Applied Linear Algebra with applications
Assignment #1

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#6.1.18. solution)

Let $e_1=(1,0,0)$, $e_2=(0,1,0)$, $e_3=(0,0,1)$ are the standard unit vectors in R^3. Since T is linear, we obtain the result in Theorem 6.1.4 as

$$T(x)=Ax \text{ where } A=\left[\,T(e_1)\ \ T(e_2)\ \ T(e_3)\right].$$

Thus, using the given information of the problem, we conclude that

$$[\,T]=\left[\,T(e_1)\ \ T(e_2)\ \ T(e_3)\right]=\begin{bmatrix}1&1&0\\1&1&3\\2&2&1\end{bmatrix}.$$

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#6.1.24. solution)

For the vector $\vec{x}=(2,-1)$, by Table 6.1.1 and Table 6.1.2,

(a) $\begin{bmatrix}-1&0\\0&1\end{bmatrix}\begin{bmatrix}2\\-1\end{bmatrix}=\begin{bmatrix}-2\\-1\end{bmatrix}$ (b) $\begin{bmatrix}0&1\\1&0\end{bmatrix}\begin{bmatrix}2\\-1\end{bmatrix}=\begin{bmatrix}-1\\2\end{bmatrix}$ (c) $\begin{bmatrix}1&0\\0&0\end{bmatrix}\begin{bmatrix}2\\-1\end{bmatrix}=\begin{bmatrix}2\\0\end{bmatrix}$ (d) $\begin{bmatrix}0&0\\0&1\end{bmatrix}\begin{bmatrix}2\\-1\end{bmatrix}=\begin{bmatrix}0\\-1\end{bmatrix}$.

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#6.1.28. solution)

(a) $H_{120^\circ}\vec{x}=\left(\begin{bmatrix}\cos 2\theta & \sin 2\theta\\ \sin 2\theta & -\cos 2\theta\end{bmatrix}_{\theta=120^\circ}\right)\begin{bmatrix}4\\3\end{bmatrix}=\begin{bmatrix}-1/2 & -\sqrt{3}/2\\ -\sqrt{3}/2 & 1/2\end{bmatrix}\begin{bmatrix}4\\3\end{bmatrix}=\begin{bmatrix}-2 & -3\sqrt{3}/2\\ -2\sqrt{3} & 3/2\end{bmatrix}\approx\begin{bmatrix}-4.598\\-1.964\end{bmatrix}$

(b) $P_{120^\circ}\vec{x}=\left(\begin{bmatrix}\cos^2\theta & \sin\theta\cos\theta\\ \sin\theta\cos\theta & -\sin^2\theta\end{bmatrix}_{\theta=120^\circ}\right)\begin{bmatrix}4\\3\end{bmatrix}=\begin{bmatrix}-1/4 & -\sqrt{3}/4\\ -\sqrt{3}/4 & 3/4\end{bmatrix}\begin{bmatrix}4\\3\end{bmatrix}=\begin{bmatrix}1 & -3\sqrt{3}/4\\ -\sqrt{3} & 9/4\end{bmatrix}\approx\begin{bmatrix}-0.299\\0.518\end{bmatrix}$

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#6.1.32. solution)

By the exercise 31, we have (a) $H_L = \dfrac{1}{1+m^2}\begin{bmatrix} 1-m^2 & 2m \\ 2m & m^2-1 \end{bmatrix}$ and (b) $P_L = \dfrac{1}{1+m^2}\begin{bmatrix} 1 & m \\ m & m^2 \end{bmatrix}$. Then,

(a) $m=2 \;\Rightarrow\; H = H_L = \dfrac{1}{5}\begin{bmatrix} -3 & 4 \\ 4 & 3 \end{bmatrix}$, and $H\begin{bmatrix} 3 \\ 4 \end{bmatrix} = \dfrac{1}{5}\begin{bmatrix} -3 & 4 \\ 4 & 3 \end{bmatrix}\begin{bmatrix} 3 \\ 4 \end{bmatrix} = \dfrac{1}{5}\begin{bmatrix} 7 \\ 24 \end{bmatrix} = \begin{bmatrix} 1.4 \\ 4.8 \end{bmatrix}$.

