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*Caveat lector.* One day I will type up the first few lectures.

## 8 1/30/23

Relevant reading: Weintraub pp. 11–13, Hatcher pp. 70–76.

### 8.1 Deck Transformations

We will first begin with an example to motivate our definition:

**Example 8.1.** Let  $S^1 = \{z \in \mathbb{C} \mid |z| = 1\} \subseteq \mathbb{C}$  be the circle regarded as a subspace of  $\mathbb{C}$ . Then we saw that  $p : \mathbb{R} \rightarrow S^1$  via  $p(t) = e^{2\pi it}$  was a covering map. Then for any  $z_0 \in S^1$  with  $p(t_0) = z_0$ , we have that  $p^{-1}(\{z_0\}) = t_0 + \mathbb{Z}$ . Equivalently,  $p(t_0 + m) = p(t_0)$  for all  $m \in \mathbb{Z}$ . Define, for  $m \in \mathbb{Z}$ ,  $T_m : \mathbb{R} \rightarrow \mathbb{R}$  via  $T_m(t) = t + m$  translation by  $m$ . Then by our discussion,  $p \circ T_m = p$ . We say that  $T_m$  is an example of a **deck transformation**.

**Definition 8.2.** Let  $p : E \rightarrow B$  be a covering projection. Then the **group of deck transformations** is the set

$$\Gamma_p := \{T \in \text{Homeo}(E) \mid p \circ T = p\},$$

where the endowed operation is function composition. That is, it is the set of all homeomorphisms such that for any  $T \in \Gamma_p$ , the following diagram commutes:

$$\begin{array}{ccc} E & \xrightarrow{T} & E \\ & \searrow p & \downarrow p \\ & & B \end{array}$$

Now given a covering projection  $p$ , we may define an equivalence relation in the following manner: for  $x, y \in E$ , we say  $x \sim y$  if and only if there exists some  $T \in \Gamma_p$  such that  $T(x) = y$ . Now we may consider the quotient space  $E/\Gamma_p$ , i.e., the topology that makes the projection  $\hat{p} : E \rightarrow E/\Gamma_p$  continuous.

**Example 8.3.** Returning to our example of  $S^1$  and  $p$  defined in Example 8.1, we now ask the question, what is  $\mathbb{R}/\Gamma_p$  with this equivalence relation?

**Claim 8.4.**  $\Gamma_p = \{T_m \mid m \in \mathbb{Z}\}$ .

To see this, proceed in the manner as we did when proving that  $p$  was a covering map. If  $e^{2\pi i T(t)} = e^{2\pi i t}$  for all  $t \in \mathbb{R}$ , then rearranging, we see that  $e^{2\pi i (T(t) - t)} = 1$  for all  $t$ . Hence  $T(t) - t \in \mathbb{Z}$  for all  $t$ , but since  $T(t) - t$  is continuous, we conclude that  $T(t) - t$  is constant, and so there exists some  $m \in \mathbb{Z}$  such that  $T(t) = t + m$  for all  $t$ .

We may now also further say that  $\Gamma_p \simeq \mathbb{Z}$ . Hence we may identify  $\mathbb{R}/\Gamma_p$  with  $\mathbb{R}/\mathbb{Z}$ , or with  $[0, 1)$ .

**Notation.** We will denote  $\Gamma_p(x) := \text{orb}_{\Gamma_p}(x) = \{T(x) \mid T \in \Gamma_p\}$ . In general, if we want to make some sort of identification for  $E/\Gamma_p$  with some set  $S$ , like we did in the previous example, we need  $\#(\Gamma_p(x) \cap S) = 1$  for all  $x$ . Indeed, this is the case for  $[0, 1)$ .

## 8.2 Discontinuous Actions

**Definition 8.5.** Let  $E$  be a topological space, and let  $\Gamma \leq \text{Homeo}(E)$ . We say that  $\Gamma$  **acts discontinuously** if for all  $x \in E$ , there exists some open neighborhood  $U_x$  of  $x$  such that if  $T \in \Gamma$  and  $T(U_x) \cap U_x \neq \emptyset$ , then  $T = \text{id}$ .

*Remark 8.6.* Some texts, like Hatcher, calls a discontinuous action as a covering space action.

One consequence of our definition is the following claim:

**Claim 8.7.** If  $\Gamma$  acts discontinuously on  $E$  and  $S_1, S_2 \in \Gamma$ , and  $S_1(U) \cap S_2(U) \neq \emptyset$  for some nonempty  $U$ , then  $S_1 = S_2$ .

*Proof of claim.* Observe that, since  $S_1$  and  $S_2$  are homeomorphisms,  $\emptyset \neq S_1(U) \cap S_2(U) = S_1(U \cap S_1^{-1} \circ S_2(U))$ . In particular, this implies that  $U \cap S_1^{-1} \circ S_2(U) \neq \emptyset$ . Since  $S_1^{-1} \circ S_2 \in \Gamma$  and  $\Gamma$  acts discontinuously, we conclude  $S_1^{-1} \circ S_2 = \text{id}$ . ■

**Lemma 8.8.** Suppose  $\Gamma$  acts discontinuously on  $E$ . Then  $p : E \rightarrow E/\Gamma$  is a covering projection, where the quotient is defined by  $x \sim_\Gamma y$  if and only if there is some  $T \in \Gamma$  such that  $Tx = y$ .

*Proof.* Given  $y \in E/\Gamma$ , take  $x \in E$  such that  $p(x) = y$ , and let  $U_x$  be the neighborhood that is granted by Definition 8.5. Then we claim that  $p(U_x)$  is open. To see this, notice that  $p^{-1}(p(U_x)) = \bigsqcup_{S \in \Gamma} S(U_x)$ , and since each  $S$  is a homeomorphism,  $S(U_x)$  is open, which implies that  $p^{-1}(p(U_x))$  is open, as desired. Moreover,  $p|_{S(U_x)} : S(U_x) \rightarrow p(U_x)$  is a homeomorphism, and thus  $p$  must be a covering map. ■

## 8.3 Universal Covering Spaces

We will state two key theorems, but we will not prove them.

**Theorem 8.9.** Let  $E$  be a simply connected space, and let  $p : E \rightarrow B$  be a covering projection. Assume further that  $B$  is semilocally simply connected. Then if  $\Gamma_p$  is the group of deck transformations, then  $\Gamma$  acts discontinuously and  $B$  is homeomorphic to  $E/\Gamma_p$ . In particular, the following diagram commutes:

$$\begin{array}{ccc} E & \xrightarrow{\text{id}} & E \\ \downarrow p & & \downarrow \hat{p} \\ B & \xrightarrow{h} & E/\Gamma_p \end{array}$$

where  $h : B \rightarrow E/\Gamma_p$  denotes the homeomorphism and  $\hat{p}$  is the projection map from  $E$  to  $E/\Gamma_p$ .

**Theorem 8.10** (Existence and Universal Property of Universal Covers). Let  $B$  be a semilocally simply connected, locally path connected, connected space. Then there exists a simply connected and connected space  $E$  such that there is a covering projection  $p : E \rightarrow B$ . Moreover, if  $q : X \rightarrow B$  is any other covering projection, with  $X$  connected, then there exists a unique continuous map  $r : E \rightarrow X$  such that the following diagram commutes:

$$\begin{array}{ccc} E & \xrightarrow{r} & X \\ \downarrow p & \swarrow q & \\ B & & \end{array}$$

The space  $E$  is unique up to homeomorphism.

**Definition 8.11.** The space  $E$  in the previous theorem is called a **universal cover**, and we will denote a universal covering space of  $B$  by  $\tilde{B}$ .

**Interpretation.** We can interpret the previous two theorems in the following way: by Theorem 8.10 we know that for any semilocally simply connected space  $B$  there is a universal cover  $\tilde{B}$ , and Theorem 8.9 tells us that  $B \approx \tilde{B}/\Gamma_p$ . Moreover, in a sense,  $\Gamma_p$  is the fundamental group.

**Proposition 8.12.** Let  $p : E \rightarrow B$  be a covering projection and assume further that  $E$  is simply connected and path connected. Suppose  $b_0 \in B$ , and  $e_0 \in E$  such that  $p(e_0) = b_0$ . Then  $\pi_1(B, b_0) \simeq \Gamma_p$ .

*Proof.* We will show that there is a one-to-one correspondence between the two groups. Let  $T \in \Gamma_p$ . Let  $\tilde{\alpha}$  be a curve in  $E$  connecting  $e_0$  and  $T(e_0)$ , and let  $\alpha := p \circ \tilde{\alpha}$ . Then observe that  $\alpha(0) = p(e_0) = b_0 = \alpha(1) = p(T(e_0))$ . So  $\alpha \in \pi_1(B, b_0)$ . Thus this gives us a way to assign a loop in  $\pi_1(B, b_0)$  for every  $T \in \Gamma_p$ . To see that this does not depend on our choice of curve  $\tilde{\alpha}$ , suppose  $\tilde{\beta}$  is another curve connecting  $e_0$  and  $T(e_0)$ , and set  $\beta = p(\tilde{\beta})$ . Then the concatenation  $\tilde{\alpha} \cdot \tilde{\beta}^{-1}$  is a loop with basepoint  $e_0$ , and since  $E$  is simply connected must be homotopically trivial. Hence there is a homotopy  $\tilde{H} : I \times I \rightarrow E$  such that  $h_0 = \tilde{\alpha} \cdot \tilde{\beta}^{-1}$ , and  $h_1 \equiv e_0$ . Now consider  $H : I \times I \rightarrow B$  given by  $H(s, t) = p(\tilde{H}(s, t))$ . Then this is a homotopy taking the concatenation  $\alpha \cdot \beta^{-1}$  to the constant path  $b_0$ , which shows that  $\alpha = \beta$  in  $\pi_1(B, b_0)$ . Hence this map is well-defined.

On the other hand, let  $\gamma \in \pi_1(B, b_0)$ . Then by the path-lifting property, there exists some  $\tilde{\gamma} : [0, 1] \rightarrow E$  such that  $\tilde{\gamma}(0) = e_0$  and  $p(\tilde{\gamma}(1)) = \gamma(1) = b_0$ . Moreover, there exists some  $T \in \Gamma_p$  such that  $T(e_0) = \tilde{\gamma}(1)$ ; this defines the inverse map. By the homotopy lifting property, it is easy to check that this map does not depend on the representative of the homotopy class. Hence it is well-defined. The two maps are clearly bijective, and it is straightforward to check that they are homomorphisms. Hence this proves the isomorphism. ■

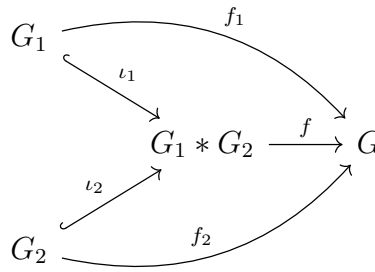
## 9 2/1/23

Today we will introduce the Seifert-van Kampen theorem. Relevant reading: Hatcher Chapter 1.2, Weintraub Section 2.3.

### 9.1 Free Group Products

**Definition 9.1** (Free Group Products). Given two groups  $G_1$  and  $G_2$ , we denote  $G_1 * G_2$  to be the **free product** of  $G_1$  and  $G_2$ , which is the coproduct of the groups  $G_1$  and  $G_2$  in the category of groups. That is, there are injective homomorphisms  $\iota_1 : G_1 \hookrightarrow G_1 * G_2$  and  $\iota_2 : G_2 \hookrightarrow G_1 * G_2$  and it satisfies the following universal property:

If  $G$  is any group and  $f_1 : G_1 \rightarrow G$  and  $f_2 : G_2 \rightarrow G$  are homomorphisms, then there exists a unique homomorphism  $f : G_1 * G_2 \rightarrow G$  such that the following diagram commutes:



**Example 9.2.**  $\mathbb{Z} * \mathbb{Z} = F_2$  the free group on two generators: alternatively, we can write  $F_2$  to be the set of all finite words on two letters  $a, b$ .

