Topology Quals

May 4, 2023

1 January 2023

1. Prove that the fundamental group U(2) of unitary 2×2 matrices is isomorphic to $\mathbb{Z}$, and describe a generator of the fundamental group.

   Proof. Let $A \in \mathrm{U}(2)$ be a unitary matrix. ∎

2. Let X be a finite CW complex. Prove the following assertion, or else provide a counterexample: for every $p \geq 0$ and every class $\alpha \in H^p(X)$ there is a p-dimensional finite CW complex Y, a class $\beta \in H^p(Y)$, and a continuous map $f : X \to Y$ such that $\alpha = f^*(\beta)$.

   Proof. Let $X = \mathbb{C}P^2$. Note that now then $H^*(\mathbb{C}P^2)$ has the ring structure of $\mathbb{Z}[\alpha]/\langle\alpha^3\rangle$, where α is degree two, i.e., in $H^2(\mathbb{C}P^2)$. We claim that for $p = 2$ for this particular α, every 2-dimensional finite CW complex Y and every $\beta \in H^2(Y)$, if $f : X \to Y$ then $\alpha \neq f^*(\beta)$.

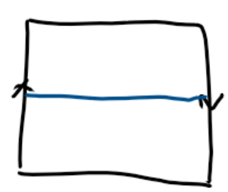
   Indeed, if $\beta \in H^2(Y)$, we have $\beta \smile \beta \in H^4(Y)$ and so $\beta^2 = 0$. But if $f^*(\beta) = \alpha$, we would have $0 = f^*(\beta^2) = f^*(\beta)^2 = \alpha^2 \neq 0$, a contradiction. This establishes the claim. ∎

3. Compute the singular homology groups of the Cantor set.

   Proof. The Cantor set is the union of uncountably many disjoint path components, and each component is a point. Since $H_i(*) = 0$ for $i \geq 1$, we see that the cantor set has trivial homology groups for $i \geq 1$, and $H_0(X) \simeq \bigoplus_{i \in I} \mathbb{Z}$, as many copies of $\mathbb{Z}$ as in the continuum. ∎

4. Prove that the boundary of the Möbius strip is not a retract of the Möbius strip.

   Proof. Suppose for a contradiction that the bouondary ∂M of the Möbius strip M is a retract of the Möbius strip M. Then there is a map $r : M \to \partial M$ such that the composition $r \circ \iota : M \to M$ is the identity map on M. Regard the Möbius strip as the quotient of a square with a pair of opposite sides identified as with opposite orientations as in the following figure:

   Now M deformation retracts onto the circle where the generator is the loop in blue; on the other hand, the boundary ∂M is also a copy of S^1, whose fundamental group has generator top and bottom lines concatenated. Thus the inclusion map $r_{\#} : \pi_1(\partial M, x_0) \to \pi_1(M, x_0)$ is given by $n \mapsto 2n$. This map is not surjective and so cannot have a right inverse, contrary to the fact that $(r \circ \iota)_{\#} = r_{\#} \circ \iota_{\#} = \mathrm{id}$. ■

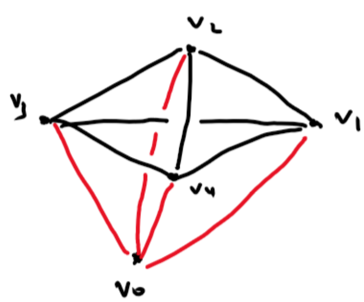
2 May 2022

1. Prove that no group of order greater than 2 can act continuously and freely on S^{2n}.

   Proof. Suppose a group G acts freely on S^{2n}. Thus, there is a map $\Phi : G \to \mathrm{Homeo}\,(S^{2n})$, $g \mapsto \varphi_g$. Since the action is free, φ_g has no fixed point for $g \neq e$. Then this implies that $\deg(\varphi_g) = -1$. Now if $h \in G$ is another nonidentity element, then $\deg(\varphi_{hg}) = \deg(\varphi_h \circ \varphi_g) = \deg(\varphi_h)\deg(\varphi_g) = 1$. Since the action is free, this implies that $hg = e$, and so $h = g^{-1}$. Thus if G is not the trivial group, necessarily G must be order 2. ■

2. Compute the homology with integer coefficients of the 2-skeleton of the 4-simplex.

   Proof. The 4-simplex is the simplex consisting of 5 vertices; its 2-skeleton consists of all the possible edges between the vertices and all possible faces for every triple of vertices. Having labelled the vertices $v_0, v_1, \ldots, v_4$, we have 10 edges. Each edge is contractible, so we can deformation retract every edge connecting v_0 and a vertex (pictured as the red edges on the figure below).



   This has the effect of identifying all the vertices with one another, and hence the remaining 1-simplices get collapsed, leaving a wedge of 4 spheres. Thus the 2-skeleton of the 4-simplex is homotopy equivalent to the wedge of 4 spheres. Using cellular homology it is now easy to see that this has the homology groups

$$H_i(X) = \begin{cases} \mathbb{Z}, & i = 0 \\ 0, & i = 1 \\ \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}, & i = 2 \\ 0, & i \geq 3. \end{cases}$$

   ■

3. Prove that a continuous map $f : \mathbb{R}P^n \to \mathbb{R}P^k$, where $n > k > 0$, induces the zero homomorphism on fundamental groups.

   Proof. Note that if $k = 1$, then the result is obvious: $\pi_1(\mathbb{R}P^n) \simeq \mathbb{Z}_2$, and $\pi_1(\mathbb{R}P^1) \simeq \mathbb{Z}$, and any homomorphism between fundamental groups must necessarily be trivial. So suppose that $k > 1$.

