a+h) - a^2 \sin a}{h}$

48. $\lim_{x \to 0} \frac{(e^{3+x} - \sin x - e^3)}{x}$

49. $\lim_{x \to 0} \frac{(e^{\tan x} - 1)}{\tan x}$

50. $\lim_{x \to 0} \frac{(e^{\tan x} - 1)}{x}$

51. $\lim_{x \to 0} \frac{ax + x\cos x}{b \sin x}$

52. $\lim_{x \to 0} \frac{\sin ax + bx}{ax + \sin bx}$, where $a, b, a+b \neq 0$

53. $\lim_{x \to \pi} \frac{\sin(\pi - x)}{\pi(\pi - x)}$

54. $\lim_{x \to \frac{\pi}{2}} \frac{\tan 2x}{x - \frac{\pi}{2}}$

55. $\lim_{x \to 0} \frac{\cos 2x - 1}{\cos x - 1}$

56. $\lim_{x \to 0} (\operatorname{cosec} x - \cot x)$

57. $\lim_{x \to 0} \frac{1 - \cos 2mx}{1 - \cos 2nx}$

58. $\lim_{x \to 0} \frac{1 - \cos mx}{1 - \cos nx}$

59. $\lim_{x \to 0} \frac{\sin^2 mx}{\sin^2 nx}$

60. $\lim_{x \to 0} \frac{\sin 2x + \sin 3x}{2x + \sin 3x}$

61. $\lim_{x \to 0} \frac{\sec 4x - \sec 2x}{\sec 3x - \sec x}$

62. $\lim_{x \to 0} \frac{\sqrt{2} - \sqrt{1+\cos x}}{\sin^2 x}$

63. $\lim_{x \to 0} \frac{\sqrt{1+\sin x} - \sqrt{1-\sin x}}{x}$

64. $\lim_{x \to \frac{\pi}{6}} \frac{2 - \sqrt{3}\cos x - \sin x}{(6x - \pi)^2}$

65. $\lim_{x \to 0} \frac{\cos ax - \cos bx}{\cos cx - 1}$

66. $\lim_{x \to a} \frac{\cos x - \cos a}{\cot x - \cot a}$

67. $\lim_{x \to \frac{\pi}{4}} \frac{\tan^3 x - \tan x}{\cos\left(x + \frac{\pi}{4}\right)}$

68. $\lim_{x \to \frac{\pi}{2}} \frac{\sqrt{2} - \sqrt{1+\sin x}}{\sqrt{2}\cos^2 x}$

69. $\lim_{x \to \frac{\pi}{6}} \frac{\cot^2 x - 3}{\operatorname{cosec} x - 2}$

70. $\lim_{x \to \pi} \frac{\sqrt{2+\cos x} - 1}{(\pi - x)^2}$

71. $\lim_{x \to \frac{\pi}{4}} \frac{1 - \tan x}{1 - \sqrt{2}\sin x}$

72. $\lim_{x \to \frac{\pi}{6}} \frac{2\sin^2 x + \sin x - 1}{2\sin^2 x - 3\sin x + 1}$

ANSWERS (EXERCISE 27B)

1. $\frac{2}{3}$ **2.** $\frac{5}{8}$ **3.** $\frac{3}{5}$ **4.** $\frac{\alpha}{\beta}$ **5.** $\frac{4}{7}$ **6.** $\frac{3}{4}$ **7.** $\frac{m}{n}$ **8.** 0

9. -3 **10.** 1 **11.** $\frac{1}{2}$ **12.** -2 **13.** 2 **14.** $\frac{1}{2}$ **15.** 1 **16.** $\frac{1}{2}$

17. $\frac{1}{3}$ **18.** $\frac{1}{4}$ **19.** $\frac{1}{6}$ **20.** $\frac{1}{2}$ **21.** $\frac{9}{2}$ **22.** $\frac{1}{2}$ **23.** $\frac{2}{3}$ **24.** $\frac{4}{9}$

25. $\frac{m^2}{n^2}$ **26.** 1 **27.** $\frac{1}{2}$ **28.** 4 **29.** $\frac{1}{2}$ **30.** -1 **31.** 4 **32.** 2

33. 2 **34.** -2 **35.** -1 **36.** $\frac{1}{2}$ **37.** $-\sin a$ **38.** $\cos a$

39. $2\sqrt{a}\cos a$ **40.** 2 **41.** 8 **42.** 4 **43.** $2\cos 2$ **44.** $\frac{1}{15}$

45. 1 **46.** 2 **47.** $(2a\sin a + a^2\cos a)$ **48.** $(e^3 - 1)$ **49.** 1

50. 1 **51.** $\frac{a+1}{b}$ **52.** 1 **53.** $\frac{1}{\pi}$ **54.** 2 **55.** 4 **56.** 0 **57.** $\frac{m^2}{n^2}$

58. $\frac{m}{n}$ **59.** $\frac{m^2}{n^2}$ **60.** 1 **61.** $\frac{3}{2}$ **62.** $\frac{1}{4\sqrt{2}}$ **63.** 1 **64.** $\frac{1}{36}$

65. $\frac{(a^2-b^2)}{c^2}$ **66.** $\sin^3 a$ **67.** -4 **68.** $\frac{1}{8}$ **69.** 4 **70.** $\frac{1}{4}$

71. 2 **72.** -3

HINTS TO SOME SELECTED QUESTIONS

8. Given limit $= \left(\lim\limits_{x\to 0}\frac{\sin x}{x}\right) - (2\times 3)\cdot\left(\lim\limits_{x\to 0}\frac{\sin 3x}{3x}\right) + 5\cdot\left(\lim\limits_{x\to 0}\frac{\sin 5x}{5x}\right)$

$= [1 - (6\times 1) + (5\times 1)] = 0.$

9. Given limit $= \lim\limits_{x\to\frac{\pi}{6}}\frac{(2\sin x - 1)(\sin x + 1)}{(2\sin x - 1)(\sin x - 1)} = \lim\limits_{x\to\frac{\pi}{6}}\frac{(\sin x + 1)}{(\sin x - 1)}$

$= \frac{\left(\sin\frac{\pi}{6} + 1\right)}{\left(\sin\frac{\pi}{6} - 1\right)} = -3.$

10. Given limit $= \frac{2\times\left(\lim\limits_{x\to 0}\frac{\sin 2x}{2x}\right) + 3}{2 + \left(3\times\lim\limits_{x\to 0}\frac{\sin 3x}{3x}\right)} = \frac{(2\times 1) + 3}{2 + (3\times 1)} = \frac{5}{5} = 1.$

11. Given limit $= \frac{2\times\left(\lim\limits_{x\to 0}\frac{\tan 2x}{2x}\right) - 1}{3 - \lim\limits_{x\to 0}\left(\frac{\tan x}{x}\right)} = \frac{(2\times 1) - 1}{(3-1)} = \frac{1}{2}.$

12. Given limit $= \lim\limits_{x\to 0}\frac{\left(x - 2\times\frac{\tan 2x}{2x}\right)}{\left(\frac{\tan x}{x}\right)} = \left\{\frac{0 - (2\times 1)}{1}\right\} = -2.$

13. Given limit $= \lim\limits_{x\to 0}\frac{\left(\cos x + \frac{\sin x}{x}\right)}{\left(x + \frac{\tan x}{x}\right)} = \frac{(1+1)}{(0+1)} = 2.$

14. Given limit $= \lim_{x\to 0} \dfrac{\sin x\,(1-\cos x)}{\cos x \sin^3 x} = \lim_{x\to 0} \dfrac{(1-\cos x)}{\cos x \sin^2 x}$

$= \lim_{x\to 0} \dfrac{(1-\cos x)}{\cos x\,(1-\cos^2 x)} = \lim_{x\to 0} \dfrac{1}{\cos x\,(1+\cos x)} = \dfrac{1}{2}\cdot$

15. Given limit $= \lim_{x\to 0} \dfrac{x}{\sin x} = 1.$

16. Given limit $= \dfrac{1}{2}\times \lim_{2x\to 0} \dfrac{2x}{\tan 2x} = \left(\dfrac{1}{2}\times 1\right) = \dfrac{1}{2}\cdot$

20. Given limit $= \lim_{x\to 0} \dfrac{(1-\cos x)}{(1-\cos^2 x)} = \lim_{x\to 0} \dfrac{1}{(1+\cos x)} = \dfrac{1}{2}\cdot$

21. Given limit $= \lim_{x\to 0} \dfrac{2\sin^2\left(\dfrac{3x}{2}\right)}{\left(\dfrac{3x}{2}\right)^2 \times \dfrac{4}{9}} = \dfrac{9}{2}\cdot$

24. Given limit $= \lim_{x\to 0} \dfrac{2\sin^2 2x}{2\sin^2 3x} = \lim_{x\to 0} \dfrac{\sin^2 2x}{\sin^2 3x}$

$= \lim_{x\to 0} \dfrac{\left(\dfrac{\sin 2x}{2x}\right)^2 \times 4x^2}{\left(\dfrac{\sin 3x}{3x}\right)^2 \times 9x^2} = \dfrac{4}{9}\cdot \dfrac{\lim_{2x\to 0}\left(\dfrac{\sin 2x}{2x}\right)^2}{\lim_{3x\to 0}\left(\dfrac{\sin 3x}{3x}\right)^2}$

$= \left(\dfrac{4}{9}\times\dfrac{1}{1}\right) = \dfrac{4}{9}\cdot$

26. Given limit $= \lim_{x\to 0} \dfrac{2\sin x(1-\cos x)}{x^3} = \lim_{x\to 0} \left[2\left(\dfrac{\sin x}{x}\right)\cdot \dfrac{2\sin^2(x/2)}{\left(\dfrac{x}{2}\right)^2 \times 4}\right] = 1.$

29. Given limit $= \lim_{x\to 0} \dfrac{(1-\cos x)}{x\sin x} = \lim_{x\to 0} \dfrac{2\sin^2(x/2)}{2x\sin(x/2)\cos(x/2)} = \lim_{x\to 0} \dfrac{\tan(x/2)}{\left(\dfrac{x}{2}\right)} \times \dfrac{1}{2} = \dfrac{1}{2}\cdot$

30. Given limit $= \lim_{x\to 0} \dfrac{(\cos 2x - 1)}{x\sin 2x} = \lim_{x\to 0} \dfrac{-(1-\cos 2x)}{x\sin 2x}$

$= \lim_{x\to 0} \dfrac{-2\sin^2 x}{2x\sin x\cos x} = -\lim_{x\to 0} \dfrac{\tan x}{x} = -1.$

35. Given limit $= \lim_{x\to \pi} \dfrac{(3\sin x - 4\sin^3 x - 3\sin x)}{(\pi - x)^3}$

$= \lim_{x\to \pi} \dfrac{-4\sin^3 x}{(\pi-x)^3} = \lim_{\theta\to 0} \dfrac{-4\sin^3(\pi+\theta)}{(-\theta)^3}$, where $(x-\pi)=\theta$

$= -4\lim_{\theta\to 0}\left(\dfrac{\sin\theta}{\theta}\right)^3 = (-4\times 1) = -4.$

36. Putting $(\pi - 2x) = t$, we have

$$\text{given limit} = \lim_{t \to 0} \frac{1 + \cos(\pi - t)}{t^2} = \lim_{t \to 0} \frac{1 - \cos t}{t^2}$$

$$= \lim_{t \to 0} \frac{2\sin^2(t/2)}{t^2} = \frac{2}{4} \cdot \lim_{t \to 0} \left\{\frac{\sin(t/2)}{(t/2)}\right\}^2 = \left(\frac{1}{2} \times 1^2\right) = \frac{1}{2}.$$

37. $$\text{Given limit} = \lim_{x \to a} \frac{-2\sin\left(\frac{x+a}{2}\right)\sin\left(\frac{x-a}{2}\right)}{(x-a)} = -\lim_{x \to a}\left\{\sin\left(\frac{x+a}{2}\right) \cdot \frac{\sin\left(\frac{x-a}{2}\right)}{\left(\frac{x-a}{2}\right)}\right\}$$

$$= -\sin a \times 1 = -\sin a.$$

39. $$\text{Given limit} = \lim_{x \to a} \frac{2\cos\left(\frac{x+a}{2}\right)\sin\left(\frac{x-a}{2}\right)}{(x-a)} \times (\sqrt{x} + \sqrt{a})$$

$$= \lim_{x \to a}\left\{\cos\left(\frac{x+a}{2}\right) \cdot \frac{\sin\left(\frac{x-a}{2}\right)}{\left(\frac{x-a}{2}\right)} \times (\sqrt{x} + \sqrt{a})\right\}$$

$$= (\cos a \times 1 \times 2\sqrt{a}) = 2\sqrt{a}\cos a.$$

41. $$\text{Given limit} = \lim_{x \to 0} \frac{2\sin 4x \sin x}{x^2} = \lim_{x \to 0}\left(\frac{\sin 4x}{4x} \times \frac{\sin x}{x} \times 8\right) = (8 \times 1 \times 1) = 8.$$

42. $$\text{Given limit} = \lim_{x \to 0} \frac{2\sin 4x \cos x}{2\cos 5x \sin x}$$

$$= \lim_{x \to 0}\left\{\frac{\left(\frac{\sin 4x}{4x} \times 4\right) \cdot \cos x}{\left(\frac{\sin x}{x}\right) \cdot \cos 5x}\right\} = \frac{(1 \times 4) \times 1}{1 \times 1} = 4.$$

43. $$\text{Given limit} = \lim_{x \to 0} \frac{2\cos\left\{\frac{(2 + x + 2 - x)}{2}\right\}\sin\left\{\frac{(2 + x - 2 + x)}{2}\right\}}{x}$$

$$= (2\cos 2) \cdot \lim_{x \to 0} \frac{\sin x}{x} = (2\cos 2) \times 1 = (2\cos 2).$$

44. $$\text{Given limit} = \lim_{x \to 0} \frac{2\sin^2 x}{2\sin 5x \sin 3x} = \lim_{x \to 0} \frac{\sin x \cdot \sin x}{\sin 5x \cdot \sin 3x}$$

$$= \frac{\lim\limits_{x \to 0}\left(\frac{\sin x}{x}\right) \cdot \lim\limits_{x \to 0}\left(\frac{\sin x}{x}\right)}{\lim\limits_{x \to 0}\left(\frac{\sin 5x}{5x} \times 5\right) \times \lim\limits_{x \to 0}\left(\frac{\sin 3x}{3x} \times 3\right)}$$

$$= \frac{1 \times 1}{(1 \times 5) \times (1 \times 3)} = \frac{1}{15}.$$

45. Put $\left(\frac{\pi}{2} - x\right) = y$ so that when $x \to \frac{\pi}{2}$ then $y \to 0$.

$\therefore$ given limit $= \lim\limits_{y \to 0} y \tan\left(\frac{\pi}{2} - y\right) = \lim\limits_{y \to 0} y \cot y = \lim\limits_{y \to 0} \frac{y}{\tan y} = 1.$

46. Given limit $= \lim\limits_{x \to 0} \frac{(\sqrt{1+2x} - \sqrt{1-2x})(\sqrt{1+2x} + \sqrt{1-2x})}{\sin x(\sqrt{1+2x} + \sqrt{1-2x})}$

$$= \lim_{x \to 0} \frac{\{(1+2x) - (1-2x)\}}{\sin x(\sqrt{1+2x} + \sqrt{1-2x})}$$

$$= 4 \cdot \lim_{x \to 0} \frac{x}{\sin x} \cdot \lim_{x \to 0} \frac{1}{(\sqrt{1+2x} + \sqrt{1-2x})}$$

$$= \left(4 \times 1 \times \frac{1}{2}\right) = 2.$$

47. Given limit $= \lim\limits_{h \to 0} \frac{(a^2 + h^2 + 2ah)\sin(a+h) - a^2 \sin a}{h}$

$$= a^2 \cdot \lim_{h \to 0} \frac{\sin(a+h) - \sin a}{h} + \lim_{h \to 0} h \sin(a+h) + \lim_{h \to 0} 2a \sin(a+h)$$

$$= a^2 \cdot \lim_{h \to 0} \frac{\cos\left(a + \frac{h}{2}\right) \cdot \sin\left(\frac{h}{2}\right)}{(h/2)} + 0 + 2a \sin a$$

$$= a^2 \cdot \lim_{h \to 0} \cos\left(a + \frac{h}{2}\right) \cdot \lim_{h \to 0} \frac{\sin(h/2)}{(h/2)} + 2a \sin a$$

$$= (a^2 \times \cos a \times 1) + 2a \sin a = (a^2 \cos a + 2a \sin a).$$

48. Given limit $= e^3 \cdot \lim\limits_{x \to 0} \left(\frac{e^x - 1}{x}\right) - \lim\limits_{x \to 0} \frac{\sin x}{x} = (e^3 \times 1 - 1) = (e^3 - 1).$

50. Given limit $= \lim\limits_{x \to 0} \frac{(e^{\tan x} - 1)}{x} = \lim\limits_{x \to 0} \left\{\frac{(e^{\tan x} - 1)}{\tan x} \times \frac{\tan x}{x}\right\}$

$$= \lim_{\tan x \to 0} \frac{(e^{\tan x} - 1)}{\tan x} \times \lim_{x \to 0} \frac{\tan x}{x} = (1 \times 1) = 1.$$

51. $\lim\limits_{x \to 0} \frac{ax + x\cos x}{b \sin x} = \lim\limits_{x \to 0} \frac{a + \cos x}{b\left(\frac{\sin x}{x}\right)}$ [dividing num. and denom. by x]

$$= \frac{(a+1)}{(b \times 1)} = \frac{(a+1)}{b}.$$

52. $\lim\limits_{x \to 0} \frac{\sin ax + bx}{ax + \sin bx} = \lim\limits_{x \to 0} \left\{\frac{\frac{\sin ax}{x} + b}{a + \frac{\sin bx}{x}}\right\}$ [dividing num. and denom. by x]

$$= \lim_{x \to 0} \frac{\left\{a\left(\frac{\sin ax}{ax}\right) + b\right\}}{\left\{a + b\left(\frac{\sin bx}{bx}\right)\right\}} = \frac{(a \times 1) + b}{a + (b \times 1)} = \frac{a+b}{a+b} = 1.$$

53. Putting $(\pi - x) = h$, we have $x \to \pi \Rightarrow \pi - x \to 0 \Rightarrow h \to 0$.

$$\therefore \quad \lim_{x\to\pi} \frac{\sin(\pi - x)}{\pi(\pi - x)} = \frac{1}{\pi} \lim_{h\to 0} \frac{\sin h}{h} = \left(\frac{1}{\pi} \times 1\right) = \frac{1}{\pi}.$$

54. Put $x - \frac{\pi}{2} = h$. Then, $x \to \frac{\pi}{2} \Rightarrow \left(x - \frac{\pi}{2}\right) \to 0 \Rightarrow h \to 0$.

$$\therefore \quad \lim_{x\to\frac{\pi}{2}} \frac{\tan 2x}{\left(x - \frac{\pi}{2}\right)} = \lim_{h\to 0} \frac{\tan 2\left(h + \frac{\pi}{2}\right)}{h} = 2 \lim_{h\to 0} \frac{\tan(2h + \pi)}{2h}$$

$$= 2 \lim_{h\to 0} \frac{\tan 2h}{2h} = (2 \times 1) = 2.$$

55. $\lim_{x\to 0} \frac{\cos 2x - 1}{\cos x - 1} = \lim_{x\to 0} \frac{2\cos^2 x - 2}{\cos x - 1} = \lim_{x\to 0} \frac{2(\cos^2 x - 1)}{(\cos x - 1)} = 2 \lim_{x\to 0} (\cos x + 1) = 4.$

56. Given limit $= \lim_{x\to 0} \left(\frac{1}{\sin x} - \frac{\cos x}{\sin x}\right) = \lim_{x\to 0} \frac{(1 - \cos x)}{\sin x} = \lim_{x\to 0} \frac{2\sin^2(x/2)}{2\sin(x/2)\cos(x/2)}$

$$= \lim_{x\to 0} \tan\left(\frac{x}{2}\right) = 0.$$

57. Given limit $= \lim_{x\to 0} \frac{2\sin^2 mx}{2\sin^2 nx} = \lim_{x\to 0} \frac{\sin^2 mx}{\sin^2 nx} = \lim_{x\to 0} \left\{\frac{\frac{\sin mx}{mx} \times mx}{\frac{\sin nx}{nx} \times nx} \times \frac{\frac{\sin mx}{mx} \times mx}{\frac{\sin nx}{nx} \times nx}\right\}$

$$= \frac{m^2}{n^2} \cdot \left\{\frac{\lim_{x\to 0} \frac{\sin mx}{mx}}{\lim_{x\to 0} \frac{\sin nx}{nx}}\right\} \times \left\{\frac{\lim_{x\to 0} \frac{\sin mx}{mx}}{\lim_{x\to 0} \frac{\sin nx}{nx}}\right\} = \left(\frac{m^2}{n^2} \times \frac{1}{1} \times \frac{1}{1}\right) = \frac{m^2}{n^2}.$$

58. Given limit $= \lim_{x\to 0} \frac{2\sin^2\left(\frac{mx}{2}\right)}{2\sin^2\left(\frac{nx}{2}\right)} = \lim_{x\to 0} \left\{\frac{\sin\frac{mx}{2}}{\sin\frac{nx}{2}}\right\}^2$

$$= \lim_{x\to 0} \left\{\frac{\frac{\sin\frac{mx}{2}}{\left(\frac{mx}{2}\right)} \times \frac{mx}{2}}{\frac{\sin\frac{nx}{x}}{\left(\frac{nx}{2}\right)} \times \left(\frac{nx}{2}\right)}\right\}^2 = \frac{m}{n} \cdot \left\{\frac{\lim_{x\to 0} \frac{\sin\frac{mx}{2}}{\frac{mx}{2}}}{\left\{\lim_{x\to 0} \frac{\sin\frac{nx}{2}}{\frac{nx}{2}}\right\}}\right\}$$

$$= \left(\frac{m}{n} \times \frac{1}{1}\right) = \frac{m}{n}.$$

59. $\lim_{x\to 0} \frac{\sin^2 mx}{\sin^2 nx} = \lim_{x\to 0} \left\{\frac{\frac{\sin mx}{mx} \times mx \cdot \frac{\sin mx}{mx} \times mx}{\frac{\sin nx}{nx} \times nx \cdot \frac{\sin nx}{nx} \times nx}\right\}$

$$= \frac{m^2}{n^2} \cdot \frac{\lim\limits_{x\to 0}\left(\dfrac{\sin mx}{mx}\cdot\dfrac{\sin mx}{mx}\right)}{\lim\limits_{x\to 0}\left(\dfrac{\sin nx}{nx}\cdot\dfrac{\sin nx}{nx}\right)} = \frac{m^2}{n^2}\cdot\frac{\lim\limits_{x\to 0}\dfrac{\sin mx}{mx}\cdot\lim\limits_{x\to 0}\dfrac{\sin mx}{mx}}{\lim\limits_{x\to 0}\dfrac{\sin nx}{nx}\cdot\lim\limits_{x\to 0}\dfrac{\sin nx}{nx}}$$

$$= \left(\frac{m^2}{n^2}\times\frac{1\times 1}{1\times 1}\right) = \frac{m^2}{n^2}.$$

60. $\lim\limits_{x\to 0}\dfrac{\sin 2x + \sin 3x}{2x + \sin 3x} = \lim\limits_{x\to 0}\left\{\dfrac{\dfrac{\sin 2x}{x}+\dfrac{\sin 3x}{x}}{2+\dfrac{\sin 3x}{x}}\right\}$ [dividing num. and denom. by x]

$$= \lim_{x\to 0}\frac{2\times\dfrac{\sin 2x}{2x}+3\times\dfrac{\sin 3x}{3x}}{2+3\times\dfrac{\sin 3x}{3x}}$$

$$= \frac{2\times\lim\limits_{x\to 0}\dfrac{\sin 2x}{2x}+3\times\lim\limits_{x\to 0}\dfrac{\sin 3x}{3x}}{2+3\times\lim\limits_{x\to 0}\dfrac{\sin 3x}{3x}} = \frac{(2\times 1+3\times 1)}{(2+3\times 1)} = \frac{5}{5} = 1.$$

61. $\lim\limits_{x\to 0}\dfrac{\sec 4x - \sec 2x}{\sec 3x - \sec x} = \lim\limits_{x\to 0}\dfrac{\left\{\dfrac{1}{\cos 4x}-\dfrac{1}{\cos 2x}\right\}}{\left\{\dfrac{1}{\cos 3x}-\dfrac{1}{\cos x}\right\}} = \lim\limits_{x\to 0}\left\{\dfrac{\dfrac{\cos 2x-\cos 4x}{\cos 4x\cos 2x}}{\dfrac{\cos x-\cos 3x}{\cos 3x\cos x}}\right\}$

$$= \lim_{x\to 0}\left\{\frac{\cos 2x-\cos 4x}{\cos x-\cos 3x}\times\frac{\cos 3x\cos x}{\cos 4x\cos 2x}\right\}$$

$$= \lim_{x\to 0}\left\{\frac{2\sin 3x\sin x}{2\sin 2x\sin x}\times\frac{\cos x\cos 3x}{\cos 2x\cos 4x}\right\}$$

$$= \lim_{x\to 0}\left\{\frac{\dfrac{\sin 3x}{3x}\times 3}{\dfrac{\sin 2x}{2x}\times 2}\right\}\times\lim_{x\to 0}\frac{\cos x\cos 3x}{\cos 2x\cos 4x}$$

$$= \left(\frac{1\times 3}{1\times 2}\times\frac{1}{1}\right) = \frac{3}{2}.$$

62. $\lim\limits_{x\to 0}\dfrac{\sqrt{2}-\sqrt{1+\cos x}}{\sin^2 x} = \lim\limits_{x\to 0}\dfrac{\sqrt{2}-\sqrt{2\cos^2(x/2)}}{(1-\cos^2 x)} = \lim\limits_{x\to 0}\dfrac{\sqrt{2}\left(1-\cos\dfrac{x}{2}\right)}{(1-\cos x)(1+\cos x)}$

$$= \lim_{x\to 0}\frac{\sqrt{2}\left(1-\cos\dfrac{x}{2}\right)}{\left(2\sin^2\dfrac{x}{2}\right)(1+\cos x)} = \frac{1}{\sqrt{2}}\lim_{x\to 0}\frac{\left(1-\cos\dfrac{x}{2}\right)}{\left(1-\cos^2\dfrac{x}{2}\right)(1+\cos x)}$$

$$= \frac{1}{\sqrt{2}}\lim_{x\to 0}\frac{1}{\left(1+\cos\dfrac{x}{2}\right)(1+\cos x)} = \frac{1}{\sqrt{2}}\times\frac{1}{(1+1)(1+1)} = \frac{1}{4\sqrt{2}}.$$

63. Given limit $= \lim\limits_{x\to 0} \dfrac{(\sqrt{1+\sin x}-\sqrt{1-\sin x})}{x} \times \dfrac{(\sqrt{1+\sin x}+\sqrt{1-\sin x})}{(\sqrt{1+\sin x}+\sqrt{1-\sin x})}$

$= \lim\limits_{x\to 0} \dfrac{(1+\sin x)-(1-\sin x)}{x(\sqrt{1+\sin x}+\sqrt{1-\sin x})}$

$= 2\times \lim\limits_{x\to 0} \dfrac{\sin x}{x} \times \lim\limits_{x\to 0} \dfrac{1}{(\sqrt{1+\sin x}+\sqrt{1-\sin x})} = \left(2\times 1\times \dfrac{1}{2}\right) = 1.$

64. Given limit $= \lim\limits_{x\to \frac{\pi}{6}} \dfrac{2-2\left(\dfrac{\sqrt{3}}{2}\cos x + \dfrac{1}{2}\sin x\right)}{(6x-\pi)^2}$

$= \lim\limits_{x\to \frac{\pi}{6}} \dfrac{2\left[1-\left(\cos\dfrac{\pi}{6}\cos x + \sin\dfrac{\pi}{6}\sin x\right)\right]}{(6x-\pi)^2}$

$= \lim\limits_{x\to \frac{\pi}{6}} \dfrac{2\left[1-\cos\left(x-\dfrac{\pi}{6}\right)\right]}{(6x-\pi)^2} = \lim\limits_{\frac{x}{2}\to \frac{\pi}{12}} \dfrac{2\times 2\sin^2\left(\dfrac{x}{2}-\dfrac{\pi}{12}\right)}{144\left(\dfrac{x}{2}-\dfrac{\pi}{12}\right)^2}$

$= \dfrac{4}{144}\times \lim\limits_{\frac{x}{2}\to \frac{\pi}{12}} \left\{\dfrac{\sin\left(\dfrac{x}{2}-\dfrac{\pi}{12}\right)}{\left(\dfrac{x}{2}-\dfrac{\pi}{12}\right)}\right\}^2 = \left(\dfrac{1}{36}\times 1^2\right) = \dfrac{1}{36}.$

65. Given limit $= \lim\limits_{x\to 0} \dfrac{-2\sin\left(\dfrac{a+b}{2}\right)x\cdot\sin\left(\dfrac{a-b}{2}\right)x}{-2\sin^2\left(\dfrac{cx}{2}\right)}$

$= \lim\limits_{x\to 0} \dfrac{\dfrac{\sin\left(\dfrac{a+b}{2}\right)x}{\left(\dfrac{a+b}{2}\right)x}\times\left(\dfrac{a+b}{2}\right)x\cdot\dfrac{\sin\left(\dfrac{a-b}{2}\right)x}{\left(\dfrac{a-b}{2}\right)x}\times\left(\dfrac{a-b}{2}\right)x}{\left\{\dfrac{\sin\left(\dfrac{cx}{2}\right)}{\left(\dfrac{cx}{2}\right)}\right\}^2}$

$= \left\{\left(\dfrac{a+b}{2}\right)\left(\dfrac{a-b}{2}\right)\times\dfrac{4}{c^2}\right\}\cdot \lim\limits_{x\to 0} \dfrac{\sin\left(\dfrac{a+b}{2}\right)x}{\left(\dfrac{a+b}{2}\right)x}\cdot \lim\limits_{x\to 0} \dfrac{\sin\left(\dfrac{a-b}{2}\right)x}{\left(\dfrac{a-b}{2}\right)x}$

$= \left\{\dfrac{(a^2-b^2)}{c^2}\times 1\times 1\right\} = \dfrac{(a^2-b^2)}{c^2}.$

66. Given limit $= \lim\limits_{x\to a} \dfrac{\cos x - \cos a}{\cos x\sin a - \sin x\cos a}\cdot \sin x\sin a$

$$= \lim_{x \to a} \frac{-2\sin\left(\frac{x+a}{2}\right)\sin\left(\frac{x-a}{2}\right)}{\sin(a-x)} \cdot \sin x \sin a$$

$$= \lim_{x \to 0} \frac{-2\sin\left(\frac{x+a}{2}\right)\sin\left(\frac{x-a}{2}\right)}{-2\sin\left(\frac{x-a}{2}\right)\cos\left(\frac{x-a}{2}\right)} \cdot \sin x \sin a$$

$$= \lim_{x \to 0} \frac{\sin\left(\frac{x+a}{2}\right)}{\cos\left(\frac{x-a}{2}\right)} \cdot \sin x \sin a = \sin^3 a.$$

67. Given limit $= \lim_{x \to \frac{\pi}{4}} \dfrac{\tan x(\tan x+1)(\tan x-1)}{\cos\left(x+\frac{\pi}{4}\right)}$

$$= \lim_{x \to \frac{\pi}{4}} \frac{\tan x(\tan x+1)(\sin x-\cos x)}{\cos x \cdot \cos\left(x+\frac{\pi}{4}\right)}$$

$$= -\lim_{x \to \frac{\pi}{4}} \frac{\tan x(\tan x+1)(\cos x-\sin x)}{\cos x \cdot \cos\left(x+\frac{\pi}{4}\right)}$$

$$= -\sqrt{2}\lim_{x \to \frac{\pi}{4}} \frac{\tan x(\tan x+1)\left(\frac{1}{\sqrt{2}}\cos x-\frac{1}{\sqrt{2}}\sin x\right)}{\cos x \cdot \cos\left(x+\frac{\pi}{4}\right)}$$

$$= -\sqrt{2}\lim_{x \to \frac{\pi}{4}} \frac{\tan x(\tan x+1)\cdot\cos\left(x+\frac{\pi}{4}\right)}{\cos x \cdot \cos\left(x+\frac{\pi}{4}\right)} = -\sqrt{2}\lim_{x \to \frac{\pi}{4}} \frac{\tan x(\tan x+1)}{\cos x}$$

$$= -\sqrt{2}\cdot\frac{1\times(1+1)}{\left(\frac{1}{\sqrt{2}}\right)} = -4.$$

68. Given limit $= \lim_{x \to \frac{\pi}{2}} \dfrac{(\sqrt{2}-\sqrt{1+\sin x})}{\sqrt{2}\cos^2 x} \times \dfrac{(\sqrt{2}+\sqrt{1+\sin x})}{(\sqrt{2}+\sqrt{1+\sin x})}$

$$= \lim_{x \to \frac{\pi}{2}} \frac{2-(1+\sin x)}{\sqrt{2}(1-\sin^2 x)} \times \frac{1}{(\sqrt{2}+\sqrt{1+\sin x})}$$

$$= \lim_{x \to \frac{\pi}{2}} \frac{(1-\sin x)}{\sqrt{2}(1-\sin x)(1+\sin x)} \times \frac{1}{(\sqrt{2}+\sqrt{1+\sin x})}$$

$$= \frac{1}{\sqrt{2}}\cdot\lim_{x \to \frac{\pi}{2}} \frac{1}{(1+\sin x)(\sqrt{2}+\sqrt{1+\sin x})}$$

$$= \frac{1}{\sqrt{2}}\times\frac{1}{2\times 2\sqrt{2}} = \frac{1}{8}\cdot$$

69. Given limit $= \lim\limits_{x \to \frac{\pi}{6}} \dfrac{\operatorname{cosec}^2 x - 4}{\operatorname{cosec} x - 2} = \lim\limits_{x \to \frac{\pi}{6}} (\operatorname{cosec} x + 2) = (2 + 2) = 4.$

70. Given limit $= \lim\limits_{x \to \pi} \dfrac{(\sqrt{2 + \cos x} - 1)(\sqrt{2 + \cos x} + 1)}{(\pi - x)^2(\sqrt{2 + \cos x} + 1)}$

$$= \lim_{x \to \pi} \frac{(1 + \cos x)}{(\pi - x)^2(\sqrt{2 + \cos x} + 1)} = \lim_{h \to 0} \frac{1 + \cos(\pi - h)}{h^2\{\sqrt{2 + \cos(\pi - h)} + 1\}}$$

[putting $\pi - x = h$]

$$= \lim_{h \to 0} \frac{(1 - \cos h)}{h^2\{\sqrt{2 - \cos h} + 1\}} = \lim_{h \to 0} \frac{2\sin^2(h/2)}{\left(\frac{h}{2}\right)^2 \times 4 \times \{\sqrt{2 - \cos h} + 1\}}$$

$$= \frac{1}{2} \lim_{h \to 0} \left\{\frac{\sin(h/2)}{(h/2)}\right\}^2 \cdot \frac{1}{\lim\limits_{h \to 0} \{\sqrt{2 - \cos h} + 1\}} = \left(\frac{1}{2} \times 1^2 \times \frac{1}{2}\right) = \frac{1}{4}.$$

71. Given limit $= \lim\limits_{x \to \frac{\pi}{4}} \dfrac{\cos x - \sin x}{\cos x \cdot \sqrt{2}\left\{\frac{1}{\sqrt{2}} - \sin x\right\}} = \lim\limits_{x \to \frac{\pi}{4}} \dfrac{\sqrt{2}\left\{\frac{1}{\sqrt{2}}\cos x - \frac{1}{\sqrt{2}}\sin x\right\}}{\sqrt{2}\cos x\left(\sin\frac{\pi}{4} - \sin x\right)}$

$$= \lim_{x \to \frac{\pi}{4}} \frac{\left\{\sin\frac{\pi}{4}\cos x - \cos\frac{\pi}{4}\sin x\right\}}{\cos x\left(\sin\frac{\pi}{4} - \sin x\right)}$$

$$= \lim_{x \to \frac{\pi}{4}} \frac{\sin\left(\frac{\pi}{4} - x\right)}{\cos x \cdot 2\cos\left(\frac{\pi}{8} + \frac{x}{2}\right)\sin\left(\frac{\pi}{8} - \frac{x}{2}\right)}$$

$$= \lim_{x \to \frac{\pi}{4}} \frac{2\sin\left(\frac{\pi}{8} - \frac{x}{2}\right)\cos\left(\frac{\pi}{8} - \frac{x}{2}\right)}{\cos x \cdot 2\cos\left(\frac{\pi}{8} + \frac{x}{2}\right)\sin\left(\frac{\pi}{8} - \frac{x}{2}\right)} = \lim_{x \to \frac{\pi}{4}} \frac{\cos\left(\frac{\pi}{8} - \frac{x}{2}\right)}{\cos x \cdot \cos\left(\frac{\pi}{8} + \frac{x}{2}\right)}$$

$$= \frac{\cos\left(\frac{\pi}{8} - \frac{\pi}{8}\right)}{\cos\frac{\pi}{4} \cdot \cos\left(\frac{\pi}{8} + \frac{\pi}{8}\right)} = \frac{\cos 0}{\left(\cos\frac{\pi}{4}\right)^2} = \frac{1}{\left(\frac{1}{\sqrt{2}}\right)^2} = 2.$$

72. Given limit $= \lim\limits_{x \to \frac{\pi}{6}} \dfrac{2\sin^2 x + 2\sin x - \sin x - 1}{2\sin^2 x - 2\sin x - \sin x + 1}$

$$= \lim_{x \to \frac{\pi}{6}} \frac{2\sin x(\sin x + 1) - (\sin x + 1)}{2\sin x(\sin x - 1) - (\sin x - 1)}$$

$$= \lim_{x \to \frac{\pi}{6}} \frac{(\sin x + 1)(2\sin x - 1)}{(\sin x - 1)(2\sin x - 1)} = \lim_{x \to \frac{\pi}{6}} \frac{(\sin x + 1)}{(\sin x - 1)}$$

$$= \frac{\left(\sin\frac{\pi}{6}+1\right)}{\left(\sin\frac{\pi}{6}-1\right)} = \frac{\left(\frac{1}{2}+1\right)}{\left(\frac{1}{2}-1\right)} = \frac{3}{2}\times(-2) = -3.$$

LEFT-HAND LIMIT

Let $f(x)$ be a given function and let x approach to a from the left. Then, the limit of $f(x)$, as x approaches to a from the left, is denoted by $\lim_{x\to a^-} f(x)$.

Method for Finding $\lim_{x\to a^-} f(x)$

Replace x by $(a-h)$ and take the limit as $h\to 0$.

RIGHT-HAND LIMIT

Let $f(x)$ be a given function and let x approach to a from the right. Then, the limit of $f(x)$, as x approaches to a from the right, is denoted by $\lim_{x\to a^+} f(x)$.

Method for Finding $\lim_{x\to a^+} f(x)$

Replace x by $(a+h)$ and take the limit as $h\to 0$.

LIMIT OF A FUNCTION *We say that $\lim_{x\to a} f(x)$ exists only when $\lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x)$ and this common value is known as $\lim_{x\to a} f(x)$.*

SOLVED EXAMPLES

EXAMPLE 1 *Let* $f(x) = \begin{cases} 2x+3, & x \le 0 \\ 3(x+1), & x > 0. \end{cases}$

Find (i) $\lim_{x\to 0} f(x)$ *(ii)* $\lim_{x\to 1} f(x)$

SOLUTION (i) We have

$$\lim_{x\to 0^+} f(x) = \lim_{h\to 0} 3(0+h+1) = \lim_{h\to 0} 3(h+1) = 3.$$

$$\lim_{x\to 0^-} f(x) = \lim_{h\to 0} 3(0-h+1) = \lim_{h\to 0} 3(-h+1) = 3.$$

$$\therefore \quad \lim_{x\to 0^+} (x) = \lim_{x\to 0^-} f(x) = 3.$$

Hence, $\lim_{x\to 0} f(x) = 3$.

(ii) We have

$$\lim_{x\to 1^+} f(x) = \lim_{h\to 0} f(1+h) = \lim_{h\to 0} 3(1+h+1) = \lim_{h\to 0} 3(2+h) = 6.$$

$$\lim_{x\to 1^-} f(x) = \lim_{h\to 0} f(1-h) = \lim_{h\to 0} 3(1-h+1) = \lim_{h\to 0} 3(2-h) = 6.$$

$\therefore \quad \lim_{x\to 1^+} f(x) = \lim_{x\to 1^-} f(x) = 6.$

Hence, $\lim_{x\to 1} f(x) = 6.$

EXAMPLE 2 *Let* $f(x) = \begin{cases} x^2 - 1, & x \le 1 \\ -x^2 - 1, & x > 1. \end{cases}$

Find $\lim_{x\to 1} f(x)$.

SOLUTION We have

$$\lim_{x\to 1^+} f(x) = \lim_{h\to 0} f(1+h) = \lim_{h\to 0} \{-(1+h)^2 - 1\} = \lim_{h\to 0} (-2 - h^2 - 2h) = -2.$$

$$\lim_{x\to 1^-} f(x) = \lim_{h\to 0} f(1-h) = \lim_{h\to 0} \{(1-h)^2 - 1\} = \lim_{h\to 0} (1 + h^2 - 2h - 1)$$

$$= \lim_{h\to 0} (h^2 - 2h) = 0.$$

$\therefore \quad \lim_{x\to 1^+} f(x) \ne \lim_{x\to 1^-} f(x).$

Hence, $\lim_{x\to 1} f(x)$ does not exist.

EXAMPLE 3 *Let* $f(x) = \begin{cases} \dfrac{|x|}{x}, & x \ne 0 \\ 0, & x = 0. \end{cases}$

Find $\lim_{x\to 0} f(x)$.

SOLUTION We have

$$\lim_{x\to 0^+} f(x) = \lim_{h\to 0} f(0+h) = \lim_{h\to 0} f(h) = \lim_{h\to 0} \frac{|h|}{h} = \lim_{h\to 0} \frac{h}{h} = 1.$$

$$\lim_{x\to 0^-} f(x) = \lim_{h\to 0} f(0-h) = \lim_{h\to 0} f(-h) = \lim_{h\to 0} \frac{|-h|}{-h} = \lim_{h\to 0} \frac{h}{-h} = -1.$$

Thus, $\lim_{x\to 0^+} f(x) \ne \lim_{x\to 0^-} f(x)$.

Hence, $\lim_{x\to 0} f(x)$ does not exist.

Caution Here note that $\dfrac{h}{h} = 1$, since $h \ne 0$.

EXAMPLE 4 *Let* $f(x) = \begin{cases} a + bx, & x < 1 \\ 4, & x = 1 \\ b - ax, & x > 1. \end{cases}$

If $\lim_{x\to 1} f(x) = f(1)$, *find the values of a and b.*

SOLUTION Clearly, $f(1) = 4$.

$$\lim_{x\to 1^+} f(x) = \lim_{h\to 0} f(1+h) = \lim_{h\to 0} \{b - a(1+h)\} = (b - a).$$

$$\lim_{x\to 1^-} f(x) = \lim_{h\to 0} f(1-h) = \lim_{h\to 0} \{a + b(1-h)\} = (a + b).$$

Now, $\lim\limits_{x\to 1} f(x) = f(1) = 4$

$\Rightarrow \quad \lim\limits_{x\to 1^+} f(x) = \lim\limits_{x\to 1^-} f(x) = f(1) = 4$

$\Rightarrow \quad (b-a) = (a+b) = 4$

$\Rightarrow \quad b-a = 4$ and $b+a = 4 \;\Rightarrow\; a = 0,\, b = 4.$

Hence, $a = 0$ and $b = 4$.

EXAMPLE 5 *Let* $f(x) = \begin{cases} |x|+1, & x < 0 \\ 0, & x = 0 \\ |x|-1, & x > 0 \end{cases}$

For what value(s) of a does $\lim\limits_{x\to a} f(x)$ *exists?*

SOLUTION **Case 1.** *When* $a < 0$.

In this case, we have

$$\lim_{x\to a^+} f(x) = \lim_{h\to 0} f(a+h) = \lim_{h\to 0} \{|a+h|+1\} = |a|+1.$$

$$\lim_{x\to a^-} f(x) = \lim_{h\to 0} f(a-h) = \lim_{h\to 0} \{|a-h|+1\} = |a|+1.$$

$$\therefore \quad \lim_{x\to a^+} f(x) = \lim_{x\to a^-} f(x) = |a|+1.$$

So, in this case, $\lim\limits_{x\to a} f(x)$ exists.

Case 2. *When* $a < 0$.

In this case, we have

$$\lim_{x\to a^+} f(x) = \lim_{h\to 0} f(a+h) = \lim_{h\to 0} \{|a+h|-1\} = |a|-1.$$

$$\lim_{x\to a^-} f(x) = \lim_{h\to 0} f(a-h) = \lim_{h\to 0} \{|a-h|-1\} = |a|-1.$$

$$\therefore \quad \lim_{x\to a^+} f(x) = \lim_{x\to a^-} f(x) = |a|-1.$$

So, in this case, $\lim\limits_{x\to a} f(x)$ exists.

Case 3. *When* $a < 0$.

In this case, we have

$$\lim_{x\to 0^+} f(x) = \lim_{h\to 0} f(0+h) = \lim_{h\to 0} \{|h|-1\} = -1.$$

$$\lim_{x\to 0^-} f(x) = \lim_{h\to 0} f(0-h) = \lim_{h\to 0} \{|-h|+1\} = 1.$$

Thus, $\lim\limits_{x\to 0^+} f(x) \neq \lim\limits_{x\to 0^-} f(x)$.

Hence, $\lim\limits_{x\to 0} f(x)$ does not exist.

Thus, $\lim\limits_{x\to a} f(x)$ exists only when $a \neq 0$.

EXAMPLE 6 *Let* $f(x) = \begin{cases} mx^2+n, & x < 0 \\ nx+m, & 0 \le x \le 1 \\ nx^3+m, & x > 1. \end{cases}$

For what values of integers m and n, $\lim\limits_{x\to 0} f(x)$ *and* $\lim\limits_{x\to 1} f(x)$ *both exist?*

SOLUTION We have

$$\lim_{x \to 0^+} f(x) = \lim_{h \to 0} f(0+h) = \lim_{h \to 0} f(h) = \lim_{h \to 0} (nh+m) = m.$$

$$\lim_{x \to 0^-} f(x) = \lim_{h \to 0} f(0-h) = \lim_{h \to 0} f(-h) = \lim_{h \to 0} \{m(-h)^2 + n\}$$

$$= \lim_{h \to 0} (mh^2 + n) = n.$$

$\therefore \quad \lim_{x \to 0} f(x)$ exists only when $m = n$.

Now, $\lim_{x \to 1^+} f(x) = \lim_{h \to 0} f(1+h) = \lim_{h \to 0} \{n(1+h)^3 + m\}$

$$= \lim_{h \to 0} \{n(1+h^3 + 3h + 3h^2) + m\} = (n+m).$$

And, $\lim_{x \to 1^-} f(x) = \lim_{h \to 0} f(1-h) = \lim_{h \to 0} \{n(1-h) + m\} = (n+m).$

$\therefore \quad \lim_{x \to 1} f(x) = (n+m).$

Hence, $\lim_{x \to 0} f(x)$ and $\lim_{x \to 1} f(x)$ both exist only when $m = n$.

EXAMPLE 7 *Let $a_1, a_2, \ldots, a_n$ be fixed real numbers and let*

$$f(x) = (x - a_1)(x - a_2)(x - a_3) \ldots (x - a_n).$$

Find $\lim_{x \to a_1} f(x)$.

If $a \neq a_1, a_2, \ldots, a_n$, compute $\lim_{x \to a} f(x)$.

SOLUTION We have

$$\lim_{x \to a_1^+} f(x) = \lim_{h \to 0} f(a_1 + h)$$

$$= \lim_{h \to 0} \{(a_1 + h - a_1)(a_1 + h - a_2)(a_1 + h - a_3) \ldots (a_1 + h - a_n)\}$$

$$= \lim_{h \to 0} \{h(a_1 + h - a_2)(a_1 + h - a_3) \ldots (a_1 + h - a_n)\}$$

$$= \{0 \times (a_1 - a_2) \times (a_1 - a_3) \times \ldots \times (a_1 - a_n) = 0.$$

$$\lim_{x \to a_1^-} f(x) = \lim_{h \to 0} f(a_1 - h)$$

$$= \lim_{h \to 0} \{(a_1 - h - a_1)(a_1 - h - a_2)(a_1 - h - a_3) \ldots (a_1 - h - a_n)\}$$

$$= \lim_{h \to 0} \{(-h)(a_1 - h - a_2)(a_1 - h - a_3) \ldots (a_1 - h - a_n)\}$$

$$= \{0 \times (a_1 - a_2) \times (a_1 - a_3) \times \ldots \times (a_1 - a_n)\} = 0.$$

$\therefore \quad \lim_{x \to a_1^+} f(x) = \lim_{x \to a_1^-} f(x) = 0.$

Hence, $\lim_{x \to a_1} f(x) = 0.$

For any $a \neq a_1, a_2, \ldots, a_n$, we have

$$\lim_{x \to a} f(x) = \lim_{x \to a} \{(x - a_1)(x - a_2)(x - a_3) \ldots (x - a_n)\}$$

$$= (a - a_1)(a - a_2)(a - a_3) \ldots (x - a_n).$$

EXAMPLE 8 *Let $f(x) = [x]$ = greatest integer less than or equal to x. For any integer k, show that $\lim_{x \to k} f(x)$ does not exist.*

SOLUTION We have

$$\lim_{x \to k^+} f(x) = \lim_{h \to 0} f(k+h) = \lim_{h \to 0} [k+h] = \lim_{h \to 0} k = k \quad \{\because\ [k+h] = k\}$$

and $\lim\limits_{x \to k^-} f(x) = \lim\limits_{h \to 0} f(k-h) = \lim\limits_{h \to 0} [k-h] = \lim\limits_{h \to 0} (k-1) = (k-1)$

$\{\because\ [k-h] = k-1\}$

Thus, $\lim\limits_{x \to k^+} f(x) \neq \lim\limits_{x \to k^-} f(x)$.

Hence, $\lim\limits_{x \to k} f(x)$ does not exist.

EXAMPLE 9 *If f is an odd function and* $\lim\limits_{x \to 0} f(x)$ *exists then prove that this limit must be* 0.

SOLUTION Let f be an odd function and let $\lim\limits_{x \to 0} f(x)$ exist. Then

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^-} f(x)$$

$$\Rightarrow \lim_{h \to 0} f(0+h) = \lim_{h \to 0} f(0-h)$$

$$\Rightarrow \lim_{h \to 0} f(h) = \lim_{h \to 0} f(-h) = -\lim_{h \to 0} f(h) \quad [\because\ f \text{ being odd, } f(-h) = -f(h)]$$

$$\Rightarrow 2\lim_{h \to 0} f(h) = 0 \Rightarrow \lim_{h \to 0} f(h) = 0 \Rightarrow f(x) = 0.$$

Hence, $\lim\limits_{x \to 0} f(x) = 0$.

EXAMPLE 10 *If f is an even function, prove that* $\lim\limits_{x \to 0^-} f(x) = \lim\limits_{x \to 0^+} f(x)$.

SOLUTION Let f be an even function. Then,

$$\lim_{x \to 0^-} f(x) = \lim_{h \to 0} f(0-h) = \lim_{h \to 0} f(-h)$$

$$= \lim_{h \to 0} f(h) \quad [\because\ f \text{ being even, } f(-h) = f(h)]$$

$$= \lim_{h \to 0} f(0+h) = \lim_{x \to 0^+} f(x).$$

Hence, $\lim\limits_{x \to 0^-} f(x) = \lim\limits_{x \to 0^+} f(x)$.

EXAMPLE 11 *Show that* $\lim\limits_{x \to 0} \left(\dfrac{e^{1/x} - 1}{e^{1/x} + 1}\right)$ *does not exist.*

SOLUTION Let $f(x) = \left(\dfrac{e^{1/x} - 1}{e^{1/x} + 1}\right)$. Then,

$$\lim_{x \to 0^+} f(x) = \lim_{h \to 0} f(0+h) = \lim_{h \to 0} f(h)$$

$$= \lim_{h \to 0} \left(\frac{e^{1/h} - 1}{e^{1/h} + 1}\right) = \lim_{h \to 0} \left(\frac{1 - \dfrac{1}{e^{1/h}}}{1 + \dfrac{1}{e^{1/h}}}\right) = \left(\frac{1-0}{1+0}\right) = 1.$$

$$\lim_{x \to 0^-} f(x) = \lim_{h \to 0} f(0-h) = \lim_{h \to 0} f(-h)$$

$$= \lim_{h \to 0} \left(\frac{e^{-1/h} - 1}{e^{-1/h} + 1} \right) = \lim_{h \to 0} \left(\frac{\frac{1}{e^{1/h}} - 1}{\frac{1}{e^{1/h}} + 1} \right) = \left(\frac{0-1}{0+1} \right) = -1.$$

Thus, $\lim_{x \to 0^+} f(x) \neq \lim_{x \to 0^-} f(x)$.

Hence, $\lim_{x \to 0} f(x)$ does not exist.

EXAMPLE 12 *Show that* $\lim_{x \to 0} \frac{x}{|x|}$ *does not exist.*

SOLUTION Let $f(x) = \frac{x}{|x|}\cdot$ Then,

$$\lim_{x \to 0^+} f(x) = \lim_{h \to 0} f(0+h) = \lim_{h \to 0} f(h) = \lim_{h \to 0} \frac{h}{|h|} = \lim_{h \to 0} \frac{h}{h} = 1.$$

$$\lim_{x \to 0^-} f(x) = \lim_{h \to 0} f(0-h) = \lim_{h \to 0} f(-h) = \lim_{h \to 0} \frac{-h}{|-h|} = \lim_{h \to 0} \frac{-h}{h} = -1.$$

$\therefore \quad \lim_{x \to 0^+} f(x) \neq \lim_{x \to 0^-} f(x)$.

Hence, $\lim_{x \to 0} f(x)$ does not exist.

EXERCISE 27C

1. If $f(x) = |x| - 3$, find $\lim_{x \to 3} f(x)$.

2. Let $f(x) = \begin{cases} \frac{x}{|x|}, & x \neq 0 \\ 0, & x = 0. \end{cases}$

Show that $\lim_{x \to 0} f(x)$ does not exist.

3. Let $f(x) = \begin{cases} \frac{|x-3|}{(x-3)}, & x \neq 3 \\ 0, & x = 3. \end{cases}$

Show that $\lim_{x \to 3} f(x)$ does not exist.

4. Let $f(x) = \begin{cases} 1 + x^2, & 0 \leq x \leq 1 \\ 2 - x, & x > 1. \end{cases}$

Show that $\lim_{x \to 1} f(x)$ does not exist.

5. Let $f(x) = \begin{cases} \frac{x - |x|}{x}, & x \neq 0 \\ 2, & x = 0. \end{cases}$

Show that $\lim_{x \to 0} f(x)$ does not exist.

6. Let $f(x) = \begin{cases} 5x - 4, & 0 < x \le 1 \\ 4x^3 - 3x, & 1 < x < 2. \end{cases}$

Find $\lim\limits_{x \to 1} f(x)$.

7. Let $f(x) = \begin{cases} 4x - 5, & x \le 2 \\ x - a, & x > 2. \end{cases}$

If $\lim\limits_{x \to 2} f(x)$ exists then find the value of a.

8. Let $f(x) = \begin{cases} \dfrac{3x}{|x| + 2x}, & x \ne 0 \\ 0, & x = 0. \end{cases}$

Show that $\lim\limits_{x \to 0} f(x)$ does not exist.

9. Let $f(x) = \begin{cases} \cos x, & x \ge 0 \\ x + k, & x < 0. \end{cases}$

Find the value of k for which $\lim\limits_{x \to 0} f(x)$ exists.

10. Show that $\lim\limits_{x \to 0} \dfrac{1}{x}$ does not exist.

11. Show that $\lim\limits_{x \to 0} \dfrac{1}{|x|} = \infty$.

12. Show that $\lim\limits_{x \to 0} e^{-1/x}$ does not exist.

13. Show that $\lim\limits_{x \to 0} \sin \dfrac{1}{x}$ does not exist.

14. Show that $\lim\limits_{x \to 2} \dfrac{x}{[x]}$ does not exist.

15. Let $f(x) = \begin{cases} \dfrac{k \cos x}{\pi - 2x}, & x \ne \dfrac{\pi}{2} \\ 3, & x = \dfrac{\pi}{2}. \end{cases}$

If $\lim\limits_{x \to \frac{\pi}{2}} f(x) = f\left(\dfrac{\pi}{2}\right)$, find the value of k.

ANSWERS (EXERCISE 27C)

1. 0 **6.** 1 **7.** $a = -1$ **9.** $k = 1$ **10.** $k = 6$

HINTS TO SOME SELECTED QUESTIONS

1. $\lim\limits_{x \to 3^+} f(x) = \lim\limits_{h \to 0} f(3 + h) = \lim\limits_{h \to 0} \{|3 + h| - 3\} = \lim\limits_{h \to 0} (3 + h - 3) = \lim\limits_{h \to 0} h = 0.$

$\lim\limits_{x \to 3^-} f(x) = \lim\limits_{h \to 0} f(3 - h) = \lim\limits_{h \to 0} \{|3 - h| - 3\} = \lim\limits_{h \to 0} (3 - h - 3) = \lim\limits_{h \to 0} (-h) = 0.$

$\therefore \quad \lim_{x \to 3^+} f(x) = \lim_{x \to 3^-} f(x) = 0 \Rightarrow \lim_{x \to 3} f(x) = 0.$

2. $\lim_{x \to 0^+} f(x) = \lim_{h \to 0} f(0+h) = \lim_{h \to 0} f(h) = \lim_{h \to 0} \frac{h}{|h|} = \lim_{h \to 0} \frac{h}{h} = 1$ $\quad [\because h > 0]$

$\lim_{x \to 0^-} f(x) = \lim_{h \to 0} f(0-h) = \lim_{h \to 0} f(-h) = \lim_{h \to 0} \frac{-h}{|-h|} = \lim_{h \to 0} \frac{-h}{h} = -1.$

$\therefore \quad \lim_{x \to 0^+} f(x) \neq \lim_{x \to 0^-} f(x)$ and so $\lim_{x \to 0} f(x)$ does not exist.

3. $\lim_{x \to 3^+} f(x) = \lim_{h \to 0} f(3+h) = \lim_{h \to 0} \frac{|3+h-3|}{(3+h-3)} = \lim_{h \to 0} \frac{|h|}{h} = \lim_{h \to 0} \frac{h}{h} = 1.$ $\quad [\because h > 0]$

$\lim_{x \to 3^-} f(x) = \lim_{h \to 0} f(3-h) = \lim_{h \to 0} \frac{|3-h-3|}{(3-h-3)} = \lim_{h \to 0} \frac{|-h|}{-h} = \lim_{h \to 0} \frac{h}{-h} = -1.$

$\therefore \quad \lim_{x \to 3^+} f(x) \neq \lim_{x \to 3^-} f(x)$ and so $\lim_{x \to 3} f(x)$ does not exist.

4. $\lim_{x \to 1^+} f(x) = \lim_{h \to 0} f(1+h) = \lim_{h \to 0} \{2-(1+h)\} = \lim_{h \to 0} (1-h) = 1.$

$\lim_{x \to 1^-} f(x) = \lim_{h \to 0} f(1-h) = \lim_{h \to 0} \{1+(1-h)^2\} = \lim_{h \to 0} (1+1+h^2-2h) = 2.$

$\therefore \quad \lim_{x \to 1^+} f(x) \neq \lim_{x \to 1^-} f(x)$ and so $\lim_{x \to 1} f(x)$ does not exist.

6. $\lim_{x \to 1^+} f(x) = \lim_{h \to 0} f(1+h) = \lim_{h \to 0} \{4(1+h)^3 - 3(1+h)\} = (4 \times 1^3 - 3) = 1.$

$\lim_{x \to 1^-} f(x) = \lim_{h \to 0} f(1-h) = \lim_{h \to 0} \{5(1-h) - 4\} = (5-4) = 1.$

$\therefore \quad \lim_{x \to 1^+} f(x) = \lim_{x \to 1^-} f(x) = 1$ and so $\lim_{x \to 1} f(x) = 1.$

7. $\lim_{x \to 2^+} f(x) = \lim_{h \to 0} f(2+h) = \lim_{h \to 0} (2+h-a) = (2-a).$

$\lim_{x \to 2^-} f(x) = \lim_{h \to 0} f(2-h) = \lim_{h \to 0} \{4(2-h)-5\} = (8-5) = 3.$

Since $\lim_{x \to 2} f(x)$ exists, we must have $\lim_{x \to 2^+} f(x) = \lim_{x \to 2^-} f(x).$

$\therefore \quad 2 - a = 3 \Rightarrow a = (2-3) = -1.$

9. $\lim_{x \to 0^+} f(x) = \lim_{h \to 0} f(0+h) = \lim_{h \to 0} \cos h = 1.$

$\lim_{x \to 0^-} f(x) = \lim_{h \to 0} f(0-h) = \lim_{h \to 0} f(-h) = \lim_{h \to 0} \{(-h)+k\} = k.$

Since $\lim_{x \to 0} f(x)$ exists, we have $\lim_{x \to 0^+} f(x) = \lim_{x \to 0^-} f(x)$ and hence $k = 1$.

10. Let $f(x) = \frac{1}{x} \cdot$ Then, $\lim_{x \to 0^+} f(x) = \lim_{h \to 0} f(0+h) = \lim_{h \to 0} f(h) = \lim_{h \to 0} \frac{1}{h} = \infty.$

And, $\lim_{x \to 0^-} f(x) = \lim_{h \to 0} f(0-h) = \lim_{h \to 0} f(-h) = \lim_{h \to 0} \frac{1}{-h} = -\infty.$

$\therefore \quad \lim_{x \to 0^+} f(x) \neq \lim_{x \to 0^-} f(x)$ and hence $\lim_{x \to 0} f(x)$ does not exist.

11. Let $f(x) = \frac{1}{|x|} \cdot$ Then,

$\lim_{x \to 0^+} f(x) = \lim_{h \to 0} f(0+h) = \lim_{h \to 0} f(h) = \lim_{h \to 0} \frac{1}{|h|} = \lim_{h \to 0} \frac{1}{h} = \infty$ $\quad [\because h > 0]$

$$\lim_{x\to 0^-} f(x) = \lim_{h\to 0} f(0-h) = \lim_{h\to 0} f(-h) = \lim_{h\to 0} \frac{1}{|-h|} = \lim_{h\to 0} \frac{1}{h} = \infty.$$

$\therefore \quad \lim_{x\to 0} \frac{1}{|x|} = \infty.$

12. Let $f(x) = e^{-1/x}$. Then,

$$\lim_{x\to 0^+} f(x) = \lim_{h\to 0} f(0+h) = \lim_{h\to 0} f(h) = \lim_{h\to 0} e^{-1/h} = \lim_{h\to 0} \frac{1}{e^{1/h}} = \frac{1}{\infty} = 0.$$

$$\lim_{x\to 0^-} f(x) = \lim_{h\to 0} f(0-h) = \lim_{h\to 0} f(-h) = \lim_{h\to 0} e^{1/h} = \infty. \qquad [\because\ e^{\infty} = \infty]$$

$\therefore \quad \lim_{x\to 0} f(x)$ does not exist.

13. Let $f(x) = \sin \frac{1}{x}\cdot$

$$\lim_{x\to 0^+} f(x) = \lim_{h\to 0} f(0+h) = \lim_{h\to 0} f(h) = \lim_{h\to 0} \sin \frac{1}{h}\cdot$$

Clearly, when h is given different values, then $\sin \frac{1}{h}$ oscillates between –1 and 1 and thus it does not approach to a definite value. So, $\lim_{h\to 0} \sin \frac{1}{h}$ does not exist.

Hence, $\lim_{x\to 0} \sin \frac{1}{x}$ does not exist.

14. Let $f(x) = \frac{x}{[x]}\cdot$ Then,

$$\lim_{x\to 2^+} f(x) = \lim_{h\to 0} f(2+h) = \lim_{h\to 0} \frac{2+h}{[2+h]} = 1 \qquad [\because\ h\to 0, [2+h] = 2]$$

$$\lim_{x\to 2^-} f(x) = \lim_{h\to 0} f(2-h) = \lim_{h\to 0} \frac{2-h}{[2-h]} = \lim_{h\to 0} \frac{2-h}{1} = 2 \qquad [\because\ [2-h] = 1]$$

$\therefore \quad \lim_{x\to 2^+} f(x) \neq \lim_{x\to 2^-} f(x)$ and so $\lim_{x\to 2} f(x)$ does not exist.

15. $$\lim_{x\to \frac{\pi}{2}^+} f(x) = \lim_{h\to 0} f\left(\frac{\pi}{2}+h\right) = \lim_{h\to 0} \frac{k\cos\left(\frac{\pi}{2}+h\right)}{\pi - 2\left(\frac{\pi}{2}+h\right)} = \lim_{h\to 0} \frac{-k\sin h}{-2h}$$

$$= \lim_{h\to 0} \frac{k}{2}\left(\frac{\sin h}{h}\right) = \frac{k}{2}\cdot$$

$\therefore \quad \frac{k}{2} = 3 \Rightarrow k = 6.$

❑

28
Differentiation

DERIVATIVE OF A FUNCTION

Let $y = f(x)$ be a given continuous function. Then, the value of y depends upon the value of x and it changes with a change in the value of x. We use the word increment to denote a small change, i.e., an increase or decrease, in the values of x and y.

Let δy be an increment in y, corresponding to an increment δx in x. Then,

$$y = f(x) \text{ and } y + \delta y = f(x + \delta x).$$

On subtraction, we get $\delta y = f(x + \delta x) - f(x)$.

$$\therefore \qquad \frac{\delta y}{\delta x} = \frac{f(x + \delta x) - f(x)}{\delta x}.$$

$$\text{So,} \qquad \lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \lim_{\delta x \to 0} \frac{f(x + \delta x) - f(x)}{\delta x}.$$

The above limit, if it exists finitely, is called the *derivative* or *differential coefficient* of $y = f(x)$ with respect to x, and it is denoted by

$$\frac{dy}{dx} \text{ or } \frac{d}{dx}\{f(x)\} \text{ or } f'(x).$$

The process of finding the derivative is known as differentiation.

REMARK $\quad \dfrac{dy}{dx} = \lim\limits_{\delta x \to 0} \dfrac{\delta y}{\delta x} = \lim\limits_{\delta x \to 0} \dfrac{f(x + \delta x) - f(x)}{\delta x}.$

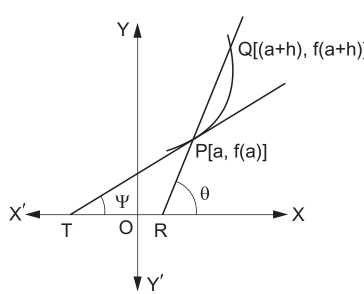
DERIVATIVE AT A POINT *The value of $f'(x)$, obtained by putting $x = a$, is called the derivative of $f(x)$ at $x = a$, and it is denoted by $f'(a)$ or $\left\{\dfrac{dy}{dx}\right\}_{x=a}$.*

If $f'(a)$ exists, we say that $f(x)$ is *differentiable* at $x = a$.

If $f'(x)$ exists for every value of x in the domain of the function, we say that $f(x)$ is differentiable.

GEOMETRICAL SIGNIFICANCE OF A DERIVATIVE

Let $y = f(x)$ be a continuous function. Then, we draw its graph. Suppose it is a curve. Let a be a point in the domain of the given function. Let $P[a, f(a)]$ be a point on this curve, and let $Q[a + h, f(a + h)]$ be some neighbouring point on it.



Slope of chord $PQ = \dfrac{f(a + h) - f(a)}{(a + h - a)}$

$$= \frac{f(a+h) - f(a)}{h}.$$

Let the point Q move along the curve such that $Q \to P$.

As Q comes closer and closer to P along the curve, the chord QP approaches the tangent at P.

This happens when $h \to 0$.

$\therefore \quad \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = \lim_{Q \to P}$ (slope of chord PQ)

or $\quad f'(a) =$ the slope of the tangent at P.

Hence, *the derivative of $f(x)$ at $x = a$ is the slope of the tangent to the curve $y = f(x)$ at the point* $[a, f(a)]$.

PHYSICAL SIGNIFICANCE OF A DERIVATIVE Let $s = f(t)$ be a function representing the distance travelled by a particle moving in a straight line in time t.

Let $(s + \delta s)$ be the distance travelled by it in time $(t + \delta t)$.

$$\therefore \quad s + \delta s = f(t + \delta t).$$

On subtraction, we get $\delta s = f(t + \delta t) - f(t)$.

$$\therefore \quad \text{average velocity} = \frac{\delta s}{\delta t} = \frac{f(t + \delta t) - f(t)}{\delta t}.$$

Now, when $\delta t \to 0$, the average velocity becomes the instantaneous velocity.

$$\therefore \quad \text{velocity at time } t = \lim_{\delta t \to 0} \frac{\delta s}{\delta t} = \lim_{\delta t \to 0} \frac{f(t + \delta t) - f(t)}{\delta t} = f'(t).$$

DIFFERENTIATION FROM THE FIRST PRINCIPLE Obtaining the derivative of a given function by using the definition is called *differentiation from the first principle* or *ab initio* or *by delta method.*

Some Important Derivatives Using the First Principle

THEOREM 1 *From the first principle, we have*

$$\frac{d}{dx}(x^n) = nx^{n-1}, \text{ where } n \text{ is a fixed number, integer or rational.}$$

PROOF Let $y = x^n$. … (i)

Let δy be an increment in y, corresponding to an increment δx in x.

Then, $\quad y + \delta y = (x + \delta x)^n$. … (ii)

On subtracting (i) from (ii), we get $\quad \delta y = (x + \delta x)^n - x^n$

$$\text{or} \quad \frac{\delta y}{\delta x} = \frac{(x + \delta x)^n - x^n}{\delta x}.$$

$$\therefore \quad \frac{dy}{dx} = \lim_{\delta x \to 0} \frac{\delta y}{\delta x}$$

$$= \lim_{(x+\delta x) \to x} \frac{(x+\delta x)^n - x^n}{(x+\delta x) - x} \quad [\because \ \delta x \to 0 \text{ means } (x+\delta x) \to x]$$

$$= nx^{n-1} \qquad \left[\because \ \lim_{x \to a} \frac{x^n - a^n}{x - a} = na^{n-1}\right].$$

Thus, $\dfrac{dy}{dx} = nx^{n-1}$, i.e., $\dfrac{d}{dx}(x^n) = nx^{n-1}$.

EXAMPLE *Find the derivatives of*

(i) x^9 *(ii)* x^{-3} *(iii)* $\sqrt[3]{x}$ *(iv)* $\dfrac{1}{\sqrt{x}}$

SOLUTION We know that $\dfrac{d}{dx}(x^n) = nx^{n-1}$. So, we have

(i) $\dfrac{d}{dx}(x^9) = 9 \cdot x^{(9-1)} = 9x^8$.

(ii) $\dfrac{d}{dx}(x^{-3}) = (-3) \cdot x^{(-3-1)} = -3x^{-4} = \dfrac{-3}{x^4}$.

(iii) $\dfrac{d}{dx}(\sqrt[3]{x}) = \dfrac{d}{dx}(x^{1/3}) = \dfrac{1}{3} x^{\left(\frac{1}{3}-1\right)} = \dfrac{1}{3} x^{-2/3}$.

(iv) $\dfrac{d}{dx}\left(\dfrac{1}{\sqrt{x}}\right) = \dfrac{d}{dx}(x^{-1/2}) = -\dfrac{1}{2} \cdot x^{\left(-\frac{1}{2}-1\right)} = -\dfrac{1}{2} x^{-3/2}$.

THEOREM 2 *From the first principle, we have* $\dfrac{d}{dx}(e^x) = e^x$.

PROOF Let $y = e^x$ and $y + \delta y = e^{x+\delta x}$. Then, $\dfrac{\delta y}{\delta x} = \dfrac{e^{(x+\delta x)} - e^x}{\delta x}$.

$$\therefore \quad \frac{dy}{dx} = \lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \lim_{\delta x \to 0} \frac{e^{(x+\delta x)} - e^x}{\delta x} = \lim_{\delta x \to 0} e^x \left(\frac{e^{\delta x} - 1}{\delta x}\right)$$

$$= e^x \cdot \lim_{\delta x \to 0} \left(\frac{e^{\delta x} - 1}{\delta x}\right) = e^x \qquad \left[\because \ \lim_{\delta x \to 0} \frac{e^{\delta x} - 1}{\delta x} = 1\right].$$

Thus, $\dfrac{dy}{dx} = e^x$, i.e., $\dfrac{d}{dx}(e^x) = e^x$.

THEOREM 3 *From the first principle, we have* $\dfrac{d}{dx}(\sin x) = \cos x$.

PROOF Let $y = \sin x$ and $y + \delta y = \sin(x + \delta x)$.

Then, $\dfrac{\delta y}{\delta x} = \dfrac{\sin(x+\delta x) - \sin x}{\delta x}$.

$$\therefore \quad \frac{dy}{dx} = \lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \lim_{\delta x \to 0} \frac{\sin(x+\delta x) - \sin x}{\delta x}$$

$$= \lim_{\delta x \to 0} \frac{2\cos\left(x + \frac{\delta x}{2}\right) \sin\left(\frac{\delta x}{2}\right)}{\delta x}$$

$$\left[\because \sin C - \sin D = 2\cos\left(\frac{C+D}{2}\right)\sin\left(\frac{C-D}{2}\right)\right]$$

$$= \lim_{\delta x \to 0} \cos\left(x + \frac{\delta x}{2}\right) \cdot \lim_{\left(\frac{\delta x}{2}\right) \to 0} \frac{\sin\left(\frac{\delta x}{2}\right)}{\left(\frac{\delta x}{2}\right)} = (\cos x \times 1) = \cos x.$$

$$\therefore \frac{dy}{dx} = \cos x, \quad \text{i.e.,} \quad \frac{d}{dx}(\sin x) = \cos x.$$

<u>THEOREM 4</u> *From the first principle, we have* $\frac{d}{dx}(\cos x) = -\sin x.$

PROOF Let $y = \cos x$ and $y + \delta y = \cos(x + \delta x)$.

Then, $\frac{\delta y}{\delta x} = \frac{\cos(x + \delta x) - \cos x}{\delta x}$.

$$\therefore \frac{dy}{dx} = \lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \lim_{\delta x \to 0} \frac{\cos(x + \delta x) - \cos x}{\delta x}$$

$$= \lim_{\delta x \to 0} \frac{-2\sin\left(x + \frac{\delta x}{2}\right) \cdot \sin\left(\frac{\delta x}{2}\right)}{\delta x}$$

$$\left[\because \cos C - \cos D = -2\sin\left(\frac{C+D}{2}\right)\sin\left(\frac{C-D}{2}\right)\right]$$

$$= -\lim_{\delta x \to 0} \sin\left(x + \frac{\delta x}{2}\right) \cdot \lim_{\delta x \to 0} \frac{\sin(\delta x/2)}{(\delta x/2)} = (-\sin x) \times 1 = -\sin x.$$

$$\therefore \frac{dy}{dx} = -\sin x, \quad \text{i.e.,} \quad \frac{d}{dx}(\cos x) = -\sin x.$$

<u>THEOREM 5</u> *From the first principle, we have* $\frac{d}{dx}(\tan x) = \sec^2 x.$

PROOF Let $y = \tan x$ and $y + \delta y = \tan(x + \delta x)$.

Then, $\frac{\delta y}{\delta x} = \frac{\tan(x + \delta x) - \tan x}{\delta x}$.

$$\therefore \frac{dy}{dx} = \lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \lim_{\delta x \to 0} \frac{\tan(x + \delta x) - \tan x}{\delta x} = \lim_{\delta x \to 0} \frac{\left\{\frac{\sin(x + \delta x)}{\cos(x + \delta x)} - \frac{\sin x}{\cos x}\right\}}{\delta x}$$

$$= \lim_{\delta x \to 0} \frac{\sin(x + \delta x)\cos x - \cos(x + \delta x)\sin x}{\delta x \cdot \cos(x + \delta x) \cdot \cos x}$$

$$= \frac{1}{\cos x} \cdot \lim_{\delta x \to 0} \frac{\sin \delta x}{\delta x \cdot \cos(x + \delta x)}$$

$$[\because \sin A \cos B - \cos A \sin B = \sin(A - B)]$$

$$= \frac{1}{\cos x} \cdot \lim_{\delta x \to 0} \frac{\sin \delta x}{\delta x} \cdot \lim_{\delta x \to 0} \frac{1}{\cos(x + \delta x)}$$

$$= \left(\frac{1}{\cos x} \times 1 \times \frac{1}{\cos x} \right) = \frac{1}{\cos^2 x} = \sec^2 x.$$

$$\therefore \frac{dy}{dx} = \sec^2 x, \quad \text{i.e.,} \quad \frac{d}{dx}(\tan x) = \sec^2 x.$$

THEOREM 6 *From the first principle, we have* $\frac{d}{dx}(\sec x) = \sec x \tan x$.

PROOF Let $y = \sec x$ and $y + \delta y = \sec(x + \delta x)$.

Then, $\frac{\delta y}{\delta x} = \frac{\sec(x + \delta x) - \sec x}{\delta x}$

$$\therefore \frac{dy}{dx} = \lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \lim_{\delta x \to 0} \frac{\sec(x + \delta x) - \sec x}{\delta x}$$

$$= \lim_{\delta x \to 0} \frac{\left\{ \frac{1}{\cos(x + \delta x)} - \frac{1}{\cos x} \right\}}{\delta x} = \lim_{\delta x \to 0} \frac{\cos x - \cos(x + \delta x)}{\delta x \cdot \cos(x + \delta x) \cdot \cos x}$$

$$= \lim_{\delta x \to 0} \frac{2 \sin\left(x + \frac{\delta x}{2}\right) \sin\left(\frac{\delta x}{2}\right)}{\cos(x + \delta x) \cdot \cos x \cdot \delta x}$$

$$\left[\because \cos C - \cos D = 2 \sin\left(\frac{C + D}{2}\right) \sin\left(\frac{D - C}{2}\right)\right]$$

$$= \frac{1}{\cos x} \cdot \lim_{\delta x \to 0} \sin\left(x + \frac{\delta x}{2}\right) \cdot \lim_{\delta x \to 0} \frac{1}{\cos(x + \delta x)} \cdot \lim_{\delta x \to 0} \frac{\sin(\delta x/2)}{(\delta x/2)}$$

$$= \left(\frac{1}{\cos x} \times \sin x \times \frac{1}{\cos x} \times 1 \right) = \sec x \tan x.$$

$$\therefore \frac{dy}{dx} = \sec x \tan x, \quad \text{i.e.,} \quad \frac{d}{dx}(\sec x) = \sec x \tan x.$$

THEOREM 7 *From the first principle, we have* $\frac{d}{dx}(\operatorname{cosec} x) = -\operatorname{cosec} x \cot x$.

PROOF Let $y = \operatorname{cosec} x$ and $y + \delta y = \operatorname{cosec}(x + \delta x)$.

Then, $\frac{\delta y}{\delta x} = \frac{\operatorname{cosec}(x + \delta x) - \operatorname{cosec} x}{\delta x}$.

$$\therefore \quad \frac{dy}{dx} = \lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \lim_{\delta x \to 0} \frac{\operatorname{cosec}(x + \delta x) - \operatorname{cosec} x}{\delta x}$$

$$= \lim_{\delta x \to 0} \frac{\left\{\dfrac{1}{\sin (x + \delta x)} - \dfrac{1}{\sin x}\right\}}{\delta x} = \lim_{\delta x \to 0} \frac{\sin x - \sin (x + \delta x)}{\sin (x + \delta x) \cdot \sin x \cdot \delta x}$$

$$= \lim_{\delta x \to 0} \frac{-2 \cos \left\{x + \dfrac{\delta x}{2}\right\} \cdot \sin \dfrac{\delta x}{2}}{\sin (x + \delta x) \cdot \sin x \cdot \delta x}$$

$$\left[\because \ \sin C - \sin D = 2 \cos \left(\frac{C + D}{2}\right) \sin \left(\frac{C - D}{2}\right)\right]$$

$$= -\frac{1}{\sin x} \cdot \lim_{\delta x \to 0} \cos \left(x + \frac{\delta x}{2}\right) \cdot \lim_{\delta x \to 0} \frac{\sin (\delta x/2)}{(\delta x/2)} \cdot \lim_{\delta x \to 0} \frac{1}{\sin (x + \delta x)}$$

$$= \left(-\frac{1}{\sin x} \times \cos x \times 1 \times \frac{1}{\sin x}\right) = -\operatorname{cosec} x \cot x.$$

$$\therefore \quad \frac{dy}{dx} = -\operatorname{cosec} x \cot x, \quad \text{i.e.,} \quad \frac{d}{dx} (\operatorname{cosec} x) = -\operatorname{cosec} x \ \cot x.$$

THEOREM 8 *From the first principle, we have* $\dfrac{d}{dx} (\cot x) = -\operatorname{cosec}^2 x.$

PROOF Let $y = \cot x$ and $y + \delta y = \cot (x + \delta x)$.

Then, $\dfrac{\delta y}{\delta x} = \dfrac{\cot (x + \delta x) - \cot x}{\delta x}.$

$$\therefore \quad \frac{dy}{dx} = \lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \lim_{\delta x \to 0} \frac{\cot (x + \delta x) - \cot x}{\delta x} = \lim_{\delta x \to 0} \frac{\left\{\dfrac{\cos (x + \delta x)}{\sin (x + \delta x)} - \dfrac{\cos x}{\sin x}\right\}}{\delta x}$$

$$= \lim_{\delta x \to 0} \frac{\sin x \ \cos (x + \delta x) - \cos x \ \sin (x + \delta x)}{\sin (x + \delta x) \cdot \sin x \cdot \delta x}$$

$$= \frac{1}{\sin x} \cdot \lim_{\delta x \to 0} \frac{-\sin \delta x}{\sin (x + \delta x) \cdot \delta x}$$

$$[\because \ \sin A \ \cos B - \cos A \ \sin B = \sin (A - B)]$$

$$= \frac{-1}{\sin x} \cdot \lim_{\delta x \to 0} \frac{\sin \delta x}{\delta x} \cdot \frac{1}{\lim\limits_{\delta x \to 0} \sin (x + \delta x)}$$

$$= \left\{-\frac{1}{\sin x} \cdot 1 \cdot \frac{1}{\sin x}\right\} = \frac{-1}{\sin^2 x} = -\operatorname{cosec}^2 x.$$

Thus, $\dfrac{dy}{dx} = -\operatorname{cosec}^2 x$, i.e., $\dfrac{d}{dx} (\cot x) = -\operatorname{cosec}^2 x.$

SUMMARY

We may summarise the above results as given below:

(i) $\frac{d}{dx}(x^n) = nx^{n-1}$ (ii) $\frac{d}{dx}(e^x) = e^x$

(iii) $\frac{d}{dx}(\sin x) = \cos x$ (iv) $\frac{d}{dx}(\cos x) = -\sin x$

(v) $\frac{d}{dx}(\tan x) = \sec^2 x$ (vi) $\frac{d}{dx}(\sec x) = \sec x \tan x$

(vii) $\frac{d}{dx}(\operatorname{cosec} x) = -\operatorname{cosec} x \cot x$ (viii) $\frac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x.$

Some more Results on Differentiation

THEOREM 9 *The derivative of a constant function is zero, i.e.,* $\frac{d}{dx}(c) = 0.$

PROOF Let $f(x) = c$ be a constant function. Then, for any change δx in x, there is no change in the value of the function.

$\therefore$ $f(x) = c$ and $f(x + \delta x) = c.$

So, from first principle, we have

$$\frac{d}{dx}(c) = f'(x) = \lim_{\delta x \to 0} \frac{f(x + \delta x) - f(x)}{\delta x} = \lim_{\delta x \to 0} \left(\frac{c - c}{\delta x}\right) = 0.$$

Hence, $\frac{d}{dx}(c) = 0$, where c is a constant.

THEOREM 10 *Let $f(x)$ be a differentiable function and let c be a fixed real number. Then, $c \cdot f(x)$ is also differentiable and* $\frac{d}{dx}\{c \cdot f(x)\} = c \cdot \frac{d}{dx}\{f(x)\}.$

PROOF Let $y = c \cdot f(x)$ and $y + \delta y = c \cdot f(x + \delta x).$

Then, $\delta y = c \cdot f(x + \delta x) - c \cdot f(x).$

$$\therefore \quad \frac{\delta y}{\delta x} = \frac{c \cdot f(x + \delta x) - c \cdot f(x)}{\delta x}.$$

So,
$$\frac{dy}{dx} = \lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \lim_{\delta x \to 0} \frac{c \cdot f(x + \delta x) - c \cdot f(x)}{\delta x}$$
$$= c \cdot \lim_{\delta x \to 0} \frac{f(x + \delta x) - f(x)}{\delta x} = c \cdot \frac{d}{dx}\{f(x)\}.$$

Thus, $\frac{dy}{dx} = c \cdot \frac{d}{dx}\{f(x)\}$, i.e., $\frac{d}{dx}\{c \cdot f(x)\} = c \cdot \frac{d}{dx}\{f(x)\}.$

Now, $f(x)$ being differentiable, it follows that $\frac{d}{dx}\{f(x)\}$ exists and therefore, $c \cdot \frac{d}{dx}\{f(x)\}$ exists.

Consequently, $\dfrac{d}{dx}\{c \cdot f(x)\}$ exists.

This shows that $c \cdot f(x)$ is differentiable.

EXAMPLE 1 *Find the derivative of*
(i) $8x^3$ *(ii)* $6\sqrt{x}$ *(iii)* $5e^x$ *(iv)* 9×2^x

SOLUTION (i) $\dfrac{d}{dx}(8x^3) = 8 \cdot \dfrac{d}{dx}(x^3) = 8 \times 3x^2 = 24x^2.$

(ii) $\dfrac{d}{dx}(6\sqrt{x}) = 6 \cdot \dfrac{d}{dx}(x^{1/2}) = 6 \cdot \dfrac{1}{2} x^{-1/2} = \dfrac{3}{\sqrt{x}} \cdot$

(iii) $\dfrac{d}{dx}(5e^x) = 5 \cdot \dfrac{d}{dx}(e^x) = 5e^x.$

(iv) $\dfrac{d}{dx}(9 \cdot 2^x) = 9 \cdot \dfrac{d}{dx}(2^x) = 9 \times 2^x (\log 2).$

THEOREM 11 *If $f(x)$ and $g(x)$ are differentiable functions then $f(x) + g(x)$ is also differentiable, and*

$$\frac{d}{dx}\{f(x) + g(x)\} = \frac{d}{dx}\{f(x)\} + \frac{d}{dx}\{g(x)\}.$$

PROOF Let $y = f(x) + g(x)$ and let $y + \delta y = f(x + \delta x) + g(x + \delta x)$.

Then, $\dfrac{\delta y}{\delta x} = \dfrac{\{f(x + \delta x) + g(x + \delta x)\} - \{f(x) + g(x)\}}{\delta x}$

$$\therefore \quad \frac{dy}{dx} = \lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \lim_{\delta x \to 0} \frac{\{f(x + \delta x) + g(x + \delta x)\} - \{f(x) + g(x)\}}{\delta x}$$

$$= \lim_{\delta x \to 0} \left\{ \frac{f(x + \delta x) - f(x)}{\delta x} + \frac{g(x + \delta x) - g(x)}{\delta x} \right\}$$

$$= \lim_{\delta x \to 0} \left\{ \frac{f(x + \delta x) - f(x)}{\delta x} \right\} + \lim_{\delta x \to 0} \left\{ \frac{g(x + \delta x) - g(x)}{\delta x} \right\}$$

$$= \frac{d}{dx}\{f(x)\} + \frac{d}{dx}\{g(x)\}.$$

$$\therefore \quad \frac{d}{dx}\{f(x) + g(x)\} = \frac{d}{dx}\{f(x)\} + \frac{d}{dx}\{g(x)\}.$$

Since $f(x)$ and $g(x)$ are differentiable, it follows that $\dfrac{d}{dx}\{f(x)\}$ as well as $\dfrac{d}{dx}\{g(x)\}$ exists.

Consequently, $\dfrac{d}{dx}\{f(x)\} + \dfrac{d}{dx}\{g(x)\}$ exists.

So, by the above result, $\dfrac{d}{dx}\{f(x) + g(x)\}$ exists.

Hence, $\{f(x) + g(x)\}$ is differentiable.

REMARK We may extend the above result for a finite number of differentiable functions. Thus, we have

$$\frac{d}{dx}[f_1(x)+f_2(x)+\ldots+f_n(x)]=\frac{d}{dx}\{f_1(x)\}+\frac{d}{dx}\{f_2(x)\}+\ldots+\frac{d}{dx}\{f_n(x)\}.$$

EXAMPLE 2 *Find the derivative of* $(x^3+e^x+3^x+\cot x)$ *with respect to* x.

SOLUTION We have

$$\frac{d}{dx}(x^3+e^x+3^x+\cot x)=\frac{d}{dx}(x^3)+\frac{d}{dx}(e^x)+\frac{d}{dx}(3^x)+\frac{d}{dx}(\cot x)$$
$$=3x^2+e^x+3^x(\log 3)-\operatorname{cosec}^2 x.$$

EXAMPLE 3 *Find the derivative of* $\left(9x^2+\frac{3}{x}+5\sin x\right)$ *with respect to* x.

SOLUTION We have $\frac{d}{dx}\left(9x^2+\frac{3}{x}+5\sin x\right)$

$$=9\cdot\frac{d}{dx}(x^2)+3\cdot\frac{d}{dx}(x^{-1})+5\cdot\frac{d}{dx}(\sin x)$$
$$=9\times 2x+3\cdot(-1)\,x^{-2}+5\cos x=18x-\frac{3}{x^2}+5\cos x.$$

THEOREM 12 *If* $f(x)$ *and* $g(x)$ *be differentiable functions then* $\{f(x)-g(x)\}$ *is also differentiable, and*

$$\frac{d}{dx}\{f(x)-g(x)\}=\frac{d}{dx}\{f(x)\}-\frac{d}{dx}\{g(x)\}.$$

PROOF We have $\frac{d}{dx}\{f(x)-g(x)\}=\frac{d}{dx}\{f(x)+(-1)\cdot g(x)\}$

$$=\frac{d}{dx}\{f(x)\}+\frac{d}{dx}\{(-1)\cdot g(x)\}$$
$$=\frac{d}{dx}\{f(x)\}+(-1)\cdot\frac{d}{dx}\{g(x)\}=\frac{d}{dx}\{f(x)\}-\frac{d}{dx}\{g(x)\}.$$

$$\therefore\quad \frac{d}{dx}\{f(x)-g(x)\}=\frac{d}{dx}\{f(x)\}-\frac{d}{dx}\{g(x)\}.$$

Now, $f(x)$ and $g(x)$ being differentiable, it follows that $\frac{d}{dx}\{f(x)\}$ and $\frac{d}{dx}\{g(x)\}$ both exist.

Consequently, $\frac{d}{dx}\{f(x)\}-\frac{d}{dx}\{g(x)\}$ exists.

So, by the above result, $\frac{d}{dx}\{f(x)-g(x)\}$ exists.

Hence, $\{f(x)-g(x)\}$ is differentiable.

SOLVED EXAMPLES

EXAMPLE 1 *Differentiate the following functions with respect to x:*

$$\left(x^2 + \frac{4}{x^2} - \frac{2}{3}\tan x + 6e\right)$$

SOLUTION $\frac{d}{dx}\left(x^2 + \frac{4}{x^2} - \frac{2}{3}\tan x + 6e\right)$

$$= \frac{d}{dx}(x^2) + 4\cdot\frac{d}{dx}(x^{-2}) - \frac{2}{3}\cdot\frac{d}{dx}(\tan x) + 6\cdot\frac{d}{dx}(e)$$

$$= 2x + 4\cdot(-2)\,x^{-3} - \frac{2}{3}\sec^2 x + 6\times 0 \qquad \left[\because\ \frac{d}{dx}(e) = 0\right]$$

$$= 2x - \frac{8}{x^3} - \frac{2}{3}\sec^2 x.$$

EXAMPLE 2 *Find the derivative of* $\left\{\frac{3}{\sqrt[3]{x}} - \frac{5}{\cos x} + \frac{6}{\sin x} - \frac{2\tan x}{\sec x} + 7\right\}$.

SOLUTION We have $\frac{d}{dx}\left\{\frac{3}{\sqrt[3]{x}} - \frac{5}{\cos x} + \frac{6}{\sin x} - \frac{2\tan x}{\sec x} + 7\right\}$

$$= \frac{d}{dx}\{3x^{-1/3} - 5\sec x + 6\operatorname{cosec} x - 2\sin x + 7\}$$

$$= 3\cdot\frac{d}{dx}(x^{-1/3}) - 5\cdot\frac{d}{dx}(\sec x) + 6\cdot\frac{d}{dx}(\operatorname{cosec} x)$$

$$-2\cdot\frac{d}{dx}(\sin x) + \frac{d}{dx}(7)$$

$$= 3\cdot\left(-\frac{1}{3}\right)x^{-4/3} - 5\sec x\tan x - 6\operatorname{cosec} x\cot x - 2\cos x$$

$$= \left\{\frac{-1}{x^{4/3}} - 5\sec x\tan x - 6\operatorname{cosec} x\cot x - 2\cos x\right\}$$

EXAMPLE 3 *Differentiate the following functions:*

(i) $(x^2 - 5x + 6)(x - 3)$ *(ii)* $\left(\sqrt{x} + \frac{1}{\sqrt{x}}\right)^2$ *(iii)* $\frac{3x^2 + 2x + 5}{\sqrt{x}}$

SOLUTION (i) $\frac{d}{dx}\{(x^2 - 5x + 6)(x - 3)\}$

$$= \frac{d}{dx}(x^3 - 8x^2 + 21x - 18)$$

$$= \frac{d}{dx}(x^3) - 8\cdot\frac{d}{dx}(x^2) + 21\cdot\frac{d}{dx}(x) - \frac{d}{dx}(18)$$

$$= 3x^2 - 8\times 2x + 21\times 1 - 0 = (3x^2 - 16x + 21).$$

(ii) $$\frac{d}{dx}\left\{\left(\sqrt{x}+\frac{1}{\sqrt{x}}\right)^2\right\}=\frac{d}{dx}\left\{\left(x+\frac{1}{x}+2\right)\right\}$$
$$=\frac{d}{dx}(x)+\frac{d}{dx}(x^{-1})+\frac{d}{dx}(2)$$
$$=1+(-1)x^{-2}+0=\left(1-\frac{1}{x^2}\right).$$

(iii) $$\frac{d}{dx}\left\{\frac{3x^2+2x+5}{\sqrt{x}}\right\}=\frac{d}{dx}\{3x^{3/2}+2x^{1/2}+5x^{-1/2}\}$$
[on dividing each term by $\sqrt{x}$]
$$=3\cdot\frac{d}{dx}(x^{3/2})+2\cdot\frac{d}{dx}(x^{1/2})+5\cdot\frac{d}{dx}(x^{-1/2})$$
$$=3\cdot\frac{3}{2}x^{1/2}+2\cdot\frac{1}{2}x^{-1/2}+5\cdot\left(-\frac{1}{2}\right)x^{-3/2}$$
$$=\frac{9}{2}\sqrt{x}+\frac{1}{\sqrt{x}}-\frac{5}{2}x^{-3/2}.$$

EXAMPLE 4 *If* $y=\sqrt{\dfrac{1-\cos 2x}{1+\cos 2x}}$, *find* $\dfrac{dy}{dx}$.

SOLUTION $$y=\sqrt{\frac{1-\cos 2x}{1+\cos 2x}}=\sqrt{\frac{2\sin^2 x}{2\cos^2 x}}=\tan x.$$
$$\therefore\quad \frac{dy}{dx}=\frac{d}{dx}(\tan x)=\sec^2 x.$$

EXAMPLE 5 *If* $y=\left(1+x+\dfrac{x^2}{2!}+\dfrac{x^3}{3!}+\ldots\infty\right)\cdot$ *show that* $\dfrac{dy}{dx}=y$.

SOLUTION We have, $y=e^x$.
$$\therefore\quad \frac{dy}{dx}=\frac{d}{dx}(e^x)=e^x=y.$$

EXAMPLE 6 *If* $u=3t^4-5t^3+2t^2-18t+4$, *find* $\dfrac{du}{dt}$ *at* $t=1$.

SOLUTION $$\frac{du}{dt}=\frac{d}{dt}(3t^4-5t^3+2t^2-18t+4)$$
$$=3\cdot\frac{d}{dt}(t^4)-5\cdot\frac{d}{dt}(t^3)+2\cdot\frac{d}{dt}(t^2)-18\cdot\frac{d}{dt}(t)+\frac{d}{dt}(4)$$
$$=3\times 4t^3-5\times 3t^2+2\times 2t-18\times 1+0$$
$$=12t^3-15t^2+4t-18.$$
$$\therefore\quad \left(\frac{du}{dt}\right)_{t=1}=(12\times 1^3-15\times 1^2+4\times 1-18)$$
$$=(12-15+4-18)=-17.$$

EXERCISE 28A

Differentiate the following functions:

1. (i) x^{-3} (ii) $\sqrt[3]{x}$ **2.** (i) $\dfrac{1}{x}$ (ii) $\dfrac{1}{\sqrt{x}}$ (iii) $\dfrac{1}{\sqrt[3]{x}}$

3. (i) $3x^{-5}$ (ii) $\dfrac{1}{5x}$ (iii) $6\cdot\sqrt[3]{x^2}$

4. (i) $6x^5 + 4x^3 - 3x^2 + 2x - 7$ (ii) $5x^{-3/2} + \dfrac{4}{\sqrt{x}} + \sqrt{x} - \dfrac{7}{x}$

(iii) $ax^3 + bx^2 + cx + d$, where a, b, c, d are constants

5. (i) $4x^3 + 3\cdot 2^x + 6\cdot\sqrt[8]{x^{-4}} + 5\cot x$

(ii) $\dfrac{x}{3} - \dfrac{3}{x} + \sqrt{x} - \dfrac{1}{\sqrt{x}} + x^2 - 2^x + 6x^{-2/3} - \dfrac{2}{3}x^6$

6. (i) $4\cot x - \dfrac{1}{2}\cos x + \dfrac{2}{\cos x} - \dfrac{3}{\sin x} + \dfrac{6\cot x}{\operatorname{cosec} x} + 9$

(ii) $-5\tan x + 4\tan x\cos x - 3\cot x\sec x + 2\sec x - 13$

7. (i) $(2x+3)(3x-5)$ (ii) $x(1+x)^3$ (iii) $\left(\sqrt{x} + \dfrac{1}{x}\right)\left(x - \dfrac{1}{\sqrt{x}}\right)$

(iv) $\left(x - \dfrac{1}{x}\right)^2$ (v) $\left(x^2 + \dfrac{1}{x^2}\right)^3$ (vi) $(2x^2 + 5x - 1)(x - 3)$

8. (i) $\dfrac{3x^2 + 4x - 5}{x}$ (ii) $\dfrac{(x^3+1)(x-2)}{x^2}$ (iii) $\dfrac{x-4}{2\sqrt{x}}$

(iv) $\dfrac{(1+x)\sqrt{x}}{\sqrt[3]{x}}$ (v) $\dfrac{ax^2 + bx + c}{\sqrt{x}}$ (vi) $\dfrac{a + b\cos x}{\sin x}$

9. (i) If $y = 6x^5 - 4x^4 - 2x^2 + 5x - 9$, find $\dfrac{dy}{dx}$ at $x = -1$.

(ii) If $y = (\sin x + \tan x)$, find $\dfrac{dy}{dx}$ at $x = \dfrac{\pi}{3}\cdot$

(iii) If $y = \dfrac{(2 - 3\cos x)}{\sin x}$, find $\dfrac{dy}{dx}$ at $x = \dfrac{\pi}{4}\cdot$

10. If $y = \left(\sqrt{x} + \dfrac{1}{\sqrt{x}}\right)$, show that $2x\cdot\dfrac{dy}{dx} + y = 2\sqrt{x}$.

11. If $y = \left(\sqrt{\dfrac{x}{a}} + \sqrt{\dfrac{a}{x}}\right)$, prove that $(2xy)\left(\dfrac{dy}{dx}\right) = \left(\dfrac{x}{a} - \dfrac{a}{x}\right)\cdot$

12. If $y = \sqrt{\dfrac{1 + \cos 2x}{1 - \cos 2x}}$, find $\dfrac{dy}{dx}\cdot$

13. $y = \dfrac{1 - \tan^2(x/2)}{1 + \tan^2(x/2)}$, find $\dfrac{dy}{dx}$.

Hint $\dfrac{1 - \tan^2(x/2)}{1 + \tan^2(x/2)} = \cos x.$

ANSWERS (EXERCISE 28A)

1. (i) $\dfrac{-3}{x^4}$ (ii) $\dfrac{1}{3}x^{-2/3}$ **2.** (i) $\dfrac{-1}{x^2}$ (ii) $-\dfrac{1}{2}x^{-3/2}$ (iii) $-\dfrac{1}{3}x^{-4/3}$

3. (i) $\dfrac{-15}{x^6}$ (ii) $\dfrac{-1}{5x^2}$ (iii) $4x^{-1/3}$

4. (i) $30x^4 + 12x^2 - 6x + 2$ (ii) $\dfrac{-15}{2}x^{-5/2} - 2x^{-3/2} + \dfrac{1}{2\sqrt{x}} + \dfrac{7}{x^2}$

(iii) $3ax^2 + 2bx + c$

5. (i) $12x^2 + (3\log 2)\cdot 2^x - 3x^{-3/2} - 5\operatorname{cosec}^2 x$

(ii) $\dfrac{1}{3} + \dfrac{3}{x^2} + \dfrac{1}{2\sqrt{x}} + \dfrac{1}{2}x^{-3/2} + 2x - 2^x(\log 2) - 4x^{-5/3} - 4x^5$

6. (i) $-4\operatorname{cosec}^2 x + \dfrac{1}{2}\sin x + 2\sec x\tan x + 3\operatorname{cosec} x\cot x - 6\sin x$

(ii) $-5\sec^2 x + 4\cos x + 3\operatorname{cosec} x\cot x + 2\sec x\tan x$

7. (i) $12x - 1$ (ii) $4x^3 + 9x^2 + 6x + 1$ (iii) $\dfrac{3}{2}(x^{1/2} + x^{-5/2})$

(iv) $2\left(x - \dfrac{1}{x^3}\right)$ (v) $6x^5 - \dfrac{6}{x^7} + 6x - \dfrac{6}{x^3}$ (vi) $6x^2 - 2x - 16$

8. (i) $\left(3 + \dfrac{5}{x^2}\right)$ (ii) $3x^2 - 2 - \dfrac{1}{x^2} + \dfrac{4}{x^3}$

(iii) $\dfrac{1}{4\sqrt{x}} + \dfrac{1}{x\sqrt{x}}$ (iv) $\dfrac{1}{6}x^{-5/6} + \dfrac{7}{6}x^{1/6}$

(v) $\dfrac{3}{2}a\sqrt{x} + \dfrac{b}{2\sqrt{x}} - \dfrac{c}{2}x^{-3/2}$ (vi) $-\operatorname{cosec} x(a\cot x + b\operatorname{cosec} x)$

9. (i) 55 (ii) $9/2$ (iii) $2(3 - \sqrt{2})$ **12.** $-\operatorname{cosec}^2 x$ **13.** $-\sin x$

Some Derivatives from the First Principle

SOLVED EXAMPLES

EXAMPLE 1 *Differentiate x^6 from the first principle.*

SOLUTION Let $y = x^6$.

Let δy be an increment in y, corresponding to an increment δx in x.

Then, $y + \delta y = (x + \delta x)^6$

$$\Rightarrow \quad \delta y = (x + \delta x)^6 - x^6$$

$$\Rightarrow \quad \frac{\delta y}{\delta x} = \frac{(x + \delta x)^6 - x^6}{\delta x}$$

$$\Rightarrow \quad \frac{dy}{dx} = \lim_{\delta x \to 0} \frac{\delta y}{\delta x}$$

$$= \lim_{\delta x \to 0} \frac{(x + \delta x)^6 - x^6}{\delta x}$$

$$= \lim_{(x + \delta x) \to 0} \frac{(x + \delta x)^6 - x^6}{(x + \delta x) - x} \quad [\because (\delta x \to 0) \Rightarrow (x + \delta x) \to x]$$

$$= 6x^{(6-1)} = 6x^5 \quad \left[\because \lim_{x \to a} \frac{x^n - a^n}{x - a} = na^{n-1}\right].$$

Hence, $\frac{d}{dx}(x^6) = 6x^5$.

EXAMPLE 2 *Differentiate e^{3x} from the first principle.*

SOLUTION Let $y = e^{3x}$.

Let δy be an increment in y, corresponding to an increment δx in x.

Then, $y + \delta y = e^{3(x + \delta x)}$

$$\Rightarrow \quad \delta y = e^{3(x + \delta x)} - e^{3x}$$

$$\Rightarrow \quad \frac{\delta y}{\delta x} = \frac{e^{3(x + \delta x)} - e^{3x}}{\delta x}$$

$$\Rightarrow \quad \frac{dy}{dx} = \lim_{\delta x \to 0} \frac{\delta y}{\delta x}$$

$$= \lim_{\delta x \to 0} \frac{e^{3(x + \delta x)} - e^{3x}}{\delta x}$$

$$= \lim_{\delta x \to 0} 3e^{3x} \cdot \left(\frac{e^{3\delta x} - 1}{3\delta x}\right)$$

$$= 3e^{3x} \cdot \lim_{z \to 0} \left(\frac{e^z - 1}{z} \right), \text{ where } z = 3\delta x$$

$$= (3e^{3x} \times 1) = 3e^{3x} \qquad \left[\because \lim_{z \to 0} \left(\frac{e^z - 1}{z} \right) = 1 \right].$$

Hence, $\frac{d}{dx}(e^{3x}) = 3e^{3x}$.

EXAMPLE 3 *Find the derivative of each of the following from the first principle.*

(i) $x^2 + 3x + 5$ *(ii)* $x^3 + 4x^2 + 3x + 2$

SOLUTION (i) Let $y = x^2 + 3x + 5$.

Let δy be an increment in y, corresponding to an increment δx in x.

Then, $y + \delta y = (x + \delta x)^2 + 3(x + \delta x) + 5$

$$\Rightarrow \delta y = [(x + \delta x)^2 + 3(x + \delta x) + 5] - (x^2 + 3x + 5)$$

$$\Rightarrow \frac{\delta y}{\delta x} = \frac{[(x + \delta x)^2 + 3(x + \delta x) + 5] - (x^2 + 3x + 5)}{\delta x}$$

$$\Rightarrow \frac{dy}{dx} = \lim_{\delta x \to 0} \frac{\delta y}{\delta x}$$

$$= \lim_{\delta x \to 0} \frac{[(x + \delta x)^2 + 3(x + \delta x) + 5] - (x^2 + 3x + 5)}{\delta x}$$

$$= \lim_{\delta x \to 0} \frac{(\delta x)^2 + 2x \cdot \delta x + 3\delta x}{\delta x} = \lim_{\delta x \to 0} [(\delta x) + 2x + 3] = (2x + 3).$$

$$\therefore \quad \frac{d}{dx}(x^2 + 3x + 5) = (2x + 3).$$

(ii) Let $y = x^3 + 4x^2 + 3x + 2$.

Let δy be an increment in y, corresponding to an increment δx in x.

Then, $y + \delta y = (x + \delta x)^3 + 4(x + \delta x)^2 + 3(x + \delta x) + 2$

$$\Rightarrow \delta y = [(x + \delta x)^3 + 4(x + \delta x)^2 + 3(x + \delta x) + 2] - (x^3 + 4x^2 + 3x + 2)$$

$$\Rightarrow \frac{\delta y}{\delta x} = \frac{[(x + \delta x)^3 + 4(x + \delta x)^2 + 3(x + \delta x) + 2] - (x^3 + 4x^2 + 3x + 2)}{\delta x}$$

$$\Rightarrow \frac{dy}{dx} = \lim_{\delta x \to 0} \frac{\delta y}{\delta x}$$

$$= \lim_{\delta x \to 0} \frac{[(x + \delta x)^3 + 4(x + \delta x)^2 + 3(x + \delta x) + 2] - (x^3 + 4x^2 + 3x + 2)}{\delta x}$$

$$= \lim_{\delta x \to 0} \frac{(\delta x)^3 + 3x \cdot \delta x\,(x + \delta x) + 4(\delta x)^2 + 8x \cdot \delta x + 3\delta x}{\delta x}$$

$$= \lim_{\delta x \to 0} [(\delta x)^2 + 3x^2 + 3x \cdot \delta x + 4(\delta x) + 8x + 3] = (3x^2 + 8x + 3).$$

$$\therefore \quad \frac{d}{dx}(x^3 + 4x^2 + 3x + 2) = 3x^2 + 8x + 3.$$

EXAMPLE 4 *Differentiate each of the following ab initio.*

(i) $(x+5)^7$ *(ii)* $\left(x - \frac{1}{x}\right)$

SOLUTION (i) Let $y = (x+5)^7$.

Let δy be an increment in y, corresponding to an increment δx in x.

Then, $y + \delta y = (x + \delta x + 5)^7$

$$\Rightarrow \delta y = (x + \delta x + 5)^7 - (x+5)^7$$

$$\Rightarrow \frac{\delta y}{\delta x} = \frac{(x + \delta x + 5)^7 - (x+5)^7}{\delta x}$$

$$\Rightarrow \frac{dy}{dx} = \lim_{\delta x \to 0} \frac{\delta y}{\delta x}$$

$$= \lim_{\delta x \to 0} \frac{(x + \delta x + 5)^7 - (x+5)^7}{\delta x}$$

$$= \lim_{(x + \delta x + 5) \to (x+5)} \frac{[(x + \delta x + 5)^7 - (x+5)^7]}{[(x + \delta x + 5) - (x+5)]}$$

$$= \lim_{u \to a} \left(\frac{u^7 - a^7}{u - a}\right), \text{ where } (x + \delta x + 5) = u \text{ and } (x+5) = a$$

$$= 7a^{(7-1)} = 7(x+5)^6 \qquad \left[\because \lim_{u \to a} \left(\frac{u^n - a^n}{u - a}\right) = na^{n-1}\right].$$

Hence, $\frac{d}{dx}(x+5)^7 = 7(x+5)^6$.

(ii) Let $y = \left(x - \frac{1}{x}\right)$.

Let δy be an increment in y, corresponding to an increment δx in x.

Then, $y + \delta y = \left[(x + \delta x) - \frac{1}{(x + \delta x)}\right]$

$$\Rightarrow \delta y = \left[(x + \delta x) - \frac{1}{(x + \delta x)}\right] - \left(x - \frac{1}{x}\right)$$

$$\Rightarrow \delta y = \delta x + \left\{\frac{1}{x} - \frac{1}{(x+\delta x)}\right\}$$

$$\Rightarrow \frac{\delta y}{\delta x} = \frac{\delta x + \left\{\frac{(x+\delta x) - x}{x(x+\delta x)}\right\}}{\delta x}$$

$$\Rightarrow \frac{dy}{dx} = \lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \lim_{\delta x \to 0} \frac{\left\{\delta x + \frac{\delta x}{x(x+\delta x)}\right\}}{\delta x}$$

$$= \lim_{\delta x \to 0} \left[1 + \frac{1}{x(x+\delta x)}\right] = \left(1 + \frac{1}{x^2}\right).$$

Hence, $\frac{d}{dx}\left(x - \frac{1}{x}\right) = \left(1 + \frac{1}{x^2}\right).$

EXAMPLE 5 *Find the derivative of each of the following from the first principle.*

(i) $\sin 2x$ *(ii)* $\cos 3x$ *(iii)* $\tan 5x$

SOLUTION (i) Let $y = \sin 2x$.

Let δy be an increment in y, corresponding to an increment δx in x.

Then, $y + \delta y = \sin 2(x + \delta x)$

$$\Rightarrow \delta y = \sin(2x + 2\delta x) - \sin 2x$$

$$\Rightarrow \frac{\delta y}{\delta x} = \frac{\sin(2x + 2\delta x) - \sin 2x}{\delta x}$$

$$\Rightarrow \frac{dy}{dx} = \lim_{\delta x \to 0} \frac{\delta y}{\delta x}$$

$$= \lim_{\delta x \to 0} \frac{[\sin(2x + 2\delta x) - \sin 2x]}{\delta x}$$

$$= \lim_{\delta x \to 0} \frac{2\cos(2x + \delta x)\sin \delta x}{\delta x}$$

$$\left\{\because (\sin C - \sin D) = 2\cos\left(\frac{C+D}{2}\right)\sin\left(\frac{C-D}{2}\right)\right\}$$

$$= 2 \cdot \lim_{\delta x \to 0} \cos(2x + \delta x) \cdot \lim_{\delta x \to 0} \frac{\sin \delta x}{\delta x}$$

$$= (2\cos 2x \times 1) = 2\cos 2x.$$

Hence, $\frac{d}{dx}(\sin 2x) = 2\cos 2x$.

(ii) Let $y = \cos 3x$.

Let δy be an increment in y, corresponding to an increment δx in x.

Then, $y + \delta y = \cos 3(x + \delta x)$

$\Rightarrow \delta y = \cos(3x + 3\delta x) - \cos 3x$

$$\Rightarrow \frac{\delta y}{\delta x} = \frac{\cos(3x + 3\delta x) - \cos 3x}{\delta x}$$

$$\Rightarrow \frac{dy}{dx} = \lim_{\delta x \to 0} \frac{\delta y}{\delta x}$$

$$= \lim_{\delta x \to 0} \frac{[\cos(3x + 3\delta x) - \cos 3x]}{\delta x}$$

$$= \lim_{\delta x \to 0} \frac{-2\sin\left(3x + \frac{3}{2}\delta x\right)\sin\left(\frac{3}{2}\delta x\right)}{\delta x}$$

$$\left\{\because (\cos C - \cos D) = -2\sin\left(\frac{C+D}{2}\right)\sin\left(\frac{C-D}{2}\right)\right\}$$

$$= -\lim_{\delta x \to 0} \sin\left(3x + \frac{3}{2}\delta x\right) \cdot \lim_{\delta x \to 0} \frac{3\sin\left(\frac{3}{2}\delta x\right)}{\left(\frac{3}{2}\delta x\right)}$$

$$= (-\sin 3x \times 3 \times 1) = -3\sin 3x.$$

Hence, $\frac{d}{dx}(\cos 3x) = -3\sin 3x$.

(iii) Let $y = \tan 5x$.

Let δy be an increment in y, corresponding to an increment δx in x.

Then, $y + \delta y = \tan 5(x + \delta x)$

$\Rightarrow \delta y = \tan(5x + 5\delta x) - \tan 5x$

$$\Rightarrow \frac{\delta y}{\delta x} = \frac{\tan(5x + 5\delta x) - \tan 5x}{\delta x}$$

$$\Rightarrow \frac{dy}{dx} = \lim_{\delta x \to 0} \frac{\delta y}{\delta x}$$

$$= \lim_{\delta x \to 0} \frac{\tan(5x + 5\delta x) - \tan 5x}{\delta x}$$

$$= \lim_{\delta x \to 0} \frac{\left\{\frac{\sin(5x + 5\delta x)}{\cos(5x + 5\delta x)} - \frac{\sin 5x}{\cos 5x}\right\}}{\delta x}$$

$$= \lim_{\delta x \to 0} \frac{\sin(5x + 5\delta x)\cos 5x - \cos(5x + 5\delta x)\sin 5x}{\cos(5x + 5\delta x)\cdot\cos 5x\cdot\delta x}$$

$$= \lim_{\delta x \to 0} \frac{\sin(5x + 5\delta x - 5x)}{\cos(5x + 5\delta x)\cdot\cos 5x\cdot\delta x}$$

$$[\because \sin A \cos B - \cos A \sin B = \sin(A - B)]$$

$$= \frac{1}{\cos 5x}\cdot\lim_{\delta x \to 0}\frac{1}{\cos(5x + 5\delta x)}\cdot\lim_{\delta x \to 0}\frac{\sin(5\delta x)\times 5}{(5\delta x)}$$

$$= \left[\frac{1}{\cos 5x}\times\frac{1}{\cos 5x}\times(1\times 5)\right] = \frac{5}{\cos^2 5x} = 5\sec^2 5x.$$

Hence, $\frac{d}{dx}(\tan 5x) = 5\sec^2 5x.$

EXAMPLE 6 *Differentiate* $x\sin x$ *from the first principle.*

SOLUTION Let $y = x\sin x$.

Let δy be an increment in y, corresponding to an increment δx in x.

Then, $y + \delta y = (x + \delta x)\sin(x + \delta x)$

$$\Rightarrow \quad \delta y = (x + \delta x)\sin(x + \delta x) - x\sin x$$

$$\Rightarrow \quad \frac{\delta y}{\delta x} = \frac{(x + \delta x)\sin(x + \delta x) - x\sin x}{\delta x}$$

$$\Rightarrow \quad \frac{dy}{dx} = \lim_{\delta x \to 0}\frac{\delta y}{\delta x}$$

$$= \lim_{\delta x \to 0}\frac{\{(x + \delta x)\sin(x + \delta x) - x\sin x\}}{\delta x}$$

$$= \lim_{\delta x \to 0}\left[\frac{x\{\sin(x + \delta x) - \sin x\}}{\delta x} + \sin(x + \delta x)\right]$$

$$= \lim_{\delta x \to 0}\frac{x\{\sin(x + \delta x) - \sin x\}}{\delta x} + \lim_{\delta x \to 0}\sin(x + \delta x)$$

$$= \lim_{\delta x \to 0}\left\{\frac{2x\cos\left(x + \frac{\delta x}{2}\right)\sin\left(\frac{\delta x}{2}\right)}{\delta x}\right\} + \sin x$$

$$= \left\{x\cdot\lim_{\delta x \to 0}\cos\left(x + \frac{\delta x}{2}\right)\cdot\lim_{\delta x \to 0}\frac{\sin(\delta x/2)}{(\delta x/2)}\right\} + \sin x$$

$$= (x\cdot\cos x\cdot 1) + \sin x = (x\cos x + \sin x).$$

Hence, $\frac{d}{dx}(x\sin x) = (x\cos x + \sin x).$

EXAMPLE 7 *Differentiate* $\dfrac{\sin x}{x}$ *from the first principle.* **[CBSE 2002]**

SOLUTION Let $y = \dfrac{\sin x}{x}$.

Let δy be an increment in y, corresponding to an increment δx in x.

Then, $y + \delta y = \dfrac{\sin(x + \delta x)}{(x + \delta x)}$

$$\Rightarrow \quad \delta y = \left\{\frac{\sin(x+\delta x)}{(x+\delta x)} - \frac{\sin x}{x}\right\} = \frac{\{x \sin(x+\delta x) - (x+\delta x)\sin x\}}{x(x+\delta x)}$$

$$\Rightarrow \quad \frac{\delta y}{\delta x} = \frac{\{x\sin(x+\delta x) - (x+\delta x)\sin x\}}{x(x+\delta x)\cdot\delta x}$$

$$\Rightarrow \quad \frac{dy}{dx} = \lim_{\delta x \to 0} \frac{\delta y}{\delta x}$$

$$= \lim_{\delta x\to 0} \frac{\{x\sin(x+\delta x) - (x+\delta x)\sin x\}}{x(x+\delta x)\cdot\delta x}$$

$$= \lim_{\delta x\to 0} \frac{x[\sin(x+\delta x) - \sin x] - (\delta x)\sin x}{x(x+\delta x)\cdot\delta x}$$

$$= \lim_{\delta x\to 0} \frac{x\left[2\cos\left(x + \frac{\delta x}{2}\right)\sin\left(\frac{\delta x}{2}\right)\right] - (\delta x)\sin x}{x(x+\delta x)\cdot\delta x}$$

$$= \left\{\lim_{\delta x\to 0}\cos\left(x+\frac{\delta x}{2}\right)\cdot\lim_{\delta x\to 0}\frac{\sin(\delta x/2)}{(\delta x/2)}\cdot\lim_{\delta x\to 0}\frac{1}{(x+\delta x)}\right\}$$
$$- \lim_{\delta x\to 0}\frac{\sin x}{x(x+\delta x)}$$

$$= \left(\cos x \times 1 \times \frac{1}{x}\right) - \frac{\sin x}{x^2} = \left(\frac{\cos x}{x} - \frac{\sin x}{x^2}\right) = \frac{(x\cos x - \sin x)}{x^2}.$$

Hence, $\dfrac{d}{dx}\left(\dfrac{\sin x}{x}\right) = \dfrac{(x\cos x - \sin x)}{x^2}$.

