Applied Linear Algebra with applications
Assignment #4

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#8.2.18. solution)

First, we have to compute the eigenvalues as

$$(\lambda-2)(\lambda-3)^2=0 \Rightarrow \lambda=2 \text{ and } \lambda=3 \text{ (multiplicity 2).}$$

The corresponding eigenvectors are

$$\lambda=2 \rightarrow v_1=\begin{bmatrix}1\\0\\0\end{bmatrix} \text{ and } \lambda=3 \rightarrow v_2=\begin{bmatrix}0\\1\\0\end{bmatrix},\ v_3=\begin{bmatrix}-2\\0\\1\end{bmatrix}.$$

In order to diagonalize A, check that the eigenvectors of A are linearly independent. (refer to the **diagonalization theorem**) Clearly, we can easily check that $\{v_1, v_2, v_3\}$ is linearly independent. Thus, A is diagonalizable and we can write $A=P^{-1}AP$, where $P=[v_1\ v_2\ v_3]$, and $P^{-1}=\begin{bmatrix}1&0&2\\0&1&0\\0&0&1\end{bmatrix}$. The result of $A=P^{-1}AP$ is $\begin{bmatrix}2&0&0\\0&3&0\\0&0&3\end{bmatrix}$.

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#8.2.24. solution)

In the same manner of #8.2.18, we can calculate the eigenvalues and eigenvectors that

$$(\lambda+2)^2(\lambda-3)^2=0 \Rightarrow \lambda=-2 \text{ (multiplicity 2) and } \lambda=3 \text{ (multiplicity 2)}$$

$$\lambda=-2 \rightarrow v_1=\begin{bmatrix}0\\1\\0\\0\end{bmatrix}, v_2=\begin{bmatrix}1\\0\\0\\0\end{bmatrix} \quad \text{and} \quad \lambda=3 \rightarrow v_3=\begin{bmatrix}0\\1\\0\\0\end{bmatrix},\ v_4=\begin{bmatrix}0\\-1\\0\\1\end{bmatrix}.$$

Then, $\{v_1, v_2, v_3, v_4\}$ is linearly independent, so that A is diagonalizable. Thus, $P=[v_1\ v_2\ v_3\ v_4]$, $P^{-1}=\begin{bmatrix}0&1&-1&1\\1&0&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}$ and $A=P^{-1}AP=\begin{bmatrix}-2&0&0&0\\0&-2&0&0\\0&0&3&0\\0&0&0&3\end{bmatrix}$.

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#8.3.10. solution)

In the same manner of #8.2.24, we can compute the eigenvalues and eigenvectors that

$$\lambda^3 + 28\lambda^2 - 1175\lambda - 3750 = (\lambda+3)(\lambda-25)(\lambda+50) = 0 \Rightarrow \lambda_1 = -3,\ \lambda_2 = 25 \text{ and } \lambda_3 = -50.$$

$$\lambda_1 = 3 \rightarrow v_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \qquad \lambda_2 = 25 \rightarrow v_2 = \begin{bmatrix} -4 \\ 0 \\ 3 \end{bmatrix} \qquad \text{and} \qquad \lambda_3 = -50 \rightarrow v_3 = \begin{bmatrix} 3 \\ 0 \\ 4 \end{bmatrix}.$$

Since the eigenvectors of A are satisfied the orthogonality (check $\vec{v} \bullet \vec{w} = 0$), the orthogonal matrix P is

$$P = \begin{bmatrix} \frac{v_1}{\| v_1 \|} & \frac{v_2}{\| v_2 \|} & \frac{v_3}{\| v_3 \|} \end{bmatrix} = \begin{bmatrix} 0 & -4/5 & 3/5 \\ 1 & 0 & 0 \\ 0 & 3/5 & 4/5 \end{bmatrix}.$$

