Caveat lector. One day I will type up the first few lectures.

8 1/30/23

Relevant reading: Weintraub pp. 11–13, Hatcher pp. 70–76.

8.1 Deck Transformations

We will first begin with an example to motivate our definition:

Example 8.1. Let $S^1 = \{z \in \mathbb{C} \mid |z| = 1\} \subseteq \mathbb{C}$ be the circle regarded as a subspace of $\mathbb{C}$. Then we saw that $p : \mathbb{R} \to S^1$ via $p(t) = e^{2\pi i t}$ was a covering map. Then for any $z_0 \in S^1$ with $p(t_0) = z_0$, we have that $p^{-1}(\{z\}) = t_0 + \mathbb{Z}$. Equivalently, $p(t_0 + m) = p(t_0)$ for all $m \in \mathbb{Z}$. Define, for $m \in \mathbb{Z}$, $T_m : \mathbb{R} \to \mathbb{R}$ via $T_m(t) = t + m$ translation by m. Then by our discussion, $p \circ T_m = p$. We say that T_m is an example of a **deck transformation**.

Definition 8.2. Let $p : E \to B$ be a covering projection. Then the **group of deck transformations** is the set

$$\Gamma_p \coloneqq \{T \in \operatorname{Homeo}(E) \mid p \circ T = p\},$$

where the endowed operation is function composition. That is, it is the set of all homeomorphisms such that for any $T \in \Gamma_p$, the following diagram commutes:

$$\begin{tikzcd} E \arrow[r, "T"] \arrow[dr, "p"'] & E \arrow[d, "p"] \\ & B \end{tikzcd}$$

Now given a covering projection p, we may define an equivalence relation in the following manner: for $x, y \in E$, we say $x \sim y$ if and only if there exists some $T \in \Gamma_p$ such that $T(x) = y$. Now we may consider the quotient space E/Γ_p, i.e., the topology that makes the projection $\hat{p} : E \to E/\Gamma_p$ continuous.

Example 8.3. Returning to our example of S^1 and p defined in Example 8.1, we now ask the question, what is $\mathbb{R}/\Gamma_p$ with this equivalence relation?

Claim 8.4. $\Gamma_p = \{T_m \mid m \in \mathbb{Z}\}$.

To see this, proceed in the manner as we did when proving that p was a covering map. If $e^{2\pi i T(t)} = e^{2\pi i t}$ for all $t \in \mathbb{R}$, then rearranging, we see that $e^{2\pi i (T(t) - t)} = 1$ for all t. Hence $T(t) - t \in \mathbb{Z}$ for all t, but since $T(t) - t$ is continuous, we conclude that $T(t) - t$ is constant, and so there exists some $m \in \mathbb{Z}$ such that $T(t) = t + m$ for all t.

We may now also further say that $\Gamma_p \simeq \mathbb{Z}$. Hence we may identify $\mathbb{R}/\Gamma_p$ with $\mathbb{R}/\mathbb{Z}$, or with $[0, 1)$.

Notation. We will denote $\Gamma_p(x) \coloneqq \operatorname{orb}_{\Gamma_p}(x) = \{T(x) \mid T \in \Gamma_p\}$. In general, if we want to make some sort of identification for E/Γ_p with some set S, like we did in the previous example, we need $\#(\Gamma_p(x) \cap S) = 1$ for all x. Indeed, this is the case for $[0, 1)$.

8.2 Discontinuous Actions

Definition 8.5. Let E be a topological space, and let $\Gamma \leq \mathrm{Homeo}(E)$. We say that Γ **acts discontinuously** if for all $x \in E$, there exists some open neighborhood U_x of x such that if $T \in \Gamma$ and $T(U_x) \cap U_x \neq \varnothing$, then $T = \mathrm{id}$.

Remark 8.6. Some texts, like Hatcher, calls a discontinuous action as a covering space action.

One consequence of our definition is the following claim:

Claim 8.7. *If Γ acts discontinuously on E and $S_1, S_2 \in \Gamma$, and $S_1(U) \cap S_2(U) \neq \varnothing$ for some nonempty U, then $S_1 = S_2$.*

Proof of claim. Observe that, since S_1 and S_2 are homeomorphisms, $\varnothing \neq S_1(U) \cap S_2(U) = S_1\left(U \cap S_1^{-1} \circ S_2(U)\right)$. In particular, this implies that $U \cap S_1^{-1} \circ S_2(U) \neq \varnothing$. Since $S_1^{-1} \circ S_2 \in \Gamma$ and Γ acts discontinuously, we conclude $S_1^{-1} \circ S_2 = \mathrm{id}$. ■

Lemma 8.8. *Suppose Γ acts discontinuously on E. Then $p : E \to E/\Gamma$ is a covering projection, where the quotient is defined by $x \sim_\Gamma y$ if and only if there is some $T \in \Gamma$ such that $Tx = y$.*

Proof. Given $y \in E/\Gamma$, take $x \in E$ such that $p(x) = y$, and let U_x be the neighborhood that is granted by Definition 8.5. Then we claim that $p(U_x)$ is open. To see this, notice that $p^{-1}(p(U_x)) = \bigsqcup_{S \in \Gamma} S(U_x)$, and since each S is a homeomorphism, $S(U_x)$ is open, which implies that $p^{-1}(p(U_x))$ is open, as desired. Moreover, $p|_{S(U_x)} : S(U_x) \to p(U_x)$ is a homeomorphism, and thus p must be a covering map. ■

8.3 Universal Covering Spaces

We will state two key theorems, but we will not prove them.

Theorem 8.9. *Let E be a simply connected space, and let $p : E \to B$ be a covering projection. Assume further that B is semilocally simply connected. Then if Γ_p is the group of deck transformations, then Γ acts discontinuously and B is homeomorphic to E/Γ_p. In particular, the following diagram commutes:*

$$\begin{tikzcd} E \arrow[r, "\mathrm{id}"] \arrow[d, "p"] & E \arrow[d, "\hat{p}"] \\ B \arrow[r, "h"] & E/\Gamma_p \end{tikzcd}$$

where $h : B \to E/\Gamma_p$ denotes the homeomorphism and $\hat{p}$ is the projection map from E to E/Γ_p.

Theorem 8.10 (Existence and Universal Property of Universal Covers)**.** *Let B be a semilocally simply connected, locally path connected, connected space. Then there exists a simply connected and connected space E such that there is a covering projection $p : E \to B$. Moreover, if $q : X \to B$ is any other covering projection, with X connected, then there exists a unique continuous map $r : E \to X$ such that the following diagram commutes:*

$$\begin{tikzcd} E \arrow[r, "r"] \arrow[d, "p"] & X \arrow[dl, "q"] \\ B & \end{tikzcd}$$

The space E is unique up to homeomorphism.

Definition 8.11. The space E in the previous theorem is called a **universal cover**, and we will denote a universal covering space of B by $\widetilde{B}$.

Interpretation. We can interpret the previous two theorems in the following way: by Theorem 8.10 we know that for any semilocally simply connected space B there is a universal cover $\widetilde{B}$, and Theorem 8.9 tells us that $B \approx \widetilde{B}/\Gamma_p$. Moreover, in a sense, Γ_p is the fundamental group.

Proposition 8.12. *Let $p : E \to B$ be a covering projection and assume further that E is simply connected and and path connected. Suppose $b_0 \in B$, and $e_0 \in E$ such that $p(e_0) = b_0$. Then $\pi_1(B, b_0) \simeq \Gamma_p$.*

Proof. We will show that there is a one-to-one correspondence between the two groups. Let $T \in \Gamma_p$. Let $\tilde{\alpha}$ be a curve in E connecting e_0 and $T(e_0)$, and let $\alpha := p \circ \tilde{\alpha}$. Then observe that $\alpha(0) = p(e_0) = b_0 = \alpha(1) = p(T(e_0))$. So $\alpha \in \pi_1(B, b_0)$. Thus this gives us a way to assign a loop in $\pi_1(B, b_0)$ for every $T \in \Gamma_p$. To see that this does not depend on our choice of curve $\tilde{\alpha}$, suppose $\tilde{\beta}$ is another curve connecting e_0 and $T(e_0)$, and set $\beta = p(\tilde{\beta})$. Then the concatenation $\tilde{\alpha} \cdot \tilde{\beta}^{-1}$ is a loop with basepoint e_0, and since E is simply connected must be homotopically trivial. Hence there is a homotopy $\tilde{H} : I \times I \to E$ such that $h_0 = \tilde{\alpha} \cdot \tilde{\beta}^{-1}$, and $h_1 \equiv e_0$. Now consider $H : I \times I \to B$ given by $H(s, t) = p(\tilde{H}(s, t))$. Then this is a homotopy taking the concatenation $\alpha \cdot \beta^{-1}$ to the constant path b_0, which shows that $\alpha = \beta$ in $\pi_1(B, b_0)$. Hence this map is well-defined.

On the other hand, let $\gamma \in \pi_1(B, b_0)$. Then by the path-lefting property, there exists some $\tilde{\gamma} : [0, 1] \to E$ such that $\tilde{\gamma}(0) = e_0$ and $p(\tilde{\beta}(1)) = \beta(1) = b_0$. Moreover, there exists some $T \in \Gamma_p$ such that $T(e_0) = \tilde{\beta}(1)$; this defines the inverse map. By the homotopy lifting property, it is easy to check that this map does not depend on the representative of the homotopy class. Hence it is well-defined. The two maps are clearly bijective, and it is straightforward to check that they are homomorphisms. Hence this proves the isomorphism. ■

9 2/1/23

Today we will introduce the Seifert-van Kampen theorem. Relevant reading: Hatcher Chapter 1.2, Weintraub Section 2.3.

9.1 Free Group Products

Definition 9.1 (Free Group Products)**.** Given two groups G_1 and G_2, we denote $G_1 * G_2$ to be the **free product** of G_1 and G_2, which is the coproduct of the groups G_1 and G_2 in the category of groups. That is, there are injective homomorphisms $\iota_1 : G_1 \hookrightarrow G_1 * G_2$ and $\iota_2 : G_2 \hookrightarrow G_1 * G_2$ and it satisfies the following universal property:

If G is any group and $f_1 : G_1 \to G$ and $f_2 : G_2 \to G$ are homomorphisms, then there exists a unique homomorphism $f : G_1 * G_2$ such that the following diagram commutes:

$$\begin{tikzcd}
G_1 \arrow[dr, hook, "\iota_1"] \arrow[drr, bend left, "f_1"] & & \\
& G_1 * G_2 \arrow[r, "f"] & G \\
G_2 \arrow[ur, hook, "\iota_2"] \arrow[urr, bend right, "f_2"'] & &
\end{tikzcd}$$

Example 9.2. $\mathbb{Z} * \mathbb{Z} = F_2$ the free group on two generators: alternatively, we can write F_2 to be the set of all finite words on two letters a, b.

Remark 9.3. In general, if $a_i \in G$, $b_i \in G$, we can write any element of $G_1 * G_2$ as $a_1 b_1 a_2 b_2 \cdots a_k b_k$.

Definition 9.4. Given a group G, and $A \subseteq G$ (not necessarily a subgroup), the **normal subgroup generated by** A is defined by $N(A) = \bigcap N$, where the intersection runs over all normal subgroups containing A: that is, it is the smallest normal subgroup of G containing A.

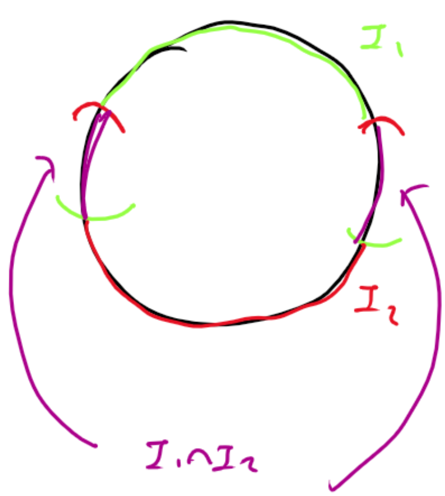
9.2 The Seifert-van Kampen Theorem and Applications

Theorem 9.5 (Seifert-van Kampen)**.** *Let X be a path-connected space, and assume $X = U_1 \cap U_2$, where both U_1 and U_2 are open and path-connected. Let $x_0 \in U_1 \cap U_2$ and assume that $U_1 \cap U_2$ is also path-connected. Then $\pi(X, x_0) \simeq (\pi_1(U_1, x_0) * \pi_1(U_2, x_0))/N(A)$, where if $(\iota_1)_*$ and $(\iota_2)_*$ are the homomorphisms induced by the inclusion map $\iota_i : U_1 \cap U_2 \to U_i$, we have*

$$A = \{(\iota_1)_*(g^{-1}) * (\iota_2)_*(g) \mid g \in \pi_1(U_1 \cap U_2, x_0)\}.$$

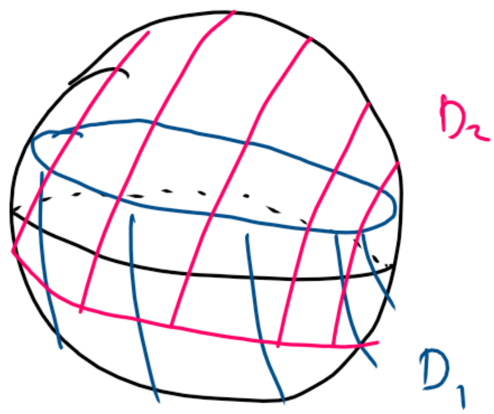
We will not prove the Seifert-van Kampen theorem today, but we will see some applications of it.

Example 9.6 (An Incorrect Application)**.** Consider S^1 as the union of two open intervals I_1 and I_2 as in the figure. But $\pi_1(S^1)$ cannot be a quotient of the free product $\pi_1(I_1) * \pi_1(I_2)$ because the two factors are both trivial, but we already know that $\pi_1(S^1) \simeq \mathbb{Z}$. The error was in that the hypothesis $U_1 \cap U_2$ is not path-connected.



Proposition 9.7 (Fundamental Group of S^n)**.** *For $n \geq 2$, S^n is simply connected.*

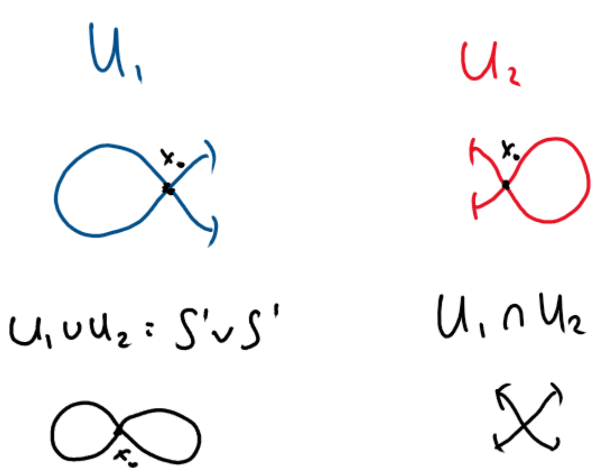
Proof. We will use the ideas from Example 9.6. Let $x_0 \in S^n$; by rotating the sphere, we may assume that x_0 is on the equator. Write $S^n = D_1 \cup D_2$, where D_1 are D_2 are the open sets in the figure below.



Note that D_1 and D_2 are both contractible, and so must have trivial fundamental group. Moreover, $D_1 \cap D_2 \approx S^{n-1} \times I$, which is also path-connected. Then applying the Seifert-van Kampen theorem, $\pi_1(S^n, x_0)$ must be a quotient of $\pi_1(D_1, x_0) * \pi_1(D_2, x_0) = \{0\}$. Hence S^n is is simply connected. ∎

Remark 9.8. We could have use the stereographic projection to map the sphere with the poles removed onto $\mathbb{R}^n$ in the previous proof.

Example 9.9 (The Figure 8)**.** Consider $E \coloneqq S^1 \vee S^1$, or the "figure 8," joined together at the point x_0. Let U_1 and U_2 be as in the figure below, so that $U_1 \cup U_2 = E$, and $E_1 \cap E_2$ is the cross in the middle.



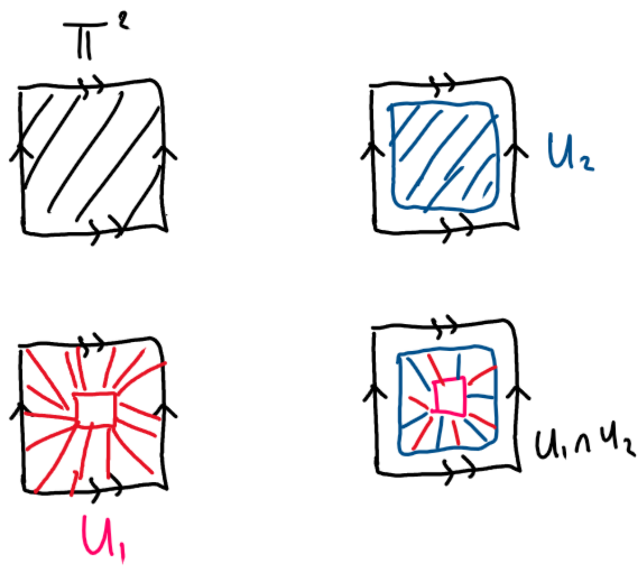
First observe that all our spaces are path-connected and so the hypotheses of the Seifert-van Kampen theorem are satisfied. Next, $U_1 \cap U_2$ is contractible, which implies that $\pi_1(U_1 \cap U_2, x_0) = \{0\}$.

Finally, observe that $U_1 \approx U_2 \approx S^1$, which implies that $\pi_1(U_1, x_0) \simeq \pi_1(U_2, x_0) \simeq \mathbb{Z}$. Appealing to the Seifert-van Kampen theorem, we conclude that $\pi(E, x_0) = \mathbb{Z} * \mathbb{Z}$.

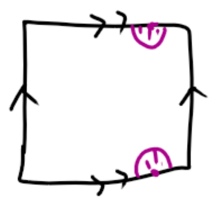
Exercise 9.10. Apply induction to the previous example to conclude that the fundamental group of the n-petal rose is F_n, the free group on n elements.

Exercise 9.11. Let X and Y be topological spaces, and suppose $X \vee Y$ be locally contractible and/or semilocally simply connected at the attaching point x_0. Show that $\pi_1(X \vee Y, x_0) \simeq \pi(X, x_0) * \pi(Y, x_0)$.

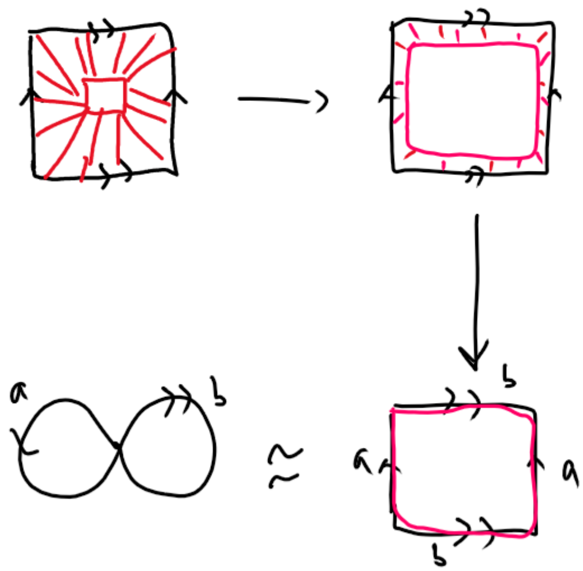
Example 9.12 (The Torus)**.** Let $\mathbb{T}^2$ denote the torus $\mathbb{T}^2 = S^1 \times S^1$. We have already noted that $\pi_1(\mathbb{T}^2) \simeq \pi_1(S^1) \times \pi_1(S^1) \simeq \mathbb{Z} \times \mathbb{Z}$. Now we will use the Seifert-van Kampen's theorem to prove this. We have shown that the torus may be considered as the quotient space of the square where the opposite edges are identified. Now let U_1 and U_2 be as in the diagram, where U_1 is the "outer" part of the square, and U_2 the "inner" part.



