Midterm Homework

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Let $\sigma(u,v)=(\cos u\cos v,\cos u\sin v,\sin u)$ and $\rho(r,s)=(-\cos r\cos s,\sin r,\cos r\sin s)$ be two patches for the unit sphere $S^2=\{(x,y,z)|x^2+y^2+z^2=1\}$ such that both σ and ρ are defined in $U=\left(-\frac{\pi}{2},\frac{\pi}{2}\right)\times(0,2\pi)$. Note that obtained from σ by applying the rotation about the $x-axis$ by 90 degrees followed by the rotation about the $z-axis$ by 180 degrees.

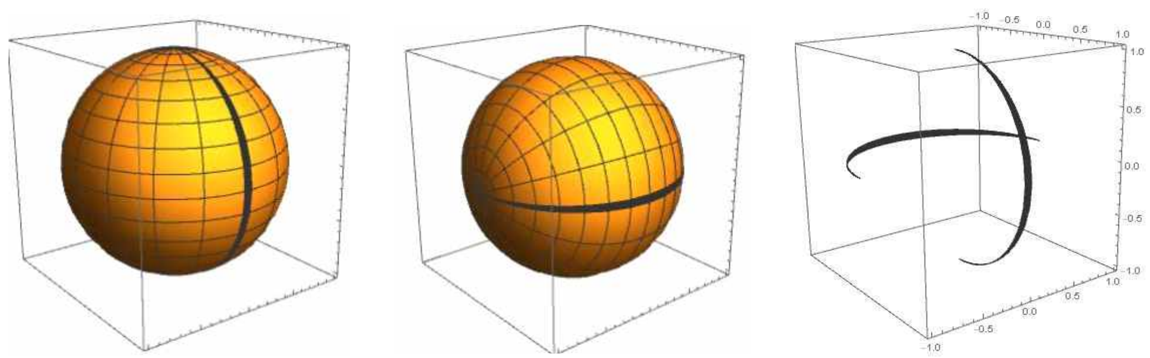
1. What is the domain V of the transition map $\Phi(r,s)=(u,v)$ from σ to ρ? What is $U-V$?

<u>Solution.</u>

The overlapping region of two patches σ,ρ is shown in Figure 1.(c). Cautions in this region is the points not in V_σ, V_ρ . Since the transition map $\Phi : V_\sigma \to V_\rho$ is homeomorphism, we know that $\sigma(U_\sigma)\cap\rho(U_\rho)=V_\sigma\cup V_\rho=V_\sigma\cap V_\rho$, and that two patches have the same domain ($\because Rot_z(180)Rot_x(90)\sigma(u,v)\equiv\rho(r,s)$).

Thus, we must eliminate the points not in $V_\sigma\cap V_\rho$ simultaneously, hence, the removing points about u : 0 and the removing points about v : $\pi/2$, $3\pi/2$. Therefore, the domain of Φ is $V=\left(-\frac{\pi}{2},0\right)\cup\left(0,\frac{\pi}{2}\right)\times\left(0,\frac{\pi}{2}\right)\cup\left(\frac{\pi}{2},\frac{3\pi}{2}\right)\cup\left(\frac{3\pi}{2},2\pi\right)$ and $U-V=\{0\}\times\left\{\frac{\pi}{2},\frac{3\pi}{2}\right\}$.

$\square$



(a) (b) (c)

Figure 1. (a) the patch $\sigma(u,v)$ of S^2, (b) the patch $\rho(r,s)$ of S^2, (c) (a) and (b).

2. Write $u(r,s)$ *and* $v(r,s)$ *using the inverse trigonometric functions* $y=\sin^{-1}x$ *(from* $[-1,1]$ *into* $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$*) and* $y=\cos^{-1}x$ *(from* $[-1,1]$ *into* $[0,\pi]$*). The formula for* $v(r,s)$ *changes according to the location of* (r,s) *in* V*. Plot* $u(r,s)$ *and* $v(r,s)$ *using Mathematica.*

Solution.