Next, observe that $\pi_1(\mathbb{R}P^n)$ and $\pi_1(\mathbb{R}P^k)$ are both $\mathbb{Z}_2$ which is abelian, so by the Hurewicz theorem $\pi_1(\mathbb{R}P^n) \simeq H_1(\mathbb{R}P^n)$; therefore it suffices to show that the induced map in homology is trivial. Now by the universal coefficient theorem, the following diagram commutes, and the top and bottom rows are split exact sequences:

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Ext}(H_0(\mathbb{R}P^n), \mathbb{Z}_2) \arrow[r] & H^1(\mathbb{R}P^n; \mathbb{Z}_2) \arrow[r] & \mathrm{Hom}(H_1(\mathbb{R}P^n), \mathbb{Z}_2) \arrow[r] & 0 \\
0 \arrow[r] & \mathrm{Ext}(H_0(\mathbb{R}P^k), \mathbb{Z}_2) \arrow[r] \arrow[u, "f^*"] & H^1(\mathbb{R}P^k; \mathbb{Z}_2) \arrow[r] \arrow[u, "f^*"] & \mathrm{Hom}(H_1(\mathbb{R}P^k), \mathbb{Z}_2) \arrow[r] \arrow[u, "f^*"] & 0
\end{tikzcd}$$

Now $H_0(\mathbb{R}P^n) = \mathbb{Z}$ for all $n \geq 1$ and $\mathrm{Ext}(\mathbb{Z}, A) = 0$ for all abelian groups A, so we have isomorphisms between the cohomology groups and Hom groups. Now the map $f^* : \mathrm{Hom}(H_1(\mathbb{R}P^k), \mathbb{Z}_2) \to \mathrm{Hom}(H_1(\mathbb{R}P^n), \mathbb{Z}_2)$ is given by $\varphi \mapsto \varphi \circ f$. This map is zero if and only if f is identically 0. By the isomorphism it suffices to show that $f^* : H^1(\mathbb{R}P^k; \mathbb{Z}_2) \to H^1(\mathbb{R}P^n; \mathbb{Z}_2)$ is identically zero, which will show that the other vertical map is also zero. Indeed, $H^*(\mathbb{R}P^k; \mathbb{Z}_2)$ has the ring structure of $\mathbb{Z}_2[\alpha]/\left\langle \alpha^{k+1} \right\rangle$ where α has degree 1, and similarly $H^*(\mathbb{R}P^n; \mathbb{Z}_2)$ has the ring structure $\mathbb{Z}_2[\beta]/\langle \beta^{n+1} \rangle$. Additionally, $H^1(\mathbb{R}P^n; \mathbb{Z}_2) \simeq \mathbb{Z}_2$ and $H^1(\mathbb{R}P^k; \mathbb{Z}_2) \simeq \mathbb{Z}_2$ also. Thus if f is not trivial then $f^*(\alpha) = \beta$. But $0 = f^*(\alpha^{k+1}) = \beta^{k+1} \neq 0$ since $k < n$. This is a contradiction, so f is trivial. ■

4. Compute the fundamental group of the space X that is obtained from the 2-sphere by identifying the North and South poles. Construct the universal cover as an explicit closed subset of $\mathbb{R}^3$, and prove that your construction really is the universal cover.

3 January 2022

1. Can an infinite, countable metric space be connected? Either prove the impossibility or construct an example.

   Proof. This is impossible. Suppose that there is a countably infinite metric space $X = \{x_i\}_{i \in \mathbb{N}}$. Then there are countably many possible values for $d(x_1, x_j)$ for $j \in \mathbb{N}$, but infinitely many values in $\mathbb{R}_{\geq 0}$. Thus there is some r such that $B_r(x_1) = \{x_i \in X \mid d(x_1, x_i) = r\} = \varnothing$. Then $\overline{B_r(x_1)} = B_r(x_1) \cup \partial B_r(x_1)$, and so $\overline{B_r(x_1)} = B_r(x_1)$. Thus $B_r(x_1)$ is a nontrivial clopen set. This implies that X is not connected. ■

2. Let ω be a nontrivial cube root of 1. Let X be the quotient space obtained from the equivalence relation on the closed unit disk in the complex plane generated by the relation $z \sim \omega z$, where $|z| = 1$. Compute the fundamental group of X.

3. Show that if a finite CW complex X is homotopy equivalent to $\mathbb{C}P^2$, then the Lefschetz number of any continuous map from X to itself is nonzero.

4. Prove that if A and B are any two measurable subsets of the 2-sphere S^2, then there is a great circle on S^2 that simultaneously divides each of A and B into subsets of equal area (No need to prove the continuity of the maps involved in your solution).

4 May 2021

1. Find the fundamental group of the Klein bottle with one point deleted.

Proof. The Klein bottle with one point deleted deformation retracts onto the 1-skeleton, which is homotopy equivalent to the wedge of two circles. Hence the fundamental group of the Klein bottle with one point deleted is $\mathbb{Z} * \mathbb{Z}$, the free group on two generators. ∎

