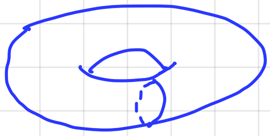
1.1.1 Show that if $M^m, N^n$ are smooth manifolds, then $M^m \times N^n$ is also a $(m+n)$ dimensional smooth manifold. Hence, the $n$-dimensional torus or simply $n$-torus

$$T^n = \underbrace{S^1 \times \cdots \times S^1}_{n} \qquad T^2 =$$



2-dim manifold

is a smooth manifold.

proof) We want to show :

[1] $M^m \times N^n$ is manifold [2] It is smooth manifold

[1] : ① Hausdorff

Since $M^m, N^n$ are smooth manifold, these are Hausdorff. Then, for any $U_M, V_M$ in $M^m$ and $U_N, V_N$ in $N^n$, let $U_M \times U_N = U \subset M^m \times N^n$ and $V_M \times V_N \subset N^n$, then $U_M \times U_N \cap V_M \times V_N = \phi$. ... (*)

($\because$)

(1) : $(U_M \times U_N) \cap V_M = V_M \cap U_M \times V_M \cap U_N = \phi$

because $V_M \cap U_M = \phi$ since $M^m$ is Hausdorff.

(2) : $(U_M \times U_N) \cap V_N = V_N \cap U_M \times V_N \cap U_N = \phi$

because $V_N \cap U_N = \phi$ since $N^n$ is Hausdorff.

(*) $= (1) \times (2) = \phi$.

Thus, $M^m \times N^n$ is Hausdorff.

② second countable

By the assumption, $M^m, N^n$ have a countable basis $\mathcal{B}_M, \mathcal{B}_N$. Then trivially $\mathcal{B}_M \times \mathcal{B}_N \subset M^m \times N^n$ and we can pick $\mathcal{B}_M \times \mathcal{B}_N$ is a countable basis for $M^m \times N^n$.

($\because$) $x \in \mathcal{B}_M$ & $y \in \mathcal{B}_N \Rightarrow (x, y) \in \mathcal{B}_M \times \mathcal{B}_N \subset M^m \times N^n$

and $\mathcal{B}_M, \mathcal{B}_N$ : open $\Rightarrow \mathcal{B}_M \times \mathcal{B}_N$ : open.

$\mathcal{B}_M \times \mathcal{B}_N$ : countable since $\mathcal{B}_M, \mathcal{B}_N$ are countable.

pf) Let $\mathcal{B}_M \times \mathcal{B}_N$ : finite $\to$ trivial.

We assume $\mathcal{B}_M, \mathcal{B}_N$ : countably infinite.

$$\begin{array}{llll}
(\mathcal{B}_M^0, \mathcal{B}_N^0) & (\mathcal{B}_M^0, \mathcal{B}_N^1) & (\mathcal{B}_M^0, \mathcal{B}_N^2) & \cdots \\
(\mathcal{B}_M^1, \mathcal{B}_N^0) & (\mathcal{B}_M^1, \mathcal{B}_N^1) & (\mathcal{B}_M^1, \mathcal{B}_N^2) & \cdots \\
(\mathcal{B}_M^2, \mathcal{B}_N^0) & (\mathcal{B}_M^2, \mathcal{B}_N^1) & (\mathcal{B}_M^2, \mathcal{B}_N^2) & \cdots \\
\vdots & \vdots & \vdots & \ddots
\end{array}$$

First, we pick $(\mathcal{B}_M^0, \mathcal{B}_N^0)$, then we pick $(\mathcal{B}_M^0, \mathcal{B}_N^1)$, $(\mathcal{B}_M^1, \mathcal{B}_N^0)$, then we pick $(\mathcal{B}_M^0, \mathcal{B}_N^2)$, $(\mathcal{B}_M^1, \mathcal{B}_N^1)$ $(\mathcal{B}_M^2, \mathcal{B}_N^0)$ $\cdots\cdots$

Continue to this processes, then we can define the one-to-one correspondence between $\mathcal{B}_M \times \mathcal{B}_N \to \mathbb{N}$ (the set of natural #).

Thus, by definition of countable, the assertion is proved.

③ Homeomorphism

Let $\varphi_M : U \to \mathbb{R}^m$ & $p \in \varphi_M(U)$ and $\varphi_N : V \to \mathbb{R}^n$ & $q \in \varphi_N(V)$, then we can define

$\varphi_{MN}(r) = (\varphi_M \times \varphi_N)(p, q) = (\varphi_M(p), \varphi_N(q))$

if $\varphi_{MN} : U \times V \to \mathbb{R}^{m+n}$.

(i) injective

$\varphi_{MN}(r_1) = \varphi_{MN}(r_2)$

$\Rightarrow (\varphi_M(p_1), \varphi_N(q_1)) = (\varphi_M(p_2), \varphi_N(q_2))$

$\Rightarrow \varphi_M(p_1) = \varphi_M(p_2)$ & $\varphi_N(q_1) = \varphi_N(q_2)$

$\Rightarrow p_1 = p_2$ & $q_1 = q_2$ since $\varphi_M, \varphi_N$ : injective.

(ii) surjective

For $\forall\, y = \varphi_{MN}(\bar{r}) \in \mathbb{R}^{m+n}$, $\exists\, (\bar{p}, \bar{q}) \in U \times V$ s.t.

$y = \varphi_{MN}(\bar{r}) = (\varphi_M(\bar{p}), \varphi_N(\bar{q}))$ since

$\varphi_M$ & $\varphi_N$ are surjective.

By (i), (ii), $\varphi_{MN}$ : bijection on $U \times V \subset M^m \times N^n$.

Thus, $\exists\, \varphi_{MN}^{-1}$ : inverse of $\varphi_{MN}$.

