Supplementary Material: Detailed Controllers for the Primal and Dual Variables

The controllers for the primal variables are:

$$\dot{H}_i^L = -k_{H_i^L}\Big(\frac{\partial L}{\partial H_i^L}\Big) = -k_{H_i^L}\Big(\frac{\omega_i^H}{\lambda_i}(H_i^L - H_i^{L,Set}) - \lambda_i^H - \psi_i^- + \psi_i^+\Big) \tag{1a}$$

$$\dot{H}_i^{CHP} = -k_{H_i^{CHP}}\Big(\frac{\partial L}{\partial H_i^{CHP}}\Big) = -k_{H_i^{CHP}}\Big(\lambda_i^{HG} - \upsilon_i^- + \upsilon_i^+ + \phi_i^- c_i^m + \phi_i^+ c_i^v\Big) \tag{1b}$$

$$\dot{H}_i^{GB} = -k_{H_i^{GB}}\Big(\frac{\partial L}{\partial H_i^{GB}}\Big) = -k_{H_i^{GB}}\Big(\frac{\lambda_i^{GB}}{\eta_i^{GB}} + \lambda_i^{HG} - \kappa_i^- + \kappa_i^+\Big) \tag{1c}$$

$$\dot{H}_i^{EB} = -k_{H_i^{EB}}\Big(\frac{\partial L}{\partial H_i^{EB}}\Big) = -k_{H_i^{EB}}\Big(\frac{\lambda_i^{EB}}{\eta_i^{EB}} + \lambda_i^{HG} - \iota_i^- + \iota_i^+\Big) \tag{1d}$$

$$\dot{H}_i^G = -k_{H_i^G}\Big(\frac{\partial L}{\partial H_i^G}\Big) = -k_{H_i^G}\Big(-\lambda_i^{HG} + \lambda_i^H\Big) \tag{1e}$$

$$\dot{G}_i^{CHP} = -k_{G_i^{CHP}}\Big(\frac{\partial L}{\partial G_i^{CHP}}\Big) = -k_{G_i^{CHP}}\Big(\lambda^{Gas} - \lambda_i^{CHP}\Big) \tag{1f}$$

$$\dot{G}_i^{GB} = -k_{G_i^{GB}}\Big(\frac{\partial L}{\partial G_i^{GB}}\Big) = -k_{G_i^{GB}}\Big(\lambda^{Gas} - \lambda_i^{GB}\Big) \tag{1g}$$

$$\dot{H}_i = -k_{H_i}\Big(\frac{\partial L}{\partial H_i}\Big) = -k_{H_i}\Big(\omega_i^{HP}(H_i - H_i^{Set}) + \sum_{ji\in\mathcal{E}_{pump}(H)}\lambda_{ji}^{pump} - \sum_{ik\in\mathcal{E}_{pump}(H)}\lambda_{ik}^{pump} + \sum_{ik\in\mathcal{E}_{flow}(H)}\lambda_{ik}^{HP} - \sum_{ji\in\mathcal{E}_{flow}(H)}\lambda_{ji}^{HP} - \sum_{ik\in\mathcal{E}_{load}(H)}\lambda_{ik}^{HD} + \sum_{ji\in\mathcal{E}_{load}(H)}\lambda_{ji}^{HD} - \sigma_i^- + \sigma_i^+\Big) \tag{1h}$$

$$\dot{H}_i^{pump} = -k_{H_i^{pump}}\Big(\frac{\partial L}{\partial H_i^{pump}}\Big) = -k_{H_i^{pump}}\Big(-\frac{\rho g}{\eta_i}M_{ji}\lambda_i^P - \sum_{ji\in\mathcal{E}_{pump}(H)}\lambda_{ji}^{pump} - \rho_i^- + \rho_i^+\Big) \tag{1i}$$