(b) $m=2 \;\Rightarrow\; P = P_L = \dfrac{1}{5}\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$, and $P\begin{bmatrix} 3 \\ 4 \end{bmatrix} = \dfrac{1}{5}\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} 3 \\ 4 \end{bmatrix} = \dfrac{1}{5}\begin{bmatrix} 11 \\ 22 \end{bmatrix} = \begin{bmatrix} 2.2 \\ 4.4 \end{bmatrix}$.

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#6.2.14. solution)

Let $e_1 = (1,0,0)$, $e_2 = (0,1,0)$, $e_3 = (0,0,1)$ are the standard unit vectors in $\mathbb{R}^3$. Then,

$$e_1 = (1,0,0) \to (1,0,0) \to (1,0,0) \to (-1,0,0) = T(e_1),$$

$$e_2 = (0,1,0) \to (0,1,0) \to (0,-1,0) \to (0,-1,0) = T(e_2),$$

$$e_3 = (0,0,1) \to (0,0,-1) \to (0,0,-1) \to (0,0,-1) = T(e_3)$$

by definition of T in this problem. Then, by using Theorem 6.1.4, we obtain $[T] = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$.

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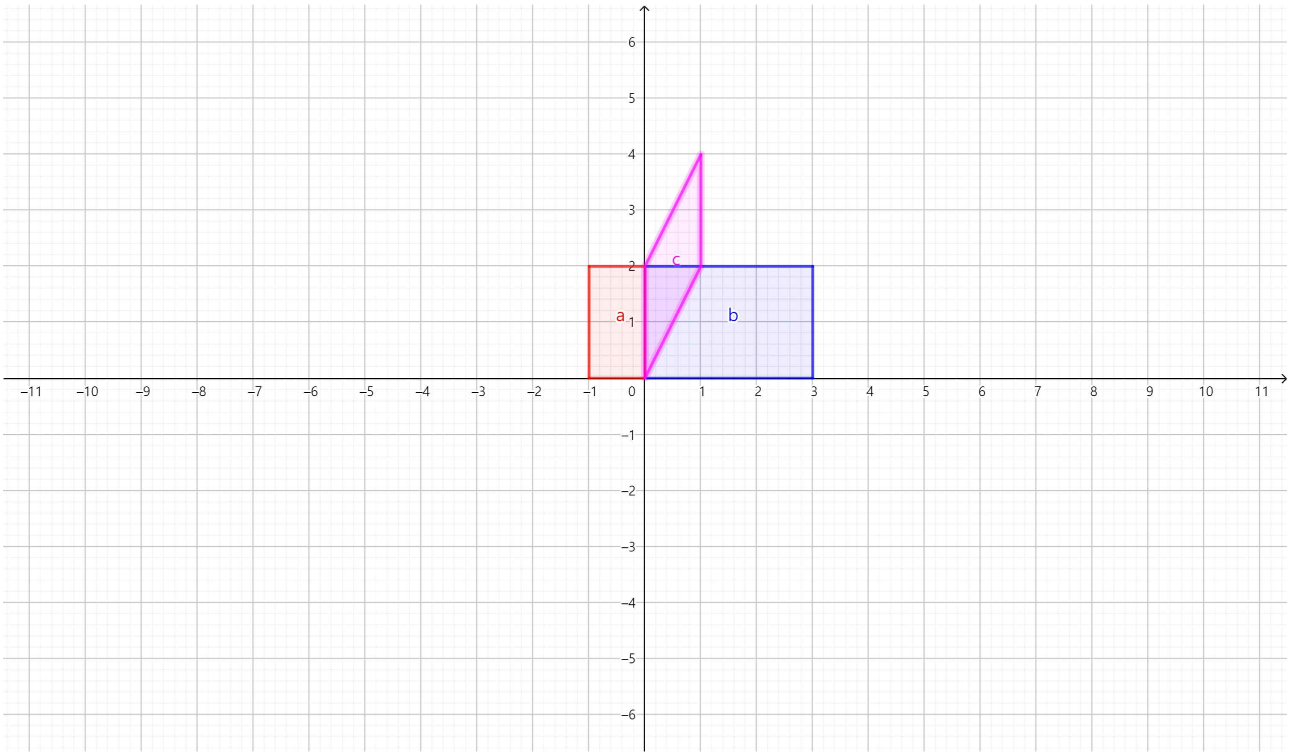
#6.2.16. solution)



