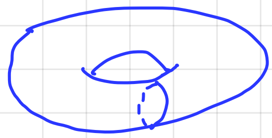
1.1.1 Show that if M^m, N^n are smooth manifolds, then $M^m \times N^n$ is also a $(m+n)$ dimensional smooth manifold. Hence, the n-dimensional torus or simply n-torus

$$T^n = \underbrace{S^1 \times \cdots \times S^1}_{n}$$ $T^2 =$  2-dim manifold

is a smooth manifold.

proof) We want to show :

[1] $M^m \times N^n$ is manifold [2] It is smooth manifold

[1] : ① Hausdorff

Since M^m, N^n are smooth manifold, these are Hausdorff. Then, for any U_M, V_M in M^m and U_N, V_N in N^n, let $U_M \times U_N = U \subset M^m \times N^n$ and $V_M \times V_N \subset N^n$, then $U_M \times U_N \cap V_M \times V_N = \phi$. ⋯ (*)

(∵)

(1) : $(U_M \times U_N) \cap V_M = V_M \cap U_M \times V_M \cap U_N = \phi$

because $V_M \cap U_M = \phi$ since M^m is Hausdorff.

(2) : $(U_M \times U_N) \cap V_N = V_N \cap U_M \times V_N \cap U_N = \phi$

because $V_N \cap U_N = \phi$ since N^n is Hausdorff.

(*) $= (1) \times (2) = \phi$.

Thus, $M^m \times N^n$ is Hausdorff.

② second countable

By the assumption, M^m, N^n have a countable basis $\mathcal{B}_M, \mathcal{B}_N$. Then trivially $\beta_M \times \beta_N \subset M^m \times N^n$ and we can pick $\beta_M \times \beta_N$ is a countable basis for $M^m \times N^n$.

($\because$) $x \in \beta_M$ & $y \in \beta_N \Rightarrow (x, y) \in \beta_M \times \beta_N \subset M^m \times N^n$

and β_M, β_N : open $\Rightarrow \beta_M \times \beta_N$: open.

$\beta_M \times \beta_N$: countable since β_M, β_N are countable.

pf) Let $\beta_M \times \beta_N$: finite $\rightarrow$ trivial.

We assume β_M, β_N : countably infinite.

$$\begin{matrix} (\beta_M^0, \beta_N^0) & (\beta_M^0, \beta_N^1) & (\beta_M^0, \beta_N^2) & \cdots \\ (\beta_M^1, \beta_N^0) & (\beta_M^1, \beta_N^1) & (\beta_M^1, \beta_N^2) & \cdots \\ (\beta_M^2, \beta_N^0) & (\beta_M^2, \beta_N^1) & (\beta_M^2, \beta_N^2) & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{matrix}$$

First, we pick (β_M^0, β_N^0), then we pick $(\beta_M^0, \beta_N^1), (\beta_M^1, \beta_N^0)$, then we pick $(\beta_M^0, \beta_N^2), (\beta_M^1, \beta_N^1)\ (\beta_M^2, \beta_N^0) \cdots\cdots$

Continue to this processes, then we can define the one-to-one correspondence between $\beta_M \times \beta_N \rightarrow \mathbb{N}$ (the set of natural #).

Thus, by definition of countable, the assertion is proved.

③ Homeomorphism

Let $\varphi_M : U \rightarrow \mathbb{R}^m$ & $p \in \varphi_M(U)$ and $\varphi_N : V \rightarrow \mathbb{R}^n$ & $q \in \varphi_N(V)$, then we can define

$\varphi_{MN}(r) = (\varphi_M \times \varphi_N)(p, q) = (\varphi_M(p), \varphi_N(q))$

if $\varphi_{MN} : U \times V \to \mathbb{R}^{m+n}$.

(i) injective

$\varphi_{MN}(r_1) = \varphi_{MN}(r_2)$

$\Rightarrow (\varphi_M(p_1), \varphi_N(q_1)) = (\varphi_M(p_2), \varphi_N(q_2))$

$\Rightarrow \varphi_M(p_1) = \varphi_M(p_2)$ & $\varphi_N(q_1) = \varphi_N(q_2)$

$\Rightarrow p_1 = p_2$ & $q_1 = q_2$ since φ_M, φ_N : injective.

(ii) surjective

For $\forall\, y = \varphi_{MN}(\bar{r}) \in \mathbb{R}^{m+n}$, $\exists\, (\bar{p}, \bar{q}) \in U \times V$ s.t.

$y = \varphi_{MN}(\bar{r}) = (\varphi_M(\bar{p}), \varphi_N(\bar{q}))$ since

φ_M & φ_N are surjective.