Thus, we can check

$$P^T A P = \begin{bmatrix} 0 & 1 & 0 \\ -4/5 & 0 & 3/5 \\ 3/5 & 0 & 4/5 \end{bmatrix} \begin{bmatrix} -2 & 0 & -36 \\ 0 & -3 & 0 \\ -36 & 0 & -23 \end{bmatrix} \begin{bmatrix} 0 & -4/5 & 3/5 \\ 1 & 0 & 0 \\ 0 & 3/5 & 4/5 \end{bmatrix} = \begin{bmatrix} -3 & 0 & 0 \\ 0 & 25 & 0 \\ 0 & 0 & -50 \end{bmatrix} = D$$

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#8.3.20. solution)

Compute the eigenvalues and eigenvectors of A that

$$896 - (30+\lambda)(30-\lambda) = 896 - 900 + \lambda^2 = \lambda^2 - 4 = (\lambda-2)(\lambda+2) = 0 \Rightarrow \lambda = 2 \text{ and } \lambda = -2.$$

$$\lambda = 2 \rightarrow v_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \qquad \text{and} \qquad \lambda = -2 \rightarrow v_2 = \begin{bmatrix} 4 \\ 7 \end{bmatrix}.$$

Thus, the matrix $P = \begin{bmatrix} 1 & 4 \\ 2 & 7 \end{bmatrix}$ and $P^{-1}AP = D = \begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix}$. Hence, we obtain the power of A:

$$A^{10} = PD^{10}P^{-1} = \begin{bmatrix} 1 & 4 \\ 2 & 7 \end{bmatrix} \begin{bmatrix} 2^{10} & 0 \\ 0 & 2^{10} \end{bmatrix} \begin{bmatrix} -7 & 4 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} 2^{10} & 0 \\ 0 & 2^{10} \end{bmatrix} = \begin{bmatrix} 1024 & 0 \\ 0 & 1024 \end{bmatrix}.$$

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#8.3.28. solution)

From #8.3.10, we have

$$P^TAP = \begin{bmatrix} 0 & 1 & 0 \\ -4/5 & 0 & 3/5 \\ 3/5 & 0 & 4/5 \end{bmatrix} \begin{bmatrix} -2 & 0 & -36 \\ 0 & -3 & 0 \\ -36 & 0 & -23 \end{bmatrix} \begin{bmatrix} 0 & -4/5 & 3/5 \\ 1 & 0 & 0 \\ 0 & 3/5 & 4/5 \end{bmatrix} = \begin{bmatrix} -3 & 0 & 0 \\ 0 & 25 & 0 \\ 0 & 0 & -50 \end{bmatrix} = D.$$

Thus, $A = PDP^T$ and

$$e^{tA} = Pe^{tD}P^T = \begin{bmatrix} 0 & -4/5 & 3/5 \\ 1 & 0 & 0 \\ 0 & 3/5 & 4/5 \end{bmatrix} \begin{bmatrix} e^{-3t} & 0 & 0 \\ 0 & e^{25t} & 0 \\ 0 & 0 & e^{-50t} \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ -4/5 & 0 & 3/5 \\ 3/5 & 0 & 4/5 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{16}{25}e^{25t} + \frac{9}{25}e^{-50t} & 0 & -\frac{12}{25}e^{25t} + \frac{12}{25}e^{-50t} \\ 0 & e^{-3t} & 0 \\ -\frac{12}{25}e^{25t} + \frac{12}{25}e^{-50t} & 0 & \frac{9}{25}e^{25t} + \frac{15}{25}e^{-50t} \end{bmatrix}$$

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#8.3.30. solution)

From #8.3.10, we have

$$P^TAP = \begin{bmatrix} 0 & 1 & 0 \\ -4/5 & 0 & 3/5 \\ 3/5 & 0 & 4/5 \end{bmatrix} \begin{bmatrix} -2 & 0 & -36 \\ 0 & -3 & 0 \\ -36 & 0 & -23 \end{bmatrix} \begin{bmatrix} 0 & -4/5 & 3/5 \\ 1 & 0 & 0 \\ 0 & 3/5 & 4/5 \end{bmatrix} = \begin{bmatrix} -3 & 0 & 0 \\ 0 & 25 & 0 \\ 0 & 0 & -50 \end{bmatrix} = D.$$