EXAMPLE 8 *Differentiate* $\cot(2x+1)$ *from the first principle.*

SOLUTION Let $y = \cot(2x+1)$.

Let δy be an increment in y, corresponding to an increment δx in x.

Then, $y + \delta y = \cot[2(x+\delta x) + 1]$

$$\Rightarrow \quad \delta y = \cot[2(x+\delta x)+1] - \cot(2x+1)$$

$$\Rightarrow \quad \frac{\delta y}{\delta x} = \frac{\cot[2(x+\delta x)+1] - \cot(2x+1)}{\delta x}$$

$$\Rightarrow \quad \frac{dy}{dx} = \lim_{\delta x\to 0}\frac{\delta y}{\delta x}$$

$$= \lim_{\delta x \to 0} \frac{\cot(2x + 2\delta x + 1) - \cot(2x + 1)}{\delta x}$$

$$= \lim_{\delta x \to 0} \frac{\left\{\dfrac{\cos(2x + 2\delta x + 1)}{\sin(2x + 2\delta x + 1)} - \dfrac{\cos(2x + 1)}{\sin(2x + 1)}\right\}}{\delta x}$$

$$= \lim_{\delta x \to 0} \frac{\sin(2x + 1)\cos(2x + 2\delta x + 1) - \cos(2x + 1)\sin(2x + 2\delta x + 1)}{\delta x \cdot \sin(2x + 2\delta x + 1) \cdot \sin(2x + 1)}$$

$$= \lim_{\delta x \to 0} \frac{\sin[(2x + 1) - (2x + 2\delta x + 1)]}{\delta x \cdot \sin(2x + 2\delta x + 1) \cdot \sin(2x + 1)}$$

$$[\because \sin A \cos B - \cos A \sin B = \sin(A - B)]$$

$$= \lim_{\delta x \to 0} \frac{\sin(-2\delta x)}{\delta x \cdot \sin(2x + 2\delta x + 1) \cdot \sin(2x + 1)}$$

$$= \frac{1}{\sin(2x+1)} \cdot \lim_{\delta x \to 0} \left\{\frac{-\sin(2\delta x)}{2\delta x} \times 2\right\} \cdot \lim_{\delta x \to 0} \frac{1}{\sin(2x + 2\delta x + 1)}$$

$$= \frac{1}{\sin(2x + 1)} \cdot (-1) \times 2 \times \frac{1}{\sin(2x + 1)} = \frac{-2}{\sin^2(2x + 1)}$$

$$= -2\operatorname{cosec}^2(2x + 1).$$

Hence, $\dfrac{d}{dx}[\cot(2x + 1)] = -2\operatorname{cosec}^2(2x + 1)$.

EXAMPLE 9 *Find the derivative of each of the following from the first principle.*

(i) $\sqrt{2x + 3}$ *(ii)* $\sqrt{4 - x}$ *(iii)* $\dfrac{1}{\sqrt{x}}$

SOLUTION (i) Let $y = \sqrt{2x + 3}$.

Let δy be an increment in y, corresponding to an increment δx in x.

Then, $y + \delta y = \sqrt{2(x + \delta x) + 3}$

$$\Rightarrow \delta y = \sqrt{2(x + \delta x) + 3} - \sqrt{2x + 3}$$

$$\Rightarrow \frac{\delta y}{\delta x} = \frac{\sqrt{2(x + \delta x) + 3} - \sqrt{2x + 3}}{\delta x}$$

$$\Rightarrow \frac{dy}{dx} = \lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \lim_{\delta x \to 0} \frac{\sqrt{2(x + \delta x) + 3} - \sqrt{2x + 3}}{\delta x}$$

$$= \lim_{\delta x \to 0} \left\{\frac{(\sqrt{2x + 2\delta x + 3} - \sqrt{2x + 3})}{\delta x} \times \frac{(\sqrt{2x + 2\delta x + 3} + \sqrt{2x + 3})}{(\sqrt{2x + 2\delta x + 3} + \sqrt{2x + 3}}\right\}$$

$$= \lim_{\delta x \to 0} \frac{\{(2x + 2\delta x + 3) - (2x + 3)\}}{\delta x \cdot (\sqrt{2x + 2\delta x + 3} + \sqrt{2x + 3})}$$

$$= \lim_{\delta x \to 0} \frac{2\delta x}{\delta x \cdot (\sqrt{2x + 2\delta x + 3} + \sqrt{2x + 3})}$$

$$= \lim_{\delta x \to 0} \frac{2}{(\sqrt{2x + 2\delta x + 3} + \sqrt{2x + 3})}$$

$$= \frac{2}{2\sqrt{2x + 3}} = \frac{1}{\sqrt{2x + 3}} \cdot$$

$$\therefore \quad \frac{d}{dx}(\sqrt{2x + 3}) = \frac{1}{\sqrt{2x + 3}} \cdot$$

(ii) Let $y = \sqrt{4 - x}$.

Let δy be an increment in y, corresponding to an increment δx in x.

Then, $y + \delta y = \sqrt{4 - (x + \delta x)}$

$$\Rightarrow \delta y = \sqrt{4 - (x + \delta x)} - \sqrt{4 - x}$$

$$\Rightarrow \frac{\delta y}{\delta x} = \frac{\sqrt{4 - (x + \delta x)} - \sqrt{4 - x}}{\delta x}$$

$$\Rightarrow \frac{dy}{dx} = \lim_{\delta x \to 0} \frac{\delta y}{\delta x}$$

$$= \lim_{\delta x \to 0} \frac{\{\sqrt{4 - (x + \delta x)} - \sqrt{4 - x}\}}{\delta x}$$

$$= \lim_{\delta x \to 0} \left\{ \frac{(\sqrt{4 - x - \delta x} - \sqrt{4 - x})}{\delta x} \times \frac{(\sqrt{4 - x - \delta x} + \sqrt{4 - x})}{(\sqrt{4 - x - \delta x} + \sqrt{4 - x})} \right\}$$

$$= \lim_{\delta x \to 0} \frac{\{(4 - x - \delta x) - (4 - x)\}}{\delta x\, (\sqrt{4 - x - \delta x} + \sqrt{4 - x})}$$

$$= \lim_{\delta x \to 0} \frac{-\delta x}{\delta x\, (\sqrt{4 - x - \delta x} + \sqrt{4 - x})}$$

$$= \lim_{\delta x \to 0} \frac{-1}{(\sqrt{4 - x - \delta x} + \sqrt{4 - x})} = \frac{-1}{2\sqrt{4 - x}} \cdot$$

$$\therefore \quad \frac{d}{dx}(\sqrt{4 - x}) = \frac{-1}{2\sqrt{4 - x}} \cdot$$

(iii) Let $y = \frac{1}{\sqrt{x}} \cdot$

Let δy be an increment in y, corresponding to an increment δx in x.

Then, $y + \delta y = \dfrac{1}{\sqrt{x + \delta x}}$

$\Rightarrow\ \delta y = \left(\dfrac{1}{\sqrt{x+\delta x}} - \dfrac{1}{\sqrt{x}}\right)$

$\Rightarrow\ \dfrac{\delta y}{\delta x} = \dfrac{1}{\delta x}\cdot\left[\dfrac{1}{\sqrt{x+\delta x}} - \dfrac{1}{\sqrt{x}}\right]$

$\Rightarrow\ \dfrac{dy}{dx} = \lim\limits_{\delta x \to 0} \dfrac{\delta y}{\delta x}$

$= \lim\limits_{\delta x \to 0} \dfrac{1}{\delta x}\cdot\left[\dfrac{1}{\sqrt{x+\delta x}} - \dfrac{1}{\sqrt{x}}\right]$

$= \lim\limits_{\delta x \to 0} \left\{\dfrac{(\sqrt{x} - \sqrt{x+\delta x})}{\delta x \cdot \sqrt{x+\delta x}\cdot\sqrt{x}} \times \dfrac{(\sqrt{x} + \sqrt{x+\delta x})}{(\sqrt{x} + \sqrt{x+\delta x})}\right\}$

$= \lim\limits_{\delta x \to 0} \left[\dfrac{\{x - (x+\delta x)\}}{\delta x \cdot \sqrt{x+\delta x}\cdot\sqrt{x}\cdot(\sqrt{x} + \sqrt{x+\delta x})}\right\}$

$= \lim\limits_{\delta x \to 0} \left[\dfrac{-\delta x}{\delta x \cdot \sqrt{x+\delta x}\cdot\sqrt{x}\cdot(\sqrt{x} + \sqrt{x+\delta x})}\right\}$

$= \lim\limits_{\delta x \to 0} \left[\dfrac{-1}{\sqrt{x+\delta x}\cdot\sqrt{x}\cdot(\sqrt{x} + \sqrt{x+\delta x})}\right\}$

$= \dfrac{-1}{\sqrt{x}\cdot\sqrt{x}\cdot(2\sqrt{x})} = \dfrac{-1}{2x^{3/2}}.$

Hence, $\dfrac{d}{dx}\left(\dfrac{1}{\sqrt{x}}\right) = \dfrac{-1}{2x^{3/2}}.$

EXAMPLE 10 *Find the derivative of $x^{-3/2}$ from the first principle.*

SOLUTION Let $y = x^{-3/2}$.

Let δy be an increment in y, corresponding to an increment δx in x.

Then, $y + \delta y = (x + \delta x)^{-3/2}$

$\Rightarrow\ \delta y = (x+\delta x)^{-3/2} - x^{-3/2}$

$\Rightarrow\ \dfrac{\delta y}{\delta x} = \dfrac{\{(x+\delta x)^{-3/2} - x^{-3/2}\}}{\delta x}$

$\Rightarrow\ \dfrac{dy}{dx} = \lim\limits_{\delta x \to 0} \dfrac{\delta y}{\delta x}$

$= \lim\limits_{\delta x \to 0} \dfrac{\{(x+\delta x)^{-3/2} - x^{-3/2}\}}{\delta x} = \lim\limits_{\delta x \to 0} \dfrac{\{(x+\delta x)^{-3/2} - x^{-3/2}\}}{\{(x+\delta x) - x\}}$

$= \lim\limits_{u \to x} \dfrac{(u^{-3/2} - x^{-3/2})}{(u - x)}$, where $u = (x + \delta x)$

$$= -\frac{3}{2} x^{\left(-\frac{3}{2}-1\right)} \qquad \left[\because \lim_{u \to a} \frac{u^n - a^n}{u - a} = na^{n-1}\right]$$

$$= -\frac{3}{2} x^{-5/2}.$$

Hence, $\frac{d}{dx}(x^{-3/2}) = -\frac{3}{2} x^{-5/2}$.

EXAMPLE 11 *Find the derivative of* $\frac{1}{x^2}$ *from the first principle.*

SOLUTION Let $y = \frac{1}{x^2} \cdot$ Then, $y = x^{-2}$.

Let δy be an increment in y, corresponding to an increment δx in x.

Then, $y + \delta y = (x + \delta x)^{-2}$

$$\Rightarrow \quad \delta y = (x + \delta x)^{-2} - x^{-2}$$

$$\Rightarrow \quad \frac{\delta y}{\delta x} = \frac{\{(x + \delta x)^{-2} - x^{-2}\}}{\delta x}$$

$$\Rightarrow \quad \frac{dy}{dx} = \lim_{\delta x \to 0} \frac{\delta y}{\delta x}$$

$$= \lim_{\delta x \to 0} \frac{\{(x + \delta x)^{-2} - x^{-2}\}}{\delta x} = \lim_{\delta x \to 0} \frac{\{(x + \delta x)^{-2} - x^{-2}\}}{[(x + \delta x) - x]}$$

$$= \lim_{u \to x} \frac{(u^{-2} - x^{-2})}{(u - x)}, \text{ where } u = (x + \delta x)$$

$$= -2x^{(-2-1)} = -2x^{-3} = \frac{-2}{x^3} \qquad \left[\because \lim_{u \to a} \frac{u^n - a^n}{u - a} = na^{n-1}\right].$$

Hence, $\frac{d}{dx}\left(\frac{1}{x^2}\right) = \frac{-2}{x^3} \cdot$

EXAMPLE 12 *Differentiate* $\sqrt{\sin 3x}$ *from the first principle.* **[CBSE 2004C]**

SOLUTION Let $y = \sqrt{\sin 3x}$.

Let δy be an increment in y, corresponding to an increment δx in x.

Then, $y + \delta y = \sqrt{\sin 3(x + \delta x)}$

$$\Rightarrow \quad \delta y = \sqrt{\sin(3x + 3\delta x)} - \sqrt{\sin 3x}$$

$$\Rightarrow \quad \frac{\delta y}{\delta x} = \frac{\sqrt{\sin(3x + 3\delta x)} - \sqrt{\sin 3x}}{\delta x}$$

$$\Rightarrow \quad \frac{dy}{dx} = \lim_{\delta x \to 0} \frac{\delta y}{\delta x}$$

$$= \lim_{\delta x \to 0} \left[\frac{\{\sqrt{\sin(3x + 3\delta x)} - \sqrt{\sin 3x}\}}{\delta x} \times \frac{\{\sqrt{\sin(3x + 3\delta x)} + \sqrt{\sin 3x}\}}{\{\sqrt{\sin(3x + 3\delta x)} + \sqrt{\sin 3x}\}} \right]$$

$$= \lim_{\delta x \to 0} \frac{[\sin(3x + 3\delta x) - \sin 3x]}{\delta x \cdot \{\sqrt{\sin(3x + 3\delta x)} + \sqrt{\sin 3x}\}}$$

$$= \lim_{\delta x \to 0} \frac{2\cos\left(3x + \frac{3}{2}\delta x\right) \sin\left(\frac{3}{2}\delta x\right)}{\delta x \cdot \{\sqrt{\sin(3x + 3\delta x)} + \sqrt{\sin 3x}\}}$$

$$\left[\because\ (\sin C - \sin D) = 2\cos\left(\frac{C+D}{2}\right)\sin\left(\frac{C-D}{2}\right)\right]$$

$$= \lim_{\delta x \to 0} \left[2\cos\left(3x + \frac{3}{2}\delta x\right) \cdot \frac{\sin\left(\frac{3}{2}\delta x\right)}{\left(\frac{3}{2}\delta x\right)} \times \frac{3}{2} \times \frac{1}{\{\sqrt{\sin(3x + 3\delta x)} + \sqrt{\sin 3x}\}} \right]$$

$$= 3 \cdot \lim_{\delta x \to 0} \cos\left(3x + \frac{3}{2}\delta x\right) \times \lim_{\frac{3}{2}\delta x \to 0} \frac{\sin\left(\frac{3}{2}\delta x\right)}{\left(\frac{3}{2}\delta x\right)} \times \lim_{\delta x \to 0} \frac{1}{\{\sqrt{\sin(3x + 3\delta x)} + \sqrt{\sin 3x}\}}$$

$$= 3\cos 3x \times 1 \times \frac{1}{2\sqrt{\sin 3x}} = \frac{3\cos 3x}{2\sqrt{\sin 3x}}.$$

Hence, $\frac{d}{dx}(\sqrt{\sin 3x}) = \frac{3\cos 3x}{2\sqrt{\sin 3x}}$.

EXAMPLE 13 *Find the derivative of* $e^{\sqrt{x}}$ *from the first principle.* **[CBSE 2003C]**

SOLUTION Let $y = e^{\sqrt{x}}$.

Let δy be an increment in y, corresponding to an increment δx in x.

Then, $y + \delta y = e^{\sqrt{x + \delta x}}$

$$\Rightarrow \quad \delta y = e^{\sqrt{x+\delta x}} - e^{\sqrt{x}}$$

$$\Rightarrow \quad \frac{\delta y}{\delta x} = \frac{e^{\sqrt{x+\delta x}} - e^{\sqrt{x}}}{\delta x}$$

$$\Rightarrow \quad \frac{dy}{dx} = \lim_{\delta x \to 0} \frac{e^{\sqrt{x+\delta x}} - e^{\sqrt{x}}}{\delta x} = \lim_{\delta x \to 0} \frac{e^{\sqrt{x}}\,[e^{\sqrt{x+\delta x} - \sqrt{x}} - 1]}{\{(x + \delta x) - x\}}$$

$$= \lim_{\delta x \to 0} \frac{e^{\sqrt{x}}\,[e^{\sqrt{x+\delta x} - \sqrt{x}} - 1]}{(\sqrt{x + \delta x} - \sqrt{x})(\sqrt{x + \delta x} + \sqrt{x})}$$

$$= e^{\sqrt{x}} \cdot \lim_{\theta \to 0} \frac{(e^{\theta} - 1)}{\theta} \cdot \lim_{\delta x \to 0} \frac{1}{(\sqrt{x + \delta x} + \sqrt{x})},$$

where $\theta = (\sqrt{x + \delta x} - \sqrt{x})$ [clearly, $\theta \to 0$ when $\delta x \to 0$]

$$= \left(e^{\sqrt{x}} \times 1 \times \frac{1}{2\sqrt{x}}\right) = \frac{e^{\sqrt{x}}}{2\sqrt{x}} \quad \left[\because \lim_{\theta \to 0} \frac{(e^{\theta} - 1)}{\theta} = 1\right].$$

Hence, $\dfrac{d}{dx}(e^{\sqrt{x}}) = \dfrac{e^{\sqrt{x}}}{2\sqrt{x}}$.

<u>EXAMPLE 14</u> *Find the derivative of e^{x^2} from the first principle.* **[CBSE 2002C]**

SOLUTION Let $y = e^{x^2}$.

Let δy be an increment in y, corresponding to an increment δx in x.

Then, $y + \delta y = e^{(x + \delta x)^2}$

$\Rightarrow \quad \delta y = e^{(x + \delta x)^2} - e^{x^2}$

$\Rightarrow \quad \dfrac{\delta y}{\delta x} = \dfrac{e^{(x + \delta x)^2} - e^{x^2}}{\delta x}$

$\Rightarrow \quad \dfrac{dy}{dx} = \lim\limits_{\delta x \to 0} \dfrac{\delta y}{\delta x}$

$$= \lim_{\delta x \to 0} \frac{e^{(x + \delta x)^2} - e^{x^2}}{\delta x} = \lim_{\delta x \to 0} \frac{e^{(x^2 + 2x\delta x + \delta x^2)} - e^{x^2}}{\delta x}$$

$$= \lim_{\delta x \to 0} \frac{e^{x^2}[e^{(2x\delta x + \delta x^2)} - 1]}{\delta x}$$

$$= e^{x^2} \cdot \lim_{\delta x \to 0} \frac{(e^{2x\delta x + \delta x^2} - 1)}{(2x\delta x + \delta x^2)} \times \frac{(2x\delta x + \delta x^2)}{\delta x}$$

$$= e^{x^2} \cdot \lim_{\theta \to 0} \left(\frac{e^{\theta} - 1}{\theta}\right) \cdot \lim_{\delta x \to 0} (2x + \delta x), \text{ where } (2x\delta x + \delta x^2) = \theta$$

[clearly, $\theta \to 0$ when $\delta x \to 0$]

$$= (e^{x^2} \times 1 \times 2x) = 2xe^{x^2} \quad \left[\because \lim_{\theta \to 0} \left(\frac{e^{\theta} - 1}{\theta}\right) = 1\right].$$

Hence, $\dfrac{d}{dx}(e^{x^2}) = 2xe^{x^2}$.

<u>EXAMPLE 15</u> *Find the derivative of $e^{\sin x}$ from the first principle.*

SOLUTION Let $y = e^{\sin x}$.

Let δy be an increment in y, corresponding to an increment δx in x.

Then, $y + \delta y = e^{\sin (x + \delta x)}$

$$\Rightarrow \quad \delta y = e^{\sin (x+\delta x)} - e^{\sin x}$$

$$\Rightarrow \quad \frac{\delta y}{\delta x} = \frac{e^{\sin (x+\delta x)} - e^{\sin x}}{\delta x}$$

$$\Rightarrow \quad \frac{dy}{dx} = \lim_{\delta x \to 0} \frac{[e^{\sin (x+\delta x)} - e^{\sin x}]}{\delta x} = \lim_{\delta x \to 0} \frac{e^{\sin x} \cdot [e^{\sin (x+\delta x)} - \sin x - 1]}{\delta x}$$

$$= e^{\sin x} \cdot \lim_{\delta x \to 0} \frac{[e^{\sin (x+\delta x) - \sin x} - 1]}{\{\sin (x+\delta x) - \sin x\}} \cdot \frac{[\sin (x+\delta x) - \sin x]}{\delta x}$$

$$= e^{\sin x} \cdot \lim_{\theta \to 0} \left(\frac{e^{\theta} - 1}{\theta}\right) \cdot \lim_{\delta x \to 0} \frac{\sin (x+\delta x) - \sin x}{\delta x},$$

where $\theta = \sin (x + \delta x) - \sin x$ [clearly, $(\delta x \to 0) \Rightarrow (\theta \to 0)$]

$$= e^{\sin x} \cdot 1 \cdot \lim_{\delta x \to 0} \frac{2\cos\left(x + \frac{\delta x}{2}\right) \sin\left(\frac{\delta x}{2}\right)}{\delta x}$$

$$= e^{\sin x} \cdot \lim_{\delta x \to 0} \cos\left(x + \frac{\delta x}{2}\right) \cdot \lim_{\delta x \to 0} \frac{\sin\left(\frac{\delta x}{2}\right)}{\left(\frac{\delta x}{2}\right)}$$

$$= (e^{\sin x} \cdot \cos x \cdot 1) = e^{\sin x} \cos x.$$

Hence, $\frac{d}{dx}(e^{\sin x}) = e^{\sin x} \cos x.$

EXAMPLE 16 *Differentiate each of the following from the first principle.*

(i) $\sqrt{\sin x}$	**[CBSE 2004C]**	*(ii)* $\sqrt{\cos x}$	**[CBSE 2004]**
(iii) $\sqrt{\tan x}$	**[CBSE 2002, '04C]**	*(iv)* $\sqrt{\operatorname{cosec} x}$	**[CBSE 1998]**

SOLUTION (i) Let $y = \sqrt{\sin x}$.

Let δy be an increment in y, corresponding to an increment δx in x.

Then, $y + \delta y = \sqrt{\sin (x + \delta x)}$

$$\Rightarrow \quad \delta y = \sqrt{\sin (x + \delta x)} - \sqrt{\sin x}$$

$$\Rightarrow \quad \frac{\delta y}{\delta x} = \frac{\sqrt{\sin (x+\delta x)} - \sqrt{\sin x}}{\delta x}$$

$$\Rightarrow \quad \frac{dy}{dx} = \lim_{\delta x \to 0} \frac{\delta y}{\delta x}$$

$$= \lim_{\delta x \to 0} \frac{\sqrt{\sin (x+\delta x)} - \sqrt{\sin x}}{\delta x}$$

$$= \lim_{\delta x \to 0} \frac{\{\sqrt{\sin (x+\delta x)} - \sqrt{\sin x}\}}{\delta x} \times \frac{\{\sqrt{\sin (x+\delta x)} + \sqrt{\sin x}\}}{\{\sqrt{\sin (x+\delta x)} + \sqrt{\sin x}\}}$$

$$= \lim_{\delta x \to 0} \frac{\{\sin(x+\delta x) - \sin x\}}{\delta x \cdot \{\sqrt{\sin(x+\delta x)} + \sqrt{\sin x}\}}$$

$$= \lim_{\delta x \to 0} \frac{2\cos\left(x + \frac{\delta x}{2}\right)\sin\left(\frac{\delta x}{2}\right)}{\{\sqrt{\sin(x+\delta x)} + \sqrt{\sin x}\}\cdot \delta x}$$

$$= \lim_{\delta x \to 0} \cos\left(x + \frac{\delta x}{2}\right) \cdot \lim_{\delta x \to 0} \frac{\sin\left(\frac{\delta x}{2}\right)}{\left(\frac{\delta x}{2}\right)} \cdot \lim_{\delta x \to 0} \frac{1}{\{\sqrt{\sin(x+\delta x)} + \sqrt{\sin x}\}}$$

$$= \left\{\cos x \cdot 1 \cdot \frac{1}{2\sqrt{\sin x}}\right\} = \frac{\cos x}{2\sqrt{\sin x}}.$$

Hence, $\frac{d}{dx}(\sqrt{\sin x}) = \frac{\cos x}{2\sqrt{\sin x}}$.

(ii) Let $y = \sqrt{\cos x}$.

Let δy be an increment in y, corresponding to an increment δx in x.

Then, $y + \delta y = \sqrt{\cos(x+\delta x)}$

$\Rightarrow \delta y = \sqrt{\cos(x+\delta x)} - \sqrt{\cos x}$

$\Rightarrow \frac{\delta y}{\delta x} = \frac{\sqrt{\cos(x+\delta x)} - \sqrt{\cos x}}{\delta x}$

$\Rightarrow \frac{dy}{dx} = \lim_{\delta x \to 0} \frac{\{\sqrt{\cos(x+\delta x)} - \sqrt{\cos x}\}}{\delta x}$

$$= \lim_{\delta x \to 0} \frac{\{\sqrt{\cos(x+\delta x)} - \sqrt{\cos x}\}}{\delta x} \times \frac{\{\sqrt{\cos(x+\delta x)} + \sqrt{\cos x}\}}{\{\sqrt{\cos(x+\delta x)} + \sqrt{\cos x}\}}$$

$$= \lim_{\delta x \to 0} \frac{\{\cos(x+\delta x) - \cos x\}}{\delta x \cdot \{\sqrt{\cos(x+\delta x)} + \sqrt{\cos x}\}}$$

$$= \lim_{\delta x \to 0} \frac{-2\sin\left(x + \frac{\delta x}{2}\right)\sin\left(\frac{\delta x}{2}\right)}{\delta x \cdot \{\sqrt{\cos(x+\delta x)} + \sqrt{\cos x}\}}$$

$$\left[\because \cos C - \cos D = -2\sin\left(\frac{C+D}{2}\right)\sin\left(\frac{C-D}{2}\right)\right]$$

$$= -\lim_{\delta x \to 0} \sin\left(x + \frac{\delta x}{2}\right) \cdot \lim_{\delta x \to 0} \frac{\sin(\delta x/2)}{(\delta x/2)} \cdot \lim_{\delta x \to 0} \frac{1}{\{\sqrt{\cos(x+\delta x)} + \sqrt{\cos x}\}}$$

$$= (-\sin x) \times 1 \times \frac{1}{2\sqrt{\cos x}} = \frac{-\sin x}{2\sqrt{\cos x}}.$$

Hence, $\dfrac{d}{dx}(\sqrt{\cos x}) = \dfrac{-\sin x}{2\sqrt{\cos x}}$.

(iii) Let $y = \sqrt{\tan x}$.

Let δy be an increment in y, corresponding to an increment δx in x.

Then, $y + \delta y = \sqrt{\tan(x + \delta x)}$

$\Rightarrow\ \delta y = \sqrt{\tan(x + \delta x)} - \sqrt{\tan x}$

$$\Rightarrow\ \frac{\delta y}{\delta x} = \frac{\sqrt{\tan(x + \delta x)} - \sqrt{\tan x}}{\delta x}$$

$$\Rightarrow\ \frac{dy}{dx} = \lim_{\delta x \to 0} \frac{\delta y}{\delta x}$$

$$= \lim_{\delta x \to 0} \frac{\sqrt{\tan(x + \delta x)} - \sqrt{\tan x}}{\delta x}$$

$$= \lim_{\delta x \to 0} \left\{ \frac{\sqrt{\tan(x+\delta x)} - \sqrt{\tan x}}{\delta x} \times \frac{\sqrt{\tan(x+\delta x)} + \sqrt{\tan x}}{\sqrt{\tan(x+\delta x)} + \sqrt{\tan x}} \right\}$$

$$= \lim_{\delta x \to 0} \frac{\tan(x + \delta x) - \tan x}{\delta x[\sqrt{\tan(x + \delta x)} + \sqrt{\tan x}]}$$

$$= \lim_{\delta x \to 0} \frac{\left\{ \dfrac{\sin(x + \delta x)}{\cos(x + \delta x)} - \dfrac{\sin x}{\cos x} \right\}}{\delta x[\sqrt{\tan(x + \delta x)} + \sqrt{\tan x}]}$$

$$= \lim_{\delta x \to 0} \frac{\sin(x + \delta x)\cos x - \cos(x + \delta x)\sin x}{\cos(x + \delta x)\cos x \cdot \delta x \cdot [\sqrt{\tan(x + \delta x)} + \sqrt{\tan x}]}$$

$$= \lim_{\delta x \to 0} \frac{\sin(x + \delta x - x)}{\cos(x + \delta x) \cdot \cos x \cdot \delta x \cdot (\sqrt{\tan(x + \delta x)} + \sqrt{\tan x})}$$

$$= \frac{1}{\cos x} \cdot \lim_{\delta x \to 0} \frac{1}{\cos(x + \delta x)} \cdot \lim_{\delta x \to 0} \frac{\sin \delta x}{\delta x} \cdot \lim_{\delta x \to 0} \frac{1}{(\sqrt{\tan(x + \delta x)} + \sqrt{\tan x})}$$

$$= \left(\frac{1}{\cos x} \cdot \frac{1}{\cos x} \cdot 1 \cdot \frac{1}{2\sqrt{\tan x}} \right) = \frac{\sec^2 x}{2\sqrt{\tan x}}.$$

Hence, $\dfrac{d}{dx}(\sqrt{\tan x}) = \dfrac{\sec^2 x}{2\sqrt{\tan x}}$.

(iv) Let $y = \sqrt{\operatorname{cosec} x}$.

Let δy be an increment in y, corresponding to an increment δx in x.

Then, $y + \delta y = \sqrt{\operatorname{cosec}(x + \delta x)}$

$$\Rightarrow \delta y = \sqrt{\operatorname{cosec}(x+\delta x)} - \sqrt{\operatorname{cosec} x}$$

$$\Rightarrow \frac{\delta y}{\delta x} = \frac{\sqrt{\operatorname{cosec}(x+\delta x)} - \sqrt{\operatorname{cosec} x}}{\delta x}$$

$$\Rightarrow \frac{dy}{dx} = \lim_{\delta x \to 0} \frac{\delta y}{\delta x}$$

$$= \lim_{\delta x \to 0} \frac{\sqrt{\operatorname{cosec}(x+\delta x)} - \sqrt{\operatorname{cosec} x}}{\delta x}$$

$$= \lim_{\delta x \to 0} \frac{\left\{\dfrac{1}{\sqrt{\sin(x+\delta x)}} - \dfrac{1}{\sqrt{\sin x}}\right\}}{\delta x}$$

$$= \lim_{\delta x \to 0} \frac{\{\sqrt{\sin x} - \sqrt{\sin(x+\delta x)}\}}{\delta x \cdot \sqrt{\sin(x+\delta x)} \cdot \sqrt{\sin x}} \times \frac{\{\sqrt{\sin x} + \sqrt{\sin(x+\delta x)}\}}{\{\sqrt{\sin x} + \sqrt{\sin(x+\delta x)}\}}$$

$$= \lim_{\delta x \to 0} \frac{\{\sin x - \sin(x+\delta x)\}}{\{\sqrt{\sin(x+\delta x)} \cdot \sqrt{\sin x}\}} \times \frac{1}{\delta x \cdot \{\sqrt{\sin x} + \sqrt{\sin(x+\delta x)}\}}$$

$$= \lim_{\delta x \to 0} \frac{-2\cos\left(x+\dfrac{\delta x}{2}\right)\sin\left(\dfrac{\delta x}{2}\right)}{\{\sqrt{\sin(x+\delta x)}\} \cdot \sqrt{\sin x}\} \cdot \delta x \cdot \{\sqrt{\sin x} + \sqrt{\sin(x+\delta x)}\}}$$

$$= -\lim_{\delta x \to 0} \cos\left(x + \frac{\delta x}{2}\right) \cdot \lim_{\delta x \to 0} \frac{\sin\left(\dfrac{\delta x}{2}\right)}{\left(\dfrac{\delta x}{2}\right)} \cdot$$

$$\lim_{\delta x \to 0} \frac{1}{\sqrt{\sin(x+\delta x)} \cdot \sqrt{\sin x}} \cdot \lim_{\delta x \to 0} \frac{1}{\{\sqrt{\sin x} + \sqrt{\sin(x+\delta x)}\}}$$

$$= -\cos x \times 1 \times \frac{1}{\sqrt{\sin x} \cdot \sqrt{\sin x}} \cdot \frac{1}{(\sqrt{\sin x} + \sqrt{\sin x})}$$

$$= \frac{-\cos x}{\sin x} \cdot \frac{1}{2\sqrt{\sin x}} = -\frac{1}{2}\sqrt{\operatorname{cosec} x}\ \cot x.$$

Hence, $\dfrac{d}{dx}(\sqrt{\operatorname{cosec} x}) = -\dfrac{1}{2}\sqrt{\operatorname{cosec} x}\ \cot x.$

EXAMPLE 17 *Differentiate each of the following from the first principle.*
(i) $\sin\sqrt{x}$ **[CBSE 2004C]** *(ii)* $\cos\sqrt{x}$
(iii) $\tan\sqrt{x}$ **[CBSE 2002, '04]**

SOLUTION (i) Let $y = \sin\sqrt{x}$.
Let δy be an increment in y, corresponding to an increment δx in x.

Then, $y + \delta y = \sin\sqrt{x + \delta x}$

$\Rightarrow\ \delta y = \sin\sqrt{x + \delta x} - \sin\sqrt{x}$

$\Rightarrow\ \dfrac{\delta y}{\delta x} = \dfrac{\sin\sqrt{x + \delta x} - \sin\sqrt{x}}{\delta x}$

$\Rightarrow\ \dfrac{dy}{dx} = \lim\limits_{\delta x \to 0} \dfrac{\delta y}{\delta x}$

$$= \lim_{\delta x \to 0} \frac{\{\sin\sqrt{x + \delta x} - \sin\sqrt{x}\}}{\delta x}$$

$$= \lim_{\delta x \to 0} \frac{2\cos\left(\dfrac{\sqrt{x + \delta x} + \sqrt{x}}{2}\right)\sin\left(\dfrac{\sqrt{x + \delta x} - \sqrt{x}}{2}\right)}{\{(x + \delta x) - x\}}$$

$$\left[\text{using } \sin C - \sin D = 2\cos\left(\frac{C + D}{2}\right)\sin\left(\frac{C - D}{2}\right)\right]$$

$$= \lim_{\delta x \to 0} \frac{2\cos\left(\dfrac{\sqrt{x + \delta x} + \sqrt{x}}{2}\right)\sin\left(\dfrac{\sqrt{x + \delta x} - \sqrt{x}}{2}\right)}{(\sqrt{x + \delta x} + \sqrt{x})\,(\sqrt{x + \delta x} - \sqrt{x})}$$

$$[\because\ (x + \delta x) - x = (\sqrt{x + \delta x} + \sqrt{x})(\sqrt{x + \delta x} - \sqrt{x})]$$

$$= \lim_{\delta x \to 0} \frac{\sin\left(\dfrac{\sqrt{x+\delta x} - \sqrt{x}}{2}\right)}{\left(\dfrac{\sqrt{x+\delta x} - \sqrt{x}}{2}\right)} \cdot \lim_{\delta x \to 0} \cos\left(\frac{\sqrt{x+\delta x} + \sqrt{x}}{2}\right) \cdot \lim_{\delta x \to 0} \frac{1}{(\sqrt{x + \delta x} + \sqrt{x})}$$

$$= \lim_{\theta \to 0} \frac{\sin\theta}{\theta} \cdot \lim_{\delta x \to 0} \cos\left(\frac{\sqrt{x+\delta x} + \sqrt{x}}{2}\right) \cdot \lim_{\delta x \to 0} \frac{1}{(\sqrt{x+\delta x} + \sqrt{x})},$$

where $\dfrac{(\sqrt{x + \delta x} - \sqrt{x})}{2} = \theta$ and $\delta x \to 0\ \Rightarrow\ \theta \to 0$

$$= 1 \times \cos\sqrt{x} \times \frac{1}{2\sqrt{x}} = \frac{\cos\sqrt{x}}{2\sqrt{x}}.$$

Hence, $\dfrac{d}{dx}(\sin\sqrt{x}) = \dfrac{\cos\sqrt{x}}{2\sqrt{x}}$.

(ii) Let $y = \cos\sqrt{x}$.

Let δy be an increment in y, corresponding to an increment δx in x.

Then, $y + \delta y = \cos\sqrt{x + \delta x}$

$\Rightarrow\ \delta y = \cos\sqrt{x + \delta x} - \cos\sqrt{x}$

$$\Rightarrow \frac{\delta y}{\delta x} = \frac{\{\cos\sqrt{x+\delta x} - \cos\sqrt{x}\}}{\delta x}$$

$$\Rightarrow \frac{dy}{dx} = \lim_{\delta x \to 0} \frac{\delta y}{\delta x}$$

$$= \lim_{\delta x \to 0} \frac{\{\cos\sqrt{x+\delta x} - \cos\sqrt{x}\}}{\delta x}$$

$$= \lim_{\delta x \to 0} \frac{-2\sin\left(\dfrac{\sqrt{x+\delta x}+\sqrt{x}}{2}\right)\sin\left(\dfrac{\sqrt{x+\delta x}-\sqrt{x}}{2}\right)}{\{(x+\delta x)-x\}}$$

$$= \lim_{\delta x \to 0} \frac{-2\sin\left(\dfrac{\sqrt{x+\delta x}+\sqrt{x}}{2}\right)\sin\left(\dfrac{\sqrt{x+\delta x}-\sqrt{x}}{2}\right)}{(\sqrt{x+\delta x}+\sqrt{x})(\sqrt{x+\delta x}-\sqrt{x})}$$

$$= -\lim_{\delta x \to 0} \sin\left(\frac{\sqrt{x+\delta x}+\sqrt{x}}{2}\right) \cdot \lim_{\delta x \to 0} \frac{\sin\left(\dfrac{\sqrt{x+\delta x}-\sqrt{x}}{2}\right)}{\left(\dfrac{\sqrt{x+\delta x}-\sqrt{x}}{2}\right)} \cdot$$

$$\lim_{\delta x \to 0} \frac{1}{(\sqrt{x+\delta x}+\sqrt{x})}$$

$$= -\lim_{\delta x \to 0} \sin\left(\frac{\sqrt{x+\delta x}+\sqrt{x}}{2}\right) \cdot \lim_{\theta \to 0} \frac{\sin\theta}{\theta}$$

$$\cdot \lim_{\delta x \to 0} \frac{1}{(\sqrt{x+\delta x}+\sqrt{x})},$$

$$\text{where } \frac{(\sqrt{x+\delta x}-\sqrt{x})}{2} = \theta$$

$$= -\sin\left(\frac{2\sqrt{x}}{2}\right) \times 1 \times \frac{1}{2\sqrt{x}} = \frac{-\sin\sqrt{x}}{2\sqrt{x}}.$$

Hence, $\dfrac{d}{dx}(\cos\sqrt{x}) = \dfrac{-\sin\sqrt{x}}{2\sqrt{x}}$.

(iii) Let $y = \tan\sqrt{x}$.

Let δy be an increment in y, corresponding to an increment δx in x.

Then, $y + \delta y = \tan\sqrt{x+\delta x}$

$$\Rightarrow \delta y = \tan\sqrt{x+\delta x} - \tan\sqrt{x}$$

$$\Rightarrow \frac{\delta y}{\delta x} = \frac{\tan\sqrt{x+\delta x} - \tan\sqrt{x}}{\delta x}$$

$$\Rightarrow \frac{dy}{dx} = \lim_{\delta x \to 0} \frac{\delta y}{\delta x}$$

$$= \lim_{\delta x \to 0} \frac{(\tan\sqrt{x+\delta x} - \tan\sqrt{x})}{\delta x}$$

$$= \lim_{\delta x \to 0} \frac{\left\{ \frac{\sin \sqrt{x + \delta x}}{\cos \sqrt{x + \delta x}} - \frac{\sin \sqrt{x}}{\cos \sqrt{x}} \right\}}{\delta x}$$

$$= \lim_{\delta x \to 0} \frac{(\sin \sqrt{x + \delta x} \cos \sqrt{x} - \cos \sqrt{x + \delta x} \sin \sqrt{x})}{\delta x \cdot \cos \sqrt{x + \delta x} \cdot \cos \sqrt{x}}$$

$$= \lim_{\delta x \to 0} \left\{ \frac{\sin (\sqrt{x + \delta x} - \sqrt{x})}{\{(x + \delta x) - x\}} \cdot \frac{1}{\cos \sqrt{x + \delta x} \cdot \cos \sqrt{x}} \right\}$$

$[\because \ \delta x = (x + \delta x) - x]$

$$= \lim_{\delta x \to 0} \left\{ \frac{\sin (\sqrt{x + \delta x} - \sqrt{x})}{(\sqrt{x + \delta x} - \sqrt{x})(\sqrt{x + \delta x} + \sqrt{x})} \cdot \frac{1}{\cos \sqrt{x + \delta x} \cdot \cos \sqrt{x}} \right\}$$

$$= \lim_{\delta x \to 0} \frac{\sin (\sqrt{x + \delta x} - \sqrt{x})}{(\sqrt{x + \delta x} - \sqrt{x})} \cdot \lim_{\delta x \to 0} \frac{1}{(\sqrt{x + \delta x} + \sqrt{x})} \cdot \lim_{\delta x \to 0} \frac{1}{\cos \sqrt{x + \delta x} \cdot \cos \sqrt{x}}$$

$$= \left(\lim_{\theta \to 0} \frac{\sin \theta}{\theta} \right) \cdot \frac{1}{2\sqrt{x}} \cdot \frac{1}{\cos^2 \sqrt{x}} = 1 \times \frac{1}{2\sqrt{x}} \times \frac{1}{\cos^2 \sqrt{x}}$$

$$= \frac{\sec^2 \sqrt{x}}{2\sqrt{x}}.$$

Hence, $\dfrac{d}{dx}(\tan \sqrt{x}) = \dfrac{\sec^2 \sqrt{x}}{2\sqrt{x}}$.

EXAMPLE 18 *Find the derivative of the following from the first principle.*

(i) $\sin^2 x$ *(ii)* $\cos^2 x$ **[CBSE 2003]**

SOLUTION (i) Let $y = \sin^2 x$.

Let δy be an increment in y, corresponding to an increment δx in x.

Then, $y + \delta y = \sin^2(x + \delta x)$

$\Rightarrow \delta y = \sin^2(x + \delta x) - \sin^2 x$

$\Rightarrow \dfrac{\delta y}{\delta x} = \dfrac{\sin^2(x + \delta x) - \sin^2 x}{\delta x}$

$\Rightarrow \dfrac{dy}{dx} = \lim_{\delta x \to 0} \dfrac{\delta y}{\delta x}$

$$= \lim_{\delta x \to 0} \frac{\sin^2(x + \delta x) - \sin^2 x}{\delta x}$$

$$= \lim_{\delta x \to 0} \frac{\sin(2x + \delta x) \sin \delta x}{\delta x}$$

$$[\because \ \sin^2 A - \sin^2 B = \sin(A + B) \cdot \sin(A - B)]$$

$$= \lim_{\delta x \to 0} \sin(2x + \delta x) \cdot \lim_{\delta x \to 0} \frac{\sin \delta x}{\delta x}$$

$$= (\sin 2x \times 1) = \sin 2x.$$

Hence, $\frac{d}{dx}(\sin^2 x) = \sin 2x.$

(ii) Let $y = \cos^2 x$.

Let δy be an increment in y, corresponding to an increment δx in x.

Then, $y + \delta y = \cos^2(x + \delta x)$

$\Rightarrow \ \delta y = \cos^2(x + \delta x) - \cos^2 x$

$\Rightarrow \ \frac{\delta y}{\delta x} = \frac{\cos^2(x + \delta x) - \cos^2 x}{\delta x}$

$\Rightarrow \ \frac{dy}{dx} = \lim_{\delta x \to 0} \frac{\delta y}{\delta x}$

$$= \lim_{\delta x \to 0} \frac{\cos^2(x + \delta x) - \cos^2 x}{\delta x}$$

$$= \lim_{\delta x \to 0} \frac{[\cos(x + \delta x) + \cos x] \times [\cos(x + \delta x) - \cos x]}{\delta x}$$

$$= \lim_{\delta x \to 0} \frac{\left\{2\cos\left(x + \frac{\delta x}{2}\right)\cos\left(\frac{\delta x}{2}\right)\right\}\left\{-2\sin\left(x + \frac{\delta x}{2}\right)\sin\left(\frac{\delta x}{2}\right)\right\}}{\delta x}$$

$$= -2 \lim_{\delta x \to 0} \cos\left(x + \frac{\delta x}{2}\right) \cdot \lim_{\delta x \to 0} \cos\left(\frac{\delta x}{2}\right) \cdot \lim_{\delta x \to 0} \sin\left(x + \frac{\delta x}{2}\right) \cdot \lim_{\delta x \to 0} \frac{\sin(\delta x/2)}{(\delta x/2)}$$

$$= (-2\cos x \times 1 \times \sin x \times 1) = -2\sin x \cos x = -\sin 2x.$$

Hence, $\frac{d}{dx}(\cos^2 x) = -\sin 2x.$

EXAMPLE 19 *Find the derivative of the following from the first principle:*

(i) $\sin x^2$ **[CBSE 2001C]** *(ii)* $\cos(x^2 + 1)$ **[CBSE 2001]**

(iii) $\tan x^2$ **[CBSE 2002]**

SOLUTION (i) Let $y = \sin x^2$.

Let δy be an increment in y, corresponding to an increment δx in x.

Then, $y + \delta y = \sin (x + \delta x)^2$

$\Rightarrow \delta y = \sin (x + \delta x)^2 - \sin x^2$

$\Rightarrow \dfrac{\delta y}{\delta x} = \dfrac{\sin (x + \delta x)^2 - \sin x^2}{\delta x}$

$\Rightarrow \dfrac{dy}{dx} = \lim\limits_{\delta x \to 0} \dfrac{\delta y}{\delta x}$

$$= \lim_{\delta x \to 0} \frac{\sin (x + \delta x)^2 - \sin x^2}{\delta x}$$

$$= \lim_{\delta x \to 0} \frac{2\cos \left[\dfrac{(x + \delta x)^2 + x^2}{2}\right] \sin \left[\dfrac{(x + \delta x)^2 - x^2}{2}\right]}{\delta x}$$

$$\left[\text{using } (\sin C - \sin D) = 2\cos\left(\frac{C + D}{2}\right) \sin\left(\frac{C - D}{2}\right)\right]$$

$$= \lim_{\delta x \to 0} 2\cos \left[\frac{(x + \delta x)^2 + x^2}{2}\right] \frac{\sin \left[\left(x + \dfrac{\delta x}{2}\right) \cdot \delta x\right]}{\left(x + \dfrac{\delta x}{2}\right) \cdot \delta x} \cdot \left(x + \frac{\delta x}{2}\right)$$

$$= 2 \cdot \lim_{\delta x \to 0} \cos \left[\frac{(x + \delta x)^2 + x^2}{2}\right] \cdot \lim_{\delta x \to 0} \frac{\sin \left[\left(x + \dfrac{\delta x}{2}\right) \cdot \delta x\right]}{\left(x + \dfrac{\delta x}{2}\right) \cdot \delta x} \cdot \lim_{\delta x \to 0} \left(x + \frac{\delta x}{2}\right)$$

$$= [2 \times \cos x^2 \times 1 \times x] = 2x \cos x^2.$$

Hence, $\dfrac{d}{dx} (\sin x^2) = 2x \cos x^2$.

(ii) Let $y = \cos (x^2 + 1)$.

Let δy be an increment in y, corresponding to an increment δx in x.

Then, $y + \delta y = \cos [(x + \delta x)^2 + 1]$

$\Rightarrow \delta y = \cos [(x + \delta x)^2 + 1] - \cos (x^2 + 1)$

$\Rightarrow \dfrac{\delta y}{\delta x} = \dfrac{\cos [(x + \delta x)^2 + 1] - \cos (x^2 + 1)}{\delta x}$

$\Rightarrow \dfrac{dy}{dx} = \lim\limits_{\delta x \to 0} \dfrac{\delta y}{\delta x}$

$$= \lim_{\delta x \to 0} \frac{\{\cos[(x+\delta x)^2+1] - \cos(x^2+1)\}}{\delta x}$$

$$= \lim_{\delta x \to 0} \frac{-2\sin\left[\dfrac{\{(x+\delta x)^2+1+(x^2+1)\}}{2}\right]\cdot \sin\left[\dfrac{\{(x+\delta x)^2+1-(x^2+1)\}}{2}\right]}{\delta x}$$

$$= -2\cdot \lim_{\delta x \to 0} \sin\left[x^2 + x\cdot\delta x + 1 + \frac{(\delta x)^2}{2}\right]\cdot \frac{\sin\left[\left(x+\dfrac{\delta x}{2}\right)\cdot\delta x\right]}{\left[\left(x+\dfrac{\delta x}{2}\right)\cdot\delta x\right]}\cdot\left(x+\frac{\delta x}{2}\right)$$

$$= -2\cdot \lim_{\delta x \to 0} \sin\left[x^2 + x\cdot\delta x + 1 + \frac{(\delta x)^2}{2}\right]\cdot \lim_{\theta \to 0}\frac{\sin\theta}{\theta}\cdot \lim_{\delta x \to 0}\left(x+\frac{\delta x}{2}\right),$$

where $\theta = \left(x + \dfrac{\delta x}{2}\right)\cdot \delta x$; clearly, $[\delta x \to 0] \Rightarrow [\theta \to 0]$

$= -2\sin(x^2+1)\times 1 \times x$

$= -2x\sin(x^2+1)$.

Hence, $\dfrac{d}{dx}[\cos(x^2+1)] = -2x\sin(x^2+1)$.

(iii) Let $y = \tan x^2$.

Let δy be an increment in y, corresponding to an increment δx in x.

Then, $y + \delta y = \tan(x+\delta x)^2$

$\Rightarrow \delta y = \tan(x+\delta x)^2 - \tan x^2$

$\Rightarrow \dfrac{\delta y}{\delta x} = \dfrac{\tan(x+\delta x)^2 - \tan x^2}{\delta x}$

$$\Rightarrow \frac{dy}{dx} = \lim_{\delta x \to 0}\frac{\delta y}{\delta x} = \lim_{\delta x \to 0}\frac{\tan(x+\delta x)^2 - \tan x^2}{\delta x}$$

$$= \lim_{\delta x \to 0}\frac{\left\{\dfrac{\sin(x+\delta x)^2}{\cos(x+\delta x)^2} - \dfrac{\sin x^2}{\cos x^2}\right\}}{\delta x}$$

$$= \lim_{\delta x \to 0}\frac{[\sin(x+\delta x)^2\cos x^2 - \cos(x+\delta x)^2\sin x^2]}{\delta x\cdot\cos(x+\delta x)^2\cdot\cos x^2}$$

$$= \lim_{\delta x \to 0}\frac{\sin[(x+\delta x)^2 - x^2]}{\delta x\cdot\cos(x+\delta x)^2\cdot\cos x^2}$$

$$= \lim_{\delta x \to 0}\frac{\sin[(2x+\delta x)\cdot\delta x]}{[(2x+\delta x)\cdot\delta x]}\cdot\frac{(2x+\delta x)}{\cos(x+\delta x)^2\cdot\cos x^2}$$

$$= \left(\lim_{\theta \to 0}\frac{\sin\theta}{\theta}\right)\cdot\lim_{\delta x \to 0}(2x+\delta x)\cdot\lim_{\delta x \to 0}\frac{1}{\cos(x+\delta x)^2\cdot\cos x^2},$$

where $(2x+\delta x)\cdot\delta x = \theta$ [clearly, $(\delta x \to 0) \Rightarrow (\theta \to 0)$]

$$= 1 \times 2x \times \frac{1}{\cos x^2 \cdot \cos x^2} = 2x \sec^2 x^2.$$

Hence, $\frac{d}{dx}(\tan x^2) = 2x \sec^2 x^2$.

EXAMPLE 20 *Differentiate xe^x from the first principle.*

SOLUTION Let $y = xe^x$.

Let δy be an increment in y, corresponding to an increment δx in x.

Then, $y + \delta y = (x + \delta x)\, e^{x + \delta x}$

$$\Rightarrow \quad \delta y = (x + \delta x)\, e^{x + \delta x} - xe^x$$

$$\Rightarrow \quad \frac{\delta y}{\delta x} = \frac{(x + \delta x)\, e^{x + \delta x} - xe^x}{\delta x}$$

$$\Rightarrow \quad \frac{dy}{dx} = \lim_{\delta x \to 0} \frac{\delta y}{\delta x}$$

$$= \lim_{\delta x \to 0} \frac{(x + \delta x)\, e^{x + \delta x} - xe^x}{\delta x}$$

$$= \lim_{\delta x \to 0} \frac{xe^{x + \delta x} - xe^x + \delta x \cdot e^{x + \delta x}}{\delta x} = \lim_{\delta x \to 0} \left[xe^x \cdot \left(\frac{e^{\delta x} - 1}{\delta x} \right) + e^{x + \delta x} \right]$$

$$= xe^x \cdot \lim_{\delta x \to 0} \left(\frac{e^{\delta x} - 1}{\delta x} \right) + \lim_{\delta x \to 0} e^{x + \delta x}$$

$$= (xe^x \times 1) + e^x \qquad \left[\because \lim_{\delta x \to 0} \left(\frac{e^{\delta x} - 1}{\delta x} \right) = 1 \right]$$

$$= (xe^x + e^x) = (x + 1)\, e^x.$$

Hence, $\frac{d}{dx}(xe^x) = (x + 1)\, e^x$.

EXAMPLE 21 *Differentiate $x^2 \cos x$ from the first principle.*

SOLUTION Let $y = x^2 \cos x$.

Let δy be an increment in y, corresponding to an increment δx in x.

Then, $y + \delta y = (x + \delta x)^2 \cos (x + \delta x)$

$$\Rightarrow \quad \delta y = (x + \delta x)^2 \cos (x + \delta x) - x^2 \cos x$$

$$\Rightarrow \quad \frac{\delta y}{\delta x} = \frac{(x + \delta x)^2 \cos (x + \delta x) - x^2 \cos x}{\delta x}$$

$$\Rightarrow \quad \frac{dy}{dx} = \lim_{\delta x \to 0} \frac{\delta y}{\delta x}$$

$$= \lim_{\delta x \to 0} \frac{(x + \delta x)^2 \cos (x + \delta x) - x^2 \cos x}{\delta x}$$

$$= \lim_{\delta x \to 0} \frac{(x^2 + \delta x^2 + 2x\delta x)\cos(x + \delta x) - x^2 \cos x}{\delta x}$$

$$= \lim_{\delta x \to 0} \frac{x^2[\cos(x + \delta x) - \cos x] + (\delta x)^2 \cos(x + \delta x) + 2x \cdot \delta x \cdot \cos(x + \delta x)}{\delta x}$$

$$= \lim_{\delta x \to 0} \left\{ \frac{x^2\left[-2\sin\left(x + \frac{\delta x}{2}\right)\sin\frac{\delta x}{2}\right] + (\delta x)^2 \cos(x + \delta x) + 2x \cdot \delta x \cdot \cos(x + \delta x)}{\delta x} \right\}$$

$$= \lim_{\delta x \to 0} \left[-x^2 \sin\left(x + \frac{\delta x}{2}\right) \cdot \frac{\sin\left(\frac{\delta x}{2}\right)}{\left(\frac{\delta x}{2}\right)} + (\delta x)\cos(x + \delta x) + 2x\cos(x + \delta x) \right]$$

$$= -x^2 \cdot \lim_{\delta x \to 0} \sin\left(x + \frac{\delta x}{2}\right) \cdot \lim_{\delta x \to 0} \frac{\sin\left(\frac{\delta x}{2}\right)}{\left(\frac{\delta x}{2}\right)} + \lim_{\delta x \to 0} (\delta x)\cos(x + \delta x) + 2x \cdot \lim_{\delta x \to 0} \cos(x + \delta x)$$

$$= (-x^2 \times \sin x \times 1) + 0 + 2x\cos x = (-x^2 \sin x + 2x \cos x).$$

Hence, $\frac{d}{dx}(x^2 \cos x) = (-x^2 \sin x + 2x \cos x)$.

EXERCISE 28B

Find the derivative of each of the following from the first principle:

1. $(ax + b)$ **2.** $\left(ax^2 + \frac{b}{x}\right)$ **3.** $3x^2 + 2x - 5$ **4.** $x^3 - 2x^2 + x + 3$

5. x^8 **6.** $\frac{1}{x^3}$ **7.** $\frac{1}{x^5}$ **8.** $\sqrt{ax + b}$

9. $\sqrt{5x - 4}$ **10.** $\frac{1}{\sqrt{x + 2}}$ **11.** $\frac{1}{\sqrt{2x + 3}}$ **12.** $\frac{1}{\sqrt{6x - 5}}$

13. $\frac{1}{\sqrt{2 - 3x}}$ **14.** $\frac{2x + 3}{3x + 2}$ **15.** $\frac{5 - x}{5 + x}$ **16.** $\frac{x^2 + 1}{x}, x \neq 0$

17. $\sqrt{\cos 3x}$ **18.** $\sqrt{\sec x}$ **19.** $\tan^2 x$ **20.** $\sin(2x + 3)$

21. $\tan(3x + 1)$

ANSWERS (EXERCISE 28B)

1. a **2.** $\left(2ax - \frac{b}{x^2}\right)$ **3.** $6x + 2$ **4.** $3x^2 - 4x + 1$

5. $8x^7$ **6.** $\frac{-3}{x^4}$ **7.** $\frac{-5}{x^6}$ **8.** $\frac{a}{2\sqrt{ax + b}}$

9. $\frac{5}{2\sqrt{5x - 4}}$ **10.** $\frac{-1}{2(x + 2)^{3/2}}$ **11.** $\frac{-1}{(2x + 3)^{3/2}}$ **12.** $\frac{-3}{(6x - 5)^{3/2}}$

13. $\frac{3}{2(2 - 3x)^{3/2}}$ **14.** $\frac{-5}{(3x + 2)^2}$ **15.** $\frac{-10}{(5 + x)^2}$ **16.** $\left(1 - \frac{1}{x^2}\right)$

17. $\frac{-3}{2} \cdot \frac{\sin 3x}{\sqrt{\cos 3x}}$ **18.** $\frac{1}{2}\tan x \sqrt{\sec x}$ **19.** $2\tan x \sec^2 x$ **20.** $2\cos(2x + 3)$

21. $3\sec^2(3x + 1)$

HINTS TO SOME SELECTED QUESTIONS

7. $y = x^{-5} \Rightarrow (y + \delta y)(x + \delta x)^{-5}$

$$\Rightarrow \frac{\delta y}{\delta x} = \frac{(x + \delta x)^{-5} - x^{-5}}{\delta x}$$

$$\therefore \frac{dy}{dx} = \lim_{\delta x \to 0} \frac{\delta y}{\delta x}$$

$$= \lim_{\delta x \to 0} \frac{(x + \delta x)^{-5} - x^{-5}}{\delta x} = \lim_{\delta x \to 0} \frac{x^{-5}\left\{\left(1 + \frac{\delta x}{x}\right)^{-5} - 1\right\}}{\delta x}$$

$$= \frac{1}{x^5} \cdot \lim_{\delta x \to 0} \frac{\left\{\left(1 + \frac{\delta x}{x}\right)^{-5} - 1\right\}}{\delta x}$$

$$= \frac{1}{x^5} \cdot \lim_{\delta x \to 0} \frac{\left\{1 + (-5) \cdot \frac{\delta x}{x} + \frac{(-5)(-6)}{2} \cdot \left(\frac{\delta x}{x}\right)^2 + \ldots - 1\right\}}{\delta x}$$

$$= \frac{1}{x^5} \cdot \lim_{\delta x \to 0} \frac{\left\{(-5) \cdot \frac{\delta x}{x} + 15 \cdot \left(\frac{\delta x}{x}\right)^2 + \ldots\right\}}{\delta x}$$

$$= \frac{1}{x^6} \cdot \lim_{\delta x \to 0} \left[(-5) + 15\left(\frac{\delta x}{x^2}\right) + \ldots\right]$$

$$= \frac{-5}{x^6}.$$

Hence, $\frac{d}{dx}(x^{-5}) = \frac{-5}{x^6}$.

9. $y = \sqrt{5x-4} \Rightarrow (y+\delta y) = \sqrt{5(x+\delta x)-4}$

$$\Rightarrow \frac{\delta y}{\delta x} = \frac{\sqrt{5(x+\delta x)-4} - \sqrt{5x-4}}{\delta x}.$$

$$\therefore \frac{dy}{dx} = \lim_{\delta x \to 0} \frac{\delta y}{\delta x}$$

$$= \lim_{\delta x \to 0} \frac{\sqrt{(5(x+\delta x)-4} - \sqrt{5x-4}}{\delta x}$$

$$= \lim_{\delta x \to 0} \frac{\{\sqrt{5(x+\delta x)-4} - \sqrt{5x-4}\}}{\delta x} \times \frac{\{\sqrt{5(x+\delta x)-4} + \sqrt{5x-4}\}}{\{\sqrt{5(x+\delta x)-4} + \sqrt{5x-4}\}}$$

$$= \lim_{\delta x \to 0} \frac{\{5(x+\delta x)-4\} - \{5x-4\}}{\delta x\{\sqrt{5(x+\delta x)-4} + \sqrt{5x-4}\}}$$

$$= \lim_{\delta x \to 0} \frac{5}{\{\sqrt{5(x+\delta x)-4} + \sqrt{5x-4}\}} = \frac{5}{2\sqrt{5x-4}}.$$

Hence, $\frac{d}{dx}(\sqrt{5x-4}) = \frac{5}{2\sqrt{5x-4}}$.

10. $y = \frac{1}{\sqrt{x+2}} \Rightarrow (y+\delta y) = \frac{1}{\sqrt{x+\delta x+2}}$

$$\Rightarrow \frac{\delta y}{\delta x} = \left\{\frac{1}{\sqrt{x+\delta x+2}} - \frac{1}{\sqrt{x+2}}\right\} \cdot \frac{1}{\delta x}$$

$$\therefore \frac{dy}{dx} = \lim_{\delta x \to 0} \frac{\delta y}{\delta x}$$

$$= \lim_{\delta x \to 0} \frac{1}{\delta x}\left\{\frac{1}{\sqrt{x+\delta x+2}} - \frac{1}{\sqrt{x+2}}\right\}$$

$$= \lim_{\delta x \to 0} \frac{1}{\delta x} \cdot \frac{(\sqrt{x+2} - \sqrt{x+\delta x+2})(\sqrt{x+2} + \sqrt{x+\delta x+2})}{(\sqrt{x+\delta x+2})(\sqrt{x+2})(\sqrt{x+2} + \sqrt{x+\delta x+2})}$$

$$= \lim_{\delta x \to 0} \frac{\{(x+2)-(x+\delta x+2)\}}{\delta x(\sqrt{x+\delta x+2})(\sqrt{x+2})} \cdot \frac{1}{(\sqrt{x+2} + \sqrt{x+\delta x+2})}$$

$$= \lim_{\delta x \to 0} \frac{-1}{(\sqrt{x+\delta x+2})(\sqrt{x+2})} \cdot \frac{1}{(\sqrt{x+2} + \sqrt{x+\delta x+2})}$$

$$= \frac{-1}{2(x+2)^{3/2}}.$$

Hence, $\frac{d}{dx}\left(\frac{1}{\sqrt{x+2}}\right) = \frac{-1}{2(x+2)^{3/2}}$.

13. $y = \frac{1}{\sqrt{2-3x}} \Rightarrow (y+\delta y) = \frac{1}{\sqrt{2-3(x+\delta x)}}$

$$\Rightarrow \frac{\delta y}{\delta x} = \frac{1}{\delta x} \cdot \left\{\frac{1}{\sqrt{2-3(x+\delta x)}} - \frac{1}{\sqrt{2-3x}}\right\}$$

$$\therefore \frac{dy}{dx} = \lim_{\delta x \to 0} \frac{\delta y}{\delta x} \lim_{\delta x \to 0} \frac{1}{\delta x} \cdot \left\{\frac{1}{\sqrt{2-3(x+\delta x)}} - \frac{1}{\sqrt{2-3x}}\right\}$$

$$= \lim_{\delta x \to 0} \frac{1}{\delta x} \cdot \frac{\{\sqrt{2-3x} - \sqrt{2-3(x+\delta x)}}{\{\sqrt{2-3(x+\delta x)}\}\{\sqrt{2-3x}\}} \cdot \frac{\{\sqrt{2-3x} + \sqrt{2-3(x+\delta x)}\}}{\{\sqrt{2-3x} + \sqrt{2-3(x+\delta x)}\}}$$

$$= \lim_{\delta x \to 0} \frac{[(2-3x) - \{2-3(x+\delta x)\}]}{\{\sqrt{2-3(x+\delta x)}\}\{\sqrt{2-3x}\}} \cdot \frac{1}{\{\sqrt{2-3x} + \sqrt{2-3(x+\delta x)}\}} \cdot \frac{1}{\delta x}$$

$$= \lim_{\delta x \to 0} \frac{3}{\{\sqrt{2-3(x+\delta x)}\}\{\sqrt{2-3x}\}} \cdot \frac{1}{\{\sqrt{2-3x} + \sqrt{2-3x(x+\delta x)}\}}$$

$$= \frac{3}{2(2-3x)\sqrt{2-3x}} = \frac{3}{2(2-3x)^{3/2}}.$$

Hence, $\dfrac{d}{dx}\left(\dfrac{1}{\sqrt{2-3x}}\right) = \dfrac{3}{2(2-3x)^{3/2}}$.

14. $y = \dfrac{2x+3}{3x+2} \Rightarrow (y+\delta y) = \dfrac{2(x+\delta x)+3}{3(x+\delta x)+2}$

$$\Rightarrow \frac{\delta y}{\delta x} = \frac{1}{\delta x} \cdot \left\{\frac{2x+3+2\delta x}{3x+2+3\delta x} - \frac{2x+3}{3x+2}\right\}.$$

$\therefore \dfrac{dy}{dx} = \lim\limits_{\delta x \to 0} \dfrac{\delta y}{\delta x}$

$$= \lim_{\delta x \to 0} \frac{(2x+3+2\delta x)(3x+2) - (2x+3)(3x+2+3\delta x)}{(3x+2+3\delta x)(3x+2)\delta x}$$

$$= \lim_{\delta x \to 0} \frac{\{(2x+3)(3x+2) - (2x+3)(3x+2)\} - 5\delta x}{(3x+2+3\delta x)(3x+2)\delta x}$$

$$= \lim_{\delta x \to 0} \frac{-5}{(3x+2+3\delta x)(3x+2)} = \frac{-5}{(3x+2)^2}.$$

Hence, $\dfrac{d}{dx}\left(\dfrac{2x+3}{3x+2}\right) = \dfrac{-5}{(3x+2)^2}$.

16. $\left(\dfrac{x^2+1}{x}\right) = \left(x + \dfrac{1}{x}\right)$.

18. $y = \sqrt{\sec x} \Rightarrow y + \delta y = \sqrt{\sec(x+\delta x)}$

$$\Rightarrow \delta y = \sqrt{\sec(x+\delta x)} - \sqrt{\sec x}.$$

$\therefore \dfrac{dy}{dx} = \lim\limits_{\delta x \to 0} \dfrac{\delta y}{\delta x}$

$$= \lim_{\delta x \to 0} \frac{\sqrt{\sec(x+\delta x)} - \sqrt{\sec x}}{\delta x} = \lim_{\delta x \to 0} \frac{\left\{\dfrac{1}{\sqrt{\cos(x+\delta x)}} - \dfrac{1}{\sqrt{\cos x}}\right\}}{\delta x}$$

$$= \lim_{\delta x \to 0} \left\{\frac{[\sqrt{\cos x} - \sqrt{\cos(x+\delta x)}]}{\sqrt{\cos(x+\delta x)} \cdot \sqrt{\cos x}} \cdot \frac{[\sqrt{\cos x} + \sqrt{\cos(x+\delta x)}}{\sqrt{\cos x} + \sqrt{\cos(x+\delta x)}} \cdot \frac{1}{\delta x}\right\}$$

$$= \lim_{\delta x \to 0} \left[\frac{[\cos x - \cos(x+\delta x)]}{\delta x[\sqrt{\cos(x+\delta x)} \cdot \sqrt{\cos x}} \cdot \frac{1}{[\sqrt{\cos x} + \sqrt{\cos(x+\delta x)}]}\right]$$

$$= \lim_{\delta x \to 0} \frac{2\sin\left(x + \dfrac{\delta x}{2}\right)\sin\dfrac{\delta x}{2}}{\delta x \cdot [\sqrt{\cos(x+\delta x)} \cdot \sqrt{\cos x}] \cdot [\sqrt{\cos x} + \sqrt{\cos(x+\delta x)}]}$$

$$= \lim_{\delta x \to 0} \sin\left(x + \frac{\delta x}{2}\right) \cdot \lim_{\delta x \to 0} \frac{\sin\left(\frac{\delta x}{2}\right)}{\left(\frac{\delta x}{2}\right)} \cdot$$

$$\lim_{\delta x \to 0} \frac{1}{\{\sqrt{\cos(x+\delta x)} \cdot \sqrt{\cos x}\}\{\sqrt{\cos x} + \sqrt{\cos(x+\delta x)}]}$$

$$= \left[\sin x \times 1 \times \frac{1}{(\cos x) \cdot 2\sqrt{\cos x}}\right] = \frac{1}{2} \tan x \sqrt{\sec x}.$$

19. $y = \tan^2 x \Rightarrow y + \delta y = \tan^2(x + \delta x)$

$$\Rightarrow \delta y = \tan^2(x+\delta x) - \tan^2 x$$

$$\Rightarrow \frac{\delta y}{\delta x} = \frac{\tan^2(x+\delta x) - \tan^2 x}{\delta x}.$$

$$\therefore \frac{dy}{dx} = \lim_{\delta x \to 0} \frac{\delta y}{\delta x}$$

$$= \lim_{\delta x \to 0} \frac{\left\{\frac{\sin^2(x+\delta x)}{\cos^2(x+\delta x)} - \frac{\sin^2 x}{\cos^2 x}\right\}}{\delta x}$$

$$= \lim_{\delta x \to 0} \frac{\sin^2(x+\delta x)\cos^2 x - \sin^2 x \cos^2(x+\delta x)}{\delta x[\cos^2(x+\delta x) \cdot \cos^2 x]}$$

$$= \lim_{\delta x \to 0} \frac{[\sin(x+\delta x)\cos x + \sin x \cos(x+\delta x)][\sin(x+\delta x)\cos x - \sin x \cos(x+\delta x)]}{\delta x[\cos^2(x+\delta x) \cdot \cos^2 x]}$$

$$= \lim_{\delta x \to 0} \frac{\sin(x+\delta x+x) \cdot \sin(x+\delta x - x)}{\delta x[\cos^2(x+\delta x) \cdot \cos^2 x]} = \lim_{\delta x \to 0} \frac{\sin(2x+\delta x) \cdot \sin \delta x}{\delta x[\cos^2(x+\delta x) \cdot \cos^2 x]}$$

$$= \lim_{\delta x \to 0} \frac{\sin(2x+\delta x)}{\cos^2(x+\delta x) \cdot \cos^2 x} \cdot \lim_{\delta x \to 0} \frac{\sin \delta x}{\delta x}$$

$$= \left(\frac{\sin 2x}{\cos^4 x} \times 1\right) = \frac{2\sin x \cos x}{\cos^4 x} = 2\tan x \sec^2 x.$$

Some More Results on Differentiation

DERIVATIVE OF THE PRODUCT OF FUNCTIONS

THEOREM (Product rule) *If $f(x)$ and $g(x)$ are two differentiable functions then $f(x) \cdot g(x)$ is also differentiable, and*

$$\frac{d}{dx}\{f(x) \cdot g(x)\} = f(x) \cdot \frac{d}{dx}\{g(x)\} + g(x) \cdot \frac{d}{dx}\{f(x)\}.$$

PROOF Let $y = f(x) \cdot g(x)$ and $y + \delta y = f(x + \delta x) \cdot g(x + \delta x)$.

Then, $\dfrac{\delta y}{\delta x} = \dfrac{f(x+\delta x)\cdot g(x+\delta x) - f(x)\cdot g(x)}{\delta x}$.

$\therefore \dfrac{dy}{dx} = \lim\limits_{\delta x \to 0} \dfrac{\delta y}{\delta x} = \lim\limits_{\delta x \to 0} \dfrac{f(x+\delta x)\cdot g(x+\delta x) - f(x)\cdot g(x)}{\delta x}$

$= \lim\limits_{\delta x \to 0} \dfrac{f(x+\delta x)\cdot g(x+\delta x) - f(x+\delta x)\cdot g(x) + f(x+\delta x)\cdot g(x) - f(x)\cdot g(x)}{\delta x}$

[adding and subtracting $f(x+\delta x)\cdot g(x)$ in num.]

$= \lim\limits_{\delta x \to 0} \dfrac{f(x+\delta x)\cdot \{g(x+\delta x) - g(x)\} + g(x)\cdot \{f(x+\delta x) - f(x)\}}{\delta x}$

$= \lim\limits_{\delta x \to 0} f(x+\delta x)\cdot \dfrac{\{g(x+\delta x) - g(x)\}}{\delta x} + \lim\limits_{\delta x \to 0} g(x)\cdot \left\{\dfrac{f(x+\delta x) - f(x)}{\delta x}\right\}$

$= \lim\limits_{\delta x \to 0} f(x+\delta x)\cdot \lim\limits_{\delta x \to 0} \dfrac{g(x+\delta x) - g(x)}{\delta x} + g(x)\cdot \lim\limits_{\delta x \to 0} \dfrac{f(x+\delta x) - f(x)}{\delta x}$

$= f(x)\cdot \dfrac{d}{dx}\{g(x)\} + g(x)\cdot \dfrac{d}{dx}\{f(x)\}$.

$\therefore \dfrac{d}{dx}\{f(x)\cdot g(x)\} = f(x)\cdot \dfrac{d}{dx}\{g(x)\} + g(x)\cdot \dfrac{d}{dx}\{f(x)\}$.

Now, $f(x)$ and $g(x)$ being differentiable, it follows that $\dfrac{d}{dx}\{f(x)\}$ as well as $\dfrac{d}{dx}\{g(x)\}$ exists.

Consequently, $f(x)\cdot \dfrac{d}{dx}\{g(x)\} + g(x)\cdot \dfrac{d}{dx}\{f(x)\}$ exists.

So, from the above result, we conclude that $\dfrac{d}{dx}\{f(x)\cdot g(x)\}$ exists and therefore, $f(x)\cdot g(x)$ is differentiable.

REMARK The above result may be expressed as $\dfrac{d}{dx}[uv] = u\cdot \dfrac{dv}{dx} + v\cdot \dfrac{du}{dx}$.
It may be remembered as
derivative of the product of two functions
= [(1*st function*) × (*derivative of* 2*nd*)]
+ [(2*nd function*) × (*derivative of* 1*st*)].

SOLVED EXAMPLES

EXAMPLE 1 *Differentiate:* (i) xe^x (ii) $x^2 e^x \sin x$

SOLUTION (i) $\dfrac{d}{dx}(xe^x) = x\cdot \dfrac{d}{dx}(e^x) + e^x\cdot \dfrac{d}{dx}(x) = (xe^x + e^x\cdot 1) = e^x(x+1)$.

$$\text{(ii)}\ \frac{d}{dx}(x^2e^x \sin x) = \frac{d}{dx}[(x^2e^x)\sin x]$$
$$= (x^2e^x)\cdot\frac{d}{dx}(\sin x) + \sin x\cdot\frac{d}{dx}(x^2e^x)$$
$$= x^2e^x\cos x + \sin x\cdot\left\{x^2\cdot\frac{d}{dx}(e^x) + e^x\cdot\frac{d}{dx}(x^2)\right\}$$
$$= x^2e^x\cos x + \sin x\cdot[x^2e^x + 2xe^x]$$
$$= x^2e^x\cos x + x^2e^x\sin x + 2xe^x\sin x$$
$$= xe^x[x\cos x + x\sin x + 2\sin x].$$

EXAMPLE 2 *Differentiate* $x^2\tan x$.

SOLUTION We have $\frac{d}{dx}(x^2\tan x)$

$$= \left\{x^2\cdot\frac{d}{dx}(\tan x) + \tan x\cdot\frac{d}{dx}(x^2)\right\}$$
$$= (x^2\sec^2 x + 2x\tan x).$$

EXAMPLE 3 *Differentiate* $(e^x\sin x + x^p\cos x)$.

SOLUTION We have $\frac{d}{dx}(e^x\sin x + x^p\cos x)$

$$= \frac{d}{dx}(e^x\sin x) + \frac{d}{dx}(x^p\cos x)$$
$$= \left\{e^x\cdot\frac{d}{dx}(\sin x) + \sin x\cdot\frac{d}{dx}(e^x)\right\} + \left\{x^p\cdot\frac{d}{dx}(\cos x) + \cos x\cdot\frac{d}{dx}(x^p)\right\}$$
$$= (e^x\cos x + e^x\sin x) + [x^p(-\sin x) + \cos x\cdot(px^{p-1})]$$
$$= e^x(\cos x + \sin x) - x^p\sin x + px^{p-1}\cos x$$
$$= e^x(\cos x + \sin x) - x^{p-1}(x\sin x - p\cos x).$$

EXAMPLE 4 *Differentiate* $e^x(x^3 + \sqrt{x})$.

SOLUTION $\frac{d}{dx}[e^x(x^3+\sqrt{x})] = e^x\cdot\frac{d}{dx}(x^3+\sqrt{x}) + (x^3+\sqrt{x})\cdot\frac{d}{dx}(e^x)$

$$= e^x\left\{3x^2 + \frac{1}{2}x^{-1/2}\right\} + (x^3+\sqrt{x})\cdot e^x$$
$$= e^x\left\{x^3 + 3x^2 + \frac{1}{2\sqrt{x}} + \sqrt{x}\right\}.$$

EXAMPLE 5 *Differentiate* $\left(\frac{e^x\cos x}{x^3}\right)$, *using the product rule.*

SOLUTION We have $\frac{d}{dx}\left\{\frac{e^x\cos x}{x^3}\right\} = \frac{d}{dx}[(e^x\cos x)\cdot x^{-3}]$

$$= e^x\cos x\cdot\frac{d}{dx}(x^{-3}) + x^{-3}\cdot\frac{d}{dx}(e^x\cos x)$$

$$= e^x \cos x \cdot (-3x^{-4}) + x^{-3} \cdot \left\{ e^x \cdot \frac{d}{dx}(\cos x) + \cos x \cdot \frac{d}{dx}(e^x) \right\}$$

$$= \frac{-3e^x \cos x}{x^4} + \frac{1}{x^3}[-e^x \sin x + e^x \cos x]$$

$$= \frac{-3e^x \cos x - xe^x \sin x + xe^x \cos x}{x^4} = \frac{e^x[(x-3)\cos x - x \sin x]}{x^4}.$$

EXAMPLE 6 *If u, v, w are differentiable functions of x, prove that*

$$\frac{d}{dx}(uvw) = (uv) \cdot \frac{dw}{dx} + (wu) \cdot \frac{dv}{dx} + (wv) \cdot \frac{du}{dx}.$$

SOLUTION We have $\frac{d}{dx}(uvw) = \frac{d}{dx}\{(uv)\, w\} = (uv) \cdot \frac{dw}{dx} + w \cdot \frac{d}{dx}(uv)$

$$= (uv) \cdot \frac{dw}{dx} + w \cdot \left\{ u \cdot \frac{dv}{dx} + v \cdot \frac{du}{dx} \right\}$$

$$= (uv) \cdot \frac{dw}{dx} + (wu) \cdot \frac{dv}{dx} + (wv) \cdot \frac{du}{dx}.$$

EXERCISE 28C

Differentiate:

1. $x^2 \sin x$

2. $e^x \cos x$

3. $e^x \cot x$

4. $x^n \cot x$

5. $x^3 \sec x$

6. $(x^2 + 3x + 1) \sin x$

7. $x^4 \tan x$

8. $(3x - 5)(4x^2 - 3 + e^x)$

9. $(x^2 - 4x + 5)(x^3 - 2)$

10. $(x^2 + 2x - 3)(x^2 + 7x + 5)$

11. $(\tan x + \sec x)(\cot x + \operatorname{cosec} x)$

12. $(x^3 \cos x - 2^x \tan x)$

ANSWERS (EXERCISE 28C)

1. $(x^2 \cos x + 2x \sin x)$

2. $e^x(\cos x - \sin x)$

3. $e^x (\cot x - \operatorname{cosec}^2 x)$

4. $x^{n-1}(n \cot x - x \operatorname{cosec}^2 x)$

5. $x^3 \sec x \tan x + 3x^2 \sec x$

6. $(x^2 + 3x + 1) \cos x + (2x + 3) \sin x$

7. $x^4 \sec^2 x + 4x^3 \tan x$

8. $(36x^2 - 40x + 3xe^x - 2e^x - 9)$

9. $5x^4 - 16x^3 + 15x^2 - 4x + 8$

10. $4x^3 + 27x^2 + 32x - 11$

11. $(\sec x - \operatorname{cosec} x)(\sec x + \tan x)(\operatorname{cosec} x + \cot x)$

12. $x^2(3 \cos x - x \sin x) - 2^x[(\log 2) \tan x + \sec^2 x]$

Derivative of the Quotient of two Functions

THEOREM (Quotient rule) *If $f(x)$ and $g(x)$ are two differentiable functions and $g(x) \neq 0$ then $\dfrac{f(x)}{g(x)}$ is also differentiable, and*

$$\frac{d}{dx}\left\{\frac{f(x)}{g(x)}\right\} = \frac{g(x)\cdot\dfrac{d}{dx}\{f(x)\} - f(x)\cdot\dfrac{d}{dx}\{g(x)\}}{[g(x)]^2}.$$

PROOF Let $y = \dfrac{f(x)}{g(x)}$ and $y + \delta y = \dfrac{f(x+\delta x)}{g(x+\delta x)}$.

Then, $\dfrac{\delta y}{\delta x} = \dfrac{1}{\delta x}\cdot\left\{\dfrac{f(x+\delta x)}{g(x+\delta x)} - \dfrac{f(x)}{g(x)}\right\}$.

$$\therefore \quad \frac{dy}{dx} = \lim_{\delta x \to 0}\frac{\delta y}{\delta x} = \lim_{\delta x \to 0}\frac{1}{\delta x}\left\{\frac{f(x+\delta x)}{g(x+\delta x)} - \frac{f(x)}{g(x)}\right\}$$

$$= \lim_{\delta x \to 0}\frac{g(x)\cdot f(x+\delta x) - g(x+\delta x)\cdot f(x)}{\delta x \cdot g(x+\delta x)\cdot g(x)}$$

$$= \lim_{\delta x \to 0}\frac{g(x)\cdot f(x+\delta x) - g(x)\cdot f(x) + g(x)\cdot f(x) - g(x+\delta x)\cdot f(x)}{\delta x \cdot g(x+\delta x)\cdot g(x)}$$

[on subtracting and adding $g(x)\cdot f(x)$ in numerator]

$$= \left[\lim_{\delta x \to 0} g(x)\cdot\left\{\frac{f(x+\delta x) - f(x)}{\delta x}\right\} - \lim_{\delta x \to 0} f(x)\cdot\left\{\frac{g(x+\delta x) - g(x)}{\delta x}\right\}\right]$$

$$\times\left[\lim_{\delta x \to 0}\frac{1}{g(x+\delta x)\cdot g(x)}\right]$$

$$= \left[g(x)\cdot\frac{d}{dx}\{f(x)\} - f(x)\cdot\frac{d}{dx}\{g(x)\}\right]\times\frac{1}{[g(x)]^2}$$

$$= \frac{g(x)\cdot\dfrac{d}{dx}\{f(x)\} - f(x)\cdot\dfrac{d}{dx}\{g(x)\}}{[g(x)]^2}$$

$$\therefore \quad \frac{d}{dx}\left\{\frac{f(x)}{g(x)}\right\} = \frac{g(x)\cdot\dfrac{d}{dx}\{f(x)\} - f(x)\cdot\dfrac{d}{dx}\{g(x)\}}{[g(x)]^2}.$$

Now, each of $f(x)$ and $g(x)$ being differentiable, it follows that $\dfrac{d}{dx}\{f(x)\}$ and $\dfrac{d}{dx}\{g(x)\}$ both exist.

So, by the above result, it follows that $\dfrac{d}{dx}\left\{\dfrac{f(x)}{g(x)}\right\}$ exists, and hence $\dfrac{f(x)}{g(x)}$ is differentiable.

REMARK The above result may be expressed as $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{\left\{v \cdot \frac{du}{dx} - u \cdot \frac{dv}{dx}\right\}}{v^2}$, i.e.,

derivative of the quotient of two functions

$$= \frac{(\textit{denom.} \times \textit{derivative of num.}) - (\textit{num.} \times \textit{derivative of denom.})}{(\textit{denominator})^2}$$

SOLVED EXAMPLES

EXAMPLE 1 *Differentiate:*

(i) $\frac{e^x}{x}$ (ii) $\left(\frac{2x+3}{x^2-5}\right)$ (iii) $\frac{e^x}{(1+\sin x)}$

SOLUTION (i) $\frac{d}{dx}\left(\frac{e^x}{x}\right) = \frac{x \cdot \frac{d}{dx}(e^x) - e^x \cdot \frac{d}{dx}(x)}{x^2} = \frac{xe^x - e^x \cdot 1}{x^2} = \frac{e^x(x-1)}{x^2}$.

(ii) $\frac{d}{dx}\left(\frac{2x+3}{x^2-5}\right) = \frac{(x^2-5) \cdot \frac{d}{dx}(2x+3) - (2x+3) \cdot \frac{d}{dx}(x^2-5)}{(x^2-5)^2}$

$$= \frac{(x^2-5) \cdot 2 - (2x+3) \cdot 2x}{(x^2-5)^2} = \frac{-2(x^2+3x+5)}{(x^2-5)^2}.$$

(iii) $\frac{d}{dx}\left(\frac{e^x}{1+\sin x}\right) = \frac{(1+\sin x) \cdot \frac{d}{dx}(e^x) - e^x \cdot \frac{d}{dx}(1+\sin x)}{(1+\sin x)^2}$

$$= \frac{(1+\sin x) \cdot e^x - e^x(\cos x)}{(1+\sin x)^2} = \frac{(1+\sin x - \cos x)\, e^x}{(1+\sin x)^2}.$$

EXAMPLE 2 *Differentiate* $\left(\frac{x^2+5x-6}{4x^2-x+3}\right)$.

SOLUTION Using the quotient rule, we have

$$\frac{d}{dx}\left(\frac{x^2+5x-6}{4x^2-x+3}\right)$$

$$= \frac{(4x^2-x+3) \cdot \frac{d}{dx}(x^2+5x-6) - (x^2+5x-6) \cdot \frac{d}{dx}(4x^2-x+3)}{(4x^2-x+3)^2}$$

$$= \frac{(4x^2-x+3)(2x+5) - (x^2+5x-6)(8x-1)}{(4x^2-x+3)^2} = \frac{(9+54x-21x^2)}{(4x^2-x+3)^2}.$$

EXAMPLE 3 *Differentiate* $\left(\frac{x^2 \sin x}{1-x}\right)$.

SOLUTION By the quotient rule, we have

$$\frac{d}{dx}\left(\frac{x^2 \sin x}{1-x}\right) = \frac{(1-x)\cdot\frac{d}{dx}(x^2 \sin x) - x^2 \sin x \cdot \frac{d}{dx}(1-x)}{(1-x)^2}$$

$$= \frac{(1-x)\left\{x^2 \cdot \frac{d}{dx}(\sin x) + \sin x \cdot \frac{d}{dx}(x^2)\right\} - (x^2 \sin x)(-1)}{(1-x)^2}$$

$$= \frac{(1-x)[x^2 \cos x + 2x \sin x] + x^2 \sin x}{(1-x)^2}$$

$$= \frac{x^2(1-x)\cos x + (x \sin x)(2-x)}{(1-x)^2}.$$

EXAMPLE 4 *If* $y = \left\{\frac{1-\tan x}{1+\tan x}\right\}$, *show that* $\frac{dy}{dx} = \frac{-2}{(1+\sin 2x)}$.

SOLUTION By the quotient rule, we have

$$\frac{dy}{dx} = \frac{(1+\tan x)\cdot\frac{d}{dx}(1-\tan x) - (1-\tan x)\cdot\frac{d}{dx}(1+\tan x)}{(1+\tan x)^2}$$

$$= \frac{(1+\tan x)(-\sec^2 x) - (1-\tan x)(\sec^2 x)}{(1+\tan x)^2}$$

$$= \frac{-2\sec^2 x}{(1+\tan x)^2} = \frac{-2}{(\cos^2 x)(1+\tan^2 x + 2\tan x)}$$

$$= \frac{-2}{(\cos^2 x)\left\{1 + \frac{\sin^2 x}{\cos^2 x} + \frac{2\sin x}{\cos x}\right\}} = \frac{-2}{(1+\sin 2x)}.$$

EXAMPLE 5 *Differentiate:*

(i) $\left(\frac{\sin x + \cos x}{\sin x - \cos x}\right)$ *(ii)* $\left(\frac{\sec x - 1}{\sec x + 1}\right)$

SOLUTION (i) $\frac{d}{dx}\left(\frac{\sin x + \cos x}{\sin x - \cos x}\right)$

$$= \frac{(\sin x - \cos x)\cdot\frac{d}{dx}(\sin x + \cos x) - (\sin x + \cos x)\cdot\frac{d}{dx}(\sin x - \cos x)}{(\sin x - \cos x)^2}$$

$$= \frac{(\sin x - \cos x)(\cos x - \sin x) - (\sin x + \cos x)(\cos x + \sin x)}{(\sin x - \cos x)^2}$$

$$= \frac{-[(\sin x - \cos x)^2 + (\sin x + \cos x)^2]}{(\sin x - \cos x)^2}$$

$$= \frac{-2(\sin^2 x + \cos^2 x)}{(\sin^2 x + \cos^2 x - 2\sin x \cos x)} = \frac{-2}{(1 - \sin 2x)}.$$

(ii) $\dfrac{d}{dx}\left(\dfrac{\sec x-1}{\sec x+1}\right)$

$$=\frac{(\sec x+1)\cdot\dfrac{d}{dx}(\sec x-1)-(\sec x-1)\cdot\dfrac{d}{dx}(\sec x+1)}{(\sec x+1)^2}$$

$$=\frac{(\sec x+1)\sec x\tan x-(\sec x-1)\sec x\tan x}{(\sec x+1)^2}=\frac{2\sec x\ \tan x}{(\sec x+1)^2}.$$

EXERCISE 28D

Differentiate:

1. $\dfrac{2^x}{x}$ **2.** $\dfrac{\log x}{x}$ **3.** $\dfrac{e^x}{(1+x)}$ **4.** $\dfrac{e^x}{(1+x^2)}$

5. $\left(\dfrac{2x^2-4}{3x^2+7}\right)$ **6.** $\left(\dfrac{x^2+3x-1}{x+2}\right)$ **7.** $\dfrac{(x^2-1)}{(x^2+7x+1)}$ **8.** $\left(\dfrac{5x^2+6x+7}{2x^2+3x+4}\right)$

9. $\dfrac{x}{(a^2+x^2)}$ **10.** $\dfrac{x^4}{\sin x}$ **11.** $\dfrac{\sqrt{a}+\sqrt{x}}{\sqrt{a}-\sqrt{x}}$ **12.** $\dfrac{\cos x}{\log x}$

13. $\dfrac{2\cot x}{\sqrt{x}}$ **14.** $\dfrac{\sin x}{(1+\cos x)}$ **15.** $\left(\dfrac{1+\sin x}{1-\sin x}\right)$ **16.** $\left(\dfrac{1-\cos x}{1+\cos x}\right)$

17. $\left(\dfrac{\sin x-\cos x}{\sin x+\cos x}\right)$ **18.** $\left(\dfrac{\sec x-\tan x}{\sec x+\tan x}\right)$ **19.** $\left(\dfrac{e^x+\sin x}{1+\log x}\right)$

20. $\dfrac{e^x\sin x}{\sec x}$ **21.** $\dfrac{2^x\cot x}{\sqrt{x}}$ **22.** $\dfrac{e^x(x-1)}{(x+1)}$

23. $\dfrac{x\tan x}{(\sec x+\tan x)}$ **24.** $\left(\dfrac{ax^2+bx+c}{px^2+qx+r}\right)$ **25.** $\left(\dfrac{\sin x-x\cos x}{x\sin x+\cos x}\right)$

26. (i) $\cot x$ (ii) $\sec x$ [**Hint** Write $\cot x=\dfrac{\cos x}{\sin x}$ and $\sec x=\dfrac{1}{\cos x}$.]

ANSWERS (EXERCISE 28D)

1. $\dfrac{2^x(x\log 2-1)}{x^2}$ **2.** $\dfrac{(1-\log x)}{x^2}$ **3.** $\dfrac{xe^x}{(1+x)^2}$ **4.** $\dfrac{e^x(1-x)^2}{(1+x^2)^2}$

5. $\dfrac{52x}{(3x^2+7)^2}$ **6.** $\dfrac{(x^2+4x+7)}{(x+2)^2}$ **7.** $\dfrac{(7x^2+4x+7)}{(x^2+7x+1)^2}$

8. $\dfrac{3(x^2+4x+1)}{(2x^2+3x+4)^2}$ **9.** $\dfrac{(a^2-x^2)}{(a^2+x^2)^2}$ **10.** $\dfrac{x^3(4\sin x-x\cos x)}{\sin^2 x}$

11. $\dfrac{\sqrt{a}}{\sqrt{x}\cdot(\sqrt{a}-\sqrt{x})^2}$

12. $\dfrac{-(x\sin x \log x + \cos x)}{x(\log x)^2}$

13. $\dfrac{-(2x\text{cosec}^2 x + \cot x)}{x^{3/2}}$

14. $\dfrac{1}{(1+\cos x)}$

15. $\dfrac{2\cos x}{(1-\sin x)^2}$

16. $\dfrac{2\sin x}{(1+\cos x)^2}$

17. $\dfrac{2}{(1+\sin 2x)}$

18. $\dfrac{2\sec x\,(\tan x - \sec x)}{(\sec x + \tan x)}$

19. $\dfrac{x(e^x+\cos x)(1+\log x)-(e^x+\sin x)}{x(1+\log x)^2}$

20. $e^x[\cos 2x + \sin x \cos x]$

21. $\dfrac{2^x[-x\text{cosec}^2 x + x(\log 2)\cot x - \frac{1}{2}\log x]}{x^{3/2}}$

22. $\dfrac{e^x(x^2+1)}{(x+1)^2}$

23. $\dfrac{x\sec x(\sec x - \tan x) + \tan x}{(\sec x + \tan x)}$

24. $\dfrac{(aq-bp)x^2 + 2(ar-pc)x + (br-cq)}{(px^2+qx+r)^2}$

25. $\dfrac{x^2}{(x\sin x + \cos x)^2}$

26. (i) $-\text{cosec}^2 x$ (ii) $\sec x \tan x$

Derivative of a Function of a Function

CHAIN RULE If $y = f(t)$ and $t = g(x)$ then $\dfrac{dy}{dx} = \left(\dfrac{dy}{dt} \times \dfrac{dt}{dx}\right)$.

This rule may be extended further.

If $y = f(t), t = g(u)$ and $u = h(x)$ then $\dfrac{dy}{dx} = \left(\dfrac{dy}{dt} \times \dfrac{dt}{du} \times \dfrac{du}{dx}\right)$.

SOLVED EXAMPLES

EXAMPLE 1 *Differentiate* *(i)* $\sin x^3$ *(ii)* $\sin^3 x$ *(iii)* $e^{\sin x}$

SOLUTION (i) Let $y = \sin x^3$.

Put $x^3 = t$, so that $y = \sin t$ and $t = x^3$.

$$\therefore \quad \frac{dy}{dt} = \cos t \text{ and } \frac{dt}{dx} = 3x^2.$$

$$\text{So,} \quad \frac{dy}{dx} = \left(\frac{dy}{dt} \times \frac{dt}{dx}\right)$$

$$= 3x^2 \cos t = 3x^2 \cos x^3 \qquad [\because\ t = x^3].$$

$$\text{Hence,} \quad \frac{d}{dx}(\sin x^3) = 3x^2 \cos x^3.$$

(ii) Let $y = \sin^3 x = (\sin x)^3$.

Put $\sin x = t$, so that $y = t^3$ and $t = \sin x$.

$\therefore \quad \frac{dy}{dt} = 3t^2$ and $\frac{dt}{dx} = \cos x.$

Hence, $\frac{dy}{dx} = \left(\frac{dy}{dt} \times \frac{dt}{dx}\right)$

$= 3t^2 \cos x = 3\sin^2 x \cos x \quad [\because \ t = \sin x].$

(iii) Let $y = e^{\sin x}$.

Put $\sin x = t$, so that $y = e^t$ and $t = \sin x$

$\therefore \quad \frac{dy}{dt} = e^t$ and $\frac{dt}{dx} = \cos x.$

Hence, $\frac{dy}{dx} = \left(\frac{dy}{dt} \times \frac{dt}{dx}\right)$

$= e^t \cos x = e^{\sin x} \cdot \cos x \quad [\because \ t = \sin x].$

EXAMPLE 2 *If* $y = \frac{1}{\sqrt{a^2 - x^2}}$, *find* $\frac{dy}{dx}$.

SOLUTION Put $(a^2 - x^2) = t$, so that $y = \frac{1}{\sqrt{t}} = t^{-1/2}$ and $t = (a^2 - x^2)$.

$\therefore \quad \frac{dy}{dt} = -\frac{1}{2}t^{-3/2}$ and $\frac{dt}{dx} = -2x.$

So, $\frac{dy}{dx} = \left(\frac{dy}{dt} \times \frac{dt}{dx}\right)$

$= \left(-\frac{1}{2}t^{-3/2}\right)(-2x) = xt^{-3/2} = x(a^2 - x^2)^{-3/2}.$

EXAMPLE 3 *Differentiate:*

(i) $(ax + b)^m$ *(ii)* $(3x + 5)^6$ *(iii)* $\sqrt{ax^2 + 2bx + c}$

SOLUTION (i) Let $y = (ax + b)^m$.

Put $(ax + b) = t$, so that $y = t^m$ and $t = (ax + b)$.

$\therefore \quad \frac{dy}{dt} = mt^{m-1}$ and $\frac{dt}{dx} = a.$

So, $\frac{dy}{dx} = \left(\frac{dy}{dt} \times \frac{dt}{dx}\right)$

$= amt^{m-1} = am\,(ax + b)^{m-1} \quad [\because \ t = (ax + b)].$

(ii) Let $y = (3x + 5)^6$.

Put $(3x + 5) = t$, so that $y = t^6$ and $t = (3x + 5)$.

$\therefore \quad \frac{dy}{dt} = 6t^5$ and $\frac{dt}{dx} = 3.$

So, $\dfrac{dy}{dx} = \left(\dfrac{dy}{dt} \times \dfrac{dt}{dx}\right)$

$= 18t^5 = 18(3x+5)^5 \qquad [\because\ t = (3x+5)].$

(iii) Let $y = \sqrt{ax^2 + 2bx + c}$.

Put $(ax^2 + 2bx + c) = t$, so that $y = \sqrt{t}$ and $t = (ax^2 + 2bx + c)$.

$\therefore \quad \dfrac{dy}{dt} = \dfrac{1}{2}t^{-1/2} = \dfrac{1}{2\sqrt{t}}$ and $\dfrac{dt}{dx} = (2ax + 2b)$.

So, $\dfrac{dy}{dx} = \left(\dfrac{dy}{dt} \times \dfrac{dt}{dx}\right) = \dfrac{1}{2\sqrt{t}} \times 2(ax+b)$

$= \dfrac{(ax+b)}{\sqrt{t}} = \dfrac{(ax+b)}{\sqrt{ax^2 + 2bx + c}}.$

EXAMPLE 4 *Differentiate* $e^{\sqrt{\cot x}}$.

SOLUTION Let $y = e^{\sqrt{\cot x}}$. Put $\cot x = t$ and $\sqrt{\cot x} = \sqrt{t} = u$, so that

$$y = e^u,\ u = \sqrt{t} \text{ and } t = \cot x.$$

$\therefore \quad \dfrac{dy}{du} = e^u,\ \dfrac{du}{dt} = \dfrac{1}{2}t^{-1/2} = \dfrac{1}{2\sqrt{t}}$ and $\dfrac{dt}{dx} = -\operatorname{cosec}^2 x$.

So, $\dfrac{dy}{dx} = \left(\dfrac{dy}{du} \times \dfrac{du}{dt} \times \dfrac{dt}{dx}\right) = -\dfrac{1}{2}\cdot\dfrac{\operatorname{cosec}^2 x}{\sqrt{t}}\ e^u$

$= \dfrac{-\operatorname{cosec}^2 x}{2\sqrt{t}} \cdot e^{\sqrt{t}} \qquad [\because\ u = \sqrt{t}]$

$= \dfrac{-\operatorname{cosec}^2 x}{2\sqrt{\cot x}} \cdot e^{\sqrt{\cot x}} \qquad [\because\ t = \cot x].$

EXAMPLE 5 *If* $y = \cos^2 x^2$, *find* $\dfrac{dy}{dx}$.

SOLUTION $y = (\cos x^2)^2$. Put $x^2 = t$ and $\cos x^2 = \cos t = u$, so that

$$y = u^2,\ u = \cos t \text{ and } t = x^2.$$

$\therefore \quad \dfrac{dy}{du} = 2u,\ \dfrac{du}{dt} = -\sin t$ and $\dfrac{dt}{dx} = 2x$.

So, $\dfrac{dy}{dx} = \left(\dfrac{dy}{du} \times \dfrac{du}{dt} \times \dfrac{dt}{dx}\right)$

$= -4ux \sin t = -4x \sin t \cos t \qquad [\because\ u = \cos t]$

$= -4x \sin x^2 \cos x^2 = -2x \sin(2x^2) \qquad [\because\ t = x^2].$

EXAMPLE 6 *Differentiate* $\sqrt{\dfrac{1 - \tan x}{1 + \tan x}}$.

SOLUTION Let $y = \sqrt{\dfrac{1 - \tan x}{1 + \tan x}}$.

Putting $\dfrac{1 - \tan x}{1 + \tan x} = t$, we get $y = \sqrt{t}$ and $t = \dfrac{1 - \tan x}{1 + \tan x}$.

$$\therefore \quad \frac{dy}{dt} = \frac{1}{2} t^{-1/2} = \frac{1}{2\sqrt{t}}.$$

$$\text{And, } \frac{dt}{dx} = \frac{(1 + \tan x) \cdot \frac{d}{dx}(1 - \tan x) - (1 - \tan x) \cdot \frac{d}{dx}(1 + \tan x)}{(1 + \tan x)^2}$$

$$= \frac{(1 + \tan x)(-\sec^2 x) - (1 - \tan x)(\sec^2 x)}{(1 + \tan x)^2} = \frac{-2 \sec^2 x}{(1 + \tan x)^2}.$$

$$\therefore \quad \frac{dy}{dx} = \left(\frac{dy}{dt} \times \frac{dt}{dx}\right) = \frac{1}{2\sqrt{t}} \times \frac{-2 \sec^2 x}{(1 + \tan x)^2}$$

$$= \frac{-\sec^2 x}{(1 + \tan x)^2} \times \frac{\sqrt{1 + \tan x}}{\sqrt{1 - \tan x}}$$

$$= \frac{-\sec^2 x}{(1 + \tan x)^{3/2} (1 - \tan x)^{1/2}}.$$

EXAMPLE 7 *If* $y = \sin(\sqrt{\sin x + \cos x})$, *find* $\dfrac{dy}{dx}$.

SOLUTION Putting $(\sin x + \cos x) = t$ and $\sqrt{(\sin x + \cos x)} = \sqrt{t} = u$, we get

$$y = \sin u, \quad u = \sqrt{t} \text{ and } t = (\sin x + \cos x).$$

$$\therefore \quad \frac{dy}{du} = \cos u, \quad \frac{du}{dt} = \frac{1}{2} t^{-1/2} = \frac{1}{2\sqrt{t}}$$

$$\text{and} \quad \frac{dt}{dx} = (\cos x - \sin x).$$

$$\text{So, } \frac{dy}{dx} = \left(\frac{dy}{du} \times \frac{du}{dt} \times \frac{dt}{dx}\right) = \frac{\cos u}{2\sqrt{t}} \cdot (\cos x - \sin x)$$

$$= \frac{\cos \sqrt{t}}{2\sqrt{t}} \cdot (\cos x - \sin x) \qquad [\because u = \sqrt{t}]$$

$$= \frac{\cos(\sqrt{\sin x + \cos x})(\cos x - \sin x)}{2\sqrt{\sin x + \cos x}} \quad [\because t = (\sin x + \cos x)].$$

EXAMPLE 8 *Differentiate* $\sin 2x \cos 3x$.

SOLUTION By using the product rule, we have

$$\frac{d}{dx}(\sin 2x \cos 3x) = (\sin 2x) \cdot \frac{d}{dx}(\cos 3x) + (\cos 3x) \cdot \frac{d}{dx}(\sin 2x)$$

$$= (\sin 2x) \cdot (-\sin 3x) \cdot 3 + (\cos 3x) \cdot (\cos 2x) \cdot 2$$

[using chain rule]

$$= (-3 \sin 2x \sin 3x + 2 \cos 3x \cos 2x).$$

EXAMPLE 9 *Differentiate* $e^{ax}\cos(bx+c)$.

SOLUTION Using the product rule, we have

$$\begin{aligned}
&\frac{d}{dx}\{e^{ax}\cos(bx+c)\}\\
&= e^{ax}\cdot\frac{d}{dx}\{\cos(bx+c)\} + \cos(bx+c)\cdot\frac{d}{dx}(e^{ax})\\
&= e^{ax}\{-\sin(bx+c)\}\cdot\frac{d}{dx}(bx+c) + \cos(bx+c)\cdot e^{ax}\cdot\frac{d}{dx}(ax)\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\text{[using the chain rule]}\\
&= -be^{ax}\sin(bx+c) + ae^{ax}\cos(bx+c)\\
&= e^{ax}[a\cos(bx+c) - b\sin(bx+c)].
\end{aligned}$$

EXAMPLE 10 *Differentiate* (*i*) $\sqrt{\dfrac{1+x}{1-x}}$ (*ii*) $\dfrac{x}{\sqrt{1-x^2}}$

SOLUTION (i) Let $y = \sqrt{\dfrac{1+x}{1-x}}\cdot$

Put $\dfrac{1+x}{1-x} = t$, so that $y = \sqrt{t}$ and $t = \dfrac{1+x}{1-x}\cdot$

$$\therefore \quad \frac{dy}{dt} = \frac{1}{2}t^{-1/2} = \frac{1}{2\sqrt{t}}\cdot$$

$$\text{And,}\ \frac{dt}{dx} = \frac{(1-x)\cdot\frac{d}{dx}(1+x) - (1+x)\cdot\frac{d}{dx}(1-x)}{(1-x)^2}$$

$$= \frac{(1-x)\cdot 1 - (1+x)(-1)}{(1-x)^2} = \frac{2}{(1-x)^2}$$

$$\therefore \quad \frac{dy}{dx} = \left(\frac{dy}{dt}\times\frac{dt}{dx}\right) = \frac{1}{2\sqrt{t}}\times\frac{2}{(1-x)^2}$$

$$= \frac{1}{(1-x)^2\sqrt{\dfrac{1+x}{1-x}}} = \frac{\sqrt{1-x}}{(1-x)^2\sqrt{1+x}}$$

$$= \frac{1}{(1-x)^{3/2}(1+x)^{1/2}}\cdot$$

(ii)
$$\frac{d}{dx}\left(\frac{x}{\sqrt{1-x^2}}\right) = \frac{\sqrt{1-x^2}\cdot\frac{d}{dx}(x) - x\cdot\frac{d}{dx}\{\sqrt{1-x^2}\}}{(1-x^2)}$$

$$= \frac{\sqrt{1-x^2}\cdot 1 - x\cdot\frac{1}{2}(1-x^2)^{-1/2}(-2x)}{(1-x^2)}$$

$$= \frac{\sqrt{1-x^2} + \dfrac{x^2}{\sqrt{1-x^2}}}{(1-x^2)} = \frac{(1-x^2)+x^2}{(1-x^2)^{3/2}} = \frac{1}{(1-x^2)^{3/2}}\cdot$$

EXERCISE 28E

Differentiate the following with respect to x:

1. $\sin 4x$
2. $\cos 5x$
3. $\tan 3x$
4. $\cos x^3$
5. $\cot^2 x$
6. $\tan^3 x$
7. $\tan \sqrt{x}$
8. e^{x^4}
9. $e^{\cot x}$
10. $\sqrt{\sin x}$
11. $(5+7x)^6$
12. $(3-4x)^5$
13. $(3x^2-x+1)^4$
14. $(ax^2+bx+c)^n$
15. $\dfrac{1}{(x^2-x+3)^3}$
16. $\sin^2(2x+3)$
17. $\cos^2 x^3$
18. $\sqrt{\sin x^3}$
19. $\sqrt{x \sin x}$
20. $\sqrt{\cot \sqrt{x}}$
21. $\cos 3x \sin 5x$
22. $\sin x \sin 2x$
23. $\cos(\sin \sqrt{ax+b})$
24. $e^{2x} \sin 3x$
25. $e^{3x} \cos 2x$
26. $e^{-5x} \cot 4x$
27. $\cos(x^3 . e^x)$
28. $e^{(x \sin x + \cos x)}$
29. $\dfrac{e^x + e^{-x}}{e^x - e^{-x}}$
30. $\dfrac{e^{2x} + e^{-2x}}{e^{2x} - e^{-2x}}$
31. $\sqrt{\dfrac{1-x^2}{1+x^2}}$
32. $\sqrt{\dfrac{a^2-x^2}{a^2+x^2}}$
33. $\sqrt{\dfrac{1+\sin x}{1-\sin x}}$
34. $\sqrt{\dfrac{1+e^x}{1-e^x}}$
35. $\dfrac{e^{2x}+x^3}{\operatorname{cosec} 2x}$

Find $\dfrac{dy}{dx}$, when

36. $y = \sin(\sqrt{\sin x + \cos x})$
37. $y = e^x \log(\sin 2x)$
38. $y = \cos\left(\dfrac{1-x^2}{1+x^2}\right)$
39. $y = \sin\left(\dfrac{1+x^2}{1-x^2}\right)$
40. $y = \dfrac{\sin x + x^2}{\cot 2x}$
41. If $y = \dfrac{\cos x - \sin x}{\cos x + \sin x}$, show that $\dfrac{dy}{dx} + y^2 + 1 = 0$.
42. If $y = \dfrac{\cos x + \sin x}{\cos x - \sin x}$, show that $\dfrac{dy}{dx} = \sec^2\left(x + \dfrac{\pi}{4}\right)$.
43. If $y = \sqrt{\dfrac{1-x}{1+x}}$, prove that $(1-x^2)\dfrac{dy}{dx} + y = 0$. **[CBSE 2004]**
44. If $y = \sqrt{\dfrac{\sec x - \tan x}{\sec x + \tan x}}$, show that $\dfrac{dy}{dx} = \sec x(\tan x + \sec x)$.

ANSWERS (EXERCISE 28E)

1. $4\cos 4x$
2. $-5\sin 5x$
3. $3\sec^2 3x$
4. $-3x^2 \sin x^3$

5. $-2 \cot x \operatorname{cosec}^2 x$ **6.** $3 \tan^2 x \sec^2 x$ **7.** $\dfrac{\sec^2 \sqrt{x}}{2\sqrt{x}}$ **8.** $4x^3(e^{x^4})$

9. $-(\operatorname{cosec}^2 x)(e^{\cot x})$ **10.** $\dfrac{\cos x}{2\sqrt{\sin x}}$ **11.** $42(5+7x)^5$ **12.** $-20(3-4x)^4$

13. $4(6x-1)(3x^2-x+1)^3$ **14.** $n(2ax+b)(ax^2+bx+c)^{n-1}$

15. $\dfrac{-3(2x-1)}{(x^2-x+3)^4}$ **16.** $2 \sin(4x+6)$ **17.** $-3x^2 \sin(2x^3)$ **18.** $\dfrac{3x^2 \cos x^3}{2\sqrt{\sin x^3}}$

19. $\dfrac{(x \cos x + \sin x)}{2\sqrt{x \sin x}}$ **20.** $\dfrac{-\operatorname{cosec}^2 \sqrt{x}}{4\sqrt{x} \,.\, \sqrt{\cot \sqrt{x}}}$

21. $5 \cos 3x \cos 5x - 3 \sin 3x \sin 5x$ **22.** $2 \sin x \cos 2x + \cos x \sin 2x$

23. $\dfrac{-a}{2\sqrt{ax+b}} (\cos \sqrt{ax+b}) \,.\, \sin(\sin \sqrt{ax+b})$ **24.** $e^{2x}(3 \cos 3x + 2 \sin 3x)$

25. $e^{3x}(3 \cos 2x - 2 \sin 2x)$ **26.** $-e^{-5x}(4\operatorname{cosec}^2 4x + 5 \cot 4x)$

27. $-x^2 e^x (x+3) \cdot \sin(x^3 e^x)$ **28.** $(x \cos x).\, e^{(x \sin x + \cos x)}$

29. $\dfrac{-4}{(e^x - e^{-x})^2}$ **30.** $\dfrac{-8}{(e^{2x} - e^{-2x})^2}$ **31.** $\dfrac{-2x}{(1+x^2)^{3/2}(1-x^2)^{1/2}}$

32. $\dfrac{-2a^2 x}{(a^2+x^2)^{3/2}(a^2-x^2)^{1/2}}$ **33.** $\dfrac{1}{(1-\sin x)}$ or $\sec x(\sec x + \tan x)$

34. $\dfrac{e^x}{(1-e^x)(1-e^{2x})^{1/2}}$ **35.** $2e^{2x}(\cos 2x + \sin 2x) + 2x^3 \cos 2x + 3x^2 \sin 2x$

36. $\dfrac{(\cos x - \sin x) \cos(\sqrt{\sin x + \cos x})}{2\sqrt{\sin x + \cos x}}$ **37.** $e^x[2 \cot 2x + \log(\sin 2x)]$

38. $\dfrac{4x}{(1+x^2)^2} \cdot \sin\left(\dfrac{1-x^2}{1+x^2}\right)^2$ **39.** $\dfrac{4x}{(1-x^2)^2} \cdot \cos\left(\dfrac{1+x^2}{1-x^2}\right)$

40. $2(\sin x + x^2) \sec^2 2x + (\cos x + 2x) \tan 2x$

HINTS TO SOME SELECTED QUESTIONS

35. Write $\dfrac{1}{\operatorname{cosec} 2x} = \sin 2x$.

42. $$y = \frac{\frac{1}{\sqrt{2}} \cos x + \frac{1}{\sqrt{2}} \sin x}{\frac{1}{\sqrt{2}} \cos x - \frac{1}{\sqrt{2}} \sin x} = \frac{\sin x \cos \frac{\pi}{4} + \cos x \sin \frac{\pi}{4}}{\cos x \cos \frac{\pi}{4} - \sin x \sin \frac{\pi}{4}} = \tan\left(x + \frac{\pi}{4}\right).$$

❑

29
Mathematical Reasoning

Introduction

Logic is the science of reasoning.

By reasoning we draw valid inferences from some well-known facts.

In mathematical language, there are two kinds of reasoning, namely *inductive* and *deductive.*

Mathematical induction is some sort of inductive reasoning, which we have studied in an earlier chapter.

Now, in this chapter we shall discuss deductive reasoning.

SENTENCES *A group of words which makes complete sense is called a sentence.*

We communicate our ideas to others through sentences.

Sentences are classified as *declarative* (or *assertive*); *imperative; exclamatory* and *interrogative*.

DECLARATIVE SENTENCES *A sentence that makes an assertion is called a declarative or assertive sentence.*

EXAMPLES *Each of the following is a declarative sentence.*

(i) The earth revolves round the sun.

(ii) The sum of the angles of a triangle is $180°$.

(iii) 2 *is the only even prime number.*

IMPERATIVE SENTENCES *A sentence that expresses a request or a command is called an imperative sentence.*

EXAMPLES *Each of the following is an imperative sentence.*

(i) Switch on the light.

(ii) Do your homework.

(iii) Give me a glass of water.

(iv) Keep quiet.

INTERROGATIVE SENTENCES *A sentence that asks some question is called an interrogative sentence.*

EXAMPLES *Each of the following is an interrogative sentence.*

(i) Where are you coming from?

(ii) How far is Mumbai from here?

(iii) Is every set finite?

EXCLAMATORY SENTENCES *A sentence that expresses some strong feeling is called an exclamatory sentence.*

EXAMPLES *Each of the following is an exclamatory sentence.*

(i) What a beautiful girl!

(ii) Listen to me, Tanvy!

(iii) May you live long!

(iv) Hurrah! We have won.

Actually, in mathematical terms, we shall be interested only in particular types of declarative sentences, called statements.

STATEMENTS *A declarative sentence which can be judged to be true or false but not both, is called a mathematically acceptable statement or simply a statement.*

EXAMPLE 1 *Check whether the following sentences are statements. Give reasons for your answer.*

(i) The earth is a planet.

(ii) There are 35 days in a month.

(iii) Mathematics is difficult.

(iv) Every square is a rectangle.

(v) Every set is a finite set.

(vi) Every real number is a complex number.

(vii) 9 is less than 7.

(viii) The square of a natural number is an even number.

SOLUTION (i) It is a scientifically established fact that the earth is a planet. So, the given sentence is true.

Hence, it is a statement.

(ii) We know that the maximum number of days in a given month is 31. So, the given sentence is false.

Therefore, it is a statement.

(iii) For some people mathematics can be easy and for some others it can be difficult.

So, we cannot judge whether the given sentence is true or false.

Hence, it is not a statement.

(iv) In a square, every angle measures 90° and its opposite sides are equal. So, it is a rectangle.

Thus, the given sentence is true.

Hence, it is a statement.

(v) Since there are sets which are not finite, so the given sentence is false.

Therefore, it is a statement.

(vi) Every real number a can be written as $a = (a + 0i)$.

So, every real number is a complex number.

Therefore, the given sentence is true.

Hence, it is a statement.

(vii) Since 9 is greater than 7, so the given sentence is false.
Hence, it is a statement.

(viii) We know that 2 and 3 are natural numbers such that $2^2 = 4$, which is even and $3^2 = 9$, which is odd.
So, the given sentence is sometimes true and sometimes false.
Therefore, it is not a statement.

REMARK (I) *Sentences involving variable time such as 'yesterday', 'today' and 'tomorrow' are not statements.*

EXAMPLES *None of the sentences given below is a statement.*
(i) Today is Sunday.
(ii) Yesterday was a windy day.
(iii) Tomorrow is a holiday.

REMARK (II) *Sentences involving variable places such as 'here', 'there', etc., are not statements.*

EXAMPLES *None of the sentences given below is a statemement.*
(i) Meerut is far from here.
(ii) This place is very hot.
(iii) Rohit will reach there in time.

REMARK (III) *Sentences involving variable pronouns such as 'he', 'she', etc., are not statements.*

EXAMPLES *None of the sentences given below is a statemement.*
(i) He is a doctor.
(ii) She is a graduate.

