8 1/30/23

Relevant reading: Weintraub pp. 11–13, Hatcher pp. 70–76.

8.1 Deck Transformations

We will first begin with an example to motivate our definition:

Example 8.1. Let $S^1 = \{z \in \mathbb{C} \mid |z| = 1\} \subseteq \mathbb{C}$ be the circle regarded as a subspace of $\mathbb{C}$. Then we saw that $p : \mathbb{R} \to S^1$ via $p(t) = e^{2\pi i t}$ was a covering map. Then for any $z_0 \in S^1$ with $p(t_0) = z_0$, we have that $p^{-1}(\{z\}) = t_0 + \mathbb{Z}$. Equivalently, $p(t_0 + m) = p(t_0)$ for all $m \in \mathbb{Z}$. Define, for $m \in \mathbb{Z}$, $T_m : \mathbb{R} \to \mathbb{R}$ via $T_m(t) = t + m$ translation by m. Then by our discussion, $p \circ T_m = p$. We say that T_m is an example of a **deck transformation**.

Definition 8.2. Let $p : E \to B$ be a covering projection. Then the **group of deck transformations** is the set

$$\Gamma_p \coloneqq \{T \in \mathrm{Homeo}(E) \mid p \circ T = p\},$$

where the endowed operation is function composition. That is, it is the set of all homeomorphisms such that for any $T \in \Gamma_p$, the following diagram commutes:

$$\begin{tikzcd} E \arrow[r, "T"] \arrow[dr, "p"'] & E \arrow[d, "p"] \\ & B \end{tikzcd}$$

Now given a covering projection p, we may define an equivalence relation in the following manner: for $x, y \in E$, we say $x \sim y$ if and only if there exists some $T \in \Gamma_p$ such that $T(x) = y$. Now we may consider the quotient space E/Γ_p, i.e., the topology that makes the projection $\hat{p} : E \to E/\Gamma_p$ continuous.

Example 8.3. Returning to our example of S^1 and p defined in Example 8.1, we now ask the question, what is $\mathbb{R}/\Gamma_p$ with this equivalence relation?

Claim 8.4. $\Gamma_p = \{T_m \mid m \in \mathbb{Z}\}$.

To see this, proceed in the manner as we did when proving that p was a covering map. If $e^{2\pi i T(t)} = e^{2\pi i t}$ for all $t \in \mathbb{R}$, then rearranging, we see that $e^{2\pi i (T(t) - t)} = 1$ for all t. Hence $T(t) - t \in \mathbb{Z}$ for all t, but since $T(t) - t$ is continuous, we conclude that $T(t) - t$ is constant, and so there exists some $m \in \mathbb{Z}$ such that $T(t) = t + m$ for all t.

We may now also further say that $\Gamma_p \simeq \mathbb{Z}$. Hence we may identify $\mathbb{R}/\Gamma_p$ with $\mathbb{R}/\mathbb{Z}$, or with $[0, 1)$.

Notation. We will denote $\Gamma_p(x) \coloneqq \mathrm{orb}_{\Gamma_p}(x) = \{T(x) \mid T \in \Gamma_p\}$. In general, if we want to make some sort of identification for E/Γ_p with some set S, like we did in the previous example, we need $\#(\Gamma_p(x) \cap S) = 1$ for all x. Indeed, this is the case for $[0, 1)$.

8.2 Discontinuous Actions

Definition 8.5. Let E be a topological space, and let $\Gamma \leq \mathrm{Homeo}(E)$. We say that Γ **acts discontinuously** if for all $x \in E$, there exists some open neighborhood U_x of x such that if $T \in \Gamma$ and $T(U_x) \cap U_x \neq \varnothing$, then $T = \mathrm{id}$.

Remark 8.6. Some texts, like Hatcher, calls a discontinuous action as a covering space action.

One consequence of our definition is the following claim:

Claim 8.7. *If Γ acts discontinuously on E and $S_1, S_2 \in \Gamma$, and $S_1(U) \cap S_2(U) \neq \varnothing$ for some nonempty U, then $S_1 = S_2$.*

Proof of claim. Observe that, since S_1 and S_2 are homeomorphisms, $\varnothing \neq S_1(U) \cap S_2(U) = S_1\left(U \cap S_1^{-1} \circ S_2(U)\right)$. In particular, this implies that $U \cap S_1^{-1} \circ S_2(U) \neq \varnothing$. Since $S_1^{-1} \circ S_2 \in \Gamma$ and Γ acts discontinuously, we conclude $S_1^{-1} \circ S_2 = \mathrm{id}$. ■

Lemma 8.8. *Suppose Γ acts discontinuously on E. Then $p : E \to E/\Gamma$ is a covering projection, where the quotient is defined by $x \sim_\Gamma y$ if and only if there is some $T \in \Gamma$ such that $Tx = y$.*

Proof. Given $y \in E/\Gamma$, take $x \in E$ such that $p(x) = y$, and let U_x be the neighborhood that is granted by Definition 8.5. Then we claim that $p(U_x)$ is open. To see this, notice that $p^{-1}(p(U_x)) = \bigsqcup_{S \in \Gamma} S(U_x)$, and since each S is a homeomorphism, $S(U_x)$ is open, which implies that $p^{-1}(p(U_x))$ is open, as desired. Moreover, $p|_{S(U_x)} : S(U_x) \to p(U_x)$ is a homeomorphism, and thus p must be a covering map. ■

8.3 Universal Covering Spaces

We will state two key theorems, but we will not prove them.

Theorem 8.9. *Let E be a simply connected space, and let $p : E \to B$ be a covering projection. Assume further that B is semilocally simply connected. Then if Γ_p is the group of deck transformations, then Γ acts discontinuously and B is homeomorphic to E/Γ_p. In particular, the following diagram commutes:*

$$\begin{tikzcd}
E \arrow[r, "\mathrm{id}"] \arrow[d, "p"] & E \arrow[d, "\hat{p}"] \\
B \arrow[r, "h"] & E/\Gamma_p
\end{tikzcd}$$

where $h : B \to E/\Gamma_p$ denotes the homeomorphism and $\hat{p}$ is the projection map from E to E/Γ_p.

Theorem 8.10 (Existence and Universal Property of Universal Covers)**.** *Let B be a semilocally simply connected, locally path connected, connected space. Then there exists a simply connected and connected space E such that there is a covering projection $p : E \to B$. Moreover, if $q : X \to B$ is any other covering projection, with X connected, then there exists a unique continuous map $r : E \to X$ such that the following diagram commutes:*

$$\begin{tikzcd}
E \arrow[r, "r"] \arrow[d, "p"] & X \arrow[dl, "q"] \\
B &
\end{tikzcd}$$

The space E is unique up to homeomorphism.

Definition 8.11. The space E in the previous theorem is called a **universal cover**, and we will denote a universal covering space of B by $\widetilde{B}$.

Interpretation. We can interpret the previous two theorems in the following way: by Theorem 8.10 we know that for any semilocally simply connected space B there is a universal cover $\widetilde{B}$, and Theorem 8.9 tells us that $B \approx \widetilde{B}/\Gamma_p$. Moreover, in a sense, Γ_p is the fundamental group.

Proposition 8.12. *Let $p : E \to B$ be a covering projection and assume further that E is simply connected and and path connected. Suppose $b_0 \in B$, and $e_0 \in E$ such that $p(e_0) = b_0$. Then $\pi_1(B, b_0) \simeq \Gamma_p$.*

Proof. We will show that there is a one-to-one correspondence between the two groups. Let $T \in \Gamma_p$. Let $\tilde{\alpha}$ be a curve in E connecting e_0 and $T(e_0)$, and let $\alpha := p \circ \tilde{\alpha}$. Then observe that $\alpha(0) = p(e_0) = b_0 = \alpha(1) = p(T(e_0))$. So $\alpha \in \pi_1(B, b_0)$. Thus this gives us a way to assign a loop in $\pi_1(B, b_0)$ for every $T \in \Gamma_p$. To see that this does not depend on our choice of curve $\tilde{\alpha}$, ■

9 2/1/23

Today we will introduce the Seifert-van Kampen theorem. Relevant reading: Hatcher Chapter 1.2, Weintraub Section 2.3.

9.1 Free Group Products

Definition 9.1 (Free Group Products)**.** Given two groups G_1 and G_2, we denote $G_1 * G_2$ to be the **free product** of G_1 and G_2, which is the coproduct of the groups G_1 and G_2 in the category of groups. That is, there are injective homomorphisms $\iota_1 : G_1 \hookrightarrow G_1 * G_2$ and $\iota_2 : G_2 \hookrightarrow G_1 * G_2$ and it satisfies the following universal property:

If G is any group and $f_1 : G_1 \to G$ and $f_2 : G_2 \to G$ are homomorphisms, then there exists a unique homomorphism $f : G_1 * G_2$ such that the following diagram commutes:

$$\begin{tikzcd}
G_1 \arrow[dr, hook, "\iota_1"] \arrow[drr, bend left, "f_1"] & & \\
& G_1 * G_2 \arrow[r, "f"] & G \\
G_2 \arrow[ur, hook, "\iota_2"] \arrow[urr, bend right, "f_2"'] & &
\end{tikzcd}$$

Example 9.2. $\mathbb{Z} * \mathbb{Z} = F_2$ the free group on two generators: alternatively, we can write F_2 to be the set of all finite words on two letters a, b.

Remark 9.3. In general, if $a_i \in G$, $b_i \in G$, we can write any element of $G_1 * G_2$ as $a_1 b_1 a_2 b_2 \cdots a_k b_k$.

Definition 9.4. Given a group G, and $A \subseteq G$ (not necessarily a subgroup), the **normal subgroup generated by** A is defined by $N(A) = \bigcap N$, where the intersection runs over all normal subgroups containing A: that is, it is the smallest normal subgroup of G containing A.