Then, $\cos(\pi A)$ is

$$\cos(\pi A) = Pcos(\pi D)P^T = \begin{bmatrix} 0 & -4/5 & 3/5 \\ 1 & 0 & 0 \\ 0 & 3/5 & 4/5 \end{bmatrix} \begin{bmatrix} \cos(-3\pi) & 0 & 0 \\ 0 & \cos(25\pi) & 0 \\ 0 & 0 & \cos(-50\pi) \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ -4/5 & 0 & 3/5 \\ 3/5 & 0 & 4/5 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -4/5 & 3/5 \\ 1 & 0 & 0 \\ 0 & 3/5 & 4/5 \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ -4/5 & 0 & 3/5 \\ 3/5 & 0 & 4/5 \end{bmatrix} = \begin{bmatrix} -7/25 & 0 & 24/25 \\ 0 & -1 & 0 \\ 24/25 & 0 & 7/25 \end{bmatrix}$$

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#8.4.6. solution)

For the matrix $A=\begin{bmatrix}5&2&0\\2&2&0\\0&0&4\end{bmatrix}$, we write $Q=\boldsymbol{x}^TA\boldsymbol{x}$ by the definition of quadratic form. Note that A is symmetric. First we compute the eigenvalues and eigenvectors of A are

$$\lambda_1=1 \rightarrow v_1=\begin{bmatrix}1\\-2\\0\end{bmatrix}, \quad \lambda_2=4 \rightarrow v_2=\begin{bmatrix}0\\0\\1\end{bmatrix}, \quad \text{and} \quad \lambda_3=6 \rightarrow v_3=\begin{bmatrix}2\\1\\0\end{bmatrix}.$$

Then, $P=\begin{bmatrix}1/\sqrt{5}&0&2/\sqrt{5}\\-2/\sqrt{5}&0&1/\sqrt{5}\\0&1&0\end{bmatrix}$ orthogonally diagonalizes A. Thus, by **Principal Axes Theorem**, we have

$$Q=\boldsymbol{x}^TA\boldsymbol{x}=\boldsymbol{y}^T(P^TAP)\boldsymbol{y}=[y_1\ y_2\ y_3]\begin{bmatrix}1&0&0\\0&4&0\\0&0&6\end{bmatrix}\begin{bmatrix}y_1\\y_2\\y_3\end{bmatrix}=y_1^2+4y_2^2+6y_3^2$$

using the change of variable $\boldsymbol{x}=P\boldsymbol{y}$.

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#8.4.20. solution)

solution (1) :

For any $\vec{x}=\begin{bmatrix}x_1\\x_2\end{bmatrix}\neq\vec{0}$, $Q=-x_1^2-3x_2^2=-(x_1^2+3x_2^2)<0$ is trivial. Thus, by the definition, Q is negative definite.

solution (2) :

Suppose that A is $n\times n$ symmetric matrix. Then, by the **principal axes theorem**, there exists an orthogonal change of variable $\boldsymbol{x}=P\boldsymbol{y}$ such that

$$Q=\boldsymbol{x}^TA\boldsymbol{x}=\boldsymbol{y}^TD\boldsymbol{y}=\lambda_1y_1^2+\lambda_2y_2^2+\cdots+\lambda_ny_n^2.$$

In our problem, the coefficients of quadratic form $Q=-x_1^2-3x_2^2$ are $a_1=-1$ and $a_2=-3$. Then, the matrix form of Q is the diagonal matrix $A=\begin{bmatrix}-1&0\\0&-3\end{bmatrix}$. Since the eigenvalues of the diagonal matrix are the elements on the diagonal, the eigenvalues of A are $\lambda_1=-1=a_1$ and $\lambda_2=-3=a_2$. Thus, by **theorem 8.4.3 (b)**, Q is negative definite.