EXAMPLE 2 *Check whether the following sentences are statements.*
(i) Answer this question.
(ii) Everyone in this room is rich.
(iii) There is no rain without clouds.
(iv) Mathematics is fun.
(v) She is a commerce graduate.
(vi) $\sqrt{2}$ *is a rational number.*
(vii) Fire is always hot.
(viii) The sides of a quadrilateral have equal length.

SOLUTION (i) It is an order and therefore, it is not a statement.

(ii) From the context it is not clear which room is referred here and also the term rich is not well defined.
So, the given sentence is not a statement.

(iii) This sentence is always true and therefore, it is a statement.

(iv) For those who like mathematics it may be a fun but for others it may not be.

So, the given sentence cannot be judged to be true or false.

Hence, it is not a statement.

(v) It is not known who she is. So, the given sentence is not a statement.

(vi) Clearly, $\sqrt{2}$ is an irrational number. So, the given sentence is false.

Therefore, it is a statement.

(vii) Clearly, the given sentence is always true.

Hence, it is a statement.

(viii) All the sides of a square as well as that of a rhombus are equal. But, the sides of a trapezium as well as that of a rectangle are inequal.

So, the given sentence is sometimes true and sometimes false.

Hence, it is not a statement.

NEGATION OF A STATEMENT

The denial of an assertion contained in a statement is called its negation.

$\sim p$ denotes the negation or denial of p.

The denial of a statement can be expressed in various ways.

EXAMPLE 3 *Let p: Kolkata is a city. Express the denial of p in three different ways.*

SOLUTION This can be done as shown below.

$\sim p$: Kolkata is not a city.

or $\sim p$: It is false that Kolkata is a city.

or $\sim p$: It is not the case that Kolkata is a city.

EXAMPLE 4 *Write the negation of each of the following statements in two different ways.*

(i) Africa is a continent.

(ii) $\sqrt{5}$ is rational.

(iii) All integers are rational numbers.

(iv) Some prime numbers are odd numbers.

(v) Everyone in Germany speaks German.

SOLUTION Given below are the required negations:

(i) Africa is not a continent.

Or, It is false that Africa is a continent.

(ii) $\sqrt{5}$ is not rational.

Or, It is false that $\sqrt{5}$ is rational.

(iii) There exists an integer which is not a rational number.

Or, It is false that all integers are rational numbers.

(iv) No prime number is an odd number.

Or, It is false that some prime numbers are odd numbers.

(v) At least one person in Germany does not speak German.

Or, It is false that everyone in Germany speaks German.

EXAMPLE 5 *Write the negation of each of the following statements:*

p: All birds have wings.

q: For every real number x, $x^2 > x$.

r: There exists a real number x such that $0 < x < 1$.

s: For every real number x, either $x > 1$ *or* $x < 1$.

SOLUTION We have:

$\sim p$: There exists a bird which has no wings.

$\sim q$: There exists a real number x such that x^2 is not greater than x.

$\sim r$: There does not exist a real number x such that $0 < x < 1$.

$\sim s$: There exists a real number x such that neither $x > 1$ nor $x < 1$.

EXERCISE 29A

1. Which of the following sentences are statements? In case of a statement mention whether it is true or false.

(i) The sun is a star.

(ii) $\sqrt{7}$ is an irrational number.

(iii) The sum of 5 and 6 is less than 10.

(iv) Go to your class.

(v) Ice is always cold.

(vi) Have you ever seen the Red Fort?

(vii) Every relation is a function.

(viii) The sum of any two sides of a triangle is always greater than the third side.

(ix) May God bless you!

2. Which of the following sentences are statements? In case of a statement, mention whether it is true or false.

(i) Paris is in France.

(ii) Each prime number has exactly two factors.

(iii) The equation $x^2 + 5|x| + 6 = 0$ has no real roots.

(iv) $(2 + \sqrt{3})$ is a complex number.

(v) Is 6 a positive integer?

(vi) The product of –3 and –2 is –6.

(vii) The angles opposite to the equal sides of an isosceles triangle are equal.

(viii) Oh! it is too hot.

(ix) Monika is a beautiful girl.

(x) Every quadratic equation has at least one real root.

3. Which of the following statements are true and which are false? In each case give a valid reason for your answer.

(i) p: $\sqrt{11}$ is an irrational number.

(ii) q: Circle is a particular case of an ellipse.

(iii) r: Each radius of a circle is a chord of the circle.

(iv) s: The centre of a circle bisects each chord of the circle.

(v) t: If a and b are integers such that $a < b$, then $-a > -b$.

(vi) u: The quadratic equation $x^2 + x + 1 = 0$ has no real roots.

4. Write the negation of each of the following statements:

(i) Every natural number is greater than 0.

(ii) Both the diagonals of a rectangle are equal.

(iii) The sum of 4 and 5 is 8.

(iv) The number 6 is greater than 4.

(v) Every natural number is an integer.

(vi) The number –5 is a rational number.

(vii) All cats scratch.

(viii) There exists a rational number x such that $x^2 = 3$.

(ix) All students study mathematics at the elementary level.

(x) Every student has paid the fees.

(xi) There is some integer k for which $2k = 6$.

(xii) None of the students of this class has passed.

ANSWERS (EXERCISE 29A)

1. (i) Statement, True (ii) Statement, True

(iii) Statement, False (iv) Imperative sentence, not a statement

(v) Statement, True (vi) Interrogative sentence, not a statement

(vii) Statement, False (viii) Statement, True

(ix) Exclamatory sentence, not a statement

2. (i) Statement, True (ii) Statement, True

(iii) Statement, True (iv) Statement, False

(v) Interrogative sentence, not a statement

(vi) Statement, False (vii) Statement, True

(viii) Exclamatory sentence, not a statement

(ix) Not a statement, the term 'beautiful' is vague.

(x) Statement, False

3. (i) True, as 11 being prime $\sqrt{11}$ is irrational.

(ii) True. The equation of an ellipse is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Putting $b = a$, it becomes $x^2 + y^2 = a^2$, which is the equation of a circle.

(iii) False, since the radius of a circle is not a chord.

(iv) False, since there are chords which do not pass through the centre

(v) True, since $a < b \Rightarrow -a > -b$ is true.

(vi) True. Since $D = (1^2 - 4 \times 1 \times 1) - 3 < 0$.

4. (i) There exists a natural number which is not greater than 0.

(ii) It is false that both the diagonals of a rectangle are equal.

(iii) It is false that the sum of 4 and 5 is 8.

(iv) The number 6 is not greater than 4.

(v) There exists a natural number which is not an integer.

(vi) The number –5 is not a rational number.

(vii) There exists a cat which does not scratch.

(viii) There does not exist a rational number x such that $x^2 = 3$.

(ix) There exists a student who does not study mathematics at the elementary level.

(x) At least one student has not paid the fees.

(xi) For all integers k, $2k \neq 6$.

(xii) At least one student of this class has passed.

LOGICAL CONNECTIVES

SIMPLE STATEMENTS *A statement which does not contain any other statement as its component part is called a simple statement.*

EXAMPLES *Each of the following is a simple statement.*

(i) Chennai is the capital of Tamil Nadu.

(ii) The set of real numbers is an infinite set.

(iii) The sum of 6 *and* 3 *is greater than* 10.

COMPOUND STATEMENTS *A compound statement is a combination of two or more simple sentences.*

The method of combining two or more simple sentences is known as *compounding*.

LOGICAL CONNECTIVES *The words or phrases which connect two or more simple sentences are called logical connectives or simply connectives.*

These connectives are:

(i) ***and*** (ii) ***or*** (iii) ***If .... then ....*** (iv) ***if and only if***

CONNECTIVES AND THEIR SYMBOLS

Connective	Symbol	Nature of compound statement
(i) and	$\wedge$	Conjunction
(ii) or	$\vee$	Disjunction
(iii) If ... then ...	$\Rightarrow$	Implication (or conditional)
(iv) if and only if	$\Leftrightarrow$	Biconditional (or equivalence)

EXAMPLES *Each of the following is a compound statement.*

(i) 2 *is a rational number and* $\sqrt{2}$ *is an irrational number.*

(ii) The sum of 3 *and* 5 *is* 7 *or* 8.

(iii) If it rains then the school may be closed.

(iv) A triangle is equiangular if and only if it is equilateral.

The connectives in (i), (ii), (iii) and (iv) are respectively and , or, If ... then, if and only if.

COMPOUNDING OF SENTENCES

1. CONJUNCTION *Two simple sentences connected by the word 'and' are said to form a conjunction.*

We use the symbol '$\wedge$' to denote **'and'**.

TRUTH TABLE FOR $p \wedge q$ *Note that '$p \wedge q$' is true only when each one of the statements p and q is true.*

Based on the above fact, we prepare the truth table for '$p \wedge q$' as under.

Truth table for $p \wedge q$

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

Remember that:

> (i) *The compound statement with 'and' is true if all its component statements are true.*
>
> (ii) *The compound statement with 'and' is false if any of its component statements is false or all of its component statements are false.*

EXAMPLE 1 *Split each of the following compound statements into simple sentences and determine whether it is true or false:*

(i) The grass is green and the sky is blue.

(ii) Agra is in Uttar Pradesh and Shimla is in Punjab.

(iii) All rational numbers are real and all real numbers are not complex numbers.

(iv) $x = 5$ *and* $x = 2$ *are the roots of the equation* $3x^2 - x - 10 = 0$.

SOLUTION (i) Let p: The grass is green.

And q: The sky is blue.

Clearly, each one of p and q is a true statement and therefore, (p and q) is true.

Hence, the given statement is true.

(ii) Let p: Agra is in Uttar Pradesh.

And q: Shimla is in Punjab.

Clearly, p is true and q is false.

Therefore, (p and q) is false.

Hence, the given statement is false.

(iii) Let p: All rational numbers are real.

And, q: All real numbers are not complex numbers.

Clearly, p is true and q is false.

$\therefore$ (p and q) is false.

Hence,the given statement is false.

(iv) Let p: $x = 5$ is a root of the equation $3x^2 - x - 10 = 0$.

And q: $x = 2$ is a root of the equation $3x^2 - x - 10 = 0$.

Clearly, $3 \times 5^2 - 5 - 10 = 60 \neq 0$. So, $x = 5$ is not a root of the equation $3x^2 - x - 10 = 0$.

And, $3 \times 2^2 - 2 - 10 = 0$. So, $x = 2$ is a root of $3x^2 - x - 10 = 0$.

$\therefore$ p is false and q is true.

Consequently, (p and q) is false.

Hence, the given statement is false.

2. DISJUNCTION *Two simple sentences connected by the word 'or' are said to form a disjunction.*

We use the symbol '$\vee$' to denote '**or**'.

Clearly, (p or q) is true when at least one of them is true.

TRUTH TABLE FOR $(p \vee q)$ Clearly, $(p \vee q)$ is true only when (p is true or q is true) or both are true.

Truth table for $(p \vee q)$

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

Remember that:

(i) *A compound statement with 'or' is true when at least one of them is true.*

(ii) *A compound statement with 'or' is false when each one of them is false.*

*In (p or q) we say that **exclusive 'or'** is used when both p and q are not true simultaneously.*

Otherwise, ***inclusive 'or'*** is used.

EXAMPLE 2 *For each of the following statements, determine whether an* ***inclusive 'or'*** *or* ***exclusive 'or'*** *is used. Give reasons for your answer.*

(i) For identification you need a passport or an Adhar Card.

(ii) The school is closed if it is a holiday or a Sunday.

(iii) $\sqrt{3}$ *is a rational number or an irrational number.*

(iv) Two lines intersect at a point or are parallel.

(v) Students can take Sanskrit or French as their third language.

SOLUTION (i) Here 'or' is inclusive since a person can have both a passport and an Adhar Card for identification.

(ii) Here 'or' is inclusive since the school is closed on holiday as well as on Sunday.

(iii) Let $p : \sqrt{3}$ is a rational number.

And, $q : \sqrt{3}$ is an irrational number.

Clearly, p is false and q is true.

$\therefore$ (p or q) is true and 'or' is *exclusive*.

(iv) Here 'or' is *exclusive* because it is not possible for two lines to intersect and be parallel together.

(v) Here 'or' is *exclusive* because a student cannot take both Sanskrit and French.

OPEN SENTENCES AND THEIR TRUTH SETS

Let $p(x)$ be a declarative sentence involving a variable x and let A be a given set such that for $a \in A$, $p(a)$ is true or false.

Then, $p(x)$ is called an *open sentence* defined on A.

The subset of A consisting of all those values of x which convert $p(x)$ into a true statement is called a *Truth Set* of the *open sentence* $p(x)$.

EXAMPLE 3 *Write down the truth set of each of the following open sentences:*

(i) $p(x): x+5<9, x \in N$.

(ii) $p(x): x+3<3, x \in N$.

(iii) $p(x): x+5>7, x \in R$.

(iv) $p(x): 2x^2+5x-3=0, x \in I$.

SOLUTION We have:

(i) Truth Set $= \{x \in N : x+5<9\}$
$= \{x \in N : x<4\} = \{1, 2, 3\}$.

(ii) Truth Set $= \{x \in N : x+3<3\}$
$= \{x \in N : x<0\} = \phi$.

(iii) Truth Set $= \{x \in R : x+5>7\}$
$= \{x \in R : x>2\} =]2, \infty[$.

(iv) $2x^2+5x-3=0 \Rightarrow 2x^2+6x-x-3=0 \Rightarrow 2x(x+3)-(x+3)=0$
$\Rightarrow (x+3)(2x-1)=0 \Rightarrow x=-3$ or $x=\frac{1}{2}\cdot$

$$\therefore \text{ truth set} = \{x \in I : 2x^2 + 5x - 3 = 0\} = \{-3\} \quad \left[\because \frac{1}{2} \notin I\right].$$

QUANTIFIERS AND QUANTIFIED STATEMENTS

Two Important Symbols:

(i) The symbol '$\forall$' stands '*for all values of*'.
This is known as universal *quantifier*.

(ii) The symbol '$\exists$' stands for '*there exists*'.
The symbol $\exists$ is known as *existential quantifier*.

QUANTIFIED STATEMENT An open sentence with a quantifier becomes a statement, called a quantified statement.

EXAMPLE 4 *Use quantifiers to convert each of the following open sentences defined on N, into a true statement:*

(i) $x + 5 = 8$ *(ii)* $x^2 > 0$ *(iii)* $x + 2 < 4$

SOLUTION (i) $\exists\, x \in N$ such that $x + 5 = 8$ is a true statement, since $x = 3 \in N$ satisfies $x + 5 = 8$.

(ii) $x^2 > 0 \,\forall\, x \in N$ is a true statement, since the square of every natural number is positive.

(iii) $\exists\, x \in N$ such that $x + 2 < 4$ is a true statement, since $x = 1 \in N$ satisfies $x + 2 < 4$.

EXAMPLE 5 *Let* $A = \{1, 2, 3, 4\}$. *Examine whether the statements given below are true or false.*

(i) $\exists\, x \in A$ *such that* $x + 3 = 8$.

(ii) $\forall\, x \in A,\ x + 2 < 7$.

(iii) $\exists\, x \leftrightarrow A$ *such that* $x + 1 < 3$.

(iv) $\forall\, x \in A,\ x + 3 \geq 5$.

SOLUTION (i) Clearly, no number in A satisfies $x + 3 = 8$.
$\therefore$ the given statement is false.

(ii) Every number in A satisfies $x + 2 < 7$.
So, the given statement is true.

(iii) Clearly, $x = 1 \leftrightarrow A$ satisfies $x + 1 < 3$.
$\therefore$ the given statement is true.

(iv) Since $x = 1 \in A$ does not satisfy $x + 3 \geq 5$, the given statement is false.

EXERCISE 29B

1. Split each of the following into simple sentences and determine whether it is true or false.

 (i) A line is straight and extends indefinitely in both the directions.

 (ii) A point occupies a position and its location can be determined.

(iii) The sand heats up quickly in the sun and does not cool down fast at night.

(iv) 32 is divisible by 8 and 12.

(v) $x = 1$ and $x = 2$ are the roots of the equation $x^2 - x - 2 = 0$.

(vi) 3 is rational and $\sqrt{3}$ is irrational.

(vii) All integers are rational numbers and all rational numbers are not real numbers.

(viii) Lucknow is in Uttar Pradesh and Kanpur is in Uttarakhand.

2. Split each of the following into simple sentences and determine whether it is true or false. Also, determine whether an *'inclusive or'* or *'exclusive or'* is used.

(i) The sum of 3 and 7 is 10 or 11.

(ii) $(1 + i)$ is a real or a complex number.

(iii) Every quadratic equation has one or two real roots.

(iv) You are wet when it rains or you are in a river.

(v) 24 is a multiple of 5 or 8.

(vi) Every integer is rational or irrational.

(vii) For getting a driving licence you should have a ration card or a passport.

(viii) 100 is a multiple of 6 or 8.

(ix) Square of an integer is positive or negative.

(x) Sun rises or Moon sets.

3. Find the truth set in case of each of the following open sentences defined on N:

(i) $x + 2 < 10$ (ii) $x + 5 < 4$ (iii) $x + 3 > 2$

4. Let $A = \{2, 3, 5, 7\}$. Examine whether the statements given below are true or false.

(i) $\exists\, x \in A$ such that $x + 3 > 9$.

(ii) $\exists\, x \in A$ such that x is even.

(iii) $\exists\, x \in A$ such that $x + 2 = 6$.

(iv) $\forall\, x \in A$, x is prime.

(v) $\forall\, x \in A$, $x + 2 < 10$

(vi) $\forall\, x \in A$, $x + 4 \geq 11$

ANSWERS (EXERCISE 29B)

1. (i) p: A line is straight.

q: A line extends indefinitely in both the directions.

(p and q) is true.

(ii) p: A point occupies a position.

q: The location of a point can be determined.

(p and q) is true.

(iii) p: The sand heats up quickly in the sun.

q: The hot sand does not cool down fast at night.

(p and q) is true.

(iv) p: 32 is divisible by 8.

q: 32 is divisible by 12.

(p and q) is not true.

(v) p: $x = 1$ is a root of the equation $x^2 - x - 2 = 0$.

q: $x = 2$ is a root of the equation $x^2 - x - 2 = 0$.

(p and q) is not true.

(vi) p: 3 is rational.

q: $\sqrt{3}$ is irrational.

(p and q) is true.

(vii) p: All integers are rational numbers.

q: All rational numbers are not real numbers.

(p and q) is false.

(viii) p: Lucknow is in Uttar Pradesh.

q: Kanpur is in Uttarakhand.

(p and q) is false.

2. (i) p: The sum of 3 and 7 is 10.

q: The sum of 3 and 7 is 11.

(p or q) is true. *'or is exclusive'*

(ii) p: $(1 + i)$ is a real number.

q: $(1 + i)$ is a complex number.

(p or q) is true. *'or is exclusive'*

(iii) p: Every quadratic equation has one real root.

q: Every quadratic equation has two real roots.

(p or q) is false.

(iv) p: You are wet when it rains.

q: You are wet when you are in a river.

(p or q) is true. *'or is inclusive'*

(v) Let p: 24 is a multiple of 5.

and q: 24 is a multiple of 8.

(p or q) is true. *'or is exclusive'*

(vi) Let p: Every integer is rational.

q: Every integer is irrational.

(p or q) is true. *'or is exclusive'*

(vii) Let p: For getting a driving licence you should have a ration card.
q: For getting a driving licence you should have a passport.
(p or q) is true. 'or is *inclusive*'

(viii) Let p: 100 is a multiple of 6.
q: 100 is a multiple of 8.
(p or q) is false.

(ix) Let p: Square of an integer is positive.
q: Square of an integer is negative.
(p or q) is false. [**Hint:** 0^2 *is not positive.*]

(x) Let p: Sun rises.
q: Moon sets.
(p or q) is true. '*or is exclusive*'

3. (i) $\{1, 2, 3, 4, 5, 6, 7\}$ (ii) ϕ (iii) N

4. (i) True (ii) True (iii) False (iv) True (v) True (vi) False

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CONDITIONAL AND BICONDITIONAL STATEMENTS

CONDITIONAL STATEMENTS *Two statements connected by 'if ... then' is called a conditional statement.*

For any statements p and q, the statement '*If p, then q*' is denoted by '$p \Rightarrow q$', which means ***'p implies q'.***

Here p is called the *antecedent* and q is called the *consequent*.

> We may write 'if p, then q' in following ways:
> (i) p implies q, written as $p \Rightarrow q$.
> (ii) p only if q.
> (iii) q is a necessary condition for p.
> (iv) p is a sufficient condition for q.
> (v) $\sim q \Rightarrow \sim p$.

EXAMPLE 1 *Rewrite the following statement, with 'if ..., then' in five different ways conveying the same meaning:*
If a natural number is even, then its square is even.

SOLUTION We may rewrite it in following ways:

(i) A natural number is even implies that its square is even.
(ii) A natural number is even only if its square is even.
(iii) For a natural number to be even it is necessary that its square is even.
(iv) For the square of a natural number to be even, it is sufficient that the number is even.
(v) If the square of a natural number is not even, then the natural number is not even.

EXAMPLE 2 *Write each of the following statements in the form 'if ... then':*

(i) A quadrilateral is a parallelogram if its diagonals bisect each other.

(ii) The banana tree will bloom if it stays warm for a month.

(iii) There is traffic jam whenever it rains.

(iv) It is necessary to have a password to log on to the server.

(v) You can access the website only if you pay a subscription fee.

SOLUTION We can write them as under:

(i) If the diagonals of a quadrialateral bisect each other, then it is a parallelogram.

(ii) If the banana tree stays warm for a month, then it will bloom.

(iii) If it rains, then there is traffic jam.

(iv) If you have a password, then you can log on to the server.

(v) If you can access the website, then you pay a subscription fee.

CONVERSE AND CONTRAPOSITIVE OF STATEMENTS

Let $p \Rightarrow q$. Then,
(i) (*converse* of $p \Rightarrow q$) is $(q \Rightarrow p)$
(ii) (*contrapositive of* $p \Rightarrow q$) *is* $(\sim q \Rightarrow \sim p)$

EXAMPLE 3 *Write the converse and contrapositive of each of the following statements:*

(i) If n is an even number, then n^2 is even.

(ii) If two integers a and b are such that $a > b$, then $(a - b)$ is always a positive integer.

(iii) If a $\triangle ABC$ is right angled at B, then $AB^2 + BC^2 = AC^2$.

(iv) If $\triangle ABC$ and $\triangle DEF$ are congruent, then they are equiangular.

(v) You cannot comprehend geometry if you do not know how to reason deductively.

(vi) Something is cold implies that it has low temperature.

SOLUTION (i) Its **converse** is:

If a number n^2 is even, then n is even.

Its **contrapositive** is:

If a number n^2 is not even, then n is not even.

(ii) Its **converse** is:

If a and b are two integers such that $(a - b)$ is a positive integer, then $a > b$.

Its **contrapositive** is:

If two integers a and b are such that $(a - b)$ is not a positive integer, then a is not greater than b.

(iii) Its **converse** is:

In a $\triangle ABC$, if $AB^2 + BC^2 = AC^2$, then it is right angled at B.

Its **contrapositive** is:

In a $\triangle ABC$, if $(AB^2 + BC^2)$ is not equal to AC^2, then it is not right angled at B.

(iv) Its **converse** is:

If $\triangle ABC$ and $\triangle DEF$ are equiangular, then they are congruent.

Its **contrapositive** is:

If $\triangle ABC$ and $\triangle DEF$ are not equiangular, then they are not congruent.

(v) Its **converse** is:

If you do not know how to reason deductively, then you cannot comprehend geometry.

Its **contrapositive** is:

If you know how to reason deductively, then you can comprehend geometry.

(vi) Given **statement** is:

Something is cold $\Rightarrow$ it has low temperature.

Its **converse** is:

If something has low temperature, then it is cold.

Its **contrapositive** is

If something does not have a low temperature, then it is not cold.

BICONDITIONAL STATEMENT Two simple sentences connected by *'if and only if'* form a *biconditional statement*.

We use the symbol '$\Leftrightarrow$' for *'if and only if'*.

$(p \Leftrightarrow q)$ is the same as $(p \Rightarrow q)$ and $(q \Rightarrow p)$.

EXAMPLE In $\triangle ABC$, $\angle B = \angle C \Leftrightarrow AC = AB$.

This statement is the same as:

'In $\triangle ABC$: $(\angle B = \angle C \Rightarrow AC = AB)$ and $(AC = AB \Rightarrow \angle B = \angle C)$'.

EXAMPLE 4 *Rewrite the following statements in the form:*

'p if and only if q'.

p: If a quadrilateral is equiangular, then it is a rectangle.

q: If a quadrilateral is a rectangle, then it is equiangular.

SOLUTION The required statement is:

A quadrilateral is equiangular if and only if it is a rectangle.

EXAMPLE 5 *Given below are pairs of statements. In each case, combine them using 'if and only if'.*

(i) p: *In* $\triangle ABC$, $\angle B = \angle C$.

q: *In* $\triangle ABC$, $AC = AB$.

(ii) p: *A and B are two sets such that* $A \subseteq B$ *and* $B \subseteq A$.

q: $A = B$.

(iii) p: *$\triangle ABC$ is equilateral.*
q: *$\triangle ABC$ is equiangular.*

(iv) p: $\{a \in R \text{ such that } |a| < 2\}$.
q: $\{a \in R \text{ such that } (a > -2 \text{ and } a < 2)\}$.

SOLUTION We can combine the given statements as given below.

(i) In $\triangle ABC$, $\angle B = \angle C \Leftrightarrow AC = AB$.

(ii) For any sets A and B, $A = B \Leftrightarrow (A \subseteq B$ and $B \subseteq A)$.

(iii) A $\triangle ABC$ is equilateral $\Leftrightarrow$ it is equiangular.

(iv) For every real a, $|a| < 2 \Leftrightarrow (a > -2$ and $a < 2)$.

VARIOUS CASES SHOWING VARIOUS CONDITIONS

SOME CONDITIONS ARE NECESSARY BUT NOT SUFFICIENT

EXAMPLES *(i) Having four sides is necessary but not sufficient for being a square (since a rectangle has four sides but it is not a square).*

(ii) Being educated is necessary but not sufficient for getting the job of a teacher (as certain minimum qualifications are required to be recruited as a teacher).

(iii) Having a difference of 2 *between two numbers is necessary but not sufficient for them to be twin-primes (since they must be prime also).*

SOME CONDITIONS ARE SUFFICIENT BUT NOT NECESSARY

EXAMPLES *(i) Having a son is sufficient but not necessary for being a parent (since a parent could have only a daughter).*

(ii) It is obvious that being a multiple of 10 *is sufficient but not necessary for a number to be a composite number.*

(iii) Having 6 *at the unit's place is sufficient but not necessary for a number to be even (since numbers having* 0, 2, 4, 8 *in the unit's place are also even).*

(iv) Having the corresponding sides equal is sufficient but not necessary for two triangles to be similar (since the triangles having the corresponding sides proportional are also similar)

SOME CONDITIONS ARE BOTH SUFFICIENT AND NECESSARY

EXAMPLES *(i) Having four sides is both sufficient and necessary for being a quadrilateral.*

(ii) Having all the three sides of equal measure is both sufficient and necessary for a triangle to be equilateral.

(iii) Having one number as zero out of two numbers to be multiplied is both sufficient and necessary for the product to be zero.

SOME CONDITIONS ARE NEITHER SUFFICIENT NOR NECESSARY

EXAMPLES *(i) Being a tall person is neither sufficient nor necessary for being a successful person.*

(ii) Being handsome is neither sufficient nor necessary for getting married.

(iii) Having both the diagonals equal is neither sufficient nor necessary for a quadrilateral to be a parallelogram.

EXERCISE 29C

1. Rewrite the following statement in five different ways conveying the same meaning.

   If a given number is a multiple of 6, then it is a multiple of 3.

2. Write each of the following statements in the form 'if .... then':

(i) A rhombus is a square only if each of its angles measures 90°.

(ii) When a number is a multiple of 9, it is necessarily a multiple of 3.

(iii) You get a job implies that your credentials are good.

(iv) Atmospheric humidity increases only if it rains.

(v) If a number is not a multiple of 3, then it is not a multiple of 6.

3. Write the converse and contrapositive of each of the following:

(i) If x is a prime number, then x is odd.

(ii) If a positive integer n is divisible by 9, then the sum of its digits is divisible by 9.

(iii) If the two lines are parallel, then they do not intersect in the same plane.

(iv) If the diagonals of a quadrilateral bisect each other, then it is a parallelogram.

(v) If A and B are subsets of X such that $A \subseteq B$, then $(X - B) \subseteq (X - A)$.

(vi) If $f(2) = 0$, then $f(x)$ is divisible by $(x - 2)$.

(vii) If you were born in India, then you are a citizen of India.

(viii) If it rains, then I stay at home.

4. Given below are some pairs of statements. Combine each pair using if and only if:

(i) p: If a quadrilateral is equiangular, then it is a rectangle.
q: If a quadrilateral is a rectangle, then it is equiangular.

(ii) p: If the sum of the digits of a number is divisible by 3, then the number is divisible by 3.
q: If a number is divisble by 3, then the sum of its digits is divisible by 3.

(iii) p: A quadrilateral is a parallelogram if its diagonals bisect each other.
q: If the diagonals of a quadrilateral bisect each other, then it is a parallelogram.

(iv) p: If $f(a) = 0$, then $(x - a)$ is a factor of polynomial $f(x)$.
q: If $(x - a)$ is a factor of polynomial $f(x)$, then $f(a) = 0$.

(v) p: If a square matrix A is invertible, then $|A|$ is nonzero.
q: If A is a square matrix such that $|A|$ is nonzero, then A is invertible.

5. Write each of the following using 'if and only if':

(i) In order to get A grade, it is necessary and sufficient that you do all the homework regularly.

(ii) If you watch television then your mind is free and if your mind is free then you watch television.

ANSWERS (EXERCISE 29C)

1. (i) A given number is a multiple of 6 implies that it is a multiple of 3.
(ii) A given number is a multiple of 6 only if it is a multiple of 3.
(iii) For a given number to be a multiple of 6, it is necessary that it is a multiple of 3.
(iv) For a given number to be a multiple of 3, it is sufficient that the number is a multiple of 6.
(v) If a given number is not a multiple of 3, then it is not a multiple of 6.

2. (i) If each angle of a rhombus measures 90°, then it is a square.
(ii) If a number is a multiple of 9, then it is a multiple of 3.
(iii) If you get a job, then your credentials are good.
(iv) If it rains, then atmospheric humidity increases.
(v) If a number is a multiple of 6, then it is a multiple of 3.

3. (i) Converse If x is an odd number, then it is prime.
Contrapositive If x is not an odd number, then it is not prime.

(ii) Converse If the sum of the digits of a positive integer n is divisible by 9, then it is divisible by 9.
Contrapositive If the sum of the digits of a positive integer n is not divisble by 9, then it is not divisible by 9.

(iii) Converse If the two lines do not intersect in the same plane, then they are parallel.
Contrapositive If the two lines intersect in the same plane, then they are not parallel.

(iv) Converse If a quadrilateral is a parallelogram, then its diagonals bisect each other.
Contrapositive If a quadrilateral is not a parallelogram, then its diagonals do not bisect each other.

(v) Converse If A and B are subsets of X such that $(X-B) \subseteq (X-A)$, then $A \subseteq B$.
Contrapositive If A and B are subsets of X such that $(X-B)$ is not a subset of $(X-A)$, then A is not a subset of B.

(vi) Converse If $f(x)$ is divisible by $(x-2)$, then $f(2)=0$.
Contrapositive If $f(x)$ is not divisible by $(x-2)$, then $f(2)$ is not equal to 0.

(vii) Converse If you are a citizen of India, then you were born in India.

Contrapositive If you are not a citizen of India, then you were not born in India.

(viii) Converse If I stay at home, then it rains.

Contrapositive If I don't stay at home, then it does not rain.

4. (i) A quadrilateral is a rectangle if and only if it is equiangular.

(ii) A number is divisible by 3 if and only if the sum of its digits is divisible by 3.

(iii) A quadrilateral is a parallelogram if and only if its diagonals bisect each other.

(iv) $(x - a)$ is a factor of polynomial $f(x)$ if and only if $f(a) = 0$.

(v) A square matrix A is invertible if and only if $|A|$ is nonzero.

5. (i) You get an A grade if and only if you do your homework regularly.

(ii) You watch television if and only if your mind is free.

VALIDATING STATEMENTS

We shall list some general rules for checking whether a given compound statement is true or not.

Rule 1. *Statements with 'And':*

Let p and q be two statements. Then, in order to prove that '*p and q*' is true, the following steps are taken.

Step 1. Show that p is true.

Step 2. Show that q is true.

Rule 2. *Statements with 'or':*

Let p and q be two statements. Then, in order to prove that '*p or q*' is true, we proceed as under.

Case 1. Assume that p is false and show that q is true.

Case 2. Assume that q is false and show that p is true.

Rule 3. *Statements with 'If .... then':*

Let p and q be two statements. Then, in order to prove that '$p \Rightarrow q$' is true, we need to prove any one of the following.

DIRECT METHOD *Assume that p is true and prove that q is true.*

CONTRAPOSITIVE METHOD *Assume that q is false and prove that p is false.*

Rule 4. *Statements with 'If and only if':*

In order to prove that '$p \Rightarrow q$', *we need to show that:*

(i) *If p is true, then q is true.*

(ii) *If q is true, then p is true.*

SOLVED EXAMPLES

EXAMPLE 1 *Prove that the following statement is true:*
If $x,\ y \in Z$ such that x and y are odd, then xy is odd.

SOLUTION Let p: $x, y \in Z$ such that x and y are odd.

And q: xy is odd.

Then, we have to prove that xy is odd.

DIRECT METHOD

We assume that p is true and show that q is true.

p is true means x and y are odd integers. Then,

$x = (2m + 1)$ for some integer m

and $y = (2n + 1)$ for some integer n.

$\therefore \quad xy = (2m + 1)(2n + 1)$

$= (4mn + 2m + 2n + 1)$

$= 2(2mn + m + n) + 1$, which is clearly odd.

Thus, $p \Rightarrow q$.

Hence, the given statement is true.

CONTRAPOSITIVE METHOD

We will show that $\sim q \Rightarrow \sim p$.

Here $\sim q$: It is false that both x and y are odd.

This means at least one of x and y is even.

Let x be even. Then, $x = 2n$ for some integer n.

$\therefore \quad xy = 2ny$ for some integer n.

This shows that xy is even.

Thus, $\sim p$ is true [$\because \sim p : xy$ is even].

$\therefore \quad \sim q \Rightarrow \sim p$.

So, the given statement is true.

EXAMPLE 2 *Prove that the necessary and sufficient condition for an integer n to be odd is that n^2 is odd.*

SOLUTION Let p: The integer n is odd.

And q: n^2 is odd.

First we prove that $p \Rightarrow q$.

Let n be odd. Then, $n = (2k + 1)$ for some integer k.

Then, $n^2 (2k + 1)^2 = (4k^2 + 4k + 1) = 2(2k^2 + 2k) + 1$

Thus, n^2 is 1 more than an even number and therefore, it is odd.

Thus, $p \Rightarrow q$.

Now, in order to prove that $q \Rightarrow p$, it is sufficient to show that $\sim p \Rightarrow \sim q$ [*Contrapositive method*].

Clearly, $\sim p$: The integer n is even.

Let $n = 2k$ for some integer k.

Then, $n^2 = 4k^2$, which is even.

$\therefore$ n is even $\Rightarrow n^2$ is even.

Consequently, $\sim p \Rightarrow \sim q$ and therefore, $q \Rightarrow p$.

Hence, $p \Leftrightarrow q$, i.e., the integer n is odd if and only if n^2 is odd.

PROVING RESULTS BY CONTRADICTION

In order to check whether a given statement p is true, we assume that p is false, i.e., ~ p is true.

From this we arrive at a result which contradicts our assumption. Hence, we conclude that p is true.

EXAMPLE 3 *By using the method of contradiction verify that p: $\sqrt{5}$ is irrational.*

PROOF If possible, let $\sqrt{5}$ be rational and let its simplest form be $\frac{a}{b}$.

Then, a and b are integers having no common factor other than 1 and $b \neq 0$.

$$\text{Now, } \sqrt{5} = \frac{a}{b} \Rightarrow 5 = \frac{a^2}{b^2} \quad \text{[on squaring both sides]}$$

$$\Rightarrow 5b^2 = a^2 \qquad \text{... (i)}$$

$$\Rightarrow 5 \text{ divides } a^2 \quad [\because\ 5 \text{ divides } 5b^2]$$

$$\Rightarrow 5 \text{ divides } a \quad [\because\ 5 \text{ is prime and divides } a^2 \Rightarrow 5 \text{ divides } a].$$

Let $a = 5c$ for some integer C.

Putting $a = 5c$ in (i), we get:

$$5b^2 = 25c^2 \Rightarrow b^2 = 5c^2$$

$$\Rightarrow 5 \text{ divides } b^2 \ [\because\ 5 \textit{ divides } 5c^2]$$

$$\Rightarrow 5 \text{ divides } b \ [\because\ 5 \text{ is prime and divides } b^2 \Rightarrow 5 \text{ divides } b].$$

Thus, 5 is a common factor of a and b.

But, this contradicts the fact that a and b have no common factor other than 1.

Hence, $\sqrt{5}$ is irrational.

EXAMPLE 4 *By using the method of contradiction, prove that the sum of an irrational number and a rational number is irrational.*

PROOF Let $\sqrt{a}$ be irrational and b be rational.

Then, we have to prove that $(\sqrt{a}+b)$ is irrational.

If possible, let $(\sqrt{a}+b)$ be rational. Then,

$(\sqrt{a}+b)$ is rational, b is rational.

$\Rightarrow \{(\sqrt{a}+b)-b\}$ is rational [$\because$ difference of rationals is rational]

$\Rightarrow \sqrt{a}$ is rational.

This contradicts the fact that $\sqrt{a}$ is irrational.

Since the contradiction arises by assuming that $(\sqrt{a}+b)$ is rational, hence $(\sqrt{a}+b)$ is irrational.

EXAMPLE 5 *Using contradiction method, check the validity of the following statement: If n is a real number with $n>3$, then $n^2>9$.*

PROOF Let n be a real number such that $n>3$ and if possible, let n^2 be not greater than 9.

Then, $n^2 \le 9$.

Let $n^2=(9-a)$, where a is a real number such that $a \ge 0$.

Then, $n=\sqrt{n^2}=\sqrt{9-a} \le 3$ [$\because$ $(9-a) \le 9$].

But, this is a contradiction since $n>3$.

Since the contradiction arises by assuming that n^2 is not greater than 9, therefore, $n^2>9$.

Hence, the given statement is valid.

EXAMPLE 6 *Consider the statement:*

p: If x a real number such that $x^3+4x=0$, then $x=0$.

Prove that p is a true statement, using:

(i) direct method

(ii) method of contradiction

(iii) method of contrapositive

SOLUTION (i) Direct Method:

Let $x^3+4x=0$, where $x \in R$. Then,

$$x^3+4x=0 \Rightarrow x(x^2+4)=0$$

$$\Rightarrow x=0 \quad [\because \quad x^2+4 \ne 0 \text{ for } x \in R]$$

Hence, p is a true statement.

(ii) Method of contradiction:

If possible, let $x^3+4x=0$ and $x \ne 0$. Then,

$$x(x^2+4)=0 \text{ and } x \ne 0 \Rightarrow x^2+4=0.$$

But, this is a contradiction, since $x^2+4 \ne 0$ for $x \in R$.

Since, the contradiction arises by assuming that

$$x^2+4=0 \text{ and } x \ne 0, \text{ so } x=0.$$

Hence, $x^3+4x=0 \Rightarrow 0$ is a true statement.

(iii) Method of contrapositive:

We have to prove that $x^3+4x=0 \Rightarrow x=0$.

Let p: $x^3+4x=0$ and q: $x=0$.

We shall prove that $\sim q \Rightarrow \sim p$.
Let $x \neq 0$. Then, $x^3 + 4x = x(x^2 + 4) \neq 0$ $[\because\ x \neq 0$ and $x^2 + 4 \neq 0]$.
Thus, $\sim q \Rightarrow \sim p$ and therefore, $p \Rightarrow q$.

EXERCISE 29D

1. Let p: *If x is an integer and x^2 is even, then x is even.*
Using the method of contrapositive, prove that p is true.

2. Consider the statement:
q: *For any real numbers a and b, $a^2 = b^2 \Rightarrow a = b$.*
By giving a counter-example, prove that q is false.

3. By giving a counter-example, show that the statement is false:
p: If n is an odd positive integer, then n is prime.

4. Use contradiction method to prove that:
p: $\sqrt{3}$ *is irrational*
is a true statement.

5. By giving a counter-example, show that the following statement is false:
p: *If all the sides of a triangle are equal, then the triangle is obtuse angled.*

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Statistics

INTRODUCTION

In earlier classes we have studied the methods of representing data graphically and in tabular form.

Mean, median and mode are three measures of central tendency, giving us a rough idea about the points where data are centred.

In order to have a better idea as to how the data are scattered, we make a study of (i) mean deviation and (ii) standard deviation.

MEAN DEVIATIONS (MD)

MEAN DEVIATION ABOUT THE MEAN *The AM of the numerical deviations of the observations from the mean of the data is called the mean deviation about the mean.*

MEAN DEVIATION ABOUT THE MEDIAN *The AM of the numerical deviations of the observations from the median of the data is called the mean deviation about the median.*

MEAN DEVIATIONS FOR UNGROUPED DATA

Let $x_1, x_2, x_3, \ldots, x_n$ be the given n observations. Let $\bar{x}$ be the AM and M be the median. Then,

(i) $\text{MD}(\bar{x}) = \dfrac{\sum\limits_{i=1}^{n} |x_i - \bar{x}|}{n}$, where $\bar{x}$ = mean.

(ii) $\text{MD}(M) = \dfrac{\sum\limits_{i=1}^{n} |x_i - M|}{n}$, where M = median.

SUMMARY

(i) $\text{MD}(\bar{x}) = \dfrac{\sum\limits_{i=1}^{n} |x_i - \bar{x}|}{n}$, where $\bar{x}$ = mean.

(ii) $\text{MD}(M) = \dfrac{\sum\limits_{i=1}^{n} |x_i - M|}{n}$, where M = median.

EXAMPLE 1 *Find the mean deviation about the mean for the following data:*

$15, 17, 10, 13, 7, 18, 9, 6, 14, 11$

SOLUTION Let the mean of the given data be $\overline{x}$. Then,

$$\overline{x} = \frac{\sum_{i=1}^{n} x_i}{n} = \frac{120}{10} = 12 \quad [\because \; n = 10].$$

The values of $(x_i - \overline{x})$ are:

$3, 5, -2, 1, -5, 6, -3, -6, 2, -1.$

So, the values of $|x_i - \overline{x}|$ are:

$3, 5, 2, 1, 5, 6, 3, 6, 2, 1.$

$$\therefore \quad \text{MD}(\overline{x}) = \frac{\sum_{i=1}^{n} |x_i - \overline{x}|}{n} = \frac{34}{10} = 3.4.$$

Hence, $\text{MD}(\overline{x}) = 3.4$.

EXAMPLE 2 *Find the mean deviation about the median for the data given below:*

$11, 3, 8, 7, 5, 14, 10, 2, 9$

SOLUTION Arranging the given data in an ascending order, we get:

$2, 3, 5, 7, 8, 9, 10, 11, 14.$

Here $n = 9$, which is odd.

$$\therefore \quad \text{median} = \frac{1}{2}(n+1)\text{th observation}$$

$$= \frac{1}{2}(9+1)\text{th observation} = 5\text{th observation} = 8.$$

Thus, $M = 8$.

The values of $(x_i - M)$ are:

$-6, -5, -3, -1, 0, 1, 2, 3, 6.$

$$\therefore \quad \sum_{i=1}^{9} |x_i - M| = (6 + 5 + 3 + 1 + 0 + 1 + 2 + 3 + 6) = 27$$

$$\Rightarrow \quad \text{MD}(M) = \frac{\sum_{i=1}^{9} |x_i - M|}{9} = \frac{27}{9} = 3.$$

Hence, $\text{MD}(M) = 3$.

EXAMPLE 3 *Find the mean deviation about the median for the data given below.*

$45, 36, 50, 60, 53, 46, 51, 48, 72, 42$

SOLUTION Arranging the given data in an ascending order, we get:

$36, 42, 45, 46, 48, 50, 51, 53, 60, 72.$

Here $n = 10$, which is even.

$$\therefore \quad \text{median} = \frac{1}{2} \cdot \left\{ \frac{n}{2}\text{th observation} + \left(\frac{n}{2} + 1\right)\text{th observation} \right\}$$

$$= \frac{1}{2}(\text{5th observation} + \text{6th observation})$$

$$= \frac{1}{2}(48 + 50) = \frac{98}{2} = 49.$$

Thus, $M = 49$.

The values of $(x_i - M)$ are

$-13, -7, -4, -3, -1, 1, 2, 4, 11, 23$.

$$\therefore \quad \sum_{i=1}^{10} |x_i - M| = (13 + 7 + 4 + 3 + 1 + 1 + 2 + 4 + 11 + 23) = 69$$

$$\Rightarrow \quad \text{MD}(M) = \frac{\sum_{i=1}^{10} |x_i - M|}{10} = \frac{69}{10} = 6.9.$$

MEAN DEVIATION FOR DISCRETE FREQUENCY DISTRIBUTION

Let the given data consist of n distinct values $x_1, x_2, \ldots, x_n$ occurring with frequencies $f_1, f_2, \ldots, f_n$ respectively.

(i) Mean deviation about mean:

We have:

$$\overline{x} = \frac{\sum_{i=1}^{n} f_i x_i}{\sum_{i=1}^{n} f_i} \Rightarrow \overline{x} = \frac{\sum_{i=1}^{n} f_i x_i}{N}, \text{ where } N = \sum_{i=1}^{n} f_i.$$

$$\therefore \quad \text{MD}(\overline{x}) = \frac{\sum_{i=1}^{n} f_i |x_i - \overline{x}|}{\sum_{i=1}^{n} f_i} = \frac{\sum_{i=1}^{n} f_i |x_i - \overline{x}|}{N}.$$

(ii) Mean deviation about median:

Let $N = \sum_{i=1}^{n} f_i$. First we find the cumulative frequencies and then, identify the observation whose cumulative frequency is equal to or just greater than $\frac{N}{2}$.

This value gives median. Then,

$$\text{MD}(M) = \frac{\sum_{i=1}^{n} f_i |x_i - M|}{N}.$$

SUMMARY

For discrete frequency distribution, we have

(i) $\text{MD}(\bar{x}) = \dfrac{\sum_{i=1}^{n} f_i |x_i - \bar{x}|}{N}$, where $\sum_{i=1}^{n} f_i = N$.

(ii) $\text{MD}(M) = \dfrac{\sum_{i=1}^{n} f_i |x_i - M|}{N}$, where $\sum_{i=1}^{n} f_i = N$.

EXAMPLE 4 *Find the mean deviation about the mean for the following data:*

x_i	3	5	7	9	11	13
f_i	6	8	15	25	8	4

SOLUTION We have

$$N = \sum_{i=1}^{6} f_i = (6 + 8 + 15 + 25 + 8 + 4) = 66.$$

$$\bar{x} = \frac{\sum_{i=1}^{6} f_i x_i}{N} = \frac{(6 \times 3) + (8 \times 5) + (15 \times 7) + (25 \times 9) + (8 \times 11) + (4 \times 13)}{66}$$

$$= \frac{528}{66} = 8.$$

Now, we prepare the table given below:

x_i	f_i	$\lvert x_i - \bar{x}\rvert$	$\sum_{i=1}^{6} f_i \lvert x_i - \bar{x}\rvert$
3	6	5	30
5	8	3	24
7	15	1	15
9	25	1	25
11	8	3	24
13	4	5	20
	$N = 66$		sum = 138

$$\therefore \quad \text{MD}(\bar{x}) = \frac{\sum_{i=1}^{6} f_i |x_i - \bar{x}|}{N} = \frac{138}{66} = \frac{23}{11} = 2.09.$$

Hence, mean deviation about the mean = 2.09.

EXAMPLE 5 *Find the mean deviation about the median for the following data:*

x_i	3	5	7	9	11	13
f_i	6	8	15	3	8	4

SOLUTION We have

$$N = \sum_{i=1}^{6} f_i = (6 + 8 + 15 + 3 + 8 + 4) = 44.$$

$$\overline{x} = \frac{\sum_{i=1}^{6} f_i x_i}{N} = \frac{(6 \times 3) + (8 \times 5) + (15 \times 7) + (3 \times 9) + (8 \times 11) + (4 \times 13)}{44}$$

$$= \frac{(18 + 40 + 105 + 27 + 88 + 52)}{44} = \frac{330}{44} = \frac{15}{2} = 7.5.$$

x_i	3	5	7	9	11	13
f_i	6	8	15	3	8	4
cf	6	14	29	32	40	44

$\therefore$ $N = 44$, which is even.

$$\therefore \quad \text{median} = \frac{1}{2} \cdot \left\{ \frac{N}{2}\text{th observation} + \left(\frac{N}{2} + 1 \right)\text{th observation} \right\}$$

$$= \frac{1}{2} \{22\text{nd observation} + 23\text{rd observation}\}$$

$$= \frac{1}{2}(7 + 7) = 7.$$

Thus, $M = 7$.

Now, we have:

$\lvert x_i - M \rvert$	4	2	0	2	4	6
f_i	6	8	15	3	8	4
$f_i \lvert x_i - M \rvert$	24	16	0	6	32	24

$$\therefore \quad \sum_{i=1}^{6} f_i = 44 \quad \text{and} \quad \sum_{i=1}^{6} f_i |x_i - M| = 102.$$

$$\therefore \quad \text{MD}(M) = \frac{\sum_{i=1}^{6} f_i |x_i - M|}{N} = \frac{102}{44} = 2.32.$$

Hence, the mean deviation about the median is 2.32.

MEAN DEVIATION ABOUT THE MEAN AND MEDIAN FOR CONTINUOUS FREQUENCY DISTRIBUTION

I. MEAN DEVIATION ABOUT THE MEAN FOR GROUPED DATA

Short Cut Method (Step-deviation method):

Step 1 *Find the class mark x_i of each class.*

Step 2 *Assume that the mean is A.*

Step 3 *Find deviations, $d_i = \frac{(x_i - A)}{h}$, where $h =$ class size.*

Step 4 *Calculate*, $\overline{x} = A + \left\{ \frac{f_i \, d_i}{N} \times h \right\}$, *where* $N = \Sigma f_i$.

Step 5 *Calculate*, $MD\,(\overline{x}) = \frac{\Sigma f_i \, |x_i - \overline{x}\,|}{N}$.

EXAMPLE 6 *Find the mean deviation about the mean for the following data:*

Marks obtained	10–20	20–30	30–40	40–50	50–60	60–70
Number of students	8	6	12	5	2	7

SOLUTION Here $h = 10$. Let the assumed mean be $A = 35$.
Then, we prepare the table given below.

Marks obtained	Number of students (f_i)	Midpoint x_i	$d_i = \frac{x_i - 35}{10}$	$f_i \, d_i$	$\|x_i - \overline{x}\,\|$	$f_i \times \|x_i - \overline{x}\,\|$
10–20	8	15	–2	–16	22	176
20–30	6	25	–1	–6	12	72
30–40	12	35 = A	0	0	2	24
40–50	5	45	1	5	8	40
50–60	2	55	2	4	18	36
60–70	7	65	3	21	28	196
	$N = \Sigma f_i = 40$			$\Sigma f_i \, d_i = 8$		$\Sigma f_i \times \|x_i - \overline{x}\,\| = 544$

$$N = \Sigma f_i = 40,\ \overline{x} = A + \left\{ \frac{\Sigma f_i \, d_i}{N} \times h \right\} = 35 + \left\{ \frac{8}{40} \times 10 \right\} = 37.$$

$$\therefore \quad MD\,(\overline{x}) = \frac{\Sigma f_i \times |x_i - \overline{x}\,|}{N} = \frac{544}{40} = \frac{136}{10} = 13.6.$$

II. MEAN DEVIATION ABOUT THE MEDIAN FOR GROUPED DATA

Short Cut Method:

Step 1 Find the class mark x_i of each class.

Step 2 Find $N = \Sigma f_i$.

Step 3 Now, median class is the class interval whose cumulative frequency is greater than or equal to $(N/2)$.

Step 4 Calculate median by using the formula

$$M = median = \left\{ L + \frac{\left(\frac{N}{2} - c\right)}{f} \times h \right\}, \text{ where}$$

L = lower limit of the median class;
f = frequency of the median class;
h = width of the median class;
and c = cumulative frequency of the class just preceding the median class.

Step 5 Calculate $\text{MD}(M) = \dfrac{\sum f_i |x_i - M|}{N}$.

EXAMPLE 7 *Calculate the mean deviation about the median for the following data:*

Height (*in* cm)	95–105	105–115	115–125	125–135	135–145	145–155
Number of boys	9	13	25	30	13	10

SOLUTION First we find the median.

Class	Frequency f_i	Cumulative frequency (cf)	Midpoint x_i
95–105	9	9	100
105–115	13	22	110
115–125	25	47	120
125–135	30	77	130
135–145	13	90	140
145–155	10	100	150
	$N = \sum f_i = 100$		

Thus $N = 100$ and therefore, $\dfrac{N}{2} = 50$

$\Rightarrow$ median class is 125–135

$\Rightarrow$ $L = 125$, $f = 30$, $h = 10$ and $c = 47$.

$$\therefore \quad \text{median} = L + \frac{\left(\frac{N}{2} - c\right)}{f} \times h$$

$$= \left\{125 + \frac{(50 - 47)}{30} \times 10\right\} = (125 + 1) = 126.$$

$\therefore$ $M = 126$.

Now, we prepare the table given below.

f_i	x_i	$\lvert x_i - M \rvert$	$f_i \times \lvert x_i - M \rvert$
9	100	26	234
13	110	16	208
25	120	6	150
30	130	4	120
13	140	14	182
10	150	24	240
$N = 100$			1134

$$\therefore \quad \text{MD}(M) = \frac{\sum f_i \times \lvert x_i - M \rvert}{N} = \frac{1134}{100} = 11.34.$$

Hence, the mean deviation about the median is 11.34.

EXAMPLE 8 *Calculate the mean deviation about the median for the following data:*

Class	16–20	21–25	26–30	31–35	36–40	41–45	46–50	51–55
Frequency	5	6	12	14	26	12	16	9

SOLUTION Converting the given series into an exclusive series, we prepare the table, given below:

Class	Frequency (f_i)	cf	Midpoint (x_i)
15.5–20.5	5	5	18
20.5–25.5	6	11	23
25.5–30.5	12	23	28
30.5–35.5	14	37	33
35.5–40.5	26	63	38
40.5–45.5	12	75	43
45.5–50.5	16	91	48
50.5–55.5	9	100	53
	$N = 100$		

Thus $N = 100$ and therefore, $\frac{N}{2} = 50$

$\Rightarrow$ median class is 35.5–40.5

$\Rightarrow$ $L = 35.5$, $f = 26$, $h = 5$ and $c = 37$.

$$\therefore \quad \text{median} = L + \frac{\left(\frac{N}{2} - c\right)}{f} \times h$$

$$= \left\{ 35.5 + \frac{(50 - 37)}{26} \times 5 \right\} = (35.5 + 2.5) = 38.$$

Thus, $M = 38$.

Now, we prepare the table given below.

f_i	x_i	$\lvert x_i - M \rvert$	$f_i \times \lvert x_i - M \rvert$
5	18	20	100
6	23	15	90
12	28	10	120
14	33	5	70
26	38	0	0
12	43	5	60
16	48	10	160
9	53	15	135
$N = 100$			735

Thus, $\sum f_i \times \lvert x_i - M \rvert = 735$ and $N = 100$.

$$\therefore \quad \text{MD}(M) = \frac{\sum f_i \times \lvert x_i - M \rvert}{N} = \frac{735}{100} = 7.35.$$

Hence, the mean deviation about the median is 7.35.

EXERCISE 30A

Find the mean deviation about the mean for the following data:

1. 7, 8, 4, 13, 9, 5, 16, 18

2. 39, 72, 48, 41, 43, 55, 60, 45, 54, 43

3. 17, 20, 12, 13, 15, 16, 12, 18, 15, 19, 12, 11

Find the mean deviation about the median for the following data:

4. 12, 5, 14, 6, 11, 13, 17, 8, 10

5. 4, 15, 9, 7, 19, 13, 6, 21, 8, 25, 11

6. 34, 23, 46, 37, 40, 28, 32, 50, 35, 44

7. 70, 34, 42, 78, 65, 45, 54, 48, 67, 50, 56, 63

Find the mean deviation about the mean for the following data:

8.

x_i	6	12	18	24	30	36
f_i	5	4	11	6	4	6

9.

x_i	2	5	6	8	10	12
f_i	2	8	10	7	8	5

10.

x_i	3	5	7	9	11	13
f_i	6	8	15	25	8	4

Find the mean deviation about the median for the following data:

11.

x_i	15	21	27	30	35
f_i	3	5	6	7	8

12.

x_i	5	7	9	11	13	15	17
f_i	2	4	6	8	10	12	8

13.

x_i	10	15	20	25	30	35	40	45
f_i	7	3	8	5	6	8	4	9

Find the mean deviation about the mean for the following data:

14.

Mark	0–10	10–20	20–30	30–40	40–50	50–60
Number of students	6	8	14	16	4	2

15.

Height (in cm)	95–105	105–115	115–125	125–135	135–145	145–155
Number of boys	9	16	23	30	12	10

16.

Class	30–40	40–50	50–60	60–70	70–80	80–90	90–100
Frequency	3	7	12	15	8	3	2

Find the mean deviation about the median for the following data:

17.

Class	0–10	10–20	20–30	30–40	40–50	50–60
Frequency	6	7	15	16	4	2

18.

Class	0–10	10–20	20–30	30–40	40–50	50–60
Frequency	6	8	11	18	5	2

ANSWERS (EXERCISE 30A)

1. 4.25	**2.** 8.2	**3.** 2.5	**4.** 3
5. 5.36	**6.** 6.5	**7.** 10.5	**8.** 8
9. 2.3	**10.** 2.09	**11.** 5.1	**12.** 2.72
13. 10.1	**14.** 10.24	**15.** 11.6	**16.** 11.36
17. 10.16	**18.** 10.8		

VARIANCE AND STANDARD DEVIATION

VARIANCE *Mean of the squares of the deviations from the mean is called the variance, to be denoted by* σ^2.

The variance of n observations $x_1, x_2, \ldots, x_n$ is given by $\sigma^2 = \frac{1}{n} \cdot \sum_{i=1}^{n} (x_i - \bar{x})^2$.

STANDARD DEVIATION *The positive square root of variance is called standard deviation and it is denoted by* σ.

$$\sigma = \sqrt{\frac{1}{n} \cdot \sum_{i=1}^{n} (x_i - \bar{x})^2}.$$

EXAMPLE 1 *Find the mean, variance and the standard deviation for the following data:*
5, 9, 8, 12, 6, 10, 6, 8

SOLUTION Here $n = 8$.

Mean, $\bar{x} = \frac{1}{8}(5 + 9 + 8 + 12 + 6 + 10 + 6 + 8) = \frac{64}{8} = 8$.

The values of $(x_i - \bar{x})$ are

$-3, 1, 0, 4, -2, 2, -2, 0$.

$\therefore \quad (x_i - \bar{x})^2 = (9 + 1 + 0 + 16 + 4 + 4 + 4 + 0) = 38$.

$\therefore \quad$ variance, $\sigma^2 = \frac{\sum(x_i - \bar{x})^2}{n} = \frac{38}{8} = \frac{19}{4} = 4.75$.

and, standard deviation, $\sigma = \sqrt{4.75} = 2.17$.

$\therefore \quad$ mean = 8, variance = 4.75 and standard deviation = 2.17.

Short Cut Method

For simple ungrouped data, we have

$$\sigma^2 = \frac{1}{n} \cdot \left\{ \sum_{i=1}^{n} (x_i - \bar{x})^2 \right\}$$

$$= \frac{1}{n} \cdot \left\{ \sum_{i=1}^{n} (x_i^2 - 2x_i \bar{x} + \bar{x}^2) \right\}$$

$$= \frac{\sum x_i^2}{n} - 2\bar{x}\left(\frac{\sum x_i}{n}\right) + \frac{n\bar{x}^2}{n}$$

$$= \frac{\sum x_i^2}{n} - 2\,\bar{x}\cdot\bar{x} + \bar{x}^2 = \frac{\sum x_i^2}{n} - \bar{x}^2.$$

Thus, for short cut method, we have

$$\sigma^2 = \frac{\sum x_i^2}{n} - \bar{x}^2.$$

EXAMPLE 2 *Find the mean, standard deviation and variance of first n natural numbers.*

SOLUTION First n natural numbers are $1, 2, 3, \ldots, n$.

$$\text{Mean, } \bar{x} = \frac{(1+2+3+\ldots+n)}{n} = \frac{1}{n}\cdot\frac{1}{2}n(n+1) = \frac{1}{2}(n+1)$$

$$\left[\because\ (1+2+3+\ldots+n) = \frac{1}{2}n(n+1)\right]$$

$$\therefore\quad \text{variance, } \sigma^2 = \frac{\sum x_i^2}{n} - \bar{x}^2$$

$$= \frac{\sum n^2}{n} - \left\{\frac{1}{2}(n+1)\right\}^2$$

$$= \frac{1}{n}\cdot\frac{n(n+1)(2n+1)}{6} - \frac{1}{4}(n+1)^2$$

$$\left[\because\ \sum n^2 = \frac{1}{6}n(n+1)(2n+1)\right]$$

$$= \left\{\frac{(n+1)(2n+1)}{6} - \frac{(n+1)^2}{4}\right\}$$

$$= (n+1)\cdot\left\{\frac{(2n+1)}{6} - \frac{(n+1)}{4}\right\}$$

$$= \frac{(n+1)(n-1)}{12} = \frac{(n^2-1)}{12}.$$

$$\therefore\quad \text{variance, } \sigma^2 = \frac{(n^2-1)}{12}.$$

$$\text{Standard deviation, } \sigma = \sqrt{\frac{n^2-1}{12}} = \frac{1}{2}\cdot\sqrt{\frac{n^2-1}{3}}.$$

EXAMPLE 3 *Find the mean, standard deviation and variance of first* 10 *multiples of* 3.

SOLUTION First 10 multiples of 3 are $3, 6, 9, 12, \ldots, 30$.

This is an AP with $a = 3$, $n = 10$ and $l = 30$.

$$\therefore\quad \text{sum} = \frac{n}{2}(a+l) = \frac{10}{2}\cdot(3+30) = 165.$$

$$\therefore\quad \text{mean, } \bar{x} = \frac{165}{10} = 16.5.$$

$$\text{Variance, } \sigma^2 = \frac{\sum x_i^2}{n} - (\bar{x})^2$$

$$= \left\{\frac{3^2 + 6^2 + 9^2 + \ldots + 30^2}{10} - (16.5)^2\right\}$$

$$= \frac{3^2 \times \{1^2 + 2^2 + 3^2 + \ldots + 10^2\}}{10} - (16.5)^2$$

$$= \frac{9 \times 10 \times (10+1)(2 \times 10 + 1)}{6 \times 10} - (16.5)^2$$

$$\left[\because \sum n^2 = \frac{n(n+1)(2n+1)}{6}\right]$$

$$= \left(\frac{9 \times 10 \times 11 \times 21}{6 \times 10} - 272.25\right) = \left(\frac{693}{2} - 272.25\right)$$

$$= (346.5 - 272.25) = 74.25.$$

$\therefore$ variance, $\sigma^2 = 74.25$.

Standard deviation, $\sigma = \sqrt{74.25} = 8.61$.

VARIANCE AND STANDARD DEVIATION OF A DISCRETE FREQUENCY DISTRIBUTION

Let the given frequency distribution be:

Variable	x_1	x_2	x_3	...	x_n
Frequency	f_1	f_2	f_3	...	f_n

Then, we define:

(i) *Variance*, $\sigma^2 = \frac{1}{N} \cdot \sum_{i=1}^{n} f_i(x_i - \bar{x})^2$, where $N = \sum_{i=1}^{n} f_i$

(ii) *Standard deviation*, $\sigma = \sqrt{\frac{1}{N} \cdot \sum f_i(x_i - \bar{x})^2}$

SUMMARY

$$\text{Variance, } \sigma^2 = \frac{1}{N} \cdot \sum_{i=1}^{n} f_i(x_i - \bar{x})^2$$

$$\text{Standard deviation, } \sigma = \sqrt{\frac{1}{N} \cdot \sum_{i=1}^{n} f_i(x_i - \bar{x})^2}$$

EXAMPLE 4 *Find the variance and standard deviation for the following data:*

x_i	10	15	18	20	25
f_i	3	2	5	8	2

SOLUTION We may prepare the table given below:

x_i	f_i	$f_i\,x_i$	$(x_i-\overline{x})=(x_i-18)$	$(x_i-\overline{x})^2$	$f_i\,(x_i-\overline{x})^2$
10	3	30	–8	64	192
15	2	30	–3	9	18
18	5	90	0	0	0
20	8	160	2	4	32
25	2	50	7	49	98
$\overline{x}=\text{Mean}=\dfrac{\Sigma f_i x_i}{\Sigma f_i}=\dfrac{360}{20}=18$					$\Sigma f_i\,(x_i-\overline{x})^2 = 340$

$\therefore \quad N=\sum_{i=1}^{5} f_i = 20,\quad \sum_{i=1}^{5} f_i\,(x_i-\overline{x})^2 = 340.$

$\therefore \quad \text{variance, } \sigma^2 = \frac{1}{N}\cdot \Sigma f_i\,(x_i-\overline{x})^2 = \left(\frac{1}{20}\times 340\right) = 17.$

Standard deviation, $\sigma = \sqrt{17} = 4.12$.

Short Cut Method:

Let A be the assumed mean and let $d_i = (x_i - A)$.

$$\text{Then, } \sigma^2 = \left\{\frac{\Sigma f_i d_i^2}{N} - \left(\frac{\Sigma f_i d_i}{N}\right)^2\right\}.$$

EXAMPLE 5 *Find the mean, variance and standard deviation for the following data using short cut method:*

x_i	60	61	62	63	64	65	66	67	68
f_i	2	1	12	29	25	12	10	4	5

SOLUTION Let the assumed mean be $A = 64$.

Then, we may present the table given below:

x_i	f_i	$d_i=(x_i-A)$ $=(x_i-64)$	d_i^2	$f_i\,d_i$	$f_i\,d_i^2$
60	2	–4	16	–8	32
61	1	–3	9	–3	9
62	12	–2	4	–24	48
63	29	–1	1	–29	29
$64=A$	25	0	0	0	0
65	12	1	1	12	12
66	10	2	4	20	40
67	4	3	9	12	36
68	5	4	16	20	80
	100			0	286

$\therefore \quad N = \Sigma f_i = 100, \ \ \Sigma f_i d_i = 0 \ \text{ and } \ \Sigma f_i d_i^2$

$$\therefore \quad \text{mean} = A + \frac{\Sigma f_i d_i}{\Sigma f_i} = \left(64 + \frac{0}{100}\right) = 64.$$

$$\text{Variance, } \sigma^2 = \frac{\Sigma f_i d_i^2}{\Sigma f_i} - \left(\frac{\Sigma f_i d_i}{\Sigma f_i}\right)^2$$

$$= \frac{286}{100} - \left(\frac{0}{100}\right)^2 = \frac{286}{100} = 2.86.$$

$\therefore \quad$ standard deviation, $\sigma = \sqrt{2.86} = 1.69.$

When Mean is a Decimal Fraction

In this case, we use the formulae given below.

(i) Variance, $\sigma^2 = \dfrac{1}{N^2} \cdot \left\{ N \cdot \sum_{i=1}^{n} f_i x_i^2 - \left(\sum_{i=1}^{n} f_i x_i \right)^2 \right\}$

(ii) Standard deviation, $\sigma = \dfrac{1}{N} \cdot \sqrt{N \cdot \sum_{i=1}^{n} f_i x_i^2 - \left(\sum_{i=1}^{n} f_i x_i \right)^2}$

EXAMPLE 6 *Find the standard deviation for the following data:*

x_i	3	8	13	18	23
f_i	6	10	14	10	10

SOLUTION We have

$$\text{Mean, } \bar{x} = \frac{\Sigma f_i x_i}{\Sigma f_i} = \frac{(18 + 80 + 182 + 180 + 230)}{50} = \frac{690}{50} = 13.8.$$

This value being in decimal form, the calculation will become tedious.

So, we use the formula, $\sigma = \dfrac{1}{N} \cdot \sqrt{N \cdot \Sigma f_i x_i^2 - (\Sigma f_i)^2}$.

Now, we prepare the table given below:

x_i	f_i	$f_i x_i$	x_i^2	$f_i x_i^2$
3	6	18	9	54
8	10	80	64	640
13	14	182	169	2366
18	10	180	324	3240
23	10	230	529	5290
	50	690		11590

$\therefore \quad N = \Sigma f_i = 50, \quad \Sigma f_i x_i = 690 \quad \text{and} \quad \Sigma f_i x_i^2 = 11590$

$\therefore \quad$ standard deviation, $\sigma = \dfrac{1}{N} \cdot \{\sqrt{N \cdot \Sigma f_i x_i^2 - (\Sigma f_i x_i)^2}\}$

$$\Rightarrow \quad \sigma = \frac{1}{50} \cdot \sqrt{50 \times 11590 - (690)^2} = \frac{1}{50} \times \sqrt{579500 - (690)^2}$$

$$= \frac{1}{50} \sqrt{579500 - 476100} = \frac{1}{50} \times \sqrt{103400} = \frac{321.5}{50} = 6.43.$$

Hence, $\sigma = 6.43$.

Shorter Method For Standard Deviation of Continuous Frequency Distribution

Let A be the assumed mean and let $y_i = \dfrac{(x_i - A)}{h}$.

Then, we have:

$$\text{Variance, } \sigma^2 = \frac{h^2}{N^2} \cdot \left\{ N \cdot \sum_{i=1}^{n} f_i y_i^2 - \left(\sum_{i=1}^{n} f_i y_i \right)^2 \right\}$$

$$\text{Standard deviation, } \sigma = \frac{h}{N} \cdot \sqrt{N \cdot \sum_{i=1}^{n} f_i y_i^2 - \left(\sum_{i=1}^{n} f_i y_i \right)^2}$$

EXAMPLE 7 *Calculate mean, variance and standard deviation for the following frequency distribution:*

Class	0–30	30–60	60–90	90–120	120–150	150–180	180–210
Frequency	2	3	5	10	3	5	2

SOLUTION Here $h = 30$. Let the assumed mean be $A = 105$.
Now, we prepare the table given below.

Class	Frequency f_i	Midpoint x_i	$y_i = \dfrac{(x_i - 105)}{30}$	y_i^2	$f_i y_i$	$f_i y_i^2$
0–30	2	15	–3	9	–6	18
30–60	3	45	–2	4	–6	12
60–90	5	75	–1	1	–5	5
90–120	10	$105 = A$	0	0	0	0
120–150	3	135	1	1	3	3
150–180	5	165	2	4	10	20
180–210	2	195	3	9	6	18
Total	$N = 30$				2	76

$\therefore \quad A = 105,\ h = 30,\ N = \Sigma f_i = 30,\ \Sigma f_i y_i = 2$ and $\Sigma f_i y_i^2 = 76.$

$\therefore \quad \overline{x} = \left(A + \dfrac{\Sigma f_i y_i}{N} \times h\right) \Rightarrow \overline{x} = \left(105 + \dfrac{2}{30} \times 30\right) = (105 + 2) = 107.$

Thus, mean = 107.

Variance, $\sigma^2 = \dfrac{h^2}{N^2} \cdot \{N \cdot \Sigma f_i y_i^2 - (\Sigma f_i y_i)^2\}$

$= \dfrac{(30)^2}{(30)^2} \cdot \{30 \times 76 - (2)^2\} = (2280 - 4) = 2276.$

Standard deviation, $\sigma = \sqrt{2276} = 47.71.$

EXAMPLE 8 *Given below are the diameters of circles (in mm) drawn in a design.*

Diameter	33–36	37–40	41–44	45–48	49–52
Number of circles	15	17	21	22	25

Calculate the mean diameter of the circles, variance and standard deviation.

SOLUTION Converting the given series into an exclusive series, we prepare the table given below.

Class	Frequency f_i	Midpoint x_i	$y_i = \dfrac{(x_i - 42.5)}{4}$	y_i^2	$f_i y_i$	$f_i y_i^2$
32.5–36.5	15	34.5	–2	4	–30	60
36.5–40.5	17	38.5	–1	1	–17	17
40.5–44.5	21	42.5 = A	0	0	0	0
44.5–48.5	22	46.5	1	1	22	22
48.5–52.5	25	50.5	2	4	50	100
	$N = 100$				25	199

$\therefore \quad A = 42.5,\ h = 4,\ N = \Sigma f_i = 100,\ \Sigma f_i y_i = 25$ and $\Sigma f_i y_i^2 = 199.$

$\therefore \quad \overline{x} = \left(A + \dfrac{\Sigma f_i y_i}{N} \times h\right) \Rightarrow \overline{x} = \left(42.5 + \dfrac{25}{100} \times 4\right) = 43.5.$

Thus, mean = 43.5 mm.

Variance, $\sigma^2 = \dfrac{h^2}{N^2} \cdot \{N \cdot \Sigma f_i y_i^2 - (\Sigma f_i y_i)^2\}$

$= \dfrac{16}{10000} \times \{100 \times 199 - (25)^2\} = \dfrac{16}{10000} \times (19900 - 625)$

$= \left(\dfrac{16}{10000} \times 19275\right) = \dfrac{3084}{100} = 30.84.$

Standard deviation, $\sigma = \sqrt{30.84} = 5.55.$

Hence, mean = 43.5 mm, variance = 30.84, and standard deviation = 5.55.

EXERCISE 30B

1. Find the mean, variance and standard deviation for the numbers 4, 6, 10, 12, 7, 8, 13, 12.
2. Find the mean, variance and standard deviation for first six odd natural numbers.

Using short cut method, find the mean, variance and standard deviation for the data:

3.

x_i	4	8	11	17	20	24	32
f_i	3	5	9	5	4	3	1

4.

x_i	6	10	14	18	24	28	30
f_i	2	4	7	12	8	4	3

5.

x_i	10	15	18	20	25
f_i	3	2	5	8	2

6.

x_i	92	93	97	98	102	104	109
f_i	3	2	3	2	6	3	3

7.

Class	0–10	10–20	20–30	30–40	40–50
Frequency	5	8	15	16	6

8.

Class	30–40	40–50	50–60	60–70	70–80	80–90	90–100
Frequency	3	7	12	15	8	3	2

9.

Class	25–35	35–45	45–55	55–65	65–75
Frequency	64	132	153	140	51

ANSWERS (EXERCISE 30B)

1. Mean = 9, Variance = 9.25 and SD = 3.04
2. Mean = 6, Variance = 11.67 and SD 3.41
3. Mean = 14, Variance = 45.8 and SD = 6.77
4. Mean = 19, Variance = 43.4 and SD = 6.59
5. Mean = 18, Variance = 17 and SD = 4.12
6. Mean = 100, Variance = 29.09 and SD = 5.39
7. Mean = 27, Variance = 132 and SD = 11.49
8. Mean = 62, Variance = 201 and SD 14.17
9. Mean = 49.67, Variance = 135.44 and SD = 11.64

MISCELLANEOUS WORD PROBLEMS

EXAMPLE 1 *The mean and variance of seven observations are* 8 *and* 16 *respectively. If five of these are* 2, 4, 10, 12, 14, *find the remaining two observations.*

SOLUTION Let the remaining two observations be x and y. Then,

$$\text{mean} = 8 \Rightarrow \frac{2 + 4 + 10 + 12 + 14 + x + y}{7} = 8$$

$$\Rightarrow 42 + x + y = 56$$

$$\Rightarrow x + y = 14. \qquad \text{... (i)}$$

Also, variance = 16

$$\Rightarrow \frac{1}{7}(2^2 + 4^2 + 10^2 + 12^2 + 14^2 + x^2 + y^2) - 8^2 = 16$$

$$\left[\because \sigma^2 = \frac{\sum x_i^2}{n} - (\bar{x})^2\right]$$

$$\Rightarrow \frac{1}{7}(460 + x^2 + y^2) = 80$$

$$\Rightarrow 460 + x^2 + y^2 = 560$$

$$\Rightarrow x^2 + y^2 = 100. \qquad \text{... (ii)}$$

Now, $(x + y)^2 + (x - y)^2 = 2(x^2 + y^2)$

$$\Rightarrow (x - y)^2 = 2(x^2 + y^2) - (x + y)^2$$

$$\Rightarrow (x - y)^2 = (2 \times 100) - (14)^2 = (200 - 196) = 4$$

$$\Rightarrow x - y = \pm 2.$$

Now, $x + y = 14,\ x - y = 2 \Rightarrow x = 8,\ y = 6;$

$x + y = 14,\ x - y = -2 \Rightarrow x = 6,\ y = 8.$

Hence, the remaining two observations are 6 and 8.

EXAMPLE 2 *The mean and variance of six observations are 8 and 16 respectively. If each observation is multiplied by 3, find the new mean and new variance of the resulting observations.*

SOLUTION Let the given observations be $x_1, x_2, x_3, x_4, x_5, x_6$.

Then, mean = 8 $\Rightarrow \frac{1}{6}(x_1 + x_2 + x_3 + x_4 + x_5 + x_6) = 8$

$$\Rightarrow x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 48. \qquad \text{... (i)}$$

Also, variance = 16

$$\Rightarrow \frac{1}{6}(x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 + x_6^2) - 8^2 = 16 \qquad \left[\because \sigma^2 = \frac{\sum x_i^2}{n} - (\bar{x})^2\right]$$

$$\Rightarrow x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 + x_6^2 = 480. \qquad \text{... (ii)}$$

When each observation is multiplied by 3, then new observations are $3x_1, 3x_2, 3x_3, 3x_4, 3x_5$ and $3x_6$.

$\therefore \quad \text{new mean} = \frac{1}{6}(3x_1 + 3x_2 + 3x_3 + 3x_4 + 3x_5 + 3x_6)$

$= \frac{3}{6}(x_1 + x_2 + x_3 + x_4 + x_5 + x_6) = \left(\frac{1}{2} \times 48\right) = 24$ [using (i)]

$\therefore \quad \text{new variance} = \frac{(3x_1)^2 + (3x_2)^2 + (3x_3)^2 + (3x_4)^2 + (3x_5)^2 + (3x_6)^2}{6} - (24)^2$

$= \frac{9}{6}(x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 + x_6^2) - 576$

$= \left(\frac{9}{6} \times 480\right) - 576 = (720 - 576) = 144$ [using (ii)].

Hence, new mean = 24 and new variance = 144.

EXAMPLE 3 *The mean and standard deviation of* 20 *observations are found to be* 10 *and* 2 *respectively. On rechecking it was found that an observation* 8 *was incorrect. Calculate the correct mean and standard deviation when*

(i) the wrong item is omitted,

(ii) the wrong item is replaced by 12.

SOLUTION Mean = 10 $\Rightarrow \frac{\sum_{i=1}^{20} x_i}{20} = 10$

$\Rightarrow \sum_{i=1}^{20} x_i = 200.$

SD = 2 $\Rightarrow \sigma^2 = 4$

$\Rightarrow \frac{\sum_{i=1}^{20} x_i^2}{20} - (10)^2 = 4$

$\Rightarrow \sum_{i=1}^{20} x_i^2 = 2080.$

Thus, incorrect $\left(\sum_{i=1}^{20} x_i^2\right) = 200$ and incorrect $\left(\sum_{i=1}^{20} x_i^2\right) = 2080$.

CASE (i) *When the wrong item is omitted*

On omitting 8, we are left with 19 observations.

$\therefore \quad \text{correct}\left(\sum_{i=1}^{19} x_i\right) = \text{incorrect}\left(\sum_{i=1}^{20} x_i\right) - 8$

$= (200 - 8) = 192.$

Thus, correct $\left(\sum_{i=1}^{19} x_i\right) = 192$

$\therefore \quad \text{correct mean} = \frac{192}{19} = 10.105.$ ... (i)

Also, correct $\left(\sum_{i=1}^{19} x_i^2\right) = \text{incorrect}\left(\sum_{i=1}^{20} x_i^2\right) - 64$

$= (2080 - 64) = 2016.$

$\therefore$ correct variance $= \frac{1}{19}\left(\text{correct } \sum_{i=1}^{19} x_i^2\right) - (\text{correct mean})^2$

$= \left(\frac{1}{19} \times 2016\right) - \left(\frac{192}{19}\right)^2 = \left(\frac{2016}{19} - \frac{36864}{361}\right)$

$= \frac{(38304 - 36864)}{361} = \frac{1440}{361}.$

$\therefore$ correct SD $= \sqrt{\frac{1440}{361}} = \frac{12\sqrt{10}}{19} = \frac{(12 \times 3.162)}{19}$

$= \frac{37.944}{19} = 1.997.$

$\therefore$ mean $= 10.105$ and SD $= 1.997$.

CASE (ii) *When incorrect observation* 8 *is replaced by* 12.

Incorrect $\left(\sum_{i=1}^{20} x_i\right) = 200$

$\Rightarrow$ correct $\left(\sum_{i=1}^{20} x_i\right) = (200 - 8 + 12) = 204$

$\Rightarrow$ correct mean $= \frac{204}{20} = 10.2.$

Also, incorrect $\left(\sum_{i=1}^{20} x_i^2\right) = 2080$

$\Rightarrow$ correct $\left(\sum_{i=1}^{20} x_i^2\right) = (2080 - 8^2 + 12^2) = 2160$

$\Rightarrow$ correct variance $= \frac{\text{correct}\left(\sum_{i=1}^{20} x_i^2\right)}{20} - (\text{correct mean})^2$

$= \left\{\frac{2160}{20} - (10.2)^2\right\} = (108 - 104.04) = 3.96.$

$\Rightarrow$ correct SD $= \sqrt{3.96} = 1.989$

Thus, in this case, we have mean = 10.2 and SD = 1.989.

EXAMPLE 4 *The mean and standard deviation of* 100 *observations were calculated as* 40 *and* 5.1 *respectively by a student who took by mistake* 50 *instead of* 40 *for one observation. What are the correct mean and standard deviation?*

SOLUTION Mean = 40 $\Rightarrow \frac{\sum_{i=1}^{100} x_i}{100} = 40$

$\Rightarrow \sum_{i=1}^{100} x_i = 4000.$

$\text{SD} = 5.1 \Rightarrow \sigma^2 = (5.1)^2$

$$\Rightarrow \frac{\sum_{i=1}^{100} x_i^2}{100} - (40)^2 = 26.01$$

$$\Rightarrow \sum_{i=1}^{100} x_i^2 = 162601.$$

Thus, incorrect $\left(\sum_{i=1}^{100} x_i\right) = 4000$ and incorrect $\left(\sum_{i=1}^{100} x_i^2\right) = 162601$

Now, incorrect $\left(\sum_{i=1}^{100} x_i\right) = 4000$

$\Rightarrow$ correct $\left(\sum_{i=1}^{100} x_i\right) = (4000 - 50 + 40) = 3990$

$\Rightarrow$ correct mean $= \dfrac{3990}{100} = 39.9.$ ... (i)

And, incorrect $\left(\sum_{i=1}^{100} x_i^2\right) = 162601$

$\Rightarrow$ correct $\left(\sum_{i=1}^{100} x_i^2\right) = \{162601 - (50)^2 + (40)^2\} = 161701$

$$\Rightarrow \text{correct variance} = \frac{\text{correct}\left(\sum_{i=1}^{100} x_i^2\right)}{100} - (\text{correct mean})^2$$

$$= \left\{\frac{161701}{100} - (39.9)^2\right\} = \{1617.01 - (40 - 0.1)^2\}$$

$$= (1617.01) - \{1600 + 0.01 - 8\}$$

$$= (1617.01 - 1592.01) = 25$$

$\Rightarrow$ correct SD $= \sqrt{25} = 5.$

Hence, correct mean = 39.9 and correct SD = 5.

EXAMPLE 5 *If each of the observations $x_1, x_2, x_3, \ldots, x_n$ is increased by an amount a, where a is a negative or positive number, then show that the variance remains unchanged.*

SOLUTION Let $\bar{x}$ be the mean of $x_1, x_2, x_3, \ldots, x_n$. Then,

$$\bar{x} = \frac{1}{n}(x_1 + x_2 + x_3 + \ldots + x_n).$$

Let $y_i = x_i + a$ for each $i = 1, 2, 3, \ldots, n$.

Let $\bar{y}$ be the mean of $y_1, y_2, y_3, \ldots, y_n$. Then,

$$\bar{y} = \frac{1}{n}(y_1 + y_2 + y_3 + \ldots + y_n)$$

$$= \frac{1}{n}(x_1 + a + x_2 + a + x_3 + a + \ldots + x_n + a)$$

$$= \frac{1}{n}(x_1 + x_2 + x_3 + \ldots + x_n) + \frac{1}{n}(a + a + a + \ldots n \text{ times})$$

$$= \overline{x} + \frac{1}{n}(na) = (\overline{x} + a).$$

$\therefore \quad \overline{y} = (\overline{x} + a).$

Now, the variance of the new observations is given by

$$\text{variance}(y) = \sigma^2 = \frac{1}{n} \cdot \sum_{i=1}^{n} (y_i - \overline{y})^2$$

$$= \frac{1}{n} \cdot \sum_{i=1}^{n} \{(x_i + a) - (\overline{x} + a)\}^2$$

$$[\because \; y_i = (x_i + a) \quad \text{and} \quad \overline{y} = (\overline{x} + a)]$$

$$= \frac{1}{n} \cdot \sum_{i=1}^{n} (x_i + a - \overline{x} - a)^2$$

$$= \frac{1}{n} \cdot \sum_{i=1}^{n} (x_i - \overline{x})^2$$

$$= \text{variance}(x).$$

Hence, the variance of the new observations is the same as the variance of the original observations.

EXAMPLE 6 *If the mean and variance of the observations $x_1, x_2, x_3, \ldots, x_n$ are $\overline{x}$ and σ^2 respectively and a be a nonzero real number, then show that the mean and variance of $ax_1, ax_2, ax_3, \ldots, ax_n$ are $a\overline{x}$ and $a^2\sigma^2$ respectively.*

SOLUTION Let $\overline{x}$ be the mean of $x_1, x_2, x_3, \ldots, x_n$ and a be a nonzero real number.

Then, $\overline{x} = \frac{1}{n}(x_1 + x_2 + x_3 + \ldots + x_n)$.

Let $y_i = ax_i$ for each $i = 1, 2, 3, \ldots, n$. Then,

$$\overline{y} = \frac{1}{n}(y_1 + y_2 + y_3 + \ldots + y_n)$$

$$= \frac{1}{n}(ax_1 + ax_2 + ax_3 + \ldots + ax_n) = a \cdot \frac{1}{n}(x_1 + x_2 + x_3 + \ldots + x_n) = a\overline{x}.$$

Thus, $\overline{y} = a\overline{x}$.

Now, the variance of new observations is given by

$$\text{variance}(y) = \sigma_1^2$$

$$= \frac{1}{n} \cdot \sum_{i=1}^{n} (y_i - \overline{y})^2$$

$$= \frac{1}{n} \cdot \sum_{i=1}^{n} (ax_i - a\overline{x})^2 \quad [\because \; y_i = ax_i \text{ for each } i \text{ and } \overline{y} = a\overline{x}\,]$$

$$= a^2 \cdot \frac{1}{n} \cdot \sum_{i=1}^{n} (x_i - \overline{x})^2$$

$$= a^2 \cdot \{\text{variance } (x)\} = a^2\sigma^2.$$

$\therefore$ new variance $= a^2\sigma^2$.

REMARK $\sigma_1 = \sqrt{a^2\sigma^2} = |a| \cdot \sigma$.

EXERCISE 30C

1. If the standard deviations of the numbers 2, 3, $2x$, 11 is 3.5, calculate the possible values of x.

2. The variance of 15 observations is 6. If each observation is increased by 8, find the variance of the resulting observations.

3. The variance of 20 observations is 5. If each observation is multiplied by 2, find the variance of the resulting observations.

4. The mean and variance of five observations are 6 and 4 respectively. If three of these are 5, 7 and 9, find the other two observations.

5. The mean and variance of five observations are 4.4 and 8.24 respectively. If three of these are 1, 2 and 6, find the other two observations.

6. The mean and standard deviation of 18 observations are found to be 7 and 4 respectively. On rechecking it was found that an observation 12 was misread as 21. Calculate the correct mean and standard deviation.

7. For a group of 200 candidates, the mean and standard deviations of scores were found to be 40 and 15 respectively. Later on it was discovered that the score of 43 was misread as 34. Find the correct mean and standard deviation.

8. The mean and standard deviations of a group of 100 observations were found to be 20 and 3 respectively. Later on it was found that three observations 21, 12 and 18 were incorrect. Find the mean and standard deviation if the incorrect observations were omitted.

ANSWERS (EXERCISE 30C)

1. 3 and $\frac{7}{3}$ **2.** 6 **3.** 20 **4.** 3 and 6

5. 4 and 9 **6.** 6.5, 2.5 **7.** 40.045, 14.995 **8.** 20.06, 14.4

HINTS TO SOME SELECTED QUESTIONS

1. $\sigma^2 = \frac{\sum x_i^2}{n} - (\overline{x})^2 \Rightarrow 3x^2 - 16x + 21 = 0.$

COEFFICIENT OF VARIATION

COEFFICIENT OF VARIATION

Let the mean and standard deviation of the given data be $\overline{x}$ and σ respectively. Then, the coefficient of variation is defined as

$$CV = \left(\frac{\sigma}{\overline{x}} \times 100\right), \text{ where } \overline{x} \neq 0.$$

For comparing the dispersion of two series, we calculate the coefficient of variation for each series.

The series whose CV is more, is more variable than the other.

COMPARISON OF TWO FREQUENCY DISTRIBUTIONS WITH SAME MEAN

Let there be two frequency distributions with the same mean $\overline{x}$ and having standard deviations σ_1 and σ_2 respectively.

Then, CV for 1st distribution $= \left(\frac{\sigma_1}{\overline{x}} \times 100\right)\%$.

And, CV for 2nd distribution $= \left(\frac{\sigma_2}{\overline{x}} \times 100\right)\%$,

Thus, $\frac{CV_1}{CV_2} = \frac{\sigma_1}{\sigma_2}$, where CV_1 and CV_2 denote the CV for 1st and 2nd distributions respectively.

Thus, *in two series having the same mean, the one having more standard deviation is more variable.*

EXAMPLE 1 *The following results show the number of workers and the wages paid to them in two factories A and B.*

Factory	*A*	*B*
Number of workers	4000	5000
Mean wages	Rs 3500	Rs 3500
Variance of distribution of wages	64	81

Which factory has more variation in wages?

SOLUTION We have:

Variance of distribution of wages in factory A, $\sigma_1^2 = 64$.

Variance of distribution of wages in factory B, $\sigma_2^2 = 81$.

$\therefore \quad (\sigma_1^2 = 64 \quad \text{and} \quad \sigma_2^2 = 81) \Rightarrow (\sigma_1 = 8 \quad \text{and} \quad \sigma_2 = 9).$

Thus, SD of distribution of wages in factory $A = 8$.

And, SD of distribution of wages in factory $B = 9$.

Since the monthly mean wages in two factories are the same and $\sigma_1 < \sigma_2$, hence, the factory *B* has more variation in wages.

EXAMPLE 2 *Coefficient of variation of two distributions are 60% and 75%, and their standard deviations are 18 and 15 respectively. Find their arithmetic means.*

SOLUTION Given: $CV_1 = 60$, $CV_2 = 75$, $\sigma_1 = 18$ and $\sigma_2 = 15$.
Let $\bar{x}_1$ and $\bar{x}_2$ be the means of 1st and 2nd distribution respectively.
Then,

$$CV_1 = \frac{\sigma_1}{\bar{x}_1} \times 100 \Rightarrow \bar{x}_1 = \frac{\sigma_1 \times 100}{CV_1}$$

$$\Rightarrow \bar{x}_1 = \frac{18 \times 100}{60} = 30.$$

$$CV_2 = \frac{\sigma_2}{x_2} \times 100 \Rightarrow \bar{x}_2 = \frac{\sigma_2 \times 100}{CV_2}$$

$$\Rightarrow \bar{x}_2 = \frac{15 \times 100}{75} = 20.$$

Hence, $\bar{x}_1 = 30$ and $\bar{x}_2 = 20$.

EXAMPLE 3 *The mean and variance of the heights and weights of the students of a class are given below.*

	Height	*Weight*
Mean	160 cm	50.4 kg
Variance	116.64 cm^2	17.64 kg^2

Show that the weights are more variable than heights.

SOLUTION *For height, we have:*
Variance, $\sigma^2 = 116.64\ cm^2 \Rightarrow$ SD, $\sigma = \sqrt{116.64}$ cm = 10.8 cm.
Also, mean height = 160 cm.

$$\therefore \quad CV_1 = \left(\frac{SD}{mean} \times 100\right)\% = \left(\frac{10.8}{160} \times 100\right)\% = 4.75\%$$

For weight, we have:
Variance, $\sigma^2 = 17.64\ kg^2 \Rightarrow$ SD, $\sigma = \sqrt{17.64}$ kg = 4.2 kg.
And, mean weight = 50.4 kg.

$$\therefore \quad CV_2 = \left(\frac{SD}{mean} \times 100\right)\% = \left(\frac{4.2}{50.4} \times 100\right)\% = \frac{25}{3}\% = 8\frac{1}{3}\%.$$

Clearly, (CV in weights) > (CV in heights).
Hence, weights are more variable than heights.

EXERCISE 30D

1. The following results show the number of workers and the wages paid to them in two factories F_1 and F_2.

Factory	A	B
Number of workers	3600	3200
Mean wages	Rs 5300	Rs 5300
Variance of distribution of wages	100	81

Which factory has more variation in wages?

2. Coefficient of variation of two distributions are 60% and 80% respectively, and their standard deviations are 21 and 16 respectively. Find their arithmetic means.

3. The mean and variance of the heights and weights of the students of a class are given below:

	Heights	Weights
Mean	63.2 inches	63.2 kg
SD	11.5 inches	5.6 kg

Which shows more variability, heights or weights?

4. The following results show the number of workers and the wages paid to them in two factories A and B of the same industry.

Firms	A	B
Number of workers	560	650
Mean monthly wages	Rs 5460	Rs 5460
Variance of distribution of wages	100	121

(i) Which firm pays larger amount as monthly wages?

(ii) Which firm shows greater variability in individual wages?

5. The sum and the sum of squares of length x (in cm) and weight y (in g) of 50 plant products are given below:

$$\sum_{i=1}^{50} x_i = 212,\ \sum_{i=1}^{50} x_i^2 = 902.8,\ \sum_{i=1}^{50} y_i = 261 \quad \text{and} \quad \sum_{i=1}^{50} y_i^2 = 1457.6.$$

Which is more variable, the length or weight?

ANSWERS (EXERCISE 30D)

1. A **2.** 35, 20 **3.** Heights **4.** (i) A (ii) B
5. Weight

HINTS TO SOME SELECTED QUESTIONS

5. $$\text{var}(x) = \frac{1}{50}\cdot\sum_{i=1}^{50} x_i^2 - \left(\frac{1}{50}\cdot\sum_{i=1}^{50} x_i\right)^2 = \left(\frac{902.8}{50}\right) - \left(\frac{212}{50}\right)^2 = \frac{451.4}{25} - \left(\frac{106}{25}\right)^2$$
$$= \left(\frac{451.4}{25} - \frac{11236}{625}\right) = \left(\frac{11285 - 11236}{625}\right)$$
$$= \frac{49}{625} = 0.078.$$

$$\text{var}(y) = \frac{1}{50}\cdot\sum_{i=1}^{50} y_i^2 - \left(\frac{1}{50}\cdot\sum_{i=1}^{50} y_i\right)^2 = \left\{\frac{1457.6}{50} - \left(\frac{261}{50}\right)^2\right\}$$
$$= \{29.152 - (5.22)^2\} = 29.152 - 27.2484$$
$$= 1.9036.$$

$\therefore$ var (x) < var (y).

❑

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Probability

INTRODUCTION Historically, the theory of probability began to develop with the study of games of chance such as roulette and cards. Apart from games, uncertainty also prevails in other walks of life such as business, economy, and even in day-to-day activities.

Probability is a concept which numerically measures the degree of uncertainty and, therefore, of certainty of the occurrence of an event.

Before defining some terms related to probability, we shall give certain concepts used therein.

EXPERIMENT *An operation which can produce some well-defined outcomes is known as an experiment.*

In science or engineering if we perform an experiment and repeat it under identical conditions, we get almost the same result every time. Such experiments are called *deterministic experiments*.

But, there are experiments, which when repeated under identical conditions, do not produce the same outcome every time. For example, if we toss a fair coin, we may get a head or a tail. Now, if we make further trials, i.e., toss the coin again and again, the outcome of each trial depends on chance, and it is not the same each time. Sometimes the head appears and sometimes, the tail. Such experiments are called *random experiments*.

RANDOM EXPERIMENT *If in each trial of an experiment, conducted under identical conditions, the outcome is not unique, but may be any of the several possible outcomes then such an experiment is known as a random experiment.*

In a random experiment, the outcome of each trial depends on chance, which is beyond our control, and as such it cannot be predicted with certainty.

EXAMPLES Tossing a fair coin, rolling an unbiased die, drawing a card from a well-shuffled pack of cards are all examples of random experiments.

NOTE 1 A *die* is a solid cube, the six faces of which are marked with 1, 2, 3, 4, 5 and 6 dots respectively. In throwing a die, the outcome is the number of dots on the uppermost face.
The plural of die is *dice*.

NOTE 2 A pack of cards consists of 52 cards in four suits, called *spades*, *clubs*, *hearts* and *diamonds*. Out of these, spades and clubs are black-faced cards, while hearts and diamonds are red-faced cards. The aces, kings, queens and jacks are known as *face cards.*

SAMPLE SPACE *The set of all possible outcomes in a random experiment is called a sample space and it is generally denoted by S.*

Each element of a sample space is called a *sample point.*

EXAMPLES OF SAMPLE SPACES

EXAMPLE 1 *List the sample space in tossing a fair coin.*

SOLUTION In tossing a fair coin, there are two possible outcomes, namely, head (H) and tail (T).

Hence, the sample space in this experiment is given by

$$S = \{H, T\}.$$

EXAMPLE 2 *List the sample space in throwing a die.*

SOLUTION When we throw a die, it can result in any of the six numbers, namely, 1, 2, 3, 4, 5, 6.

$\therefore$ the sample space is $S = \{1, 2, 3, 4, 5, 6\}$.

EXAMPLE 3 *Two coins are tossed together. List the sample space.*

SOLUTION When two coins are tossed together, the sample space is

$$S = \{HT, TH, HH, TT\}.$$

Here, HT shows a head on the first coin and a tail on the second. Similarly, TH means a tail on the first and a head on the second; HH means a head on each, and TT means a tail on each.

EXAMPLE 4 *Three coins are tossed simultaneously. List the sample space for the event.* **[CBSE 1999]**

SOLUTION When three coins are tossed simultaneously, the sample space is given by

$$S = \{HHH, HHT, HTH, THH, HTT, TTH, THT, TTT\}.$$

EXAMPLE 5 *List the sample space in a simultaneous toss of a die and a coin.*

SOLUTION Clearly, the sample space is given by

$$S = \{(1, H), (2, H), (3, H), (4, H), (5, H), (6, H), (1, T), (2, T), (3, T), (4, T), (5, T), (6, T)\}.$$

EXAMPLE 6 *Two well-balanced dice are rolled and the numbers that turn up are observed. Determine the sample space.* **[CBSE 1999]**

SOLUTION When two dice are rolled, there are $(6 \times 6) = 36$ outcomes. The set of all these outcomes is the sample space, given by

$$\begin{aligned}S = \{&(1, 1), (2, 1), (3, 1), (4, 1), (5, 1), (6, 1),\\ &(1, 2), (2, 2), (3, 2), (4, 2), (5, 2), (6, 2),\\ &(1, 3), (2, 3), (3, 3), (4, 3), (5, 3), (6, 3),\\ &(1, 4), (2, 4), (3, 4), (4, 4), (5, 4), (6, 4),\\ &(1, 5), (2, 5), (3, 5), (4, 5), (5, 5), (6, 5),\\ &(1, 6), (2, 6), (3, 6), (4, 6), (5, 6), (6, 6)\}.\end{aligned}$$

EXAMPLE 7 *From a group of* 3 *boys and* 2 *girls, we select two children. What would be the sample space for this experiment?*

SOLUTION Let us name the boys as B_1, B_2 and B_3, and the girls as G_1 and G_2. Then the sample space is given by

$$S = \{B_1B_2, B_1B_3, B_1G_1, B_1G_2, B_2B_3, B_2G_1, B_2G_2, B_3G_1, B_3G_2, G_1G_2\}.$$

EXAMPLE 8 *A coin is tossed twice. If the second throw results in a tail then a die is thrown. Describe the sample space.*

SOLUTION Clearly, the sample space is given by

$$S = \{HH, TH, HT1, HT2, HT3, HT4, HT5, HT6, TT1, TT2, TT3, TT4, TT5, TT6\}.$$

EXAMPLE 9 *A coin is tossed. If it shows a tail, we draw a ball from a box which contains* 2 *red and* 3 *black balls; if it shows a head, we throw a die. Find the sample space for this experiment.*

SOLUTION Let the red balls be named as R_1 and R_2, and the black balls be named as B_1, B_2 and B_3.
Then, the sample space is given by

$$S = \{TR_1, TR_2, TB_1, TB_2, TB_3, H1, H2, H3, H4, H5, H6\}.$$

EXAMPLE 10 *An experiment consists of rolling a die and then tossing a coin once if the number on the die is even. If the number on the die is odd, the coin is tossed twice. Write the sample space for this experiment.*

SOLUTION Clearly, the sample space is given by

$$S = \{2H, 2T, 4H, 4T, 6H, 6T, 1HH, 1HT, 1TH, 1TT, 3HH, 3HT, 3TH, 3TT, 5HH, 5HT, 5TH, 5TT\}.$$

EXAMPLE 11 *A bag contains* 4 *identical red balls and* 3 *identical black balls. The experiment consists of drawing one ball, then putting it into the bag and again drawing a ball. What are the possible outcomes of this experiment?*

SOLUTION Let us denote a red ball by R and a black ball by B. Then, clearly the sample space is given by

$$S = \{RR, RB, BR, BB\}.$$

Now, we define some terms related to probability.

EVENT *Every subset of a sample space is called an event.*

EXAMPLE *In a single throw of a die, we have*
sample space $S = \{1, 2, 3, 4, 5, 6\}$.

The event of getting a prime number is given by

$$E = \{2, 3, 5\}.$$

Clearly, $E \subseteq S$.

IMPOSSIBLE EVENT Let S be a sample space.

Since $\phi \subset S$, ϕ is an event, called an *impossible event.*

SURE EVENT Let S be a sample space.

Since $S \subseteq S$, S is an event, called a *sure event.*

EXAMPLE In a throw of a die, we have

sample space $S = \{1, 2, 3, 4, 5, 6\}$.

Let E_1 = event of getting a number less than 1.

And, E_2 = event of getting a number less than 7.

Clearly, no outcome can be less than 1.

$\therefore$ E_1 is an *impossible event.*

Also, each outcome is a number less than 7.

$\therefore$ E_2 is a *sure event.*

SIMPLE EVENT *An event containing only a single element of the sample space is called a simple or an elementary event.*

COMPOUND EVENT *An event which is not simple is called a compound or composite or mixed event.*

EXAMPLE In a simultaneous toss of two coins, we have

sample space $S = \{HH, HT, TH, TT\}$.

Then E_1 = event of getting a tail on both the coins = $\{TT\}$,

is a simple event.

And, E_2 = event of getting at least 1 tail = $\{HT, TH, TT\}$,

is a compound event.

MUTUALLY EXCLUSIVE EVENTS *Two events E_1 and E_2 are said to be mutually exclusive if $E_1 \cap E_2 = \phi$.*

However, if $E_1 \cap E_2 \neq \phi$ then E_1 and E_2 are called *compatible events.*

EXAMPLES (i) In throwing a die, we have $S = \{1, 2, 3, 4, 5, 6\}$.

Let E_1 = event of getting a number less than 3.

And, E_2 = event of getting a number more than 4.

Then $E_1 = \{1, 2\}$ and $E_2 = \{5, 6\}$.

Clearly, $E_1 \cap E_2 = \phi$.

Hence, E_1 and E_2 are mutually exclusive.

(ii) In a simultaneous toss of two coins, we have

$$S = \{HH, HT, TH, TT\}.$$

Let E_1 = event of getting a head on the first coin = $\{HH, HT\}$.
And, E_2 = event of getting a tail on the second coin = $\{HT, TT\}$.
Clearly, $E_1 \cap E_2 \neq \phi$.
Hence, E_1 and E_2 are compatible events.

EXAMPLE 1 *Two dice are rolled. Let A, B, C be the events of getting a sum of 2, a sum of 3 and a sum of 4 respectively. Then, show that*
(i) A is a simple event
(ii) B and C are compound events
(iii) A and B are mutually exclusive

SOLUTION Clearly, we have

$$A = \{(1, 1)\},\ B = \{(1, 2), (2, 1)\}, \text{ and } C = \{(1, 3), (3, 1), (2, 2)\}.$$

(i) Since A consists of a single sample point, it is a simple event.
(ii) Since both B and C contain more than one sample point, each one of them is a compound event.
(iii) Since $A \cap B = \phi$, A and B are mutually exclusive.

EXAMPLE 2 *From a group of 2 boys and 3 girls, two children are selected at random. Describe the events:*
(i) A = event that both the selected children are girls
(ii) B = event that the selected group consists of one boy and one girl
(iii) C = event that at least one boy is selected
Which pairs of events are mutually exclusive?

SOLUTION Let us name the boys as B_1 and B_2, and the girls as G_1, G_2 and G_3. Then

$$S = \{B_1B_2, B_1G_1, B_1G_2, B_1G_3, B_2G_1, B_2G_2, B_2G_3, G_1G_2, G_1G_3, G_2G_3\}$$

We have
(i) $A = \{G_1G_2, G_1G_3, G_2G_3\}$
(ii) $B = \{B_1G_1, B_1G_2, B_1G_3, B_2G_1, B_2G_2, B_2G_3\}$
(iii) $C = \{B_1G_1, B_1G_2, B_1G_3, B_2G_1, B_2G_2, B_2G_3, B_1B_2\}$

Clearly, $A \cap B = \phi$ and $A \cap C = \phi$.
Hence, (A, B) and (A, C) are mutually exclusive events.

MUTUALLY EXCLUSIVE AND EXHAUSTIVE SYSTEM OF EVENTS Let $E_1, E_2, \ldots, E_n$ be subsets of a sample space S. Then, we say that

the events $E_1, E_2, \ldots, E_n$ form a mutually exclusive and exhaustive system if
(i) $E_i \cap E_j \neq \phi$ for $i \neq j$, and (ii) $E_1 \cup E_2 \cup \ldots \cup E_n = S$.

EXAMPLE Suppose we draw a card from a well-shuffled pack of 52 cards.
Let E_1, E_2, E_3 and E_4 be the events of *drawing a spade, drawing a club, drawing a heart* and *drawing a diamond* respectively.

As the card drawn is necessarily one of the four types, one of these events is sure to occur.

When one of these events occurs then none of the others occur.

Thus, E_1, E_2, E_3, E_4 form a mutually exclusive and exhaustive system of events.

EXAMPLE *Two dice are rolled. A is the event that the sum of the numbers shown on the two dice is* 5*, and B is the event that at least one of the dice shows up a* 3*. Are the two events (i) mutually exclusive, (ii) exhaustive? Give arguments in support of your answer.* **[CBSE 2001C]**

SOLUTION When two dice are rolled, we have $n(S) = (6 \times 6) = 36$.

Now, $A = \{(1, 4), (2, 3), (4, 1), (3, 2)\}$, and

$$B = \{(3, 1), (3, 2), (3, 3), (3, 4), (3, 5), (3, 6)\ (1, 3), (2, 3), (4, 3), (5, 3), (6, 3)\}.$$

(i) $A \cap B = \{(2, 3), (3, 2)\} \neq \phi$.
Hence, A and B are not mutually exclusive.

(ii) Also, $A \cup B \neq S$.
$\therefore$ A and B are not exhaustive events.

INDEPENDENT EVENTS *Two events E_1 and E_2 are said to be independent if the occurrence of one does not depend upon the occurrence of the other.*

If the two events are not independent, they are known as *dependent events.*

EXAMPLE Suppose we toss two unbiased coins.
Let E_1 = event of getting a head on the first coin.
And, E_2 = event of getting a head on the second coin.
Clearly, the occurrence of a head on the second coin does not depend upon the occurrence of a head on the first coin.
$\therefore$ E_1 and E_2 are independent events.

OCCURRENCE OF AN EVENT In a random experiment, let S be the sample space and let $E \subseteq S$. Then, E is an event.

Let w be an outcome of a trial.

If $w \in E$, we say that the event E *has occurred.*

If $w \notin E$, we say that the event E *has not occurred.*

EXAMPLE In a throw of a die, we have $S = \{1, 2, 3, 4, 5, 6\}$.
Let E be the event of getting an even number.
Then $E = \{2, 4, 6\}$.
In a trial, let the outcome be 4.
Since $4 \in E$, we say that the event E has occurred.
In another trial, let the outcome be 5.
Since $5 \notin E$, in this case, the event E has not occurred.

EQUALLY LIKELY EVENTS *A given number of events are said to be equally likely if none of them is expected to occur in preference to the others.*

EXAMPLE If we roll an unbiased die, each outcome is equally likely to happen.

If, however, a die is so formed that a particular face occurs most often then the die is biased. In this case, the outcomes are not equally likely to happen.

COMPLEMENTARY EVENTS In a random experiment, let S be the sample space and let E be an event. Then $E \subseteq S$.

Now, $E \subseteq S \Rightarrow E^c \subseteq S$.

Thus, E^c is also an event, called the *complement of E*.

We denote it by E^c or E' or $\overline{E}$ and call it *not E*.

Clearly, *$\overline{E}$ occurs when E does not occur.*

And, *E occurs when $\overline{E}$ does not occur.*

Thus, in a trial, one of two events, E or $\overline{E}$, is sure to occur.

ALGEBRA OF EVENTS *In a random experiment, let S be the sample space.*

Let $E \subseteq S$ and $F \subseteq S$.

Then, E as well as F is an event.

We say that

(i) $(E \cap F)$ is an event that occurs only when each one of E and F occurs

(ii) $(E \cup F)$ is an event that occurs only when at least one of E and F occurs

(iii) $\overline{E}$ is an event that occurs only when E does not occur

PROBABILITY OF AN EVENT

INTRODUCTION Suppose a bag contains 90 red and 10 white balls, which are similar in shape and size.

If the balls are mixed thoroughly and then one ball is drawn at random, it will be either red or white.

Clearly, *the ball drawn is more likely to be red than white.*

We express it by saying that the event of drawing a red ball is more probable than the event of drawing a white ball.

To every event associated with a random experiment, we try to attach a numerical value, called its *probability*, in such a manner that for any two events, the event which is more likely to happen has a higher probability.

PROBABILITY OF AN EVENT *In a random experiment, let S be the sample space and let $E \subseteq S$. Then, E is an event.*

The probability of occurrence of E is defined as

$$P(E) = \frac{\text{number of outcomes favourable to occurrence of } E}{\text{number of all possible outcomes}}$$

$$= \frac{\text{number of distinct elements in E}}{\text{number of distinct elements in } S}$$

$$= \frac{n(E)}{n(S)}.$$

$$\therefore \quad P(E) = \frac{n(E)}{n(S)}.$$

SOLVED EXAMPLES

EXAMPLE 1 *A coin is tossed once. Find the probability of getting a head.*

SOLUTION When a coin is tossed once, the sample space is given by $S = \{H, T\}$.

Let E be the event of getting a head.

Then, $E = \{H\}$.

$\therefore \; n(E) = 1$ and $n(S) = 2$

$$\Rightarrow P \text{ (getting a head)} = P(E) = \frac{n(E)}{n(S)} = \frac{1}{2}.$$

EXAMPLE 2 *Two coins are tossed once. Find the probability of*

(i) getting 2 heads *(ii) getting at least 1 head*

(iii) getting no head *(iv) getting 1 head and 1 tail*

SOLUTION When two coins are tossed once, the sample space is given by $S = \{HH, HT, TH, TT\}$ and, therefore, $n(S) = 4$.

(i) Let E_1 = event of getting 2 heads. Then,

$E_1 = \{HH\}$ and, therefore, $n(E_1) = 1$.

$$\therefore \quad P(\text{getting 2 heads}) = P(E_1) = \frac{n(E_1)}{n(S)} = \frac{1}{4}.$$

(ii) Let E_2 = event of getting at least 1 head. Then,

$E_2 = \{HT, TH, HH\}$ and, therefore, $n(E_2) = 3$.

$$\therefore \quad P(\text{getting at least 1 head}) = P(E_2) = \frac{n(E_2)}{n(S)} = \frac{3}{4}.$$

(iii) Let E_3 = event of getting no head. Then,

$E_3 = \{TT\}$ and, therefore, $n(E_3) = 1$.

$$\therefore \quad P(\text{getting no head}) = P(E_3) = \frac{n(E_3)}{n(S)} = \frac{1}{4}.$$

(iv) Let E_4 = event of getting 1 head and 1 tail. Then,

$E_4 = \{HT, TH\}$ and, therefore, $n(E_4) = 2$.

$\therefore$ $P(\text{getting 1 head and 1 tail}) = P(E_4) = \dfrac{n(E_4)}{n(S)} = \dfrac{2}{4} = \dfrac{1}{2}\cdot$

EXAMPLE 3 *Three unbiased coins are tossed once. What is the probability of getting*
(i) all heads? *(ii) two heads?* *(iii) one head?*
(iv) at least 1 *head?* *(v) at least* 2 *heads?*

SOLUTION In tossing three coins, the sample space is given by

$S = \{HHH, HHT, HTH, THH, HTT, THT, TTH, TTT\}$.

And, therefore, $n(S) = 8$.

(i) Let E_1 = event of getting all heads. Then,

$E_1 = \{HHH\}$ and, therefore, $n(E_1) = 1$.

$\therefore$ $P(\text{getting all heads}) = P(E_1) = \dfrac{n(E_1)}{n(S)} = \dfrac{1}{8}\cdot$

(ii) Let E_2 = event of getting 2 heads. Then,

$E_2 = \{HHT, HTH, THH\}$ and, therefore, $n(E_2) = 3$.

$\therefore$ $P(\text{getting 2 heads}) = P(E_2) = \dfrac{n(E_2)}{n(S)} = \dfrac{3}{8}\cdot$

(iii) Let E_3 = event of getting 1 head. Then,

$E_3 = \{HTT, THT, TTH\}$ and, therefore, $n(E_3) = 3$.

$\therefore$ $P(\text{getting 1 head}) = P(E_3) = \dfrac{n(E_3)}{n(S)} = \dfrac{3}{8}\cdot$

(iv) Let E_4 = event of getting at least 1 head. Then,

$E_4 = \{HTT, THT, TTH, HHT, HTH, THH, HHH\}$.

And, therefore, $n(E_4) = 7$.

$\therefore$ $P(\text{getting at least 1 head}) = P(E_4) = \dfrac{n(E_4)}{n(S)} = \dfrac{7}{8}\cdot$

(v) Let E_5 = event of getting at least 2 heads. Then,

$E_5 = \{HHT, HTH, THH, HHH\}$ and, therefore, $n(E_5) = 4$.

$\therefore$ $P(\text{getting at least 2 heads}) = P(E_5) = \dfrac{n(E_5)}{n(S)} = \dfrac{4}{8} = \dfrac{1}{2}\cdot$

EXAMPLE 4 *A die is tossed once. What is the probability of getting*
(i) the number 4*?* *(ii) an even number?*
(iii) a number less than 5*?* *(iv) a number greater than* 4*?*
(v) the number 8*?* *(vi) a number less than* 8*?*

SOLUTION In tossing a die once, the sample space is given by

$S = \{1, 2, 3, 4, 5, 6\}$ and, therefore, $n(S) = 6$.

(i) Let E_1 = event of getting the number 4.
Then, $E_1 = \{4\}$ and, therefore, $n(E_1) = 1$.

$\therefore$ $P(\text{getting the number 4}) = P(E_1) = \dfrac{n(E_1)}{n(S)} = \dfrac{1}{6}\cdot$

(ii) Let E_2 = event of getting an even number. Then,
$E_2 = \{2, 4, 6\}$ and, therefore, $n(E_2) = 3$.

$\therefore$ $P(\text{getting an even number}) = P(E_2) = \dfrac{n(E_2)}{n(S)} = \dfrac{3}{6} = \dfrac{1}{2}\cdot$

(iii) Let E_3 = event of getting a number less than 5. Then,
$E_3 = \{1, 2, 3, 4\}$ and, therefore, $n(E_3) = 4$.

$\therefore$ $P(\text{getting a number less than 5}) = P(E_3) = \dfrac{n(E_3)}{n(S)} = \dfrac{4}{6} = \dfrac{2}{3}\cdot$

(iv) Let E_4 = event of getting a number greater than 4. Then,
$E_4 = \{5, 6\}$ and, therefore, $n(E_4) = 2$.

$\therefore$ $P(\text{getting a number greater than 4})$

$$= P(E_4) = \frac{n(E_4)}{n(S)} = \frac{2}{6} = \frac{1}{3}\cdot$$

(v) Let E_5 = event of getting the number 8.
Since no face of the die is marked with the number 8,
we have $E_5 = \phi$ and, therefore, $n(E_5) = 0$.

$\therefore$ $P(\text{getting the number 8}) = P(E_5) = \dfrac{n(E_5)}{n(S)} = \dfrac{0}{8} = 0.$

(vi) Let E_6 = event of getting a number less than 8. Then,
$E_6 = \{1, 2, 3, 4, 5, 6\}$ and, therefore, $n(E_6) = 6$.

$\therefore$ $P(\text{getting a number less than 8}) = P(E_6) = \dfrac{n(E_6)}{n(S)} = \dfrac{6}{6} = 1.$

EXAMPLE 5 *In a single throw of two dice, find the probability of obtaining 'a total of 8'.*

SOLUTION We know that in a single throw of two dice, the total number of possible outcomes is $(6 \times 6) = 36$.

Let S be the sample space. Then, $n(S) = 36$.

Let E = event of getting a total of 8. Then,

$E = \{(2, 6), (3, 5), (4, 4), (5, 3), (6, 2)\}$ and, therefore, $n(E) = 5$.

$\therefore$ $P(\text{getting a total of 8}) = P(E) = \dfrac{n(E)}{n(S)} = \dfrac{5}{36}\cdot$

EXAMPLE 6 *Two dice are thrown simultaneously. Find the probability of getting*
(i) a doublet
(ii) an even number as the sum
(iii) a prime number as the sum
(iv) a multiple of 3 as the sum

(v) a total of at least 10
(vi) a doublet of even numbers
(vii) a multiple of 2 *on one die and a multiple of* 3 *on the other die*

SOLUTION We know that in a single throw of two dice, the total number of possible outcomes is $(6 \times 6) = 36$.

Let S be the sample space. Then, $n(S) = 36$.

(i) Let E_1 = event of getting a doublet. Then,

$E_1 = \{(1, 1), (2, 2), (3, 3), (4, 4), (5, 5), (6, 6)\}$.

$\therefore\ n(E_1) = 6$.

$$\therefore\ P(E_1) = \frac{n(E_1)}{n(S)} = \frac{6}{36} = \frac{1}{6}.$$

(ii) Let E_2 = event of getting an even number as the sum. Then,

$E_2 = \{(1, 1), (1, 3), (1, 5), (2, 2), (2, 4), (2, 6), (3, 3), (3, 1), (3, 5), (4, 2), (4, 4), (4, 6), (5, 1), (5, 3), (5, 5), (6, 2), (6, 4), (6, 6)\}$.