2. Let $f : S^4 \to \mathbb{C}P^2$ be a continuous map. Show that $\deg f = 0$.

Proof. Let f be as prescribed. Then by the Universal Coefficient Theorem for cohomology groups, the we have the following diagram commutes where the top and bottom rows are exact sequences that split, and the vertical arrows are maps induced by f:

$$\begin{tikzcd}
0 \arrow[r] & \operatorname{Ext}(H_3(S^4), \mathbb{Z}) \arrow[r] & H^4(S^4; \mathbb{Z}) \arrow[r] & \operatorname{Hom}(H_4(S^4); \mathbb{Z}) \arrow[r] & 0 \\
0 \arrow[r] & \operatorname{Ext}(H_3(\mathbb{C}P^2), \mathbb{Z}) \arrow[r] \arrow[u, "f^*"] & H^4(\mathbb{C}P^2; \mathbb{Z}) \arrow[r] \arrow[u, "f^*"] & \operatorname{Hom}(H_4(\mathbb{C}P^2); \mathbb{Z}) \arrow[r] \arrow[u, "f^*"] & 0
\end{tikzcd}$$

Now note that $H_3(S^4) = H_3(\mathbb{C}P^4) = 0$, so the Ext groups vanish. Thus we get an isomorphism between the cohomology groups and the Homs for both the top and bottom rows. Moreover, if $\deg f = d$, then the map between the Homs is multiplication by d: for the map $f : H_4(S^4) \to H_4(\mathbb{C}P^2)$ is multiplication by d, and so the induced map in Hom is $\varphi \mapsto \varphi \circ f$ for $\varphi \in \operatorname{Hom}(H_4(\mathbb{C}P^2); \mathbb{Z})$, and so the claim follows.

Since the diagram commutes, this implies that the map in the middle vertical column must also be multiplication by d. Now the cohomology ring of $\mathbb{C}P^2$, $H^*(\mathbb{C}P^2)$ is isomorphic to $\mathbb{Z}[\alpha]/(\alpha^3)$ where α is degree 2. That is, $\alpha \in H^2(\mathbb{C}P^2)$ is the generator of the ring. Note that since $H^2(\mathbb{C}P^2) = 0$, $f^*(\alpha) \in H^2(S^4) = 0$. Further note that $\alpha^2 \in H^4(\mathbb{C}P^2)$ is the generator of that group; therefore $f^*(\alpha^2) = f^*(\alpha)^2 = 0$ since f^* is a ring homomorphism. This immediately implies that $f^* : H^4(\mathbb{C}P^2) \to H^4(S^4)$ is the multiplication by zero map, and so $\deg f = 0$. ∎

3. Suppose that a topological space X is the union of open subsets $U_1, \ldots, U_k$ (with k at least 2) and suppose that each of these sets, and all the intersections of any number of them, are contractible or empty. Show that $H_p(X)$ is zero for all $p > k - 2$.

4. Prove that for every $n \geq 0$ every continuous map $f : \mathbb{R}P^{2n} \to \mathbb{R}P^{2n}$ has a fixed point, but that for every $n \geq 0$ there exists a continuous map $f : \mathbb{R}P^{2n+1} \to \mathbb{R}P^{2n+1}$ that is fixed-point free.

Proof. Using that S^{2n} is the universal cover of $\mathbb{R}P^{2n}$ with $p : S^{2n} \to \mathbb{R}P^{2n}$ being the covering map, consider the composition $f \circ p : S^{2n} \to \mathbb{R}P^{2n}$. Then the induced homomorphism $(f \circ p)_\# : \pi_1(S^{2n}) \to \pi_1(\mathbb{R}P^{2n})$ is trivial since S^{2n} is simply connected. Thus by the lifting criterion, $f \circ p$ lifts to a map $g : S^{2n} \to S^{2n}$, such that the following diagram commutes:

$$\begin{tikzcd}
S^{2n} \arrow[r, "g"] \arrow[d, "p"] & S^{2n} \arrow[d, "p"] \\
\mathbb{R}P^{2n} \arrow[r, "f"] & \mathbb{R}P^{2n}
\end{tikzcd}$$

Now $g : S^{2n} \to S^{2n}$ has a point $x \in S^{2n}$ such that $g(x) = \pm x$; writing $p(x) = [x] \in \mathbb{R}P^{2n}$, then $(p \circ g)(x) = p(\pm x) = [x]$, and on the other hand $(f \circ p)(x) = f([x]) = [x]$. Thus $[x]$ is a fixed point.

On the other hand, treat S^{2n+1} as a subset of $\mathbb{R}^{2n+2}$ and consider the map

$$(x_1, x_2, \ldots, x_{2n+2}) \mapsto (x_1 \cos\theta, x_2 \sin\theta, x_3, \ldots, x_{2n+2}),$$

where $\theta \neq 0, \pi/2, \pi, 3\pi/2$. Then the induced map in the quotient, $\mathbb{R}P^{2n+1} \to \mathbb{R}P^{2n+1}$ has no fixed point. ■

5 January 2021

1. Is S^4 a covering space of $\mathbb{C}P^2$? Prove your answer.

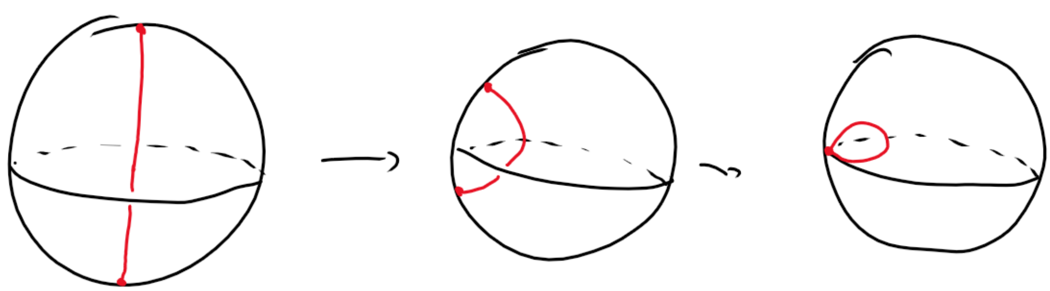
   Proof. Note that S^4 and $\mathbb{C}P^2$ are both CW complexes, and S^4 and $\mathbb{C}P^2$ are both compact. Thus if $p : S^4 \to \mathbb{C}P^2$ is a covering map, it has only finitely many sheets, say k sheets, and moreover $k\chi(\mathbb{C}P^2) = \chi(S^4)$. But $\chi(S^4) = 2$ and $\chi(\mathbb{C}P^2) = 3$, so no such k can exist. ■