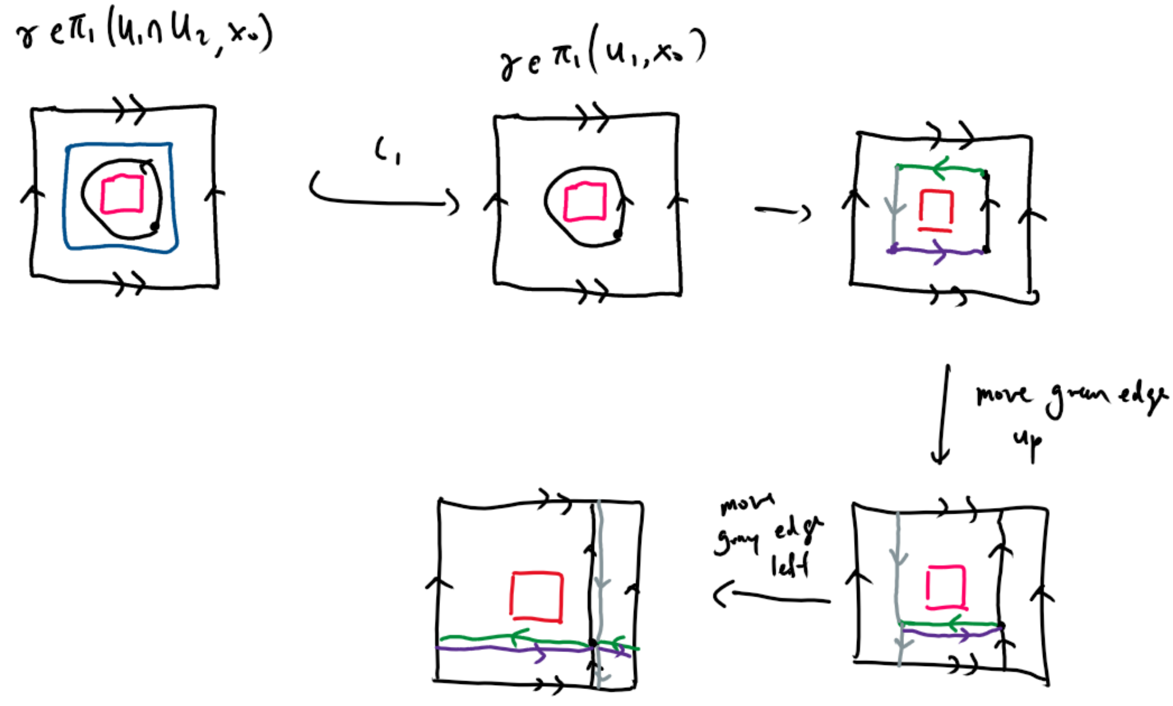
To see that U_1 is open, note that on the edges, any ball would "bleed over" to the opposite edge, as in the following figure:



It is now easy to see that U_1 and U_2 are both open, $U_1 \cup U_2 = \mathbb{T}^2$, and U_1, U_2, and $U_1 \cap U_2$ are all path-connected. The hypotheses of the Seifert-van Kampen theorem are now satisfied. Fix $x_0 \in U_1 \cap U_2$. First observe that U_2 is contractible, and so $\pi_1(U_2, x_0) = \{0\}$. On the other hand, we see that U_1 deformation retracts onto the boundary of the square, and then identified with the figure 8 in the following manner:



Since deformation retracts induce an isomorphism of fundamental groups, we have from Example 9.9 $\pi_1(U_1, x_0) \simeq \pi_1(S^1 \vee S^1) \simeq \mathbb{Z} * \mathbb{Z}$. Now $U_1 \cap U_2$ is the annulus, which deformation retracts onto the circle S^1, so its fundamental group is the free group on one generator, the loop going around the annulus once counterclockwise. The following figure shows its image under $(\iota_1)_*$:



Now after the deformation retract, we see that in the image this loop is exactly the commutator $aba^{-1}b^{-1}$. But this was the image of the generator, and so we conclude that $N(A)$ (in the statement of the theorem) must be the commutator subgroup inside $\pi_1(U_1, x_0)$. Therefore $\pi(\mathbb{T}^2, x_0) \simeq \mathbb{Z} * \mathbb{Z}/\langle aba^{-1}b^{-1}\rangle = \mathbb{Z} \times \mathbb{Z}$.

Remark 9.13. The above proof can be adapted to compute $\pi_1(\mathbb{T}^n, x_0)$ with induction.

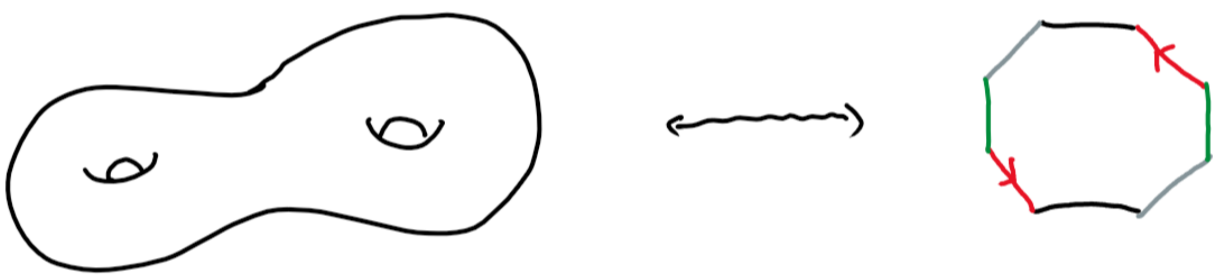
10 2/3/23

Today we will continue with examples of van Kampen's Theorem.

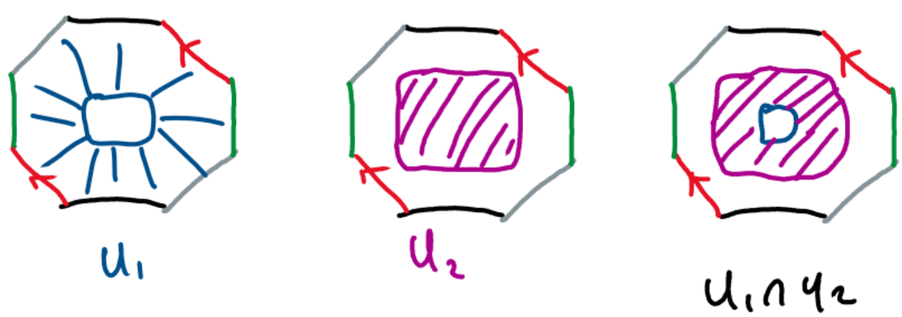
10.1 The Genus 2 Surface

Recall when we computed the fundamental group of the torus via Seifert-van Kampen theorem, we used the quotient of a square that is homeomorphic to the torus.

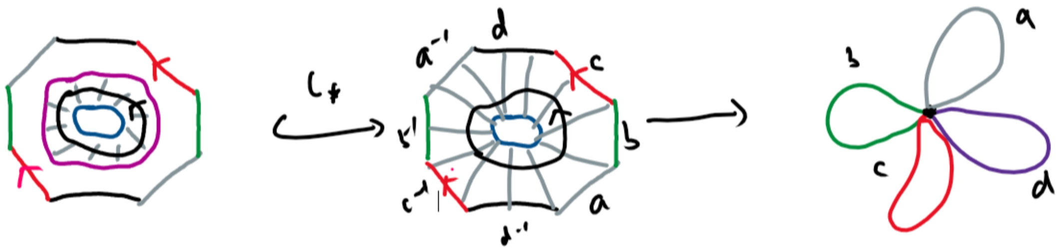
Method 1. For the genus 2 surface S, we will consider the quotient of an octagon as follows:



Then just as we did for the torus, decompose the octagon into following pieces:



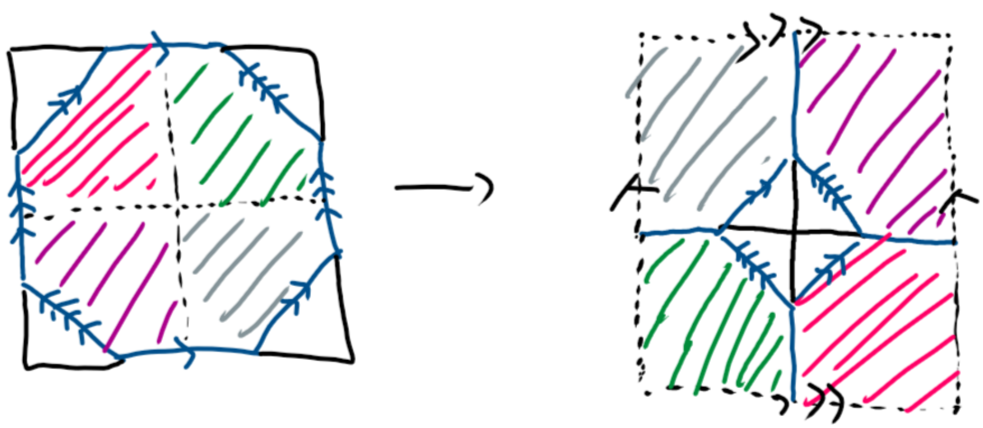
First note that all these sets are path-connected, so the hypotheses of the van Kampen theorem are satisfied. Then notice that U_1 deformation retracts onto the boundary, which is homeomorphic to the 4-petal rose; thus $\pi_1(U_1) \simeq F_4$, the free group on four elements. Moreover, U_2 is contractible and so has trivial fundamental group. Finally, $U_1 \cap U_2$ is the annulus, which deformation retracts onto S^1, so has fundamental group the free group on one generator. Then by the Seifert-van Kampen theorem, we have that $\pi_1(S) \simeq \pi_1(U_1)/N(\iota_1(g) \mid g \in \pi_1(U_1 \cap U_2))$. Let g be a loop in $U_1 \cap U_2$, like in the diagram below. Then considered as a loop in U_1 and its image in the deformation retract, its image is $abcda^{-1}b^{-1}c^{-1}d^{-1}$.



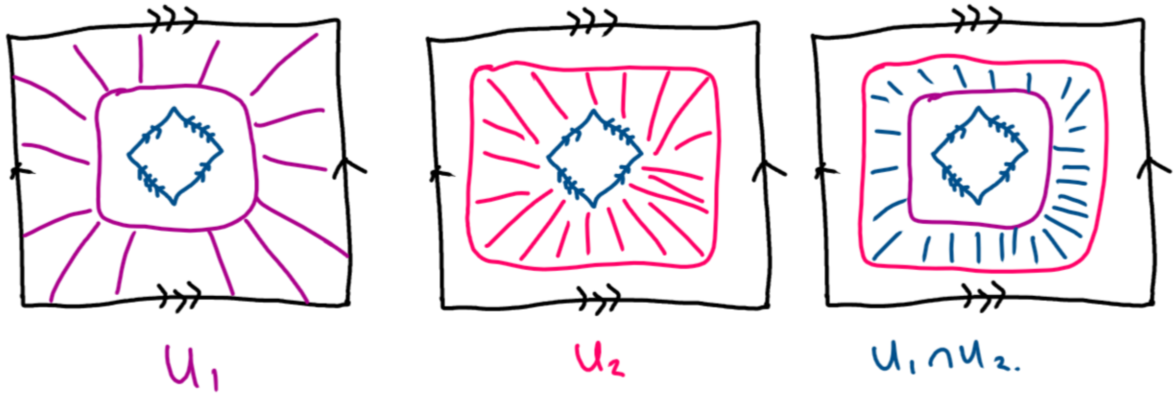
Therefore $\pi_1(S) \simeq F_4/\left\langle abcda^{-1}b^{-1}c^{-1}d^{-1}\right\rangle \simeq \langle a, b, c, d \mid abcd = dcba\rangle$.

Method 2. The idea will be to decompose the surface into two parts, just like below:

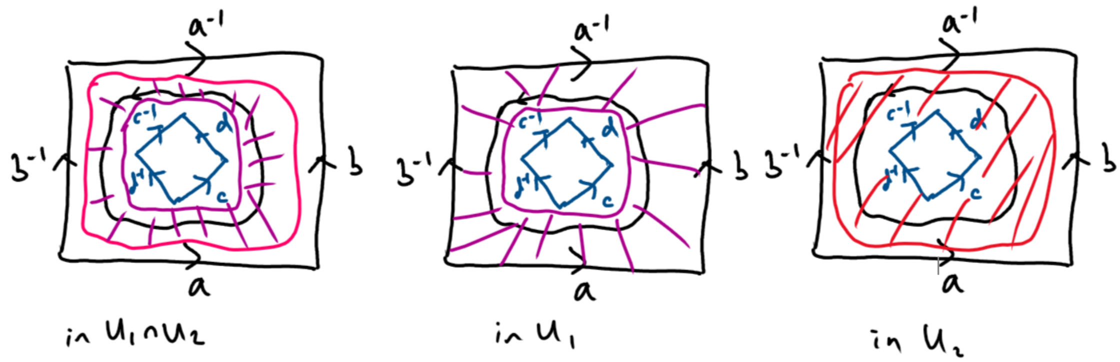
Consider the octagon again, but this time as a subspace of the square below after cutting and pasting.



Just as above, consider U_1 and U_2 defined as in the figure below.



Then both U_1 and U_2 deformation retracts onto the figure 8, and $U_1 \cap U_2$ is an annulus which deformation retracts onto S^1. Hence $\pi_1(U_1) \simeq \pi_1(U_2) \simeq F_2$ the free group on two generators, and $\pi_1(U_1 \cap U_2) \simeq \mathbb{Z}$. Now consider the single loop $g \in \pi_1(U_1 \cap U_2)$ given by the generator: that is, the loop that goes around once in the annulus. Then considered as a loop in U_1 and U_2 respectively, the diagram below shows that in U_1 it deformation retracts onto the loop $aba^{-1}b^{-1}$, and in U_2 it deformation retracts onto the loop $cdc^{-1}d^{-1}$.

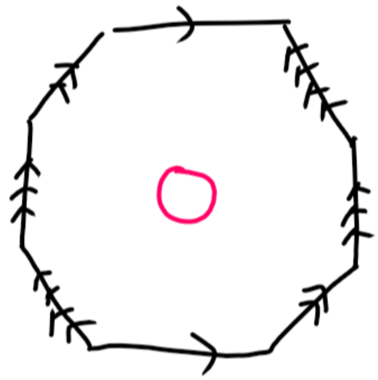


Therefore $(\iota_1)_*(g) = aba^{-1}b^{-1}$ and $(\iota_2)_*(g) = cdc^{-1}d^{-1}$. Thus applying van Kampen's theorem, we conclude that $\pi_1(S) \simeq F_2 * F_2/N\left(aba^{-1}b^{-1}\left(cdc^{-1}d^{-1}\right)^{-1}\right) \simeq \langle a, b, c, d \mid [a, b] = [c, d]\rangle$.

Remark 10.1. We can compute the fundamental group of a genus g surface by induction.

Corollary 10.2. *The fundamental group of a genus 2 surface with a point deleted is the free group on 4 elements. In general, the fundamental group of a genus g surface, with $g \geq 2$, is the free group on $2g$ elements.*

Proof. The genus 2 surface with a point deleted can be identified with the quotient space of the octagon in Method 1 with a neighborhood deleted in the interior, as in the diagram. Then this deformation retracts onto the boundary, which is homeomorphic to the 4-petal rose.

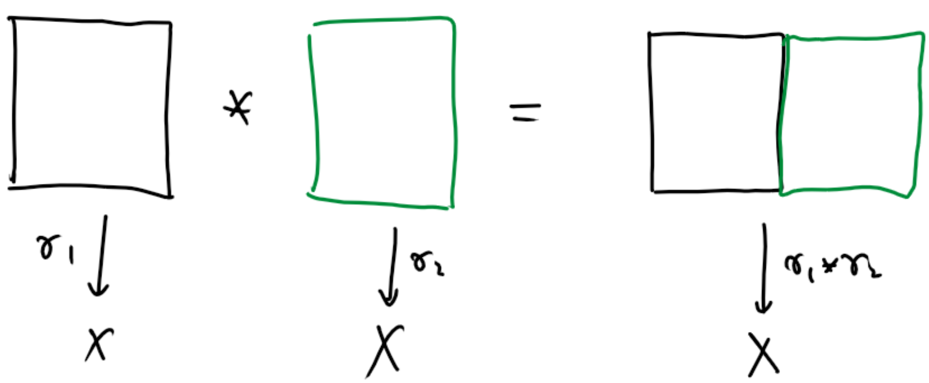


The general case is handled by induction. ∎

10.2 Higher Homotopy Groups

Recall that the fundamental group $\pi_1(X, x_0)$ was all about maps of the form $\gamma : (S^1, 1) \to (X, x_0)$, or equivalently maps of the form $([0,1], \{0,1\}) \to (X, x_0)$. Now similarly, we define the **higher homotopy groups** in the following manner: $\pi_n(X, x_0) := [(S^n, 1); (x, x_0)]$ where the bracket denotes the homotopy classes of maps. Equivalently, we may define $\pi_n(X, x_0)$ as $[(I_n, \partial I_n), (X, x_0)]$ where $I_n = [0,1]^n$.

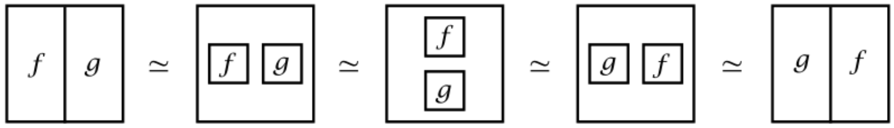
The group operation is defined as follows: as an example, we will use π_2, and analogize.



Because we stipulate that the boundary gets mapped to x_0, the multiplication is well-defined by the pasting lemma. The identity element is the constant map mapping to x_0. Another way of writing the multiplication is as follows: if $\gamma_1, \gamma_2 : (t_1, \ldots, t_n) \to X$, then we may write their product to be

$$(\gamma_1 * \gamma_2)(t_1, \ldots, t_n) = \begin{cases} \gamma_1(2t_1, t_2, \ldots, t_n), & t \in [0, 1/2] \\ \gamma_2(2t_1 - 1, t_2, \ldots, t_n), & t \in [1/2, 1]. \end{cases}$$

Next, the following figure from Hatcher illustrates the following lemma:

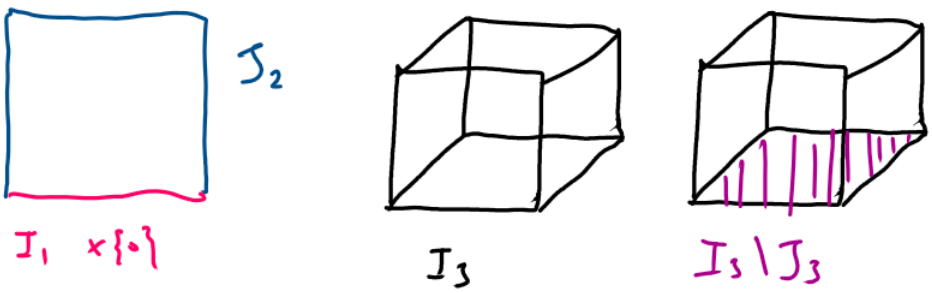


Lemma 10.3. *$\pi_n(X, x_0)$ is abelian for $n \geq 2$.*

11 2/6/23

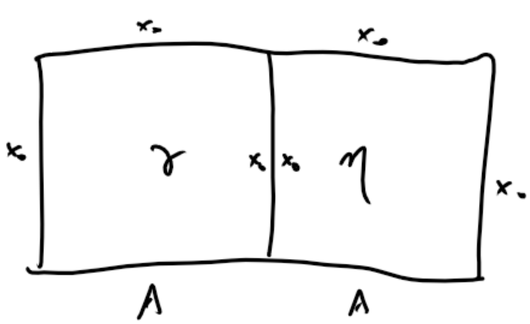
11.1 Relative Homotopy Groups

Definition 11.1. As we did last time, define $I_n = [0,1]^n$, ∂I_n the boundary of I_n, and let $J_n \coloneqq \partial I_n \setminus (I_{n-1} \times \{0\})$, as in the following diagram:



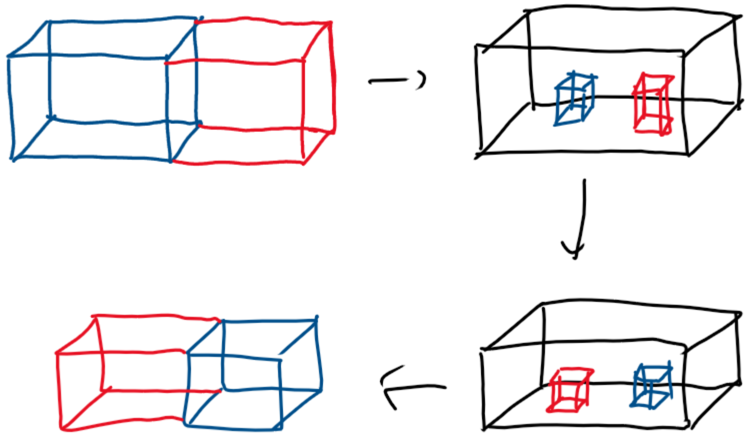
Then we define the **relative homotopy groups** as following: $\pi_n(X, A, x_0) \coloneqq [(I_n, \partial I_n, J_n); (X, A, x_0)]$.