*Remark 9.3.* In general, if  $a_i \in G, b_i \in G$ , we can write any element of  $G_1 * G_2$  as  $a_1 b_1 a_2 b_2 \cdots a_k b_k$ .

**Definition 9.4.** Given a group  $G$ , and  $A \subseteq G$  (not necessarily a subgroup), the **normal subgroup generated by  $A$**  is defined by  $N(A) = \bigcap N$ , where the intersection runs over all normal subgroups containing  $A$ : that is, it is the smallest normal subgroup of  $G$  containing  $A$ .

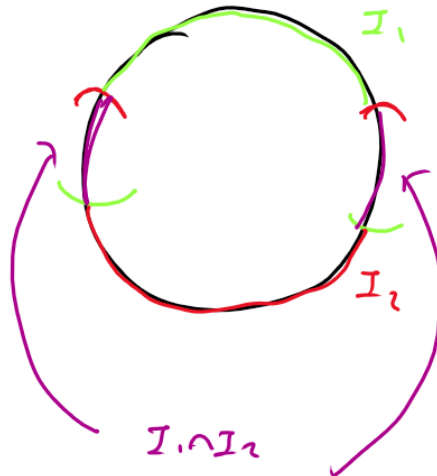
## 9.2 The Seifert-van Kampen Theorem and Applications

**Theorem 9.5** (Seifert-van Kampen). *Let  $X$  be a path-connected space, and assume  $X = U_1 \cup U_2$ , where both  $U_1$  and  $U_2$  are open and path-connected. Let  $x_0 \in U_1 \cap U_2$  and assume that  $U_1 \cap U_2$  is also path-connected. Then  $\pi(X, x_0) \simeq (\pi_1(U_1, x_0) * \pi_1(U_2, x_0)) / N(A)$ , where if  $(\iota_1)_*$  and  $(\iota_2)_*$  are the homomorphisms induced by the inclusion map  $\iota_i : U_i \rightarrow X$ , we have*

$$A = \{(\iota_1)_*(g^{-1}) * (\iota_2)_*(g) \mid g \in \pi_1(U_1 \cap U_2, x_0)\}.$$

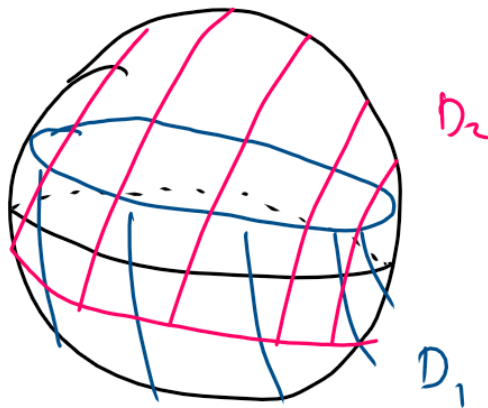
We will not prove the Seifert-van Kampen theorem today, but we will see some applications of it.

**Example 9.6** (An Incorrect Application). Consider  $S^1$  as the union of two open intervals  $I_1$  and  $I_2$  as in the figure. But  $\pi_1(S^1)$  cannot be a quotient of the free product  $\pi_1(I_1) * \pi_1(I_2)$  because the two factors are both trivial, but we already know that  $\pi_1(S^1) \simeq \mathbb{Z}$ . The error was in that the hypothesis  $U_1 \cap U_2$  is not path-connected.



**Proposition 9.7** (Fundamental Group of  $S^n$ ). *For  $n \geq 2$ ,  $S^n$  is simply connected.*

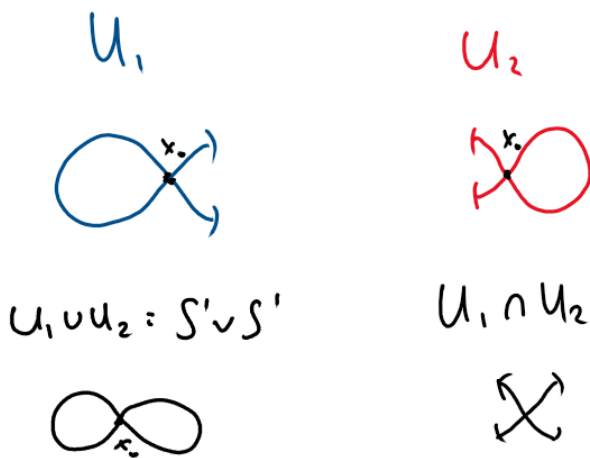
*Proof.* We will use the ideas from Example 9.6. Let  $x_0 \in S^n$ ; by rotating the sphere, we may assume that  $x_0$  is on the equator. Write  $S^n = D_1 \cup D_2$ , where  $D_1$  and  $D_2$  are the open sets in the figure below.



Note that  $D_1$  and  $D_2$  are both contractible, and so must have trivial fundamental group. Moreover,  $D_1 \cap D_2 \approx S^{n-1} \times I$ , which is also path-connected. Then applying the Seifert-van Kampen theorem,  $\pi_1(S^n, x_0)$  must be a quotient of  $\pi_1(D_1, x_0) * \pi_1(D_2, x_0) = \{0\}$ . Hence  $S^n$  is simply connected. ■

*Remark 9.8.* We could have used the stereographic projection to map the sphere with the poles removed onto  $\mathbb{R}^n$  in the previous proof.

**Example 9.9** (The Figure 8). Consider  $E := S^1 \vee S^1$ , or the “figure 8,” joined together at the point  $x_0$ . Let  $U_1$  and  $U_2$  be as in the figure below, so that  $U_1 \cup U_2 = E$ , and  $U_1 \cap U_2$  is the cross in the middle.



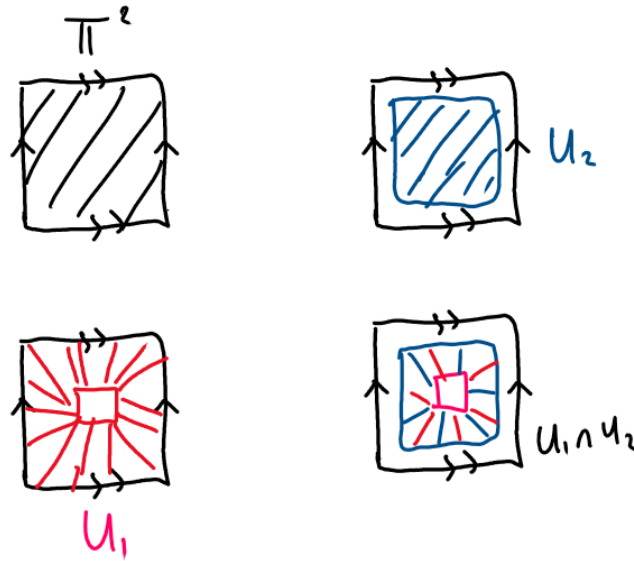
First observe that all our spaces are path-connected and so the hypotheses of the Seifert-van Kampen theorem are satisfied. Next,  $U_1 \cap U_2$  is contractible, which implies that  $\pi_1(U_1 \cap U_2, x_0) = \{0\}$ .

Finally, observe that  $U_1 \approx U_2 \approx S^1$ , which implies that  $\pi_1(U_1, x_0) \simeq \pi_1(U_2, x_0) \simeq \mathbb{Z}$ . Appealing to the Seifert-van Kampen theorem, we conclude that  $\pi(E, x_0) = \mathbb{Z} * \mathbb{Z}$ .

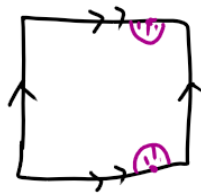
**Exercise 9.10.** Apply induction to the previous example to conclude that the fundamental group of the  $n$ -petal rose is  $F_n$ , the free group on  $n$  elements.

**Exercise 9.11.** Let  $X$  and  $Y$  be topological spaces, and suppose  $X \vee Y$  be locally contractible and/or semilocally simply connected at the attaching point  $x_0$ . Show that  $\pi_1(X \vee Y, x_0) \simeq \pi_1(X, x_0) * \pi_1(Y, x_0)$ .

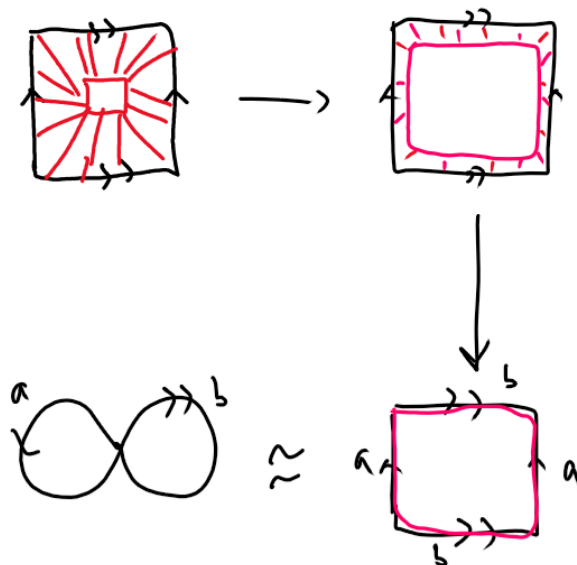
**Example 9.12 (The Torus).** Let  $\mathbb{T}^2$  denote the torus  $\mathbb{T}^2 = S^1 \times S^1$ . We have already noted that  $\pi_1(\mathbb{T}^2) \simeq \pi_1(S^1) \times \pi_1(S^1) \simeq \mathbb{Z} \times \mathbb{Z}$ . Now we will use the Seifert-van Kampen's theorem to prove this. We have shown that the torus may be considered as the quotient space of the square where the opposite edges are identified. Now let  $U_1$  and  $U_2$  be as in the diagram, where  $U_1$  is the “outer” part of the square, and  $U_2$  the “inner” part.



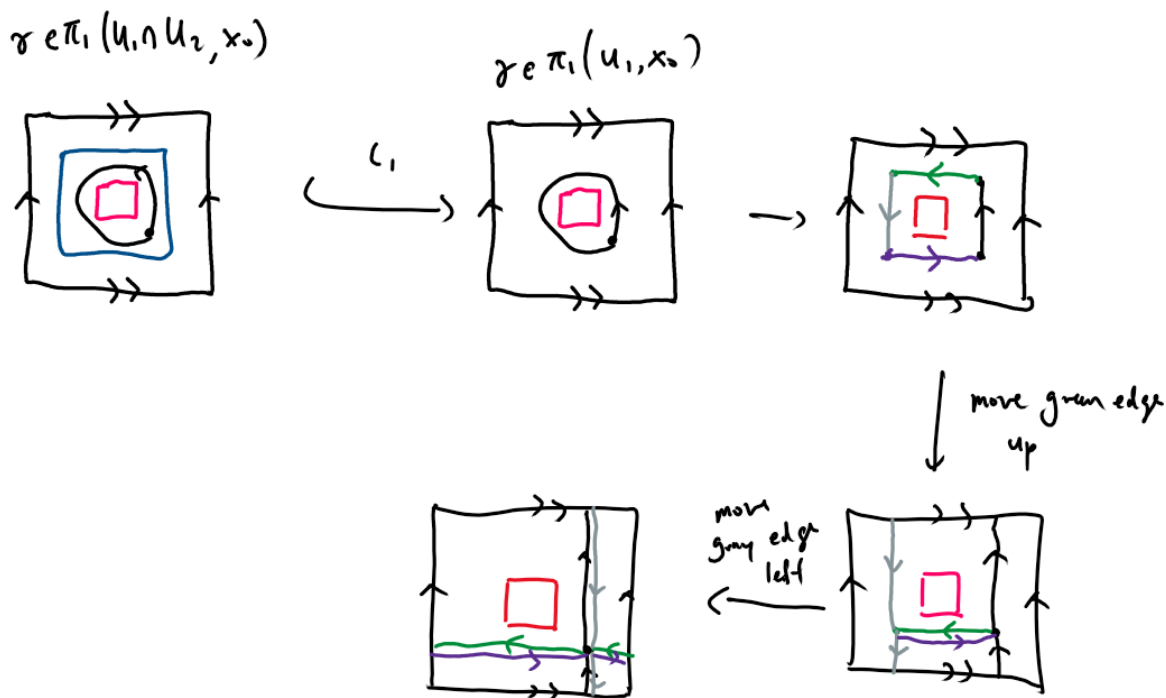
To see that  $U_1$  is open, note that on the edges, any ball would “bleed over” to the opposite edge, as in the following figure:



It is now easy to see that  $U_1$  and  $U_2$  are both open,  $U_1 \cup U_2 = \mathbb{T}^2$ , and  $U_1, U_2$ , and  $U_1 \cap U_2$  are all path-connected. The hypotheses of the Seifert-van Kampen theorem are now satisfied. Fix  $x_0 \in U_1 \cap U_2$ . First observe that  $U_2$  is contractible, and so  $\pi_1(U_2, x_0) = \{0\}$ . On the other hand, we see that  $U_1$  deformation retracts onto the boundary of the square, and then identified with the figure 8 in the following manner:



Since deformation retracts induce an isomorphism of fundamental groups, we have from Example 9.9  $\pi_1(U_1, x_0) \simeq \pi_1(S^1 \vee S^1) \simeq \mathbb{Z} * \mathbb{Z}$ . Now  $U_1 \cap U_2$  is the annulus, which deformation retracts onto the circle  $S^1$ , so its fundamental group is the free group on one generator, the loop going around the annulus once counterclockwise. The following figure shows its image under  $(\iota_1)_*$ :



Now after the deformation retract, we see that in the image this loop is exactly the commutator  $aba^{-1}b^{-1}$ . But this was the image of the generator, and so we conclude that  $N(A)$  (in the statement of the theorem) must be the commutator subgroup inside  $\pi_1(U_1, x_0)$ . Therefore  $\pi(\mathbb{T}^2, x_0) \simeq \mathbb{Z} * \mathbb{Z} / \langle aba^{-1}b^{-1} \rangle = \mathbb{Z} \times \mathbb{Z}$ .

*Remark 9.13.* The above proof can be adapted to compute  $\pi_1(\mathbb{T}^n, x_0)$  with induction.

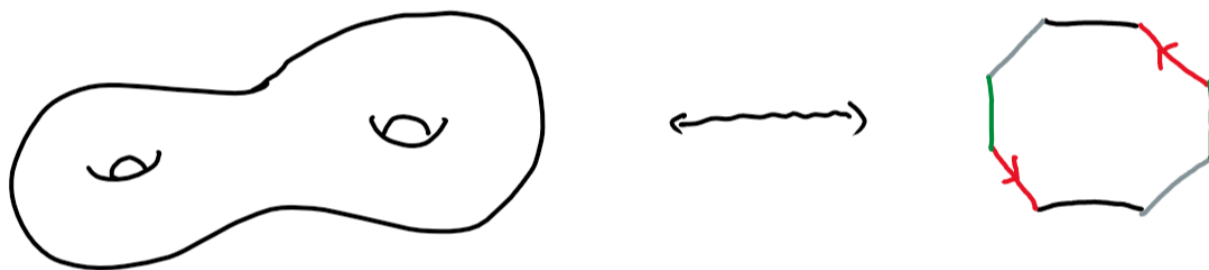
## 10 2/3/23

Today we will continue with examples of van Kampen's Theorem.

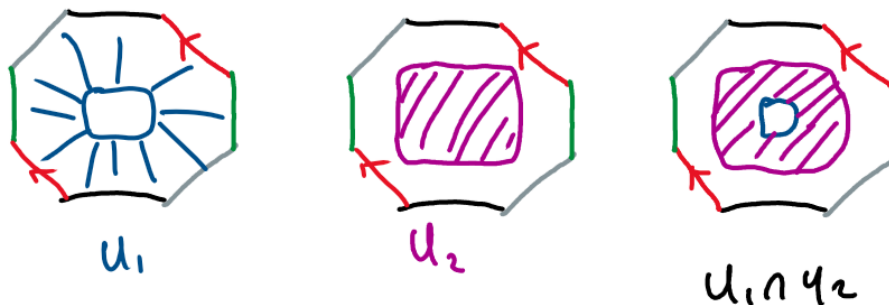
### 10.1 The Genus 2 Surface

Recall when we computed the fundamental group of the torus via Seifert-van Kampen theorem, we used the quotient of a square that is homeomorphic to the torus.

**Method 1.** For the genus 2 surface  $S$ , we will consider the quotient of an octagon as follows:

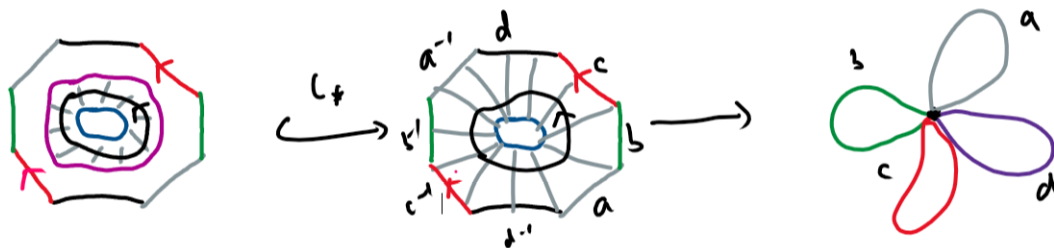


Then just as we did for the torus, decompose the octagon into following pieces:



First note that all these sets are path-connected, so the hypotheses of the van Kampen theorem are satisfied. Then notice that  $U_1$  deformation retracts onto the boundary, which is homeomorphic to the 4-petal rose; thus  $\pi_1(U_1) \simeq F_4$ , the free group on four elements. Moreover,  $U_2$  is contractible and so has trivial fundamental group. Finally,  $U_1 \cap U_2$  is the annulus, which deformation retracts onto  $S^1$ , so has fundamental group the free group on one generator. Then by the Seifert-van Kampen theorem, we have that  $\pi_1(S) \simeq \pi_1(U_1)/N(\iota_1(g) \mid g \in \pi_1(U_1 \cap U_2))$ . Let  $g$  be a loop in  $U_1 \cap U_2$ , like in the diagram below. Then considered as a loop in  $U_1$  and its image in the deformation retract, its image is  $abcd a^{-1} b^{-1} c^{-1} d^{-1}$ .



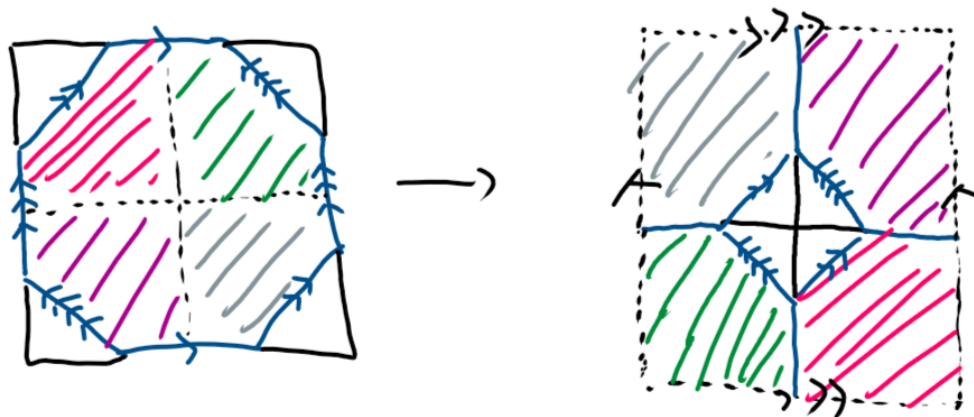


Therefore  $\pi_1(S) \simeq F_4 / \langle abcd a^{-1} b^{-1} c^{-1} d^{-1} \rangle \simeq \langle a, b, c, d \mid abcd = dcba \rangle$ .

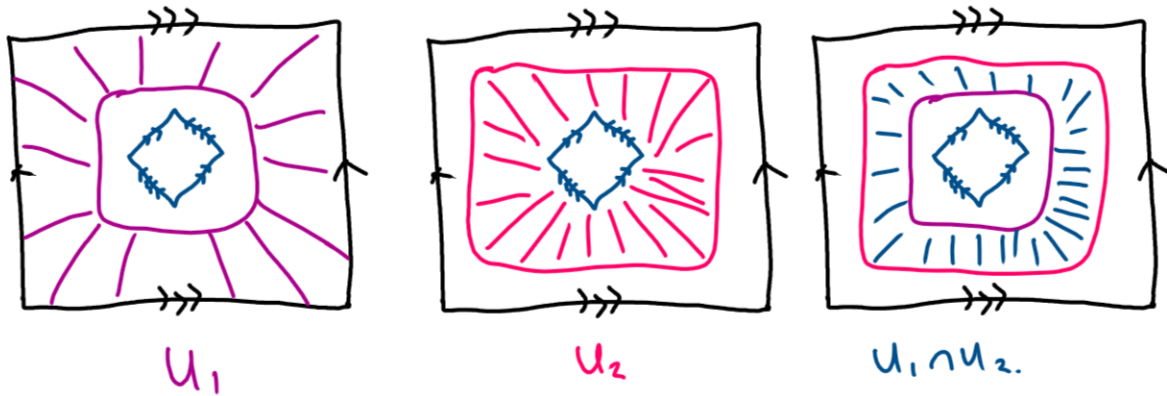
**Method 2.** The idea will be to decompose the surface into two parts, just like below:



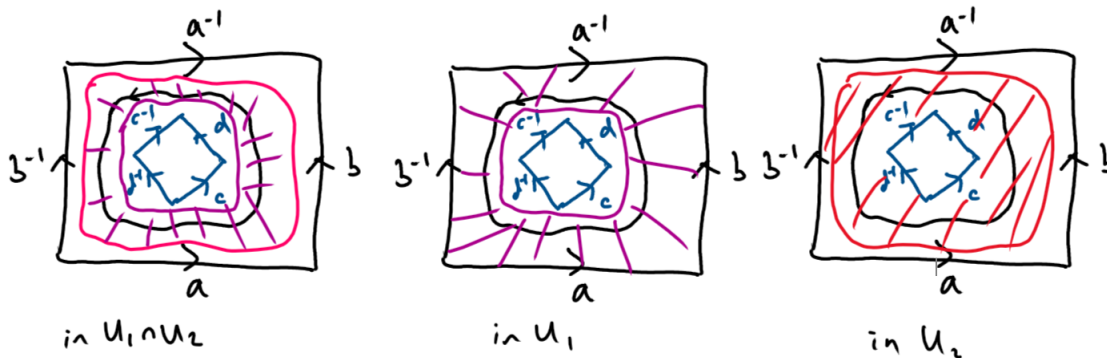
Consider the octagon again, but this time as a subspace of the square below after cutting and pasting.



Just as above, consider  $U_1$  and  $U_2$  defined as in the figure below.



Then both  $U_1$  and  $U_2$  deformation retract onto the figure 8, and  $U_1 \cap U_2$  is an annulus which deformation retract onto  $S^1$ . Hence  $\pi_1(U_1) \simeq \pi_1(U_2) \simeq F_2$  the free group on two generators, and  $\pi_1(U_1 \cap U_2) \simeq \mathbb{Z}$ . Now consider the single loop  $g \in \pi_1(U_1 \cap U_2)$  given by the generator: that is, the loop that goes around once in the annulus. Then considered as a loop in  $U_1$  and  $U_2$  respectively, the diagram below shows that in  $U_1$  it deformation retracts onto the loop  $aba^{-1}b^{-1}$ , and in  $U_2$  it deformation retracts onto the loop  $cdc^{-1}d^{-1}$ .