$$\dot{T}_i = -k_{T_i}\Big(\frac{\partial L}{\partial T_i}\Big) = -k_{T_i}\Bigg(-\sum_{k\in\mathcal{N}_{out}(i)}\lambda_i^H cM_{ik} + \sum_{k\in\mathcal{N}_{out}(i)}\lambda_{ik}^T \exp(-\frac{\nu L_{ik}}{cM_{ik}}) + \nu_i^+ - \nu_i^-\Bigg) \tag{1j}$$

$$\dot{T}_{ji} = -k_{T_{ji}}\Big(\frac{\partial L}{\partial T_{ji}}\Big) = -k_{T_{ji}}\Big(\lambda_i^H cM_{ji} - \lambda_{ji}^T + \xi_{ji}^+ - \xi_{ji}^-\Big) \tag{1k}$$

$$\dot{P}_i^{CHP} = -k_{P_i^{CHP}}\Big(\frac{\partial L}{\partial P_i^{CHP}}\Big) = -k_{P_i^{CHP}}\Big(\lambda_i^{PG} + \frac{\lambda_i^{CHP}}{\eta_i^{CHP}} - \varsigma_i^- + \varsigma_i^+ - \phi_i^- + \phi_i^+\Big) \tag{1l}$$

$$\dot{P}_i^{pump} = -k_{P_i^{pump}}\Big(\frac{\partial L}{\partial P_i^{pump}}\Big) = -k_{P_i^{pump}}\Big(-\lambda_i^{PG} + \lambda_i^P\Big) \tag{1m}$$

$$\dot{P}_i^{EB} = -k_{P_i^{EB}}\Big(\frac{\partial L}{\partial P_i^{EB}}\Big) = -k_{P_i^{EB}}\Big(-\lambda_i^{PG} - \lambda_i^{EB}\Big) \tag{1n}$$

$$\dot{P}_1^{Grid} = -k_{P_1^{Grid}}\Big(\frac{\partial L}{\partial P_1^{Grid}}\Big) = -k_{P_1^{Grid}}\Big(\lambda^{Grid}(P_1^{Grid} - P_1^{Grid*}) + \lambda_i^{PG}\Big) \tag{1o}$$

$$\dot{Q}^{Grid} = -k_{Q^{Grid}}\Big(\frac{\partial L}{\partial Q^{Grid}}\Big) = -k_{Q^{Grid}}\Big(\lambda_i^{QG}\Big) \tag{1p}$$

$$\dot{P}_i = -k_{P_i}\Big(\frac{\partial L}{\partial P_i}\Big) = -k_{P_i}\Big(-\lambda_i^{PG} + \lambda_i^{PF}\Big) \tag{1q}$$

$$\dot{Q}_i = -k_{Q_i}\Big(\frac{\partial L}{\partial Q_i}\Big) = -k_{Q_i}\Big(-\lambda_i^{QG} + \lambda_i^{QF}\Big) \tag{1r}$$

$$\dot{Q}_i^G = -k_{Q_i^G}\Big(\frac{\partial L}{\partial Q_i^G}\Big) = -k_{Q_i^G}\Big(\lambda_i^{QG} - \chi_i^- + \chi_i^+\Big) \tag{1s}$$

$$\dot{P}_{ij} = -k_{P_{ij}}\Big(\frac{\partial L}{\partial P_{ij}}\Big) = -k_{P_{ij}}\Big(\lambda_j^{PF} - \lambda_i^{PF} - 2\lambda_i^V r_{ij} - \vartheta_{ij}^- + \vartheta_{ij}^+\Big) \tag{1t}$$

$$\dot{Q}_{ij} = -k_{Q_{ij}}\Big(\frac{\partial L}{\partial Q_{ij}}\Big) = -k_{Q_{ij}}\Big(\lambda_j^{QF} - \lambda_i^{QF} - 2\lambda_i^V x_{ij} - \epsilon_{ij}^- + \epsilon_{ij}^+\Big) \tag{1u}$$

$$\dot{V}_i = -k_{V_i}\Big(\frac{\partial L}{\partial V_i}\Big) = -k_{V_i}\Big(\sum_{ij\in\mathcal{E}(\mathcal{P})}\lambda_i^V - \sum_{mi\in\mathcal{E}(\mathcal{P})}\lambda_m^V - \theta_i^- + \theta_i^+\Big). \tag{1v}$$