$\therefore\ n(E_2) = 18$.

$$\therefore\ P(E_2) = \frac{n(E_2)}{n(S)} = \frac{18}{36} = \frac{1}{2}.$$

(iii) Let E_3 = event of getting a prime number as the sum. Then,

$E_3 = \{(1, 1), (1, 2), (1, 4), (1, 6), (2, 1), (2, 3), (2, 5), (3, 2), (3, 4), (4, 1), (4, 3), (5, 2), (5, 6), (6, 1), (6, 5)\}$.

$\therefore\ n(E_3) = 15$.

$$\therefore\ P(E_3) = \frac{n(E_3)}{n(S)} = \frac{15}{36} = \frac{5}{12}.$$

(iv) Let E_4 = event of getting a multiple of 3 as the sum. Then,

$E_4 = \{(1, 2), (1, 5), (2, 1), (2, 4), (3, 3), (3, 6), (4, 2), (4, 5), (5, 1), (5, 4), (6, 3), (6, 6)\}$.

$\therefore\ n(E_4) = 12$.

$$\therefore\ P(E_4) = \frac{n(E_4)}{n(S)} = \frac{12}{36} = \frac{1}{3}.$$

(v) Let E_5 = event of getting a total of at least 10. Then,

E_5 = event of getting a total of 10 or 11 or 12

$= \{(4, 6), (5, 5), (5, 6), (6, 4), (6, 5), (6, 6)\}$.

$\therefore\ n(E_5) = 6$.

$$\therefore\ P(E_5) = \frac{n(E_5)}{n(S)} = \frac{6}{36} = \frac{1}{6}.$$

(vi) Let E_6 = event of getting a doublet of even numbers. Then,

$E_6 = \{(2, 2), (4, 4), (6, 6)\}$.

$\therefore \quad n(E_6) = 3.$

$$\therefore \quad P(E_6) = \frac{n(E_6)}{n(S)} = \frac{3}{36} = \frac{1}{12}\cdot$$

(vii) Let E_7 = event of getting a multiple of 2 on one die and a multiple of 3 on the other die. Then,

$E_7 = \{(2, 3), (2, 6), (4, 3), (4, 6), (6, 3), (6, 6), (3, 2), (3, 4), (3, 6), (6, 2), (6, 4)\}.$

$\therefore \quad n(E_7) = 11.$

$$\therefore \quad P(E_7) = \frac{n(E_7)}{n(S)} = \frac{11}{36}\cdot$$

EXAMPLE 7 20 *cards are numbered from* 1 *to* 20*. One card is then drawn at random. What is the probability that the number on the card drawn is*

(i) a prime number? *(ii) an odd number?*

(iii) a multiple of 5*?* *(iv) not divisible by* 3*?*

SOLUTION Clearly, the sample space is given by

$S = \{1, 2, 3, 4, 5, \ldots, 19, 20\}$ and, therefore, $n(S) = 20$.

(i) Let E_1 = event of getting a prime number. Then,

$E_1 = \{2, 3, 5, 7, 11, 13, 17, 19\}$ and, therefore, $n(E_1) = 8$.

$$\therefore \quad P(\text{getting a prime number}) = P(E_1) = \frac{n(E_1)}{n(S)} = \frac{8}{20} = \frac{2}{5}\cdot$$

(ii) Let E_2 = event of getting an odd number. Then,

$E_2 = \{1, 3, 5, 7, 9, 11, 13, 15, 17, 19\}$ and, therefore, $n(E_2) = 10$.

$$\therefore \quad P(\text{getting an odd number}) = P(E_2) = \frac{n(E_2)}{n(S)} = \frac{10}{20} = \frac{1}{2}\cdot$$

(iii) Let E_3 = event of getting a multiple of 5. Then,

$E_3 = \{5, 10, 15, 20\}$ and, therefore, $n(E_3) = 4$.

$$\therefore \quad P(\text{getting a multiple of 5}) = P(E_3) = \frac{n(E_3)}{n(S)} = \frac{4}{20} = \frac{1}{5}\cdot$$

(iv) Let E_4 = event of getting a number which is not divisible by 3. Then, $E_4 = \{1, 2, 4, 5, 7, 8, 10, 11, 13, 14, 16, 17, 19, 20\}$ and so, $n(E_4) = 14$.

$\therefore \quad P(\text{getting a number which is not divisible by 3})$

$$= P(E_4) = \frac{n(E_4)}{n(S)} = \frac{14}{20} = \frac{7}{10}\cdot$$

EXAMPLE 8 *From a well-shuffled deck of* 52 *cards, a card is drawn at random. Find the probability of getting*

(i) an ace *(ii) a heart* *(iii) an eight of hearts*

(iv) a club *(v) a red card* *(vi) a face card*

(vii) a diamond *(viii) a jack* *(ix) a black card*

SOLUTION Let S denote the sample space. Then, $n(S) = 52$.

(i) Let E_1 = event of drawing an ace.

Since, the number of all aces is 4, so $n(E_1) = 4$.

$\therefore$ $P(\text{getting an ace}) = P(E_1) = \dfrac{n(E_1)}{n(S)} = \dfrac{4}{52} = \dfrac{1}{13}\cdot$

(ii) Let E_2 = event of getting a heart. Then, $n(E_2) = 13$.

$\therefore$ $P(\text{getting a heart}) = P(E_2) = \dfrac{n(E_2)}{n(S)} = \dfrac{13}{52} = \dfrac{1}{4}\cdot$

(iii) Let E_3 = event of getting an eight of hearts. Then, $n(E_3) = 1$.

$\therefore$ $P(\text{getting an eight of hearts}) = P(E_3) = \dfrac{n(E_3)}{n(S)} = \dfrac{1}{52}\cdot$

(iv) Let E_4 = event of getting a club. Then, $n(E_4) = 13$.

$\therefore$ $P(\text{getting a club}) = P(E_4) = \dfrac{n(E_4)}{n(S)} = \dfrac{13}{52} = \dfrac{1}{4}\cdot$

(v) Let E_5 = event of getting a red card. Then, $n(E_5) = 26$.

$\therefore$ $P(\text{getting a red card}) = P(E_5) = \dfrac{n(E_5)}{n(S)} = \dfrac{26}{52} = \dfrac{1}{2}\cdot$

(vi) Let E_6 = event of getting a face card. Then, $n(E_6) = 16$.

$\therefore$ $P(\text{getting a face card}) = P(E_6) = \dfrac{n(E_6)}{n(S)} = \dfrac{16}{52} = \dfrac{4}{13}\cdot$

(vii) Let E_7 = event of getting a diamond. Then, $n(E_7) = 13$.

$\therefore$ $P(\text{getting a diamond}) = P(E_7) = \dfrac{n(E_7)}{n(S)} = \dfrac{13}{52} = \dfrac{1}{4}\cdot$

(viii) Let E_8 = event of getting a jack. Then, $n(E_8) = 4$.

$\therefore$ $P(\text{getting a jack}) = P(E_8) = \dfrac{n(E_8)}{n(S)} = \dfrac{4}{52} = \dfrac{1}{13}\cdot$

(ix) Let E_9 = event of getting a black card. Then, $n(E_9) = 26$.

$\therefore$ $P(\text{getting a black card}) = P(E_9) = \dfrac{n(E_9)}{n(S)} = \dfrac{26}{52} = \dfrac{1}{2}\cdot$

EXAMPLE 9 *From a well-shuffled deck of* 52 *cards, a card is drawn at random. Find the probability that the card drawn is*

(i) red and a king *(ii) either red or a king*

SOLUTION Let S denote the sample space. Then, $n(S) = 52$.

(i) Let E_1 = event of drawing a red card which is a king.

We know that the number of red kings is 2. So, $n(E_1) = 2$.

$\therefore \quad P(\text{getting a red king}) = P(E_1) = \dfrac{n(E_1)}{n(S)} = \dfrac{2}{52} = \dfrac{1}{26}\cdot$

(ii) Let E_2 = event of drawing a card which is either red or a king. There are 26 red cards (including 2 red kings) and there are 2 more kings.

$\therefore \quad n(E_2) = (26 + 2) = 28$

$\therefore \quad P(\text{getting a red card or a king}) = P(E_2) = \dfrac{n(E_2)}{n(S)} = \dfrac{28}{52} = \dfrac{7}{13}\cdot$

EXAMPLE 10 *A bag contains 9 red and 12 white balls. One ball is drawn at random. Find the probability that the ball drawn is red.*

SOLUTION Total number of balls = (9 + 12) = 21.

Let S be the sample space. Then,

$n(S)$ = number of ways of selecting 1 ball out of 21 = 21.

Let E be the event of drawing a red ball. Then,

$n(E)$ = number of ways of selecting 1 red ball out of 9 = 9.

$\therefore \quad P(\text{getting a red ball}) = P(E) = \dfrac{n(E)}{n(S)} = \dfrac{9}{21} = \dfrac{3}{7}\cdot$

ODDS OF AN EVENT *Let there be m outcomes favourable to an event E and n outcomes unfavourable to E. Then,*

(i) odds in favour of $E = \dfrac{m}{n}$ *or* $(m : n)$

(ii) odds against $E = \dfrac{n}{m}$ *or* $(n : m)$

(iii) $P(E) = \dfrac{m}{(m+n)}$

EXAMPLE 11 *The odds in favour of occurrence of an event are* 5 : 12. *Find the probability of the occurrence of this event.*

SOLUTION Number of favourable outcomes = 5.

Number of unfavourable outcomes = 12.

Total number of outcomes = (5 + 12) = 17.

Let E be the event. Then,

$$P(E) = \frac{\text{number of favourable outcomes}}{\textit{total number of outcomes}} = \frac{5}{17}\cdot$$

EXAMPLE 12 *If the probability of the occurrence of a certain event E is* $3/11$, *find (i) the odds in favour of its occurrence, and (ii) the odds against its occurrence.*

SOLUTION Let there be m outcomes favourable to E and n outcomes unfavourable to E. Then,

$$P(E) = \frac{m}{m+n} \Rightarrow \frac{m}{m+n} = \frac{3}{11} \qquad \left[\because P(E) = \frac{3}{11} \text{ (given)}\right]$$

$$\Rightarrow m = 3 \text{ and } n = 8.$$

(i) odds in favour of $E = \frac{m}{n} = \frac{3}{8}\cdot$

(ii) odds against $E = \frac{n}{m} = \frac{8}{3}\cdot$

PROBABILITY OF OCCURRENCE OF A COMPLEMENTARY EVENT

THEOREM 1 *Let E be an event and $\overline{E}$ be the complement of E. Then, $P(\overline{E}) = 1 - P(E)$.*

PROOF Let there be m outcomes favourable to E and n outcomes unfavourable to E. Then,

number of possible outcomes $= (m + n)$.

$$\therefore \quad P(E) = \frac{m}{(m+n)} \text{ and } P(\overline{E}) = P(\text{not } E) = \frac{n}{(m+n)}$$

$$\Rightarrow P(E) + P(\overline{E}) = \left(\frac{m}{m+n} + \frac{n}{m+n}\right) = \frac{(m+n)}{(m+n)} = 1$$

$$\Rightarrow P(\overline{E}) = 1 - P(E).$$

Hence, $P(\overline{E}) = 1 - P(E)$.

NOTE There is an alternative proof of this theorem, which we will soon discuss.

EXAMPLE 13 *If $3/10$ is the probability that an event will happen, what is the probability that it will not happen?*

SOLUTION Let E be the event. Then,

$$P(E) = \frac{3}{10} \Rightarrow P(\overline{E}) = \{1 - P(E)\} = \left(1 - \frac{3}{10}\right) = \frac{7}{10}\cdot$$

Hence, $P(\text{not } E) = \frac{7}{10}\cdot$

EXAMPLE 14 *Three dice are thrown together. Find the probability of getting a total of at least 6.*

SOLUTION In throwing 3 dice together, the number of all possible outcomes is $(6 \times 6 \times 6) = 216$.

Let E = event of getting a total of at least 6.

Then, $\overline{E}$ = event of getting a total of less than 6

= event of getting a total of 3 or 4 or 5.

$= \{(1, 1, 1), (1, 1, 2), (1, 2, 1), (2, 1, 1), (1, 1, 3), (1, 3, 1), (3, 1, 1), (1, 2, 2), (2, 1, 2), (2, 2, 1)\}$

Now, $n(\overline{E}) = 10 \Rightarrow P(\text{not } E) = P(\overline{E}) = \frac{n(\overline{E})}{n(S)} = \frac{10}{216} = \frac{5}{108}\cdot$

$$\Rightarrow P(E) = 1 - P(\text{not } E) = \left(1 - \frac{5}{108}\right) = \frac{103}{108}\cdot$$

Hence, the required probability is $\frac{103}{108}\cdot$

THEOREM 2 *For an event E,*

(i) odds in favour of $E = \dfrac{P(E)}{1 - P(E)}$

(ii) odds against $E = \dfrac{1 - P(E)}{P(E)}$

PROOF Let there be m outcomes favourable to E and n outcomes unfavourable to E.

Then, there are m outcomes favourable to E and n outcomes favourable to $\overline{E}$.

$$\therefore \quad P(E) = \frac{n(E)}{n(S)} = \frac{m}{(m+n)} \text{ and } P(\overline{E}) = \frac{n}{(m+n)}.$$

(i) Odds in favour of $E = \dfrac{m}{n} = \dfrac{\frac{m}{(m+n)}}{\frac{n}{(m+n)}} = \dfrac{P(E)}{P(\overline{E})} = \dfrac{P(E)}{1 - P(E)}.$

(ii) Odds against $E = \dfrac{n}{m} = \dfrac{1 - P(E)}{P(E)}.$

EXAMPLE 15 *If the odds in favour of an event be* $3/5$, *find the probability of the occurrence of the event.*

SOLUTION Let the given event be E and let $P(E) = x$. Then,

odds in favour of $E = \dfrac{P(E)}{1 - P(E)}$

$$\Leftrightarrow \quad \frac{P(E)}{1 - P(E)} = \frac{3}{5} \quad \Leftrightarrow \quad \frac{x}{(1-x)} = \frac{3}{5}$$

$$\Leftrightarrow \; 5x = 3 - 3x \;\; \Leftrightarrow \;\; 8x = 3 \;\; \Leftrightarrow \;\; x = \frac{3}{8}.$$

$\therefore$ required probability $= \dfrac{3}{8}.$

EXAMPLE 16 *Two dice are thrown. Find (i) the odds in favour of getting the sum* 5, *and (ii) the odds against getting the sum* 6.

SOLUTION Let S be the sample space. Then, $n(S) = 36$.

(i) Let E_1 be the event of getting the sum 5. Then,

$E_1 = \{(1, 4), (2, 3), (3, 2), (4, 1)\} \;\Rightarrow\; n(E_1) = 4.$

$$\therefore \quad P(E_1) = \frac{n(E_1)}{n(S)} = \frac{4}{36} = \frac{1}{9}$$

$$\Rightarrow \text{ odds in favour of } E_1 = \frac{P(E_1)}{1 - P(E_1)} = \frac{(1/9)}{(1 - 1/9)} = \frac{1}{8}.$$

(ii) Let E_2 be the event of getting the sum 6. Then,

$$E_2 = \{(1, 5), (2, 4), (3, 3), (4, 2), (5, 1)\} \Rightarrow n(E_2) = 5.$$

$$\therefore \quad P(E_2) = \frac{n(E_2)}{n(S)} = \frac{5}{36}$$

$$\Rightarrow \text{ odds against } E_2 = \frac{1 - P(E_2)}{P(E_2)} = \frac{\left(1 - \frac{5}{36}\right)}{(5/36)} = \frac{31}{5}.$$

EXAMPLE 17 *A card is drawn from a well-shuffled deck of* 52 *cards. Find (i) the odds in favour of getting a face card, and (ii) the odds against getting a spade.*

SOLUTION When a card is drawn from a well-shuffled deck of 52 cards, the number of possible outcomes is 52.

Now, let S be the sample space. Then, $n(S) = 52$.

(i) Let E_1 be the event of getting a face card.

Since there are 16 face cards, we have $n(E_1) = 16$.

$$\therefore \quad P(E_1) = \frac{n(E_1)}{n(S)} = \frac{16}{52} = \frac{4}{13}$$

$\Rightarrow$ odds in favour of getting a face card

$$= \frac{P(E_1)}{\{1 - P(E_1)\}} = \frac{(4/13)}{\left(1 - \frac{4}{13}\right)} = \frac{4}{9}.$$

(ii) Let E_2 be the event of getting a spade. Then, $n(E_2) = 13$.

$$\therefore \quad P(E_2) = \frac{n(E_2)}{n(S)} = \frac{13}{52} = \frac{1}{4}$$

$$\Rightarrow \text{ odds against getting a spade} = \frac{\{1 - P(E_2)\}}{P(E_2)}$$

$$= \frac{\left(1 - \frac{1}{4}\right)}{(1/4)} = \frac{3}{1}.$$

EXAMPLE 18 *Two cards are drawn at random from a pack of* 52 *cards. What is the probability that both the drawn cards are aces?*

SOLUTION Let S be the sample space. Then,

$$n(S) = \text{number of ways of selecting 2 cards out of 52}$$

$$= {}^{52}C_2 = \frac{(52 \times 51)}{(2 \times 1)} = 1326.$$

Let E = event that both the cards drawn are aces. Then,

$$n(E) = \text{number of ways of drawing 2 aces out of 4}$$

$$= {}^{4}C_2 = \frac{(4 \times 3)}{(2 \times 1)} = 6.$$

$$P(\text{getting 2 aces}) = P(E) = \frac{n(E)}{n(S)} = \frac{6}{1326} = \frac{1}{221}.$$

EXERCISE 31A

1. A coin is tossed once. Find the probability of getting a tail.
2. A die is thrown. Find the probability of
   (i) getting a 5 (ii) getting a 2 or a 3
   (iii) getting an odd number (iv) getting a prime number
   (v) getting a multiple of 3 (vi) getting a number between 3 and 6
3. In a single throw of two dice, find the probability of
   (i) getting a sum less than 6
   (ii) getting a doublet of odd numbers
   (iii) getting the sum as a prime number
4. In a single throw of two dice, find
   (i) P (an odd number on the first die and a 6 on the second)
   (ii) P (a number greater than 3 on each die)
   (iii) P (a total of 10)
   (iv) P (a total greater than 8)
   (v) P (a total of 9 or 11)
5. A bag contains 4 white and 5 black balls. A ball is drawn at random from the bag. Find the probability that the ball drawn is white.
6. An urn contains 9 red, 7 white and 4 black balls. A ball is drawn at random. Find the probability that the ball drawn is
   (i) red (ii) white (iii) red or white
   (iv) white or black (v) not white
7. In a lottery, there are 10 prizes and 25 blanks. Find the probability of getting a prize.
8. If there are two children in a family, find the probability that there is at least one boy in the family.
9. Three unbiased coins are tossed once. Find the probability of getting
   (i) exactly 2 tails (ii) exactly one tail (iii) at most 2 tails
   (iv) at least 2 tails (v) at most 2 tails or at least 2 heads **[CBSE 2005]**
10. In a single throw of two dice, determine the probability of not getting the same number on the two dice.
11. If a letter is chosen at random from the English alphabet, find the probability that the letter chosen is (i) a vowel, and (ii) a consonant.

12. A card is drawn at random from a well-shuffled pack of 52 cards. What is the probability that the card bears a number greater than 3 and less than 10?

13. Tickets numbered from 1 to 12 are mixed up together and then a ticket is withdrawn at random. Find the probability that the ticket has a number which is a multiple of 2 or 3.

14. What is the probability that an ordinary year has 53 Tuesdays?

15. What is the probability that a leap year has 53 Sundays?

16. What is the probability that in a group of two people, both will have the same birthday, assuming that there are 365 days in a year and no one has his/her birthday on 29th February?

17. Which of the following cannot be the probability of occurrence of an event?

(i) 0 (ii) $\frac{-3}{4}$ (iii) $\frac{3}{4}$ (iv) $\frac{4}{3}$

18. If $7/10$ is the probability of occurrence of an event, what is the probability that it does not occur?

19. The odds in favour of the occurrence of an event are 8 : 13. Find the probability that the event will occur.

20. If the odds against the occurrence of an event be 4 : 7, find the probability of the occurrence of the event.

21. If $5/14$ is the probability of occurrence of an event, find

(i) the odds in favour of its occurrence

(ii) the odds against its occurrence

22. Two dice are thrown. Find

(i) the odds in favour of getting the sum 6

(ii) the odds against getting the sum 7

23. A combination lock on a suitcase has 3 wheels, each labelled with nine digits from 1 to 9. If an opening combination is a particular sequence of three digits with no repeats, what is the probability of a person guessing the right combination?

24. In a lottery, a person chooses six different numbers at random from 1 to 20. If these six numbers match with the six numbers already fixed by the lottery committee, he wins the prize. What is the probability of winning the prize in the game?

25. In a single throw of three dice, find the probability of getting (i) a total of 5 (ii) a total of at most 5. **[CBSE 2005]**

ANSWERS (EXERCISE 31A)

1. $\frac{1}{2}$ **2.** (i) $\frac{1}{6}$ (ii) $\frac{1}{3}$ (iii) $\frac{1}{2}$ (iv) $\frac{1}{2}$ (v) $\frac{1}{3}$ (vi) $\frac{1}{3}$

3. (i) $\frac{5}{18}$ (ii) $\frac{1}{12}$ (iii) $\frac{5}{12}$ **4.** (i) $\frac{1}{12}$ (ii) $\frac{1}{4}$ (iii) $\frac{1}{12}$ (iv) $\frac{5}{18}$ (v) $\frac{1}{6}$

5. $\frac{4}{9}$ **6.** (i) $\frac{9}{20}$ (ii) $\frac{7}{20}$ (iii) $\frac{4}{5}$ (iv) $\frac{11}{20}$ (v) $\frac{13}{20}$ **7.** $\frac{2}{7}$ **8.** $\frac{3}{4}$

9. (i) $\frac{3}{8}$ (ii) $\frac{3}{8}$ (iii) $\frac{7}{8}$ (iv) $\frac{1}{2}$ (v) $\frac{7}{8}$ **10.** $\frac{5}{6}$ **11.** (i) $\frac{5}{26}$ (ii) $\frac{21}{26}$ **12.** $\frac{6}{13}$

13. $\frac{2}{3}$ **14.** $\frac{1}{7}$ **15.** $\frac{2}{7}$ **16.** $\frac{1}{365}$ **17.** (ii) and (iv) **18.** $\frac{3}{10}$ **19.** $\frac{8}{21}$

20. $\frac{7}{11}$ **21.** (i) 5 : 9 (ii) 9 : 5 **22.** (i) 5 : 31 (ii) 5 : 1 **23.** $\frac{1}{504}$

24. $\frac{1}{38760}$ **25.** (i) $\frac{1}{36}$ (ii) $\frac{5}{108}$

HINTS TO SOME SELECTED QUESTIONS

7. There are 10 prizes and the total number of outcomes is 35.

8. $S = \{B_1B_2, B_1G_2, G_1B_2, G_1G_2\}$ and $E = \{B_1G_2, G_1B_2, B_1B_2\}$.

$\therefore$ required probability $= \frac{n(E)}{n(S)}$

10. $n(S) = (6 \times 6) = 36$.
Let E = event of getting the same number on two dice
or $E = \{(1, 1), (2, 2), (3, 3), (4, 4), (5, 5), (6, 6)\}$.

$\therefore \quad P(E) = \frac{n(E)}{n(S)} = \frac{6}{36} = \frac{1}{6}$

$\Rightarrow$ P(not getting the same number on the two dice) $= \left(1 - \frac{1}{6}\right) = \frac{5}{6}$.

11. There are 26 letters in the English alphabet, out of which there are 5 vowels and 21 consonants.

12. There are four suits in all, and the number of cards bearing a number greater than 3 and less than 10 in each suit is 6.

$\therefore$ total number of favourable cases $= (6 \times 4) = 24$.
Total number of cards = 52.

13. Clearly, $n(S) = 12$ and $E = \{2, 4, 6, 8, 10, 12, 3, 9\} \Rightarrow n(E) = 8$.

14. An ordinary year has 365 days, i.e., 52 weeks and 1 day.
Now, 52 weeks have 52 Tuesdays and the remaining one day can be any of the 7 days.

$\therefore$ required probability = probability of this day being a Tuesday $= \frac{1}{7}$.

15. A leap year has 366 days, and, therefore, 52 weeks and 2 days. These 2 days may be
(i) Sunday and Monday or (ii) Monday and Tuesday
or (iii) Tuesday and Wednesday or (iv) Wednesday and Thursday
or (v) Thursday and Friday or (vi) Friday and Saturday
or (vii) Saturday and Sunday.

Thus, out of 7 possibilities, 2 favour the event that one of the two days is a Sunday.

$\therefore$ required probability $= \frac{2}{7}$.

22. $n(S) = (6 \times 6) = 36.$

(i) Let E_1 = event of getting the sum 6

$= \{(1,5), (2,4), (3,3), (4,2), (5,1)\}$

$\Rightarrow n(E_1) = 5.$

Thus, E_1 occurs in 5 ways and does not occur in $(36 - 5) = 31$ ways.

$\therefore$ odds in favour of getting the sum $6 = 5 : 31$.

(ii) Let E_2 = event of getting the sum 7

$= \{(1,6), (2,5), (3,4), (4,3), (5,2), (6,1)\}$

$\Rightarrow n(E_2) = 6.$

Thus, E_2 occurs in 6 ways and does not occur in $(36 - 6) = 30$ ways.

$\therefore$ odds against getting the sum $7 = 30 : 6 = 5 : 1$.

23. Let us fill 3 places with different digits from 1 to 9.

Total number of ways $= (9 \times 8 \times 7) = 504$.

Thus, we have in all 504 combinations.

P(choosing one opening combination) $= \dfrac{1}{504}\cdot$

24. $n(S)$ = number of ways of choosing 6 numbers out of 20

$= {}^{20}C_6.$

There is a fixed set of 6 numbers, which can be chosen in 1 way only. Let E be the event of getting the fixed set of 6 numbers.

Then, $n(E) = 1$.

$\therefore \quad P(E) = \dfrac{n(E)}{n(S)} = \dfrac{1}{{}^{20}C_6}\cdot$

25. $n(S) = (6 \times 6 \times 6) = 216$

(i) $E_1 = \{(1,1,3), (1,3,1), (3,1,1), (1,2,2), (2,1,2), (2,2,1)\}$

$\therefore P(E_1) = \dfrac{6}{216} = \dfrac{1}{26}\cdot$

(ii) $E_2 = \{(1,1,1), (1,1,2), (1,2,1), (2,1,1), (1,1,3), (1,3,1), (3,1,1), (1,2,2), (2,1,2),$
$(2,2,1)\}$

$\therefore P(E_2) = \dfrac{10}{216} = \dfrac{5}{108}\cdot$

RESULTS ON PROBABILITY

THEOREM 1 *Let S be the sample space and let E be an event. Then,*
(i) $P(E) \geq 0$ *(ii)* $P(\phi) = 0$ *(iii)* $P(S) = 1$

PROOF (i) Since E is an event, we have $E \subseteq S$.

$$\therefore \quad P(E) = \frac{n(E)}{n(S)} \geq 0 \quad [\because n(E) \geq 0].$$

(ii) $P(\phi) = \dfrac{n(\phi)}{n(S)} = \dfrac{0}{n(S)} = 0.$

(iii) $P(S) = \dfrac{n(S)}{n(S)} = 1.$

<u>THEOREM 2</u> *If E_1 and E_2 are mutually exclusive events,*

(i) $P(E_1$ and $E_2) = 0$, i.e., $P(E_1 \cap E_2) = 0$

(ii) $P(E_1$ or $E_2) = P(E_1) + P(E_2)$, i.e., $P(E_1 \cup E_2) = P(E_1) + P(E_2)$

PROOF Let S be the sample space, and let E_1 and E_2 be two mutually exclusive events. Then, $E_1 \cap E_2 = \phi$.

(i) $P(E_1$ and $E_2) = P(E_1 \cap E_2) = P(\phi) = 0$.

(ii) $P(E_1$ or $E_2) = P(E_1 \cup E_2)$

$$= \frac{n(E_1 \cup E_2)}{n(S)}$$

$$= \frac{n(E_1) + n(E_2)}{n(S)} \quad [\because E_1 \cap E_2 = \phi \Rightarrow n(E_1 \cup E_2) = n(E_1) + n(E_2)]$$

$$= \frac{n(E_1)}{n(S)} + \frac{n(E_2)}{n(S)} = P(E_1) + P(E_2).$$

$\therefore$ $P(E_1$ or $E_2) = P(E_1) + P(E_2)$, when E_1 and E_2 are mutually exclusive.

REMARK If $E_1, E_2, \ldots, E_n$ are pairwise disjoint then

$P(E_1 \cup E_2 \cup \ldots \cup E_n) = P(E_1) + P(E_2) + \ldots + P(E_n)$.

ADDITION THEOREM FOR TWO EVENTS

<u>THEOREM 3</u> *For any two events E_1 and E_2,*

$$P(E_1 \cup E_2) = P(E_1) + P(E_2) - P(E_1 \cap E_2),$$

i.e., $P(E_1$ or $E_2) = P(E_1) + P(E_2) - P(E_1$ and $E_2)$.

PROOF Let S be the sample space. Then, $E_1 \subseteq S$ and $E_2 \subseteq S$.

From set theory, we know that

$$n(E_1 \cup E_2) = n(E_1) + n(E_2) - n(E_1 \cap E_2)$$

$$\Rightarrow \frac{n(E_1 \cup E_2)}{n(S)} = \frac{n(E_1) + n(E_2) - n(E_1 \cap E_2)}{n(S)}$$

$$= \frac{n(E_1)}{n(S)} + \frac{n(E_2)}{n(S)} - \frac{n(E_1 \cap E_2)}{n(S)}$$

$\Rightarrow$ $P(E_1 \cup E_2) = P(E_1) + P(E_2) - P(E_1 \cap E_2)$

$\Rightarrow$ $P(E_1$ or $E_2) = P(E_1) + P(E_2) - P(E_1$ and $E_2)$.

REMARK If E_1 and E_2 are mutually exclusive events then $E_1 \cap E_2 = \phi$, and so $P(E_1 \cap E_2) = P(\phi) = 0$.

So, in this case, $P(E_1 \cup E_2) = P(E_1) + P(E_2)$.

ADDITION THEOREM FOR THREE EVENTS

THEOREM 4 *For any three events E_1, E_2, E_3,*

$$P(E_1 \cup E_2 \cup E_3) = P(E_1) + P(E_2) + P(E_3) - P(E_1 \cap E_2) - P(E_2 \cap E_3) - P(E_3 \cap E_1) + P(E_1 \cap E_2 \cap E_3).$$

PROOF We have

$$\begin{aligned}
&P(E_1 \cup E_2 \cup E_3) \\
&= P[(E_1 \cup E_2) \cup E_3] \\
&= P(E_1 \cup E_2) + P(E_3) - P[(E_1 \cup E_2) \cap E_3] \\
&\qquad\qquad \text{[by the addition theorem for two events]} \\
&= P(E_1) + P(E_2) - P(E_1 \cap E_2) + P(E_3) - P[(E_1 \cap E_3) \cup (E_2 \cap E_3)] \\
&\qquad\qquad \text{[by the addition theorem on } P(E_1 \cup E_2)\text{]} \\
&= P(E_1) + P(E_2) + P(E_3) - P(E_1 \cap E_2) - [P(E_1 \cap E_3) + P(E_2 \cap E_3) \\
&\qquad - P(E_1 \cap E_3 \cap E_2 \cap E_3)] \\
&= P(E_1) + P(E_2) + P(E_3) - P(E_1 \cap E_2) - P(E_1 \cap E_3) - P(E_2 \cap E_3) \\
&\qquad + P(E_1 \cap E_2 \cap E_3)
\end{aligned}$$

Hence, $P(E_1 \cup E_2 \cup E_3) = P(E_1) + P(E_2) + P(E_3) - P(E_1 \cap E_2) - P(E_2 \cap E_3) - P(E_3 \cap E_1) + P(E_1 \cap E_2 \cap E_3)$.

THEOREM 5 *If E_1 and E_2 are two mutually exclusive exhaustive events then $P(E_1) + P(E_2) = 1$.*

PROOF Let S be the sample space. Then, E_1 and E_2 being mutually exclusive exhaustive events, we have

$$E_1 \cap E_2 = \phi \text{ and } E_1 \cup E_2 = S.$$

$$\begin{aligned}
\therefore\quad P(E_1) + P(E_2) &= P(E_1 \cup E_2) && [\because E_1 \cap E_2 = \phi] \\
&= P(S) = 1 && [\because E_1 \cup E_2 = S].
\end{aligned}$$

$\therefore\quad (E_1 \cap E_2) = \phi$ and $(E_1 \cup E_2) = S \Rightarrow P(E_1) + P(E_2) = 1$.

THEOREM 6 *For any event E, $P(\overline{E}) = 1 - P(E)$.*

PROOF Let S be the sample space.

Now, $(E \cap \overline{E}) = \phi$ and $(E \cup \overline{E}) = S$

$$\begin{aligned}
\Rightarrow\quad P(E) + P(\overline{E}) &= P(E \cup \overline{E}) && [\because E \cap \overline{E} = \phi] \\
&= P(S) = 1 && [\because E \cup \overline{E} = S].
\end{aligned}$$

Hence, $P(\overline{E}) = 1 - P(E)$.

NOTE For an alternative proof of this theorem, see P-17.

THEOREM 7 *For any two compatible events E_1 and E_2,*

$$P(E_1 - E_2) = P(E_1) - P(E_1 \cap E_2).$$

PROOF Let E_1 and E_2 be two compatible events.

Then, $E_1 \cap E_2 \neq \phi$.

From the adjoining Venn diagram, we have

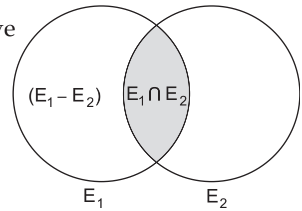
$(E_1 - E_2) \cap (E_1 \cap E_2) = \phi$, ... (i)

and $(E_1 - E_2) \cup (E_1 \cap E_2) = E_1$. ... (ii)

$\therefore \quad P(E_1) = P[(E_1 - E_2) \cup (E_1 \cap E_2)]$

$\quad = P(E_1 - E_2) + P(E_1 \cap E_2)$

$\Rightarrow \quad P(E_1 - E_2) = P(E_1) - P(E_1 \cap E_2)$.



THEOREM 8 *If E_1 and E_2 be two events such that $E_1 \subseteq E_2$ then $P(E_1) \leq P(E_2)$.*

PROOF Let $E_1 \subseteq E_2$. Then,

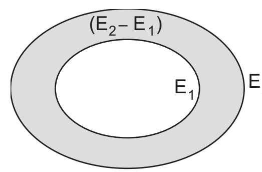
$E_1 \cap (E_2 - E_1) = \phi$ and $E_1 \cup (E_2 - E_1) = E_2$.

$\therefore \quad P(E_2) = P[E_1 \cup (E_2 - E_1)]$

$\quad = P(E_1) + P(E_2 - E_1)$

$[\because E_1 \cap (E_2 - E_1) = \phi]$.

$\therefore \quad P(E_1) \leq P(E_2) \quad [\because P(E_2 - E_1) \geq 0]$.



THEOREM 9 *For any event E, $0 \leq P(E) \leq 1$.*

PROOF Let S be the sample space and let E be an event.

Then, $\phi \subseteq E$ and $E \subseteq S$

$\Rightarrow P(\phi) \leq P(E)$ and $P(E) \leq P(S)$

$\Rightarrow P(\phi) \leq P(E) \leq P(S)$

$\Rightarrow 0 \leq P(E) \leq 1 \quad [\because P(\phi) = 0$ and $P(S) = 1]$.

Hence, $0 \leq P(E) \leq 1$, for every event E.

SOLVED EXAMPLES

EXAMPLE 1 *If E_1 and E_2 are two events associated with a random experiment such that $P(E_2) = 0.35$, $P(E_1$ or $E_2) = 0.85$ and $P(E_1$ and $E_2) = 0.15$, find $P(E_1)$.*

SOLUTION Let $P(E_1) = x$. Then,

$P(E_1 \text{ or } E_2) = P(E_1) + P(E_2) - P(E_1 \text{ and } E_2)$

$\Rightarrow 0.85 = x + 0.35 - 0.15$

$\Rightarrow x = (0.85 - 0.35 + 0.15) = 0.65$.

Hence, $P(E_1) = 0.65$.

EXAMPLE 2 *Two dice are tossed together. Find the probability of getting a doublet or a total of 6.* **[CBSE 2004C]**

SOLUTION When two dice are thrown together, there are (6×6) outcomes. Let S be the sample space. Then, $n(S) = 36$.

Let E_1 = event of getting a doublet,

and E_2 = event of getting a total of 6.

Then, $E_1 = \{(1, 1), (2, 2), (3, 3), (4, 4), (5, 5), (6, 6)\}$

and $E_2 = \{(2, 4), (3, 3), (4, 2)\}$.

$\therefore\ (E_1 \cap E_2) = \{(3, 3)\}$.

Thus, $n(E_1) = 6$, $n(E_2) = 3$ and $n(E_1 \cap E_2) = 1$.

$$\therefore\ P(E_1) = \frac{n(E_1)}{n(S)} = \frac{6}{36} = \frac{1}{6},\ P(E_2) = \frac{n(E_2)}{n(S)} = \frac{3}{36} = \frac{1}{12}$$

and $P(E_1 \cap E_2) = \dfrac{n(E_1 \cap E_2)}{n(S)} = \dfrac{1}{36}$.

$\therefore\ P$(getting a doublet or a total of 6)

$= P(E_1 \text{ or } E_2) = P(E_1 \cup E_2)$

$= P(E_1) + P(E_2) - P(E_1 \cap E_2)$ [by the addition theorem]

$$= \left(\frac{1}{6} + \frac{1}{12} - \frac{1}{36}\right) = \frac{8}{36} = \frac{2}{9}.$$

Hence, the required probability is $\dfrac{2}{9}$.

EXAMPLE 3 *In a single throw of two dice, find the probability that neither a doublet nor a total of* 10 *will appear.* **[CBSE 1998, 2003]**

SOLUTION Let S be the sample space. Then, $n(S) = 36$.

Let E_1 = event that a doublet appears,

and E_2 = event of getting a total of 10.

Then, $E_1 = \{(1, 1), (2, 2), (3, 3), (4, 4), (5, 5), (6, 6)\}$,

and $E_2 = \{(4, 6), (5, 5), (6, 4)\}$.

$\therefore\ (E_1 \cap E_2) = \{(5, 5)\}$.

Thus, $n(E_1) = 6$, $n(E_2) = 3$ and $n(E_1 \cap E_2) = 1$.

$$\therefore\ P(E_1) = \frac{n(E_1)}{n(S)} = \frac{6}{36} = \frac{1}{6},\ P(E_2) = \frac{n(E_2)}{n(S)} = \frac{3}{36} = \frac{1}{12}$$

and $P(E_1 \cap E_2) = \dfrac{n(E_1 \cap E_2)}{n(S)} = \dfrac{1}{36}$.

$\therefore\ P$(getting a doublet or a total of 10)

$= P(E_1 \text{ or } E_2) = P(E_1 \cup E_2)$

$= P(E_1) + P(E_2) - P(E_1 \cap E_2)$

$$= \left(\frac{1}{6} + \frac{1}{12} - \frac{1}{36}\right) = \frac{8}{36} = \frac{2}{9}.$$

$\therefore$ P(getting neither a doublet nor a total of 10)

$= P(\overline{E}_1 \text{ and } \overline{E}_2) = P(\overline{E}_1 \cap \overline{E}_2)$

$= P(\overline{E_1 \cup E_2}) = 1 - P(E_1 \cup E_2) = \left(1 - \frac{2}{9}\right) = \frac{7}{9}\cdot$

Hence, the required probability is $\frac{7}{9}\cdot$

EXAMPLE 4 *A natural number is chosen at random from among the first* 500. *What is the probability that the number so chosen is divisible by* 3 *or* 5?

SOLUTION Let S be the sample space. Then, clearly $n(S) = 500$.

Let E_1 = event of getting a number divisible by 3,

and E_2 = event of getting a number divisible by 5. Then,

$(E_1 \cap E_2)$ = event of getting a number divisible by both 3 and 5

= event of getting a number divisible by 15.

$\therefore$ $E_1 = \{3, 6, 9, \ldots, 495, 498\}$, $E_2 = \{5, 10, 15, \ldots, 495, 500\}$

and $(E_1 \cap E_2) = \{15, 30, 45, \ldots, 495\}$.

$\therefore$ $n(E_1) = \left(\frac{498}{3}\right) = 166$, $n(E_2) = \left(\frac{500}{5}\right) = 100$

and $n(E_1 \cap E_2) = \left(\frac{495}{15}\right) = 33$.

$\therefore$ $P(E_1) = \frac{n(E_1)}{n(S)} = \frac{166}{500} = \frac{83}{250}$, $P(E_2) = \frac{n(E_2)}{n(S)} = \frac{100}{500} = \frac{1}{5}$

and $P(E_1 \cap E_2) = \frac{n(E_1 \cap E_2)}{n(S)} = \frac{33}{500}\cdot$

$\therefore$ P(the chosen number is divisible by 3 or 5)

$= P(E_1 \text{ or } E_2) = P(E_1 \cup E_2)$

$= P(E_1) + P(E_2) - P(E_1 \cap E_2)$

$= \left(\frac{83}{250} + \frac{1}{5} - \frac{33}{500}\right) = \frac{233}{500}\cdot$

Hence, the required probability is $\frac{233}{500}\cdot$

EXAMPLE 5 *A card is drawn at random from a well-shuffled deck of* 52 *cards. Find the probability of its being a spade or a king.*

SOLUTION Let S be the sample space. Then, $n(S) = 52$.

Let E_1 = event of getting a spade,

and E_2 = event of getting a king.

Then, $(E_1 \cap E_2)$ = event of getting a king of spades.

Clearly, $n(E_1) = 13$, $n(E_2) = 4$ and $n(E_1 \cap E_2) = 1$.

$\therefore\ P(E_1) = \dfrac{n(E_1)}{n(S)} = \dfrac{13}{52} = \dfrac{1}{4},\ P(E_2) = \dfrac{n(E_2)}{n(S)} = \dfrac{4}{52} = \dfrac{1}{13}$

and $P(E_1 \cap E_2) = \dfrac{n(E_1 \cap E_2)}{n(S)} = \dfrac{1}{52}\cdot$

$\therefore$ P(getting a spade or a king)

$= P(E_1 \text{ or } E_2) = P(E_1 \cup E_2)$

$= P(E_1) + P(E_2) - P(E_1 \cap E_2)$ [by the addition theorem for two events]

$= \left(\dfrac{1}{4} + \dfrac{1}{13} - \dfrac{1}{52}\right) = \dfrac{16}{52} = \dfrac{4}{13}\cdot$

Hence, the required probability is $\dfrac{4}{13}\cdot$

EXAMPLE 6 *Two cards are drawn at random from a well-shuffled pack of* 52 *cards. What is the probability that either both are red or both are kings?*

SOLUTION Let S be the sample space. Then,

$n(S) =$ number of ways of drawing 2 cards out of 52

$= {}^{52}C_2.$

Let $E_1 =$ event that both are red cards,

and $E_2 =$ event that both are kings.

Then, $(E_1 \cap E_2) =$ event of getting 2 red kings.

$\therefore$ $n(E_1) =$ number of ways of drawing 2 red cards out of 26 red cards $= {}^{26}C_2.$

$n(E_2) =$ number of ways of drawing 2 kings out of 4 kings

$= {}^{4}C_2.$

$\therefore$ $n(E_1 \cap E_2) =$ number of ways of drawing 2 red kings out of 2 red kings $= {}^{2}C_2 = 1.$

$\therefore\ P(E_1) = \dfrac{n(E_1)}{n(S)} = \dfrac{{}^{26}C_2}{{}^{52}C_2};\ P(E_2) = \dfrac{n(E_2)}{n(S)} = \dfrac{{}^{4}C_2}{{}^{52}C_2}$

and $P(E_1 \cap E_2) = \dfrac{n(E_1 \cap E_2)}{n(S)} = \dfrac{1}{{}^{52}C_2}\cdot$

$\therefore$ P(drawing both red cards or both kings)

$= P(E_1 \text{ or } E_2) = P(E_1 \cup E_2)$

$= P(E_1) + P(E_2) - P(E_1 \cap E_2)$

$= \left(\dfrac{{}^{26}C_2}{{}^{52}C_2} + \dfrac{{}^{4}C_2}{{}^{52}C_2} - \dfrac{1}{{}^{52}C_2}\right) = \dfrac{({}^{26}C_2 + {}^{4}C_2 - 1)}{{}^{52}C_2}$

$$= \frac{(325 + 6 - 1)}{1326} = \frac{330}{1326} = \frac{55}{221}.$$

Hence, the required probability is $\frac{55}{221}$.

EXAMPLE 7 *A box contains* 100 *bolts and* 50 *nuts. It is given that* 50% *bolts and* 50% *nuts are rusted. Two objects are selected from the box at random. Find the probability that either both are bolts or both are rusted.* **[CBSE 2003C]**

SOLUTION Total number of objects $= (100 + 50) = 150$.

Let S be the sample space. Then,

$n(S) =$ number of ways of selecting 2 objects out of 150 $= {}^{150}C_2$.

Number of rusted objects

$= (50\% \text{ of } 100) + (50\% \text{ of } 50) = (50 + 25) = 75$.

Let $E_1 =$ event of selecting 2 bolts out of 100 bolts,

and $E_2 =$ event of selecting 2 rusted objects out of 75 rusted objects.

$\therefore$ $(E_1 \cap E_2) =$ event of selecting 2 rusted bolts out of 50 rusted bolts

$\therefore$ $n(E_1) =$ number of ways of selecting 2 bolts out of 100 $= {}^{100}C_2$.

$\therefore$ $n(E_2) =$ number of ways of selecting 2 rusted objects out of 75 $= {}^{75}C_2$.

$\therefore$ $n(E_1 \cap E_2) =$ number of ways of selecting 2 rusted bolts out of 50 $= {}^{50}C_2$.

$$\therefore \; P(E_1) = \frac{n(E_1)}{n(S)} = \frac{{}^{100}C_2}{{}^{150}C_2}, \; P(E_2) = \frac{n(E_2)}{n(S)} = \frac{{}^{75}C_2}{{}^{150}C_2}$$

and $P(E_1 \cap E_2) = \dfrac{n(E_1 \cap E_2)}{n(S)} = \dfrac{{}^{50}C_2}{{}^{150}C_2}$.

P(selecting both bolts or both rusted objects)

$$= P(E_1 \text{ or } E_2) = P(E_1 \cup E_2)$$

$$= P(E_1) + P(E_2) - P(E_1 \cap E_2)$$

$$= \frac{{}^{100}C_2}{{}^{150}C_2} + \frac{{}^{75}C_2}{{}^{150}C_2} - \frac{{}^{50}C_2}{{}^{150}C_2} = \frac{({}^{100}C_2 + {}^{75}C_2 - {}^{50}C_2)}{{}^{150}C_2}$$

$$= \frac{(4950 + 2775 - 1225)}{11175} = \frac{6500}{11175} = \frac{260}{447} = 0.58.$$

Hence, the required probability is 0.58.

EXAMPLE 8 *If E_1 and E_2 are two events such that $P(E_1) = 0.5$, $P(E_2) = 0.3$ and $P(E_1 \text{ and } E_2) = 0.1$, find*

(i) $P(E_1 \text{ or } E_2)$ (ii) $P(E_1 \text{ but not } E_2)$
(iii) $P(E_2 \text{ but not } E_1)$ (iv) $P(\text{neither } E_1 \text{ nor } E_2)$

SOLUTION We have

$$P(E_1) = 0.5,\ P(E_2) = 0.3 \text{ and } P(E_1 \cap E_2) = P(E_1 \text{ and } E_2) = 0.1.$$

$\therefore\ P(\overline{E}_1) = \{1 - P(E_1)\} = (1 - 0.5) = 0.5,$

and $P(\overline{E}_2) = \{1 - P(E_2)\} = (1 - 0.3) = 0.7.$

Thus, we have

(i) $P(E_1 \text{ or } E_2) = P(E_1 \cup E_2)$
$= P(E_1) + P(E_2) - P(E_1 \cap E_2)$
$= (0.5 + 0.3 - 0.1) = 0.7.$

(ii) $P(E_1 \text{ but not } E_2) = P(E_1 \cap \overline{E}_2)$
$= P(E_1) - P(E_1 \cap E_2)$
$= (0.5 - 0.1) = 0.4.$

(iii) $P(E_2 \text{ but not } E_1) = P(E_2 \cap \overline{E}_1)$
$= P(E_2) - P(E_2 \cap E_1)$
$= P(E_2) - P(E_1 \cap E_2) = (0.3 - 0.1) = 0.2.$

(iv) $P(\text{neither } E_1 \text{ nor } E_2) = P(\text{not } E_1 \text{ and not } E_2)$
$= P(\overline{E}_1 \text{ and } \overline{E}_2) = P(\overline{E}_1 \cap \overline{E}_2)$
$= P(\overline{E_1 \cup E_2}) = 1 - P(E_1 \cup E_2)$
$= 1 - P(E_1 \text{ or } E_2)$
$= (1 - 0.7) = 0.3$ [using (i)].

EXAMPLE 9 *The probability that at least one of the events E_1 and E_2 occurs is 0.6. If the probability of the simultaneous occurrence of E_1 and E_2 is 0.2, find $P(\overline{E}_1) + P(\overline{E}_2)$.*

SOLUTION Given, $P(E_1 \cup E_2) = 0.6$ and $P(E_1 \cap E_2) = 0.2$.

$\therefore\ P(E_1 \cup E_2) = P(E_1) + P(E_2) - P(E_1 \cap E_2)$

$\Rightarrow\ P(E_1) + P(E_2) = P(E_1 \cup E_2) + P(E_1 \cap E_2) = (0.6 + 0.2) = 0.8$

$\Rightarrow\ P(E_1) + P(E_2) = 0.8$

$\Rightarrow\ \{1 - P(\overline{E}_1)\} + \{1 - P(\overline{E}_2)\} = 0.8$

$\Rightarrow\ P(\overline{E}_1) + P(\overline{E}_2) = (2 - 0.8) = 1.2.$

Hence, $P(\overline{E}_1) + P(\overline{E}_2) = 1.2.$

EXAMPLE 10 *The probabilities of the occurrences of two events E_1 and E_2 are 0.25 and 0.50 respectively. The probability of their simultaneous occurrence is 0.14. Find the probability that neither E_1 nor E_2 occurs.*

SOLUTION Given, $P(E_1) = 0.25$, $P(E_2) = 0.50$
and $P(E_1 \text{ and } E_2) = P(E_1 \cap E_2) = 0.14$.

$\therefore\ P(\overline{E}_1) = 1 - P(E_1) = (1 - 0.25) = 0.75$

and $P(\overline{E}_2) = 1 - P(E_2) = (1 - 0.50) = 0.50$.

$\therefore\ P(\text{neither } E_1 \text{ nor } E_2)$

$= P(\text{not } E_1 \text{ and not } E_2) = P(\overline{E}_1 \text{ and } \overline{E}_2) = P(\overline{E}_1 \cap \overline{E}_2)$

$= P(\overline{E_1 \cup E_2}) = \{1 - P(E_1 \cup E_2)\}$

$= 1 - \{P(E_1) + P(E_2) - P(E_1 \cap E_2)\}$

$= 1 - (0.25 + 0.50 - 0.14) = (1 - 0.61) = 0.39$.

Hence, the required probability is 0.39.

EXAMPLE 11 *A card is drawn from a deck of* 52 *cards. Find the probability of getting a king or a heart or a red card.*

SOLUTION Let S be the sample space. Then, $n(S) = 52$.

Let E_1, E_2 and E_3 be the events of getting a king, a heart and a red card respectively. Then,

$n(E_1) = 4$, $n(E_2) = 13$ and $n(E_3) = 26$.

$(E_1 \cap E_2) =$ event of getting a king of hearts;

$(E_2 \cap E_3) =$ event of getting a heart [$\because$ a heart is a red card also];

$(E_3 \cap E_1) =$ event of getting a red king;

and $(E_1 \cap E_2 \cap E_3) =$ event of getting a king of hearts.

$\therefore\ n(E_1 \cap E_2) = 1$, $n(E_2 \cap E_3) = 13$, $n(E_3 \cap E_1) = 2$

and $n(E_1 \cap E_2 \cap E_3) = 1$.

$$\therefore\ P(E_1) = \frac{n(E_1)}{n(S)} = \frac{4}{52} = \frac{1}{13};\ P(E_2) = \frac{n(E_2)}{n(S)} = \frac{13}{52} = \frac{1}{4};$$

$$P(E_3) = \frac{n(E_3)}{n(S)} = \frac{26}{52} = \frac{1}{2};\ P(E_1 \cap E_2) = \frac{n(E_1 \cap E_2)}{n(S)} = \frac{1}{52};$$

$$P(E_2 \cap E_3) = \frac{n(E_2 \cap E_3)}{n(S)} = \frac{13}{52} = \frac{1}{4};$$

$$P(E_3 \cap E_1) = \frac{n(E_3 \cap E_1)}{n(S)} = \frac{2}{52} = \frac{1}{26}$$

$$\text{and } P(E_1 \cap E_2 \cap E_3) = \frac{n(E_1 \cap E_2 \cap E_3)}{n(S)} = \frac{1}{52}.$$

$\therefore\ P(\text{getting a king or a heart or a red card})$

$= P(E_1 \text{ or } E_2 \text{ or } E_3) = P(E_1 \cup E_2 \cup E_3)$

$= P(E_1) + P(E_2) + P(E_3) - P(E_1 \cap E_2) - P(E_2 \cap E_3) - P(E_3 \cap E_1)$
$+ P(E_1 \cap E_2 \cap E_3)$

$$= \left(\frac{1}{13} + \frac{1}{4} + \frac{1}{2} - \frac{1}{52} - \frac{1}{4} - \frac{1}{26} + \frac{1}{52}\right) = \frac{28}{52} = \frac{7}{13}.$$

Hence, the required probability is $\frac{7}{13}$.

EXERCISE 31B

1. If A and B are two events associated with a random experiment for which $P(A) = 0.60$, $P(A \text{ or } B) = 0.85$ and $P(A \text{ and } B) = 0.42$, find $P(B)$. **[CBSE 1998]**

2. Let A and B be two events associated with a random experiment for which $P(A) = 0.4$, $P(B) = 0.5$ and $P(A \text{ or } B) = 0.6$. Find $P(A \text{ and } B)$.

3. In a random experiment, let A and B be events such that $P(A \text{ or } B) = 0.7$, $P(A \text{ and } B) = 0.3$ and $P(\overline{A}) = 0.4$. Find $P(B)$.

4. If A and B are two events associated with a random experiment such that $P(A) = 0.25$, $P(B) = 0.4$ and $P(A \text{ or } B) = 0.5$, find the values of
   (i) $P(A \text{ and } B)$ (ii) $P(A \text{ and } \overline{B})$

5. If A and B be two events associated with a random experiment such that $P(A) = 0.3$, $P(B) = 0.2$ and $P(A \cap B) = 0.1$, find
   (i) $P(\overline{A} \cap B)$ (ii) $P(A \cap \overline{B})$

6. If A and B are two mutually exclusive events such that $P(A) = (1/2)$ and $P(B) = (1/3)$, find $P(A \text{ or } B)$.

7. Let A and B be two mutually exclusive events of a random experiment such that $P(\text{not } A) = 0.65$ and $P(A \text{ or } B) = 0.65$, find $P(B)$.

8. A, B, C are three mutually exclusive and exhaustive events associated with a random experiment.
   If $P(B) = (3/2)\ P(A)$ and $P(C) = (1/2)\ P(B)$, find $P(A)$.

9. The probability that a company executive will travel by plane is $(2/5)$ and that he will travel by train is $(1/3)$. Find the probability of his travelling by plane or train.

10. From a well-shuffled pack of 52 cards, a card is drawn at random. Find the probability of its being a king or a queen.

11. From a well-shuffled pack of cards, a card is drawn at random. Find the probability of its being either a queen or a heart.

12. A card is drawn at random from a well-shuffled deck of 52 cards. Find the probability of its being a spade or a king.

13. A number is chosen from the numbers 1 to 100. Find the probability of its being divisible by 4 or 6.

14. A die is thrown twice. What is the probability that at least one of the two throws comes up with the number 4?

15. Two dice are tossed once. Find the probability of getting an even number on the first die or a total of 8. **[CBSE 2004]**

16. Two dice are thrown together. What is the probability that the sum of the numbers on the two faces is neither divisible by 3 nor by 4.

17. In a class, 30% of the students offered mathematics, 20% offered chemistry and 10% offered both. If a student is selected at random, find the probability that he has offered mathematics or chemistry.

18. The probability that Hemant passes in English is (2/3) and the probability that he passes in Hindi is (5/9). If the probability of his passing both the subjects is (2/5), find the probability that he will pass in at least one of these subjects.

19. The probability that a person will get an electrification contract is (2/5) and the probability that he will not get a plumbing contract is (4/7). If the probability of getting at least one contract is (2/3), what is the probability that he will get both? **[CBSE 2005C]**

20. The probability that a patient visiting a dentist will have a tooth extracted is 0.06, the probability that he will have a cavity filled is 0.2, and the probability that he will have a tooth extracted or a cavity filled is 0.23. What is the probability that he will have a tooth extracted as well as a cavity filled?

21. In a town of 6000 people, 1200 are over 50 years old and 2000 are females. It is known that 30% of the females are over 50 years. What is the probability that a randomly chosen individual from the town is either female or over 50 years?

ANSWERS (EXERCISE 31B)

1. 0.67 **2.** 0.3 **3.** 0.4 **4.** (i) 0.15 (ii) 0.1 **5.** (i) 0.1 (ii) 0.2 **6.** $\frac{5}{6}$

7. 0.3 **8.** $\frac{4}{13}$ **9.** $\frac{11}{15}$ **10.** $\frac{2}{13}$ **11.** $\frac{4}{13}$ **12.** $\frac{4}{13}$ **13.** $\frac{33}{100}$ **14.** $\frac{11}{36}$

15. $\frac{5}{9}$ **16.** $\frac{4}{9}$ **17.** $\frac{2}{5}$ **18.** $\frac{37}{45}$ **19.** $\frac{17}{105}$ **20.** 0.03 **21.** $\frac{13}{30}$

HINTS TO SOME SELECTED QUESTIONS

4. (ii) $P(A \text{ and } \overline{B}) = P(A \cap \overline{B}) = P(A) - P(A \cap B)$.

5. (i) $P(\overline{A} \cap B) = P(B) - P(A \cap B)$.

6. $A \cap B = \phi \Rightarrow P(A \cup B) = P(A) + P(B)$.

8. A, B, C being mutually exclusive exhaustive events, we have

$$P(A) + P(B) + P(C) = 1 \Rightarrow P(A) + \frac{3}{2}P(A) + \frac{1}{2}\cdot\left\{\frac{3}{2}P(A)\right\} = 1$$

$$\Rightarrow \frac{13}{4}\cdot P(A) = 1 \Rightarrow P(A) = \frac{4}{13}\cdot$$

9. The events of travelling by plane and that by train are mutually exclusive.

$\therefore\ P(E_1 \cup E_2) = P(E_1) + P(E_2)$.

10. The probability of getting a king and that of getting a queen are mutually exclusive.

$\therefore\ P(\text{a king or a queen}) = P(\text{a king}) + P(\text{a queen}) = \left(\frac{4}{52} + \frac{4}{52}\right)$.

11. Clearly, $n(S) = 52$.

Let E_1 = event of getting a queen,

and E_2 = event of getting a heart.

Then, $E_1 \cap E_2$ = event of getting a queen of hearts.

$\therefore\ n(E_1) = 4, n(E_2) = 13$ and $n(E_1 \cap E_2) = 1$.

Now, use $P(E_1 \cup E_2) = P(E_1) + P(E_2) - P(E_1 \cap E_2)$.

13. Here, $S = \{1, 2, 3, 4, \dots, 99, 100\} \Rightarrow n(S) = 100$.

Let E_1 = event of getting a number divisible by 4.

And, E_2 = event of getting a number divisible by 6.

Then, $E_1 \cap E_2$ = event of getting a number divisible by both 4 and 6, i.e., divisible by 12 (LCM of 4 and 6).

$\therefore\ E_1 = \{4, 8, 12, \dots, 100\} \Rightarrow n(E_1) = 25$.

$E_2 = \{6, 12, 18, \dots, 96\} \Rightarrow n(E_2) = 16$.

$(E_1 \cap E_2) = \{12, 24, 36, \dots, 96\} \Rightarrow n(E_1 \cap E_2) = 8$.

Now, use $P(E_1 \cup E_2) = P(E_1) + P(E_2) - P(E_1 \cap E_2)$.

14. Clearly, $n(S) = (6 \times 6) = 36$.

Let E_1 = event that the first throw shows a 4,

and E_2 = event that the second throw shows a 4.

Then, $(E_1 \cap E_2)$ = event that each throw shows a 4.

$\therefore\ E_1 = \{(4, 1), (4, 2), (4, 3), (4, 4), (4, 5), (4, 6)\}$,

and $E_2 = \{(1, 4), (2, 4), (3, 4), (4, 4), (5, 4), (6, 4)\}$.

$\therefore\ (E_1 \cap E_2) = \{(4, 4)\}$.

Now, use $P(E_1 \cup E_2) = P(E_1) + P(E_2) - P(E_1 \cap E_2)$.

15. Clearly, $n(S) = (6 \times 6) = 36$.

Let E_1 = event of getting an even number on the first die,

and E_2 = event of getting a total of 8 on both dice. Then,

$E_1 = \{(2, 1), (2, 2), \dots, (2, 6), (4, 1), (4, 2), \dots, (4, 6), (6, 1), (6, 2), \dots, (6, 6)\}$,

$E_2 = \{(2, 6), (3, 5), (4, 4), (5, 3), (6, 2)\}$.

$\therefore\ (E_1 \cap E_2) = \{(2, 6), (4, 4), (6, 2)\}$.

$\therefore\ n(E_1) = 18, n(E_2) = 5$ and $n(E_1 \cap E_2) = 3$.

Now, use $P(E_1 \cup E_2) = P(E_1) + P(E_2) - P(E_1 \cap E_2)$.

16. Clearly, $n(S) = (6 \times 6) = 36$.

Let E_1 = event that the sum of numbers on two faces is divisible by 3,

and E_2 = event that the sum of numbers on two faces is divisible by 4.

Then, $E_1 \cap E_2$ = event that the sum of the numbers on two faces is divisible by both 3 and 4, i.e., by 12.

$\therefore \quad E_1 = \{(1,2),(1,5),(2,1),(2,4),(3,3),(3,6),(4,2),(4,5),(5,1),(5,4),(6,3),(6,6)\}$,

$E_2 = \{1,3),(2,2),(2,6),(3,1),(3,5),(4,4),(5,3),(6,2),(6,6)\}$.

$\therefore \quad E_1 \cap E_2 = \{(6,6)\}$.

Thus, $n(E_1) = 12, n(E_2) = 9$ and $n(E_1 \cap E_2) = 1$.

Required probability $= P[(\text{not } E_1) \text{ and } (\text{not } E_2)] = P(\overline{E}_1 \cap \overline{E}_2)$

$= P(\overline{E_1 \cup E_2}) = 1 - P(E_1 \cup E_2)$

$= 1 - \{P(E_1) + P(E_2) - P(E_1 \cap E_2)\}$.

17. Here, $n(S) = 100, n(M) = 30, n(C) = 20$ and $n(M \cap C) = 10$.

$\therefore \quad P(M) = \frac{30}{100} = \frac{3}{10}; P(C) = \frac{20}{100} = \frac{1}{5}$ and $P(M \cap C) = \frac{10}{100} = \frac{1}{10}\cdot$

Use $P(M \cup C) = P(M) + P(C) + P(M \cap C)$.

18. $P(E) = \frac{2}{3}, P(H) = \frac{5}{9}$ and $P(E \cap H) = \frac{2}{5}\cdot$ Find $P(E \cup H)$.

19. Let E_1 = event of getting an electrification contract,

and E_2 = event of getting a plumbing contract.

Then, $P(E_1) = \frac{2}{5}, P(\text{not } E_2) = \frac{4}{7} \Rightarrow P(E_2) = \left(1 - \frac{4}{7}\right) = \frac{3}{7}\cdot$

And, $P(E_1 \cup E_2) = P(E_1 \text{ or } E_2) = \frac{2}{3}\cdot$

$\therefore$ required probability $= P(E_1 \cap E_2)$

$= P(E_1) + P(E_2) - P(E_1 \cup E_2)$.

20. $P(E_1) = 0.06, P(E_2) = 0.2$ and $P(E_1 \cup E_2) = 0.23$.

$P(E_1 \cap E_2) = P(E_1) + P(E_2) - P(E_1 \cup E_2)$.

21. Let E_1 = event of person being a female,

and E_2 = event of person being 50 years old.

Then, $n(E_1) = 2000, n(E_2) = 1200$,

and $n(E_1 \cap E_2) = (30\% \text{ of } 2000) = 600$.

Now, $n(E_1 \cup E_2) = n(E_1) + n(E_2) - n(E_1 \cap E_2) = 2600$.

$\therefore \quad P(E_1 \cup E_2) = \frac{2600}{6000} = \frac{13}{30}\cdot$

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Logarithm

LOGARITHM

If a is a positive real number and $a \neq 1$ such that $a^x = m$, then x is called the logarithm of m to the base a, written as $\log_a m$.

$\therefore \quad a^x = m \iff \log_a m = x.$

LAWS OF LOGARITHM

(i) $\log_a(mn) = (\log_a m) + (\log_a n)$

(ii) $\log_a\left(\frac{m}{n}\right) = (\log_a m) - (\log_a n)$

(iii) $\log_a(m^k) = k \cdot (\log_a m)$

(iv) $\log_a 1 = 0$

(v) $\log_a a = 1$

(vi) $\log_a m = \left(\frac{\log m}{\log a}\right) = \left(\frac{\log_b m}{\log_b a}\right)$

(vii) $\log_a m = \frac{1}{\log_b a}$

COMMON LOGARITHMS

$\log_{10} a$ is called common logarithm, written as $\log a$.

EXAMPLES *(i)* $\log 10 = 1$ *(ii)* $\log 100 = \log (10)^2 = 2$

(iii) $\log\left(\frac{1}{10}\right) = \log (10)^{-1} = -1$, *etc.*

Standard Form of a Decimal

Any number in decimal form expressed as the product of a number between 1 and 10, and an integral power of 10, is in standard form.

	Number	Standard form
(i)	27.3	2.73×10^1
(ii)	106.34	1.0634×10^2
(iii)	0.49	4.9×10^{-1}
(iv)	0.08	8×10^{-2}
(v)	0.00037	3.7×10^{-4}

CHARACTERISTIC AND MANTISSA OF A LOGARITHM

log *m consists of two parts:*

(i) the integral part is called *characteristic*,

(ii) the decimal part is called *mantissa.*

Mantissa is always taken as positive, while the characteristic may be positive or negative.

Instead of –2, we write, $\bar{2}$.

REMARK $\bar{2}.7384$ means $(-2 + 0.7384)$.

CHARACTERISTIC OF log *m*

(i) If $m \geq 1$, then its characteristic is one less than the number of digits to the left of the decimal point.

(ii) If $m < 1$, then its characteristic is a negative number whose numerical value is one more than the number of zeros between the decimal point and the first nonzero figure.

ANOTHER METHOD Write the number in standard form: $m \times 10^p$. Then, its characteristic $= p$.

EXAMPLES

Number	Standard form	Characteristic
5376.4	5.3764×10^3	3
53.764	5.3764×10^1	1
5.3764	5.3764×10^0	0
0.53764	5.3764×10^{-1}	–1
0.00537	5.37×10^{-3}	–3

MANTISSA OF log *m*

We find mantissa from log table.

Position of the decimal point in a number is immaterial for mantissa. Mantissa of log (56), log (.56), log (.0056) is the same.

For finding mantissa of 4385 from log table, we proceed in the row headed by 43 and in this row, we find the number under the column headed by 8. Now, to this number, we add the mean difference headed by 5 and in the same row headed by 43.

Thus, mantissa for 4385 is (6415 + 5) = .6420.

REMARK 1 For finding the mantissa of 43 from log table, we find the number in the row headed by 43 under the column 0. It is 0.6335.

REMARK 2 For finding the mantissa of 4 from log table, we find the number in the row headed by 40 and under the column 0. It is 0.6021.

Thus, we have:

(i) $\log 4385 = 3.6420$ (ii) $\log 43.85 = 1.6420$

(iii) $\log 4.385 = 0.6420$ (iv) $\log 0.4385 = \bar{1}.6420$

(v) $\log 0.04385 = \bar{2}.6420$, etc.

ANTILOGARITHM If $\log m = n$, then $m = \text{antilog}\, n$.

EXAMPLE $\log 1000 = 3 \Rightarrow \text{antilog}\; 3 = 1000$.

Finding Antilogarithm of a Number

For finding the antilog of a number, we use its decimal part and read the antilog table in the same manner as log table.

After finding the number from antilog table, we insert the decimal point as under.

(i) When the characteristic is n, the decimal point is inserted after $(n+1)$th digit.

(ii) When the characteristic is $\overline{n}$, the decimal point is inserted in such a way that the first significant digit is at the nth place.

EXAMPLES

Number	Antilog
0.2364	1.724
1.4216	26.40
2.6019	399.08
5.2612	182500

Number	Antilog
$\overline{1}.4356$	0.2727
$\overline{2}.0318$	0.01076
$\overline{3}.1459$	0.001399
$\overline{4}.7912$	0.0006183

SOME EXAMPLES

EXAMPLE 1 *Find the value of:*

(i) $\log 3$ *(ii)* $\log 23.45$ *(iii)* $\log 4.17$

(iv) $\log 0.325$ *(v)* $\log 0.0954$ *(vi)* $\log 0.00056$

SOLUTION Using log table, we have

(i) $\log 3 = 0.4771$

(ii) $\log 23.45 = 1.3701$

(iii) $\log 4.17 = 0.6201$

(iv) $\log 0.325 = \overline{1}.5119$

(v) $\log 0.0954 = \overline{2}.9795$

(vi) $\log 0.00056 = \overline{4}.7482$

EXAMPLE 2 *Find the value of:*

(i) antilog 0.654 *(ii) antilog* 1.204

(iii) antilog $\overline{1}.3612$ *(iv) antilog* $\overline{3}.4568$

SOLUTION We have

(i) $\text{antilog}(0.654) = 4.508$

(ii) $\text{antilog}(1.204) = 16.00$

(iii) $\text{antilog}(\overline{1}.3612) = 0.2297$

(iv) $\text{antilog}(\overline{3}.4568) = 0.002863$

EXAMPLE 3 *Evaluate* $\dfrac{8492 \times 3.72}{47.8 \times 52.24}$.

SOLUTION Let $x = \dfrac{8492 \times 3.72}{47.8 \times 52.24}$. Then,

$$\log x = \log 8492 + \log 3.72 - \log 47.8 - \log 52.24$$
$$= 3.9290 + 0.5705 - 1.6794 - 1.7180$$
$$= 4.4995 - 3.3974 = 1.1021$$
$$\Rightarrow \quad x = \text{antilog}\,(1.1021) = 12.65.$$

Hence, the value of the given expression is 12.65.

EXAMPLE 4 *Evaluate* $\dfrac{563.4 \times \sqrt[3]{0.4573}}{(6.15)^3}$.

SOLUTION Let $x = \dfrac{563.4 \times \sqrt[3]{0.4573}}{(6.15)^3}$. Then,

$$\log x = \log 563.4 + \frac{1}{3}\log(0.4573) - 3\log(6.15)$$
$$= 2.7508 + \frac{1}{3}(\bar{1}.6602) - 3 \times 0.7889$$
$$= 2.7508 + \frac{1}{3}(-0.3398) - 2.3667$$
$$= 2.7508 - 0.1133 - 2.3667$$
$$= 0.2708$$
$$\Rightarrow \quad x = \text{antilog}\,(0.2708) = 1.865.$$

Hence, the value of the given expression is 1.865.

EXAMPLE 5 *Evaluate* $\sqrt[3]{0.08034}$.

SOLUTION Let $x = \sqrt[3]{0.08034}$. Then,

$$\log x = \frac{1}{3}\log(0.08034).$$
$$= \frac{1}{3}(\bar{2}.9049) = \frac{1}{3}(-2 + 0.9049) = \frac{1}{3}(-1.0951)$$
$$= -0.3650 = -1 + (1 - 0.3650) = \bar{1}.6350$$
$$\Rightarrow \quad x = \text{antilog}\,(\bar{1}.6350) = 0.4315.$$

EXAMPLE 6 *Evaluate* $\sqrt[7]{0.00003587}$.

SOLUTION Let $x = \sqrt[7]{0.00003587}$. Then,

$$\log x = \frac{1}{7}\log(0.00003587)$$
$$= \frac{1}{7}(\bar{5}.5548) = \frac{1}{7}(-5 + 0.5548)$$
$$= \frac{1}{7}(-4.4452) = -0.6350$$
$$= -1 + (1 - 0.6350) = \bar{1}.3650$$

$\Rightarrow \quad x = \text{antilog}\,(\bar{1}.3650) = 0.2317.$

Hence, the required value is 0.2317.

EXAMPLE 7 *Evaluate* $\sqrt{\dfrac{(76.24)^5 \times \sqrt[3]{65}}{(3.2)^7 \times \sqrt{17}}}\,.$

SOLUTION Let $x = \sqrt{\dfrac{(76.24)^5 \times \sqrt[3]{65}}{(3.2)^7 \times \sqrt{17}}}\cdot$ Then,

$$\log x = \frac{1}{2}\cdot \log \left\{\frac{(76.24)^5 \times (65)^{\frac{1}{3}}}{(3.2)^7 \times (17)^{\frac{1}{2}}}\right\}$$

$$= \frac{1}{2}\cdot\left\{5\log(76.24) + \frac{1}{3}\log 65 - 7\log(3.2) - \frac{1}{2}\log 17\right\}$$

$$= \frac{1}{2}\cdot\left\{5 \times 1.8822 + \frac{1}{3}\times 1.8129 - 7 \times 0.5051 - \frac{1}{2}\times 1.2304\right\}$$

$$= \frac{1}{2}\cdot\{9.4110 + 0.6043 - 3.5357 - 0.6152\}$$

$$= \frac{1}{2}\times(10.0153 - 4.1509) = \frac{1}{2}\times 5.8644 = 2.9322$$

$\Rightarrow \quad x = \text{antilog}\,(2.9322) = 855.5.$

Hence, the required value of the given expression is 855.5.

EXERCISE 1

Using log table evaluate the following:

1. $69.13 \times 0.34 \times 0.014$

2. 0.7625×0.000357

3. $358.6 \times 0.078 \times 0.5943$

4. $\dfrac{8.67 \times 99}{1.78}$

5. $(0.09634)^3$

6. $\sqrt[5]{8.0125}$

7. $\sqrt[7]{142.71}$

8. $\dfrac{0.9876 \times (16.42)^2}{(4.567)^{\frac{1}{3}}}$

9. $\dfrac{(6.45)^3 \times (0.00034)^{\frac{1}{3}} \times (981.4)}{(9.37)^2 \times (8.93)^{\frac{1}{4}} \times (0.0617)}$

ANSWERS (EXERCISE 1)

1. 0.3291 **2.** 0.0002723 **3.** 16.63 **4.** 19.19

5. 0.0008941 **6.** 7.517 **7.** 2.032 **8.** 160.4

9. 1963

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Appendix Mathematical Modelling

WHAT IS MATHEMATICAL MODELLING?

Mathematical modelling consists of translating a real-world problem into a mathematical problem, solving it and interpreting the solution in the language of real world.

There are four steps involved in mathematical modelling.

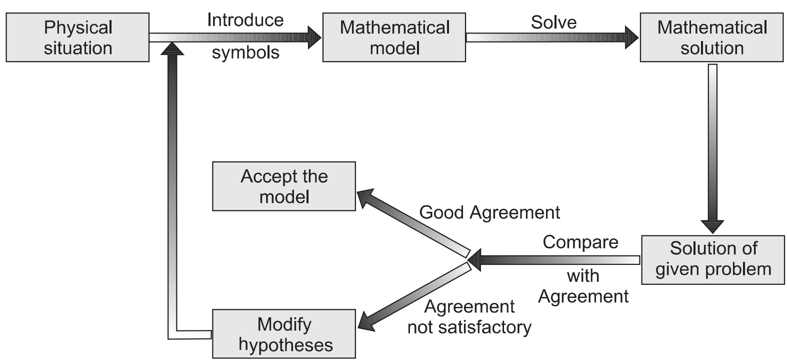
Step 1 *Convert the given problem into a mathematical problem.*

Step 2 *Solve the mathematical problem.*

Step 3 *Interpret the result for the real situation.*

Step 4 *If need arises, modify the model.*

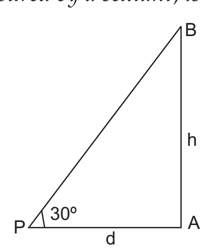
The technique of modelling can be expressed through the diagram given below.



Technique of Modelling

SOLVED EXAMPLES

EXAMPLE 1 *The angle of elevation of the top of a tower from a point P on the ground, which is* 600 m *away from the foot of the tower, measured by a sextant, is* $30°$. *Find the height of the tower.*



SOLUTION **Step 1.** First we try to understand the problem.

AB is a tower. Let its height be h metres.

Then, $AB = h$ metres.

Let P be the position of the observer.

Then, $PA = 600$ m.

Let B be the point on the top of the tower. Then, $\angle APB = 30°$.

The real problem is to find the value of h in metres.

Step 2. The three quantities involved in the problem are height $AB = h$ metres, distance $PA = 600$ m and $\angle APB = 30°$

We use the formula: $\tan\theta = \dfrac{h}{d} \Leftrightarrow h = d\tan\theta$.

Step 3. In equation (i), we put $d = 600$ m and $\theta = 30°$.

$$\therefore\ h = 600 \text{ m} \times \tan 30° = \left(600 \times \frac{1}{\sqrt{3}}\right) \text{m} = \left(600 \times \frac{1}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}}\right) \text{m}$$

$$= (200 \times \sqrt{3}) \text{ m} = (200 \times 1.732) \text{ m} = 346.4 \text{ m}.$$

Step 4. Hence, the height of the tower is 346.4 m.

OUR PROCEEDINGS

Step 1. Studied the real-life problem and identified three parameters: height $= h$ m, distance $AP = d$ m and $\angle APB = \theta°$.

Step 2. Using trigonometry, we applied the formula

$$h = d\tan\theta.$$

Step 3. Putting the known values of d and θ, we find h.

This gives us the solution of the problem.

EXAMPLE 2 *Suppose the present population of a city is* 100000 *and we want to find its population, say after* 10 *years.*

FORMULATION Let $P(t)$ be the population in a certain year t.

Let $B(t)$ be the number of births and $D(t)$ be the number of deaths between the years t and $t+1$. Then,

$$P(t+1) = P(t) + B(t) - D(t). \quad \ldots \text{(i)}$$

Let $\dfrac{B(t)}{P(t)} = b$ and $\dfrac{D(t)}{P(t)} = d$. Then,

$$P(t+1) = P(t) + b\,P(t) - d\,P(t)$$

$$\Rightarrow\ P(t+1) = (1+b-d)\,P(t). \quad \ldots \text{(ii)}$$

Putting $t = 0$ in (ii), we get $P(1) = (1+b-d)\,P(0)$. ... (iii)

Putting $t = 1$ in (ii), we get $P(2) = (1+b-d)\,P(1)$

$$= (1+b-d)^2 P(0) \quad \text{[using (iii)]}.$$

Thus, $P(2) = (1+b-d)^2 P(0)$.

Continuing in this way, we get:

$$P(t) = (1+b-d)^t P(0) \text{ for } t = 0, 1, 2, \ldots$$

$$\Rightarrow\ P(t) = P(0) \times r^t, \text{ where } (1+b-d) = r.$$

SOLUTION OF GIVEN PROBLEM

Suppose it is given that $P(0) = 100000$, $b = 0.02$ and $d = 0.01$.

Then, $P(10) = (1.01)^{10} \times 100000$ [let $(1.01)^{10} = 1.104622125$ be given]

$= (1.104622125 \times 100000) = 1104622.125.$

VALIDATION AND INTERPRETATION

Since we cannot have the number of persons in decimal fraction, the above result is not valid.

So, we take the population as 1104622 approximately.

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Complex Numbers

EXERCISE 1

Mark (✓) against the correct answer in each of the following.

1. $i^{91} = ?$

(a) –1 (b) 1 (c) i (d) $-i$

2. $i^{326} = ?$

(a) 1 (b) –1 (c) i (d) $-i$

3. $i^{273} = ?$

(a) i (b) $-i$ (c) 1 (d) –1

4. $i^{124} = ?$

(a) –1 (b) 1 (c) i (d) $-i$

5. $i^{-75} = ?$

(a) 1 (b) –1 (c) i (d) $-i$

6. $i^{-38} = ?$

(a) i (b) $-i$ (c) 1 (d) –1

7. For any positive integer n, $(-\sqrt{-1})^{4n+3} = ?$

(a) 1 (b) –1 (c) i (d) $-i$

8. If $i = \sqrt{-1}$ and n is a positive integer then $(i^n + i^{n+1} + i^{n+2} + i^{n+3}) = ?$

(a) i (b) i^n (c) 1 (d) 0

9. $(i^{109} + i^{114} + i^{119} + i^{124}) = ?$

(a) 0 (b) i (c) $-2i$ (d) 2

10. $3i^{34} + 5i^{27} - 2i^{38} + 5i^{41} = ?$

(a) 1 (b) –1 (c) $-4i$ (d) $10i$

11. If a and b are integers then $\sqrt{a} \times \sqrt{b} = \sqrt{ab}$ is true only when

(a) a and b are both positive

(b) a and b are both negative

(c) a and b are both zero

(d) at least one of a and b is non-negative

12. $\sqrt{-9} \times \sqrt{-25} = ?$

(a) 15 (b) –15 (c) $15i$ (d) $-15i$

13. Which of the following statements is correct?

(a) $(5+7i) > (3+4i)$ (b) $(5+7i) < (3+4i)$

(c) $(3+5i) > (4+3i)$ (d) none of these

14. Which of the following statements is correct?

(a) $3i > 2i$ (b) $1+3i > 2i$ (c) $i > 0$ (d) none of these

15. Which of the following statements is correct?

(a) $(2+3i) > (2-3i)$ (b) $(3+2i) > (-3+2i)$

(c) $(5+4i) > (-5-4i)$ (d) none of these

16. $\left(\frac{1-i}{1+i}\right)^2 = ?$

(a) 1 (b) –1 (c) $\frac{-1}{2}$ (d) $\frac{1}{\sqrt{2}}$

17. If $(a+ib) = \sqrt{\frac{1+i}{1-i}}$ then the value of (a^2+b^2) is

(a) 1 (b) –1 (c) 2 (d) –2

18. The smallest positive integer n for which $\left(\frac{1+i}{1-i}\right)^n = 1$ is

(a) 2 (b) 3 (c) 4 (d) 6

19. If $(a^2+b^2) = 1$ then $\frac{(1+b+ia)}{(1+b-ia)} = ?$

(a) $(a+ib)$ (b) $(b+ia)$ (c) $i(a+b)$ (d) none of these

20. $\left(\frac{1+2i}{1-i}\right)$ lies in

(a) quadrant I (b) quadrant II (c) quadrant III (d) quadrant IV

21. If $(x+iy) = \left(\frac{a+ib}{c+id}\right)$ then $(x^2+y^2) = ?$

(a) $\frac{(a^2+b^2)}{(c^2+d^2)}$ (b) $\frac{(a^2-b^2)}{(c^2+d^2)}$ (c) $\frac{(a^2+b^2)}{(c^2-d^2)}$ (d) none of these

22. $(1+i)^{-1} = ?$

(a) $(2-i)$ (b) $\left(\frac{-1}{2}+\frac{1}{2}i\right)$ (c) $\left(\frac{1}{2}-\frac{1}{2}i\right)$ (d) none of these

23. $(1-2i)^{-2} = ?$

(a) $\left(\frac{3}{25}-\frac{4}{25}i\right)$ (b) $\left(\frac{-3}{25}+\frac{4}{25}i\right)$ (c) $\left(\frac{-3}{25}-\frac{4}{25}i\right)$ (d) none of these

24. $(1-i)^{-3}=?$

(a) $\left(\frac{1}{4}-\frac{1}{4}i\right)$ (b) $\left(\frac{-1}{4}+\frac{1}{4}i\right)$ (c) $\left(\frac{-1}{4}-\frac{1}{4}i\right)$ (d) none of these

25. $(2-3i)(-3+4i)=?$

(a) $(6+17i)$ (b) $(6-17i)$ (c) $(-6+17i)$ (d) none of these

26. $(3-5i)\div(-2+3i)=?$

(a) $\left(\frac{21}{13}-\frac{1}{13}i\right)$ (b) $\left(\frac{-21}{13}+\frac{1}{13}i\right)$ (c) $\left(\frac{21}{13}+\frac{1}{13}i\right)$ (d) none of these

27. If $\left(\frac{2-\sqrt{-9}}{1-\sqrt{-4}}\right)=(x+iy)$ then

(a) $x=\frac{2}{5},\ y=\frac{3}{5}$ (b) $x=\frac{3}{5},\ y=\frac{2}{5}$

(c) $x=\frac{8}{5},\ y=\frac{1}{5}$ (d) none of these

28. $(1-\sqrt{-1})(1+\sqrt{-1})(5-\sqrt{-7})(5+\sqrt{-7})=?$

(a) $(25+7i)$ (b) $(32+5i)$ (c) $(29-3i)$ (d) none of these

29. The multiplicative inverse of $(-2+5i)$ is

(a) $\left(\frac{-2}{29}+\frac{5}{29}i\right)$ (b) $\left(\frac{2}{29}-\frac{5}{29}i\right)$ (c) $\left(\frac{-2}{29}-\frac{5}{29}i\right)$ (d) $\left(\frac{2}{29}+\frac{5}{29}i\right)$

30. The multiplicative inverse of $(3+2i)^2$ is

(a) $\left(\frac{-5}{169}+\frac{12}{169}i\right)$ (b) $\left(\frac{5}{169}-\frac{12}{169}i\right)$ (c) $\left(\frac{5}{169}+\frac{12}{169}i\right)$ (d) none of these

31. The complex number z such that $\left|\frac{z-i}{z+i}\right|=1$ lies on

(a) the x-axis (b) the line $y=1$ (c) a circle (d) none of these

32. The complex number z such that $\left|\frac{z-5i}{z+5i}\right|=1$ lies on

(a) the x-axis (b) the y-axis (c) a circle (d) none of these

33. The complex number z such that $\left|\frac{z-2}{z+2}\right|=2$ lies on

(a) the x-axis (b) the y-axis (c) a circle (d) none of these

34. -2, when expressed in polar form, is

(a) $-2\left(\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}\right)$ (b) $2(\cos\pi+i\sin\pi)$

(c) $2(\cos 2\pi+i\sin 2\pi)$ (d) none of these

35. $-3i$, when expressed in polar form, is

(a) $3\left[\cos\left(-\frac{\pi}{2}\right)+i\sin\left(-\frac{\pi}{2}\right)\right]$ (b) $3\left[\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}\right]$

(c) $-3\left[\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}\right]$ (d) none of these

36. $(1+i)$, when expressed in polar form, is

(a) $2\left[\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}\right]$ (b) $2\left[\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right]$

(c) $\sqrt{2}\left[\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right]$ (d) none of these

37. $(-1+i\sqrt{3})$, when expressed in polar form, is

(a) $2(\cos\pi+i\sin\pi)$ (b) $2\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)$

(c) $2\left(\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3}\right)$ (d) $2\left[\cos\left(\frac{-2\pi}{3}\right)+i\sin\left(\frac{-2\pi}{3}\right)\right]$

38. $(\sqrt{3}+i)$, when expressed in polar form, is

(a) $2\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)$ (b) $2\left(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}\right)$

(c) $2\left(\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3}\right)$ (d) none of these

39. $(1+i)$, when expressed in polar form, is

(a) $\sqrt{2}\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)$ (b) $2\left(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}\right)$

(c) $2\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)$ (d) none of these

40. $(-1-i)$, when expressed in polar form, is

(a) $\sqrt{2}\left\{\cos\left(\frac{-\pi}{4}\right)+i\sin\left(\frac{-\pi}{4}\right)\right\}$ (b) $\sqrt{2}\left\{\cos\left(\frac{-3\pi}{4}\right)+i\sin\left(\frac{-3\pi}{4}\right)\right\}$

(c) $\sqrt{2}\left\{\cos\left(\frac{-2\pi}{3}\right)+i\sin\left(\frac{-2\pi}{3}\right)\right\}$ (d) $\sqrt{2}\left\{\cos\left(\frac{-\pi}{2}\right)+i\sin\left(\frac{-\pi}{2}\right)\right\}$

41. $(-1-\sqrt{3}i)$, when expressed in polar form, is

(a) $2\left[\cos\left(\frac{-\pi}{3}\right)+i\sin\left(\frac{-\pi}{3}\right)\right]$ (b) $2\left[\cos\left(\frac{-2\pi}{3}\right)+i\sin\left(\frac{-2\pi}{3}\right)\right]$

(c) $2\left[\cos\left(\frac{-3\pi}{4}\right)+i\sin\left(\frac{-3\pi}{4}\right)\right]$ (d) none of these

42. $(1-i)$, when expressed in polar form, is

(a) $\sqrt{2}\left(\cos\frac{\pi}{4}-i\sin\frac{\pi}{4}\right)$ (b) $\sqrt{2}\left[\cos\left(\frac{-\pi}{4}\right)+i\sin\left(\frac{-\pi}{4}\right)\right]$

(c) $\sqrt{2}\left(\cos\frac{3\pi}{4}+i\sin\frac{3\pi}{4}\right)$ (d) none of these

43. $\dfrac{(5-i)}{(2-3i)}$, when expressed in polar form, is

(a) $2\left[\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right]$ (b) $3\left[\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3}\right]$

(c) $\sqrt{2}\left[\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right]$ (d) $2\left[\cos\frac{3\pi}{4}+i\sin\frac{3\pi}{4}\right]$

44. $\dfrac{(1-3i)}{(1+2i)}$, when expressed in polar form, is

(a) $\sqrt{2}\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)$ (b) $\sqrt{2}\left(\cos\frac{5\pi}{4}+i\sin\frac{5\pi}{4}\right)$

(c) $\sqrt{2}\left(\cos\frac{3\pi}{4}+i\sin\frac{3\pi}{4}\right)$ (d) none of these

45. $(-4+4\sqrt{3}i)$, when expressed in polar form, is

(a) $4\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)$ (b) $8\left(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}\right)$

(c) $8\left(\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3}\right)$ (d) none of these

46. $(\sin 30°+i\cos 30°)$, when expressed in polar form, is

(a) $(\cos 60°+i\sin 60°)$ (b) $(\cos 60°-i\sin 60°)$

(c) $(\cos 150°+i\sin 150°)$ (d) none of these

47. $(\sin 120°-i\cos 120°)$, when expressed in polar form, is

(a) $(\cos 60°+i\sin 60°)$ (b) $(\cos 30°+i\sin 30°)$

(c) $(\cos 150°+i\sin 150°)$ (d) none of these

48. $3(\cos 300°-i\sin 30°)$, when expressed in polar form, is

(a) $\frac{3}{\sqrt{2}}\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)$ (b) $\frac{3}{\sqrt{2}}\left[\cos\left(\frac{-\pi}{4}\right)+i\sin\left(\frac{-\pi}{4}\right)\right]$

(c) $\frac{3}{\sqrt{2}}\left(\cos\frac{3\pi}{4}+i\sin\frac{3\pi}{4}\right)$ (d) none of these

49. If $z=(2+\sqrt{-5})$ then $|z|=?$

(a) 9 (b) 7 (c) 3 (d) none of these

50. If $z=(3i-1)^2$ then $|z|=?$

(a) 8 (b) 10 (c) 4 (d) $\sqrt{10}$

51. If $z = \dfrac{(1 - i\sqrt{3})}{2(1 - i)}$ then $|z| = ?$

(a) $\dfrac{1}{\sqrt{2}}$ (b) $\dfrac{1}{2\sqrt{2}}$ (c) $\dfrac{1}{\sqrt{3}}$ (d) none of these

52. If $z = (3 + \sqrt{2}i)$ then $z\bar{z} = ?$

(a) 5 (b) 7 (c) 11 (d) $\sqrt{11}$

53. $\arg(1 + i) = ?$

(a) π (b) $\dfrac{\pi}{2}$ (c) $\dfrac{\pi}{3}$ (d) $\dfrac{\pi}{4}$

54. $\arg(-1 + i\sqrt{3}) = ?$

(a) $\dfrac{\pi}{3}$ (b) $\dfrac{2\pi}{3}$ (c) π (d) none of these

55. $\arg\left(\dfrac{2 + 6\sqrt{3}i}{5 + \sqrt{3}i}\right) = ?$

(a) $\dfrac{\pi}{3}$ (b) $\dfrac{\pi}{4}$ (c) $\dfrac{2\pi}{3}$ (d) π

56. For any complex number z, $\arg(z) + \arg(\bar{z}) = ?$

(a) 0 (b) $\dfrac{\pi}{2}$ (c) π (d) 2π

57. If z_1 and z_2 be two complex numbers such that $|z_1 + z_2| = |z_1 - z_2|$ then $\text{amp}(z_1) - \text{amp}(z_2) = ?$

(a) $\dfrac{\pi}{4}$ (b) $\dfrac{\pi}{3}$ (c) $\dfrac{\pi}{2}$ (d) π

58. $\left\{\dfrac{1}{(1 - 2i)} + \dfrac{3}{(1 + i)}\right\}\left(\dfrac{3 + 4i}{2 - 4i}\right) = ?$

(a) $\left(\dfrac{3}{4} + \dfrac{9}{4}i\right)$ (b) $\left(\dfrac{3}{4} - \dfrac{5}{4}i\right)$ (c) $\left(\dfrac{1}{2} + \dfrac{3}{2}i\right)$ (d) $\left(\dfrac{1}{4} + \dfrac{9}{4}i\right)$

59. Compare List I with List II and choose the correct answer using codes given below:

List I (Complex number)	List II (Conjugate)
(i) $3 + \sqrt{-4}$	(p) $(-2 - 3i)$
(ii) $-3 - 2i$	(q) $(2 + 3i)$
(iii) $-3i - 2$	(r) $(3 - 2i)$
(iv) $-3i + 2$	(s) $(-3 + 2i)$

(a) (i)–(s), (ii)–(r), (iii)–(p), (iv)–(q) (b) (i)–(r), (ii)–(s), (iii)–(q), (iv)–(p)
(c) (i)–(r), (ii)–(s), (iii)–(p), (iv)–(q) (d) (i)–(p), (ii)–(s), (iii)–(r), (iv)–(q)

60. Compare List I with List II and choose the correct answer using codes given below:

List I (Complex number)	List II (Its modulus)
(i) $(4-3i)$	(p) 10
(ii) $(8+6i)$	(q) $\frac{1}{5}$
(iii) $\frac{1}{(3+4i)}$	(r) 1
(iv) $\frac{(3-4i)}{(3+4i)}$	(s) 5

(a) (i)–(p), (ii)–(s), (iii)–(r), (iv)–(q) (b) (i)–(s), (ii)–(p), (iii)–(q), (iv)–(r)
(c) (i)–(s), (ii)–(p), (iii)–(r), (iv)–(q) (d) (i)–(r), (ii)–(p), (iii)–(s), (iv)–(q)

61. For any complex numbers z_1 and z_2, compare List I with List II and choose the correct answer, using codes given below:

List I	List II
(i) $\arg(z_1 z_2)$	(p) $\frac{\pi}{2}$
(ii) $\arg\left(\frac{z_1}{z_2}\right)$	(q) $\arg(z_1) - \arg(z_2)$
(iii) $\arg(z) + \arg(\bar{z})$	(r) $\arg(z_1) + \arg(z_2)$
(iv) $\arg(i)$	(s) 2π

(a) (i)–(q), (ii)–(r), (iii)–(s), (iv)–(p) (b) (i)–(r), (ii)–(q), (iii)–(p), (iv)–(s)
(c) (i)–(r), (ii)–(q), (iii)–(s), (iv)–(p) (d) none of these

62. The solution set of $x^2 + 2 = 0$ is
(a) $\{\sqrt{2}, -\sqrt{2}\}$ (b) $\{\sqrt{2}i, -\sqrt{2}\}$ (c) $\{\sqrt{2}, -\sqrt{2}i\}$ (d) $\{\sqrt{2}i, -\sqrt{2}i\}$

63. The solution set of $x^2 + 13 = 4x$ is
(a) $\{3+2i, 3-2i\}$ (b) $\{2+3i, 2-3i\}$ (c) $\{3i, -3i\}$ (d) none of these

64. The solution set of $x^2 + 2x + 2 = 0$ is
(a) $\{i, -i\}$ (b) $\{1+i, -i\}$ (c) $\{1+i, 1-i\}$ (d) $\{-1+i, -1-i\}$

ANSWERS (EXERCISE 1)

1. (d)	**2.** (b)	**3.** (a)	**4.** (b)	**5.** (c)	**6.** (d)	**7.** (c)	**8.** (d)
9. (a)	**10.** (b)	**11.** (d)	**12.** (b)	**13.** (d)	**14.** (d)	**15.** (d)	**16.** (b)
17. (a)	**18.** (c)	**19.** (b)	**20.** (b)	**21.** (a)	**22.** (c)	**23.** (b)	**24.** (b)
25. (a)	**26.** (b)	**27.** (c)	**28.** (d)	**29.** (c)	**30.** (b)	**31.** (a)	**32.** (a)
33. (c)	**34.** (b)	**35.** (a)	**36.** (c)	**37.** (c)	**38.** (b)	**39.** (a)	**40.** (b)

41. (b) **42.** (b) **43.** (c) **44.** (b) **45.** (c) **46.** (a) **47.** (b) **48.** (b)
49. (c) **50.** (b) **51.** (a) **52.** (c) **53.** (d) **54.** (b) **55.** (a) **56.** (d)
57. (c) **58.** (d) **59.** (c) **60.** (b) **61.** (c) **62.** (d) **63.** (b) **64.** (d)

HINTS TO SOME SELECTED QUESTIONS

1. $i^{91} = (i^4)^{22} \times i^3 = 1 \times i^2 \times i = 1 \times (-1) \times i = -i.$

2. $i^{326} = (i^4)^{81} \times i^2 = (1)^{81} \times (-1) = 1 \times (-1) = -1.$

3. $i^{273} = (i^4)^{68} \times i = (1)^{68} \times i = (1 \times i) = i.$

4. $i^{124} = (i^4)^{31} = (1)^{31} = 1.$

5. $i^{-75} = \dfrac{1}{i^{75}} = \dfrac{1}{i^{75} \times i} \times i = \dfrac{1}{i^{76}} \times i = \dfrac{1}{(i^4)^{19}} \times i = 1 \times i = i.$

6. $i^{-38} = \dfrac{1}{i^{38}} \times \dfrac{i^2}{i^2} = \dfrac{-1}{i^{40}} = \dfrac{-1}{(i^4)^{10}} = \dfrac{-1}{(1)^{10}} = \dfrac{-1}{1} = -1.$

7. $(-\sqrt{-1})^{4n+3} = (-i)^{4n+3} = \{(-i)^4\}^n \cdot (-i)^3 = 1 \times (-i) \times (-i) \times (-i) = i^2 \times (-i) = -1 \times (-i) = i.$

8. $(i^n + i^{n+1} + i^{n+2} + i^{n+3}) = i^n(1 + i + i^2 + i^3) = i^n(1 + i - 1 - i) = i^n \times 0 = 0.$

9. $i^{109} + i^{114} + i^{119} + i^{124} = i^{109}(1 + i^5 + i^{10} + i^{15})$

$= i^{109} \times [1 + i^4 \times i + (i^4)^2 \times i^2 + (i^4)^3 \times i^3] = i^{109}[1 + i + i^2 + i^3]$

$= i^{109} \times [1 + i - 1 - i] = i^{109} \times 0 = 0.$

10. $3i^{34} + 5i^{27} - 2i^{38} + 5i^{41} = 3 \times (i^4)^8 \times i^2 + 5 \times (i^4)^6 \times i^3 - 2 \times (i^4)^9 \times i^2 + 5 \times (i^4)^{10} \times i$

$= 3 \times 1 \times (-1) + 5 \times 1 \times (-i) - 2 \times 1 \times (-1) + 5 \times 1 \times i$

$= -3 - 5i + 2 + 5i = -1.$

11. $\sqrt{a} \times \sqrt{b} = \sqrt{ab}$ is true only when at least one of a and b is non-negative.

12. $\sqrt{-9} \times \sqrt{-25} = (3i) \times (5i) = (15 \times i^2) = [15 \times (-1)] = -15.$

13. Two complex numbers cannot be compared.

14. Two complex numbers cannot be compared.

15. Two complex numbers cannot be compared.

16. $\dfrac{(1-i)}{(1+i)} = \dfrac{(1-i)}{(1+i)} \times \dfrac{(1-i)}{(1-i)} = \dfrac{(1-i)^2}{(1-i^2)} = \dfrac{1 + i^2 - 2i}{(1+1)} = \dfrac{1 - 1 - 2i}{2} = \dfrac{-2i}{2} = -i$

$\Rightarrow \quad \left(\dfrac{1-i}{1+i}\right)^2 = (-i)^2 = i^2 = -1.$

17. $(a + ib) = \dfrac{\sqrt{1+i}}{\sqrt{1-i}} \Rightarrow (a - ib) = \dfrac{\sqrt{1-i}}{\sqrt{1+i}}$

$\Rightarrow \quad (a + ib)(a - ib) = \dfrac{\sqrt{1+i}}{\sqrt{1-i}} \times \dfrac{\sqrt{1-i}}{\sqrt{1+i}} = 1 \Rightarrow (a^2 + b^2) = 1.$

18. $\dfrac{(1+i)}{(1-i)} = \dfrac{(1+i)}{(1-i)} \times \dfrac{(1+i)}{(1+i)} = \dfrac{(1+i)^2}{(1-i^2)} = \dfrac{1 + i^2 + 2i}{(1+1)} = \dfrac{1 - 1 + 2i}{2} = \dfrac{2i}{2} = i.$

Clearly, 4 is the smallest positive integer for which $i^4 = 1$.

$\therefore \quad n = 4.$

19. $\dfrac{(1+b+ia)}{(1+b-ia)}=\dfrac{(1+b+ia)}{(1+b-ia)}\times\dfrac{(1+b+ia)}{(1+b+ia)}=\dfrac{(1+b+ia)^2}{(1+b)^2-(ia)^2}$

$$=\frac{1+b^2-a^2+2b+2iab+2ia}{1+b^2+2b+a^2}=\frac{2b^2+2b+2iab+2ia}{2b+2} \qquad [\because\ a^2+b^2=1]$$

$$=\frac{2b(b+1)+2ia(b+1)}{2(b+1)}=\frac{2(b+1)(b+ia)}{2(b+1)}=(b+ia).$$

20. $\dfrac{(1+2i)}{(1-i)}=\dfrac{(1+2i)}{(1-i)}\times\dfrac{(1+i)}{(1+i)}=\dfrac{(1+2i)(1+i)}{(1^2-i^2)}=\dfrac{1+3i+2i^2}{(1+1)}$

$$=\frac{1-2+3i}{2}=\frac{-1+3i}{2}=\left(\frac{-1}{2}+\frac{3}{2}i\right).$$

Clearly, $\left(\dfrac{-1}{2},\dfrac{3}{2}\right)$ lies in quadrant II.

21. $(x+iy)=\dfrac{(a+ib)}{(a+id)} \Rightarrow |x+iy|=\left|\dfrac{a+ib}{c+id}\right|=\dfrac{|a+ib|}{|c+id|}$

$$\Rightarrow |x+iy|^2=\frac{|a+ib|^2}{|c+id|^2} \Rightarrow (x^2+y^2)=\frac{(a^2+b^2)}{(c^2+d^2)}.$$

22. $(1+i)^{-1}=\dfrac{1}{(1+i)}=\dfrac{1}{(1+i)}\times\dfrac{(1-i)}{(1-i)}=\dfrac{(1-i)}{(1^2-i^2)}=\dfrac{(1-i)}{2}=\left(\dfrac{1}{2}-\dfrac{1}{2}i\right).$

23. $(1-2i)^{-2}=\dfrac{1}{(1-2i)^2}=\dfrac{1}{(1+4i^2-4i)}=\dfrac{1}{(1-4-4i)}=\dfrac{1}{(-3-4i)}\times\dfrac{(-3+4i)}{(-3+4i)}$

$$=\frac{(-3+4i)}{(9-16i^2)}=\frac{(-3+4i)}{25}=\left(\frac{-3}{25}+\frac{4}{25}i\right).$$

24. $(1-i)^{-3}=\dfrac{1}{(1-i)^3}=\dfrac{1}{1-i^3-3i(1-i)}=\dfrac{1}{1+i-3i-3}$

$$=\frac{1}{(-2-2i)}\times\frac{(-2+2i)}{(-2+2i)}=\frac{(-2+2i)}{4-4i^2}=\frac{(-2+2i)}{8}=\left(\frac{-1}{4}+\frac{1}{4}i\right).$$

25. $(2-3i)(-3+4i)=(-6+8i+9i-12i^2)=(-6+17i+12)=(6+17i).$

26. $\dfrac{(3-5i)}{(-2+3i)}=\dfrac{(3-5i)}{(-2+3i)}\times\dfrac{(-2-3i)}{(-2-3i)}=\dfrac{(3-5i)(-2-3i)}{(-2)^2-(3i)^2}$

$$=\frac{-6-9i+10i+15i^2}{(4-9i^2)}=\frac{-6+i-15}{(4+9)}=\frac{(-21+i)}{13}=\left(\frac{-21}{13}+\frac{1}{13}i\right).$$

27. $\left(\dfrac{2-\sqrt{-9}}{1-\sqrt{-4}}\right)=\left(\dfrac{2-3i}{1-2i}\right)=\dfrac{(2-3i)}{(1-2i)}\times\dfrac{(1+2i)}{(1+2i)}=\dfrac{(2-3i)(1+2i)}{(1-2i)(1+2i)}=\dfrac{(2+4i-3i-6i^2)}{(1-4i^2)}$

$$=\frac{(2+i+6)}{(1+4)}=\frac{(8+i)}{5}=\left(\frac{8}{5}+\frac{1}{5}i\right).$$

28. Given expression $=(1-i)(1+i)(5-\sqrt{7}i)(5+\sqrt{7}i)$

$$=(1-i^2)(25-7i^2)=(1+1)(25+7)=(2\times 32)=64.$$

29. $z = (-2+5i) \Rightarrow z^{-1} = \frac{1}{z} = \frac{1}{(-2+5i)} \times \frac{(-2-5i)}{(-2-5i)} = \frac{(-2-5i)}{(4-25i^2)} = \frac{(-2-5i)}{(4+25)} = \frac{(-2-5i)}{29}$

$$\Rightarrow z^{-1} = \left(\frac{-2}{29} - \frac{5}{29}i\right).$$

30. $z = (3+2i)^2 = (9+4i^2+12i) = (9-4+12i) = (5+12i)$

$$\Rightarrow \quad z^{-1} = \frac{1}{(5+12i)} \times \frac{(5-12i)}{(5-12i)} = \frac{(5-12i)}{(25-144i^2)} = \frac{(5-12i)}{(25+144)} = \frac{(5-12i)}{(169)}$$

$$\Rightarrow \quad z^{-1} = \left(\frac{5}{169} - \frac{12}{169}i\right).$$

31. $\left|\frac{z-i}{z+i}\right| = 1 \Rightarrow \left|\frac{z-i}{z+i}\right|^2 = 1$

$$\Rightarrow \quad \left|\frac{x+iy-i}{x+iy+i}\right|^2 = 1 \Rightarrow \left|\frac{x+i(y-1)}{x+i(y+1)}\right|^2 = 1 \Rightarrow \frac{|x+i(y-1)|^2}{|x+i(y+1)|^2} = 1$$

$$\Rightarrow \quad \frac{x^2+(y-1)^2}{x^2+(y+1)^2} = 1 \Rightarrow x^2+(y-1)^2 = x^2+(y+1)^2$$

$$\Rightarrow \quad (y+1)^2-(y-1)^2 = 0 \Rightarrow 4y = 0 \Rightarrow y = 0$$

$\Rightarrow$ z lies on the x-axis.

32. $\left|\frac{z-5i}{z+5i}\right| = 1 \Rightarrow \left|\frac{z-5i}{z+5i}\right|^2 = 1 \Rightarrow \frac{|z-5i|^2}{|z+5i|^2} = 1$

$$\Rightarrow \quad |z-5i|^2 = |z+5i|^2 \Rightarrow |x+iy-5i|^2 = |x+iy+5i|^2$$

$$\Rightarrow \quad |x+i(y-5)|^2 = |x+i(y+5)|^2 \Rightarrow x^2+(y-5)^2 = x^2+(y+5)^2$$

$$\Rightarrow \quad (y+5)^2-(y-5)^2 = 0 \Rightarrow 20y = 0 \Rightarrow y = 0$$

$\Rightarrow$ z lies on the x-axis.

33. $\left|\frac{z-2}{z+2}\right| = 2 \Rightarrow \left|\frac{z-2}{z+2}\right|^2 = 4 \Rightarrow \frac{|z-2|^2}{|z+2|^2} = 4$

$$\Rightarrow \quad |x+iy-2|^2 = 4|x+iy+2|^2 \Rightarrow (x-2)^2+y^2 = 4\{(x+2)^2+y^2\}$$

$$\Rightarrow \quad x^2-4x+4+y^2 = 4\{x^2+4x+4+y^2\}$$

$\Rightarrow$ $3(x^2+y^2)+20x+12 = 0$, which is a circle.

34. Let $z = -2+0\cdot i$ and let $z = r(\cos\theta + i\sin\theta)$.

Then, $r\cos\theta = -2$ and $r\sin\theta = 0$.

Squaring and adding, we get $r^2 = 4 \Rightarrow r = 2$.

$\therefore$ $[\cos\theta = -1$ and $\sin\theta = 0] \Rightarrow \theta = \pi$.

$\therefore$ $-2 = 2[\cos\pi + i\sin\pi]$.

35. Let $z = 0-3i$ and let $z = r(\cos\theta + i\sin\theta)$.

Then, $r\cos\theta = 0$ and $r\sin\theta = -3$.

Squaring and adding, we get $r^2 = 9 \Rightarrow r = 3$.

$\therefore \quad [\cos\theta = 0 \text{ and } \sin\theta = -1] \Rightarrow \theta = -\frac{\pi}{2}\cdot$

$\therefore \quad -3i = 3\left[\cos\left(\frac{-\pi}{2}\right) + i\sin\left(\frac{-\pi}{2}\right)\right]\cdot$

36. Let $z = (1+i)$ and let $z = r(\cos\theta + i\sin\theta)$.

Then, $r\cos\theta = 1$ and $r\sin\theta = 1$.

Squaring and adding, we get $r^2 = 2 \Rightarrow r = \sqrt{2}$.

$\therefore \quad \left\{\cos\theta = \frac{1}{\sqrt{2}} \text{ and } \sin\theta = \frac{1}{\sqrt{2}}\right\} \Rightarrow \theta = \frac{\pi}{4}\cdot$

$\therefore \quad (1+i) = \sqrt{2}\left[\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right]\cdot$

37. Let $z = (-1 + i\sqrt{3})$ and let $z = r(\cos\theta + i\sin\theta)$.

Then, $r\cos\theta = -1$ and $r\sin\theta = \sqrt{3}$.

Squaring and adding, we get $r^2 = 4 \Rightarrow r = 2$.

$\therefore \quad \left[\cos\theta = \frac{-1}{2} \text{ and } \sin\theta = \frac{\sqrt{3}}{2}\right] \Rightarrow \theta = \frac{2\pi}{3}\cdot$

$\therefore \quad (-1 + i\sqrt{3}) = 2\left[\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right]\cdot$

38. Let $z = (\sqrt{3} + i)$ and let $z = r(\cos\theta + i\sin\theta)$.

Then, $r\cos\theta = \sqrt{3}$ and $r\sin\theta = 1$.

Squaring and adding, we get $r^2 = 4 \Rightarrow r = 2$.

$\therefore \quad \left[\cos\theta = \frac{\sqrt{3}}{2} \text{ and } \sin\theta = \frac{1}{2}\right] \Rightarrow \theta = \frac{\pi}{6}\cdot$

$\therefore \quad (\sqrt{3} + i) = 2\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)\cdot$

39. Let $z = (1+i)$ and let $z = r(\cos\theta + i\sin\theta)$.

Then, $r\cos\theta = 1$ and $r\sin\theta = 1$.

Squaring and adding, we get $r^2 = 2 \Rightarrow r = \sqrt{2}$.

$\therefore \quad \left[\cos\theta = \frac{1}{\sqrt{2}} \text{ and } \sin\theta = \frac{1}{\sqrt{2}}\right] \Rightarrow \theta = \frac{\pi}{4}\cdot$

$\therefore \quad (1+i) = 2\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)\cdot$

40. Let $z = (-1 - i)$ and let $z = r(\cos\theta + i\sin\theta)$.

Then, $r\cos\theta = -1$ and $r\sin\theta = -1$.

Squaring and adding, we get $r^2 = 2 \Rightarrow r = \sqrt{2}$.

$\therefore \quad \left[\cos\theta = \frac{-1}{\sqrt{2}} \text{ and } \sin\theta = \frac{-1}{\sqrt{2}}\right] \Rightarrow \theta = \frac{-3\pi}{4}\cdot$

$\therefore \quad (-1 - i) = \sqrt{2}\left\{\cos\left(\frac{-3\pi}{4}\right) + i\sin\left(\frac{-3\pi}{4}\right)\right\}\cdot$

46. $(\sin 30^\circ + i\cos 30^\circ) = \sin(90^\circ - 60^\circ) + i\cos(90^\circ - 60^\circ) = (\cos 60^\circ + i\sin 60^\circ)$.

47. $(\sin 120° - i\cos 120°) = \sin(90° + 30°) - i\cos(90° + 30°) = (\cos 30° + i\sin 30°)$.

48. $3(\cos 300° - i\sin 30°) = 3[\cos(360° - 60°) - i\sin 30°]$

$$= 3(\cos 60° - i\sin 30°) = 3\left(\frac{1}{2} - \frac{1}{2}i\right) = \frac{3}{2}(1-i)$$

$$= \frac{3}{2}\cdot\sqrt{2}\left(\cos\frac{\pi}{4} - i\sin\frac{\pi}{4}\right) = \frac{3}{\sqrt{2}}\left[\cos\left(\frac{-\pi}{4}\right) + i\sin\left(\frac{-\pi}{4}\right)\right].$$

49. $z = (2 + \sqrt{5}i) \Rightarrow |z|^2 = \{2^2 + (\sqrt{5})^2\} = (4+5) = 9 \Rightarrow |z| = 3.$

50. $z = (3i - 1)^2 = (9i^2 + 1 - 6i) = (-9 + 1 - 6i) = (-8 - 6i)$

$\Rightarrow \quad |z|^2 = \{(-8)^2 + (-6)^2\} = (64 + 36) = 100 \Rightarrow |z| = \sqrt{100} = 10.$

51. $|z|^2 = \dfrac{|1 - i\sqrt{3}|^2}{|(2-2i)|^2} = \dfrac{\{1^2 + (\sqrt{3})^2\}}{\{2^2 + 2^2\}} = \dfrac{(1+3)}{(4+4)} = \dfrac{4}{8} = \dfrac{1}{2} \Rightarrow |z| = \dfrac{1}{\sqrt{2}}.$

52. $z\bar{z} = |z|^2 = \{3^2 + (\sqrt{2})^2\} = (9+2) = 11.$

53. $(1+i) = \sqrt{2}\left(\cos\dfrac{\pi}{4} + i\sin\dfrac{\pi}{4}\right) \Rightarrow \arg(1+i) = \dfrac{\pi}{4}.$

54. $(-1 + i\sqrt{3}) = 2\left(\cos\dfrac{2\pi}{3} + i\sin\dfrac{2\pi}{3}\right) \Rightarrow \arg(-1 + i\sqrt{3}) = \dfrac{2\pi}{3}.$

55. $\dfrac{(2 + 6\sqrt{3}i)}{(5 + \sqrt{3}i)} = \dfrac{(2 + 6\sqrt{3}i)}{(5 + \sqrt{3}i)} \times \dfrac{(5 - \sqrt{3}i)}{(5 - \sqrt{3}i)} = \dfrac{(2 + 6\sqrt{3}i)(5 - \sqrt{3}i)}{(5 + \sqrt{3}i)(5 - \sqrt{3}i)}$

$$= \frac{(28 + 28\sqrt{3}i)}{28} = \frac{28(1 + \sqrt{3}i)}{28} = (1 + \sqrt{3}i) = 2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right).$$

$\therefore \quad \arg\left(\dfrac{2 + 6\sqrt{3}i}{5 + \sqrt{3}i}\right) = \dfrac{\pi}{3}.$

56. Let $z = r(\cos\theta + i\sin\theta)$. Then,

$\bar{z} = r(\cos\theta - i\sin\theta) = r[\cos(2\pi - \theta) + i\sin(2\pi - \theta)]$

$\therefore \quad \arg(z) + \arg(\bar{z}) = \theta + (2\pi - \theta) = 2\pi.$

57. Let $z_1 = (x_1 + iy_1)$ and $z_2 = (x_2 + iy_2)$. Then,

$(z_1 + z_2) = (x_1 + x_2) + i(y_1 + y_2)$ and $(z_1 - z_2) = (x_1 - x_2) + i(y_1 - y_2)$.

Now, $|z_1 + z_2| = |z_1 - z_2|$

$\Rightarrow \quad |z_1 + z_2|^2 = |z_1 - z_2|^2 \Rightarrow (x_1 + x_2)^2 + (y_1 + y_2)^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2$

$\Rightarrow \quad (x_1 + x_2)^2 - (x_1 - x_2)^2 + (y_1 + y_2)^2 - (y_1 - y_2)^2 = 0$

$\Rightarrow \quad 4(x_1x_2 + y_1y_2) = 0 \Rightarrow (x_1x_2 + y_1y_2) = 0.$

$\therefore \quad \text{amp}(z_1) - \text{amp}(z_2) = \tan^{-1}\dfrac{y_1}{x_1} - \tan^{-1}\dfrac{y_2}{x_2}$

$$= \tan^{-1}\left\{\frac{\left(\dfrac{y_1}{x_1} - \dfrac{y_2}{x_2}\right)}{1 + \left(\dfrac{y_1}{x_1}\cdot\dfrac{y_2}{x_2}\right)}\right\} = \tan^{-1}\left\{\frac{(x_2y_1 - x_1y_2)}{(x_1x_2 + y_1y_2)}\right\}$$

$$= \tan^{-1}\infty = \frac{\pi}{2} \qquad [\because\ x_1x_2 + y_1y_2 = 0].$$

58. $\dfrac{1}{(1-2i)}=\dfrac{1}{(1-2i)}\times\dfrac{(1+2i)}{(1+2i)}=\dfrac{(1+2i)}{(1-4i^2)}=\left(\dfrac{1}{5}+\dfrac{3}{5}i\right)$

$\dfrac{3}{(1+i)}=\dfrac{3}{(1+i)}\times\dfrac{(1-i)}{(1-i)}=\dfrac{(3-3i)}{(1-i^2)}=\dfrac{(3-3i)}{(1+1)}=\left(\dfrac{3}{2}-\dfrac{3}{2}i\right)$

$\dfrac{(3+4i)}{(2-4i)}=\dfrac{(3+4i)}{(2-4i)}\times\dfrac{(2+4i)}{(2+4i)}=\dfrac{-10+20i}{(4-16i^2)}=\dfrac{-10+20i}{(4+16)}=\left(\dfrac{-10}{20}+\dfrac{20}{20}i\right)=\left(\dfrac{-1}{2}+i\right)$.