In case of continuity, for any $\theta$, $\beta$ be open in $\mathbb{R}^m, \mathbb{R}^n$,

$\varphi_M^{-1}(\theta), \varphi_N^{-1}(\beta)$ are open by the assumption.

Since $\theta \times \beta$ : open and its pre image

$\varphi_{MN}^{-1}(\theta \times \beta)$ is open.

$\therefore$ $\varphi_{MN}$ is continuous.

($\because$) if $\theta = \varphi_M(\alpha)$, $\beta = \varphi_N(\beta)$ for any open sets

$\alpha$, $\beta$ in $U, V$, then

$\varphi_{MN}^{-1}(\varphi_{MN}(\theta \times \beta)) = \theta \times \beta = \varphi_M(\alpha) \times \varphi_N(\beta)$ : open.

For $\varphi_{MN}^{-1}$ : inverse of $\varphi_{MN}$, $\varphi_{MN}^{-1}(\Theta \times \beta)$ is open

$\Rightarrow \varphi_{MN}(\varphi_{MN}^{-1}(\Theta \times \beta)) = \Theta \times \beta$ is open

$\therefore$ $\varphi_{MN}^{-1}$ is continuous.

$\therefore$ $\varphi_{MN}$ is Homeomorphism.

Therefore, $M^m \times N^n$ is a manifold.

[2] : Since $M^m$, $N^n$ are smooth manifold, they have a $C^\infty$-structure, so that the coordinate charts $(U, \varphi_M)$, $(V, \varphi_N)$ is $C^\infty$-compatible with all charts in the atlas of $M^m$, $N^n$, respectively.

By [1], we defined the homeomorphism $\varphi_{MN}$, hence we can write the coordinate chart of $M^m \times N^n$ that $(U \times V, \varphi_{MN})$. Consider another chart $(U' \times V', \varphi_{MN}^*)$, then

$$\varphi_{MN} \circ \varphi_{MN}^* = (\varphi_M \times \varphi_N) \circ (\varphi_M^* \times \varphi_N^*)^{-1}$$

$$= \varphi_M \circ \varphi_M^{*-1} \times \varphi_N \circ \varphi_N^{*-1}.$$

Since $\varphi_M$, $\varphi_N$, $\varphi_M^{*-1}$, $\varphi_N^{*-1}$ are $C^\infty$,

$\varphi_{MN} \circ \varphi_{MN}^*$ is $C^\infty$.

$\therefore$ $M^m \times N^n$ is a smooth manifold.

Thus, By the proof above, $T^n$ is smooth manifold. $\square$

1.1.2 Let $U \subset \mathbb{R}^n$ be open and $f: U \to \mathbb{R}^m$ be continuous. Show that the graph of $f$

$$\Gamma_f = \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^m : x \in U \text{ and } y = f(x)\}$$

is an $n$-dimensional manifold.

proof) By the example 1.7. (i) of the lecture note of professor Han, $\mathbb{R}^n$, $\mathbb{R}^m$ are $n$, $m$ dimensional smooth manifold and hence $\mathbb{R}^n \times \mathbb{R}^m$ is smooth manifold by the exercise 1.1.1.

Thus, the graph of $f$ $\Gamma_f$ is the subspace topology of $\mathbb{R}^n \times \mathbb{R}^m$. Hence, $\Gamma_f$ is Hausdorff and 2nd-countable space.

So, we want to show that $\Gamma_f$ has the locally Euclidean property only.

Let $\pi_x : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is the projection onto $x$, and let $\varphi : \Gamma_f \longrightarrow U$ be the restriction of $\pi_x$ to $\Gamma_f$ that $\varphi(x, y) = x$, $(x, y) \in \Gamma_f$.

Since $\pi_x$ is continuous (clearly),

the restriction of $\pi_x$ $\varphi$ is continuous, and bijective also. Thus $\exists\, \varphi^{-1}$ : inverse of $\varphi$ and since $\varphi^{-1}(x) = (x, f(x))$, $\varphi^{-1}$ is continuous.

$\therefore$ $\varphi$ : Homeomorphism.

$\therefore$ $\Gamma_f$ is $n$-dimensional manifold. $\square$

1.2.1 Complete the proof of proposition 1.14 :
Suppose that $\pi : M \to M/\sim$ is an open map. Then
(ii) $M/\sim$ is Hausdorff $\Rightarrow R = \{(p,q) : p \sim q\}$ is closed in $M \times M$.

proof) Note that :

$[x]_\sim = \{x \in M : x \sim \alpha, \alpha \in M\}$.

$M/\sim = \{[x]_\sim : x \in M\}$

$\mathcal{O} \subset M/\sim$ is open $\Leftrightarrow \pi^{-1}(\mathcal{O}) = \{x : \pi(x) = [x] \in \mathcal{O}\}$ is open in $M$.

Assume that $M/\sim$ is Hausdorff.

Claim : $R \subset M \times M$ is closed

$\Leftrightarrow$ $M \times M - R$ is open.

Let $(p,q) \in M \times M - R$, then $\pi(p) \neq \pi(q)$

$\Rightarrow (p,q) \notin R$. Thus we can take the disjoint open sets $\pi(p) \in U_1$, $\pi(q) \in U_2$ since $M/\sim$ is Hausdorff.

Let $V_1 = \pi^{-1}(U_1)$ & $V_2 = \pi^{-1}(U_2)$.

If $(V_1 \times V_2) \cap R \neq \phi$, then $^\exists (v_1, v_2) \in V_1 \times V_2$ such that $\pi(v_1) = \pi(v_2)$, $\pi(v_1) \in U_1$, $\pi(v_2) \in U_2$.

But, $U_1 \cap U_2 = \phi$, that is contradiction.

$\therefore$ $R$ is closed in $M \times M$. $\square$