Excluding a Bipartite Circle Graph
from Line Graphs

Sang-il Oum[*][†]

School of Mathematics

Georgia Institute of Technology

Atlanta, Georgia, USA

June 16, 2006

Abstract

We prove that for fixed bipartite circle graph H, all line graphs with sufficiently large rank-width (or clique-width) must contain an isomorphic copy of H as a pivot-minor. To prove this, we introduce graphic delta-matroids. Graphic delta-matroids are minors of delta-matroids of line graphs and they generalize graphic or cographic matroids.

1   Introduction

Robertson and Seymour [20] proved that every graph of sufficiently large tree-width must contain a minor isomorphic to a fixed planar graph. Their theorem was generalized to a theorem on representable matroids by Geelen, Gerards, and Whittle [9], stating that every matroid representable over a fixed finite field of sufficiently large branch-width must contain a minor isomorphic to a fixed planar matroid. (A *planar matroid* is a cycle matroid of a planar graph.)

We aim to prove the following conjecture, that is another generalization of Robertson and Seymour's grid theorem. Rank-width is a graph width parameter, like tree-width, introduced by Oum and Seymour [18] to investigate clique-width [5]. Pivot-minors of a graph G are graphs obtained from G by repeatedly applying certain operations, like minors [17]. We defer definitions of rank-width and pivot-minor to Section 2. A *circle graph* is the intersection graph of chords in a circle, see Figure 1. Bipartite circle graphs are related to planar graphs, shown by de Fraysseix [6]. (See Lemma 23.)
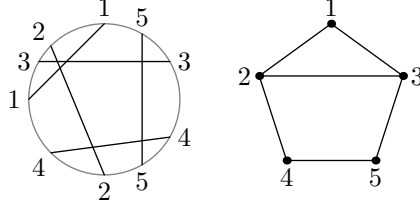

1

Figure 1: Chord diagrams and circle graphs

Conjecture. *Let H be a bipartite circle graph. Every graph G with sufficiently large rank-width must have a pivot-minor isomorphic to H.*

Surprisingly this conjecture would, if true, imply Robertson and Seymour's grid theorem as well as its matroidal version for binary matroids of Geelen et al. This conjecture is still open. Using the relation between bipartite graphs and binary matroids, Oum [17] showed that when G is bipartite, the conjecture is implied by the grid theorem for binary matroids by Geelen et al. We will also discuss in Section 9 that when G is a circle graph, the conjecture is implied by a theorem in the PhD thesis of Johnson [13] on 4-regular Eulerian digraphs.

Our main theorem is that the conjecture is true when G is a line graph. It has been known that rank-width of line graphs are bounded if and only if tree-width of their primal graphs are bounded, for instance [10, 11]. So ultimately we apply Robertson and Seymour's grid theorem on the primal graphs. But the difficulty arises because pivot-minors of a line graph may not be a line graph at all. Compare this to the fact that both bipartite graphs and circle graphs — above-discussed two valid cases — are closed under pivot-minors.

To overcome this, we define *graphic delta-matroids*. Delta-matroids are generalizations of matroids introduced by Bouchet [2]. While matroids capture combinatorial properties on linear independence of column vectors of matrices, delta-matroids capture combinatorial properties on nonsingular principal submatrices of skew-symmetric or symmetric matrices. Delta-matroids that are representable by skew-symmetric matrices over GF(2) are called *even binary delta-matroids*. Since the skew-symmetric matrices over GF(2) are equivalent to simple graphs, we observe that even binary delta-matroids naturally correspond to graphs, which we call *fundamental graphs* of delta-matroids. Graphic delta-matroids are defined in such a way that pivot-minors of line graphs are exactly fundamental graphs of graphic delta-matroids. So far this is a vague explanation and the detail will be discussed later.

We define graphic delta-matroids from a *graft*, which is a pair (G, T) of a graph G and a subset T of $V(G)$ (Section 4). We will show that graft minors naturally correspond to delta-matroid minors (Section 5) and if the branch-width of the graph in a graft is small, then the rank-width of the fundamental graph of the graphic delta-matroid given by the graft is small (Section 6). Then we use Robertson and Seymour's grid theorem on the graph in a graft and deduce our main theorem (Section 7 and 8).

2

2 Preliminaries

In our paper, graphs (sometimes called multigraphs) may have parallel edges or loops. A graph is *simple* if it has no loops and no parallel edges. All graphs in this paper will be finite.

Rank-width and branch-width. For an $A \times B$ matrix $M = (a_{ij})_{i \in A, j \in B}$ and subsets $X \subseteq A, Y \subseteq B$, let $M[X, Y]$ be the $X \times Y$ submatrix of M, that is $(a_{ij})_{i \in X, j \in Y}$ and let $M[X] = M[X, X]$. The *adjacency matrix* $A(G)$ of a graph $G = (V, E)$ is a $V \times V$ 0-1 matrix $(a_{ij})_{i,j \in V}$ over GF(2) such that $a_{ij} = 1$ if $ij \in E$ and $a_{ij} = 0$ otherwise.

For a graph G and a subset F of $E(G)$, let $\beta_G(F)$ be the number of vertices incident to both F and $E(G) \setminus F$. For a simple graph G and a subset X of $V(G)$, let $\rho_G(X)$ be $\mathrm{rank}(A(G)[X, V(G) \setminus X])$, called the *cut-rank* function. Note that when measuring the matrix rank, the matrix $A(G)[X, V(G) \setminus X]$ is considered on GF(2) because we define $A(G)$ as a matrix over GF(2). Both β_G and ρ_G are *symmetric* and *submodular*; in other words,

$$\beta_G(F) = \beta_G(E(G) \setminus F), \quad \beta_G(F_1) + \beta_G(F_2) \geq \beta_G(F_1 \cap F_2) + \beta_G(F_1 \cup F_2),$$
$$\rho_G(X) = \rho_G(V(G) \setminus X), \quad \rho_G(X_1) + \rho_G(X_2) \geq \rho_G(X_1 \cap X_2) + \rho_G(X_1 \cup X_2). \text{ (see [18])}$$

For a symmetric submodular function $f : 2^V \to \mathbb{Z}$ on a finite set V, the branch-width is defined as follows. A tree is *subcubic* if all vertices have degree 1 or 3. A *branch-decomposition* of the symmetric submodular function f is a pair (T, μ) of a subcubic tree T and a bijective function $\mu : V \to \{t : t \text{ is a leaf of } T\}$. (If $|V| \leq 1$ then f admits no branch-decomposition.) For an edge e of T, the connected components of $T \setminus e$ induce a partition (X, Y) of the set of leaves of T. The *width* of an edge e of a branch-decomposition (T, μ) is $f(\mu^{-1}(X))$. The *width* of (T, μ) is the maximum width of all edges of T. The *branch-width* of f is the minimum of the width of all branch-decompositions of f. (If $|V| \leq 1$, we define that the branch-width of f is $f(\emptyset)$.)

Branch-width of graphs was defined by Robertson and Seymour [20]. Branch-width, branch-decomposition of a graph G is defined as branch-width, branch-decomposition of β_G respectively. Oum and Seymour [18] defined rank-width of simple graphs as follows: rank-width, rank-decomposition of a simple graph G is branch-width, branch-decomposition of ρ_G respectively. Rank-width is related to clique-width (introduced by Courcelle and Olariu [5]) in the following manner: rank-width is at most clique-width, and clique-width is at most $2^{1+k} - 1$ in which k is the rank-width (Oum and Seymour [18]).

Matroids. Let us review matroid theory. For general matroid theory, we refer to Oxley's book [19]. We call $\mathcal{M} = (E, \mathcal{I})$ a *matroid* if E is a finite set and $\mathcal{I}$ is a collection of subsets of E, satisfying

(i) $\emptyset \in \mathcal{I}$

(ii) If $A \in \mathcal{I}$ and $B \subseteq A$ then $B \in \mathcal{I}$.

(iii) For every $Z \subseteq E$, maximal subsets of Z in $\mathcal{I}$ all have the same size $r(Z)$. We call $r(Z)$ the *rank* of Z.

An element of $\mathcal{I}$ is called *independent* in $\mathcal{M}$. We let $E(\mathcal{M}) = E$. A *base* is a maximally independent set. A matroid $\mathcal{M} = (E, \mathcal{I})$ is *binary* if there exists a matrix N over GF(2) such that E is a set of column vectors of N and $\mathcal{I} = \{X \subseteq E : X$ is linearly independent$\}$. The *connectivity* function $\lambda_{\mathcal{M}}$ of $\mathcal{M}$ is $\lambda_{\mathcal{M}}(X) = r(X) + r(E \setminus X) - r(E) + 1$. Branch-width, branch-decomposition of a matroid $\mathcal{M}$ is defined as branch-width, branch-decomposition of $\lambda_{\mathcal{M}}$, respectively.