The controllers for the dual variables related to the equality constraints are:

$$\dot{\lambda}_i^{GB} = k_{\lambda_i^{GB}}\Big(\frac{\partial L}{\partial \lambda_i^{GB}}\Big) = k_{\lambda_i^{GB}}\left(\frac{H_i^{GB}}{\eta_i^{GB}} - G_i^{GB}\right) \tag{2a}$$

$$\dot{\lambda}_i^{EB} = k_{\lambda_i^{EB}}\Big(\frac{\partial L}{\partial \lambda_i^{EB}}\Big) = k_{\lambda_i^{EB}}\left(\frac{H_i^{EB}}{\eta_i^{EB}} - P_i^{EB}\right) \tag{2b}$$

$$\dot{\lambda}_i^{CHP} = k_{\lambda_i^{CHP}}\Big(\frac{\partial L}{\partial \lambda_i^{CHP}}\Big) = k_{\lambda_i^{CHP}}\left(\frac{P_i^{CHP}}{\eta_i^{CHP}} - G_i^{CHP}\right) \tag{2c}$$

$$\begin{aligned}\dot{\lambda}_i^{PG} =&k_{\lambda_i^{PG}}\Big(\frac{\partial L}{\partial \lambda_i^{PG}}\Big) = k_{\lambda_i^{PG}}\Bigg(P_i^{CHP}+P_i^{WT}\\&+P_i^{PV}+P_i^{Grid}-P_i^{L}-P_i^{pump}-P_i^{EB}-P_i\Bigg)\end{aligned} \tag{2d}$$

$$\dot{\lambda}_i^{QG} =k_{\lambda_i^{QG}}\Big(\frac{\partial L}{\partial \lambda_i^{QG}}\Big) = k_{\lambda_i^{QG}}\Bigg(Q_i^{G}+Q_i^{Grid}-Q_i^{L}-Q_i\Bigg) \tag{2e}$$

$$\begin{aligned}\dot{\lambda}_j^{PF} =&k_{\lambda_j^{PF}}\Big(\frac{\partial L}{\partial \lambda_j^{PF}}\Big) = k_{\lambda_j^{PF}}\Bigg(\sum_{i\in\Omega_{i\to j}}P_{ij}+P_j\\&-\sum_{k\in\Omega_{j\to k}}P_{jk}\Bigg)\end{aligned} \tag{2f}$$

$$\begin{aligned}\dot{\lambda}_j^{QF} =&k_{\lambda_j^{QF}}\Big(\frac{\partial L}{\partial \lambda_j^{QF}}\Big) = k_{\lambda_j^{QF}}\Bigg(\sum_{i\in\Omega_{i\to j}}Q_{ij}+Q_j\\&-\sum_{k\in\Omega_{j\to k}}Q_{jk}\Bigg)\end{aligned} \tag{2g}$$

$$\begin{aligned}\dot{\lambda}_i^{V} =&k_{\lambda_i^{V}}\Big(\frac{\partial L}{\partial \lambda_i^{V}}\Big) = k_{\lambda_i^{V}}\Bigg(V_i-V_j\\&-2(r_{ij}P_{ij}+x_{ij}Q_{ij})\Bigg),\ ij\in\mathcal{E}(\mathcal{P})\end{aligned} \tag{2h}$$

$$\begin{aligned}\dot{\lambda}_i^{HG} =&k_{\lambda_i^{HG}}\Big(\frac{\partial L}{\partial \lambda_i^{HG}}\Big) = k_{\lambda_i^{HG}}\Bigg(H_i^{GB}+H_i^{EB}+H_i^{CHP}\\&-H_i^{G}\Bigg)\end{aligned} \tag{2i}$$