We want to find a function Φ such that $\sigma(\Phi(u,v))=\rho(r,s)$ when $\Phi(r,s)=(u(r,s),v(r,s))$, Φ : transition map, and so note that $\cos(\sin^{-1}(x))=\sqrt{1-x^2}$ and $\sin(\cos^{-1}(x))=\sqrt{1-x^2}$, then we can let :

$$u(r,s)=\sin^{-1}(\cos r\sin s) \Rightarrow v(r,s)=\cos^{-1}\left(\frac{-\cos r\cos s}{\sqrt{1-(-\cos r\sin s)^2}}\right)$$

$$\text{where } u,v\in\left\{\left(-\frac{\pi}{2},\frac{\pi}{2}\right)-\{0\}\right\}\times\left(0,\frac{\pi}{2}\right)\cup\left(\frac{\pi}{2},\pi\right].$$

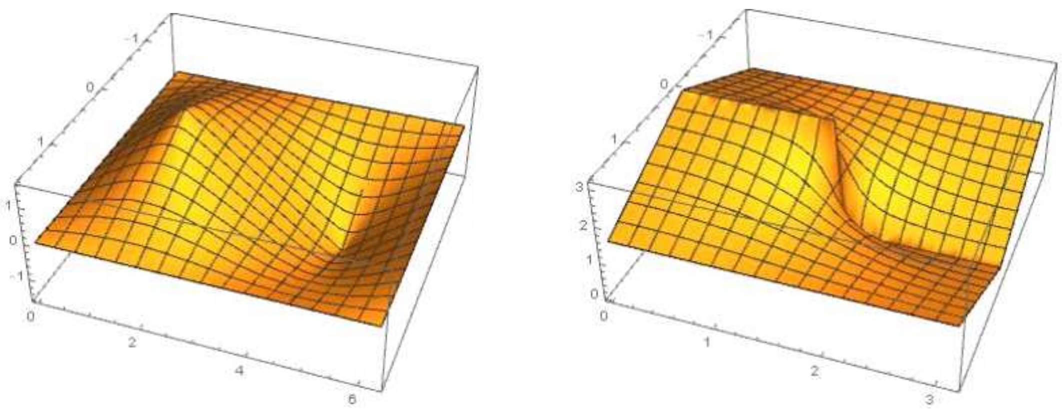
Since $V=\left(-\frac{\pi}{2},0\right)\cup\left(0,\frac{\pi}{2}\right)\times\left(0,\frac{\pi}{2}\right)\cup\left(\frac{\pi}{2},\frac{3\pi}{2}\right)\cup\left(\frac{3\pi}{2},2\pi\right)$, also $v\in\left(\pi,\frac{3\pi}{2}\right)\cup\left(\frac{3\pi}{2},2\pi\right)$. So we now let :

$$v(r,s)=\begin{cases}\cos^{-1}\left(\dfrac{-\cos r\cos s}{\sqrt{1-(-\cos r\sin s)^2}}\right) & \text{for } \left(0,\frac{\pi}{2}\right)\cup\left(\frac{\pi}{2},\pi\right] \\ -\cos^{-1}\left(\dfrac{-\cos r\cos s}{\sqrt{1-(-\cos r\sin s)^2}}\right) & \text{for } \left(\pi,\frac{3\pi}{2}\right)\cup\left(\frac{3\pi}{2},2\pi\right)\end{cases}$$

Then, the equation of σ and ρ is satisfied the condition of transition formula in V :

$$\sigma(\Phi(u(r,s),v(r,s)))=\rho(r,s).$$

$\square$



(a) (b)

Figure 2. The graph of u,v using mathematica, **(a)** $u(r,s)$ in V , **(b)** $v(r,s)$ in V

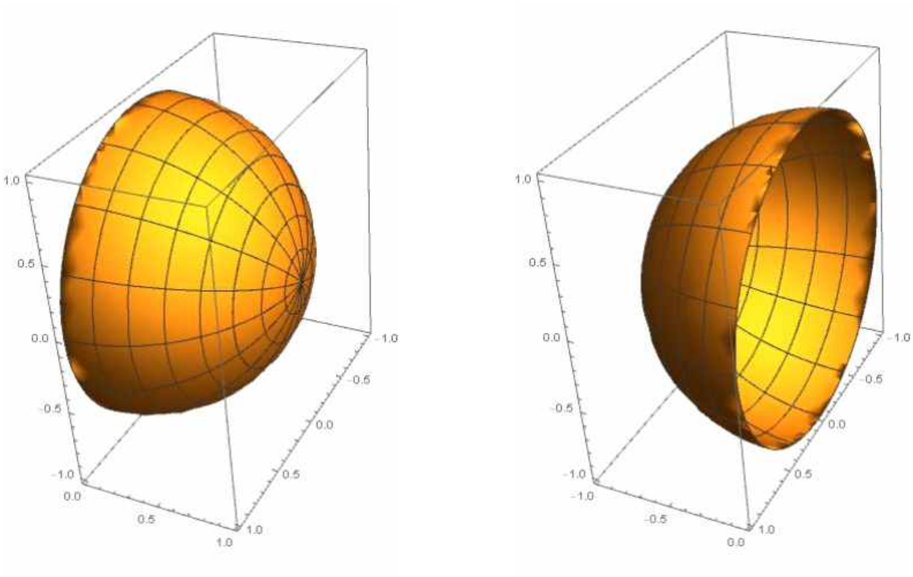
3. Verify your answer by plotting $\sigma(\Phi(r,s))$. Do you get the whole sphere (except a few semicircles)?