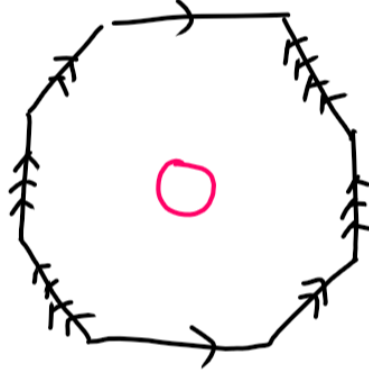


Therefore  $(\iota_1)_*(g) = aba^{-1}b^{-1}$  and  $(\iota_2)_*(g) = cdc^{-1}d^{-1}$ . Thus applying van Kampen's theorem, we conclude that  $\pi_1(S) \simeq F_2 * F_2 / N \left( aba^{-1}b^{-1} (cdc^{-1}d^{-1})^{-1} \right) \simeq \langle a, b, c, d \mid [a, b] = [c, d] \rangle$ .

*Remark 10.1.* We can compute the fundamental group of a genus  $g$  surface by induction.

**Corollary 10.2.** *The fundamental group of a genus 2 surface with a point deleted is the free group on 4 elements. In general, the fundamental group of a genus  $g$  surface, with  $g \geq 2$ , is the free group on  $2g$  elements.*

*Proof.* The genus 2 surface with a point deleted can be identified with the quotient space of the octagon in Method 1 with a neighborhood deleted in the interior, as in the diagram. Then this deformation retracts onto the boundary, which is homeomorphic to the 4-petal rose.

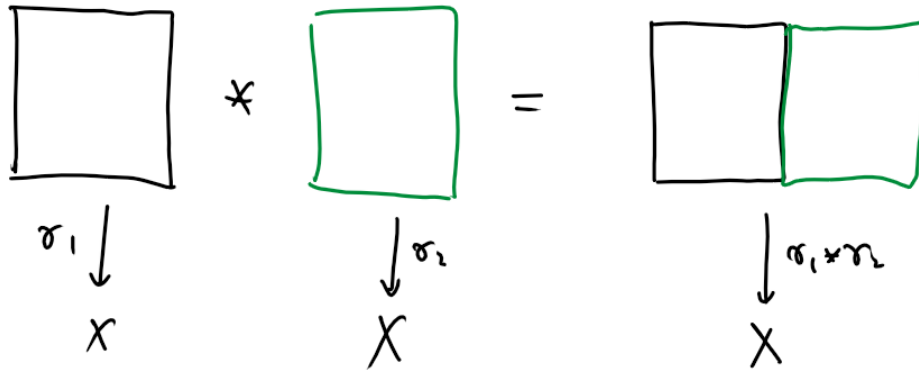


The general case is handled by induction. ■

## 10.2 Higher Homotopy Groups

Recall that the fundamental group  $\pi_1(X, x_0)$  was all about maps of the form  $\gamma : (S^1, 1) \rightarrow (X, x_0)$ , or equivalently maps of the form  $([0, 1], \{0, 1\}) \rightarrow (X, x_0)$ . Now similarly, we define the **higher homotopy groups** in the following manner:  $\pi_n(X, x_0) := [(S^n, 1); (x, x_0)]$  where the bracket denotes the homotopy classes of maps. Equivalently, we may define  $\pi_n(X, x_0)$  as  $[(I_n, \partial I_n), (X, x_0)]$  where  $I_n = [0, 1]^n$ .

The group operation is defined as follows: as an example, we will use  $\pi_2$ , and analogize.



Because we stipulate that the boundary gets mapped to  $x_0$ , the multiplication is well-defined by the pasting lemma. The identity element is the constant map mapping to  $x_0$ . Another way of writing the multiplication is as follows: if  $\gamma_1, \gamma_2 : (t_1, \dots, t_n) \rightarrow X$ , then we may write their product to be

$$(\gamma_1 * \gamma_2)(t_1, \dots, t_n) = \begin{cases} \gamma_1(2t_1, t_2, \dots, t_n), & t \in [0, 1/2] \\ \gamma_2(2t_1 - 1, t_2, \dots, t_n), & t \in [1/2, 1]. \end{cases}$$

Next, the following figure from Hatcher illustrates the following lemma:

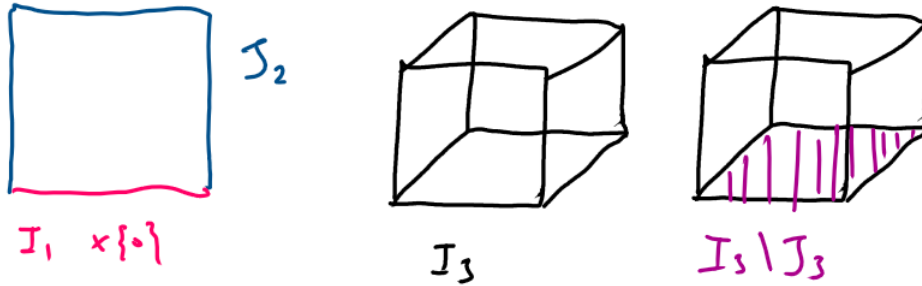
$$\begin{array}{|c|c|} \hline f & g \\ \hline \end{array} \simeq \begin{array}{|c|c|} \hline \begin{array}{|c|c|} \hline f & g \\ \hline \end{array} & \\ \hline \end{array} \simeq \begin{array}{|c|} \hline \begin{array}{|c|} \hline f \\ \hline g \\ \hline \end{array} \\ \hline \end{array} \simeq \begin{array}{|c|c|} \hline \begin{array}{|c|} \hline g \\ \hline f \\ \hline \end{array} & \\ \hline \end{array} \simeq \begin{array}{|c|c|} \hline g & f \\ \hline \end{array}$$

**Lemma 10.3.**  $\pi_n(X, x_0)$  is abelian for  $n \geq 2$ .

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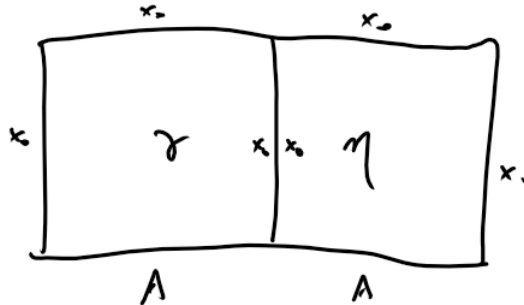
### 11.1 Relative Homotopy Groups

**Definition 11.1.** As we did last time, define  $I_n = [0, 1]^n$ ,  $\partial I_n$  the boundary of  $I_n$ , and let  $J_n := \partial I_n \setminus (I_{n-1} \times \{0\})$ , as in the following diagram:



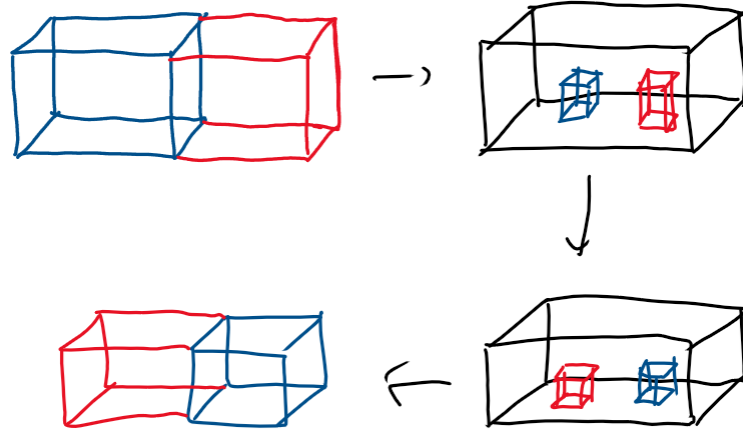
Then we define the **relative homotopy groups** as following:  $\pi_n(X, A, x_0) := [(I_n, \partial I_n, J_n); (X, A, x_0)]$ .

Because the elements of the homotopy groups are equivalence classes, we will write what it means for two elements to be equivalent. We say that for  $\gamma, \eta \in \pi_n(X, A, x_0)$ ,  $\gamma \sim \eta$  if and only if there exists  $F : (I_n, \partial I_n, J_n) \times [0, 1] = (I_n \times [0, 1], \partial I_n \times [0, 1], J_n \times [0, 1]) \rightarrow (X, A, x_0)$  such that  $f_0 = F(\cdot, 0) = \gamma$  and  $f_1 = F(\cdot, 1) = \eta$ . Similar as was done in homotopy groups, the product  $\gamma \cdot \eta$  is defined in the following way:

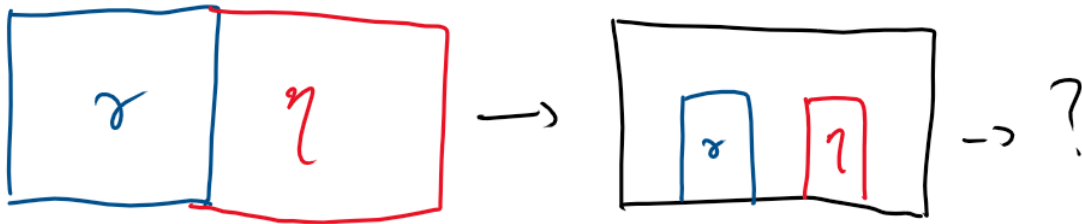


**Lemma 11.2.** This product makes  $\pi_n(X, A, x_0)$  into a group, and for  $n \geq 3$ , is abelian.

*Proof.* To see that  $\pi_n(X, A, x_0)$  is obvious. To see that it is obvious, consider the following figure:



It is important that the two cubes inside the big cube in the figure above do not have their bases taken off of  $\partial I_n \setminus J_n$ . The fact that this is not possible in dimension two illustrates why  $\pi_n(X, A, x_0)$  is not abelian:



■

## 11.2 Exact Sequences

**Definition 11.3.** Given groups  $G_1, G_2, \dots$  and homomorphisms  $L_n : G_n \rightarrow G_{n+1}$ , we say that the sequence

$$G_1 \xrightarrow{L_1} G_2 \xrightarrow{L_2} \cdots \rightarrow G_n \xrightarrow{L_n} \cdots \rightarrow 0$$

is **exact** if  $\ker L_{n+1} = \operatorname{im} L_n$  for each  $n$ .

Observe that if we have a sequence of groups,  $L_{n+1} \circ L_n \equiv 0$  if and only if  $\ker L_{n+1} \supseteq \operatorname{im} L_n$ .

**Example 11.4.** Consider the sequence  $0 \xrightarrow{L_1} G \xrightarrow{L_2} 0$ . Certainly  $L_2 \circ L_1 \equiv 0$ . But  $\operatorname{im} L_1 = \{0\}$  since  $L_1$  is a homomorphism. Thus this sequence must be exact if and only if  $G$  is trivial.

**Example 11.5.** Consider the sequence  $0 \xrightarrow{L_1} G \xrightarrow{L_2} H \xrightarrow{L_3} 0$ , and suppose that it is exact. Then  $\operatorname{im} L_1 = \{0\} = \ker L_2$ , which implies that  $L_2$  is injective. On the other hand,  $H = \ker L_3 = \operatorname{im} L_2$ , and so  $L_2$  is surjective. Thus  $L_2$  is a group isomorphism.