Because the elements of the homotopy groups are equivalence classes, we will write what it means for two elements to be equivalent. We say that for $\gamma, \eta \in \pi_n(X, A, x_0)$, $\gamma \sim \eta$ if and only if there exists $F : (I_n, \partial I_n, J_n) \times [0,1] = (I_n \times [0,1], \partial I_n \times [0,1], J_n \times [0,1]) \to (X, A, x_0)$ such that $f_0 = F(\cdot, 0) = \gamma$ and $f_1 = F(\cdot, 1) = \eta$. Similar as was done in homotopy groups, the product $\gamma \cdot \eta$ is defined in the following way:

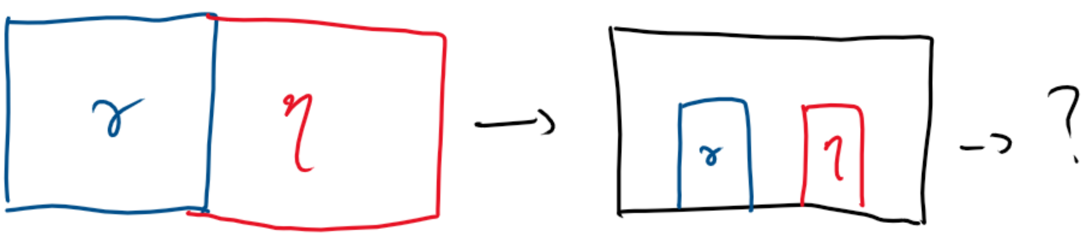


Lemma 11.2. *This product makes $\pi_n(X, A, x_0)$ into a group, and for $n \geq 3$, is abelian.*

Proof. To see that $\pi_n(X, A, x_0)$ is obvious. To see that it is obvious, consider the following figure:



It is important that the two cubes inside the big cube in the figure above do not have their bases taken off of $\partial I_n \setminus J_n$. The fact that this is not possible in dimension two illustrates why $\pi_n(X, A, x_0)$ is not abelian:



■

11.2 Exact Sequences

Definition 11.3. Given groups $G_1, G_2, \dots$ and homomorphisms $L_n : G_n \to G_{n+1}$, we say that the sequence

$$G_1 \xrightarrow{L_1} G_2 \xrightarrow{L_2} \cdots \to G_n \xrightarrow{L_n} \cdots \to 0$$

is **exact** if $\ker L_{n+1} = \operatorname{im} L_n$ for each n.

Observe that if we have a sequence of groups, $L_{n+1} \circ L_n \equiv 0$ if and only if $\ker L_{n+1} \supseteq \operatorname{im} L_n$.

Example 11.4. Consider the sequence $0 \xrightarrow{L_1} G \xrightarrow{L_2} 0$. Certainly $L_2 \circ L_1 \equiv 0$. But $\operatorname{im} L_1 = \{0\}$ since L_1 is a homomorphism. Thus this sequence must be exact if and only if G is trivial.

Example 11.5. Consider the sequence $0 \xrightarrow{L_1} G \xrightarrow{L_2} H \xrightarrow{L_3} 0$, and suppose that it is exact. Then $\operatorname{im} L_1 = \{0\} = \ker L_2$, which implies that L_2 is injective. On the other hand, $H = \ker L_3 = \operatorname{im} L_2$, and so L_2 is surjective. Thus L_2 is a group isomorphism.

Example 11.6. Consider the exact sequence $0 \xrightarrow{L_1} N \xrightarrow{\iota} G \xrightarrow{\pi} H \xrightarrow{L_3} 0$. Since $H = \ker L_3 = \operatorname{im} \pi$, we have that π is surjective. On the other hand, $\operatorname{im} L_1 = \ker \iota = \{0\}$ and so ι is injective. Thus by the first isomorphism theorem, $G/\ker \pi = G/\iota(N) \simeq H$. Identifying N with its image under ι, we conclude that $G/N \simeq H$.

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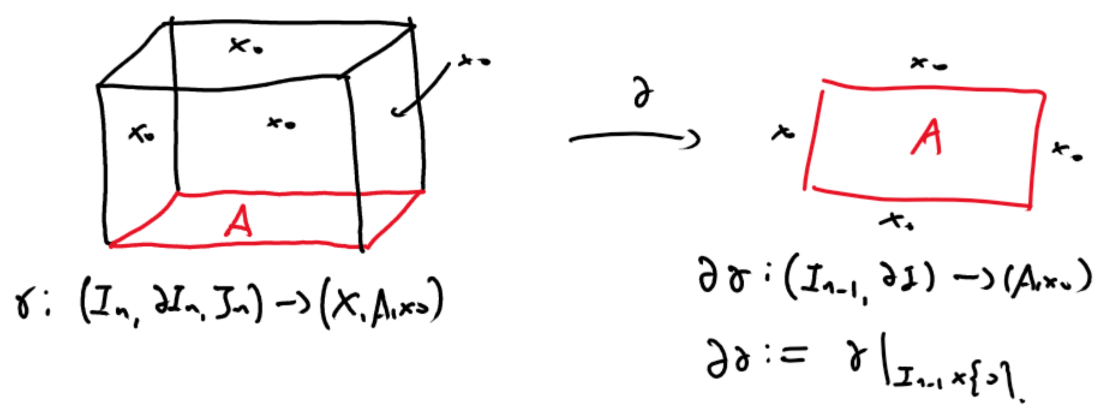
12.1 Long Exact Sequences of Relative Homotopy Groups

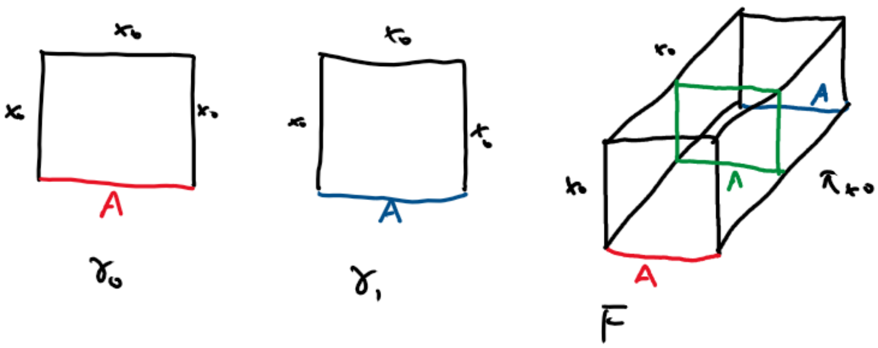
Theorem 12.1 (Long Exact Sequences of Relative Homotopy Groups)**.** *Let $J : (X, x_0, x_0) \hookrightarrow (X, A, x_0)$ be the inclusion. Then there is a long exact sequence*

$$\to \pi_n(A, x_0) \xrightarrow{\iota_\#} \pi_n(X, x_0) \xrightarrow{J_\#} \pi_n(X, A, x_0) \xrightarrow{\partial} \pi_{n-1}(A, x_0) \to \cdots \to \pi_0(X, A, x_0) \to 0.$$

The boundary map $\partial : \pi_n(X, A, x_0) \to \pi_{n-1}(A, x_0)$ is defined in the following manner: for $\gamma \in \pi_n(X, A, x_0)$, say $\gamma : (I_n, \partial I_n, J_n) \to (X, A, x_0)$, the restriction $\gamma|_{I_{n-1}\times\{0\}}$ can be regarded as a map $(I_{n-1}, \partial I) \to (A, x_0)$. Then $\partial\gamma$ is precisely this restriction.

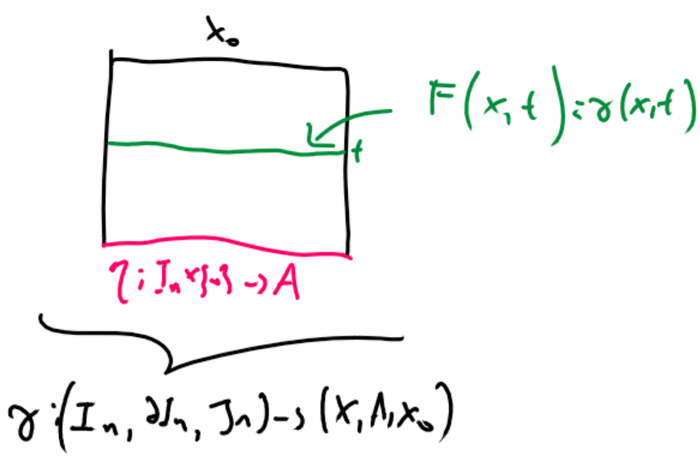


Proof. First we check that ∂ is well-defined. To this end suppose $\gamma_0 \sim \gamma_1$. Then there exists some $F : (I_n, \partial I_n, J_n) \times [0,1] \to (X, A, x_0)$ such that $F(\cdot, 0) = f_0 = \gamma_0$ and $F(\cdot, 1) = f_1 = \gamma_1$ (a 2-dimensional schematic diagram is below).



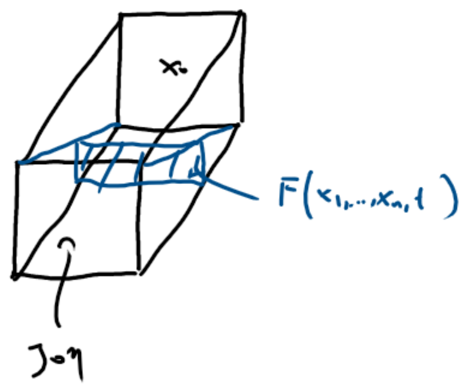
Then restricting F to $I_{n-1} \times \{0\} \approx I_{n-1}$ gives a homotopy $F|_{I_{n-1}\times\{0\}\times[0,1]} : I_{n-1} \times [0,1] \to A$. Then this gives a homotopy $\partial(\gamma_1) = \gamma_1|_{I_{n-1}\times\{0\}}$ to $\partial(\gamma_2) = \gamma_2|_{I_{n-1}\times\{0\}}$. Thus $\partial(\gamma_1) \sim \partial(\gamma_2)$ in homotopy classes.

Now we check exactness. First, we will show that $\ker \iota_\# = \operatorname{im} \partial$. To prove one direction, suppose that $\eta \in \operatorname{im} \partial$. Then there exists some $\gamma \in \pi_{n+1}(X, A, x_0)$, $\gamma : (I_{n+1}, \partial I_{n+1}, J_{n+1}) \to (X, A, x_0)$ such that $\partial\gamma = \gamma|_{I_n\times\{0\}} = \eta$. In order to prove that $\eta \in \ker \iota_\#$, we must show that $\iota_\#(\eta) \sim 0$ in $\pi_n(X, x_0)$, that is, there exists some homotopy $F : I_n \times [0,1] \to X$ such that $f_0 = \eta$ and $f_1 = x_0$. Define $F(x,t) = \gamma(x,t)$, regarded as a map from $I_{n+1} = I_n \times [0,1]$ to X. Indeed, $F(x,0) = \gamma(x,0) = \gamma|_{I_n\times\{0\}}(x) = \eta(x)$. On the other hand, $F(x,1) = \gamma(x,1) = x_0$ since $\gamma \in \pi_{n+1}(X, A, x_0)$ (see the figure below). Hence $\iota_\#(\eta)$ is homotopic to the constant map in $\pi_n(X, x_0)$, so $\eta \in \ker \iota_\#$.



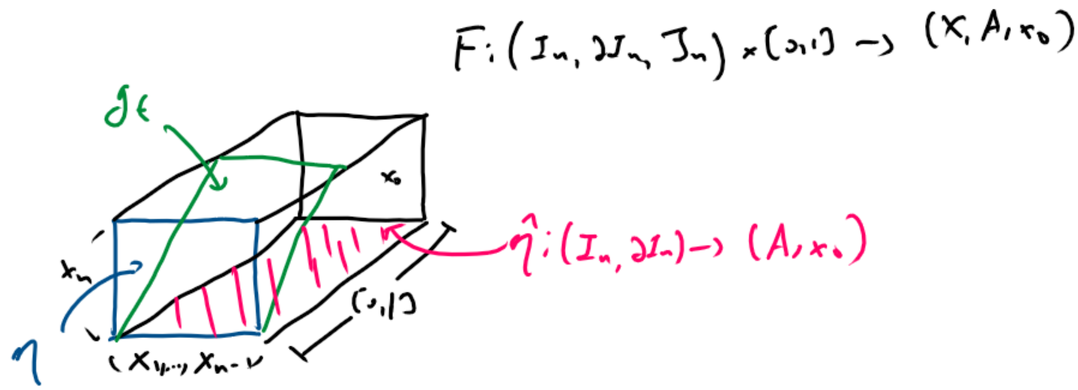
Conversely, suppose that $\eta \in \ker \iota_\#$. Then $\iota_\#(\eta)$ is homotopic to the constant map in $\pi_n(X, x_0)$, that is, there exists some $F : I_n \times [0,1] \to X$ such that $f_0(x) = \iota_\#(\eta)$ and $f_1(x) \equiv x_0$. Then proceeding as in the other direction, defining $\gamma : (I_{n+1}, \partial I_{n+1}, J_{n+1}) \to (X, A, x_0)$, with the identification $I_{n+1} = I_n \times [0,1] \to X$ via $\gamma(x,t) = F(x,t)$ we have that clearly $\partial\gamma = \eta$.

For the next part, we will show that $\operatorname{im}\iota_\# = \ker J_\#$. Let $\eta \in \operatorname{im}\iota_\#$. Then there exists some $\widetilde{\eta} \in \pi_n(A, x_0)$, $\widetilde{\eta} : (I_n, \partial I_n) \to (A, x_0)$ such that $\eta = \iota \circ \widetilde{\eta}$. Now consider the map $J \circ \eta : (I_n, \partial I_n, J_n) \to (X, A, x_0)$, which obtained by changing the domain and codomain: note that $\eta = \iota \circ \widetilde{\eta}$, and so the image of η is completely contained in A, and moreover for any $x \in \partial I_n$, we have $\eta(x) = \widetilde{\eta}(x) = x_0$ since $\widetilde{\eta} \in \pi_n(A, x_0)$. Thus $J \circ \eta$ as a map from $(I_n, \partial I_n, J_n) \to (X, A, x_0)$ makes sense. To show that $\eta \in \ker J_\#$, we will show that there is a homotopy $F : (I_n, \partial I_n, J_n) \times [0, 1] \to (X, A, x_0)$ such that $f_0 = J \circ \eta$ and $f_1 \equiv x_0$. Next, consider $F(x_1, \ldots, x_{n-1}, x_n, t) \coloneqq (J \circ \eta)(x_1, \ldots, x_{n-1}, (1-t)x_n)$.



Indeed, $f_0(x_1, \ldots, x_n) = F(x_1, \ldots, x_n, 0) = J \circ \eta$, and $f_1(x_1, \ldots, x_n) = F(x_1, \ldots, x_n, 1) = (J \circ \eta)(x_1, \ldots, x_{n-1}, 0) = \widetilde{\eta}(x_1, \ldots, x_{n-1}, 0) = x_0$. The schematic figure above shows that F indeed is the map of the desired form. Hence F is the desired homotopy.

Conversely, suppose $\eta \in \ker J_\#$. Then $\eta : (I_n, \partial I_n) \to (X, x_0)$ and $J \circ \eta : (I_n, \partial I_n, J_n) \to (X, A, x_0)$ is homotopically trivial. Thus there exists some $F : (I_n, \partial I_n, J_n) \times [0, 1] \to (X, A, x_0)$ such that $f_0 = J \circ \eta$ and $f_1 \equiv x_0$. Define $\widetilde{\eta} : (I_n, \partial I_n) \to (A, x_0)$ by $\widetilde{\eta}(x_1, \ldots, x_n) = F(x_1, \ldots, x_{n-1}, 0, x_n)$. Clearly by definition of F, $\widetilde{\eta}$ takes image in A, and its boundary takes value in x_0 (see figure below for an illustration). We claim that $\iota \circ \widetilde{\eta} \sim \eta$ in $\pi_n(X, x_0)$. Indeed, let $G : (I_n, \partial I_n) \times [0, 1] \to (X, x_0)$ via $G(x_1, \ldots, x_n, t) = F(x_1, \ldots, x_{n-1}, (1-t)x_n, tx_n)$. Then $g_0(x_1, \ldots, x_n) = F(x_1, \ldots, x_n, 0) = (J \circ \eta)(x_1, \ldots, x_n) = \eta(x_1, \ldots, x_n)$ and $g_1(x_1, \ldots, x_n) = F(x_1, \ldots, 0, x_n) = \widetilde{\eta}(x_1, \ldots, x_n)$. Pictorially, the green slanted rectangle depicts g_t during a time between 0 and 1, in the middle of the homotopy. Thus $\iota \circ \widetilde{\iota} \sim \eta$ in $\pi_n(X, x_0)$, as desired.



The remainder of checking exactness is straightforward and will be omitted. ■

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13.1 Serre Fibrations and Hurewicz Fibrations

First, recall that $p : E \to B$ has the homotopy lifting property with respect to X if for all homotopies $F : X \times I \to B$, and $h : X \to E$ such that $(p \circ h)(x) = f_0(x)$, there exists a unique $\widetilde{F} : X \times I \to E$ such that $p \circ \widetilde{F} = F$ and $\widetilde{f}_0 = \widetilde{F}(x, 0) = h(x)$ for all x. That is, the following diagram commutes:

$$\begin{tikzcd} X \arrow[r, "h"] \arrow[d, hook, "\iota"] & E \arrow[d, "p"] \\ X \times I \arrow[r, "F"] \arrow[ur, dashed, "\widetilde{F}"] & B \end{tikzcd}$$

Definition 13.1. A continuous map $p : E \to B$ is called a **Serre fibration** if it has the homotopy lifting property with respect to I_n for all n. We say that p is a **Hurewicz Fibration** if instead of I_n, it has the homotopy lifting property for all spaces X.

Lemma 13.2. *Let $p : E \to B$ be a continuous map, $\mathcal{U} = \{U_i\}_{i \in I}$ be an open cover of B, and let $p_i \coloneqq p|_{U_i}$, that is, $p_i : p^{-1}(U_i) \to U_i$. If p_i has the homotopy lifting property for each i, then p has the homotopy lifting property for E.*

Definition 13.3. A map $p : E \to B$ is called a **fiber bundle** with fiber F if there exists an open cover $\mathcal{U} = \{U_i\}_{i \in I}$ of B and a family $\{H_i : F \times U_i \to p^{-1}(U_i)\}_{i \in I}$ of homeomorphisms, called **local trivializations**, such that for all $i \in I$ and $x \in U_i$, $p(H_i(f, x)) = x$ for all $x \in U_i$, for all i. Then $p \circ H_i : F \times U_i \to U_i$ is the projection onto the second coordinate: that is, the following diagram commutes:

$$\begin{tikzcd} F \times U_i \arrow[r, "H_i"] \arrow[dr, "p_2"] & p^{-1}(U_i) \arrow[d, "p"] \\ & U_i \end{tikzcd}$$

Example 13.4. The tangent bundle with the natural projection is a fiber bundle.

Theorem 13.5 (Long Exact Sequences of Fibrations)**.** *Given a Serre fibration $p : E \to B$, $p(e_0) = x_0$, $p^{-1}(x_0) = F$ and $e_0 \in F$, there is a long exact sequence*

$$\cdots \to \pi_n(F, e_0) \xrightarrow{\iota_\#} \pi_n(E, e_0) \xrightarrow{p_\#} \pi_n(B, b_0) \xrightarrow{\partial} \pi_{n-1}(F, E_0) \to \cdots,$$

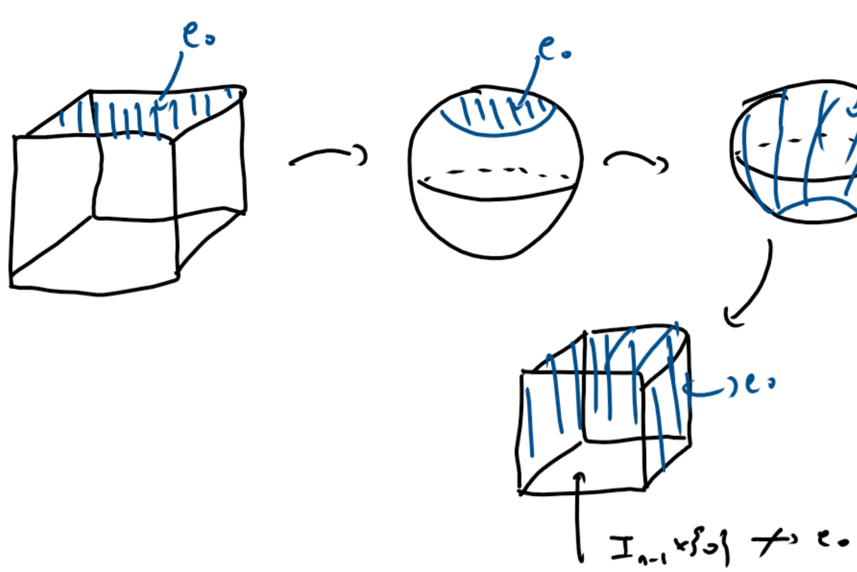
where the boundary map is defined in the following manner: if $\gamma : (I_n, \partial I_n) \to (B, b_0) \in \pi_n(B, b_0)$, then define $H : I_{n-1} \times I \to B$ by viewing γ as a homotopy. That is, $H(x_1, \ldots, x_{n-1}, t) = \gamma(x_1, \ldots, x_{n-1}, t)$. Then notice that $h_1 \equiv b_0$. Then since p is a Serre fibration, p satisfies the homotopy lifting property, so H lifts to a unique homotopy $\widetilde{H} : X \times I \to E$ such that $p \circ \widetilde{H} = H$ and $\widetilde{h}_1 \equiv e_0$. Then we define $\partial\gamma = \widetilde{H}|_{I_{n-1}\times\{0\}} : I_{n-1} \to E$. Then $p(\partial\gamma(x)) = p(\widetilde{H}(x, 0)) = H(x_1, \ldots, x_{n-1}, 0) = b_0$. Hence $\partial\gamma(x) \in p^{-1}(b_0) = F$, so $\partial\gamma \in \pi_{n-1}(F, e_0)$.

Proof Idea. We will show that there is a natural isomorphism between $\pi_n(B, x_0)$ and $\pi_n(X, A, x_0)$, and then appeal to Theorem 12.1, so that we can fit this long exact sequence into the previous one.

