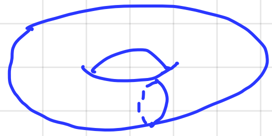
1.1.1 Show that if M^m, N^n are smooth manifolds, then $M^m \times N^n$ is also a $(m+n)$ dimensional smooth manifold. Hence, the n-dimensional torus or simply n-torus

$$T^n = \underbrace{S^1 \times \cdots \times S^1}_{n}$$

$T^2 =$  2-dim manifold

is a smooth manifold.

proof) We want to show :

[1] $M^m \times N^n$ is manifold [2] It is smooth manifold

[1] : ① Hausdorff

Since M^m, N^n are smooth manifold, these are Hausdorff. Then, for any U_M, V_M in M^m and U_N, V_N in N^n, let $U_M \times U_N = U \subset M^m \times N^n$ and $V_M \times V_N \subset N^n$, then $U_M \times U_N \cap V_M \times V_N = \phi$. $\cdots$ $(*)$

$(\because)$

(1) : $(U_M \times U_N) \cap V_M = V_M \cap U_M \times V_M \cap U_N = \phi$

because $V_M \cap U_M = \phi$ since M^m is Hausdorff.

(2) : $(U_M \times U_N) \cap V_N = V_N \cap U_M \times V_N \cap U_N = \phi$

because $V_N \cap U_N = \phi$ since N^n is Hausdorff.

$(*) = (1) \times (2) = \phi$.

Thus, $M^m \times N^n$ is Hausdorff.

② second countable

By the assumption, M^m, N^n have a countable basis $\mathcal{B}_M, \mathcal{B}_N$. Then trivially $\beta_M \times \beta_N \subset M^m \times N^n$ and we can pick

$\beta_M \times \beta_N$ is a countable basis for $M^m \times N^n$.

(∵) $x \in \beta_M$ & $y \in \beta_N \Rightarrow (x, y) \in \beta_M \times \beta_N \subset M^m \times N^n$

and β_M, β_N : open $\Rightarrow \beta_M \times \beta_N$: open.

$\beta_M \times \beta_N$: countable since β_M, β_N are countable.

pf) Let $\beta_M \times \beta_N$: finite $\rightarrow$ trivial.

We assume β_M, β_N : countably infinite.

$$\begin{array}{cccc}
(\beta_M^0, \beta_N^0) & (\beta_M^0, \beta_N^1) & (\beta_M^0, \beta_N^2) & \cdots \\
(\beta_M^1, \beta_N^0) & (\beta_M^1, \beta_N^1) & (\beta_M^1, \beta_N^2) & \cdots \\
(\beta_M^2, \beta_N^0) & (\beta_M^2, \beta_N^1) & (\beta_M^2, \beta_N^2) & \cdots \\
\vdots & \vdots & \vdots & \ddots
\end{array}$$

First, we pick (β_M^0, β_N^0), then we pick

$(\beta_M^0, \beta_N^1), (\beta_M^1, \beta_N^0)$, then we pick

$(\beta_M^0, \beta_N^2), (\beta_M^1, \beta_N^1)$ (β_M^2, β_N^0) ......

Continue to this processes, then we can

define the one-to-one correspondence

between $\beta_M \times \beta_N \rightarrow \mathbb{N}$ (the set of natural #).

Thus, by definition of countable,

the assertion is proved.

③ Homeomorphism

Let $\varphi_M : U \rightarrow \mathbb{R}^m$ & $p \in \varphi_M(U)$ and

$\varphi_N : V \rightarrow \mathbb{R}^n$ & $q \in \varphi_N(V)$, then we can define

$\varphi_{MN}(r) = (\varphi_M \times \varphi_N)(p, q) = (\varphi_M(p), \varphi_N(q))$

if $\varphi_{MN} : U \times V \to \mathbb{R}^{m+n}$.

(i) injective

$\varphi_{MN}(r_1) = \varphi_{MN}(r_2)$

$\Rightarrow (\varphi_M(p_1), \varphi_N(q_1)) = (\varphi_M(p_2), \varphi_N(q_2))$

$\Rightarrow \varphi_M(p_1) = \varphi_M(p_2)$ & $\varphi_N(q_1) = \varphi_N(q_2)$

$\Rightarrow p_1 = p_2$ & $q_1 = q_2$ since φ_M, φ_N : injective.

(ii) surjective

For $\forall\, y = \varphi_{MN}(\bar{r}) \in \mathbb{R}^{m+n}$, $\exists\, (\bar{p}, \bar{q}) \in U \times V$ s.t.

$y = \varphi_{MN}(\bar{r}) = (\varphi_M(\bar{p}), \varphi_N(\bar{q}))$ since

φ_M & φ_N are surjective.

By (i), (ii), φ_{MN} : bijection on $U \times V \subset M^m \times N^n$.

Thus, $\exists\, \varphi_{MN}^{-1}$: inverse of φ_{MN}.

be open

In case of continuity, for any θ, β in $\mathbb{R}^m, \mathbb{R}^n$,

$\varphi_M^{-1}(\theta), \varphi_N^{-1}(\beta)$ are open by the assumption.

Since $\theta \times \beta$: open and its pre image

$\varphi_{MN}^{-1}(\theta \times \beta)$ is open.

$\therefore\ \varphi_{MN}$ is continuous.

($\because$) if $\theta = \varphi_M(\alpha)$, $\beta = \varphi_N(\beta)$ for any open sets

α, β in U, V, then

$\varphi_{MN}^{-1}(\varphi_{MN}(\theta \times \beta)) = \theta \times \beta = \varphi_M(\alpha) \times \varphi_N(\beta)$: open.

For φ_{MN}^{-1} : inverse of φ_{MN}, $\varphi_{MN}^{-1}(\mathcal{O}\times\beta)$ is open

$\Rightarrow \varphi_{MN}(\varphi_{MN}^{-1}(\mathcal{O}\times\beta)) = \mathcal{O}\times\beta$ is open

$\therefore$ φ_{MN}^{-1} is continuous.

$\therefore$ φ_{MN} is Homeomorphism.

Therefore, $M^m \times N^n$ is a manifold.

[2] : Since M^m, N^n are smooth manifold, they have a C^∞-structure, so that the coordinate charts (U, φ_M), (V, φ_N) is C^∞-compatible with all charts in the atlas of M^m, N^n, respectively.

