Is a typical biPerron number a pseudo-Anosov dilatation?

HYUNGRYUL BAIK, AHMAD RAFIQI, AND CHENXI WU

Abstract. We give a statistical answer to a variation of the question in the title.

Let $n \geq 2$ be fixed. Definte $\mathcal{B}_n(R)$ to be the set of bi-Perron numbers no larger than R whose characteristic polynomial has degree at most $2n$ and, $\mathcal{D}_n(R)$ be the set of dilatations of pseudo-Anosov maps with orientable invariant foliations on a surface of genus n.

We remark that $\mathcal{D}_n(R)$ is contained in $\mathcal{B}_n(R)$. The pseudo-Anosov dilatation of genus n surface is a root of an integral palindromic polynomial of degree at most $2n$ if its invariant foliations are orientable. The reason is that it is the leading eigenvalue of the action on the homology group, which is $\mathbb{Z}^{2n}$. (If we do not require the invariant foliation to be orientable, the upper bound on degree is $6n-6$: we can reduce this case to the case of orientable foliation by a double cover, and this follows from the fact that a quadratic differential on a surface of genus n has at most $2n$ zeros due to Gauss-Bonnet together with Riemann-Hurwitz formula).

On the other hand, Fried's conjecture says that $\mathcal{B}_n(R)$ is contained in $\mathcal{D}_m(R)$ for some large enough m. But a priori, m could be arbitrarily large and we do not know how to show this claim. Instead we show the following

Theorem 1. *Let $\mathcal{B}_n(R)$ and $\mathcal{D}_n(R)$ be as above. Then*

$$\limsup_{R\to\infty} \frac{|\mathcal{D}_n(R)|}{|\mathcal{B}_n(R)|} < 1.$$

Here $|A|$ means the cardinality of A for a finite set A.

Theorem 1 says that asymptotically some positive proportion of the set of bi-Perron numbers whose characteristic polynomial has degree at most $2n$ is not dilatations of pseudo-Anosov maps with orientable invariant foliations on a surface of genus n. One would disprove Fried's conjecture if one could drop the assumption on orientability of invariant foliations and the genus restriction. But this seems to be out of reach at the moment.

Let $P_n(R)$ be the set of Perron polynomials of degree n with roots no larger than R. By $f \sim g$ we mean $\exists C$ such that $\frac{1}{C} f(x) \leq g(x) \leq C f(x)$ for $x \gg 0$. By $f \lesssim g$ we mean $f = O(g)$ as $x \to \infty$.

Lemma 2. $|P_n(R)| \sim R^{n(n+1)/2}$.

Proof. $|P_n(R)| \lesssim R^{n(n+1)/2}$: Because the k-th coefficient of a monic, degree n polynomial with all roots no larger than R is no larger than $O(R^{n-k})$. The total number of such polynomials must be $O(\prod_k R^{n-k}) = O(R^{n(n+1)/2})$.

$R^{n(n+1)/2} \lesssim |P_n(R)|$: Let a_k be the k-th coefficient of a degree n monic polynimial. By Rouché's theorem, as long as

$$R^n > |a_0| + R|a_1| + \dots R^{n-1}|a_{n-1}|$$
$$\left(\frac{R}{2}\right)^{n-1}|a_{n-1}| > |a_0| + \dots \left(\frac{R}{2}\right)^n$$

This polynomial has a real root λ with the greatest magnitude which is in $(-R, R)$. Half of those polynomials (those for which this leading real root is positive) would be in $P_n(R)$.

□

Lemma 3. $\lim_{R\to\infty} \frac{|\{\textit{reducible elements in } P_n(R)\}|}{|P_n(R)|} = 0$

Proof. $|\{\text{reducible elements in } P_n(R)\}| \leq \sum_k |b_k(R)||b_{n-k}(R)|$, where $b_k(R)$ is the set of monic polynomials with roots bounded by R. By the proof of the previous lemma, the right-hand-side is $o(R^{n(n+1)/2})$. □

Let $\mathcal{P}_n(R)$ be the set of Perron numbers of level n no larger than R. The previous two lemmas imply that $|\mathcal{P}_n(R)| \sim R^{n(n+1)/2}$.

Lemma 4. $|\{x : x + 1/x \in \mathcal{P}_n(R)\}\backslash\mathcal{B}_n(R)| = o(|\mathcal{P}_n(R)|)$.

Proof. This is because when x is large, $x \sim x + 1/x$. □

Hence, $|\mathcal{B}_n(R)| \sim R^{n(n+1)/2}$.

Let $\mathcal{G}_n(R, c)$ be the set of Perron numbers in $(R/2, R)$ of level n such that all its conjugates are less than c of itself. Here $c \ll 1$.

Lemma 5. $|\mathcal{G}_n(R, c)| \sim R^{n(n+1)/2} \sim |\mathcal{B}_n(R)|$.

Proof. Let the defining polynomial of a Perron number be $x^n + a_{n-1}x^{n-1} + \dots x_0$. By Rouché's theorem, as long as

$$R^n > |a_0| + R|a_1| + \dots R^{n-1}|a_{n-1}|$$
$$\left(\frac{R}{2}\right)^{n-1}|a_{n-1}| > |a_0| + \dots \left(\frac{R}{2}\right)^n$$
$$\left(\frac{cR}{2}\right)^{n-1}|a_{n-1}| > |a_0| + \dots \left(\frac{cR}{2}\right)^n$$

And the polynomial is irreducible (which, according to Lemma 3, excludes a negligible portion of elements) with positive leading root (which excludes half of those that satisfy the other conditions), its leading root would be in $\mathcal{G}_n(R, c)$. □

Lemma 6. $\{x \in \mathcal{B}_n(R) : x + 1/x \in \mathcal{G}_n(R, c)\} \sim |\mathcal{G}_n(R, c)|$.

Proof. $\lesssim$ follows from the previous two lemmas. $\gtrsim$ is because all elements in $\{x : x + 1/x \in \mathcal{G}_n(R, c)\}$ are in $\mathcal{B}_n(R)$ for $R \gg 0$. □

The Theorem follows from Lemma 6 and [Hamen15].

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References

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Mathematisches Institut, Rheinische Friedrich-Wilhelms-Universität Bonn, Endenicher Allee 60, 53115 Bonn, Germany
E-mail address: baik@math.uni-bonn.de

Department of Mathematics, Cornell University, Malott Hall, Ithaca, NY 14853, USA
E-mail address: ar776@cornell.edu

Department of Mathematics, Cornell University, Malott Hall, Ithaca, NY 14853, USA
E-mail address: cw538@cornell.edu