For a graph G, the *cycle matroid* $\mathcal{M}(G)$ is a matroid on $E(G)$ where a set F of edges is independent if and only if F induces no cycles in G. The *fundamental graph* of a matroid $\mathcal{M} = (E, \mathcal{I})$ with respect to a base B is a bipartite graph on E such that $a \in B, b \notin B$ are adjacent if and only if $(B \setminus \{a\}) \cup \{b\}$ is independent. Oum [17] showed that the rank-width of a fundamental graph of a binary matroid $\mathcal{M}$ is exactly one less than the branch-width of $\mathcal{M}$.

Delta-matroids. A matrix A is *symmetric* if $A^t = A$, and *skew-symmetric* if $A^t = -A$ and the diagonal entries are zero. Let V be a finite set. For a $V \times V$ skew-symmetric or symmetric matrix A, let $\mathcal{F}(A) = \{X \subseteq V : A[X]$ is nonsingular$\}$. For two sets A, B, let $A \Delta B = (A \setminus B) \cup (B \setminus A)$. Bouchet [3] showed that $\mathcal{F} = \mathcal{F}(A)$ satisfies the *symmetric exchange axiom*:

$$\text{If } X, Y \in \mathcal{F} \text{ and } x \in X \Delta Y, \text{ then there is } y \in X \Delta Y \text{ such that } X \Delta \{x, y\} \in \mathcal{F}. \tag{1}$$

A *delta-matroid* is a pair $(V, \mathcal{F})$ of a finite set V and the *nonempty* collection $\mathcal{F}$ of subsets of V such that (1) is satisfied. Elements of $\mathcal{F}$ are called *feasible* sets. If $\mathcal{M} = (V, \mathcal{F})$ is a delta-matroid, then for a subset X of V, $\mathcal{M} \Delta X = (V, \mathcal{F} \Delta X)$ where $\mathcal{F} \Delta X = \{F \Delta X : F \in \mathcal{F}\}$ is a delta-matroid. This operation is called *twisting*. A *representable delta-matroid* is a delta-matroid whose set of feasible sets is $\mathcal{F}(A) \Delta X$ for a skew-symmetric or symmetric matrix A and a subset X of V. A representable delta-matroid is *binary* if the underlying field of the matrix is GF(2). A delta-matroid is *even* if $|X \Delta Y|$ is even for all feasible sets X and Y. Consequently when a delta-matroid $\mathcal{M} = (V, \mathcal{F})$ is an even binary delta-matroid, there is a skew-symmetric matrix A over GF(2) and a subset X of V such that $\mathcal{F} = \mathcal{F}(A) \Delta X$. The simple graph having A as the adjacency matrix is called a *fundamental graph* of the even binary delta-matroid $\mathcal{M}$. It is easy to construct a fundamental graph of even binary delta-matroids: pick a feasible set X and construct a simple graph on V such that two vertices $a, b \in V$ are adjacent if and only if $X \Delta \{a, b\}$ is feasible.

If there is a feasible subset of $V \setminus X$, then $\mathcal{M} \setminus X = (V \setminus X, \{F : F \in \mathcal{F}, F \cap X = \emptyset\})$ is a delta-matroid. This operation is called *deletion*. A *minor* of a delta-matroid $\mathcal{M}$ is a delta-matroid obtained from $\mathcal{M}$ by repeatedly applying twisting and deletion.
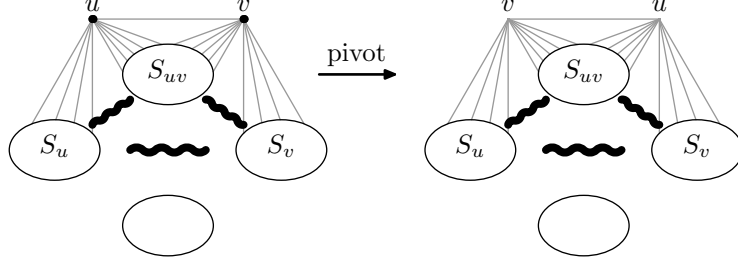
4

Figure 2: Pivoting (edges between three sets are 'toggled' and u, v are switched)

Pivoting. For a symmetric or skew-symmetric matrix A, if A is partitioned as

$$A = \begin{matrix} & \begin{matrix} X & Y \end{matrix} \\ \begin{matrix} X \\ Y \end{matrix} & \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \end{matrix}$$

and $A[X] = \alpha$ is nonsingular, then let

$$A * X = \begin{matrix} & \begin{matrix} X & Y \end{matrix} \\ \begin{matrix} X \\ Y \end{matrix} & \begin{pmatrix} \alpha^{-1} & \alpha^{-1}\beta \\ -\gamma\alpha^{-1} & \delta - \gamma\alpha^{-1}\beta \end{pmatrix} \end{matrix}.$$

This operation is called *pivoting*. Tucker [22] showed that when $A[X]$ is nonsingular, $A*X[Y]$ is nonsingular if and only if $A[X\Delta Y]$ is nonsingular. Consequently if $\mathcal{M} = (V, \mathcal{F}(A))$ is a delta-matroid and X is a feasible set of $\mathcal{M}$, then $\mathcal{M}\Delta X = (V, \mathcal{F}(A * X))$.

This implies that all fundamental graphs of an even binary delta-matroid are obtained from one fundamental graph by pivoting (in the adjacency matrix). In addition, all fundamental graphs of an even binary delta-matroid are generated by repeatedly pivoting pairs of adjacent vertices, because every $k \times k$ nonsingular skew-symmetric matrix has a 2×2 nonsingular principal submatrix if $k \geq 2$. So when considering fundamental graphs of even binary delta-matroids, it is enough to consider the pivoting operations on a pair of vertices. Let $G \wedge uv$ denote the simple graph obtained from a simple graph G by pivoting a pair of adjacent vertices u, v of G in the adjacency matrix. This operation is also called *pivoting* an edge of simple graphs.

We can describe $G \wedge uv$ as follows. Let $G = (V, E)$ be a simple graph. For disjoint subsets S, T of V, let $[S, T] = \{st : s \in S, t \in T\}$. Let S_u be the set of vertices (other than u, v) that are adjacent to u but nonadjacent to v. Let S_v be the set of vertices (other than u, v) that are adjacent to v but nonadjacent to u. Let S_{uv} be the set of common neighbors of v and w. Let $G' = (V, E\Delta[S_u, S_v]\Delta[S_u, S_{uv}]\Delta[S_v, S_{uv}])$. Then $G \wedge uv$ is a graph obtained from G' by switching the labels of v and w. See Figure 2.

Minors of delta-matroids and pivot-minors of graphs. A *pivot-minor* of a simple graph G is a graph obtained from G by repeatedly applying pivoting edges and deleting

vertices. Pivoting preserves the rank-width [17] and therefore the rank-width of pivot-minors of a simple graph G is at most the rank-width of G. Note that pivot-minors of bipartite graphs are bipartite and pivot-minors of circle graphs are circle graphs, see Bouchet [4]. (Bouchet [4] called it *p-reductions*.)

We will now explain why pivot-minors of fundamental graphs of an even binary delta-matroid are fundamental graphs of minors of the delta-matroid. Suppose that $\mathcal{M}$ is an even binary delta-matroid and $\mathcal{M}' = \mathcal{M} \Delta X \setminus Y$ is a minor of $\mathcal{M}$. We may assume that $\mathcal{M} = (V, \mathcal{F}(A))$ for a skew-symmetric matrix A over GF(2), which is the adjacency matrix of the fundamental graph G. If $Y = \emptyset$, then $\mathcal{M}'$ is an even binary delta-matroid and its fundamental graph can be obtained from a fundamental graph of $\mathcal{M}$ by pivoting as discussed above. If Y is nonempty, we may assume that $|Y| = 1$ by the induction argument. Let $Y = \{v\}$. If $v \notin X$, then $\mathcal{M}' = (V \setminus \{v\}, \mathcal{F}(A[V \setminus \{v\}]) \Delta X)$ and therefore $G \setminus v$ is a fundamental graph of $\mathcal{M}'$. If $v \in X$, then there is a feasible set Z in $\mathcal{M}$ containing v, otherwise $\mathcal{M} \Delta X \setminus Y$ is not defined. We can pick $v \in Z$ so that $w \neq v$ and $\{v, w\}$ is feasible. Then $\mathcal{M}' = \mathcal{M} \Delta \{v, w\} \setminus \{v\} \Delta (X \setminus \{w\})$ and $G \wedge vw \setminus v$ is its fundamental graph. We conclude that pivot-minors of simple graphs exactly correspond to minors of even binary delta-matroids. This also implies that minors of even binary delta-matroids are even binary. In fact, minors of representable or binary delta-matroids are representable or binary, respectively, shown by Bouchet [3].