Proof. Consider $\pi_n(E, F, e_0) \xrightarrow{p_\#} \pi_n(B, b_0, b_0) \simeq \pi_n(B, b_0)$. We claim that $p_\#$ is an isomorphism, which will allow $\pi_n(B, b_0)$ to naturally fit into the long exact sequence for relative homotopy groups. To see this we will construct an inverse for $p_\#$. Take $\gamma : (I_n, \partial I_n, J_n) \to (E, F, e_0) \in \pi_n(E, F, e_0)$. Then this fits into the diagram

$$\begin{tikzcd}
(I_n, \partial I_n, J_n) \arrow[r, "\gamma"] \arrow[dr, "p\circ\gamma"] & (E, F, e_0) \arrow[d, "p"] \\
& (B, b_0, b_0)
\end{tikzcd}$$

Take H and $\widetilde{H}$ as in the statement of Theorem 13.5. Then $\widetilde{H}$ is a homotopy $I_{n-1} \times [0, 1] \to E$ such that $\widetilde{h}_1 = \widetilde{H}(\cdot, 1) \equiv e_0$, and moreover $\widetilde{H}(\partial I_n) \subseteq F = p^{-1}(b_0)$. Now consider the following deformation:



This gives us a new map $\hat{H} : I_{n-1} \times \hat{[}0,1] \to E$ homotopic to $\widetilde{H}$. Then this induces a map hat : $\pi_n(B, b_0, b_0) \to \pi_n(E, F, e_0)$, $\gamma \mapsto \hat{H}$. It is (presumably) straightforward to check that hat is the inverse of $p_\#$, which concludes the proof. ■

Example 13.6. Let $p : E \to B$ be a covering projection. Then $F = p^{-1}(b_0)$ is discrete; hence $\pi_n(F, e_0) = \{0\}$ for all $n \neq 0$. Then we have the long exact sequence

$$\cdots \to \underbrace{\pi_n(F, e_0)}_{=0} \to \pi_n(E, e_0) \to \pi_n(B, b_0) \to \underbrace{\pi_{n-1}(F, e_0)}_{=0}.$$

for $n - 1 \geq 1$, that is, $n \geq 2$. Thus we have proven the following:

Corollary 13.7. *For $n \geq 2$, and $p : E \to B$ a covering projection, then $p_\# : \pi_n(E, e_0) \to \pi_n(B, b_0)$ is an isomorphism.*

Corollary 13.8. $\pi_n(S^1, 1) = 0$ *for all $n \geq 2$.*

Proof. Contractible spaces have trivial homotopy groups. ■

Definition 13.9. Let G be a given group and $n \in \mathbb{Z}$ an integer. A space (X, x_0) is called **Eilenberg-Maclane Space**, and we write $K(G, n)$, if $\pi_n(X, x_0) = G$ and $\pi_\ell(X, x_0) = 0$ for all $\ell \neq 0$.

One observation to make is that for $\ell \geq 2$, we need the group G to be abelian, for $\pi_n(X, x_0)$ is abelian for $n \geq 2$. Moreover, Corollary 13.8 shows that $K(\mathbb{Z}, 1) = S^1$.

Now one might be wondering what $K(\mathbb{Z}, 2)$ might be. Continuing the above discussion, with the facts that $\pi_1(S^2) = 0$ and $\pi_2(S^2) \simeq \mathbb{Z}$, one might wonder if $S^2 = K(\mathbb{Z}, 2)$, but this is not the case, for $\pi_3(S^2) = \mathbb{Z}$; one way to see this is to use something called the Hopf fibration.

14 2/13/23

14.1 Homotopy Groups of S^n

One of our goals today will be to give a partial answer about computing a subset of all homotopy groups of S^n. Computing *all* of the homotopy groups of S^n, however, is still an open question!

Theorem 14.1. *For all $n \geq 1$ and $0 \leq k \leq n - 1$, $\pi_k(S^n) = 0$.*

But before we move to the proof of this theorem, let's begin with a warm-up.

Proposition 14.2. *For all $n \geq 1$, S^n is path-connected.*

Proof. Fix $x, y \in S^n$, and let $\overline{\gamma}_{x,y} : [0, 1] \to D^{n+1}$ via $\overline{\gamma}_{x,y}(t) = tx + (1 - t)y$, which is the straight line through the $n + 1$-dimensional ball connecting x and y. Now consider $\gamma_{x,y} : [0, 1] \to S^n$ via $\gamma_{x,y}(t) := \overline{\gamma}_{x,y}(t)/|\overline{\gamma}_{x,y}(t)|$. Now this path $\gamma_{x,y}$ if well-defined and connects x and y, as long as x and y are not antipodal: that is, $x \neq -y$. In the case that x and y are antipodal, choose $z \in S^n$ such that z is not antipodal to x and y. Then $\gamma_{x,z}$ and $\gamma z, y$ is well-defined, and their concatenation is a path connecting x and y. ■

Lemma 14.3. *If $f : M \to S^n$ is a continuous map that is not surjective, then it is homotopically trivial.*

Proof. Suppose f is not onto, say $p \notin f(M)$. Now by stereographic projection $h : S^n \setminus \{p\} \to \mathbb{R}^n$, we have the homeomorphism $S^n \setminus \{p\} \approx \mathbb{R}^n$. Note that $\mathbb{R}^n$ is contractible: the map $c : \mathbb{R}^n \times I \to \mathbb{R}^n$ defined by $c(x,t) = (1-t)x$ is the homotopy that contracts the identity map to a constant map. Now consider $F : M \to S^n$ defined by $F(u,t) = h^{-1}(c(t, h(f(u))))$: clearly this is a composition of continuous functions and is continuous. Moreover, $f_0(u) = F(u,0) = h^{-1}(c(0,h(f(u)))) = h^{-1}(h(f(u))) = f(u)$, and $f_1(u) = h^{-1}(0)$ which is constant. Hence F is the desired homotopy. ■

Lemma 14.4. *If M is a manifold and $f : M \to S^n$ and $g : M \to S^n$ satisfy $|f(x) - g(x)| < 2$ for all $x \in M$, then f is homotopic to g.*

Proof. Consider $F : M \times [0,1] \to S^n$ given by

$$F(x,t) \coloneqq \frac{tf(x) + (1-t)g(x)}{|tf(x) + (1-t)g(x)|}.$$

Since $|f(x) - g(x)| < 2$ for all x, it follows that $f(x)$ can never be antipodal to $g(x)$. Hence F is well defined for all x, t, and so this is a homotopy. ■

Next, we will need two results (actually, corollaries) from analysis and smooth manifolds, which we will take as given.

Lemma 14.5 (Stone-Weierstrass)**.** *Given $f : S^k \to S^n$ and an $\epsilon > 0$, there exists a polynomial $p : \mathbb{R}^k \to \mathbb{R}^n$ such that $|p(x) - f(x)| < \epsilon$ for all $x \in S^k$.*

Lemma 14.6 (Sard's Theorem)**.** *If $f : M \to N$ and $f \in C^\infty$, and $\dim M < \dim N$, then f is not onto.*

Exercise 14.7. Construct a continuous function $\gamma : [0,1] \to [0,1]^m$ that is surjective.

Lemma 14.8. *Suppose $f : S^k \to S^n$ is continuous and $k < n$. Then f is homotopic to a map that is not surjective.*

Proof. Suppose f is as prescribed. Then by Lemma 14.5, there exists a polynomial $p : \mathbb{R}^k \to \mathbb{R}^n$ such that $|p(x) - f(x)| < 2$. But applying Lemma 14.4, f is homotopic to p. But p is a polynomial and hence smooth; appealing to Lemma 14.6 yields the result. ■

Now we have enough machinery to accomplish what we set out to do in the beginning of the section.

Proof of Theorem 14.1. Suppose $\gamma : (I_k, \partial I_k) \to (S^n, x_0)$. Note that $\gamma(\partial I_k) = x_0$, and $I_k/\partial I_k \approx S^k$. Now define $\pi : I_k \to S^k$ via the natural projection from the homeomorphism, $\pi(\partial I_k) = b_0 \in S^k$. Now define $f : S^k \to S^n$ by $f(x) = \gamma(\pi^{-1}(x))$. We need to check that f is well-defined: the only problematic point is when $x \neq b_0$, which has ∂I_k as the preimage. But indeed, γ maps ∂I_k to one point, so f is well-defined. Further, we have that f is continuous. Now Lemma 14.8 implies that there exists a homotopy $F : S^k \times [0,1] \to S^n$ such that $f_0 = f$ and $f_1 \equiv \text{const}$.

Consider the path $\eta(t) \coloneqq F(b_0, t)$. Then $\eta(0) = f_0(b_0) = f(b_0) = \gamma(\pi^{-1}(b_0)) = x_0$. We want to modify F to another homotopy $\widetilde{F}$ such that $\widetilde{F}(b_0, t) = x_0$ for all t. Since $\mathrm{SO}(n, \mathbb{R})$ acts transitively on the sphere, i.e., for all $P, Q \in S^n$, there exists some $O \in \mathrm{SO}(n,\mathbb{R})$ such that $OP = Q$, and $O(S^n) = S^n$. Moreover, the choice of O varies continuously with respect to P and Q. Therefore we can choose O_t such that $O_t(\eta(t)) = x_0$ for all t continuously and $O_0 = \mathrm{id}$, and define $\widetilde{F}(x,t) = O_t(F(x,t))$. Then it is easy to verify that $\widetilde{F}(\cdot, 0) = F(\cdot, 0) = f_0 = f$, $\widetilde{F}(\cdot, 1) = O_1(F(\cdot, 1)) =$

$O_1(x_0) = x_0$, and $\widetilde{F}(b_0, t) = x_0$ for all t by construction. Now set $H : (I_k, \partial I_k) \times I \to (S, x_0)$ by $H(x,t) \coloneqq \widetilde{F}(\pi(x), t)$; then with the identification made earlier, this is the desired homotopy, and so γ is trivial in $\pi_k(S^n, x_0)$. ■

Remark 14.9. A similar argument will work for any connected manifold.

14.2 Fiber Bundles and Lie Groups

Note that $\mathrm{SO}(n)$ acts transitively on S^n, and consider $\mathbf{1} = (1, 0, \ldots, 0)^t$. Then we consider $\mathrm{stab}(\mathbf{1}) = \{O \in \mathrm{SO}(n) \mid O\mathbf{1} = \mathbf{1}\}$. Note that if $O \in \mathrm{stab}(\mathbf{1})$, then O takes $\mathbf{1}$ to $\mathbf{1}$, so must have the first column be $\begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$: that is,

$$O = \left(\begin{array}{c|ccc} 1 & 0 & \cdots & 0 \\ \hline 0 & & & \\ \vdots & & \mathrm{SO}(n-1) & \\ 0 & & & \end{array}\right)$$

We get a map $p : \mathrm{SO}(n) \to S^n$, $O \mapsto O \cdot \mathbf{1}$. Then $p^{-1}(\mathbf{1}) \simeq \mathrm{SO}(n-1)$.

Lemma 14.10. *This map p defines a fiber bundle.*

This would imply that p satisfies the path and homotopy lifting property. To see the lemma we will appeal to the following theorem:

Theorem 14.11. *Let G be a Lie group. If H is a closed subgroup, then $p : G \to G/H$ defines a fiber bundle with fiber is H.*

We will prove this theorem in the next two lectures.

We will now consider an application of the above theorem. To do this we define a new type of space. Take $\lambda \in \mathbb{C} \setminus \{0\}$. Then the map $M_\lambda : z \mapsto \lambda z$ acts on $\mathbb{C}^n \setminus \{0\}$. Note that $M_\lambda(S^{2n+1}) = S^{2n+1}$ if $|\lambda| = 1$. Next, consider the relation $z \sim w$ if and only if there exists some $\lambda \in \mathbb{C} \setminus \{0\}$ such that $\lambda z = w$ on $\mathbb{C}^n \setminus \{0\}$. Then the **complex projective space** is defined as $\mathbb{C}P^n \coloneqq (\mathbb{C}^n \setminus \{0\})/\sim = S^{2n+1}/\sim$. Then this gets us a projection map $p : S^{2n+1} \to \mathbb{C}P^n$, where each fiber is S^1, for the same reason as in the theorem.

Corollary 14.12. *We have a long exact sequence*

$$\cdots \to \pi_k(S^1) \to \pi_k(S^{2n+1}) \to \pi_k(\mathbb{C}P^n) \to \pi_{k-1}(S^1) \to \cdots .$$

Thus if $k - 1 \geq 2$, that is, $k \geq 3$, $\pi_k(S^{2n+1}) \simeq \pi_k(\mathbb{C}P^n)$. For $k = 2$, we have $\pi_2(\mathbb{C}P^n) \simeq \mathbb{Z}$.

In general, $\pi_{2n+1}(\mathbb{C}P^n) \simeq \pi_{2n+1}(S^{2n+1}) \simeq \mathbb{Z}$. To answer the question posed in the previous lecture, we note that $\mathbb{C}P^\infty$ is the Eilenberg-Maclane space needed: it is the sought for $K(\mathbb{Z}, 2)$.

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Today we will start the proof of a theorem that was stated and unproved in the last lecture. We will state it again for a reminder.

Theorem 15.1. *Let G be a Lie group and H a closed subgroup of G. Then the projection $p : G \to G/H$ is a defines a fiber bundle with fiber H. That is, for every $x \in G/H$, there exists an open neighborhood U of x and a homeomorphism $Q : U \times H \to p^{-1}(U)$ such that $p(Q(u, h)) = u$ for all $u \in U$.*

15.1 A Short Course in Lie Groups

Let G be a Lie group. Then we will denote $\mathrm{Lie}(G) := \mathfrak{g}$ to be the tangent space at the identity. Then the **exponential map** $\exp : \mathrm{Lie}(G) \to G$ is defined in the following manner: if $v \in T_eG$ and X is a left-invariant vector field of G such that $X(0) = v$, $\alpha : \mathbb{R} \to G$ the associated one-parameter subgroup with $\alpha'(0) = X(0)$, then $v \mapsto \alpha(1)$.

Example 15.2. As an example $\mathrm{GL}(n, \mathbb{R})$ is a Lie group, and $\mathrm{Lie}(\mathrm{GL}(n, \mathbb{R})) = \mathrm{Mat}_{n\times n}(\mathbb{R})$. Then the exponential map coincides with the usual matrix exponential: $\exp(A) = \sum_{n=0}^{\infty} \frac{A^n}{n!}$. Then $\exp(0) = \mathrm{id}$ and $d(\exp)_0 : \mathrm{Mat}_{n\times n}(\mathbb{R}) \to T_e\mathrm{GL}(n, \mathbb{R}) = \mathrm{Mat}_{n\times n}(\mathbb{R})$, where $d(\exp)_0 \equiv \mathrm{id}_{\mathrm{Lie}(G)}$.

Now by the inverse function theorem, exp is a local diffeomorphism; hence we can use the exponential map for charts for G. Moreover, since G is a Lie group, left multiplication $L_g : G \to G$ and $x \mapsto gx$ is a diffeomorphism, so once we know open sets near the identity, we know it everywhere. Thus studying the Lie algebra lets us study Lie groups.

Exercise 15.3. Let G be a path-connected topological group. Then $\pi_1(G, e_0)$ is abelian.

Suppose $\mathrm{Lie}(G) = E \oplus F$ a direct sum of vector spaces. Then one can use $\exp|_E$ and $\exp|_F$ to define a new map $r_{E,F} : \mathrm{Lie}(G) \to G$, given by

$$v = v_E + v_F \mapsto \exp(v_e)\exp(v_F).$$

Warning. $\exp(x + y) \neq \exp(x)\exp(y)$ in general, unless G is abelian.

However, $d(r_{E,F})_0 = \mathrm{id}$, so the inverse function theorem tells us that $r_{E,F}$ is a local diffeomorphism.

Theorem 15.4 (Closed Subgroup Theorem)**.** *Every closed subgroup of a Lie group is an embedded submanifold, and hence a Lie group.*

Note that if H is a closed subgroup of G, then $\mathrm{Lie}(H)$ is a vector subspace of $\mathrm{Lie}(G)$, and we say that $\mathrm{Lie}(H)$ is a Lie subalgebra. Moreover, $\exp(\mathrm{Lie}(H))$ is a subgroup of H, and is the connected component of H at the identity, sometimes also called the **analytic subgroup** associated to $\mathrm{Lie}(H)$.

Now let $T \subseteq \mathrm{Lie}(G)$ be a vector subspace such that $T \oplus \mathrm{Lie}(G) = \mathrm{Lie}(G)$. Then from linear algebra, naturally we have that $\mathrm{Lie}(G)/\mathrm{Lie}(H) \simeq T$. Hence $(\mathrm{Lie}(G)/\mathrm{Lie}(H)) \times \mathrm{Lie}(H) \simeq \mathrm{Lie}(G)$. This sketches out the proof of a special case of Theorem 15.1, when G is a vector space. In particular, it proves the theorem for $\mathrm{Lie}(G)$ the tangent space at the identity.

Next we will state a theorem from Lie theory, which we take as given:

Theorem 15.5. *There exists a neighborhood W of the identity $e \in G$ and a neighborhood $V \subseteq \mathrm{Lie}(G)$ of 0 such that the following hold:*

(i) $\exp : V \to W$ *is a diffeomorphism.*

(ii) if $h \in H \cap W$, then $(\exp)^{-1}(h) \in \mathrm{Lie}(H)$; equivalently, $H \cap W \subseteq H^0$, where H^0 denotes the connected component of H with the identity.

Example 15.6. Let $G = \mathbb{R}^2$, and $H = \{(t, n_1 + n_2\sqrt{2}) \mid t \in \mathbb{R}, n_1, n_2 \in \mathbb{Z}\}$. Then $H^0 = \{(t, 0) \mid t \in \mathbb{R}\}$. But for any open $W \subseteq G$ with $0 \in W$, $H \cap W$ is not a subset of H^0, since H is not a closed subgroup of G. In fact, H is dense in G!

Exercise 15.7. Show that, in the above example, H^0 is a normal subgroup of H.

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16.1 The Proof of the Main Theorem

Today we will prove Theorem 15.1. Our goal is to show that the projection $p : G \to G/H$ defines a fiber bundle. That is, we want to show that for all $x \in G/H$, there exists some U an open neighborhood of x and a homeomorphism $Q : U \times H \to p^{-1}(U)$ such that $p(Q(u, h)) = u$. That is, the following diagram commutes:

$$\begin{tikzcd}
U \times H \arrow[r, "Q"] \arrow[d, "p_1"] & p^{-1}(U) \arrow[r, hook, "\iota"] & G \arrow[dl, "p"] \\
U \arrow[r, hook, "\iota"] & G/H &
\end{tikzcd}$$

Exercise 16.1. Let S be the genus 2 surface. Then there is no nontrivial $p : S \to B$, where B is a topological space, such that S together with p defines a fiber bundle. That is, B cannot be the singleton set and B cannot be equal to S.

Let G be a Lie group and H a closed subgroup of G; consider $\mathrm{Lie}(H)$, and $T \subseteq \mathrm{Lie}(G)$ such that $\mathrm{Lie}(G) = T \oplus \mathrm{Lie}(H)$. Take $\Sigma \subseteq T$ and $B_H \subseteq \mathrm{Lie}(H)$ both open sets such that $\Sigma \oplus B_H \sim \Sigma \times B_H$ are open sets. After shrinking if necessary, WLOG assume that $\Sigma = -\Sigma$ and the map

$$\begin{aligned} r : \Sigma \times B_H &\to G \\ (s, v) &\mapsto \exp(s)\exp(v) \end{aligned}$$

is a local diffeomorphism (this is a consequence of the fact that exp is a local diffeomorphism); let $W \coloneqq r(\Sigma \times B_H) \subseteq G$, and note that this is a neighborhood of the origin.

Now define

$$\begin{aligned} \widehat{Q} : \exp(\Sigma) \times H &\to G \\ (\sigma, h) &\mapsto \sigma h. \end{aligned}$$

Lemma 16.2. *Consider $\widehat{Q}$ as above. Then:*

(a) $\widehat{Q}$ is one-to-one.

(b) $\widehat{Q}$ is continuous.

(c) $\widehat{Q}$ is an open map.

Proof. To prove (a), suppose $\sigma_1 h_1 = \sigma_2 h_2$. Then rearranging, $\sigma_2^{-1}\sigma_1 = h_2 h_1^{-1}$. Clearly $h_2 h_1^{-1} \in H$. On the other hand, $\exp(0) = e$, so $e \in \exp(\Sigma)$; hence $h_2 h_1^{-1} = e \cdot (h_2 h_1^{-1}) \in W$. Thus applying Theorem 15.5, there exists some $v \in \mathrm{Lie}(H)$ such that $h_2 h^{-1} = \exp(v)$. Let $s_1, s_2 \in \Sigma$ be such that $\exp(s_1) = \sigma_1$ and $\exp(s_2) = \sigma_2$. Then we have $\sigma_1 = \sigma_2 \exp(v_2)$, and looking the definition of r again, $r(s_1, e) = r(s_2, v)$. Now using the fact that r and exp are local diffeomorphisms, we conclude that $0 = v$ and $s_1 = s_2$, and hence $\sigma_1 = \sigma_2$ and $h_1 = h_2$. This shows that $\widehat{Q}$ is one-to-one.