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#6.2.24. solution)

For the vector $(-2,1,2)$, using Table 6.2.6, then

(a) $R_{60^\circ} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}_{\theta=60^\circ} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/2 & -\sqrt{3}/2 \\ 0 & \sqrt{3}/2 & 1/2 \end{bmatrix}$. Then,

$$R_{60^\circ}\begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/2 & -\sqrt{3}/2 \\ 0 & \sqrt{3}/2 & 1/2 \end{bmatrix}\begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -2 \\ 1/2-\sqrt{3} \\ \sqrt{3}/2+1 \end{bmatrix}.$$

(b) $R_{30^\circ} = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}_{\theta=30^\circ} = \begin{bmatrix} \sqrt{3}/2 & 0 & 1/2 \\ 0 & 1 & 0 \\ -1/2 & 0 & \sqrt{3}/2 \end{bmatrix}$. Then,

$$R_{30^\circ}\begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} \sqrt{3}/2 & 0 & 1/2 \\ 0 & 1 & 0 \\ -1/2 & 0 & \sqrt{3}/2 \end{bmatrix}\begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -\sqrt{3}+1 \\ 1 \\ 1+\sqrt{3} \end{bmatrix}.$$

(c) $R_{-45^\circ} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}_{\theta=-45^\circ} = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ -1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$. Then,

$$R_{-45^\circ}\begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ -1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -1/\sqrt{2} \\ 3/\sqrt{2} \\ 2 \end{bmatrix}.$$

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#6.2.30. solution)

(a) Define the linear transformation $T: \mathrm{R}^3 \to \mathrm{R}^3$ by $T(x,y,z) = (x+kz, y+kz, z)$, and let $e_1 = (1,0,0)$, $e_2 = (0,1,0)$, $e_3 = (0,0,1)$ are the standard unit vectors in R^3. Then, we obtain that

$$T(e_1) = (1,0,0),\ T(e_2) = (0,1,0),\ T(e_3) = (k,k,1).$$

Thus, the standard matrix for T is $[T] = \begin{bmatrix} 1 & 0 & k \\ 0 & 1 & k \\ 0 & 0 & 1 \end{bmatrix}$ by using Theorem 6.1.4.

(b) If we define T as $T_{xz}(x,y,z) = (x+ky,\ y,\ z+ky)$, $T_{yz}(x,y,z) = (x,\ y+kx,\ z+kx)$, respectively, we can get the results that the standard matrix of each linear transformation is

$$[T_{xz}] = \begin{bmatrix} 1 & k & 0 \\ 0 & 1 & 0 \\ 0 & k & 1 \end{bmatrix},\quad [T_{yz}] = \begin{bmatrix} 1 & 0 & 0 \\ k & 1 & 0 \\ k & 0 & 1 \end{bmatrix}.$$

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#6.3.6. solution)

Let $T: \mathrm{R}^3 \to \mathrm{R}^4$ be the linear transformation has the standard matrix A. Then the kernel of T is the solution set of the linear system

$$A\vec{x} = \vec{0} \iff \begin{bmatrix} 2 & 1 & -1 \\ 1 & -2 & 1 \\ 1 & -7 & 4 \\ 3 & 4 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

Since the reduced row echelon form of the system is $\begin{bmatrix} 1 & 0 & -1/5 & 0 \\ 0 & 1 & 3/5 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, $x_1 = 1/5$, $x_2 = -3/5$ and x_3 is a free variable. Thus, the solution set is represented as

$$\vec{x} = t\begin{bmatrix} 1/5 \\ -3/5 \\ 1 \end{bmatrix} \text{ or } \vec{x} = t\begin{bmatrix} 1 \\ -3 \\ 5 \end{bmatrix} \text{ where } -\infty < t < \infty.$$

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#6.3.10. solution)

(a) According Theorem 6.3.15, we can easily know that $\vec{b} \in \mathrm{Col(A)} \iff A\vec{x} = \vec{b}$ is consistent for $\forall\, \vec{b} \in \mathrm{R}^n$. Hence, given A and $\vec{b}$, check $A\vec{x} = \vec{b}$ that

$$\begin{bmatrix} 3 & -2 & 1 & 5 \\ 1 & 4 & 5 & -3 \\ 0 & 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix};$$

$\therefore$ this linear system is inconsistent. Thus, $\vec{b} \notin \mathrm{Col(A)}$.