2. Let $f : \mathbb{R}P^3 \to \mathbb{R}P^2$ be a continuous map. Compute the induced cohomology homomorphisms $f^* : H^i(\mathbb{R}P^2, \mathbb{Z}_2) \to H^i(\mathbb{R}P^3, \mathbb{Z}_2)$ for $i = 1, 2$.

   Proof. We recall that $\mathbb{R}P^n$ has the cohomology ring structure of $H^*(\mathbb{R}P^n; \mathbb{Z}_2) \simeq \mathbb{Z}_2[x]/\langle x^{n+1}\rangle$, where $x \in H^1(\mathbb{R}P^n; \mathbb{Z}_2)$ is the generator. With this in mind let α be the generator of $H^*(\mathbb{R}P^3; \mathbb{Z}_2)$ and β that of $H^*(\mathbb{R}P^2; \mathbb{Z}_2)$. Now since $H^1(\mathbb{R}P^2; \mathbb{Z}_2) \simeq \mathbb{Z}_2$ and $H^1(\mathbb{R}P^3; \mathbb{Z}_2) \simeq \mathbb{Z}_2$, the only possible homomorphisms $f^* : H^1(\mathbb{R}P^2; \mathbb{Z}_2) \to H^1(\mathbb{R}P^3; \mathbb{Z}_2)$ are either the zero homomorphism or $\beta \mapsto \alpha$. But if we were in the latter case, then $0 = f^*(\beta^3) = f^*(\beta)^3 = \alpha^3 \neq 0$, a contradiction. Thus f^* can only be the zero homomorphism; since it is a ring homomorphism, it must also be the zero homomorphism on H^2 as well. ■

3. Let A be a diameter of the sphere S^2. Compute $H_i(S^2 \cup A)$ for all $i \geq 0$.

   Proof. The space A is homotopy equivalent to $S^1 \vee S^2$ via the following figure:

   One can give a CW structure to $S^1 \vee S^2$ with a 1-cell and a 2-cell attached to a point. Then we have the cell complex structure $C_0^{\text{cell}}(A) = \mathbb{Z}$, $C_1^{\text{cell}}(A) = \mathbb{Z}$, and $C_2^{\text{cell}}(A) = \mathbb{Z}$, and the boundary homomorphisms are zero. Thus we have $H_i(A) \simeq \mathbb{Z}$ for $i = 0, 1, 2$, and $H_i(A) = 0$ for $i \geq 3$. One can also use the Mayer-Vietoris sequence to compute the homology of $S^1 \vee S^2$ as well. ■

4. Prove that S^4 is not a topological group.

   Proof. Any topological group acts on itself via left multiplication. Thus if S^4 were a topological group, for every $g \in S^4$ we get a corresponding $\varphi_g \in \text{Homeo}(S^4)$, where $\varphi_g(x) = g \cdot x$. Now suppose that for some $g \in S^4$, φ_g has a fixed point, i.e., there exists some $x \in S^4$ such that $\varphi_g(x) = x$. Then $g \circ x = x$, which implies that $g = e$. Therefore for $g \neq e$, $\deg(\varphi_g) = -1$. But then for all $g \in S^4$, $\deg(\varphi_{g^2}) = \deg(\varphi_g \circ \varphi_g) = \deg(\varphi_g)\deg(\varphi_g) = 1$, which implies that $g^2 = e$ for all S^4, a contradiction. ■

6 August 2020

1. Let $\mathbb{R}^{k+1}$ be a proper subspace in $\mathbb{R}^{n+1}$, $k < n$. Consider the corresponding projective spaces $\mathbb{R}P^k \subset \mathbb{R}P^n$, and let $X = \mathbb{R}P^n - \mathbb{R}P^k$. Find $\pi_1(X)$ and $H_i(X, \mathbb{Z}_2)$ for all i.

2. Show that every continuous map $\mathbb{R}P^4 \to T^4$ is nullhomotopic (where T^4 is the 4-dimensional torus).

   Proof. Since $T^4 = S^1 \times S^1 \times S^1 \times S^1$ and $\mathbb{R}$ is the universal cover of S^1, $\mathbb{R}^4$ is the universal cover of T^4 with $p : \mathbb{R}^4 \to T^4$ being the covering map. Now if $f : \mathbb{R}P^4 \to T^4$ is a continuous map, then the induced homomorphism of fundamental groups $f_\# : \pi_1(\mathbb{R}P^4) \to \pi_1(T^4)$ is trivial, since $\pi_1(\mathbb{R}P^4) \simeq \mathbb{Z}_2$ while $\pi_1(T^4) = \mathbb{Z}^4$. Thus by the lifting criterion, we have $f_\#(\pi_1(\mathbb{R}P^4)) \subseteq p_\#(\pi_1(\mathbb{R}^4))$, and so there is a lifting $\widetilde{f}$ such that the following diagram commutes:

$$\begin{tikzcd} & \mathbb{R}^4 \arrow[d, "p"] \\ \mathbb{R}P^4 \arrow[ur, "\widetilde{f}"] \arrow[r, "f"'] & T^4 \end{tikzcd}$$

   Now define the homotopy $F : \mathbb{R}P^4 \times [0,1] \to T^4$ given by $F(x,t) = p\left((1-t)\widetilde{f}(x)\right)$. Then $F(x,0) = p\left(\widetilde{f}(x)\right) = f(x)$ while $F(x,1) = p(0)$ which is constant; hence f is nullhomotopic. ∎

3. Consider a closed disc $D^n \subseteq \mathbb{R}^n$ and let $f : D^n \to \mathbb{R}^n$ be a continuous map such that the restriction to the boundary S^{n-1} is the identity map. Prove that the image of f contains D^n.