$$\begin{aligned}\dot{\lambda}_i^{H} =&k_{\lambda_i^{H}}\Big(\frac{\partial L}{\partial \lambda_i^{H}}\Big) = k_{\lambda_i^{H}}\Big(H_i^{G}-H_i^{L}-c\sum_{k\in\mathcal{N}_{out}(i)}M_{ik}T_i\\&+c\sum_{j\in\mathcal{N}_{in}(i)}M_{ji}T_{ji}\Big)\end{aligned} \tag{2j}$$

$$\begin{aligned}\dot{\lambda}_{ji}^{T} =&k_{\lambda_{ji}^{T}}\Big(\frac{\partial L}{\partial \lambda_{ji}^{T}}\Big) = k_{\lambda_{ji}^{T}}\Big(-T_{ji}\\&+(T_j-T_a)\exp(-\frac{\nu L_{ji}}{cM_{ji}})+T_a\Big)\end{aligned} \tag{2k}$$

$$\dot{\lambda}_{ji}^{pump} =k_{\lambda_{ji}^{pump}}\Big(\frac{\partial L}{\partial \lambda_{ji}^{pump}}\Big) = k_{\lambda_{ji}^{pump}}\Big(H_i-H_j-H_i^{pump}\Big) \tag{2l}$$

$$\dot{\lambda}_{ji}^{HP} =k_{\lambda_{ji}^{HP}}\Big(\frac{\partial L}{\partial \lambda_{ji}^{HP}}\Big) = k_{\lambda_{ji}^{HP}}\Big(H_j-H_i-F_{ji}M_{ji}^2\Big) \tag{2m}$$

$$\dot{\lambda}_i^{P} =k_{\lambda_i^{P}}\Big(\frac{\partial L}{\partial \lambda_i^{P}}\Big) = k_{\lambda_i^{P}}\Bigg(P_i^{pump}-\frac{\rho g}{\eta_i}H_i^{pump}M_{ji}\Bigg). \tag{2n}$$

The controllers for the dual variables related to the inequality constraints are:

$$\dot{\lambda}_{ji}^{HD} =k_{\lambda_{ji}^{HD}}\Big(\frac{\partial L}{\partial \lambda_{ji}^{HD}}\Big) = k_{\lambda_{ji}^{HD}}\Big(-H_j+H_i+H_{i,\min}^{c}\Big)^+_{\lambda_{ji}^{HD}} \tag{3a}$$

$$\dot{\kappa}_i^{+} =k_{\kappa_i^{+}}\Big(\frac{\partial L}{\partial \kappa_i^{+}}\Big) = k_{\kappa_i^{+}}\Big(H_i^{GB}-H_{i,\max}^{GB}\Big)^+_{\kappa_i^{+}} \tag{3b}$$

$$\dot{\kappa}_i^{-} =k_{\kappa_i^{-}}\Big(\frac{\partial L}{\partial \kappa_i^{-}}\Big) = k_{\kappa_i^{-}}\Big(H_{i,\min}^{GB}-H_i^{GB}\Big)^+_{\kappa_i^{-}} \tag{3c}$$

$$\dot{\iota}_i^{+} =k_{\iota_i^{+}}\Big(\frac{\partial L}{\partial \iota_i^{+}}\Big) = k_{\iota_i^{+}}\Big(H_i^{EB}-H_{i,\max}^{EB}\Big)^+_{\iota_i^{+}} \tag{3d}$$

$$\dot{\iota}_i^{-} =k_{\iota_i^{-}}\Big(\frac{\partial L}{\partial \iota_i^{-}}\Big) = k_{\iota_i^{-}}\Big(H_{i,\min}^{EB}-H_i^{EB}\Big)^+_{\iota_i^{-}} \tag{3e}$$