Now to prove (b), obviously $\widehat{Q}$ is continuous since multiplication is a continuous operator, coming from the fact that G is a Lie group.

Finally, to prove (c), let $U_1 \subseteq \exp(\Sigma)$ be open and $U_2 \subseteq H$ be open, and further assume that $e \in U_2$. Then defining $\widehat{U_1} := (\exp|_\Sigma)^{-1}(U_1)$ and $\widehat{U_2} := (\exp|_{B_H})^{-1}(U_2)$, we see that $Q(U_1, U_2) = r\left(\widehat{U_1}, \widehat{U_2}\right)$. Since exp is a local diffeomorphism, we have $\widehat{U_1}$ and $\widehat{U_2}$ are both open; similarly, since r is a local diffeomorphism, $r(\widehat{U_1}, \widehat{U_2})$ must also be open. This proves the claim for the single case where U_2 is a neighborhood around the identity.

Now to handle the general case, take U_1 open in $\exp(\Sigma)$ and U_2 open in H without the assumption that $e \in H$. Let h and $\widetilde{U_2}$ be defined as $U_2 = \widetilde{U_2} \cdot h$, and $\widetilde{U_2}$ is a neighborhood of e; that is, $\widetilde{U_2} = R_{h^{-1}}(U_2)$. Then

$$\widehat{Q}\left(U_1, U_2\right) = U_1 \cdot U_2 = U_1 \cdot \left(\widetilde{U_2} \cdot h\right) = \widehat{Q}\left(U_1, \widetilde{U_2}\right) \cdot h,$$

and since by the special case $\widehat{Q}\left(U_1, \widetilde{U_2}\right)$ is open and right multiplication is a diffeomorphism, we have that $\widehat{Q}\left(U_1, U_2\right)$ is open. ■

Corollary 16.3. *The image $\widehat{Q}\left(\exp(\Sigma) \times H\right)$ is open in G and $\widehat{Q}$ is a homeomorphism onto its image.*

Now we return to studying p and the quotient.

Lemma 16.4. *Consider the restriction of the map p to $\exp(\Sigma)$, that is, $p : \exp(\Sigma) \to G/H$. Then:*

(a) p is one-to-one,

(b) p is continuous,

(c) p is an open map.

Proof. To prove (a), suppose that $p(\sigma_1) = p(\sigma_2)$ for some $\sigma_1, \sigma_2 \in \exp(\Sigma)$. Then by definition, $\sigma_1 H = \sigma_2 H$. Then there exist $h_1, h_2 \in H$ such that $\sigma_1 h_1 = \sigma_2 h_2$, that is, $\widehat{Q}(\sigma_1, h_1) = \widehat{Q}(\sigma_2, h_2)$. But by Lemma 16.2 we see that $\sigma_1 = \sigma_2$.

Statement (b) is trivial: this is the restriction of a continuous map.

Finally, to prove (c), take any open set $U \subseteq \exp(\Sigma)$. Note that $p(U)$ is open in G/H if and only if $p^{-1}(p(U))$ is open in G, by the definition of quotient topology. But $p^{-1}(p(U)) = UH = \widehat{Q}(U \times H)$, which is open again by Lemma 16.2. ■

Corollary 16.5. *The image $p(\exp(\Sigma))$ is open in G/H and $p|_{\exp(\Sigma)}$ is a homeomorphism onto its image.*

Now we finish the proof of Theorem 15.1. Let $A \coloneqq p\left(\exp(\Sigma)\right)$. Then A is open by the previous corollary. Define

$$\begin{aligned} Q &: A \times H \to p^{-1}(A), \\ &(a,h) \mapsto \left(p|_{\exp(\Sigma)}\right)^{-1}(a) \cdot h = \widehat{Q}\left(\left(p|_{\exp(\Sigma)}\right)^{-1}(a), h\right). \end{aligned}$$

Then now defining

$$\begin{aligned} f &: A \times H \to \exp(\Sigma) \times H, \\ &(a,h) \mapsto \left(\left(p|_{\exp(\Sigma)}\right)^{-1}(a), h\right), \end{aligned}$$

we have that $Q = \widehat{Q} \circ f$; if you like commutative diagrams, this means that the following diagram commutes:

$$\begin{tikzcd} A \times H \arrow[r, "f"] \arrow[dr, "Q"] & \exp(\Sigma) \times H \arrow[d, "\widehat{Q}"] \\ & p^{-1}(A) \end{tikzcd}$$

By Corollary 16.5 f is a homeomorphism, and hence Q is a homeomorphism. And indeed, for $(a,h) \in A \times H$,

$$\begin{aligned} (p \circ Q)(a,h) &= p\left(\widehat{Q}\left((p|_{\exp(\Sigma)})^{-1}(a), h\right)\right) \\ &= p\left(\left(p|_{\exp(\Sigma)}\right)^{-1}(a) \cdot h\right) \\ &= a. \end{aligned}$$

Finally, noting that left multiplication is a diffeomorphism from G onto itself, we can translate any neighborhood about the identity and functions on it diffeomorphically. Hence this defines a fiber bundle, as desired. ■

Remark 16.6. We did not really use the full power of G being a Lie group, but we only used the local structure: that is, the existence of $W \subseteq G$ a neighborhood and a homeomorphism to $\Sigma \times B_H \to W$, where $e \in \Sigma \subseteq G$, and B_H also containing e. Then if G was any topological group with the above property, the above proof would have gone through.

In fact, we can prove the following theorem for topological groups using similar ideas as above:

Definition 16.7. A subgroup $H \leq G$ is **discrete** if $\{e\}$ is an open set in H.

Theorem 16.8. *If H is discrete, then $p : G \to G/H$ is a covering map.*

17 2/20/23

Today we start discussing homology. First we will approach it axiomatically.

17.1 The Eilenberg-Steenrod Axioms

To any topological pair (X, A), we associate a sequence of abelian groups $\{H_i(X, A)\}_{i\in\mathbb{Z}}$ and a sequence of homomorphisms $\partial : H_i(X, A) \to H_{i-1}(A, \varnothing)$, called the **boundary map**.

Notation. We will denote $H_i(X, \varnothing) = H_i(X)$ for brevity.

To each continuous map $f : (X, A) \to (Y, B)$, we associate a family of homomorphisms $\{f_i : H_i(X, A) \to H_i(Y, B)\}_{i\in\mathbb{Z}}$. Then we have a **homology theory** if the following axioms are satisfied:

(H1) if $f : (X, A) \to (X, A)$ is the identity map, then $f_i \equiv \mathrm{id}_{H_i(X,A)}$ for each i.

(H2) if $f : (X, A) \to (Y, B)$ and $g : (Y, B) \to (Z, C)$ then $(g \circ f)_i = g_i \circ f_i$ for each i.

(H3) If $f : (X, A) \to (Y, B)$ then the following diagram commutes:

$$\begin{array}{ccc} H_i(X,A) & \xrightarrow{f_i} & H_i(Y,B) \\ \big\downarrow \partial & & \big\downarrow \partial \\ H_{i-1}(A) & \xrightarrow{(f|_A)_{i-1}} & H_{i-1}(B) \end{array}$$

(H4) Letting $\iota : (A, \varnothing) \hookrightarrow (X, \varnothing)$ and $J : (X, \varnothing) \hookrightarrow (X, A)$ be the inclusion maps, the following sequence is exact:

$$\cdots \to H_i(A) \xrightarrow{\iota_i} H_i(X) \xrightarrow{J_i} H_i(X, A) \xrightarrow{\partial_i} H_{i-1}(A) \to \cdots .$$

(H5) If $f : (X, A) \to (Y, B)$ and $g : (X, A) \to (Y, B)$ are homotopic, then $f_i = g_i$ for all i.

(H6) If $U \subseteq A$ with $\overline{U} \subseteq \operatorname{int} A$, then $\iota_i : H_i(X \setminus U, A \setminus U) \to H_i(X, A)$ is an isomorphism for all i, where $\iota : (X \setminus U, A \setminus U) \hookrightarrow (X, A)$ is the inclusion map.

(**Warning.** It is not true in general that $H_i(X, A) \simeq H_i(X \setminus A)$.)

(H7) If $X = \{x_0\}$, then $H_i(X) = 0$ for all $i \neq 0$, and $H_0(X)$ is called the **coefficient group**.

Remark 17.1. Often times we would have $H_0(X) \simeq \mathbb{Z}$.

Lemma 17.2. *If $f : (X, A) \to (Y, B)$ is a homotopy equivalence, then $f_i : H_i(X, A) \to H_i(Y, B)$ is an isomorphism for all i.*

Proof. If f is a homotopy equivalence, there is a $g : (Y, B) \to (X, A)$ such that $g \circ f$ is homotopic to $\mathrm{id}_{(X,A)}$. Then by Axiom (H5), $(g \circ f)_i : H_i(X, A) \to H_i(X, A)$ is equal to $\mathrm{id}_{H_i(X,A)}$ for all i. But by Axiom (H2), $g_i \circ f_i = \mathrm{id}_{H_i(X,A)}$. By the same argument, *mutatis mutandis*, we have $f_i \circ g_i = \mathrm{id}_{H_i(Y,B)}$. Hence f_i is invertible for each i, and so must be an isomorphism. ■

Definition 17.3. Given a pair of subsets U and A such that $U \subseteq A \subseteq X$, we say that the inclusion $\iota : (X \setminus U, A \setminus U) \hookrightarrow (X, A)$ is **excisive** if it induces isomorphisms in homology.

Equivalently, given X and a subset $A \subseteq B \subseteq X$, and $A \subseteq C$, we say that the inclusion of pairs $\iota : (B, A) \hookrightarrow (X, C)$ is excisive if $\iota_i : H_i(B, A) \to H_i(X, C)$ is an isomorphism for all i. To see the equivalence, consider $B = U^c$ and set $C = A$. Using this new definition, we can reformulate Axiom (H6) in the following way: if $\overline{U} \subseteq \operatorname{int} A$, the inclusion $J : (X \setminus U, A \setminus U) \hookrightarrow (X, A)$ is excisive.

Definition 17.4. If $A, B \subseteq X$, then we say that A and B are an **excisive pair** if the inclusion $\iota : (B, A \cap B) \hookrightarrow (A \cup B, A)$ is excisive.

Definition 17.5. A **generalized homology theory** is a theory satisfying Axioms (H1) through (H6).

Definition 17.6. Let

$$0 \to M_1 \to N \to M_2 \to 0$$

be a short exact sequence. We say that this short exact sequence **splits** if there is an isomorphism $\varphi : N \to M_1 \oplus M_2$ such that the following diagram commutes:

$$\begin{tikzcd}
0 \arrow[r] & M_1 \arrow[r] \arrow[d, "\mathrm{id}"] & N \arrow[r] \arrow[d, "\varphi"] & M_2 \arrow[r] \arrow[d, "\mathrm{id}"] & 0 \\
0 \arrow[r] & M_1 \arrow[r, "\iota"] & M_1 \oplus M_2 \arrow[r, "p"] & M_2 \arrow[r] & 0
\end{tikzcd}$$

where $\iota : M_1 \to M_1 \oplus M_2$ is natural injection and $p : M_1 \oplus M_2 \to M_2$ is the projection onto the second factor.

Lemma 17.7 (Splitting Lemma)**.** *Suppose we have a short exact sequence*

$$0 \to M_1 \xrightarrow{r} N \xrightarrow{s} M_2 \to 0.$$

Then the following are equivalent:

(a) r has a left-inverse.

(b) s has a right-inverse.

(c) This sequence splits.

Lemma 17.8. *(a) $H_i(\varnothing) = 0$ for all i.*

(b) If $A \subseteq X$ is a retract of X, that is, there exists some continuous $r : X \to A$ such that $r \circ \iota = \mathrm{id}_A$, then $\iota_i : H_i(A) \to H_i(X)$ is one-to-one and $r_i : H_i(X) \to H_i(A)$ is onto for all i. Moreover, $H_i(X) \simeq H_i(A) \oplus H_i(X, A)$ and $H_i(X, A) \simeq \ker r_i$.

(c) Let $X = X_1 \sqcup X_2$ be a disjoint union of X_1 and X_2, with X_1 and X_2 open. Then $J^l : H_i(X_l) \to H_i(X)$ are one-to-one for each l and $(J^1 + J^2)_i : H_i(X_1) \oplus H_i(X_2) \to H_i(X)$ is an isomorphism.

Proof. Given in class as an exercise, but we will prove it.

To see part (a), note that $\varnothing \subseteq \{x_0\}$, and $\varnothing$ is open. So by Axiom (H4) and (H7) we have the following long exact sequence:

$$\cdots \to H_i(\varnothing) \xrightarrow{\iota_i} \underbrace{H_i(\{x_0\})}_{=0} \xrightarrow{J_i} \underbrace{H_i(\{x_0\}, \varnothing)}_{=0} \xrightarrow{\partial} H_{i-1}(\varnothing) \to \cdots,$$

from which it is immediate.

To see part (b), suppose $A \subseteq X$ is as prescribed. Then as a consequence of Axiom (H1) and (H3), $r_i \circ \iota_i = \mathrm{id}_{H_i(A)}$ for each i. This proves the first portion. Next, consider the exact sequence given by Axiom (H4). Since ι_i is injective for each i, by exactness $0 = \ker \iota_{i-1} = \operatorname{im} \partial_i$. Thus ∂_i is the trivial map, and so $H_i(X, A) = \ker \partial_i = \operatorname{im} J_i$, and so J_i is surjective for each i. Then this implies that we have the following short exact sequence:

$$0 \to H_i(A) \xrightarrow{\iota_i} H_i(X) \xrightarrow{J_i} H_i(X, A) \to 0.$$

Then appealing to the Splitting Lemma we deduce that $H_i(X) \simeq H_i(A) \oplus H_i(X, A)$. On the other hand, again by the Splitting Lemma the following diagram commutes:

$$\begin{tikzcd}
H_i(A) \arrow[r, "\iota_i"] \arrow[d, "id"] & H_i(X) \arrow[d, "\varphi"] \\
H_i(A) \arrow[r, "\iota"] & H_i(A) \oplus H_i(X, A)
\end{tikzcd}$$

where $\varphi : H_i(X) \to H_i(A) \oplus H_i(X, A)$ is an isomorphism. Then defining $p_1 : H_i(A) \oplus H_i(X, A) \to H_i(A)$ to be the projection onto the first factor, we see that $p_1 = r_i \circ \varphi^{-1}$, and since φ^{-1} is an isomorphism $\ker p_1 \simeq \ker r_i$, but $\ker p_1 \simeq H_i(X, A)$. This proves (b).

To prove (c), fix $x_0 \in X_1$ and consider the map $r : X \to X_1$ given by

$$r(x) = \begin{cases} x, & x \in X_1 \\ x_0, & x \in X_2. \end{cases}$$

Then r is continuous since any open set $U \subseteq X_1$ not containing x_0 has preimage U, and if $x_0 \in U$ then $r^{-1}(U) = U \cup X_2$ which is open. Moreover, if $J^1 : X_1 \hookrightarrow X$ is the inclusion then $r \circ J^1 = \mathrm{id}_A$, so part (b) we have $J^1 : H_i(X_1) \to H_i(X_2)$ is one-to-one for each i. Furthermore, $H_i(X) \simeq H_i(X_1) \oplus H_i(X, X_1)$. But by Axiom (H6), we have the inclusion $J^2 : (X_2, \varnothing) = (X \setminus X_1, X_1 \setminus X_1) \hookrightarrow (X, X_1)$ is excisive so $H_i(X_2) \simeq H_i(X, X_1)$ for each i via this isomorphism. Hence $J^1 + J^2$ induces the desired isomorphism.

■

17.2 Some Foreshadowing

Our goal is to show next Monday that $H_n(S^n) \simeq H_0(S^n) \simeq \mathbb{Z}$ for all n. This would imply that by Lemma 17.2 that S^n is not contractible. Therefore, $\pi_n(S^n, e_0) \neq 0$, for $\mathrm{id}_{S^n} \in \pi_n(S^n, e_0)$ but is nontrivial. We are also going to prove Invariance of Domain: that if $\mathbb{R}^n \approx \mathbb{R}^k$, then $n = k$, and if $f : U \to \mathbb{R}^n$ is continuous and one-to-one with $U \subseteq \mathbb{R}^n$ open, then $f(U)$ is open in $\mathbb{R}^n$.

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18.1 Reduced Homology Groups

Definition 18.1. Denote $\widetilde{H}_i(X) \coloneqq \ker (f_i : H_i(X) \to H_i(x_0))$ where $f(X) = x_0$. Then $\widetilde{H}_i(X)$ are called the **reduced homology groups**.

Observation. Note that by Axiom (H7), $H_i(x_0) = 0$ for all $i \neq 0$, so by definition $\widetilde{H}_i(X) = H_i(X)$ for all $i \neq 0$. Thus $\widetilde{H}_0(X)$ is the only group that could be different from H_0.

Theorem 18.2. *(a) For all $x_0 \in X$, $\widetilde{H}_i(X) \simeq H_i(X, x_0)$.*

(b) $H_i(X) \simeq H_i(x_0) \oplus \widetilde{H}_i(X)$.

Proof. Note that $f : X \to x_0$ given by $f(x) = x_0$ is a retract of X. Then applying Lemma 17.6 part (b), we conclude that $H_i(X) \simeq H_i(x_0) \oplus H_i(X, x_0)$ with $H_i(X, x_0) \simeq \ker f_i = \widetilde{H}_i(X)$. ■

Note that in part (b) of this theorem, $H_0(X) = H_0(x_0) \oplus \widetilde{H}_0(X)$ is the only interesting part.

Corollary 18.3. *If $f : X \to Y$, then there exists some $f_i : \widetilde{H}_i(X) \to \widetilde{H}_i(Y)$.*

18.2 Mayer-Vietoris Sequence

Theorem 18.4 (Mayer-Vietoris)**.** *Let X be a space, and assume that $X = X_1 \cup X_2$, with $X_1, X_2 \subseteq X$, and let $A \coloneqq X_1 \cap X_2$. Assume further that $(X_1, A) \hookrightarrow (X, X_2)$ is excisive. Then we have the following long exact sequence:*

$$\cdots \to H_i(A) \xrightarrow{\alpha} H_i(X_1) \oplus H_i(X_2) \xrightarrow{\beta} H_i(X) \xrightarrow{\Delta} H_{i-1}(A) \to \cdots . \tag{18.5}$$

Remark 18.6. Leopold Vietoris, one of the people for whom this theorem is named, lived until he was 110 years old.

We will finish the proof in the next lecture, and just outline the main ideas for the proof. Let $\alpha_j : A \hookrightarrow X_j$, $\beta_j : X_j \hookrightarrow X$ be the inclusion maps, and also let $\gamma_1 : (X_1, \varnothing) \hookrightarrow (X, A)$ and $\gamma_2 : (X, \varnothing) \hookrightarrow (X, X_2)$ also be the inclusion maps. Then Axiom (H4) gives us exact sequences

$$\cdots \to H_i(A) \xrightarrow{\alpha_1} H_i(X_1) \xrightarrow{\gamma_1} H_i(X_1) \xrightarrow{\partial_1} H_{i-1}(A) \to \cdots$$

and

$$\cdots \to H_i(X_2) \xrightarrow{\beta_2} H_i(X) \xrightarrow{\gamma_2} H_i(X, X_2) \xrightarrow{\partial_2} H_{i-1}(X_2) \to \cdots .$$

By the excision axiom (H6) we have the existence of an isomorphism $\varepsilon : H_i(X_1, A) \to H_i(X, X_2)$. Then we can link them up so that the following diagram commutes, by Axiom (H3) and the observation that α_j, β_j, γ_j, and ε are all induced by inclusion maps:

$$\begin{tikzcd}
\cdots \arrow[r] & H_i(A) \arrow[r, "\alpha_1"] \arrow[d, "\alpha_2"] & H_i(X_1) \arrow[r, "\gamma_1"] \arrow[d, "\beta_1"] & H_i(X_1, A) \arrow[r, "\partial_1"] \arrow[d, "\varepsilon"] & H_{i-1}(A) \arrow[r] \arrow[d, "\alpha_2"] & \cdots \\
\cdots \arrow[r] & H_i(X_2) \arrow[r, "\beta_2"] & H_i(X) \arrow[r, "\gamma_2"] & H_i(X, X_2) \arrow[r, "\partial_2"] & H_{i-1}(X_2) \arrow[r] & \cdots
\end{tikzcd}$$

Now we want to make the sequence in (18.5) exact using the α_j and β_j. First, take the obvious map $\beta : H_i(X_1) \oplus H_i(X_2) \to H_i(X)$ via $\beta(v_1 + v_2) = \beta_1(v_1) + \beta_2(v_2)$. Similarly, the temptation is to define α as $\alpha_1 + \alpha_2$, but this will not guarantee exactness. Thus we define $\alpha : H_i(A) \to H_i(X_1) \oplus H_i(X_2)$ via $\alpha(v) = \alpha_1(v) - \alpha_2(v)$, so that $\beta \circ \alpha = 0$. Indeed,

$$\beta(\alpha(v)) = \beta(\alpha_1(v) - \alpha_2(v)) = \beta_1(\alpha_1(v)) - \beta_2(\alpha_2(v)) = 0$$

where the last equality is by the commutativity of the diagram. Hence $\operatorname{im} \alpha \subseteq \ker \beta$. Now the next step is to define the map Δ, which we will postpone until the next time.