3   Adjacency matrices of line graphs

Kishi and Uetake [15] proved the following theorem, which later appeared in the papers by Doob [8] and Deo, Krishnamoorthy, and Pai [7].

Theorem 1 (Kishi and Uetake [15]). *For every connected simple graph G, the rank of $A(L(G))$ is either $|V(G)| - 1$ if $|V(G)|$ is odd, or $|V(G)| - 2$ otherwise.*

From Theorem 1, we deduce the following lemma.

Proposition 2. *The adjacency matrix of the line graph of a simple graph G is nonsingular if and only if every component of G is a tree with odd number of vertices.*

Proof. We may assume that G is connected. Then we have $|E(G)| \geq |V(G)| - 1 \geq \operatorname{rank}(A(L(G))$. The conclusion follows from Theorem 1, because $A(L(G))$ is nonsingular if and only if $\operatorname{rank}(A(L(G))) = |E(G)|$. $\qquad\square$

In order to extend Proposition 2 to graphs having parallel edges or loops, we define *binary line graph $BL(G)$* of a graph G as a graph on $E(G)$ such that two edges e, f of G are adjacent in $BL(G)$ if and only if both e and f are not loops and e and f share exactly one vertex as an end. We note that if G is simple, then $L(G) = BL(G)$.

Lemma 3. *The adjacency matrix of the binary line graph of a graph G is nonsingular if and only if every component of G is a tree with odd number of vertices.*

Proof. If every component of G is a tree with odd number of vertices, then G has no loops and parallel edges. Hence Proposition 2 implies that the adjacency matrix of $BL(G) = L(G)$ is nonsingular.

Now suppose that $A(BL(G))$ is nonsingular. If G has a loop e, then e has no neighbors in $BL(G)$ and therefore $A(BL(G))$ is singular. If G has parallel edges e and f, then e and f are nonadjacent in $BL(G)$ (unlike in $L(G)$) and they have the same set of neighbors. So the row of e and f are identical in $A(BL(G))$. So $BL(G)$ is nonsingular. So G is simple. Then lemma is implied by Proposition 2. $\qquad\qquad\square$

Lemma 4. *Let $G = (V, E)$ be a graph. Let $\mathcal{F}$ be the set of edge sets of spanning subgraphs H of G such that each component of H has odd number of vertices and no cycles. Then $\mathcal{M} = (E, \mathcal{F})$ is an even binary delta-matroid.*

Proof. We claim that $\mathcal{F} = \mathcal{F}(A(BL(G)))$. Let $H = (V(G), F)$ be one of such spanning subgraph. It is enough to show that $F \in \mathcal{F}$ if and only if $A(L(G))[F]$ is nonsingular. It is easy to observe that $A(L(G))[F] = A(L(H))$ and therefore $A(L(G))[F]$ is nonsingular if and only if $A(L(H))$ is nonsingular. Then our conclusion follows from Lemma 3. $\qquad\square$

4 Graphic delta-matroids from grafts

Let $G = (V, E)$ be a graph and T be a subset of vertices of G. We allow graphs G to have parallel edges, or loops. A subgraph H of G is called a *T-spanning* subgraph of G if $V(G) = V(H)$ and for each component C of H, either

  (i) $|V(C) \cap T|$ is odd, or

  (ii) $V(C) \cap T = \emptyset$ and $G[V(C)]$ is a component of G.

A *graft* is a pair (G, T) of a graph G and a subset T of vertices of G. A set F of edges of G is *feasible* in (G, T) if it is the edge set of a T-spanning forest of G. Let $\mathcal{G}(G, T) = (E(G), \mathcal{F})$ such that $\mathcal{F}$ is the set of all feasible sets of (G, T).

Theorem 5. *Let (G, T) be a graft. Then $\mathcal{G}(G, T) = (E(G), \mathcal{F})$ is an even binary delta-matroid.*

Before proving Theorem 5, we will name such delta-matroids and prove two necessary lemmas. A delta-matroid $\mathcal{M}$ is called a *graphic delta-matroid* of a graft (G, T) if there is a subset X of vertices of G such that $\mathcal{M} = \mathcal{G}(G, T)\Delta X$. Any delta-matroid $\mathcal{M}$ that is a graphic delta-matroid of a graft is called *graphic*.

Lemma 6. *If $T = V(G)$ then $\mathcal{G}(G, T)$ is an even binary delta-matroid having $BL(G)$ as a fundamental graph.*
    If $T = \emptyset$ then $\mathcal{G}(G, T)$ is the cycle matroid of G (so that each feasible set is a base of the cycle matroid) and therefore is an even binary delta-matroid.
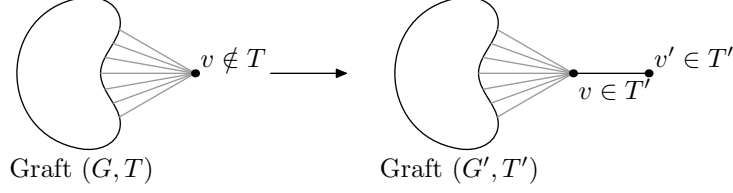
Figure 3: Illustration of proof of Theorem 5

Proof. If $T = V(G)$ then by Lemma 4, $\mathcal{G}(G, T) = (E(G), \mathcal{F}(A(BL(G))))$ and therefore it is an even binary delta-matroid.

If $T = \emptyset$ then every T-spanning forest is a spanning forest and therefore each feasible set is a base of the cycle matroid. $\square$

Lemma 7. *Every graph G has a T-spanning forest.*

Proof. Let R be a spanning tree of G. If $|T|$ is odd, then R is a T-spanning forest. If $|T|$ is even, then pick an edge $e \in E(R)$ such that $R \setminus e$ splits the tree R into two subtrees so that one of them contains exactly one vertex in T. Then $R \setminus e$ is a T-spanning forest. So the claim is proved. $\square$

Proof of Theorem 5. Suppose there is a counterexample (G, T). We may assume that $|V(G) \setminus T|$ is minimum. In addition to that, we may assume that $|V(G)|$ is minimum. By the hypothesis, G is connected, because the direct sum of two even binary delta-matroids is an even binary delta-matroid. Let $\mathcal{M} = \mathcal{G}(G, T) = (E(G), \mathcal{F})$. We may assume that $T \neq \emptyset$ and $T \neq V(G)$ by Lemma 6.

We will construct another graft (G', T') such that $|V(G)| - |T| > |V(G')| - |T'|$. Pick $v \in V(G) \setminus T$. Let G' be a graph obtained from G by adding a new vertex v' adjacent only to v. Let $T' = T \cup \{v, v'\}$. (See Figure 3.) Let $\mathcal{F}'$ be the set of all feasible sets of G'. Since $|V(G')| - |T'| = |V(G)| - |T| - 1$, we deduce that $\mathcal{M}' = (E(G) \cup \{vv'\}, \mathcal{F}')$ is an even binary delta-matroid.

The first condition for being a delta-matroid is that $\mathcal{F} \neq \emptyset$. It follow from Lemma 7.

We now aim to show that $\mathcal{M}$ is indeed a minor of $\mathcal{M}'$. We claim that $\mathcal{M} = (\mathcal{M}' \Delta \{vv'\}) \setminus \{vv'\}$. (Notice that if $T = \emptyset$ then this does not hold.) First let us assume that $F \in \mathcal{F}$. Let $C_0, C_1, \ldots, C_m$ be components of the graph $(V(G), F)$ such that $v \in V(C_0)$. Since F is feasible and T is nonempty, $|V(C_i) \cap T|$ is odd for all i. We deduce that $F \cup \{vv'\}$ is feasible in $\mathcal{M}'$ because $(V(C_0) \cup \{v'\}, E(C_0) \cup \{vv'\}), C_1, \ldots, C_m$ are components of the graph $(V(G) \cup \{v'\}, F \cup \{vv'\})$. Conversely let us assume that $F \cup \{vv'\}$ is feasible in $\mathcal{M}'$. Let C' be the component of the graph $(V(G'), F \cup \{vv'\})$ containing v. Then $|V(C') \cap T'|$ is odd. Let C be the component of $(V(G), F)$ containing v. Since $C' \setminus v' = C$ and $T = T' \setminus \{v, v'\}$, $|V(C) \cap T|$ is odd. So $F \in \mathcal{F}$. Since $\mathcal{F}$ is nonempty, $\mathcal{M}$ is a minor of $\mathcal{M}'$ and therefore it is an even binary delta-matroid, contradictary to our assumption. So the theorem is proved. $\square$
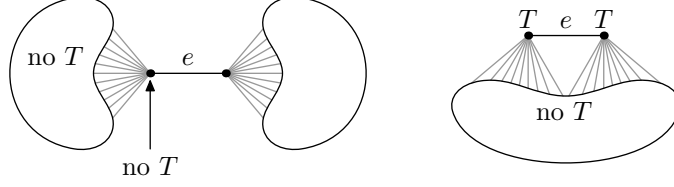
8

Figure 4: T-bridge and T-tunnel

5 Graft minors

We now define a graft minor. Let (G,T) be a graft. For an edge e of G, we define $(G,T) \setminus e$ to be $(G \setminus e, T)$. This operation is called a *deletion* of e in a graft (G,T). For an isolated vertex v of G, let $(G,T) \setminus v$ be $(G \setminus v, T \setminus \{v\})$. This is a deletion of a vertex v. We define $(G,T)/e$ as $(G/e, T')$ where T' is described as follows. Let e^* be the vertex in G/e obtained by identifying both ends u and v of e. Then

$$T' = \begin{cases} (T \setminus \{u,v\}) \cup \{e^*\} & \text{if exactly one of } u \text{ or } v \text{ is in } T, \\ T \setminus \{u,v\} & \text{otherwise.} \end{cases}$$

This operation is called a *contraction* of e in a graft (G,T). Note that if e is a loop, then we define $(G,T)/e = (G,T) \setminus e$. We say that (G',T') is a *minor* of a graft (G,T) if (G',T') is obtained from (G,T) by a sequence of deletions and contractions.