By (i), (ii), φ_{MN} : bijection on $U \times V \subset M^m \times N^n$.

Thus, $\exists\, \varphi_{MN}^{-1}$: inverse of φ_{MN}.

In case of continuity, for any θ, β be open in $\mathbb{R}^m$, $\mathbb{R}^n$,

$\varphi_M^{-1}(\theta), \varphi_N^{-1}(\beta)$ are open by the assumption.

Since $\theta \times \beta$: open and its pre image

$\varphi_{MN}^{-1}(\theta \times \beta)$ is open.

$\therefore\ \varphi_{MN}$ is continuous.

($\because$) if $\theta = \varphi_M(\alpha)$, $\beta = \varphi_N(\beta)$ for any open sets

α, β in U, V, then

$\varphi_{MN}^{-1}(\varphi_{MN}(\theta \times \beta)) = \theta \times \beta = \varphi_M(\alpha) \times \varphi_N(\beta)$: open.

For φ_{MN}^{-1} : inverse of φ_{MN}, $\varphi_{MN}^{-1}(\theta \times \beta)$ is open

$\Rightarrow \varphi_{MN}(\varphi_{MN}^{-1}(\theta \times \beta)) = \theta \times \beta$ is open

$\therefore$ φ_{MN}^{-1} is continuous.

$\therefore$ φ_{MN} is Homeomorphism.

Therefore, $M^m \times N^n$ is a manifold.

[2] : Since M^m, N^n are smooth manifold, they have a C^∞-structure, so that the coordinate charts (U, φ_M), (V, φ_N) is C^∞-compatible with all charts in the atlas of M^m, N^n, respectively.

By [1], we defined the homeomorphism φ_{MN}, hence we can write the coordinate chart of $M^m \times N^n$ that $(U \times V, \varphi_{MN})$. Consider another chart $(U' \times V', \varphi_{MN}^*)$, then

$$\varphi_{MN} \circ \varphi_{MN}^* = (\varphi_M \times \varphi_N) \circ (\varphi_M^* \times \varphi_N^*)^{-1}$$

$$= \varphi_M \circ \varphi_M^{*-1} \times \varphi_N \circ \varphi_N^{*-1}.$$

Since φ_M, φ_N, φ_M^{*-1}, φ_N^{*-1} are C^∞,

$\varphi_{MN} \circ \varphi_{MN}^*$ is C^∞.

$\therefore$ $M^m \times N^n$ is a smooth manifold.

Thus, By the proof above, T^n is smooth manifold. $\blacksquare$

1.1.2 Let $U \subset \mathbb{R}^n$ be open and $f : U \to \mathbb{R}^m$ be continuous. Show that the graph of f

$$\Gamma_f = \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^m : x \in U \text{ and } y = f(x)\}$$

is an n-dimensional manifold.

proof) By the example 1.7. (i) of the lecture note of professor Han, $\mathbb{R}^n$, $\mathbb{R}^m$ are n, m dimensional smooth manifold and hence $\mathbb{R}^n \times \mathbb{R}^m$ is smooth manifold by the exercise 1.1.1.

Thus, the graph of f Γ_f is the subspace topology of $\mathbb{R}^n \times \mathbb{R}^m$. Hence, Γ_f is Hausdorff and 2nd-countable space.

So, we want to show that Γ_f has the locally Euclidean property only.

Let $\pi_x : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is the projection onto x, and let $\varphi : \Gamma_f \longrightarrow U$ be the restriction of π_x to Γ_f that $\varphi(x, y) = x$, $(x, y) \in \Gamma_f$.

Since π_x is continuous (clearly), the restriction of π_x φ is continuous, and bijective also. Thus $\exists\, \varphi^{-1}$: inverse of φ and since $\varphi^{-1}(x) = (x, f(x))$, φ^{-1} is continuous.

$\therefore$ φ : Homeomorphism.

$\therefore$ Γ_f is n-dimensional manifold. $\square$

1.2.1 Complete the proof of proposition 1.14 :
Suppose that $\pi : M \to M/\sim$ is an open map. Then
(ii) $M/\sim$ is Hausdorff $\Rightarrow$ $R = \{(p,q) : p \sim q\}$ is closed in $M \times M$.

proof) Note that :

$$[x]_\sim = \{ x \in M : x \sim \alpha, \alpha \in M \}.$$
$$M/\sim = \{ [x]_\sim : x \in M \}$$
$\mathcal{O} \subset M/\sim$ is open $\Leftrightarrow$ $\pi^{-1}(\mathcal{O}) = \{ x : \pi(x) = [x] \in \mathcal{O} \}$ is open in M.

Assume that $M/\sim$ is Hausdorff.