$\therefore$ given expression $=\left\{\left(\dfrac{1}{5}+\dfrac{2}{5}i\right)+\left(\dfrac{3}{2}-\dfrac{3}{2}i\right)\right\}\left(\dfrac{-1}{2}+i\right)$

$=\left\{\left(\dfrac{1}{5}+\dfrac{3}{2}\right)+\left(\dfrac{2}{5}-\dfrac{3}{2}i\right)\right\}\left(\dfrac{-1}{2}+i\right)=\left(\dfrac{17}{10}-\dfrac{11i}{10}\right)\left(\dfrac{-1}{2}+i\right)$

$=\left(\dfrac{-17}{20}+\dfrac{11}{10}\right)+\left(\dfrac{17}{10}+\dfrac{11}{20}\right)i=\left(\dfrac{5}{20}+\dfrac{45}{20}i\right)=\left(\dfrac{1}{4}+\dfrac{9}{4}i\right)$.

59. (i) $\rightarrow \overline{(3+2i)}=(3-2i)$, (ii) $\rightarrow \overline{(-3-2i)}=(-3+2i)$, (iii) $\rightarrow \overline{(-2-3i)}=(-2-3i)$ and (iv) $\rightarrow \overline{(2-3i)}=(2+3i)$.

$\therefore$ the correct answer is (i)–(r), (ii)–(s), (iii)–(p), (iv)–(q).

60. (i) $\rightarrow |4-3i|=\sqrt{16+9}=\sqrt{25}=5$, (ii) $\rightarrow |8+6i|=\sqrt{64+36}=\sqrt{100}=10$,

(iii) $\rightarrow \dfrac{1}{|3+4i|}=\dfrac{1}{\sqrt{9+16}}=\dfrac{1}{\sqrt{25}}=\dfrac{1}{5}$, (iv) $\rightarrow \left|\dfrac{3-4i}{3+4i}\right|=\dfrac{|3-4i|}{|3+4i|}=\dfrac{\sqrt{25}}{\sqrt{25}}=\dfrac{5}{5}=1$.

$\therefore$ the correct answer is (i)–(s), (ii)–(p), (iii)–(q), (iv)–(r).

61. (i) $\rightarrow \{\arg(z_1z_2)=\arg(z_1)+\arg(z_2)\}\rightarrow 3$, (ii) $\rightarrow \left\{\arg\left(\dfrac{z_1}{z_2}\right)=\arg(z_1)-\arg(z_2)\right\}\rightarrow 2$,

(iii) $\rightarrow \{\arg(z)+\arg(\bar{z})=2\pi\}\rightarrow 4$, (iv) $\rightarrow \left\{\arg(i)=\dfrac{\pi}{2}\right\}\rightarrow 1$.

$\therefore$ the correct answer is (i)–(r), (ii)–(q), (iii)–(s), (iv)–(p).

62. $x^2+2=0 \Rightarrow x^2=-2 \Rightarrow x=\pm\sqrt{2}i$.

Solution set is $\{\sqrt{2}i, -\sqrt{2}i\}$.

63. $x^2+13=4x \Rightarrow x^2-4x+13=0$

$\therefore \quad x=\dfrac{4\pm\sqrt{16-52}}{2}=\dfrac{4\pm\sqrt{-36}}{2}=\dfrac{4\pm 6i}{2}=(2\pm 3i)$

$\therefore$ solution set $=\{2+3i, 2-3i\}$.

64. $x^2+2x+2=0 \Rightarrow x=\dfrac{-2\pm\sqrt{4-8}}{2}=\dfrac{-2\pm\sqrt{-4}}{2}=\dfrac{-2\pm 2i}{2} \Rightarrow x=-1\pm i$

$\therefore$ solution set $=\{-1+i, -1-i\}$.

❑

Permutations and Combinations

REVIEW OF FACTS AND FORMULAE

1. **The Factorial**

   We define, $n\,! = n(n-1)(n-2)\,\ldots\,3\cdot 2\cdot 1.$

   Examples (i) $5\,! = (5\times 4\times 3\times 2\times 1) = 120.$

   (ii) $6\times 7\times 8\times 9 = \dfrac{(5\,!)\times 6\times 7\times 8\times 9}{5\,!} = \dfrac{9\,!}{5\,!}\cdot$

2. **Fundamental Principles of Counting**

   (i) **Multiplication Principle**

   If an event can occur in m different ways and if following it, a second event can occur in n different ways, then the two events in succession can occur in $(m\times n)$ different ways.

   (ii) **Addition Principle**

   If two events can occur independently in exactly m ways and n ways respectively, then either of the two events can occur in $(m+n)$ ways.

3. (i) Number of all permutations of n dissimilar things taken r at a time is given by ${}^nP_r = n(n-1)(n-2)\,\ldots\,(n-r+1) = \dfrac{n\,!}{(n-r)\,!}\cdot$

   (ii) ${}^nP_n = n\,!.$ (iii) $0\,! = 1.$

   (iv) ${}^nP_n = {}^nP_{n-1}.$ (v) ${}^nP_r = n\cdot {}^{n-1}P_{r-1}.$

   (vi) ${}^nP_r = {}^{n-1}P_r + r\cdot {}^{n-1}P_{r-1}.$

4. (i) Number of all permutations of n different things taken r at a time when a particular thing is always included in each arrangement $= r\cdot {}^{n-1}P_{r-1}.$

   (ii) Number of all permutations of n different things taken r at a time when a particular thing is never taken $= {}^{n-1}P_r$.

   (iii) If there are n objects out of which m are alike of one kind and $(n-m)$ objects are alike of another kind, then total number of permutations $= \dfrac{n\,!}{(m\,!)\cdot(n-m)\,!}\cdot$

5. (i) Number of ways in which n persons can be seated round a table $= (n-1)\,!.$

   (ii) Number of ways in which n different beads can be arranged $= \dfrac{1}{2}(n-1)\,!.$

COMBINATIONS

Each of the different groups or selections which can be formed by taking some or all of a number of objects, irrespective of their arrangements, is called a combination.

Examples (i) The different combinations formed of three letters a, b, c taken two at a time are ab, bc, ac.

(ii) The only combination formed of three letters a, b, c taken all at a time is abc.

(iii) Groups of 2 out of 4 persons A, B, C, D are AB, AC, AD, BC, BD, CD.

NOTE *ab* and *ba* are two different permutations but each gives the same combination.

1. (i) Number of all combinations of n things taken r at a time $= {}^nC_r$.

(ii) ${}^nC_r = \dfrac{n!}{(r!)\times(n-r)!}$.

(iii) ${}^nC_r = \dfrac{{}^nP_r}{r!}$.

(iv) ${}^nC_r = {}^nC_{n-r}$.

(v) ${}^nC_r + {}^nC_{r-1} = {}^{n+1}C_r$.

(vi) $\dfrac{{}^nC_r}{{}^nC_{r-1}} = \dfrac{n-r+1}{r}$.

(vii) ${}^nC_r + {}^nC_{r+1} = {}^{n+1}C_{r+1}$.

(viii) ${}^nC_p = {}^nC_q \Rightarrow p+q=n$.

EXERCISE 1

Mark (✓) against the correct answer in each of the following.

1. If $\dfrac{1}{6!}+\dfrac{1}{7!}=\dfrac{x}{8!}$, then $x=?$

(a) 32 (b) 48 (c) 56 (d) 64

2. If ${}^{n-1}P_3 : {}^nP_4 = 1:9$, then $n=?$

(a) 12 (b) 11 (c) 9 (d) 10

3. $\dfrac{{}^nP_n}{{}^nP_{n-2}} = ?$

(a) $\dfrac{1}{2}$ (b) 2 (c) $\dfrac{1}{n-2}$ (d) $n(n-1)$

4. If ${}^{15}P_r = 2730$, then $r=?$

(a) 3 (b) 4 (c) 5 (d) 6

5. ${}^7P_3 = ?$

(a) 105 (b) 140 (c) 210 (d) 175

6. If ${}^nP_5 = 20\cdot{}^nP_3$, then $n=?$

(a) 8 (b) 9 (c) 10 (d) 11

7. If ${}^{15}P_{r-1} : {}^{16}P_{r-2} = 3:4$, then $r = ?$

(a) 8 (b) 10 (c) 12 (d) 14

8. If ${}^{n}C_{10} = {}^{n}C_{14}$, then $n = ?$

(a) 4 (b) 24 (c) 14 (d) 10

9. If ${}^{n}C_{3} = 220$, then $n = ?$

(a) 9 (b) 10 (c) 11 (d) 12

10. If ${}^{n}C_{r} + {}^{n}C_{r+1} = {}^{n+1}C_{x}$, then $x = ?$

(a) $r-1$ (b) r (c) $r+1$ (d) n

11. ${}^{36}C_{34} = ?$

(a) 1224 (b) 612 (c) 630 (d) none of these

12. $\dfrac{{}^{n}C_{r}}{{}^{n}C_{r-1}} = ?$

(a) $\dfrac{n-r}{r}$ (b) $\dfrac{n-r-1}{r}$ (c) $\dfrac{n-r+1}{r}$ (d) none of these

13. If ${}^{n}C_{18} = {}^{n}C_{12}$, then ${}^{32}C_{n} = ?$

(a) 248 (b) 496 (c) 992 (d) none of these

14. ${}^{60}C_{60} = ?$

(a) 60 ! (b) 1 (c) $\dfrac{1}{60}$ (d) none of these

15. In how many ways can 5 persons occupy 3 seats?

(a) 15 (b) 20 (c) 30 (d) 60

16. In how many ways can 5 children stand in a queue?

(a) 5 (b) 25 (c) 60 (d) 120

17. In how many ways can 4 different books be arranged on a shelf?

(a) 4 (b) 8 (c) 24 (d) 16

18. Ten students are participating in a race. In how many different ways can the first prize be won?

(a) 30 (b) 60 (c) 120 (d) 720

19. Three different rings are to be worn in 4 fingers. In how many ways can this be done?

(a) 12 (b) 24 (c) 64 (d) 81

20. There are 6 periods on each working day of a school. In how many ways can one arrange 5 subjects such that each subject is allowed at least one period?

(a) 360 (b) 720 (c) 3600 (d) 1800

21. How many words with or without meaning can be formed by using all the letters of the word, 'DELHI', using each letter exactly once?

(a) 20 (b) 60 (c) 120 (d) 5

22. It is required to seat 5 men and 4 women in a row so that the women occupy the even places. How many such arrangements are possible?

(a) 24 (b) 120 (c) 720 (d) 2880

23. How many 3-digit numbers are there?

(a) 648 (b) 729 (c) 900 (d) 1000

24. How many 3-digit numbers are there with no digit repeated?

(a) 648 (b) 720 (c) 729 (d) none of these

25. How many 4-digit numbers can be formed with no digit repeated by using the digits 3, 4, 5, 6, 7, 8 and 0?

(a) 280 (b) 560 (c) 720 (d) 840

26. How many 3-digit even numbers can be formed with no digit repeated by using the digits 0, 1, 2, 3, 4 and 5?

(a) 50 (b) 52 (c) 54 (d) 56

27. The number of positive integers greater than 6000 and less than 7000 which are divisible by 5 with no digit repeated, is

(a) 28 (b) 56 (c) 84 (d) 112

28. How many 10-digit numbers can be formed by using the digits 1 and 2?

(a) ${}^{10}P_2$ (b) ${}^{10}C_2$ (c) 2^{10} (d) 10 !

29. How many words beginning with T and ending with E can be made with no letter repeated out of the letters of the word 'TRIANGLE'?

(a) ${}^{8}P_6$ (b) 720 (c) 722 (d) 1440

30. How many words can be formed from the letters of the word 'DAUGHTER' so that the vowels always come together?

(a) 720 (b) 2160 (c) 4320 (d) none of these

31. How many words can be formed from the letters of the word 'LAUGHTER' so that the vowels are never together?

(a) 3600 (b) 4320 (c) 36000 (d) 40320

32. In how many ways can the letters of the word 'MACHINE' be arranged so that the vowels may occupy only odd positions?

(a) 288 (b) 576 (c) 5040 (d) none of these

33. In how many ways can the letters of the word 'PENCIL' be arranged so that N is always next to E?

(a) 120 (b) 240 (c) 720 (d) 1440

34. In how many ways can the letters of the word 'APPLE' be arranged?

(a) 6 (b) 60 (c) 90 (d) 120

35. How many words can be formed by using all the letters of the word 'ALLAHABAD'?

(a) 9 ! (b) 1890 (c) 3780 (d) 7560

36. How many words can be formed using the letter A thrice, the letter B twice and the letter C once?

(a) 6 (b) 60 (c) 90 (d) 120

37. In how many ways can 10 books be arranged on a shelf so that a particular pair of books shall be always together?

(a) $8!$ (b) $9!$ (c) $2\times 8!$ (d) $2\times 9!$

38. In how many ways can 10 books be arranged on a shelf so that a particular pair of books shall be never together?

(a) $8!$ (b) $9!$ (c) $2\times 9!$ (d) $8\times 9!$

39. How many 4-digit numbers are there when a digit may be repeated any number of times in each number?

(a) 5040 (b) 9000 (c) 10000 (d) 4500

40. In how many ways can 6 boys be arranged in a row?

(a) $5!$ (b) $6!$ (c) 6 (d) $2\times 6!$

41. In how many ways can 6 girls be seated in a circle?

(a) 6 (b) $6!$ (c) $5!$ (d) $2\times 5!$

42. How many diagonals are there in a polygon of n sides?

(a) $\frac{1}{2}n(n-1)$ (b) $\frac{1}{2}n(n-2)$ (c) $\frac{1}{2}n(n-3)$ (d) $\frac{1}{2}n(n+1)$

43. How many diagonals are there in an octagon?

(a) 28 (b) 24 (c) 20 (d) 36

44. A polygon has 54 diagonals. Number of sides in this polygon is

(a) 9 (b) 12 (c) 15 (d) 16

45. There are 10 points in a plane, out of which 4 points are collinear. The number of line segments obtained from the pairs of these points is

(a) 39 (b) 40 (c) 41 (d) 45

46. There are 10 points in a plane, out of which 4 points are collinear. The number of triangles formed with vertices as these points is

(a) 20 (b) 120 (c) 116 (d) none of these

47. Out of 7 consonants and 4 vowels, how many words of 3 consonants and 2 vowels can be formed?

(a) 330 (b) 1050 (c) 6300 (d) 25200

48. In how many ways can a committee of 5 members be selected from 6 men and 5 ladies, consisting of 3 men and 2 ladies?

(a) 25 (b) 50 (c) 100 (d) 200

49. Out of 5 men and 2 women, a committee of 3 is to be formed. In how many ways can it be formed if at least one woman is included in each committee?

(a) 21 (b) 25 (c) 32 (d) 50

50. A committee of 5 is to be formed out of 6 gents and 4 ladies. In how many ways can this be done when each committee may have at the most 2 ladies?
(a) 120 (b) 160 (c) 180 (d) 186

51. How many different teams of 7 players can be chosen out of 10 players?
(a) 720 (b) 70 (c) 120 (d) none of these

52. 12 persons meet in a room and each shakes hands with all the others. How many handshakes are there?
(a) 144 (b) 132 (c) 72 (d) 66

53. In how many ways can we select 9 balls out of 6 red balls, 5 white balls and 5 blue balls if 3 balls of each colour are selected?
(a) 40 (b) 200 (c) 2000 (d) 400

54. In how many ways can a cricket team be chosen out of a batch of 15 players, if a particular player is always chosen?
(a) 1364 (b) 364 (c) 1001 (d) none of these

55. In how many ways can a cricket team be chosen out of a batch of 15 players, if a particular player is never chosen?
(a) 364 (b) 1001 (c) 1364 (d) none of these

56. For the post of 5 teachers, there are 23 applicants. 2 posts are reserved for SC candidates and there are 7 SC candidates among the applicants. In how many ways can the selection be made?
(a) 5880 (b) 11760 (c) 3920 (d) none of these

57. In how many ways can 5 white balls and 3 black balls be arranged in a row so that no two black balls are together?
(a) 192 (b) 40 (c) 20 (d) 120

58. In an examination, a candidate has to pass in each of the five subjects. In how many ways can he fail?
(a) 5 (b) 10 (c) 21 (d) 31

59. An examination paper contains 12 questions consisting of two parts, A and B. Part A contains 7 questions and part B contains 5 questions. A candidate is required to attempt 8 questions, selecting at least 3 from each part. In how many ways can the candidate select the questions?
(a) 210 (b) 175 (c) 420 (d) none of these

ANSWERS (EXERCISE 1)

1. (d) **2.** (c) **3.** (b) **4.** (a) **5.** (c) **6.** (a) **7.** (d) **8.** (b) **9.** (d) **10.** (c)
11. (c) **12.** (c) **13.** (b) **14.** (b) **15.** (d) **16.** (d) **17.** (c) **18.** (d) **19.** (b) **20.** (c)
21. (c) **22.** (d) **23.** (c) **24.** (a) **25.** (c) **26.** (b) **27.** (d) **28.** (c) **29.** (b) **30.** (c)
31. (c) **32.** (b) **33.** (a) **34.** (b) **35.** (d) **36.** (b) **37.** (d) **38.** (d) **39.** (b) **40.** (b)

41. (c) **42.** (c) **43.** (c) **44.** (b) **45.** (b) **46.** (c) **47.** (d) **48.** (d) **49.** (b) **50.** (c)
51. (c) **52.** (d) **53.** (c) **54.** (c) **55.** (a) **56.** (b) **57.** (c) **58.** (d) **59.** (c)

HINTS TO SOME SELECTED QUESTIONS

1. $\dfrac{1}{6!}+\dfrac{1}{7!}=\dfrac{x}{8!} \Rightarrow \dfrac{8\times 7}{8\times 7\times(6!)}+\dfrac{8}{8\times(7!)}=\dfrac{x}{8!}$

$$\Rightarrow \frac{56}{8!}+\frac{8}{8!}=\frac{x}{8!} \Rightarrow x=56+8=64.$$

2. $\dfrac{{}^{n-1}P_3}{{}^nP_4}=\dfrac{1}{9} \Rightarrow \dfrac{(n-1)!}{(n-1-3)!}\times\dfrac{(n-4)!}{n!}=\dfrac{1}{9} \Rightarrow \dfrac{1}{n}=\dfrac{1}{9} \Rightarrow n=9.$

3. $\dfrac{{}^nP_n}{{}^nP_{n-2}}=(n!)\times\dfrac{\{n-(n-2)\}!}{n!}=2!=2.$

4. $2730=(15\times 14\times 13)={}^{15}P_3 \Rightarrow r=3.$

5. ${}^7P_3=\dfrac{7!}{(7-3)!}=\dfrac{7\times 6\times 5\times(4!)}{4!}=210.$

6. $\dfrac{{}^nP_5}{{}^nP_3}=20 \Rightarrow \dfrac{n!}{(n-5)!}\times\dfrac{(n-3)!}{n!}=20$

$$\Rightarrow \frac{(n-3)(n-4)\{(n-5)!\}}{(n-5)!}=20 \Rightarrow (n-3)(n-4)=20$$

$$\Rightarrow n^2-7n-8=0 \Rightarrow (n-8)(n+1)=0 \Rightarrow n=8.$$

7. $\dfrac{{}^{15}P_{r-1}}{{}^{16}P_{r-2}}=\dfrac{3}{4} \Rightarrow \dfrac{15!}{\{15-(r-1)\}!}\times\dfrac{\{16-(r-2)\}!}{16!}=\dfrac{3}{4}$

$$\Rightarrow \frac{1}{16}\times\frac{(18-r)!}{(16-r)!}=\frac{3}{4} \Rightarrow (18-r)(17-r)=12$$

$$\Rightarrow r^2-35r+294=0 \Rightarrow (r-21)(r-14)=0 \Rightarrow r=14 \quad [\because r\le 16].$$

8. ${}^nC_p={}^nC_q \Rightarrow p+q=n.$

$\therefore \quad {}^nC_{10}={}^nC_{14} \Rightarrow n=(10+14)=24.$

9. ${}^nC_3=220 \Rightarrow \dfrac{n(n-1)(n-2)}{6}=220$

$\Rightarrow \quad n(n-1)(n-2)=1320 \Rightarrow n=12 \quad [\because 12\times 11\times 10=1320].$

10. We know that ${}^nC_r+{}^nC_{r+1}={}^{n+1}C_{r+1}$.

So, $x=(r+1)$.

11. ${}^{36}C_{34}={}^{36}C_{(36-34)}={}^{36}C_2=\dfrac{36\times 35}{2}=630.$

12. $\dfrac{{}^nC_r}{{}^nC_{r-1}}=\dfrac{n!}{(r!)\times(n-r)!}\times\dfrac{(r-1)!\times(n-r+1)!}{n!}$

$$=\frac{(r-1)!\times(n-r+1)\times(n-r)!}{r\cdot(r-1)!\times(n-r)!}=\frac{(n-r+1)}{r}.$$

13. ${}^nC_{18} = {}^nC_{12} \Rightarrow n = (18 + 12) = 30.$

$\therefore \quad {}^{32}C_n = {}^{32}C_{30} = {}^{32}C_2 = \dfrac{32 \times 31}{2} = 496.$

14. ${}^{60}C_{60} = 1 \quad [\because\ {}^nC_n = 1].$

15. Required number of ways $= {}^5P_3 = \dfrac{5!}{(5-3)!} = \dfrac{5 \times 4 \times 3 \times 2 \times 1}{2 \times 1} = 60.$

16. Required number of ways $= {}^5P_5 = 5! = (5 \times 4 \times 3 \times 2 \times 1) = 120.$

17. Required number of ways $= 4! = (4 \times 3 \times 2 \times 1) = 24.$

18. Required number of ways $= {}^{10}P_3 = (10 \times 9 \times 8) = 720.$

19. Required number of ways $= (4 \times 3 \times 2) = 24.$

20. Out of 6 periods, 5 may be arranged for 5 subjects in ${}^6P_5$ ways.

Remaining 1 period may be arranged for any one of the five subjects in ${}^5P_1$ ways.

$\therefore$ required number of ways $= ({}^6P_5 \times {}^5P_1) = (6 \times 5 \times 4 \times 3 \times 2 \times 5) = 3600.$

21. Required number of words = number of arrangements of 5 letters taken all at a time

$= {}^5P_5 = 5! = (5 \times 4 \times 3 \times 2 \times 1) = 120.$

22. In a row of 9 seats, the 2nd, 4th, 6th and 8th are the even places.

These 4 places can be occupied by 4 women in ${}^4P_4$ ways = 24 ways.

Remaining 5 places can be occupied by 5 men in ${}^5P_5$ ways = 120 ways.

$\therefore$ total number of seating arrangements $= (24 \times 120) = 2880.$

23. The hundreds place can be filled by any of the 9 nonzero digits.

So, there are 9 ways of filling this place. The tens place can be filled by any of the 10 digits. So, there are 10 ways of filling it.

The units place can be filled by any of the 10 digits. So, there are 10 ways of filling it.

$\therefore$ total number of 3-digit numbers $= (9 \times 10 \times 10) = 900.$

24. The hundreds place can be filled by any of the 9 nonzero digits.

So, there are 9 ways of filling the hundreds place. The tens digit can be filled by any of the remaining 9 digits. So, there are 9 ways of filling the tens place.

The units place can now be filled by any of the remaining 8 digits.

So, there are 8 ways of filling the units digit.

Required number of numbers $= (9 \times 9 \times 8) = 648.$

25. Thousands place can be filled by any of the 6 nonzero digits.

So, there are 6 ways to fill this place.

Hundreds place can be filled by any of the remaining 6 digits.

So, there are 6 ways to fill this place.

Tens place can be filled by any of the remaining 5 digits.

So, there are 5 ways to fill this place.

Units place can be filled by any of the remaining 4 digits.

So, there are 4 ways to fill this place.

Required number of numbers $= (6 \times 6 \times 5 \times 4) = 720.$

26. Numbers with 0 at units place $= (5\times 4\times 1) = 20$.

Numbers with 2 at units place $= (4\times 4\times 1) = 16$.

Numbers with 4 at units place $= (4\times 4\times 1) = 16$.

Total numbers = (20 + 16 + 16) = 52.

27. Clearly, thousands digit is 6.

Number of numbers with units digit $0 = (1\times 8\times 7\times 1) = 56$.

Number of numbers with units digit $5 = (1\times 8\times 7\times 1) = 56$.

Required number of numbers = (56 + 56) = 112.

28. Each place of the number can be filled in 2 ways.

$\therefore$ required number of numbers $= 2^{10}$.

29. Fixing T at the beginning and E at the end, the remaining 6 letters can be arranged in 6 places in 6 ! = 720 ways.

$\therefore$ required number of words = 720.

30. Take all the vowels A U E together and take them as one letter.

Then, the letters to be arranged are D, G, H, T, R, (A U E).

These 6 letters can be arranged in 6 places in 6 ! ways.

Now, 3 letters A, U, E among themselves can be arranged in 3 ! = 6 ways.

$\therefore$ required number of words $= (6!)\times 6 = (6\times 5\times 4\times 3\times 2\times 1\times 6) = 4320$.

31. Total number of words formed by using all the 8 letters at a time $= {}^8P_8 = 8! = 40320$.

Number of words in which vowels are never together

= (total number of words) – (no. of words in which vowels are always together)

= (40320 – 4320) = 36000.

32. The given word has 7 letters out of which there are 3 vowels and 4 consonants. Let us mark the positions of these letters as (1) (2) (3) (4) (5) (6) (7).

Now, the 3 vowels can be placed at any of the 3 places out of four marked 1, 3, 5, 7.

$\therefore$ number of ways of arranging vowels $= {}^4P_3 = (4\times 3\times 2) = 24$.

Now, 4 consonants may be arranged at the remaining four positions in ${}^4P_4 = 4! = 24$ ways.

Required number of ways $= (24\times 24) = 576$.

33. Keeping EN together and considering it as one letter, we have to arrange 5 letters at 5 places.

This can be done in ${}^5P_5 = 5! = 120$ ways.

34. There are in all 5 letters out of which there are 2P, 1A, 1L and 1E.

$\therefore$ required number of ways $= \dfrac{5!}{(2!)(1!)(1!)(1!)} = 60$.

35. There are 9 letters in all. Out of these A is repeated 4 times, L is repeated 2 times and the rest are different.

Required number of words $= \dfrac{9!}{(4!)\cdot(2!)} = 7560$.

36. There are 6 letters in all. Out of these A is repeated thrice, 8 is repeated twice and C is taken only once.

$\therefore$ required number of words $= \dfrac{6!}{(3!)(2!)(1!)} = \dfrac{6\times5\times4\times3\times2\times1}{3\times2\times2} = 60.$

37. Let us keep the two particular books together and treat them as one.

Now, 9 books can be arranged among themselves in 9 ! ways.

Also, 2 books can be arranged among themselves in 2 ! = 2 ways.

$\therefore$ required number of ways $= 2\times 9!$.

38. Number of ways in which 10 books may be arranged = 10 !.

Number of ways in which 10 books may be arranged with two particular books together $= (2\times 9!)$.

Required number of ways in which 2 particular books are never together $= (10!) - (2\times 9!) = (10\times 9!) - (2\times 9!) = (8\times 9!)$.

39. Clearly, 0 cannot be placed at the thousands place.

So, this place can be filled in 9 ways.

Each of the hundreds, tens and units digits can be filled in 10 ways.

$\therefore$ required number of numbers $= (9\times 10\times 10\times 10) = 9000$.

40. 6 boys can be arranged in a row in 6 ! ways.

41. 6 girls can be arranged in a circle in 5 ! ways.

42. In a polygon of n sides, number of diagonals $= \frac{1}{2}n(n-3)$.

43. Putting $n = 8$ in $\frac{1}{2}n(n-3)$, we get 20.

$\therefore$ number of diagonals in an octagon = 20.

44. $\frac{1}{2}n(n-3) = 54 \Rightarrow n(n-3) = 108 \Rightarrow n^2 - 3n - 108 = 0$

$$\Rightarrow n^2 - 12n + 9n - 108 = 0 \Rightarrow n(n-12) + 9(n-12) = 0$$

$$\Rightarrow (n-12)(n+9) = 0 \Rightarrow n = 12.$$

45. Number of line segments formed by joining pairs of points out of 10 $= {}^{10}C_2 = \dfrac{10\times 9}{2} = 45$.

Number of line segments formed by joining pairs of 4 points $= {}^4C_2 = \dfrac{4\times 3}{2} = 6$.

But, these points being collinear give only one line.

$\therefore$ required number of line segments $= (45 - 6 + 1) = 40$.

46. Number of triangles obtained from 10 points $= {}^{10}C_3 = \dfrac{10\times 9\times 8}{3\times 2\times 1} = 120$.

Number of triangles obtained from 4 points $= {}^4C_3 = {}^4C_1 = 4$.

But, these 4 points being collinear will give no triangle.

$\therefore$ required number of triangles $= (120 - 4) = 116$.

47. Number of ways of selecting 3 consonants out of 7 and 2 vowels out of 4
$= ({}^7C_3 \times {}^4C_2) = \left(\frac{7 \times 6 \times 5}{3 \times 2 \times 1} \times \frac{4 \times 3}{2 \times 1}\right) = 210.$

Now, 5 letters can be arranged among themselves in 5 ! ways = 120 ways.

Required number of words $= (210 \times 120) = 25200.$

48. Number of ways of selecting 3 men out of 6 and 2 ladies out of $5 = ({}^6C_3 \times {}^5C_2)$
$= \left(\frac{6 \times 5 \times 4}{3 \times 2 \times 1} \times \frac{5 \times 4}{2 \times 1}\right) = 200.$

49. We may have:

(i) 1 woman and 2 men or (ii) 2 women and 1 man.

$\therefore$ required number of ways $= ({}^2C_1 \times {}^5C_2) + ({}^2C_2 \times {}^5C_1) = \left(2 \times \frac{5 \times 4}{2 \times 1}\right) + (1 \times 5)$

$= (20 + 5) = 25.$

50. We may have:

(i) (1 lady out of 4) and (4 gents out of 6)

or (ii) (2 ladies out of 4) and (3 gents out of 6).

$\therefore$ required number of ways

$$= ({}^4C_1 \times {}^6C_4) + ({}^4C_2 \times {}^6C_3) = (4 \times {}^6C_2) + \left(\frac{4 \times 3}{2 \times 1} \times \frac{6 \times 5 \times 4}{3 \times 2 \times 1}\right)$$

$$= \left(4 \times \frac{6 \times 5}{2 \times 1}\right) + (6 \times 20) = (60 + 120) = 180.$$

51. Required number of teams $= {}^{10}C_7 = {}^{10}C_3 = \frac{10 \times 9 \times 8}{3 \times 2 \times 1} = 120.$

52. Number of handshakes $= {}^{12}C_2 = \frac{12 \times 11}{2} = 66.$

53. Required number of ways $= ({}^6C_3 \times {}^5C_3 \times {}^5C_3)$

$$= ({}^6C_3 \times {}^5C_2 \times {}^5C_2) = \left(\frac{6 \times 5 \times 4}{3 \times 2 \times 1} \times \frac{5 \times 4}{2 \times 1} \times \frac{5 \times 4}{2 \times 1}\right) = 2000.$$

54. When a particular player is always chosen, then we have to select 10 players out of 14.

$\therefore$ required number of ways $= {}^{14}C_{10} = {}^{14}C_4 = \frac{14 \times 13 \times 12 \times 11}{4 \times 3 \times 2 \times 1} = 1001.$

55. When a particular player is never chosen, then we have to select 11 players out of 14.

$\therefore$ required number of ways $= {}^{14}C_{11} = {}^{14}C_3 = \frac{14 \times 13 \times 12}{3 \times 2 \times 1} = 364.$

56. We have to select 2 posts out of 7 SC and 3 posts out of 16.

Required number of ways $= ({}^7C_2 \times {}^{16}C_3) = \left(\frac{7 \times 6}{2} \times \frac{16 \times 15 \times 14}{3 \times 2 \times 1}\right) = 11760.$

57. Let us arrange the white balls (shown by W) and leave a space in between every pair as shown below.

X W X W X W X W X W X

Now, 3 black balls may be arranged in 6 places in ${}^6C_3$ ways $= \frac{6 \times 5 \times 4}{3 \times 2 \times 1} = 20.$

58. The candidate can fail by failing in 1 or 2 or 3 or 4 or 5 subjects out of 5 in each case.

$\therefore$ required number of ways $= {}^5C_1 + {}^5C_2 + {}^5C_3 + {}^5C_4 + {}^5C_5$

$$= {}^5C_1 + {}^5C_2 + {}^5C_{(5-3)} + {}^5C_{(5-4)} + 1$$

$$= {}^5C_1 + {}^5C_2 + {}^5C_2 + {}^5C_1 + 1$$

$$= 2({}^5C_1 + {}^5C_2) + 1 = 2\left(5 + \frac{5\times 4}{2\times 1}\right) + 1 = (30 + 1) = 31.$$

59. He may select:

(i) (3 out of 7 from A) and (5 out of 5 from B)

or (ii) (4 out of 7 from A) and (4 out of 5 from B)

or (iii) (5 out of 7 from A) and (3 out of 5 from B).

The number of ways of these selections are:

(i) ${}^7C_3 \times {}^5C_5 = (35 \times 1) = 35$

(ii) ${}^7C_4 \times {}^5C_4 = (35 \times 5) = 175$

(iii) ${}^7C_5 \times {}^5C_3 = (21 \times 10) = 210.$

$\therefore$ required number of ways = (35 + 175 + 210) = 420.

❑

Binomial Theorem

REVIEW OF FACTS AND FORMULAE

BINOMIAL EXPANSIONS

1. (i) $\boldsymbol{(x+y)^n = \sum_{r=0}^{n} {}^nC_r\, x^{n-r} y^r}$.

   (ii) $\boldsymbol{T_{r+1} = {}^nC_r\, x^{n-r} y^r}$.

   (iii) Number of terms in the expansion of $(x+y)^n = (n+1)$.

   (iv) When n is even, **middle term** in the expansion of $(x+y)^n$ is $\boldsymbol{\left(\frac{n}{2}+1\right)}$***th term.***

   (v) When n is odd, **middle terms** in the expansion of $(x+y)^n$ are $\boldsymbol{\left(\frac{n+1}{2}\right)}$***th term*** and $\boldsymbol{\left(\frac{n+3}{2}\right)}$***th term.***

   (vi) ***pth term from the end*** $=$ $\boldsymbol{(n-p+2)}$***th term.***

2. (i) $\boldsymbol{(1+x)^n = \sum_{r=0}^{n} {}^nC_r\, x^r}$.

   (ii) $\boldsymbol{T_{r+1} = {}^nC_r\, x^r}$.

3. (i) $\boldsymbol{(x-y)^n = \sum_{r=0}^{n} (-1)^r\, {}^nC_r\, x^{n-r} y^r}$.

   (ii) $\boldsymbol{T_{r+1} = (-1)^r\, {}^nC_r\, x^{n-r} y^r}$.

4. (i) $\boldsymbol{(1-x)^n = \sum_{r=0}^{n} (-1)^r \cdot {}^nC_r\, x^r}$.

   (ii) $\boldsymbol{T_{r+1} = (-1)^r \cdot {}^nC_r\, x^r}$.

5. (i) $\boldsymbol{(x+y)^n + (x-y)^n = 2[{}^nC_0\, x^n y^0 + {}^nC_2\, x^{n-2} y^2 + {}^nC_4\, x^{n-4} y^4 + \ldots]}$.

   (ii) $\boldsymbol{(x+y)^n - (x-y)^n = 2[{}^nC_1\, x^{n-1} y + {}^nC_3\, x^{n-3} y^3 + \ldots]}$.

6. (i) When n is odd, then each one of $[(x+y)^n + (x-y)^n]$ and $[(x+y)^n - (x-y)^n]$ has $\boldsymbol{\frac{n+1}{2}}$ ***terms.***

   (ii) When n is even, then $[(x+y)^n + (x-y)^n]$ has $\boldsymbol{\left(\frac{n}{2}+1\right)}$ ***terms*** and $[(x+y)^n - (x-y)^n]$ has $\boldsymbol{\frac{n}{2}}$ ***terms.***

7. **RESULTS ON BINOMIAL COEFFICIENTS**

   We denote ${}^nC_0$ by C_0; ${}^nC_1$ by C_1; ${}^nC_2$ by C_2, ... and ${}^nC_r$ by C_r.

   (i) $C_0 + C_1 + C_2 + \ldots + C_n = 2^n$.

(ii) $C_0 + C_2 + C_4 + \ldots = C_1 + C_3 + C_5 + \ldots = 2^{n-1}$.

(iii) $C_0^2 + C_1^2 + C_2^2 + \ldots + C_n^2 = \dfrac{(2n)!}{(n!)^2}$.

(iv) $C_0 C_1 + C_1 C_2 + C_2 C_3 + \ldots + C_{n-1} C_n = \dfrac{n \cdot 2^n \cdot [1 \cdot 3 \cdot 5 \cdot \ldots \cdot (2n-1)]}{(n+1)!}$.

EXERCISE 1

Mark (✓) against the correct answer in each of the following.

1. The number of terms in the expansion of $(2x + 3y)^{17}$ is

(a) 16 (b) 17 (c) 18 (d) 34

2. The number of terms in the expansion of $\left(3x - \dfrac{5}{y}\right)^{10}$ is

(a) 9 (b) 11 (c) 10 (d) 8

3. The number of terms in the expansion of $(\sqrt{x} + \sqrt{y})^8 + (\sqrt{x} - \sqrt{y})^8$ is

(a) 8 (b) 7 (c) 5 (d) 9

4. The number of terms in the expansion of $\{(2x + 3y)^9 + (2x - 3y)^9\}$ is

(a) 10 (b) 8 (c) 4 (d) 5

5. The number of terms in the expansion of $\{(5x + 2y)^7 - (5x - 2y)^7\}$ is

(a) 4 (b) 8 (c) 6 (d) 3

6. The number of terms in the expansion of $\{(\sqrt{2}x + \sqrt{3}y)^{10} - (\sqrt{2}x - \sqrt{3}y)^{10}\}$ is

(a) 11 (b) 9 (c) 6 (d) 5

7. The number of terms in the expansion of $\{(x + a)^{16} + (x - a)^{16}\}$ is

(a) 7 (b) 8 (c) 9 (d) 17

8. The 7th term in the expansion of $(\sqrt{x} + \sqrt{y})^{10}$ is

(a) $120xy^2$ (b) $210x^2y^3$ (c) $210x^3y^4$ (d) none of these

9. The 10th term in the expansion of $\left(2x^2 + \dfrac{1}{x^2}\right)^{12}$ is

(a) $264x^{-16}$ (b) $1760x^{-12}$ (c) $792x^{-14}$ (d) none of these

10. The 4th term in the expansion of $(x - 2y)^{12}$ is

(a) $1760x^8y^6$ (b) $-440x^7y^5$ (c) $-1760x^9y^3$ (d) none of these

11. The 13th term in the expansion of $\left(9x - \dfrac{1}{3\sqrt{x}}\right)^{18}$, $x \neq 0$ is

(a) 16854 (b) 18564 (c) 17954 (d) 18832

12. The 6th term in the expansion of $\left(\dfrac{4x}{5} - \dfrac{5}{2x}\right)^9$ is

(a) $-2520x$ (b) $-5040x$ (c) $\dfrac{-5040}{x}$ (d) $\dfrac{-560}{x^2}$

13. The 5th term from the end in the expansion of $\left(\dfrac{x^3}{2}-\dfrac{2}{x^2}\right)^9$ is

(a) $252x^2$ (b) $-252x^2$ (c) $-2520x^3$ (d) $2520x^3$

14. The 3rd term from the end in the expansion of $\left(x+\dfrac{1}{x}\right)^6$ is

(a) $\dfrac{15}{x^2}$ (b) $\dfrac{30}{x^3}$ (c) $\dfrac{12}{x^2}$ (d) $\dfrac{24}{x^3}$

15. The 5th term from the end in the expansion of $\left(x-\dfrac{1}{x}\right)^{12}$ is

(a) $99x^4$ (b) $495x^4$ (c) $-99x^4$ (d) $\dfrac{495}{x^4}$

16. The 14th term from the end in the expansion of $(\sqrt{x}-\sqrt{y})^{17}$ is
(a) ${}^{17}C_6(\sqrt{x})^{11}\cdot y^3$ (b) $-{}^{17}C_5x^6(\sqrt{y})^5$ (c) ${}^{17}C_4x^{13/2}\cdot y^2$ (d) none of these

17. The middle term in the expansion of $\left(x-\dfrac{1}{2y}\right)^{10}$ is

(a) $\dfrac{-63}{8}x^5y^{-5}$ (b) $\dfrac{-21}{4}x^6y^{-6}$ (c) $\dfrac{63}{8}x^4y^{-4}$ (d) $\dfrac{-63}{8}x^4y^{-4}$

18. The middle term in the expansion of $\left(x^2-\dfrac{2}{x}\right)^{10}$ is

(a) $8064x^5$ (b) $-8064x^5$ (c) $6720x^4$ (d) $-6720x^4$

19. The coefficient of x^7 in the expansion of $\left(x^2+\dfrac{1}{x}\right)^{11}$ is

(a) 154 (b) 231 (c) 462 (d) 924

20. The coefficient of x^2 in the expansion of $\left(3x-\dfrac{1}{x}\right)^6$ is

(a) 405 (b) 1215 (c) 2430 (d) 3645

21. The coefficient of x^6 in the expansion of $\left(3x^2-\dfrac{1}{3x}\right)^9$ is

(a) 378 (b) 756 (c) 189 (d) 567

22. The term independent of x in the expansion of $\left(x+\dfrac{1}{x}\right)^{10}$ is

(a) 210 (b) 252 (c) 504 (d) 756

23. The term independent of x in the expansion of $\left(\sqrt{x}+\dfrac{1}{3x^2}\right)^{10}$ is

(a) 35 (b) 32 (c) 15 (d) 5

24. The term independent of x in the expansion of $\left(x-\frac{1}{x}\right)^{12}$ is

(a) 924 (b) 462 (c) 231 (d) 693

25. The coefficient of x^{32} in the expansion of $\left(x^4-\frac{1}{x^3}\right)^{15}$ is

(a) 273 (b) 546 (c) 1365 (d) 1092

26. The coefficient of x in the expansion of $\left(x^2+\frac{a}{x}\right)^5$ is

(a) $12a^2$ (b) $10a^3$ (c) $9a^4$ (d) none of these

27. The term independent of x in the expansion of $\left(\frac{3}{2}x^2-\frac{1}{3x}\right)^9$ is

(a) $\frac{1}{24}$ (b) $\frac{7}{18}$ (c) $\frac{8}{27}$ (d) $\frac{5}{36}$

28. Which term contains x^7 in the expansion of $\left(2x^2-\frac{3}{x}\right)^{11}$?

(a) 6th (b) 7th (c) 8th (d) none of these

29. Which term contains x^3 in the expansion of $\left(3x-\frac{1}{2x}\right)^8$?

(a) 2nd (b) 3rd (c) 4th (d) none of these

30. The coefficient of x^{-4} in the expansion of $\left(\frac{4x}{5}+\frac{5}{2x}\right)^8$ is

(a) 625 (b) 1875 (c) 4375 (d) none of these

31. The coefficient of x^{-15} in the expansion of $\left(3x^2-\frac{1}{3x^3}\right)^{10}$ is

(a) $\frac{40}{27}$ (b) $\frac{-40}{27}$ (c) $\frac{80}{9}$ (d) $\frac{-80}{9}$

32. The total number of terms in the expansion of $(x+k)^{100}+(x-k)^{100}$ after simplification is

(a) 50 (b) 51 (c) 101 (d) 202

33. The number of terms in the expansion of $(1+3\sqrt{2}x)^9+(1-3\sqrt{2}x)^9$ is

(a) 5 (b) 7 (c) 9 (d) 10

34. If the coefficients of the second, third and fourth terms in the expansion of $(1+x)^n$ are in AP, then $n=?$

(a) 5 (b) 6 (c) 7 (d) 9

35. If p and q are positive integers, then the coefficients of x^p and x^q in the expansion of $(1+x)^{p+q}$ are

(a) equal (b) equal with opposite signs

(c) reciprocal to each other (d) none of these

36. Let the coefficient of x^n in the expansion of $(1+x)^{2n}$ be p and the coefficient of x^n in the expansion of $(1+x)^{2n-1}$ be q. Then,

(a) $p = 2q$ (b) $2p = q$ (c) $2p = 3q$ (d) $3p = 2q$

37. In the expansion of $(1+x)^n$, the binomial coefficients of three consecutive terms are respectively 220, 495 and 792. The value of n is

(a) 10 (b) 11 (c) 12 (d) 13

38. If the coefficients of x^7 and x^8 are equal in the expansion of $\left(2+\frac{x}{3}\right)^n$, then $n = ?$

(a) 15 (b) 45 (c) 55 (d) 56

39. Total number of terms in the expansion of $(a+b+c)^n$ is

(a) $(n+1)$ (b) $(2n+1)$ (c) $\frac{1}{2}n(n+1)$ (d) $\frac{1}{2}(n+1)(n+2)$

40. Total number of terms in the expansion of $(a+b+c)^{10}$ is

(a) 11 (b) 21 (c) 55 (d) 66

41. If the total number of terms in the expansion of $(a-b+2c)^n$ is 45, then $n = ?$

(a) 7 (b) 8 (c) 9 (d) 22

42. The coefficient of x^3 in the expansion of $(1+2x)^6(1-x)^7$ is

(a) 43 (b) -43 (c) 63 (d) -63

43. In the expansion of $(x+a)^n$, if the sum of odd terms be P and the sum of even terms be Q, then $4PQ = ?$

(a) $(x+a)^n + (x-a)^n$ (b) $(x+a)^n - (x-a)^n$

(c) $(x+a)^{2n} - (x-a)^{2n}$ (d) none of these

44. If the coefficients of the second, third and fourth terms in the expansion of $(1+x)^{2n}$ are in AP, then

(a) $n^2 - 3n + 8 = 0$ (b) $2n^2 - 7n + 5 = 0$

(c) $2n^2 - 9n + 7 = 0$ (d) none of these

45. If C_r denotes ${}^nC_r$ in the expansion of $(1+x)^n$, then $C_0 + C_1 + C_2 + \ldots + C_n = ?$

(a) $2n$ (b) $\frac{1}{2}n(n+1)$ (c) $\frac{1}{3}n(2n+1)$ (d) 2^n

46. $(C_0 + C_2 + C_4 + C_6 + \ldots) = ?$

(a) 2^{n-1} (b) 2^n (c) 4^n (d) 4^{n-1}

47. $\left\{\frac{C_1}{C_0} + 2\frac{C_2}{C_1} + 3\frac{C_3}{C_2} + \ldots + n\cdot\frac{C_n}{C_{n-1}}\right\} = ?$

(a) 2^n (b) 2^{n-1} (c) $2n$ (d) $\frac{1}{2}n(n+1)$

48. $\{C_1 + 2C_2 + 3C_3 + \ldots + nC_n\} = ?$

(a) $n\cdot 2^n$ (b) $n\cdot 2^{n-1}$ (c) $(n-1)\cdot 2^n$ (d) $(n+1)\cdot 2^n$

49. $\{C_0 + 2C_1 + 3C_2 + \ldots + (n+1)C_n\} = ?$

(a) $(n+1)\cdot 2^n$ (b) $n\cdot 2^{n-1}$ (c) $(n+2)\cdot 2^{n-1}$ (d) $(n+2)\cdot 2^{n+1}$

50. $\{C_0 + 3C_1 + 5C_2 + \ldots + (2n+1)C_n\} = ?$

(a) $(n+1)2^n$ (b) $(n+2)\cdot 2^{n-1}$ (c) $(n-1)(n+2)$ (d) none of these

ANSWERS (EXERCISE 1)

1. (c)	**2.** (b)	**3.** (c)	**4.** (d)	**5.** (a)	**6.** (d)	**7.** (c)	**8.** (b)	**9.** (b)	**10.** (c)
11. (b)	**12.** (c)	**13.** (b)	**14.** (a)	**15.** (d)	**16.** (c)	**17.** (a)	**18.** (b)	**19.** (c)	**20.** (b)
21. (a)	**22.** (b)	**23.** (d)	**24.** (a)	**25.** (c)	**26.** (b)	**27.** (b)	**28.** (a)	**29.** (d)	**30.** (c)
31. (b)	**32.** (b)	**33.** (a)	**34.** (c)	**35.** (a)	**36.** (a)	**37.** (c)	**38.** (c)	**39.** (d)	**40.** (d)
41. (b)	**42.** (b)	**43.** (c)	**44.** (c)	**45.** (d)	**46.** (a)	**47.** (d)	**48.** (b)	**49.** (c)	**50.** (a)

HINTS TO SOME SELECTED QUESTIONS

1. The expansion of $(2x + 3y)^{17}$ has 18 terms.

2. The expansion of $\left(3x - \frac{5}{y}\right)^{10}$ has 11 terms.

3. The expansion of $\{(\sqrt{x} + \sqrt{y})^8 + (\sqrt{x} - \sqrt{y})^8\}$ has $\left(\frac{8}{2} + 1\right) = 5$ terms.

4. The expansion of $\{(2x + 3y)^9 + (2x - 3y)^9\}$ has $\left(\frac{9+1}{2}\right) = 5$ terms.

5. The expansion of $\{(5x + 2y)^7 - (5x - 2y)^7\}$ has $\left(\frac{7+1}{2}\right) = 4$ terms.

6. The expansion of $\{(\sqrt{2}x + \sqrt{3}y)^{10} - (\sqrt{2}x - \sqrt{3}y)^{10}\}$ has $\frac{10}{2} = 5$ terms.

7. The expansion of $\{(x + a)^{16} + (x - a)^{16}\}$ has $\left(\frac{16}{2} + 1\right) = 9$ terms.

8. In the expansion of $(\sqrt{x} + \sqrt{y})^{10}$, we have

$$T_{r+1} = {}^{10}C_r \cdot (\sqrt{x})^{10-r} \cdot (\sqrt{y})^r$$

$$\Rightarrow \quad T_7 = T_{6+1} = {}^{10}C_6(\sqrt{x})^4 \cdot (\sqrt{y})^6 = {}^{10}C_4\, x^2y^3 = \frac{10\times 9\times 8\times 7}{4\times 3\times 2\times 1}x^2y^3 = 210x^2y^3.$$

9. In the expansion of $\left(2x^2+\frac{1}{x^2}\right)^{12}$, we have

$$T_{r+1}={}^{12}C_r\cdot(2x^2)^{12-r}\cdot\left(\frac{1}{x^2}\right)^r$$

$$\Rightarrow\quad T_{10}=T_{9+1}={}^{12}C_9\cdot(2x^2)^{(12-9)}\cdot\left(\frac{1}{x^2}\right)^9={}^{12}C_3\cdot(2x^2)^3\cdot\left(\frac{1}{x^2}\right)^9$$

$$=\frac{12\times11\times10}{3\times2\times1}\times8x^6\cdot\frac{1}{x^{18}}=\frac{1760}{x^{12}}=1760x^{-12}.$$

10. In the expansion of $(x-2y)^{12}$, we have

$$T_{r+1}={}^{12}C_r\cdot x^{(12-r)}\cdot(-2y)^r$$

$$\Rightarrow\quad T_4=T_{3+1}={}^{12}C_3\cdot x^{(12-3)}\cdot(-2y)^3=\frac{12\times11\times10}{3\times2\times1}\times(-8)x^9y^3=-1760x^9y^3.$$

11. In the expansion of $\left(9x-\frac{1}{3\sqrt{x}}\right)^{18}$, we have

$$T_{r+1}={}^{18}C_r(9x)^{(18-r)}\left(\frac{-1}{3\sqrt{x}}\right)^r$$

$$\Rightarrow\quad T_{13}=T_{12+1}={}^{18}C_{12}(9x)^{(18-12)}\cdot\left(\frac{-1}{3\sqrt{x}}\right)^{12}={}^{18}C_6(9x)^6\cdot\left(\frac{-1}{3\sqrt{x}}\right)^{12}$$

$$=\frac{18\times17\times16\times15\times14\times13}{6\times5\times4\times3\times2\times1}\times9^6x^6\cdot\frac{1}{3^{12}x^6}=18564.$$

12. In the expansion of $\left(\frac{4x}{5}-\frac{5}{2x}\right)^9$, we have

$$T_{r+1}={}^9C_r\cdot\left(\frac{4x}{5}\right)^{(9-r)}\cdot\left(\frac{-5}{2x}\right)^r$$

$$\Rightarrow\quad T_6=T_{5+1}={}^9C_5\cdot\left(\frac{4x}{5}\right)^{(9-5)}\cdot\left(\frac{-5}{2x}\right)^5={}^9C_4\cdot\left(\frac{4x}{5}\right)^4\times\frac{-5^5}{2^5x^5}$$

$$=\frac{9\times8\times7\times6}{4\times3\times2\times1}\times\frac{256x^4}{625}\times\frac{(-3125)}{32x^5}=\frac{-5040}{x}.$$

13. Given expansion is $\left(\frac{x^3}{2}-\frac{2}{x^2}\right)^9$.

pth term from the end $=(n-p+2)$th term from the beginning.

$\therefore$ 5th term from the end $=(9-5+2)$th term $=$ 6th term.

$$T_{r+1}=(-1)^r\cdot{}^9C_r\left(\frac{x^3}{2}\right)^{(9-r)}\cdot\left(\frac{2}{x^2}\right)^r$$

$$\Rightarrow\quad T_6=T_{(5+1)}=(-1)^5\cdot{}^9C_5\cdot\left(\frac{x^3}{2}\right)^{(9-5)}\cdot\left(\frac{2}{x^2}\right)^5=-{}^9C_4\cdot\left(\frac{x^3}{2}\right)^4\cdot\left(\frac{2}{x^2}\right)^5$$

$$=-\frac{9\times8\times7\times6}{4\times3\times2\times1}\cdot\frac{x^{12}}{16}\times\frac{32}{x^{10}}=-252x^2.$$

14. Given expansion is $\left(x+\frac{1}{x}\right)^6$.

pth term from the end = $(n-p+2)$th term from the beginning.

3rd term from the end = $(6-3+2)$th term = 5th term.

$$T_{r+1} = {}^6C_r\, x^{(6-r)} \cdot \left(\frac{1}{x}\right)^r$$

$$\Rightarrow \quad T_5 = T_{4+1} = {}^6C_4 \cdot x^{(6-4)} \cdot \left(\frac{1}{x}\right)^4 = {}^6C_2 \cdot x^2 \cdot \frac{1}{x^4} = \frac{6\times 5}{2\times 1}\times\frac{1}{x^2} = \frac{15}{x^2}.$$

15. Given expansion is $\left(x-\frac{1}{x}\right)^{12}$.

pth term from the end = $(n-p+2)$th term from the beginning.

5th term from the end = $(12-5+2)$th term = 9th term.

$$T_{r+1} = (-1)^r \cdot {}^{12}C_r\, x^{(12-r)} \cdot \left(\frac{1}{x}\right)^r$$

$$\Rightarrow \quad T_9 = T_{8+1} = (-1)^8 \cdot {}^{12}C_8 \cdot x^{(12-8)} \cdot \frac{1}{x^8} = {}^{12}C_4 \cdot x^4 \cdot \frac{1}{x^8} = \frac{12\times 11\times 10\times 9}{4\times 3\times 2\times 1}\cdot\frac{1}{x^4} = \frac{495}{x^4}.$$

16. Given expansion is $(\sqrt{x}-\sqrt{y})^{17}$.

pth term from the end = $(n-p+2)$th term.

14th term from the end = $(17-14+2)$th term = 5th term.

$$T_{r+1} = (-1)^r \cdot {}^{17}C_r \cdot (\sqrt{x})^{(17-r)} \cdot (\sqrt{y})^r$$

$$\Rightarrow \quad T_5 = T_{4+1} = (-1)^4 \cdot {}^{17}C_4 \cdot (\sqrt{x})^{(17-4)} \cdot (\sqrt{y})^4 = {}^{17}C_4 x^{13/2} \cdot y^2.$$

17. Given expansion is $\left(x-\frac{1}{2y}\right)^{10}$.

Total number of terms in this expansion is 11.

$$T_{r+1} = (-1)^r \cdot {}^{10}C_r\, x^{(10-r)} \cdot \left(\frac{1}{2y}\right)^r$$

$$\therefore \quad \text{middle term} = \text{6th term} = T_{5+1} = (-1)^5 \cdot {}^{10}C_5 \cdot x^{(10-5)} \cdot \left(\frac{1}{2y}\right)^5 = \frac{-63}{8}x^5y^{-5}.$$

18. Given expansion is $\left(x^2-\frac{2}{x}\right)^{10}$.

Total number of terms = 11.

Middle term = 6th term.

Now, $T_{r+1} = (-1)^r \cdot {}^{10}C_r \cdot (x^2)^{(10-r)} \cdot \left(\frac{2}{x}\right)^r$

$$\therefore \quad \text{middle term} = T_6 = T_{5+1} = (-1)^5 \cdot {}^{10}C_5 \cdot (x^2)^{(10-5)} \cdot \left(\frac{2}{x}\right)^5$$

$$= -{}^{10}C_5 \cdot x^{10} \cdot \frac{2^5}{x^5} = -\frac{10\times 9\times 8\times 7\times 6}{5\times 4\times 3\times 2\times 1}\cdot x^{10}\cdot\frac{32}{x^5} = -8064x^5.$$

19. General term in the expansion of $\left(x^2+\frac{1}{x}\right)^{11}$ is

$$T_{r+1}={}^{11}C_r\,(x^2)^{(11-r)}\cdot\left(\frac{1}{x}\right)^r={}^{11}C_r\,x^{(22-2r-r)}={}^{11}C_r\,x^{(22-3r)}.$$

Putting $22-3r=7$, we get $3r=15 \Rightarrow r=5$.

$$T_5={}^{11}C_5x^{(22-15)}={}^{11}C_5x^7.$$

$\therefore$ coefficient of $x^7={}^{11}C_5=\dfrac{11\times10\times9\times8\times7}{5\times4\times3\times2\times1}=462.$

20. General term in the expansion of $\left(3x-\frac{1}{x}\right)^6$ is

$$T_{r+1}=(-1)^r\cdot{}^6C_r\,(3x)^{(6-r)}\cdot\left(\frac{1}{x}\right)^r=(-1)^r\cdot{}^6C_r\cdot3^{(6-r)}x^{(6-2r)}.$$

Putting $6-2r=2$, we get $2r=4 \Rightarrow r=2$.

$\therefore$ $T_3=(-1)^2\cdot{}^6C_2\cdot3^{(6-2)}\cdot x^2={}^6C_2\times3^4=\dfrac{6\times5}{2\times1}\times81=1215.$

21. In the expansion of $\left(3x^2-\frac{1}{3x}\right)^9$, we have

$$T_{r+1}=(-1)^r\cdot{}^9C_r\cdot(3x^2)^{(9-r)}\cdot\left(\frac{1}{3x}\right)^r=(-1)^r\cdot{}^9C_r\cdot3^{(9-2r)}\cdot x^{(18-2r-r)}.$$

Putting $18-3r=6$, we get $3r=12 \Rightarrow r=4$.

$\therefore$ $T_5=(-1)^4\cdot{}^9C_4\cdot3^1\cdot x^6$

$\Rightarrow$ coefficient of $x^6=3\times\dfrac{9\times8\times7\times6}{4\times3\times2\times1}=378.$

22. In the expansion of $\left(x+\frac{1}{x}\right)^{10}$, we have

$$T_{r+1}={}^{10}C_r\cdot x^{(10-r)}\cdot\left(\frac{1}{x}\right)^r={}^{10}C_r\,x^{(10-2r)}.$$

Putting $10-2r=0$, we get $2r=10 \Rightarrow r=5$.

$$T_6={}^{10}C_5=\frac{10\times9\times8\times7\times6}{5\times4\times3\times2\times1}=252.$$

23. In the expansion of $\left(\sqrt{x}+\frac{1}{3x^2}\right)^{10}$, we have

$$T_{r+1}={}^{10}C_r\,(\sqrt{x})^{(10-r)}\cdot\left(\frac{1}{3x^2}\right)^r={}^{10}C_r\,x^{\left(5-\frac{1}{2}r-2r\right)}\cdot\frac{1}{3^r}.$$

Now, $5-\frac{1}{2}r-2r=0 \Rightarrow 5-\frac{5r}{2}=0 \Rightarrow \frac{5r}{2}=5 \Rightarrow r=2.$

$\therefore$ $T_3={}^{10}C_2x^0\cdot\dfrac{1}{3^2}=\dfrac{10\times9}{2\times1}\times\dfrac{1}{9}=5.$

24. In the expansion of $\left(x-\frac{1}{x}\right)^{12}$, we have

$$T_{r+1}=(-1)^r\cdot{}^{12}C_r\,x^{(12-r)}\cdot\left(\frac{1}{x}\right)^r=(-1)^r\cdot{}^{12}C_r-x^{(12-2r)}.$$

Now, $12-2r=0 \Rightarrow 2r=12 \Rightarrow r=6.$

$$T_7=(-1)^6\cdot{}^{12}C_6={}^{12}C_6=\frac{12\times 11\times 10\times 9\times 8\times 7}{6\times 5\times 4\times 3\times 2\times 1}=924.$$

25. In the expansion of $\left(x^4-\dfrac{1}{x^3}\right)^{15}$, we have

$$T_{r+1}=(-1)^r\cdot{}^{15}C_r(x^4)^{(15-r)}\cdot\left(\frac{1}{x^3}\right)^r=(-1)^r\cdot{}^{15}C_r\,x^{(60-4r-3r)}.$$

Putting $60-7r=32$, we get $7r=28 \Rightarrow r=4.$

$\therefore\quad T_5=(-1)^4\cdot{}^{15}C_4\,x^{32}.$

$\therefore\quad$ coefficient of $x^{32}={}^{15}C_4=\dfrac{15\times 14\times 13\times 12}{4\times 3\times 2\times 1}=1365.$

26. In the expansion of $\left(x^2+\dfrac{a}{x}\right)^5$, we have

$$T_{r+1}={}^5C_r\cdot(x^2)^{(5-r)}\cdot\left(\frac{a}{x}\right)^r={}^5C_r\cdot x^{(10-3r)}\cdot a^r.$$

Putting $10-3r=1$, we get $3r=9 \Rightarrow r=3.$

$\therefore\quad$ coefficient of $x={}^5C_3\cdot a^3=\dfrac{5\times 4\times 3}{3\times 2\times 1}a^3=10a^3.$

27. In the expansion of $\left(\dfrac{3}{2}x^2-\dfrac{1}{3x}\right)^9$, we have

$$T_{r+1}={}^9C_r\cdot\left(\frac{3}{2}x^2\right)^{(9-r)}\cdot\left(\frac{1}{3x}\right)^r\cdot(-1)^r$$

$$\Rightarrow\quad T_{r+1}=(-1)^r\cdot{}^9C_r\cdot 3^{(9-2r)}\cdot\frac{1}{2^{(9-r)}}\cdot x^{(18-3r)}.$$

Putting $18-3r=0$, we get $r=6.$

$\therefore\quad$ the term independent of $x=T_7=(-1)^6\cdot{}^9C_6\cdot 3^{-3}\cdot\dfrac{1}{2^3}$

$$={}^9C_3\cdot\frac{1}{8\times 27}=\frac{9\times 8\times 7}{3\times 2\times 1}\times\frac{1}{8\times 27}=\frac{7}{18}.$$

28. In the expansion of $\left(2x^2-\dfrac{3}{x}\right)^{11}$, we have

$$T_{r+1}=(-1)^r\cdot{}^{11}C_r\cdot(2x^2)^{(11-r)}\cdot\left(\frac{3}{x}\right)^r=(-1)^r\cdot{}^{11}C_r\cdot 2^{(11-r)}\cdot 3^r\cdot x^{22-3r}.$$

Putting $22-3r=7$, we get $3r=15 \Rightarrow r=5.$

Hence, the 6th term contains x^7.

29. In the expansion of $\left(3x-\dfrac{1}{2x}\right)^8$, we have

$$T_{r+1}=(-1)^r\cdot{}^8C_r(3x)^{(8-r)}\cdot\left(\frac{1}{2x}\right)^r=(-1)^r\cdot{}^8C_r\cdot 3^{(8-r)}\cdot\frac{1}{2^r}x^{(8-2r)}.$$

Putting $8 - 2r = 3$, we get $2r = 5 \Rightarrow r = \frac{5}{2}$.

$\therefore$ no term contains x^3.

30. In the expansion of $\left(\frac{4x}{5} + \frac{5}{2x}\right)^8$, we have

$$T_{r+1} = {}^8C_r \cdot \left(\frac{4x}{5}\right)^{(8-r)} \cdot \left(\frac{5}{2x}\right)^r = {}^8C_r\, x^{(8-2r)} \cdot \frac{2^{(16-2r-r)}}{5^{(8-2r)}}.$$

Putting $8 - 2r = -4$, we get $2r = 12 \Rightarrow r = 6$.

$$T_7 = {}^8C_6 x^{-4} \cdot \frac{2^{(16-3\times 6)}}{5^{(8-2\times 6)}} = {}^8C_2 x^{-4} \cdot \frac{5^4}{2^2} = \frac{8\times 7}{2\times 1} \times \frac{625}{4} x^{-4} = 4375x^{-4}.$$

$\therefore$ coefficient of x^{-4} is 4375.

31. In the expansion of $\left(3x^2 - \frac{1}{3x^3}\right)^{10}$, we have

$$T_{r+1} = (-1)^r \cdot {}^{10}C_r \cdot (3x^2)^{(10-r)} \cdot \left(\frac{1}{3x^3}\right)^r = (-1)^r \cdot {}^{10}C_r \cdot 3^{(10-2r)} \cdot x^{(20-2r-3r)}.$$

Putting $20 - 5r = -15$, we get $5r = 35 \Rightarrow r = 7$.

$$T_8 = (-1)^7 \cdot {}^{10}C_7 \cdot 3^{(10-14)} \cdot x^{-15} = -{}^{10}C_3 \cdot \frac{1}{3^4} x^{-15} = \frac{-(10\times 9\times 8)}{(3\times 2\times 1)} \times \frac{1}{81} x^{-15} = \frac{-40}{27} x^{-15}.$$

$\therefore$ coefficient of x^{-15} is $\frac{-40}{27}$.

32. When n is even, the number of terms in $\{(x+y)^n + (x-y)^n\}$ is $\left(\frac{n}{2} + 1\right)$.

$\therefore$ $(x+k)^{100} + (x-k)^{100}$ has $\left(\frac{100}{2} + 1\right) = 51$ terms.

33. When n is odd, the number of terms in $\{(x+y)^n + (x-y)^n\}$ is $\left(\frac{n+1}{2}\right)$.

$\therefore$ $(1 + 3\sqrt{2}x)^9 + (1 - 3\sqrt{2}x)^9$ contains $\frac{(9+1)}{2} = 5$ terms.

34. In the expansion of $(1+x)^n$, we have $T_{r+1} = {}^nC_r\, x^r$.

$\therefore$ $T_2 = {}^nC_1 x^1 = nx$, $T_3 = {}^nC_2 x^2$ and $T_4 = {}^nC_3 x^3$.

$\therefore$ $2 \times {}^nC_2 = {}^nC_1 + {}^nC_3 \Rightarrow n(n-1) = n + \frac{n(n-1)(n-2)}{6}$

$\Rightarrow$ $(n-2)(n-7) = 0 \Rightarrow n = 7$ [$\because$ number of terms is 4 or more].

35. $T_{p+1} = {}^{(p+q)}C_p \cdot x^p$ and $T_{q+1} = {}^{(p+q)}C_q \cdot x^q$.

But, ${}^{(p+q)}C_p = {}^{(p+q)}C_{(p+q)-p} = {}^{(p+q)}C_q$.

So, the coefficients of x^p and x^q are equal.

36. In the expansion of $(1+x)^{2n}$, we have

$$t_{r+1} = {}^{2n}C_r\, x^{2n-r} \Rightarrow T_{n+1} = {}^{2n}C_n x^n \Rightarrow {}^{2n}C_n = p.$$

In the expansion of $(1+x)^{2n-1}$, we have $T_{r+1} = {}^{2n-1}C_r\, x^{2n-1-r}$.

Putting $2n-1-r=n$, we get $r=n-1$.

$\therefore \quad T_n = T_{(n-1)+1} = {}^{2n-1}C_{n-1}x^n \Rightarrow {}^{2n-1}C_{n-1} = q.$

$\therefore \quad \dfrac{p}{q} = \dfrac{{}^{2n}C_n}{{}^{2n-1}C_{n-1}} = \dfrac{(2n)!}{(n!)(n!)} \times \dfrac{\{(n-1)!\}(n!)}{(2n-1)!} = \dfrac{2n}{n} = 2 \Rightarrow p = 2q.$

37. Let the three successive terms be T_r, T_{r+1} and T_{r+2}. Then,

$${}^nC_{r-1} = 220,\ {}^nC_r = 495 \text{ and } {}^nC_{r+1} = 792$$

$\Rightarrow \quad \dfrac{{}^nC_r}{{}^nC_{r-1}} = \dfrac{495}{220} = \dfrac{9}{4}$ and $\dfrac{{}^nC_{r+1}}{{}^nC_r} = \dfrac{792}{495} = \dfrac{8}{5}$

$\Rightarrow \quad \dfrac{n-r+1}{r} = \dfrac{9}{4}$ and $\dfrac{n-(r+1)+1}{r+1} = \dfrac{8}{5}$

$\Rightarrow \quad 4n - 13r + 4 = 0$ and $5n - 13r - 8 = 0$

$\Rightarrow \quad n = 12$ and $r = 4$.

38. In the expansion of $\left(2 + \dfrac{x}{3}\right)^n$, we have:

$$T_{r+1} = {}^nC_r \cdot 2^{(n-r)} \cdot \left(\frac{x}{3}\right)^r = {}^nC_r \cdot 2^{(n-r)} \cdot 3^{-r} \cdot x^r.$$

Putting $r=7$ and 8, we get

$$T_8 = {}^nC_7 \cdot 2^{(n-7)} \cdot 3^{-7} \cdot x^7 \text{ and } T_9 = {}^nC_8 \cdot 2^{(n-8)} \cdot 3^{-8} \cdot x^8.$$

$\therefore \quad {}^nC_7 \cdot 2^{(n-7)} \cdot 3^{-7} = {}^nC_8 \cdot 2^{(n-8)} \cdot 3^{-8} \Rightarrow \dfrac{{}^nC_8}{{}^nC_7} \cdot \dfrac{2^{(n-8)}}{2^{(n-7)}} \cdot \dfrac{3^{-8}}{3^{-7}} = 1$

$\Rightarrow \quad \dfrac{(n-8+1)}{8} \times \dfrac{1}{2} \times \dfrac{1}{3} = 1 \Rightarrow n-7 = 48 \Rightarrow n = 55.$

39. $(a+b+c)^n = \{a+(b+c)\}^n$

$$= a^n + {}^nC_1 a^{n-1}(b+c) + {}^nC_2 \cdot a^{n-2}(b+c)^2 + \dots + {}^nC_n(b+c)^n.$$

Total number of terms in this expansion $= 1+2+3+\dots+(n+1) = \dfrac{1}{2}(n+1)(n+2).$

40. Putting $n=10$, required number of terms $= \dfrac{1}{2} \times 11 \times 12 = 66.$

41. $\dfrac{1}{2}(n+1)(n+2) = 45 \Rightarrow (n+1)(n+2) = 90$

$\Rightarrow \quad n^2 + 3n - 88 = 0 \Rightarrow (n-8)(n+11) = 0 \Rightarrow n = 8.$

42. $(1+2x)^6(1-x)^7$

$= \{1 + {}^6C_1 \cdot 2x + {}^6C_2 \cdot (2x)^2 + {}^6C_3 \cdot (2x)^3 + \dots\} \times \{1 - {}^7C_1 \cdot x + {}^7C_2 \cdot x^2 - {}^7C_3 \cdot x^3 + \dots\}$

$= (1 + 12x + 60x^2 + 160x^3 + \dots)(1 - 7x + 21x^2 - 35x^3 + \dots).$

$\therefore \quad$ coefficient of $x^3 = 1 \times (-35) + (12 \times 21) + 60 \times (-7) + (160 \times 1)$

$$= (-35 + 252 - 420 + 160) = -43.$$

43. $P + Q = (x+a)^n$ and $P - Q = (x-a)^n$

$\Rightarrow \quad 4PQ = (P+Q)^2 - (P-Q)^2 = (x+a)^{2n} - (x-a)^{2n}.$

44. $T_{r+1} = {}^{2n}C_r x^r \Rightarrow T_2 = {}^{2n}C_1 x,\ T_3 = {}^{2n}C_2 x^2$ and $T_4 = {}^{2n}C_3 x^3.$

Now, ${}^{2n}C_1$, ${}^{2n}C_2$ and ${}^{2n}C_3$ are in AP

$$\Rightarrow \quad 2 \cdot \frac{2n(2n-1)}{2} = 2n + \frac{2n(2n-1)(2n-2)}{6}$$

$$\Rightarrow \quad 2n^2 - 9n + 7 = 0.$$

45. We know that $C_0 + C_1 + C_2 + \ldots + C_n = 2^n$.

46. We know that $C_0 + C_2 + C_4 + \ldots = 2^{n-1}$.

47. We know that $\dfrac{C_r}{C_{r-1}} = \dfrac{n-r+1}{r}$.

Putting $r = 1, 2, 3, \ldots, n$, we get

$$\frac{C_1}{C_0} + 2 \cdot \frac{C_2}{C_1} + 3 \cdot \frac{C_3}{C_2} + \ldots + n \cdot \frac{C_n}{C_{n-1}} = n + (n-1) + (n-2) + \ldots + 1 = \frac{1}{2} n(n+1).$$

48. $C_1 + 2C_2 + 3C_3 + \ldots + nC_n = n + 2 \cdot \dfrac{n(n-1)}{2} + 3 \cdot \dfrac{n(n-1)(n-2)}{3!} + \ldots + n$

$$= n \cdot \left[1 + (n-1) + \frac{(n-1)(n-2)}{2!} + \ldots + 1 \right]$$

$$= n \cdot [{}^{(n-1)}C_0 + {}^{(n-1)}C_1 + {}^{(n-1)}C_2 + \ldots + {}^{(n-1)}C_{n-1}]$$

$$= n \cdot (1+1)^{n-1} = n \cdot 2^{n-1}.$$

49. $C_0 + 2C_1 + 3C_2 + \ldots + (n+1)C_n$

$$= (C_0 + C_1 + C_2 + \ldots + C_n) + (C_1 + 2C_2 + 3C_3 + \ldots + nC_n]$$

$$= 2^n + n \cdot 2^{n-1} = (n+2) \cdot 2^{n-1}.$$

50. $C_0 + 3C_1 + 5C_2 + \ldots + (2n+1)C_n$

$$= (C_0 + C_1 + C_2 + \ldots + C_n) + 2(C_1 + 2C_2 + \ldots + nC_n)$$

$$= 2^n + 2(n \cdot 2^{n-1}) = (n+1) \cdot 2^n.$$

❑

Sequences and Series (AP, GP, HP)

REVIEW OF FACTS AND FORMULAE

1. ARITHMETIC PROGRESSION (AP)

I. $a, a+d, a+2d, \ldots, a+(n-1)d, \ldots$ is an AP in which

first term $= a$ and ***common difference*** $= d$.

In this AP, we have:

(i) nth term, $T_n = a+(n-1)d$.

(ii) nth term from the end $= l-(n-1)d$.

(iii) sum to n terms, $S_n = \frac{n}{2}\cdot\{2a+(n-1)d\}$.

(iv) sum to n terms, $S_n = \frac{n}{2}(a+l)$, where l is the last term.

II. (i) Three numbers in AP are taken as $(a-d), a, (a+d)$.

(ii) Four numbers in AP are taken as $(a-3d), (a-d), (a+d), (a+3d)$.

(iii) Five numbers in AP are taken as $(a-2d), (a-d), a, (a+d), (a+2d)$.

III. (i) AM between a and $b = \frac{1}{2}(a+b)$.

(ii) If a, A_1, A_2, A_3, b are in AP, we say that A_1, A_2, A_3 are the three arithmetic means between a and b.

2. GEOMETRICAL PROGRESSION (GP)

I. $a, ar, ar^2, \ldots, ar^{n-1}, \ldots$ is a GP in which

first term $= a$ and ***common ratio*** $= r$.

In this GP, we have

(i) nth term, $T_n = ar^{n-1}$.

(ii) nth term from the end $= \frac{l}{r^{n-1}}\cdot$

(iii) sum to n terms of the GP is given by

$$S_n = \begin{cases} \dfrac{a(1-r^n)}{(1-r)}, & \text{when } r<1 \\ \dfrac{a(r^n-1)}{(r-1)}, & \text{when } r>1. \end{cases}$$

II. (i) Three numbers in GP are taken as $\frac{a}{r}, a, ar$.

(ii) Four numbers in GP are taken as $\frac{a}{r^3}, \frac{a}{r}, ar, ar^3$.

III. (i) If a, G, b are in GP then G is the GM between a and b and $G = \sqrt{ab}$.

(ii) AM $\geq$ GM.

3. HARMONIC PROGRESSION (HP)

a, b, c are in HP $\Rightarrow \frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ are in AP.

4. (i) $(1 + 2 + 3 + \ldots + n) = \frac{1}{2}n(n+1)$

(ii) $(1^2 + 2^2 + 3^2 + \ldots + n^2) = \frac{1}{6}n(n+1)(2n+1)$

(iii) $(1^3 + 2^3 + 3^3 + \ldots + n^3) = \left\{\frac{1}{2}n(n+1)\right\}^2$

EXERCISE 1

Mark (✓) against the correct answer in each of the following.

1. If $\frac{3}{4}, a, 2$ are in AP then $a = ?$

(a) $\frac{3}{8}$ (b) $\frac{5}{8}$ (c) $\frac{7}{8}$ (d) $\frac{11}{8}$

2. What is the next term of the AP $\sqrt{8}, \sqrt{18}, \sqrt{32}, \ldots$?

(a) $\sqrt{40}$ (b) $\sqrt{48}$ (c) $\sqrt{50}$ (d) $\sqrt{54}$

3. What is the next term of the AP $\sqrt{2}, \sqrt{8}, \sqrt{18}, \ldots$?

(a) $\sqrt{24}$ (b) $\sqrt{28}$ (c) $\sqrt{30}$ (d) $\sqrt{32}$

4. Which term of the AP 5, 8, 11, 14, … is 323?

(a) 108th (b) 107th (c) 106th (d) 54th

5. Which term of the AP 92, 88, 84, … is 0?

(a) 24th (b) 27th (c) 23rd (d) 28th

6. Which term of the AP 27, 24, 21, 18, … is –81?

(a) 39th (b) 38th (c) 37th (d) 29th

7. Which term of the AP 40, 35, 30, … is the first negative term?

(a) 9th (b) 10th (c) 12th (d) 14th

8. Which term of the AP $\frac{5}{6}, 1, 1\frac{1}{6}, \ldots$ is $4\frac{2}{3}$?

(a) 26th (b) 25th (c) 24th (d) 19th

9. The 5th and 13th terms of an AP are 5 and –3 respectively. The 24th term of this AP is

(a) –6 (b) –8 (c) –11 (d) –14

10. The 2nd, 31st and the last terms of an AP are $7\frac{3}{4}, \frac{1}{2}$ and $-6\frac{1}{2}$ respectively. How many terms are there in this AP?

(a) 53 (b) 56 (c) 59 (d) 62

11. The 10th term from the end of the AP 7, 10, 13, …, 184 is

(a) 151 (b) 154 (c) 160 (d) 157

12. The 12th term from the end of the AP 17, 14, 11, …, −61 is

(a) −31 (b) −28 (c) −34 (d) −37

13. If 4, x_1, x_2, x_3, 28 are in AP then $x_3 = ?$

(a) 24 (b) 22

(c) 20 (d) cannot be determined

14. The sides of a right triangle are in AP. The ratio of their lengths is

(a) 1 : 2 : 3 (b) 2 : 3 : 4 (c) 3 : 4 : 5 (d) 5 : 8 : 3

15. How many 2-digit numbers are there which are divisible by 6?

(a) 14 (b) 15 (c) 16 (d) 17

16. How many numbers are there between 102 and 750 which are divisible by 8?

(a) 75 (b) 78 (c) 81 (d) 84

17. If the nth term of a progression be a linear expression in n then the given progression is

(a) an AP (b) a GP (c) an HP (d) none of these

18. In a given AP if pth term is q and the qth term is p then its nth term is

(a) $(p+q+n)$ (b) $(p+q-n)$ (c) $(p-q+n)$ (d) $(p-q-n)$

19. In a given AP if mth term is $\frac{1}{n}$ and nth term is $\frac{1}{m}$ then its (mn)th term is

(a) $\left(\frac{1}{m}+\frac{1}{n}\right)$ (b) mn (c) $\frac{1}{mn}$ (d) 1

20. In an AP if m times the mth term is equal to n times the nth term then its $(m+n)$th term is

(a) $-(m+n)$ (b) −1 (c) 1 (d) 0

21. (5 + 9 + 13 + 17 + … up to 23 terms) = ?

(a) 1123 (b) 1127 (c) 1131 (d) 1135

22. (0.7 + 0.71 + 0.72 + … up to 100 terms) = ?

(a) 117.5 (b) 118.5 (c) 119.5 (d) 121.5

23. (25 + 28 + 31 + … + 100) = ?

(a) 1545 (b) 1585 (c) 1625 (d) 1525

24. (1 + 3 + 5 + 7 + … + 999) = ?

(a) 251001 (b) 249500 (c) 249496 (d) 250000

25. $(101 + 99 + 97 + \ldots + 47) = ?$

(a) 2076 (b) 2072 (c) 2177 (d) 2173

26. If $(1 + 6 + 11 + \ldots + x) = 148$ then $x = ?$

(a) 8 (b) 48 (c) 36 (d) 54

27. If $(26 + 21 + 16 + \ldots + x) = 11$ then $x = ?$

(a) –12 (b) –18 (c) –24 (d) –30

28. How many terms of the AP $-5, \frac{-9}{2}, -4, \ldots$ will give the sum 0?

(a) 21 (b) 18 (c) 23 (d) 16

29. The 3rd term of an AP is 1 and its 6th term is –11. The sum of 32 terms of this AP is

(a) 1696 (b) –1696 (c) 848 (d) –848

30. The sum of first 7 terms of an AP is 10 and the sum of next 7 terms is 17. What is the 3rd term of the AP?

(a) $1\frac{2}{7}$ (b) $1\frac{3}{7}$ (c) $1\frac{5}{7}$ (d) 2

31. The sum of first 80 natural numbers is

(a) 3236 (b) 3240 (c) 3248 (d) 3250

32. The sum of all even natural numbers between 300 and 400 is

(a) 17350 (b) 17250 (c) 17150 (d) 17400

33. The sum of all odd numbers between 100 and 200 is

(a) 7500 (b) 7450 (c) 7560 (d) 7600

34. The sum of all positive integral multiples of 3 less than 100 is

(a) 1686 (b) 1683 (c) 1680 (d) 1677

35. How many terms of the AP 6, 12, 18, 24, … must be taken to make the sum 816?

(a) 16 (b) 18 (c) 14 (d) 22

36. The sum of all 3-digit numbers divisible by 5 is

(a) 97650 (b) 98550 (c) 95850 (d) 96950

37. The sum of n terms of an AP is $(3n^2 + 2n)$. Its common difference is

(a) 5 (b) –5 (c) 6 (d) –6

38. The sum of n terms of an AP is $(3n^2 + 5n)$. Which of its terms is 164?

(a) 28th (b) 27th (c) 26th (d) 29th

39. If the sum of first m terms of an AP is the same as the sum of its first n terms then the sum of its first $(m + n)$ terms is

(a) 0 (b) 1 (c) $2(m + n)$ (d) none of these

40. If the mth term of an AP is $\left(\frac{1}{n}\right)$ and the nth term is $\left(\frac{1}{m}\right)$ then the sum of its mn terms is

(a) $\frac{1}{2}(mn+1)$ (b) $2(m+n)$ (c) $-(m+n)$ (d) none of these

41. If the sum of first m terms is n and the sum of first n terms is m, then the sum of first $(m+n)$ terms is

(a) 0 (b) $(m+n)$ (c) $-(m+n)$ (d) $-2(m+n)$

42. If the sum of n terms of a progression be a quadratic expression in n then it is

(a) an AP (b) a GP (c) an HP (d) none of these

43. The sum of n terms of an AP is given by $S_n = (3n^2 + 4n)$. Its rth term is

(a) $(3r+4)$ (b) $(4r+3)$ (c) $(5r+2)$ (d) $(6r+1)$

44. If S_1, S_2, S_3 be the sums of n, $2n$ and $3n$ terms of an AP respectively and $(S_2 - S_1) = kS_3$ then $k = ?$

(a) $\frac{1}{2}$ (b) $\frac{1}{3}$ (c) 2 (d) 3

45. The sum of all 2-digit numbers is

(a) 4750 (b) 4895 (c) 4905 (d) 4850

46. The second and 7th terms of an AP are 2 and 22 respectively. The sum of its first 35 terms is

(a) 2160 (b) 2240 (c) 2310 (d) 2470

47. The first and 5th terms of an AP are –14 and 2 respectively and the sum of its n terms is 40. Then, $n = ?$

(a) 8 (b) 10 (c) 12 (d) 13

48. If $\frac{(a^{n+1} + b^{n+1})}{(a^n + b^n)}$ is the arithmetic mean between unequal numbers a and b then $n = ?$

(a) 0 (b) 1 (c) 2 (d) 4

49. The ratio between the sums of n terms of two arithmetic progressions is $(7n+1):(4n+27)$. The ratio of their 11th terms is

(a) 136 : 117 (b) 124 : 105 (c) 148 : 111 (d) 78 : 71

50. The ratio between the sums of n terms of two arithmetic progressions is $(3n+8):(7n+15)$. The ratio of their 12th terms is

(a) 9 : 14 (b) 7 : 16 (c) 8 : 15 (d) 6 : 11

51. If there are $(2n+1)$ terms in an AP then the ratio of the sum of all odd terms and that of all even terms is

(a) $(n+1):n$ (b) $(n+2):n$ (c) $(n-1):n$ (d) $1:1$

52. If the ratio of the sums of m and n terms of an AP be $(m^2 : n^2)$ then the ratio of mth and nth terms is

(a) $m : n$ (b) $(2m+1) : (2n+1)$
(c) $(2m-1) : (2n-1)$ (d) $m^2 : n^2$

53. Three numbers are in AP whose sum is 24 and whose product is 440. The largest of these numbers is

(a) 13 (b) 11 (c) 9 (d) 12

54. The angles of a quadrilateral are in AP whose common difference is 10°. The largest of these angles is

(a) 110° (b) 120° (c) 125° (d) 130°

55. If $\dfrac{(3+5+7+\ldots \text{ to } n \text{ terms})}{(5+8+11+\ldots \text{ to 10 terms})} = 7$ then $n = ?$

(a) 35 (b) 34 (c) 32 (d) 30

56. How many numbers lie between 10 and 300, which when divided by 4 leave a remainder 3?

(a) 71 (b) 72 (c) 73 (d) 74

57. Two APs have the same common difference and their first terms are –1 and –8 respectively. The difference between their 4th terms is

(a) 4 (b) 5 (c) 6 (d) 7

58. If $(k-1)$, $(2k+1)$, $(6k+3)$ are in GP then $k = ?$

(a) 7 (b) 4 (c) –2 (d) 0

59. For what values of x are the numbers $\dfrac{-2}{7}, x, \dfrac{-7}{2}$ in GP?

(a) –1 and 1 (b) –1 and 2 (c) –2 and 1 (d) –2 and 2

60. The 17th term of the GP $2, \sqrt{8}, 4, \sqrt{32}, \ldots$ is

(a) 256 (b) 512 (c) $128\sqrt{2}$ (d) $256\sqrt{2}$

61. The nth term of the GP $12, 4, \dfrac{4}{3}, \dfrac{4}{9}, \ldots$ is

(a) $\dfrac{4}{3^{n-1}}$ (b) $\dfrac{4}{3^{n-2}}$ (c) $\dfrac{4}{3^{n-3}}$ (d) none of these

62. Which term of the GP 5, 10, 20, 40, … is 5120?

(a) 9th (b) 10th (c) 11th (d) 12th

63. Which term of the GP $\sqrt{3}, 3, 3\sqrt{3}, \ldots$ is 729?

(a) 11th (b) 12th (c) 10th (d) 13th

64. If the 4th and 9th terms of a GP are 54 and 13122 respectively then its 6th term is

(a) 243 (b) 1458 (c) 486 (d) 729

65. The 4th and 7th terms of a GP are $\frac{1}{18}$ and $\frac{-1}{486}$ respectively. Its first term is

(a) $\frac{2}{3}$ (b) $\frac{-2}{3}$ (c) $\frac{3}{2}$ (d) $\frac{-3}{2}$

66. The 8th term from the end of the GP 3, 6, 12, 24, …, 12288 is

(a) 96 (b) 192 (c) 48 (d) 288

67. The 6th term from the end of the GP 8, 4, 2, …, $\frac{1}{1024}$ is

(a) $\frac{1}{64}$ (b) $\frac{1}{32}$ (c) $\frac{1}{128}$ (d) $\frac{1}{16}$

68. The 4th, 7th and 10th terms of a GP are in

(a) AP (b) GP (c) HP (d) none of these

69. If a, x, b are in GP then

(a) $x = ab$ (b) $x^2 = ab$ (c) $x = \frac{1}{2}ab$ (d) $d = \frac{1}{2}(a+b)$

70. The arithmetic mean of two numbers is 34 and their geometric mean is 16. The numbers are

(a) 64 and 4 (b) 52 and 16 (c) 56 and 12 (d) 60 and 8

71. If $\frac{1}{3}, x_1, x_2, 9$ are in GP then $x_2 = ?$

(a) 1 (b) 3
(c) 6 (d) cannot be determined

72. If the nth term of a GP is 2^n, the sum of its first 6 terms is

(a) 124 (b) 126 (c) 190 (d) 254

73. $(1 + \sqrt{3} + 3 + 3\sqrt{3} + \ldots \text{ up to 10 terms}) = ?$

(a) $81(\sqrt{3}+1)$ (b) $100(\sqrt{3}+1)$ (c) $121(\sqrt{3}+1)$ (d) none of these

74. $(0.15 + 0.015 + 0.0015 + \ldots \text{ to 8 terms}) = ?$

(a) $\frac{1}{6}\left(1-\frac{1}{10^7}\right)$ (b) $\frac{1}{6}\left(1+\frac{1}{10^8}\right)$ (c) $\frac{1}{6}\left(1-\frac{1}{10^8}\right)$ (d) $\frac{1}{6}\left(1+\frac{1}{10^7}\right)$

75. $\left(1-\frac{1}{2}+\frac{1}{4}-\frac{1}{8}+\ldots \text{ to 9 terms}\right) = ?$

(a) $\frac{151}{196}$ (b) $\frac{161}{225}$ (c) $\frac{171}{256}$ (d) $\frac{181}{256}$

76. $(3 + 6 + 12 + \ldots + 1536) = ?$

(a) 1023 (b) 2046 (c) 3069 (d) 4092

77. $(2 + 6 + 18 + 54 + \ldots + 4374) = ?$

(a) 6450 (b) 6560 (c) 6670 (d) 6380

78. In a GP it is given that $a = 3$, $T_n = 96$ and $S_n = 189$. The value of n is

(a) 7 (b) 8 (c) 6 (d) 5

79. How many terms of the GP 2, 6, 18, … will make the sum 728?

(a) 6 (b) 9 (c) 8 (d) 7

80. How many terms of the GP $\frac{2}{9}-\frac{1}{3}+\frac{1}{2}-\ldots$ must be taken to make the sum $\frac{55}{72}$?

(a) 6 (b) 5 (c) 7 (d) 8

81. If the sum of n terms of a GP is (2^n-1) then its common ratio is

(a) 2 (b) 3 (c) $\frac{1}{2}$ (d) $\frac{-1}{2}$

82. In a GP, the ratio between the sum of first 3 terms and the sum of first 6 terms is 125 : 152. The common ratio is

(a) $\frac{1}{2}$ (b) $\frac{2}{3}$ (c) $\frac{3}{5}$ (d) $\frac{5}{6}$

83. If the nth term of the GP $3, \sqrt{3}, 1, \ldots$ is $\frac{1}{243}$ then $n=?$

(a) 12 (b) 13 (c) 14 (d) 15

84. For any two positive numbers, we have

(a) AM $\le$ GM (b) AM $\ge$ GM (c) AM $=\frac{3}{4}$GM (d) none of these

85. The AM between two positive numbers a and b $(a>b)$ is twice their GM. Then, $a:b=?$

(a) $(3+\sqrt{2}):(3-\sqrt{2})$ (b) $(2+\sqrt{3}):(2-\sqrt{3})$

(c) 2 : 3 (d) none of these

86. GM between 27 and 243 is

(a) 135 (b) $3\sqrt{30}$ (c) 81 (d) 40.5

87. GM between 0.15 and 0.0015 is

(a) 1.5 (b) 0.015 (c) 0.15 (d) none of these

88. Sum of the infinite GP $\left(1+\frac{1}{3}+\frac{1}{9}+\frac{1}{27}+\ldots\infty\right)=?$

(a) $\frac{2}{3}$ (b) $\frac{5}{9}$ (c) $\frac{3}{2}$ (d) $\frac{4}{9}$

89. The sum of the infinite geometric series $\left(1-\frac{1}{3}+\frac{1}{3^2}-\frac{1}{3^3}+\ldots\infty\right)=?$

(a) $\frac{3}{4}$ (b) $\frac{2}{3}$ (c) $\frac{2}{9}$ (d) $\frac{4}{9}$

90. The sum of the infinite geometric series $\left(\frac{-5}{4}+\frac{5}{16}-\frac{5}{64}+\ldots\infty\right)=?$

(a) $\frac{1}{4}$ (b) $\frac{-1}{4}$ (c) $\frac{5}{8}$ (d) -1

91. The sum of the infinite geometric series $\{(\sqrt{2}+1)+1+(\sqrt{2}-1)+\ldots\infty\}=?$

(a) $\frac{(2+3\sqrt{2})}{2}$ (b) $\frac{(4+3\sqrt{2})}{2}$ (c) $\frac{(3+2\sqrt{2})}{2}$ (d) $\frac{(3+\sqrt{2})}{2}$

92. The sum of the infinite series $\left(\frac{2}{5}+\frac{3}{5^2}+\frac{2}{5^3}+\frac{3}{5^4}+\ldots\infty\right)=?$

(a) $\frac{11}{15}$ (b) $\frac{13}{15}$ (c) $\frac{11}{24}$ (d) $\frac{13}{24}$

93. The sum of an infinite GP is $\frac{80}{9}$ and its common ratio is $-\frac{4}{5}\cdot$ The first term of the GP is

(a) 8 (b) 12 (c) 16 (d) 20

94. The sum of an infinite series is 8 and its second term is 2. Its common ratio is

(a) $\frac{1}{2}$ (b) $\frac{1}{4}$ (c) $\frac{2}{3}$ (d) $\frac{3}{4}$

95. The sum of an infinite geometric series is 15 and the sum of the squares of these terms is 45. The first term of the series is

(a) 6 (b) 7 (c) 5 (d) 9

96. If a, b, c are in GP and $a^{1/x}=b^{1/y}=c^{1/z}$ then x, y, z are in

(a) AP (b) GP (c) HP (d) none of these

97. If a, b, c are in GP then $\log a$, $\log b$, $\log c$ are in

(a) AP (b) GP (c) HP (d) none of these

98. If $y=x+x^2+x^3+\ldots\infty$ then $x=?$

(a) $\frac{y}{(1-y)}$ (b) $\frac{y}{(1+y)}$ (c) $\frac{1}{y}$ (d) $\frac{1+y}{y}$

99. If $x=1+a+a^2+\ldots\infty$ and $y=1+b+b^2+\ldots\infty$, where $|a|<1$ and $|b|<1$ then $(1+ab+a^2b^2+\ldots\infty)=?$

(a) $\frac{xy}{x+y}$ (b) $\frac{x+y}{xy}$ (c) $\frac{xy}{x+y+1}$ (d) $\frac{xy}{x+y-1}$

100. If $x=\left(a+\frac{a}{r}+\frac{a}{r^2}+\ldots\infty\right)$, $y=\left(b-\frac{b}{r}+\frac{b}{r^2}-\ldots\infty\right)$ and $z=\left(c+\frac{c}{r^2}+\frac{c}{r^4}+\ldots\infty\right)$ then $\frac{xy}{z}=?$

(a) $\frac{ab}{c}$ (b) $\frac{c}{ab}$ (c) $c\sqrt{ab}$ (d) $\frac{c(a+b)}{ab}$

101. If a, b, c are in AP; a, x, b and b, y, c are in GP then x^2, b^2, y^2 are in

(a) AP (b) GP (c) HP (d) none of these

102. If a, b, c are in AP and a, b, d are in GP then $a, (a-b), (d-c)$ are in

(a) AP (b) GP (c) HP (d) none of these

103. If pth, qth and rth terms of a GP be a, b, c respectively then $a^{(q-r)}$, $b^{(r-p)}$, $c^{(p-q)} = ?$

(a) 0 (b) 1 (c) –1 (d) none of these

104. If a, b, c are in AP then bc, ca, ab are in

(a) AP (b) GP (c) HP (d) none of these

105. If a^2, b^2, c^2 are in AP then $(b+c), (c+a), (a+b)$ are in

(a) AP (b) GP (c) HP (d) none of these

106. If a, b, c are in AP then $\frac{a(b+c)}{bc}, \frac{b(c+a)}{ca}, \frac{c(a+b)}{ab}$ are in

(a) AP (b) GP (c) HP (d) none of these

107. If $\frac{(b+c-a)}{a}, \frac{(c+a-b)}{b}, \frac{(a+b-c)}{c}$ are in AP then a, b, c are in

(a) AP (b) GP (c) HP (d) none of these

108. $(1 + 2 + 3 + \dots + 80) = ?$

(a) 6480 (b) 3240 (c) 1620 (d) 2430

109. $(1^2 + 2^2 + 3^2 + \dots + 60^2) = ?$

(a) 73810 (b) 36905 (c) 63920 (d) 71930

110. $(1^3 + 2^3 + 3^3 + \dots + 40^3) = ?$

(a) 635400 (b) 628400 (c) 672400 (d) none of these

ANSWERS (EXERCISE 1)

1. (d)	**2.** (c)	**3.** (d)	**4.** (b)	**5.** (a)	**6.** (c)	**7.** (b)	**8.** (c)	**9.** (d)	**10.** (c)
11. (d)	**12.** (b)	**13.** (b)	**14.** (c)	**15.** (b)	**16.** (c)	**17.** (a)	**18.** (b)	**19.** (d)	**20.** (d)
21. (b)	**22.** (c)	**23.** (c)	**24.** (d)	**25.** (b)	**26.** (c)	**27.** (c)	**28.** (a)	**29.** (b)	**30.** (a)
31. (b)	**32.** (c)	**33.** (a)	**34.** (b)	**35.** (a)	**36.** (b)	**37.** (c)	**38.** (b)	**39.** (a)	**40.** (a)
41. (c)	**42.** (a)	**43.** (d)	**44.** (b)	**45.** (b)	**46.** (c)	**47.** (b)	**48.** (a)	**49.** (c)	**50.** (b)
51. (a)	**52.** (c)	**53.** (b)	**54.** (b)	**55.** (a)	**56.** (c)	**57.** (d)	**58.** (b)	**59.** (a)	**60.** (b)
61. (b)	**62.** (c)	**63.** (b)	**64.** (c)	**65.** (d)	**66.** (a)	**67.** (b)	**68.** (b)	**69.** (b)	**70.** (a)
71. (b)	**72.** (b)	**73.** (c)	**74.** (c)	**75.** (c)	**76.** (c)	**77.** (b)	**78.** (c)	**79.** (a)	**80.** (b)
81. (a)	**82.** (c)	**83.** (b)	**84.** (b)	**85.** (b)	**86.** (c)	**87.** (b)	**88.** (c)	**89.** (a)	**90.** (d)
91. (b)	**92.** (d)	**93.** (c)	**94.** (a)	**95.** (c)	**96.** (a)	**97.** (a)	**98.** (b)	**99.** (d)	**100.** (a)
101. (a)	**102.** (b)	**103.** (b)	**104.** (c)	**105.** (c)	**106.** (a)	**107.** (c)	**108.** (b)	**109.** (a)	**110.** (c)

HINTS TO SOME SELECTED QUESTIONS

1. Since $\frac{3}{4}, a, 2$ are in AP, we have

$$a - \frac{3}{4} = 2 - a \Rightarrow 2a = \left(2 + \frac{3}{4}\right) = \frac{11}{4} \Rightarrow a = \frac{11}{8}.$$

2. Given AP is $2\sqrt{2}, 3\sqrt{2}, 4\sqrt{2}$. Its next term is $5\sqrt{2} = \sqrt{25 \times 2} = \sqrt{50}$.

3. Given AP is $\sqrt{2}, 2\sqrt{2}, 3\sqrt{2}$. Its next term is $4\sqrt{2} = \sqrt{16 \times 2} = \sqrt{32}$.

4. Let $T_n = 323$. Then, $a + (n-1)d = 323$, where $a = 5$ and $d = 3$.

$\therefore \quad 5 + (n-1) \times 3 = 323 \Rightarrow (n-1) = \dfrac{318}{3} = 106 \Rightarrow n = 107.$

$\therefore \quad$ 107th term is 323.

5. Let $T_n = 0$. Then, $a + (n-1)d = 0$, where $a = 92$ and $d = -4$.

$\therefore \quad 92 + (n-1) \times (-4) = 0 \Rightarrow 4n = 96 \Rightarrow n = 24.$

$\therefore \quad$ 24th term is 0.

6. Let $T_n = -81$. Then, $a + (n-1)d = -81$, where $a = 27$ and $d = -3$.

$\therefore \quad 27 + (n-1) \times (-3) = -81 \Rightarrow 3n = (30 + 81) = 111 \Rightarrow n = 37.$

$\therefore \quad$ 37th term is –81.

7. Let $T_n < 0$. Then, $\{a + (n-1)d\} < 0$, where $a = 40$ and $d = -5$.

$\therefore \quad \{40 + (n-1) \times (-5)\} < 0 \Rightarrow 45 < 5n \Rightarrow 5n > 45 \Rightarrow n > 9.$

$\therefore \quad$ 10th term is the first negative term.

8. This is an AP in which $a = \dfrac{5}{6}, d = \dfrac{1}{6}$ and $T_n = \dfrac{14}{3}$.

$\therefore \quad a + (n-1)d = \dfrac{14}{3} \Rightarrow \dfrac{5}{6} + (n-1) \times \dfrac{1}{6} = \dfrac{14}{3}$

$\therefore \quad \dfrac{1}{6}n = \left(\dfrac{14}{3} - \dfrac{5}{6} + \dfrac{1}{6}\right) = \dfrac{(28 - 5 + 1)}{6} = \dfrac{24}{6} = 4 \Rightarrow n = 24.$

Hence, 24th term is $4\dfrac{2}{3}$.

9. $a + 4d = 5 \quad \ldots \text{(i)} \qquad$ and $a + 12d = -3 \quad \ldots \text{(ii)}$

On solving (i) and (ii), we get $a = 9$ and $d = -1$.

$\therefore \quad T_{24} = a + 23d = (9 - 23) = -14.$

10. $a + d = \dfrac{31}{4} \quad \ldots \text{(i)} \qquad$ and $a + 30d = \dfrac{1}{2} \quad \ldots \text{(ii)}$

On solving (i) and (ii), we get $a = 8$ and $d = \dfrac{-1}{4}$.

$$T_n = \frac{-13}{2} \Rightarrow a + (n-1)d = \frac{-13}{2}$$

$$\Rightarrow 8 + (n-1) \times \left(\frac{-1}{4}\right) = \frac{-13}{2} \Rightarrow (n-1) \times \frac{(-1)}{4} = \left(\frac{-13}{2} - 8\right) = \frac{-29}{2}$$

$$\Rightarrow (n-1) = \frac{-29}{2} \times \frac{4}{(-1)} = 58 \Rightarrow n = 59.$$

$\therefore \quad$ the given AP has 59 terms.

11. 10th term from the end $= l - (n-1)d$, where $l = 184, n = 10, d = 3$

$= (184 - 9 \times 3) = (184 - 27) = 157.$

12. 12th term from the end $= l - (n-1)d$, where $l = -61, n = 12, d = -3$

$= -61 - 11 \times (-3) = (-61 + 33) = -28.$

13. $a = 4$ and $T_5 = 28 \Rightarrow a + 4d = 28 \Rightarrow 4 + 4d = 28 \Rightarrow d = 6.$

$\therefore \quad x_3 = a + 3d = 4 + 3 \times 6 = 22.$

14. Let the sides of the triangle be $a - d$, a and $a + d$.

Then, $(a + d)^2 = (a - d)^2 + a^2$

$\Rightarrow \quad a^2 = (a + d)^2 - (a - d)^2 \Rightarrow a^2 = 4ad \Rightarrow a = 4d$

$\Rightarrow$ the sides are $(4d - d), (4d)$ and $(4d + d)$, i.e., $3d, 4d, 5d$

$\Rightarrow$ ratio of the sides $= 3 : 4 : 5$.

15. The required numbers are 12, 18, 24, …, 96.

This is an AP in which $a = 12$, $d = 6$ and $t_n = 96$.

$$a + (n - 1)d = 96 \Rightarrow 12 + (n - 1) \times 6 = 96 \Rightarrow (n - 1) = \frac{84}{6} = 14 \Rightarrow n = 15.$$

So, there are 15 such numbers.

16. The required numbers are 104, 112, 120, …, 744.

This is an AP in which $a = 104$, $d = 8$ and $T_n = 744$.

$$a + (n - 1)d = 744 \Rightarrow 104 + (n - 1) \times 8 = 744 \Rightarrow (n - 1) = \frac{640}{8} = 80 \Rightarrow n = 81.$$

So, there are 81 such numbers.

17. Let $T_n = an + b$, where a and b are constants.

$\therefore \quad (T_n - T_{n-1}) = (an + b) - [a(n - 1) + b] = a$, which is a constant.

$\therefore$ the given progression is an AP.

18. Given $a + (p - 1)d = q$ ... (i) and $a + (q - 1)d = p$ ... (ii)

On subtracting, we get $(p - q)d = (q - p) \Rightarrow d = -1$.

Putting $d = -1$ in (i), we get $a = (p + q - 1)$.

$\therefore \quad T_n = a + (n - 1)d \Rightarrow T_n = (p + q - 1) + (n - 1) \times (-1) = (p + q - n).$

19. $a + (m - 1)d = \dfrac{1}{n}$ ... (i) and $a + (n - 1)d = \dfrac{1}{m}$ ... (ii)

On solving (i) and (ii), we get $d = \dfrac{1}{mn}$ and $a = \dfrac{1}{mn}$.

$\therefore \quad T_{mn} = a + (mn - 1)d = \dfrac{1}{mn} + (mn - 1) \times \dfrac{1}{mn} = 1.$

Hence, (mn)th term is 1.

20. $m \cdot T_m = n \cdot T_n$

$\Rightarrow \quad m \cdot [a + (m - 1)d] = n \cdot [a + (n - 1)d]$

$\Rightarrow \quad [(m^2 - n^2) - (m - n)] \cdot d = (n - m)a$

$\Rightarrow \quad (m + n - 1)d = -a \Rightarrow a + (m + n - 1)d = 0 \Rightarrow T_{m+n} = 0.$

Hence, the $(m + n)$th term is 0.

21. In this AP, we have $a = 5$ and $d = 4$.

$$S_n = \frac{n}{2}[2a + (n - 1)d] \Rightarrow S_{23} = \frac{23}{2} \cdot [2 \times 5 + 22 \times 4]$$

$$\Rightarrow S_{23} = \left(\frac{23}{2} \times 98\right) = (23 \times 49) = 1127.$$

22. In this AP, we have $a = 0.7$ and $d = (0.71 - 0.70) = 0.01$.

$$S_n = \frac{n}{2}[2a + (n-1)d] \Rightarrow S_{100} = \frac{100}{2}\cdot[2\times 0.7 + 99\times 0.1]$$

$$\Rightarrow S_{100} = 50\times(1.4 + 0.99) = (50\times 2.39) = 119.5.$$

23. In this AP, we have $a = 25$, $d = 3$ and $l = 100$.

Let $T_n = 100 \Rightarrow a + (n-1)d = 100 \Rightarrow 25 + (n-1)\times 3 = 100$

$$\Rightarrow (n-1) = \frac{75}{3} = 25 \Rightarrow n = 26.$$

$$S_{26} = \frac{n}{2}(a + l) = \frac{26}{2}\cdot(25 + 100) = (13\times 125) = 1625.$$

24. This is an AP in which $a = 1$, $d = 2$ and $l = 999$.

Let $T_n = 999 \Rightarrow a + (n-1)d = 999$

$$\Rightarrow 1 + (n-1)\times d = 999 \Rightarrow (n-1) = \frac{998}{2} = 499 \Rightarrow n = 500.$$

$$\therefore \quad S_n = \frac{n}{2}(a + l) = \frac{500}{2}\cdot(1 + 999) = (250\times 1000) = 250000.$$

25. This is an AP in which $a = 101$, $d = -2$ and $l = 47$.

Let $T_n = 47$. Then, $a + (n-1)d = 47$.

$$\therefore \quad 101 + (n-1)\times(-2) = 47 \Rightarrow 2n = (103 - 47) = 56 \Rightarrow n = 28.$$

$$\therefore \quad S_{28} = \frac{n}{2}(a + l) = \frac{28}{2}\cdot(101 + 47) = (14\times 148) = 2072.$$

26. Let the number of terms be n. Then,

$$\frac{n}{2}\cdot[2a + (n-1)d] = 148$$

$$\Rightarrow \frac{n}{2}\cdot[2\times 1 + (n-1)\times 5] = 148 \Rightarrow n(5n - 3) = 296$$

$$\Rightarrow 5n^2 - 3n - 296 = 0 \Rightarrow n = \frac{3 \pm \sqrt{9 + 5920}}{10} = \frac{3 \pm 77}{10}$$

$$\Rightarrow n = \frac{(3 + 77)}{10} \text{ or } n = \frac{(3 - 77)}{10} \Rightarrow n = 8 \text{ or } n = \frac{-37}{5}$$

$$\Rightarrow n = 8 \qquad [\because \text{ number of terms cannot be a fraction}]$$

$$T_8 = (a + 7d) = (1 + 7\times 5) = 36 \Rightarrow x = 36.$$

27. Let the number of terms be n. Then,

$$\frac{n}{2}\cdot[2a + (n-1)d] = 11$$

$$\Rightarrow \frac{n}{2}\cdot[2\times 26 + (n-1)\times(-5)] = 11 \Rightarrow n(57 - 5n) = 22$$

$$\Rightarrow 5n^2 - 57n + 22 = 0 \Rightarrow 5n^2 - 55n - 2n + 22 = 0$$

$$\Rightarrow 5n(n - 11) - 2(n - 11) = 0 \Rightarrow (n - 11)(5n - 2) = 0 \Rightarrow n = 11 \qquad \left[\because n \neq \frac{2}{5}\right]$$

$$T_{11} = (a + 10d) = [26 + 10\times(-5)] = (26 - 50) = -24 \Rightarrow x = -24.$$

28. Let $S_n = 0$. Then, $\frac{n}{2}\cdot[2a+(n-1)d]=0$.

$\therefore \quad \frac{n}{2}\cdot\left[2\times(-5)+(n-1)\times\frac{1}{2}\right]=0 \Rightarrow n\cdot\left[\frac{-21}{2}+\frac{1}{2}n\right]=0$

$\Rightarrow n=0 \text{ or } \left\{\frac{1}{2}n-\frac{21}{2}=0 \Rightarrow \frac{1}{2}n=\frac{21}{2} \Rightarrow n=21\right\}$

$\therefore$ sum of 21 terms is 0.

29. $T_3 = 1$ and $T_6 = -11$

$\Rightarrow \quad a+2d=1 \quad$ ... (i) $\quad a+5d=-11 \quad$ ... (ii)

$\Rightarrow \quad d=-4$ and $a=9$

$\Rightarrow \quad S_{32}=\frac{32}{2}\cdot[2a+(32-1)d]=16\times[2\times9+31\times(-4)]=16\times(-106)=-1696.$

30. Given $S_7 = 10$ and $S_{14} = 27$.

$\therefore \quad \frac{7}{2}\cdot(2a+6d)=10$ and $\frac{14}{2}\cdot(2a+13d)=27$

$\Rightarrow \quad a+3d=\frac{10}{7}$ and $2a+13d=\frac{27}{7}$

$\Rightarrow \quad d=\frac{1}{7}$ and $a=1$

$\therefore \quad T_3=(a+2d)=\left(1+\frac{2}{7}\right)=1\frac{2}{7}\cdot$

31. Given sum $= 1+2+3+\ldots+80$.

$$S=\frac{n}{2}(a+l)=\frac{80}{2}\times(1+80)=(40\times81)=3240.$$

32. Let $S_n = 302+304+\ldots+398$.

Let the number of terms be n. Then,

$a+(n-1)d=398 \Rightarrow 302+(n-1)\times2=398 \Rightarrow n-1=48 \Rightarrow n=49.$

$\therefore \quad S_n=\frac{n}{2}(a+l)=\frac{49}{2}(302+398)=\left(\frac{49}{2}\times700\right)=(49\times350)=17150.$

33. Let $S_n = 101+103+105+\ldots+199$.

Let the number of terms be n. Then,

$a+(n-1)d=199 \Rightarrow 101+(n-1)\times2=199 \Rightarrow n-1=49 \Rightarrow n=50.$

$\therefore \quad S_n=\frac{n}{2}(a+l)=\frac{50}{2}\cdot(101+199)=(25\times300)=7500.$

34. Let $S_n = 3+6+9+12+\ldots+99$.

Let the number of terms be n. Then,

$a+(n-1)d=99 \Rightarrow 3+(n-1)\times3=99 \Rightarrow (n-1)=32 \Rightarrow n=33.$

$\therefore \quad S_n=\frac{n}{2}(a+l)=\frac{33}{2}\cdot(3+99)=\left(\frac{33}{2}\times102\right)=(33\times51)=1683.$

35. Here $a=6$ and $d=6$. Let $S_n = 816$. Then,

$$\frac{n}{2}\cdot[2a+(n-1)d]=816 \Rightarrow \frac{n}{2}\cdot[2\times6+(n-1)\times6]=816$$

$\Rightarrow \quad n(n+1) = 272 \Rightarrow n^2 + n - 272 = 0 \Rightarrow n^2 + 17n - 16n - 272 = 0$

$\Rightarrow \quad n(n+17) - 16(n+17) = 0 \Rightarrow (n+17)(n-16) = 0 \Rightarrow n = 16.$

Required number of terms = 16.

36. $S_n = 100 + 105 + 110 + \ldots + 995$

$a + (n-1)d = 995 \Rightarrow 100 + (n-1) \times 5 = 995 \Rightarrow (n-1) = \dfrac{895}{5} = 179 \Rightarrow n = 180.$

$S_n = \dfrac{n}{2}(a + l) = \dfrac{180}{2} \times (100 + 995) = (90 \times 1095) = 98550.$

37. $S_n = (3n^2 + 2n) \Rightarrow S_{n-1} = 3(n-1)^2 + 2(n-1) = (3n^2 - 4n + 1)$

$\therefore \quad T_n = (S_n - S_{n-1}) = (3n^2 + 2n) - (3n^2 - 4n + 1) = (6n - 1)$

$\Rightarrow \quad T_1 = (6 \times 1 - 1) = 5$ and $T_2 = (6 \times 2 - 1) = 11$

$\Rightarrow \quad d = (T_2 - T_1) = (11 - 5) = 6.$

38. $S_n = (3n^2 + 5n) \Rightarrow S_{n-1} = 3(n-1)^2 + 5(n-1) = (3n^2 - n - 2)$

$\therefore \quad T_n = (S_n - S_{n-1}) = (3n^2 + 5n) - (3n^2 - n - 2) = (6n + 2)$

$\therefore \quad T_1 = 8, T_2 = 14$ and $d = (14 - 8) = 6.$

Let $T_n = 164$. Then $a + (n-1)d = 164.$

$\therefore \quad 8 + (n-1) \times 6 = 164 \Rightarrow 6n = 162 \Rightarrow n = 27.$

$\therefore \quad$ 27th term is 164.

39. $S_m = S_n \Rightarrow \dfrac{m}{2} \cdot [2a + (m-1)d] = \dfrac{n}{2} \cdot [2a + (n-1)d]$

$\Rightarrow ma + \dfrac{1}{2}m(m-1)d = na + \dfrac{1}{2}n(n-1)d$

$\Rightarrow \dfrac{1}{2}\{(m^2 - n^2) - (m-n)\} \cdot d = (n-m)a$

$\Rightarrow (m + n - 1)d = -2a.$

$S_{m+n} = \dfrac{(m+n)}{2} \cdot [2a + (m+n-1)d] = \dfrac{(m+n)}{2} \cdot [2a - 2a] = 0.$

40. $a + (m-1)d = \dfrac{1}{n}$ … (i) and $a + (n-1)d = \dfrac{1}{m}$ ... (ii)

On solving (i) and (ii), we get $d = \dfrac{1}{mn}$ and $a = \dfrac{1}{mn}$.

$\therefore \quad S_{mn} = \left(\dfrac{mn}{2}\right)[2a + (mn-1)d] = \left(\dfrac{mn}{2}\right)\left[\dfrac{2}{mn} + \dfrac{(mn-1)}{mn}\right] = \dfrac{1}{2}(mn + 1).$

41. $\dfrac{m}{2} \cdot [2a + (m-1)d] = n$ and $\dfrac{n}{2} \cdot [2a + (n-1)d] = m$

$\Rightarrow \quad m \cdot [2a + (m-1)d] = 2n$ ... (i) and $n \cdot [2a + (n-1)d] = 2m$ ... (ii)

On subtracting, we get

$2a(m-n) + [(m^2 - n^2) - (m-n)] \cdot d = 2(n-m)$

$\Rightarrow \quad 2a(m-n) + (m-n)(m+n-1)d = 2(n-m)$

$\Rightarrow \quad 2a + (m+n-1)d = -2$ ... (iii)

$\therefore \quad S_{m+n} = \left(\frac{m+n}{2}\right) \cdot [2a + (m+n-1)d]$

$= \frac{(m+n)}{2} \cdot (-2) = -(m+n) \qquad$ [using (iii)].

42. Let $S_n = an^2 + bn + c$. Then,

$S_{n-1} = a(n-1)^2 + b(n-1) + c$

$\therefore \quad T_n = (S_n - S_{n-1}) = a[n^2 - (n-1)^2] + b[n - (n-1)] = a(2n-1) + b$

$= 2an + (b - a)$, which is a linear expression in n.

Hence, the given progression is an AP.

43. $S_n = (3n^2 + 4n)$ and $S_{n-1} = 3(n-1)^2 + 4(n-1)$

$\therefore \quad T_n = (S_n - S_{n-1}) = 3\{n^2 - (n-1)^2\} + 4\{n - (n-1)\} = 3(2n-1) + 4 = (6n+1)$.

$\therefore \quad r$th term $= T_r = (6r + 1)$.