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#8.4.22. solution)

solution (1) :

For $\vec{x}=\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \neq \vec{0}$, if $x_1 \neq x_2$, $Q=-(x_1-x_2)^2<0$ is trivial. If we suppose $x_1=x_2$, then $Q=0$. Thus, by the definition (the remark at page 489 in section 8.4), Q is negative semidefinite.

solution (2) :

The given quadratic form is

$$Q=-(x_1-x_2)^2=-\left[x_1^2-2x_1x_2+x_2^2\right]=-x_1^2+2x_1x_2-x_2^2$$

This can be written as the matrix form that $A=\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$. The eigenvalues of A are $\lambda=0$ and $\lambda=-2$. Thus, by the definition (the remark at page 489 in section 8.4), Q is negative semidefinite.

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#8.4.24. solution)

solution (1) :

For $\vec{x}=\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \neq \vec{0}$, if $x_1 \neq x_2$ and $x_1=-x_2$, then $Q=x_1x_2<0$ is trivial. If we suppose $x_1=x_2$, then $Q=x_1x_2=x_1^2>0$. Thus, by the definition, Q is indefinite.

solution (2) :

The given quadratic form is $Q=x_1x_2$. This can be written as the matrix form that $A=\begin{bmatrix} 0 & 1/2 \\ 1/2 & 0 \end{bmatrix}$. The eigenvalues of A are $\lambda=1/2$ and $\lambda=-1/2$. Thus, by the **theorem 8.4.3**, Q is indefinite.

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#8.6.8. solution)

The eigenvalues and unit eigenvectors of $A^TA=\begin{bmatrix}18 & 18\\ 18 & 18\end{bmatrix}$ are

$$\lambda_1=36 \rightarrow v_1=\begin{bmatrix}1/\sqrt{2}\\ 1/\sqrt{2}\end{bmatrix} \quad \text{and} \quad \lambda_2=0 \rightarrow v_2=\begin{bmatrix}-1/\sqrt{2}\\ 1/\sqrt{2}\end{bmatrix}.$$

Then, the only singular value of A is $\sigma_1=\sqrt{36}=6$ since $\sigma_i>0$, and we have

$$u_1=\frac{1}{\sigma_1}Av_1=\frac{1}{6}\begin{bmatrix}3 & 3\\ 3 & 3\end{bmatrix}\begin{bmatrix}1/\sqrt{2}\\ 1/\sqrt{2}\end{bmatrix}=\begin{bmatrix}1/\sqrt{2}\\ 1/\sqrt{2}\end{bmatrix}.$$

Since $\{u_1,u_2\}$ is an orthonormal basis for $\mathrm{Col}(A)$, the vector u_2 must be chosen that $u_2=\begin{bmatrix}-1/\sqrt{2}\\ 1/\sqrt{2}\end{bmatrix}$. Therefore, the singular value decomposition of A is

$$A=\begin{bmatrix}3 & 3\\ 3 & 3\end{bmatrix}=\begin{bmatrix}\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\end{bmatrix}\begin{bmatrix}6 & 0\\ 0 & 0\end{bmatrix}\begin{bmatrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\end{bmatrix}=U\Sigma V^T.$$

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#8.6.10. solution)

The eigenvalues and unit eigenvectors of $A^TA=\begin{bmatrix}8 & 4 & -8\\ 4 & 2 & -4\\ -8 & -4 & 8\end{bmatrix}$ are

$$\lambda_1=18 \rightarrow v_1=\begin{bmatrix}2/3\\ 1/3\\ -2/3\end{bmatrix} \quad \text{and} \quad \lambda_2=\lambda_3=0 \rightarrow v_2=\begin{bmatrix}1\\ -2\\ 0\end{bmatrix},\ v_3=\begin{bmatrix}0\\ 2\\ 1\end{bmatrix}.$$