   Proof. Toward a contradiction suppose the image of f does not contain D^n; WLOG we can assume and consider the map on $g : \mathbb{R}^n \setminus D^n \to \mathbb{R}^n \setminus D^n$ given by $g : \mathbb{R}^n \to \mathbb{R}^n$ given by $g(x) = \frac{x}{\|x\|}$, which sends all of $\mathbb{R}^n$ outside the disc to the boundary of the disc, S^{n-1}. Now extend g to all of $\mathbb{R}^n$ by letting it be the identity in the closure of D^n. By the pasting lemma this map is continuous. Next, suppose that $p \in D^n$ is not in the image of f. Then for each $x \in S^{n-1}$, we can draw a line segment connecting each x to p. Note that as a result of this, every point in $D^n \setminus \{p\}$ is in some line segment; for if $q \in D^n \setminus \{p\}$, the ray starting at p going through q will pass through the sphere at some point, and that defines the desired line segment. Next, define the map $h : D^n \setminus \{p\} \to S^{n-1}$ given by sending each point q to the endpoint in S^{n-1} corresponding to the line segment on which q is. Then the composition $h \circ g \circ f : D^n \to S^{n-1}$ is a retraction, but no such retraction can exist. For this now defines a map in homology $H_{n-1}(D^n) \to H^{(}_{n-1}S^{n-1})$ ∎

4. Let n be even. Prove that every homeomorphism of $\mathbb{C}P^n$ has degree 1.

   Proof. Let $f : \mathbb{C}P^n \to \mathbb{C}P^n$ be a homeomorphism, and say that $\deg f = d$. Then $f_* : H_{2n}(\mathbb{C}P^n) \to H_{2n}(\mathbb{C}P^n)$ is given by multiplication by d. Then by the Universal Coefficient theorem, the following diagram commutes:

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{Ext}(H_{2n-1}(\mathbb{C}P^n), \mathbb{Z}) \arrow[r] & H^{2n}(\mathbb{C}P^n; \mathbb{Z}) \arrow[r] & \mathrm{Hom}(H_n(\mathbb{C}P^n); \mathbb{Z}) \arrow[r] & 0 \\ 0 \arrow[r] & \mathrm{Ext}(H_{2n-1}(\mathbb{C}P^n), \mathbb{Z}) \arrow[r] \arrow[u, "f^*"] & H^{2n}(\mathbb{C}P^n; \mathbb{Z}) \arrow[r] \arrow[u, "f^*"] & \mathrm{Hom}(H_{2n}(\mathbb{C}P^n); \mathbb{Z}) \arrow[r] \arrow[u, "f^*"] & 0 \end{tikzcd}$$

   But $H_{2n-1}(\mathbb{C}P^n) = 0$, and so the ext group vanishes. Thus $H^{2n}(\mathbb{C}P^n)$ is isomorphic to

$\mathrm{Hom}(H_{2n}(\mathbb{C}P^n);\mathbb{Z})$, and the map f^* between Homs is given by multiplication by d. Therefore the map between cohomology groups is also multiplication by d. We will show that this map is necessarily multiplication by 1.

Indeed, $H^*(\mathbb{C}P^n) \simeq \mathbb{Z}[\alpha]/\langle \alpha^{n+1}\rangle$ where α has degree 2 and can be identified with an element in $H^2(\mathbb{C}P^n)$. Then since f is a homeomorphism, $f^*(\alpha) = \pm\alpha$. But then $\alpha^n \in H^{2n}(\mathbb{C}P^n)$, and $f^*(\alpha^n) = (f^*(\alpha))^n = (\pm\alpha)^n$. But if n is even, say $n = 2k$ for some k, $(\pm\alpha)^n = (\pm\alpha)^{2k} = \alpha$. Thus the f^* map on $H^{2n}(\mathbb{C}P^n)$ is multiplication by 1; hence f has degree 1. ∎

7 January 2020

1. Let X be the complement of the coordinate axes in $\mathbb{R}^3$. Compute the fundamental group of X.

   Proof. Note that if the coordinate axes in $\mathbb{R}^3$ are removed, the origin is also removed. Hence we may apply the deformation retract $F(x,t) = t\frac{x}{|x|} + (1-t)x$ to the sphere $S^2 \subseteq \mathbb{R}^3$. Note that the image in the sphere avoids six points. Via the stereographic projection, this is homeomorphic to $\mathbb{R}^2$ minus five points, which is homotopy equivalent to the wedge of five circles. Hence the fundamental group of X is isomorphic to the free group on five generators. ∎

2. Let X be the space obtained by attaching a Möbius band M to $\mathbb{R}P^2$ via a homeomorphism from the boundary S^1 of the band M to $\mathbb{R}P^1 \subseteq \mathbb{R}P^2$. Compute the homology groups of X using the Mayer-Vietoris sequence.