By [1], we defined the homeomorphism φ_{MN}, hence we can write the coordinate chart of $M^m \times N^n$ that $(U\times V, \varphi_{MN})$. Consider another chart $(U'\times V', \varphi_{MN}^*)$, then

$$\varphi_{MN} \circ \varphi_{MN}^* = (\varphi_M \times \varphi_N)\circ(\varphi_M^* \times \varphi_N^*)^{-1}$$

$$= \varphi_M \circ \varphi_M^{*-1} \times \varphi_N \circ \varphi_N^{*-1}.$$

Since φ_M, φ_N, φ_M^{*-1}, φ_N^{*-1} are C^∞,

$\varphi_{MN} \circ \varphi_{MN}^*$ is C^∞.

$\therefore$ $M^m \times N^n$ is a smooth manifold.

Thus, By the proof above, T^n is smooth manifold. $\blacksquare$

1.1.2 Let $U \subset \mathbb{R}^n$ be open and $f : U \to \mathbb{R}^m$ be continuous. Show that the graph of f

$$\Gamma_f = \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^m : x \in U \text{ and } y = f(x)\}$$

is an n-dimensional manifold.

proof) By the example 1.7. (i) of the lecture note of professor Han, $\mathbb{R}^n$, $\mathbb{R}^m$ are n, m dimensional smooth manifold and hence $\mathbb{R}^n \times \mathbb{R}^m$ is smooth manifold by the exercise 1.1.1.

Thus, the graph of f Γ_f is the subspace topology of $\mathbb{R}^n \times \mathbb{R}^m$. Hence, Γ_f is Hausdorff and 2nd-countable space.

So, we want to show that Γ_f has the locally Euclidean property only.

Let $\pi_x : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is the projection onto x, and let $\varphi : \Gamma_f \to U$ be the restriction of π_x to Γ_f that $\varphi(x, y) = x$, $(x, y) \in \Gamma_f$.

Since π_x is continuous (clearly), the restriction of π_x φ is continuous, and bijective also. Thus $\exists\, \varphi^{-1}$: inverse of φ and since $\varphi^{-1}(x) = (x, f(x))$, φ^{-1} is continuous.

$\therefore$ φ : Homeomorphism.

$\therefore$ Γ_f is n-dimensional manifold. $\square$

1.2.1 Complete the proof of proposition 1.14 :

Suppose that $\pi : M \to M/\sim$ is an open map. Then

(ii) $M/\sim$ is Hausdorff $\Rightarrow$ $R = \{(p,q) : p \sim q\}$ is closed in $M \times M$.

proof) Note that :

$$[x]_\sim = \{ x \in M : x \sim \alpha, \alpha \in M \}.$$

$$M/\sim = \{ [x]_\sim : x \in M \}$$

$\mathcal{O} \subset M/\sim$ is open $\Leftrightarrow$ $\pi^{-1}(\mathcal{O}) = \{ x : \pi(x) = [x] \in \mathcal{O} \}$ is open in M.

Assume that $M/\sim$ is Hausdorff.

Claim : $R \subset M \times M$ is closed

$\Leftrightarrow$ $M \times M - R$ is open.

Let $(p,q) \in M \times M - R$, then $\pi(p) \neq \pi(q)$

$\Rightarrow$ $(p,q) \notin R$. Thus we can take the disjoint open sets $\pi(p) \in U_1$, $\pi(q) \in U_2$ since $M/\sim$ is Hausdorff.

Let $V_1 = \pi^{-1}(U_1)$ & $V_2 = \pi^{-1}(U_2)$.

If $(V_1 \times V_2) \cap R \neq \phi$, then $^\exists (v_1, v_2) \in V_1 \times V_2$ such that $\pi(v_1) = \pi(v_2)$, $\pi(v_1) \in U_1$, $\pi(v_2) \in U_2$.

But, $U_1 \cap U_2 = \phi$, that is contradiction.

$\therefore$ R is closed in $M \times M$. $\blacksquare$

1.2.2 Let $f : S^n \to S^n$ be the antipodal map defined by $f(x) = -x$. Define an relation $\sim$ on S^n by $x \sim y$ iff $y = x$ or $y = f(x)$. Show that $\sim$ is an equivalence relation and $S^n/\sim \; = \mathbb{R}P^n$.

proof) ① Equivalence relation

「 $x \sim y \iff y = x \iff y - x = 0$ 」

(i) $x \sim x$ since $x - x = 0$

(ii) if $x \sim y$, then

$$y = x \iff y - x = 0 \iff -(x - y) = 0$$

$$\iff x - y = 0 \iff x = y$$

$$\iff y \sim x.$$

$$y = -x \iff -y = x \iff x = f(y)$$

$$\iff y \sim x$$

(iii) if $x \sim y$ & $y \sim z$, then

$y = x$ and $z = y$ and so

$$z = y = x \iff x \sim z.$$

$y = -x$ & $z = -y$, then

$$z = -y = -(-x) = x \iff x \sim z.$$

② $S^n/\sim \; = \mathbb{R}P^n$

$$[x]_M = \bigcup_i [x_i]_{S^n}. \quad (\because [x]_{S^n} \subset [x]_M)$$

$$\Rightarrow \bigcup_j \left(\bigcup_i [x_i]_{S^n} \right)_j = M/\sim \; = \mathbb{R}P^n$$

For arbitrary $r \in \mathbb{R}^{n+1} - \{0\}$, we can let

$S^n = \{ \vec{x} : \|\vec{x}\| = r \}$, and thus

$S^n/\sim \; = \mathbb{R}P^n$.