**Example 11.6.** Consider the exact sequence  $0 \xrightarrow{L_1} N \xrightarrow{\iota} G \xrightarrow{\pi} H \xrightarrow{L_3} 0$ . Since  $H = \ker L_3 = \text{im } \pi$ , we have that  $\pi$  is surjective. On the other hand,  $\text{im } L_1 = \ker \iota = \{0\}$  and so  $\iota$  is injective. Thus by the first isomorphism theorem,  $G/\ker \pi = G/\iota(N) \simeq H$ . Identifying  $N$  with its image under  $\iota$ , we conclude that  $G/N \simeq H$ .

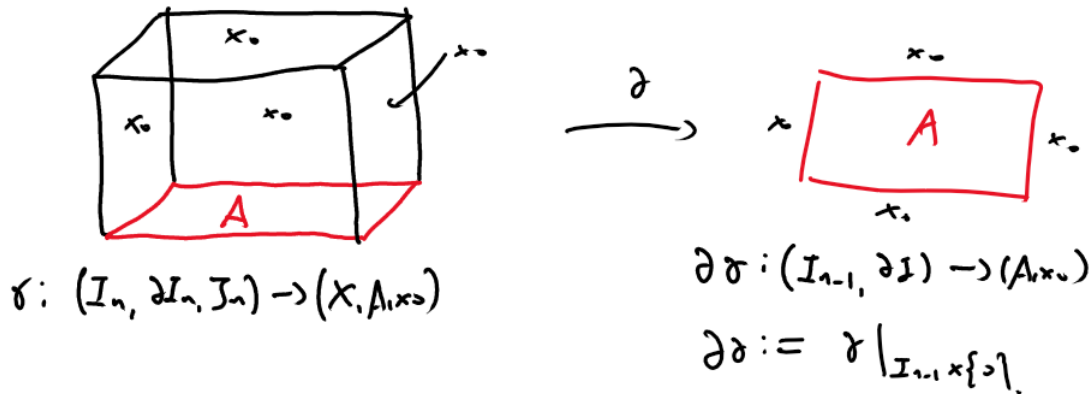
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### 12.1 Long Exact Sequences of Relative Homotopy Groups

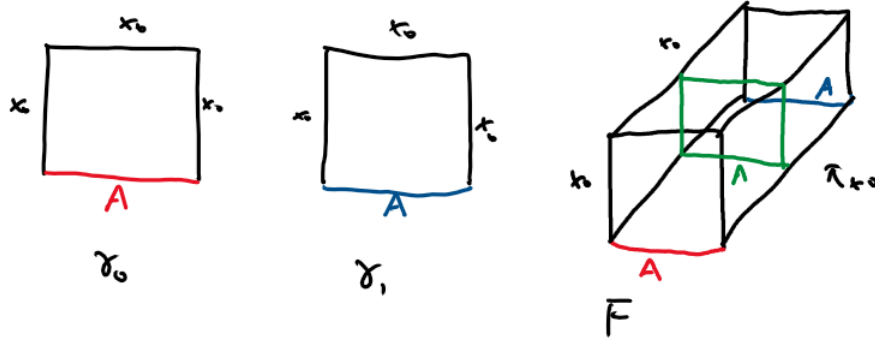
**Theorem 12.1** (Long Exact Sequences of Relative Homotopy Groups). *Let  $J : (X, x_0, x_0) \hookrightarrow (X, A, x_0)$  be the inclusion. Then there is a long exact sequence*

$$\rightarrow \pi_n(A, x_0) \xrightarrow{\iota_{\#}} \pi_n(X, x_0) \xrightarrow{J_{\#}} \pi_n(X, A, x_0) \xrightarrow{\partial} \pi_{n-1}(A, x_0) \rightarrow \cdots \rightarrow \pi_0(X, A, x_0) \rightarrow 0.$$

The boundary map  $\partial : \pi_n(X, A, x_0) \rightarrow \pi_{n-1}(A, x_0)$  is defined in the following manner: for  $\gamma \in \pi_n(X, A, x_0)$ , say  $\gamma : (I_n, \partial I_n, J_n) \rightarrow (X, A, x_0)$ , the restriction  $\gamma|_{I_{n-1} \times \{0\}}$  can be regarded as a map  $(I_{n-1}, \partial I) \rightarrow (A, x_0)$ . Then  $\partial\gamma$  is precisely this restriction.

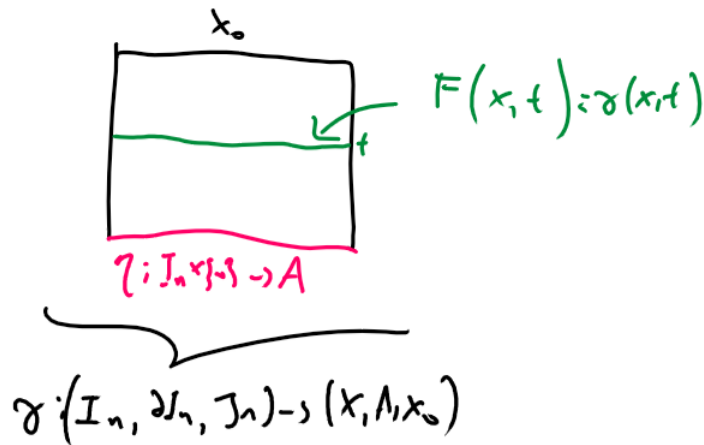


*Proof.* First we check that  $\partial$  is well-defined. To this end suppose  $\gamma_0 \sim \gamma_1$ . Then there exists some  $F : (I_n, \partial I_n, J_n) \times [0, 1] \rightarrow (X, A, x_0)$  such that  $F(\cdot, 0) = f_0 = \gamma_0$  and  $F(\cdot, 1) = f_1 = \gamma_1$  (a 2-dimensional schematic diagram is below).



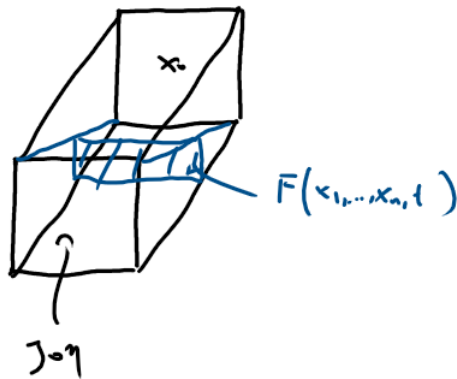
Then restricting  $F$  to  $I_{n-1} \times \{0\} \approx I_{n-1}$  gives a homotopy  $F|_{I_{n-1} \times \{0\} \times [0,1]} : I_{n-1} \times [0,1] \rightarrow A$ . Then this gives a homotopy  $\partial(\gamma_1) = \gamma_1|_{I_{n-1} \times \{0\}}$  to  $\partial(\gamma_2) = \gamma_2|_{I_{n-1} \times \{0\}}$ . Thus  $\partial(\gamma_1) \sim \partial(\gamma_2)$  in homotopy classes.

Now we check exactness. First, we will show that  $\ker \iota_{\#} = \text{im } \partial$ . To prove one direction, suppose that  $\eta \in \text{im } \partial$ . Then there exists some  $\gamma \in \pi_{n+1}(X, A, x_0)$ ,  $\gamma : (I_{n+1}, \partial I_{n+1}, J_{n+1}) \rightarrow (X, A, x_0)$  such that  $\partial\gamma = \gamma|_{I_n \times \{0\}} = \eta$ . In order to prove that  $\eta \in \ker \iota_{\#}$ , we must show that  $\iota_{\#}(\eta) \sim 0$  in  $\pi_n(X, x_0)$ , that is, there exists some homotopy  $F : I_n \times [0,1] \rightarrow X$  such that  $f_0 = \eta$  and  $f_1 = x_0$ . Define  $F(x, t) = \gamma(x, t)$ , regarded as a map from  $I_{n+1} = I_n \times [0,1]$  to  $X$ . Indeed,  $F(x, 0) = \gamma(x, 0) = \gamma|_{I_n \times \{0\}}(x) = \eta(x)$ . On the other hand,  $F(x, 1) = \gamma(x, 1) = x_0$  since  $\gamma \in \pi_{n+1}(X, A, x_0)$  (see the figure below). Hence  $\iota_{\#}(\eta)$  is homotopic to the constant map in  $\pi_n(X, x_0)$ , so  $\eta \in \ker \iota_{\#}$ .



Conversely, suppose that  $\eta \in \ker \iota_{\#}$ . Then  $\iota_{\#}(\eta)$  is homotopic to the constant map in  $\pi_n(X, x_0)$ , that is, there exists some  $F : I_n \times [0,1] \rightarrow X$  such that  $f_0(x) = \iota_{\#}(\eta)$  and  $f_1(x) \equiv x_0$ . Then proceeding as in the other direction, defining  $\gamma : (I_{n+1}, \partial I_{n+1}, J_{n+1}) \rightarrow (X, A, x_0)$ , with the identification  $I_{n+1} = I_n \times [0,1] \rightarrow X$  via  $\gamma(x, t) = F(x, t)$  we have that clearly  $\partial\gamma = \eta$ .

For the next part, we will show that  $\text{im } \iota_{\#} = \ker J_{\#}$ . Let  $\eta \in \text{im } \iota_{\#}$ . Then there exists some  $\tilde{\eta} \in \pi_n(A, x_0)$ ,  $\tilde{\eta} : (I_n, \partial I_n) \rightarrow (A, x_0)$  such that  $\eta = \iota \circ \tilde{\eta}$ . Now consider the map  $J \circ \eta : (I_n, \partial I_n, J_n) \rightarrow (X, A, x_0)$ , which obtained by changing the domain and codomain: note that  $\eta = \iota \circ \tilde{\eta}$ , and so the image of  $\eta$  is completely contained in  $A$ , and moreover for any  $x \in \partial I_n$ , we have  $\eta(x) = \tilde{\eta}(x) = x_0$  since  $\tilde{\eta} \in \pi_n(A, x_0)$ . Thus  $J \circ \eta$  as a map from  $(I_n, \partial I_n, J_n) \rightarrow (X, A, x_0)$  makes sense. To show that  $\eta \in \ker J_{\#}$ , we will show that there is a homotopy  $F : (I_n, \partial I_n, J_n) \times [0, 1] \rightarrow (X, A, x_0)$  such that  $f_0 = J \circ \eta$  and  $f_1 \equiv x_0$ . Next, consider  $F(x_1, \dots, x_{n-1}, x_n, t) := (J \circ \eta)(x_1, \dots, x_{n-1}, (1-t)x_n)$ .



Indeed,  $f_0(x_1, \dots, x_n) = F(x_1, \dots, x_n, 0) = J \circ \eta$ , and  $f_1(x_1, \dots, x_n) = F(x_1, \dots, x_n, 1) = (J \circ \eta)(x_1, \dots, x_{n-1}, 0) = \tilde{\eta}(x_1, \dots, x_{n-1}, 0) = x_0$ . The schematic figure above shows that  $F$  indeed is the map of the desired form. Hence  $F$  is the desired homotopy.