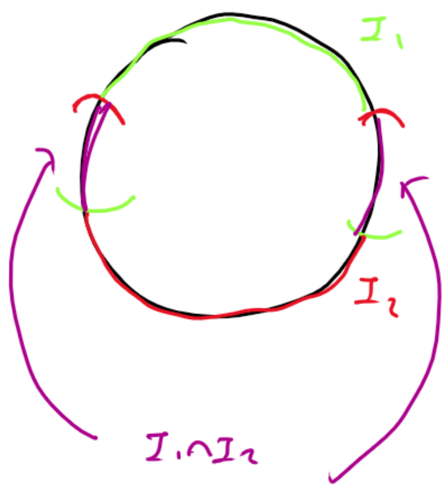
9.2 The Seifert-van Kampen Theorem and Applications

Theorem 9.5 (Seifert-van Kampen)**.** *Let X be a path-connected space, and assume $X = U_1 \cap U_2$, where both U_1 and U_2 are open and path-connected. Let $x_0 \in U_1 \cap U_2$ and assume that $U_1 \cap U_2$ is* *also path-connected. Then* $\pi(X, x_0) \simeq (\pi_1(U_1, x_0) * \pi_1(U_2, x_0))/N(A)$*, where if* $(\iota_1)_*$ *and* $(\iota_2)_*$ *are the homomorphisms induced by the inclusion map* $\iota_i : U_1 \cap U_2 \to U_i$*, we have*

$$A = \{(\iota_1)_*(g^{-1}) * (\iota_2)_*(g) \mid g \in \pi_1(U_1 \cap U_2, x_0)\}.$$

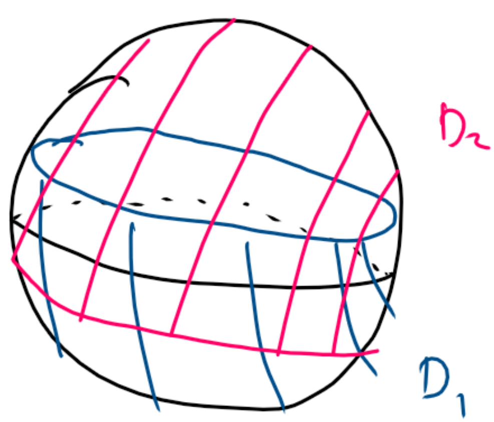
We will not prove the Seifert-van Kampen theorem today, but we will see some applications of it.

Example 9.6 (An Incorrect Application)**.** Consider S^1 as the union of two open intervals I_1 and I_2 as in the figure. But $\pi_1(S^1)$ cannot be a quotient of the free product $\pi_1(I_1) * \pi_1(I_2)$ because the two factors are both trivial, but we already know that $\pi_1(S^1) \simeq \mathbb{Z}$. The error was in that the hypothesis $U_1 \cap U_2$ is not path-connected.



Proposition 9.7 (Fundamental Group of S^n)**.** *For* $n \geq 2$*,* S^n *is simply connected.*

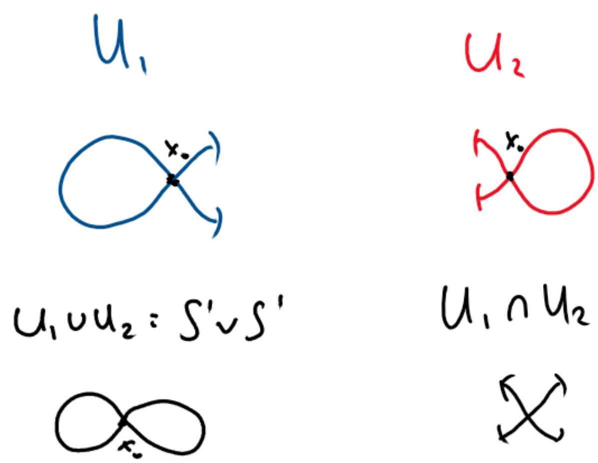
Proof. We will use the ideas from Example 9.6. Let $x_0 \in S^n$; by rotating the sphere, we may assume that x_0 is on the equator. Write $S^n = D_1 \cup D_2$, where D_1 are D_2 are the open sets in the figure below.



Note that D_1 and D_2 are both contractible, and so must have trivial fundamental group. Moreover, $D_1 \cap D_2 \approx S^{n-1} \times I$, which is also path-connected. Then applying the Seifert-van Kampen theorem, $\pi_1(S^n, x_0)$ must be a quotient of $\pi_1(D_1, x_0) * \pi_1(D_2, x_0) = \{0\}$. Hence S^n is is simply connected. ■

Remark 9.8. We could have use the stereographic projection to map the sphere with the poles removed onto $\mathbb{R}^n$ in the previous proof.

Example 9.9 (The Figure 8)**.** Consider $E \coloneqq S^1 \vee S^1$, or the "figure 8," joined together at the point x_0. Let U_1 and U_2 be as in the figure below, so that $U_1 \cup U_2 = E$, and $E_1 \cap E_2$ is the cross in the middle.

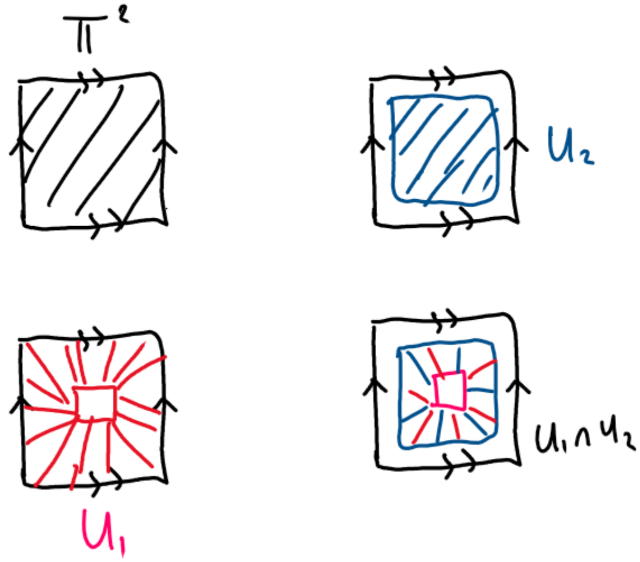


First observe that all our spaces are path-connected and so the hypotheses of the Seifert-van Kampen theorem are satisfied. Next, $U_1 \cap U_2$ is contractible, which implies that $\pi_1(U_1 \cap U_2, x_0) = \{0\}$. Finally, observe that $U_1 \approx U_2 \approx S^1$, which implies that $\pi_1(U_1, x_0) \simeq \pi_1(U_2, x_0) \simeq \mathbb{Z}$. Appealing to the Seifert-van Kampen theorem, we conclude that $\pi(E, x_0) = \mathbb{Z} * \mathbb{Z}$.

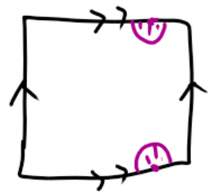
Exercise 9.10. Apply induction to the previous example to conclude that the fundamental group of the n-petal rose is F_n, the free group on n elements.

Exercise 9.11. Let X and Y be topological spaces, and suppose $X \vee Y$ be locally contractible and/or semilocally simply connected at the attaching point x_0. Show that $\pi_1(X \vee Y, x_0) \simeq \pi(X, x_0) * \pi(Y, x_0)$.

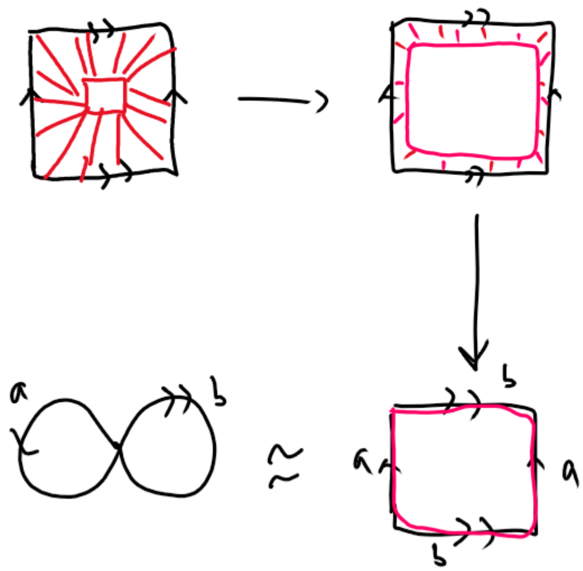
Example 9.12 (The Torus)**.** Let $\mathbb{T}^2$ denote the torus $\mathbb{T}^2 = S^1 \times S^1$. We have already noted that $\pi_1(\mathbb{T}^2) \simeq \pi_1(S^1) \times \pi_1(S^1) \simeq \mathbb{Z} \times \mathbb{Z}$. Now we will use the Seifert-van Kampen's theorem to prove this. We have shown that the torus may be considered as the quotient space of the square where the opposite edges are identified. Now let U_1 and U_2 be as in the diagram, where U_1 is the "outer" part of the square, and U_2 the "inner" part.



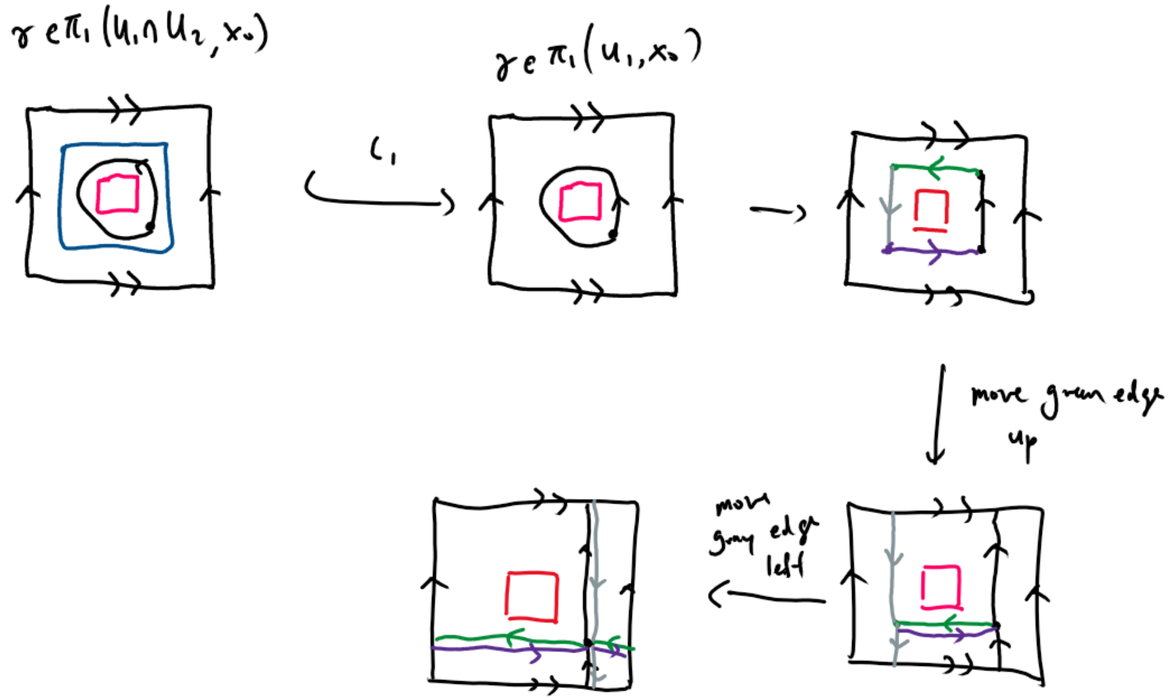
To see that U_1 is open, note that on the edges, any ball would "bleed over" to the opposite edge, as in the following figure:



It is now easy to see that U_1 and U_2 are both open, $U_1 \cup U_2 = \mathbb{T}^2$, and U_1, U_2, and $U_1 \cap U_2$ are all path-connected. The hypotheses of the Seifert-van Kampen theorem are now satisfied. Fix $x_0 \in U_1 \cap U_2$. First observe that U_2 is contractible, and so $\pi_1(U_2, x_0) = \{0\}$. On the other hand, we see that U_1 deformation retracts onto the boundary of the square, and then identified with the figure 8 in the following manner:



Since deformation retracts induce an isomorphism of fundamental groups, we have from Example 9.9 $\pi_1(U_1, x_0) \simeq \pi_1(S^1 \vee S^1) \simeq \mathbb{Z} * \mathbb{Z}$. Now $U_1 \cap U_2$ is the annulus, which deformation retracts onto the circle S^1, so its fundamental group is the free group on one generator, the loop going around the annulus once counterclockwise. The following figure shows its image under $(\iota_1)_*$:



Now after the deformation retract, we see that in the image this loop is exactly the commutator $aba^{-1}b^{-1}$. But this was the image of the generator, and so we conclude that $N(A)$ (in the statement of the theorem) must be the commutator subgroup inside $\pi_1(U_1, x_0)$. Therefore $\pi(\mathbb{T}^2, x_0) \simeq \mathbb{Z} * \mathbb{Z}/\left\langle aba^{-1}b^{-1}\right\rangle = \mathbb{Z} \times \mathbb{Z}$.

Remark 9.13. The above proof can be adapted to compute $\pi_1(\mathbb{T}^n, x_0)$ with induction.

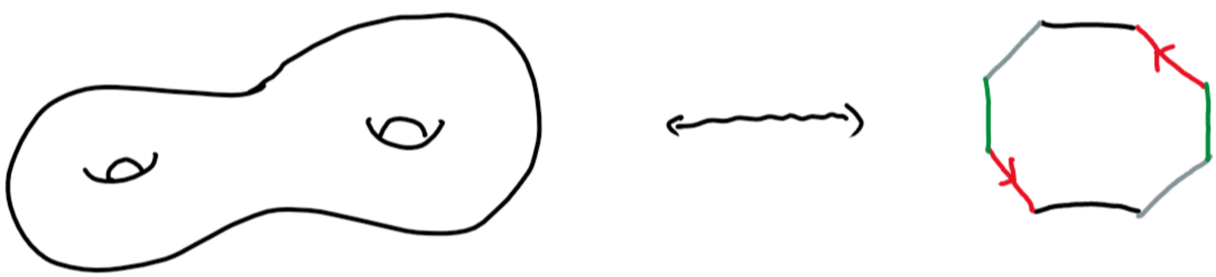
10 2/3/23

Today we will continue with examples of van Kampen's Theorem.

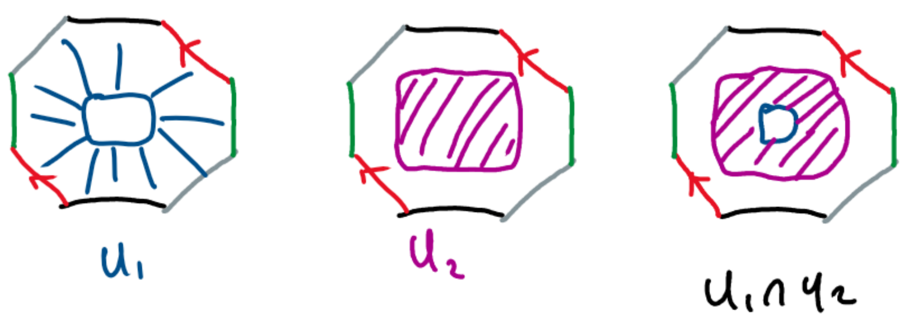
10.1 The Genus 2 Surface

Recall when we computed the fundamental group of the torus via Seifert-van Kampen theorem, we used the quotient of a square that is homeomorphic to the torus.

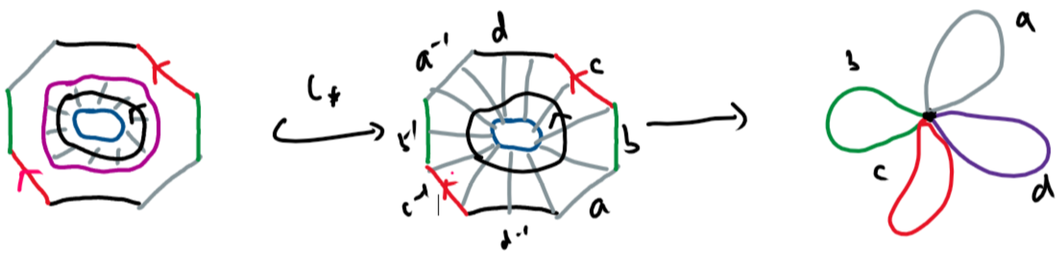
Method 1. For the genus 2 surface S, we will consider the quotient of an octagon as follows:



Then just as we did for the torus, decompose the octagon into following pieces:

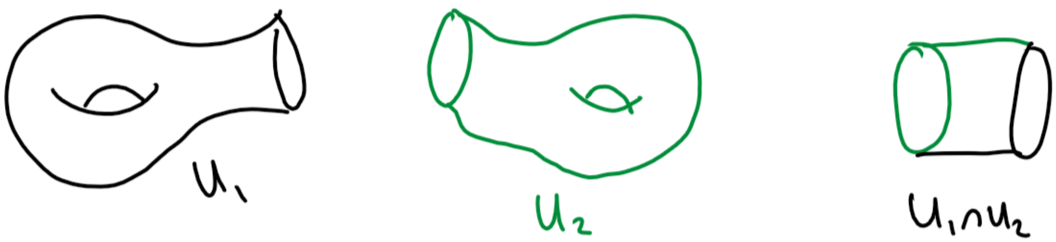


First note that all these sets are path-connected, so the hypotheses of the van Kampen theorem are satisfied. Then notice that U_1 deformation retracts onto the boundary, which is homeomorphic to the 4-petal rose; thus $\pi_1(U_1) \simeq F_4$, the free group on four elements. Moreover, U_2 is contractible and so has trivial fundamental group. Finally, $U_1 \cap U_2$ is the annulus, which deformation retracts onto S^1, so has fundamental group the free group on one generator. Then by the Seifert-van Kampen theorem, we have that $\pi_1(S) \simeq \pi_1(U_1)/N(\iota_1(g) \mid g \in \pi_1(U_1 \cap U_2))$. Let g be a loop in $U_1 \cap U_2$, like in the diagram below. Then considered as a loop in U_1 and its image in the deformation retract, its image is $abcda^{-1}b^{-1}c^{-1}d^{-1}$.

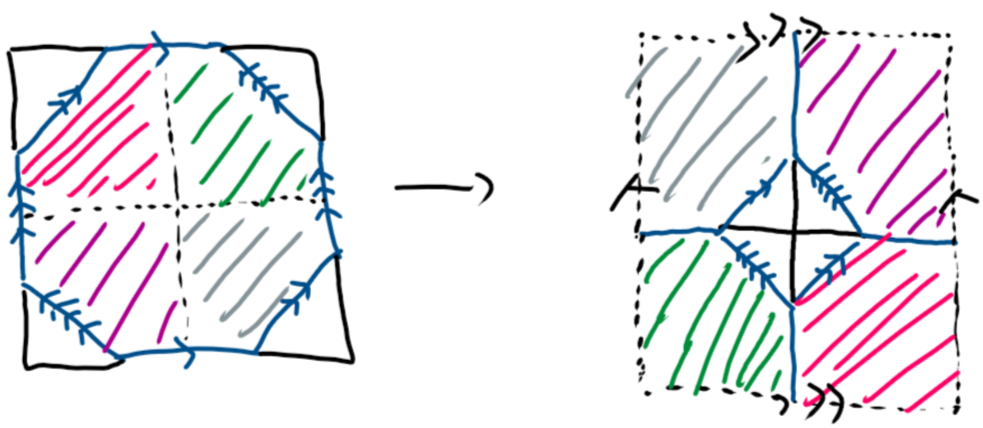


Therefore $\pi_1(S) \simeq F_4/\left\langle abcda^{-1}b^{-1}c^{-1}d^{-1}\right\rangle \simeq \langle a,b,c,d \mid abcd = dcba\rangle$.

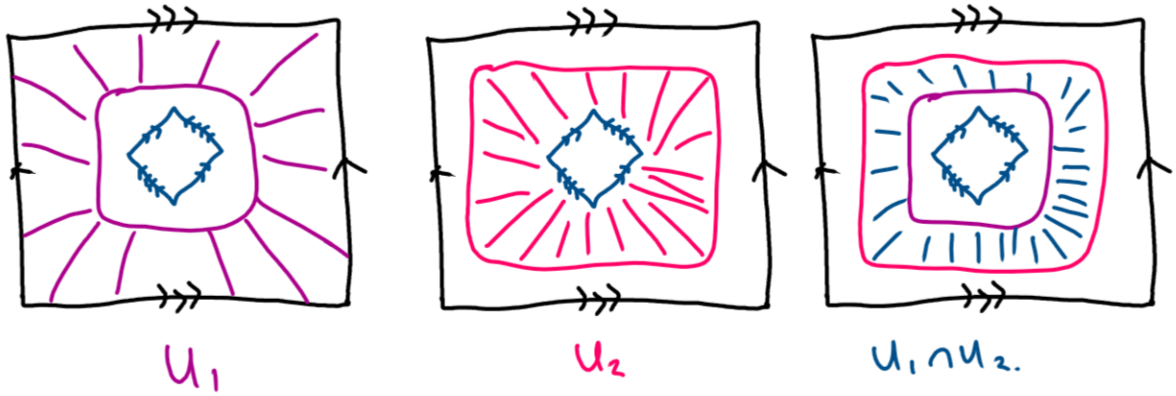
Method 2. The idea will be to decompose the surface into two parts, just like below:



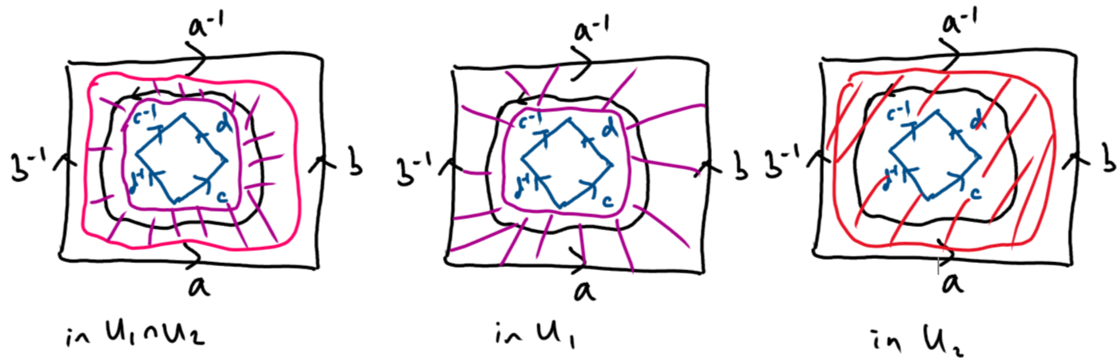
Consider the octagon again, but this time as a subspace of the square below after cutting and pasting.