(Additional Information i thought)

$$\begin{cases} S^n = \{ (x_1, \cdots, x_{n+1}) : x_1^2 + \cdots + x_{n+1}^2 = r^2 \}. \\ (0, 0, \cdots, 0) \notin S^n \subset M = \mathbb{R}^{n+1} - \{0\}. \\ [x]_M = \{ x \in M : x \sim y \iff y = tx \text{ for some } t \neq 0 \} \\ [x]_{S^n} = \{ x \in S^n : x \sim y \iff y = \pm x \} \\ \Rightarrow \quad [x]_{S^n} \subset [x]_M. \quad (\because [x]_{S^n} = \{-x, x\}) \\ \Rightarrow [x]_{S^n} \in M/\sim \; = \mathbb{R}P^n. \\ \therefore \; S^n/\sim \; \subset M/\sim \; = \mathbb{R}P^n. \end{cases}$$

$\square$

1.2.3. The complex projective space $\mathbb{C}P^n$ is the set of all line through the origin in $\mathbb{C}^{n+1}$, i.e., the set of 1-dimensional subspaces of $\mathbb{C}^{n+1}$. If we define an equivalence relation on $M = \mathbb{C}^{n+1} - \{0\}$ by $z \sim w \Leftrightarrow w = \lambda z$ for some $\lambda \in \mathbb{C}^*$, then $\mathbb{C}P^n = M/\sim$. Show that $\mathbb{C}P^n$ is a $2n$-dimensional smooth manifold.

proof) ① 2nd-countable

Since M is 2nd-countable, the quotient set of M is 2nd-countable.

② Hausdorff.

$[z_1], [z_2] \in U_j$ for some j

$\Rightarrow [z_1]$ and $[z_2]$ are disjoint open set,

$(\because)$ $\varphi_j(z_1), \varphi_j(z_2) \in \mathbb{C}^n$.

Claim: $\nexists\, U_j$ containing $[z_1]$ & $[z_2]$.

Given $j \neq k$, let

$A_{j,k} = \{[z] : |z^j| > |z^k|\} \subset \mathbb{C}P^n$.

Then $A_{j,k}$ is open since

$\pi^{-1}(A_{j,k})$ is open in $\mathbb{C}^{n+1} - \{0\}$.

By the assumption, $\exists\, j \neq k$ s.t.

$[z_1] \in U_j$ and $[z_2] \in U_k$, but

$z_1^j = z_2^k = 0$.

$\therefore$ $z_1 \in A_{j,k}$, $z_2 \in A_{k,j}$.

$\therefore A_{j,k} \cap A_{k,j} = \phi$.

③ local Euclidean

For $z = (z^0, \cdots, z^n) \in \mathbb{C}^{n+1}$, define

$U_i = \{[z] : z^i \neq 0\} \subset \mathbb{CP}^n$, then we can

define $\varphi_i : U_i \to \mathbb{C}^n$ by

$$\varphi_i([z]) = \left(\frac{z^0}{z^i}, \cdots, \frac{z^{i-1}}{z^i}, \frac{z^{i+1}}{z^i}, \cdots, \frac{z^n}{z^i}\right).$$

continuous

For a projection $\pi : M \to M/_\sim$ by $\pi(z) = [z]$,

$\varphi_i^{-1}(V)$ is open for any open $V \subset \mathbb{C}^n$

$\Leftrightarrow \pi^{-1} \circ \varphi_i^{-1}(V) = (\varphi_i \circ \pi)^{-1}(V)$ is open in $\mathbb{C}^{n+1}$.

Since $\varphi_i \circ \pi$ is clearly continuous, $(\because z^i \neq 0)$

$(\varphi_i \circ \pi)^{-1}(V)$ is open in $\mathbb{C}^{n+1}$.

① injective

$$\varphi_i([z_1]) = \varphi_i([z_2]) \Rightarrow \frac{z_1^j}{z_2^j} = \frac{z_1^i}{z_2^i}, \quad j \neq i$$

$\Rightarrow [z_1] = [z_2]$. $(\because z_1^j = t z_2^j$ for $\forall j)$

②

$v = (v^1, \cdots, v^n) \in \mathbb{C}^n$, then

$\varphi_i^{-1}(v) = \pi(v^1, \cdots, v^{i-1}, 1, v^i, \cdots, v^n)$.

Since π is continuous, φ_i^{-1} is continuous.

$(\because)$ $\varphi_i(\pi(v^1, \cdots, v^{i-1}, 1, v^i, \cdots, v^n))$

$$= \left(\frac{v^1}{1}, \cdots, \frac{v^{i-1}}{1}, \frac{v^{i+1}}{1}, \cdots, \frac{v^n}{1}\right) = v$$

By ①, ③, φ_i is a homeomorphism.

③ assuming w.r.o.g, $i < j$, the transition maps

$$\varphi_j \circ \varphi_i^{-1} : \varphi(U_i \cap U_j) = \{ z = (z^1, \cdots, z^n) \in \mathbb{C}^n : z^j \neq 0 \}$$

$$\to \varphi(U_i \cap U_j)$$

$$\varphi_j \circ \varphi_i^{-1}(z^1, \cdots, z^n) = \varphi_j([(z^1, \cdots, z^{i-1}, 1, z^{i+1}, \cdots, z^n)])$$

$$= \left(\frac{z^1}{z^j}, \cdots, \frac{z^i}{z^j}, \frac{1}{z^i}, \frac{z^{i+1}}{z^j}, \cdots, \frac{z^{j-1}}{z^j}, \frac{z^{j+1}}{z^j}, \cdots, \frac{z^n}{z^j} \right)$$

is smooth,

Thus, for $\{(U_i, \varphi_i) \mid i = 1, \cdots, n+1\} = \mathcal{A}$,

$\mathcal{A}$ is C^∞ atlas.

$\therefore$ $\mathbb{CP}^n$ is a $2n$-dimensional smooth manifold. $\blacksquare$

1.3.1. Let $N = M = \mathbb{R}P^1$ and write a point in $\mathbb{R}P^1$ as $[(x,y)]$ for $(x,y) \in \mathbb{R}^2$. Show that the map $F : N \to M$ given by $F([(x,y)]) = [(x^2, y^2)]$ is smooth.

proof) For $\{(U, \varphi)\} = \mathcal{A}_N$ and $\{(V, \psi)\} = \mathcal{A}_M$,

let $\varphi = \psi = \pi^{-1}$, i.e.

$\varphi : N \to \mathbb{R}^2$ by $\varphi([[(x,y)]]) = (x,y)$

$\psi : M \to \mathbb{R}^2$ by $\psi([x,y]) = (x,y)$

Then, These can be a homeomorphism.

Thus,

$$f \equiv \psi \circ F \circ \varphi^{-1} = \psi(F([(x,y)]))$$

$\mathbb{R}^2 \to \mathbb{R}^2$

$$= \psi([(x^2, y^2)]) = (x^2, y^2).$$

Since the components of f is smooth,

$f = \psi \circ F \circ \varphi^{-1}$ is smooth, and therefore,

$F : N \to M$ is smooth mapping.