Conversely, suppose  $\eta \in \ker J_{\#}$ . Then  $\eta : (I_n, \partial I_n) \rightarrow (X, x_0)$  and  $J \circ \eta : (I_n, \partial I_n, J_n) \rightarrow (X, A, x_0)$  is homotopically trivial. Thus there exists some  $F : (I_n, \partial I_n, J_n) \times [0, 1] \rightarrow (X, A, x_0)$  such that  $f_0 = J \circ \eta$  and  $f_1 \equiv x_0$ . Define  $\tilde{\eta} : (I_n, \partial I_n) \rightarrow (A, x_0)$  by  $\tilde{\eta}(x_1, \dots, x_n) = F(x_1, \dots, x_{n-1}, 0, x_n)$ . Clearly by definition of  $F$ ,  $\tilde{\eta}$  takes image in  $A$ , and its boundary takes value in  $x_0$  (see figure below for an illustration). We claim that  $\iota \circ \tilde{\eta} \sim \eta$  in  $\pi_n(X, x_0)$ . Indeed, let  $G : (I_n, \partial I_n) \times [0, 1] \rightarrow (X, x_0)$  via  $G(x_1, \dots, x_n, t) = F(x_1, \dots, x_{n-1}, (1-t)x_n, tx_n)$ . Then  $g_0(x_1, \dots, x_n) = F(x_1, \dots, x_n, 0) = (J \circ \eta)(x_1, \dots, x_n) = \eta(x_1, \dots, x_n)$  and  $g_1(x_1, \dots, x_n) = F(x_1, \dots, 0, x_n) = \tilde{\eta}(x_1, \dots, x_n)$ . Pictorially, the green slanted rectangle depicts  $g_t$  during a time between 0 and 1, in the middle of the homotopy. Thus  $\iota \circ \tilde{\eta} \sim \eta$  in  $\pi_n(X, x_0)$ , as desired.





### 13.1 Serre Fibrations and Hurewicz Fibrations

$$\begin{array}{ccc} X & \xrightarrow{h} & E \\ \downarrow \iota & \nearrow \tilde{F} & \downarrow p \\ X \times I & \xrightarrow{F} & B \end{array}$$

**Lemma 13.2.** *Let  $p : E \rightarrow B$  be a continuous map,  $\mathcal{U} = \{U_i\}_{i \in I}$  be an open cover of  $B$ , and let  $p_i := p|_{U_i}$ , that is,  $p_i : p^{-1}(U_i) \rightarrow U_i$ . If  $p_i$  has the homotopy lifting property for each  $i$ , then  $p$  has the homotopy lifting property for  $E$ .*

$$\begin{array}{ccc} F \times U_i & \xrightarrow{H_i} & p^{-1}(U_i) \\ & \searrow p_2 & \downarrow p \\ & & U_i \end{array}$$

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**Theorem 13.5** (Long Exact Sequences of Fibrations). *Given a Serre fibration  $p : E \rightarrow B$ ,  $p(e_0) = x_0$ ,  $p^{-1}(x_0) = F$  and  $e_0 \in F$ , there is a long exact sequence*

$$\cdots \rightarrow \pi_n(F, e_0) \xrightarrow{\iota_{\#}} \pi_n(E, e_0) \xrightarrow{p_{\#}} \pi_n(B, b_0) \xrightarrow{\partial} \pi_{n-1}(F, e_0) \rightarrow \cdots,$$

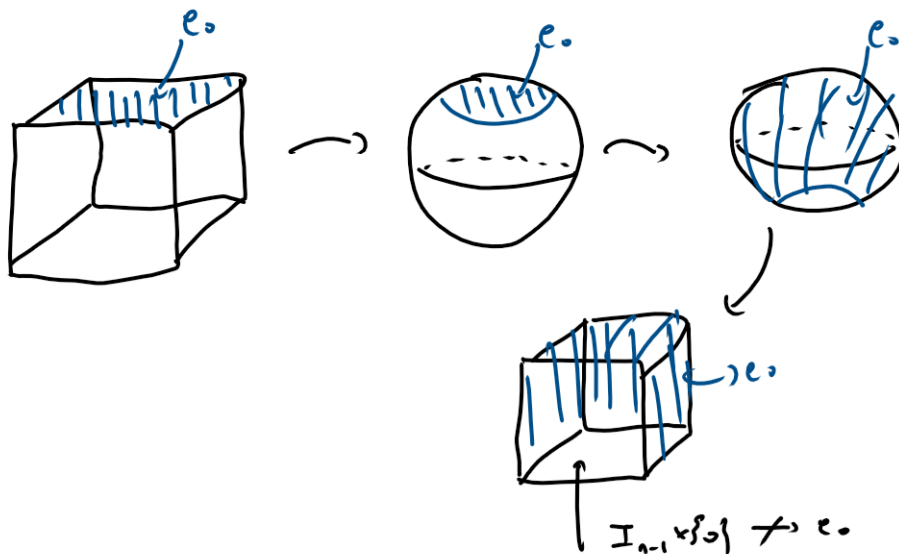
where the boundary map is defined in the following manner: if  $\gamma : (I_n, \partial I_n) \rightarrow (B, b_0) \in \pi_n(B, b_0)$ , then define  $H : I_{n-1} \times I \rightarrow B$  by viewing  $\gamma$  as a homotopy. That is,  $H(x_1, \dots, x_{n-1}, t) = \gamma(x_1, \dots, x_{n-1}, t)$ . Then notice that  $h_1 \equiv b_0$ . Then since  $p$  is a Serre fibration,  $p$  satisfies the homotopy lifting property, so  $H$  lifts to a unique homotopy  $\tilde{H} : X \times I \rightarrow E$  such that  $p \circ \tilde{H} = H$  and  $\tilde{h}_1 \equiv e_0$ . Then we define  $\partial\gamma = \tilde{H}|_{I_{n-1} \times \{0\}} : I_{n-1} \rightarrow E$ . Then  $p(\partial\gamma(x)) = p(\tilde{H}(x, 0)) = H(x_1, \dots, x_{n-1}, 0) = b_0$ . Hence  $\partial\gamma(x) \in p^{-1}(b_0) = F$ , so  $\partial\gamma \in \pi_{n-1}(F, e_0)$ .

**Proof Idea.** We will show that there is a natural isomorphism between  $\pi_n(B, x_0)$  and  $\pi_n(X, A, x_0)$ , and then appeal to Theorem 12.1, so that we can fit this long exact sequence into the previous one.

*Proof.* Consider  $\pi_n(E, F, e_0) \xrightarrow{p_{\#}} \pi_n(B, b_0, b_0) \simeq \pi_n(B, b_0)$ . We claim that  $p_{\#}$  is an isomorphism, which will allow  $\pi_n(B, b_0)$  to naturally fit into the long exact sequence for relative homotopy groups. To see this we will construct an inverse for  $p_{\#}$ . Take  $\gamma : (I_n, \partial I_n, J_n) \rightarrow (E, F, e_0) \in \pi_n(E, F, e_0)$ . Then this fits into the diagram

$$\begin{array}{ccc} (I_n, \partial I_n, J_n) & \xrightarrow{\gamma} & (E, F, e_0) \\ & \searrow p \circ \gamma & \downarrow p \\ & & (B, b_0, b_0) \end{array}$$

Take  $H$  and  $\tilde{H}$  as in the statement of Theorem 13.5. Then  $\tilde{H}$  is a homotopy  $I_{n-1} \times [0, 1] \rightarrow E$  such that  $\tilde{h}_1 = \tilde{H}(\cdot, 1) \equiv e_0$ , and moreover  $\tilde{H}(\partial I_n) \subseteq F = p^{-1}(b_0)$ . Now consider the following deformation:



This gives us a new map  $\hat{H} : I_{n-1} \times [0, 1] \rightarrow E$  homotopic to  $\tilde{H}$ . Then this induces a map  $\text{hat} : \pi_n(B, b_0, b_0) \rightarrow \pi_n(E, F, e_0)$ ,  $\gamma \mapsto \hat{H}$ . It is (presumably) straightforward to check that  $\text{hat}$  is the inverse of  $p_\#$ , which concludes the proof. ■

**Example 13.6.** Let  $p : E \rightarrow B$  be a covering projection. Then  $F = p^{-1}(b_0)$  is discrete; hence  $\pi_n(F, e_0) = \{0\}$  for all  $n \neq 0$ . Then we have the long exact sequence

$$\cdots \rightarrow \underbrace{\pi_n(F, e_0)}_{=0} \rightarrow \pi_n(E, e_0) \rightarrow \pi_n(B, b_0) \rightarrow \underbrace{\pi_{n-1}(F, e_0)}_{=0}.$$

for  $n - 1 \geq 1$ , that is,  $n \geq 2$ . Thus we have proven the following:

**Corollary 13.7.** For  $n \geq 2$ , and  $p : E \rightarrow B$  a covering projection, then  $p_\# : \pi_n(E, e_0) \rightarrow \pi_n(B, b_0)$  is an isomorphism.

**Corollary 13.8.**  $\pi_n(S^1, 1) = 0$  for all  $n \geq 2$ .

*Proof.* Contractible spaces have trivial homotopy groups. ■

**Definition 13.9.** Let  $G$  be a given group and  $n \in \mathbb{Z}$  an integer. A space  $(X, x_0)$  is called **Eilenberg-MacLane Space**, and we write  $K(G, n)$ , if  $\pi_n(X, x_0) = G$  and  $\pi_\ell(X, x_0) = 0$  for all  $\ell \neq n$ .

One observation to make is that for  $\ell \geq 2$ , we need the group  $G$  to be abelian, for  $\pi_n(X, x_0)$  is abelian for  $n \geq 2$ . Moreover, Corollary 13.8 shows that  $K(\mathbb{Z}, 1) = S^1$ .

Now one might be wondering what  $K(\mathbb{Z}, 2)$  might be. Continuing the above discussion, with the facts that  $\pi_1(S^2) = 0$  and  $\pi_2(S^2) \simeq \mathbb{Z}$ , one might wonder if  $S^2 = K(\mathbb{Z}, 2)$ , but this is not the case, for  $\pi_3(S^2) = \mathbb{Z}$ ; one way to see this is to use something called the Hopf fibration.

## 14 2/13/23

### 14.1 Homotopy Groups of $S^n$

One of our goals today will be to give a partial answer about computing a subset of all homotopy groups of  $S^n$ . Computing *all* of the homotopy groups of  $S^n$ , however, is still an open question!

**Theorem 14.1.** For all  $n \geq 1$  and  $0 \leq k \leq n - 1$ ,  $\pi_k(S^n) = 0$ .

But before we move to the proof of this theorem, let's begin with a warm-up.

**Proposition 14.2.** For all  $n \geq 1$ ,  $S^n$  is path-connected.

*Proof.* Fix  $x, y \in S^n$ , and let  $\bar{\gamma}_{x,y} : [0, 1] \rightarrow D^{n+1}$  via  $\bar{\gamma}_{x,y}(t) = tx + (1 - t)y$ , which is the straight line through the  $n + 1$ -dimensional ball connecting  $x$  and  $y$ . Now consider  $\gamma_{x,y} : [0, 1] \rightarrow S^n$  via  $\gamma_{x,y}(t) := \bar{\gamma}_{x,y}(t)/|\bar{\gamma}_{x,y}(t)|$ . Now this path  $\gamma_{x,y}$  is well-defined and connects  $x$  and  $y$ , as long as  $x$  and  $y$  are not antipodal: that is,  $x \neq -y$ . In the case that  $x$  and  $y$  are antipodal, choose  $z \in S^n$  such that  $z$  is not antipodal to  $x$  and  $y$ . Then  $\gamma_{x,z}$  and  $\gamma_{z,y}$  is well-defined, and their concatenation is a path connecting  $x$  and  $y$ . ■

**Lemma 14.3.** If  $f : M \rightarrow S^n$  is a continuous map that is not surjective, then it is homotopically trivial.