Just as above, consider U_1 and U_2 defined as in the figure below.



Then both U_1 and U_2 deformation retracts onto the figure 8, and $U_1 \cap U_2$ is an annulus which deformation retracts onto S^1. Hence $\pi_1(U_1) \simeq \pi_1(U_2) \simeq F_2$ the free group on two generators, and $\pi_1(U_1 \cap U_2) \simeq \mathbb{Z}$. Now consider the single loop $g \in \pi_1(U_1 \cap U_2)$ given by the generator: that is, the loop that goes around once in the annulus. Then considered as a loop in U_1 and U_2 respectively, the diagram below shows that in U_1 it deformation retracts onto the loop $aba^{-1}b^{-1}$, and in U_2 it deformation retracts onto the loop $cdc^{-1}d^{-1}$.

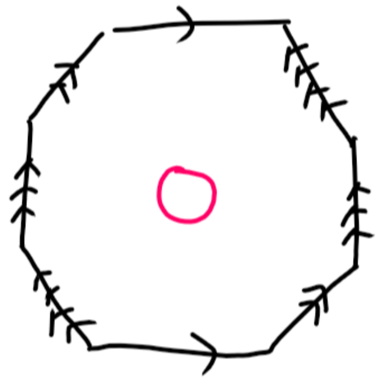


Therefore $(\iota_1)_*(g) = aba^{-1}b^{-1}$ and $(\iota_2)_*(g) = cdc^{-1}d^{-1}$. Thus applying van Kampen's theorem, we conclude that $\pi_1(S) \simeq F_2 * F_2 / N\left(aba^{-1}b^{-1}\,(cdc^{-1}d^{-1})^{-1}\right) \simeq \langle a, b, c, d \mid [a, b] = [c, d] \rangle$.

Remark 10.1. We can compute the fundamental group of a genus g surface by induction.

Corollary 10.2. *The fundamental group of a genus 2 surface with a point deleted is the free group on 4 elements. In general, the fundamental group of a genus g surface, with $g \geq 2$, is the free group on $2g$ elements.*

Proof. The genus 2 surface with a point deleted can be identified with the quotient space of the octagon in Method 1 with a neighborhood deleted in the interior, as in the diagram. Then this deformation retracts onto the boundary, which is homeomorphic to the 4-petal rose.

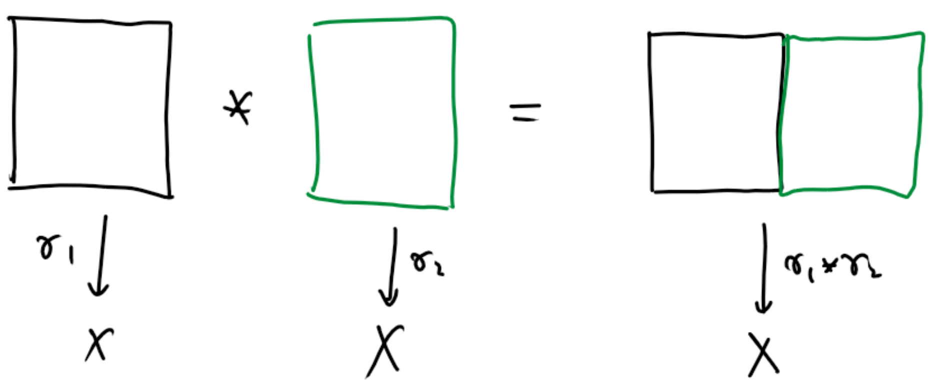


The general case is handled by induction. ■

10.2 Higher Homotopy Groups

Recall that the fundamental group $\pi_1(X, x_0)$ was all about maps of the form $\gamma : (S^1, 1) \to (X, x_0)$, or equivalently maps of the form $([0,1], \{0,1\}) \to (X, x_0)$. Now similarly, we define the **higher homotopy groups** in the following manner: $\pi_n(X, x_0) := [(S^n, 1); (x, x_0)]$ where the bracket denotes the homotopy classes of maps. Equivalently, we may define $\pi_n(X, x_0)$ as $[(I_n, \partial I_n), (X, x_0)]$ where $I_n = [0,1]^n$.

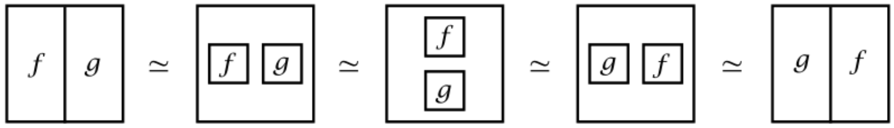
The group operation is defined as follows: as an example, we will use π_2, and analogize.



Because we stipulate that the boundary gets mapped to x_0, the multiplication is well-defined by the pasting lemma. The identity element is the constant map mapping to x_0. Another way of writing the multiplication is as follows: if $\gamma_1, \gamma_2 : (t_1, \ldots, t_n) \to X$, then we may write their product to be

$$(\gamma_1 * \gamma_2)(t_1, \ldots, t_n) = \begin{cases} \gamma_1(2t_1, t_2, \ldots, t_n), & t \in [0, 1/2] \\ \gamma_2(2t_1 - 1, t_2, \ldots, t_n), & t \in [1/2, 1]. \end{cases}$$

Next, the following figure from Hatcher illustrates the following lemma:

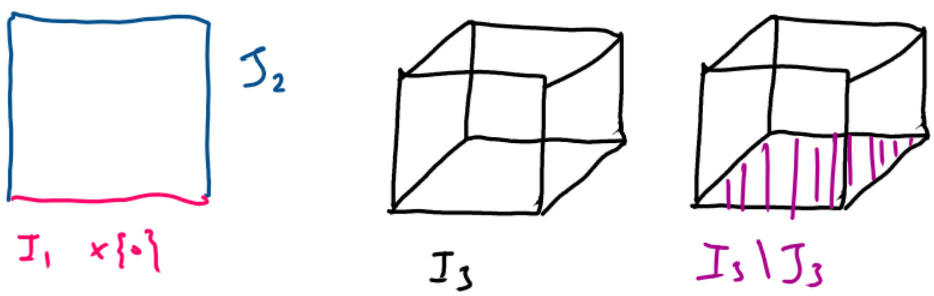


Lemma 10.3. *$\pi_n(X, x_0)$ is abelian for $n \geq 2$.*

11 2/6/23

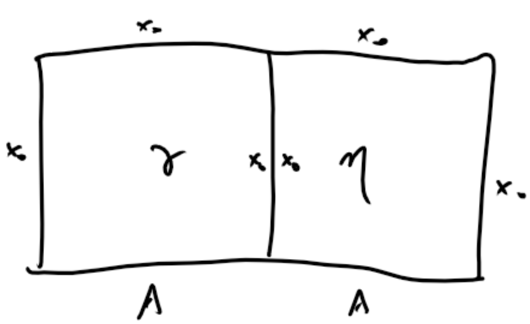
11.1 Relative Homotopy Groups

Definition 11.1. As we did last time, define $I_n = [0,1]^n$, ∂I_n the boundary of I_n, and let $J_n := \partial I_n \setminus (I_{n-1} \times \{0\})$, as in the following diagram:



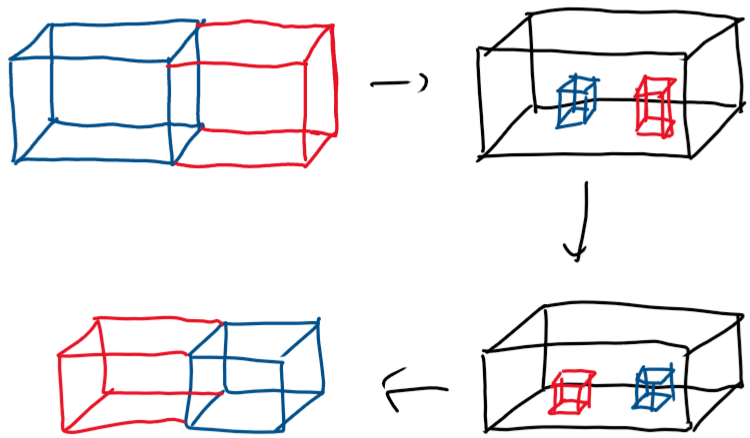
Then we define the **relative homotopy groups** as following: $\pi_n(X, A, x_0) := [(I_n, \partial I_n, J_n); (X, A, x_0)]$.