$$\dot{\theta}_i^{+} =k_{\theta_i^{+}}\Big(\frac{\partial L}{\partial \theta_i^{+}}\Big) = k_{\theta_i^{+}}\Big(V_i-V_{i,\max}\Big)^+_{\theta_i^{+}} \tag{3f}$$

$$\dot{\theta}_i^{-} =k_{\theta_i^{-}}\Big(\frac{\partial L}{\partial \theta_i^{-}}\Big) = k_{\theta_i^{-}}\Big(V_{i,\min}-V_i\Big)^+_{\theta_i^{-}} \tag{3g}$$

$$\dot{\vartheta}_{ij}^{+} =k_{\vartheta_{ij}^{+}}\Big(\frac{\partial L}{\partial \vartheta_{ij}^{+}}\Big) = k_{\vartheta_{ij}^{+}}\Big(P_{ij}-P_{ij,\max}\Big)^+_{\vartheta_{ij}^{+}} \tag{3h}$$

$$\dot{\vartheta}_{ij}^{-} =k_{\vartheta_{ij}^{-}}\Big(\frac{\partial L}{\partial \vartheta_{ij}^{-}}\Big) = k_{\vartheta_{ij}^{-}}\Big(P_{ij,\min}-P_{ij}\Big)^+_{\vartheta_{ij}^{-}} \tag{3i}$$

$$\dot{\epsilon}_{ij}^{+} =k_{\epsilon_{ij}^{+}}\Big(\frac{\partial L}{\partial \epsilon_{ij}^{+}}\Big) = k_{\epsilon_{ij}^{+}}\Big(Q_{ij}-Q_{ij,\max}\Big)^+_{\epsilon_{ij}^{+}} \tag{3j}$$

$$\dot{\epsilon}_{ij}^{-} =k_{\epsilon_{ij}^{-}}\Big(\frac{\partial L}{\partial \epsilon_{ij}^{-}}\Big) = k_{\epsilon_{ij}^{-}}\Big(Q_{ij,\min}-Q_{ij}\Big)^+_{\epsilon_{ij}^{-}} \tag{3k}$$

$$\dot{\chi}_i^{-} =k_{\chi_i^{-}}\Big(\frac{\partial L}{\partial \chi_i^{-}}\Big) = k_{\chi_i^{-}}\Big(Q_{i,\min}^{G}-Q_i^{G}\Big)^+_{\chi_i^{-}} \tag{3l}$$

$$\dot{\chi}_i^{+} =k_{\chi_i^{+}}\Big(\frac{\partial L}{\partial \chi_i^{+}}\Big) = k_{\chi_i^{+}}\Big(Q_i^{G}-Q_{i,\max}^{G}\Big)^+_{\chi_i^{+}}. \tag{3m}$$

$$\dot{\sigma}_i^{+} =k_{\sigma_i^{+}}\Big(\frac{\partial L}{\partial \sigma_i^{+}}\Big) = k_{\sigma_i^{+}}\Big(H_i-H_{i,\max}\Big)^+_{\sigma_i^{+}} \tag{4a}$$

$$\dot{\sigma}_i^{-} =k_{\sigma_i^{-}}\Big(\frac{\partial L}{\partial \sigma_i^{-}}\Big) = k_{\sigma_i^{-}}\Big(H_{i,\min}-H_i\Big)^+_{\sigma_i^{-}} \tag{4b}$$

$$\dot{\psi}_i^{+} =k_{\psi_i^{+}}\Big(\frac{\partial L}{\partial \psi_i^{+}}\Big) = k_{\psi_i^{+}}\Big(H_i^{L}-H_{i,\max}^{L}\Big)^+_{\psi_i^{+}} \tag{4c}$$

$$\dot{\psi}_i^{-} =k_{\psi_i^{-}}\Big(\frac{\partial L}{\partial \psi_i^{-}}\Big) = k_{\psi_i^{-}}\Big(H_{i,\min}^{L}-H_i^{L}\Big)^+_{\psi_i^{-}} \tag{4d}$$