Let $\kappa(G,T)$ be the number of components of G having no vertices in T. An edge e is a *T-bridge* of a graft (G,T) if $\kappa(G \setminus e, T) > \kappa(G,T)$. (See Figure 4.) We now relate graft minors to delta-matroid minors. Our goal is to show that the set of graphic delta-matroids is minor-closed.

Proposition 8. *Every feasible set of (G,T) contains an edge e if and only if e is a T-bridge.*

Proof. Suppose $e = uv$ is a T-bridge. Let $H = (V(G), F)$ be a T-spanning forest of G. Since $\kappa(G \setminus e, T) > \kappa(G,T)$, there is a component C of $G \setminus e$ containing one of the ends of e such that $V(C) \cap T = \emptyset$. We may assume that $u \in V(C)$, $v \notin V(C)$. Since H is a subgraph of $G \setminus e$, H has a component D that is a subgraph of C. So $V(D) \cap T \subseteq V(C) \cap T = \emptyset$. Since F is feasible, D should span a component of G. But D does not contain v. Contradiction.

Conversely suppose e is not a T-bridge. By Lemma 7, $G \setminus e$ has at least one T-spanning forest H. It is easy to see that H is a T-spanning forest of G. $\qquad\square$

Proposition 9. *Let (G,T) be a graft. For an edge e of G, we have*

$$\mathcal{G}((G,T) \setminus e) = \begin{cases} \mathcal{G}(G,T) \setminus \{e\} & \text{if } e \text{ is not a } T\text{-bridge,} \\ \mathcal{G}(G,T) \triangle \{e\} \setminus \{e\} & \text{otherwise.} \end{cases}$$

Proof. Let us first show the proposition when e is not a T-bridge. If $H = (V(G), F)$ is a T-spanning forest of $G \setminus e$, then H is a T-spanning forest of G as well. Conversely, if H

is a T-spanning forest of G such that $e \notin E(H)$, then H is a T-spanning forest of $G \setminus e$. Therefore $\mathcal{G}(G \setminus e, T) = \mathcal{G}(G, T) \setminus \{e\}$.

Now it remains to consider the case when e is a T-bridge. If $H = (V(G), F)$ is a T-spanning forest of G, then $e \in F$ by Proposition 8. Moreover $H \setminus e$ is a T-spanning forest of $G \setminus e$. Conversely, if $H = (V(G), F)$ is a T-spanning forest of $G \setminus e$ then $H' = (V(G), F \cup \{e\})$ is a T-spanning forest of G. $\qquad \square$

An edge $e = uv$ of a graft (G, T) is called a T-*tunnel* if $V(C) \cap T = \{u, v\}$ for the component C of G containing e. (See Figure 4.)

Proposition 10. *No feasible set of (G, T) contains an edge e if and only if e is a loop or a T-tunnel.*

Proof. If e is a loop, then it is trivial. Let e be a T-tunnel and C be the component of G containing e. Suppose that there is a T-spanning forest H containing e. Then $|V(C) \cap T| = 2$. Contradiction. So no T-spanning forest contains e.

Conversely we claim that if e is neither a T-tunnel nor a loop, then there is a T-spanning forest of G containing e. We proceed by induction on $|V(G)|$. If $|V(G)| = 2$, then $|T| \leq 1$. Then $\{e\}$ is feasible in (G, T).

Now let us assume that $|V(G)| > 2$. If G is disconnected then disjoint union of T-spanning forests of each component is a T-spanning forest of G. Therefore we may assume that G is connected.

Let R be a spanning tree of G containing e. If $T = \emptyset$ or $|T|$ is odd, then R is a T-spanning forest of G. So we may assume that $|T| \geq 2$ and $|T|$ is even.

If $R \setminus e$ has a component C such that $|V(C) \cap T| \geq 2$, then pick $v \in V(C) \cap T$ such that the distance from e to v on R is maximum. Let $f \in E(R)$ be an edge incident to v such that $R \setminus f$ disconnects v from e. Then $R \setminus f$ has a component D such that $V(D) \cap T = \{v\}$ and therefore $|V(D) \cap T|$ is odd. Since T is even, for the component D' of $R \setminus f$ other than D, $|V(D') \cap T|$ is odd as well. So $R \setminus f$ is a T-spanning forest of G containing e. Therefore we may assume that both components of $R \setminus e$ has exactly one vertex in T.

Let $T = \{v_1, v_2\}$. Let P be a path from v_1 to v_2 on R. Since e is not a T-tunnel, either $e \notin E(P)$ or the length of P is at least two. So we pick an edge f on P such that $f \neq e$. Then $R \setminus f$ is a T-spanning forest of G containing e. $\qquad \square$

Proposition 11. *Let (G, T) be a graft. For an edge e of G, let e^* be the vertex in G/e obtained by identifying both ends of e in G. Then we have*

$$\mathcal{G}((G, T)/e) = \begin{cases} \mathcal{G}(G, T) \triangle \{e\} \setminus \{e\} & \text{if } e \text{ is neither a } T\text{-tunnel nor a loop,} \\ \mathcal{G}(G, T) \setminus \{e\} & \text{otherwise.} \end{cases}$$

Proof. We may assume that e is not a loop. If e is a loop, then we use Proposition 9.

Suppose that e is not a T-tunnel. Let u and v be ends of e. Let T^* be either $T \setminus \{v, w\} \cup \{e^*\}$ or $T \setminus \{v, w\}$ so that $|T| \equiv |T^*| \pmod 2$.

Let H^* be a T^*-spanning forest of G/e. The edge set of H^* is a feasible set of $(G, T)/e$. We claim that $H = (V(G), E(H^*) \cup \{e\})$ is a T-spanning forest of G. Let C^* be the component

10

of H^* containing e^*. Let C be the component of H corresponding to C^*. It is enough to show that either $|V(C) \cap T|$ is odd or $V(C) \cap T = \emptyset$ and $G[V(C)]$ is a component of G. Suppose that $|V(C) \cap T|$ is even. Then $|V(C^*) \cap T^*|$ is even because $u, v \in V(C)$. Since H^* is a T^*-spanning forest, $(G/e)[V(C^*)]$ is a component of G/e and therefore $G[V(C)]$ is a component of G. Moreover $V(C^*) \cap T^* = \emptyset$ because $V(C) \cap T = \emptyset$.

Conversely let H be a T-spanning forest containing e. We aim to show that H/e is a T^*-spanning forest of G/e. Let C be the component of H containing e. Let C^* be the component of H/e containing e^*. Suppose that $|V(C^*) \cap T^*|$ is even. Then $|V(C) \cap T|$ is even and therefore $G[V(C)]$ is a component of G and $V(C) \cap T = \emptyset$. It follows that $V(C^*) \cap T^* = \emptyset$ and $(G/e)[V(C^*)]$ is a component of G. Therefore H/e is a T^*-spanning forest of G/e.

Now it remains to consider the case when e is a T-tunnel. Let H^* be a T^*-spanning forest of G/e. Let $H = (V(G), E(H^*))$ be a graph. We claim that H is a T-spanning forest of G. Let C^* be the component of G/e containing e^*. Since e is a T-tunnel, $V(C^*) \cap T^* = \emptyset$ and therefore $(G/e)[V(C^*)]$ is a component of G. Let $C = (V(G), E(C^*))$ be a subgraph of H corresponding to C^*. Since $e \notin E(C)$, C is a disjoint union of two trees C_1 and C_2 such that $v \in V(C_1)$ and $w \in V(C_2)$. Because e is a T-tunnel, $V(C_1) \cap T = \{v\}$ and $V(C_2) \cap T = \{w\}$. It follows that H is a T-spanning forest of G.