44. $(S_2 - S_1) = \frac{2n}{2} \cdot [2a + (2n-1)d] - \frac{n}{2} \cdot [2a + (n-1)d]$

$= \frac{n}{2} \cdot [4a + (4n-2)d] - \frac{n}{2} \cdot [2a + (n-1)d]$

$= \frac{n}{2} \cdot [(4a - 2a) + \{(4n-2) - (n-1)\}d]$

$= \frac{n}{2} \cdot [2a + (3n-1)d] = \frac{1}{3} \cdot \left\{\frac{3n}{2} \cdot [2a + (3n-1)d]\right\} = \frac{1}{3} S_3.$

$\therefore \quad k = \frac{1}{3}\cdot$

45. $S_n = 11 + 12 + 13 + \ldots + 99$.

Here, $n = (99 - 10) = 89$, $d = 1$.

$\therefore \quad S_n = \frac{n}{2}(a + l) = \frac{89}{2} \cdot (11 + 99) = \left(\frac{89 \times 110}{2}\right) = (89 \times 55) = 4895.$

46. $a + d = 2$ and $a + 6d = 22$.

On solving, we get $d = 4$ and $a = -2$.

$\therefore \quad S_{35} = \frac{35}{2} \cdot [2a + 34d] = \frac{35}{2} \cdot [2 \times (-2) + 34 \times 4]$

$= \left(\frac{35}{2} \times 132\right) = (35 \times 66) = 2310.$

47. $a = -14$ and $(a + 4d = 2 \Rightarrow -14 + 4d = 2 \Rightarrow 4d = 16 \Rightarrow d = 4)$

$\therefore \quad a = -14$ and $d = 4$.

$S_n = 40 \Rightarrow \frac{n}{2} \cdot [2a + (n-1)d] = 40$

$\Rightarrow \quad \frac{n}{2} \cdot [2 \times (-14) + (n-1) \times 4] = 40$

$\Rightarrow \quad n(4n - 32) = 80 \Rightarrow n(n-8) = 20$

$\Rightarrow \quad n^2 - 8n - 20 = 0 \Rightarrow n^2 - 10n + 2n - 20 = 0$

$\Rightarrow \quad n(n-10) + 2(n-10) = 0 \Rightarrow (n-10)(n+2) = 0 \Rightarrow n = 10.$

Hence, $n = 10$.

48. Given $\dfrac{(a^{n+1}+b^{n+1})}{(a^n+b^n)}=\dfrac{a+b}{2}$

$\Rightarrow\quad (a^n+b^n)(a+b)=2(a^{n+1}+b^{n+1})$

$\Rightarrow\quad a^{n+1}+b^{n+1}+a^nb+b^na=2a^{n+1}+2b^{n+1}$

$\Rightarrow\quad a^{n+1}-a^nb+b^{n+1}-b^na=0 \Rightarrow a^n(a-b)-b^n(a-b)=0$

$\Rightarrow\quad (a-b)(a^n-b^n)=0 \Rightarrow a^n-b^n=0 \quad [\because\ a-b\neq 0]$

$\Rightarrow\quad a^n=b^n$ and $a\neq b \Rightarrow n=0.$

49. Let $S_n=\dfrac{n}{2}[2a+(n-1)d].$

And, $S_n'=\dfrac{n}{2}[2A+(n-1)D].$

$$\therefore\quad \frac{S_n}{S_n'}=\frac{7n+1}{4n+27} \Rightarrow \frac{2a+(n-1)d}{2A+(n-1)D}=\frac{7n+1}{4n+27} \qquad \text{... (i)}$$

$$\frac{T_{11}}{t_{11}}=\frac{a+10d}{A+10D}=\frac{2a+20d}{2A+20D}=\frac{2a+(21-1)d}{2A+(21-1)D}$$

$$=\frac{7\times 21+1}{4\times 21+27}=\frac{148}{111} \quad [\text{putting } n=21 \text{ in (i)}].$$

$\therefore\quad$ required ratio $=148:111.$

50. Let $S_n=\dfrac{n}{2}\cdot[2a+(n-1)d].$

And, $S_n'=\dfrac{n}{2}\cdot[2A+(n-1)D].$

$$\text{Now, } \frac{S_n}{S_n'}=\frac{3n+8}{7n+15} \Rightarrow \frac{2a+(n-1)d}{2A+(n-1)D}=\frac{3n+8}{7n+15} \qquad \text{... (i)}$$

$$\therefore\quad \frac{T_{12}}{t_{12}}=\frac{a+11d}{A+11D}=\frac{2a+22d}{2A+22D}=\frac{2a+(23-1)d}{2A+(23-1)D}$$

$$=\frac{3\times 23+8}{7\times 23+15}=\frac{77}{176}=\frac{7}{16} \quad [\text{putting } n=23 \text{ in (i)}].$$

$\therefore\quad$ required ratio $=7:16.$

51. Let a be the first term and d be the common difference of the given AP.

Then, $a_k=a+(k-1)d$ ... (i)

Let $S_1=$ sum of all odd terms and $S_2=$ sum of all even terms.

Then, $S_1=a_1+a_3+a_5+\ldots+a_{2n+1}$

$$=\frac{(n+1)}{2}\cdot\{a_1+a_{2n+1}\}=\frac{(n+1)}{2}\cdot\{a+a+(2n+1-1)d\}$$

$$=\frac{(n+1)}{2}\cdot 2(a+nd)=(n+1)(a+nd) \qquad \text{... (ii)}$$

And, $S_2=a_2+a_4+a_6+\ldots+a_{2n}$

$$=\frac{n}{2}\cdot\{a_2+a_{2n}\}=\frac{n}{2}\cdot\{(a+d)+a+(2n-1)d\}$$

$$=\frac{n}{2}\cdot 2\{a+nd\}=n(a+nd) \qquad \text{... (iii)}$$

$$\therefore \quad \frac{S_1}{S_2} = \frac{(n+1)(a+nd)}{n(a+nd)} = \frac{n+1}{n} \Rightarrow S_1 : S_2 = (n+1) : n.$$

52. Let a be the first term and d be the common difference.

Then, $S_m = \frac{m}{2} \cdot [2a + (m-1)d]$ and $S_n = \frac{n}{2} \cdot [2a + (n-1)d]$.

$$\therefore \quad \frac{\frac{m}{2}[2a+(m-1)d]}{\frac{n}{2}[2a+(n-1)d]} = \frac{S_m}{S_n} = \frac{m^2}{n^2}$$

$$\Rightarrow \quad \frac{2a+(m-1)d}{2a+(n-1)d} = \frac{m}{n} \Rightarrow 2an + n(m-1)d = 2am + m(n-1)d$$

$$\Rightarrow \quad (m-n)d = 2a(m-n) \Rightarrow d = 2a.$$

$$\therefore \quad \frac{T_m}{T_n} = \frac{a+(m-1)d}{a+(n-1)d} = \frac{a+(m-1)\times 2a}{a+(n-1)\times 2a} \quad [\because d = 2a]$$

$$= \frac{1+2(m-1)}{1+2(n-1)} = \frac{2m-1}{2n-1}.$$

$$\Rightarrow \quad T_m : T_n = (2m-1) : (2n-1).$$

53. Let the numbers be $(a-d)$, a and $(a+d)$. Then,

$$(a-d) + a + (a+d) = 24 \Rightarrow 3a = 24 \Rightarrow a = 8.$$

And, $(8-d) \times 8 \times (8+d) = 440 \Rightarrow (64 - d^2) = 55 \Rightarrow d^2 = 9 \Rightarrow d = \pm 3.$

So, the numbers are $\{(8-3), 8 \text{ and } (8+3)\}$ or $\{(8+3), 8 \text{ and } (8-3)\}$.

The largest of these numbers is 11.

54. Let the given angles be $(a-3d), (a-d), (a+d), (a+3d)$.

Then, $a - 3d + a - d + a + d + a + 3d = 360 \Rightarrow 4a = 360 \Rightarrow a = 90.$

Also, $d = 10°$ (given)

$\therefore$ the angles are $(90° - 3 \times 10°), (90° - 10°), (90° + 10°), (90° + 3 \times 10°)$

i.e., $60°, 80°, 100°$ and $120°$.

The largest of these angles is $120°$.

55. $\dfrac{(3+5+7+\ldots \text{ to } n \text{ terms})}{(5+8+11+\ldots \text{ to } 10 \text{ terms})} = 7$

$$\Rightarrow \quad \frac{\frac{n}{2} \cdot [6 + (n-1) \times 2]}{\frac{10}{2} \cdot [10 + 9 \times 3]} = 7 \Rightarrow n(n+2) = 1295$$

$$\Rightarrow \quad n^2 + 2n - 1295 = 0 \Rightarrow n^2 + 37n - 35n - 1295 = 0$$

$$\Rightarrow \quad n(n+37) - 35(n+37) = 0 \Rightarrow (n+37)(n-35) = 0 \Rightarrow n = 35.$$

56. The required numbers are 11, 15, 19, ..., 299.

This is an AP in which $a = 11$, $d = 4$ and $T_n = 299$.

$$T_n = 299 \Rightarrow a + (n-1)d = 299$$

$$\Rightarrow 11 + (n-1) \times 4 = 299 \Rightarrow (n-1) \times 4 = 288 \Rightarrow n - 1 = 72 \Rightarrow n = 73.$$

So, there are 73 such numbers.

57. Let the common difference of each AP be d. Then,

$$T_4 = -1 + 3d \text{ and } t_4 = -8 + 3d$$

$$\Rightarrow \quad (T_4 - t_4) = (-1 + 3d) - (-8 + 3d) = 7.$$

58. $(k-1), (2k+1), (6k+3)$ are in GP

$$\Rightarrow \quad \frac{(2k+1)}{(k-1)} = \frac{(6k+3)}{(2k+1)} = \frac{3(2k+1)}{(2k+1)} = 3$$

$$\Rightarrow \quad 2k + 1 = 3k - 3 \Rightarrow k = 4.$$

59. $\frac{-2}{7}, x, \frac{-7}{2}$ are in GP $\Rightarrow \dfrac{x}{\left(\frac{-2}{7}\right)} = \dfrac{\left(\frac{-7}{2}\right)}{x}$

$$\Rightarrow x^2 = \left(\frac{-2}{7}\right) \times \left(\frac{-7}{2}\right) = 1 \Rightarrow x = -1 \quad \text{or} \quad x = 1.$$

$\therefore$ required values of x are -1 and 1.

60. Given GP is $2, 2\sqrt{2}, 4, 4\sqrt{2}, \ldots$.

Here, $a = 2$ and $r = \frac{2\sqrt{2}}{2} = \sqrt{2}.$

$$\therefore \quad T_{17} = ar^{16} = 2 \times (\sqrt{2})^{16} = 2 \times 2^8 = 2^9 = 512.$$

Hence, 17th term is 512.

61. Given GP is $12, 4, \frac{4}{3}, \frac{4}{9}, \ldots$.

Here, $a = 12$ and $r = \frac{4}{12} = \frac{1}{3}.$

$$\therefore \quad T_n = ar^{n-1} = 12 \times \left(\frac{1}{3}\right)^{n-1} = 4 \times 3 \times \frac{1}{3^{n-1}} = \frac{4}{3^{(n-2)}}.$$

62. Given GP is 5, 10, 20, 40, … .

Here, $a = 5$ and $r = \frac{10}{5} = 2.$

Let $T_n = 5120$. Then, $ar^{n-1} = 5120 \Rightarrow 5 \times 2^{n-1} = 5120$

$$\therefore \quad 2^{n-1} = \frac{5120}{5} = 1024 = 2^{10} \Rightarrow n - 1 = 10 \Rightarrow n = 11.$$

Hence, the 11th term is 5120.

63. Given GP is $\sqrt{3}, 3, 3\sqrt{3}, \ldots$.

Here, $a = \sqrt{3}$ and $r = \frac{3}{\sqrt{3}} = \sqrt{3}.$

Let $T_n = 729$. Then, $ar^{n-1} = 729 \Rightarrow \sqrt{3} \times (\sqrt{3})^{n-1} = 729 = 3^6$

$$\therefore \quad (\sqrt{3})^n = (\sqrt{3})^{12} \Rightarrow n = 12.$$

Hence, the 12th term is 729.

64. Given $T_4 = 54$ and $T_9 = 13122$.

$$\therefore \quad ar^3 = 54 \text{ and } ar^8 = 13122$$

$\Rightarrow \quad \dfrac{ar^8}{ar^3} = \dfrac{13122}{54} = 243 \Rightarrow r^5 = 3^5 \Rightarrow r = 3.$

$\therefore \quad a \times 3^3 = 54 \Rightarrow a = \dfrac{54}{27} = 2.$

Thus, $a = 2$ and $r = 3$.

$\therefore \quad T_6 = ar^5 = (2 \times 3^5) = (2 \times 243) = 486.$

Hence, the 6th term is 486.

65. Given $T_4 = \dfrac{1}{18}$ and $T_7 = \dfrac{-1}{486}\cdot$

$\therefore \quad ar^3 = \dfrac{1}{18}$ and $ar^6 = \dfrac{-1}{486}$

$\Rightarrow \quad \dfrac{ar^6}{ar^3} = \dfrac{-1}{486} \times 18 = \dfrac{-1}{27} \Rightarrow r^3 = \dfrac{-1}{27} = \left(\dfrac{-1}{3}\right)^3 \Rightarrow r = \dfrac{-1}{3}\cdot$

$\therefore \quad a \times \left(\dfrac{-1}{3}\right)^3 = \dfrac{1}{18} \Rightarrow a \times \left(\dfrac{-1}{27}\right) = \dfrac{1}{18} \Rightarrow a = \dfrac{1}{18} \times (-27) = \dfrac{-3}{2}\cdot$

Hence, the first term is $\dfrac{-3}{2}\cdot$

66. Here, $a = 3, r = \dfrac{6}{3} = 2$ and $l = 12288$.

$\therefore \quad$ 8th term from the end $= \dfrac{l}{r^{(8-1)}} = \dfrac{l}{r^7} = \dfrac{12288}{2^7} = \dfrac{12288}{128} = 96.$

Hence, the 8th term from the end is 96.

67. Given GP is $8, 4, 2, \ldots, \dfrac{1}{1024}\cdot$

Here, $r = \dfrac{4}{8} = \dfrac{1}{2}$ and $l = \dfrac{1}{1024}\cdot$

$\therefore \quad$ 6th term from the end $= \dfrac{l}{r^{(6-1)}} = \dfrac{l}{r^5} = \dfrac{1}{1024} \cdot \dfrac{1}{\left(\dfrac{1}{2}\right)^5} = \dfrac{2^5}{1024} = \dfrac{32}{1024} = \dfrac{1}{32}\cdot$

68. Let a be the first term and r be the common ratio of the given GP. Then, $T_4 = ar^3, T_7 = ar^6$ and $T_{10} = ar^9$.

Clearly, $\dfrac{T_7}{T_4} = r^3$ and $\dfrac{T_{10}}{T_7} = r^3 \Rightarrow \dfrac{T_7}{T_4} = \dfrac{T_{10}}{T_7}\cdot$

$\therefore \quad T_4, T_7$ and T_{10} are in GP.

69. a, x, b are in GP $\Rightarrow \dfrac{x}{a} = \dfrac{b}{x} \Rightarrow x^2 = ab.$

70. Let the required numbers be a and b. Then,

$$\left(\frac{a+b}{2} = 34 \Rightarrow a + b = 68\right) \text{ and } \sqrt{ab} = 16 \Rightarrow ab = (16)^2 = 256.$$

$$(a-b)^2 = (a+b)^2 - 4ab = (68)^2 - 4 \times 256 = (4624 - 1024) = 3600$$

$\Rightarrow \quad a - b = \sqrt{3600} = 60.$

On solving $a + b = 68$ and $a - b = 60$, we get $a = 64, b = 4$.

$\therefore \quad$ required numbers are 64 and 4.

71. Here, $a = \frac{1}{3}$ and let r be the common ratio.

Then, $x_1 = ar = \frac{1}{3}r$, $x_2 = ar^2 = \frac{1}{3}r^2$ and $9 = ar^3 = \frac{1}{3}r^3$.

$\therefore \quad r^3 = 27 = 3^3 \Rightarrow r = 3.$

Hence, $x_2 = \frac{1}{3} \times 3^2 = \frac{1}{3} \times 9 = 3.$

72. $S_6 = 2 + 2^2 + 2^3 + 2^4 + 2^5 + 2^6.$

Here, $a = 2$ and $r = 2$.

$$S_n = \frac{a(r^n - 1)}{(r-1)} \Rightarrow S_6 = \frac{2(2^6 - 1)}{(2-1)} = 2 \times (64 - 1) = 2 \times 63 = 126.$$

73. Here, $a = 1$ and $r = \frac{\sqrt{3}}{1} = \sqrt{3} > 1.$

$$\therefore \quad S_n = \frac{a(r^n - 1)}{(r-1)} \Rightarrow S_{10} = \frac{a(r^{10} - 1)}{(r-1)} = \frac{1 \times [(\sqrt{3})^{10} - 1]}{(\sqrt{3} - 1)} = \frac{(3^5 - 1)}{(\sqrt{3} - 1)} \times \frac{(\sqrt{3} + 1)}{(\sqrt{3} + 1)}$$

$$\Rightarrow S_{10} = \frac{(243 - 1)}{2} \times (\sqrt{3} + 1) = \frac{242}{2} \cdot (\sqrt{3} + 1) = 121(\sqrt{3} + 1).$$

74. Here, $a = 0.15$ and $r = \frac{0.015}{0.15} = \frac{0.015}{0.015 \times 10} = \frac{1}{10} < 1.$

$$\therefore \quad S_n = \frac{a(1 - r^n)}{(1 - r)} = \frac{0.15\left[1 - \frac{1}{10^n}\right]}{\left(1 - \frac{1}{10}\right)} = \left(\frac{15}{100} \times \frac{10}{9}\right)\left(1 - \frac{1}{10^n}\right) = \frac{1}{6}\left(1 - \frac{1}{10^n}\right).$$

$$\therefore \quad S_8 = \frac{1}{6}\left(1 - \frac{1}{10^8}\right).$$

75. This is a GP in which $a = 1$ and $r = \frac{-1}{2}$.

$$S_n = \frac{a(1 - r^n)}{(1 - r)} \Rightarrow S_9 = \frac{a(1 - r^9)}{(1 - r)} = \frac{1 \times \left[1 - \left(\frac{-1}{2}\right)^9\right]}{\left(1 + \frac{1}{2}\right)} = \frac{2}{3} \times \left(1 + \frac{1}{2^9}\right)$$

$$\Rightarrow S_9 = \frac{2}{3} \times \left(1 + \frac{1}{512}\right) = \frac{2}{3} \times \frac{513}{512} = \frac{171}{256}.$$

76. This is a GP in which $a = 3, r = \frac{6}{3} = 2$ and $l = 1536$.

$$\therefore \quad \text{required sum} = \frac{(lr - a)}{(r - 1)} = \frac{(1536 \times 2 - 3)}{(2 - 1)} = (3072 - 3) = 3069.$$

77. This is a GP in which $a = 2, r = \frac{6}{2} = 3$ and $l = 4374$.

$$\therefore \quad \text{required sum} = \frac{(lr - a)}{(r - 1)} = \frac{(4374 \times 3 - 2)}{(3 - 1)} = \frac{(13122 - 2)}{2} = \frac{13120}{2} = 6560.$$

78. Let the common ratio be r. Then,

$$S_n = \frac{(lr - a)}{(r - 1)} \Rightarrow \frac{(96r - 3)}{(r - 1)} = 189$$

$\therefore \quad (96r - 3) = 189r - 189 \Rightarrow (189r - 96r) = 186 \Rightarrow 93r = 186 \Rightarrow r = 2.$

Now, $l = ar^{n-1} \Rightarrow 3 \times 2^{(n-1)} = 96 \Rightarrow 2^{(n-1)} = 32 = 2^5 \Rightarrow n - 1 = 5 \Rightarrow n = 6.$

Hence, $n = 6$.

79. Let the required number of terms be n.

Here, $a = 2, r = \frac{6}{2} = 3 > 1$ and $S_n = 728$.

$$S_n = \frac{a(r^n - 1)}{(r - 1)} \Rightarrow \frac{2(3^n - 1)}{(3 - 1)} = 728$$

$\therefore \quad 3^n - 1 = 728 \Rightarrow 3^n = 729 = 3^6 \Rightarrow n = 6.$

$\therefore \quad$ required number of terms = 6.

80. Let the required number of terms be n.

Here, $a = \frac{2}{9}, r = \frac{-1}{3} \times \frac{9}{2} = \frac{-3}{2} < 1$ and $S_n = \frac{55}{72}\cdot$

$$S_n = \frac{a(1 - r^n)}{(1 - r)} \Rightarrow \frac{\frac{2}{9}\left\{1 - \left(\frac{-3}{2}\right)^n\right\}}{\left(1 + \frac{3}{2}\right)} = \frac{55}{72}$$

$$\Rightarrow \left(\frac{2}{9} \times \frac{2}{5}\right)\left\{1 - \left(\frac{-3}{2}\right)^n\right\} = \frac{55}{72} \Rightarrow 1 - \left(\frac{-3}{2}\right)^n = \frac{55}{72} \times \frac{45}{4} = \frac{275}{32}$$

$$\Rightarrow \left(\frac{-3}{2}\right)^n = \left(1 - \frac{275}{32}\right) = \frac{(32 - 275)}{32} = \frac{-243}{32} = \left(\frac{-3}{2}\right)^5 \Rightarrow n = 5.$$

$\therefore \quad$ required number of terms = 5.

81. $S_n = (2^n - 1) \Rightarrow S_1 = (2^1 - 1) = 1, S_2 = (2^2 - 1) = (4 - 1) = 3.$

$$\Rightarrow T_1 = 1, T_2 = (S_2 - S_1) = (3 - 1) = 2$$

$$\Rightarrow r = \frac{T_2}{T_1} = \frac{2}{1} = 2.$$

Hence, the common ratio is 2.

82. $S_3 = \frac{a(r^3 - 1)}{(r - 1)}$ and $S_6 = \frac{a(r^6 - 1)}{(r - 1)}\cdot$

$$\therefore \quad \frac{S_3}{S_6} = \frac{125}{152} \Rightarrow \frac{a(r^3 - 1)}{(r - 1)} \times \frac{(r - 1)}{a(r^6 - 1)} = \frac{125}{152} \Rightarrow \frac{1}{(r^3 + 1)} = \frac{125}{152}$$

$$\Rightarrow 125r^3 + 125 = 152 \Rightarrow 125r^3 = (152 - 125) = 27$$

$$\Rightarrow r^3 = \frac{27}{125} = \left(\frac{3}{5}\right)^3 \Rightarrow r = \frac{3}{5}\cdot$$

$\therefore \quad$ the common ratio is $\frac{3}{5}\cdot$

83. In this GP, we have $a = 3$ and $r = \frac{\sqrt{3}}{3} = \frac{1}{\sqrt{3}}\cdot$

$$\therefore \quad T_n = \frac{1}{243} \Rightarrow ar^{n-1} = \frac{1}{243} \Rightarrow 3\times\left(\frac{1}{\sqrt{3}}\right)^{n-1} = \frac{1}{243}$$

$$\Rightarrow \left(\frac{1}{\sqrt{3}}\right)^{n-1} = \frac{1}{729} \Rightarrow \left(\frac{1}{3}\right)^{\left(\frac{n-1}{2}\right)} = \left(\frac{1}{3}\right)^6$$

$$\Rightarrow \frac{n-1}{2} = 6 \Rightarrow n = 13.$$

84. AM $\geq$ GM is always true.

85. $\text{AM} = 2\times \text{GM} \Rightarrow \frac{1}{2}(a+b) = 2\sqrt{ab} \Rightarrow \frac{a+b}{2\sqrt{ab}} = \frac{2}{1}$

$$\Rightarrow \frac{a+b+2\sqrt{ab}}{a+b-2\sqrt{ab}} = \frac{2+1}{2-1} = \frac{3}{1}$$

$$\Rightarrow \frac{(\sqrt{a}+\sqrt{b})^2}{(\sqrt{a}-\sqrt{b})^2} = \frac{(\sqrt{3})^2}{(1)^2} \Rightarrow \frac{\sqrt{a}+\sqrt{b}}{\sqrt{a}-\sqrt{b}} = \frac{\sqrt{3}}{1}$$

$$\Rightarrow \frac{(\sqrt{a}+\sqrt{b})+(\sqrt{a}-\sqrt{b})}{(\sqrt{a}+\sqrt{b})-(\sqrt{a}-\sqrt{b})} = \frac{\sqrt{3}+1}{\sqrt{3}-1} \Rightarrow \frac{2\sqrt{a}}{2\sqrt{b}} = \frac{\sqrt{3}+1}{\sqrt{3}-1}$$

$$\Rightarrow \frac{\sqrt{a}}{\sqrt{b}} = \frac{\sqrt{3}+1}{\sqrt{3}-1} \Rightarrow \frac{a}{b} = \frac{(\sqrt{3}+1)^2}{(\sqrt{3}-1)^2} = \frac{2+\sqrt{3}}{2-\sqrt{3}}\cdot$$

86. $\text{GM} = \sqrt{27\times 243} = \sqrt{81\times 81} = 81.$

87. $\text{GM} = \sqrt{0.15\times 0.0015} = \sqrt{\frac{15}{100}\times\frac{15}{10000}} = \sqrt{\frac{15\times 15}{10^6}} = \frac{15}{10^3} = \frac{15}{1000} = 0.015.$

88. In the given GP, we have $a = 1$ and $r = \frac{1}{3}\cdot$

$$\therefore \quad S_\infty = \frac{a}{(1-r)} = \frac{1}{\left(1-\frac{1}{3}\right)} = \frac{3}{2}\cdot$$

89. In the given geometric series $a = 1$ and $r = \frac{-1}{3}\cdot$ Clearly, $|r| = \frac{1}{3} < 1.$

$$\therefore \quad S_\infty = \frac{a}{(1-r)} = \frac{1}{\left(1+\frac{1}{3}\right)} = \frac{3}{4}\cdot$$

90. In the given geometric series $a = \frac{-5}{4}$ and $r = \frac{5}{16}\times\frac{(-4)}{5} = \frac{-1}{4}\cdot$

Clearly, $|r| = \frac{1}{4} < 1.$

$$\therefore \quad S_\infty = \frac{a}{(1-r)} = \frac{\left(\frac{-5}{4}\right)}{\left(1+\frac{1}{4}\right)} = \left(\frac{-5}{4}\times\frac{4}{5}\right) = -1.$$

91. Here $a = (\sqrt{2} + 1)$ and $r = \dfrac{1}{(\sqrt{2}+1)} \times \dfrac{(\sqrt{2}-1)}{(\sqrt{2}-1)} = \dfrac{(\sqrt{2}-1)}{1} = (\sqrt{2}-1)$ and $|r| < 1$.

$$\therefore \quad S_\infty = \frac{a}{(1-r)} = \frac{(\sqrt{2}+1)}{\{1-(\sqrt{2}-1)\}} = \frac{(\sqrt{2}+1)}{(2-\sqrt{2})} \times \frac{(2+\sqrt{2})}{(2+\sqrt{2})} = \frac{4+3\sqrt{2}}{(4-2)} = \frac{4+3\sqrt{2}}{2}.$$

92. Given series $= \left\{\dfrac{2}{5} + \dfrac{2}{5^3} + \dfrac{2}{5^5} + \ldots \infty\right\} + \left\{\dfrac{3}{5^2} + \dfrac{3}{5^4} + \ldots \infty\right\}$

$$= \frac{\left(\frac{2}{5}\right)}{\left\{1 - \frac{2}{125} \times \frac{5}{2}\right\}} + \frac{\left(\frac{3}{5^2}\right)}{\left\{1 - \frac{3}{5^4} \times \frac{5^2}{3}\right\}} = \frac{\left(\frac{2}{5}\right)}{\left(1 - \frac{1}{25}\right)} + \frac{\left(\frac{3}{25}\right)}{\left(1 - \frac{1}{25}\right)}$$

$$= \left(\frac{2}{5} \times \frac{25}{24}\right) + \left(\frac{3}{25} \times \frac{25}{24}\right) = \left(\frac{5}{12} + \frac{1}{8}\right) = \frac{(10+3)}{24} = \frac{13}{24}.$$

93. Here, $r = \dfrac{-4}{5}$ and so $|r| = \dfrac{4}{5} < 1$.

$$S_\infty = \frac{80}{9} \Rightarrow \frac{a}{(1-r)} = \frac{80}{9} \Rightarrow \frac{a}{\left(1 + \frac{4}{5}\right)} = \frac{80}{9}$$

$$\Rightarrow \frac{5a}{9} = \frac{80}{9} \Rightarrow a = \left(\frac{80}{9} \times \frac{9}{5}\right) = 16.$$

$\therefore$ the first term = 16.

94. Let the given geometric series be $(a + ar + ar^2 + \ldots \infty)$.

Then, $ar = 2$ and $\dfrac{a}{(1-r)} = 8$.

Putting, $r = \dfrac{2}{a}$, we get

$$\frac{a}{\left(1 - \frac{2}{a}\right)} = 8 \Rightarrow \frac{a^2}{(a-2)} = 8 \Rightarrow a^2 = 8a - 16$$

$$\Rightarrow a^2 - 8a + 16 = 0 \Rightarrow (a-4)^2 = 0 \Rightarrow a - 4 = 0 \Rightarrow a = 4.$$

$$\therefore \quad r = \frac{2}{4} = \frac{1}{2}.$$

Hence, the common ratio is $\dfrac{1}{2}$.

95. Let $(a + ar + ar^2 + \ldots \infty) = 15$ and $(a^2 + a^2r^2 + a^2r^4 + \ldots \infty) = 45$.

Then, $\dfrac{a}{(1-r)} = 15$ and $\dfrac{a^2}{(1-r^2)} = 45$.

On dividing, we get $\dfrac{a^2}{(1-r^2)} \times \dfrac{(1-r)}{a} = \dfrac{45}{15} \Rightarrow \dfrac{a}{(1+r)} = 3$

$$\Rightarrow \frac{15(1-r)}{(1+r)} = 3 \qquad \left[\text{using } \frac{a}{(1-r)} = 15\right]$$

$\Rightarrow \quad 3+3r = 15-15r \Rightarrow 18r = 12 \Rightarrow r = \frac{12}{18} = \frac{2}{3}.$

$\therefore \quad \frac{a}{\left(1-\frac{2}{3}\right)} = 15 \Rightarrow 3a = 15 \Rightarrow a = 5.$

Hence, the first term is 5.

96. Let $a^{1/x} = b^{1/y} = c^{1/z} = k$. Then, $a = k^x$, $b = k^y$ and $c = k^z$.

Now, a, b, c are in GP.

$\Rightarrow \quad b^2 = ac \Rightarrow (k^y)^2 = (k^x \times k^z)$

$\Rightarrow \quad k^{2y} = k^{(x+z)} \Rightarrow 2y = x+z$

$\Rightarrow \quad x, y, z$ are in AP.

97. a, b, c are in GP

$\Rightarrow \quad b^2 = ac \Rightarrow \log(b^2) - \log(ac)$

$\Rightarrow \quad 2\log b = \log a + \log c \Rightarrow \log a, \log b, \log c$ are in AP.

98. $y = x(1+x+x^2+\ldots\infty) = x\cdot\frac{1}{(1-x)}$ [sum of infinite GP]

$\Rightarrow \quad (1-x)y = x \Rightarrow y - xy = x \Rightarrow y = x+xy = x(1+y)$

$\Rightarrow \quad x = \frac{y}{(1+y)}.$

99. $x = \frac{1}{(1-a)}$ and $y = \frac{1}{(1-b)}.$

$\therefore \quad (1+ab+a^2b^2+\ldots\infty) = \frac{1}{(1-ab)}.$

$$\text{Now, } \frac{xy}{(x+y-1)} = \frac{\left\{\frac{1}{(1-a)}\cdot\frac{1}{(1-b)}\right\}}{\left\{\frac{1}{(1-a)}+\frac{1}{(1-b)}-1\right\}}$$

$$= \left\{\frac{1}{(1-a)(1-b)} \times \frac{(1-a)(1-b)}{(1-b)+(1-a)-(1-a)(1-b)}\right\} = \frac{1}{(1-ab)}.$$

$\therefore \quad (1+ab+a^2b^2+\ldots\infty) = \frac{xy}{(x+y-1)}.$

100. $x = \frac{a}{\left(1-\frac{1}{r}\right)} = \frac{ar}{(r-1)}$, $y = \frac{b}{\left(1+\frac{1}{r}\right)} = \frac{br}{(r+1)}$ and $z = \frac{c}{\left(1-\frac{1}{r^2}\right)} = \frac{cr^2}{(r^2-1)}.$

$\therefore \quad \frac{xy}{z} = \frac{ar}{(r-1)} \times \frac{br}{(r+1)} \times \frac{(r^2-1)}{cr^2} = \frac{ab}{c}.$

101. a, b, c are in AP $\Rightarrow 2b = (a+c)$.

a, x, b are in GP $\Rightarrow x^2 = ab$.

b, y, c are in GP $\Rightarrow y^2 = bc$.

$\therefore \quad x^2 + y^2 = ab + bc = b(a + c) = 2b^2$

$\Rightarrow \quad x^2, b^2, y^2$ are in AP.

102. a, b, c are in AP $\Rightarrow 2b = (a + c)$.

a, b, d are in GP $\Rightarrow b^2 = ad$.

$\therefore \quad a(d - c) = ad - ac = b^2 - ac = b^2 - a(2b - a) = a^2 + b^2 - 2ab = (a - b)^2$.

$\therefore \quad a, (a - b), (d - c)$ are in GP.

103. Let A be the first term and R be the common ratio of the GP.

Then, $T_p = AR^{(p-1)}, T_q = AR^{(q-1)}$ and $T_r = AR^{(r-1)}$.

$\therefore \quad a = AR^{(p-1)}, b = AR^{(q-1)}$ and $c = AR^{(r-1)}$.

$$\therefore \quad a^{(q-r)} \cdot b^{(r-p)} \cdot c^{(p-q)} = A^{(q-r)} \cdot R^{(p-1)(q-r)} \cdot A^{(r-p)} R^{(r-p)(q-1)} \cdot A^{(p-q)} R^{(r-1)(p-q)}$$

$$= A^{(p-q)} \cdot R^{(p-q)(r-1)} = A^0 R^0 = 1.$$

104. a, b, c are in AP $\Rightarrow \dfrac{a}{abc}, \dfrac{b}{abc}, \dfrac{c}{abc}$ are in AP

$\Rightarrow \dfrac{1}{bc}, \dfrac{1}{ca}, \dfrac{1}{ab}$ are in AP

$\Rightarrow bc, ca, ab$ are in HP.

105. a^2, b^2, c^2 are in AP $\Rightarrow (a^2 + ab + bc + ca), (b^2 + ab + bc + ca), (c^2 + ab + bc + ca)$ are in AP

$\Rightarrow (a + b)(c + a), (a + b)(b + c), (c + a)(b + c)$ are in AP

$\Rightarrow \dfrac{1}{b + c}, \dfrac{1}{c + a}, \dfrac{1}{a + b}$ are in AP

$\Rightarrow (b + c), (c + a), (a + b)$ are in HP.

106. a, b, c are in AP $\Rightarrow \dfrac{a}{abc}, \dfrac{b}{abc}, \dfrac{c}{abc}$ are in AP

$\Rightarrow \dfrac{1}{bc}, \dfrac{1}{ca}, \dfrac{1}{ab}$ are in AP

$\Rightarrow \dfrac{(ab + bc + ca)}{bc}, \dfrac{(ab + bc + ca)}{ca}, \dfrac{(ab + bc + ca)}{ab}$ are in AP

$\Rightarrow \left\{\dfrac{(ab + bc + ca)}{bc} - 1\right\}, \left\{\dfrac{(ab + bc + ca)}{ca} - 1\right\}, \left\{\dfrac{(ab + bc + ca)}{ab} - 1\right\}$

are in AP

$\Rightarrow \dfrac{a(b + c)}{bc}, \dfrac{b(c + a)}{ca}, \dfrac{c(a + b)}{ab}$ are in AP.

107. $\dfrac{b + c - a}{a}, \dfrac{c + a - b}{b}, \dfrac{a + b - c}{c}$ are in AP

$\Rightarrow \left\{\dfrac{b + c - a}{a} + 2\right\}, \left\{\dfrac{c + a - b}{b} + 2\right\}, \left\{\dfrac{a + b - c}{c} + 2\right\}$ are in AP

$\Rightarrow \dfrac{(a + b + c)}{a}, \dfrac{(a + b + c)}{b}, \dfrac{(a + b + c)}{c}$ are in AP $\Rightarrow \dfrac{1}{a}, \dfrac{1}{b}, \dfrac{1}{c}$ are in AP

$\Rightarrow a, b, c$ are in HP.

108. $(1+2+3+\ldots+n)=\frac{1}{2}n(n+1)$.

$\therefore$ the given sum $=(1+2+3+\ldots+80)=\left(\frac{1}{2}\times 80\times 81\right)=3240.$

109. $(1^2+2^2+3^2+\ldots+n^2)=\frac{1}{6}n(n+1)(2n+1)$.

$\therefore$ the given sum $=(1^2+2^2+3^2+\ldots+60^2)=\left(\frac{1}{6}\times 60\times 61\times 121\right)=73810.$

110. $(1^3+2^3+3^3+\ldots+n^3)=\left\{\frac{1}{2}n(n+1)\right\}^2$

$\therefore$ the given sum $=(1^3+2^3+3^3+\ldots+40^3)=\left(\frac{1}{2}\times 40\times 41\right)^2=(820)^2=672400.$

❑

Trigonometry

REVIEW OF FACTS AND FORMULAE

1. **Measurement of angles**

   I. **Sexagesimal system**

   1 right angle = 90°.

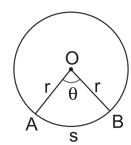
   $1° = 60$ minutes $= 60'$.

   $1' = 60$ seconds $= 60''$.

   II. **Circular measures**

   π radian, written as $\pi^c = 180°$.

2. $\theta = \dfrac{s}{r} = \dfrac{\text{arc}}{\text{radius}}\cdot$



3. In a clock, we have:

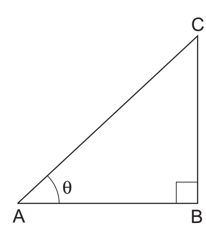
   (i) Angle between two consecutive digits

   $= 30° = \dfrac{\pi}{6}$ radian.

   (ii) Angle traced by the hour hand in 1 hr = 30°.

   (iii) Angle traced by the minute hand in 1 min = 6°.

4. **T-ratios**



   (i) $\sin\theta = \dfrac{\text{perpendicular}}{\text{hypotenuse}} = \dfrac{BC}{AC}\cdot$

   (ii) $\cos\theta = \dfrac{\text{base}}{\text{hypotenuse}} = \dfrac{AB}{AC}\cdot$

   (iii) $\tan\theta = \dfrac{\text{perpendicular}}{\text{base}} = \dfrac{BC}{AB}\cdot$

   (iv) $\text{cosec}\,\theta = \dfrac{1}{\sin\theta}\cdot$ (v) $\sec\theta = \dfrac{1}{\cos\theta}\cdot$ (vi) $\cot\theta = \dfrac{1}{\tan\theta}\cdot$

5. (i) $\sin^2\theta + \cos^2\theta = 1$. (ii) $1 + \tan^2\theta = \sec^2\theta$. (iii) $1 + \cot^2\theta = \text{cosec}^2\theta$.

6. **Extremum values of T-ratios**

   (i) $-1 \le \sin\theta \le 1 \;\Rightarrow\; |\sin\theta| \le 1$. (ii) $-1 \le \cos\theta \le 1 \;\Rightarrow\; |\cos\theta| \le 1$.

   (iii) Either $\sec\theta \le -1$ or $\sec\theta \ge 1$.

   $\therefore \quad |\sec\theta| \ge 1$.

   (iv) Either $\text{cosec}\,\theta \le -1$ or $\text{cosec}\,\theta \ge 1$.

   $\therefore \quad |\text{cosec}\,\theta| \ge 1$.

7. **Angles in various quadrants**

Quadrant		Angle
(i)	I	$0° < \theta < 90°$
(ii)	II	$90° < \theta < 180°$
(iii)	III	$180° < \theta < 270°$
(iv)	IV	$270° < \theta < 360°$

8. **Signs of T-ratios**

Quadrant		Signs
(i)	I	All +*ve*
(ii)	II	$\sin\theta > 0$, $\operatorname{cosec}\theta > 0$
(iii)	III	$\tan\theta > 0$ and $\cot\theta > 0$
(iv)	IV	$\cos\theta > 0$ and $\sec\theta > 0$.

Remember:

II sin	I All
III tan	IV cos

$\boxed{\text{All: sin, tan, cos}}$

9. **Values of T-ratios**

	0°	**(π/6)** **30°**	**(π/4)** **45°**	**(π/3)** **60°**	**(π/2)** **90°**
$\sin\theta$	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1
$\cos\theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0
$\tan\theta$	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	not defined

10.

	$(90° - \theta)$	$(90° + \theta)$	$(180° - \theta)$	$(180° + \theta)$
sin	$\cos\theta$	$\cos\theta$	$\sin\theta$	$-\sin\theta$
cos	$\sin\theta$	$-\sin\theta$	$-\cos\theta$	$-\cos\theta$
tan	$\cot\theta$	$-\cot\theta$	$-\tan\theta$	$\tan\theta$

11. (i) $\sin(-\theta) = -\sin\theta$. (ii) $\cos(-\theta) = \cos\theta$. (iii) $\tan(-\theta) = -\tan\theta$.

12. (i) $\sin 15° = \cos 75° = \frac{(\sqrt{3}-1)}{2\sqrt{2}}$. (ii) $\cos 15° = \sin 75° = \frac{(\sqrt{3}+1)}{2\sqrt{2}}$.

(iii) $\tan 15° = \cot 75° = (2-\sqrt{3})$. (iv) $\cot 15° = \tan 75° = (2+\sqrt{3})$.

13. (i) $\sin 18° = \cos 72° = \frac{(\sqrt{5}-1)}{4}$. (ii) $\cos 18° = \sin 72° = \frac{\sqrt{10+2\sqrt{5}}}{4}$.

(iii) $\sin 36° = \cos 54° = \frac{\sqrt{10-2\sqrt{5}}}{4}$. (iv) $\cos 36° = \sin 54° = \frac{(\sqrt{5}+1)}{4}$.

14. (i) $\sin(A+B) = \sin A \cos B + \cos A \sin B$.

(ii) $\cos(A+B) = \cos A \cos B - \sin A \sin B$.

(iii) $\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$.

(iv) $\cot(A+B) = \frac{\cot A \cot B - 1}{\cot A + \cot B}$.

15. (i) $\sin(A-B) = \sin A \cos B - \cos A \sin B$.

(ii) $\cos(A-B) = \cos A \cos B + \sin A \sin B$.

(iii) $\tan(A-B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}$.

(iv) $\cot(A-B) = \frac{\cot A \cot B + 1}{\cot B - \cot A}$.

16. (i) $\sin(A+B) \cdot \sin(A-B) = \sin^2 A - \sin^2 B = \cos^2 B - \sin^2 A$.

(ii) $\cos(A+B) \cdot \cos(A-B) = \cos^2 A - \sin^2 B = \cos^2 B - \sin^2 A$.

17. (i) $\sin(A+B) + \sin(A-B) = 2\sin A \cos B$.

(ii) $\sin(A+B) - \sin(A-B) = 2\cos A \sin B$.

(iii) $\cos(A+B) + \cos(A-B) = 2\cos A \cos B$.

(iv) $\cos(A-B) - \cos(A+B) = 2\sin A \sin B$.

18. (i) $\sin C + \sin D = 2\sin\frac{(C+D)}{2}\cos\frac{(C-D)}{2}$.

(ii) $\cos C + \cos D = 2\cos\frac{(C+D)}{2}\cos\frac{(C-D)}{2}$.

19. (i) $\sin C - \sin D = 2\cos\frac{(C+D)}{2}\sin\frac{(C-D)}{2}$.

(ii) $\cos C - \cos D = 2\sin\frac{(C+D)}{2}\sin\frac{(D-C)}{2}$.

T-RATIOS OF MULTIPLE ANGLES

20. (i) $\sin 2A = 2\sin A \cos A = \frac{2\tan A}{1+\tan^2 A}$.

(ii) $\cos 2A = (\cos^2 A - \sin^2 A) = (1 - 2\sin^2 A) = (2\cos^2 A - 1) = \dfrac{1 - \tan^2 A}{1 + \tan^2 A}$.

(iii) $\tan 2A = \dfrac{2\tan A}{1 - \tan^2 A}$.

21. (i) $\sin 3A = 3\sin A - 4\sin^3 A$. (ii) $\cos 3A = 4\cos^3 A - 3\cos A$.

(iii) $\tan 3A = \dfrac{3\tan A - \tan^3 A}{1 - 3\tan^2 A}$.

TRIGONOMETRIC EQUATIONS

22. (i) $\sin\theta = 0 \Leftrightarrow \theta = n\pi,\ n \in I$.

(ii) $\cos\theta = 0 \Leftrightarrow \theta = (2n+1)\dfrac{\pi}{2},\ n \in I$.

(iii) $\tan\theta = 0 \Leftrightarrow \theta = n\pi,\ n \in I$.

23. (i) $\sin\theta = \sin\alpha \Rightarrow \theta = n\pi + (-1)^n\alpha,\ n \in I$.

(ii) $\cos\theta = \cos\alpha \Rightarrow \theta = 2n\pi \pm \alpha,\ n \in I$.

(iii) $\tan\theta = \tan\alpha \Rightarrow \theta = n\pi + \alpha,\ n \in I$.

24. (i) $\sin^2\theta = \sin^2\alpha \Rightarrow \theta = n\pi \pm \alpha,\ n \in I$.

(ii) $\cos^2\theta = \cos^2\alpha \Rightarrow \theta = n\pi \pm \alpha,\ n \in I$.

(iii) $\tan^2\theta = \tan^2\alpha \Rightarrow \theta = n\pi \pm \alpha,\ n \in I$.

SOLUTION OF TRIANGLES

25. **Sine rule**

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}.$$

26. **Cosine formulae**

(i) $a^2 = b^2 + c^2 - 2bc\cos A$. (ii) $b^2 = c^2 + a^2 - 2ca\cos B$.

(iii) $c^2 = a^2 + b^2 - 2ab\cos C$.

27. **Projection formulae**

(i) $a = a\cos B + b\cos C$. (ii) $b = a\cos C + c\cos A$.

(iii) $c = a\cos B + b\cos A$.

28. **Half-angle formulae**

(i) $\sin\dfrac{A}{2} = \sqrt{\dfrac{(s-b)(s-c)}{bc}}$. (ii) $\sin\dfrac{B}{2} = \sqrt{\dfrac{(s-c)(s-a)}{ca}}$.

(iii) $\sin\dfrac{C}{2} = \sqrt{\dfrac{(s-a)(s-b)}{ab}}$. (iv) $\cos\dfrac{A}{2} = \sqrt{\dfrac{s(s-a)}{bc}}$.

(v) $\cos\dfrac{B}{2} = \sqrt{\dfrac{s(s-b)}{ca}}$. (vi) $\cos\dfrac{C}{2} = \sqrt{\dfrac{s(s-c)}{ab}}$.

29. (i) $\tan \frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}$. (ii) $\tan \frac{B}{2} = \sqrt{\frac{(s-c)(s-a)}{s(s-b)}}$.

(iii) $\tan \frac{C}{2} = \sqrt{\frac{(s-a)(s-b)}{s(s-c)}}$.

30. **Area of a triangle is:**

$$\Delta = \frac{1}{2} ab \sin C = \frac{1}{2} bc \sin A = \frac{1}{2} ac \sin B.$$

31. **Napier's analogies**

(i) $\tan \frac{(B-C)}{2} = \frac{(b-c)}{(b+c)} \cot \frac{A}{2}$. (ii) $\tan \frac{(C-A)}{2} = \frac{(c-a)}{(c+a)} \cot \frac{B}{2}$.

(iii) $\tan \frac{(A-B)}{2} = \frac{(a-b)}{(a+b)} \cot \frac{C}{2}$.

EXERCISE 1

Mark (✓) against the correct answer in each of the following.

1. $25°$ when measured in radians is

(a) $\left(\frac{5\pi}{18}\right)^c$ (b) $\left(\frac{5\pi}{24}\right)^c$ (c) $\left(\frac{5\pi}{36}\right)^c$ (d) none of these

2. $162°$ when measured in radians is

(a) $\left(\frac{7\pi}{10}\right)^c$ (b) $\left(\frac{9\pi}{10}\right)^c$ (c) $\left(\frac{4\pi}{3}\right)^c$ (d) $\left(\frac{5\pi}{4}\right)^c$

3. $\left(\frac{8\pi}{5}\right)^c = ?$

(a) $272°$ (b) $302°$ (c) $288°$ (d) $316°$

4. $11^c = ?$

(a) $315°$ (b) $372°$ (c) $418°$ (d) $630°$

5. $1^c = ?$

(a) $56°27'22''$ (b) $57°16'22''$ (c) $55°18'32''$ (d) $57°26'32''$

6. $3°45'$ expressed in radians is

(a) $\left(\frac{\pi}{36}\right)^c$ (b) $\left(\frac{\pi}{54}\right)^c$ (c) $\left(\frac{\pi}{48}\right)^c$ (d) $\left(\frac{5\pi}{96}\right)^c$

7. $50°37'30'' = ?$

(a) $\left(\frac{5\pi}{16}\right)^c$ (b) $\left(\frac{7\pi}{18}\right)^c$ (c) $\left(\frac{9\pi}{32}\right)^c$ (d) $\left(\frac{11\pi}{36}\right)^c$

8. In a right triangle, the difference between two acute angles is $\left(\frac{\pi}{15}\right)^c$. The measure of the smallest angle is

(a) 40° (b) 45° (c) 36° (d) 39°

9. The angles of a triangle are in AP and the greatest angle is double the least. The largest angle measures

(a) 60° (b) 80° (c) 75° (d) 90°

10. The angles of a triangle are in AP and the ratio of the number of degrees in the least to the number of radians in the greatest is $60 : \pi$. The smallest angle is

(a) 15° (b) 30° (c) 45° (d) 60°

11. In a circle, the central angle of 45° intercepts an arc of length 33 cm. The radius of the circle is

(a) 21 cm (b) 35 cm (c) 42 cm (d) 14 cm

12. In a circle of radius 14 cm an arc subtends an angle of 36° at the centre. The length of the arc is

(a) 6.6 cm (b) 7.7 cm (c) 8.8 cm (d) 9.1 cm

13. The minute hand of a watch is 1.4 cm long. How far does its tip move in 45 minutes?

(a) 6 cm (b) 6.3 cm (c) 6.6 cm (d) 7 cm

14. If the arcs of the same length in two circles subtend angles of 60° and 75° at their respective centres, the ratio of their radii is

(a) 4 : 5 (b) 5 : 4 (c) 3 : 5 (d) 5 : 3

15. A wire of length 121 cm is bent to form an arc of a circle of radius 180 cm. The angle subtended at the centre by the arc is

(a) $36°20'$ (b) $34°40'$ (c) $38°30'$ (d) $39°10'$

16. A horse is tied to a post by a rope. If the horse moves along a circular path, always keeping the rope tight and describes 88 m when it traces 72° at the centre, then the length of the rope is

(a) 35 m (b) 70 m (c) 17.5 m (d) 22 m

17. A pendulum swings through an angle of 42° in describing an arc of length 55 cm. The length of the pendulum is

(a) 56 cm (b) 60 cm (c) 75 cm (d) 88 cm

18. The radius of a circle is 30 cm. The length of the arc of this circle whose chord is 30 cm long, is

(a) 9π cm (b) 10π cm (c) 12π cm (d) 13.6π cm

19. A wheel makes 180 revolutions in 1 minute. How many radians does it turn in 1 second?

(a) $(3\pi)^c$ (b) $(4\pi)^c$ (c) $(6\pi)^c$ (d) $(12\pi)^c$

20. A railway train is moving on a circular curve of radius 1500 m at a speed of 90 km/hr. Through what angle has it turned in 11 seconds?

(a) $10°30'$ (b) $11°40'$ (c) $12°$ (d) $16°30'$

21. When a clock shows the time 7 : 20, what is the angle between its minute hand and the hour hand?

(a) $60°$ (b) $80°$ (c) $100°$ (d) $120°$

22. The angle between the hour hand and the minute hand of a clock at half past three is

(a) $54°$ (b) $63°$ (c) $72°$ (d) $75°$

23. The angle between the minute hand and the hour hand of a clock when the time is 8 : 25 am, is

(a) $107°15'$ (b) $105°$ (c) $102°30'$ (d) $92°45'$

24. The length of a pendulum is 60 cm. The angle through which it swings when its tip describes an arc of length 16.5 cm is

(a) $15°30'$ (b) $15°45'$ (c) $16°15'$ (d) $16°30'$

25. The angles of a quadrilateral in degrees are in AP and the greatest angle is 120°. The smallest angle is

(a) $\left(\frac{\pi}{4}\right)^c$ (b) $\left(\frac{\pi}{3}\right)^c$ (c) $\left(\frac{\pi}{5}\right)^c$ (d) $\left(\frac{\pi}{6}\right)^c$

26. The perimeter of a sector of a circle is equal to half the circumference of the circle. The angle of the sector is

(a) $\left(\frac{\pi}{4}\right)^c$ (b) $\left(\frac{\pi}{2}\right)^c$ (c) $(\pi - 2)^c$ (d) $(\pi + 2)^c$

27. $\sin \frac{25\pi}{3} = ?$

(a) $\frac{1}{2}$ (b) $\frac{1}{\sqrt{2}}$ (c) $\frac{\sqrt{3}}{2}$ (d) $\frac{-\sqrt{3}}{2}$

28. $\cos \frac{41\pi}{4} = ?$

(a) $\frac{1}{\sqrt{2}}$ (b) $\frac{-1}{\sqrt{2}}$ (c) $\frac{\sqrt{3}}{2}$ (d) $\frac{-\sqrt{3}}{2}$

29. $\tan\left(\frac{-16\pi}{3}\right) = ?$

(a) $\sqrt{3}$ (b) $-\sqrt{3}$ (c) $\frac{1}{\sqrt{3}}$ (d) $\frac{-1}{\sqrt{3}}$

30. $\cot\left(\frac{29\pi}{4}\right) = ?$

(a) -1 (b) 1 (c) $\sqrt{3}$ (d) $\frac{1}{\sqrt{3}}$

31. $\sec\left(\frac{-19\pi}{3}\right) = ?$

(a) -2 (b) $\frac{-1}{2}$ (c) 2 (d) $\frac{1}{2}$

32. $\text{cosec}\left(\frac{-33\pi}{4}\right) = ?$

(a) $-\sqrt{2}$ (b) $\sqrt{2}$ (c) $\frac{1}{\sqrt{2}}$ (d) $\frac{-1}{\sqrt{2}}$

33. $\cos 15\pi = ?$

(a) 1 (b) -1 (c) 0 (d) none of these

34. $\sec 6\pi = ?$

(a) 1 (b) -1 (c) $\sqrt{2}$ (d) $-\sqrt{2}$

35. $\tan \frac{5\pi}{4} = ?$

(a) $\sqrt{3}$ (b) $\frac{1}{\sqrt{3}}$ (c) 1 (d) -1

36. $\sin(765°) = ?$

(a) $\frac{\sqrt{3}}{2}$ (b) $\frac{-\sqrt{3}}{2}$ (c) $\frac{1}{\sqrt{2}}$ (d) $\frac{-1}{\sqrt{2}}$

37. $\cot(-600°) = ?$

(a) -1 (b) $-\sqrt{3}$ (c) $\frac{-1}{\sqrt{3}}$ (d) none of these

38. If $\sin x = \frac{-2\sqrt{6}}{5}$ and x lies in quadrant III, then $\cot x = ?$

(a) $\frac{1}{2\sqrt{6}}$ (b) $\frac{-1}{2\sqrt{6}}$ (c) $\frac{3}{2\sqrt{6}}$ (d) $\frac{-3}{2\sqrt{6}}$

39. If $\cos x = \frac{-\sqrt{15}}{4}$ and $\frac{\pi}{2} < x < \pi$, then $\sin x = ?$

(a) $\frac{3}{4}$ (b) $\frac{-3}{4}$ (c) $\frac{1}{4}$ (d) $\frac{-1}{4}$

40. If $\sec x = -2$ and $\pi < x < \frac{3\pi}{2}$, then $\sin x = ?$

(a) $\frac{\sqrt{3}}{2}$ (b) $\frac{-\sqrt{3}}{2}$ (c) $\frac{1}{2}$ (d) $\frac{-1}{2}$

41. If $\text{cosec}\, x = \frac{-2}{\sqrt{3}}$ and x lies in quadrant IV, then $\tan x = ?$

(a) $\frac{1}{\sqrt{3}}$ (b) $\frac{-1}{\sqrt{3}}$ (c) $\sqrt{3}$ (d) $-\sqrt{3}$

42. If $\cot\theta = \sqrt{5}$ and θ does not lie in quadrant I, then the values of $\operatorname{cosec}\theta$ and $\sec\theta$ are respectively

(a) $\sqrt{6}, \sqrt{\frac{6}{5}}$ (b) $-\sqrt{6}, \sqrt{\frac{6}{5}}$ (c) $-\sqrt{6}, -\sqrt{\frac{6}{5}}$ (d) $\sqrt{6}, -\sqrt{\frac{6}{5}}$

43. If $\cos\theta = \frac{-1}{2}$ and θ lies in quadrant II, then $(2\sin\theta + \tan\theta) = ?$

(a) 0 (b) $\frac{-\sqrt{3}}{2}$ (c) $\frac{3\sqrt{3}}{2}$ (d) none of these

44. If $\cos\theta = \frac{-3}{5}$ and $\pi < \theta < \frac{3\pi}{2}$, then $\frac{(\operatorname{cosec}\theta + \cot\theta)}{(\sec\theta - \tan\theta)} = ?$

(a) $\frac{1}{6}$ (b) $\frac{2}{3}$ (c) $\frac{3}{2}$ (d) $\frac{1}{3}$

45. If $\sin\theta = \frac{3}{5}$ and $\frac{\pi}{2} < \theta < \pi$, then $(2\sec\theta - 3\cot\theta) = ?$

(a) $\frac{3}{2}$ (b) $\frac{-3}{2}$ (c) $\frac{13}{2}$ (d) $\frac{-13}{2}$

46. If $\sec\theta = \sqrt{2}$ and $\frac{3\pi}{2} < \theta < 2\pi$, then $\frac{(1 + \tan\theta + \operatorname{cosec}\theta)}{(1 + \cot\theta - \operatorname{cosec}\theta)} = ?$

(a) $\frac{\sqrt{3}}{2}$ (b) -1 (c) $\frac{-3}{8}$ (d) $\frac{3}{4}$

47. If $\cos\theta = \frac{-12}{13}$ and $\pi < \theta < \frac{3\pi}{2}$, then $(\cot\theta + \operatorname{cosec}\theta) = ?$

(a) $\frac{1}{5}$ (b) $\frac{-1}{5}$ (c) $\frac{3}{5}$ (d) $\frac{-3}{5}$

48. If $\cot\theta = \frac{-12}{5}$ and $\frac{\pi}{2} < \theta < \pi$, then $\frac{(1 + \sin\theta - \cos\theta)}{(1 - \sin\theta + \cos\theta)} = ?$

(a) $\frac{13}{4}$ (b) $\frac{-13}{4}$ (c) $\frac{15}{2}$ (d) $\frac{-15}{2}$

49. If $\sin\theta = \frac{4}{5}$ and $\frac{\pi}{2} < \theta < \pi$, then $\frac{(5\cos\theta + 4\operatorname{cosec}\theta + 3\tan\theta)}{(4\cot\theta + 3\sec\theta + 5\sin\theta)} = ?$

(a) $\frac{-1}{2}$ (b) 1 (c) $\frac{1}{2}$ (d) -1

50. If $\sec\theta = \frac{13}{5}$ and θ is acute, then $\frac{(4 - 3\cot\theta)}{(3 + 4\tan\theta)} = ?$

(a) $\frac{55}{252}$ (b) $\frac{44}{305}$ (c) $\frac{54}{255}$ (d) $\frac{33}{215}$

51. $\cos 135° = ?$

(a) $\frac{1}{\sqrt{2}}$ (b) $\frac{-1}{\sqrt{2}}$ (c) $\frac{1}{2}$ (d) $\frac{-1}{2}$

52. $\sec 120° = ?$

(a) $\sqrt{2}$ (b) $-\sqrt{2}$ (c) 2 (d) –2

53. $\operatorname{cosec} 150° = ?$

(a) –2 (b) 2 (c) $\sqrt{2}$ (d) $-\sqrt{2}$

54. $\sin 315° = ?$

(a) $\frac{1}{\sqrt{2}}$ (b) $\frac{-1}{\sqrt{2}}$ (c) $\sqrt{2}$ (d) $-\sqrt{2}$

55. $\cos 405° = ?$

(a) $\sqrt{2}$ (b) $\frac{1}{\sqrt{2}}$ (c) $-\sqrt{2}$ (d) $\frac{-1}{\sqrt{2}}$

56. $\tan \frac{11\pi}{6} = ?$

(a) $\frac{-1}{\sqrt{3}}$ (b) $\frac{1}{\sqrt{3}}$ (c) $\sqrt{3}$ (d) $-\sqrt{3}$

57. $\cot 675° = ?$

(a) –1 (b) 1 (c) $-\sqrt{3}$ (d) $\sqrt{3}$

58. $\sin\left(\frac{31\pi}{3}\right) = ?$

(a) $\frac{1}{2}$ (b) $\frac{-1}{2}$ (c) $\frac{\sqrt{3}}{2}$ (d) $\frac{-\sqrt{3}}{2}$

59. $\cot(-600°) = ?$

(a) $-\sqrt{3}$ (b) $\frac{-1}{\sqrt{3}}$ (c) $\sqrt{3}$ (d) $\frac{1}{\sqrt{3}}$

60. $\operatorname{cosec}(-1110°) = ?$

(a) $\frac{2}{\sqrt{3}}$ (b) $\frac{-2}{\sqrt{3}}$ (c) 2 (d) –2

61. $\sec\left(\frac{-33\pi}{4}\right) = ?$

(a) $-\sqrt{2}$ (b) $\sqrt{2}$ (c) $\frac{-\sqrt{3}}{2}$ (d) $\frac{\sqrt{3}}{2}$

62. $\tan\left(\frac{-25\pi}{3}\right) = ?$

(a) $-\sqrt{3}$ (b) $\sqrt{3}$ (c) $\frac{-1}{\sqrt{3}}$ (d) $\frac{1}{\sqrt{3}}$

63. The values of $\cot\frac{\pi}{3}$, $\cot\frac{\pi}{4}$, $\cot\frac{\pi}{6}$ are in

(a) AP (b) GP (c) HP (d) none of these

64. Which is smaller, $\sin 64°$ or $\cos 64°$?

(a) $\sin 64°$ (b) $\cos 64°$

(c) both are equal (d) cannot be compared

65. Which is larger, $\sin 24°$ or $\cos 24°$?

(a) $\sin 24°$ (b) $\cos 24°$

(c) both are equal (d) cannot be compared

66. The extremum values of $\sin\theta$ are

(a) 0 and 1 (b) –1 and 0 (c) –1 and 1 (d) $\frac{-\sqrt{3}}{2}$ and $\frac{1}{\sqrt{2}}$

67. The extremum values of $\cos\theta$ are

(a) 0 and 1 (b) –1 and 0 (c) –1 and 1 (d) $\frac{-\sqrt{3}}{2}$ and $\sqrt{2}$

68. The value of $\sec\theta$ can

(a) never be greater than 1 (b) never be less than 1

(c) never be equal to 1 (d) never lie between –1 and 1

69. $\tan 150° = ?$

(a) $\frac{1}{\sqrt{3}}$ (b) $\frac{-1}{\sqrt{3}}$ (c) $\sqrt{3}$ (d) $-\sqrt{3}$

70. $\sec 150° = ?$

(a) $\frac{2}{\sqrt{3}}$ (b) 2 (c) $\frac{-2}{\sqrt{3}}$ (d) –2

71. $\cot 120° = ?$

(a) $\frac{-1}{\sqrt{3}}$ (b) $\frac{1}{\sqrt{3}}$ (c) $-\sqrt{3}$ (d) $\sqrt{3}$

72. $(\sin 105° + \cos 105°) = ?$

(a) $\sqrt{2}$ (b) $\frac{1}{\sqrt{2}}$ (c) $\frac{1}{\sqrt{3}}$ (d) $\frac{2}{\sqrt{3}}$

73. $\sin 15° = ?$

(a) $\frac{\sqrt{3}}{2\sqrt{2}}$ (b) $\frac{(\sqrt{3}+1)}{2\sqrt{2}}$ (c) $\frac{(\sqrt{3}-1)}{2\sqrt{2}}$ (d) $\frac{(\sqrt{2}-1)}{\sqrt{2}}$

74. $\cos 15° = ?$

(a) $\frac{(\sqrt{3}+1)}{2\sqrt{2}}$ (b) $\frac{(\sqrt{3}-1)}{2\sqrt{2}}$ (c) $\frac{(\sqrt{3}+1)}{\sqrt{2}}$ (d) $\frac{(\sqrt{3}-1)}{\sqrt{2}}$

75. $\tan 15^\circ = ?$

(a) $\frac{(\sqrt{3}+1)}{(\sqrt{3}-1)}$ (b) $\frac{(\sqrt{3}-1)}{(\sqrt{3}+1)}$ (c) $\frac{(\sqrt{2}+1)}{(\sqrt{2}-1)}$ (d) $\frac{(\sqrt{2}-1)}{(\sqrt{2}+1)}$

76. $\sin 75^\circ = ?$

(a) $\frac{(\sqrt{3}+1)}{2\sqrt{2}}$ (b) $\frac{(\sqrt{3}-1)}{2\sqrt{2}}$ (c) $\frac{(\sqrt{2}+1)}{2\sqrt{2}}$ (d) $\frac{(\sqrt{2}-1)}{2\sqrt{2}}$

77. $\cos 75^\circ = ?$

(a) $\frac{(\sqrt{3}+1)}{2\sqrt{2}}$ (b) $\frac{(\sqrt{3}-1)}{2\sqrt{2}}$ (c) $\frac{(\sqrt{2}-1)}{2\sqrt{2}}$ (d) $\frac{(\sqrt{2}+1)}{2\sqrt{2}}$

78. $\tan \frac{13\pi}{12} = ?$

(a) $(2+\sqrt{3})$ (b) $(1+\sqrt{2})$ (c) $(2-\sqrt{3})$ (d) $(\sqrt{2}-1)$

79. $(\sin 70^\circ \cos 10^\circ - \cos 70^\circ \sin 10^\circ) = ?$

(a) $\frac{1}{\sqrt{2}}$ (b) $\frac{1}{2}$ (c) $\frac{\sqrt{3}}{2}$ (d) $\frac{-\sqrt{3}}{2}$

80. $(\sin 36^\circ \cos 9^\circ + \cos 36^\circ \sin 9^\circ) = ?$

(a) $\frac{1}{\sqrt{2}}$ (b) $\frac{1}{2}$ (c) $\frac{\sqrt{3}}{2}$ (d) 1

81. $\cos 80^\circ \cos 20^\circ + \sin 80^\circ \sin 20^\circ = ?$

(a) $\frac{1}{2}$ (b) $\frac{\sqrt{3}}{2}$ (c) $\frac{1}{\sqrt{2}}$ (d) 1

82. $\cos 50^\circ \cos 10^\circ - \sin 50^\circ \sin 10^\circ = ?$

(a) $\frac{1}{\sqrt{2}}$ (b) $\frac{\sqrt{3}}{2}$ (c) $\frac{1}{2}$ (d) 1

83. $\sin (40^\circ + \theta) \cos (10^\circ + \theta) - \cos (40^\circ + \theta) \sin (10^\circ + \theta) = ?$

(a) $\frac{\sqrt{3}}{2}$ (b) $\frac{1}{2}$ (c) $\sqrt{2}$ (d) none of these

84. $\sin \frac{7\pi}{12} \cos \frac{\pi}{4} - \cos \frac{7\pi}{12} \sin \frac{\pi}{4} = ?$

(a) $\sqrt{2}$ (b) $\frac{1}{\sqrt{2}}$ (c) $\frac{\sqrt{3}}{2}$ (d) none of these

85. $\sin \frac{\pi}{4} \cos \frac{\pi}{12} + \cos \frac{\pi}{4} \sin \frac{\pi}{12} = ?$

(a) $\frac{\sqrt{3}}{2}$ (b) $\sqrt{2}$ (c) $\frac{1}{2}$ (d) none of these

86. $\cos\frac{2\pi}{3}\cos\frac{\pi}{4}-\sin\frac{2\pi}{3}\sin\frac{\pi}{4}=?$

(a) $\frac{(\sqrt{3}+1)}{\sqrt{2}}$ (b) $\frac{-(\sqrt{3}+1)}{2\sqrt{2}}$ (c) $\frac{(\sqrt{3}+1)}{2\sqrt{2}}$ (d) $\frac{-(\sqrt{3}+1)}{2\sqrt{2}}$

87. $\sin\frac{\pi}{12}=?$

(a) $\frac{(\sqrt{3}-1)}{2\sqrt{2}}$ (b) $\frac{-(\sqrt{3}-1)}{2\sqrt{2}}$ (c) $\frac{(\sqrt{3}+1)}{2\sqrt{2}}$ (d) none of these

88. $\sin\frac{5\pi}{12}=?$

(a) $\frac{(\sqrt{3}-1)}{2\sqrt{2}}$ (b) $\frac{(\sqrt{3}+1)}{2\sqrt{2}}$ (c) $\frac{-(\sqrt{3}-1)}{2\sqrt{2}}$ (d) $\frac{-(\sqrt{3}+1)}{2\sqrt{2}}$

89. If $\sin\theta=\frac{15}{17}$ and $\cos\phi=\frac{12}{13}$, where θ and ϕ both lie in quadrant I, then $\sin(\theta+\phi)=?$

(a) $\frac{171}{221}$ (b) $\frac{180}{221}$ (c) $\frac{220}{221}$ (d) $\frac{181}{221}$

90. If $\sin\theta=\frac{3}{5}$ and $\cos\phi=\frac{-12}{13}$, where θ and ϕ both lie in quadrant II, then $\sin(\theta-\phi)=?$

(a) $\frac{16}{65}$ (b) $\frac{-16}{65}$ (c) $\frac{33}{65}$ (d) $\frac{-33}{65}$

91. If $\cos\theta=\frac{4}{5}$ and $\cos\phi=\frac{12}{13}$, where θ and ϕ both lie in quadrant IV, then $\cos(\theta+\phi)=?$

(a) $\frac{33}{65}$ (b) $\frac{-33}{65}$ (c) $\frac{16}{65}$ (d) $\frac{-16}{65}$

92. If $\cot\theta=\frac{1}{2}$ and $\sec\phi=\frac{-5}{3}$, where θ lies in quadrant III and ϕ lies in quadrant II, then $\tan(\theta+\phi)=?$

(a) $\frac{5}{11}$ (b) $\frac{2}{11}$ (c) $\frac{-6}{11}$ (d) $\frac{10}{11}$

93. $\cos 15^\circ-\sin 15^\circ=?$

(a) $\frac{1}{2}$ (b) $\frac{1}{\sqrt{2}}$ (c) $\frac{(\sqrt{2}-1)}{\sqrt{2}}$ (d) $\frac{(\sqrt{2}+1)}{\sqrt{2}}$

94. $\cot 105^\circ-\tan 105^\circ=?$

(a) $\sqrt{3}$ (b) $2\sqrt{3}$ (c) $\frac{\sqrt{3}}{2}$ (d) $\frac{(\sqrt{3}+1)}{(\sqrt{3}-1)}$

95. $2\sin\dfrac{5\pi}{12}\sin\dfrac{\pi}{12}=?$

(a) $\dfrac{1}{\sqrt{2}}$ (b) $\sqrt{2}$ (c) $\dfrac{1}{2}$ (d) $\dfrac{\sqrt{3}}{2}$

96. $2\cos\dfrac{5\pi}{12}\cos\dfrac{\pi}{12}=?$

(a) $\dfrac{1}{2}$ (b) $\dfrac{1}{\sqrt{2}}$ (c) $\dfrac{\sqrt{3}}{2}$ (d) $\sqrt{2}$

97. $2\sin\dfrac{5\pi}{12}\cos\dfrac{\pi}{12}=?$

(a) $\dfrac{1}{2}$ (b) $\dfrac{\sqrt{3}+1}{2}$ (c) $\dfrac{(2+\sqrt{3})}{2}$ (d) $\dfrac{\sqrt{3}}{2}$

98. $\dfrac{\sin(180^\circ+\theta)\cos(90^\circ+\theta)\tan(270^\circ-\theta)\cot(360^\circ-\theta)}{\sin(360^\circ-\theta)\cos(360^\circ+\theta)\operatorname{cosec}(-\theta)\sin(270^\circ+\theta)}=?$

(a) $\sqrt{2}$ (b) $\dfrac{\sqrt{3}}{2}$ (c) $\dfrac{3}{2}$ (d) 1

99. $\sin(40^\circ+\theta)\cos(10^\circ+\theta)-\cos(40^\circ+\theta)\sin(10^\circ+\theta)=?$

(a) 2 (b) $\dfrac{1}{2}$ (c) $\dfrac{\sqrt{3}}{2}$ (d) none of these

100. $\dfrac{\cos(90^\circ+\theta)\sec(270^\circ+\theta)\sin(180^\circ+\theta)}{\operatorname{cosec}(-\theta)\cos(270^\circ-\theta)\tan(180^\circ+\theta)}=?$

(a) $\cos\theta$ (b) $\sec\theta$ (c) $\cot\theta$ (d) none of these

101. $\cos\theta+\sin(270^\circ+\theta)-\sin(270^\circ-\theta)+\cos(180^\circ+\theta)=?$

(a) $2\cos\theta$ (b) $2\sin\theta$ (c) 0 (d) none of these

102. $\dfrac{\cos 8^\circ-\sin 8^\circ}{\cos 8^\circ+\sin 8^\circ}=?$

(a) $\tan 8^\circ$ (b) $\tan 37^\circ$ (c) $\tan 52^\circ$ (d) none of these

103. $\dfrac{\cos(\pi+\theta)\cos(-\theta)}{\cos(\pi-\theta)\cos\left(\dfrac{\pi}{2}+\theta\right)}=?$

(a) $-\cot\theta$ (b) $\cot\theta$ (c) $-\tan\theta$ (d) $\tan\theta$

104. $\cos\left(\dfrac{\pi}{4}+x\right)+\cos\left(\dfrac{\pi}{4}-x\right)=?$

(a) $2\cos x$ (b) $\sqrt{2}\cos x$ (c) $2\sin x$ (d) $\sqrt{2}\sin x$

105. $\cos\left(\frac{3\pi}{4}+x\right)-\cos\left(\frac{3\pi}{4}-x\right)=?$

(a) $\sqrt{2}\sin x$ (b) $2\sin x$ (c) $-\sqrt{2}\sin x$ (d) $\frac{1}{\sqrt{2}}\sin x$

106. $\frac{\sin 3x-\sin x}{\cos x-\cos 3x}=?$

(a) $\tan 2x$ (b) $\cot 2x$ (c) $-\tan 2x$ (d) $-\cot 2x$

107. $\frac{\cos 6x+\cos 4x}{\sin 6x-\sin 4x}=?$

(a) $\cot x$ (b) $\tan x$ (c) $-\cot x$ (d) $-\tan x$

108. $\frac{\cos 4x+\cos 3x+\cos 2x}{\sin 4x+\sin 3x+\sin 2x}=?$

(a) $\tan 2x$ (b) $\tan 3x$ (c) $\cot 2x$ (d) $\cot 3x$

109. $\frac{\sin 7x-\sin 5x}{\cos 7x+\cos 5x}=?$

(a) $\tan x$ (b) $\cot x$ (c) $\tan 2x$ (d) $\cot 2x$

110. $(\sin^2 6x-\sin^2 4x)=?$

(a) $\sin 10x$ (b) $\sin 2x$ (c) $\sin 10x\sin 2x$ (d) none of these

111. $\cos 20^\circ\cos 40^\circ\cos 80^\circ=?$

(a) $\frac{1}{16}$ (b) $\frac{1}{8}$ (c) $\frac{\sqrt{3}}{8}$ (d) $\frac{\sqrt{3}}{16}$

112. $\sin 10^\circ\sin 50^\circ\sin 70^\circ=?$

(a) $\frac{\sqrt{3}}{4}$ (b) $\frac{\sqrt{3}}{8}$ (c) $\frac{1}{8}$ (d) $\frac{1}{16}$

113. $2\cos 45^\circ\cos 15^\circ=?$

(a) $\frac{\sqrt{3}}{2}$ (b) $\frac{(\sqrt{3}-1)}{2}$ (c) $\frac{(\sqrt{3}+1)}{2}$ (d) $\frac{3}{2}$

114. $2\sin 75^\circ\sin 15^\circ=?$

(a) $\frac{1}{2}$ (b) $\frac{\sqrt{3}}{2}$ (c) $\frac{(\sqrt{3}+1)}{2}$ (d) none of these

115. $\cos 15^\circ-\sin 15^\circ=?$

(a) $\frac{1}{2}$ (b) $\frac{1}{\sqrt{2}}$ (c) $\frac{1}{2\sqrt{2}}$ (d) $\frac{\sqrt{3}}{2}$

116. If $\sin x=\frac{-1}{2}$ and x lies in quadrant III, then $\sin 2x=?$

(a) $\frac{1}{2}$ (b) $\sqrt{3}$ (c) $\frac{\sqrt{3}}{2}$ (d) $\frac{1}{2\sqrt{3}}$

117. If $\sec x = \frac{-13}{12}$ and $\frac{\pi}{2} < x < \pi$, then $\cos 2x = ?$

(a) $\frac{-120}{169}$ (b) $\frac{119}{169}$ (c) $\frac{-120}{119}$ (d) none of these

118. If $\tan x = \frac{-3}{4}$ and $\frac{3\pi}{2} < x < 2\pi$, then $\tan 2x = ?$

(a) $\frac{-24}{25}$ (b) $\frac{7}{25}$ (c) $\frac{24}{7}$ (d) $\frac{-24}{7}$

119. If $\sin x = \frac{1}{3}$, then $\sin 3x = ?$

(a) 1 (b) $\frac{1}{9}$ (c) $\frac{7}{9}$ (d) $\frac{23}{27}$

120. If $\cos x = \frac{1}{2}$, then $\cos 3x = ?$

(a) $\frac{3}{2}$ (b) $\frac{1}{6}$ (c) -1 (d) $\frac{2}{3}$

121. $\left(\cos^2 \frac{\pi}{12} - \sin^2 \frac{\pi}{12}\right) = ?$

(a) $\frac{1}{2}$ (b) $\frac{\sqrt{3}}{2}$ (c) $\frac{3}{2}$ (d) $\frac{-1}{2}$

122. $\frac{1 + \sin 2x - \cos 2x}{1 + \sin 2x + \cos 2x} = ?$

(a) $\tan 2x$ (b) $\tan x$ (c) $\cot 2x$ (d) $\cot x$

123. If $\cos x = \frac{-3}{5}$ and $\frac{\pi}{2} < x < \pi$, then $\sin \frac{x}{2} = ?$

(a) $\frac{-2}{\sqrt{5}}$ (b) $\frac{2}{\sqrt{5}}$ (c) $\frac{-1}{\sqrt{5}}$ (d) $\frac{1}{\sqrt{5}}$

124. If $\cos x = \frac{-3}{5}$ and $\frac{\pi}{2} < x < \pi$, then $\cos \frac{x}{2} = ?$

(a) $\frac{1}{\sqrt{5}}$ (b) $\frac{-1}{\sqrt{5}}$ (c) $\frac{2}{\sqrt{5}}$ (d) $\frac{-2}{\sqrt{5}}$

125. If $\cos x = \frac{-4}{5}$ and $\pi < x < \frac{3\pi}{2}$, then $\cos \frac{x}{2} = ?$

(a) $\frac{1}{\sqrt{10}}$ (b) $\frac{-1}{\sqrt{10}}$ (c) $\frac{3}{\sqrt{10}}$ (d) $\frac{-3}{\sqrt{10}}$

126. If $\tan x = \frac{3}{4}$ and $\pi < x < \frac{3\pi}{2}$, then $\tan \frac{x}{2} = ?$

(a) 3 (b) $\frac{1}{3}$ (c) -3 (d) $\frac{-1}{3}$

127. If $\cos x = \frac{-1}{3}$ and x lies in quadrant III, then $\tan \frac{x}{2} = ?$

(a) $\sqrt{2}$ (b) $-\sqrt{2}$ (c) $\sqrt{3}$ (d) $-\sqrt{3}$

128. If $\sin x = \frac{-1}{2}$ and x lies in quadrant IV, then $\sin \frac{x}{2} = ?$

(a) $\frac{\sqrt{2+\sqrt{3}}}{2}$ (b) $\frac{\sqrt{3+\sqrt{2}}}{2}$ (c) $\frac{\sqrt{2-\sqrt{3}}}{2}$ (d) none of these

129. $\frac{1+\cos x}{1-\cos x} = ?$

(a) $\tan^2 \frac{x}{2}$ (b) $\cot^2 \frac{x}{2}$ (c) $\sec^2 \frac{x}{2}$ (d) $\operatorname{cosec}^2 \frac{x}{2}$

130. $\sqrt{\frac{1+\sin x}{1-\sin x}} = ?$

(a) $\tan \frac{x}{2}$ (b) $\cot \frac{x}{2}$ (c) $\tan\left(\frac{\pi}{4}+\frac{x}{2}\right)$ (d) $\cot\left(\frac{\pi}{4}+\frac{x}{2}\right)$

131. $\frac{\sin x}{1+\cos x} = ?$

(a) $\tan \frac{x}{2}$ (b) $\cot \frac{x}{2}$ (c) $\tan\left(\frac{\pi}{4}+\frac{x}{2}\right)$ (d) none of these

132. $\cot \frac{x}{2} - \tan \frac{x}{2} = ?$

(a) $2\tan x$ (b) $2\cot x$ (c) $2\sin x$ (d) $2\cos x$

133. $\sin 18° = ?$

(a) $\frac{(\sqrt{5}-1)}{4}$ (b) $\frac{(\sqrt{5}+1)}{4}$ (c) $\frac{(\sqrt{3}+1)}{2}$ (d) $\frac{(\sqrt{3}-1)}{2}$

134. $\cos 18° = ?$

(a) $\frac{\sqrt{10-2\sqrt{5}}}{4}$ (b) $\frac{\sqrt{10+2\sqrt{5}}}{4}$ (c) $\frac{(\sqrt{5}+1)}{4}$ (d) none of these

135. $\cos 36° = ?$

(a) $\frac{(\sqrt{5}-1)}{4}$ (b) $\frac{(\sqrt{5}+1)}{4}$ (c) $\frac{(\sqrt{3}+1)}{2}$ (d) $\frac{(\sqrt{3}-1)}{2}$

136. $\sin 36° = ?$

(a) $\frac{(\sqrt{5}-1)}{4}$ (b) $\frac{\sqrt{10+2\sqrt{5}}}{4}$ (c) $\frac{\sqrt{10-2\sqrt{5}}}{4}$ (d) none of these

137. $\sin 54° = ?$

(a) $\frac{(\sqrt{5}+1)}{4}$ (b) $\frac{(\sqrt{5}-1)}{4}$ (c) $\frac{\sqrt{10+2\sqrt{5}}}{4}$ (d) $\frac{\sqrt{10-2\sqrt{5}}}{4}$

138. $\cos 72^\circ = ?$

(a) $\frac{(\sqrt{5}+1)}{4}$ (b) $\frac{(\sqrt{5}-1)}{4}$ (c) $\frac{(2+\sqrt{5})}{3}$ (d) $\frac{(2-\sqrt{5})}{3}$

139. $\cos 54^\circ = ?$

(a) $\frac{\sqrt{10+2\sqrt{5}}}{4}$ (b) $\frac{\sqrt{10-2\sqrt{5}}}{4}$ (c) $\frac{(\sqrt{5}+1)}{4}$ (d) $\frac{(\sqrt{5}-1)}{4}$

140. $\sin 72^\circ = ?$

(a) $\frac{\sqrt{10+2\sqrt{5}}}{4}$ (b) $\frac{\sqrt{10-2\sqrt{5}}}{4}$ (c) $\frac{(\sqrt{5}-1)}{4}$ (d) $\frac{(\sqrt{5}+1)}{4}$

141. $2\sin 22\frac{1}{2}^\circ \cos 22\frac{1}{2}^\circ = ?$

(a) 1 (b) $\frac{1}{2}$ (c) $\frac{1}{\sqrt{2}}$ (d) $\sqrt{2}$

142. $(2\cos^2 15^\circ - 1) = ?$

(a) $\frac{3}{2}$ (b) $\frac{\sqrt{3}}{2}$ (c) $2\sqrt{3}$ (d) $\frac{\sqrt{3}}{\sqrt{2}}$

143. $(3\sin 40^\circ - 4\sin^3 40^\circ) = ?$

(a) $\frac{\sqrt{3}}{2}$ (b) $\frac{3}{2}$ (c) $3\sqrt{3}$ (d) none of these

144. $(8\cos^3 20^\circ - 6\cos 20^\circ) = ?$

(a) $\frac{5}{2}$ (b) $\frac{5}{3}$ (c) 1 (d) $\frac{\sqrt{3}}{2}$

145. If $\tan\theta = \frac{a}{b}$, then $a\sin 2\theta + b\cos 2\theta = ?$

(a) a (b) b (c) $a+b$ (d) $a-b$

146. $\cot x - 2\cot 2x = ?$

(a) $\tan x$ (b) $\cos x$ (c) $\sin x$ (d) $\cos 2x$

147. $\cos 2x + 2\sin^2 x = ?$

(a) 2 (b) $\frac{1}{2}$ (c) 1 (d) $\frac{3}{2}$

148. $\frac{\sin 2x}{1-\cos 2x} = ?$

(a) $\tan x$ (b) $\cot x$ (c) $\sec x$ (d) $\operatorname{cosec} x$

149. $\frac{\sin 2x}{1+\cos 2x} = ?$

(a) $\tan x$ (b) $\operatorname{cosec} x$ (c) $\sec x$ (d) $\cot x$

150. $\dfrac{\tan 2x}{1+\sec 2x}=?$

(a) $\sin x$ (b) $\cos x$ (c) $\tan x$ (d) $\cot x$

151. $\dfrac{1-\cos 2x+\sin x}{\sin 2x+\cos x}=?$

(a) $\tan x$ (b) $\cot x$ (c) $\sec x$ (d) $\operatorname{cosec} x$

152. $\dfrac{\cos x}{(1-\sin x)}=?$

(a) $\tan \dfrac{x}{2}$ (b) $\tan\left(\dfrac{\pi}{4}+\dfrac{x}{2}\right)$ (c) $\tan\left(\dfrac{\pi}{4}-\dfrac{x}{2}\right)$ (d) none of these

153. $(\sin^2\alpha\cos^2\beta+\cos^2\alpha\sin^2\beta+\sin^2\alpha\sin^2\beta+\cos^2\alpha\cos^2\beta)=?$

(a) 1 (b) –1 (c) 0 (d) 3

154. If x is acute, then $\sqrt{\dfrac{1+\sin x}{1-\sin x}}=?$

(a) $\sec x+\operatorname{cosec} x$ (b) $\sec x+\tan x$ (c) $\operatorname{cosec} x+\cot x$ (d) $\tan x+\cot x$

155. $(\sec\theta-\cos\theta)(\operatorname{cosec}\theta-\sin\theta)(\cot\theta+\tan\theta)=?$

(a) 1 (b) –1 (c) 0 (d) none of these

156. $\sin\theta\left(\dfrac{\sin\theta}{1+\cos\theta}+\dfrac{1+\cos\theta}{\sin\theta}\right)=?$

(a) 1 (b) 2 (c) 3 (d) 4

157. $(1+\sin\theta)\left(\dfrac{\sin\theta+\cos\theta-1}{\sin\theta-\cos\theta+1}\right)=?$

(a) $\tan\theta$ (b) $\cot\theta$ (c) $\sin\theta$ (d) $\cos\theta$

158. $\dfrac{(\cot\theta-\operatorname{cosec}\theta+1)}{(\cot\theta+\operatorname{cosec}\theta-1)}=?$

(a) $\operatorname{cosec}\theta+\cot\theta$ (b) $\operatorname{cosec}\theta-\cot\theta$ (c) 1 (d) –1

159. $\left(\dfrac{\sec\theta+\tan\theta-1}{\tan\theta-\sec\theta+1}\right)^2=?$

(a) $\dfrac{(1+\sin\theta)}{(1-\sin\theta)}$ (b) $\dfrac{(1-\sin\theta)}{(1+\sin\theta)}$ (c) 1 (d) none of these

160. If $x=r\cos\alpha\cos\beta$, $y=r\cos\alpha\sin\beta$ and $z=r\sin\alpha$, then $x^2+y^2+z^2=?$

(a) 1 (b) r^2 (c) r^4 (d) none of these

161. If $a\tan\theta=b$, then $\left(\dfrac{b\sin\theta-a\cos\theta}{b\sin\theta+a\cos\theta}\right)=?$

(a) $\dfrac{(a^2-b^2)}{(a^2+b^2)}$ (b) $\dfrac{(a^2+b^2)}{(a^2-b^2)}$ (c) $\dfrac{(b^2-a^2)}{(b^2+a^2)}$ (d) none of these

162. If $5\cot\theta = 4$, then $\left(\dfrac{5\sin\theta - 3\cos\theta}{\sin\theta + 2\cos\theta}\right) = ?$

(a) 1 (b) $\dfrac{3}{4}$ (c) $\dfrac{5}{14}$ (d) $\dfrac{3}{14}$

163. If $\tan\theta + \cot\theta = 2$, then $\sin\theta = ?$

(a) $\dfrac{1}{2}$ (b) $\dfrac{1}{\sqrt{2}}$ (c) $\dfrac{\sqrt{3}}{2}$ (d) $\dfrac{1}{\sqrt{3}}$

164. If θ lies in quadrant II, then $\sqrt{\dfrac{1-\sin\theta}{1+\sin\theta}} - \sqrt{\dfrac{1+\sin\theta}{1-\sin\theta}} = ?$

(a) $\tan\theta$ (b) $2\tan\theta$ (c) $\cot\theta$ (d) $2\cot\theta$

165. If $\pi < \theta < \dfrac{3\pi}{2}$, then $\sqrt{\dfrac{1+\cos\theta}{1-\cos\theta}} + \sqrt{\dfrac{1-\cos\theta}{1+\cos\theta}} = ?$

(a) $2\sec\theta$ (b) $-2\sec\theta$ (c) $2\operatorname{cosec}\theta$ (d) $-2\operatorname{cosec}\theta$

166. If $7\sin^2\theta + 3\cos^2\theta = 4$, then $\tan\theta = ?$

(a) $\pm\dfrac{1}{\sqrt{2}}$ (b) $\pm\dfrac{1}{\sqrt{3}}$ (c) $\pm\dfrac{1}{2}$ (d) $\pm\dfrac{1}{3}$

167. If $\sin\theta + \sin^2\theta = 1$, then $(\cos^2\theta + \cos^4\theta) = ?$

(a) 0 (b) 1 (c) 2 (d) none of these

168. If $\sec\theta - \tan\theta = \dfrac{2}{3}$, then which one of the following is true?

(a) $\sec\theta = \dfrac{5}{6}$, $\tan\theta = \dfrac{3}{2}$ (b) $\sec\theta = \dfrac{13}{12}$, $\tan\theta = \dfrac{5}{12}$

(c) $\sec\theta = \dfrac{8}{9}$, $\tan\theta = \dfrac{4}{5}$ (d) none of these

169. If $\operatorname{cosec}\theta + \cot\theta = 6$, then which one of the following is true?

(a) $\operatorname{cosec}\theta = \dfrac{35}{12}$, $\cot\theta = \dfrac{37}{12}$ (b) $\operatorname{cosec}\theta = \dfrac{37}{12}$, $\cot\theta = \dfrac{35}{12}$

(c) $\operatorname{cosec}\theta = \dfrac{41}{12}$, $\cot\theta = \dfrac{31}{12}$ (d) none of these

170. If $(3\sin\theta + 5\cos\theta) = 5$, then $(5\sin\theta - 3\cos\theta) = ?$

(a) 4 (b) 2 (c) 1 (d) 3

171. $\dfrac{(1+\tan 15°)}{(1-\tan 15°)} = ?$

(a) $\dfrac{\sqrt{3}}{2}$ (b) $\sqrt{3}$ (c) $\dfrac{-2}{\sqrt{3}}$ (d) $\dfrac{1}{2}$

172. $(3\sin 20° - 4\sin^3 20°) = ?$

(a) -1 (b) $\frac{1}{\sqrt{2}}$ (c) $\sqrt{3}$ (d) $\frac{\sqrt{3}}{2}$

173. $(4\cos^3 15° - 3\cos 15°) = ?$

(a) 1 (b) -1 (c) $\frac{1}{\sqrt{2}}$ (d) 0

174. $(2\cos 75° \cos 15°) = ?$

(a) $\frac{1}{2}$ (b) $\sqrt{2}$ (c) 2 (d) 4

175. $(\sin 105° \sin 75°) = ?$

(a) $\frac{(2-\sqrt{3})}{4}$ (b) $\frac{(2+\sqrt{3})}{4}$ (c) $\frac{(\sqrt{5}+1)}{4}$ (d) $\frac{(\sqrt{5}-1)}{4}$

176. $(\sin 105° + \cos 105°) = ?$

(a) $\frac{1}{\sqrt{2}}$ (b) $\sqrt{2}$ (c) $\frac{1}{2}$ (d) $\frac{3}{2}$

177. If $0° < \theta < 90°$ and $(\sin\theta + \cos\theta) = \sqrt{2}$, then $\theta = ?$

(a) $\frac{\pi}{3}$ (b) $\frac{\pi}{4}$ (c) $\frac{\pi}{6}$ (d) π

178. If $0° < \theta < 90°$ and $(\sin\theta + \operatorname{cosec}\theta) = \frac{5}{2}$, then $\theta = ?$

(a) $\frac{\pi}{3}$ (b) $\frac{\pi}{4}$ (c) $\frac{\pi}{6}$ (d) π

179. If $0° < \theta < 90°$ and $\sin 2\theta = \frac{1}{5}$, then $(\sin\theta + \cos\theta) = ?$

(a) $\sqrt{\frac{2}{5}}$ (b) $\sqrt{\frac{3}{5}}$ (c) $\sqrt{\frac{6}{5}}$ (d) $\sqrt{\frac{7}{5}}$

180. If θ is acute and $\sin\theta = \cos 2\theta$, then $(\sin\theta + \cos\theta) = ?$

(a) $\frac{(\sqrt{3}-1)}{2}$ (b) $\frac{(\sqrt{3}+1)}{2}$ (c) $\frac{(3+\sqrt{2})}{2}$ (d) $\frac{(3-\sqrt{2})}{2}$

181. If θ is acute and $(\cos\theta - \sin\theta) > 0$, then $(\cos\theta + \sin\theta)$ cannot be greater than

(a) $\frac{1}{\sqrt{2}}$ (b) $\frac{1}{2}$ (c) $\frac{1}{3}$ (d) $\sqrt{2}$

182. $\sqrt{2+\sqrt{2+2\cos 4\theta}} = ?$

(a) $2\sin\theta$ (b) $2\cos\theta$ (c) $\sin 2\theta$ (d) $\cos 2\theta$

183. If $\sin 2\theta = \cos 3\theta$ and θ is acute, then $\sin\theta = ?$

(a) $\frac{(\sqrt{5}-1)}{4}$ (b) $\frac{(\sqrt{5}+1)}{4}$ (c) $\frac{(\sqrt{3}+2)}{4}$ (d) $\frac{(\sqrt{3}-1)}{4}$

184. The least value of $(2\sin^2\theta + 3\cos^2\theta)$ is

(a) 1 (b) 2 (c) 3 (d) 5

185. The greatest value of $(\sin^4\theta + \cos^4\theta)$ is

(a) $\frac{1}{2}$ (b) 1 (c) 2 (d) 3

186. If $\cos^2\theta + \sec^2\theta = p$, then

(a) $p < 1$ (b) $p = 1$ (c) $1 < p < 2$ (d) $p \geq 2$

187. If $\sin^2\theta + \cos^4\theta = p$, then

(a) $1 \leq p \leq 2$ (b) $\frac{3}{4} \leq p \leq 1$ (c) $\frac{3}{4} \leq p \leq \frac{13}{16}$ (d) $\frac{13}{16} \leq p \leq 1$

188. If $2\sin^2\theta - \cos\theta = p$, then

(a) $1 \leq p \leq 2$ (b) $-1 \leq p \leq 3$ (c) $-2 \leq p \leq 3$ (d) $-2 \leq p \leq 4$

189. $\left\{\sin^2\left(\frac{\pi}{8} + \frac{A}{2}\right) - \sin^2\left(\frac{\pi}{8} - \frac{A}{2}\right)\right\} = ?$

(a) $\frac{1}{\sqrt{2}}\sin A$ (b) $\frac{1}{2}\sin A$ (c) $2\sin\frac{\pi}{8}$ (d) $\sin A$

190. $(\cos 1^\circ \cos 2^\circ \cos 3^\circ \ldots\ldots \cos 178^\circ \cos 179^\circ) = ?$

(a) -1 (b) 1 (c) 0 (d) $\sqrt{2}$

191. $(\tan 1^\circ \tan 2^\circ \tan 3^\circ \ldots\ldots \tan 89^\circ) = ?$

(a) 0 (b) 1 (c) $\frac{1}{2}$ (d) none of these

192. $(\tan 15^\circ \tan 25^\circ \tan 45^\circ \tan 65^\circ \tan 75^\circ) = ?$

(a) 1 (b) 2 (c) 3 (d) $\sqrt{3}$

193. $\left(\tan\frac{\pi}{20}\tan\frac{3\pi}{20}\tan\frac{5\pi}{20}\tan\frac{7\pi}{20}\tan\frac{9\pi}{20}\right) = ?$

(a) -1 (b) 1 (c) $\frac{1}{2}$ (d) none of these

194. The general solution of the equation $\sin\theta = 0$ is

(a) $\theta = 2n\pi,\ n \in I$ (b) $\theta = n\pi,\ n \in I$

(c) $\theta = (2n+1)\frac{\pi}{2},\ n \in I$ (d) $\theta = \frac{n\pi}{2},\ n \in I$

195. The general solution of the equation $\cos\theta = 0$ is

(a) $\theta = n\pi,\ n \in I$ (b) $\theta = 2n\pi,\ n \in I$

(c) $\theta = (2n+1)\frac{\pi}{2},\ n \in I$ (d) none of these

196. The general solution of the equation $\tan\theta = 0$ is

(a) $\theta = n\pi,\ n \in I$ (b) $\theta = 2n\pi,\ n \in I$

(c) $\theta = (2n+1)\frac{\pi}{2},\ n \in I$ (d) none of these

197. The general solution of the equation $\sin 3x = 0$ is

(a) $x = 3n\pi, n \in I$ (b) $x = \frac{n\pi}{3}, n \in I$

(c) $x = (2n+1)\frac{\pi}{6}, n \in I$ (d) none of these

198. The general solution of the equation $\cos 2x = 0$ is

(a) $x = \frac{n\pi}{2}, n \in I$ (b) $x = (2n+1)\frac{\pi}{4}, n \in I$

(c) $x = n\pi, n \in I$ (d) none of these

199. The general solution of the equation $\sin\theta = \sin\alpha$ is

(a) $\theta = \alpha$ (b) $\theta = n\pi \pm \alpha, n \in I$

(c) $\theta = n\pi + (-1)^n\alpha, n \in I$ (d) none of these

200. The general solution of the equation $\cos\theta = \cos\alpha$ is

(a) $\theta = \alpha$ (b) $\theta = n\pi \pm \alpha, n \in I$

(c) $\theta = 2n\pi \pm \alpha, n \in I$ (d) none of these

201. The general solution of the equation $\tan\theta = \tan\alpha$ is

(a) $\theta = n\pi + \alpha, n \in I$ (b) $\theta = 2n\pi + \alpha, n \in I$

(c) $\theta = n\pi \pm \alpha, n \in I$ (d) $\theta = 2n\pi \pm \alpha, n \in I$

202. The general solution of the equation $\sin^2\theta = \sin^2\alpha$ is

(a) $\theta = n\pi + \alpha, n \in I$ (b) $\theta = n\pi \pm \alpha, n \in I$

(c) $\theta = 2n\pi + \alpha, n \in I$ (d) $\theta = 2n\pi \pm \alpha, n \in I$

203. The general solution of the equation $\cos^2\theta = \cos^2\alpha$ is

(a) $\theta = n\pi + \alpha, n \in I$ (b) $\theta = 2n\pi + \alpha, n \in I$

(c) $\theta = n\pi \pm \alpha, n \in I$ (d) $\theta = 2n\pi \pm \alpha, n \in I$

204. The general solution of the equation $\tan^2\theta = \tan^2\alpha$ is

(a) $\theta = n\pi + \alpha, n \in I$ (b) $\theta = 2n\pi + \alpha, n \in I$

(c) $\theta = n\pi \pm \alpha, n \in I$ (d) $\theta = 2n\pi \pm \alpha, n \in I$

205. The general solution of the equation $\sin\theta = \frac{1}{\sqrt{2}}$ is

(a) $\theta = n\pi + \frac{\pi}{4}, n \in I$ (b) $\theta = 2n\pi + \frac{\pi}{4}, n \in I$

(c) $\theta = n\pi + (-1)^n\frac{\pi}{4}, n \in I$ (d) none of these

206. The general solution of the equation $\cos\theta = \frac{1}{2}$ is

(a) $\theta = n\pi + \frac{\pi}{3}, n \in I$ (b) $\theta = 2n\pi + \frac{\pi}{3}, n \in I$

(c) $\theta = 2n\pi \pm \frac{\pi}{3}, n \in I$ (d) none of these

207. The general solution of the equation $\tan\theta = \dfrac{1}{\sqrt{3}}$ is

(a) $\theta = n\pi + \dfrac{\pi}{6}, n \in I$ (b) $\theta = 2n\pi + \dfrac{\pi}{6}, n \in I$

(c) $\theta = 2n\pi \pm \dfrac{\pi}{6}, n \in I$ (d) none of these

208. The general solution of the equation $\sin\theta = \dfrac{-\sqrt{3}}{2}$ is

(a) $\theta = n\pi + \dfrac{4\pi}{3}, n \in I$ (b) $\theta = 2n\pi + \dfrac{4\pi}{3}, n \in I$

(c) $\theta = n\pi + (-1)^n \dfrac{4\pi}{3}, n \in I$ (d) none of these

209. The general solution of the equation $\cos\theta = \dfrac{-1}{2}$ is

(a) $\theta = n\pi \pm \dfrac{2\pi}{3}, n \in I$ (b) $\theta = 2n\pi \pm \dfrac{\pi}{3}, n \in I$

(c) $\theta = 2n\pi \pm \dfrac{2\pi}{3}, n \in I$ (d) none of these

210. The general solution of the equation $\cot\theta = -\sqrt{3}$ is

(a) $\theta = n\pi + \dfrac{5\pi}{6}, n \in I$ (b) $\theta = 2n\pi + \dfrac{5\pi}{6}, n \in I$

(c) $\theta = n\pi + \dfrac{2\pi}{3}, n \in I$ (d) none of these

211. The general solution of the equation $\operatorname{cosec}\theta + \sqrt{2} = 0$ is

(a) $\theta = n\pi + \dfrac{5\pi}{4}, n \in I$ (b) $\theta = n\pi - \dfrac{5\pi}{4}, n \in I$

(c) $\theta = n\pi + (-1)^n \dfrac{5\pi}{4}, n \in I$ (d) none of these

212. The general solution of the equation $\tan 3\theta = -1$ is

(a) $\theta = \dfrac{n\pi}{3} + \dfrac{\pi}{4}, n \in I$ (b) $\theta = \dfrac{2n\pi}{3} + \dfrac{\pi}{4}, n \in I$

(c) $\theta = \dfrac{n\pi}{3} \pm \dfrac{\pi}{4}, n \in I$ (d) none of these

213. The general solution of the equation $\sin 2\theta = \dfrac{-1}{2}$ is

(a) $\theta = \dfrac{n\pi}{4} + \dfrac{\pi}{24}, n \in I$ (b) $\dfrac{n\pi}{2} + (-1)^n \dfrac{7\pi}{12}, n \in N$

(c) $\theta = \dfrac{n\pi}{2} \pm \dfrac{\pi}{12}, n \in I$ (d) none of these

214. The general solution of the equation $4\sin^2\theta = 1$ is

(a) $\theta = n\pi \pm \frac{\pi}{6}, n \in I$ (b) $\theta = 2n\pi \pm \frac{\pi}{6}, n \in I$

(c) $\theta = \frac{n\pi}{4} + \frac{\pi}{24}, n \in I$ (d) none of these

215. The general solution of the equation $2\cos^2\theta = 1$ is

(a) $\theta = 2n\pi + \frac{\pi}{4}, n \in I$ (b) $\theta = \frac{n\pi}{2} \pm \frac{\pi}{8}, n \in I$

(c) $\theta = n\pi \pm \frac{\pi}{4}, n \in I$ (d) none of these

216. The general solution of the equation $\cot^2\theta = 3$ is

(a) $\theta = n\pi + \frac{\pi}{6}, n \in I$ (b) $\theta = n\pi \pm \frac{\pi}{6}, n \in I$

(c) $\theta = 2n\pi + \frac{\pi}{6}, n \in I$ (d) none of these

217. In a $\triangle ABC$, if $a = 2$, $b = 3$, $c = 4$, then $\cos A = ?$

(a) $\frac{2}{9}$ (b) $\frac{5}{7}$ (c) $\frac{6}{7}$ (d) $\frac{7}{8}$

218. In a $\triangle ABC$, if $a = 2$, $b = \sqrt{6}$ and $c = (\sqrt{3}+1)$, then $\angle A = ?$

(a) 18° (b) 30° (c) 45° (d) 60°

219. In a $\triangle ABC$, if $a = \sqrt{2}$, $b = \sqrt{6}$, $c = \sqrt{8}$, then $\angle A = ?$

(a) 60° (b) 45° (c) 30° (d) 90°

220. The angles of a $\triangle ABC$ are in the ratio 1 : 2 : 3. The ratio of their corresponding sides is

(a) $1 : 2 : 3$ (b) $3 : 2 : 1$ (c) $1 : \sqrt{2} : \sqrt{3}$ (d) $1 : \sqrt{3} : 2$

221. In a $\triangle ABC$, if $a = (\sqrt{3}+1)$, $b = 2$ and $\angle C = 60°$, then $c = ?$

(a) $\sqrt{2}$ (b) $\sqrt{3}$ (c) $\sqrt{5}$ (d) $\sqrt{6}$

222. In a $\triangle ABC$, if $a = 2$, $b = 3$ and $\sin A = \left(\frac{2}{3}\right)$, then $\angle B = ?$

(a) 30° (b) 60° (c) 90° (d) 120°

223. The angles of a $\triangle ABC$ are in AP and $b : c = \sqrt{3} : \sqrt{2}$, then $\angle C = ?$

(a) 30° (b) 45° (c) 60° (d) 90°

224. In a $\triangle ABC$, if $\angle A = 30°$ and $b : c = 2 : \sqrt{3}$, then $\angle B = ?$

(a) 30° (b) 45° (c) 60° (d) 90°

225. In a right $\triangle ABC$, the sides are in AP, then their ratio is

(a) $2 : 3 : 4$ (b) $3 : 4 : 5$ (c) $4 : 5 : 6$ (d) $2 : 3 : 5$

226. In a $\triangle ABC$, if $\angle A = 45°$, $\angle B = 60°$, $\angle C = 75°$, then $a : b : c = ?$

(a) $\sqrt{2} : \sqrt{3} : \sqrt{5}$ (b) $3 : 4 : 5$ (c) $2 : \sqrt{6} : (\sqrt{3}+1)$ (d) none of these

227. In a $\triangle ABC$, if $\angle A = 30°$, $\angle C = 105°$ and $b = 3\sqrt{2}$, then $a = ?$

(a) 2 (b) 3 (c) $\sqrt{2}$ (d) $3\sqrt{2}$

228. In a $\triangle ABC$, if $a = 5$, $c = 2\sqrt{2}$ and $\angle B = 45°$, then $b = ?$

(a) 6 (b) $\sqrt{3}$ (c) $\sqrt{13}$ (d) $2\sqrt{3}$

229. In a $\triangle ABC$, if $a = (\sqrt{3}+1)$, $\angle B = 30°$, $\angle C = 45°$, then $c = ?$

(a) 5 (b) 4 (c) 3 (d) 2

230. In a $\triangle ABC$, $b = 5$, $c = 5\sqrt{3}$ and $\angle A = 30°$, then $\triangle ABC$ is

(a) isosceles (b) equilateral (c) right angled (d) scalene

231. In a $\triangle ABC$, it is given that $\angle A : \angle B : \angle C = 3 : 4 : 5$, then $a : c = ?$

(a) $3 : 5$ (b) $\sqrt{3} : \sqrt{5}$ (c) $2 : (\sqrt{3}+1)$ (d) $3 : (\sqrt{2}+1)$

232. In a $\triangle ABC$, if $b = 20$, $c = 15$ and $\angle A = 150°$, then ar($\triangle ABC$) = ?

(a) 60 sq units (b) 75 sq units (c) 90 sq units (d) 120 sq units

233. In a $\triangle ABC$, if $a = 18$, $b = 24$ and $c = 30$, then ar($\triangle ABC$) = ?

(a) 180 sq units (b) 210 sq units (c) 216 sq units (d) 225 sq units

234. In a $\triangle ABC$, if $a = 16$, $c = 9$ and $\angle B = 30°$, then ar($\triangle ABC$) = ?

(a) 36 sq units (b) 48 sq units (c) $18\sqrt{3}$ sq units (d) 72 sq units

235. In a $\triangle ABC$, if $a = 15$, $b = 14$, $c = 13$, then its circumradius is

(a) $\frac{21}{2}$ (b) $\frac{17}{2}$ (c) $\frac{55}{4}$ (d) $\frac{65}{8}$

236. In a $\triangle ABC$, if $a = 18$, $b = 24$, $c = 30$, then the area of its circumcircle is

(a) 121π sq units (b) 144π sq units (c) 225π sq units (d) 256π sq units

237. In a $\triangle ABC$, if $a = 4$, $b = 13$, $c = 15$, then the radius of the incircle is

(a) 3 (b) 2 (c) $\frac{3}{2}$ (d) $\frac{5}{2}$

238. In a $\triangle ABC$, if $a = 13$, $b = 14$ and $c = 15$, then the area of the incircle is

(a) (16π) sq units (b) (25π) sq units (c) (36π) sq units (d) none of these

239. In an equilateral $\triangle ABC$, we have:

(a) $R = 2r$ (b) $R = 3r$ (c) $R = 4r$ (d) $2R = 3r$

240. In an equilateral triangle of side $2\sqrt{3}$, the circumradius is

(a) 1 (b) 2 (c) 3 (d) $\sqrt{3}$

241. The sides of a $\triangle ABC$ are in the ratio $3 : 7 : 8$. Then, $(R : r) = ?$

(a) $7 : 5$ (b) $8 : 5$ (c) $7 : 2$ (d) $7 : 3$

242. The perimeter of a $\triangle ABC$ is 27 cm and its area is 81 cm^2. Its inradius is

(a) 4.5 cm (b) 6 cm (c) 7.5 cm (d) 7 cm

243. In a right $\triangle ABC$, we have: $\sin^2 A + \sin^2 B + \sin^2 C = ?$

(a) 1 (b) 2 (c) 3 (d) 0

244. In a $\triangle ABC$, we have: $\dfrac{\sin(B-C)}{\sin(B+C)} = ?$

(a) $\dfrac{(b^2-c^2)}{a^2}$ (b) $\dfrac{(b^2+c^2)}{a^2}$ (c) $\dfrac{b^2-c^2}{b^2+c^2}$ (d) none of these

245. In a $\triangle ABC$, we have: $a(b\cos C - c\cos B) = ?$

(a) a^2 (b) $a(b^2-c^2)$ (c) (b^2-c^2) (d) 0

246. In a $\triangle ABC$, we have: $\dfrac{1+\cos(A-B)\cos C}{1+\cos(A-C)\cos B} = ?$

(a) $\dfrac{a^2+b^2}{a^2+c^2}$ (b) $\dfrac{b^2+c^2}{c^2+a^2}$ (c) $\dfrac{c^2-a^2}{a^2+b^2}$ (d) none of these

247. In a $\triangle ABC$, $(a-b)^2\cos^2\dfrac{C}{2} + (a+b)^2\sin^2\dfrac{C}{2} = ?$

(a) a^2 (b) b^2 (c) c^2 (d) $2(a^2+b^2)$

248. In a $\triangle ABC$, $(b+c)\cos A + (c+a)\cos B + (a+b)\cos C = ?$

(a) $a+b+c$ (b) abc (c) $ab+bc+ca$ (d) none of these

249. In a $\triangle ABC$, $2(bc\cos A + ca\cos B + ab\cos C) = ?$

(a) $(a^2+b^2-c^2)$ (b) $(a^2+c^2-b^2)$ (c) $(b^2+c^2-a^2)$ (d) $(a^2+b^2+c^2)$

250. In a $\triangle ABC$, $(ac\cos B - bc\cos A) = ?$

(a) (a^2-b^2) (b) (b^2-c^2) (c) (c^2-a^2) (d) none of these

251. In a $\triangle ABC$, $\dfrac{a-b}{a+b} = ?$

(a) $\dfrac{\sin(A-B)}{\sin(A+B)}$ (b) $\dfrac{\cos(A-B)}{\cos(A+B)}$ (c) $\dfrac{\tan\dfrac{(A-B)}{2}}{\tan\dfrac{(A+B)}{2}}$ (d) none of these

252. In a $\triangle ABC$, $c\cos^2\dfrac{A}{2} + a\cos^2\dfrac{C}{2} = ?$

(a) s (b) $2s$ (c) $3s$ (d) $4s$

253. In a $\triangle ABC$, $\dfrac{(b-c)}{a}\cos^2\dfrac{A}{2} + \dfrac{(c-a)}{b}\cos^2\dfrac{B}{2} + \dfrac{(a-b)}{c}\cdot\cos^2\dfrac{C}{2} = ?$

(a) $a+b+c$ (b) abc (c) 0 (d) none of these

254. In a $\triangle ABC$, $\triangle\left(\cot\frac{A}{2}+\cot\frac{B}{2}+\cot\frac{C}{2}\right)=?$

(a) s (b) s^2 (c) $2s$ (d) $2s^2$

255. In a $\triangle ABC$, $bc\cos^2\frac{A}{2}+ca\cos^2\frac{B}{2}+ab\cos^2\frac{C}{2}=?$

(a) s^2 (b) $(s-a)^2$ (c) $(s-b)^2$ (d) $(s-c)^2$

256. In a $\triangle ABC$, $2abc\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}=?$

(a) $(a+b-c)\triangle$ (b) $(b+c-a)\triangle$ (c) $(c+a-b)\triangle$ (d) $(a+b+c)\triangle$

257. In a $\triangle ABC$, if $\cos\frac{A}{2}=\sqrt{\frac{b+c}{2c}}$, then

(a) $c^2+a^2=b^2$ (b) $a^2+b^2=c^2$ (c) $b^2+c^2=a^2$ (d) none of these

258. In a $\triangle ABC$, if $\cot\frac{A}{2}:\cot\frac{B}{2}:\cot\frac{C}{2}=3:5:7$, then $a:b:c=?$

(a) $6:5:4$ (b) $4:5:6$ (c) $5:6:4$ (d) $5:4:6$

259. If $a=16$, $b=24$ and $c=20$, then $\cos\frac{B}{2}=?$

(a) $\frac{1}{2}$ (b) $\frac{1}{3}$ (c) $\frac{1}{4}$ (d) $\frac{3}{4}$

260. In a $\triangle ABC$, if $a\cos A=b\cos B$, then $\triangle ABC$ is

(a) either equilateral or right angled
(b) either isosceles or right angled
(c) isosceles as well as right angled
(d) none of these

ANSWERS (EXERCISE 1)

1. (c) **2.** (b) **3.** (c) **4.** (d) **5.** (b) **6.** (c) **7.** (c) **8.** (d) **9.** (b) **10.** (b)
11. (c) **12.** (c) **13.** (c) **14.** (b) **15.** (c) **16.** (b) **17.** (c) **18.** (b) **19.** (c) **20.** (a)
21. (c) **22.** (d) **23.** (c) **24.** (b) **25.** (b) **26.** (c) **27.** (c) **28.** (a) **29.** (b) **30.** (b)
31. (c) **32.** (a) **33.** (b) **34.** (a) **35.** (c) **36.** (c) **37.** (c) **38.** (a) **39.** (c) **40.** (b)
41. (d) **42.** (c) **43.** (a) **44.** (a) **45.** (a) **46.** (b) **47.** (b) **48.** (d) **49.** (c) **50.** (a)
51. (b) **52.** (d) **53.** (b) **54.** (b) **55.** (b) **56.** (a) **57.** (a) **58.** (c) **59.** (b) **60.** (d)
61. (b) **62.** (a) **63.** (b) **64.** (b) **65.** (b) **66.** (c) **67.** (c) **68.** (d) **69.** (b) **70.** (c)
71. (a) **72.** (b) **73.** (c) **74.** (a) **75.** (b) **76.** (a) **77.** (b) **78.** (c) **79.** (c) **80.** (a)

81. (a) **82.** (c) **83.** (b) **84.** (c) **85.** (a) **86.** (d) **87.** (a) **88.** (b) **89.** (c) **90.** (b)

91. (a) **92.** (b) **93.** (b) **94.** (b) **95.** (c) **96.** (a) **97.** (c) **98.** (d) **99.** (b) **100.** (a)

101. (c) **102.** (b) **103.** (a) **104.** (b) **105.** (c) **106.** (b) **107.** (a) **108.** (d) **109.** (a) **110.** (c)

111. (b) **112.** (c) **113.** (c) **114.** (a) **115.** (b) **116.** (c) **117.** (b) **118.** (d) **119.** (d) **120.** (c)

121. (b) **122.** (b) **123.** (b) **124.** (a) **125.** (b) **126.** (c) **127.** (b) **128.** (c) **129.** (b) **130.** (c)

131. (a) **132.** (b) **133.** (a) **134.** (b) **135.** (b) **136.** (c) **137.** (a) **138.** (b) **139.** (b) **140.** (a)

141. (c) **142.** (b) **143.** (a) **144.** (c) **145.** (b) **146.** (a) **147.** (c) **148.** (b) **149.** (a) **150.** (c)

151. (a) **152.** (b) **153.** (a) **154.** (b) **155.** (a) **156.** (b) **157.** (d) **158.** (b) **159.** (a) **160.** (b)

161. (c) **162.** (a) **163.** (b) **164.** (b) **165.** (d) **166.** (b) **167.** (b) **168.** (b) **169.** (b) **170.** (d)

171. (b) **172.** (d) **173.** (c) **174.** (a) **175.** (b) **176.** (a) **177.** (b) **178.** (c) **179.** (c) **180.** (b)

181. (d) **182.** (b) **183.** (a) **184.** (b) **185.** (b) **186.** (d) **187.** (b) **188.** (b) **189.** (a) **190.** (c)

191. (b) **192.** (a) **193.** (b) **194.** (b) **195.** (c) **196.** (a) **197.** (b) **198.** (b) **199.** (b) **200.** (c)

201. (a) **202.** (b) **203.** (c) **204.** (c) **205.** (c) **206.** (c) **207.** (a) **208.** (c) **209.** (c) **210.** (a)

211. (c) **212.** (a) **213.** (b) **214.** (a) **215.** (c) **216.** (b) **217.** (d) **218.** (c) **219.** (c) **220.** (d)

221. (d) **222.** (c) **223.** (b) **224.** (d) **225.** (b) **226.** (c) **227.** (b) **228.** (c) **229.** (d) **230.** (a)

231. (c) **232.** (b) **233.** (c) **234.** (a) **235.** (d) **236.** (c) **237.** (c) **238.** (a) **239.** (a) **240.** (b)

241. (c) **242.** (b) **243.** (b) **244.** (a) **245.** (c) **246.** (a) **247.** (c) **248.** (a) **249.** (d) **250.** (a)

251. (c) **252.** (a) **253.** (c) **254.** (b) **255.** (a) **256.** (d) **257.** (b) **258.** (a) **259.** (d) **260.** (b)

HINTS TO SOME SELECTED QUESTIONS

1. $180° = \pi^c \Rightarrow 1° = \left(\frac{\pi}{180}\right)^c \Rightarrow 25° = \left(\frac{\pi}{180} \times 25\right)^c = \left(\frac{5\pi}{36}\right)^c.$

2. $180° = \pi^c \Rightarrow 1° = \left(\frac{\pi}{180}\right)^c \Rightarrow 162° = \left(\frac{\pi}{180} \times 162\right)^c = \left(\frac{9\pi}{10}\right)^c.$

3. $\pi^c = 180° \Rightarrow 1^c = \left(\frac{180}{\pi}\right)^\circ \Rightarrow \left(\frac{8\pi}{5}\right)^c = \left(\frac{180}{\pi} \times \frac{8\pi}{5}\right)^\circ = 288°.$

4. $\pi^c = 180° \Rightarrow 1^c = \left(\frac{180}{\pi}\right)^\circ \Rightarrow 11^c = \left(\frac{180}{\pi} \times 11\right)^\circ = \left(180 \times \frac{7}{22} \times 11\right)^\circ = 630°.$

5. $\pi^c = 180° \Rightarrow 1^c = \left(\frac{180}{\pi}\right)^\circ = \left(180 \times \frac{7}{22}\right)^\circ = \left(\frac{630}{11}\right)^\circ = 57°16'22''$ (app.).