   Proof. The space X can be decomposed into M unioned with a small open set in $\mathbb{R}P^2$ (call it A) and $\mathbb{R}P^2$ unioned with a small open set in M (call it B) such that the boundary S^1 is contained in $A \cap B$; these sets can be chosen so that A strongly deformation retracts onto M and likewise with B, and $A \cap B$. Then the interiors of A and B now cover X, and moreover $A \cap B \neq \varnothing$. Then using the Mayer-Vietoris long exact sequence:

$$\begin{tikzcd} 0 \arrow[r] & \widetilde{H}_2(A\cap B) \arrow[r] & \widetilde{H}_2(A)\oplus\widetilde{H}_2(B) \arrow[r] & \widetilde{H}_2(X) \arrow[dll] \\ & \widetilde{H}_1(A\cap B) \arrow[r] & \widetilde{H}_1(A)\oplus\widetilde{H}_1(B) \arrow[r] & \widetilde{H}_1(X) \arrow[dll] \\ & \widetilde{H}_0(A\cap B) \arrow[r] & \widetilde{H}_0(A)\oplus\widetilde{H}_0(B) \arrow[r] & \widetilde{H}_0(X) \arrow[r] & 0 \end{tikzcd}$$

   Observe that $H_i(X) = 0$ for $i \geq 3$ since $H_i(A\cap B) \simeq H_i(S^1) = 0$ and $H_i(A) \simeq H_i(M)0$ and $H_i(B) \simeq H_i(\mathbb{R}P^2) \simeq 0$ in that case. Now, $H_2(A\cap B) \simeq H_2(S^1) = 0$, and $H_2(A) \oplus H_2(B) \simeq H_2(M) \oplus H_2(\mathbb{R}P^2) \simeq 0$ since $H_2(M) \simeq H_2(S^1) = 0$ and $H_2(\mathbb{R}P^2) \simeq 0$. Next, $H_1(A\cap B) \simeq H_1(S^1) \simeq \mathbb{Z}$, and $H_1(A)\oplus H_1(B) \simeq \mathbb{Z}\oplus\mathbb{Z}_2$.

   From the bottom row it is clear that $\widetilde{H}_0(X) = 0$ and hence $H_0(X) \simeq \mathbb{Z}$, which lines up with the observation that X is path-connected. To compute $H_1(X)$ and $H_2(X)$, we need to compute the homomorphisms in this sequence. Now from $H_2(A)\oplus H_2(B) = 0$, we have that the map $H_2(X) \to H_1(A\cap B)$ must be injective. Next, from $H_1(A\cap B) \simeq \mathbb{Z}$, a cycle wrapping around the circle once in $A \cap B$ wraps around the central circle in the Möbius band twice, and the boundary circle of $\mathbb{R}P^2$ once. Therefore the map $H_1(A\cap B) \to H_1(A)\oplus H_1(B)$ is

$1 \mapsto (2,1)$. In particular, this map is injective, which implies that the image of the map $H_2(X) \to H_1(A \cap B)$ is 0, but since this map was determined to be injective, it follows that $H_2(X) = 0$.

Next, the image of the map $H_1(A \cap B) \to H_1(A) \oplus H_1(B)$ is $2\mathbb{Z} \oplus \mathbb{Z}_2$, which by exactness is the kernel of the map $H_1(A) \oplus H_1(B) \to H_1(X)$. Then we deduce the map must be mod 2 in the $\mathbb{Z}$ component and killing the $\mathbb{Z}_2$ component. Therefore the image in $H_1(X)$ must be $\mathbb{Z}_2$. But the map $H_1(X) \to \widetilde{H}_0(A \cap B)$ is the zero map, which implies that $H_1(X) \simeq \mathbb{Z}_2$ by exactness. Indeed, the fundamental group of X is also $\mathbb{Z}_2$ using an application of van Kampen's theorem, as expected. ■

3. Construct a connected CW-complex such that the cellular homology group $H_1(X;\mathbb{Z}) = \mathbb{Z} \times \mathbb{Z}_3$, $H_q(X;\mathbb{Z}) = 0$ for all $q \geq 1$, and show that your complex has the desired homology.

   Proof. Let A be the circle S^1, and let B be the CW-complex obtained by attaching a 2-cell to S^1 via the map $z \mapsto z^3$. Finally, let $X = A \vee B$. We claim that X has the desired homology.

   To see this we will appeal to the Mayer-Vietoris exact sequence. Let A' be an open neighborhood containing A homotopy equivalent to A, and likewise with B', so that $A' \cap B'$ is nonempty and has the homotopy type of a point. Now removing the primes for notation's sake, we now have

$$\begin{tikzcd}
\cdots \arrow[r] & \underbrace{\widetilde{H}_2(A \cap B)}_{=0} \arrow[r] & \widetilde{H}_2(A) \oplus \widetilde{H}_2(B) \arrow[r] & \widetilde{H}_2(X) \arrow[dll, rounded corners] & \\
 & \underbrace{\widetilde{H}_1(A \cap B)}_{=0} \arrow[r] & \widetilde{H}_1(A) \oplus \widetilde{H}_1(B) \arrow[r] & \widetilde{H}_1(X) \arrow[dll, rounded corners] & \\
 & \underbrace{\widetilde{H}_0(A \cap B)}_{=0} \arrow[r] & \widetilde{H}_0(A) \oplus \widetilde{H}_0(B) \arrow[r] & \widetilde{H}_0(X) \arrow[r] & 0
\end{tikzcd}$$

   Thus for each i, $\widetilde{H}_i(X) \simeq \widetilde{H}_i(A) \oplus \widetilde{H}_i(B)$, and for $i \geq 1$ the reduced homology agrees with the ordinary homology. But $H_1(A) \simeq \mathbb{Z}$ and $H_1(B) \simeq \mathbb{Z}_3$, while $H_i(A) = H_i(B) = 0$ for $i \geq 2$. The desired claim now follows. ■

4. Use the Künneth formula to compute the ($\mathbb{Z}$) homology of $\mathbb{R}P^2 \times \mathbb{R}P^2$. You may take the homology groups of $\mathbb{R}P^2$ as given; $H_0(\mathbb{R}P^2) = \mathbb{Z}$, $H_1(\mathbb{R}P^2) = \mathbb{Z}_2$, and the rest are zero.