<u>Claim</u> : $R \subset M \times M$ is closed

$\Leftrightarrow$ $M \times M - R$ is open.

Let $(p,q) \in M \times M - R$, then $\pi(p) \neq \pi(q)$

$\Rightarrow$ $(p,q) \notin R$. Thus we can take the disjoint open sets $\pi(p) \in U_1$, $\pi(q) \in U_2$ since $M/\sim$ is Hausdorff.

Let $V_1 = \pi^{-1}(U_1)$ & $V_2 = \pi^{-1}(U_2)$.

If $(V_1 \times V_2) \cap R \neq \phi$, then $\exists (v_1, v_2) \in V_1 \times V_2$ such that $\pi(v_1) = \pi(v_2)$, $\pi(v_1) \in U_1$, $\pi(v_2) \in U_2$.

But, $U_1 \cap U_2 = \phi$, that is contradiction.

$\therefore$ R is closed in $M \times M$. $\square$

1.2.2 Let $f : S^n \to S^n$ be the antipodal map defined by $f(x) = -x$. Define an relation $\sim$ on S^n by $x \sim y$ iff $y = x$ or $y = f(x)$. Show that $\sim$ is an equivalence relation and $S^n/\sim\ = \mathbb{R}P^n$.

proof) ① Equivalence relation

「 $x \sim y \iff y = x \iff y - x = 0$ 」

(i) $x \sim x$ since $x - x = 0$

(ii) if $x \sim y$, then

$y = x \iff y - x = 0 \iff -(x-y) = 0$

$\iff x - y = 0 \iff x = y$

$\iff y \sim x$.

$y = -x \iff -y = x \iff x = f(y)$

$\iff y \sim x$

(iii) if $x \sim y$ & $y \sim z$, then

$y = x$ and $z = y$ and so

$z = y = x \iff x \sim z$.

$y = -x$ & $z = -y$, then

$z = -y = -(-x) = x \iff x \sim z$.

② $S^n/\sim \; = \mathbb{R}P^n$

$$[x]_M = \bigcup_i [x_i]_{S^n}. \quad (\because [x]_{S^n} \subset [x]_M)$$

$$\Rightarrow \bigcup_j \left(\bigcup_i [x_i]_{S^n} \right)_j = M/\sim \; = \mathbb{R}P^n$$

For arbitrary $r \in \mathbb{R}^{n+1} - \{0\}$, we can let

$S^n = \{ \vec{x} : \|\vec{x}\| = r \}$, and thus

$S^n/\sim \; = \mathbb{R}P^n$.

(Additional Information i thought)

$S^n = \{ (x_1, \cdots, x_{n+1}) : x_1^2 + \cdots + x_{n+1}^2 = r^2 \}$.

$(0, 0, \cdots, 0) \notin S^n \subset M = \mathbb{R}^{n+1} - \{0\}$.

$[x]_M = \{ x \in M : x \sim y \iff y = tx \text{ for some } t \neq 0 \}$

$[x]_{S^n} = \{ x \in S^n : x \sim y \iff y = \pm x \}$

$\Rightarrow \quad [x]_{S^n} \subset [x]_M. \quad (\because [x]_{S^n} = \{-x, x\})$

$\Rightarrow [x]_{S^n} \in M/\sim \; = \mathbb{R}P^n$.

$\therefore \quad S^n/\sim \; \subset M/\sim \; = \mathbb{R}P^n$. $\square$

1.2.3. The complex projective space $\mathbb{C}P^n$ is the set of all line through the origin in $\mathbb{C}^{n+1}$, i.e., the set of 1-dimensional subspaces of $\mathbb{C}^{n+1}$. If we define an equivalence relation on $M = \mathbb{C}^{n+1} - \{0\}$ by $z \sim w \Leftrightarrow w = \lambda z$ for some $\lambda \in \mathbb{C}^*$, then $\mathbb{C}P^n = M/\sim$. Show that $\mathbb{C}P^n$ is a $2n$-dimensional smooth manifold.

proof) ① 2nd-countable

Since M is 2nd-countable, the quotient set of M is 2nd-countable.

② Hausdorff.

$[z_1], [z_2] \in U_j$ for some j

$\Rightarrow [z_1]$ and $[z_2]$ are disjoint open set,

$(\because)\ \varphi_j(z_1), \varphi_j(z_2) \in \mathbb{C}^n$.

Claim: $\nexists\, U_j$ containing $[z_1]$ & $[z_2]$.

Given $j \neq k$, let

$A_{j,k} = \{[z] : |z^j| > |z^k|\} \subset \mathbb{C}P^n$.

Then $A_{j,k}$ is open since

$\pi^{-1}(A_{j,k})$ is open in $\mathbb{C}^{n+1} - \{0\}$.