$\square$

1.3.2 Prove that (ii) of Theorem 1.28.

proof) For each $q \in A$, $\exists (\varphi, U)$ near q

that $U_q \subset U$ & $\varphi(U_q) \subset V$,

$V = \{$ contains the open ball $B_3(0)$ $\}$.

$B_3(0) \equiv$ open ball of radius 3 centered at 0.

Let $\tilde{U}_q = \varphi^{-1}(B_1(0))$ and

define the function called "Bump function" that

$$g(t) = \begin{cases} 1 & \text{for } t \le 1 \\ 0 & \text{for } t \ge 2 \end{cases},$$

and let

$$f(p) = \begin{cases} g(\varphi(p)) & p \in U_q \\ 0 & p \notin U_q \end{cases}.$$

Then, $f \in C^\infty(M)$ such that $0 \le f \le 1$,

$f \equiv 1$ on $\tilde{U}_q \subset A$ and $\mathrm{supp}(f) \subset U_p \subset U$. $\blacksquare$

2.1.1 For a smooth map $F : N^n \to M^m$, the push forward $F_{*p} : T_pN \to T_{F(p)}M$ at $p \in N$ was defined in terms of derivations. This can be also defined by the equivalence class of curves as

$$F_{*p}([\gamma]) = [F \circ \gamma] .$$

Show that this definition is well defined. In other words, $\gamma_1 \sim \gamma_2$ implies that $F \circ \gamma_1 \sim F \circ \gamma_2$.

proof) For a coordinate chart (U, φ) at p,

$$\gamma_1 \sim \gamma_2 \iff (\varphi \circ \gamma_1)'(0) = (\varphi \circ \gamma_2)'(0).$$

$$F \circ \gamma = F \circ \varphi^{-1} \circ \varphi \circ \gamma \quad \text{since } \varphi : \text{homeomorphism}.$$

$$\begin{aligned}
\Rightarrow (F \circ \gamma_1)'(0) &= (F \circ \varphi^{-1} \circ \varphi \circ \gamma_1)'(0) \\
&= (F \circ \varphi^{-1})'(\varphi \circ \gamma_1)(0). \\
(\because \gamma_1 \sim \gamma_2) \quad &= (F \circ \varphi^{-1})'(\varphi \circ \gamma_2)(0) \\
&= (F \circ \varphi^{-1} \circ \varphi \circ \gamma_2)'(0) \\
&= (F \circ \gamma_2)'(0)
\end{aligned}$$

$$\iff F \circ \gamma_1 \sim F \circ \gamma_2 \qquad \blacksquare$$

2.1.2. Let $F : N^n \to M^m$ be a smooth map.
Prove that if N is connected and $F_{*p} = 0$ for any $p \in N$, then F is constant map.

proof) Let $f \in C^\infty(M)$ and let $X_p \in T_pN$.
By the assumption, $F_{*p}[f] = X_p(f \circ F) = 0$.

Let (U, φ) : smooth chart containing p. Then

$$X_p = \sum_i X_p^i \frac{\partial}{\partial x^i}\Big|_p = \sum_i X^i (\varphi^{-1})_{*p} \frac{\partial}{\partial x^i}\Big|_{\varphi(p)}$$

$$\Rightarrow \left(\sum_i X^i (\varphi^{-1})_{*p} \frac{\partial}{\partial x^i}\Big|_{\varphi(p)}\right)(f \circ F) = \sum_i X^i \frac{\partial}{\partial x^i}\Big|_{\varphi(p)} (f \circ F \circ \varphi^{-1}) = 0$$

$\Rightarrow$ F is constant on U.

Since N is connected, N : path connected.
Let $q \in N$ & let $\gamma : [0,1] \to N$ be a path connecting p & q.
Since F is constant on each smooth chart $(U_{\gamma(x)}, \varphi_{\gamma(x)})$ containing $\gamma(x)$ for every $x \in [0,1]$, $F \equiv c$ on N since $F(p) = c$ & γ is continuous. $\square$

2.1.3. Prove that for any $p \in S^n$,

$$T_p S^n = \{ X \in \mathbb{R}^{n+1} : \langle p, X \rangle = 0 \}.$$

proof) Let $p \in S^n \subset \mathbb{R}^{n+1} - \{0\}$ and let $X \in \mathbb{R}^{n+1}$.

$\ulcorner$

By the example 2.2, $T_p \mathbb{R}^m = \mathbb{R}^m$.

$m = n+1 \Rightarrow T_p \mathbb{R}^{n+1} = \mathbb{R}^{n+1} \supset S^n$.

$[X]_{S^n} = \{ X \in \mathbb{R}^{n+1} : X \sim p \Leftrightarrow p = tX$ for some $t \neq 0 \}$.

$\langle p, X \rangle = 0 \Rightarrow p = -X$. $\lrcorner$

Define

$$\gamma_{p,X}(t) = \begin{cases} p + tX & t \neq 1 \\ p & t = 1 \end{cases}.$$

Then, $\left.\frac{d}{dt} \gamma_{p,X}(t) \right|_{t=0} = p + X$, $\gamma_{p,X}(0) = p = -X$

$\Rightarrow [\gamma_{p,X}] \in T_p S^n$.

$\therefore T_p S^n = \{ X \in \mathbb{R}^{n+1} : \langle p, X \rangle = 0 \}$. $\blacksquare$

2.3.1. Let $F : \mathbb{R}^2 \to \mathbb{R}^2$ such that $F(x, y) = (x^2 - 2y,\ 4x^3y^2)$. For $X = 4x\frac{\partial}{\partial x} + 3y^2\frac{\partial}{\partial y}$, compute $F_* X$.

proof) For the vector field $X = 4x\frac{\partial}{\partial x} + 3y^2\frac{\partial}{\partial y}$,

let $x' = x^2 - 2y$ & $y' = 4x^3y^2$, then

$$F_*\left(\frac{\partial}{\partial x}\right) = \frac{\partial x'}{\partial x}\frac{\partial}{\partial x} + \frac{\partial y}{\partial x}\frac{\partial}{\partial y} = 12x^2y^2\frac{\partial}{\partial y}$$

$$F_*\left(\frac{\partial}{\partial y}\right) = \frac{\partial x'}{\partial y}\frac{\partial}{\partial x} + \frac{\partial y'}{\partial y}\frac{\partial}{\partial y} = -2\frac{\partial}{\partial x} + 8x^3y\frac{\partial}{\partial y}$$

$$F_* X = 4x\,F_*\left(\frac{\partial}{\partial x}\right) + 3y^2\,F_*\left(\frac{\partial}{\partial y}\right)$$

$$= -6y^2\frac{\partial}{\partial x} + x^3y^2(24y + 48)\frac{\partial}{\partial y}$$