We now aim to prove the converse. Let H be a T-spanning forest. We claim that $H^* = (V(G/e), E(H))$ is a T^*-spanning forest of G/e. Let C_1 be the component of H containing u and C_2 be the component of H containing v. Since e is a T-tunnel, $V(C_1) \cap T = \{v\}$ and $V(C_2) \cap T = \{w\}$. Moreover $G[V(C_1) \cup V(C_2) \cup \{e\}]$ is a component of G because the component of G containing e has exactly two vertices u and v in T. Let C^* be the component of H^* containing e^*. It follows that $V(C^*) \cap T^* = \emptyset$ and $(G/e)[V(C^*)]$ is a component of G/e. Therefore H^* is a T^*-spanning forest of G/e. $\qquad\square$

Proposition 12. *Let (G, T) be a graft and v be an isolated vertex of G. Then*

$$\mathcal{G}((G, T) \setminus v) = \mathcal{G}(G \setminus v, T \setminus \{v\}).$$

Proof. This proof is easy and and so omitted. $\qquad\square$

Theorem 13. *The graphic delta-matroid of a minor of a graft (G, T) is a minor of the graphic delta-matroid of (G, T). Conversely, a minor of a graphic delta-matroid is graphic.*

Proof. The first statement follows from Proposition 9, 11, and 12.

Let us prove the converse. Let us show that if $\mathcal{M}_1 = (E_1, \mathcal{F}_1)$ is a minor of a graphic delta-matroid $\mathcal{M}_2 = (E_2, \mathcal{F}_2)$, then $\mathcal{M}_1$ is graphic. We will proceed by induction on $|E_1 \setminus E_2|$. If $E_1 = E_2$, then $\mathcal{M}_1 = \mathcal{M}_2 \Delta X$ for some $X \subseteq E_2$. So $\mathcal{M}_1$ is graphic. If $E_1 \neq E_2$, we may assume that $|E_2 \setminus E_1| = 1$. Let $E_2 \setminus E_1 = \{e\}$. By twisting, we may assume that $\mathcal{M}_2 = \mathcal{G}(G, T)$ for a graft (G, T). Without loss of generality, we may assume that either $\mathcal{M}_1 = \mathcal{M}_2 \Delta \{e\} \setminus \{e\}$ or $\mathcal{M}_1 = \mathcal{M}_2 \setminus \{e\}$. Since $\mathcal{M}_1$ is a delta-matroid, $\mathcal{F}_1$ is nonempty.

If e is neither a T-tunnel nor a T-bridge, then $\mathcal{M}_1$ is either $\mathcal{G}((G, T)/e)$ or $\mathcal{G}((G, T) \setminus e)$.

If e is a T-tunnel, then $\mathcal{M}_1 = \mathcal{M}_2 \setminus \{e\}$ because no feasible set of $\mathcal{M}_2$ contains e by Proposition 10 and $\mathcal{F}_1$ is nonempty. So $\mathcal{M}_1 = \mathcal{G}((G, T)/e)$.

If e is a T-bridge, then $\mathcal{M}_1 = \mathcal{M}_2 \Delta \{e\} \setminus \{e\}$ because every feasible set of $\mathcal{M}$ contains e by Proposition 8. So $\mathcal{M}_1 = \mathcal{G}((G, T) \setminus e)$. Therefore the claim is proved. $\qquad\square$

6   Branch-width and rank-width

We would like to find a connection from the rank-width of a graphic delta-matroid of a graft (G, T) to the branch-width of a graph G. The *rank-width* of an even binary delta-matroid is defined as the rank-width of its fundamental graph.

For a graft (G, T) and a subset X of $E(G)$, let $\rho_{(G,T)}(X) = \rho_H(X)$ where H is a fundamental graph of $\mathcal{G}(G, T)$. Since all fundamental graphs are obtainable from one fundamental graph by pivoting and pivoting preserves the cut-rank functions, $\rho_{(G,T)}(X)$ is well-defined.

Lemma 14. *Let (G, T) be a graph. If X is a subset of $E(G)$, then $\rho_{(G,T)}(X) \leq \beta_G(X)$.*

Proof. Suppose the lemma is false. Pick a counterexample (G, T) and X so that $\beta_G(X)$ is minimum and among those, $|T| + 3|V(G) \setminus T|$ is minimum. Clearly $X \neq \emptyset$ and $X \neq V(G)$.

First we claim that $T = V(G)$. Suppose $v \in V(G) \setminus T$. Then let G' be a graph obtained by adding a new vertex v' and a new edge vv' to G (Figure 3). Let $T' = T \cup \{v, v'\}$. We may assume that v is incident with an edge in $E(G) \setminus X$ or v is an isolated vertex in G; otherwise we replace X with $E(G) \setminus X$. Then $\beta_{G'}(X) = \beta_G(X)$. and $|T| + 3|V(G') \setminus T'| = |T| + 3|V(G) \setminus T| - 1$. By assumption, $\rho_{(G',T')}(X) \leq \beta_{G'}(X)$. Since (G, T) is a minor of (G', T'), $\rho_{(G,T)}(X) \leq \rho_{(G',T')}(X)$. So $\rho_{(G,T)}(X) \leq \beta_G(X)$. A contradiction.

Let M be the set of vertices meeting both X and $E(G) \setminus X$. By definition, $\beta_G(X) = M$.

We claim that no edges have both ends in M. Suppose that there is an edge e in $G[M]$. We may assume that $e \notin X$ by replacing X with $E(G) \setminus X$ if necessary. Then $\beta_{G/e}(X) = \beta_G(X) - 1$. By assumption, $\rho_{(G,T)/e}(X) \leq \beta_{G/e}(X)$. Since a fundamental graph of $(G, T)/e$ is obtained from a fundamental graph of (G, T) by some number of pivoting and exactly one deleting, we have $\rho_{(G,T)}(X) \leq \rho_{(G,T)/e}(X) + 1$. Therefore $\rho_{(G,T)}(X) \leq \beta_G(X)$. A contradiction to our assumption.

Since $T = V(G)$, $BL(G)$ is a fundamental graph of (G, T) by Lemma 6. Let $M = \{a_1, a_2, \ldots, a_k\}$. For $i \in \{1, 2, \ldots, k\}$, let X_i (Y_i) be the set of non-loop edges in X ($E(G) \setminus X$) incident to a_i, respectively. We know that $e \in X_i$ and $f \in Y_j$ are adjacent in $BL(G)$ if and only if $i = j$. Therefore $\rho_{(G,T)}(X) = \rho_{BL(G)}(X) \leq k = \beta_G(X)$. (We have an inequality here because some X_i or Y_i could be an empty set.)   □

Theorem 15. *Let (G, T) be a graft. Then the rank-width of a fundamental graph of $\mathcal{G}(G, T)$ is at most the branch-width of G.*

Proof. This is implied by Lemma 14.   □

The following discussion until the end of this section is irrelevant to our main theorem but it may be of independent interest. The previous theorem implies that for a graph G, the rank-width of $BL(G)$ is at most the branch-width of G. If G is a simple graph, then the rank-width of $L(G)$ is at most the branch-width of G. What could happen if G is not a simple graph?

Corollary 16. *Let G be a graph. Then the rank-width of the line graph of G is at most the branch-width of G.*

Proof. Suppose not. Let G be a counterexample with minimum number of loops and minimum number of parallel edges. If G is a simple graph, then it is implied by Theorem 15 when $T = V(G)$.

If G has a loop e, then we consider another graph G' obtained by adding a new vertex v' and replace e by a new edge joining the end of e with v'. Then $L(G) = L(G')$. Since the number of loops in G' is less than the number of loops in G, the rank-width of $L(G')$ is at most the branch-width of G'. This is a contradiction because branch-width of G' is equal to branch-width of G'. So G has no loops.

Thus, G has an edge e parallel to f. The branch-width of $G \setminus e$ is smaller than or equal to the branch-width of G. So the rank-width of $L(G \setminus e)$ is smaller than or equal to the branch-width of G. In order to obtain $L(G)$ from $L(G \setminus e)$, we can add a vertex e as a copy of f and then make e and f adjacent. This operation (creating *twins*) preserves the rank-width, shown by Oum [17]. So the rank-width of $L(G)$ is equal to the rank-width of $L(G \setminus e)$. A contradiction. $\square$

Let us now prove that if the rank-width of a fundamental graph of a graphic delta-matroid is small, then the graph in the graft has small branch-width. Recall that $\lambda_{\mathcal{M}(G)}$ is the matroid connectivity function defined as $r(X) + r(E(G) \setminus X) - r(E(G)) + 1$ where r is the rank function of $\mathcal{M}(G)$.