By the assumption, $\exists\, j \neq k$ s.t.

$[z_1] \in U_j$ and $[z_2] \in U_k$, but

$z_1^j = z_2^k = 0$.

$\therefore\ z_1 \in A_{j,k},\ z_2 \in A_{k,j}$.

$\therefore A_{j,k} \cap A_{k,j} = \phi$.

③ local Euclidean

For $z = (z^0, \cdots, z^n) \in \mathbb{C}^{n+1}$, define

$U_i = \{[z] : z^i \neq 0\} \subset \mathbb{CP}^n$, then we can

define $\varphi_i : U_i \to \mathbb{C}^n$ by

$$\varphi_i([z]) = \left(\frac{z^0}{z^i}, \cdots, \frac{z^{i-1}}{z^i}, \frac{z^{i+1}}{z^i}, \cdots, \frac{z^n}{z^i}\right).$$

continuous

For a projection $\pi : M \to M/\sim$ by $\pi(z) = [z]$,

$\varphi_i^{-1}(V)$ is open for any open $V \subset \mathbb{C}^n$

$\Leftrightarrow \pi^{-1} \circ \varphi_i^{-1}(V) = (\varphi_i \circ \pi)^{-1}(V)$ is open in $\mathbb{C}^{n+1}$.

Since $\varphi_i \circ \pi$ is clearly continuous, $(\because z^i \neq 0)$

$(\varphi_i \circ \pi)^{-1}(V)$ is open in $\mathbb{C}^{n+1}$.

① injective

$$\varphi_i([z_1]) = \varphi_i([z_2]) \Rightarrow \frac{z_1^j}{z_2^j} = \frac{z_1^i}{z_2^i}, \quad j \neq i$$

$\Rightarrow [z_1] = [z_2]$. $(\because z_1^j = t z_2^j$ for $\forall j)$

②

$v = (v^1, \cdots, v^n) \in \mathbb{C}^n$, then

$$\varphi_i^{-1}(v) = \pi(v^1, \cdots, v^{i-1}, 1, v^i, \cdots, v^n).$$

Since π is continuous, φ_i^{-1} is continuous.

$(\because)$ $\varphi_i(\pi(v^1, \cdots, v^{i-1}, 1, v^i, \cdots, v^n))$

$$= \left(\frac{v^1}{1}, \cdots, \frac{v^{i-1}}{1}, \frac{v^{i+1}}{1}, \cdots, \frac{v^n}{1}\right) = v$$

By ①, ②, φ_i is a homeomorphism.

③ assuming w.r.o.g, $i < j$, the transition maps

$$\varphi_j \circ \varphi_i^{-1} : \varphi(U_i \cap U_j) = \{ z = (z^1, \cdots, z^n) \in \mathbb{C}^n : z^j \neq 0 \}$$
$$\longrightarrow \varphi(U_i \cap U_j)$$

$$\varphi_j \circ \varphi_i^{-1}(z^1, \cdots, z^n) = \varphi_j([(z^1, \cdots, z^{i-1}, 1, z^{i+1}, \cdots, z^n)])$$
$$= \left(\frac{z^1}{z^j}, \cdots, \frac{z^i}{z^j}, \frac{1}{z^j}, \frac{z^{i+1}}{z^j}, \cdots, \frac{z^{j-1}}{z^j}, \frac{z^{j+1}}{z^j}, \cdots, \frac{z^n}{z^j} \right)$$

is smooth.

Thus, for $\{(U_i, \varphi_i) \mid i = 1, \cdots, n+1\} = \mathcal{A}$,

$\mathcal{A}$ is C^∞ atlas.

$\therefore$ $\mathbb{CP}^n$ is a $2n$-dimensional smooth manifold.

$\square$

1.3.1. Let $N = M = \mathbb{R}P^1$ and write a point in $\mathbb{R}P^1$ as $[(x, y)]$ for $(x, y) \in \mathbb{R}^2$. Show that the map $F : N \to M$ given by $F([(x,y)]) = [(x^2, y^2)]$ is smooth.

proof) For $\{(U, \varphi)\} = \mathcal{A}_N$ and $\{(V, \psi)\} = \mathcal{A}_M$,

let $\varphi = \psi = \pi^{-1}$, i.e.

$\varphi : N \to \mathbb{R}^2$ by $\varphi([(x,y)]) = (x, y)$

$\psi : M \to \mathbb{R}^2$ by $\psi([x, y]) = (x, y)$

Then, These can be a homeomorphism.