$\square$

2.3.2. Express the following planar vector fields in polar coordinates.

$$X = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}, \quad Y = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}.$$

What is $[X, Y]$?

proof) Let $x = r\cos\theta$, $y = r\sin\theta$, then

$$\frac{\partial}{\partial r} = \frac{\partial x}{\partial r}\frac{\partial}{\partial x} + \frac{\partial y}{\partial r}\frac{\partial}{\partial y} = \cos\theta\frac{\partial}{\partial x} + \sin\theta\frac{\partial}{\partial y}$$

$$= \frac{x}{r}\frac{\partial}{\partial x} + \frac{y}{r}\frac{\partial}{\partial y} = \frac{1}{r}X$$

$$\frac{\partial}{\partial\theta} = \frac{\partial x}{\partial\theta}\frac{\partial}{\partial x} + \frac{\partial y}{\partial\theta}\frac{\partial}{\partial y} = -r\sin\theta\frac{\partial}{\partial x} + r\cos\theta\frac{\partial}{\partial y}$$

$$= -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y} = Y$$

$$\therefore X = r\frac{\partial}{\partial r}, \quad Y = \frac{\partial}{\partial\theta}.$$

By definition 2.21.(iii), $[X, Y] = \left[r\frac{\partial}{\partial r}, \frac{\partial}{\partial\theta}\right] = 0$.

↑

professor Han's note

$\square$

2.3.3. In $\mathbb{R}^3$, let

$$X = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x} \quad \text{and} \quad Y = y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}.$$

Compute $[X, Y]$.

proof) By definition 2.22, we have

$$\begin{aligned}
[X,Y] &= \left[x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x},\ y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right] \\
&= \underbrace{\left[x\frac{\partial}{\partial y},\ y\frac{\partial}{\partial z}\right]}_{①} + \underbrace{\left[x\frac{\partial}{\partial y},\ -z\frac{\partial}{\partial y}\right]}_{②} + \\
&\quad \underbrace{\left[-y\frac{\partial}{\partial x},\ y\frac{\partial}{\partial z}\right]}_{③} + \underbrace{\left[-y\frac{\partial}{\partial x},\ -z\frac{\partial}{\partial y}\right]}_{④}
\end{aligned}$$

By definition 2.21 (iii), we have

$$① = x\cdot 0\frac{\partial}{\partial z} + y\cdot 0\frac{\partial}{\partial y} = 0$$

$$② = x\cdot 0\frac{\partial}{\partial y} + (-z)\cdot 0\frac{\partial}{\partial y} = 0$$

$$③ = -y\cdot 0\frac{\partial}{\partial z} + y\cdot 0\frac{\partial}{\partial x} = 0$$

$$④ = -y\cdot 0\frac{\partial}{\partial y} - z\cdot 0\frac{\partial}{\partial x} = 0$$

$\therefore [X, Y] = 0$ $\square$

2.3.4. Verify Example 2.23.

proof) (i) $\mathbb{R}^n$ is a Lie algebra.

Since $[a,b] = (a+b)-(b+a) = 0$,

bilinear & skew summetric satisfied.

Check Jacobi identity condition. For $c \in \mathbb{R}^n$,

$[a,[b,c]] + [b,[c,a]] + [c,[a,b]]$

$= a + [b,c] - ([b,c]+a) + b + [c,a] - ([c,a]+b)$
$+ c + [a,b] - ([a,b]+c)$

$= a+0 - (0+a) + b + 0 - (0+b) + c + 0 - (0+c)$

$= a-a + b-b + c-c = 0.$

$\therefore$ $\mathbb{R}^n$ is a Lie algebra.

(ii) $GL(n,\mathbb{R})$ is a Lie algebra.

Check only the condition for Jacobi.

$[A,B] = AB - BA$ for $A, B \in GL(n,\mathbb{R})$,

pick $C \in GL(n,\mathbb{R})$, then

$[A,[B,C]] + [B,[C,A]] + [C,[A,B]]$

$= A([B,C]) - ([B,C]A) + B([C,A]) - ([C,A]B)$
$+ C([A,B]) - ([A,B]C)$

$= A(BC-CB) - (BC-CB)A + B(CA-AC) - (CA-AC)B$
$+ C(AB-BA) - (AB-BA)C$

$= ABC - ACB - BCA + CBA + BCA - BAC - CAB + ACB$
$+ CAB - CBA - ABC + BAC$

$= 0$. $\therefore$ $GL(n,\mathbb{R})$ is a Lie algebra.

(iii) $\mathbb{R}^3$ is a Lie algebra with $[u,v] = u \times v$, $u, v \in \mathbb{R}^3$

Cross product satisfy the skew symmetric condition

($\because$ $u \times v = -v \times u$). and bilinear condition also.

($\because$ Let $u, v, w \in \mathbb{R}^3$, i, j, k : standard basis of $\mathbb{R}^3$.

$u = u_1 i + u_2 j + u_3 k$, $v = v_1 i + v_2 j + v_3 k$,

$w = w_1 i + w_2 j + w_3 k$. Then, for $c \in \mathbb{R}$,

$(cu + v) \times w = c(u \times w) + v \times w$.

pf)
$$(cu+v) \times w = \begin{vmatrix} i & j & k \\ cu_1+v_1 & cu_2+v_2 & cu_3+v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$

$$= \begin{vmatrix} i & j & k \\ cu_1 & cu_2 & cu_3 \\ w_1 & w_2 & w_3 \end{vmatrix} + \begin{vmatrix} i & j & k \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$

$$= c \begin{vmatrix} i & j & k \\ u_1 & u_2 & u_3 \\ w_1 & w_2 & w_3 \end{vmatrix} + \begin{vmatrix} i & j & k \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$

$= c(u \times w) + v \times w$. Q.E.D)

Check Jacobi identity condition. For $u, v, w \in \mathbb{R}^3$,

$[u,[v,w]] + [v,[w,u]] + [w,[u,v]]$

$= u \times (v \times w) + v \times (w \times u) + w \times (u \times v)$

$= (u \cdot w)v - (u \cdot v)w + (v \cdot u)w - (v \cdot w)u$

$\quad + (w \cdot v)u - (w \cdot u)v$

$= 0$.