Lemma 17. *Let (G, T) be a graft. If G is connected and $|T|$ is odd or $|T| = 0$, then $\lambda_{\mathcal{M}(G)}(X) - 1 \leq \rho_{(G,T)}(X)$.*

Proof. Let T be a spanning tree of G and F be the set of edges of T. Let $M = (m_{ij} : i \in F, j \in E(G))$ be an $F \times E(G)$ matrix over GF(2) such that $m_{ij} = 1$ if and only if i is in the fundamental cycle of j with respect to T. It is known that

$$
\begin{array}{cc}
 & F \quad\; E(G) \setminus F \\
F & \left(\begin{array}{cc} I_F & M \end{array} \right)
\end{array}
$$

is the matrix representation of $\mathcal{M}(G)$, which means that a set of edges is independent in $\mathcal{M}(G)$ if and only if corresponding column vectors are linearly independent. (The matrix I_F is the $F \times F$ identity matrix.) Now notice that $r(Z) = \text{rank}(M[F \setminus Z, Z \setminus F]) + |Z \cap F|$ for all subsets Z of $E(G)$. Let $Y = E(G) \setminus X$. So $\lambda_{\mathcal{M}(G)}(X) = r(X) + r(Y) - r(E(G)) + 1 = \text{rank}(M[F \setminus X, X \setminus F]) + \text{rank}(M[F \setminus Y, Y \setminus F]) + 1$.

Since $|T|$ is odd or $|T| = 0$, F is feasible in $\mathcal{G}(G, T)$. Let H be the fundamental graph of $\mathcal{G}(G, T)$ with respect to F. Let $N = A(H)[X, E(G) \setminus X]$, and then $\rho_{(G,T)}(X) = \text{rank}(N)$. Since F is maximally feasible in $\mathcal{G}(G, T)$, $F\Delta\{x, y\}$ is not feasible when $x, y \in F$ and therefore $N[X \setminus F, E(G) \setminus (X \cup F)]$ is a zero matrix. By definition, $N[X \setminus F, F \setminus X] = M[X \setminus F, F \setminus X]$, $(N[F \cap X, E(G) \setminus (X \cup F)])^t = M[F \setminus Y, Y \setminus F]$. We have the following inequality: $\text{rank}(N) \geq \text{rank}(M[X\setminus F, F\setminus X]) + \text{rank}(N[F\cap X, E(G)\setminus(X\cup F)]) = \text{rank}(M[F\setminus X, X \setminus F]) + \text{rank}(M[F \setminus Y, Y \setminus F]) = \lambda_{\mathcal{M}(G)} - 1$. $\square$

Lemma 18. *Let (G, T) be a graft. If G is connected, then $\lambda_{\mathcal{M}(G)}(X) - 2 \leq \rho_{(G,T)}(X)$.*

Proof. If $|T|$ is odd, then it is implied by Lemma 17. So we may assume that $|T|$ is even.

Let v be a vertex of G. Let G' be a graph obtained from G by adding a new vertex v' and a new edge vv'. Let $T' = T \cup \{v'\}$. Then (G', T') is a graft such that $|T'|$ is even. Moreover (G, T) is a minor of (G', T'), obtained by deleting vv' and deleting v'. By symmetry of $\lambda_{\mathcal{M}(G)}$, we may assume that $\lambda_{\mathcal{M}(G')}(X) = \lambda_{\mathcal{M}(G)}(X)$. By Lemma 17, $\lambda_{\mathcal{M}(G')}(X) - 1 \leq \rho_{(G', T')}(X)$. Since $\mathcal{G}(G, T)$ is obtained by some pivoting and one deleting from $\mathcal{G}(G', T')$, $\rho_{(G', T')}(X) \leq \rho_{(G, T)}(X) + 1$. So $\lambda_{\mathcal{M}(G)}(X) - 2 \leq \rho_{(G, T)}(X)$. $\qquad\square$

The *branch-width of a matroid* $\mathcal{M}$ is defined as the branch-width of the matroid connectivity function $\lambda_{\mathcal{M}}$.

Theorem 19. *Let (G, T) be a graft. If the branch-width of the cycle matroid of G is k, then the rank-width of a fundamental graph of $\mathcal{G}(G, T)$ is at least $k - 2$.*

Proof. We may assume that G is connected. Then this is implied by Lemma 18. $\qquad\square$

The following was a conjecture for a while, but Hicks and McMurray [12] and Mazoit and Thomassé [16] independently claimed that they proved this.

Conjecture. *Let G be a graph with no cutedge. Then branch-width of G is equal to branch-width of $\mathcal{M}(G)$.*

Assuming this conjecture is true, we obtain the following: for a graft (G, T), if G is a graph with no cutedge and branch-width k, then the rank-width of a fundamental graph of $\mathcal{G}(G, T)$ is $k - 2$, $k - 1$, or k.

7 Obtaining the grid minor

We are now going towards the final theorem. The following lemma is a direct consequence of Robertson and Seymour's theorem on graphs of large tree-width.

Lemma 20. *For each positive integer n, there is a constant c such that if the rank-width of a fundamental graph of the graphic delta-matroid of a graft (G, T) is larger than c, then there is a minor (G', T') of (G, T) such that G' is isomorphic to the $3n \times 3n$ grid.*

Proof. Let N be a constant so that if the branch-width of a graph G is larger than N then G contains a minor isomorphic to the $3n \times 3n$ grid. By Theorem 15 if the rank-width of a fundamental graph of $\mathcal{G}(G, T)$ is larger than N then branch-width of G is larger than N and therefore G has a minor isomorphic to the $3n \times 3n$ grid. Since every graph minor operation corresponds to graft minor operations by Proposition 9, 11, and 12, we deduce that (G, T) has a minor (G', T') such that G' is isomorphic to the $3n \times 3n$ grid. $\qquad\square$

In Lemma 20, we obtained a graft (G, T) such that G is the $3n \times 3n$ grid but T could be arbitrary. We aim to obtain a minor (G', T') of (G, T) such that G' is an $n \times n$ grid and $T' = \emptyset$.

Lemma 21. *Let $G = (V, E)$ be the $3n \times 3n$ grid so that*

$$V = \{(i, j) : 0 \leq i, j \leq 3n - 1, i, j \in \mathbb{Z}\}$$

and two vertices (a, b) and (c, d) are adjacent if $|a - c| + |b - d| = 1$. Let T be a subset of V. Then there is a minor (G', T') of a graft (G, T) such that G' is isomorphic to the $n \times n$ grid and $T' = \emptyset$.

Proof. We will partition V into 3×3 blocks, delete at most one vertex from each block, and then contract each block to a vertex. Each block consists of vertices $(3i + a, 3j + b)$ where $a, b \in \{0, 1, 2\}$.

If a block has an even number of vertices in T, then we do not delete any vertex in the block. We contract the block into a vertex v and then v will not belong to T.

If a block has an odd number of vertices in T, then we pick one vertex w in T and delete all incident edges to w and then delete w. After removal of v, the block is still connected because the 3×3 grid is 2-connected. Furthermore it has an even number of vertices in T. We now contract the block into a single vertex and then the vertex will not belong to T.

It is evident that after treating each block, we get $n \times n$ vertices. It remains to show that edges are present in the way of the $n \times n$ grid. There are three edges connecting two adjacent blocks. We remove at most one vertex from each block and therefore at least one edge remains. So after contraction, the graph is indeed isomorphic to the $n \times n$ grid. $\square$

8   Excluding a bipartite circle graph from line graphs

Now we are ready to prove our main theorem.

Theorem 22. *Let H be a simple bipartite circle graph. Then there is a constant $c(H)$ such that if (G, T) is a graft and a fundamental graph K of $\mathcal{G}(G, T)$ has rank-width larger than $c(T)$, then H is isomorphic to a pivot-minor of K.*

We will need the following two lemmas.

Lemma 23 (de Fraysseix [6]). *A simple graph is a bipartite circle graph if and only if it is a fundamental graph of a planar matroid.*

Lemma 24 (Robertson, Seymour, and Thomas [21, (1.5)]). *If H is a planar graph with $|V(H)| + 2|E(H)| \leq n$, then H is isomorphic to a minor of the $2n \times 2n$ grid.*

Proof of Theorem 22. By Lemma 23, H is a fundamental graph of a cycle matroid of a planar graph J. Then H is a fundamental graph of $\mathcal{G}(J, \emptyset)$. Let n be the minimum number so that J is a minor of the $n \times n$ grid G_n. (By Lemma 24, the number n exists.) It follows that $(J, \emptyset)$ is a minor of $(G_n, \emptyset)$.

By Lemma 20, there is a constant c such that if the rank-width of K is larger than c then there is a minor (G', T') such that G' is isomorphic to the $3n \times 3n$ grid. By Lemma 21, (G', T') has a minor $(G'', \emptyset)$, where G'' is isomorphic to G_n.

Since $(J, \emptyset)$ is a minor of $(G_n, \emptyset)$, we deduce that there is a minor $(J', \emptyset)$ of (G, T) such that J' is isomorphic to J.