Because the elements of the homotopy groups are equivalence classes, we will write what it means for two elements to be equivalent. We say that for $\gamma, \eta \in \pi_n(X, A, x_0)$, $\gamma \sim \eta$ if and only if there exists $F : (I_n, \partial I_n, J_n) \times [0,1] = (I_n \times [0,1], \partial I_n \times [0,1], J_n \times [0,1]) \to (X, A, x_0)$ such that $f_0 = F(\cdot, 0) = \gamma$ and $f_1 = F(\cdot, 1) = \eta$. Similar as was done in homotopy groups, the product $\gamma \cdot \eta$ is defined in the following way:

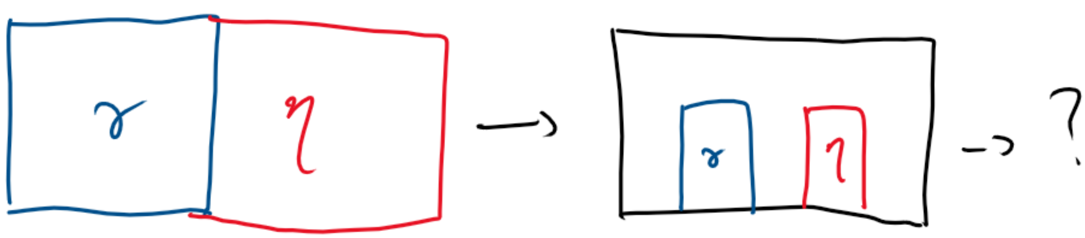


Lemma 11.2. *This product makes $\pi_n(X, A, x_0)$ into a group, and for $n \geq 3$, is abelian.*

Proof. To see that $\pi_n(X, A, x_0)$ is obvious. To see that it is obvious, consider the following figure:



It is important that the two cubes inside the big cube in the figure above do not have their bases taken off of $\partial I_n \setminus J_n$. The fact that this is not possible in dimension two illustrates why $\pi_n(X, A, x_0)$ is not abelian:



■

11.2 Exact Sequences

Definition 11.3. Given groups $G_1, G_2, \ldots$ and homomorphisms $L_n : G_n \to G_{n+1}$, we say that the sequence

$$G_1 \xrightarrow{L_1} G_2 \xrightarrow{L_2} \cdots \to G_n \xrightarrow{L_n} \cdots \to 0$$

is **exact** if $\ker L_{n+1} = \operatorname{im} L_n$ for each n.

Observe that if we have a sequence of groups, $L_{n+1} \circ L_n \equiv 0$ if and only if $\ker L_{n+1} \supseteq \operatorname{im} L_n$.

Example 11.4. Consider the sequence $0 \xrightarrow{L_1} G \xrightarrow{L_2} 0$. Certainly $L_2 \circ L_1 \equiv 0$. But $\operatorname{im} L_1 = \{0\}$ since L_1 is a homomorphism. Thus this sequence must be exact if and only if G is trivial.

Example 11.5. Consider the sequence $0 \xrightarrow{L_1} G \xrightarrow{L_2} H \xrightarrow{L_3} 0$, and suppose that it is exact. Then $\operatorname{im} L_1 = \{0\} = \ker L_2$, which implies that L_2 is injective. On the other hand, $H = \ker L_3 = \operatorname{im} L_2$, and so L_2 is surjective. Thus L_2 is a group isomorphism.

Example 11.6. Consider the exact sequence $0 \xrightarrow{L_1} N \xrightarrow{\iota} G \xrightarrow{\pi} H \xrightarrow{L_3} 0$. Since $H = \ker L_3 = \operatorname{im} \pi$, we have that π is surjective. On the other hand, $\operatorname{im} L_1 = \ker \iota = \{0\}$ and so ι is injective. Thus by the first isomorphism theorem, $G/\ker \pi = G/\iota(N) \simeq H$. Identifying N with its image under ι, we conclude that $G/N \simeq H$.

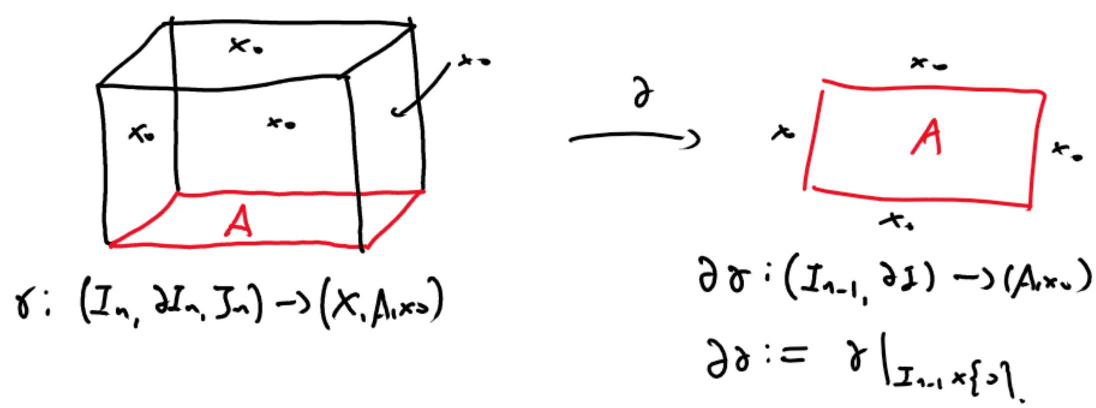
12 2/8/23

12.1 Long Exact Sequences of Relative Homotopy Groups

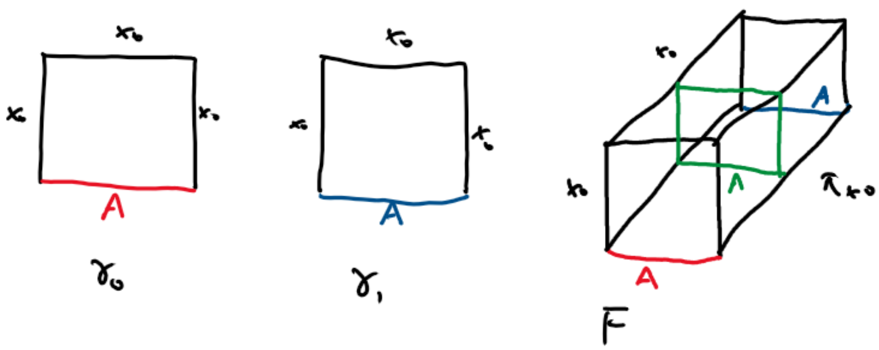
Theorem 12.1 (Long Exact Sequences of Relative Homotopy Groups)**.** *Let $J : (X, x_0, x_0) \hookrightarrow (X, A, x_0)$ be the inclusion. Then there is a long exact sequence*

$$\to \pi_n(A, x_0) \xrightarrow{\iota_\#} \pi_n(X, x_0) \xrightarrow{J_\#} \pi_n(X, A, x_0) \xrightarrow{\partial} \pi_{n-1}(A, x_0) \to \cdots \to \pi_0(X, A, x_0) \to 0.$$

The boundary map $\partial : \pi_n(X, A, x_0) \to \pi_{n-1}(A, x_0)$ is defined in the following manner: for $\gamma \in \pi_n(X, A, x_0)$, say $\gamma : (I_n, \partial I_n, J_n) \to (X, A, x_0)$, the restriction $\gamma|_{I_{n-1}\times\{0\}}$ can be regarded as a map $(I_{n-1}, \partial I) \to (A, x_0)$. Then $\partial\gamma$ is precisely this restriction.

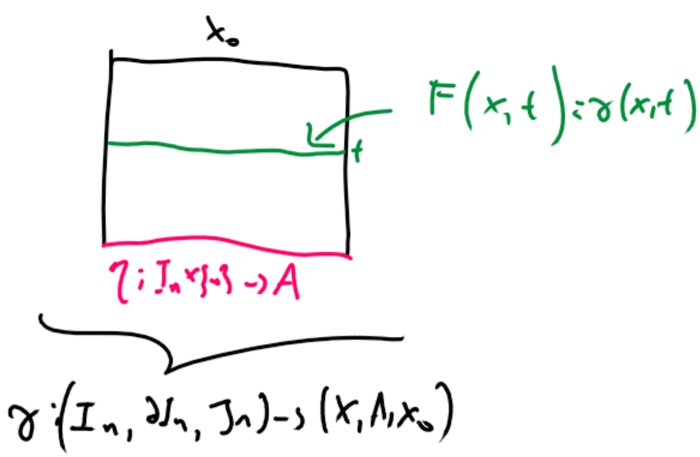


Proof. First we check that ∂ is well-defined. To this end suppose $\gamma_0 \sim \gamma_1$. Then there exists some $F : (I_n, \partial I_n, J_n) \times [0,1] \to (X, A, x_0)$ such that $F(\cdot, 0) = f_0 = \gamma_0$ and $F(\cdot, 1) = f_1 = \gamma_1$ (a 2-dimensional schematic diagram is below).



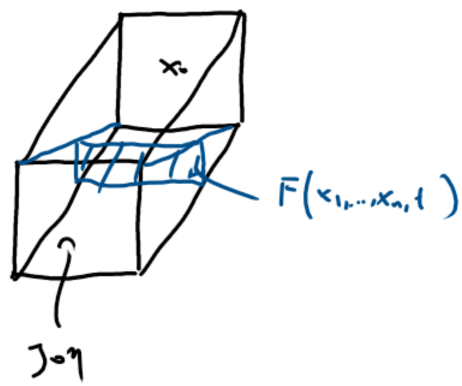
Then restricting F to $I_{n-1} \times \{0\} \approx I_{n-1}$ gives a homotopy $F|_{I_{n-1}\times\{0\}\times[0,1]} : I_{n-1} \times [0,1] \to A$. Then this gives a homotopy $\partial(\gamma_1) = \gamma_1|_{I_{n-1}\times\{0\}}$ to $\partial(\gamma_2) = \gamma_2|_{I_{n-1}\times\{0\}}$. Thus $\partial(\gamma_1) \sim \partial(\gamma_2)$ in homotopy classes.

Now we check exactness. First, we will show that $\ker \iota_\# = \operatorname{im} \partial$. To prove one direction, suppose that $\eta \in \operatorname{im} \partial$. Then there exists some $\gamma \in \pi_{n+1}(X, A, x_0)$, $\gamma : (I_{n+1}, \partial I_{n+1}, J_{n+1}) \to (X, A, x_0)$ such that $\partial\gamma = \gamma|_{I_n\times\{0\}} = \eta$. In order to prove that $\eta \in \ker \iota_\#$, we must show that $\iota_\#(\eta) \sim 0$ in $\pi_n(X, x_0)$, that is, there exists some homotopy $F : I_n \times [0,1] \to X$ such that $f_0 = \eta$ and $f_1 = x_0$. Define $F(x,t) = \gamma(x,t)$, regarded as a map from $I_{n+1} = I_n \times [0,1]$ to X. Indeed, $F(x,0) = \gamma(x,0) = \gamma|_{I_n\times\{0\}}(x) = \eta(x)$. On the other hand, $F(x,1) = \gamma(x,1) = x_0$ since $\gamma \in \pi_{n+1}(X, A, x_0)$ (see the figure below). Hence $\iota_\#(\eta)$ is homotopic to the constant map in $\pi_n(X, x_0)$, so $\eta \in \ker \iota_\#$.