6. $3°45' = \left(3\frac{45}{60}\right)^\circ = \left(3\frac{3}{4}\right)^\circ = \left(\frac{15}{4}\right)^\circ.$

$180° = \pi^c \Rightarrow 1° = \left(\frac{\pi}{180}\right)^c \Rightarrow \left(\frac{15}{4}\right)^\circ = \left(\frac{\pi}{180} \times \frac{15}{4}\right)^c = \left(\frac{\pi}{48}\right)^c.$

7. $50°\,37'\,30'' = 50° + \left(37\frac{30}{60}\right)' = 50° + \left(\frac{75}{2}\right)' = 50° + \left(\frac{75}{2\times 60}\right)° = \left(50\frac{5}{8}\right)° = \left(\frac{405}{8}\right)°$.

$180° = \pi^c \Rightarrow 1° = \left(\frac{\pi}{180}\right)^c \Rightarrow \left(\frac{405}{8}\right)° = \left(\frac{\pi}{180}\times\frac{405}{8}\right)^c = \left(\frac{9\pi}{32}\right)^c$.

8. Let $\angle C = 90°$. Then, $A + B = 90°$.

$A - B = \left(\frac{\pi}{15}\times\frac{180}{\pi}\right)° = 12° \Rightarrow A - B = 12°$.

On solving $A + B = 90$ and $A - B = 12$, we get: $A = 51°$ and $B = 39°$.

$\therefore$ the measure of the smallest angle is $39°$.

9. Let the angles be $(a - d)°$, $a°$ and $(a + d)°$. Then,

$a - d + a + a + d = 180 \Rightarrow 3a = 180 \Rightarrow a = 60$.

So, the angles are $(60 - d)°$, $60°$ and $(60 + d)°$.

$\therefore$ $60 + d = 2(60 - d) \Rightarrow 3d = 60 \Rightarrow d = 20$.

$\therefore$ the largest angle $= (60 + 20)° = 80°$.

10. Let the angles be $(a - d)°$, $a°$ and $(a + d)°$. Then,

$a - d + a + a + d = 180 \Rightarrow 3a = 180 \Rightarrow a = 60$.

So, the angles are $(60 - d)°$, $60°$ and $(60 + d)°$.

$180° = \pi^c \Rightarrow (60 + d)° = \left\{\frac{\pi}{180}\times(60 + d)\right\}^c = \left\{\frac{(60 + d)\pi}{180}\right\}^c$

$\therefore$ $\frac{(60 - d)}{(60 + d)\cdot\frac{\pi}{180}} = \frac{60}{\pi} \Rightarrow \frac{3(60 - d)}{(60 + d)} = 1$

$\Rightarrow 180 - 3d = 60 + d \Rightarrow 4d = 120 \Rightarrow d = 30$.

$\therefore$ the smallest angle $= (60 - 30)° = 30°$.

11. Here, $l = 33$ cm and $\theta = \left(\frac{\pi}{180}\times 45\right)^c = \left(\frac{\pi}{4}\right)^c$.

$\theta = \frac{l}{r} \Rightarrow r = \frac{l}{\theta} = \left(33\times\frac{4}{\pi}\right) = \left(33\times 4\times\frac{7}{22}\right) = 42$ cm.

12. $\theta = \left(36\times\frac{\pi}{180}\right)^c = \left(\frac{\pi}{5}\right)^c$ and $r = 14$ cm.

$\therefore$ $l = r\theta = \left(14\times\frac{\pi}{5}\right)$ cm $= \left(14\times\frac{22}{7}\times\frac{1}{5}\right)$ cm $= \frac{44}{5}$ cm $= 8.8$ cm.

13. Angle traced by the minute hand in 60 min $= (2\pi)^c$.

Angle traced by the minute hand in 45 min $= \left(\frac{2\pi}{60}\times 45\right)^c = \left(\frac{3\pi}{2}\right)^c$.

$\therefore$ $r = 1.4$ cm and $\theta = \left(\frac{3\pi}{2}\right)^c$

$\Rightarrow$ $l = r\theta = \left(1.4\times\frac{3\pi}{2}\right)$ cm $= \left(1.4\times\frac{3}{2}\times\frac{22}{7}\right)$ cm $= 6.6$ cm.

14. $\theta_1 = 60° = \left(60 \times \frac{\pi}{180}\right)^c = \left(\frac{\pi}{3}\right)^c$ and $\theta_2 = 75° = \left(75 \times \frac{\pi}{180}\right)^c = \left(\frac{5\pi}{12}\right)^c$.

$\therefore \quad l = r_1\theta_1 = r_2\theta_2$

$\Rightarrow \quad r_1 \times \frac{\pi}{3} = r_2 \times \frac{5\pi}{12} \Rightarrow \frac{r_1}{r_2} = \left(\frac{5}{12} \times 3\right) = \frac{5}{4} \Rightarrow r_1 : r_2 = 5:4.$

15. $l = 121$ cm and $r = 180$ cm.

$\therefore \quad \theta = \frac{l}{r} = \left(\frac{121}{180}\right)^c = \left(\frac{121}{180} \times \frac{180}{\pi}\right)^\circ = \left(121 \times \frac{7}{22}\right)^\circ = \left(\frac{77}{2}\right)^\circ = 38°\,30'.$

16. $\theta = 72° = \left(72 \times \frac{\pi}{180}\right)^c = \left(\frac{2\pi}{5}\right)^c$ and $l = 88$ m.

$\therefore \quad r = \frac{l}{\theta} = \left(88 \times \frac{5}{2\pi}\right) \text{m} = \left(88 \times \frac{5}{2} \times \frac{7}{22}\right) \text{m} = 70 \text{ m}.$

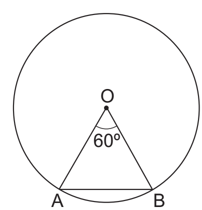
17. $\theta = 42° = \left(42 \times \frac{\pi}{180}\right)^c = \left(\frac{7\pi}{30}\right)^c$ and $l = 55$ cm.

$\therefore \quad r = \frac{l}{\theta} = \left(55 \times \frac{30}{7\pi}\right) \text{cm} = \left(55 \times \frac{30}{7} \times \frac{7}{22}\right) \text{cm} = 75 \text{ cm}.$

18. Let O be the centre of the circle and AB be the chord.

Then, $OA = OB = AB = 30$ cm.

$\therefore \quad \triangle OAB$ is an equilateral triangle and therefore each of its angles is 60°.



$\therefore \quad \theta = \angle AOB = 60° = \left(60 \times \frac{\pi}{180}\right)^c = \left(\frac{\pi}{3}\right)^c$ and $r = 30$ cm.

$\therefore \quad l = r\theta = \left(30 \times \frac{\pi}{3}\right) \text{cm} = (10\pi) \text{ cm}.$

19. Number of revolutions made in 1 second $= \frac{180}{60} = 3$.

Angle turned in 1 revolution $= (2\pi)^c$.

Angle turned in 3 revolutions $= (3 \times 2\pi)^c = (6\pi)^c$.

20. Speed = 90 km/hr $= \left(90 \times \frac{5}{18}\right)$ m/sec = 25 m/sec.

Distance moved in 11 sec = (25×11) m = 275 m.

$\therefore \quad l = 275$ m and $r = 1500$ m.

$\theta = \frac{l}{r} = \left(\frac{275}{1500}\right)^c = \left(\frac{275}{1500} \times \frac{180}{\pi}\right)^\circ = \left(33 \times \frac{7}{22}\right)^\circ = \left(\frac{21}{2}\right)^\circ = 10°\,30'.$

21. Angle traced by the hour hand in 12 hours = 360°.

Angle traced by the hour hand in $\frac{22}{3}$ hours $= \left(\frac{360}{12} \times \frac{22}{3}\right)^\circ = 220°$.

Angle traced by the minute hand in 60 min = 360°.

Angle traced by the minute hand in 20 min $= \left(\frac{360}{60} \times 20\right)^\circ = 120°$.

Angle between the two hands $= (220° - 120°) = 100°$.

22. Angle traced by the hour hand in 12 hours = 360°.

Angle traced by the hour hand in $\frac{7}{2}$ hours $= \left(\frac{360}{12} \times \frac{7}{2}\right)^{\circ} = 105°$.

Angle traced by the minute hand in 60 min = 360°.

Angle traced by the minute hand in 30 min $= \left(\frac{360}{60} \times 30\right)^{\circ} = 180°$.

Angle between the two hands $= (180° - 105°) = 75°$.

23. Angle traced by the hour hand in 12 hours = 360°.

Angle traced by the hour hand in $8\frac{25}{60}$ hours $= \left(\frac{360}{12} \times \frac{101}{12}\right)^{\circ} = \left(\frac{505}{2}\right)^{\circ} = 252.5°$.

Angle traced by the minute hand in 60 min = 360°.

Angle traced by the minute hand in 25 min $= \left(\frac{360}{60} \times 25\right)^{\circ} = 150°$.

Angle between the two hands $= (252.5 - 150)° = 102° 30'$.

24. Here, $r = 60$ cm and $l = 16.5$ cm.

$$\therefore \quad \theta = \frac{l}{r} = \left(\frac{16.5}{60}\right)^{c} = \left(\frac{16.5}{60} \times \frac{180}{\pi}\right)^{\circ} = \left(\frac{16.5}{60} \times 180 \times \frac{7}{22}\right)^{\circ} = \left(\frac{63}{4}\right)^{\circ} = 15°45'.$$

25. Let the angles be $(a - 3d)°, (a - d)°, (a + d)°$ and $(a + 3d)°$. Then,

$$a - 3d + a - d + a + d + a + 3d = 360 \Rightarrow 4a = 360 \Rightarrow a = 90.$$

$\therefore$ the angles are $(90 - 3d)°, (90 - d)°, (90 + d)°$ and $(90 + 3d)°$.

$\therefore$ $90 + 3d = 120 \Rightarrow 3d = 30 \Rightarrow d = 10$.

$\therefore$ the smallest angle $= (a - 3d)° = (90 - 3 \times 10)° = 60° = \left(\frac{\pi}{3}\right)^{c}$.

26. Let the radius be r cm. Then,

$$r + r + \text{arc length} = \frac{1}{2} \times 2\pi r = \pi r.$$

$$\therefore \quad l = \pi r - 2r = (\pi - 2)r \Rightarrow \theta = \frac{l}{r} = (\pi - 2)^{c}.$$

27. $\sin \frac{25\pi}{3} = \sin\left(8\pi + \frac{\pi}{3}\right) = \sin \frac{\pi}{3} = \frac{\sqrt{3}}{2}$ $\quad [\because \sin(2n\pi + \theta) = \sin\theta]$.

28. $\cos \frac{41\pi}{3} = \cos\left(10\pi + \frac{\pi}{4}\right) = \cos \frac{\pi}{4} = \frac{1}{\sqrt{2}}$ $\quad [\because \cos(2n\pi + \theta) = \cos\theta]$.

29. $\tan\left(\frac{-16\pi}{3}\right) = -\tan \frac{16\pi}{3}$ $\quad [\because \tan(-\theta) = -\tan\theta]$

$= -\tan\left(5\pi + \frac{\pi}{3}\right) = -\tan \frac{\pi}{3} = -\sqrt{3}$ $\quad [\because \tan(n\pi + \theta) = \tan\theta]$.

30. $\cot\left(\frac{29\pi}{4}\right) = \cot\left(7\pi + \frac{\pi}{4}\right) = \cot \frac{\pi}{4} = 1$ $\quad [\because \cot(n\pi + \theta) = \cot\theta]$.

31. $\sec\left(\frac{-19\pi}{3}\right) = \sec \frac{19\pi}{3}$ $\quad [\because \sec(-\theta) = \sec\theta]$

$= \sec\left(6\pi + \frac{\pi}{3}\right) = \sec \frac{\pi}{3} = 2$ $\quad [\because \sec(2n\pi + \theta) = \sec\theta]$.

32. $\operatorname{cosec}\left(\dfrac{-33\pi}{4}\right) = -\operatorname{cosec}\dfrac{33\pi}{4} = -\operatorname{cosec}\left(8\pi + \dfrac{\pi}{4}\right)$

$= -\operatorname{cosec}\dfrac{\pi}{4} = -\sqrt{2}$ $\quad [\because \operatorname{cosec}(2n\pi + \theta) = \operatorname{cosec}\theta]$.

33. $\cos 15\pi = \cos(14\pi + \pi) = \cos\pi = -1$ $\quad [\because \cos(2n\pi + \theta) = \cos\theta]$.

34. $\sec 6\pi = \sec(6\pi + 0) = \sec 0 = 1$ $\quad [\because \sec(2n\pi + \theta) = \sec\theta]$.

35. $\tan\dfrac{5\pi}{4} = \tan\left(\pi + \dfrac{\pi}{4}\right) = \tan\dfrac{\pi}{4} = 1$ $\quad [\because \tan(\pi + \theta) = \tan\theta]$.

36. $\sin(765°) = \sin(360 \times 2 + 45)° = \sin 45° = \dfrac{1}{\sqrt{2}}\cdot$

37. $\cot(-600°) = -\cot 600° = -\cot(360 + 240)° = -\cot 240°$

$= -\cot(180° + 60°) = -\cot 60° = \dfrac{-1}{\sqrt{3}}\cdot$

38. In quadrant III, we know that $\cos x$ is negative.

$\cos^2 x = (1 - \sin^2 x) = \left(1 - \dfrac{24}{25}\right) = \dfrac{1}{25} \Rightarrow \cos x = \dfrac{-1}{5}\cdot$

$\therefore \quad \cot x = \dfrac{\cos x}{\sin x} = \dfrac{-1}{5} \times \dfrac{5}{-2\sqrt{6}} = \dfrac{1}{2\sqrt{6}}\cdot$

39. $\sin^2 x = (1 - \cos^2 x) = \left(1 - \dfrac{15}{16}\right) = \dfrac{1}{16}\cdot$

When $\dfrac{\pi}{2} < x < \pi$, then x lies in quadrant II.

And $\sin x > 0$ in quadrant II.

$\therefore \quad \sin x = \sqrt{\dfrac{1}{16}} = \dfrac{1}{4}\cdot$

40. $\pi < x < \dfrac{3\pi}{2} \Rightarrow x$ lies in quadrant III and $\sin x < 0$ in quadrant III.

$\sec x = -2 \Rightarrow \cos x = \dfrac{-1}{2}\cdot$

$\sin^2 x = (1 - \cos^2 x) = \left(1 - \dfrac{1}{4}\right) = \dfrac{3}{4}\cdot$

$\therefore \quad \sin x = -\sqrt{\dfrac{3}{4}} = \dfrac{-\sqrt{3}}{2}\cdot$

41. In quadrant IV, $\cos x > 0$.

Now, $\operatorname{cosec} x = \dfrac{-2}{\sqrt{3}} \Rightarrow \sin x = \dfrac{-\sqrt{3}}{2}\cdot$

$\cos^2 x = (1 - \sin^2 x) = \left(1 - \dfrac{3}{4}\right) = \dfrac{1}{4} \Rightarrow \cos x = +\sqrt{\dfrac{1}{4}} = \dfrac{1}{2}\cdot$

$\therefore \quad \tan x = \dfrac{\sin x}{\cos x} = \left(\dfrac{-\sqrt{3}}{2} \times \dfrac{2}{1}\right) = -\sqrt{3}.$

42. Since cot θ is *+ve* and θ does not lie in quadrant I, so θ lies in quadrant III. In this quadrant, $\sin\theta < 0$ and $\cos\theta < 0$.

$\text{cosec}^2\theta = (1+\cot^2\theta) = (1+5) = 6 \Rightarrow \text{cosec}\,\theta = -\sqrt{6}$ $\quad [\because \text{cosec}\,\theta < 0]$.

Now, $\tan\theta = \frac{1}{\sqrt{5}}$.

$\therefore \quad \sec^2\theta = (1+\tan^2\theta) = \left(1+\frac{1}{5}\right) = \frac{6}{5} \Rightarrow \sec\theta = -\sqrt{\frac{6}{5}}$ $\quad [\because \sec\theta < 0]$.

Thus, $\text{cosec}\,\theta = -\sqrt{6}$ and $\sec\theta = -\sqrt{\frac{6}{5}}$.

43. In quadrant II, $\sin\theta > 0$ and $\tan\theta < 0$.

$\sin^2\theta = (1-\cos^2\theta) = \left(1-\frac{1}{4}\right) = \frac{3}{4} \Rightarrow \sin\theta = \sqrt{\frac{3}{4}} = \frac{\sqrt{3}}{2}$.

$\therefore \quad \tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{\sqrt{3}}{2}\times(-2) = -\sqrt{3}$.

$\therefore \quad (2\sin\theta + \tan\theta) = \left(2\times\frac{\sqrt{3}}{2} - \sqrt{3}\right) = (\sqrt{3}-\sqrt{3}) = 0.$

44. In quadrant III, $\sin\theta < 0$.

$\sin^2\theta = (1-\cos^2\theta) = \left(1-\frac{9}{25}\right) = \frac{16}{25} \Rightarrow \sin\theta = -\sqrt{\frac{16}{25}} = -\frac{4}{5}$.

$\therefore \quad \tan\theta = \left(\frac{-4}{5}\times\frac{5}{-3}\right) = \frac{4}{3}$ and $\cot\theta = \frac{3}{4}$.

$\text{cosec}\,\theta = \frac{-5}{4}$ and $\sec\theta = \frac{-5}{3}$.

$$\therefore \quad \frac{(\text{cosec}\,\theta + \cot\theta)}{(\sec\theta - \tan\theta)} = \frac{\left(\frac{-5}{4}+\frac{3}{4}\right)}{\left(\frac{-5}{3}-\frac{4}{3}\right)} = \frac{\left(\frac{-2}{4}\right)}{\left(\frac{-9}{3}\right)} = \frac{-2}{4}\times\frac{3}{-9} = \frac{1}{6}.$$

45. In quadrant II, $\cos\theta < 0$.

Now, $\cos^2\theta = (1-\sin^2\theta) = \left(1-\frac{9}{25}\right) = \frac{16}{25} \Rightarrow \cos\theta = -\sqrt{\frac{16}{25}} = \frac{-4}{5}$.

$\therefore \quad \sec\theta = \frac{-5}{4}$ and $\cot\theta = \frac{\cos\theta}{\sin\theta} = \frac{-4}{5}\times\frac{5}{3} = \frac{-4}{3}$.

$\therefore \quad (2\sec\theta - 3\cot\theta) = \left\{2\times\frac{(-5)}{4} - 3\times\frac{(-4)}{3}\right\} = \left(\frac{-5}{2}+4\right) = \frac{3}{2}.$

46. In quadrant IV, $\sin\theta < 0$.

$\sec\theta = \sqrt{2} \Rightarrow \cos\theta = \frac{1}{\sqrt{2}}$.

$\therefore \quad \sin^2\theta = (1-\cos^2\theta) = \left(1-\frac{1}{2}\right) = \frac{1}{2} \Rightarrow \sin\theta = \frac{-1}{\sqrt{2}}$.

$\therefore \quad \tan\theta = \dfrac{\sin\theta}{\cos\theta} = -1, \cot\theta = -1, \operatorname{cosec}\theta = -\sqrt{2}.$

$\therefore \quad \dfrac{(1+\tan\theta+\operatorname{cosec}\theta)}{(1+\cot\theta-\operatorname{cosec}\theta)} = \dfrac{(1-1-\sqrt{2})}{(1-1+\sqrt{2})} = \dfrac{-\sqrt{2}}{\sqrt{2}} = -1.$

47. In quadrant III, $\sin\theta < 0$.

$\sin^2\theta = (1-\cos^2\theta) = \left(1-\dfrac{144}{169}\right) = \dfrac{25}{169} \Rightarrow \sin\theta = -\sqrt{\dfrac{25}{169}} = \dfrac{-5}{13}\cdot$

$\therefore \quad \cot\theta = \dfrac{\cos\theta}{\sin\theta} = \dfrac{-12}{13}\times\dfrac{13}{-5} = \dfrac{12}{5}\cdot$

$\operatorname{cosec}^2\theta = (1+\cot^2\theta) = \left(1+\dfrac{144}{25}\right) = \dfrac{169}{25} \Rightarrow \operatorname{cosec}\theta = -\sqrt{\dfrac{169}{25}} = \dfrac{-13}{5}\cdot$

$\therefore \quad (\cot\theta+\operatorname{cosec}\theta) = \left(\dfrac{12}{5}-\dfrac{13}{5}\right) = \dfrac{-1}{5}\cdot$

48. In quadrant II, $\sin\theta > 0, \cos\theta < 0.$ So, $\operatorname{cosec}\theta > 0$ and $\sec\theta < 0.$

$\operatorname{cosec}^2\theta = (1+\cot^2\theta) = \left(1+\dfrac{144}{25}\right) = \dfrac{169}{25} \Rightarrow \operatorname{cosec}\theta = \sqrt{\dfrac{169}{25}} = \dfrac{13}{5}\cdot$

$\cos^2\theta = (1-\sin^2\theta) = \left(1-\dfrac{25}{169}\right) = \dfrac{144}{169} \Rightarrow \cos\theta = -\sqrt{\dfrac{144}{169}} = -\dfrac{12}{13}\cdot$

$\therefore \quad \sin\theta = \dfrac{5}{13}$ and $\cos\theta = \dfrac{-12}{13}\cdot$

$\therefore \quad \dfrac{(1+\sin\theta-\cos\theta)}{(1-\sin\theta+\cos\theta)} = \dfrac{\left(1+\dfrac{5}{13}+\dfrac{12}{13}\right)}{\left(1-\dfrac{5}{13}-\dfrac{12}{13}\right)} = \left(\dfrac{30}{-4}\right) = \dfrac{-15}{2}\cdot$

49. In quadrant II, $\cos\theta < 0$.

$\sin\theta = \dfrac{4}{5} \Rightarrow \cos^2\theta = (1-\sin^2\theta) = \left(1-\dfrac{16}{25}\right) = \dfrac{9}{25}$

$\Rightarrow \cos\theta = -\sqrt{\dfrac{9}{25}} = \dfrac{-3}{5}\cdot$

$\therefore \quad \tan\theta = \dfrac{\sin\theta}{\cos\theta} = \dfrac{4}{5}\times\dfrac{5}{(-3)} = \dfrac{-4}{3} \Rightarrow \cot\theta = \dfrac{-3}{4}\cdot$

$\operatorname{cosec}\theta = \dfrac{1}{\sin\theta} = \dfrac{5}{4}$ and $\sec\theta = \dfrac{1}{\cos\theta} = \dfrac{-5}{3}\cdot$

$$\therefore \quad \frac{(5\cos\theta+4\operatorname{cosec}\theta+3\tan\theta)}{(4\cot\theta+3\sec\theta+5\sin\theta)} = \frac{\left\{5\times\dfrac{(-3)}{5}\right\}+\left(4\times\dfrac{5}{4}\right)+\left\{3\times\dfrac{(-4)}{3}\right\}}{\left\{4\times\dfrac{(-3)}{4}\right\}+\left\{3\times\dfrac{(-5)}{3}\right\}+\left\{5\times\dfrac{4}{5}\right\}}$$

$$= \frac{(-3+5-4)}{(-3-5+4)} = \frac{-2}{-4} = \frac{1}{2}\cdot$$

50. $\tan^2\theta = (\sec^2\theta - 1) = \left\{\left(\frac{13}{5}\right)^2 - 1\right\} = \left(\frac{169}{25} - 1\right) = \frac{144}{25}.$

$\therefore \quad \tan\theta = +\sqrt{\frac{144}{25}} = \frac{12}{5}$ and $\cot\theta = \frac{5}{12}.$

$$\therefore \quad \frac{(4 - 3\cot\theta)}{(3 + 4\tan\theta)} = \frac{\left(4 - 3\times\frac{5}{12}\right)}{\left(3 + 4\times\frac{12}{5}\right)} = \frac{\left(\frac{33}{12}\right)}{\left(\frac{63}{5}\right)} = \left(\frac{33}{12}\times\frac{5}{63}\right) = \frac{55}{252}.$$

51. $\cos 135^\circ = \cos(90^\circ + 45^\circ) = -\sin 45^\circ = \frac{-1}{\sqrt{2}}.$

52. $\sec 120^\circ = \sec(180^\circ - 60^\circ) = -\sec 60^\circ = -2.$

53. $\operatorname{cosec} 150^\circ = \operatorname{cosec}(180^\circ - 30^\circ) = \operatorname{cosec} 30^\circ = 2.$

54. $\sin 315^\circ = \sin(360^\circ - 45^\circ) = -\sin 45^\circ = \frac{-1}{\sqrt{2}}.$

55. $\cos 405^\circ = \cos(360^\circ + 45^\circ) = \cos 45^\circ = \frac{1}{\sqrt{2}}.$

56. $\tan\frac{11\pi}{6} = \tan\left(2\pi - \frac{\pi}{6}\right) = -\tan\frac{\pi}{6} = \frac{-1}{\sqrt{3}}.$

57. $\cot 675^\circ = \cot(720^\circ - 45^\circ) = \cot(2\times 360^\circ - 45^\circ)$

$= \cot(-45^\circ) = -\cot 45^\circ = -1.$

58. $\sin\left(\frac{31\pi}{3}\right) = \sin\left(10\pi + \frac{\pi}{3}\right) = \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}.$

59. $180^\circ = \pi^c \Rightarrow 600^\circ = \left(\frac{\pi}{180}\times 600\right)^c = \left(\frac{10\pi}{3}\right)^c.$

$\therefore \quad \cot(-600^\circ) = -\cot 600^\circ = -\cot\frac{10\pi}{3}$

$= -\cot\left(3\pi + \frac{\pi}{3}\right) = -\cot\frac{\pi}{3} = \frac{-1}{\sqrt{3}} \quad [\because \cot(n\pi + \theta) = \cot\theta].$

60. $180^\circ = \pi^c \Rightarrow 1110^\circ = \left(\frac{\pi}{180}\times 1110\right)^c = \left(\frac{37\pi}{6}\right)^c.$

$\therefore \quad \operatorname{cosec}(-1110^\circ) = -\operatorname{cosec} 1110^\circ = -\operatorname{cosec}\frac{37\pi}{6}$

$= -\operatorname{cosec}\left(6\pi + \frac{\pi}{6}\right) = -\operatorname{cosec}\frac{\pi}{6} = -2 \quad [\because \operatorname{cosec}(2n\pi + \theta) = \operatorname{cosec}\theta].$

61. $\sec\left(\frac{-33\pi}{4}\right) = \sec\frac{33\pi}{4} = \sec\left(8\pi + \frac{\pi}{4}\right)$

$= \sec\frac{\pi}{4} = \sqrt{2} \quad [\because \sec(2n\pi + \theta) = \sec\theta].$

62. $\tan\left(\frac{-25\pi}{3}\right) = -\tan\frac{25\pi}{3} = -\tan\left(8\pi + \frac{\pi}{3}\right)$

$= -\tan\frac{\pi}{3} = -\sqrt{3} \quad [\because \tan(2n\pi + \theta) = \tan\theta].$

63. $\cot\frac{\pi}{3}=\frac{1}{\sqrt{3}}$, $\cot\frac{\pi}{4}=1$, $\cot\frac{\pi}{6}=\sqrt{3}$.

Now, $\frac{1}{\sqrt{3}}, 1, \sqrt{3}$ are in GP with common ratio $\sqrt{3}$.

64. In quadrant I, $\sin\theta$ is increasing.

Now, $\cos 64° = \cos(90° - 26°) = \sin 26°$.

Clearly, $\sin 26° < \sin 64° \Rightarrow \cos 64° < \sin 64°$.

65. $\cos 24° = \cos(90° - 66°) = \sin 66°$.

In quadrant I, $\sin\theta$ is increasing.

$\therefore \quad \sin 66° > \sin 24° \Rightarrow \cos 24° > \sin 24°$.

66. The extremum values of $\sin\theta$ are –1 and 1.

67. The extremum values of $\cos\theta$ are –1 and 1.

68. $|\sec\theta| \geq 1 \Rightarrow (\sec\theta \leq -1)$ or $(\sec\theta \geq 1)$.

$\therefore$ value of $\sec\theta$ can never lie between –1 and 1.

69. $\tan 150° = \tan(180° - 30°) = -\tan 30° = \frac{-1}{\sqrt{3}}\cdot$

70. $\sec 150° = \sec(180° - 30°) = -\sec 30° = \frac{-2}{\sqrt{3}}\cdot$

71. $\cot 120° = \cot(180° - 60°) = -\cot 60° = \frac{-1}{\sqrt{3}}\cdot$

72. $\sin 105° + \cos 105° = \sin(60° + 45°) + \cos(60° + 45°)$

$= (\sin 60°\cos 45° + \cos 60°\sin 45°) + (\cos 60°\cos 45° - \sin 60°\sin 45°)$

$$= \left(\frac{\sqrt{3}}{2}\times\frac{1}{\sqrt{2}}\right) + \left(\frac{1}{2}\times\frac{1}{\sqrt{2}}\right) + \left(\frac{1}{2}\times\frac{1}{\sqrt{2}}\right) - \left(\frac{\sqrt{3}}{2}\times\frac{1}{\sqrt{2}}\right)$$

$$= \frac{\sqrt{3}}{2\sqrt{2}} + \frac{1}{2\sqrt{2}} + \frac{1}{2\sqrt{2}} - \frac{\sqrt{3}}{2\sqrt{2}} = \frac{2}{2\sqrt{2}} = \frac{1}{\sqrt{2}}\cdot$$

73. $\sin 15° = \sin(45° - 30°) = (\sin 45°\cos 30° - \cos 45°\sin 30°)$

$$= \left(\frac{1}{\sqrt{2}}\times\frac{\sqrt{3}}{2}\right) - \left(\frac{1}{\sqrt{2}}\times\frac{1}{2}\right) = \left(\frac{\sqrt{3}}{2\sqrt{2}} - \frac{1}{2\sqrt{2}}\right) = \frac{(\sqrt{3}-1)}{2\sqrt{2}}\cdot$$

74. $\cos 15° = \cos(45° - 30°) = \cos 45°\cos 30° + \sin 45°\sin 30°$

$$= \left(\frac{1}{\sqrt{2}}\times\frac{\sqrt{3}}{2}\right) + \left(\frac{1}{\sqrt{2}}\times\frac{1}{2}\right) = \frac{(\sqrt{3}+1)}{2\sqrt{2}}\cdot$$

75. $$\tan 15° = \tan(45° - 30°) = \frac{\tan 45° - \tan 30°}{1 + \tan 45°\tan 30°} = \frac{\left(1 - \frac{1}{\sqrt{3}}\right)}{\left(1 + \frac{1}{\sqrt{3}}\right)} = \frac{(\sqrt{3}-1)}{(\sqrt{3}+1)}\cdot$$

76. $\sin 75° = \sin(90° - 15°) = \cos 15° = \frac{(\sqrt{3}+1)}{2\sqrt{2}}\cdot$

77. $\cos 75° = \cos(90° - 15°) = \sin 15° = \frac{(\sqrt{3}-1)}{2\sqrt{2}}\cdot$

78. $\tan \frac{13\pi}{12} = \tan\left(\pi + \frac{\pi}{12}\right) = \tan \frac{\pi}{12}$ $\quad [\because \tan(\pi + \theta) = \tan\theta]$

$$= \tan 15° = \frac{(\sqrt{3}-1)}{(\sqrt{3}+1)} = \frac{(\sqrt{3}-1)}{(\sqrt{3}+1)} \times \frac{(\sqrt{3}-1)}{(\sqrt{3}-1)} = \frac{(\sqrt{3}-1)^2}{(3-1)}$$

$$= \frac{(3+1-2\sqrt{3})}{2} = \frac{(4-2\sqrt{3})}{2} = (2-\sqrt{3}).$$

79. $(\sin 70° \cos 10° - \cos 70° \sin 10°) = \sin(70° - 10°)$

$[\because \sin A \cos B - \cos A \sin B = \sin(A-B)]$

$$= \sin 60° = \frac{\sqrt{3}}{2}\cdot$$

80. $(\sin 36° \cos 9° + \cos 36° \sin 9°) = \sin(36° + 9°)$

$[\because \sin A \cos B + \cos A \sin B = \sin(A+B)]$

$$= \sin 45° = \frac{1}{\sqrt{2}}\cdot$$

81. $\cos 80° \cos 20° + \sin 80° \sin 20° = \cos(80° - 20°)$

$[\because \cos A \cos B + \sin A \sin B = \cos(A-B)]$

$$= \cos 60° = \frac{1}{2}\cdot$$

82. $\cos 50° \cos 10° - \sin 50° \sin 10° = \cos(50° + 10°)$

$[\because \cos A \cos B - \sin A \sin B = \cos(A+B)]$

$$= \cos 60° = \frac{1}{2}\cdot$$

83. $\sin(40° + \theta)\cos(10° + \theta) - \cos(40° + \theta)\sin(10° + \theta)$

$= \sin\{(40°+\theta) - (10° + \theta)\}$ $\quad [\because \sin A \cos B - \cos A \sin B = \sin(A-B)]$

$= \sin 30° = \frac{1}{2}\cdot$

84. $\sin \frac{7\pi}{12} \cos \frac{\pi}{4} - \cos \frac{7\pi}{12} \sin \frac{\pi}{4} = \sin\left(\frac{7\pi}{12} - \frac{\pi}{4}\right)$ $[\because \sin A \cos B - \cos A \sin B = \sin(A-B)]$

$$= \sin \frac{4\pi}{12} = \sin \frac{\pi}{3} = \frac{\sqrt{3}}{2}\cdot$$

85. $\sin \frac{\pi}{4} \cos \frac{\pi}{12} + \cos \frac{\pi}{4} \sin \frac{\pi}{12} = \sin\left(\frac{\pi}{4} + \frac{\pi}{12}\right) = \sin \frac{4\pi}{12} = \sin \frac{\pi}{3} = \frac{\sqrt{3}}{2}\cdot$

86. $\cos \frac{2\pi}{3} \cos \frac{\pi}{4} - \sin \frac{2\pi}{3} \sin \frac{\pi}{4} = \cos\left(\pi - \frac{\pi}{3}\right)\cos \frac{\pi}{4} - \sin\left(\pi - \frac{\pi}{3}\right)\sin \frac{\pi}{4}$

$$= -\cos \frac{\pi}{3} \cos \frac{\pi}{4} - \sin \frac{\pi}{3} \sin \frac{\pi}{4}$$

$$= \left(-\frac{1}{2} \times \frac{1}{\sqrt{2}}\right) - \left(\frac{\sqrt{3}}{2} \times \frac{1}{\sqrt{2}}\right) = \frac{-(\sqrt{3}+1)}{2\sqrt{2}}\cdot$$

87. $\sin \frac{\pi}{12} = \sin\left(\frac{\pi}{4} - \frac{\pi}{6}\right) = \sin \frac{\pi}{4} \cos \frac{\pi}{6} - \cos \frac{\pi}{4} \sin \frac{\pi}{6}$

$$= \left(\frac{1}{\sqrt{2}} \times \frac{\sqrt{3}}{2}\right) - \left(\frac{1}{\sqrt{2}} \times \frac{1}{2}\right) = \frac{(\sqrt{3}-1)}{2\sqrt{2}}\cdot$$

88. $\sin\frac{5\pi}{12} = \sin\left(\frac{\pi}{4}+\frac{\pi}{6}\right) = \sin\frac{\pi}{4}\cos\frac{\pi}{6} + \cos\frac{\pi}{4}\sin\frac{\pi}{6}$

$$= \left(\frac{1}{\sqrt{2}}\times\frac{\sqrt{3}}{2}\right) + \left(\frac{1}{\sqrt{2}}\times\frac{1}{2}\right) = \frac{(\sqrt{3}+1)}{2\sqrt{2}}.$$

89. $\sin\theta = \frac{15}{17} \Rightarrow \cos^2\theta = (1-\sin^2\theta) = \left(1-\frac{225}{289}\right) = \frac{64}{289}$

$$\Rightarrow \cos\theta = \sqrt{\frac{64}{289}} = \frac{8}{17}.$$

$\cos\phi = \frac{12}{13} \Rightarrow \sin^2\phi = (1-\cos^2\phi) = \left(1-\frac{144}{169}\right) = \frac{25}{169}$

$$\Rightarrow \sin\phi = \sqrt{\frac{25}{169}} = \frac{5}{13}.$$

$\therefore \quad \sin(\theta+\phi) = \sin\theta\cos\phi + \cos\theta\sin\phi$

$$= \left(\frac{15}{17}\times\frac{12}{13}\right) + \left(\frac{8}{17}\times\frac{5}{13}\right) = \left(\frac{180}{221}+\frac{40}{221}\right) = \frac{(180+40)}{221} = \frac{220}{221}.$$

90. In quadrant II, $\sin\theta > 0$, $\cos\theta < 0$, $\sin\phi > 0$ and $\cos\phi < 0$.

Now, $\sin\theta = \frac{3}{5} \Rightarrow \cos^2\theta = (1-\sin^2\theta) = \left(1-\frac{9}{25}\right) = \frac{16}{25}$

$$\Rightarrow \cos\theta = -\sqrt{\frac{16}{25}} = \frac{-4}{5}.$$

And, $\cos\phi = \frac{-12}{13} \Rightarrow \sin^2\phi = (1-\cos^2\phi) = \left(1-\frac{144}{169}\right) = \frac{25}{169}$

$$\Rightarrow \sin\phi = \sqrt{\frac{25}{169}} = \frac{5}{13}.$$

$\therefore \quad \sin(\theta-\phi) = \sin\theta\cos\phi - \cos\theta\sin\phi$

$$= \left(\frac{3}{5}\times\frac{-12}{13}\right) - \left(\frac{-4}{5}\times\frac{5}{13}\right) = \left(\frac{-36}{65}+\frac{20}{65}\right) = \frac{-16}{65}.$$

91. In quadrant IV, $\cos\theta > 0$, $\sin\theta < 0$, $\cos\phi > 0$ and $\sin\phi < 0$.

Now, $\cos\theta = \frac{4}{5} \Rightarrow \sin^2\theta = (1-\cos^2\theta) = \left(1-\frac{16}{25}\right) = \frac{9}{25}$

$$\Rightarrow \sin\theta = -\sqrt{\frac{9}{25}} = \frac{-3}{5}.$$

$\cos\phi = \frac{12}{13} \Rightarrow \sin^2\phi = (1-\cos^2\phi) = \left(1-\frac{144}{169}\right) = \frac{25}{169}$

$$\Rightarrow \sin\phi = -\sqrt{\frac{25}{169}} = \frac{-5}{13}.$$

$\therefore \quad \cos(\theta+\phi) = \cos\theta\cos\phi - \sin\theta\sin\phi$

$$= \left(\frac{4}{5}\times\frac{12}{13}\right) - \left(\frac{-3}{5}\times\frac{-5}{13}\right) = \left(\frac{48}{65}-\frac{15}{65}\right) = \frac{33}{65}.$$

92. In quadrant III, $\sin\theta < 0$, $\cos\theta < 0$ and $\tan\theta > 0$.

In quadrant II, $\sin\phi > 0$, $\cos\phi < 0$ and $\tan\phi < 0$.

Now, $\cot\theta = \frac{1}{2} \Rightarrow \tan\theta = 2$

$$\sec\phi = \frac{-5}{3} \Rightarrow \cos\phi = \frac{-3}{5}.$$

$$\therefore \quad \sin^2\phi = (1-\cos^2\phi) = \left(1-\frac{9}{25}\right) = \frac{16}{25} \Rightarrow \sin\phi = \sqrt{\frac{16}{25}} = \frac{4}{5}.$$

$$\therefore \quad \tan\phi = \left(\frac{4}{5}\times\frac{5}{-3}\right) = \frac{-4}{3}.$$

$$\therefore \quad \tan(\theta+\phi) = \frac{\tan\theta+\tan\phi}{1-\tan\theta\tan\phi} = \frac{\left(2-\frac{4}{3}\right)}{\left\{1-\left(2\times\frac{-4}{3}\right)\right\}} = \frac{\left(\frac{2}{3}\right)}{\left(1+\frac{8}{3}\right)} = \left(\frac{2}{3}\times\frac{3}{11}\right) = \frac{2}{11}.$$

93. $\cos 15° - \sin 15° = \cos(45° - 30°) - \sin(45° - 30°)$

$= (\cos 45°\cos 30° + \sin 45°\sin 30°) - (\sin 45°\cos 30° - \cos 45°\sin 30°)$

$$= \left\{\left(\frac{1}{\sqrt{2}}\times\frac{\sqrt{3}}{2}\right)+\left(\frac{1}{\sqrt{2}}\times\frac{1}{2}\right)\right\} - \left\{\left(\frac{1}{\sqrt{2}}\times\frac{\sqrt{3}}{2}\right)-\left(\frac{1}{\sqrt{2}}\times\frac{1}{2}\right)\right\}$$

$$= \frac{(\sqrt{3}+1)}{2\sqrt{2}} - \frac{(\sqrt{3}-1)}{2\sqrt{2}} = \frac{2}{2\sqrt{2}} = \frac{1}{\sqrt{2}}.$$

94. $$\cot 105° = \cot(60° + 45°) = \frac{\cot 60°\cot 45° - 1}{\cot 60° + \cot 45°} = \frac{\left(\frac{1}{\sqrt{3}}\times 1 - 1\right)}{\left(\frac{1}{\sqrt{3}}+1\right)} = \frac{(1-\sqrt{3})}{(1+\sqrt{3})}.$$

$$\tan 105° = \tan(60° + 45°) = \frac{(\tan 60° + \tan 45°)}{1-\tan 60°\tan 45°} = \frac{(\sqrt{3}+1)}{(1-\sqrt{3})}.$$

$$\therefore \quad \cot 105° - \tan 105° = \frac{(1-\sqrt{3})}{(1+\sqrt{3})} - \frac{(1+\sqrt{3})}{(1-\sqrt{3})}$$

$$= \frac{(1-\sqrt{3})^2-(1+\sqrt{3})^2}{(1-3)} = \frac{-4\sqrt{3}}{-2} = 2\sqrt{3}.$$

95. $2\sin\frac{5\pi}{12}\sin\frac{\pi}{12} = \cos(A-B) - \cos(A+B)$

$$= \left\{\cos\left(\frac{5\pi}{12}-\frac{\pi}{12}\right) - \cos\left(\frac{5\pi}{12}+\frac{\pi}{12}\right)\right\}$$

$$= \left(\cos\frac{\pi}{3} - \cos\frac{\pi}{2}\right) = \left(\frac{1}{2}-0\right) = \frac{1}{2}.$$

96. $$2\cos\frac{5\pi}{12}\cos\frac{\pi}{12} = \cos(A+B) + \cos(A-B) = \cos\left(\frac{5\pi}{12}+\frac{\pi}{12}\right) + \cos\left(\frac{5\pi}{12}-\frac{\pi}{12}\right)$$

$$= \left(\cos\frac{\pi}{2} + \cos\frac{\pi}{3}\right) = \left(0+\frac{1}{2}\right) = \frac{1}{2}.$$

97. $2\sin\frac{5\pi}{12}\cos\frac{\pi}{12} = \sin(A+B)+\sin(A-B)$

$$= \sin\left(\frac{5\pi}{12}+\frac{\pi}{12}\right)+\sin\left(\frac{5\pi}{12}-\frac{\pi}{12}\right)$$

$$= \left(\sin\frac{\pi}{2}+\sin\frac{\pi}{3}\right) = \left(1+\frac{\sqrt{3}}{2}\right) = \frac{(2+\sqrt{3})}{2}.$$

98. $\sin(180°+\theta) = -\sin\theta$, $\cos(90°+\theta) = -\sin\theta$, $\tan(270°-\theta) = \cot\theta$,

$\cot(360°-\theta) = -\cot\theta$, $\sin(360°-\theta) = -\sin\theta$, $\cos(360°+\theta) = \cos\theta$,

$\operatorname{cosec}(-\theta) = -\operatorname{cosec}\theta$, $\sin(270°+\theta) = -\cos\theta$.

$$\therefore \quad \text{given exp.} = \frac{(-\sin\theta)\cdot(-\sin\theta)\cdot\cot\theta(-\cot\theta)}{(-\sin\theta)\cdot\cos\theta(-\operatorname{cosec}\theta)(-\cos\theta)} = \frac{-\sin^2\theta\times\dfrac{\cos^2\theta}{\sin^2\theta}}{-\cos^2\theta} = 1.$$

99. $\sin(40°+\theta)\cos(10°+\theta) - \cos(40°+\theta)\sin(10°+\theta)$

$= \sin A\cos B - \cos A\sin B$, where $(40°+\theta) = A$ and $(10°+\theta) = B$

$= \sin(A-B) = \sin\{(40°+\theta)-(10°+\theta)\} = \sin 30° = \frac{1}{2}.$

100. $\cos(90°+\theta) = -\sin\theta$, $\sec(270°+\theta) = \operatorname{cosec}\theta$, $\sin(180°+\theta) = -\sin\theta$,

$\operatorname{cosec}(-\theta) = -\operatorname{cosec}\theta$, $\cos(270°-\theta) = -\sin\theta$, $\tan(180°+\theta) = \tan\theta$.

$$\therefore \quad \text{given exp.} = \frac{(-\sin\theta)(\operatorname{cosec}\theta)(-\sin\theta)}{(-\operatorname{cosec}\theta)(-\sin\theta)\tan\theta} = \cos\theta.$$

101. $\sin(270°+\theta) = -\cos\theta$, $\sin(270°-\theta) = -\cos\theta$, $\cos(180°+\theta) = -\cos\theta$

$\therefore$ given exp. $= \cos\theta - \cos\theta + \cos\theta - \cos\theta = 0.$

102. $\dfrac{\cos 8° - \sin 8°}{\cos 8° + \sin 8°} = \dfrac{1-\tan 8°}{1+\tan 8°} = \dfrac{1-\tan 8°}{1+\tan 45°\tan 8°}$ $\quad [\because\ 1 = \tan 45°]$

$$= \tan(45°-8°) = \tan 37°.$$

103. $\cos(\pi+\theta) = -\cos\theta$, $\cos(-\theta) = \cos\theta$, $\cos(\pi-\theta) = -\cos\theta$, $\cos\left(\frac{\pi}{2}+\theta\right) = -\sin\theta.$

$$\therefore \quad \text{given exp.} = \frac{(-\cos\theta)(\cos\theta)}{(-\cos\theta)(-\sin\theta)} = -\cot\theta.$$

104. Using $\cos(A+B)+\cos(A-B) = 2\cos A\cos B$, we get:

$$\cos\left(\frac{\pi}{4}+x\right)+\cos\left(\frac{\pi}{4}-x\right) = 2\cos\frac{\pi}{4}\cos x = \left(2\times\frac{1}{\sqrt{2}}\times\cos x\right) = \sqrt{2}\cos x.$$

105. Using $\cos(A+B)-\cos(A-B) = -2\sin A\sin B$, we get:

$$\cos\left(\frac{3\pi}{4}+x\right)-\cos\left(\frac{3\pi}{4}-x\right) = -2\sin\frac{3\pi}{4}\sin x = -2\sin\left(\pi-\frac{\pi}{4}\right)\sin x$$

$$= -2\sin\frac{\pi}{4}\sin x = -2\times\frac{1}{\sqrt{2}}\cdot\sin x = -\sqrt{2}\sin x.$$

106. Using $\sin C - \sin D = 2\cos\frac{(C+D)}{2}\sin\frac{(C-D)}{2}$

and $\quad \cos C - \cos D = -2\sin\frac{(C+D)}{2}\sin\frac{(C-D)}{2}$, we get:

$$\frac{\sin 3x - \sin x}{\cos x - \cos 3x} = \frac{2\cos\left(\frac{4x}{2}\right)\sin\left(\frac{2x}{2}\right)}{2\sin\left(\frac{4x}{2}\right)\sin\left(\frac{2x}{2}\right)} = \frac{\cos 2x \sin x}{\sin 2x \sin x} = \cot 2x.$$

107. Using $\cos C + \cos D = 2\cos\frac{(C+D)}{2}\cos\frac{(C-D)}{2}$

and $\sin C - \sin D = 2\cos\frac{(C+D)}{2}\sin\frac{(C-D)}{2}$, we get:

$$\frac{\cos 6x + \cos 4x}{\sin 6x - \sin 4x} = \frac{2\cos\left(\frac{10x}{2}\right)\cos\left(\frac{2x}{2}\right)}{2\cos\left(\frac{10x}{2}\right)\sin\left(\frac{2x}{2}\right)} = \frac{\cos 5x \cos x}{\cos 5x \sin x} = \frac{\cos x}{\sin x} = \cot x.$$

108. Given exp. $= \dfrac{(\cos 4x + \cos 2x) + \cos 3x}{(\sin 4x + \sin 2x) + \sin 3x} = \dfrac{2\cos\frac{(4x+2x)}{2}\cos\frac{(4x-2x)}{2} + \cos 3x}{2\sin\frac{(4x+2x)}{2}\cos\frac{(4x-2x)}{2} + \sin 3x}$

$$= \frac{2\cos 3x \cos x + \cos 3x}{2\sin 3x \cos x + \sin 3x} = \frac{\cos 3x(2\cos x + 1)}{\sin 3x(2\cos x + 1)} = \frac{\cos 3x}{\sin 3x} = \cot 3x.$$

109. Given exp. $= \dfrac{\sin 7x - \sin 5x}{\cos 7x + \cos 5x} = \dfrac{2\cos\frac{(7x+5x)}{2}\sin\frac{(7x-5x)}{2}}{2\cos\frac{(7x+5x)}{2}\cos\frac{(7x-5x)}{2}} = \dfrac{\sin x}{\cos x} = \tan x.$

110. $\sin^2 6x - \sin^2 4x = \sin(6x + 4x)\sin(6x - 4x)$ $[\because \sin^2 x - \sin^2 y = \sin(x+y)\sin(x-y)]$

$= \sin 10x \sin 2x.$

111. Given exp. $= \frac{1}{2}(2\cos 20° \cos 80°)\cos 40°$

$= \frac{1}{2}[\cos(80° + 20°) + \cos(80° - 20°)]\cos 40°$

$= \frac{1}{2}[(\cos 100° + \cos 60°)\cos 40°] = \frac{1}{2}\left[\left(\cos 100° + \frac{1}{2}\right)\cos 40°\right]$

$= \frac{1}{4}(2\cos 100° \cos 40°) + \frac{1}{4}\cos 40°$

$= \frac{1}{4}[\cos(100° + 40°) + \cos(100° - 40°)] + \frac{1}{4}\cos 40°$

$= \frac{1}{4}(\cos 140° + \cos 60°) + \frac{1}{4}\cos 40° = \frac{1}{4}(\cos 140° + \cos 40°) + \left(\frac{1}{4} \times \frac{1}{2}\right)$

$= \frac{1}{4}[\cos(180° - 40°) + \cos 40°] + \frac{1}{8} = \frac{1}{4}(-\cos 40° + \cos 40°) + \frac{1}{8} = \frac{1}{8}.$

112. Given exp. $= \frac{1}{2}(2\sin 70° \sin 50°) \cdot \sin 10° = \frac{1}{2}[\cos(70° - 50°) - \cos(70° + 50°)]\sin 10°$

$= \frac{1}{2}(\cos 20° - \cos 120°)\sin 10° = \frac{1}{2}\left(\cos 20° + \frac{1}{2}\right)\sin 10°$

$= \frac{1}{4}(2\cos 20° \sin 10°) + \frac{1}{4}\sin 10°$

$$= \frac{1}{4}[\sin(20° + 10°) - \sin(20° - 10°)] + \frac{1}{4}\sin 10°$$

$$= \frac{1}{4}\sin 30° = \left(\frac{1}{4} \times \frac{1}{2}\right) = \frac{1}{8}.$$

113. Using $2\cos A \cos B = \cos(A + B) + \cos(A - B)$, we get:

$$2\cos 45° \cos 15° = \cos(45° + 15°) + \cos(45° - 15°)$$

$$= (\cos 60° + \cos 30°) = \left(\frac{1}{2} + \frac{\sqrt{3}}{2}\right) = \frac{(\sqrt{3} + 1)}{2}.$$

114. Using $2\sin A \sin B = \cos(A - B) - \cos(A + B)$, we get:

$$2\sin 75° \sin 15° = \cos(75° - 15°) - \cos(75° + 15°)$$

$$= (\cos 60° - \cos 90°) = \left(\frac{1}{2} - 0\right) = \frac{1}{2}.$$

115. $\cos 15° - \sin 15° = \cos 15° - \cos(90° - 15°) = \cos 15° - \cos 75°$

$$= 2\sin\frac{(15° + 75°)}{2}\sin\frac{(75° - 15°)}{2}$$

$$\left[\because\ \cos C - \cos D = 2\sin\frac{(C + D)}{2}\sin\frac{(D - C)}{2}\right]$$

$$= 2\sin 45° \sin 30° = \left(2 \times \frac{1}{\sqrt{2}} \times \frac{1}{2}\right) = \frac{1}{\sqrt{2}}.$$

116. In quadrant III, we have: $\sin x < 0$ and $\cos x < 0$.

$$\sin x = \frac{-1}{2} \Rightarrow \cos x = (1 - \sin^2 x) = \left(1 - \frac{1}{4}\right) = \frac{3}{4}$$

$$\Rightarrow \cos x = -\sqrt{\frac{3}{4}} = -\frac{\sqrt{3}}{2}.$$

$$\therefore \quad \sin 2x = 2\sin x \cos x = 2 \times \frac{(-1)}{2} \times \frac{(-\sqrt{3})}{2} = \frac{\sqrt{3}}{2}.$$

117. Here, x lies in quadrant II, where $\sin x > 0$ and $\cos x < 0$.

$$\sec x = \frac{-13}{12} \Rightarrow \cos x = \frac{-12}{13}.$$

Now, $\cos 2x = (2\cos^2 x - 1) = \left\{2 \times \left(\frac{-12}{13}\right)^2 - 1\right\} = \left(2 \times \frac{144}{169} - 1\right) = \frac{(288 - 169)}{169} = \frac{119}{169}.$

118. Here, x lies in quadrant IV, where $\cos x > 0$, $\sin x < 0$ and $\tan x < 0$.

$$\tan x = \frac{-3}{4} \Rightarrow \sec^2 x = (1 + \tan^2 x) = \left(1 + \frac{9}{16}\right) = \frac{25}{16}$$

$$\Rightarrow \sec x = +\sqrt{\frac{25}{16}} = \frac{5}{4} \Rightarrow \cos x = \frac{4}{5}.$$

Now, $\sin^2 x = (1 - \cos^2 x) = \left(1 - \frac{16}{25}\right) = \frac{9}{25} \Rightarrow \sin x = -\sqrt{\frac{9}{25}} = \frac{-3}{5}.$

$$\therefore \quad \tan x = \frac{\sin x}{\cos x} = \left(\frac{-3}{5} \times \frac{5}{4}\right) = \frac{-3}{4}.$$

$$\therefore \quad \tan 2x = \frac{2\tan x}{1-\tan^2 x} = \frac{2\times\frac{(-3)}{4}}{\left(1-\frac{9}{16}\right)} = \frac{\left(\frac{-3}{2}\right)}{\left(\frac{7}{16}\right)} = \frac{-3}{\cancel{2}_1}\times\frac{\cancel{16}^{8}}{7} = \frac{-24}{7}.$$

119. $\sin 3x = (3\sin x - 4\sin^3 x) = \left(3\times\frac{1}{3} - 4\times\frac{1}{27}\right) = \left(1-\frac{4}{27}\right) = \frac{23}{27}.$

120. $\cos 3x = (4\cos^3 x - 3\cos x) = \left(4\times\frac{1}{8} - 3\times\frac{1}{2}\right) = \left(\frac{1}{2}-\frac{3}{2}\right) = -1.$

121. $\cos^2\frac{\pi}{12} - \sin^2\frac{\pi}{12} = \cos\left(2\times\frac{\pi}{12}\right) \qquad [\because \cos^2 x - \sin^2 x = \cos 2x]$

$$= \cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}.$$

122. Given exp. $= \dfrac{(1-\cos 2x)+\sin 2x}{(1+\cos 2x)+\sin 2x} = \dfrac{2\sin^2 x + 2\sin x\cos x}{2\cos^2 x + 2\sin x\cos x}$

$$= \frac{2\sin x(\sin x + \cos x)}{2\cos x(\cos x + \sin x)} = \frac{\sin x}{\cos x} = \tan x.$$

123. $\frac{\pi}{2} < x < \pi \Rightarrow \frac{\pi}{4} < \frac{x}{2} < \frac{\pi}{2} \Rightarrow \frac{\pi}{2}$ lies in quadrant I $\Rightarrow \sin\frac{x}{2} > 0.$

Now, $2\sin^2\frac{x}{2} = (1-\cos x) = \left(1+\frac{3}{5}\right) = \frac{8}{5} \Rightarrow \sin^2\frac{x}{2} = \frac{8}{10} = \frac{4}{5}.$

$$\therefore \quad \sin\frac{x}{2} = \sqrt{\frac{4}{5}} = \frac{2}{\sqrt{5}}.$$

124. $\frac{\pi}{2} < x < \pi \Rightarrow \frac{\pi}{4} < \frac{x}{2} < \frac{\pi}{2} \Rightarrow \frac{x}{2}$ lies in quadrant I $\Rightarrow \cos\frac{x}{2} > 0.$

$2\cos^2\frac{x}{2} = (1+\cos x) = \left(1-\frac{3}{5}\right) = \frac{2}{5} \Rightarrow \cos^2\frac{x}{2} = \frac{1}{5} \Rightarrow \cos\frac{x}{2} = \frac{1}{\sqrt{5}}.$

125. $\pi < x < \frac{3\pi}{2} \Rightarrow \frac{\pi}{2} < \frac{x}{2} < \frac{3\pi}{4} \Rightarrow \frac{x}{2}$ lies in quadrant II $\Rightarrow \cos\frac{x}{2} < 0.$

$2\cos^2\frac{x}{2} = (1+\cos x) = \left(1-\frac{4}{5}\right) = \frac{1}{5} \Rightarrow \cos^2\frac{x}{2} = \frac{1}{10}$

$$\Rightarrow \cos\frac{x}{2} = \frac{-1}{\sqrt{10}} \qquad \left[\because \cos\frac{x}{2} < 0\right].$$

126. Since x lies in quadrant III, we have: $\cos x < 0$.

Now, $\tan x = \frac{3}{4} \Rightarrow \sec^2 x = (1+\tan^2 x) = \left(1+\frac{9}{16}\right) = \frac{25}{16}$

$$\Rightarrow \cos^2 x = \frac{16}{25} \Rightarrow \cos x = -\sqrt{\frac{16}{25}} = \frac{-4}{5}.$$

Also, $\pi < x < \frac{3\pi}{2} \Rightarrow \frac{\pi}{2} < \frac{x}{2} < \frac{3\pi}{4} \Rightarrow \frac{x}{2}$ lies in quadrant II

$$\Rightarrow \sin\frac{x}{2} > 0 \text{ and } \cos\frac{x}{2} < 0.$$

$$2\sin^2\frac{x}{2}=(1-\cos x)=\left(1+\frac{4}{5}\right)=\frac{9}{5} \;\Rightarrow\; \sin^2\frac{x}{2}=\frac{9}{10}\cdot$$

$$\therefore \quad \sin\frac{x}{2}=+\sqrt{\frac{9}{10}}=\frac{3}{\sqrt{10}} \qquad \left[\because\ \sin\frac{x}{2}>0\right]\cdot$$

$$2\cos^2\frac{x}{2}=(1+\cos x)=\left(1-\frac{4}{5}\right)=\frac{1}{5} \;\Rightarrow\; \cos^2\frac{x}{2}=\frac{1}{10}$$

$$\therefore \quad \cos\frac{x}{2}=-\sqrt{\frac{1}{10}}=\frac{-1}{\sqrt{10}}\cdot$$

$$\therefore \quad \tan\frac{x}{2}=\frac{\sin\left(\frac{x}{2}\right)}{\cos\left(\frac{x}{2}\right)}=\frac{3}{\sqrt{10}}\times\frac{\sqrt{10}}{-1}=-3.$$

127. Since x lies in quadrant III, we have: $\pi < x < \frac{3\pi}{2}\cdot$

$$\therefore \quad \frac{\pi}{2}<\frac{x}{2}<\frac{3\pi}{4} \;\Rightarrow\; \frac{x}{2} \text{ lies in quadrant II}$$

$$\Rightarrow\; \sin\frac{x}{2}>0,\ \cos\frac{x}{2}<0 \text{ and } \tan\frac{x}{2}<0.$$

$$2\sin^2\frac{x}{2}=(1-\cos x)=\left(1+\frac{1}{3}\right)=\frac{4}{3} \;\Rightarrow\; \sin^2\frac{x}{2}=\frac{2}{3}$$

$$\therefore \quad \sin\frac{x}{2}=\sqrt{\frac{2}{3}}\cdot$$

$$2\cos^2\frac{x}{2}=(1+\cos x)=\left(1-\frac{1}{3}\right)=\frac{2}{3} \;\Rightarrow\; \cos^2\frac{x}{2}=\frac{1}{3}$$

$$\therefore \quad \cos\frac{x}{2}=-\frac{1}{\sqrt{3}}\cdot$$

$$\therefore \quad \tan\frac{x}{2}=\frac{\sin\left(\frac{x}{2}\right)}{\cos\left(\frac{x}{2}\right)}=\frac{\sqrt{2}}{\sqrt{3}}\times(-\sqrt{3})=-\sqrt{2}.$$

128. Since x lies in quadrant IV, we have: $\cos x > 0$.

Now, $\cos^2 x=(1-\sin^2 x)=\left(1-\frac{1}{4}\right)=\frac{3}{4} \;\Rightarrow\; \cos x=+\sqrt{\frac{3}{4}}=\frac{\sqrt{3}}{2}\cdot$

Now, $\frac{3\pi}{2}<x<2\pi \;\Rightarrow\; \frac{3\pi}{4}<\frac{x}{2}<\pi \;\Rightarrow\; \frac{x}{2}$ lies in quadrant II.

$$\therefore \quad \sin\frac{x}{2}>0.$$

$$2\sin^2\frac{x}{2}=(1-\cos x)=\left(1-\frac{\sqrt{3}}{2}\right)=\frac{(2-\sqrt{3})}{2}$$

$$\Rightarrow\; \sin^2\frac{x}{2}=\frac{(2-\sqrt{3})}{4} \;\Rightarrow\; \sin\frac{x}{2}=\frac{\sqrt{2-\sqrt{3}}}{2}\cdot$$

129. $\dfrac{1+\cos x}{1-\cos x}=\dfrac{2\cos^2\left(\dfrac{x}{2}\right)}{2\sin^2\left(\dfrac{x}{2}\right)}=\cot^2\dfrac{x}{2}\cdot$

130. $\sqrt{\dfrac{1+\sin x}{1-\sin x}}=\sqrt{\dfrac{1-\cos\left(\dfrac{\pi}{2}+x\right)}{1+\cos\left(\dfrac{\pi}{2}+x\right)}}=\left\{\dfrac{2\sin^2\left(\dfrac{\pi}{4}+\dfrac{x}{2}\right)}{2\cos^2\left(\dfrac{\pi}{4}+\dfrac{x}{2}\right)}\right\}^{\frac{1}{2}}=\left\{\tan^2\left(\dfrac{\pi}{4}+\dfrac{x}{2}\right)\right\}^{\frac{1}{2}}=\tan\left(\dfrac{\pi}{4}+\dfrac{x}{2}\right)\cdot$

131. $\dfrac{\sin x}{1+\cos x}=\dfrac{2\sin\dfrac{x}{2}\cos\dfrac{x}{2}}{2\cos^2\dfrac{x}{2}}=\tan\dfrac{x}{2}\cdot$

132. $\cot\dfrac{x}{2}-\tan\dfrac{x}{2}=\dfrac{\cos\left(\dfrac{x}{2}\right)}{\sin\left(\dfrac{x}{2}\right)}-\dfrac{\sin\left(\dfrac{x}{2}\right)}{\cos\left(\dfrac{x}{2}\right)}=\dfrac{2\left\{\cos^2\left(\dfrac{x}{2}\right)-\sin^2\left(\dfrac{x}{2}\right)\right\}}{2\sin\dfrac{x}{2}\cos\dfrac{x}{2}}=\dfrac{2\cos x}{\sin x}=2\cot x.$

133. Remember $\sin 18^\circ=\dfrac{(\sqrt{5}-1)}{4}\cdot$

134. $\cos^2 18^\circ=(1-\sin^2 18^\circ)=\left\{1-\dfrac{(\sqrt{5}-1)^2}{16}\right\}=\left\{1-\dfrac{(6-2\sqrt{5})}{16}\right\}=\dfrac{(10+2\sqrt{5})}{16}$

$\Rightarrow\quad \cos 18^\circ=\dfrac{\sqrt{10+2\sqrt{5}}}{4}\cdot$

135. $\cos 36^\circ=(1-2\sin^2 18^\circ)=\left\{1-2\cdot\dfrac{(\sqrt{5}-1)^2}{16}\right\}=\left\{1-\dfrac{(6-2\sqrt{5})}{8}\right\}=\dfrac{(\sqrt{5}+1)}{4}\cdot$

136. $\sin^2 36^\circ=(1-\cos^2 36^\circ)=\left\{1-\dfrac{(\sqrt{5}+1)^2}{16}\right\}=\dfrac{(10-2\sqrt{5})}{16}\Rightarrow\ \sin 36^\circ=\dfrac{\sqrt{10-2\sqrt{5}}}{4}\cdot$

137. $\sin 54^\circ=\sin(90^\circ-36^\circ)=\cos 36^\circ=\dfrac{(\sqrt{5}+1)}{4}\cdot$

138. $\cos 72^\circ=\cos(90^\circ-18^\circ)=\sin 18^\circ=\dfrac{(\sqrt{5}-1)}{4}\cdot$

139. $\cos 54^\circ=\cos(90^\circ-36^\circ)=\sin 36^\circ=\dfrac{\sqrt{10-2\sqrt{5}}}{4}\cdot$

140. $\sin 72^\circ=\sin(90^\circ-18^\circ)=\cos 18^\circ=\dfrac{\sqrt{10+2\sqrt{5}}}{4}\cdot$

141. Using $2\sin A\cos A=\sin 2A$, we get

$$2\sin 22\frac{1}{2}^\circ\cos 22\frac{1}{2}^\circ=\sin\left(2\times\frac{45}{2}\right)^\circ=\sin 45^\circ=\frac{1}{\sqrt{2}}\cdot$$

142. Using $(2\cos^2\theta-1)=\cos 2\theta$, we get

$$(2\cos^2 15^\circ-1)=\cos(2\times 15^\circ)=\cos 30^\circ=\frac{\sqrt{3}}{2}\cdot$$

143. Using $(3\sin\theta - 4\sin^3\theta) = \sin 3\theta$, we get

$$(3\sin 40° - 4\sin^3 40°) = \sin(3\times 40°) = \sin 120°$$
$$= \sin(180° - 60°) = \sin 60° = \frac{\sqrt{3}}{2}.$$

144. Using $(4\cos^3\theta - 3\cos\theta) = \cos 3\theta$, we get

$$(8\cos^3 20° - 6\cos 20°) = 2(4\cos^3 20° - 3\cos 20°)$$
$$= 2\times\cos(3\times 20°) = 2\cos 60° = \left(2\times\frac{1}{2}\right) = 1.$$

145. Using $\sin 2\theta = \dfrac{2\tan\theta}{(1+\tan^2\theta)}$ and $\cos 2\theta = \dfrac{(1-\tan^2\theta)}{(1+\tan^2\theta)}$, we get

$$a\sin 2\theta + b\cos 2\theta = \frac{2a\tan\theta}{(1+\tan^2\theta)} + \frac{b(1-\tan^2\theta)}{(1+\tan^2\theta)}$$
$$= \frac{2a\tan\theta + b(1-\tan^2\theta)}{(1+\tan^2\theta)} = \frac{\left(2a\times\frac{a}{b}\right) + b\left(1-\frac{a^2}{b^2}\right)}{\left(1+\frac{a^2}{b^2}\right)}$$
$$= \frac{\left(\frac{2a^2}{b} - \frac{a^2}{b}\right) + b}{\frac{(a^2+b^2)}{b^2}} = \frac{\left(\frac{a^2}{b} + b\right)b^2}{(a^2+b^2)} = \frac{(a^2+b^2)b}{(a^2+b^2)} = b.$$

146. $\cot x - 2\cot 2x = \dfrac{1}{\tan x} - \dfrac{2}{\tan 2x} = \left\{\dfrac{1}{\tan x} - \dfrac{2(1-\tan^2 x)}{2\tan x}\right\} = \dfrac{2\tan^2 x}{2\tan x} = \tan x.$

147. $\cos 2x + 2\sin^2 x = \cos^2 x - \sin^2 x + 2\sin^2 x = \cos^2 x + \sin^2 x = 1.$

148. $\dfrac{\sin 2x}{1-\cos 2x} = \dfrac{2\sin x\cos x}{2\sin^2 x} = \dfrac{\cos x}{\sin x} = \cot x.$

149. $\dfrac{\sin 2x}{1+\cos 2x} = \dfrac{2\sin x\cos x}{2\cos^2 x} = \dfrac{\sin x}{\cos x} = \tan x.$

150. $\dfrac{\tan 2x}{1+\sec 2x} = \dfrac{\left(\dfrac{\sin 2x}{\cos 2x}\right)}{\left(1+\dfrac{1}{\cos 2x}\right)} = \dfrac{\sin 2x}{(1+\cos 2x)} = \dfrac{2\sin x\cos x}{2\cos^2 x} = \dfrac{\sin x}{\cos x} = \tan x.$

151. $\dfrac{1-\cos 2x + \sin x}{\sin 2x + \cos x} = \dfrac{2\sin^2 x + \sin x}{2\sin x\cos x + \cos x} = \dfrac{\sin x(2\sin x + 1)}{\cos x(2\sin x + 1)} = \tan x.$

152. $\dfrac{\cos x}{(1-\sin x)} = \dfrac{\sin\left(\dfrac{\pi}{2}+x\right)}{1+\cos\left(\dfrac{\pi}{2}+x\right)} = \dfrac{2\sin\left(\dfrac{\pi}{4}+\dfrac{x}{2}\right)\cos\left(\dfrac{\pi}{4}+\dfrac{x}{2}\right)}{2\cos^2\left(\dfrac{\pi}{4}+\dfrac{x}{2}\right)} = \tan\left(\dfrac{\pi}{4}+\dfrac{x}{2}\right).$

153. $(\sin^2\alpha\cos^2\beta + \cos^2\alpha\sin^2\beta + \sin^2\alpha\sin^2\beta + \cos^2\alpha\cos^2\beta)$

$= (\sin^2\alpha\cos^2\beta + \sin^2\alpha\sin^2\beta) + (\cos^2\alpha\sin^2\beta + \cos^2\alpha\cos^2\beta)$

$= \sin^2\alpha(\cos^2\beta + \sin^2\beta) + \cos^2\alpha(\sin^2\beta + \cos^2\beta) = (\sin^2\alpha + \cos^2\alpha) = 1.$

154. $\sqrt{\dfrac{1+\sin x}{1-\sin x}} = \dfrac{\sqrt{1+\sin x}}{\sqrt{1-\sin x}} \times \dfrac{\sqrt{1+\sin x}}{\sqrt{1+\sin x}} = \dfrac{(1+\sin x)}{\sqrt{1-\sin^2 x}} = \dfrac{(1+\sin x)}{\cos x}$

$$= \left(\frac{1}{\cos x} + \frac{\sin x}{\cos x}\right) = (\sec x + \tan x).$$

155. Given exp. $= \left(\dfrac{1}{\cos\theta} - \cos\theta\right)\left(\dfrac{1}{\sin\theta} - \sin\theta\right)\left(\dfrac{\cos\theta}{\sin\theta} + \dfrac{\sin\theta}{\cos\theta}\right)$

$$= \frac{(1-\cos^2\theta)}{\cos\theta} \cdot \frac{(1-\sin^2\theta)}{\sin\theta} \cdot \frac{(\cos^2\theta + \sin^2\theta)}{\sin\theta\cos\theta} = \frac{\sin^2\theta\cos^2\theta}{\sin^2\theta\cos^2\theta} = 1.$$

156. Given exp. $= \dfrac{\sin\theta[\sin^2\theta + (1+\cos\theta)^2]}{\sin\theta(1+\cos\theta)} = \dfrac{(\sin^2\theta + \cos^2\theta + 1 + 2\cos\theta)}{(1+\cos\theta)}$

$$= \frac{(2+2\cos\theta)}{(1+\cos\theta)} = \frac{2(1+\cos\theta)}{(1+\cos\theta)} = 2.$$

157. Given exp. $= \dfrac{(1+\sin\theta)(\sin\theta + \cos\theta - 1)}{(\sin\theta - \cos\theta + 1)}$

$$= \frac{(\sin\theta + \cos\theta - 1 + \sin^2\theta + \sin\theta\cos\theta - \sin\theta)}{(\sin\theta - \cos\theta + 1)}$$

$$= \frac{\cos\theta - (1-\sin^2\theta) + \sin\theta\cos\theta}{(\sin\theta - \cos\theta + 1)} = \frac{\cos\theta - \cos^2\theta + \sin\theta\cos\theta}{(\sin\theta - \cos\theta + 1)}$$

$$= \frac{\cos\theta(\sin\theta - \cos\theta + 1)}{(\sin\theta - \cos\theta + 1)} = \cos\theta.$$

158. Given exp. $= \dfrac{(\cot\theta - \operatorname{cosec}\theta) + (\operatorname{cosec}^2\theta - \cot^2\theta)}{(\cot\theta + \operatorname{cosec}\theta - 1)}$

$$= \frac{(\operatorname{cosec}\theta - \cot\theta)(\operatorname{cosec}\theta + \cot\theta) - (\operatorname{cosec}\theta - \cot\theta)}{(\cot\theta + \operatorname{cosec}\theta - 1)}$$

$$= \frac{(\operatorname{cosec}\theta - \cot\theta)(\operatorname{cosec}\theta + \cot\theta - 1)}{(\operatorname{cosec}\theta + \cot\theta - 1)} = (\operatorname{cosec}\theta - \cot\theta).$$

159. Given exp. $= \left\{\dfrac{(\sec\theta + \tan\theta) - (\sec^2\theta - \tan^2\theta)}{(\tan\theta - \sec\theta + 1)}\right\}^2$

$$= \left\{\frac{(\sec\theta + \tan\theta)[1 - (\sec\theta - \tan\theta)]}{\tan\theta - \sec\theta + 1}\right\}^2$$

$$= \left\{\frac{(\sec\theta + \tan\theta)(\tan\theta - \sec\theta + 1)}{(\tan\theta - \sec\theta + 1)}\right\}^2 = (\sec\theta + \tan\theta)^2$$

$$= \left(\frac{1}{\cos\theta} + \frac{\sin\theta}{\cos\theta}\right)^2 = \frac{(1+\sin\theta)^2}{\cos^2\theta} = \frac{(1+\sin\theta)^2}{(1-\sin^2\theta)} = \frac{(1+\sin\theta)}{(1-\sin\theta)}.$$

160. $(x^2 + y^2 + z^2) = r^2\cos^2\alpha\cos^2\beta + r^2\cos^2\alpha\sin^2\beta + r^2\sin^2\alpha$

$$= r^2\cos^2\alpha(\cos^2\beta + \sin^2\beta) + r^2\sin^2\alpha$$

$$= r^2\cos^2\alpha + r^2\sin^2\alpha = r^2(\cos^2\alpha + \sin^2\alpha) = r^2.$$

161. On dividing num. and denom. by $\cos\theta$, we get

$$\text{the given exp.} = \frac{b\tan\theta - a}{b\tan\theta + a} = \frac{\left(b\times\frac{b}{a}\right) - a}{\left(b\times\frac{b}{a}\right) + a} = \frac{(b^2 - a^2)}{(b^2 + a^2)}.$$

162. $$\text{Given exp.} = \frac{(5 - 3\cot\theta)}{(1 + 2\cot\theta)} = \frac{\left(5 - 3\times\frac{4}{5}\right)}{\left(1 + 2\times\frac{4}{5}\right)} = \frac{(25 - 12)}{(5 + 8)} = \frac{13}{13} = 1.$$

163. $\tan\theta + \cot\theta = 2 \Rightarrow \tan\theta + \dfrac{1}{\tan\theta} = 2$

$$\Rightarrow \tan^2\theta + 1 = 2\tan\theta \Rightarrow 1 + \tan^2\theta - 2\tan\theta = 0$$

$$\Rightarrow (1 - \tan\theta)^2 = 0 \Rightarrow 1 - \tan\theta = 0 \Rightarrow \tan\theta = 1 \Rightarrow \theta = \frac{\pi}{4}.$$

$\therefore \quad \sin\theta = \sin\dfrac{\pi}{4} = \dfrac{1}{\sqrt{2}}.$

164. $$\text{Given exp.} = \frac{\sqrt{1 - \sin\theta}}{\sqrt{1 + \sin\theta}} - \frac{\sqrt{1 + \sin\theta}}{\sqrt{1 - \sin\theta}} = \frac{(1 - \sin\theta) - (1 + \sin\theta)}{\sqrt{1 - \sin^2\theta}} = \frac{-2\sin\theta}{|\cos\theta|}$$

$$= \frac{-2\sin\theta}{-\cos\theta} = 2\tan\theta \qquad \left[\begin{array}{l}\because \text{ in quadrant II, } \cos\theta < 0 \\ \Rightarrow |\cos\theta| = -\cos\theta\end{array}\right].$$

165. $$\text{Given exp.} = \frac{\sqrt{1 + \cos\theta}}{\sqrt{1 - \cos\theta}} + \frac{\sqrt{1 - \cos\theta}}{\sqrt{1 + \cos\theta}} = \frac{(1 + \cos\theta) + (1 - \cos\theta)}{\sqrt{1 - \cos^2\theta}}$$

$$= \frac{2}{\sqrt{\sin^2\theta}} = \frac{2}{|\sin\theta|} \qquad \left[\begin{array}{l}\because \ \theta \text{ lies in quadrant III} \\ \Rightarrow \sin\theta < 0 \Rightarrow |\sin\theta| = -\sin\theta\end{array}\right]$$

$$= \frac{2}{-\sin\theta} = -2\operatorname{cosec}\theta.$$

166. $7\sin^2\theta + 3\cos^2\theta = 4$

$\Rightarrow 7\tan^2\theta + 3 = 4\sec^2\theta \qquad$ [dividing by $\cos^2\theta$]

$\Rightarrow 7\tan^2\theta - 4(1 + \tan^2\theta) + 3 = 0$

$\Rightarrow 7\tan^2\theta - 4\tan^2\theta - 1 = 0 \Rightarrow 3\tan^2\theta = 1$

$\Rightarrow \tan^2\theta = \dfrac{1}{3} \Rightarrow \tan\theta = \pm\dfrac{1}{\sqrt{3}}.$

167. $\sin\theta + \sin^2\theta = 1 \Rightarrow \sin\theta = (1 - \sin^2\theta) = \cos^2\theta \Rightarrow \sin^2\theta = \cos^4\theta$

$\therefore \quad 1 - \cos^2\theta = \cos^4\theta \Rightarrow \cos^2\theta + \cos^4\theta = 1.$

168. $\sec^2\theta - \tan^2\theta = 1 \Rightarrow (\sec\theta - \tan\theta)(\sec\theta + \tan\theta) = 1$

$\therefore \quad \dfrac{2}{3}(\sec\theta + \tan\theta) = 1 \Rightarrow (\sec\theta + \tan\theta) = \dfrac{3}{2}.$

On solving $(\sec\theta - \tan\theta) = \dfrac{2}{3}$ and $(\sec\theta + \tan\theta) = \dfrac{3}{2}$, we get

$\sec\theta = \dfrac{13}{12}$ and $\tan\theta = \dfrac{5}{12}.$

169. $\operatorname{cosec}^2\theta - \cot^2\theta = 1 \Rightarrow (\operatorname{cosec}\theta - \cot\theta)(\operatorname{cosec}\theta + \cot\theta) = 1$

$\therefore \quad (\operatorname{cosec}\theta - \cot\theta) \times 6 = 1 \Rightarrow (\operatorname{cosec}\theta - \cot\theta) = \frac{1}{6}\cdot$

On solving $(\operatorname{cosec}\theta + \cot\theta) = 6$ and $(\operatorname{cosec}\theta - \cot\theta) = \frac{1}{6}$, we get

$$\operatorname{cosec}\theta = \frac{37}{12} \text{ and } \cot\theta = \frac{35}{12}\cdot$$

170. $(3\sin\theta + 5\cos\theta)^2 + (5\sin\theta - 3\cos\theta)^2 = 9(\sin^2\theta + \cos^2\theta) + 25(\cos^2\theta + \sin^2\theta)$

$$= (9 \times 1) + (25 \times 1) = 34.$$

$\Rightarrow \quad (5)^2 + (5\sin\theta - 3\cos\theta)^2 = 34 \Rightarrow (5\sin\theta - 3\cos\theta)^2 = (34 - 25) = 9$

$\Rightarrow \quad (5\sin\theta - 3\cos\theta) = 3.$

171. $\dfrac{(1 + \tan 15°)}{(1 - \tan 15°)} = \dfrac{(\tan 45° + \tan 15°)}{(1 - \tan 45° \tan 15°)} = \tan(45° + 15°) = \tan 60° = \sqrt{3}.$

172. Using $(3\sin\theta - 4\sin^3\theta) = \sin 3\theta$, we get

$$(3\sin 20° - 4\sin^3 20°) = \sin(3 \times 20°) = \sin 60° = \frac{\sqrt{3}}{2}\cdot$$

173. We know that $(4\cos^3\theta - 3\cos\theta) = \cos 3\theta$.

$\therefore \quad (4\cos^3 15° - 3\cos 15°) = \cos(3 \times 15°) = \cos 45° = \frac{1}{\sqrt{2}}\cdot$

174. Using $2\cos A \cos B = \cos(A + B) + \cos(A - B)$, we get

$$2\cos 75° \cos 15° = \cos(75° + 15°) + \cos(75° - 15°)$$
$$= \cos 90° + \cos 60° = 0 + \frac{1}{2} = \frac{1}{2}\cdot$$

175. Using $2\sin A \sin B = \cos(A - B) - \cos(A + B)$, we get

$$2\sin 105° \sin 75° = \cos(105° - 75°) - \cos(105° + 75°)$$
$$= (\cos 30° - \cos 180°) = \frac{\sqrt{3}}{2} + \cos 0 = \left(\frac{\sqrt{3}}{2} + 1\right) = \frac{(\sqrt{3} + 2)}{2}$$

$\Rightarrow \quad \sin 105° \sin 75° = \dfrac{(\sqrt{3} + 2)}{4}\cdot$

176. $\sin 105° + \cos 105° = \sin 105° + \cos(90° + 15°)$

$$= \sin 105° - \sin 15° = 2\cos\frac{(105° + 15°)}{2}\sin\frac{(105° - 15°)}{2}$$
$$= 2\cos 60° \sin 45° = \left(2 \times \frac{1}{2} \times \frac{1}{\sqrt{2}}\right) = \frac{1}{\sqrt{2}}\cdot$$

177. $\sin\theta + \cos\theta = \sqrt{2} \Rightarrow \sin\theta + \sin(90° - \theta) = \sqrt{2}$

$\Rightarrow \quad 2\sin\dfrac{\{\theta + (90° - \theta)\}}{\sin\theta}\cos\dfrac{\{(90° - \theta) + \theta\}}{2} = \sqrt{2}$

$\Rightarrow \quad 2\sin 45° \cos(45° - \theta) = \sqrt{2} \Rightarrow 2 \times \dfrac{1}{\sqrt{2}}\cos(45° - \theta) = \sqrt{2}$

$\Rightarrow \quad \cos(45° - \theta) = 1 \Rightarrow 45° - \theta = 0 \Rightarrow \theta = 45° = \dfrac{\pi}{4}\cdot$

178. $\sin\theta + \operatorname{cosec}\theta = \frac{5}{2} \Rightarrow \sin\theta + \frac{1}{\sin\theta} = \frac{5}{2}$

$\Rightarrow \frac{(\sin^2\theta + 1)}{\sin\theta} = \frac{5}{2} \Rightarrow 2\sin^2\theta - 5\sin\theta + 2 = 0$

$\Rightarrow (2\sin\theta - 1)(\sin\theta - 2) = 0 \Rightarrow \sin\theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{6}$ $[\because \sin\theta \neq 2]$.

179. $\sin 2\theta = \frac{1}{5} \Rightarrow 1 + \sin 2\theta = \left(1 + \frac{1}{5}\right) = \frac{6}{5}$

$\Rightarrow \sin^2\theta + \cos^2\theta + 2\sin\theta\cos\theta = \frac{6}{5} \Rightarrow (\sin\theta + \cos\theta)^2 = \frac{6}{5}$

$\Rightarrow (\sin\theta + \cos\theta) = \sqrt{\frac{6}{5}}$.

180. $\sin\theta = \cos 2\theta \Rightarrow \sin\theta = \sin(90° - 2\theta)$

$\Rightarrow 90° - 2\theta = \theta \Rightarrow 3\theta = 90° \Rightarrow \theta = 30°$.

$\therefore (\sin\theta + \cos\theta) = (\sin 30° + \cos 30°) = \left(\frac{1}{2} + \frac{\sqrt{3}}{2}\right) = \frac{(\sqrt{3} + 1)}{2}$.

181. $(\cos\theta - \sin\theta) > 0 \Rightarrow \cos\theta > \sin\theta \Rightarrow \sin\theta < \cos\theta \Rightarrow \tan\theta < 1 \Rightarrow \theta < 45°$.

$\therefore (\sin\theta + \cos\theta) \leq (\sin 45° + \cos 45°) = \left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right) = \frac{2}{\sqrt{2}} = \sqrt{2}$.

182. $\sqrt{2 + \sqrt{2 + 2\cos 4\theta}} = \sqrt{2 + \sqrt{2(1 + \cos 4\theta)}} = \sqrt{2 + \sqrt{4\cos^2 2\theta}} = \sqrt{2 + 2\cos 2\theta}$

$= \sqrt{2(1 + \cos 2\theta)} = \sqrt{2 \times 2\cos^2\theta} = \sqrt{4\cos^2\theta} = 2\cos\theta$.

183. $\sin 2\theta = \cos 3\theta \Rightarrow 2\sin\theta\cos\theta = 4\cos^3\theta - 3\cos\theta$

$\Rightarrow 2\sin\theta = 4\cos^2\theta - 3 \Rightarrow 2\sin\theta = 4(1 - \sin^2\theta) - 3$

$\Rightarrow 4\sin^2\theta + 2\sin\theta - 1 = 0$

$\Rightarrow \sin\theta = \frac{-2 \pm \sqrt{4 + 16}}{8} = \frac{(-1 \pm \sqrt{5})}{4}$

$\Rightarrow \sin\theta = \frac{(\sqrt{5} - 1)}{4}$ $[\because \theta \text{ is acute} \Rightarrow \sin\theta > 0]$.

184. $2\sin^2\theta + 3\cos^2\theta = 2\sin^2\theta + 2\cos^2\theta + \cos^2\theta$

$= 2(\sin^2\theta + \cos^2\theta) + \cos^2\theta = 2 + \cos^2\theta \geq 2$.

185. $(\sin^4\theta + \cos^4\theta) = (\sin^2\theta + \cos^2\theta)^2 - 2\sin^2\theta\cos^2\theta = (1 - 2\sin^2\theta\cos^2\theta) \leq 1$.

186. $p = (\cos^2\theta + \sec^2\theta) = (\cos\theta - \sec\theta)^2 + 2\cos\theta\sec\theta = (\cos\theta - \sec\theta)^2 + 2 \geq 2$.

187. $p = (\sin^2\theta + \cos^4\theta)$

$= \sin^2\theta + \cos^2\theta \cdot \cos^2\theta \leq \sin^2\theta + \cos^2\theta$ $[\because \cos^2\theta \leq 1]$

$\Rightarrow p \leq 1$.

Also, $p = (\sin^2\theta + \cos^4\theta) = (1 - \cos^2\theta + \cos^4\theta)$

$= 1 + \left(\cos^4\theta - \cos^2\theta + \frac{1}{4}\right) - \frac{1}{4} = \frac{3}{4} + \left(\cos^2\theta - \frac{1}{2}\right)^2 \geq \frac{3}{4}$.

$\therefore \frac{3}{4} \leq p \leq 1$.

188. $p = 2\sin^2\theta - \cos 2\theta = (1 - \cos 2\theta) - \cos 2\theta = (1 - 2\cos 2\theta) \Rightarrow \cos 2\theta = \frac{1}{2}(1 - p).$

Now, $-1 \le \cos 2\theta \le 1 \Rightarrow -1 \le \frac{1}{2}(1-p) \le 1 \Rightarrow -2 \le (1-p) \le 2$

$\Rightarrow -3 \le -p \le 1 \Rightarrow 3 \ge p \ge -1 \Rightarrow -1 \le p \le 3.$

189. Given expression $= \frac{1}{2}\left[2\sin^2\left(\frac{\pi}{8} + \frac{A}{2}\right) - 2\sin^2\left(\frac{\pi}{8} - \frac{A}{2}\right)\right]$

$= \frac{1}{2}\left[\left\{1 - \cos\left(\frac{\pi}{4} + A\right)\right\} - \left\{1 - \cos\left(\frac{\pi}{4} - A\right)\right\}\right]$

$= \frac{1}{2}\left[\cos\left(\frac{\pi}{4} - A\right) - \cos\left(\frac{\pi}{4} + A\right)\right] = \frac{1}{2} \times 2\sin\frac{\pi}{4}\sin A = \frac{1}{\sqrt{2}}\sin A.$

190. $\cos 1^\circ \cos 2^\circ \ldots \cos 90^\circ \ldots \cos 178^\circ \cos 179^\circ = 0$ $\quad [\because \cos 90^\circ = 0].$

191. Given exp. $= (\tan 1^\circ \tan 89^\circ)(\tan 2^\circ \tan 88^\circ)(\tan 3^\circ \tan 87^\circ) \ldots (\tan 44^\circ \tan 46^\circ) \tan 45^\circ$

$= (\tan 1^\circ \cot 1^\circ)(\tan 2^\circ \cot 2^\circ) \ldots (\tan 44^\circ \cot 44^\circ) \tan 45^\circ = 1.$

192. Given exp. $= (\tan 15^\circ \tan 75^\circ)(\tan 25^\circ \tan 65^\circ) \tan 45^\circ$

$= (\tan 15^\circ \cot 15^\circ)(\tan 25^\circ \cot 25^\circ) \times 1 = 1.$

193. Given exp. $= \left(\tan\frac{\pi}{20}\tan\frac{9\pi}{20}\right)\left(\tan\frac{3\pi}{20}\tan\frac{7\pi}{20}\right)\tan\frac{\pi}{4}$

$= \tan\frac{\pi}{20}\tan\left(\frac{\pi}{2} - \frac{\pi}{20}\right)\tan\frac{3\pi}{20}\tan\left(\frac{\pi}{2} - \frac{3\pi}{20}\right)\tan\frac{\pi}{4}$

$= \tan\frac{\pi}{20}\cot\frac{\pi}{20}\tan\frac{3\pi}{20}\cot\frac{3\pi}{20} \times 1 = 1.$

194. $\sin\theta = 0 \Leftrightarrow \theta = 0, \pm\pi, \pm 2\pi, \pm 3\pi, \ldots \Leftrightarrow \theta = n\pi, n \in I.$

195. $\cos\theta = 0 \Leftrightarrow \theta = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}, \ldots \Leftrightarrow \theta = (2n+1)\frac{\pi}{2}, n \in I.$

196. $\tan\theta = 0 \Leftrightarrow \frac{\sin\theta}{\cos\theta} = 0 \Leftrightarrow \sin\theta = 0 \Leftrightarrow \theta = n\pi, n \in I.$

197. $\sin 3x = 0 \Leftrightarrow 3x = n\pi \Leftrightarrow x = \frac{n\pi}{3}\cdot$

198. $\cos 2x = 0 \Leftrightarrow 2x = (2n+1)\frac{\pi}{2} \Leftrightarrow x = (2n+1)\frac{\pi}{4}\cdot$

199. Remember: $\sin\theta = \sin\alpha \Rightarrow \theta = n\pi \pm \alpha, n \in I.$

200. Remember: $\cos\theta = \cos\alpha \Rightarrow \theta = 2n\pi \pm \alpha, n \in I.$

201. Remember: $\tan\theta = \tan\alpha \Rightarrow \theta = n\pi + \alpha, n \in I.$

202. $\sin^2\theta = \sin^2\alpha \Rightarrow 2\sin^2\theta = 2\sin^2\alpha$

$\Rightarrow (1 - \cos 2\theta) = (1 - \cos 2\alpha) \Rightarrow \cos 2\theta = \cos 2\alpha$

$\Rightarrow 2\theta = 2n\pi \pm 2\alpha \Rightarrow \theta = n\pi \pm \alpha, n \in I.$

203. $\cos^2\theta = \cos^2\alpha \Rightarrow 2\cos^2\theta = 2\cos^2\alpha$

$\Rightarrow (1 + \cos 2\theta) = (1 + \cos 2\alpha) \Rightarrow \cos 2\theta = \cos 2\alpha$

$\Rightarrow 2\theta = 2n\pi \pm 2\alpha \Rightarrow \theta = n\pi \pm \alpha, n \in I.$

204. $\tan^2\theta = \tan^2\alpha \Rightarrow \frac{1 - \tan^2\theta}{1 + \tan^2\theta} = \frac{1 - \tan^2\alpha}{1 + \tan^2\alpha}$

$\Rightarrow \cos 2\theta = \cos 2\alpha \Rightarrow 2\theta = 2n\pi \pm 2\alpha \Rightarrow \theta = n\pi \pm \alpha, n \in I.$

205. $\sin\theta = \dfrac{1}{\sqrt{2}} = \sin\dfrac{\pi}{4} \Rightarrow \theta = n\pi + (-1)^n \dfrac{\pi}{4}, n \in I.$

206. $\cos\theta = \dfrac{1}{2} = \cos\dfrac{\pi}{3} \Rightarrow \theta = 2n\pi \pm \dfrac{\pi}{3}, n \in I.$

207. $\tan\theta = \dfrac{1}{\sqrt{3}} = \tan\dfrac{\pi}{6} \Rightarrow \theta = n\pi + \dfrac{\pi}{6}, n \in I.$

208. $\sin\theta = \dfrac{-\sqrt{3}}{2} = -\sin\dfrac{\pi}{3} = \sin\left(\pi + \dfrac{\pi}{3}\right) = \sin\dfrac{4\pi}{3}$

$\therefore \quad \theta = n\pi + (-1)^n \dfrac{4\pi}{3}, n \in I.$

209. $\cos\theta = \dfrac{-1}{2} = -\cos\dfrac{\pi}{3} = \cos\left(\pi - \dfrac{\pi}{3}\right) = \cos\dfrac{2\pi}{3}$

$\therefore \quad \theta = \left(2n\pi \pm \dfrac{2\pi}{3}\right), n \in I.$

210. $\cot\theta = -\sqrt{3} \Rightarrow \tan\theta = \dfrac{-1}{\sqrt{3}} = -\tan\dfrac{\pi}{6} = \tan\left(\pi - \dfrac{\pi}{6}\right) = \tan\dfrac{5\pi}{6}$

$\Rightarrow \quad \theta = \left(n\pi + \dfrac{5\pi}{6}\right), n \in I.$

211. $\operatorname{cosec}\theta = -\sqrt{2} \Rightarrow \sin\theta = \dfrac{-1}{\sqrt{2}} = -\sin\dfrac{\pi}{4} = \sin\left(\pi + \dfrac{\pi}{4}\right) = \sin\dfrac{5\pi}{4}$

$\Rightarrow \quad \theta = n\pi + (-1)^n \cdot \dfrac{5\pi}{4}, n \in N.$

212. $\tan 3\theta = -1 = -\tan\dfrac{\pi}{4} = \tan\left(\pi - \dfrac{\pi}{4}\right) = \tan\dfrac{3\pi}{4}$

$\Rightarrow \quad 3\theta = n\pi + \dfrac{3\pi}{4}, n \in I \Rightarrow \theta = \dfrac{n\pi}{3} + \dfrac{\pi}{4}, n \in I.$

213. $\sin 2\theta = \dfrac{-1}{2} = -\sin\dfrac{\pi}{6} = \sin\left(\pi + \dfrac{\pi}{6}\right) = \sin\dfrac{7\pi}{6}$

$\Rightarrow \quad 2\theta = n\pi + (-1)^n \dfrac{7\pi}{6}, n \in I \Rightarrow \theta = \dfrac{n\pi}{2} + (-1)^n \dfrac{7\pi}{12}, n \in I.$

214. $4\sin^2\theta = 1 \Rightarrow \sin^2\theta = \dfrac{1}{4} = \left(\dfrac{1}{2}\right)^2 = \sin^2\dfrac{\pi}{6}$

$\Rightarrow \quad x = n\pi \pm \dfrac{\pi}{6}, n \in I.$

215. $2\cos^2\theta = 1 \Rightarrow \cos^2\theta = \dfrac{1}{2} = \left(\dfrac{1}{\sqrt{2}}\right)^2 = \cos^2\dfrac{\pi}{4}$

$\therefore \quad \theta = n\pi \pm \dfrac{\pi}{4}, n \in N.$

216. $\cot^2\theta = 3 \Rightarrow \tan^2\theta = \dfrac{1}{3} = \left(\dfrac{1}{\sqrt{3}}\right)^2 = \tan^2\dfrac{\pi}{6} \Rightarrow \theta = n\pi \pm \dfrac{\pi}{6}, n \in I.$

217. $\cos A = \dfrac{(b^2 + c^2 - a^2)}{2bc} = \dfrac{(9 + 16 - 4)}{(2 \times 3 \times 4)} = \dfrac{21}{24} = \dfrac{7}{8}.$

218. $\cos A = \dfrac{(b^2 + c^2 - a^2)}{2bc} = \dfrac{6 + (\sqrt{3} + 1)^2 - 4}{2\sqrt{6}(\sqrt{3} + 1)} = \dfrac{6 + 2\sqrt{3}}{2\sqrt{6}(\sqrt{3} + 1)} = \dfrac{2\sqrt{3}(\sqrt{3} + 1)}{2\sqrt{6}(\sqrt{3} + 1)} = \dfrac{1}{\sqrt{2}}$

$\Rightarrow \quad A = 45°.$

219. $\cos A = \dfrac{(b^2 + c^2 - a^2)}{2bc} = \dfrac{(6 + 8 - 2)}{2\sqrt{6}\sqrt{8}} = \dfrac{12}{4\sqrt{12}} = \dfrac{3}{2\sqrt{3}} = \dfrac{\sqrt{3}}{2} \Rightarrow \theta = 30°.$

220. Let the angles of $\triangle ABC$ be $x°$, $2x°$ and $3x°$.

Then, $x + 2x + 3x = 180 \Rightarrow 6x = 180 \Rightarrow x = 30.$

So, the angles are $30°$, $60°$ and $90°$.

$\therefore$ ratio of sides $= \sin 30° : \sin 60° : \sin 90° = \dfrac{1}{2} : \dfrac{\sqrt{3}}{2} : 1 = 1 : \sqrt{3} : 2.$

221. $\cos C = \dfrac{(a^2 + b^2 - c^2)}{2ab} \Rightarrow \cos 60° = \dfrac{(4 + 2\sqrt{3} + 4 - c^2)}{4(\sqrt{3} + 1)}$

$\Rightarrow \dfrac{8 + 2\sqrt{3} - c^2}{4(\sqrt{3} + 1)} = \dfrac{1}{2} \Rightarrow 16 + 4\sqrt{3} - 2c^2 = 4\sqrt{3} + 4$

$\Rightarrow 2c^2 = 12 \Rightarrow c^2 = 6 \Rightarrow c = \sqrt{6}.$

222. $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} \Rightarrow \dfrac{2}{\left(\dfrac{2}{3}\right)} = \dfrac{3}{\sin B} \Rightarrow \sin B = 1 \Rightarrow B = 90°.$

223. A, B, C are in AP $\Rightarrow 2B = A + C \Rightarrow 3B = A + B + C = 180° \Rightarrow B = 60°.$

$\therefore \quad \dfrac{b}{\sin B} = \dfrac{c}{\sin C} \Rightarrow \dfrac{\sqrt{3}}{\sin 60°} = \dfrac{\sqrt{2}}{\sin C} \Rightarrow \dfrac{2}{\sqrt{3}} \times \sqrt{3} = \dfrac{\sqrt{2}}{\sin C}$

$\Rightarrow \sin C = \dfrac{\sqrt{2}}{2} = \dfrac{1}{\sqrt{2}} \Rightarrow C = 45°.$

224. Let $b = 2k$ and $c = \sqrt{3}k$. Then,

$\cos A = \dfrac{(b^2 + c^2 - a^2)}{2bc} \Rightarrow \cos 30° = \dfrac{4k^2 + 3k^2 - a^2}{4\sqrt{3}k^2}$

$\therefore \quad 7k^2 - a^2 = 4\sqrt{3}k^2 \times \dfrac{\sqrt{3}}{2} \qquad \left[\because \cos 30° = \dfrac{\sqrt{3}}{2}\right]$

$\Rightarrow k^2 = a^2 \Rightarrow a = k \Rightarrow b = 2a, c = \sqrt{3}a.$

Now, $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} \Rightarrow \dfrac{a}{\sin 30°} = \dfrac{2a}{\sin B} \Rightarrow \sin B = 1 \Rightarrow B = 90°.$

225. Let the sides be $(a - d)$, a and $(a + d)$. Then,

$(a + d)^2 = (a - d)^2 + a^2 \Rightarrow a^2 = 4ad \Rightarrow a = 4d.$

$\therefore$ the sides are $3d, 4d, 5d$. Their ratio is $3 : 4 : 5$.

226. $a : b : c = \sin A : \sin B : \sin C = \sin 45° : \sin 60° : \sin 75°$

$= \dfrac{1}{\sqrt{2}} : \dfrac{\sqrt{3}}{2} : \dfrac{(\sqrt{3} + 1)}{2\sqrt{2}} = 2 : \sqrt{6} : (\sqrt{3} + 1).$

227. $\angle A = 30°, \angle C = 105°, \angle B = 180° - (30° + 105°) = 45°.$

$$\therefore \quad \frac{a}{\sin A} = \frac{b}{\sin B} \Rightarrow \frac{a}{\sin 30°} = \frac{3\sqrt{2}}{\sin 45°} \Rightarrow 2a = 6 \Rightarrow a = 3.$$

228. $\cos B = \dfrac{a^2 + c^2 - b^2}{2ac} \Rightarrow \cos 45° = \dfrac{25 + 8 - b^2}{2 \times 5 \times 2\sqrt{2}} \Rightarrow \dfrac{1}{\sqrt{2}} = \dfrac{33 - b^2}{20\sqrt{2}}$

$$\therefore \quad 33 - b^2 = 20 \Rightarrow b^2 = 13 \Rightarrow b = \sqrt{13}.$$

229. $\angle B = 30°, \angle C = 45°, \angle A = 180° - (30° + 45°) = 105°.$

$$\therefore \quad \frac{a}{\sin A} = \frac{c}{\sin C} \Rightarrow \frac{(\sqrt{3}+1)}{\sin 105°} = \frac{c}{\sin 45°} \Rightarrow \frac{(\sqrt{3}+1)}{\cos 15°} = \frac{c}{\left(\frac{1}{\sqrt{2}}\right)}$$

$$\therefore \quad c = (\sqrt{3}+1) \cdot \frac{1}{\sqrt{2}} \cdot \frac{2\sqrt{2}}{(\sqrt{3}+1)} = 2.$$

230. $b = 5, c = 5\sqrt{3}$ and $\angle A = 30°.$

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc} \Rightarrow \cos 30° = \frac{25 + 75 - a^2}{50\sqrt{3}}$$

$$\Rightarrow \frac{100 - a^2}{50\sqrt{3}} = \frac{\sqrt{3}}{2} \Rightarrow 200 - 2a^2 = 150 \Rightarrow a^2 = 25 \Rightarrow a = 5.$$

Thus, $a = b$ and hence $\triangle ABC$ is isosceles.

231. Let the angles be $3x°, 4x°$ and $5x°$. Then,

$$3x + 4x + 5x = 180 \Rightarrow 12x = 180 \Rightarrow x = 15.$$

$$\therefore \quad \angle A = 45°, \angle B = 60° \text{ and } \angle C = 75°.$$

$$\therefore \quad a : c = \sin 45° : \sin 75° = \frac{1}{\sqrt{2}} : \frac{\sqrt{3}+1}{2\sqrt{2}} = 2 : (\sqrt{3}+1).$$

232. Here, $b = 20, c = 15, \angle A = 150°.$

$$\therefore \quad \Delta = \frac{1}{2} bc \sin A = \frac{1}{2} \times 20 \times 15 \times \sin 150° = 150 \times \sin 30° = \left(150 \times \frac{1}{2}\right) = 75 \text{ sq units.}$$

233. $s = \frac{1}{2}(18 + 24 + 30) = 36, (s - a) = (36 - 18) = 18, (s - b) = (36 - 24) = 12$

and $(s - c) = (36 - 30) = 6.$

$$\therefore \quad \Delta = \sqrt{s(s-a)(s-b)(s-c)} = \sqrt{36 \times 18 \times 12 \times 6}$$

$$= \sqrt{36 \times 36 \times 36} = (36 \times 6) = 216 \text{ sq units.}$$

234. Here, $a = 16, c = 9$ and $\angle B = 30°.$

$$\therefore \quad \Delta = \frac{1}{2} ac \sin B = \left(\frac{1}{2} \times 16 \times 9 \times \sin 30°\right) = \left(72 \times \frac{1}{2}\right) = 36 \text{ sq units.}$$

235. $s = \frac{1}{2}(15 + 14 + 13) = 21, (s - a) = 6, (s - b) = 7, (s - c) = 8.$

$$\therefore \quad \Delta = \sqrt{s(s-a)(s-b)(s-c)} = \sqrt{21 \times 6 \times 7 \times 8} = (21 \times 4) = 84 \text{ sq units.}$$

$$\therefore \quad R = \frac{abc}{4\Delta} = \frac{15 \times 14 \times 13}{4 \times 84} = \frac{65}{8}.$$

236. $s = \frac{1}{2}(18 + 24 + 30) = 36, (s-a) = (36-18) = 18, (s-b) = (36-24) = 12$

and $(s-c) = (36-30) = 6$.

$\therefore \quad \Delta = \sqrt{s(s-a)(s-b)(s-c)} = \sqrt{36 \times 18 \times 12 \times 6} = 216$ sq units.

$\therefore \quad R = \frac{abc}{4\Delta} = \frac{(18 \times 24 \times 30)}{4 \times 216} = 15.$

Area of the circumcircle $= \pi R^2 = (\pi \times 15 \times 15) = (225\pi)$ sq units.

237. $s = \frac{1}{2}(4 + 13 + 15) = 16, (s-a) = 12, (s-b) = 3, (s-c) = 1.$

$\therefore \quad \Delta = \sqrt{s(s-a)(s-b)(s-c)} = \sqrt{16 \times 12 \times 3 \times 1} = 24$ sq units.

$\therefore \quad r = \frac{\Delta}{s} = \frac{24}{16} = \frac{3}{2} \cdot$

238. $s = \frac{1}{2}(13 + 14 + 15) = 21, (s-a) = 8, (s-b) = 7, (s-c) = 6.$

$\therefore \quad \Delta = \sqrt{s(s-a)(s-b)(s-c)} = \sqrt{21 \times 8 \times 7 \times 6} = \sqrt{21 \times 21 \times 16} = (21 \times 4) = 84.$

$\therefore \quad r = \frac{\Delta}{s} = \frac{84}{21} = 4 \cdot$

Area of the incircle $= \pi r^2 = (\pi \times 4 \times 4) = (16\pi)$ sq units.

239. $r = 4R \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}$

$\Rightarrow \quad r = 4R \sin 30° \cdot \sin 30° \cdot \sin 30° \qquad [\because\ A = B = C = 60°]$

$\Rightarrow \quad r = 4R \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \;\Rightarrow\; R = 2r.$

240. $\Delta = \frac{1}{2} \times a \times a \times \sin 60° = \left(\frac{1}{2} \times 2\sqrt{3} \times 2\sqrt{3} \times \frac{\sqrt{3}}{2}\right) = 3\sqrt{3}.$

$\therefore \quad R = \frac{abc}{4\Delta} = \frac{a^3}{4\Delta} = \frac{(2\sqrt{3})^3}{4 \times 3\sqrt{3}} = \frac{24\sqrt{3}}{12\sqrt{3}} = 2.$

241. Let $a = 3k, b = 7k, c = 8k$. Then, $s = 9k, (s-a) = 6k, (s-b) = 2k, (s-c) = k$.

$$\frac{R}{r} = \frac{abc}{4\Delta} \cdot \frac{s}{\Delta} = \frac{abc}{4(s-a)(s-b)(s-c)} = \left(\frac{3k \times 7k \times 8k}{4 \times 6k \times 2k \times k}\right) = \frac{7}{2} \cdot$$

Hence, $R : r = 7 : 2$.

242. $s = \frac{27}{2}$ cm and $\Delta = 81 \text{ cm}^2$.

$\therefore \quad r = \frac{\Delta}{s} = \left(81 \times \frac{2}{27}\right) = 6$ cm.

243. Let $C = 90°$. Then, $B = (90° - A)$.

$$\sin^2 A + \sin^2 B + \sin^2 C = \sin^2 A + \sin^2(90° - A) + \sin^2 90°$$

$$= (\sin^2 A + \cos^2 A + 1) = 2.$$

244. $\dfrac{\sin(B-C)}{\sin(B+C)}=\dfrac{\sin(B-C)\cdot\sin(B+C)}{\sin^2(B+C)}=\dfrac{\sin^2 B-\sin^2 C}{\sin^2(\pi-A)}$

$$=\frac{\sin^2 B-\sin^2 C}{\sin^2 A}=\frac{(b^2-c^2)}{a^2}\qquad\left[\because\ \frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}\right].$$

245. $a(b\cos C-c\cos B)=(b\cos C+c\cos B)(b\cos C-c\cos B)$

$$=(b^2\cos^2 C-c^2\cos^2 B)=b^2(1-\sin^2 C)-c^2(1-\sin^2 B)$$

$$=(b^2-c^2)+(c^2\sin^2 B-b^2\sin^2 C)=(b^2-c^2)\left[\because\ \frac{b}{\sin B}=\frac{c}{\sin C}\right].$$

246. $\dfrac{1+\cos(A-B)\cos C}{1+\cos(A-C)\cos B}=\dfrac{1+\cos(A-B)\cos[\pi-(A+B)]}{1+\cos(A-C)\cos[\pi-(A+C)]}$

$$=\frac{1-\cos(A-B)\cos(A+B)}{1-\cos(A-C)\cos(A+C)}=\frac{1-(\cos^2 A-\sin^2 B)}{1-(\cos^2 A-\sin^2 C)}$$

$$=\frac{\sin^2 A+\sin^2 B}{\sin^2 A+\sin^2 C}=\frac{a^2+b^2}{a^2+c^2}.$$

247. Given exp. $=(a^2+b^2-2ab)\cos^2\dfrac{C}{2}+(a^2+b^2+2ab)\sin^2\dfrac{C}{2}$

$$=a^2\left(\cos^2\frac{C}{2}+\sin^2\frac{C}{2}\right)+b^2\left(\cos^2\frac{C}{2}+\sin^2\frac{C}{2}\right)-2ab\left(\cos^2\frac{C}{2}-\sin^2\frac{C}{2}\right)$$

$$=a^2+b^2-2ab\cos C\qquad\left[\because\ \cos^2\frac{C}{2}-\sin^2\frac{C}{2}=\cos C\right]$$

$$=a^2+b^2-2ab\cdot\frac{(a^2+b^2-c^2)}{2ab}=c^2.$$

248. Given exp. $=(b\cos C+c\cos B)+(c\cos A+a\cos C)+(a\cos B+b\cos A)$

$=(a+b+c)$ [by projection formulae].

249. Given exp. $=2bc\cos A+2ca\cos B+2ab\cos C$

$$=2bc\cdot\frac{(b^2+c^2-a^2)}{2bc}+2ca\cdot\frac{(c^2+a^2-b^2)}{2ca}+2ab\cdot\frac{(a^2+b^2-c^2)}{2ab}$$

$$=(b^2+c^2-a^2)+(c^2+a^2-b^2)+(a^2+b^2-c^2)=(a^2+b^2+c^2).$$

250. Given exp. $=ac\cos B-bc\cos A=ac\cdot\dfrac{(c^2+a^2-b^2)}{2ac}-bc\cdot\dfrac{(b^2+c^2-a^2)}{2bc}$

$$=\frac{(c^2+a^2-b^2)-(b^2+c^2-a^2)}{2}=(a^2-b^2).$$

251. $\dfrac{a-b}{a+b}=\dfrac{k\sin A-k\sin B}{k\sin A+k\sin B}=\dfrac{k(\sin A-\sin B)}{k(\sin A+\sin B)}$

$$=\frac{(\sin A-\sin B)}{(\sin A+\sin B)}=\frac{2\cos\frac{(A+B)}{2}\sin\frac{(A-B)}{2}}{2\sin\frac{(A+B)}{2}\cos\frac{(A-B)}{2}}=\frac{\tan\frac{(A-B)}{2}}{\tan\frac{(A+B)}{2}}.$$

252. $c\cos^2\frac{A}{2}+a\cos^2\frac{C}{2}=c\cdot\frac{s(s-a)}{bc}+a\cdot\frac{s(s-c)}{ab}$

$$=\left\{\frac{s(s-a)}{b}+\frac{s(s-c)}{b}\right\}=\frac{s}{b}\cdot\{(s-a)+(s-c)\}$$

$$=\frac{s}{b}\cdot\{2s-(a+c)\}=\frac{s}{b}\cdot\{(a+b+c)-(a+c)\}=\frac{s}{b}\times b=s.$$

253. $\frac{(b-c)}{a}\cdot\cos^2\frac{A}{2}=\frac{(b-c)}{a}\cdot\frac{s(s-a)}{bc}=\frac{s(s-a)(b-c)}{abc}.$

Similary, $\frac{(c-a)}{b}\cdot\cos^2\frac{B}{2}=\frac{(c-a)}{b}\cdot\frac{s(s-b)}{ac}=\frac{s(s-b)(c-a)}{abc}.$

And, $\frac{(a-b)}{c}\cdot\cos^2\frac{C}{2}=\frac{(a-b)}{c}\cdot\frac{s(s-c)}{ab}=\frac{s(s-c)(a-b)}{abc}.$

$\therefore$ given exp. $=\frac{s(s-a)(b-c)}{abc}+\frac{s(s-b)(c-a)}{abc}+\frac{s(s-c)(a-b)}{abc}$

$$=\frac{s}{abc}\cdot[(s-a)(b-c)+(s-b)(c-a)+(s-c)(a-b)]$$

$$=\frac{s}{abc}\cdot[s(b-c+c-a+a-b)-\{a(b-c)-b(c-a)-c(a-b)\}]=0.$$

254. Given exp. $=\triangle\cdot\left[\frac{s(s-a)}{\triangle}+\frac{s(s-b)}{\triangle}+\frac{s(s-c)}{\triangle}\right]$

$$=s(s-a)+s(s-b)+s(s-c)=s\cdot[(s-a)+(s-b)+(s-c)]$$

$$=s[3s-(a+b+c)]=s[3s-2s]=(s\times s)=s^2.$$

255. $bc\cos^2\frac{A}{2}+ca\cos^2\frac{B}{2}+ab\cos^2\frac{C}{2}=bc\cdot\frac{s(s-a)}{bc}+ca\cdot\frac{s(s-b)}{ca}+ab\cdot\frac{s(s-c)}{ab}$

$$=s(s-a)+s(s-b)+s(s-c)$$

$$=s\cdot[(s-a)+(s-b)+(s-c)]=s\cdot[3s-(a+b+c)]$$

$$=s\cdot[3s-2s]=(s\times s)=s^2.$$

256. $2abc\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}=2abc\cdot\sqrt{\frac{s(s-a)}{bc}}\cdot\sqrt{\frac{s(s-b)}{ac}}\cdot\sqrt{\frac{s(s-c)}{ab}}$

$$=2abc\cdot\frac{1}{abc}\cdot\sqrt{s(s-a)(s-b)(s-c)}\cdot s=2s\triangle=(a+b+c)\triangle.$$

257. $\cos^2\frac{A}{2}=\frac{b+c}{2c}\Rightarrow\frac{s(s-a)}{bc}=\frac{b+c}{2c}\Rightarrow 2s(2s-2a)=2b(b+c)$

$\Rightarrow (a+b+c)(a+b+c-2a)=2b(b+c)$

$\Rightarrow (b+c+a)(b+c-a)=2b^2+2bc$

$\Rightarrow (b+c)^2-a^2-2b^2-2bc=0$

$\Rightarrow c^2-a^2=b^2\Rightarrow a^2+b^2=c^2.$

258. $\cot\frac{A}{2}:\cot\frac{B}{2}:\cot\frac{C}{2}=3:5:7$

$\Rightarrow \frac{s(s-a)}{\triangle}:\frac{s(s-b)}{\triangle}:\frac{s(s-c)}{\triangle}=3:5:7$

$\Rightarrow \quad (s-a):(s-b):(s-c) = 3:5:7$

$\Rightarrow \quad \dfrac{b+c-a}{2} : \dfrac{c+a-b}{2} : \dfrac{a+b-c}{2} = 3:5:7$

$\Rightarrow \quad (b+c-a):(c+a-b):(a+b-c) = 3:5:7.$

Let $b+c-a = 3k$, $c+a-b = 5k$ and $a+b-c = 7k$.

Adding, we get: $a+b+c = 15k$.

$\therefore \quad 2a = 12k$, $2b = 10k$ and $2c = 8k$

$\Rightarrow \quad a = 6k$, $b = 5k$ and $c = 4k$

$\Rightarrow \quad a:b:c = 6:5:4.$

259. $s = \dfrac{1}{2}(16+24+20) = 30$, $(s-a) = 14$, $(s-b) = 6$ and $(s-c) = 10$.