Thus,

$$f \equiv \psi \circ F \circ \varphi^{-1} = \psi(F([(x,y)]))$$

$$\mathbb{R}^2 \to \mathbb{R}^2 \qquad = \psi([(x^2, y^2)]) = (x^2, y^2).$$

Since the components of f is smooth,

$f = \psi \circ F \circ \varphi^{-1}$ is smooth, and therefore,

$F : N \to M$ is smooth mapping.

$\square$

1.3.2 Prove that (ii) of Theorem 1.28.

proof) For each $q \in A$, $\exists(\varphi, U)$ near q

that $U_q \subset U$ & $\varphi(U_q) \subset V$,

$V = \{$ contains the open ball $B_3(0)$ $\}$.

$B_3(0) \equiv$ open ball of radius 3 centered at 0.

Let $\tilde{U}_q = \varphi^{-1}(B_1(0))$ and

define the function called "Bump function" that

$$g(t) = \begin{cases} 1 & \text{for } t \leq 1 \\ 0 & \text{for } t \geq 2 \end{cases},$$

and let

$$f(p) = \begin{cases} g(\varphi(p)) & p \in U_q \\ 0 & p \notin U_q \end{cases}.$$

Then, $f \in C^\infty(M)$ such that $0 \leq f \leq 1$,

$f \equiv 1$ on $\tilde{U}_q \subset A$ and $\mathrm{supp}(f) \subset U_p \subset U$. $\blacksquare$

2.1.1 For a smooth map $F : N^n \to M^m$, the push forward $F_{*p} : T_pN \to T_{F(p)}M$ at $p \in N$ was defined in terms of derivations. This can be also defined by the equivalence class of curves as

$$F_{*p}([\gamma]) = [F \circ \gamma].$$

Show that this definition is well defined. In other words, $\gamma_1 \sim \gamma_2$ implies that $F \circ \gamma_1 \sim F \circ \gamma_2$.

proof) For a coordinate chart (U, φ) at p,

$$\gamma_1 \sim \gamma_2 \Leftrightarrow (\varphi \circ \gamma_1)'(0) = (\varphi \circ \gamma_2)'(0).$$

$F \circ \gamma = F \circ \varphi^{-1} \circ \varphi \circ \gamma$ since φ : homeomorphism.

$$\begin{aligned}
\Rightarrow (F \circ \gamma_1)'(0) &= (F \circ \varphi^{-1} \circ \varphi \circ \gamma_1)'(0) \\
&= (F \circ \varphi^{-1})'(\varphi \circ \gamma_1)(0). \\
(\because \gamma_1 \sim \gamma_2) \quad &= (F \circ \varphi^{-1})'(\varphi \circ \gamma_2)(0) \\
&= (F \circ \varphi^{-1} \circ \varphi \circ \gamma_2)'(0) \\
&= (F \circ \gamma_2)'(0)
\end{aligned}$$

$\Leftrightarrow F \circ \gamma_1 \sim F \circ \gamma_2$ $\square$

2.1.2. Let $F : N^n \to M^m$ be a smooth map.
Prove that if N is connected and $F_{*p} = 0$ for any $p \in N$, then F is constant map.

proof) Let $f \in C^\infty(M)$ and let $X_p \in T_p N$.

By the assumption, $F_{*p}[f] = X_p(f \circ F) = 0$.

Let (U, φ) : smooth chart containing p. Then

$$X_p = \sum_i X_p^i \frac{\partial}{\partial x^i}\Big|_p = \sum_i X^i (\varphi^{-1})_{*p} \frac{\partial}{\partial x^i}\Big|_{\varphi(p)}$$

$$\Rightarrow \left(\sum_i X^i (\varphi^{-1})_{*p} \frac{\partial}{\partial x^i}\Big|_{\varphi(p)}\right)(f \circ F) = \sum_i X^i \frac{\partial}{\partial x^i}\Big|_{\varphi(p)} (f \circ F \circ \varphi^{-1}) = 0$$

$\Rightarrow$ F is constant on U.

Since N is connected, N : path connected.
Let $q \in N$ & let $\gamma : [0,1] \to N$ be a path connecting p & q.
Since F is constant on each smooth chart $(U_{\gamma(x)}, \varphi_{\gamma(x)})$ containing $\gamma(x)$ for every $x \in [0,1]$, $F \equiv c$ on N since $F(p) = c$ & γ is continuous. $\square$

2.1.3. Prove that for any $p \in S^n$,

$$T_p S^n = \{ X \in \mathbb{R}^{n+1} : \langle p, X \rangle = 0 \}.$$

proof) Let $p \in S^n \subset \mathbb{R}^{n+1} - \{0\}$ and let $X \in \mathbb{R}^{n+1}$.

$\ulcorner$

By the example 2.2, $T_p \mathbb{R}^m = \mathbb{R}^m$.

