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Rudin 3.5 For any two real sequences $\{a_n\}$, $\{b_n\}$, prove that

$$\limsup_{n\to\infty} (a_n + b_n) \leq \limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n,$$

provided the sum on the right is not of the form $\infty - \infty$.

Suppose that $\limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n \neq \infty - \infty$, so that this sum is determinate. Define

$$A_n = \sup_{k\geq n} a_n, \quad B_n = \sup_{k\geq n} b_n, \quad \text{and} \quad C_n = \sup_{k\geq n} (a_n + b_n).$$

We first show that $C_n \leq A_n + B_n$ for all n. For k and n such that $k \geq n$, we have that $a_k \leq A_n$ and $b_k \leq B_n$. Then $a_k + b_k \leq A_n + B_n$ for all $k \geq n$, so $C_n = \sup_{k\geq n} (a_n + b_n) \leq A_n + B_n$. Thus, using the alternate definition of the lim sup, we have

$$\begin{aligned} \limsup_{n\to\infty} (a_n + b_n) &= \lim_{n\to\infty} C_n \\ &\leq \lim_{n\to\infty} (A_n + B_n) = \lim_{n\to\infty} A_n + \lim_{n\to\infty} B_n = \limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n. \end{aligned}$$

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SS 1.5.11 Solve the equation $(z + 1)^5 = z^5$.

Taking fifth roots of the equation yields

$$z + 1 = ze^{ik\frac{2\pi}{5}},$$

where $k \in \mathbb{Z}$. We note that $k = 0$ (and all other multiples of 5) yields $z + 1 = z$, which reduces to $1 = 0$, an inconsistent equation. Isolating z, we therefore have the solutions

$$\boxed{z = \frac{1}{e^{ik\frac{2\pi}{5}} - 1},}$$

with four unique solutions obtained using $k = 1, 2, 3, 4$. We expect 4 unique solutions because $(z + 1)^5 - z^5$ is a fourth-degree polynomial. ■