$\therefore \quad \cos\dfrac{B}{2} = \sqrt{\dfrac{s(s-b)}{ca}} = \sqrt{\dfrac{30\times 6}{20\times 16}} = \sqrt{\dfrac{9}{16}} = \dfrac{3}{4}.$

260. $a\cos A = b\cos B \Rightarrow k\sin A\cos A = k\sin B\cos B$

$\Rightarrow \quad \sin A\cos A = \sin B\cos B \Rightarrow 2\sin A\cos A = 2\sin B\cos B$

$\Rightarrow \quad \sin 2A - \sin 2B = 0 \Rightarrow 2\cos(A+B)\sin(A-B) = 0$

$\Rightarrow \quad \cos(A+B) = 0 \quad \text{or} \quad \sin(A-B) = 0$

$\Rightarrow \quad A+B = 90^\circ \quad \text{or} \quad A-B = 0 \Rightarrow \angle C = 90^\circ \quad \text{or} \quad A = B$

$\Rightarrow \quad \triangle ABC$ is either right angled or isosceles.

❑

Coordinate Geometry

SUMMARY OF THE RESULTS

1. If $A(x_1, y_1)$ and $B(x_2, y_2)$ be two given points, then

$$AB^2 = \{(x_2 - x_1)^2 + (y_2 - y_1)^2\}.$$

2. (i) Let $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ be the vertices of $\triangle ABC$, then

$$\triangle = \frac{1}{2} \cdot |x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)| \text{ sq units.}$$

(ii) Points $A(x_1, y_1)$, $B(x_2, y_2)$, $C(x_3, y_3)$ are collinear $\Leftrightarrow$ ar$(\triangle ABC) = 0$.

3. (i) If $P(x, y)$ divides the join of $A(x_1, y_1)$ and $B(x_2, y_2)$ in the ratio $m : n$, then $x = \dfrac{(mx_2 + nx_1)}{(m+n)}$, $y = \dfrac{(my_2 + ny_1)}{(m+n)}$.

(ii) If $P(x, y)$ is the midpoint of the join of $A(x_1, y_1)$ and $B(x_2, y_2)$, then $x = \dfrac{(x_1 + x_2)}{2}$ and $y = \dfrac{(y_1 + y_2)}{2}$.

4. A quadrilateral is a

(i) ***rectangle,*** if its **opposite sides are equal** and **diagonals are equal.**

(ii) ***square,*** if its **all sides are equal** and **diagonals are equal.**

(iii) **||gm but not a rectangle,** if its opposite sides are equal and diagonals are not equal.

(iv) **rhombus but not a square,** if its all sides are equal and the diagonals are not equal.

5. (i) **Inclination of a line:**

The angle θ which the given line makes with the positive direction of the x-axis. Clearly, $0° \le \theta \le 180°$.

(ii) **Slope of a line:** $m = \tan\theta$.

(iii) Slope of a horizontal line is 0.

(iv) Slope of a vertical line is not defined.

(v) Slope of a line passing through the points $A(x_1, y_1)$ and $B(x_2, y_2)$ is $m = \dfrac{(y_2 - y_1)}{(x_2 - x_1)}$.

(vi) $L_1 \parallel L_2 \;\Leftrightarrow\; m_1 = m_2$.

(vii) $L_1 \perp L_2 \;\Leftrightarrow\; m_1 m_2 - 1$.

(viii) $\tan\alpha = \left|\dfrac{m_1 - m_2}{1 + m_1 m_2}\right|$, where $1 + m_1 m_2 \ne 0$.

6. (i) Equation of the x-axis is $y = 0$.

(ii) Equation of the y-axis is $x = 0$.

(iii) Equation of a line parallel to the y-axis, $x = a$.

(iv) Equation of a line parallel to the x-axis, $y = b$.

7. **Equation of a line**

(i) **Point-slope form:** $m = \dfrac{(y - y_1)}{(x - x_1)}$.

(ii) **Two-point form:** $\dfrac{(y - y_1)}{(x - x_1)} = \dfrac{(y_2 - y_1)}{(x_2 - x_1)}$.

(iii) **Slope-intercept form:** $y = mx + c$.

(iv) Slope $= m$, x-intercept $= d$, then $y = m(x - d)$.

(v) **Intercept form:** $\dfrac{x}{a} + \dfrac{y}{b} = 1$.

(vi) **Normal form:** $x \cos \alpha + y \sin \alpha = p$.

8. **Reduction of general form $Ax + By + C = 0$ to**

(i) slope intercept form: $y = \dfrac{(-A)}{B} \cdot x + \dfrac{(-C)}{B}$.

(ii) intercept form: $\dfrac{x}{\left(\dfrac{-C}{A}\right)} + \dfrac{y}{\left(\dfrac{-C}{B}\right)} = 1$.

(iii) normal form: $\dfrac{-Ax}{\sqrt{A^2 + B^2}} + \dfrac{-By}{\sqrt{A^2 + B^2}} = \dfrac{C}{\sqrt{A^2 + B^2}}$.

9. (i) Length of perpendicular from a point $P(x_1, y_1)$ on a line $Ax + By + C = 0$ is given by

$$d = \frac{|Ax_1 + By_1 + C|}{\sqrt{A^2 + B^2}}.$$

(ii) Distance between two parallel lines $Ax + By + C_1 = 0$ and $Ax + By + C_2 = 0$ is given by

$$d = \frac{|C_2 - C_1|}{\sqrt{A^2 + B^2}}.$$

(iii) Distance between two parallel lines $y = mx + C_1$ and $y = mx + C_2$ is given by

$$d = \frac{|C_2 - C_1|}{\sqrt{1 + m^2}}.$$

10. (i) Equation of a circle with centre $C(h, k)$ and radius $= r$ is given by $(x-h)^2 + (y-k)^2 = r^2$.

(ii) Equation of a circle described on the line joining $A(x_1, y_1)$ and $B(x_2, y_2)$ as diameter is $(x-x_1)(x-x_2)+(y-y_1)(y-y_2)=0$.

11. **Parabolas**

I. **Right-handed parabola ($y^2 = 4ax$, $a > 0$)**

(i) Vertex is $O(0, 0)$.

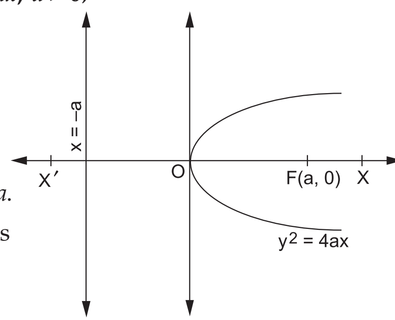
(ii) Focus is $F(a, 0)$.

(iii) Directrix is $x + a = 0$.

(iv) Axis is $y = 0$.

(v) Length of latus rectum $= 4a$.

(vi) Equation of latus rectum is $x = a$.



II. **Left-handed parabola ($y^2 = -4ax$, $a > 0$)**

(i) Vertex is $O(0, 0)$.

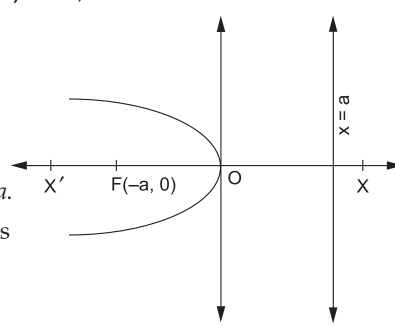
(ii) Focus is $F(-a, 0)$.

(iii) Directrix is $x - a = 0$.

(iv) Axis is $y = 0$.

(v) Length of latus rectum $= 4a$.

(vi) Equation of latus rectum is $x = -a$.



III. **Upward parabola ($x^2 = 4ay$, $a > 0$)**

(i) Vertex is $O(0, 0)$.

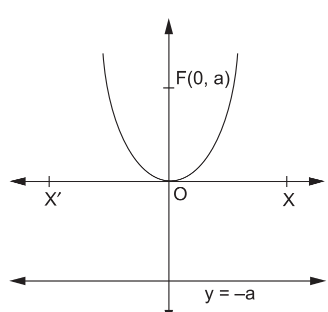
(ii) Focus is $F(0, a)$.

(iii) Directrix is $y + a = 0$.

(iv) Axis is $x = 0$.

(v) Length of latus rectum $= 4a$.

(vi) Equation of latus rectum is $y = a$.



IV. **Downward parabola ($x^2 = -4ay$, $a > 0$)**

(i) Vertex is $O(0, 0)$.

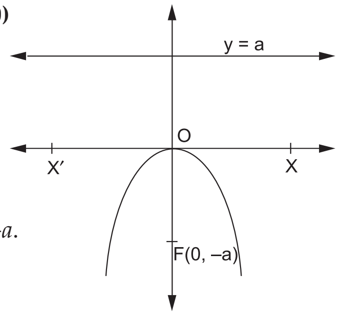
(ii) Focus is $F(0, -a)$.

(iii) Directrix is $y = a$.

(iv) Axis is $x = 0$.

(v) Length of latus rectum $= 4a$.

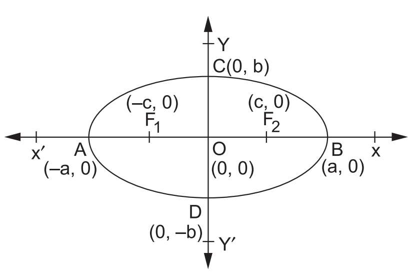
(vi) Equation of latus rectum is $y = -a$.



MAIN FACTS ABOUT ALL TYPES OF PARABOLAS

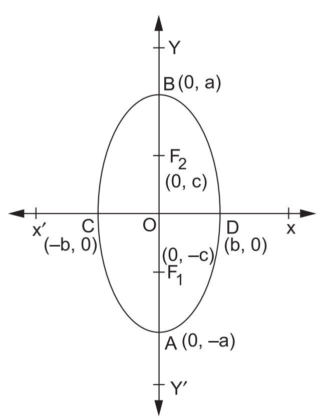
Parabola (Equation)	Focus	Vertex	Equation of directrix	Equation of axis	Length of latus rectum	Equation of latus rectum
(i) $y^2 = 4ax$, $a > 0$ (Right-handed)	$(a, 0)$	$(0, 0)$	$x + a = 0$	$y = 0$	$4a$	$x - a = 0$
(ii) $y^2 = -4ax$, $a > 0$ (Left-handed)	$(-a, 0)$	$(0, 0)$	$x - a = 0$	$y = 0$	$4a$	$x + a = 0$
(iii) $x^2 = 4ay$, $a > 0$ (Upward)	$(0, a)$	$(0, 0)$	$y + a = 0$	$x = 0$	$4a$	$y - a = 0$
(iv) $x^2 = -4ay$, $a > 0$ (Downward)	$(0, -a)$	$(0, 0)$	$y - a = 0$	$x = 0$	$4a$	$y + a = 0$

12. **Ellipse**



HORIZONTAL ELLIPSE

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,\ b < a.$$



VERTICAL ELLIPSE

$$\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1,\ b < a.$$

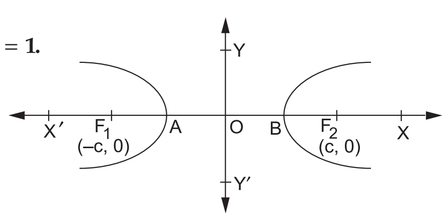
MAIN FACTS ABOUT HORIZONTAL AND VERTICAL ELLIPSES

Properties	Horizontal Ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ $0<b<a$ and $c^2=(a^2-b^2)$	Vertical Ellipse $\frac{x^2}{b^2}+\frac{y^2}{a^2}=1$ $0<b<a$ and $c^2=(a^2-b^2)$
(i) Centre	(0, 0)	(0, 0)
(ii) Vertices	$A(-a, 0)$, $B(a, 0)$	$A(0, -a)$, $B(0, a)$
(iii) Foci	$F_1(-c, 0)$ and $F_2(c, 0)$ or $(-ae, 0)$ and $(ae, 0)$	$F_1(0, -c)$ and $F_2(0, c)$ or $(0, -ae)$ and $(0, ae)$
(iv) Length of the major axis	$2a$	$2a$
(v) Length of the minor axis	$2b$	$2b$
(vi) Equation of the major axis	$y = 0$	$x = 0$
(vii) Equation of the minor axis	$x = 0$	$y = 0$
(viii) Length of the latus rectum	$\frac{2b^2}{a}$	$\frac{2b^2}{a}$
(ix) Eccentricity	$e=\frac{c}{a}=\frac{\sqrt{a^2-b^2}}{a}$	$e=\frac{c}{a}=\frac{\sqrt{a^2-b^2}}{a}$

13. **Hyperbola**

I. **Horizontal hyperbola**

(i) Its equation is $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1.$



(ii) Centre is $O(0, 0)$.

(iii) $X'OX$ is the transverse axis and YOY' is the conjugate axis.

(iv) Length of the transverse axis = $2a$.

(v) Length of the conjugate axis = $2b$.

(vi) Foci are $F_1(-c, 0)$ and $F_2(c, 0)$, where $c = ae$.

(vii) Vertices are $A(-a, 0)$ and $B(a, 0)$.

(viii) Eccentricity, $e = \frac{c}{a} = \frac{\sqrt{a^2 + b^2}}{a}$.

(ix) Length of latus rectum $= \frac{2b^2}{a}$.

II. **Vertical hyperbola**

(i) Its equation is $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$.

(ii) Its centre is $O(0, 0)$.

(iii) YOY' is the transverse axis and $X'OX$ is the conjugate axis.

(iv) Length of transverse axis $= 2a$.

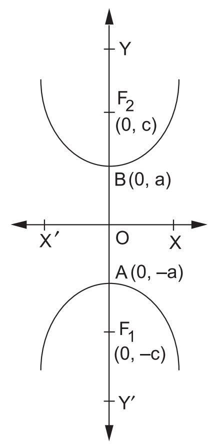
(v) Length of conjugate axis $= 2b$.

(vi) Its vertices are $A(0, -a)$ and $B(0, a)$.

(vii) Its foci are $F_1(0, -c)$ and $F_2(0, c)$, where $c = ae$.

(viii) Eccentricity, $e = \frac{c}{a} = \frac{\sqrt{a^2 + b^2}}{a}$.

(ix) Length of latus rectum $= \frac{2b^2}{a}$.



EXERCISE 1

Mark (✓) against the correct answer in each of the following.

1. The distance between the points $A(2, -3)$ and $B(-6, 3)$ is

(a) 4 units (b) 6 units (c) 8 units (d) 10 units

2. The distance between the points $P(\sqrt{5}+1, \sqrt{3}-1)$ and $Q(\sqrt{5}-2, \sqrt{3}+2)$ is

(a) $3\sqrt{2}$ units (b) $4\sqrt{3}$ units (c) $3\sqrt{5}$ units (d) $2\sqrt{6}$ units

3. If the distance between the points $A(-3, 4)$ and $B(x, 7)$ is 5 units, then $x = ?$

(a) –1 or 7 (b) 1 or –7 (c) 5 or –3 (d) –5 or 3

4. If the distance between the points $A(x, 3)$ and $B(3, 5)$ is 4 units, then $x = ?$

(a) $2 \pm 3\sqrt{3}$ (b) $3 \pm 2\sqrt{3}$ (c) $3 \pm 3\sqrt{2}$ (d) $2 \pm 2\sqrt{2}$

5. The distance of the point $A(6, -6)$ from the origin is

(a) $2\sqrt{3}$ units (b) 6 units (c) 12 units (d) $6\sqrt{2}$ units

6. If $P(x, y)$ is equidistant from $A(6, -1)$ and $B(2, 3)$, then

(a) $3x - y = 3$ (b) $x - 3y = 3$ (c) $x - y = 3$ (d) $x + y = 3$

7. A point P on the x-axis which is equidistant from the points $A(7, 6)$ and $B(-3, 4)$ is

(a) $P(3, 0)$ (b) $P(0, 3)$ (c) $P(-3, 0)$ (d) $P(0, -3)$

8. A point P on the y-axis which is equidistant from the points $A(-4, 3)$ and $B(5, 2)$ is

(a) $P(-2, 0)$ (b) $P(0, -2)$ (c) $P(2, 0)$ (d) $P(0, 2)$

9. A is a point on the x-axis with abscissa -8 and B is a point on the y-axis with ordinate 15. Distance $AB = ?$

(a) 13 units (b) 15 units (c) 17 units (d) 23 units

10. If the points $A(-2, 3)$, $B(1, 2)$ and $C(k, 0)$ are collinear, then $k = ?$

(a) 5 (b) 6 (c) 7 (d) 8

11. The vertices of a $\triangle ABC$ are $A(3, 8)$, $B(-4, 2)$ and $C(5, -1)$. The area of $\triangle ABC$ is

(a) 57 sq units (b) 75 sq units (c) $28\frac{1}{2}$ sq units (d) $37\frac{1}{2}$ sq units

12. The area of quadrilateral $ABCD$ with $A(1, 1)$, $B(7, -3)$, $C(12, 2)$ and $D(7, 21)$ as its vertices is

(a) 35 sq units (b) 65 sq units (c) 85 sq units (d) 115 sq units

13. In $\triangle ABC$ having vertices $A(-1, 3)$, $B(1, -1)$ and $C(5, 1)$, the length of the median AD is

(a) 3 units (b) 4 units (c) 5 units (d) 6 units

14. In $\triangle ABC$, its vertices are $A(7, -3)$, $B(3, -1)$ and $C(5, 3)$. If BE is one of its medians, then $BE = ?$

(a) $2\sqrt{2}$ units (b) 3 units (c) $\sqrt{10}$ units (d) none of these

15. In $\triangle ABC$ with vertices $A(-1, 0)$, $B(5, -2)$ and $C(8, 2)$, the centroid is

(a) $(4, 0)$ (b) $(0, 4)$ (c) $(6, 0)$ (d) $(0, 6)$

16. In $\triangle ABC$ having vertices $A(5, 2)$ and $B(-7, -4)$, the centroid is $G(-2, 3)$. Then, the third vertex C is

(a) $C(-6, 9)$ (b) $C(8, 11)$ (c) $C(-4, 11)$ (d) $C(-6, 11)$

17. The points $A(1, 1)$, $B(-1, -1)$ and $C(-\sqrt{3}, \sqrt{3})$ are the vertices of

(a) an equilateral triangle (b) an isosceles triangle

(c) a right triangle (d) none of these

18. The points $A(2a, 4a)$, $B(2a, 6a)$ and $C\{(2+\sqrt{3})a, 5a\}$ are the vertices of

(a) an isosceles triangle (b) an equilateral triangle

(c) a right triangle (d) none of these

19. The points $A(7, 10)$, $B(-2, 5)$ and $C(3, -4)$ are the vertices of

(a) an isosceles right triangle (b) a scalene right triangle

(c) an equilateral triangle (d) none of these

20. The points $A(2, 6)$, $B(5, 1)$, $C(0, -2)$ and $D(-3, 3)$ are the vertices of

(a) a rhombus (b) a square (c) a trapezium (d) none of these

21. The points $A(-1, 0)$, $B(0, 3)$, $C(1, 3)$ and $D(0, 0)$ are the vertices of

(a) a parallelogram (b) a rectangle

(c) a rhombus (d) none of these

22. Three consecutive vertices of a parallelogram $ABCD$ are $A(3, -2)$, $B(4, 0)$ and $C(6, -3)$. The 4th vertex D is

(a) $D(5, -5)$ (b) $D(-5, 5)$ (c) $D(-3, 8)$ (d) $D(3, -8)$

23. The vertices of a $\triangle ABC$ are $A(8, 6)$, $B(8, -2)$ and $C(2, -2)$. The coordinates of its circumcentre are

(a) $(5, -2)$ (b) $(5, 2)$ (c) $(-5, 2)$ (d) $(-5, -2)$

24. A point P divides the join of $A(5, -2)$ and $B(9, 6)$ in the ratio $3 : 1$. The coordinates of P are

(a) $(4, -7)$ (b) $(-4, 7)$ (c) $\left(\frac{7}{2}, 4\right)$ (d) $(8, 4)$

25. The ratio in which the point $P(1, 2)$ divides the join of $A(-2, 1)$ and $B(7, 4)$ is

(a) $1 : 2$ (b) $2 : 1$ (c) $3 : 2$ (d) $2 : 3$

26. The ratio in which the line segment joining the points $A(-12, 2)$ and $B(8, 3)$ is divided by the y-axis is

(a) $2 : 1$ (b) $1 : 2$ (c) $2 : 3$ (d) $3 : 2$

27. The ratio in which the line segment joining the points $P(4, 5)$ and $Q(-10, -2)$ is divided by the x-axis is

(a) $3 : 2$ (b) $2 : 3$ (c) $5 : 2$ (d) $2 : 5$

28. The vertices of $\triangle ABC$ are $A(0, 6)$, $B(8, 12)$ and $C(8, 0)$. The coordinates of its incentre are

(a) $(-4, 3)$ (b) $(5, 6)$ (c) $(8, 11)$ (d) $\left(\frac{16}{3}, 6\right)$

29. A circle passes through the points $A(2, -9)$, $B(5, -8)$ and $C(2, 1)$. The centre of the circle is

(a) $(2, -4)$ (b) $(-3, 4)$ (c) $\left(3, \frac{-16}{3}\right)$ (d) none of these

30. One end of the diameter of a circle is $A(4, 1)$ and its centre is $C(3, 3)$. The coordinates of the other end of the diameter is

(a) $B(5, 2)$ (b) $B(-4, 1)$ (c) $B(4, -1)$ (d) $B(2, 5)$

31. The slope of the line AB passing through the points $A(-2, 3)$ and $B(8, -5)$ is

(a) $\frac{4}{5}$ (b) $\frac{-4}{5}$ (c) $\frac{5}{4}$ (d) $\frac{-5}{4}$

32. The inclination of the line joining the points $A(x, -3)$ and $B(2, 5)$ is $135°$. Then, the value of x is

(a) 8 (b) -8 (c) 10 (d) -10

33. The angle between the x-axis and the line joining the points $A(3, -1)$ and $B(4, -2)$ is

(a) 45° (b) 90° (c) 135° (d) 150°

34. If the line AB with $A(-2, 6)$ and $B(4, 8)$ is perpendicular to the line CD with $C(8, 12)$ and $D(x, 24)$, then $x = ?$

(a) –4 (b) 4 (c) –6 (d) 6

35. If the points $A(x, -1)$, $B(2, 1)$ and $C(4, 5)$ are collinear, then $x = ?$

(a) 1 (b) –2 (c) –4 (d) –5

36. If $A(-2, 1)$, $B(2, 3)$ and $C(-2, -4)$ be the vertices of a $\triangle ABC$, then $\tan B = ?$

(a) $\frac{1}{3}$ (b) $\frac{3}{4}$ (c) $\frac{2}{3}$ (d) $\frac{4}{5}$

37. If the angle between two lines is $\frac{\pi}{4}$ and the slope of one line is $\frac{1}{2}$, then the slope of the other line is

(a) 2 or $\frac{-1}{2}$ (b) 3 or $\frac{-1}{3}$ (c) 4 or $\frac{-1}{4}$ (d) $\frac{1}{2}$ or $\frac{-1}{3}$

38. If θ is the angle between AB and CD with $A(0, 0)$, $B(2, 3)$, $C(2, -2)$ and $D(3, 5)$, then $\tan\theta = ?$

(a) $\frac{9}{13}$ (b) $\frac{11}{23}$ (c) $\frac{8}{17}$ (d) $\frac{10}{19}$

39. The slopes of two lines AB and CD are $(2-\sqrt{3})$ and $(2+\sqrt{3})$ respectively. The angle between these lines is

(a) 30° (b) 45° (c) 60° (d) 120°

40. If the slope of the line joining the points $A(x, 2)$ and $B(6, -8)$ is $\frac{-5}{4}$, then $x = ?$

(a) –2 (b) 2 (c) –3 (d) 3

41. If $A(3, x)$, $B(2, 7)$, $C(-1, 4)$ and $D(0, 6)$ are the points such that $AB \parallel CD$, then $x = ?$

(a) 6 (b) 8 (c) 9 (d) 12

42. If $A(-2, 6)$, $B(4, x)$, $C(3, -3)$ and $D(5, -9)$ are the points such that AB is perpendicular to CD, then $x = ?$

(a) 9 (b) 8 (c) 7 (d) 6

43. The vertices of a parallelogram $ABCD$ are $A(0, 2)$, $B(-2, -1)$, $C(4, 0)$ and $D(2, 3)$ and θ is the angle between the diagonals AC and BD. Then, $\tan\theta = ?$

(a) $\frac{3}{2}$ (b) $\frac{2}{3}$ (c) 2 (d) 3

44. If $A(-1, 8)$, $B(4, -2)$ and $C(-5, -3)$ are the vertices of a $\triangle ABC$, then equation of AB is

(a) $2x - y + 6 = 0$ (b) $2x + y - 6 = 0$ (c) $3x - y + 9 = 0$ (d) $3x + y - 9 = 0$

45. The vertices of a $\triangle ABC$ are $A(2, 5)$, $B(-4, 9)$ and $C(-2, -1)$. The equation of the median BE is

(a) $x - 5y + 23 = 0$ (b) $8x - y + 15 = 0$ (c) $7x + 4y - 8 = 0$ (d) none of these

46. A line passes through the point $P(0, 5)$ and its inclination with the x-axis is 30°. The equation of the line is

(a) $x + \sqrt{3}y - 5\sqrt{3} = 0$ (b) $x - \sqrt{3}y - 5\sqrt{3} = 0$
(c) $x - \sqrt{3}y + 5\sqrt{3} = 0$ (d) none of these

47. The equation of the perpendicular bisector of the line joining the points $A(2, 3)$ and $B(6, -5)$ is

(a) $x + 2y - 6 = 0$ (b) $x - 2y - 6 = 0$ (c) $x + 2y + 6$ (d) $x - 2y + 6 = 0$

48. A line passes through the points $A(1, \sqrt{3})$ and $B(\sqrt{2}, \sqrt{6})$. The angle which AB makes with the x-axis is

(a) 30° (b) 45° (c) 60° (d) 90°

49. If $A(2, 1)$, $B(-2, 3)$ and $C(4, 5)$ are the vertices of a $\triangle ABC$, then the equation of altitude through B is

(a) $x - 2y + 4 = 0$ (b) $x + 2y - 4 = 0$ (c) $x - 2y - 4 = 0$ (d) $x + 2y + 4 = 0$

50. If $A(1, 4)$, $B(2, -3)$ and $C(-1, -2)$ are the vertices of a $\triangle ABC$, then the equation of the right bisector of BC is

(a) $3x - y - 4 = 0$ (b) $3x + y - 4 = 0$ (c) $3x - y + 4 = 0$ (d) none of these

51. The equation of a line which is equidistant from the lines $x = -2$ and $x = 6$ is

(a) $x = 4$ (b) $x = 2$ (c) $x = 3$ (d) none of these

52. The equation of a line which is equidistant from the lines $y = 8$ and $y = -2$ is

(a) $y = 6$ (b) $y = 4$ (c) $y = 3$ (d) none of these

53. The equation of a line passing through the point $(3, -4)$ and parallel to the x-axis is

(a) $x - 4 = 0$ (b) $x + 4 = 0$ (c) $y - 4 = 0$ (d) $y + 4 = 0$

54. The equation of a line with slope $\frac{1}{2}$ and y-intercept $\frac{-5}{4}$ is

(a) $2x - 4y - 5 = 0$ (b) $2x - 4y + 5 = 0$ (c) $2x + 4y - 5 = 0$ (d) $2x + 4y + 5 = 0$

55. A line cuts the y-axis at a distance of 3 units from the origin and makes an angle of 30° with the positive direction of the x-axis. The equation of the line is

(a) $y - \sqrt{3}x + 3\sqrt{3} = 0$ (b) $x - \sqrt{3}y + 3\sqrt{3} = 0$
(c) $x + \sqrt{3}y - 3\sqrt{3} = 0$ (d) none of these

56. A line cuts off an intercept of 4 units on negative direction of the y-axis and makes an angle of 120° with the positive direction of the x-axis. The equation of the line is

(a) $\sqrt{3}x - y - 4 = 0$ (b) $\sqrt{3}x + y - 4 = 0$
(c) $\sqrt{3}x + y + 4 = 0$ (d) none of these

57. The equation of a line for which $\tan\theta = \frac{1}{3}$ and x-intercept = 5 is

(a) $x - 3y - 5 = 0$ (b) $x + 3y - 5 = 0$ (c) $x + 3y + 5 = 0$ (d) none of these

58. A line cuts the x-axis at a distance of 3 units to the left of the origin and has slope –2. The equation of the line is

(a) $2x - y + 6 = 0$ (b) $2x + y - 6 = 0$ (c) $2x + y + 6 = 0$ (d) $2x - y - 6 = 0$

59. The lines $2x + 3y + 7 = 0$ and $27x - 18y + 25 = 0$ are

(a) parallel to each other (b) coincident
(c) perpendicular to each other (d) none of these

60. The lines $x + 2y - 9 = 0$ and $2x + 4y + 5 = 0$ are

(a) parallel to each other (b) coincident
(c) perpendicular to each other (d) none of these

61. The angle made by the line $x + \sqrt{3}y - 6 = 0$ with the positive direction of the x-axis is

(a) 45° (b) 60° (c) 120° (d) 150°

62. A line passes through the point (2, –5) and is parallel to the line $2x - 3y = 7$. The equation of the line is

(a) $2x + 3y - 14 = 0$ (b) $2x - 3y - 21 = 0$
(c) $2x - 3y - 19 = 0$ (d) none of these

63. A line passes through the point (–2, –4) and is perpendicular to the line $3x - y + 5 = 0$. The equation of the line is

(a) $x + 3y + 15 = 0$ (b) $3x + y + 15 = 0$
(c) $x + 3y - 14 = 0$ (d) $x + 3y + 14 = 0$

64. The y-intercept of a line is –3 and it is perpendicular to the line $3x - 2y + 5 = 0$. The equation of the line is

(a) $2x + 3y + 9 = 0$ (b) $2x - 3y + 9 = 0$ (c) $2x - 3y - 9 = 0$ (d) none of these

65. The equation of the line perpendicular to the line $x - 7y + 5 = 0$ and having x-intercept 3 is

(a) $7x + y - 21 = 0$ (b) $x + 7y - 21 = 0$ (c) $7x + y - 15 = 0$ (d) none of these

66. The equation of the line making intercepts 2 and –3 on the x-axis and y-axis respectively is

(a) $3x + 2y - 6 = 0$ (b) $3x - 2y + 6 = 0$ (c) $3x - 2y - 6 = 0$ (d) none of these

67. The equation of the line so that the line segment intercepted between the axes is bisected at the point (2, 3), is

(a) $3x + 2y - 12 = 0$ (b) $2x + 3y - 12 = 0$ (c) $3x + 2y + 12 = 0$ (d) none of these

68. If the straight line $\frac{x}{a} + \frac{y}{b} = 1$ passes through the points $A(8, -9)$ and $B(12, -15)$, then

(a) $a = 3, b = 2$ (b) $a = 2, b = 3$ (c) $a = 4, b = 9$ (d) $a = 9, b = 4$

69. The equation of the line passing through the point $P(2, 2)$ and cutting off intercepts on the axis having sum 9 is

(a) $x + 2y - 6 = 0$ or $2x + y - 6 = 0$ (b) $x - 2y + 6 = 0$ or $2x + y + 6 = 0$
(c) $x - 2y - 6 = 0$ or $2x - y + 6 = 0$ (d) none of these

70. The equation of a line for which $p = 5$ and $\alpha = 135°$ is

(a) $x + y + 3\sqrt{2} = 0$ (b) $x - y + 5\sqrt{2} = 0$
(c) $x + y + 5\sqrt{2} = 0$ (d) none of these

71. The equation of a line for which $p = 8$ and $\alpha = 150°$ is

(a) $\sqrt{3}x - y + 8 = 0$ (b) $\sqrt{3}x + y - 16 = 0$
(c) $\sqrt{3}x - y + 16 = 0$ (d) none of these

72. The slope of the line $\sqrt{3}x + y + 2 = 0$ is

(a) $\frac{1}{\sqrt{3}}$ (b) $\frac{-1}{\sqrt{3}}$ (c) $\sqrt{3}$ (d) $-\sqrt{3}$

73. The perpendicular distance of a line from the origin is 5 units and the angle between the positive direction of the x-axis and the perpendicular is 30°. The equation of the line is

(a) $\sqrt{3}x + y - 10 = 0$ (b) $\sqrt{3}x - y + 10 = 0$
(c) $\sqrt{3}x - y - 10 = 0$ (d) none of these

74. The equation $3x - 2y + 4 = 0$ when reduced to intercept form takes the form $\frac{x}{a} + \frac{y}{b} = 1$, where

(a) $a = \frac{4}{3}, b = 2$ (b) $a = \frac{-4}{3}, b = -2$ (c) $a = \frac{-4}{3}, b = 2$ (d) none of these

75. The distance of the point $P(4, 1)$ from the line $3x - 4y + 12 = 0$ is

(a) 4 units (b) 5 units (c) 3 units (d) 6 units

76. The distance of the point $P(-1, 1)$ from the line $12x - 5y + 82 = 0$ is

(a) 8 units (b) 6 units (c) 5 units (d) 7 units

77. The length of perpendicular from the origin to the line $4x + 3y - 2 = 0$ is

(a) $\frac{2}{3}$ unit (b) $\frac{2}{5}$ unit (c) $\frac{4}{3}$ unit (d) $\frac{4}{5}$ unit

78. The distance between the parallel lines $3x - 4y + 9 = 0$ and $6x - 8y - 17 = 0$ is

(a) $\frac{5}{2}$ units (b) 3 units (c) $\frac{7}{2}$ units (d) 4 units

79. What are the points on the x-axis whose perpendicular distance from the line $\frac{x}{3} + \frac{y}{4} = 1$ is 4 units?

(a) (8, 0) and (2, 0) (b) (–8, 0) and (2, 0)
(c) (8, 0) and (–2, 0) (d) (–8, 0) and (–2, 0)

80. The perpendicular distance of a line from the origin is 5 units and its slope is –1. The equation of the line is

(a) $x - y + 5\sqrt{2} = 0$ or $x - y - 5\sqrt{2} = 0$ (b) $x + y + 5\sqrt{2} = 0$ or $x + y - 5\sqrt{2} = 0$

(c) $x - y + 2\sqrt{5} = 0$ or $x - y - 2\sqrt{5} = 0$ (d) none of these

81. The distance between the parallel lines $p(x+y)+q=0$ and $p(x+y)-r=0$ is

(a) $\frac{|q+r|}{2p}$ (b) $\frac{|q+r|}{\sqrt{2}p}$ (c) $\frac{|q-r|}{2p}$ (d) $\frac{|q-r|}{\sqrt{2}p}$

82. The point of intersection of the lines $5x + 7y = 3$ and $2x - 3y = 7$ is

(a) (2, –1) (b) (–2, 1) (c) (–2, –1) (d) (2, 1)

83. The equation of the line parallel to the y-axis and drawn through the point of intersection of the lines $x - 7y + 15 = 0$ and $2x + y = 0$ is

(a) $x - 1 = 0$ (b) $x + 1 = 0$ (c) $x - 2 = 0$ (d) $x + 2 = 0$

84. The equation of the line passing through the point of intersection of the lines $x + 2y + 3 = 0$ and $3x + 4y + 7 = 0$ and parallel to the line $y - x = 6$ is

(a) $x - y = 0$ (b) $x + y = 1$ (c) $x + 2y - 3 = 0$ (d) $2x + y - 3 = 0$

85. The image of the point $P(3, 8)$ in the line $x + 3y - 7 = 0$ is

(a) (1, 4) (b) (–1, 4) (c) (1, –4) (d) (–1, –4)

86. The point of intersection of the lines $x - y = 6$, $4x - 3y = 20$ and $6x + 5y + 8 = 0$ is

(a) (2, 4) (b) (2, –4) (c) (–2, 4) (d) (–2, –4)

87. The area of the triangle formed by the lines $y - x = 0$, $x + y = 0$ and $x - c = 0$ is

(a) $\frac{1}{2}c^2$ sq units (b) $2c^2$ sq units (c) c^2 sq units (d) $3c^2$ sq units

88. If the lines $3x + y = 2$, $kx + 2y = 3$ and $2x - y = 3$ are concurrent, then $k = ?$

(a) –5 (b) 5 (c) –3 (d) 3

89. The centre of the circle $x^2 + y^2 - 6x + 4y - 12 = 0$ is

(a) (–3, 2) (b) (3, 2) (c) (3, –2) (d) (–3, –2)

90. Radius of the circle $x^2 + y^2 - 4x + 2y - 45 = 0$ is

(a) $5\sqrt{2}$ units (b) $4\sqrt{2}$ units (c) $3\sqrt{5}$ units (d) $4\sqrt{5}$ units

91. The end points of a diameter of a circle are $A(2, -3)$ and $B(-3, 5)$. The equation of the circle is

(a) $x^2 + y^2 + 2x - y - 21 = 0$ (b) $x^2 + y^2 + x - 2y - 21 = 0$

(c) $x^2 + y^2 + x - 2y + 21 = 0$ (d) none of these

92. The centre of a circle is $C(2, -5)$ and the circle passes through the point $A(3, 2)$. The equation of the circle is

(a) $x^2 + y^2 - 4x + 10y - 21 = 0$ (b) $x^2 + y^2 + 4x + 6y - 21 = 0$

(c) $x^2 + y^2 + 4x - 10y + 21 = 0$ (d) none of these

93. For the circle $(x+5)^2+(y-3)^2=20$, the centre and radius are respectively
(a) $(5, -3), 2\sqrt{5}$ (b) $(-5, 3), 5\sqrt{2}$ (c) $(-5, 3), 2\sqrt{5}$ (d) none of these

94. If $A(-1, 3)$ and $B(\alpha, \beta)$ be the extremities of the diameter of the circle $x^2+y^2-6x+5y-7=0$, then
(a) $\alpha=-7, \beta=8$ (b) $\alpha=7, \beta=-8$ (c) $\alpha=-6, \beta=7$ (d) $\alpha=6, \beta=-7$

95. For the parabola $y^2=8x$, the focus and vertex are respectively
(a) $F(2, 0)$ and $O(0, 0)$ (b) $F(-2, 0)$ and $O(0, 0)$
(c) $F(4, 0)$ and $O(0, 0)$ (d) $F(-4, 0)$ and $O(0, 0)$

96. In the parabola $y^2=-12x$, the focus and the equation of directrix are respectively
(a) $F(3, 0), x=-3$ (b) $F(-3, 0), x=3$
(c) $F(-3, 0), x=-3$ (d) none of these

97. For the parabola $y^2=-8x$, the focus and the equation of directrix are respectively
(a) $F(-2, 0), x=2$ (b) $F(2, 0), x=-2$
(c) $F(2, 0), x=2$ (d) $F(-2, 0), x=-2$

98. For the parabola $x^2=-16y$, the focus and the equation of directrix are respectively
(a) $F(0, 4), y=4$ (b) $F(0, -4), y=4$
(c) $F(0, 4), y=-4$ (d) none of these

99. For the parabola $x^2=6y$, the focus and the equation of directrix are respectively
(a) $F\left(0, \frac{-3}{2}\right), y=\frac{3}{2}$ (b) $F\left(0, \frac{3}{2}\right), y=\frac{3}{2}$
(c) $F\left(0, \frac{3}{2}\right), y=\frac{-3}{2}$ (d) none of these

100. If the parabola $y^2=4ax$ passes through the point $P(3, 2)$, then the length of its latus rectum is
(a) $\frac{1}{3}$ (b) $\frac{2}{3}$ (c) $\frac{4}{3}$ (d) 4

101. If $A(2, 0)$ is the vertex and the y-axis is the directrix of a parabola, then its focus is
(a) $F(2, 0)$ (b) $F(-2, 0)$ (c) $F(4, 0)$ (d) $F(-4, 0)$

102. An equilateral triangle is inscribed in a parabola $y^2=4ax$, whose vertex is at the vertex of the parabola. The length of each side of the triangle is
(a) $2a\sqrt{3}$ (b) $4a\sqrt{3}$ (c) $6a\sqrt{3}$ (d) $8a\sqrt{3}$

103. In the parabola $y^2 = 4ax$, the length of the chord passing through the vertex and inclined to the axis at an angle $\left(\frac{\pi}{4}\right)$ is

(a) $\sqrt{2}a$ (b) $2a\sqrt{2}$ (c) $2a$ (d) $4a\sqrt{2}$

104. The focal distance of a point P on the parabola $y^2 = 12x$ is 4. The abscissa of P is

(a) –1 (b) 1 (c) –2 (d) 2

105. The equation of a parabola with vertex at $O(0, 0)$ and focus at $F(0, 2)$ is

(a) $y^2 = 8x$ (b) $x^2 = 8y$ (c) $y^2 = 4x$ (d) $x^2 = 4y$

106. A parabola has its vertex at the origin; its axis lies along the x-axis and it passes through the point $P(2, 3)$. The equation of the parabola is

(a) $4y^2 = 9x$ (b) $y^2 = \frac{4}{9}x$ (c) $y^2 = \frac{9}{2}x$ (d) $y^2 = \frac{3}{2}x$

107. A parabola has its vertex at the origin and passes through the point $P(3, -4)$ and it is symmetric about the y-axis. Its equation is

(a) $4x^2 + 9y = 0$ (b) $4x + 9y^2 = 0$ (c) $9x^2 + 4y = 0$ (d) none of these

108. The equation of an ellipse is $\frac{x^2}{16} + \frac{y^2}{9} = 1$.

Fill in the blanks given below:

(i) Length of the major axis = …… units.

(ii) Length of the minor axis = …… units.

(iii) Coordinates of its vertices are A(……, ……) and B(……, ……).

(iv) Coordinates of its foci are F_1(……, ……) and F_2(……, ……).

(v) Eccentricity, e = …… .

(vi) Length of the latus rectum = …… units.

109. The equation of an ellipse is $4x^2 + 9y^2 = 144$.

Fill in the blanks given below:

(i) Length of the major axis = …… units.

(ii) Length of the minor axis = …… units.

(iii) Coordinates of its vertices are A(……, ……) and B(……, ……).

(iv) Coordinates of its foci are F_1(……, ……) and F_2(……, ……).

(v) Eccentricity, e = …… .

(vi) Length of the latus rectum = …… units.

110. The equation of an ellipse is $\frac{x^2}{4} + \frac{y^2}{36} = 1$.

Fill in the blanks given below:

(i) Length of the major axis = …… units.

(ii) Length of the minor axis = …… units.

(iii) Coordinates of its vertices are $A(\ldots\ldots, \ldots\ldots)$ and $B(\ldots\ldots, \ldots\ldots)$.
(iv) Coordinates of its foci are $F_1(\ldots\ldots, \ldots\ldots)$ and $F_2(\ldots\ldots, \ldots\ldots)$.
(v) Eccentricity, $e = \ldots\ldots$.
(vi) Length of the latus rectum = …… units.

111. The equation of an ellipse is $4x^2 + y^2 = 100$.
Fill in the blanks given below:
(i) Length of the major axis = …… units.
(ii) Length of the minor axis = …… units.
(iii) Coordinates of its vertices are $A(\ldots\ldots, \ldots\ldots)$ and $B(\ldots\ldots, \ldots\ldots)$.
(iv) Coordinates of its foci are $F_1(\ldots\ldots, \ldots\ldots)$ and $F_2(\ldots\ldots, \ldots\ldots)$.
(v) Eccentricity, $e = \ldots\ldots$.
(vi) Length of the latus rectum = …… units.

Mark (✓) against the correct answer in each of the following.

112. The vertices of an ellipse are $(\pm 5, 0)$ and its foci are $(\pm 4, 0)$. The equation of the ellipse is

(a) $\frac{x^2}{25} + \frac{y^2}{9} = 1$ (b) $\frac{x^2}{9} + \frac{y^2}{25} = 1$ (c) $\frac{x^2}{25} + \frac{y^2}{16} = 1$ (d) $\frac{x^2}{16} + \frac{y^2}{25} = 1$

113. The foci of an ellipse are $(\pm 4, 0)$ and its eccentricity is $\frac{1}{3}\cdot$ The equation of the ellipse is

(a) $\frac{x^2}{16} + \frac{y^2}{9} = 1$ (b) $\frac{x^2}{144} + \frac{y^2}{128} = 1$ (c) $\frac{x^2}{144} + \frac{y^2}{81} = 1$ (d) none of these

114. The foci of an ellipse are $(0, \pm 5)$ and its vertices are $(0, \pm 13)$. The equation of the ellipse is

(a) $\frac{x^2}{169} + \frac{y^2}{144} = 1$ (b) $\frac{x^2}{169} + \frac{y^2}{25} = 1$ (c) $\frac{x^2}{144} + \frac{y^2}{169} = 1$ (d) none of these

115. The foci of an ellipse are $(0, \pm 6)$ and the length of its minor axis is 16. The equation of the ellipse is

(a) $\frac{x^2}{16} + \frac{y^2}{36} = 1$ (b) $\frac{x^2}{36} + \frac{y^2}{64} = 1$ (c) $\frac{x^2}{100} + \frac{y^2}{64} = 1$ (d) $\frac{x^2}{64} + \frac{y^2}{100} = 1$

116. The foci of an ellipse are $(0, \pm 5)$ and the length of its major axis is 20. The equation of the ellipse is

(a) $\frac{x^2}{25} + \frac{y^2}{40} = 1$ (b) $\frac{x^2}{25} + \frac{y^2}{100} = 1$ (c) $\frac{x^2}{75} + \frac{y^2}{100} = 1$ (d) none of these

117. The vertices of an ellipse are $(0, \pm 10)$ and $e = \frac{4}{5}\cdot$ The equation of the ellipse is

(a) $\frac{x^2}{100} + \frac{y^2}{36} = 1$ (b) $\frac{x^2}{36} + \frac{y^2}{100} = 1$ (c) $\frac{x^2}{36} + \frac{y^2}{64} = 1$ (d) none of these

118. The equation of a hyperbola is $\frac{x^2}{36}-\frac{y^2}{64}=1.$

Fill in the blanks given below:

(i) Length of its transverse axis = ... units.

(ii) Length of its conjugate axis = ... units.

(iii) Coordinates of its vertices are A(......, ...) and B(......, ...).

(iv) Coordinates of its foci are F_1(......, ...) and F_2(......, ...).

(v) Eccentricity, e = ...

(vi) Length of the latus rectum = ... units.

119. The equation of a hyperbola is $9x^2-16y^2=144.$
Fill in the blanks given below:

(i) Length of its transverse axis = ... units.

(ii) Length of its conjugate axis = ... units.

(iii) Coordinates of its vertices are A(......, ...) and B(......, ...).

(iv) Coordinates of its foci are F_1(......, ...) and F_2(......, ...).

(v) Eccentricity, e = ...

(vi) Length of the latus rectum = ... units.

Mark (✓) against the correct answer in each of the following.

120. The vertices of a hyperbola are $(\pm 2, 0)$ and its foci are $(\pm 3, 0)$. The equation of the hyperbola is

(a) $\frac{x^2}{2}-\frac{y^2}{3}=1$ (b) $\frac{x^2}{3}-\frac{y^2}{4}=1$ (c) $\frac{x^2}{4}-\frac{y^2}{5}=1$ (d) none of these

121. The foci of a hyperbola are $(\pm 5, 0)$ and its transverse axis is of length 8. The equation of the hyperbola is

(a) $\frac{x^2}{16}-\frac{y^2}{9}=1$ (b) $\frac{x^2}{9}-\frac{y^2}{16}=1$ (c) $\frac{x^2}{25}-\frac{y^2}{16}=1$ (d) $\frac{x^2}{16}-\frac{y^2}{25}=1$

122. The foci of a hyperbola are $(\pm 4, 0)$ and the length of its latus rectum is 12 units. The equation of the hyperbola is

(a) $\frac{x^2}{4}-\frac{y^2}{6}=1$ (b) $\frac{x^2}{4}-\frac{y^2}{12}=1$ (c) $\frac{x^2}{4}-\frac{y^2}{8}=1$ (d) none of these

123. The equation of a hyperbola is $\frac{y^2}{4}-\frac{x^2}{9}=1.$

Fill in the blanks given below:

(i) Length of its transverse axis = ... units.

(ii) Length of its conjugate axis = ... units.

(iii) The coordinates of its vertices are A(......, ...) and B(......, ...).

(iv) The coordinates of its foci are F_1(......, ...) and F_2(......, ...).

(v) Its eccentricity, $e = \ldots\ldots$.

(vi) Length of its latus rectum = …… units.

Mark (✓) against the correct answer in each of the following.

124. The vertices of a hyperbola are $(0, \pm 3)$ and its foci are $(0, \pm 5)$. The equation of the hyperbola is

(a) $\frac{y^2}{16} - \frac{x^2}{9} = 1$ (b) $\frac{y^2}{9} - \frac{x^2}{16} = 1$ (c) $\frac{y^2}{9} - \frac{x^2}{25} = 1$ (d) none of these

125. The vertices of a hyperbola are $(0, \pm 6)$ and its eccentricity is $\frac{5}{3}\cdot$ The equation of the hyperbola is

(a) $\frac{y^2}{36} - \frac{x^2}{64} = 1$ (b) $\frac{y^2}{36} - \frac{x^2}{25} = 1$ (c) $\frac{x^2}{36} - \frac{y^2}{64} = 1$ (d) none of these

126. One focus of a hyperbola is at (0, 4) and the length of its transverse axis is 6. The equation of the hyperbola is

(a) $\frac{x^2}{7} - \frac{y^2}{9} = 1$ (b) $\frac{y^2}{9} - \frac{x^2}{7} = 1$ (c) $\frac{y^2}{4} - \frac{x^2}{9} = 1$ (d) none of these

127. Foci of a hyperbola are $(0, \pm 6)$ and the length of its conjugate axis is $2\sqrt{11}$. The equation of the hyperbola is

(a) $\frac{y^2}{36} - \frac{x^2}{11} = 1$ (b) $\frac{x^2}{36} - \frac{y^2}{11} = 1$ (c) $\frac{y^2}{25} - \frac{x^2}{11} = 1$ (d) none of these

ANSWERS (EXERCISE 1)

1. (d)	**2.** (a)	**3.** (b)	**4.** (b)	**5.** (d)	**6.** (c)	**7.** (a)	**8.** (b)	**9.** (c)	**10.** (c)
11. (d)	**12.** (c)	**13.** (c)	**14.** (c)	**15.** (a)	**16.** (c)	**17.** (a)	**18.** (b)	**19.** (a)	**20.** (b)
21. (a)	**22.** (a)	**23.** (b)	**24.** (d)	**25.** (b)	**26.** (d)	**27.** (c)	**28.** (b)	**29.** (a)	**30.** (d)
31. (b)	**32.** (c)	**33.** (c)	**34.** (b)	**35.** (a)	**36.** (c)	**37.** (b)	**38.** (b)	**39.** (c)	**40.** (a)
41. (c)	**42.** (b)	**43.** (d)	**44.** (b)	**45.** (c)	**46.** (c)	**47.** (b)	**48.** (c)	**49.** (b)	**50.** (a)
51. (b)	**52.** (c)	**53.** (d)	**54.** (a)	**55.** (b)	**56.** (c)	**57.** (a)	**58.** (c)	**59.** (c)	**60.** (a)
61. (d)	**62.** (c)	**63.** (d)	**64.** (a)	**65.** (a)	**66.** (c)	**67.** (a)	**68.** (b)	**69.** (a)	**70.** (b)
71. (c)	**72.** (d)	**73.** (a)	**74.** (c)	**75.** (a)	**76.** (c)	**77.** (b)	**78.** (c)	**79.** (c)	**80.** (b)
81. (b)	**82.** (a)	**83.** (b)	**84.** (a)	**85.** (d)	**86.** (b)	**87.** (c)	**88.** (b)	**89.** (c)	**90.** (a)
91. (b)	**92.** (a)	**93.** (c)	**94.** (b)	**95.** (a)	**96.** (b)	**97.** (a)	**98.** (b)	**99.** (c)	**100.** (c)
101. (c)	**102.** (d)	**103.** (d)	**104.** (b)	**105.** (b)	**106.** (c)	**107.** (a)			

108. (i) 8 (ii) 6 (iii) $A(-4, 0)$ and $B(4, 0)$ (iv) $F_1(-\sqrt{7}, 0)$, $F_2(\sqrt{7}, 0)$ (v) $e = \frac{\sqrt{7}}{4}$

(vi) 6

109. (i) 6 (ii) 4 (iii) $A(-3, 0)$ and $B(3, 0)$ (iv) $F_1(\sqrt{5}, 0)$, $F_2(-\sqrt{5}, 0)$ (v) $e = \frac{\sqrt{5}}{3}$

(vi) $\frac{8}{3}$

110. (i) 12 (ii) 4 (iii) $A(0, -6)$ and $B(0, 6)$ (iv) $F_1(0, -4\sqrt{2})$, $F_2(0, 4\sqrt{2})$

(v) $e = \frac{2\sqrt{2}}{3}$ (vi) $\frac{4}{3}$

111. (i) 20 (ii) 10 (iii) $A(0, -10)$ and $B(0, 10)$ (iv) $F_1(0, -5\sqrt{3})$, $F_2(0, 5\sqrt{3})$

(v) $e = \frac{\sqrt{3}}{2}$ (vi) 5

112. (a) **113.** (b) **114.** (c) **115.** (d) **116.** (c) **117.** (b)

118. (i) 12 (ii) 16 (iii) $A(-6, 0)$ and $B(6, 0)$ (iv) $F_1(-10, 0)$, $F_2(10, 0)$ (v) $e = \frac{5}{3}$

(vi) $\frac{64}{3}$

119. (i) 8 (ii) 6 (iii) $A(-4, 0)$ and $B(4, 0)$ (iv) $F_1(-5, 0)$, $F_2(5, 0)$ (v) $e = \frac{5}{4}$

(vi) $\frac{9}{2}$

120. (c) **121.** (a) **122.** (b)

123. (i) 4 (ii) 6 (iii) $A(0, -2)$ and $B(0, 2)$ (iv) $F_1(0, -\sqrt{13})$, $F_2(0, \sqrt{13})$

(v) $e = \frac{\sqrt{13}}{2}$ (vi) 9

124. (b) **125.** (a) **126.** (b) **127.** (c)

HINTS TO SOME SELECTED QUESTIONS

1. $AB^2 = (-6-2)^2 + (3+3)^2 = (-8)^2 + 6^2 = (64+36) = 100$

$\Rightarrow \quad AB = \sqrt{100}$ units $= 10$ units.

2. $PQ^2 = \{(\sqrt{5}-2) - (\sqrt{5}+1)\}^2 + \{(\sqrt{3}+2) - (\sqrt{3}-1)\}^2 = (-3)^2 + 3^2 = (9+9) = 18$

$\Rightarrow \quad PQ = \sqrt{18}$ units $= 3\sqrt{2}$ units.

3. $AB^2 = 5^2 \Rightarrow (x+3)^2 + (7-4)^2 = 25$

$\Rightarrow x^2 + 6x + 9 + 9 = 25 \Rightarrow x^2 + 6x - 7 = 0$

$\Rightarrow (x+7)(x-1) = 0 \Rightarrow x = 1$ or -7.

4. $AB^2 = 4^2 \Rightarrow (3-x)^2 + (5-3)^2 = 4^2$

$\Rightarrow x^2 - 6x + 9 + 4 = 16 \Rightarrow x^2 - 6x - 3 = 0$

$\Rightarrow x = \frac{6 \pm \sqrt{36+12}}{2} = \frac{6 \pm \sqrt{48}}{2} = \frac{6 \pm 4\sqrt{3}}{2} = 3 \pm 2\sqrt{3}$.

5. $OA^2 = (6-0)^2 + (-6-0)^2 = 6^2 + (-6)^2 = (36+36) = 72$

$\Rightarrow \quad OA = \sqrt{72} = 6\sqrt{2}$ units.

6. $AP^2 = BP^2 \Rightarrow (x-6)^2 + (y+1)^2 = (x-2)^2 + (y-3)^2$

$\Rightarrow x^2 - 12x + 36 + y^2 + 2y + 1 = x^2 - 4x + 4 + y^2 - 6y + 9$

$\Rightarrow 37 - 12x + 2y = 13 - 4x - 6y \Rightarrow 8x - 8y = 24 \Rightarrow x - y = 3.$

7. Let the required point be $P(x, 0)$. Then,

$AP^2 = BP^2 \Rightarrow (x-7)^2 + (6-0)^2 = (x+3)^2 + (0-4)^2$

$\Rightarrow x^2 - 14x + 49 + 36 = x^2 + 6x + 9 + 16$

$\Rightarrow 20x = (85-25) = 60 \Rightarrow x = 3.$

$\therefore$ the point is $P(3, 0)$.

8. Let the required point be $P(0, y)$. Then,

$AP^2 = BP^2 \Rightarrow (-4-0)^2 + (3-y)^2 = (5-0)^2 + (2-y)^2$

$\Rightarrow (-4)^2 + (3-y)^2 = 5^2 + (2-y)^2$

$\Rightarrow 16 + 9 - 6y + y^2 = 25 + 4 + y^2 - 4y \Rightarrow 2y = -4 \Rightarrow y = -2.$

Hence, the required point is $P(0, -2)$.

9. The given points are $A(-8, 0)$ and $B(0, 15)$.

$\therefore \quad AB^2 = (0+8)^2 + (15-0)^2 = 8^2 + (15)^2 = (64+225) = 289$

$\Rightarrow \quad AB = \sqrt{289}$ units $= 17$ units.

10. Here, $(x_1 = -2, y_1 = 3)$; $(x_2 = 1, y_2 = 2)$ and $(x_3 = k, y_3 = 0)$.

$\text{ar}(\triangle ABC) = 0 \Rightarrow x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2) = 0$

$\Rightarrow -2(2-0) + 1\cdot(0-3) + k\cdot(3-2) = 0 \Rightarrow -4 - 3 + k = 0 \Rightarrow k = 7.$

11. Here, $(x_1 = 3, y_1 = 8)$, $(x_2 = -4, y_2 = 2)$ and $(x_3 = 5, y_3 = -1)$.

$\therefore \quad \text{ar}(\triangle ABC) = \frac{1}{2}\{x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)\}$

$= \frac{1}{2}\{3(2+1) - 4(-1-8) + 5(8-2)\} = \frac{1}{2}(9 + 36 + 30)$

$= \frac{75}{2}$ sq units $= 37\frac{1}{2}$ sq units.

12. Area of the $\triangle ABC = \frac{1}{2}\cdot[1\cdot(-3-2) + 7\cdot(2-1) + 12\cdot(1+3)] = 25$ sq units,

area of the $\triangle BCD = \frac{1}{2}\cdot[7\cdot(2-21) + 12\cdot(21+3) + 7\cdot(-3-2)] = 60$ sq units.

$\therefore$ area of the quad. $ABCD = \text{ar}(\triangle ABC) + \text{ar}(\triangle BCD) = 85$ sq units.

13. Midpoint of BC is $D\left(\frac{1+5}{2}, \frac{-1+1}{2}\right)$, i.e., $D(3, 0)$.

$\therefore \quad$ median $AD = \sqrt{(-1-3)^2 + (3-0)^2} = \sqrt{(-4)^2 + 3^2} = \sqrt{16+9} = \sqrt{25}$ units $= 5$ units.

14. Midpoint of AC is $E\left(\frac{7+5}{2}, \frac{-3+3}{2}\right)$, i.e., $E(6, 0)$.

$\therefore \quad$ median $BE = \sqrt{(6-3)^2 + (0+1)^2} = \sqrt{3^2 + 1^2} = \sqrt{9+1} = \sqrt{10}$ units.

15. Coordinates of centroid are $G\left(\dfrac{-1+5+8}{3}, \dfrac{0-2+2}{3}\right)$, i.e., $G(4, 0)$.

16. Let the 3rd vertex be $C(x, y)$. Then,

centroid is $G\left(\dfrac{5-7+x}{3}, \dfrac{2-4+y}{3}\right)$, i.e., $\left(\dfrac{x-2}{3}, \dfrac{y-2}{3}\right)$.

$\therefore \quad \dfrac{x-2}{3} = -2$ and $\dfrac{y-2}{3} = 3 \Rightarrow x = (-6+2) = -4, y = (9+2) = 11$.

Hence, the 3rd vertex of $\triangle ABC$ is $C(-4, 11)$.

17. $AB^2 = (-1-1)^2 + (-1-1)^2 = (-2)^2 + (-2)^2 = (4+4) = 8;$

$BC^2 = (-1+\sqrt{3})^2 + (-1-\sqrt{3})^2 = (1+3-2\sqrt{3}) + (1+3+2\sqrt{3}) = 8;$

$AC^2 = (-\sqrt{3}-1)^2 + (\sqrt{3}-1)^2 = (3+1+2\sqrt{3}) + (3+1-2\sqrt{3}) = 8.$

$\therefore \quad AB = BC = AC = \sqrt{8}$.

Hence, $\triangle ABC$ is equilateral.

18. $AB^2 = (2a-2a)^2 + (6a-4a)^2 = 0^2 + (2a)^2 = 4a^2;$

$BC^2 = (2a+\sqrt{3}a-2a)^2 + (5a-6a)^2 = (\sqrt{3}a)^2 + (-a)^2 = (3a^2+a^2) = 4a^2;$

$AC^2 = (2a+\sqrt{3}a-2a)^2 + (5a-4a)^2 = (\sqrt{3}a)^2 + a^2 = (3a^2+a^2) = 4a^2.$

$\therefore \quad AB = BC = AC = 2a$.

Hence, $\triangle ABC$ is an equilateral triangle.

19. $AB^2 = (7+2)^2 + (10-5)^2 = (9^2+5^2) = (81+25) = 106;$

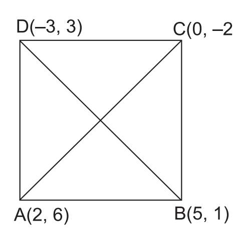
$BC^2 = (-2-3)^2 + (5+4)^2 = (-5)^2 + 9^2 = (25+81) = 106;$

$AC^2 = (7-3)^2 + (10+4)^2 = 4^2 + (14)^2 = (16+196) = 212.$

$\therefore \quad AB^2 + BC^2 = AC^2$ and $AB = BC \neq AC$.

Hence, $\triangle ABC$ is an isosceles right triangle.

20. $AB^2 = (5-2)^2 + (1-6)^2 = 3^2 + (-5)^2 = (9+25) = 34;$



$BC^2 = (0-5)^2 + (-2-1)^2 = (-5)^2 + (-3)^2 = (25+9) = 34;$

$CD^2 = (-3-0)^2 + (3+2)^2 = (-3)^2 + 5^2 = (9+25) = 34;$

$AD^2 = (-3-2)^2 + (3-6)^2 = (-5)^2 + (-3)^2 = (25+9) = 34.$

$\therefore \quad AB = BC = CD = AD = \sqrt{34}$.

Diag. $AC^2 = (0-2)^2 + (-2-6)^2$

$= (-2)^2 + (-8)^2 = (4+64) = 68.$

Diag. $BD^2 = (-3-5)^2 + (3-1)^2 = (-8)^2 + 2^2 = (64+4) = 68.$

$\therefore \quad$ diagonal AC = diagonal BD.

Hence, $ABCD$ is a square.

21. $AB^2 = (0+1)^2 + (3-0)^2 = (1^2+3^2) = (1+9) = 10;$

$BC^2 = (1-0)^2 + (3-3)^2 = (1^2+0^2) = (1+0) = 1;$

$CD^2 = (0-1)^2 + (0-3)^2 = (-1)^2 + (-3)^2 = (1+9) = 10;$

$AD^2 = (-1-0)^2 + (0-0)^2 = (-1)^2 + 0^2 = (1+0) = 1.$

$\therefore \quad AB = CD = \sqrt{10}$ and $BC = AD = \sqrt{1} = 1$.

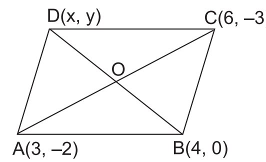
Diag. $AC^2 = (1+1)^2 + (3-0)^2 = 2^2 + 3^2 = (4+9) = 13;$

Diag. $BD^2 = (-3-5)^2 + (3-1)^2 = (-8)^2 + 2^2 = (64+4) = 68.$

$\therefore$ diag. $AC = \sqrt{13}$ and diag. $BD = \sqrt{68}$.

Hence, $ABCD$ is a ||gm.

22. Let the 4th vertex be $D(x, y)$. Then,



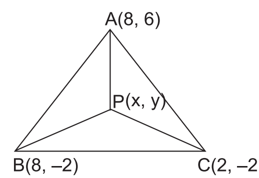
midpoint of AC = midpoint of BD.

$\therefore \quad \dfrac{4+x}{2} = \dfrac{3+6}{2}$ and $\dfrac{0+y}{2} = \dfrac{-2-3}{2}$

$\Rightarrow \quad x = (9-4) = 5$ and $y = -5$

$\Rightarrow$ point D is $D(5, -5)$.

23. Let the circumcentre be $P(x, y)$.



$PA^2 = PB^2 \Rightarrow (x-8)^2 + (y-6)^2 = (x-8)^2 + (y+2)^2$

$\Rightarrow (y-6)^2 - (y+2)^2 = 0$

$\Rightarrow (y-6+y+2)(y-6-y-2) = 0$

$\Rightarrow -8(2y-4) = 0 \Rightarrow 2y-4 = 0 \Rightarrow y = 2.$

$PB^2 = PC^2 \Rightarrow (x-8)^2 + (y+2)^2 = (x-2)^2 + (y+2)^2$

$\Rightarrow (x-8)^2 - (x-2)^2 = 0 \Rightarrow (x-8+x-2)(x-8-x+2) = 0$

$\Rightarrow (2x-10)(-6) = 0 \Rightarrow 2x-10 = 0 \Rightarrow x = 5.$

$\therefore$ the required point is $P(5, 2)$.

24. Coordinates of P are $P(x, y)$, where

$$x = \frac{3\times 9 + 1\times 5}{3+1} = \frac{32}{4} = 8, y = \frac{3\times 6 + 1\times(-2)}{3+1} = \frac{16}{4} = 4.$$

$\therefore$ the required point is $P(8, 4)$.

25. Let the required ratio be $\lambda : 1$. Then,

$$\frac{-2\lambda+7}{\lambda+1} = 1 \Rightarrow -2\lambda+7 = \lambda+1 \Rightarrow 3\lambda = 6 \Rightarrow \lambda = 2.$$

$\therefore$ the required ratio is $2:1$.

26. Let the required ratio be $k:1$.

Then, abscissa of the point lying on the y-axis is 0.

$\therefore \quad \dfrac{8k-12}{k+1} = 0 \Rightarrow 8k-12 = 0 \Rightarrow 8k = 12 \Rightarrow k = \dfrac{3}{2}\cdot$

$\therefore$ the required ratio $= \dfrac{3}{2} : 1 = 3:2.$

27. Let the required ratio be $k:1$.

Then, the ordinate of the point lying on the x-axis is 0.

$\therefore \quad \dfrac{-2k+5}{k+1} = 0 \Rightarrow -2k+5 = 0 \Rightarrow 2k = 5 \Rightarrow k = \dfrac{5}{2}\cdot$

$\therefore$ the required ratio is $\dfrac{5}{2} : 1$, i.e., $5:2$.

28. $a = BC = \sqrt{(8-8)^2 + (12-0)^2} = \sqrt{0+144} = \sqrt{144} = 12.$

$b = AC = \sqrt{(8-0)^2 + (0-6)^2} = \sqrt{8^2 + (-6)^2} = \sqrt{64+36} = \sqrt{100} = 10.$

$c = AB = \sqrt{(8-0)^2 + (12-6)^2} = \sqrt{8^2 + 6^2} = \sqrt{64 + 36} = \sqrt{100} = 10.$

Incentre is $\left(\dfrac{ax_1 + bx_2 + cx_3}{a+b+c}, \dfrac{ay_1 + by_2 + cy_3}{a+b+c} \right)$

$$\left(\frac{12 \times 0 + 10 \times 8 + 10 \times 8}{12 + 10 + 10}, \frac{12 \times 6 + 10 \times 12 + 10 \times 0}{12 + 10 + 10} \right) = (5, 6).$$

29. Let $O(x, y)$ be the centre of the circle. Then,

$$OA^2 = OB^2 = OC^2 \Rightarrow (x-2)^2 + (y+9)^2 = (x-5)^2 + (y+8)^2$$

$$\Rightarrow -4x + 4 + 18y + 81 = -10x + 25 + 16y + 64$$

$$\Rightarrow 6x + 2y = 4 \Rightarrow 3x + y = 2 \qquad \text{... (i)}$$

And, $(x-2)^2 + (y+9)^2 = (x-2)^2 + (y-1)^2$

$\Rightarrow \quad -4x + 4 + 18y + 81 = -4x + 4 + 1 - 2y$

$\Rightarrow \quad 20y = -80 \Rightarrow y = -4.$

Putting $y = -4$ in (i), we get $x = 2$.

Hence, the centre is $O(2, -4)$.

30. Let the other end of the dimeter be $B(x, y)$.

Then, $\dfrac{4+x}{2} = 3$ and $\dfrac{1+y}{2} = 3 \Rightarrow x = 2, y = 5.$

$\therefore \quad$ the other end of the diameter is $B(2, 5)$.

31. Slope of $AB = \dfrac{(y_2 - y_1)}{(x_2 - x_1)} = \dfrac{(-5-3)}{(8+2)} = \dfrac{-8}{10} = \dfrac{-4}{5}.$

32. Slope of $AB = \dfrac{5-(-3)}{2-x} = \dfrac{8}{(2-x)}.$

$\therefore \quad \dfrac{8}{2-x} = \tan 135° = \tan(180° - 45°) = -\tan 45° = -1$

$\Rightarrow \quad x - 2 = 8 \Rightarrow x = 10.$

33. Slope, $m = \dfrac{-2-(-1)}{4-3} = \dfrac{-2+1}{1} = -1.$

$\tan\theta = m = -1 = -\tan 45° = \tan(180° - 45°) = \tan 135° \Rightarrow \theta = 135°.$

34. $m_1 = $ slope of $AB = \dfrac{(8-6)}{4-(-2)} = \dfrac{2}{6} = \dfrac{1}{3}.$

$m_2 = $ slope of $CD = \dfrac{(24-12)}{(x-8)} = \dfrac{12}{(x-8)}.$

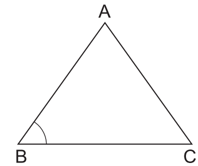
$AB \perp CD \Rightarrow m_1 m_2 = -1 \Rightarrow \dfrac{1}{3} \times \dfrac{12}{(x-8)} = -1 \Rightarrow -x + 8 = 4 \Rightarrow x = 4.$

35. Since the given points are collinear, we have

$$\text{slope of } AB = \text{slope of } BC \Rightarrow \frac{1-(-1)}{2-x} = \frac{(5-1)}{(4-2)} = \frac{4}{2} = 2$$

$\therefore \quad \dfrac{2}{2-x} = 2 \Rightarrow 2 = 4 - 2x \Rightarrow 2x = 2 \Rightarrow x = 1.$

36. Clearly, B is the angle between BA and BC.



$$m_1 = \text{slope of } BA = \frac{3-1}{2+2} = \frac{2}{4} = \frac{1}{2}.$$

$$m_2 = \text{slope of } BC = \frac{-4-3}{-2-2} = \frac{-7}{-4} = \frac{7}{4}.$$

$$\therefore \quad \tan B = \left|\frac{m_2 - m_1}{1 + m_1 m_2}\right| = \left|\frac{\frac{7}{4} - \frac{1}{2}}{1 + \frac{7}{4} \times \frac{1}{2}}\right| = \frac{2}{3}.$$

37. $\tan\theta = \left|\dfrac{m_2 - m_1}{1 + m_1 m_2}\right|$, where $m_1 = \dfrac{1}{2}$, $m_2 = m$ and $\theta = \dfrac{\pi}{4}$.

$$\therefore \quad \left|\frac{m - \frac{1}{2}}{1 + \frac{1}{2}m}\right| = \tan\frac{\pi}{4} = 1 \Rightarrow \left|\frac{2m-1}{2+m}\right| = 1$$

$$\therefore \quad \frac{2m-1}{2+m} = +1 \quad \text{or} \quad \frac{2m-1}{2+m} = -1$$

$$\Rightarrow \quad 2m - 1 = 2 + m \quad \text{or} \quad 2m - 1 = -2 - m \Rightarrow m = 3 \quad \text{or} \quad m = \frac{-1}{3}.$$

38. $m_1 = \dfrac{(3-0)}{(2-0)} = \dfrac{3}{2}$ and $m_2 = \dfrac{5+2}{3-2} = \dfrac{7}{1}$

$$\tan\theta = \left|\frac{m_2 - m_1}{1 + m_1 m_2}\right| = \left|\frac{7 - \frac{3}{2}}{1 + \frac{3}{2} \times 7}\right| = \frac{11}{2} \times \frac{2}{23} = \frac{11}{23}.$$

39. Here, $m_1 = (2 - \sqrt{3})$ and $m_2 = (2 + \sqrt{3})$.

$$\therefore \quad \tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right| = \left|\frac{(2-\sqrt{3}) - (2+\sqrt{3})}{1 + (2-\sqrt{3})(2+\sqrt{3})}\right| = \left|\frac{-2\sqrt{3}}{1 + (4-3)}\right| = \frac{2\sqrt{3}}{2} = \sqrt{3}$$

$$\Rightarrow \quad \theta = 60°.$$

40. $m = \dfrac{-8-2}{6-x} \Rightarrow \dfrac{-10}{6-x} = \dfrac{-5}{4} \Rightarrow 5(6-x) = 40$

$$\Rightarrow 30 - 5x = 40 \Rightarrow 5x = -10 \Rightarrow x = -2.$$

41. Since $AB \parallel CD$, we have: slope of AB = slope of CD.

$$\therefore \quad \frac{7-x}{2-3} = \frac{6-4}{0+1} \Rightarrow \frac{7-x}{-1} = \frac{2}{1} \Rightarrow 7 - x = -2 \Rightarrow x = 9.$$

42. $m_1 = \dfrac{x-6}{4+2} = \dfrac{x-6}{6}$ and $m_2 = \dfrac{-9+3}{5-3} = \dfrac{-6}{2} = -3.$

Since $AB \perp CD$, we have: $m_1 m_2 = -1$.

$$\therefore \quad \frac{x-6}{6} \times (-3) = -1 \Rightarrow x - 6 = 2 \Rightarrow x = 8.$$

43. Let m_1 = slope of the diagonal $AC = \frac{0-2}{4-0} = \frac{-2}{4} = \frac{-1}{2}$

and m_2 = slope of the diagonal $BD = \frac{(3+1)}{(2+2)} = \frac{4}{4} = 1.$

$$\therefore \quad \tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right| = \left|\frac{-\frac{1}{2} - 1}{1 + \left(\frac{-1}{2}\right) \times 1}\right| = \left|\frac{\left(\frac{-3}{2}\right)}{\left(\frac{1}{2}\right)}\right| = 3 \Rightarrow \tan\theta = 3.$$

44. Equation of AB is $\frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1}$

i.e., $\frac{y-8}{x+1} = \frac{-2-8}{4+1} \Rightarrow \frac{y-8}{x+1} = \frac{-10}{5} = -2$

$\Rightarrow \quad y - 8 = -2(x+1) \Rightarrow y - 8 = -2x - 2 \Rightarrow 2x + y - 6 = 0.$

45. Clearly, E is the midpoint of AC given by $E\left(\frac{2-2}{2}, \frac{5-1}{2}\right)$, i.e., $E(0, 2)$.

Equation of the median BE is $\frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1}$

i.e., $\frac{y-9}{x+4} = \frac{2-9}{0+4} = \frac{-7}{4} \Rightarrow 4(y-9) = -7(x+4)$

$\Rightarrow \quad 4y - 36 = -7x - 28 \Rightarrow 7x + 4y - 8 = 0.$

46. The equation of the line is $\frac{y-5}{x-0} = \tan 30° = \frac{1}{\sqrt{3}}$

$\Rightarrow \quad \sqrt{3}y - 5\sqrt{3} = x \Rightarrow x - \sqrt{3}y + 5\sqrt{3} = 0.$

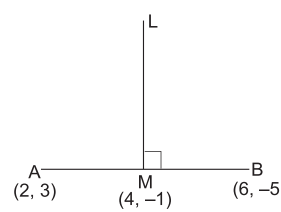
47. End points of AB are $A(2, 3)$ and $B(6, -5)$.

Midpoint of AB is $M\left(\frac{2+6}{2}, \frac{3-5}{2}\right)$, i.e., $M(4, -1)$.

Slope of $AB = \frac{-5-3}{6-2} = \frac{-8}{4} = -2.$

Let the slope of LM be m. Then,

$LM \perp AB \Rightarrow -2 \times m = -1 \Rightarrow m = \frac{1}{2}.$



$\therefore \quad$ slope of $LM = \frac{1}{2}.$

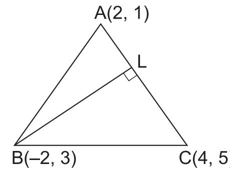
Required equation is $\frac{y-(-1)}{x-4} = \frac{1}{2} \Rightarrow x - 4 = 2(y+1) \Rightarrow x - 2y - 6 = 0.$

48. Slope of $AB = \frac{\sqrt{6} - \sqrt{3}}{\sqrt{2} - 1} = \frac{\sqrt{3}(\sqrt{2} - 1)}{(\sqrt{2} - 1)} = \sqrt{3}$

$\therefore \quad \tan\theta = \sqrt{3} \Rightarrow \theta = 60°.$

Hence, AB makes an angle of $60°$ with the x-axis.

49. Let $A(2, 1)$, $B(-2, 3)$ and $C(4,5)$ be the vertices of $\triangle ABC$.



Let $BL \perp AC$.

Slope of $AC = \dfrac{(5-1)}{(4-2)} = \dfrac{4}{2} = 2.$

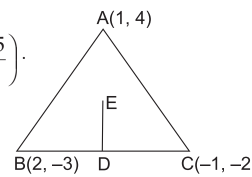
Let the slope of BL be m. Then,

$$2 \times m = -1 \Rightarrow m = \frac{-1}{2}\cdot$$

$\therefore$ equation of BL is $\dfrac{y-3}{x+2} = \dfrac{-1}{2}$

$\Rightarrow$ $2y - 6 = -x - 2 \Rightarrow x + 2y - 4 = 0.$

50. Let $A(1, 4)$, $B(2, -3)$ and $C(-1, -2)$ be the vertices of a $\triangle ABC$.



Let D be the midpoint of BC and let $DE \perp BC$.

Then, the coordinates of D are $\left(\dfrac{2-1}{2}, \dfrac{-3-2}{2}\right)$, i.e., $\left(\dfrac{1}{2}, \dfrac{-5}{2}\right)\cdot$

Slope of $BC = \dfrac{-2+3}{-1-2} = \dfrac{-1}{3}\cdot$ Let the slope of DE be m.

Then, $m \times \dfrac{-1}{3} = -1 \Rightarrow m = 3$. Thus, slope of DE is 3.

$\therefore$ equation of DE is $\dfrac{y+\frac{5}{2}}{x-\frac{1}{2}} = 3 \Rightarrow \dfrac{2y+5}{2x-1} = 3$

$\Rightarrow$ $2y + 5 = 6x - 3 \Rightarrow 2y - 6x = -8 \Rightarrow y - 3x + 4 = 0 \Rightarrow 3x - y - 4 = 0.$

51. The midpoint of the join of $A(-2, 0)$ and $B(6, 0)$ is $M\left(\dfrac{-2+6}{3}, 0\right)$, i.e., $M(2, 0)$.

So, the required line is $x = 2$.

52. The midpoint of the join of $A(0, 8)$ and $B(0, -2)$ is $M\left(0, \dfrac{8-2}{2}\right)$, i.e., $M(0, 3)$.

So, the required line is $y = 3$.

53. Since the given line is parallel to the x-axis, we have $m = 0$.

So, the required equation is $\dfrac{y+4}{y-3} = 0 \Rightarrow y + 4 = 0.$

54. Here, $m = \dfrac{1}{2}$ and $c = \dfrac{-5}{4}\cdot$

$\therefore$ the required equation is $y = \dfrac{1}{2}x - \dfrac{5}{4} \Rightarrow 4y = 2x - 5 \Rightarrow 2x - 4y - 5 = 0.$

55. Here, $m = \tan 30° = \dfrac{1}{\sqrt{3}}$ and $c = 3$.

$\therefore$ the equation of the line is $y = \dfrac{1}{\sqrt{3}}x + 3$

or $\sqrt{3}y = x + 3\sqrt{3} \Rightarrow x - \sqrt{3}y + 3\sqrt{3} = 0.$

56. Here, $m = \tan 120° = \tan(180° - 60°) = -\tan 60° = -\sqrt{3}$ and $c = -4$.

$\therefore$ the required equation is $y = -\sqrt{3}x - 4 \Rightarrow \sqrt{3}x + y + 4 = 0.$

57. Here, $m = \tan\theta = \frac{1}{3}$ and x-intercept, $d = 5$.

So, the equation of the line is $y = m(x - d)$.

$\therefore$ the required equation is $y = \frac{1}{3}(x-5) \Rightarrow x - 5 = 3y \Rightarrow x - 3y - 5 = 0.$

58. Here, $m = -2$ and $d = -3$.

$\therefore$ the required equation is $y = m(x - d)$

or $y = -2(x+3) \Rightarrow 2x + y + 6 = 0.$

59. $2x + 3y + 7 = 0 \Rightarrow 3y = -2x - 7 \Rightarrow y = \frac{-2}{3}x - \frac{7}{3}\cdot$

$27x - 18y + 25 = 0 \Rightarrow 18y = 27x + 25 \Rightarrow y = \frac{3}{2}x + \frac{25}{18}\cdot$

$\therefore \quad m_1 = \frac{-2}{3}$ and $m_2 = \frac{3}{2}\cdot$

$\Rightarrow \quad m_1 m_2 = \frac{-2}{3} \times \frac{3}{2} = -1$

$\Rightarrow$ the given lines are perpendicular to each other.

60. $x + 2y - 9 = 0 \Rightarrow 2y = 9 - x \Rightarrow y = -\frac{1}{2}x + \frac{9}{2}\cdot$

$2x + 4y + 5 = 0 \Rightarrow 4y = -2x - 5 \Rightarrow y = \frac{-1}{2}x - \frac{5}{4}\cdot$

$\therefore \quad m_1 = m_2 = -\frac{1}{2}\cdot$

Hence, the given lines are parallel to each other.

61. $x + \sqrt{3}y - 6 = 0 \Rightarrow \sqrt{3}y = -x + 6 \Rightarrow y = \frac{-1}{\sqrt{3}}x + \frac{6}{\sqrt{3}}\cdot$

$\therefore \quad m = \frac{-1}{\sqrt{3}} \Rightarrow \tan\theta = \frac{-1}{\sqrt{3}} = -\tan 30° = \tan(180° - 30°) = \tan 150°.$

$\therefore \quad \theta = 150°.$

62. $2x - 3y = 7 \Rightarrow 3y = 2x - 7 \Rightarrow y = \frac{2}{3}x - \frac{7}{3}\cdot$

$\therefore$ slope of the required line = slope of given line $= \frac{2}{3}\cdot$

$\therefore$ the required equation is $\frac{y-(-5)}{x-2} = \frac{2}{3}$

$\Rightarrow \quad \frac{y+5}{x-2} = \frac{2}{3} \Rightarrow 2x - 4 = 3y + 15 \Rightarrow 2x - 3y - 19 = 0.$

63. $3x - y = 5 = 0 \Rightarrow y = 3x + 5.$

Slope of the given line = 3.

Let the slope of the required line be m.

Then, $3m = -1 \Rightarrow m = \frac{-1}{3}\cdot$

$\therefore$ the required equation is $\frac{y-(-4)}{x-(-2)} = \frac{-1}{3} \Rightarrow \frac{y+4}{x+2} = \frac{-1}{3} \Rightarrow 3y + 12 = -x - 2$

$\Rightarrow x + 3y + 14 = 0.$

64. $3x - 2y + 5 = 0 \Rightarrow 2y = 3x + 5 \Rightarrow y = \frac{3}{2}x + \frac{5}{2}\cdot$

Slope of the given line $= \frac{3}{2}\cdot$

Let the slope of the required line be m.

Then, $m \times \frac{3}{2} = -1 \Rightarrow m = \frac{-2}{3}\cdot$

Also, y-intercept $= -3$.

$\therefore$ the required equation is $y = \frac{-2}{3}x - 3 \Rightarrow 2x + 3y + 9 = 0$.

65. $x - 7y + 5 = 0 \Rightarrow 7y = x + 5 \Rightarrow y = \frac{1}{7}x + \frac{5}{7}\cdot$

Slope of this line $= \frac{1}{7}\cdot$

Let the slope of the required line be m.

Then, $m \times \frac{1}{7} = -1 \Rightarrow m = -7$.

Also, x-intercept, $d = 3$.

$\therefore$ the required equation is $y = m(x - d)$, i.e., $y = -7(x - 3)$, i.e., $y + 7x - 21 = 0$.

66. The required equation is $\frac{x}{a} + \frac{y}{b} = 1$, where $a = 2$ and $b = -3$.

$\therefore \quad \frac{x}{2} - \frac{y}{3} = 1 \Rightarrow 3x - 2y - 6 = 0$.

67. Let the equation be $\frac{x}{a} + \frac{y}{b} = 1$.

Then, the end points of the line segment are $A(a, 0)$ and $B(0, b)$ and the midpoint of AB is $M(2, 3)$.

$\therefore \quad \frac{a+0}{2} = 2$ and $\frac{0+b}{2} = 3 \Rightarrow \frac{a}{2} = 2$ and $\frac{b}{2} = 3 \Rightarrow a = 4$ and $b = 6$.

$\therefore$ the required equation is $\frac{x}{4} + \frac{y}{6} = 1 \Rightarrow 6x + 4y - 24 = 0 \Rightarrow 3x + 2y - 12 = 0$.

68. Since $\frac{x}{a} + \frac{y}{b} = 1$ passes through $A(8, -9)$ and $B(12, -15)$, we have:

$$\frac{8}{a} - \frac{9}{b} = 1 \quad \text{... (i)} \qquad \text{and} \qquad \frac{12}{a} - \frac{15}{b} = 1 \quad \text{... (ii)}$$

On solving, we get: $a = 2, b = 3$.

69. Let the required equation be $\frac{x}{a} + \frac{y}{(9-a)} = 1$.

Since it passes through $(2, 2)$, we have:

$$\frac{2}{a} + \frac{2}{9-a} = 1 \Rightarrow 2(9-a) + 2a = a(9-a)$$

$$\Rightarrow 9a - a^2 = 18 \Rightarrow a^2 - 9a + 18 = 0$$

$$\Rightarrow (a-6)(a-3) = 0 \Rightarrow a = 6 \quad \text{or} \quad a = 3.$$

$\therefore$ the required equation is $\frac{x}{6}+\frac{y}{3}=1$ or $\frac{x}{3}+\frac{y}{6}=1$

$\Rightarrow$ $x+2y-6=0$ or $2x+y-6=0$.

70. The required equation is $x\cos\alpha+y\sin\alpha=p$, where $\alpha=135°$ and $p=5$

i.e., $x\cos 135°+y\sin 135°=5$

$$\Rightarrow \quad x\times\left(\frac{-1}{\sqrt{2}}\right)+y\times\left(\frac{1}{\sqrt{2}}\right)=5$$

$$\Rightarrow \quad -x+y=5\sqrt{2} \Rightarrow x-y+5\sqrt{2}=0.$$

71. The required equation is $x\cos\alpha+y\sin\alpha=p$, where $\alpha=150°$ and $p=8$.

$$\therefore \quad x\cos 150°+y\sin 150°=8 \Rightarrow x\cos(180°-30°)+y\sin(180°-30°)=8$$

$$\Rightarrow \quad -x\cos 30°+y\sin 30°=8 \Rightarrow -x\times\frac{\sqrt{3}}{2}+y\times\frac{1}{2}=8$$

$$\Rightarrow \quad -\sqrt{3}x+y=16 \Rightarrow \sqrt{3}x-y+16=0.$$

72. $\sqrt{3}x+y+2=0 \Rightarrow y=-\sqrt{3}x-2$

$\therefore$ $y=mx+c$, where $m=-\sqrt{3}$.

73. Here, $p=5$ and $\alpha=30°$.

So, the equation of the line is

$$x\cos 30°+y\sin 30°=5 \Rightarrow x\cdot\frac{\sqrt{3}}{2}+y\cdot\frac{1}{2}=5$$

$$\Rightarrow \sqrt{3}x+y-10=0.$$

74. $3x-2y=-4 \Rightarrow \frac{3}{(-4)}x+\left(\frac{-2}{-4}\right)y=1 \Rightarrow \frac{x}{\frac{(-4)}{3}}+\frac{y}{2}=1.$

$\therefore$ x-intercept $=\frac{-4}{3}$, y-intercept $=2$.

Hence, $a=\frac{-4}{3}$ and $b=2$.

75. Distance of the point $P(4, 1)$ from the line $3x-4y+12=0$ is

$$d=\frac{|3\times 4-4\times 1+12|}{\sqrt{3^2+(-4)^2}}=\frac{20}{5}\text{ units}=4\text{ units.}$$

76. Distance of the point $P(-1, 1)$ from the line $12x-5y+82=0$ is

$$\frac{|12\times(-1)-5\times 1+82|}{\sqrt{(12)^2+(-5)^2}}=\frac{65}{\sqrt{169}}=\frac{65}{13}=5\text{ units.}$$

77. Length of the perpendicular from $O(0, 0)$ to the line $4x+3y-2=0$ is

$$d=\frac{|4\times 0+3\times 0-2|}{\sqrt{4^2+3^2}}=\frac{2}{\sqrt{25}}=\frac{2}{5}\text{ unit.}$$

78. Converting each of the given lines to the form $y=mx+c$, we get

$$y=\frac{3}{4}x+\frac{9}{4}\text{ and }y=\frac{3}{4}x-\frac{17}{8}\cdot$$

Here, $m = \frac{3}{4}$, $c_1 = \frac{9}{4}$ and $c_2 = \frac{-17}{8}$.

Distance between given lines $= \frac{|c_2 - c_1|}{\sqrt{1+m^2}} = \frac{\left|\frac{-17}{8} - \frac{9}{4}\right|}{\sqrt{1+\frac{9}{16}}} = \frac{\left|\frac{-35}{8}\right|}{\sqrt{\frac{25}{16}}} = \left(\frac{35}{8} \times \frac{4}{5}\right) = \frac{7}{2}$ units.

79. Let the required point be $P(x, 0)$.

Then, $\frac{|4x + 3 \times 0 - 12|}{\sqrt{4^2 + 3^2}} = 4 \Rightarrow |x - 3| = 5$

$\Rightarrow \quad x - 3 = 5 \quad$ or $\quad x - 3 = -5 \Rightarrow x = 8 \quad$ or $\quad x = -2$.

Hence, the required points are (8, 0) and (–2, 0).

80. Let the required equation be $y = -x + c$. Then,

$$\frac{|0 + 0 - c|}{\sqrt{1^2 + 1^2}} = 5 \Rightarrow |c| = 5\sqrt{2} \Rightarrow c = 5\sqrt{2} \quad \text{or} \quad c = -5\sqrt{2}.$$

$\therefore$ required equation is $x + y = -5\sqrt{2}$ or $x + y = 5\sqrt{2}$

i.e., $x + y + 5\sqrt{2} = 0$ or $x + y - 5\sqrt{2} = 0$.

81. Putting $x = 0$ in $px + py + q = 0$, we get $y = \left(\frac{-q}{p}\right)$.

$\therefore$ required distance = length of perp. from $P\left(0, \frac{-q}{p}\right)$ on $px + py - r = 0$

$$= \frac{\left|p \times 0 + p \times \frac{(-q)}{p} - r\right|}{\sqrt{p^2 + p^2}} = \frac{|-q-r|}{\sqrt{2}p} = \frac{|q+r|}{\sqrt{2}p}.$$

82. On solving $5x + 7y - 3 = 0$ ... (i)

$2x - 3y - 7 = 0$ ... (ii)

we get $\frac{x}{-49 - 9} = \frac{y}{-6 + 35} = \frac{1}{-15 - 14} \Rightarrow \frac{x}{-58} = \frac{y}{29} = \frac{1}{-29}$.

$\therefore \quad x = \left(-58 \times \frac{1}{-29}\right) = 2$ and $y = \left(29 \times \frac{1}{-29}\right) = -1$.

Hence, the required point is (2, –1).

83. On solving $x - 7y + 15 = 0$ and $2x + y = 0$, we get $x = -1$ and $y = 2$.

So, the given lines intersect at the point $P(-1, 2)$.

The line parallel to the y-axis and passing through $P(-1, 2)$ is $x = -1$, i.e., $x + 1 = 0$.

84. On solving $x + 2y + 3 = 0$ and $3x + 4y + 7 = 0$, we get $x = -1$ and $y = -1$.

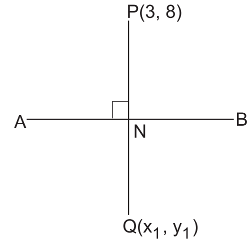
So, the given lines intersect at $P(-1, -1)$.

Now, $y - x = 8 \Rightarrow y = x + 8$.

Slope of this line = 1 = slope of the required line.

$\therefore$ the required equation is $\frac{y + 1}{x + 1} = 1 \Rightarrow y + 1 = x + 1 \Rightarrow x - y = 0$.

85. Equation of the line AB is $y = \frac{-1}{3}x + \frac{7}{3}\cdot$



Slope of $AB = \frac{-1}{3}\cdot$

Let $PN \perp AB$ and let $PQ = 2\times PN$.

Let the slope of PQ be m.

Then, $m\times\frac{-1}{3} = -1 \Rightarrow m = 3$.

Equation of PQ is $\frac{y-8}{x-3} = 3 \Rightarrow y - 8 = 3x - 9 \Rightarrow 3x - y - 1 = 0$.

On solving $x + 3y = 7$ and $3x - y = 1$, we get $x = 1, y = 2$.

Thus, AB and PQ intersect at $N(1, 2)$.

Since N bisects PQ, we have $\frac{3 + x_1}{2} = 1$ and $\frac{8 + y_1}{2} = 2$.

$\therefore\quad x_1 = -1, y_1 = -4$.

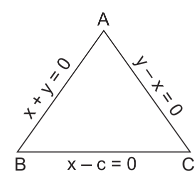
Required image is $Q(-1, -4)$.

86. On solving $x - y - 6 = 0$ and $4x - 3y - 20 = 0$, we get $x = 2$ and $y = -4$.

Also, $x = 2, y - 4$ satisfies $6x + 5y + 8 = 0$ as $(6\times 2) + [5\times(-4)] + 8 = 0$.

$\therefore\quad$ point of intersection of the given lines is $(2, -4)$.

87. Let the sides AC, AB and BC of $\triangle ABC$ be given by $y - x = 0, x + y = 0$ and $x - c = 0$ respectively.



On solving these equations pairwise, we get $A(0, 0), B(c, -c)$ and $C(c, c)$.

area of $\triangle ABC = \frac{1}{2}\cdot|x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)|$

$= \frac{1}{2}\cdot|0\cdot(-c - c) + c(c - 0) + c(0 + c)| = c^2$ sq units.

88. On solving $3x + y = 2$ and $2x - y = 3$, we get $x = 1$ and $y = -1$.

Putting $x = 1$ and $y = -1$ in $kx + 2y = 3$, we get $k = 5$.

89. Given circle is $x^2 + y^2 - 6x + 4y - 12 = 0$.

This is of the form $x^2 + y^2 + 2gx + 2fy + c = 0$, where

$(2g = -6$ and $2f = 4) \Rightarrow (g = -3$ and $f = 2)$.

$\therefore\quad$ centre is $(-g, -f) = (3, -2)$.

90. Given circle is $x^2 + y^2 - 4x + 2y - 45 = 0$.

This is of the form $x^2 + y^2 + 2gx + 2fy + c = 0$, where

$(2g = -4$ and $2f = 2) \Rightarrow (g = -2$ and $f = 1)$. Also, $c = -45$.

$\therefore\quad$ radius $= \sqrt{g^2 + f^2 - c} = \sqrt{(-2)^2 + 1^2 + 45} = \sqrt{50} = 5\sqrt{2}$.

91. Given end points are $(x_1, y_1) = (2, -3)$ and $(x_2, y_2) = (-3, 5)$.

$\therefore\quad$ equation of the circle is $(x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0$

$\Rightarrow\quad (x - 2)(x + 3) + (y + 3)(y - 5) = 0$

$\Rightarrow\quad x^2 + y^2 + x - 2y - 21 = 0$.

92. Radius of the circle is $AC = \sqrt{(3-2)^2 + (2+5)^2} = \sqrt{1+49} = \sqrt{50}.$

$\therefore$ equation of the circle is $(x-2)^2 + (y+5)^2 = 50$

$\Rightarrow$ $x^2 + y^2 - 4x + 10y - 21 = 0.$

93. Given circle is $(x+5)^2 + (y-3)^2 = 20.$

Its centre is $(-5, 3)$ and radius $= \sqrt{20} = 2\sqrt{5}.$

94. Given equation is $x^2 + y^2 - 6x + 5y - 7 = 0.$

This is of the form $x^2 + y^2 + 2gx + 2fy + c = 0$, where

$$(2g = -6, 2f = 5 \text{ and } c = -7) \Rightarrow \left(g = -3, f = \frac{5}{2}, c = -7\right)$$

$\therefore$ centre is $(-g, -f) = \left(3, \frac{-5}{2}\right)\cdot$

But, C is the midpoint of AB, where $A(-1, 3)$ and $B(\alpha, \beta)$ are given.

$\therefore$ $\left(3 = \frac{-1+\alpha}{2} \text{ and } \frac{-5}{2} = \frac{3+\beta}{2}\right) \Rightarrow (\alpha = 7 \text{ and } \beta = -8).$

95. General equation is $y^2 = 4ax$. Here, $4a = 8 \Rightarrow a = 2.$

$\therefore$ focus of the parabola is $F(a, 0) = F(2, 0)$ and its vertex is $O(0, 0).$

96. Given equation is $y^2 = -4ax$, where $a = 3.$

$\therefore$ focus is $F(-a, 0) = F(-3, 0).$

And, the directrix is $x = a \Rightarrow x = 3.$

97. Given equation is $y^2 = -4ax$, where $a = 2.$

$\therefore$ its focus is $F(-a, 0) = F(-2, 0).$

Its directrix is $x = a \Rightarrow x = 2.$

98. Given equation is $x^2 = -4ay$, where $a = 4.$

Its focus is $F(0, -a) = F(0, -4).$

Its directrix is $y = a \Rightarrow y = 4.$

99. Given equation is $x^2 = 4ay$, where $4a = 6 \Rightarrow a = \frac{3}{2}\cdot$

Its focus is $F(0, a) = F\left(0, \frac{3}{2}\right)\cdot$

Its directrix is $y = -a \Rightarrow y = \frac{-3}{2}\cdot$

100. Since the point $P(3, 2)$ lies on $y^2 = 4ax$, we have

$$4a \times 3 = 2^2 \Rightarrow a = \frac{1}{3}\cdot$$

$\therefore$ latus rectum $= 4a = \left(4 \times \frac{1}{3}\right) = \frac{4}{3}\cdot$

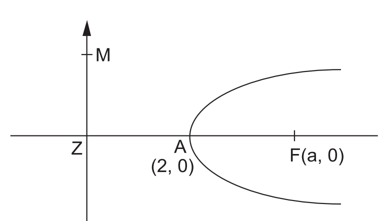
101. Let ZM be the directrix, where coordinates of Z are $(0, 0)$.

Let the focus be $F(a, 0)$.

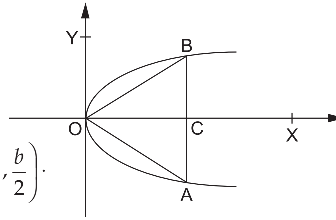
Then, vertex A is the midpoint of ZF.

$\therefore$ $\frac{a+0}{2} = 2 \Rightarrow a = 4.$

$\therefore$ focus is $F(4, 0).$



102. Let $\triangle OAB$ be the equilateral triangle inscribed in a parabola $y^2 = 4ax$.



Then, $\angle COB = 30°$.

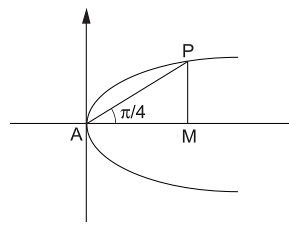
Let $OA = OB = AB = b$.

Then, we have $B(b\cos 30°, b\sin 30°) = B\left(\frac{\sqrt{3}b}{2}, \frac{b}{2}\right)$.

This point B lies on $y^2 = 4ax$.

$\therefore \quad \frac{b^2}{4} = 2\sqrt{3}\,ab \Rightarrow b = 8\sqrt{3}\,a$.

103. Let AP be the chord of length p, making an angle of $\left(\frac{\pi}{4}\right)$ with the axis of the parabola.



Then, coordinates of P are

$$P\left(p\cos\frac{\pi}{4}, p\sin\frac{\pi}{4}\right) = P\left(\frac{p}{\sqrt{2}}, \frac{p}{\sqrt{2}}\right).$$

Since $P\left(\frac{p}{\sqrt{2}}, \frac{p}{\sqrt{2}}\right)$ lies on $y^2 = 4ax$, we have

$$\frac{p^2}{2} = 4a \times \frac{p}{\sqrt{2}} \Rightarrow p = 4a\sqrt{2}.$$

104. Given parabola is $y^2 = 4ax$, where $a = 3$.

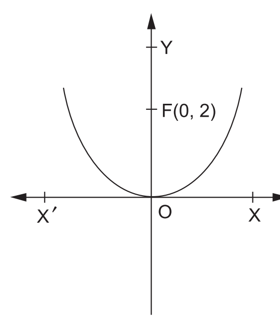
Its directrix is $x = -a \Rightarrow x = -3 \Rightarrow x + 3 = 0$.

Focal distance of $P(x_1, y_1)$ = distance of $P(x_1, y_1)$ from $x + 3 = 0$

$$= \frac{x_1 + 3}{\sqrt{1^2}} = \frac{x_1 + 3}{1} = x_1 + 3.$$

$\therefore \quad x_1 + 3 = 4 \Rightarrow x_1 = 1$.

105. Let $O(0, 0)$ be the vertex and $F(0, 2)$ be the focus.



Clearly, it is a case of upward parabola.

Let its equation be $x^2 = 4ay$.

Since focus is $F(0, a)$, so $a = 2$.

$\therefore \quad$ its equation is $x^2 = 4 \times 2 \times y \Rightarrow x^2 = 8y$.

106. The vertex of the parabola is $O(0, 0)$ and its axis lies along the x-axis. So, its equation is $y^2 = 4ax$ or $y^2 = -4ax$.

But, it passes through the point $P(2, 3)$, so its equation is $y^2 = 4ax$..

Since it passes through $P(2, 3)$, we have $4a \times 2 = 9 \Rightarrow a = \frac{9}{8}$.

$\therefore \quad$ its equation is $y^2 = 4 \times \frac{9}{8}x \Rightarrow y^2 = \frac{9}{2}x$.

107. It is given that the vertex of the parabola is $O(0, 0)$ and it is symmetric about the y-axis.

So, its equation is $x^2 = 4ay$ or $x^2 = -4ay$.

Since it passes through the point $P(3, -4)$, so it lies in quadrant IV.

$\therefore$ it is a downward parabola.

Let its equation be $x^2 = -4ay$.

Since it passes through $P(3, -4)$, we have $16a = 9 \Rightarrow a = \frac{9}{16}$.

$\therefore$ its equation is $x^2 = -4 \times \frac{9}{16} y \Rightarrow 4x^2 + 9y = 0$.

108. Given equation is $\frac{x^2}{16} + \frac{y^2}{9} = 1$.

This is of the form $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, where $a^2 > b^2$.

Here, $(a^2 = 16 \Rightarrow a = 4)$ and $(b^2 = 9 \Rightarrow b = 3)$.

$\therefore$ $c = \sqrt{a^2 - b^2} = \sqrt{16 - 9} = \sqrt{7}$.

Thus, $a = 4$, $b = 3$, $c = \sqrt{7}$.

(i) Length of the major axis $= 2a = 8$ units.

(ii) Length of the minor axis $= 2b = 6$ units.

(iii) Coordinates of its vertices are $A(-a, 0)$ and $B(a, 0)$, i.e., $A(-4, 0)$ and $B(4, 0)$.

(iv) Coordinates of its foci are $F_1(-c, 0)$ and $F_2(c, 0)$, i.e., $F_1(-\sqrt{7}, 0)$ and $F_2(\sqrt{7}, 0)$.

(v) Eccentricity, $e = \frac{c}{a} = \frac{\sqrt{7}}{4}$.

(vi) Length of the latus rectum $= \frac{2b^2}{a} = \frac{(2 \times 9)}{3} = 6$ units.

109. Given equation is $\frac{x^2}{9} + \frac{y^2}{4} = 1$.

This is of the form $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, where $a^2 > b^2$.

Here, $(a^2 = 9 \Rightarrow a = 3)$ and $(b^2 = 4 \Rightarrow b = 2)$.

$\therefore$ $c = \sqrt{a^2 - b^2} = \sqrt{9 - 4} = \sqrt{5}$.

Thus, $a = 3$, $b = 2$ and $c = \sqrt{5}$.

(i) Length of the major axis $= 2a = 6$ units.

(ii) Length of the minor axis $= 2b = 4$ units.

(iii) Coordinates of its vertices are $A(-a, 0)$ and $B(a, 0)$, i.e., $A(-3, 0)$ and $B(3, 0)$.

(iv) Coordinates of its foci are $F_1(-c, 0)$ and $F_2(c, 0)$, i.e., $F_1(\sqrt{5}, 0)$ and $F_2(-\sqrt{5}, 0)$.

(v) Eccentricity, $e = \frac{c}{a} = \frac{\sqrt{5}}{3}$.

(vi) Length of the latus rectum $= \frac{2b^2}{a} = \frac{2 \times 4}{3} = \frac{8}{3}$ units.

110. Given equation is $\dfrac{x^2}{4} + \dfrac{y^2}{36} = 1.$

This is of the form $\dfrac{x^2}{b^2} + \dfrac{y^2}{a^2} = 1$, where $a^2 > b^2$.

So, it is an equation of a *vertical ellipse.*

Now, $(b^2 = 4 \text{ and } a^2 = 36) \Rightarrow (b = 2 \text{ and } a = 6)$.

$\therefore \quad c = \sqrt{a^2 - b^2} = \sqrt{36 - 4} = \sqrt{32} = 4\sqrt{2}.$

Thus, $a = 6, b = 2, c = 4\sqrt{2}$.

(i) Length of the major axis $= 2a = 12$ units.

(ii) Length of the minor axis $= 2b = 4$ units.

(iii) Coordinates of its vertices are $A(0, -a)$ and $B(0, a)$, i.e., $A(0, -6)$ and $B(0, 6)$.

(iv) Coordinates of its foci are $F_1(0, -c)$ and $F_2(0, c)$, i.e., $F_1(0, -4\sqrt{2})$ and $F_2(0, 4\sqrt{2})$.

(v) Eccentricity, $e = \dfrac{c}{a} = \dfrac{4\sqrt{2}}{6} = \dfrac{2\sqrt{2}}{3}.$

(vi) Length of the latus rectum $= \dfrac{2b^2}{a} = \dfrac{(2 \times 4)}{6} = \dfrac{4}{3}$ units.

111. Given equation is $\dfrac{x^2}{25} + \dfrac{y^2}{100} = 1.$

This is of the form $\dfrac{x^2}{b^2} + \dfrac{y^2}{a^2} = 1$, where $a^2 > b^2$.

$\therefore$ it is an equation of a *vertical ellipse.*

Now, $(b^2 = 25, a^2 = 100) \Rightarrow (b = 5, a = 10)$.

$\therefore \quad c = \sqrt{a^2 - b^2} = \sqrt{100 - 25} = \sqrt{75} = 5\sqrt{3}.$

Thus, $a = 10, b = 5$ and $c = 5\sqrt{3}$.

(i) Length of the major axis $= 2a = 20$ units.

(ii) Length of the minor axis $= 2b = 10$ units.

(iii) Coordinates of its vertices are $A(0, -a)$ and $B(0, a)$, i.e., $A(0, -10)$ and $B(0, 10)$.

(iv) Coordinates of its foci are $F_1(0, -c)$ and $F_2(0, c)$, i.e., $F_1(0, -5\sqrt{3})$ and $F_2(0, 5\sqrt{3})$.

(v) Eccentricity, $e = \dfrac{c}{a} = \dfrac{5\sqrt{3}}{10} = \dfrac{\sqrt{3}}{2}.$

(vi) Length of the latus rectum $= \dfrac{2b^2}{a} = \dfrac{(2 \times 25)}{10} = 5$ units.

112. Since the vertices of the given ellipse are on the x-axis, so it is a horizontal ellipse.

Let the equation of the ellipse be $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$, where $a^2 > b^2$.

Its vertices are $(\pm a, 0) = (\pm 5, 0) \Rightarrow a = 5$.

Its foci are $(\pm c, 0) = (\pm 4, 0) \Rightarrow c = 4$.

$$c^2 = (a^2 - b^2) \Rightarrow b^2 = (a^2 - c^2) = (25 - 16) = 9 \Rightarrow b = 3.$$

$\therefore$ required equation of the ellipse is $\dfrac{x^2}{25} + \dfrac{y^2}{9} = 1.$

113. Since the foci of the given ellipse are on the x-axis, so it is a horizontal ellipse.

Let its equation be $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, where $a^2 > b^2$.

Let its foci be $(\pm c, 0)$. Then, $c = 4$.

Also, $e = \frac{c}{a} \Rightarrow a = \frac{c}{e} = \frac{4}{\left(\frac{1}{3}\right)} = 12.$

$$c^2 = (a^2 - b^2) \Rightarrow b^2 = (a^2 - c^2) = (144 - 16) = 128.$$

$\therefore \quad a^2 = (12)^2 = 144$ and $b^2 = 128$.

$\therefore \quad$ the required equation of the ellipse is $\frac{x^2}{144} + \frac{y^2}{128} = 1.$

114. Since the vertices of the given ellipse lie on the y-axis, it is a vertical ellipse.

Let the required equation be $\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$, where $a^2 > b^2$.

Its vertices are $(0, \pm a) = (0, \pm 13) \Rightarrow a = 13$.

Its foci are $(0, \pm c) = (0, \pm 5) \Rightarrow c = 5$.

$\therefore \quad b^2 = (a^2 - c^2) = (169 - 25) = 144.$

Thus, $b^2 = 144$ and $a^2 = 169$.

$\therefore \quad$ the required equation is $\frac{x^2}{144} + \frac{y^2}{169} = 1.$

115. Since the foci of the given ellipse lie on the y-axis, it is a vertical ellipse.

Let its equation be $\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$, where $a^2 > b^2$.

Let $c^2 = (a^2 - b^2)$.

Then, its foci are $(0, \pm c) = (0, \pm 6) \Rightarrow c = 6$.

$b =$ length of semi-minor axis $= \left(\frac{1}{2} \times 16\right) = 8.$

$\therefore \quad a^2 = (c^2 + b^2) = (36 + 64) = 100.$

Thus, $b^2 = 64$ and $a^2 = 100$.

Hence, the required equaton is $\frac{x^2}{64} + \frac{y^2}{100} = 1.$

116. Since the foci of the given ellipse lie on the y-axis, so it is a vertical ellipse.

Let its equation be $\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$, where $a^2 > b^2$.

Let $c^2 = (a^2 - b^2)$.

Its foci are $(0, \pm c) = (0, \pm 5) \Rightarrow c = 5$.

$a =$ length of semi-major axis $= \left(\frac{1}{2} \times 20\right) = 10.$

$c^2 = (a^2 - b^2) \Rightarrow b^2 = (a^2 - c^2) = (100 - 25) = 75.$

$\therefore \quad a^2 = (10)^2 = 100$ and $b^2 = 75$.

Hence, the required equation is $\frac{x^2}{75} + \frac{y^2}{100} = 1$.

117. Since the vertices of the ellipse lie on the y-axis, it is a vertical ellipse.

Let the required equation be $\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$, where $a^2 > b^2$.

Its vertices are $(0, \pm a) = (0, \pm 10) \Rightarrow a = 10$.

Let $c^2 = (a^2 - b^2)$.

Now, $e = \frac{c}{a} \Rightarrow c = ae = \left(10 \times \frac{4}{5}\right) = 8$.

$\therefore \quad c^2 = (a^2 - b^2) \Rightarrow b^2 = (a^2 - c^2) = (100 - 64) = 36$.

$\therefore \quad a^2 = (10)^2 = 100$ and $b^2 = 36$.

Hence, the required equation is $\frac{x^2}{36} + \frac{y^2}{100} = 1$.

118. Given equation is $\frac{x^2}{36} - \frac{y^2}{64} = 1$.

Comparing it with $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, we get $a = 6, b = 8$.

$\therefore \quad c = \sqrt{a^2 + b^2} = \sqrt{36 + 44} = \sqrt{100} = 10$.

Thus, we have

(i) Length of the transverse axis $= 2a = 12$ units.

(ii) Length of the conjugate axis $= 2b = 16$ units.

(iii) Coordinates of its vertices are $A(-a, 0)$ and $B(a, 0)$, i.e., $A(-6, 0)$ and $B(6, 0)$.

(iv) Coordinates of its foci are $F_1(-c, 0)$ and $F_2(c, 0)$, i.e., $F_1(-10, 0)$ and $F_2(10, 0)$.

(v) Eccentricity, $e = \frac{c}{a} = \frac{10}{6} = \frac{5}{3}\cdot$

(vi) Length of the latus rectum $= \frac{2b^2}{a} = \frac{2 \times 64}{6} = \frac{64}{3}$ units.

119. $9x^2 - 16y^2 = 144 \Rightarrow \frac{x^2}{16} - \frac{y^2}{9} = 1$.

Comparing it with $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, we get $a = 4, b = 3$.

$\therefore \quad c = \sqrt{a^2 + b^2} = \sqrt{16 + 9} = \sqrt{25} = 5$.

Thus, we have

(i) Length of the transverse axis $= 2a = 8$ units.

(ii) Length of the conjugate axis $= 2b = 6$ units.

(iii) Its vertices are $A(-a, 0)$ and $B(a, 0)$, i.e., $A(-4, 0)$ and $B(4, 0)$.

(iv) Its foci are $F_1(-c, 0)$ and $F_2(c, 0)$, i.e., $F_1(-5, 0)$ and $F_2(5, 0)$.

(v) Eccentricity, $e = \frac{c}{a} = \frac{5}{4}\cdot$

(vi) Length of the latus rectum $= \frac{2b^2}{a} = \frac{2 \times 9}{4} = \frac{9}{2}$ units.

120. Since the vertices of the given hyperbola are of the form $(\pm a, 0)$, so it is a horizontal hyperbola.

Let the required equation be $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

Then, its vertices are $(\pm a, 0) = (\pm 2, 0) \Rightarrow a = 2$.

Let its foci be $(\pm c, 0) = (\pm 3, 0) \Rightarrow c = 3$.

$\therefore \quad b^2 = (c^2 - a^2) = (3^2 - 2^2) = (9 - 4) = 5$.

$\therefore \quad a^2 = 2^2 = 4, b^2 = 5$.

Hence, the required equation is $\frac{x^2}{4} - \frac{y^2}{5} = 1$.

121. Since the foci of the given hyperbola are $(\pm 5, 0)$, so it is a horizontal hyperbola.

Let the required equation be $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

Now, $2a = 8 \Rightarrow a = 4 \Rightarrow a^2 = 16$.

Let its foci be $(\pm c, 0) = (\pm 5, 0) \Rightarrow c = 5$.

$\therefore \quad b^2 = (c^2 - a^2) = (5^2 - 4^2) = (25 - 16) = 9$.

Thus, $a^2 = 16$ and $b^2 = 9$.

Hence, the required equation is $\frac{x^2}{16} - \frac{y^2}{9} = 1$.

122. Since the foci of the given hyperbola are $(\pm 4, 0)$, so it is a horizontal hyperbola.

Let its equation be $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

Its foci are $(\pm c, 0) = (\pm 4, 0) \Rightarrow c = 4$.

$\therefore \quad c^2 = a^2 + b^2 \Rightarrow a^2 + b^2 = 4^2 = 16$.

Length of the latus rectum $= 12 \Rightarrow \frac{2b^2}{a} = 12 \Rightarrow b^2 = 6a$.

$\therefore \quad a^2 + 6a = 16 \Rightarrow a^2 + 6a - 16 = 0 \Rightarrow (a + 8)(a - 2) = 0 \Rightarrow a = 2$.

$\therefore \quad b^2 = (6 \times 2) = 12$ and $a^2 = 2^2 = 4$.

Hence, the required equation is $\frac{x^2}{4} - \frac{y^2}{12} = 1$.

123. Given equation is $\frac{y^2}{4} - \frac{x^2}{9} = 1$, which is a vertical hyperbola.

Here, $(a^2 = 4, b^2 = 9) \Rightarrow (a = 2, b = 3)$.

$\therefore \quad c = \sqrt{a^2 + b^2} = \sqrt{4 + 9} = \sqrt{13}$.

(i) Length of the transverse axis $= 2a = 4$ units.

(ii) Length of the conjugate axis $= 2b = 6$ units.

(iii) Its vertices are $A(0, -a)$ and $B(0, a)$, i.e., $A(0, -2)$ and $B(0, 2)$.

(iv) Its foci are $F_1(0, -c)$ and $F_2(0, c)$, i.e., $F_1(0, -\sqrt{13})$ and $F_2(0, \sqrt{13})$.

(v) Eccentricity, $e = \frac{c}{a} = \frac{\sqrt{13}}{2}$.

(vi) Length of the latus rectum $= \frac{2b^2}{a} = \frac{2 \times 9}{2} = 9$ units.

124. Since the vertices of the given hyperbola are of the form $(0, \pm a)$, it is a vertical hyperbola. Let its equation be $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$.

Its vertices are $(0, \pm a) = (0, \pm 3) \Rightarrow a = 3$.

Its foci are $(0, \pm c) = (0, \pm 5) \Rightarrow c = 5$.

$b^2 = (c^2 - a^2) = (25 - 9) = 16, a^2 = 9$.

$\therefore$ its equation is $\frac{y^2}{9} - \frac{x^2}{16} = 1$.

125. Since the vertices of the given hyperbola are $(0, \pm a)$, so it is a vertical hyperbola. Let its equation be $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$.

Its vertices are $(0, \pm a) = (0, \pm 6) \Rightarrow a = 6 \Rightarrow a^2 = 36$.

Now, $e = \frac{5}{3}$ and $c = ae = \left(6 \times \frac{5}{3}\right) = 10 \Rightarrow c^2 = 100$.

$b^2 = (c^2 - a^2) = (100 - 36) = 64$.

$\therefore$ the required equation is $\frac{y^2}{36} - \frac{x^2}{64} = 1$.

126. Since the foci are $(0, \pm c)$, so it is a case of vertical hyperbola.

Let its equation be $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$.

Length of the transverse axis $= 6 \Rightarrow 2a = 6 \Rightarrow a = 3$.

Also, $c = 4$ (given).

$\therefore$ $c^2 = (a^2 + b^2) \Rightarrow b^2 = (c^2 - a^2) = (4^2 - 3^2) = (16 - 9) = 7$.

Thus, $a^2 = 3^2 = 9$ and $b^2 = 7$.

Hence, the required equation is $\frac{y^2}{9} - \frac{x^2}{7} = 1$.

127. Since the foci of the hyperbola are $(0, \pm c)$, so it is a vertical hyperbola.

Let its equation be $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$.

Its foci are $(0, \pm c) = (0, \pm 6) \Rightarrow c = 6$.

$2b = 2\sqrt{11} \Rightarrow b = \sqrt{11} \Rightarrow b^2 = 11$.

$c^2 = (a^2 + b^2) \Rightarrow a^2 = (c^2 - b^2) = (36 - 11) = 25$.

$\therefore$ $a^2 = 25$ and $b^2 = 11$.

$\therefore$ the required equation is $\frac{y^2}{25} - \frac{x^2}{11} = 1$.

❑