Research Statement

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1 Introduction

My research has been mostly focused on the dynamics and geometry of translation surfaces, tropical geometry, and low dimensional topology. Below are some of my ongoing projects. More background and details will be provided in the next section.

- **Constructing pseudo-Anosov maps, Teichmüller polynomials.** Harry Baik, Ahmad Rafiqi and I [BRW] gave a sufficient condition for constructing pseudo-Anosov maps by "thickening" a post-critically finite continuous interval map. In the process, following a suggestion of Danny Calegari, we built many infinite sequences of pseudo-Anosov maps that converges to end-periodic maps, and related them to the Teichmüller polynomials for 3-manifolds with toric boundaries. We are trying to generalize this construction to the case of closed 3-manifolds and also find more explicit relationship between the interval map we started with and the resulting Teichmüller polynomial.
- **A Kazhdan's theorem for metric graphs.** Farbod Shokrieh and I proved a result on finite metric graphs that is analogous to Kazhdan's theorem on compact Riemann surfaces (c.f. [M]), which has application on Mumford curves. We are working on generalizing it further to metrized simplicial complexes, in particular this will give us a discrete version of the original Kazhdan's theorem.
- **A Borel-Serre type compactification of the strata.** I am also working with my PhD advisor Prof. John Smillie on developing a way to compactify the strata of finite translation surfaces, and exploring its possible application on the dynamics of unipotent flow.

2 Ongoing Projects

2.1 Constructing pseudo-Anosov maps

The works of de Carvalho and T. Hall [CH] described a way of constructing a pseudo-Anosov map by "thickening" a post-critically finite continuous interval map into a piecewise linear map in 2 dimension, then gluing the boundary of its ω-limit set to make it a continuous surface map. A similar construction was also described by Thurston [T]. However, in general, the translation surface one gets in this way is not a finite translation surface. Harry Baik, Ahmad Rafiqi and I gave a sufficient condition for this construction to give us a finite translation surface in [BRW]. We also implemented an algorithm to check our condition efficiently, and obtained many examples.

Following the suggestion of Danny Calegari, we built many examples of pseudo-Anosov maps on finite surfaces from those interval maps that give us infinite translation surfaces, by going through end-periodic maps. We start by "blowing up" the ends of the finite translation surface to reverse the Handel-Miller construction, so that the affine map lifts to an end-periodic map of a larger surface. Then, we build a 3-manifold with toric boundaries with a depth 1 foliation as in [F] and perturb it to get a sequence of pseudo-Anosov maps. In particular, we showed that the sequence described by Bowman in [B] is an example of this construction.

We are currently working on the following questions related to this project:

(i) Can we relax the condition in our paper [BRW] if we do not assume the interval map we start with is continuous?

(ii) Given an infinite translation surface or half translation surface with finite area and an affine diffeomorphism, when can we "blow it up" into an end-periodic map? How do we calculate the resulting sequence of pseudo-Anosov maps or their characteristic polynomials efficiently? Can we define the analogue of the Teichmüller polynomial in this situation?

(iii) In [B], Bowman defined a concept of "limit surface" on the set of translation surfaces, and showed that the infinite Arnoux-Yoccoz surface is the limit of the finite ones. Is this true for all the sequences of pseudo-Anosov maps obtained from the construction described above?

(iv) Can we generalize our construction of the sequence of pseudo-Anosov maps so that it may produce closed atoridal 3-manifolds?

2.2 Problems on finite metric graphs

Kazhdan's theorem for compact Riemann surface theory says that the hyperbolic metric from uniformization can be obtained alternatively via the Jacobians of finite covers, as the limit of Bergman metric inherited from increasingly larger finite Galois over. c.f [M]. Farbod Shokrieh and I found a way to obtain a similar result to Zhang's canonical measures [Z, BF] on finite metric graph. We showed the following:

Theorem 1
Let $\phi\colon \Gamma' \to \Gamma$ be a infinite Galois covering of compact metric graph Γ. Let $\phi_n\colon \Gamma_n \to \Gamma$ (for $n \geq 1$) be an ascending sequence of finite Galois covers con-

verging to Γ', in the sense that the equality

$$\bigcap_{n\geq 1} \pi_1(\Gamma_n) = \pi_1(\Gamma') \tag{1}$$

holds in $\pi_1(\Gamma)$. Let μ'_{can} be the canonical measure on Γ', μ^i_{can} be the canonical metric on Γ_i. Let μ_ϕ and μ_{ϕ_i} be measures on Γ whose pull-backs are μ'_{can} and μ^i_{can}, respectively. Then

$$\lim_{n\to\infty} \mu_{\phi_n} = \mu_\phi \ .$$

A key lemma for our theorem is the "Gauss-Bonnet's theorem" for canonical measures on infinite metric graph:

Theorem 2
Let $\phi\colon \Gamma' \to \Gamma$ be a infinite Galois covering of compact metric graph Γ. Let μ'_{can} denote the canonical measure on Γ', and let μ_ϕ be the measure on Γ whose pull-back is μ'_{can}. Then

$$\mu_\phi(\Gamma) = -\chi(\Gamma) \ .$$

which can be proved through either harmonic analysis on infinite metric graphs or von-Neumann algebra. Currently, we are trying to generalize it to the case of metric 2-complexes as well as working on extending other result on Riemann surface to the case of graphs, for example the conjecture by Thomas Koberda [K] on the expansion factor of outer automorphism of free groups.