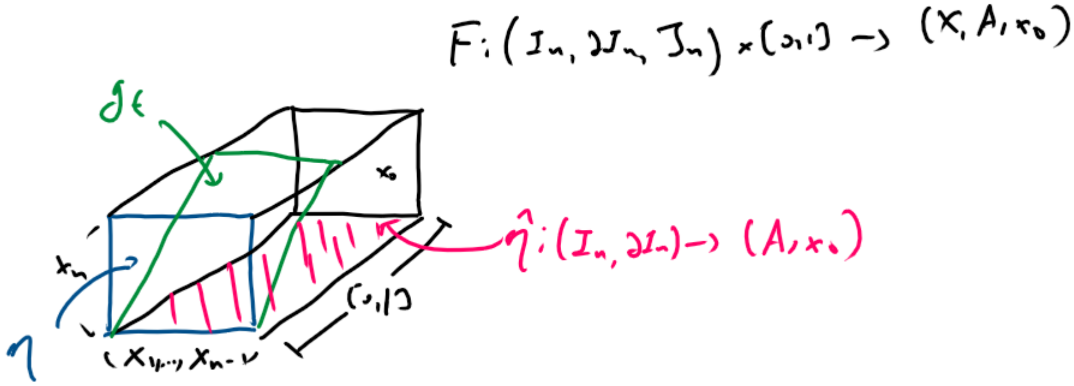
Conversely, suppose that $\eta \in \ker \iota_\#$. Then $\iota_\#(\eta)$ is homotopic to the constant map in $\pi_n(X, x_0)$, that is, there exists some $F : I_n \times [0,1] \to X$ such that $f_0(x) = \iota_\#(\eta)$ and $f_1(x) \equiv x_0$. Then proceeding as in the other direction, defining $\gamma : (I_{n+1}, \partial I_{n+1}, J_{n+1}) \to (X, A, x_0)$, with the identification $I_{n+1} = I_n \times [0,1] \to X$ via $\gamma(x,t) = F(x,t)$ we have that clearly $\partial\gamma = \eta$.

For the next part, we will show that $\operatorname{im}\iota_\# = \ker J_\#$. Let $\eta \in \operatorname{im}\iota_\#$. Then there exists some $\widetilde{\eta} \in \pi_n(A, x_0)$, $\widetilde{\eta} : (I_n, \partial I_n) \to (A, x_0)$ such that $\eta = \iota \circ \widetilde{\eta}$. Now consider the map $J \circ \eta : (I_n, \partial I_n, J_n) \to (X, A, x_0)$, which obtained by changing the domain and codomain: note that $\eta = \iota \circ \widetilde{\eta}$, and so the image of η is completely contained in A, and moreover for any $x \in \partial I_n$, we have $\eta(x) = \widetilde{\eta}(x) = x_0$ since $\widetilde{\eta} \in \pi_n(A, x_0)$. Thus $J \circ \eta$ as a map from $(I_n, \partial I_n, J_n) \to (X, A, x_0)$ makes sense. To show that $\eta \in \ker J_\#$, we will show that there is a homotopy $F : (I_n, \partial I_n, J_n) \times [0,1] \to (X, A, x_0)$ such that $f_0 = J \circ \eta$ and $f_1 \equiv x_0$. Next, consider $F(x_1, \ldots, x_{n-1}, x_n, t) \coloneqq (J \circ \eta)(x_1, \ldots, x_{n-1}, (1-t)x_n)$.



Indeed, $f_0(x_1, \ldots, x_n) = F(x_1, \ldots, x_n, 0) = J \circ \eta$, and $f_1(x_1, \ldots, x_n) = F(x_1, \ldots, x_n, 1) = (J \circ \eta)(x_1, \ldots, x_{n-1}, 0) = \widetilde{\eta}(x_1, \ldots, x_{n-1}, 0) = x_0$. The schematic figure above shows that F indeed is the map of the desired form. Hence F is the desired homotopy.

Conversely, suppose $\eta \in \ker J_\#$. Then $\eta : (I_n, \partial I_n) \to (X, x_0)$ and $J \circ \eta : (I_n, \partial I_n, J_n) \to (X, A, x_0)$ is homotopically trivial. Thus there exists some $F : (I_n, \partial I_n, J_n) \times [0,1] \to (X, A, x_0)$ such that $f_0 = J \circ \eta$ and $f_1 \equiv x_0$. Define $\widetilde{\eta} : (I_n, \partial I_n) \to (A, x_0)$ by $\widetilde{\eta}(x_1, \ldots, x_n) = F(x_1, \ldots, x_{n-1}, 0, x_n)$. Clearly by definition of F, $\widetilde{\eta}$ takes image in A, and its boundary takes value in x_0 (see figure below for an illustration). We claim that $\iota \circ \widetilde{\eta} \sim \eta$ in $\pi_n(X, x_0)$. Indeed, let $G : (I_n, \partial I_n) \times [0,1] \to (X, x_0)$ via $G(x_1, \ldots, x_n, t) = F(x_1, \ldots, x_{n-1}, (1-t)x_n, tx_n)$. Then $g_0(x_1, \ldots, x_n) = F(x_1, \ldots, x_n, 0) = (J \circ \eta)(x_1, \ldots, x_n) = \eta(x_1, \ldots, x_n)$ and $g_1(x_1, \ldots, x_n) = F(x_1, \ldots, 0, x_n) = \widetilde{\eta}(x_1, \ldots, x_n)$. Pictorially, the green slanted rectangle depicts g_t during a time between 0 and 1, in the middle of the homotopy. Thus $\iota \circ \widetilde{\iota} \sim \eta$ in $\pi_n(X, x_0)$, as desired.



The remainder of checking exactness is straightforward and will be omitted. ■

13 2/10/23

13.1 Serre Fibrations and Hurewicz Fibrations

First, recall that $p : E \to B$ has the homotopy lifting property with respect to X if for all homotopies $F : X \times I \to B$, and $h : X \to E$ such that $(p \circ h)(x) = f_0(x)$, there exists a unique $\widetilde{F} : X \times I \to E$ such that $p \circ \widetilde{F} = F$ and $\widetilde{f}_0 = \widetilde{F}(x, 0) = h(x)$ for all x. That is, the following diagram commutes:

$$\begin{tikzcd} X \arrow[r, "h"] \arrow[d, hook, "\iota"'] & E \arrow[d, "p"] \\ X \times I \arrow[r, "F"'] \arrow[ur, dashed, "\widetilde{F}"] & B \end{tikzcd}$$

Definition 13.1. A continuous map $p : E \to B$ is called a **Serre fibration** if it has the homotopy lifting property with respect to I_n for all n. We say that p is a **Hurewicz Fibration** if instead of I_n, it has the homotopy lifting property for all spaces X.

Lemma 13.2. *Let $p : E \to B$ be a continuous map, $\mathcal{U} = \{U_i\}_{i \in I}$ be an open cover of B, and let $p_i := p|_{U_i}$, that is, $p_i : p^{-1}(U_i) \to U_i$. If p_i has the homotopy lifting property for each i, then p has the homotopy lifting property for E.*

Definition 13.3. A map $p : E \to B$ is called a **fibration** with fiber F if there exists an open cover $\mathcal{U} = \{U_i\}_{i \in I}$ of B and a family $\{H_i : F \times U_i \to p^{-1}(U_i)\}_{i \in I}$ of homeomorphisms such that for all $i \in I$ and $x \in U_i$, $p(H_i(f, x)) = x$ for all $x \in U_i$, for all i. Then $p \circ H_i : F \times U_i \to U_i$ is the projection onto the second coordinate.

Example 13.4. The tangent bundle with the natural projection is a fibration.

Theorem 13.5 (Long Exact Sequences of Fibrations)**.** *Given a Serre fibration $p : E \to B$, $p(e_0) = x_0$, $p^{-1}(x_0) = F$ and $e_0 \in F$, there is a long exact sequence*

$$\cdots \to \pi_n(F, e_0) \xrightarrow{\iota_\#} \pi_n(E, e_0) \xrightarrow{p_\#} \pi_n(B, b_0) \xrightarrow{\partial} \pi_{n-1}(F, E_0) \to \cdots,$$

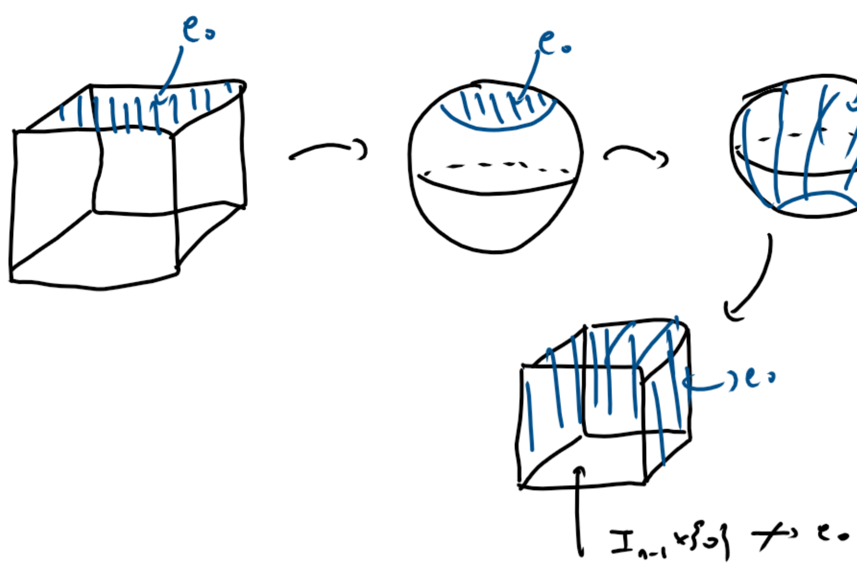
where the boundary map is defined in the following manner: if $\gamma : (I_n, \partial I_n) \to (B, b_0) \in \pi_n(B, b_0)$, then define $H : I_{n-1} \times I \to B$ by viewing γ as a homotopy. That is, $H(x_1, \ldots, x_{n-1}, t) = \gamma(x_1, \ldots, x_{n-1}, t)$. Then notice that $h_1 \equiv b_0$. Then since p is a Serre fibration, p satisfies the homotopy lifting property, so H lifts to a unique homotopy $\widetilde{H} : X \times I \to E$ such that $p \circ \widetilde{H} = H$ and $\widetilde{h}_1 \equiv e_0$. Then we define $\partial\gamma = \widetilde{H}|_{I_{n-1}\times\{0\}} : I_{n-1} \to E$. Then $p(\partial\gamma(x)) = p(\widetilde{H}(x, 0)) = H(x_1, \ldots, x_{n-1}, 0) = b_0$. Hence $\partial\gamma(x) \in p^{-1}(b_0) = F$, so $\partial\gamma \in \pi_{n-1}(F, e_0)$.

Proof Idea. We will show that there is a natural isomorphism between $\pi_n(B, x_0)$ and $\pi_n(X, A, x_0)$, and then appeal to Theorem 12.1, so that we can fit this long exact sequence into the previous one.