(b) Check $A\vec{x} = \vec{b}$ that

$$\begin{bmatrix} 3 & -2 & 1 & 5 \\ 1 & 4 & 5 & -3 \\ 0 & 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 6 \\ 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix};$$

$\therefore$ this linear system is consistent. The general solution is

$$\vec{x} = \begin{bmatrix} 2-s-t \\ 1-s+t \\ s \\ t \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + s\begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \end{bmatrix}.$$

Let $s = t = 0$, then we can express $\vec{b}$ as a linear combination of the column vectors of A as follows.

$$\begin{bmatrix} 4 \\ 6 \\ 1 \end{bmatrix} = \vec{b} = 2c_1(A) + (1)c_2(A) + 0c_3(A) + 0c_4(A) = 2\begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} -2 \\ 4 \\ 1 \end{bmatrix}, \text{ where } \mathrm{Col(A)} = \mathrm{span}\{c_1, c_2, c_3, c_4\}.$$

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#6.3.16. solution)

The system of equations is represented as the matrix representation

$$\begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}.$$

Thus, the standard matrix of the operator is $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{bmatrix}$. Since $\det(A) = -1 \neq 0$, the operator is both 1-1 and onto. (refer to Theorem 6.3.15)

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#6.3.20. solution)

(a) By Theorem 6.3.13, $T_A : \mathbb{R}^2 \to \mathbb{R}^3$ is onto $\Leftrightarrow$ $A\vec{x} = \vec{b}$ is consistent for every $\vec{b} \in \mathbb{R}^2$. The augmented matrix of $A\vec{x} = \vec{b}$ is

$$\begin{bmatrix} 1 & -1 & b_1 \\ 2 & 0 & b_2 \\ 3 & -4 & b_3 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & b_1 \\ 0 & 2 & b_2 - 2b_1 \\ 0 & -1 & b_3 - 3b_1 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & b_1 \\ 0 & 2 & b_2 - 2b_1 \\ 0 & -2 & 2b_3 - 6b_1 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & b_1 \\ 0 & 2 & b_2 - 2b_1 \\ 0 & 0 & 2b_3 + b_2 - 8b_1 \end{bmatrix}.$$

From this, we conclude that $A\vec{x} = \vec{b}$ is consistent $\Leftrightarrow$ $2b_3 + b_2 - 8b_1 = 0$. Thus T_A is not onto.

(b) Similarly, the augmented matrix of $A\vec{x} = \vec{b}$ is

$$\begin{bmatrix} 1 & 2 & 3 & b_1 \\ -1 & 0 & -4 & b_2 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 3 & b_1 \\ 0 & 2 & -1 & b_2 + b_1 \end{bmatrix}.$$

This satisfies that $A\vec{x} = \vec{b}$ is consistent for every $\vec{b} \in \mathbb{R}^2$. Thus, T_A is onto.

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#6.4.6. solution)

By Theorem 6.4.1 and 6.4.2, $[T_3 \circ T_2 \circ T_1] = [T_3][T_2][T_1]$. Thus,

(a) $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1/2 & -\sqrt{3}/2 \end{bmatrix}$

(b) $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} -\sqrt{2} & \sqrt{2} \\ \sqrt{2} & \sqrt{2} \end{bmatrix}$

(c) $R_{60^\circ} R_{105^\circ} R_{15^\circ} = R_{60^\circ + 105^\circ + 15^\circ} = R_{180^\circ} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$

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#6.4.8. solution)

Similar to exercise 6.4.6, we can calculate

(a) $\begin{bmatrix} 1&0&0 \\ 0&1&0 \\ 0&0&0 \end{bmatrix}\begin{bmatrix} 1&0&0 \\ 0&-1&0 \\ 0&0&-1 \end{bmatrix}=\begin{bmatrix} 1&0&0 \\ 0&-1&0 \\ 0&0&0 \end{bmatrix}$ is the standard matrix for the reflection followed by the projection.