2.3 Compactification of strata

A *translation surface* is a surface with a translation structure, which is a local chart that covers the surface except for a discrete set Σ, for which the change of coordinate maps are translations. For example, holomorphic differentials on compact Riemann surfaces make them translation surfaces. The translation surfaces constructed in this way are called *finite translation surfaces*, while the other translation surfaces are called *infinite translation surfaces*. Translation surfaces arise naturally in the study of certain dynamical systems, Teichmüller theory, and the theory of hyperbolic 3-manifolds.

The set of holomorphic differentials with zeros of orders $k_1, \ldots, k_j$ up to permutation is called the stratum $\mathcal{H}(k_1, \ldots, k_j)$. The stratum is a complex affine manifold through the period coordinates. The group $GL(2, \mathbb{R})$ acts on the stratum by post-composition with the charts of the translation surfaces. This action preserves the measure induced by the periodic coordinates, and is ergodic with respect to an invariant measure defined through period coordinate. Since [EMM], there have been many major breakthroughs on the study of orbit closures of this group action by Wright, Filip, and others.

An approach to study the strata of translation surfaces is through its boundary, which requires a "good" compactification. My PhD advisor John Smillie developed a way to compactify the stratum similar to the Borel-Serre compactification, which embedded the strata into a compact real orbifold with corners.

We are currently working on understanding the properties of this compactification. For example, we described the strata of meromorphic differentials that contain differentials with zero residue at all the cone points, and showed that:

Theorem 3
Let $\mathcal{H} = \mathcal{H}(n_1, \ldots, n_r, -p_1, \ldots, -p_s)$ *with genus* $g = 1 + \frac{1}{2}(\sum_i n_i - \sum_j p_j)$.

(i) When $g > 0$, $\mathcal{H}$ *contains elements with zero residue at all poles iff* $p_j > 1$ *for all* j.

(ii) When $g = 0$, $\mathcal{H}$ *contains elements with zero residue at all poles iff* $p_j > 1$ *for all* j, *and* $n_i \leq \sum_j p_j - s - 1$ *for all* i.

We also built affine manifold models of this compactification in a few small strata. This compactification can also be used to give an alternative proof of Theorem 6 in Section 3.1.

3 Previous Projects

3.1 Abelian cover of the flat pillowcase

A finite translation surface whose $GL(2,\mathbb{R})$ orbit is closed is called a *lattice surface*, and the discrete subgroup of $SL(2,\mathbb{R})$ formed by the derivatives of its affine diffeomorphisms is called its *Veech group*. One of the first classes of examples of lattice surfaces is the square-tiled surfaces, which are branched covers of the flat torus with one single branch point. The Veech groups of square-tiled surfaces are finite index subgroups of $SL(2,\mathbb{Z})$, and the algorithm to calculate them was described in [S].

A class of square-tiled surfaces that has been extensively studied is the abelian branched covers of the flat pillowcase. For example, [Wr1] studied the Veech group action on their cohomology; [MY] calculated the full affine diffeomorphism group action on the relative homology of two well-known translation surfaces in this class, the Wollmilchsau and the Ornithorynque; and [BM] used them to construct and study a class of non-square-tiled lattice surfaces which was further described in [Wr2]. In order to answer a question raised by Smillie and Weiss in their study of horocycle orbit closure, I calculated the affine diffeomorphism group action on the relative cohomology of an abelian branched cover of the flat pillowcase, and showed the following:

Theorem 4 ([Wu1])

(i) Let M *be a branched cover of the pillowcase with abelian deck group* G. *Let* $\Sigma \subset M$ *be the preimage of the four cone points of the pillowcase. Let* Δ *be the set of irreducible representations of a finite abelian group* G. *The deck transformations give an action of* G *on* $H^1(M, \Sigma; \mathbb{C})$. *Let* $H^1(\rho)$ *be the sum of all irreducible sub-representations of* $H^1(M, \Sigma; \mathbb{C})$ *that are isomorphic to* ρ. *Then, we have decomposition* $H^1(M, \Sigma; \mathbb{C}) = \bigoplus_{\rho\in\Delta} H^1(\rho)$, *which is preserved by the action of the affine group* **Aff**, *though the factors may be permuted.*

(ii) Let g_i, $i = 1, 2, 3, 4$ be the elements in deck group G that correspond to the counterclockwise loops around the four cone points of the pillowcase. Then, the projection $r : H^1(M, \Sigma; \mathbb{C}) \to H^1(M; \mathbb{C})$ splits as an $\mathbf{Aff}$-module if and only if deck group G does not have a representation ρ such that only one of the four $\rho(g_i)$ is 1.