$m = n+1 \Rightarrow T_p \mathbb{R}^{n+1} = \mathbb{R}^{n+1} \supset S^n$.

$[X]_{S^n} = \{ X \in \mathbb{R}^{n+1} : X \sim p \Leftrightarrow p = tX$ for some $t \neq 0 \}$.

$\langle p, X \rangle = 0 \Rightarrow p = -X$. $\lrcorner$

Define

$$\gamma_{p,X}(t) = \begin{cases} p + tX & t \neq 1 \\ p & t = 1 \end{cases}.$$

Then, $\frac{d}{dt} \gamma_{p,X}(t) \Big|_{t=0} = p + X$, $\gamma_{p,X}(0) = p = -X$

$\Rightarrow [\gamma_{p,X}] \in T_p S^n$.

$\therefore T_p S^n = \{ X \in \mathbb{R}^{n+1} : \langle p, X \rangle = 0 \}$. $\blacksquare$

2.3.1. Let $F : \mathbb{R}^2 \to \mathbb{R}^2$ such that

$F(x, y) = (a^2 - 2y,\ 4x^3y^2)$. For $X = 4x\frac{\partial}{\partial x} + 3y^2\frac{\partial}{\partial y}$,

compute $F_* X$.

proof) For the vector field $X = 4x\frac{\partial}{\partial x} + 3y^2\frac{\partial}{\partial y}$,

let $x' = a^2 - 2y$ & $y' = 4x^3y^2$, then

$$F_*\left(\frac{\partial}{\partial x}\right) = \frac{\partial x'}{\partial x}\frac{\partial}{\partial x} + \frac{\partial y'}{\partial x}\frac{\partial}{\partial y} = 12x^2y^2\frac{\partial}{\partial y}$$

$$F_*\left(\frac{\partial}{\partial y}\right) = \frac{\partial x'}{\partial y}\frac{\partial}{\partial x} + \frac{\partial y'}{\partial y}\frac{\partial}{\partial y} = -2\frac{\partial}{\partial x} + 8x^3y\frac{\partial}{\partial y}$$

$$F_* X = 4x\, F_*\left(\frac{\partial}{\partial x}\right) + 3y^2\, F_*\left(\frac{\partial}{\partial y}\right)$$

$$= -6y^2\frac{\partial}{\partial x} + x^3y^2(24y + 48)\frac{\partial}{\partial y}$$

$\square$

2.3.2. Express the following planar vector fields in polar coordinates.

$$X = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}, \quad Y = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}.$$

What is $[X, Y]$?

proof) Let $x = r\cos\theta$, $y = r\sin\theta$, then

$$\frac{\partial}{\partial r} = \frac{\partial x}{\partial r}\frac{\partial}{\partial x} + \frac{\partial y}{\partial r}\frac{\partial}{\partial y} = \cos\theta\frac{\partial}{\partial x} + \sin\theta\frac{\partial}{\partial y}$$

$$= \frac{x}{r}\frac{\partial}{\partial x} + \frac{y}{r}\frac{\partial}{\partial y} = \frac{1}{r}X$$

$$\frac{\partial}{\partial \theta} = \frac{\partial x}{\partial \theta}\frac{\partial}{\partial x} + \frac{\partial y}{\partial \theta}\frac{\partial}{\partial y} = -r\sin\theta\frac{\partial}{\partial x} + r\cos\theta\frac{\partial}{\partial y}$$

$$= -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y} = Y$$

$$\therefore X = r\frac{\partial}{\partial r}, \quad Y = \frac{\partial}{\partial \theta}.$$

By definition 2.21.(iii), $[X, Y] = \left[r\frac{\partial}{\partial r}, \frac{\partial}{\partial \theta}\right] = 0$.

professor Han's note

$\square$

2.3.3. In $\mathbb{R}^3$, let

$$X = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x} \quad \text{and} \quad Y = y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}.$$

Compute $[X, Y]$.

proof) By definition 2.22, we have

$$\begin{aligned}[X,Y] &= \left[x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x},\ y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right] \\ &= \left[x\frac{\partial}{\partial y},\ y\frac{\partial}{\partial z}\right]^{①} + \left[x\frac{\partial}{\partial y},\ -z\frac{\partial}{\partial y}\right]^{②} + \\ &\quad \left[-y\frac{\partial}{\partial x},\ y\frac{\partial}{\partial z}\right]^{③} + \left[-y\frac{\partial}{\partial x},\ -z\frac{\partial}{\partial y}\right]^{④}\end{aligned}$$

By definition 2.21 (iii), we have

$$① = x\cdot 0\frac{\partial}{\partial z} + y\cdot 0\frac{\partial}{\partial y} = 0$$

$$② = x\cdot 0\frac{\partial}{\partial y} + (-z)\cdot 0\frac{\partial}{\partial y} = 0$$

$$③ = -y\cdot 0\frac{\partial}{\partial z} + y\cdot 0\frac{\partial}{\partial x} = 0$$

$$④ = -y\cdot 0\frac{\partial}{\partial y} - z\cdot 0\frac{\partial}{\partial x} = 0$$

$\therefore [X, Y] = 0$ $\square$

2.3.4. Verify Example 2.23.

proof) (i) $\mathbb{R}^n$ is a Lie algebra.