Since $\lambda=0$ has the multiplicity 2, use the **Gram-Schmidt process** in order to obtain the following orthonormal vectors:

$$v_2=\begin{bmatrix}1/\sqrt{5}\\ -2/\sqrt{5}\\ 0\end{bmatrix} \text{ and } v_3=\begin{bmatrix}4/3\sqrt{5}\\ 2/3\sqrt{5}\\ 5/3\sqrt{5}\end{bmatrix}.$$

The only singular value of A is $\sigma_1=\sqrt{18}=3\sqrt{2}$ since $\sigma_i>0$, and we have

$$u_1=\frac{1}{\sigma_1}Av_1=\frac{1}{3\sqrt{2}}\begin{bmatrix}-2 & -1 & 2\\ 2 & 1 & -2\end{bmatrix}\begin{bmatrix}2/3\\ 1/3\\ -2/3\end{bmatrix}=\begin{bmatrix}-1/\sqrt{2}\\ 1/\sqrt{2}\end{bmatrix}.$$

Since $\{u_1,u_2\}$ is an orthonormal basis for $\mathrm{Col}(A)$, the vector u_2 must be chosen that

$u_2 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$. Therefore, the singular value decomposition of A is

$$A = \begin{bmatrix} -2 & -1 & 2 \\ 2 & 1 & -2 \end{bmatrix} = \begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 3\sqrt{2} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 2/3 & 1/3 & -2/3 \\ 1/\sqrt{5} & -2/\sqrt{5} & 0 \\ 4/3\sqrt{5} & 2/3\sqrt{5} & 5/3\sqrt{5} \end{bmatrix} = U\Sigma V^T.$$

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#8.8.16. solution)

The eigenvalues of A are $\lambda = 3 \pm 2i$. Then, the basis vectors corresponding to each eigenvalue are as follows.

$$\lambda = 3+2i \rightarrow \begin{bmatrix} -(4+2i) & 5 & 0 \\ 4 & 4-2i & 0 \end{bmatrix} \sim \begin{bmatrix} -(4+2i) & 5 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow v_1 = \begin{bmatrix} i/2-1 \\ 1 \end{bmatrix}.$$

$$\lambda = 3+2i \rightarrow \begin{bmatrix} -(4+2i) & 5 & 0 \\ 4 & 4-2i & 0 \end{bmatrix} \sim \begin{bmatrix} -(4+2i) & 5 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow v_2 = \begin{bmatrix} -(i/2+1) \\ 1 \end{bmatrix}.$$

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#8.8.18. solution)

The eigenvalues of A are $\lambda = 5 \pm 3i$. Then, the basis vectors corresponding to each eigenvalue are as follows.

$$\lambda = 5+3i \rightarrow \begin{bmatrix} 1-i & 2 & 0 \\ -1 & -1-i & 0 \end{bmatrix} \sim \begin{bmatrix} 1-i & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow v_1 = \begin{bmatrix} -1-i \\ 1 \end{bmatrix}.$$

$$\lambda = 5-3i \rightarrow \begin{bmatrix} 1+i & 2 & 0 \\ -1 & -1+i & 0 \end{bmatrix} \sim \begin{bmatrix} 1+i & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow v_1 = \begin{bmatrix} 1-i \\ 1 \end{bmatrix}.$$

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#8.10.6. solution)

For given the system of differential equations, we have the matrix form $\overrightarrow{y'} = A\vec{y}$, where $A = \begin{bmatrix} 1 & 3 \\ 4 & 5 \end{bmatrix}$. The eigenvalues and the eigenvectors of A are as follows.