Proof. Consider $\pi_n(E, F, e_0) \xrightarrow{p_\#} \pi_n(B, b_0, b_0) \simeq \pi_n(B, b_0)$. We claim that $p_\#$ is an isomorphism, which will allow $\pi_n(B, b_0)$ to naturally fit into the long exact sequence for relative homotopy groups. To see this we will construct an inverse for $p_\#$. Take $\gamma : (I_n, \partial I_n, J_n) \to (E, F, e_0) \in \pi_n(E, F, e_0)$. Then this fits into the diagram

$$\begin{tikzcd}
(I_n, \partial I_n, J_n) \arrow[r, "\gamma"] \arrow[dr, "p\circ\gamma"'] & (E, F, e_0) \arrow[d, "p"] \\
& (B, b_0, b_0)
\end{tikzcd}$$

Take H and $\widetilde{H}$ as in the statement of Theorem 13.5. Then $\widetilde{H}$ is a homotopy $I_{n-1} \times [0, 1] \to E$ such that $\widetilde{h}_1 = \widetilde{H}(\cdot, 1) \equiv e_0$, and moreover $\widetilde{H}(\partial I_n) \subseteq F = p^{-1}(b_0)$. Now consider the following deformation:



This gives us a new map $\hat{H} : I_{n-1} \times [0, 1] \to E$ homotopic to $\widetilde{H}$. Then this induces a map hat : $\pi_n(B, b_0, b_0) \to \pi_n(E, F, e_0)$, $\gamma \mapsto \hat{H}$. It is (presumably) straightforward to check that hat is the inverse of $p_\#$, which concludes the proof. ∎

Example 13.6. Let $p : E \to B$ be a covering projection. Then $F = p^{-1}(b_0)$ is discrete; hence $\pi_n(F, e_0) = \{0\}$ for all $n \neq 0$. Then we have the long exact sequence

$$\cdots \to \underbrace{\pi_n(F, e_0)}_{=0} \to \pi_n(E, e_0) \to \pi_n(B, b_0) \to \underbrace{\pi_{n-1}(F, e_0)}_{=0}.$$

for $n - 1 \geq 1$, that is, $n \geq 2$. Thus we have proven the following:

Corollary 13.7. *For $n \geq 2$, and $p : E \to B$ a covering projection, then $p_\# : \pi_n(E, e_0) \to \pi_n(B, b_0)$ is an isomorphism.*

Corollary 13.8. $\pi_n(S^1, 1) = 0$ *for all* $n \geq 2$.

Proof. Contractible spaces have trivial homotopy groups. ■

Definition 13.9. Let G be a given group and $n \in \mathbb{Z}$ an integer. A space (X, x_0) is called **Eilenberg-Maclane Space**, and we write $K(G, n)$, if $\pi_n(X, x_0) = G$ and $\pi_\ell(X, x_0) = 0$ for all $\ell \neq 0$.

One observation to make is that for $\ell \geq 2$, we need the group G to be abelian, for $\pi_n(X, x_0)$ is abelian for $n \geq 2$. Moreover, Corollary 13.8 shows that $K(\mathbb{Z}, 1) = S^1$.

Now one might be wondering what $K(\mathbb{Z}, 2)$ might be. Continuing the above discussion, with the facts that $\pi_1(S^2) = 0$ and $\pi_2(S^2) \simeq \mathbb{Z}$, one might wonder if $S^2 = K(\mathbb{Z}, 2)$, but this is not the case, for $\pi_3(S^2) = \mathbb{Z}$; one way to see this is to use something called the Hopf fibration.

14 2/13/23

14.1 Homotopy Groups of S^n

One of our goals today will be to give a partial answer about computing a subset of all homotopy groups of S^n. Computing *all* of the homotopy groups of S^n, however, is still an open question!

Theorem 14.1. *For all $n \geq 1$ and $0 \leq k \leq n - 1$, $\pi_k(S^n) = 0$.*

But before we move to the proof of this theorem, let's begin with a warm-up.

Proposition 14.2. *For all $n \geq 1$, S^n is path-connected.*

Proof. Fix $x, y \in S^n$, and let $\overline{\gamma}_{x,y} : [0, 1] \to D^{n+1}$ via $\overline{\gamma}_{x,y}(t) = tx + (1 - t)y$, which is the straight line through the $n + 1$-dimensional ball connecting x and y. Now consider $\gamma_{x,y} : [0, 1] \to S^n$ via $\gamma_{x,y}(t) := \overline{\gamma}_{x,y}(t)/|\overline{\gamma}_{x,y}(t)|$. Now this path $\gamma_{x,y}$ if well-defined and connects x and y, as long as x and y are not antipodal: that is, $x \neq -y$. In the case that x and y are antipodal, choose $z \in S^n$ such that z is not antipodal to x and y. Then $\gamma_{x,z}$ and $\gamma z, y$ is well-defined, and their concatenation is a path connecting x and y. ■

Lemma 14.3. *If $f : M \to S^n$ is a continuous map that is not surjective, then it is homotopically trivial.*

Proof. Suppose f is not onto, say $p \notin f(M)$. Now by stereographic projection $h : S^n \setminus \{p\} \to \mathbb{R}^n$, we have the homeomorphism $S^n \setminus \{p\} \approx \mathbb{R}^n$. Note that $\mathbb{R}^n$ is contractible: the map $c : \mathbb{R}^n \times I \to \mathbb{R}^n$ defined by $c(x, t) = (1 - t)x$ is the homotopy that contracts the identity map to a constant map.

Now consider $F : M \to S^n$ defined by $F(u,t) = h^{-1}(c(t, h(f(u))))$: clearly this is a composition of continuous functions and is continuous. Moreover, $f_0(u) = F(u,0) = h^{-1}(c(0, h(f(u)))) = h^{-1}(h(f(u))) = f(u)$, and $f_1(u) = h^{-1}(0)$ which is constant. Hence F is the desired homotopy. ∎

Lemma 14.4. *If M is a manifold and $f : M \to S^n$ and $g : M \to S^n$ satisfy $|f(x) - g(x)| < 2$ for all $x \in M$, then f is homotopic to g.*

Proof. Consider $F : M \times [0,1] \to S^n$ given by

$$F(x,t) \coloneqq \frac{tf(x) + (1-t)g(x)}{|tf(x) + (1-t)g(x)|}.$$

Since $|f(x) - g(x)| < 2$ for all x, it follows that $f(x)$ can never be antipodal to $g(x)$. Hence F is well defined for all x, t, and so this is a homotopy. ∎

Next, we will need two results (actually, corollaries) from analysis and smooth manifolds, which we will take as given.

Lemma 14.5 (Stone-Weierstrass)**.** *Given $f : S^k \to S^n$ and an $\epsilon > 0$, there exists a polynomial $p : \mathbb{R}^k \to \mathbb{R}^n$ such that $|p(x) - f(x)| < \epsilon$ for all $x \in S^k$.*

Lemma 14.6 (Sard's Theorem)**.** *If $f : M \to N$ and $f \in C^\infty$, and $\dim M < \dim N$, then f is not onto.*

Exercise 14.7. Construct a continuous function $\gamma : [0,1] \to [0,1]^m$ that is surjective.

Lemma 14.8. *Suppose $f : S^k \to S^n$ is continuous and $k < n$. Then f is homotopic to a map that is not surjective.*

Proof. Suppose f is as prescribed. Then by Lemma 14.5, there exists a polynomial $p : \mathbb{R}^k \to \mathbb{R}^n$ such that $|p(x) - f(x)| < 2$. But applying Lemma 14.4, f is homotopic to p. But p is a polynomial and hence smooth; appealing to Lemma 14.6 yields the result. ∎

Now we have enough machinery to accomplish what we set out to do in the beginning of the section.

Proof of Theorem 14.1. Suppose $\gamma : (I_k, \partial I_k) \to (S^n, x_0)$. Note that $\gamma(\partial I_k) = x_0$, and $I_k/\partial I_k \approx S^k$. Now define $\pi : I_k \to S^k$ via the natural projection from the homeomorphism, $\pi(\partial I_k) = b_0 \in S^k$. Now define $f : S^k \to S^n$ by $f(x) = \gamma(\pi^{-1}(x))$. We need to check that f is well-defined: the only problematic point is when $x \neq b_0$, which has ∂I_k as the preimage. But indeed, γ maps ∂I_k to one point, so f is well-defined. Further, we have that f is continuous. Now Lemma 14.8 implies that there exists a homotopy $F : S^k \times [0,1] \to S^n$ such that $f_0 = f$ and $f_1 \equiv \text{const}$.

Consider the path $\eta(t) \coloneqq F(b_0, t)$. Then $\eta(0) = f_0(b_0) = f(b_0) = \gamma(\pi^{-1}(b_0)) = x_0$. We want to modify F to another homotopy $\widetilde{F}$ such that $\widetilde{F}(b_0, t) = x_0$ for all t. Since $\mathrm{SO}(n, \mathbb{R})$ acts transitively on the sphere, i.e., for all $P, Q \in S^n$, there exists some $O \in \mathrm{SO}(n, \mathbb{R})$ such that $OP = Q$, and $O(S^n) = S^n$. Moreover, the choice of O varies continuously with respect to P and Q. Therefore we can choose O_t such that $O_t(\eta(t)) = x_0$ for all t continuously and $O_0 = \mathrm{id}$, and define $\widetilde{F}(x,t) = O_t(F(x,t))$. Then it is easy to verify that $\widetilde{F}(\cdot, 0) = F(\cdot, 0) = f_0 = f$, $\widetilde{F}(\cdot, 1) = O_1(F(\cdot, 1)) = O_1(x_0) = x_0$, and $\widetilde{F}(b_0, t) = x_0$ for all t by construction. Now set $H : (I_k, \partial I_k) \times I \to (S, x_0)$ by $H(x,t) \coloneqq \widetilde{F}(\pi(x), t)$; then with the identification made earlier, this is the desired homotopy, and so γ is trivial in $\pi_k(S^n, x_0)$. ∎

Remark 14.9. A similar argument will work for any connected manifold.