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Today we will finish up the proof of Theorem 18.4.

19.1 Mayer-Vietoris Sequence, Part Two

Now it is time to define Δ in (18.5), and to this end we will refer back to the diagram we drew in the last lecture. Notice how $H_i(X)$ fits in the bottom row of the diagram, and we can get to $H_{i-1}(A)$ by defining $\Delta : H_i(X) \to H_{i-1}(A)$ by $\Delta \coloneqq \partial_1 \circ \varepsilon^{-1} \circ \gamma_2$, which is well-defined since ε is an isomorphism and so has a defined inverse.

We now verify that defining Δ this way gives us an exact sequence. First we will show that $\operatorname{im} \beta = \ker \Delta$. First suppose that $v \in \operatorname{im} \beta$, that is, $v = \beta(w_1 + w_2) = \beta_1(w_1) + \beta_2(w_2)$ for $w_1 + w_2 \in H_i(X_1) \oplus H_i(X_2)$. Then

$$\gamma_2(v) = \gamma_2(\beta_1(w_1)) + \underbrace{\gamma_2(\beta_2(w_2))}_{=0} = \varepsilon(\gamma_1(w_1)),$$

where the first equality is by the exactness of the sequence and the second equality is by the commutativity of the diagram. Thus rearranging, $\gamma_1(w_1) = (\varepsilon^{-1} \circ \gamma_2)(v)$. Then $\Delta(v) = (\partial_1 \circ \varepsilon^{-1} \circ \gamma_2)(v) = (\partial_1 \circ \gamma_1)(w_1) = 0$ by exactness. Hence $v \in \ker \Delta$.

Conversely, suppose that $v \in \ker \Delta$. Then $(\partial_1 \circ \varepsilon^{-1} \circ \gamma_2)(v) = 0$, and so $(\varepsilon^{-1} \circ \gamma_2)(v) \in \ker \partial_1 = \operatorname{im} \gamma_1$ by exactness. So there exists somse $v_1 \in H_i(X_1)$ such that

$$\left(\varepsilon^{-1} \circ \gamma_2\right)(v) = \gamma_1(v_1). \tag{19.1}$$

Define $\widehat{v}_2 \coloneqq v - \beta_1(v_1)$. Then by construction we have $v = \beta_1(v_1) + \widehat{v}_2$. To show that $v \in \operatorname{im} \beta$, we need to show that $\widehat{v}_2 = \beta(v_2)$ for some $v_2 \in H_i(X_2)$: that is, $\widehat{v}_2 \in \operatorname{im} \beta = \ker \gamma_2$. So it suffices to show that $\gamma_2(\widehat{v}_2) = 0$. Indeed,

$$\gamma_2(\widehat{v}_2) = \gamma_2(v) - \gamma_2(\beta_1(v_1)) = \gamma_2(v) - \varepsilon(\gamma_1(v_1)) = 0,$$

where the second equality is by commutativity of the diagram and the last inequality is by (19.1).

Next, we will verify that $\operatorname{im} \alpha = \ker \beta$. One direction has already been proven in the last section. Suppose that $v_1 + v_2 \in \ker \beta$. Then $\beta(v_1 + v_2) = \beta_1(v_1) + \beta_2(v_2) = 0$, and so $\beta_2(v_2) = -\beta_1(v_1)$. Then $\gamma_2(\beta_2(v_2)) = 0 = \gamma_2(\beta_1(v_1))$. But by commutativity of the diagram, $\gamma_2(\beta_1(v_1)) = \varepsilon(\gamma_1(v_1))$, but since ε is an isomorphism, this implies that $v_1 \in \ker \gamma_1 = \operatorname{im} \alpha_1$. Hence there exists some $w_1 \in H_i(A)$ such that $v_1 = \alpha_1(w_1)$. Now by the commutativity of the diagram, $(\beta_2 \circ \alpha_2)(w_1) = (\beta_1 \circ \alpha_1)(w_1) = -\beta_2(v_2)$ and so $\beta_2(v_2 + \alpha_2(w_1)) = 0$, that is, $v_2 + \alpha_2(w_1) \in \ker \beta_2 = \operatorname{im} \partial_2$. Then there exists some $u \in H_{i+1}(X, X_2)$ such that $\partial_2(u) = v_2 + \alpha_2(w_1)$. Now since ε is an isomorphism, consider $w_2 \coloneqq (\partial_1 \circ \varepsilon^{-1})(u)$. Then

$$\alpha_2(w_2) = \left(\alpha_2 \circ \partial_1 \circ \varepsilon^{-1}\right)(u) = \left(\partial_2 \circ \varepsilon \circ \varepsilon^{-1}\right)(u) = \partial_2(u) = v_2 + \alpha_2(w_1).$$

Now, noting that $w_2 \in \operatorname{im} \partial_1 = \ker \alpha_1$, we have

$$\alpha_1(w_1 - w_2) = \alpha_1(w_1) - \alpha_2(w_2) = v_1,$$

and

$$\alpha_2(w_1 - w_2) = \alpha_2(w_1) - (v_2 + \alpha_2(w_1)) = -v_2.$$

Thus we conclude that

$$\alpha(w_1 - w_2) = \alpha_1(w_1 - w_2) - \alpha_2(w_1 - w_2) = v_1 + v_2,$$

as needed.

The final step is to show that $\operatorname{im}\Delta = \ker\alpha$. Suppose that $\alpha(r) = 0$ for some $r \in H_i(A)$. Then $\alpha_1(r) = \alpha_2(r) = 0$. But by exactness $\ker\alpha_1 = \operatorname{im}\partial_1$ and so there exists some $s \in H_{i+1}(X_1, A)$ such that $\partial_1(s) = r$. Now by the commutativity of the diagram, we have

$$0 = \alpha_2(r) = (\alpha_2 \circ \partial_1)(s) = (\partial_2 \circ \varepsilon)(s).$$

Hence $\varepsilon(s) \in \ker\partial_2 = \operatorname{im}\gamma_2$. Hence there exists some $q \in H_i(X)$ such that $\gamma_2(q) = \varepsilon(s)$. Now observe:

$$\Delta(q) = \left(\partial_1 \circ \varepsilon^{-1} \circ \gamma_2\right)(q) = \left(\partial_1 \circ \varepsilon^{-1}\right)(\varepsilon(s)) = \partial_1(s) = r,$$

so $r \in \operatorname{im}\Delta$.

Conversely, let $r \in \operatorname{im}\Delta$, so $r = \Delta(q)$ for some $q \in H_{i+1}(X)$. Then $\alpha_1(r) = \left(\alpha_1 \circ \partial_1 \circ \varepsilon^{-1} \circ \gamma_2\right)(q) = 0$ since $\alpha_1 \circ \partial_1 = 0$ by exactness. On the other hand,

$$\alpha_2(r) = \left(\alpha_2 \circ \partial_1 \circ \varepsilon^{-1} \circ \gamma_2\right)(q) = (\partial_2 \circ \gamma_2)(q) = 0$$

by the commutativity of the diagram. Hence $\alpha(r) = \alpha_1(r) - \alpha_2(r) = 0$, concluding the proof of exactness. ∎

19.2 Consequences of the Mayer-Vietoris Sequence

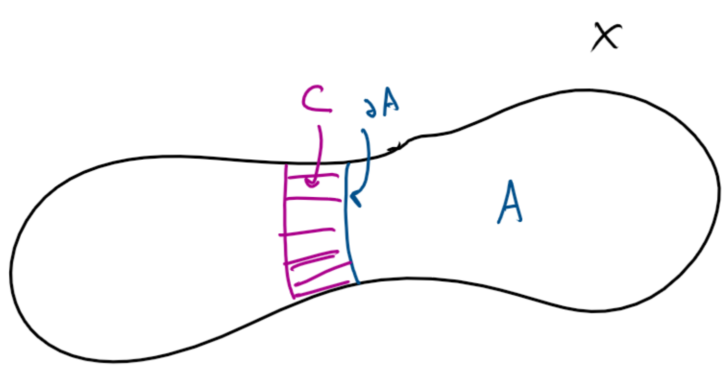
Corollary 19.2 (Mayer-Vietoris Sequence for Reduced Homology Groups)**.** *Assume the same hypothesis as in Theorem 18.4 with the addition that $A = X_1 \cap X_2 \neq \varnothing$. Then we have the long exact sequence*

$$\cdots \to \widetilde{H}_i(A) \xrightarrow{\alpha} \widetilde{H}_i(X_1) \oplus \widetilde{H}_i(X_2) \xrightarrow{\beta} \widetilde{H}_i(X) \xrightarrow{\Delta} \widetilde{H}_{i-1} \to \cdots.$$

Remark 19.3. Recall that $\widetilde{H}_i(X) \leq H_i(X)$ for each i, and moreover the Mayer-Vietoris sequence for reduced homology groups is identical to the usual Mayer-Vietoris sequence, except for H_0.

Proposition 19.4. *Let A be a nonempty closed subset of X, and suppose that ∂A has an open neighborhood $C \subseteq X \setminus \operatorname{int} A$ such that the inclusions $A \hookrightarrow C \cup A$ and $\partial A \hookrightarrow C$ are both homotopy equivalences. Then $(X \setminus \operatorname{int} A, \partial A) \hookrightarrow (X, A)$ is excisive.*

Proof. Since A is closed subset of $A \cup C$, we see that by Axiom (H6), the inclusion $(X \setminus \operatorname{int} A, (C \cup A) \setminus \operatorname{int} A) \hookrightarrow (X, C \cup A)$ is excisive. Now $(C \cup A) \setminus \operatorname{int} A = C$, so now we have $(X \setminus \operatorname{int} A, C) \hookrightarrow (X, C \cup A)$ is excisive; applying the hypothesis, the homotopy equivalences induce isomorphisms of homology groups. Thus $(X \setminus \operatorname{int} A, \partial A) \hookrightarrow (X, A)$ is excisive.



$\blacksquare$

Also recall that the suspension of a space was defined in lecture 7; if X is a space, then its suspension ΣX was the space $X \times [-1,1]$ quotiented by identifying $X \times \{-1\}$ with a point c_- and $X \times \{1\}$ with another point c_+.

Theorem 19.5 (Suspension Theorem). *Let X be a space.*

(a) There exists an isomorphism $\Sigma : \widetilde{H}_{i+1}(\Sigma X) \to \widetilde{H}_i(X)$.

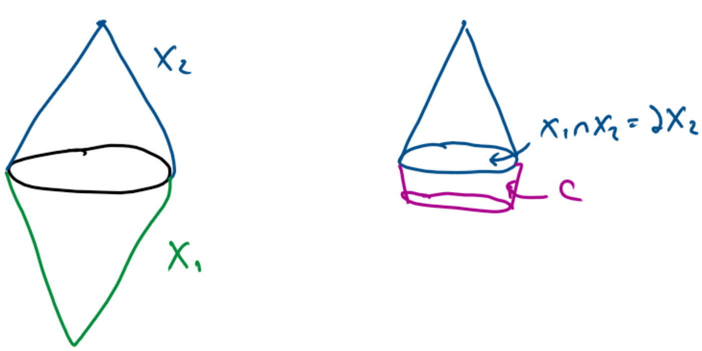
(b) If $f : X \to Y$, then we can define $\Sigma f : \Sigma X \to \Sigma Y$, where $\Sigma[x,t] = [f(x), t]$ where $[x,t]$ is the equivalence class under the quotient defining the suspension, such that Σf is well-defined and the following diagram commutes:

$$\begin{tikzcd}
\widetilde{H}_{i+1}(\Sigma X) \arrow[r, "(\Sigma f)_i"] \arrow[d, "\Sigma"] & \widetilde{H}_{i+1}(\Sigma Y) \arrow[d, "\Sigma"] \\
\widetilde{H}_i(X) \arrow[r, "f_i"] & \widetilde{H}_i(Y)
\end{tikzcd}$$

Proof. Let $\epsilon > 0$ be small, and define $X_1 := X \times [-1,0]/(X \times \{-1\} \sim c_-)$, $X_2 := X \times [0,1]/(X \times \{1\} \sim c_+)$. Then $A = X_1 \cap X_2 = X \times \{0\}$ which is homotopy equivalent to X in the obvious way. Now if the hypotheses for the Mayer-Vietoris sequence held, then we would have the following exact sequence:

$$\begin{tikzcd}
\cdots \arrow[r] & \widetilde{H}_{i+1}(X \times \{0\}) \arrow[r] & \widetilde{H}_{i+1}(X_1) \oplus \widetilde{H}_{i+1}(X_2) \arrow[r] & \widetilde{H}_{i+1}(\Sigma X) \arrow[dll, out=-30, in=150] & \\
& \widetilde{H}_i(X \times \{0\}) \arrow[r] & \widetilde{H}_i(X_1) \oplus \widetilde{H}_i(X_2) \arrow[r] & \widetilde{H}_i(\Sigma X) \arrow[r] & \cdots
\end{tikzcd}$$

But since X_1 and X_2 are both contractible, by Lemma 17.2 we have that $\widetilde{H}_i(X_1) = \widetilde{H}_i(X_2) = 0$ for all i. Therefore this implies that $\widetilde{H}_{i+1}(\Sigma X) \simeq \widetilde{H}_i(X \times \{0\}) \simeq \widetilde{H}_i(X)$. Thus it suffices to verify excision; to do this, we will appeal to Proposition 19.4: indeed, X_2 is a closed subspace of ΣX, $\partial X_2 = X \times \{0\} = X_1 \cap X_2$, and $X \setminus \operatorname{int} X_2 = X_1$. Let $\epsilon > 0$, and let $C := X \times (-\epsilon, 0]$ as in the figure below.



Then clearly $C \cup X_2$ deformation retracts onto X_2 and C deformation retracts onto $X_1 \cap X_2$; hence the inclusions are homotopy equivalences. We conclude that $(X_1, X_1 \cap X_2) \hookrightarrow (X, X_2)$ is excisive, as desired. ■

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20.1 Consequences of the Suspension Theorem

Theorem 20.1. *The homology groups of S^n are given as follows:*

$$H_i(S^n) = \begin{cases} H_0(x_0), & n \neq 0, i = n \text{ or } i = 0 \\ H_0(x_0) \oplus H_0(x_0), & i = n = 0 \\ 0 & \text{otherwise}, \end{cases}$$

and the reduced homology groups of S^n are

$$\widetilde{H}_i(S^n) = \begin{cases} H_0(x_0), & i = n \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Recall that $\Sigma S^n \approx S^{n+1}$. Then by Theorem 19.5 and using induction, we have $\widetilde{H}_{i+k}\left(\Sigma^k X\right) \simeq \widetilde{H}_i(X)$ for a space X, and hence $\widetilde{H}_{i+k}(S^k) \simeq \widetilde{H}_i(S^0)$. But $S^0 = \{-1, 1\}$, and so by Lemma 17.8 we have $H_i(S^0) \simeq H_i(x_0) \oplus H_i(x_0)$, which is 0 unless $i = 0$, and $H_0(S^0) \simeq H_0(x_0) \oplus H_0(x_0)$. Hence $\widetilde{H}_i(S^0) = 0$ if $i \neq 0$. On the other hand, $\widetilde{H}_0(S^0) = \ker\left(f_i : H_0(S^0) \to H_0(x_0)\right) \simeq H_0(x_0)$.

Now if $n \neq 0$, we observe that

$$\widetilde{H}_{i+n}(S^n) \simeq \widetilde{H}_i(S^0) = \begin{cases} 0, & i \neq 0 \\ H_0(x_0), & i = 0. \end{cases}$$

From here we easily deduce the statement of the theorem for reduced homology groups. Now in order to compute the nonreduced homology groups, we only need to worry about the case $i = 0$. We only consider $f : H_0(S^n) \to H_0(x_0)$ for $n \neq 0$ since the $n = 0$ has been computed above; this map is one-to-one since the reduced homology group $\widetilde{H}_0(S^n) = 0$ and this is the kernel of f. On the other hand, the singleton point is a retract of every space: if $\iota : \{x_0\} \hookrightarrow S^n$ is the inclusion, then $f \circ \iota = \mathrm{id}_{\{x_0\}}$. Hence by Lemma 17.8, $f : H_n(S^n) \to H_n(x_0)$ is onto. Hence f must be an isomorphism. ■

Theorem 20.2 (Brouwer Fixed Point Theorem)**.** *If $f : \overline{B_n} \to \overline{B_n}$ is a continuous map, where $\overline{B_n} \subseteq \mathbb{R}^n$ is the closed n-dimensional ball, then there exists some $x_0 \in \overline{B_n}$ such that $f(x_0) = x_0$.*

Proof. The proof mirrors that of the strategy of what we did when we proved this for $n = 2$, *mutatis mutandis*. The only thing that changes is the following proposition. ■

Theorem 20.3 (Brouwer No-Retract Theorem)**.** *There is no retraction $r : \overline{B_n} \to S^{n-1}$.*

Proof. Towards a contradiction suppose there is a retract $r : \overline{B_n} \to S^{n-1}$, so if $\iota : S^{n-1} \hookrightarrow \overline{B_n}$ is the inclusion, then $r \circ \iota = \mathrm{id}_{S^{n-1}}$. Then by Lemma 17.8, $r_i : H_i\left(\overline{B_n}\right) \to H_i(S^n)$ is onto. But if $i = n-1$, then $\operatorname{im} r_{n-1} = H_{n-1}(S^{n-1}) \simeq H_0(x_0) \neq 0$. But on the other hand $\overline{B_n}$ is contractible, so $H_{n-1}\left(\overline{B_n}\right) \simeq H_{n-1}(x_0)$, which is trivial unless $n = 1$, a contradiction. If $n = 1$, then $H_0(\overline{B_n}) \simeq H_0(x_0)$ and $H_0(S^n) \simeq H_0(x_0) \oplus H_0(x_0)$, which also yields a contradiction. ■

Theorem 20.4 (Invariance of Dimension). *If $h : \mathbb{R}^n \to \mathbb{R}^m$ is a homeomorphism, then $n = m$.*

Proof. We have that $\mathbb{R}^n \setminus \{0\}$ is homeomorphic to $\mathbb{R}^m \setminus \{0\}$, via the map $h|_{\mathbb{R}^n \setminus \{0\}}$. But S^{n-1} is homotopy equivalent to $\mathbb{R}^n\{0\}$ via a deformation retract. Then we have the following isomorphisms of homology groups for all i:

$$H_i(S^{m-1}) \simeq H_i(\mathbb{R}^m \setminus \{0\}) \simeq H_i(\mathbb{R}^n \setminus \{0\}) \simeq H_i(S^{n-1}).$$

Thus it follows that $n = m$ by Theorem 20.1. ■

Remark 20.5. Using excision, we can extend the above result to hold for any open sets: that is, if $U \subseteq \mathbb{R}^n$ and $V \subseteq \mathbb{R}^m$ and $f : U \to V$ is a homeomorphism, then $m = n$.

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21.1 Invariance of Domain

Recall in the last lecture we proved the invariance of dimension theorem, Theorem 20.4. In this lecture we will go a step further and generalize the result to arbitrary topological manifolds.

Definition 21.1. A **topological manifold** is a topological space X such that for all $x \in X$, there is an open neighborhood U containing x such that U is homeomorphic to an open subset $V \subseteq \mathbb{R}^n$.

Theorem 21.2 (Invariance of Domain). *If M is a connected topological manifold, then its dimension is well-defined: it is the unique integer $n = \dim M$ such that $H_n(M, M \setminus \{x\}) \neq 0$ for all $x \in M$, and $H_k(M, M \setminus \{x\}) = 0$ for all $k \neq n$. Furthermore, if M is homeomorphic to N and both are connected, then $\dim M = \dim N$.*

Proof. **Step 1.** Using the excision axiom, for all k and any open $U \subseteq M$, $H_k(M, M \setminus \{x\}) \simeq H_k(U, U \setminus \{x\})$.

Step 2. Let $x \in M$ be arbitrary, and choose an open neighborhood of x $U \subseteq M$ that is homeomorphic to some $V \subseteq \mathbb{R}^{n(x)}$, where $n(x)$ may be dependent on x. Now using excision and writing φ to be the homeomorphism, $H_k(V, V \setminus \{\varphi(x)\}) \simeq H_k(B, B \setminus \{\varphi(x)\})$ for some open ball contained in V also containing $\varphi(x)$. But an open ball in $\mathbb{R}^{n(x)}$ is homeomorphic to $\mathbb{R}^{n(x)}$ itself. To conclude: $H_k(U, U \setminus \{x\}) \simeq H_k(\mathbb{R}^{n(x)}, \mathbb{R}^{n(x)} \setminus \{\varphi(x)\})$ where φ is the final homomorphism, after abusing notation. But by Theorem 20.4, we cnclude that

$$H_k(M, M \setminus \{x\}) \simeq H_k(\mathbb{R}^{n(x)}, \mathbb{R}^{n(x)} \setminus \{\varphi(x)\}) \simeq \begin{cases} 0, & k \neq n(x) \\ H_0(x_0), & k = n(x). \end{cases}$$

Step 3. Step 2 gives us a function $n : M \to \mathbb{N}$. Our aim is to now show that n is constant;. Indeed, if $y \in U$, then $n(y) = n(x)$, since

$$H_k(U, U \setminus \{y\}) \simeq H_k(\mathbb{R}^{n(x)}, \mathbb{R}^{n(x)} \setminus \{\varphi(y)\}) \simeq H_k(\mathbb{R}^{n(y)}, \mathbb{R}^{n(y)} \setminus \{\varphi(x)\}),$$

and hence n is a locally constant function on M. Now with the observation that M is connected, we conclude that $n(x) = n(y)$ for all $x, y \in M$.