$\therefore$ $\mathbb{R}^3$ is a Lie algebra with $[u,v] = u \times v$ for $u, v \in \mathbb{R}^3$.

(iv) $\mathfrak{G}, \mathfrak{h}$ are Lie algebras $\Rightarrow$ $\mathfrak{G} \times \mathfrak{h}$ is also a Lie algebra under the bracket

$$[(X_1, Y_1), (X_2, Y_2)] = ([X_1, Y_1], [X_2, Y_2]).$$

① Bilinearity

Clearly we obtain the property after complicate calculation.

Note that $[X_1, Y_1], [X_2, Y_2]$ are satisfy the bilinearity in $\mathfrak{G}, \mathfrak{h}$, respectively.

② skew - symmetric.

For the simplicity, denote $X_i, Y_i \equiv x_i, y_i$.

$$\begin{aligned}
[(x_1, y_1), (x_2, y_2)] &= ([x_1, y_1], [x_2, y_2]) \\
&= (-[y_1, x_1], -[y_2, x_2]) \\
&= -([y_1, x_1], [y_2, x_2]) \\
&= -[(y_1, x_1), (y_2, x_2)]
\end{aligned}$$

$\mathfrak{G}, \mathfrak{h}$: Lie algebra $\longrightarrow$ (second equality)

③ Jacobi identity

$$\begin{aligned}
&[(x_1, y_1), [(x_2, y_2), (x_3, y_3)]] \\
&+ [(x_2, y_2), [(x_3, y_3), (x_1, y_1)]] \\
&+ [(x_3, y_3), [(x_1, y_1), (x_2, y_2)]] = 0
\end{aligned}$$

using the previous results we proved and our Lie bracket. $\square$

2.3.5. Prove Theorem 2.24.

proof) Check the Jacobi identity.

For the simplicity, denote $X, Y, \cdots \equiv x, y, \cdots$.

$$[x,[y,z]]+[y,[z,x]]+[z,[x,y]]$$

$$= x(yz-zy)-(yz-zy)x+y(zx-xz)-(zx-xz)y$$
$$+z(xy-yx)-(xy-yx)z$$

$$= xyz-xzy-yzx+zyx+yzx-yxz-zxy+xzy$$
$$+zxy-zyx-xyz+yxz$$

$$=0. \qquad \square$$

2.3.6. Prove Theorem 2.26.

proof) By the assumption, X_i, Y_i are F-related.

i.e. $F_*(X_i) = Y_i$ by definition 2.25.

<u>Claim</u>: $F_*([X_1, X_2]) = [Y_1, Y_2]$.

Choose $g \in C^\infty(M)$ and $x \in N$, then

$$(Y_i g)(F(x)) = (F_*)_x (X_i)(g) = X_i(g \circ F)$$

Thus, $(Y_i g) \circ F = X_i(g \circ F)$ $\cdots$ (*)

Let $f \in C^\infty(M)$ be arbitrary. Using (*)

$\Rightarrow Y_1(Y_2 f) \circ F = X_1((Y_2 f) \circ F)$. $\cdots$ (**)

By (*), we also obtain

$$(Y_2 f) \circ F = X_2(f \circ F) \text{ and thus}$$

$$(**) = Y_1(Y_2 f) \circ F = X_1(X_2(f \circ F)).$$

Likewise, we get

$$Y_2(Y_1 f) \circ F = X_2(X_1(f \circ F)).$$

Hence, $([Y_1, Y_2] f) \circ F = [X_1, X_2](f \circ F)$.

Therefore, $[Y_1, Y_2]$ is F-related to $[X_1, X_2]$ $\square$

2.3.7. Let $F : N \longrightarrow M$ be a diffeomorphism. Prove that for any $Y \in \mathfrak{X}(M)$, there is a unique $X \in \mathfrak{X}(N)$ such that X is F-related to Y.

proof) Assume that X is F-related to Y.

i.e. $X_{F(p)} = F_{*p}(Y_p)$.

If F is a diffeomorphism, we define X by

$$X_q = F_{*F^{-1}(q)}(Y_{F^{-1}(q)})$$

Then, it is clear that X is the unique vector field such that F-related to Y.

Note that

$X : N \rightarrow TN$, N : manifold, TN : tangent bundle.

Then X is the composition that

$$N \xrightarrow{F^{-1}} M \xrightarrow{X} TM \xrightarrow{dF} TN$$

$\Rightarrow X$ is smooth.

$\square$

2.3.8. Express the planar 1-form $\omega = x\,dx + y\,dy$ in polar coordinates.

proof) Let $x = r\cos\theta$ and $y = r\sin\theta$.
By Chain rule,

$$dx = \frac{\partial x}{\partial r} dr + \frac{\partial x}{\partial \theta} d\theta = \cos\theta\, dr - r\sin\theta\, d\theta$$

$$dy = \frac{\partial y}{\partial r} dr + \frac{\partial y}{\partial \theta} d\theta = \sin\theta\, dr + r\cos\theta\, d\theta$$

The differential 1-form ω is expressed by

$$\omega = \left(\frac{\partial x}{\partial r} x + \frac{\partial y}{\partial r} y\right) dr + \left(\frac{\partial x}{\partial \theta} x + \frac{\partial y}{\partial \theta} y\right) d\theta$$

$$= \left(r\cos^2\theta + r\sin^2\theta\right) dr + \left(-r^2\sin\theta\cos\theta + r^2\cos\theta\sin\theta\right) d\theta$$

$$= r\,dr + 0\,d\theta$$

$$= r\,dr.$$

$\square$

3.2.1. Let $M = \{(x,y,z) \in \mathbb{R}^3 : x^3+y^3+z^3-3xyz = 1\}$. Prove that M is a 2-dimensional regular submanifold of $\mathbb{R}^3$. What is T_pM at $p = (0,0,1)$?

proof) Let $F : \mathbb{R}^3 \to \mathbb{R}$ be defined by

$$F(x,y,z) = x^3+y^3+z^3-3xyz-1.$$

Note that :

Definition

The rank of a smooth map $f : N \to M$ between two manifolds at a point $p \in N$ is the rank of the derivative of f at p.