14.2 Fibrations and Lie Groups

Note that $\mathrm{SO}(n)$ acts transitively on S^n, and consider $\mathbf{1} = (1, 0, \ldots, 0)^t$. Then we consider $\mathrm{stab}(\mathbf{1}) = \{O \in \mathrm{SO}(n) \mid O\mathbf{1} = \mathbf{1}\}$. Note that if $O \in \mathrm{stab}(\mathbf{1})$, then O takes $\mathbf{1}$ to $\mathbf{1}$, so must have the first column be $\begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$: that is,

$$O = \left(\begin{array}{c|ccc} 1 & 0 & \cdots & 0 \\ \hline 0 & & & \\ \vdots & & \mathrm{SO}(n-1) & \\ 0 & & & \end{array}\right)$$

We get a map $p : \mathrm{SO}(n) \to S^n$, $O \mapsto O \cdot \mathbf{1}$. Then $p^{-1}(\mathbf{1}) \simeq \mathrm{SO}(n-1)$.

Lemma 14.10. *This map p is a fibration.*

This would imply that p satisfies the path and homotopy lifting property. To see that p is a fibration we will appeal to the following theorem:

Theorem 14.11. *Let G be a Lie group. If H is a closed subgroup, then $p : G \to G/H$ is a fibration, and the fiber is H.*

We will prove this theorem in the next lecture.

We will now consider an application of the above theorem. To do this we define a new type of space. Take $\lambda \in \mathbb{C} \setminus \{0\}$. Then the map $M_\lambda : z \mapsto \lambda z$ acts on $\mathbb{C}^n \setminus \{0\}$. Note that $M_\lambda(S^{2n+1}) = S^{2n+1}$ if $|\lambda| = 1$. Next, consider the relation $z \sim w$ if and only if there exists some $\lambda \in \mathbb{C} \setminus \{0\}$ such that $\lambda z = w$ on $\mathbb{C}^n \setminus \{0\}$. Then the **complex projective space** is defined as $\mathbb{C}P^n := (\mathbb{C}^n \setminus \{0\})/\sim = S^{2n+1}/\sim$. Then this gets us a projection map $p : S^{2n+1} \to \mathbb{C}P^n$, where each fiber is S^1, for the same reason as in the theorem.

Corollary 14.12. *We have a long exact sequence*

$$\cdots \to \pi_k(S^1) \to \pi_k(S^{2n+1}) \to \pi_k(\mathbb{C}P^n) \to \pi_{k-1}(S^1) \to \cdots .$$

Thus if $k - 1 \geq 2$, that is, $k \geq 3$, $\pi_k(S^{2n+1}) \simeq \pi_k(\mathbb{C}P^n)$. For $k = 2$, we have $\pi_2(\mathbb{C}P^n) \simeq \mathbb{Z}$.

In general, $\pi_{2n+1}(\mathbb{C}P^n) \simeq \pi_{2n+1}(S^{2n+1}) \simeq \mathbb{Z}$. To answer the question posed in the previous lecture, we note that $\mathbb{C}P^\infty$ is the Eilenberg-Maclane space needed: it is the sought for $K(\mathbb{Z}, 2)$.

15 2/15/23

Today we will start the proof of a theorem that was stated and unproved in the last lecture. We will state it again for a reminder.

Theorem 15.1. *Let G be a Lie group and H a closed subgroup of G. Then the projection $p : G \to G/H$ is a locally trivial fibration. That is, for every $x \in G/H$, there exists an open neighborhood U of x and a homeomorphism $Q : U \times H \to p^{-1}(U)$ such that $p(Q(u, h)) = u$ for all $u \in U$.*

15.1 A Short Course in Lie Groups

Let G be a Lie group. Then we will denote $\mathrm{Lie}(G) \coloneqq \mathfrak{g}$ to be the tangent space at the identity. Then the **exponential map** $\exp : \mathrm{Lie}(G) \to G$ is defined in the following manner: if $v \in T_eG$ and X is a left-invariant vector field of G such that $X(0) = v$, $\alpha : \mathbb{R} \to G$ the associated one-parameter subgroup with $\alpha'(0) = X(0)$, then $v \mapsto \alpha(1)$.

Example 15.2. As an example $\mathrm{GL}(n, \mathbb{R})$ is a Lie group, and $\mathrm{Lie}(\mathrm{GL}(n, \mathbb{R})) = \mathrm{Mat}_{n\times n}(\mathbb{R})$. Then the exponential map coincides with the usual matrix exponential: $\exp(A) = \sum_{n=0}^{\infty} \frac{A^n}{n!}$. Then $\exp(0) = \mathrm{id}$ and $(d\exp)_0 : \mathrm{Mat}_{n\times n}(\mathbb{R}) \to T_e\mathrm{GL}(n, \mathbb{R}) = \mathrm{Mat}_{n\times n}(\mathbb{R})$, where $(d\exp)_0 \equiv \mathrm{id}_{\mathrm{Lie}(G)}$.

Now by the inverse function theorem, exp is a local diffeomorphism; hence we can use the exponential map for charts for G. Moreover, since G is a Lie group, left multiplication $L_g : G \to G$ and $x \mapsto gx$ is a diffeomorphism, so once we know open sets near the identity, we know it everywhere. Thus studying the Lie algebra lets us study Lie groups.

Exercise 15.3. Let G be a path-connected topological group. Then $\pi_1(G, e_0)$ is abelian.

Suppose $\mathrm{Lie}(G) = E \oplus F$ a direct sum of vector spaces. Then one can use $\exp|_E$ and $\exp|_F$ to define a new map $r_{E,F} : \mathrm{Lie}(G) \to G$, given by

$$v = v_E + v_F \mapsto \exp(v_e)\exp(v_F).$$

Warning. $\exp(x + y) = \exp(x)\exp(y)$ in general, unless G is abelian.

However, $(dr_{E,F})_0 = \mathrm{id}$, so the inverse function theorem tells us that $r_{E,F}$ is a local diffeomorphism.

Theorem 15.4 (Closed Subgroup Theorem)**.** *Every closed subgroup of a Lie group is an embedded submanifold, and hence a Lie group.*

Note that if H is a closed subgroup of G, then $\mathrm{Lie}(H)$ is a vector subspace of $\mathrm{Lie}(G)$, and we say that $\mathrm{Lie}(H)$ is a Lie subalgebra. Moreover, $\exp(\mathrm{Lie}(H))$ is a subgroup of H, and is the connected component of H at the identity, sometimes also called the **analytic subgroup** associated to $\mathrm{Lie}(H)$.

Now let $T \subseteq \mathrm{Lie}(G)$ be a vector subspace such that $T \oplus \mathrm{Lie}(G) = \mathrm{Lie}(G)$. Then from linear algebra, naturally we have that $\mathrm{Lie}(G)/\mathrm{Lie}(H) \simeq T$. Hence $(\mathrm{Lie}(G)/\mathrm{Lie}(H)) \times \mathrm{Lie}(H) \simeq \mathrm{Lie}(G)$. This sketches out the proof of a special case of Theorem 15.1, when G is a vector space. In particular, it proves the theorem for $\mathrm{Lie}(G)$ the tangent space at the identity.

Next we will state a theorem from Lie theory, which we take as given:

Theorem 15.5. *There exists a neighborhood W of the identity $e \in G$ and a neighborhood $V \subseteq \mathrm{Lie}(G)$ of 0 such that the following hold:*

(i) $\exp : V \to W$ *is a diffeomorphism.*

(ii) if $h \in H \cap W$, then $(\exp)^{-1}(h) \in \mathrm{Lie}(H)$; equivalently, $H \cap W \subseteq H^0$, where H^0 denotes the connected component of H with the identity.

Example 15.6. Let $G = \mathbb{R}^2$, and $H = \{(t, n_1 + n_2\sqrt{2}) \mid t \in \mathbb{R}, n_1, n_2 \in \mathbb{Z}\}$. Then $H^0 = \{(t, 0) \mid t \in \mathbb{R}\}$. But for any open $W \subseteq G$ with $0 \in W$, $H \cap W$ is not a subset of H^0, since H is not a closed subgroup of G. In fact, H is dense in G!

Exercise 15.7. Show that, in the above example, H^0 is a normal subgroup of H.

15.2 Towards the Proof of the Main Theorem

Our goal is to show that p is a fibration. Let G be a Lie group and H a closed subgroup of G; consider $\mathrm{Lie}(H)$, and $T \subseteq \mathrm{Lie}(G)$ such that $\mathrm{Lie}(G) = T \oplus \mathrm{Lie}(H)$. take Σ a small open subset of T containing 0. Letting $V \subseteq \mathrm{Lie}(G)$ and $W \subseteq G$ as in the statement of Theorem 15.5, WLOG, by shrinking if necessary, we can assume that $V = \{v_1 + v_2 \mid v_1 \in \Sigma, v_2 \in V_2\}$, where $V_2 \subseteq \mathrm{Lie}(H)$: that is, $V = \Sigma \oplus V_2$.

Lemma 15.8. *Suppose $r : V \to G$ is defined by*

$$v = v_1 + v_2 \mapsto \exp(v_1)\exp(v_2),$$

and let $\hat{W} = r(V)$. Then the same statement with r instead of exp *holds as in Theorem 15.5.*

Our goal is to define an open set $U \subseteq G/H$ and a map $Q : H \times U \to G$ such that Q is a local homeomorphism. To this end, define $\hat{Q} : H \times \exp(\Sigma) \to G$ via $(h, \sigma) \mapsto \sigma h$.

Claim 15.9.

1. *$\hat{Q}$ is a homeomorphism onto its image.*
2. *$\hat{Q}(H \times \exp(\Sigma))$ is open.*
3. *$\hat{Q}(H \times \exp(\Sigma)) = p^{-1}(p(\exp(\Sigma)))$.*
4. *$p|_{\exp(\Sigma)}$ is one-to-one.*
5. *$p(\exp(\Sigma))$ is open, and $p^{-1} : p(\exp(\Sigma)) \to \exp(\Sigma)$ is continuous.*

Assuming the claim, we may define $Q : H \times p(\exp(\Sigma)) \to p^{-1}(p(\exp(\Sigma)))$ given by

$$Q(h, p(\sigma)) = \sigma h = \hat{Q}(h, \sigma).$$

Then define $U = p(\exp(\Sigma))$ to conclude. Since the claim tells us that every map involved is a homeomorphism, Q is a homeomorphism. Now having defined U at the identity, via left multiplication U can be translated around G, which would form an open cover $\{U_i\}_{i \in I}$ of G, and similarly Q_i defined for each i. Since Q is a homeomorphism satisfying $p(Q(h, p(\sigma))) = p(h\sigma) = p(\sigma)$, each Q_i satisfies the same, which would indeed show that p is a fibration (check with Definition 13.3). So all we need to do now is to verify the claim.

Proof of 1. We need to show that $\hat{Q}$ is one-to-one. Suppose $\sigma_1 h_1 = \sigma_2 h_2$. Then rearranging, we have $\sigma_2^{-1}\sigma_1 =$