<u>Solution.</u>

Look at the Figure 3. we can find the whole sphere except a few semicircles when

$$v(r,s)=\begin{cases} \cos^{-1}\left(\dfrac{-\cos r\cos s}{\sqrt{1-(-\cos r\sin s)^2}}\right) & \text{for}\left(0,\dfrac{\pi}{2}\right)\cup\left(\dfrac{\pi}{2},\pi\right] \\ -\cos^{-1}\left(\dfrac{-\cos r\cos s}{\sqrt{1-(-\cos r\sin s)^2}}\right) & \text{for}\left(\pi,\dfrac{3\pi}{2}\right)\cup\left(\dfrac{3\pi}{2},2\pi\right) \end{cases}$$

□



(a) (b)

Figure 3. The graph of $\sigma(\Phi(r,s))$. (a) $v(r,s)$ in $0\sim\pi$, (b) $v(r,s)$ in $\pi\sim 2\pi$

Mathematica programming codes in this homework

❖ ***Figure 1 :*** (a) the patch $\sigma(u,v)$ of S^2, (b) the patch $\rho(r,s)$ of S^2, (c) (a) and (b).

```
Show[sigma,sigmaline,PlotRange -> All]
Show[rho,rholine,PlotRange -> All]
sigma=ParametricPlot3D[{Cos[u]*Cos[v],Cos[u]*Sin[v],Sin[u]},{u,-Pi/2,Pi/2},{v,0,2*Pi}];

sigmaline=ParametricPlot3D[{Cos[u]*Cos[v],Cos[u]*Sin[v],Sin[u]},{u,-Pi/2,Pi/2},{v,-0.04,0.04}];
rho=ParametricPlot3D[{-Cos[r]*Cos[s],Sin[r],Cos[r]*Sin[s]},{r,-Pi/2,Pi/2},{s,0,2*Pi}];
rholine=ParametricPlot3D[{-Cos[r]*Cos[s],Sin[r],Cos[r]*Sin[s]},{r,-Pi/2,Pi/2},{s,-0.04,0.04}];
```

❖ ***Figure 2 :*** The graph of u,v using mathematica, (a) $u(r,s)$ in V , (b) $v(r,s)$ in V

```
u=ArcSin[Cos[r]*Sin[s]]
Plot3D[u,{r,-Pi/2,Pi/2},{s,0,2*Pi}]
v=ArcCos[(-Cos[r]*Cos[s])/Sqrt[1-(-Cos[r]Sin[s])^2]]
Plot3D[v, {r,-Pi/2,Pi/2},{s,0,Pi}]
```

❖ ***Figure 3 :*** The graph of $\sigma(\Phi(r,s))$. (a) $v(r,s)$ in $0 \sim \pi$, (b) $v(r,s)$ in $\pi \sim 2\pi$

```
sigma_PHI1=ParametricPlot3D[{Cos[ArcSin[Cos[r]*Sin[s]]]*Cos[ArcCos[(-Cos[r]*Cos[s])/Sqrt[
1-(-Cos[r]Sin[s])^2]]],Cos[ArcSin[Cos[r]*Sin[s]]]*Sin[ArcCos[(-Cos[r]*Cos[s])/Sqrt[1-(-Cos[r]
Sin[s])^2]]],Sin[ArcSin[Cos[r]*Sin[s]]]},{r,-Pi/2,Pi/2},{s,0,2*Pi}]

sigma_PHI2=ParametricPlot3D[{Cos[ArcSin[Cos[r]*Sin[s]]]*Cos[-ArcCos[(-Cos[r]*Cos[s])/Sqr
t[1-(-Cos[r]Sin[s])^2]]],Cos[ArcSin[Cos[r]*Sin[s]]]*Sin[-ArcCos[(-Cos[r]*Cos[s])/Sqrt[1-(-Co
s[r]Sin[s])^2]]],Sin[ArcSin[Cos[r]*Sin[s]]]},{r,-Pi/2,Pi/2},{s,0,2*Pi}]
```