Step 4. To prove the last statement, if $\psi : M \to N$ is a homeomorphism, then we have the isomorphisms $H_k(M, M \setminus \{x\}) \simeq H_k(N, N \setminus \{\psi(x)\})$ for all k. ■

We can generalize the definition of manifold a little further via the previous theorem.

Definition 21.3. A space X is called a **homological manifold** at x if there exists some n_x such that $H_k(X, X \setminus \{x\}) = 0$ for all $k \neq n_x$, $H_{n_x}(X, X \setminus \{x\}) \simeq H_0(x_0)$. We call n_x the **pointwise homological dimension**, and we write $\dim xX = n_x$.

21.2 Orientability

For this section, we will assume that $H_0(x_0) \simeq \mathbb{Z}$. Notice that if so, there are two choices of isomorphisms: either we can send the generator of $H_0(x_0)$ to 1 in $\mathbb{Z}$, or we can send it to -1 in $\mathbb{Z}$.

Remark 21.4. While two distinct choices for an isomorphisms exist for $\mathbb{Z}$, no two distinct choices for an isomorphism can exist for the case of $\mathbb{Z}_2$: that is, if $H_0(x_0) \simeq \mathbb{Z}_2$, then there cannot be more than one isomorphism.

Now suppose that M is a connected manifold, with $\dim M = n$. Then as we saw in Theorem 21.2, $H_n(X, X \setminus \{x\}) \simeq H_0(x_0) \simeq \mathbb{Z}$. Then in order to define a notion of orientability, we want to be able to pick an element $1 \in H_n(X, X \setminus \{x\})$ to choose an isomorphism.

Exercise 21.5. Define $f : \mathbb{R}^n \to \mathbb{R}^n$ via

$$f(x_1, \ldots, x_n) = (-x_1, x_2, \ldots, x_n).$$

Then consider the induced maps on homology $f_n : H_n(\mathbb{R}^n, R^n \setminus \{0\}) \to H_n(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\})$. Show that $f_n = -\mathrm{id}_{H_n(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\})}$.

Definition 21.6. A manifold M is **orientable** if there exists a covering of M by open sets $\mathcal{U} = \{U_i\}_{i \in I}$ and a family of charts $\varphi_i : \mathbb{R}^n \to U_i$ homeomorphisms such that if $U_i \cap U_j \neq \varnothing$, then $\varphi_i^{-1} \circ \varphi_j : \varphi_j^{-1}(U_i \cap U_j) \to \varphi_i^{-1}(U_i \cap U_j)$ is **orientation-preserving**. That is, $\varphi_i^{-1} \circ \varphi_j$ induces a map $\left(\varphi_i^{-1} \circ \varphi_j\right)_n : H_n\left(\varphi_j^{-1}(U_i \cap U_j), \varphi_j^{-1}(U_i \cap U_j) \setminus \{a\}\right) \to H_n\left(\varphi_i^{-1}(U_i \cap U_j), \varphi_i^{-1}(U_i \cap U_j) \setminus \left\{\varphi_i^{-1}(\varphi_j(a))\right\}\right)$; then the inclusion map $\iota : \varphi_j^{-1}(U_i \cap U_j) \hookrightarrow \mathbb{R}^n$ and similarly for $\varphi_i^{-1}(U_i \cap U_j)$ induces a map $\psi : H_n(\mathbb{R}^n, \mathbb{R}^n \setminus \{a\}) \to H_n(\mathbb{R}^n, \mathbb{R}^n \setminus \{\varphi_i^{-1}(\varphi_j(a))\})$. Now by translating $\varphi^{-1}(\varphi_j(a))$ to a in $\mathbb{R}^n$, we induce a map from $H_n(\mathbb{R}^n, \mathbb{R}^n \setminus \{a\})$ to itself. If this map is the identity, then we say that $\varphi_i^{-1} \circ \varphi_j$ is orientation preserving. To summarize, if the isomorphism on the bottom diagonal arrow in the following diagram is the identity map on $H_n(\mathbb{R}^n, \mathbb{R}^n \setminus \{a\})$, then we say $\varphi_i^{-1} \circ \varphi_j$ is orientation-preserving.

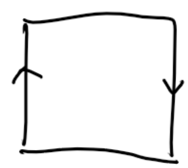
$$
\begin{tikzcd}
H_n\left(\varphi_j^{-1}(U_i \cap U_j), \varphi_j^{-1}(U_i \cap U_j) \setminus \{a\}\right) \arrow[r, "(\varphi_i^{-1} \circ \varphi_j)_n"] \arrow[d, "\iota"] & H_n\left(\varphi_i^{-1}(U_i \cap U_j), \varphi_i^{-1}(U_i \cap U_j) \setminus \left\{\varphi_i^{-1}(\varphi_j(a))\right\}\right) \arrow[d, "\iota"] \\
H_n(\mathbb{R}^n, \mathbb{R}^n \setminus \{a\}) \arrow[r, "\psi"] \arrow[dr, "\simeq"'] & H_n\left(\mathbb{R}^n, \mathbb{R}^n \setminus \{\varphi_j^{-1}(\varphi_i(a))\}\right) \arrow[d, "\text{trans.}"] \\
& H_n(\mathbb{R}^n, \mathbb{R}^n \setminus \{a\})
\end{tikzcd}
$$

Remark 21.7. If M is orientable, then there is a coherent set of generators $1 \in H_n(M, M \setminus \{x\})$ for all x. That is, if x is contained in two different charts, then both charts induce the same generators in H_n.

Example 21.8. $\mathbb{R}^n$ is orientable: $\mathbb{R}^n$ itself is an atlas for $\mathbb{R}^n$.

Example 21.9. Any open, connected subset of $\mathbb{R}^n$ is orientable.

Example 21.10. The Möbius strip is is not orientable.



Exercise 21.11. Prove the above statement. [Hint: covering spaces.]

22 3/3/23

22.1 Axioms of Cohomology

Today we will discuss the axioms of cohomology and some of their consequences. Many of the results will mirror what we did in homology and the proofs are similar, so will be omitted.

For any pair of spaces (X, A), we associate abelian groups $H^i(X, A)$ and boundary maps $\partial : H^i(A) \to H^{i+1}(X, A)$ and as before identify $H^i(X, \varnothing) = H^i(X)$, and for every map $f : (X, A) \to (Y, B)$ we have maps $f^i : H^i(Y, B) \to H^i(X, A)$, satisfying the following axioms:

(C1) if $f : (X, A) \to (X, A)$ is the identity map, then f^i is the identity map for each i.

(C2) If $(X, A) \xrightarrow{f} (Y, B) \xrightarrow{g} (Z, C)$ then the following diagram commutes:

$$
\begin{tikzcd}
H^i(Z, C) \arrow[r, "g^i"] \arrow[rr, bend right, "(g \circ f)^i"'] & H^i(Y, B) \arrow[r, "f^i"] & H^i(X, A)
\end{tikzcd}
$$

That is, $(g \circ f)^i = f^i \circ g^i$ for each i.

(C3) The following diagram commutes:

$$\begin{array}{ccc} H^{i+1}(Y,B) & \xrightarrow{f^{i+1}} & H^{i+1}(X,A) \\ \partial\uparrow & & \partial\uparrow \\ H^i(B) & \xrightarrow{f^i} & H^i(A) \end{array}$$

(C4) We have the following long exact sequence:

$$\cdots \leftarrow H^i(A) \xleftarrow{J} H^i(X) \xleftarrow{\iota} H^i(X,A) \xleftarrow{\partial} H^{i-1}(A) \leftarrow \cdots,$$

where $\iota : (X, \varnothing) \hookrightarrow (X, A)$ and $J : A \hookrightarrow X$ are the inclusion maps.

(C5) If f is homotopic to g, then $f^i = g^i$ for all i.

(C6) If $U \subseteq A$ and $\overline{U} \subseteq \operatorname{int} A$, then $\iota : (X \backslash U, A \backslash U) \hookrightarrow (X, A)$ induces isomorphisms of cohomology groups.

(C7) If $i \neq 0$, $H^i(x_0) = 0$.

Theorem 22.1 (Mayer-Vietoris)**.** *Let $X = X_1 \cup X_2$ and $A = X_1 \cap X_2$, and assume that $\iota : (X, A) \hookrightarrow (X, X_2)$ is excisive, that is, $H^i(X, X_2) \to H^i(X, A)$ is an isomorphism. Then there is a long exact sequence*

$$\cdots \leftarrow H^i(A) \xleftarrow{\alpha} H^i(X_1) \oplus H^i(X_2) \xleftarrow{\beta} H^i(X) \xleftarrow{\Delta} H^{i-1}(A) \leftarrow \cdots.$$

Definition 22.2. If $f : X \to \{x_0\}$ a map into the one-point space, then this induces maps $f^i : H^i(x_0) \to H^i(X)$, and $\operatorname{im} f^i \subseteq H^i(X)$. Then the **reduced cohomology groups** are the cokernels of these homomorphisms, $\widetilde{H}^i(X) \coloneqq \operatorname{coker} f^i = H^i(X)/\operatorname{im}(f^i)$.

Corollary 22.3. *The exact sequence in Theorem 22.1 holds for reduced cohomology groups.*

Theorem 22.4 (Suspension Theorem)**.** *For any n, $\widetilde{H}^n(\Sigma X) \simeq \widetilde{H}^{n-1}(X)$.*

Theorem 22.5. *$\widetilde{H}^n(S^n) \simeq H^0(x_0)$ and $\widetilde{H}^k(S^n) = 0$ for $k \neq n$.*

22.2 Eilenberg-Maclane Cohomology (Optional)

Remember the definition of Eilenberg-Maclane spaces (Definition 13.9). Here we will concern ourselves with a special case: path connected spaces B_n such that $\pi_k(B_n, x_0) = 0$ for every $k \neq n$ and $\pi_n(B_n, 0) \simeq \mathbb{Z}$. Thus as seen before, $B_1 = S^1$ and $B_2 = \mathbb{C}P^\infty = \bigcup_{n \geq 0} \mathbb{C}P^n$.

Define a sort of cohomology using these spaces as follows: let $H^n(X) \coloneqq [X, B_n]$, the set of homotopy classes of maps $X \to B_n$. Now if $f : X \to Y$, this induces maps $f^n : H^n(Y) \to H^n(X)$ via $f^n(v) = v \circ f$, i.e., if $v : Y \to B_n$, then $v \circ f : X \to B_n$. These maps are well-defined because we can compose homotopy. Moreover, it maps the identity to the identity, where the identity is the constant map. Finally, $H^0(x_0) = \mathbb{Z}$ by definition.

Now we ask ourselves, what is $H^1(X)$? If $v \in H^1(X)$, then we have $v : X \to S^1$. Take $x_0 \in X$. Then v induces a map $v_\# : \pi_1(X, x_0) \to \pi_1(S^1, v(x_0)) \simeq \mathbb{Z}$ in the obvious manner: if $\alpha : [0,1] \to X$ is a loop based at x_0, then $(v_\#(\alpha))(t) = v(\alpha(t))$. It can be checked that this map is well-defined, that is, it only depends on the homotopy class. Then we get a map

$$\begin{aligned} L : H^1(X) = [X, S^1] &\to \operatorname{Hom}(\pi_1(X, x_0) \to \mathbb{Z}) \\ v &\mapsto v_\#. \end{aligned}$$

Now if we assume that X is path-connected and has a universal covering, then it can be shown that L is an isomorphism!

23 3/13/23

Today we start our journey towards building up cellular homology by giving the necessary background to define CW complexes.

23.1 Attaching Cells

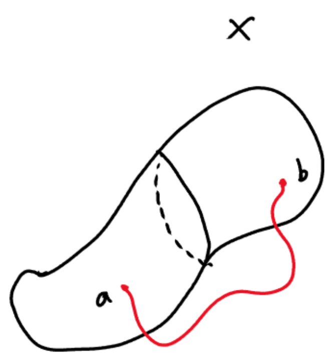
Definition 23.1. An n**-cell** is an n-dimensional closed disc $e^n := D^n \approx [0,1]^n$.

Example 23.2. A 0-cell is the single point $\{0\}$, a 1-cell is the segment $[0,1]$, a 2-cell is the disc $D^2 = \{x \in \mathbb{R}^2 \mid \|x\| \leq 1\}$, etc.

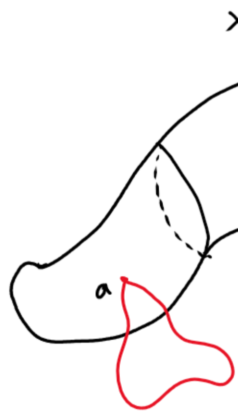
Definition 23.3. Given a space X and a pair of spaces (Z, A) where $A \subsetneq Z$ with A closed, we can **attach** Z **to** X **via** f if $f : A \to X$ is continuous and by forming a new space $X \sqcup_f Z := X \sqcup Z/\sim$, where the identifications are $f(a) \sim a$ for all $a \in A$, and $X \sqcup_f Z$ is endowed with the quotient topology.

Definition 23.4. If X is a space, then we may **attach an** n**-cell** by attaching (D^n, S^{n-1}) to X via the map $f : S^{n-1} \to X$ and forming $X \sqcup_f D^n$.

Example 23.5. Let X be a space with $a, b \in X$. Let $f : [0,1] \to X$ with $f(0) = a$ and $f(1) = b$. Then we may attach a 1-cell via f and form $X \sqcup_f D^1$. If $a \neq b$, then we get the following figure:

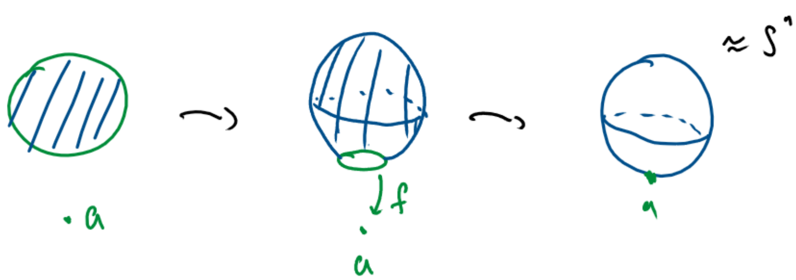


Otherwise, if $a = b$, then we can imagine attaching a loop based at a:



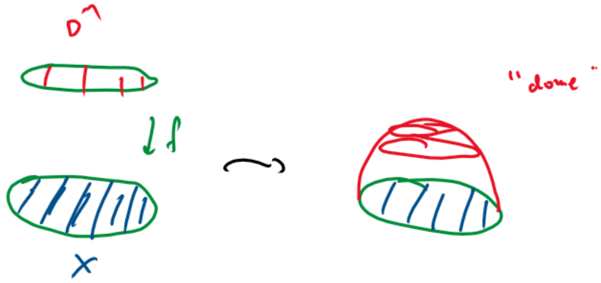
Example 23.6. If we let $X = \{a, b\}$, then attaching a 1-cell via $f : \{0, 1\} \to X$ with $f(0) = a$ and $f(1) = b$ gives us the compact inverval. If $X = \{a\}$, then attaching a 1-cell gives us a loop based at a.

Example 23.7. Let $X = \{a\}$, then we may attach D^n by $f : S^{n-1} \to X$ by identifying S^{n-1} with a single point. Then this can be seen to be homeomorphic to S^n.

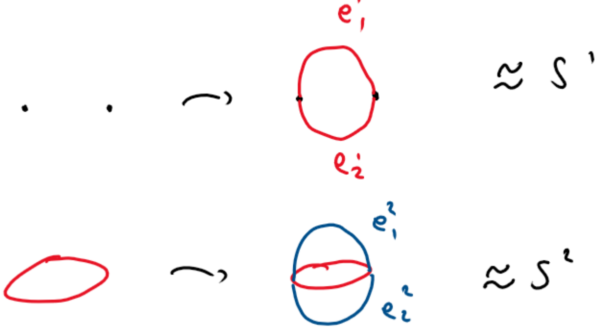


Example 23.8. Let $X = S^{n-1}$, and attach D^n via $f : S^{n-1} \to S^{n-1}$, $f = \mathrm{id}_{S^n}$. Then $X \sqcup_f D^n \approx D^n$.

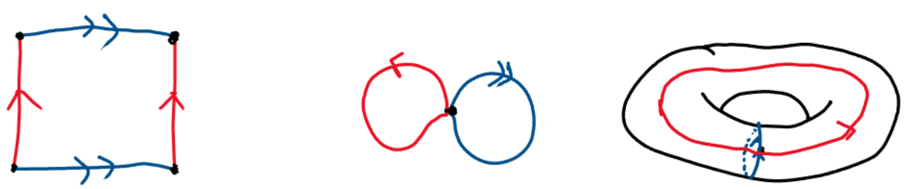
Example 23.9. Let $X = D^n$, and let $f : S^{n-1} \to D^n$ by $f(x) = x$. Then we can imagine attaching a copy of a disc on top of a disc by gluing the boundaries of the two discs together. This is then homeomorphic to S^n.



Example 23.10. In general, we can attach S^n by inductively adding 2 k-cells for each $k = 1, \dots, n$ in the following manner:



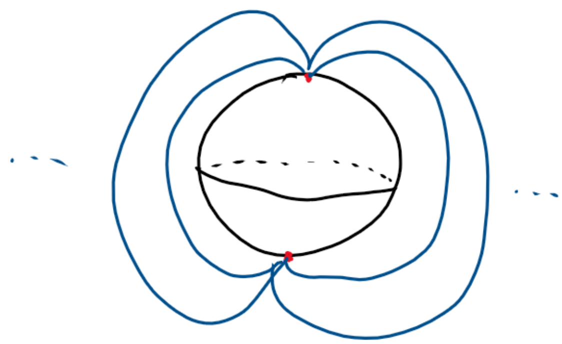
Example 23.11. Recall that we can view the torus as the quotient of a square where the opposite sides are identified with each other. In this light, we can build the 2-torus in the following way. First attach two 1-cells to a point to form the wedge of two circles, $S^1 \wedge S^1$, and then attach a 2-cell (visualized as $[0,1]^2$) by gluing one edge of the rectangle to one of the circles, and then wrapping the rectangle around the other circle.



In general, we can build any g-genus surface by attaching a 2-cell to the wedge of $g-1$ 1-cells.

The above example suggests that we can make more than one attachment simultaneously. Indeed, given a space X and a family of maps $\{f_\lambda : S_\lambda^{n-1} \to X\}_{\lambda \in \Lambda}$, where each $S_\lambda^{n-1} = S^{n-1}$ is indexed by λ, we can attach all of them simultaneously to form $X \cup_{f_\lambda} \left(\bigsqcup_{\lambda \in \Lambda} D_\lambda^n\right)$ to be the same quotient space as before. Note that in this construction, the (images of) the interiors of the n-cells are all disjoint.

Example 23.12. Let $\Lambda = \mathbb{N}$ and $X = S^2$, and let S be the the south pole and N be the south pole. Let $f_n : \{0,1\} \to S^2$ be given by $f_n(0) = S$, $f_n(1)$ for each n. Then this is attaching a 1-cell for each $n \in \mathbb{N}$, and can be viewed as "magnetic fields" on the sphere.



23.2 CW Complexes

Definition 23.13. A space X is a **CW-complex** if $X = \bigcup_{n \geq 0} X_n$ such that

(i) $X_n \subseteq X_{n+1}$ for each n.

(ii) X_0 is a discrete space (i.e., all points are open).

(iii) X_{n+1} is built from X_n by attaching $(n+1)$-cells via maps $f_\lambda : S^n_\lambda \to X_n$, $\lambda \in \Lambda$ and giving it the quotient topology. Note that Λ is allowed to be empty.

Each X_n is called the n**th skeleton of** X. The space X is endowed with the **weak topology** determined by $\{X_n\}_{n \in \mathbb{N}}$; that is, $A \subseteq X$ is closed if and only if $A \cap X_n$ is closed in X_n for each n. If $X_m = X_n$ for every $m \geq n$ and $X_{n-1} \neq X_n$, then we say that X has **dimension** n. We call the image of each int D^n_λ via the map $\pi_\lambda : D^n_\lambda \sqcup X_{n-1} \to X_n$ in X to be a **cell**.

Lemma 23.14. *If* $\dim X = n$*, then each* n*-cell is an open subsets of* X*.*

Definition 23.15. If $X_0 \sqcup \left(\bigsqcup_{n \geq n} \Lambda_n\right)$ is finite, then we say that X is a **finite CW-complex**.