8 May 2019

1. Suppose $X_1, \dots, X_n$ are convex open sets in $\mathbb{R}^m$ such that any triple intersection $X_i \cap X_j \cap X_k$ is nonempty. Show that the fundamental group of $X_1 \cup \cdots \cup X_n$ is trivial.

2. Let Δ^n be the n-simplex $[v_0, \dots, v_n]$ and identify all faces of the same dimension such that the induced order on vertices of the faces agree. Compute the homology groups of the resulting space X.

3. Let X be the CW complex obtained by identifying opposite sides of a 3-dimensional cube. This is without changing orientation, i.e., if $x, y, z \in [0,1]$, $(x, y, 0)$ is identified with $(x, y, 1)$. Compute the cellular homology groups and cellular boundary maps.

4. Show that for any two points x, y in an arcwise (pathwise) connected manifold M, there is a smooth diffeomorphism h of M such that $h(x) = y$.

9 December 2018

1. (Tube Lemma) Let X and Y be topological spaces and suppose that X is compact. Suppose that $y \in Y$ is such that $X \times \{y\} \subseteq U$ for some $U \subseteq X \times Y$. Prove that there exists an open set $W \subseteq Y$ such that $X \times \{y\} \subseteq X \times W \subseteq U$.

   Proof. Consider the family of open sets

   $$\mathcal{A} := \{A \times N \subseteq U \mid A \subseteq X \text{ open, } B \subseteq Y \text{ open, } y \in B\}$$

   such that $\mathcal{A}$ is an open cover for $X \times \{y\}$. Letting $p_1 : X \times Y \to X$ be the projection onto the first factor, take the collection of open sets $\{p_1(A \times B) \mid A \times B \in \mathcal{A}\}$, which forms an open cover for X. Then by compactness of X we may now extract a finite subcover $A_1 \dots, A_n$. Then the corresponding sets in $\mathcal{A}$, $A_1 \times B_1, \dots, A_n \times B_n$ form a finite cover for $X \times \{y\}$. Moreover, each $A_n \times B_n \subseteq U$ by construction. Now let $B := B_1 \cap B_2 \cap \dots \cap B_n$, we see that $X \times \{y\} \subseteq X \times B \subseteq U$, since each $y \in B_j$ for all j, and $B_1 \cap \dots \cap B_n \subseteq B_j$ for each j, and $A_j \times B_j \subseteq U$ for each j also. ∎

2. Prove that the fundamental group of a topological group is abelian.

   Proof. Let $*$ denote the concatenation of paths and $\cdot$ denote pointwise multiplication. Thus if γ, η are paths then $\gamma * \eta$ will denote the concatenation of the two paths and $\gamma \cdot \eta$ will be given by $(\gamma \cdot \eta)(t) = \gamma(t)\eta(t)$. Let 0 be the identity element of the topological group X, and let $\gamma, \eta, \alpha, \beta \in \pi_1(X, 0)$. Then:

   $$\begin{aligned} ((\gamma * \eta) \cdot (\alpha * \beta))(t) &= \begin{cases} \gamma(t) \cdot \alpha(t), & 0 \le t \le \frac{1}{2} \\ \eta(2t-1) \cdot \beta(2t-1), & \frac{1}{2} \le t \le 1 \end{cases} \\ &= ((\gamma \cdot \alpha) * (\eta \cdot \beta))(t). \end{aligned}$$

   Using this, we observe that if e is the constant path at the identity, we have:

   $$\begin{aligned} \alpha * \beta &= (\alpha \cdot e) * (e \cdot \beta) \\ &= (\alpha * e) \cdot (e * \beta) \\ &\approx (e * \alpha) \cdot (\beta * e) \\ &= (e \cdot \beta) * (\alpha \cdot e) = \beta * \alpha, \end{aligned}$$

   where the middle $\approx$ denotes that $(\alpha * e)$ is homotopic to $(e * \alpha)$, and similarly with β. ∎

3. Let S^n denote the n-sphere and let $f : S^n \to S^n$ be a continuous map with degree $\deg f \neq (-1)^{n+1}$. Prove that f has a fixed point.

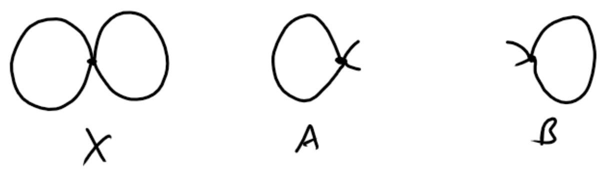
Proof. We will prove the contrapositive. Suppose that f has no fixed point. We will show that f is homotopic to the antipodal map. Indeed, consider the map $F : S^n \times I \to S^n$ given by

$$F(x,t) = \frac{-tx + (1-t)f(x)}{|-tx + (1-t)f(x)|}.$$

Since f has no fixed points, $f(x) \neq x$ for all x. In particular the denominator in F is never 0, because $-tx+(1-t)f(x)$ defines a line connecting $f(x)$ and $-x$, and so that is 0 possibly only at $t = 1/2$ if it goes through the center of the sphere. Therefore this is a homotopy defined for all x and t, and we conclude that f has the same degree as the antipodal map, which is $(-1)^{n+1}$. ∎

4. Let p be a point in the two-torus T^2. Compute the homology of $T^2 \setminus \{p\}$ using the Mayer-Vietoris sequence.