Since $[a,b] = (a+b)-(b+a) = 0$,

bilinear & skew summetric satisfied.

Check Jacobi identity condition. For $c \in \mathbb{R}^n$,

$[a,[b,c]]+[b,[c,a]]+[c,[a,b]]$

$= a+[b,c]-([b,c]+a)+b+[c,a]-([c,a]+b)$
$\quad +c+[a,b]-([a,b]+c)$

$= a+0-(0+a)+b+0-(0+b)+c+0-(0+c)$

$= a-a+b-b+c-c = 0.$

$\therefore$ $\mathbb{R}^n$ is a Lie algebra.

(ii) $GL(n,\mathbb{R})$ is a Lie algebra.

Check only the condition for Jacobi.

$[A,B] = AB-BA$ for $A, B \in GL(n,\mathbb{R})$,

pick $C \in GL(n,\mathbb{R})$, then

$[A,[B,C]]+[B,[C,A]]+[C,[A,B]]$

$= A([B,C])-([B,C]A)+B([C,A])-([C,A]B)$
$\quad + C([A,B))-([A,B]C)$

$= A(BC-CB)-(BC-CB)A+B(CA-AC)-(CA-AC)B$
$\quad + C(AB-BA)-(AB-BA)C$

$= ABC-ACB-BCA+CBA+BCA-BAC-CAB+ACB$
$\quad +CAB-CBA-ABC+BAC$

$= 0.$ $\therefore$ $GL(n,\mathbb{R})$ is a Lie algebra.

(iii) $\mathbb{R}^3$ is a Lie algebra with $[u,v] = u \times v$, $u, v \in \mathbb{R}^3$

Cross product satisfy the skew symmetric condition

($\because$ $u \times v = -v \times u$). and bilinear condition also.

($\because$ Let $u, v, w \in \mathbb{R}^3$, i, j, k : standard basis of $\mathbb{R}^3$.

$u = u_1 i + u_2 j + u_3 k$, $v = v_1 i + v_2 j + v_3 k$,

$w = w_1 i + w_2 j + w_3 k$. Then, for $c \in \mathbb{R}$,

$(cu + v) \times w = c(u \times w) + v \times w$.

pf) $$(cu+v) \times w = \begin{vmatrix} i & j & k \\ cu_1 + v_1 & cu_2 + v_2 & cu_3 + v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$

$$= \begin{vmatrix} i & j & k \\ cu_1 & cu_2 & cu_3 \\ w_1 & w_2 & w_3 \end{vmatrix} + \begin{vmatrix} i & j & k \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$

$$= c \begin{vmatrix} i & j & k \\ u_1 & u_2 & u_3 \\ w_1 & w_2 & w_3 \end{vmatrix} + \begin{vmatrix} i & j & k \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$

$$= c(u \times w) + v \times w.$$ Q.E.D)

Check Jacobi identity condition. For $u, v, w \in \mathbb{R}^3$,

$[u,[v,w]] + [v,[w,u]] + [w,[u,v]]$

$= u \times (v \times w) + v \times (w \times u) + w \times (u \times v)$

$= (u \cdot w)v - (u \cdot v)w + (v \cdot u)w - (v \cdot w)u$

$\quad + (w \cdot v)u - (w \cdot u)v$

$= 0$.

$\therefore$ $\mathbb{R}^3$ is a Lie algebra with $[u,v] = u \times v$ for $u, v \in \mathbb{R}^3$.

(iv) $\mathfrak{G}, \mathfrak{h}$ are Lie algebras $\Rightarrow$ $\mathfrak{G} \times \mathfrak{h}$ is also a Lie algebra under the bracket

$$[(X_1, Y_1), (X_2, Y_2)] = ([X_1, Y_1], [X_2, Y_2]).$$

① Bilinearity

Clearly we obtain the property after complicate calculation.

Note that $[X_1, Y_1]$, $[X_2, Y_2]$ are satisfy the bilinearity in $\mathfrak{G}, \mathfrak{h}$, respectively.

② skew - symmetric.