(iii) In the case when $\mathbf{Aff}$ action on $H^1(\rho)$ preserves a positive definite Hermitian norm, this action is discrete if and only if $\rho(g_1) = \rho(g_2)$, $\rho(g_3) = \rho(g_4)$ up to permutation, or if it is one of the other sporadic 28 cases.

As a consequence, I gave a positive answer to the question by Smillie and Weiss:

Theorem 5 ([Wu1])
There is a square-tiled surface M with the set of cone points Σ constructed as a normal abelian branched cover of the flat pillowcase, such that:

(i) There is a direct sum decomposition $H^1(M, \Sigma; \mathbb{R}) = N \oplus H$ preserved by the action of the group of orientation preserving affine diffeomorphisms.

(ii) There is a positive definite norm on N invariant under the affine diffeomorphism group action.

(iii) The affine diffeomorphism group action on N does not factor through a discrete group.

Following [BM, Wr2], I used the constructions in [Wu1] to study the Bouw-Möller surfaces, and found an alternative characterization of them as well as a more explicit construction of Hooper's "grid graph". In [Wu2], I defined a concept of "deformation" and showed that the lattice surfaces arise from "deforming" an abelian branched cover of flat pillowcase which also satisfy two other conditions must be branched covers of the Bouw-Möller surfaces.

As another application, Lucien Clavier and I did some calculation on horocycle orbit closures arise from "pushing" the $SL(2, \mathbb{R})$ orbit of some square-tiled surfaces. Some of the results are summarized below:

Theorem 6
There is a horocycle orbit closure in $\mathcal{H}(1, 1, 1, 1)$, which is an immersed affine submanifold with boundary, such that there are infinitely many boundary components within finite distance, and that it intersects itself transversely.

This result is proved by using the relationship of interval exchange maps and translation surfaces. It can also be understood from the perspective of Borel-Serre type compactification of the strata c.f. Section 1.1.

3.2 Singularity of infinite translation surface

An isolated singularity of a translation surface must be one of the 3 cases: a cone point of finite cone angle, where locally it is a $\mathbb{Z}/n$-branched cover of an open disc branched at the center; a cone point of infinite cone angle, where locally it is a $\mathbb{Z}$-branched cover of an open disc branched at the center; or neither of the above, in which case it is called a *wild singularity.* To study the topology of a neighborhood of a wild singularity, [BV] introduced the concept of linear approaches, which are divided into rotational components. [BV] also defined a topology on the set of linear approaches. Anja Randecker, Lucien Clavier and I investigated the quotient topology on the set of rotational components at one wild singularity, and its relationship with the topology of the set of linear approaches, and found many interesting examples. In particular, we showed the following:

Theorem 7 ([CKW])

(i) Any topological space of finitely many points is homeomorphic to the topology of rotational components at some wild singularity.

(ii) There are uncountably many wild singularities with the same topology on the space of rotational components but different topologies on the space of linear approaches.

There are still many remaining questions regarding this project. In particular, we hope to have a criteria for which kind of topologies the set of rotational component can or can not have, and we hope to use these concepts to develop a concept of strata for infinite translation surfaces. Furthermore, it seems to us that there may be some connection between the concept of rotational component and the "blow up" construction described in Section 1.2.

3.3 Area of smallest triangle and virtual triangle

Vorobets, Smillie and Weiss [SW] showed that a surface is a lattice surface if and only if there is a lower bound on the areas of embedded triangles such that all vertices are cone points, or, if and only if there is a lower bound on the length of the cross product of two non-parallel saddle connections. The former is called the area of the smallest triangle while one half of the latter is called the area of the smallest virtual triangle. In [Wu3], I calculated the area of the smallest triangle and virtual triangle of lattice surfaces in $\mathbb{H}(2)$, in the Prym loci of $\mathcal{H}(4)$, as well as the Bouw-Möller family. Furthermore, I optimized the algorithm described in [SW] and used it to show that all lattice surfaces with the area of smallest virtual triangle greater than $1/20$ are already known.

3.4 Uniform discreteness of holonomy vectors

There is a question posed by Barak Weiss on if the set of holonomy vectors of translation surfaces is always uniformly discrete. I gave a negative answer by showing:

Theorem 8 ([Wu4])

The holonomy vectors of a non-square-tiled Veech surface can not be uniformly discrete.

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