For $q = 0 \in \mathbb{R}$, $F^{-1}(q) = M$ and Jacobian is

$$DF = F_* = \left[\, 3x^2-3yz \quad 3y^2-3xz \quad 3z^2-3xy \,\right], \vec{0} \notin M.$$

Then, $(0,0,0)$ is the only critical point of F.

Thus, q is a regular value of F.

By Theorem 3.14 (ii) in Han's lecture note,

$F^{-1}(q) = M$ is a 2-dimensional regular submanifold.

T_pM at $p = (0,0,1)$ is :

we have the equation of T_pM that

$$0(x-0) + 0(y-0) + 3(z-1) = 3(z-1) = 0.$$

$\therefore$ $T_pM = \{(x,y,z) \in \mathbb{R}^3 : 3(z-1) = 0\}$. $\square$

3.2.2. Show that $F : \mathbb{RP}^2 \to \mathbb{R}^4$ be defined by

$$F[(x,y,z)] = \frac{1}{x^2+y^2+z^2}(x^2-y^2, xy, xz, yz)$$

is a smooth embedding.

proof) $\ulcorner$ Consider the quotient projection $\pi : p \mapsto [p]$.
We know that π is local diffeomorphism and $\mathbb{RP}^2$ has the quotient topology of S^2 via π.

$$\begin{tikzcd} S^2 \arrow[r, "f"] \arrow[d, "\pi"'] & \mathbb{R}^4 \\ \mathbb{RP}^2 \arrow[ur, "F"'] & \end{tikzcd}$$

$\lrcorner$

Define f by

$f : S^2 \to \mathbb{R}^4$, $f(x,y,z) = (x^2-y^2, xy, xz, yz)$.

Then, $f = F \circ \pi$ since

$\pi(p_1) = \pi(p_2) \Rightarrow p_1 = \pm p_2 \Rightarrow f(p_1) = f(p_2)$.

$f(p) = f(-p)$.

Calculate the Jacobian of f :

$$Df = \begin{bmatrix} 2x & -2y & 0 \\ y & x & 0 \\ z & 0 & x \\ 0 & z & y \end{bmatrix}$$

If $x \neq 0, y \neq 0$, then $\operatorname{rank}(Df) = 3$.

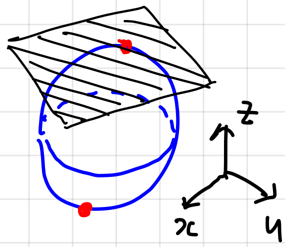
$\Rightarrow$ it is injective linear map $: \mathbb{R}^3 \to \mathbb{R}^4$.

(in this, restricted to the tangent plane of S^2 at $p \in S^2$ is still injective.

$\Rightarrow$ the map has $\operatorname{rank} = 2$ at p $\to$ a map of S^2.)

If $x = y = 0$, then $p = (0, 0, \pm 1)$

$\Rightarrow$ $T_p S^2$ is xy-plane in $\mathbb{R}^3$.



$\Rightarrow$ $T_p S^2$ is mapped injectively into $\mathbb{R}^4$

$\Rightarrow$ $\operatorname{rank}(f) = 2$.

Thus, $F = f \circ \pi^{-1}$, $\operatorname{rank}(F) = 2$ locally.

Now, we claim $f(p) = f(q)$ for $p, q \in S^2$, then $p = \pm q$.

Consider $\varphi : \mathbb{R}^2 \to \mathbb{R}^2$ by $(x, y) \mapsto (x^2 - y^2, xy)$.

Let $x^2 - y^2 = a$, $xy = b$. Then

$$a^2 + 4b^2 = x^4 + 2x^2y^2 + y^4 = (x^2+y^2)^2$$
$$x^2 + y^2 = \sqrt{a^2+4b^2}$$
$$x^2 = \frac{1}{2}(a + \sqrt{a^2+4b^2})$$
$$y^2 = \frac{1}{2}(-a + \sqrt{a^2+4b^2})$$

$\Rightarrow$ $\pm x, \pm y$: uniquely determined by a, b

xy already determined by b.

$\Rightarrow$ only $\pm(x, y)$ are mapped to (a, b).

Now, (x_1, y_1, z_1), (x_2, y_2, z_2) are mapped to the same point, then we know that $(x_1, y_1) = \pm(x_2, y_2)$, so $z_1 = \pm z_2$.

$\Rightarrow (x_1, y_1, z_1) = \pm(x_2, y_2, z_2)$ in S^2

$\rightsquigarrow$ the two points are antipodal.

Thus, F is injective, so F is injective immersion.

Since $\mathbb{RP}^2$ is compact because

it is the image of S^2 by the continuous map π.

Therefore, by Theorem 3.6.(iii),

F is a smooth embedding. $\square$

3.2.3. Let $F : \mathbb{R}^3 \to \mathbb{R}^2$ be defined by $F(x,y,z) = (x^2+y,\ x^2+y^2+z^2+y)$. Show that $q = (0,1)$ is regular value of F and $F^{-1}(q)$ is diffeomorphic to S^1.

proof) Let $f_1 = x^2+y$, $f_2 = x^2+y^2+z^2+y$.

$f_1 = 0 \Rightarrow y = -x^2$ and $f_2 = 1 \Rightarrow y^2+z^2 = 1$.

$$\underbrace{F^{-1}(0,1)}_{\text{Level set.}} = \{(x,y,z) \in \mathbb{R}^3 : y = -x^2 \text{ and } y^2+z^2 = 1\}$$

$$DF = \begin{bmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} & \frac{\partial f_1}{\partial z} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} & \frac{\partial f_2}{\partial z} \end{bmatrix} = \begin{bmatrix} 2x & 1 & 0 \\ 2x & 2y+1 & 2z \end{bmatrix}$$

consider $x=y=0$ or $x\neq 0, y\neq 0$

$\Rightarrow$ rank $DF = 2$ for all $(x,y,z) \in F^{-1}(0,1)$.

By the definition 3.13, $q = (1,0)$ is regular value.