It follows that there is a pivot-minor H' of a fundamental graph of $\mathcal{G}(G, T)$ such that H' is isomorphic to H. In particular, there is a pivot-minor H' of the line graph $L(G)$ such that H' is isomorphic to H. $\qquad\square$

The above theorem implies that for a fixed bipartite circle graph H, every *binary line graph* of sufficiently large rank-width must have a pivot-minor isomorphic to H. In order to prove this for line graphs, it remains to consider the remaining little case when the graph has parallel edges or loops.

Theorem 25. *Let H be a fixed bipartite circle graph. There is a constant $c(H)$ such that if the line graph $L(G)$ of a graph G has rank-width larger than $c(H)$, then H is isomorphic to a pivot-minor of $L(G)$.*

Proof. Suppose not. Let $c(H)$ be the constant given by Theorem 22 for H. Let G be a counterexample with minimum number of loops and minimum number of parallel edges. Then $L(G)$ has rank-width larger than $c(H)$. If G is simple then $BL(G) = L(G)$ and therefore the theorem is implied by Theorem 22 and Lemma 6.

If G has a loop e, then we consider another graph G' obtained by adding a new vertex v' and replace e by a new edge joining the end of e with v'. Then $L(G) = L(G')$. By the assumption that G is a counterexample with minimum number of loops, H is isomorphic to a pivot-minor of $L(G') = L(G)$. So we may assume that G has no loops.

So G has an edge e parallel to f. Then the rank-width of $L(G \setminus e)$ is equal to the rank-width of $L(G)$, because $L(G)$ is obtained from $L(G \setminus e)$ by adding a twin of f and adding a twin preserve rank-width, shown by Oum [17]. Then $L(G \setminus e)$ has a pivot-minor isomorphic to H. Since $L(G \setminus e) = L(G) \setminus e$, $L(G) \setminus e$ is a pivot-minor of $L(G)$. Thus, H is isomorphic to a pivot-minor of $L(G)$. A contradiction. $\qquad\square$

9   Excluding a bipartite circle graph from circle graphs

In Section 1, we mentioned that the conjecture has been proved for bipartite graphs by Oum [17]. In this section, we would like to present how the PhD thesis of Johnson [13] implies the conjecture for circle graphs. This is the theorem in his thesis.

Theorem 26 (Johnson [13, Theorem 2.5]). *Let n be a positive integer. There is a positive integer t such that if D is an Eulerian digraph with tree-width at least t, then the medial grid of size n is immersed in D.*

We need a few definitions to decode this theorem properly. An *Eulerian digraph* is a digraph such that the in-degree is equal to the out-degree at each vertex. For our purpose it is enough to consider *4-regular Eulerian digraphs* such that it is connected and each vertex has in-degree 2 and out-degree 2, even though Johnson's theorem works for general Eulerian digraphs.
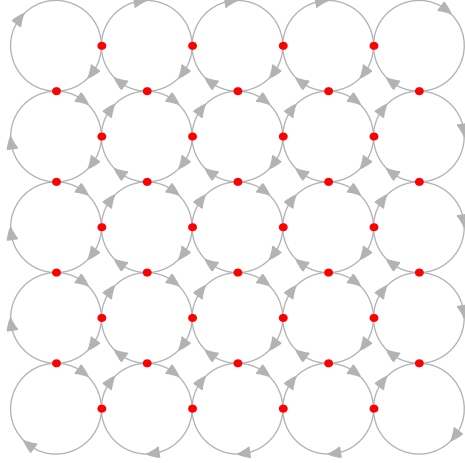
Figure 5: The medical grid of size 5

Unlike the usual tree-width of graphs, the *tree-width* of an Eulerian digraph [13, page 8] is defined more like branch-width. In fact, the tree-width of an Eulerian digraph D is defined as the branch-width of the following symmetric submodular function $\kappa : 2^{V(D)} \to \mathbb{Z}$ where $\kappa(X) = |\delta(X)|$, that is the number of non-loop edges having exactly one end in X.

The medial grid of size n is the medial graph of the $n \times n$ grid with all edges directed clockwise around the vertex of the grid (see Figure 5).

An Eulerian digraph H is *immersed* in an Eulerian digraph D if H is obtained from D by repeatedly applying the following operations [13, Theorem 2.2]:

 (i) Deleting a vertex of out-degree 0,

 (ii) Deleting a loop,

(iii) (*Supressing*) Contracting an edge incident to a vertex of out-degree 1,

(iv) (*Splitting*) Replacing a vertex v by two new vertices v_1 and v_2 and partitioning the edges incident with v between v_1 and v_2 so that each of v_1 and v_2 has in-degree equal to and out-degree. (See Figure 6.)

To deduce the conjecture for circle graphs from Theorem 26, we will discuss

- how the 4-regular Eulerian digraphs are associated to circle graphs,

- why the immersion is related to pivot-minors of associated circle graphs,

- relation of tree-width of the 4-regular Eulerian digraphs and rank-width of associated circle graphs,

- and why medial grids are associated to the fundamental graph of the cycle matroid of the grid.
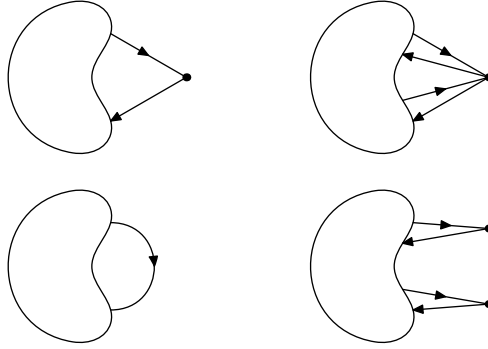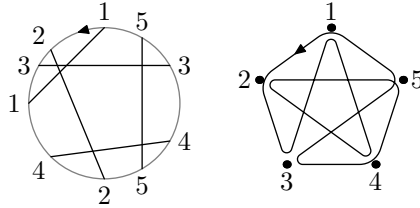
17

Figure 6: Supressing (left) and splitting (right)

Figure 7: Chord diagrams and 4-regular Eulerian digraphs

4-regular Eulerian graphs and circle graphs. Let us discuss how the 4-regular Eulerian digraph D is associated to circle graphs. For each Eulerian circuit C of D, we will associate the chord diagram as follows. If D has n vertices, then put $2n$ vertices on the circle. Following the orientation of the circle and the Eulerian circuit C, associate each vertex on the circle to the vertex on the Eulerian circuit. Then put the chord joining two vertices on the circle associated to the same vertex. The obtained diagram is the chord diagram of the circle graph. (See Figure 7.) It is easy to obtain the 4-regular Eulerian digraph from circle graph; we orient the circle and then contract chords.

Immersion and pivot-minors. We will discuss why the immersion is related to pivot-minors. Let D be a 4-regular Eulerian digraph. Let C be an Eulerian circuit. Then we have the circle graph G associated with D and C. It is enough to consider splitting operations on D. Suppose that H is a 4-regular Eulerian digraph obtained from D by splitting a vertex v and supressing vertices of degree 1. We claim that the circle graph associated to H is a pivot-minor of G.

We first explain that for every Eulerian circuit C' of D, the associated circle graph G' can be obtained from G by repeatedly applying pivoting. This is implied by the following theorem, appeared in [1]. They have a simple proof based on induction. Based on the pivoting operation on circle graphs, we can define the pivoting operation on 4-regular Eulerian digraphs. Let a, b be two vertices of D so that in the Eulerian circuit C of D, a and b are *interlaced* meaning that C visits them in the sequence $\ldots a \ldots b \ldots a \ldots b \ldots$. (This is equivalent to say that a and b are adjacent in the circle graph associated with D and C.)
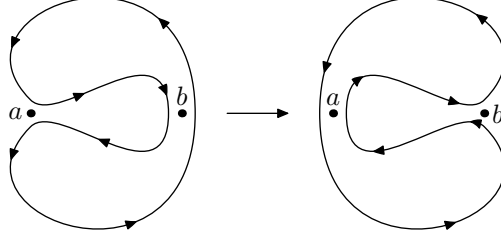
18

Figure 8: Pivoting ab on Eulerian circuits of 4-regular Eulerian digraphs

By pivoting ab on C, we obtain another Eulerian circuit of D as in Figure 8, by changing transitions at a and b locally.

Lemma 27 (Arratia, Bollobás, and Sorkin [1, Lemma 4]). *For a 4-regular Eulerian digraph D, all Eulerian circuits form a single orbit under pivoting.*

By the previous lemma, the choice of Eulerian circuits in 4-regular Eulerian digraphs does not affect pivot-minors of the associated circle graphs. When the splitting follows C, then it is easy to see that C induces an Eulerian circuit C' of H and therefore the circle graph associated with H and C' is $G \setminus v$. It remains to consider the case when the splitting does not follow C. Let D' be the digraph obtained by splitting a vertex in a way not compatible with C. Since H is connected, D' is connected and therefore D' has an Eulerian circuit C'_0. By lifting the Eulerian circuit C'_0, we obtain an Eulerian circuit C_0 of D. By the previous lemma, C_0 is obtained from C by repeatedly applying pivoting and so we obtain a circle graph G' associated to D and C' that is obtained from G by repeatedly applying pivoting. Now it is clear that H is a pivot-minor of G.