Example 23.16. $\mathbb{C}P^n$ is a CW-complex for every n.

Example 23.17. Example 23.10 shows that S^n is a CW-complex for each n.

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24.1 CW-Complexes and their Homologies

Lemma 24.1. *Let* X *be a CW-complex and let* e_n *be an* n*-cell in* X*. Then:*

(i) $e_n \subseteq X_n$.

(ii) (Closure-finiteness) $\overline{e_n} \subseteq X_n$ *and* $\overline{e_n}$ *intersects finitely many other cells of any dimension.*

Proof. We will prove (ii). Note that

$$\pi_n\left(D^n_\lambda\right) = \overline{\pi_\lambda(\text{int } D^n_\lambda)} = \pi_\lambda(\text{int } D^n_\lambda) \cup \pi_\lambda(\partial D^n_\lambda).$$

Then π_λ maps D^n_λ homeomorphically into its image in X_n, and the second set in the union is contained in X_{n-1} as the image of the attaching map f_λ. Thus $\overline{e_n} = \pi_n(D^n_\lambda)$ is contained in X_n.

■

Lemma 24.2. *Assume that X is obtained from A by attaching an n-cell, say $X = A \sqcup_f D^n$. Then $H_i(X, A) = 0$ unless $i = n$, and, assuming $H_0(x_0) = \mathbb{Z}$, we have $H_n(X, A) \simeq \mathbb{Z}$.*

Lemma 24.3. *Let X be a CW-complex. Then:*

(i) $H_i(X_n, X_{n-1}) = 0$ *for all* $i \neq n$.

(ii) $f_\lambda : (D^n_\lambda, S^n_\lambda) \to (X_n, X_{n-1})$ *induces a one-to-one homomorphism on H_i for each i, and we have the isomorphism*
$$H_n(X_n, X_{n-1}) \simeq \bigoplus_\lambda (f_\lambda)_* \underbrace{H_n\left(D^n_\lambda, S^n_\lambda\right)}_{\simeq \mathbb{Z}}.$$

(iii) *If $i > n$, then $H_i(X_n) = 0$.*

(iv) *The inclusion $\iota : X_{n-1} \hookrightarrow X_n$ induces isomorphisms $H_i(X_{n-1}) \to H_i(X_n)$ if $i < n - 1$ or $i > n$.*

(v) *There is an exact sequence*
$$0 \to H_n(X_n) \to H_n(X_n, X_{n-1}) \to H_{n-1}(X_{n-1}) \to H_{n-1}(X_n) \to 0.$$

24.2 Attaching a Circle

Example 24.4. Let $X_0 = \{x_0\}$ and let $X_1 = S^1$, attached in the obvious way. Next, attach a 2-cell D_2 in the following way: for each $d \geq 1$, we have a family of maps $f_d : S^1 = \partial D_2 \to X_1 = S^1$ given by $f_d(z) = z^d$, taken as a complex number. We want to compute the homology groups of $X = X_2$. Lemma 24.3 tells us that since $\dim X = 2$, $H_i(X) = 0$ for $i > 2$, $H_2(X) = H_2(X_2)$ since $X = X_2$. Moreover, $H_1(X) = H_1(X_1) = H_1(S^1) \simeq \mathbb{Z}$, and $H_0(X) = H_0(x_0)$. Now the question is to compute $H_2(X_2)$. Lemma 24.3 gives us the sxact sequence
$$0 \to H_2(X_2) \to H_2(X_2, X_1) \to \underbrace{H_1(X_1)}_{\mathbb{Z}} \to \underbrace{H_1(X_2)}_{\mathbb{Z}} \to 0.$$

We need to somehow gather more information about $H_2(X_2, X_1)$ to make some conclusions about $H_2(X_2)$. To do this we will build up cellular homology.

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25.1 Cellular Homology

Let X be a CW-complex. Define $C^{\text{cell}}_n(X) \coloneqq H_n(X_n, X_{n-1})$ be the group of n**-cellular chains**. Then we define the **boundary map** $\partial_n : C^{\text{cell}}_n(X) \to C^{\text{cell}}_{n-1}(X)$ is defined by $\partial_n \coloneqq \iota_n \circ \partial$:

$$\begin{tikzcd} H_n(X_n, X_{n-1}) \arrow[r, "\partial"] \arrow[rr, bend right, "\partial_n"'] & H_{n-1}(X_{n-1}) \arrow[r, "\iota_n"] & H_{n-1}(X_{n-1}, X_{n-2}) \end{tikzcd}$$

where $\partial : H_n(X_n, X_{n-1}) \to H_{n-1}(X_{n-1})$ comes from Axiom (H4), and ι_n is induced by the incusion $(X_{n-1}, \varnothing) \hookrightarrow (X_{n-1}, X_{n-2})$.

Proposition 25.1. $\partial_{n-1} \circ \partial_n = 0$*; that is,* $\operatorname{im} \partial_n \subseteq \ker \partial_{n-1}$.

Hence the n-cellular chains define a chain complex. Define the **cellular homology groups** as $H_n^{\text{cell}}(X) \coloneqq \ker \partial_n / \operatorname{im} \partial_{n+1}$.

Theorem 25.2. *If X is a finite-dimensional CW-complex, then $H_n^{\text{cell}} \simeq H_n(X)$ for all n.*

So if we want to understand $H_n(X)$, we can start by understanding $C_n^{\text{cell}}(X)$ and ∂_n. So what is $C_n^{\text{cell}}(X)$? From Lemma 24.3 we have the isomorphism $C_n^{\text{cell}}(X) \simeq \bigoplus_{\lambda \in \Lambda_n} (f_\lambda)_* H_n\left(D_\lambda^n, S_\lambda^{n-1}\right)$. Moreover, $H_n(D^n, S^{n-1}) \simeq \mathbb{Z}$, since $H_i(D^n) = 0$ for all $i \neq 0$ since D^n is contractible, and $H_i(S^{n-1}) = 0$ for $i \neq n-1$, $i \neq 0$, and $H_{n-1}(S^{n-1}) \simeq \mathbb{Z}$. Then there is a long exact sequence by Axiom (H4)

$$\cdots \to H_n(D^n) \to H_n(D^n, S^{n-1}) \to H_{n-1}(S^{n-1}) \to H_{n-1}(D^n) \to \cdots,$$

which immediately gives us what we want. Hence $C_n^{\text{cell}}(X)$ is the free abelian group spanned by e_λ^n where e_λ^n are the n-cells of X_n. That is,

$$e_\lambda^n \coloneqq (f_\lambda)_* \left(D_\lambda^n\right),$$

where by some abuse of language we have $D_\lambda^n \in H_n\left(D_\lambda^n, S_\lambda\right) \simeq \mathbb{Z}$ is the generator. If we abuse notation and write D_λ^n to be the generator of $H_n\left(D_\lambda^n, S_\lambda^{n-1}\right) \simeq \mathbb{Z}$, then $e_\lambda^n = (f_\lambda)_* \left(D_\lambda^n\right)$.

Question. So then how do we compute $\partial_n e_\lambda^n$?

Proposition 25.3. $\partial_n e_\lambda^n = (\iota \circ f_\lambda \circ \partial)\left(D_\lambda^n\right)$.

Proof. The following diagram commutes:

$$\begin{tikzcd}
H_n(D_\lambda^n, S_\lambda^{n-1}) \arrow[r, "\partial"] \arrow[d, "f_\lambda"] & H_{n-1}(S_\lambda^{n-1}) \arrow[d, "f_\lambda"] & \\
H_n\left(X_n, X_{n-1}\right) \arrow[r, "\partial"] \arrow[rr, bend right, "\partial_n"] & H_{n-1}(X_{n-1}) \arrow[r, "\iota"] & H_{n-1}(X_{n-1}, X_{n-2})
\end{tikzcd}$$

■

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26.1 A Return to our Motivating Example

Recall the setup in Example 24.4. Our goal will be to compute the homology groups $H_i^{\text{cell}}(X)$. First we shall consider $H_2^{\text{cell}}(X)$. Note that $C_2^{\text{cell}}(X)$ is the span of $(f_d)_*(D^2) \simeq \mathbb{Z}$, where $f_d : \partial D^2 \to S^1$ is the attaching map, extended to $f_d : D^2 \to X_2$ where f_d is a homeomorphism on $\operatorname{int} D^2$. Similarly, we have $C_1^{\text{cell}}(X) \simeq \mathbb{Z}$. Then $\partial_2 : C_2^{\text{cell}}(X) \to C_1^{\text{cell}}(X)$ is a homomorphism of integers. Later we will see that we can compute this homomorphism ∂_2 with the degree of a map, but for now we shall take it for granted that ∂_2 is given by $e^2 \mapsto de^1$, where e^2 and e^1 are the images of D^2 and D^1 respectively under f_d.

A Detour. It needs some justification as to why $C_1^{\text{cell}}(X) \simeq \mathbb{Z}$ and $C_0^{\text{cell}}(X) \simeq \mathbb{Z}$. For $n \geq 2$, we computed $C_n^{\text{cell}}(X) = H_n\left(D^n, S^{n-1}\right)$ via the long exact sequence of homology groups:

$$\cdots \to H_n(S^{n-1}) \to H_n(D^n) \to H_n\left(D^n, S^{n-1}\right) \to H_{n-1}\left(S^{n-1}\right) \to H_{n-1}\left(D^n\right) \to \cdots .$$

Now by Theorem 20.1, $H_{n-1}\left(S^{n-1}\right) \simeq H_0(x_0) \simeq \mathbb{Z}$; additionally, D^n is contractible so $H_i(D^n) \simeq 0$ for all i; hence $H_n\left(D^n, S^{n-1}\right) \simeq H_{n-1}\left(S^{n-1}\right) \simeq \mathbb{Z}$.

Notice, however, that this approach breaks down when $n = 1$ or $n = 0$, since if $n = 1$ we have the long exact sequence

$$\cdots \to \underbrace{H_1(D^1)}_{0} \to H_1(D^1, S^0) \to \underbrace{H_0(S^0)}_{\mathbb{Z}\oplus\mathbb{Z}} \to \underbrace{H_0(D^1)}_{\mathbb{Z}} \to H_0\left(D^1, S^0\right) \to 0,$$

so we can't make the same conclusion as before. The idea here is to use reduced homology groups, where we would now have the long exact sequence

$$\cdots \to \underbrace{\widetilde{H}_1\left(D^1\right)}_{0} \to \widetilde{H}_1\left(D^1, S^0\right) \to \underbrace{\widetilde{H}_0\left(S^0\right)}_{\mathbb{Z}} \to \underbrace{\widetilde{H}_0\left(D^1\right)}_{0} \to \widetilde{H}_0\left(D^1, S^0\right) \to 0.$$

Thus we now have $H_1\left(D^1, S^0\right) \simeq \widetilde{H}_0\left(S^0\right) \simeq \mathbb{Z}$. Finally, note that $H_0\left(D^0, \partial D^0\right) = H_0(x_0) \simeq \mathbb{Z}$. Thus for the example in this section, $C_1^{\text{cell}}(X) \simeq C_0^{\text{cell}}(X) \simeq \mathbb{Z}$.

We return to computing $H_2^{\text{cell}}(X)$. Note that since $\dim X = 2$, we have $C_n^{\text{cell}}(X) = 0$ for all $n \geq 3$. In particular, $\partial_3 : C_3^{\text{cell}}(X) \to C_2^{\text{cell}}(X)$ must be the trivial map. Hence $H_2^{\text{cell}}(X) = \ker \partial_2 / \operatorname{im} \partial_3 = \ker \partial_2 = 0$ since ∂_2 was shown to be injective earlier in this section.

We move on to computing $H_1^{\text{cell}}(X)$. To this end we investigate the nature of $\partial_1 : C_1^{\text{cell}}(X) \to C_0^{\text{cell}}(X)$. Writing D^1 as the generator of $H_1\left(D^1, S^0\right)$, we need to evaluate $\partial_1\left(f(D^1)\right)$. Applying Proposition 25.3 and the observation that $H_0(X_0) = H_0(X_0, \varnothing) = C_0^{\text{cell}}(X)$, to compute $\ker \partial_1$ we compute $\partial_1\left(f(D^1)\right) = f(\partial(D^1))$, where we are writing $f : \underbrace{H_0(S^0)}_{\simeq\mathbb{Z}\oplus\mathbb{Z}} \to H_0(X_0) = \underbrace{H_0(x_0)}_{\simeq\mathbb{Z}}$. Thus we must determine if $f(\partial D^1) = 0$, that is, if $\partial D^1 \in \ker f = \widetilde{H}_0(S^0)$ or not. Consider the exact sequence

$$0 \to H_1(D^1, S^0) \xrightarrow{\partial} H_0(S^0) \xrightarrow{\iota} H_0(D^1) \to H_0(D^1, S^0) \to 0$$

from Axiom (H4). Observe that $(\iota \circ \partial)\left(D^1\right) = 0$, and $H_0(D^1) \simeq H_0(x_0)$. Hence $\partial D^1 \in \widetilde{H}_0(S^0) = \ker f$. But this now implies that $0 = f(\partial D^1) = \partial_1(f(D^1))$. Hence ∂_1 maps the generator $e^1 = f(D^1)$ to 0, and $\ker \partial_1 = C_1^{\text{cell}}(X) \simeq \mathbb{Z}$. Hence $H_1^{\text{cell}}(X) = \ker \partial_1 / \operatorname{im} \partial_2 \simeq \mathbb{Z}/d\mathbb{Z} = \mathbb{Z}_d$.

Exercise 26.1. Finish the computation by verifying that $H_0(X) \simeq \mathbb{Z}$.

Now we remark on what our space X looks like. If $d = 1$, then we are attaching a disc to the circle via the identity, in which case X is the open disc. Then $H_1^{\text{cell}}(X) = H_2^{\text{cell}}(X) = 0$, which matches what we already know.

If $d = 0$, then we have a circle and a disc attached to the circle at a point, which is the sphere. This space looks like the following picture:

If $d = 2$, then $z \mapsto z^2$ is the attaching map. This map maps $-z$ to z^2 also; thus it "glues" a point on the circle with its antipode while attaching the disc. Thus the resulting space is $\mathbb{R}P^2 = S^2/\{\pm\}$, the space obtained by taking the 2-sphere and identifying the antipodes. Equivalently, this is the space of lines in $\mathbb{R}^3$ going through the origin.

27 3/24/23

27.1 Equivalence of Cellular Homology and Homology

The goal of today's lecture is to furnish a proof for Theorem 25.2. First we recall the general setup: we have the chain complexes $C_n^{\text{cell}}(X) \coloneqq H_n(X_n, X_{n-1})$, and boundary maps $\partial_n : C_n^{\text{cell}}(X) \to C_{n-1}^{\text{cell}}(X)$ given by $\partial_n \coloneqq J_{n-1} \circ \partial$:

$$\begin{tikzcd} H_n(X_n, X_{n-1}) \arrow[r, "\partial"] \arrow[rr, bend right, "\partial_n"'] & H_{n-1}(X_{n-1}) \arrow[r, "J_{n-1}"] & H_{n-1}(X_{n-1}, X_{n-2}) \end{tikzcd}$$

Then writing $Z_n^{\text{cell}}(X) \coloneqq \ker \partial_n$ and $B_n^{\text{cell}}(X) \coloneqq \operatorname{im} \partial_{n+1}$, we have $H_n^{\text{cell}}(X) = Z_n^{\text{cell}}(X)/B_n^{\text{cell}}(X) = \ker \partial_n / \operatorname{im} \partial_{n+1}$.

Claim 27.1. *The map $k_n : H_n(X_n) \to H_n(X)$ induced by inclusion is surjective.*

To see this, we have the exact sequence by Lemma 24.3

$$0 \to H_n(X_n) \to H_n(X_n, X_{n-1}) \to H_{n-1}(X_{n-1}) \to H_{n-1}(X_n) \to 0,$$

and we see by exactness the map (induced by inclusion) $H_{n-1}(X_{n-1}) \to H_{n-1}(X_n)$ is surjective. Since in the hypothesis of the theorem we assume that X is finite dimensional, in the sequence

$$H_n(X_n) \to H_n(X_{n+1}) \to \cdots \to H_n(X),$$

every map is surjective, and so $H_n(X_n) \to H_n(X)$ must also be surjective.

Claim 27.2. *The map $J_n : H_n(X_n) \to H_n(X_n, X_{n-1})$ is injective.*

To prove this claim, consider the exact sequence by Axiom (H4)

$$\cdots \to H_{n+1}(X_n, X_{n-1}) \xrightarrow{\partial} H_n(X_{n-1}) \xrightarrow{\iota} H_n(X_n) \xrightarrow{J_n} H_n(X_n, X_{n-1}) \xrightarrow{\partial} H_{n-1}(X_{n-1}) \to \cdots . \quad (27.3)$$

Then by exactness, $\ker J_n = \operatorname{im} \iota$. But $H_n(X_{n-1}) = 0$ by Lemma 24.3, and so $\operatorname{im} \iota = 0$, from which we get the claim.

Claim 27.4. *For each n, $\operatorname{im} J_n = Z_n^{\text{cell}}(X)$.*

To prove this claim, we modify the exact sequence in (27.3) by adding in a J_{n-1} after the boundary map, losing exactness:

$$\cdots \to H_n(X_n) \xrightarrow{J_n} H_n(X_n, X_{n-1}) \xrightarrow{\partial} H_{n-1}(X_{n-1}) \xrightarrow{J_{n-1}} H_{n-1}(X_{n-1}, X_{n-2}) \to \cdots .$$

By the previous claim, J_{n-1} is also injective; hence $\ker \partial = \ker(J_{n-1} \circ \partial) = \ker \partial_n = Z_n^{\text{cell}}(X)$. But by exactness of the first half of the sequence, we have $\ker \partial = \operatorname{im} J_n$. The claim now follows.

Now having proved the previous two claims, we may now consider the inverse map $J_n^{-1} : Z_n^{\text{cell}}(X) \to H_n(X_n)$, and by composing with k_n we may consider $k_n \circ J_n^{-1} : Z_n^{\text{cell}}(X) \to H_n(X)$ as a well-defined map.

Claim 27.5. *There exists some $B \subseteq Z_n^{\text{cell}}(X)$ such that $J_n^{-1}(B) = \ker(k_n)$, and in fact, $B = B_n^{\text{cell}}(X) = \operatorname{im} \partial_{n+1}$.*

Recapping what we did so far, we have the following commutative diagram:

$$\begin{tikzcd} & Z_n^{\text{cell}}(X) \arrow[dr, "k_n \circ J_n^{-1}"] & \\ H_{n+1}(X_{n+1}, X_n) \arrow[ur, "\partial_{n+1}"] \arrow[r, "\partial"] & H_n(X_n) \arrow[u, "J_n"] \arrow[r, "k_n"] & H_n(X) \end{tikzcd}$$

Now let $v \in H_{n+1}(X_{n+1}, X_n)$. Then $\partial_{n+1}(v) = (J_n \circ \partial)v \in \operatorname{im} \partial_{n+1} = B_n^{\text{cell}}(X)$, and by the commutativity of the diagram we have $(k_n \circ J_n^{-1})\,(\partial_{n+1}(v)) = k_n\,(\partial(v))$. Our goal is to show that $J_n^{-1}(B_n^{\text{cell}}(X)) \subseteq \ker k_n$, that is, $k_n\,(J_n^{-1}\,(J_n\,(\partial(v)))) = 0$. But notice that this is just $(k_n \circ \partial)(v)$. So if $\partial(v) \in \ker k_n$, we would be done.

We can augment the diagram by adding in $H_n(X_{n+1})$ and maps induced by inclusion:

$$\begin{tikzcd} & Z_n^{\text{cell}}(X) \arrow[dr, "k_n \circ J_n^{-1}"] & \\ H_{n+1}(X_{n+1}, X_n) \arrow[ur, "\partial_{n+1}"] \arrow[r, "\partial"] & H_n(X_n) \arrow[u, "J_n"] \arrow[r, "k_n"] \arrow[d, "\iota_1"] & H_n(X) \\ & H_n(X_{n+1}) \arrow[ur, "\iota_2"] & \end{tikzcd}$$

By Lemma 24.3, the inclusion $\iota_2 : H_n(X_{n+1}) \to H_n(X)$ is an isomorphism, and the sequence $H_n(X_{n+1}, X_n) \xrightarrow{\partial} H_n(X_n) \xrightarrow{\iota_1} H_n(X_{n+1}) \simeq H_n(X)$ is exact. Hence $\iota_1(\partial(v)) = 0$ by exactness; but then that implies $k_n(\partial(v)) = \iota_2(\iota_1(\partial(v))) = 0$ also.

Conversely, if $w \in \ker k_n$, then $k_n(w) = 0$. But then $\iota_2(\iota_1(w)) = 0$, and so $w \in \ker \iota_1 = \operatorname{im} \partial$ by exactness. Hence $w = J_n^{-1}\,(J_n \circ \partial)\,(v)$ for some $v \in H_{n+1}(X_{n+1}, X_n)$. But then $w = J_n^{-1}(\partial_{n+1}(v))$, and so $w \in J_n^{-1}(\operatorname{im} \partial_{n+1}) = J_n^{-1}(Z_n^{\text{cell}}(X))$. ■