*Proof.* Suppose  $f$  is not onto, say  $p \notin f(M)$ . Now by stereographic projection  $h : S^n \setminus \{p\} \rightarrow \mathbb{R}^n$ , we have the homeomorphism  $S^n \setminus \{p\} \approx \mathbb{R}^n$ . Note that  $\mathbb{R}^n$  is contractible: the map  $c : \mathbb{R}^n \times I \rightarrow \mathbb{R}^n$  defined by  $c(x, t) = (1 - t)x$  is the homotopy that contracts the identity map to a constant map. Now consider  $F : M \rightarrow S^n$  defined by  $F(u, t) = h^{-1}(c(t, h(f(u))))$ : clearly this is a composition of continuous functions and is continuous. Moreover,  $f_0(u) = F(u, 0) = h^{-1}(c(0, h(f(u)))) = h^{-1}(h(f(u))) = f(u)$ , and  $f_1(u) = h^{-1}(0)$  which is constant. Hence  $F$  is the desired homotopy. ■

**Lemma 14.4.** *If  $M$  is a manifold and  $f : M \rightarrow S^n$  and  $g : M \rightarrow S^n$  satisfy  $|f(x) - g(x)| < 2$  for all  $x \in M$ , then  $f$  is homotopic to  $g$ .*

*Proof.* Consider  $F : M \times [0, 1] \rightarrow S^n$  given by

$$F(x, t) := \frac{tf(x) + (1 - t)g(x)}{|tf(x) + (1 - t)g(x)|}.$$

Since  $|f(x) - g(x)| < 2$  for all  $x$ , it follows that  $f(x)$  can never be antipodal to  $g(x)$ . Hence  $F$  is well defined for all  $x, t$ , and so this is a homotopy. ■

Next, we will need two results (actually, corollaries) from analysis and smooth manifolds, which we will take as given.

**Lemma 14.5** (Stone-Weierstrass). *Given  $f : S^k \rightarrow S^n$  and an  $\epsilon > 0$ , there exists a polynomial  $p : \mathbb{R}^k \rightarrow \mathbb{R}^n$  such that  $|p(x) - f(x)| < \epsilon$  for all  $x \in S^k$ .*

**Lemma 14.6** (Sard's Theorem). *If  $f : M \rightarrow N$  and  $f \in C^\infty$ , and  $\dim M < \dim N$ , then  $f$  is not onto.*

**Exercise 14.7.** Construct a continuous function  $\gamma : [0, 1] \rightarrow [0, 1]^m$  that is surjective.

**Lemma 14.8.** *Suppose  $f : S^k \rightarrow S^n$  is continuous and  $k < n$ . Then  $f$  is homotopic to a map that is not surjective.*

*Proof.* Suppose  $f$  is as prescribed. Then by Lemma 14.5, there exists a polynomial  $p : \mathbb{R}^k \rightarrow \mathbb{R}^n$  such that  $|p(x) - f(x)| < 2$ . But applying Lemma 14.4,  $f$  is homotopic to  $p$ . But  $p$  is a polynomial and hence smooth; appealing to Lemma 14.6 yields the result. ■

Now we have enough machinery to accomplish what we set out to do in the beginning of the section.

*Proof of Theorem 14.1.* Suppose  $\gamma : (I_k, \partial I_k) \rightarrow (S^n, x_0)$ . Note that  $\gamma(\partial I_k) = x_0$ , and  $I_k/\partial I_k \approx S^k$ . Now define  $\pi : I_k \rightarrow S^k$  via the natural projection from the homeomorphism,  $\pi(\partial I_k) = b_0 \in S^k$ . Now define  $f : S^k \rightarrow S^n$  by  $f(x) = \gamma(\pi^{-1}(x))$ . We need to check that  $f$  is well-defined: the only problematic point is when  $x \neq b_0$ , which has  $\partial I_k$  as the preimage. But indeed,  $\gamma$  maps  $\partial I_k$  to one point, so  $f$  is well-defined. Further, we have that  $f$  is continuous. Now Lemma 14.8 implies that there exists a homotopy  $F : S^k \times [0, 1] \rightarrow S^n$  such that  $f_0 = f$  and  $f_1 \equiv \text{const}$ .

Consider the path  $\eta(t) := F(b_0, t)$ . Then  $\eta(0) = f_0(b_0) = f(b_0) = \gamma(\pi^{-1}(b_0)) = x_0$ . We want to modify  $F$  to another homotopy  $\tilde{F}$  such that  $\tilde{F}(b_0, t) = x_0$  for all  $t$ . Since  $\text{SO}(n, \mathbb{R})$  acts transitively on the sphere, i.e., for all  $P, Q \in S^n$ , there exists some  $O \in \text{SO}(n, \mathbb{R})$  such that  $OP = Q$ , and  $O(S^n) = S^n$ . Moreover, the choice of  $O$  varies continuously with respect to  $P$  and  $Q$ . Therefore we can choose  $O_t$  such that  $O_t(\eta(t)) = x_0$  for all  $t$  continuously and  $O_0 = \text{id}$ , and define  $\tilde{F}(x, t) = O_t(F(x, t))$ . Then it is easy to verify that  $\tilde{F}(\cdot, 0) = F(\cdot, 0) = f_0 = f$ ,  $\tilde{F}(\cdot, 1) = O_1(F(\cdot, 1)) =$

$O_1(x_0) = x_0$ , and  $\tilde{F}(b_0, t) = x_0$  for all  $t$  by construction. Now set  $H : (I_k, \partial I_k) \times I \rightarrow (S, x_0)$  by  $H(x, t) := \tilde{F}(\pi(x), t)$ ; then with the identification made earlier, this is the desired homotopy, and so  $\gamma$  is trivial in  $\pi_k(S^n, x_0)$ . ■

*Remark 14.9.* A similar argument will work for any connected manifold.

## 14.2 Fiber Bundles and Lie Groups

Note that  $\text{SO}(n)$  acts transitively on  $S^n$ , and consider  $\mathbf{1} = (1, 0, \dots, 0)^t$ . Then we consider  $\text{stab}(\mathbf{1}) = \{O \in \text{SO}(n) \mid O\mathbf{1} = \mathbf{1}\}$ . Note that if  $O \in \text{stab}(\mathbf{1})$ , then  $O$  takes  $\mathbf{1}$  to  $\mathbf{1}$ , so must have the first

column be  $\begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$ : that is,

$$O = \left( \begin{array}{c|ccc} 1 & 0 & \cdots & 0 \\ \hline 0 & & & \\ \vdots & & \text{SO}(n-1) & \\ 0 & & & \end{array} \right)$$

We get a map  $p : \text{SO}(n) \rightarrow S^n$ ,  $O \mapsto O \cdot \mathbf{1}$ . Then  $p^{-1}(\mathbf{1}) \simeq \text{SO}(n-1)$ .

**Lemma 14.10.** *This map  $p$  defines a fiber bundle.*

This would imply that  $p$  satisfies the path and homotopy lifting property. To see the lemma we will appeal to the following theorem:

**Theorem 14.11.** *Let  $G$  be a Lie group. If  $H$  is a closed subgroup, then  $p : G \rightarrow G/H$  defines a fiber bundle with fiber is  $H$ .*

We will prove this theorem in the next two lectures.

We will now consider an application of the above theorem. To do this we define a new type of space. Take  $\lambda \in \mathbb{C} \setminus \{0\}$ . Then the map  $M_\lambda : z \mapsto \lambda z$  acts on  $\mathbb{C}^n \setminus \{0\}$ . Note that  $M_\lambda(S^{2n+1}) = S^{2n+1}$  if  $|\lambda| = 1$ . Next, consider the relation  $z \sim w$  if and only if there exists some  $\lambda \in \mathbb{C} \setminus \{0\}$  such that  $\lambda z = w$  on  $\mathbb{C}^n \setminus \{0\}$ . Then the **complex projective space** is defined as  $\mathbb{C}P^n := (\mathbb{C}^n \setminus \{0\}) / \sim = S^{2n+1} / \sim$ . Then this gets us a projection map  $p : S^{2n+1} \rightarrow \mathbb{C}P^n$ , where each fiber is  $S^1$ , for the same reason as in the theorem.

**Corollary 14.12.** *We have a long exact sequence*

$$\cdots \rightarrow \pi_k(S^1) \rightarrow \pi_k(S^{2n+1}) \rightarrow \pi_k(\mathbb{C}P^n) \rightarrow \pi_{k-1}(S^1) \rightarrow \cdots$$

Thus if  $k-1 \geq 2$ , that is,  $k \geq 3$ ,  $\pi_k(S^{2n+1}) \simeq \pi_k(\mathbb{C}P^n)$ . For  $k=2$ , we have  $\pi_2(\mathbb{C}P^n) \simeq \mathbb{Z}$ .

In general,  $\pi_{2n+1}(\mathbb{C}P^n) \simeq \pi_{2n+1}(S^{2n+1}) \simeq \mathbb{Z}$ . To answer the question posed in the previous lecture, we note that  $\mathbb{C}P^\infty$  is the Eilenberg-MacLane space needed: it is the sought for  $K(\mathbb{Z}, 2)$ .

## 15 2/15/23

Today we will start the proof of a theorem that was stated and unproved in the last lecture. We will state it again for a reminder.

**Theorem 15.1.** *Let  $G$  be a Lie group and  $H$  a closed subgroup of  $G$ . Then the projection  $p : G \rightarrow G/H$  defines a fiber bundle with fiber  $H$ . That is, for every  $x \in G/H$ , there exists an open neighborhood  $U$  of  $x$  and a homeomorphism  $Q : U \times H \rightarrow p^{-1}(U)$  such that  $p(Q(u, h)) = u$  for all  $u \in U$ .*

## 15.1 A Short Course in Lie Groups

Let  $G$  be a Lie group. Then we will denote  $\text{Lie}(G) := \mathfrak{g}$  to be the tangent space at the identity. Then the **exponential map**  $\exp : \text{Lie}(G) \rightarrow G$  is defined in the following manner: if  $v \in T_e G$  and  $X$  is a left-invariant vector field of  $G$  such that  $X(0) = v$ ,  $\alpha : \mathbb{R} \rightarrow G$  the associated one-parameter subgroup with  $\alpha'(0) = X(0)$ , then  $v \mapsto \alpha(1)$ .

**Example 15.2.** As an example  $\text{GL}(n, \mathbb{R})$  is a Lie group, and  $\text{Lie}(\text{GL}(n, \mathbb{R})) = \text{Mat}_{n \times n}(\mathbb{R})$ . Then the exponential map coincides with the usual matrix exponential:  $\exp(A) = \sum_{n=0}^{\infty} \frac{A^n}{n!}$ . Then  $\exp(0) = \text{id}$  and  $d(\exp)_0 : \text{Mat}_{n \times n}(\mathbb{R}) \rightarrow T_e \text{GL}(n, \mathbb{R}) = \text{Mat}_{n \times n}(\mathbb{R})$ , where  $d(\exp)_0 \equiv \text{id}_{\text{Lie}(G)}$ .

Now by the inverse function theorem,  $\exp$  is a local diffeomorphism; hence we can use the exponential map for charts for  $G$ . Moreover, since  $G$  is a Lie group, left multiplication  $L_g : G \rightarrow G$  and  $x \mapsto gx$  is a diffeomorphism, so once we know open sets near the identity, we know it everywhere. Thus studying the Lie algebra lets us study Lie groups.

**Exercise 15.3.** Let  $G$  be a path-connected topological group. Then  $\pi_1(G, e_0)$  is abelian.

Suppose  $\text{Lie}(G) = E \oplus F$  a direct sum of vector spaces. Then one can use  $\exp|_E$  and  $\exp|_F$  to define a new map  $r_{E,F} : \text{Lie}(G) \rightarrow G$ , given by

$$v = v_E + v_F \mapsto \exp(v_E) \exp(v_F).$$

**Warning.**  $\exp(x + y) \neq \exp(x) \exp(y)$  in general, unless  $G$  is abelian.

However,  $d(r_{E,F})_0 = \text{id}$ , so the inverse function theorem tells us that  $r_{E,F}$  is a local diffeomorphism.