$$\dot{\nu}_i^{+} =k_{\nu_i^{+}}\Big(\frac{\partial L}{\partial \nu_i^{+}}\Big) = k_{\nu_i^{+}}\Big(T_i-T_{i,\max}\Big)^+_{\nu_i^{+}} \tag{4e}$$

$$\dot{\nu}_i^{-} =k_{\nu_i^{-}}\Big(\frac{\partial L}{\partial \nu_i^{-}}\Big) = k_{\nu_i^{-}}\Big(T_{i,\min}-T_i\Big)^+_{\nu_i^{-}} \tag{4f}$$

$$\dot{\xi}_{ji}^{+} =k_{\xi_{ji}^{+}}\Big(\frac{\partial L}{\partial \xi_{ji}^{+}}\Big) = k_{\xi_{ji}^{+}}\Big(T_{ji}-T_{ji,\max}\Big)^+_{\xi_{ji}^{+}} \tag{4g}$$

$$\dot{\xi}_{ji}^{-} =k_{\xi_{ji}^{-}}\Big(\frac{\partial L}{\partial \xi_{ji}^{-}}\Big) = k_{\xi_{ji}^{-}}\Big(T_{ji,\min}-T_{ji}\Big)^+_{\xi_{ji}^{-}} \tag{4h}$$

$$\dot{\rho}_i^{+} =k_{\rho_i^{+}}\Big(\frac{\partial L}{\partial \rho_i^{+}}\Big) = k_{\rho_i^{+}}\Big(H_i^{pump}-H_{i,max}^{pump}\Big)^+_{\rho_i^{+}} \tag{4i}$$

$$\dot{\rho}_i^{-} =k_{\rho_i^{-}}\Big(\frac{\partial L}{\partial \rho_i^{-}}\Big) = k_{\rho_i^{-}}\Big(H_{i,min}^{pump}-H_i^{pump}\Big)^+_{\rho_i^{-}} \tag{4j}$$

$$\dot{\varsigma}_i^{-} =k_{\varsigma_i^{-}}\Big(\frac{\partial L}{\partial \varsigma_i^{-}}\Big) = k_{\varsigma_i^{-}}\Big(P_{i,\min}^{CHP}-P_i^{CHP}\Big)^+_{\varsigma_i^{-}} \tag{4k}$$

$$\dot{\varsigma}_i^+ = k_{\varsigma_i^+}\left(\frac{\partial L}{\partial \varsigma_i^+}\right) = k_{\varsigma_i^+}\left(P_i^{CHP} - P_{i,\max}^{CHP}\right)_{\varsigma_i^+}^+ \tag{4l}$$

$$\dot{\upsilon}_i^- = k_{\upsilon_i^-}\left(\frac{\partial L}{\partial \upsilon_i^-}\right) = k_{\upsilon_i^-}\left(H_{i,\min}^{CHP} - H_i^{CHP}\right)_{\upsilon_i^-}^+ \tag{4m}$$

$$\dot{\upsilon}_i^+ = k_{\upsilon_i^+}\left(\frac{\partial L}{\partial \upsilon_i^+}\right) = k_{\upsilon_i^+}\left(H_i^{CHP} - H_{i,\max}^{CHP}\right)_{\upsilon_i^+}^+ \tag{4n}$$

$$\dot{\phi}_i^- = k_{\phi_i^-}\left(\frac{\partial L}{\partial \phi_i^-}\right) = k_{\phi_i^-}\left(c_i^m H_i^{CHP} + P_{i,\min}^{CHP} - P_i^{CHP}\right)_{\phi_i^-}^+ \tag{4o}$$

$$\dot{\phi}_i^+ = k_{\phi_i^+}\left(\frac{\partial L}{\partial \phi_i^+}\right) = k_{\phi_i^+}\left(P_i^{CHP} - P_{i,\max}^{CHP} + c_i^v H_i^{CHP}\right)_{\phi_i^+}^+ \tag{4p}$$