Tree-width and rank-width. Now we will show that if a 4-regular Eulerian digraph has small tree-width, then its associated circle graph has small rank-width. We warn that the notion of tree-width for 4-regular Eulerian digraphs [13] is different from the usual tree-width or the directed tree-width [14]. Note that the tree-width of a 4-regular Eulerian digraph is always even.

Lemma 28. *Let D be a 4-regular Eulerian digraph and G be the associated circle graph with respect to an Eulerian circuit C of D. If the tree-width of D is $2t$, then the rank-width of G is at most $t(t-1)/2$.*

Proof. Let (T, μ) be the tree-decomposition of D of width $2t$, that is, in fact, the branch-decomposition of κ_D or the rank-decomposition of G. (So T is a subcubic tree and μ is a bijection from $V(G) = V(D)$ to the set of leaves of D.) We claim that the width of the rank-decomposition (T, μ) is at most $t(t-1)/2$.

To prove the claim, it is enough to show that for a subset X of $V(D)$, if $\kappa(X) = 2k$, then $\rho_G(X) \le k(k-1)/2$. Because D has an Eulerian circuit, $|\delta^+(X)| = |\delta^-(X)|$ and so $\kappa(X) = |\delta(X)|$ is always even. Let M be the chord diagram of G corresponding to D so that
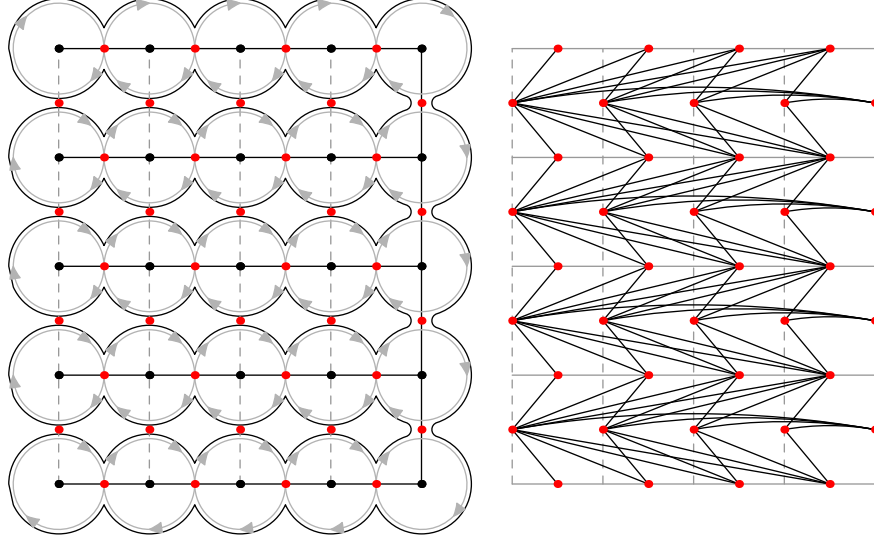
Figure 9: Medial grids and fundamental graphs of grids

the circle corresponds to the Eulerian circuit C. By deleting $2k$ edges in $\delta(X)$, we obtain $2k$ segments of the circle in M. (A segment can be a single point.)

We color k segments having vertices in X by blue and other k segments by red. Then the colors of segments of the circle of M alternate between red and blue. We color each chord of M by blue if it belongs to X and red otherwise. In the chord diagram M, the color of each chord is equal to the color of segments that it meets. The cut-rank function $\rho_G(X)$ depends on the adjacency of blue and red chords. For distinct $i, j \in \{1, 2, \ldots, k\}$, let $R_{i,j}$ be the set of red chords of M incident with the i-th and the j-th segments. It is easy to observe that for each blue chord x, the set of red chords intersecting with x is the disjoint union $\cup_{(i,j)\in L} R_{i,j}$ for some L. Therefore $\rho_G(X) \leq \binom{k}{2}$. $\qquad\square$

Medical grids and fundamental graphs of cycle matroids of grids. We now show that the circle graph associated with the medial grid is a fundamental graph of the cycle matroid of the grid. Figure 9 will describe it clearly. To see this, we pick a spanning tree of the grid. The spanning tree will induce the Eulerian circuit on the medial grid naturally as in the figure. Then it is easy to see that the circle graphs associated to this Eulerian circuit is equal to the fundamental graph of the cycle matroid with respect to the spanning tree.

Deducing the conjecture for circle graphs. Combining all the above arguments, we obtain the following theorem.

Theorem 29. *For a simple bipartite circle H, there is a constant $c(H)$ such that if a circle graph G has rank-width larger than $c(H)$, then G has a pivot-minor isomorphic to H.*

Proof. Suppose G is a circle graph with sufficiently large rank-width. Let D be the corresponding 4-regular Eulerian digraph. Then by Lemma 28, D has large tree-width. Theorem

26 implies that D immerses a large medial grid. So the fundamental graph of a grid is a pivot-minor of G. By Lemma 24 and Lemma 23, the fundamental graph of a large grid contains H as a pivot-minor. $\square$

References

[1] R. Arratia, B. Bollabás, and G. B. Sorkin. The interlace polynomial of a graph. *J. Combin. Theory Ser. B*, 92(2):199–233, 2004.

[2] A. Bouchet. Greedy algorithm and symmetric matroids. *Math. Programming*, 38(2):147–159, 1987.

[3] A. Bouchet. Representability of Δ-matroids. In *Combinatorics (Eger, 1987)*, volume 52 of *Colloq. Math. Soc. János Bolyai*, pages 167–182. North-Holland, Amsterdam, 1988.

[4] A. Bouchet. Circle graph obstructions. *J. Combin. Theory Ser. B*, 60(1):107–144, 1994.

[5] B. Courcelle and S. Olariu. Upper bounds to the clique width of graphs. *Discrete Appl. Math.*, 101(1-3):77–114, 2000.

[6] H. de Fraysseix. Local complementation and interlacement graphs. *Discrete Math.*, 33(1):29–35, 1981.

[7] N. Deo, M. S. Krishnamoorthy, and A. B. Pai. Generalizations of line graphs and applications. *Information Processing Lett.*, 6(1):14–17, 1977.

[8] M. Doob. An interrelation between line graphs, eigenvalues, and matroids. *J. Combinatorial Theory Ser. B.*, 15:40–50, 1973.

[9] J. F. Geelen, A. M. H. Gerards, and G. Whittle. Excluding a planar graph from GF(q)-representable matroids. Research Report 03-4, School of Mathematical and Computing Sciences, Victoria University of Wellington, 2003.

[10] F. Gurski and E. Wanke. Vertex disjoint paths on clique-width bounded graphs (extended abstract). In *LATIN 2004: Theoretical informatics*, volume 2976 of *Lecture Notes in Comput. Sci.*, pages 119–128. Springer, Berlin, 2004.

[11] F. Gurski and E. Wanke. Line graphs of bounded clique-width. Submitted, 2005.

[12] I. V. Hicks and N. B. McMurray Jr. The branchwidth of graphs and their cycle matroids. Preprint, 2005.

[13] C. D. T. Johnson. *Eulerian digraph immersion*. PhD thesis, Princeton University, 2002.

[14] T. Johnson, N. Robertson, P. Seymour, and R. Thomas. Directed tree-width. *J. Combin. Theory Ser. B*, 82(1):138–154, 2001.

[15] G. Kishi and Y. Uetake. Rank of edge incidence matrix. *IEEE Trans. Circuit Theory*, CT-16:230–232, 1969.

[16] F. Mazoit and S. Thomassé. Branchwidth of graphic matroids. Manuscript, 2005.

[17] S. Oum. Rank-width and vertex-minors. *J. Combin. Theory Ser. B*, 95(1):79–100, 2005.

[18] S. Oum and P. Seymour. Approximating clique-width and branch-width. *J. Combin. Theory Ser. B*, 96(4):514–528, 2006.

[19] J. G. Oxley. *Matroid theory*. Oxford University Press, New York, 1992.

[20] N. Robertson and P. Seymour. Graph minors. X. Obstructions to tree-decomposition. *J. Combin. Theory Ser. B*, 52(2):153–190, 1991.

[21] N. Robertson, P. Seymour, and R. Thomas. Quickly excluding a planar graph. *J. Combin. Theory Ser. B*, 62(2):323–348, 1994.

[22] A. W. Tucker. A combinatorial equivalence of matrices. In R. Bellman and M. Hall, Jr., editors, *Combinatorial Analysis*, pages 129–140. American Mathematical Society, Providence, R.I., 1960.