$$\lambda = 7 \rightarrow v_1 = \begin{bmatrix} 1/2 \\ 1 \end{bmatrix} \quad \text{and} \quad \lambda = -1 \rightarrow v_2 = \begin{bmatrix} -3/2 \\ 1 \end{bmatrix}.$$

Then, A is diagonalizable since $\{v_1, v_2\}$ is linearly independent, and the matrix P and its inverse are

$$P = \begin{bmatrix} 1/2 & -3/2 \\ 1 & 1 \end{bmatrix}, \ P^{-1} = \begin{bmatrix} 1/2 & 3/4 \\ -1/2 & 1/4 \end{bmatrix}.$$

By theorem 8.10.5, the general solution of the system $\overrightarrow{y'} = A\vec{y}$ is

$$\vec{y} = c_1 e^{7t} v_1 + c_2 e^{-t} v_2 = \begin{bmatrix} c_1 e^{7t}/2 - 3c_2 e^{-t}/2 \\ c_1 e^{7t} + c_2 e^{-t} \end{bmatrix}.$$

Since A is diagonalizable, by the **theorem 8.10.6**, the solution that satisfies the initial conditions is $\vec{y} = e^{tA}\overrightarrow{y_0}$. Compute the exponential of A that

$$e^{tA} = P\begin{bmatrix} e^{7t} & 0 \\ 0 & e^{-t} \end{bmatrix} P^{-1} = \begin{bmatrix} 1/2 & -3/2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} e^{7t} & 0 \\ 0 & e^{-t} \end{bmatrix} \begin{bmatrix} 1/2 & 3/4 \\ -1/2 & 1/4 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{4}e^{7t} + \frac{3}{4}e^{-t} & \frac{3}{8}e^{7t} - \frac{3}{8}e^{-t} \\ \frac{1}{2}e^{7t} - \frac{1}{2}e^{-t} & \frac{3}{4}e^{7t} + \frac{1}{4}e^{-t} \end{bmatrix}$$

Then, the final result is

$$\vec{y} = e^{tA}\overrightarrow{y_0} = \begin{bmatrix} \frac{1}{4}e^{7t} + \frac{3}{4}e^{-t} & \frac{3}{8}e^{7t} - \frac{3}{8}e^{-t} \\ \frac{1}{2}e^{7t} - \frac{1}{2}e^{-t} & \frac{3}{4}e^{7t} + \frac{1}{4}e^{-t} \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{7}{8}e^{7t} + \frac{9}{8}e^{-t} \\ \frac{7}{4}e^{7t} - \frac{3}{4}e^{-t} \end{bmatrix}.$$

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#8.10.12. solution)

We easily know that $A = \begin{bmatrix} 1 & 2 \\ -1 & 4 \end{bmatrix}$ and $y(0) = \overrightarrow{y_0} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. By **theorem 8.10.6**, the solution of initial problem can be $\vec{y} = e^{tA}\overrightarrow{y_0}$. Now we can check that A is diagonalizable, as computing the eigenvalues $\lambda = 2, 3$ and the eigenvectors $v_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, $v_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Thus the matrix $P = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$ and its inverse $P^{-1} = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$. Compute the exponential of A that

$$e^{tA} = P\begin{bmatrix} e^{2t} & 0 \\ 0 & e^{3t} \end{bmatrix} P^{-1} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} e^{2t} & 0 \\ 0 & e^{3t} \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 2e^{2t} - e^{3t} & -2e^{2t} + 2e^{3t} \\ e^{2t} - e^{3t} & -e^{2t} + 2e^{3t} \end{bmatrix}$$

Then, the final result is

$$\vec{y} = e^{tA}\overrightarrow{y_0} = \begin{bmatrix} 2e^{2t} - e^{3t} & -2e^{2t} + 2e^{3t} \\ e^{2t} - e^{3t} & -e^{2t} + 2e^{3t} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -2e^{2t} + 2e^{3t} \\ -e^{2t} + 2e^{3t} \end{bmatrix}.$$

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