(b) $\begin{bmatrix} 1/3&0&0 \\ 0&1/3&0 \\ 0&0&1/3 \end{bmatrix}\begin{bmatrix} 1&0&0 \\ 0&\sqrt{3}/2&-1/2 \\ 0&1/2&\sqrt{3}/2 \end{bmatrix}=\begin{bmatrix} 1/3&0&0 \\ 0&\sqrt{3}/6&-1/6 \\ 0&1/6&\sqrt{3}/6 \end{bmatrix}$ is the standard matrix for the rotation followed by the contraction.

(c) $\begin{bmatrix} 1&0&0 \\ 0&1&0 \\ 0&0&-1 \end{bmatrix}\begin{bmatrix} 1&0&0 \\ 0&0&0 \\ 0&0&1 \end{bmatrix}=\begin{bmatrix} 1&0&0 \\ 0&0&0 \\ 0&0&-1 \end{bmatrix}$ is the standard matrix for the projection followed by the reflection.

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#6.4.14. solution)

By Theorem 6.4.6, we obtain the formula $[T^{-1}]=[T]^{-1}$, T is a linear operator. Using this, we conclude that

(a) Reflection of R^2 about the y-axis (Reflection → Reflection in the same y-axis)
(b) Rotation of R^2 about the origin through an angle of $\pi/6$ (Rotation → Rotation, $-\pi/6 \to \pi/6$)
(c) Dilation by a factor of 5. (Contraction → Dilation, $1/5 \to 5$)
(d) Compression in the x-direction with factor $1/7$. (Expansion → Compression, $7 \to 1/7$)

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#6.4.22. solution)

We can find the standard matrix of T is $A=\begin{bmatrix} 1&2&1 \\ 2&1&4 \\ 7&4&5 \end{bmatrix}$ (refer to the exercise 6.3.16). Since $\det A \neq 0$, A is invertible by Theorem 6.3.15. Also, by Theorem 6.3.15, T is one-to-one.

By Theorem 6.4.6, we obtain the standard matrix of T^{-1} is

$$A^{-1}=\begin{bmatrix} -\frac{11}{26} & -\frac{3}{13} & \frac{7}{26} \\ \frac{9}{13} & -\frac{1}{13} & -\frac{1}{13} \\ \frac{1}{26} & \frac{5}{13} & -\frac{3}{26} \end{bmatrix}.$$

Thus, the formula for T^{-1} is

$$T^{-1}(w_1,w_2,w_3)=\left(-\frac{11}{26}w_1-\frac{3}{13}w_2+\frac{7}{26}w_3,\ \frac{9}{13}w_1-\frac{1}{13}w_2-\frac{1}{13}w_3,\ \frac{1}{26}w_1+\frac{5}{13}w_2-\frac{3}{26}w_3\right)$$

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#6.4.24. solution)

Note that $[T_1 \circ T_2] = [T_1][T_2]$ by Theorem 6.4.2.

(a) $[T_1][T_2] = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$, $[T_2][T_1] = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$.

$\therefore\ [T_1 \circ T_2] = [T_1][T_2] = [T_2][T_1] = [T_2 \circ T_1] \Rightarrow T_1 \circ T_2 = T_2 \circ T_1$

(b) $[T_1][T_2] = \begin{bmatrix} \cos\theta & -\sin\theta \\ 0 & 0 \end{bmatrix}$ and $[T_2][T_1] = \begin{bmatrix} \cos\theta & 0 \\ \sin\theta & 0 \end{bmatrix}$ where $[T_1] = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $[T_2] = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$.

$\therefore\ [T_1 \circ T_2] = [T_1][T_2] \neq [T_2][T_1] = [T_2 \circ T_1] \Rightarrow T_1 \circ T_2 \neq T_2 \circ T_1$

(c) $[T_1] = \begin{bmatrix} k & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & k \end{bmatrix} = kI$, where I is identity. Then $[T_1][T_2] = (kI)[T_2]$ and $[T_2][T_1] = [T_2](kI) = k[T_2]$.

$\therefore\ [T_1 \circ T_2] = [T_1][T_2] = [T_2][T_1] = [T_2 \circ T_1] \Rightarrow T_1 \circ T_2 = T_2 \circ T_1 = kT_2$

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