Proof. Expressing T^2 via the square with opposite sides identified, observe that $T^2 \setminus \{p\}$ deformation retracts onto the boundary which is homotopy equivalent to a wedge of two circles. Thus we will now compute the homology groups of $S^1 \vee S^1$ via the Mayer-Vietoris sequence. Let A be one of the circles unioned with a small neighborhood in the opposite circle, and B defined similarly as in the figure below.



Then X is now the union of the interiors of A and B, and moreover $A \cap B \neq \varnothing$, so we may now use the Mayer-Vietoris sequence for reduced homology groups. Moreover, observe that X is a CW complex with no 2-cells, so the homology groups vanish for $i \geq 2$. Thus we only need to look at the sequence for $i = 0, 1$. Indeed, we have

$$\begin{array}{l} 0 \longrightarrow \widetilde{H_1}(A \cap B) \longrightarrow \widetilde{H}_1(A) \oplus \widetilde{H}_1(B) \longrightarrow \widetilde{H}_1(X) \longrightarrow \\ \longrightarrow \widetilde{H}_0(A \cap B) \longrightarrow \widetilde{H}_0(A) \oplus \widetilde{H}_0(B) \longrightarrow \widetilde{H}_0(X) \longrightarrow 0 \end{array}$$

Now $A \cap B$ is contractible, so it has the homotopy type of a point; thus its reduced homology groups now vanish. Next, A and B are homotopy equivalent to S^1 respectively, so $\mathbb{Z} \oplus \mathbb{Z} \simeq \widetilde{H}_1(A) \oplus \widetilde{H}_1(B) \simeq \widetilde{H}_1(X)$. Similarly, we see that $0 \simeq \widetilde{H}_0(A) \oplus \widetilde{H}_0(B) \simeq \widetilde{H}_0(X)$. Now for $i \geq 1$, $\widetilde{H}_i(X) \simeq H_i(X)$, and $H_0(X) \simeq \widetilde{H}_0(X) \oplus \mathbb{Z}$. We now conclude:

$$H_i(X) \simeq \begin{cases} \mathbb{Z}, & i = 0 \\ \mathbb{Z} \oplus \mathbb{Z}, & i = 1 \\ 0, & i \geq 2. \end{cases}$$

∎

10 May 2018

1. Show that a continuous and bijective map $f : X \to Y$, where X is compact and Y is Hausdorff, is a homeomorphism.

2. Prove that if $p : (E, e) \to (B, b)$ is a covering map from a topological space E to a topological space B, then the induced group homomorphisms $p_* : \pi_n(E, e) \to \pi_n(B, b)$ are isomorphisms for $n \geq 2$, $n \in \mathbb{N}$. Recall that $\pi_n(X)$ is the nth homotopy group of a topological space X consisting of the homotopy classes of maps from S^n to X.

   Proof. A covering map is in particular a Serre fibration since it satisfies the homotopy lifting property for any space. Thus we have the long exact sequence of homotopy groups

$$\begin{tikzcd}
\cdots \arrow[r] & \pi_{n+1}(p^{-1}(b)) \arrow[r] & \pi_{n+1}(E,e) \arrow[r] & \pi_{n+1}(B,b) \arrow[dll, rounded corners] & \\
 & \pi_n(p^{-1}(b)) \arrow[r] & \pi_n(E,e) \arrow[r] & \pi_n(B,b) \arrow[r] & \cdots
\end{tikzcd}$$

   But if p is a covering space, $p^{-1}(b)$ is discrete, so has trivial homotopy groups for $n \geq 1$. The desired isomorphisms now follow. ■

3. Prove that a continuous map from S^2 into itself has a fixed point or sends some point to its antipode.

   Proof. If not, then f is homotopic to both the identity map and the antipodal map via the following homotopies:

$$F_1(x,t) = \frac{tx + (1-t)f(x)}{|tx + (1-t)f(x)|}, \qquad F_2(x,t) = \frac{-tx + (1-t)f(x)}{|-tx + (1-t)f(x)|}.$$

   The denominators for F_1 and F_2 are never 0 for all t. But on S^2 the degree of the antipodal map is -1 while the identity has degree 1, so they cannot be homotopic. ■

4. Prove that any differential 1-form S^2 which is invariant under SO(3) (the group of orientation-preserving isometries of the sphere) is identically zero.

11 December 2017

1. Show that there are embeddings of two solid tori $A, B \in S^3$ such that S^3 is equal to the union of A and B with their boundaries, which are diffeomorphic to T^2, idnetified.

2. Give a topological proof of the Fundamental Theorem of Algebra: every non-constant complex polynomial has a zero.

3. Consider a 2×2 matrix A with integer entries. This matrix determines a map f from the torus T^2 identified with $\mathbb{R}^2/\mathbb{Z}^2$ to itself, given by $f(v) = Av \bmod \mathbb{Z}^2$ if $v \in \mathbb{R}^2 \bmod \mathbb{Z}^2$. Find the degree of the map f.

4. Find the fundamental group of the punctured torus, that is, of T^2 with a point deleted.

   Proof. Writing the torus as a square with the opposite sides identified, we see that T^2 with a point deleted deformation retracts onto the boundary. The boundary is homotopy equivalent to the wedge of two circles, so the desired fundamental group is $\mathbb{Z} * \mathbb{Z}$. ■