**Theorem 15.4** (Closed Subgroup Theorem). *Every closed subgroup of a Lie group is an embedded submanifold, and hence a Lie group.*

Note that if  $H$  is a closed subgroup of  $G$ , then  $\text{Lie}(H)$  is a vector subspace of  $\text{Lie}(G)$ , and we say that  $\text{Lie}(H)$  is a Lie subalgebra. Moreover,  $\exp(\text{Lie}(H))$  is a subgroup of  $H$ , and is the connected component of  $H$  at the identity, sometimes also called the **analytic subgroup** associated to  $\text{Lie}(H)$ .

Now let  $T \subseteq \text{Lie}(G)$  be a vector subspace such that  $T \oplus \text{Lie}(H) = \text{Lie}(G)$ . Then from linear algebra, naturally we have that  $\text{Lie}(G)/\text{Lie}(H) \simeq T$ . Hence  $(\text{Lie}(G)/\text{Lie}(H)) \times \text{Lie}(H) \simeq \text{Lie}(G)$ . This sketches out the proof of a special case of Theorem 15.1, when  $G$  is a vector space. In particular, it proves the theorem for  $\text{Lie}(G)$  the tangent space at the identity.

Next we will state a theorem from Lie theory, which we take as given:

**Theorem 15.5.** *There exists a neighborhood  $W$  of the identity  $e \in G$  and a neighborhood  $V \subseteq \text{Lie}(G)$  of 0 such that the following hold:*

- (i)  $\exp : V \rightarrow W$  is a diffeomorphism.
- (ii) if  $h \in H \cap W$ , then  $(\exp)^{-1}(h) \in \text{Lie}(H)$ ; equivalently,  $H \cap W \subseteq H^0$ , where  $H^0$  denotes the connected component of  $H$  with the identity.

**Example 15.6.** Let  $G = \mathbb{R}^2$ , and  $H = \{(t, n_1 + n_2\sqrt{2}) \mid t \in \mathbb{R}, n_1, n_2 \in \mathbb{Z}\}$ . Then  $H^0 = \{(t, 0) \mid t \in \mathbb{R}\}$ . But for any open  $W \subseteq G$  with  $0 \in W$ ,  $H \cap W$  is not a subset of  $H^0$ , since  $H$  is not a closed subgroup of  $G$ . In fact,  $H$  is dense in  $G$ !

**Exercise 15.7.** Show that, in the above example,  $H^0$  is a normal subgroup of  $H$ .

## 16 2/17/23

### 16.1 The Proof of the Main Theorem

Today we will prove Theorem 15.1. Our goal is to show that the projection  $p : G \rightarrow G/H$  defines a fiber bundle. That is, we want to show that for all  $x \in G/H$ , there exists some  $U$  an open neighborhood of  $x$  and a homeomorphism  $Q : U \times H \rightarrow p^{-1}(U)$  such that  $p(Q(u, h)) = u$ . That is, the following diagram commutes:

$$\begin{array}{ccccc} U \times H & \xrightarrow{Q} & p^{-1}(U) & \xhookrightarrow{\iota} & G \\ \downarrow p_1 & & & \swarrow p & \\ U & \xhookrightarrow{\iota} & G/H & & \end{array}$$

**Exercise 16.1.** Let  $S$  be the genus 2 surface. Then there is no nontrivial  $p : S \rightarrow B$ , where  $B$  is a topological space, such that  $S$  together with  $p$  defines a fiber bundle. That is,  $B$  cannot be the singleton set and  $B$  cannot be equal to  $S$ .

Let  $G$  be a Lie group and  $H$  a closed subgroup of  $G$ ; consider  $\text{Lie}(H)$ , and  $T \subseteq \text{Lie}(G)$  such that  $\text{Lie}(G) = T \oplus \text{Lie}(H)$ . Take  $\Sigma \subseteq T$  and  $B_H \subseteq \text{Lie}(H)$  both open sets such that  $\Sigma \oplus B_H \sim \Sigma \times B_H$  are open sets. After shrinking if necessary, WLOG assume that  $\Sigma = -\Sigma$  and the map

$$\begin{aligned} r : \Sigma \times B_H &\rightarrow G \\ (s, v) &\mapsto \exp(s) \exp(v) \end{aligned}$$

is a local diffeomorphism (this is a consequence of the fact that  $\exp$  is a local diffeomorphism); let  $W := r(\Sigma \times B_H) \subseteq G$ , and note that this is a neighborhood of the origin.

Now define

$$\begin{aligned} \widehat{Q} : \exp(\Sigma) \times H &\rightarrow G \\ (\sigma, h) &\mapsto \sigma h. \end{aligned}$$

**Lemma 16.2.** Consider  $\widehat{Q}$  as above. Then:

- (a)  $\widehat{Q}$  is one-to-one.
- (b)  $\widehat{Q}$  is continuous.
- (c)  $\widehat{Q}$  is an open map.

*Proof.* To prove (a), suppose  $\sigma_1 h_1 = \sigma_2 h_2$ . Then rearranging,  $\sigma_2^{-1} \sigma_1 = h_2 h_1^{-1}$ . Clearly  $h_2 h_1^{-1} \in H$ . On the other hand,  $\exp(0) = e$ , so  $e \in \exp(\Sigma)$ ; hence  $h_2 h_1^{-1} = e \cdot (h_2 h_1^{-1}) \in W$ . Thus applying Theorem 15.5, there exists some  $v \in \text{Lie}(H)$  such that  $h_2 h_1^{-1} = \exp(v)$ . Let  $s_1, s_2 \in \Sigma$  be such

that  $\exp(s_1) = \sigma_1$  and  $\exp(s_2) = \sigma_2$ . Then we have  $\sigma_1 = \sigma_2 \exp(v_2)$ , and looking the definition of  $r$  again,  $r(s_1, e) = r(s_2, v)$ . Now using the fact that  $r$  and  $\exp$  are local diffeomorphisms, we conclude that  $0 = v$  and  $s_1 = s_2$ , and hence  $\sigma_1 = \sigma_2$  and  $h_1 = h_2$ . This shows that  $\widehat{Q}$  is one-to-one.

Now to prove (b), obviously  $\widehat{Q}$  is continuous since multiplication is a continuous operator, coming from the fact that  $G$  is a Lie group.

Finally, to prove (c), let  $U_1 \subseteq \exp(\Sigma)$  be open and  $U_2 \subseteq H$  be open, and further assume that  $e \in U_2$ . Then defining  $\widehat{U}_1 := (\exp|_{\Sigma})^{-1}(U_1)$  and  $\widehat{U}_2 := (\exp|_{B_H})^{-1}(U_2)$ , we see that  $Q(U_1, U_2) = r(\widehat{U}_1, \widehat{U}_2)$ . Since  $\exp$  is a local diffeomorphism, we have  $\widehat{U}_1$  and  $\widehat{U}_2$  are both open; similarly, since  $r$  is a local diffeomorphism,  $r(\widehat{U}_1, \widehat{U}_2)$  must also be open. This proves the claim for the single case where  $U_2$  is a neighborhood around the identity.

Now to handle the general case, take  $U_1$  open in  $\exp(\Sigma)$  and  $U_2$  open in  $H$  without the assumption that  $e \in U_2$ . Let  $h$  and  $\widetilde{U}_2$  be defined as  $U_2 = \widetilde{U}_2 \cdot h$ , and  $\widetilde{U}_2$  is a neighborhood of  $e$ ; that is,  $\widetilde{U}_2 = R_{h^{-1}}(U_2)$ . Then

$$\widehat{Q}(U_1, U_2) = U_1 \cdot U_2 = U_1 \cdot (\widetilde{U}_2 \cdot h) = \widehat{Q}(U_1, \widetilde{U}_2) \cdot h,$$

and since by the special case  $\widehat{Q}(U_1, \widetilde{U}_2)$  is open and right multiplication is a diffeomorphism, we have that  $\widehat{Q}(U_1, U_2)$  is open. ■

**Corollary 16.3.** *The image  $\widehat{Q}(\exp(\Sigma) \times H)$  is open in  $G$  and  $\widehat{Q}$  is a homeomorphism onto its image.*

Now we return to studying  $p$  and the quotient.

**Lemma 16.4.** *Consider the restriction of the map  $p$  to  $\exp(\Sigma)$ , that is,  $p : \exp(\Sigma) \rightarrow G/H$ . Then:*

- (a)  $p$  is one-to-one,
- (b)  $p$  is continuous,
- (c)  $p$  is an open map.

*Proof.* To prove (a), suppose that  $p(\sigma_1) = p(\sigma_2)$  for some  $\sigma_1, \sigma_2 \in \exp(\Sigma)$ . Then by definition,  $\sigma_1 H = \sigma_2 H$ . Then there exist  $h_1, h_2 \in H$  such that  $\sigma_1 h_1 = \sigma_2 h_2$ , that is,  $\widehat{Q}(\sigma_1, h_1) = \widehat{Q}(\sigma_2, h_2)$ . But by Lemma 16.2 we see that  $\sigma_1 = \sigma_2$ .

Statement (b) is trivial: this is the restriction of a continuous map.

Finally, to prove (c), take any open set  $U \subseteq \exp(\Sigma)$ . Note that  $p(U)$  is open in  $G/H$  if and only if  $p^{-1}(p(U))$  is open in  $G$ , by the definition of quotient topology. But  $p^{-1}(p(U)) = UH = \widehat{Q}(U \times H)$ , which is open again by Lemma 16.2. ■

**Corollary 16.5.** *The image  $p(\exp(\Sigma))$  is open in  $G/H$  and  $p|_{\exp(\Sigma)}$  is a homeomorphism onto its image.*



Now we finish the proof of Theorem 15.1. Let  $A := p(\exp(\Sigma))$ . Then  $A$  is open by the previous corollary. Define

$$\begin{aligned} Q : A \times H &\rightarrow p^{-1}(A), \\ (a, h) &\mapsto (p|_{\exp(\Sigma)})^{-1}(a) \cdot h = \widehat{Q}\left((p|_{\exp(\Sigma)})^{-1}(a), h\right). \end{aligned}$$

Then now defining

$$\begin{aligned} f : A \times H &\rightarrow \exp(\Sigma) \times H, \\ (a, h) &\mapsto \left((p|_{\exp(\Sigma)})^{-1}(a), h\right), \end{aligned}$$

we have that  $Q = \widehat{Q} \circ f$ ; if you like commutative diagrams, this means that the following diagram commutes:

$$\begin{array}{ccc} A \times H & \xrightarrow{f} & \exp(\Sigma) \times H \\ & \searrow Q & \downarrow \widehat{Q} \\ & & p^{-1}(A) \end{array}$$

By Corollary 16.5  $f$  is a homeomorphism, and hence  $Q$  is a homeomorphism. And indeed, for  $(a, h) \in A \times H$ ,

$$\begin{aligned} (p \circ Q)(a, h) &= p\left(\widehat{Q}\left((p|_{\exp(\Sigma)})^{-1}(a), h\right)\right) \\ &= p\left((p|_{\exp(\Sigma)})^{-1}(a) \cdot h\right) \\ &= a. \end{aligned}$$

Finally, noting that left multiplication is a diffeomorphism from  $G$  onto itself, we can translate any neighborhood about the identity and functions on it diffeomorphically. Hence this defines a fiber bundle, as desired.  $\blacksquare$

*Remark 16.6.* We did not really use the full power of  $G$  being a Lie group, but we only used the local structure: that is, the existence of  $W \subseteq G$  a neighborhood and a homeomorphism to  $\Sigma \times B_H \rightarrow W$ , where  $e \in \Sigma \subseteq G$ , and  $B_H$  also containing  $e$ . Then if  $G$  was any topological group with the above property, the above proof would have gone through.

In fact, we can prove the following theorem for topological groups using similar ideas as above:

**Definition 16.7.** A subgroup  $H \leq G$  is **discrete** if  $\{e\}$  is an open set in  $H$ .

**Theorem 16.8.** If  $H$  is discrete, then  $p : G \rightarrow G/H$  is a covering map.