For the simplicity, denote $X_i, Y_i \equiv x_i, y_i$.

$$\begin{aligned}
[(x_1, y_1), (x_2, y_2)] &= ([x_1, y_1], [x_2, y_2]) \\
\mathfrak{G}, \mathfrak{h} : \text{Lie algebra} \longrightarrow &= (-[y_1, x_1], -[y_2, x_2]) \\
&= -([y_1, x_1], [y_2, x_2]) \\
&= -[(y_1, x_1), (y_2, x_2)]
\end{aligned}$$

③ Jacobi identity

$$\begin{aligned}
&[(x_1, y_1), [(x_2, y_2), (x_3, y_3)]] \\
&+ [(x_2, y_2), [(x_3, y_3), (x_1, y_1)]] \\
&+ [(x_3, y_3), [(x_1, y_1), (x_2, y_2)]] = 0
\end{aligned}$$

using the previous results we proved and our Lie bracket. $\square$

2.3.5. Prove Theorem 2.24.

proof) Check the Jacobi identity.

For the simplicity, denote $X, Y, \cdots \equiv x, y, \cdots$.

$$[x,[y,z]]+[y,[z,x]]+[z,[x,y]]$$

$$= x(yz-zy)-(yz-zy)x+y(zx-xz)-(zx-xz)y$$
$$+z(xy-yx)-(xy-yx)z$$

$$= xyz-xzy-yzx+zyx+yzx-yxz-zxy+xzy$$
$$+zxy-zyx-xyz+yxz$$

$$=0. \qquad \square$$

2.3.6. Prove Theorem 2.26.

proof) By the assumption, X_i, Y_i are F-related.

i.e. $F_*(X_i) = Y_i$ by definition 2.25.

<u>Claim</u> : $F_*([X_1, X_2]) = [Y_1, Y_2]$.

Choose $g \in C^\infty(M)$ and $x \in N$, then

$$(Y_i g)(F(x)) = (F_*)_x (X_i)(g) = X_i(g \circ F)$$

Thus, $(Y_i g) \circ F = X_i(g \circ F)$ ⋯ (*)

Let $f \in C^\infty(M)$ be arbitrary. Using (*)

$\Rightarrow Y_1(Y_2 f) \circ F = X_1((Y_2 f) \circ F)$. ⋯ (**)

By (*), we also obtain

$$(Y_2 f) \circ F = X_2(f \circ F) \text{ and thus}$$

$$(**) = Y_1(Y_2 f) \circ F = X_1(X_2(f \circ F)).$$

Likewise, we get

$$Y_2(Y_1 f) \circ F = X_2(X_1(f \circ F)).$$

Hence, $([Y_1, Y_2] f) \circ F = [X_1, X_2](f \circ F)$.

Therefore, $[Y_1, Y_2]$ is F-related to $[X_1, X_2]$ $\square$

2.3.7. Let $F: N \to M$ be a diffeomorphism. Prove that for any $Y \in \mathfrak{X}(M)$, there is a unique $X \in \mathfrak{X}(N)$ such that X is F-related to Y.

proof) Assume that X is F-related to Y.

i.e. $X_{F(p)} = F_{*p}(Y_p)$.

If F is a diffeomorphism, we define X by

$$X_q = F_{*F^{-1}(q)}\left(Y_{F^{-1}(q)}\right)$$

Then, it is clear that X is the unique vector field such that F-related to Y.

⌜ Note that

$X : N \to TN$, N : manifold, TN : tangent bundle.

Then X is the composition that

$$N \xrightarrow{F^{-1}} M \xrightarrow{X} TM \xrightarrow{dF} TN$$

$\Rightarrow X$ is smooth. ⌟

$\square$

2.3.8. Express the planar 1-form $\omega = x\,dx + y\,dy$ in polar coordinates.

proof) Let $x = r\cos\theta$ and $y = r\sin\theta$.

By Chain rule,

$$dx = \frac{\partial x}{\partial r} dr + \frac{\partial x}{\partial \theta} d\theta = \cos\theta\, dr - r\sin\theta\, d\theta$$

$$dy = \frac{\partial y}{\partial r} dr + \frac{\partial y}{\partial \theta} d\theta = \sin\theta\, dr + r\cos\theta\, d\theta$$

The differential 1-form ω is expressed by

$$\omega = \left(\frac{\partial x}{\partial r} x + \frac{\partial y}{\partial r} y\right) dr + \left(\frac{\partial x}{\partial \theta} x + \frac{\partial y}{\partial \theta} y\right) d\theta$$

$$= \left(r\cos^2\theta + r\sin^2\theta\right) dr + \left(-r^2\sin\theta\cos\theta + r^2\cos\theta\sin\theta\right) d\theta$$

$$= r\,dr + 0\,d\theta$$

$$= r\,dr.$$

$\square$