Thus, by theorem 3.14 (ii), $F^{-1}(q)$ is an 1-dimensional regular submanifold of $\mathbb{R}^3$.

Now, we want to show that $F^{-1}(q)$ is diffeomorphic to S^1.

Recall the definition of diffeomorphic.

Two manifolds M, N are diffeomorphic if there is a diffeomorphism f from M to N

($f : C^\infty$, homeo, $f^{-1} : C^\infty$)

Define $\varphi : F^{-1}(q) \to S^1$ by $\varphi(x,y,z) = (y,z)$.

Then, clearly, φ is smooth and bijective.

$(\because)$ φ : surjective $\to$ trivial.

$\varphi(x_1, y_1, z_1) = \varphi(x_2, y_2, z_2)$

$\Rightarrow (y_1, z_1) = (y_2, z_2) \Rightarrow y_1 = y_2$ & $z_1 = z_2$

Since $y = -x^2$, $x_1 = x_2$.

$\therefore$ φ : injective.

Thus, there exists an inverse of $\varphi \equiv \varphi^{-1}$.

Now, we only check that φ^{-1} is smooth.

Consider the inclusion map $i : F^{-1}(q) \to \mathbb{R}^3$ of submanifold $F^{-1}(q)$ and the projection $\pi : \mathbb{R}^3 \to \mathbb{R}^2$. Then i is smooth and π is also smooth. Thus, $\varphi = \pi \circ i$.

Hence, $\varphi^{-1} = (\pi \circ i)^{-1} = i^{-1} \circ \pi^{-1}$ is smooth.

$\therefore$ φ is a diffeomorphism.

$\therefore$ the level set $F^{-1}(q)$ is diffeomorphic to S^1. $\blacksquare$

3.2.4. Let $F : N \to M$ be a smooth map of constant rank. Prove that if F is injective, then it is an immersion.

proof) Let $\dim M = m$ and $\dim N = n$ and suppose that F has constant rank r.

Suppose that F is not an immersion, i.e. $r < n$. By the rank theorem, for each $p \in N$, $\exists$ (U, φ) for N centered at p & (V, ψ) for M centered at $F(p)$ such that F has the coordinate representation

$$\psi \circ F \circ \varphi^{-1}(x^1, \cdots, x^n) = (x^1, \cdots, x^r, 0, \cdots, 0).$$

It follows that $F(0, \cdots, 0, \varepsilon) = F(0, \cdots, 0, 0)$ for any sufficiently small ε.

$\therefore$ F is not injective. $\blacksquare$

~~3.2.2.~~ Show that $F : \mathbb{RP}^2 \to \mathbb{R}^4$ be defined by

$$F[(x,y,z)] = \frac{1}{x^2+y^2+z^2}(x^2-y^2, xy, xz, yz)$$

is a smooth embedding.

~~proof)~~ Note that $\mathbb{RP}^2 = S^2/\{\pm 1\}$ of S^2 which is obtained by identifying antipodal points.

reference 3 생길.

Then, F naturally reduced to the map

$$f : S^2 \to \mathbb{R}^4, \quad f(x,y,z) = (x^2-y^2, xy, xz, yz).$$

In order to apply Theorem 3.6 (Han's note), we have to check two conditions that ① injective immersion ② $\mathbb{RP}^2$ is compact.

Claim : ① → (1) : f is immersion.

Γ f is immersion $\Longleftrightarrow$ $Df : T_pM \to T_{f(p)}M$ is injective.
$(f : M \to N)$

pf) We know that $\dim M = \dim T_pM$ and the fact $\dim V = \operatorname{rank} A + \dim \ker A$, for any linear map A on vector space V.

Since Df is linear, by definition, we obtain $\dim T_pM = \operatorname{rank} Df + \dim \ker Df$.
Now, by the definition of immersion (3.2, (i)), we have :

$$
\begin{aligned}
f : \text{immersion} &\iff \dim M = \operatorname{rank} Df \\
&\iff \dim \ker Df = 0 \\
&\iff \ker Df = \{0\} \\
&\iff Df \text{ is injective.}
\end{aligned}
$$

Clearly, $\dim \mathbb{R}P^2 = 2$.

We know that $\dim M = \dim T_pM$ and the fact $\dim V = \operatorname{rank} A + \dim \ker A$, for any linear map A on vector space V.

Hence, f : injective $\Rightarrow$ Df : injective.

Since Df is linear, by definition, we obtain $\dim T_pM = \operatorname{rank} Df + \dim \ker Df$.

(∵) T_pM is a vector space spanned by a basis $\left\{ \frac{\partial}{\partial x^i}\Big|_p : i \in \{1, \cdots, n\} \right\}$, $n \in \mathbb{N}$.

Now, by the definition of immersion (3.2, (i)), we have :

$$
\begin{aligned}
Df \text{ is injective} &\iff \ker Df = \{0\} \\
&\iff \dim \ker Df = 0 \\
&\iff \dim M = \operatorname{rank} Df \\
&\iff f : \text{immersion}
\end{aligned}
$$

3.2.5. Give an example of an immersion $\iota : N \hookrightarrow M$ and $\omega \in \Gamma(M)$ such that $\iota^*\omega \equiv 0$ on N although $\omega \neq 0$ everywhere on M.

proof) Let $M = \mathbb{R}^2$, $\omega := dy \in \Gamma(M)$.

Consider S be the x-axis $\equiv N$.

(Note that S : embedded submanifold of $\mathbb{R}^2$.)

Then, as a covector field on M,

ω is nonzero everywhere since one of its component functions is always 1.

But, for the restriction $\iota^*\omega$ where $\iota : N \hookrightarrow M$ be an immersion,

$$\iota^*\omega = \iota^* dy = d(y \circ \iota) = 0$$

proposition 2.17 (iii) in Han's note.

y vanishes identically on S.

$\blacksquare$

4.1.1. Let G be a manifold with a group structure. Prove that if the map $G \times G \to G$ defined by $(g, h) \mapsto gh^{-1}$ is smooth, then G is a Lie group.

proof) Let $\mu : G \times G \to G$, $\mu(g, h) = gh^{-1}$.

By the assumption, μ is smooth for all (g, h).

Thus, consider the restriction map defined by

$\mu_g = \mu(e, h) = h^{-1}$ and $\mu_h = \mu(g, e) = g$

for the identity $e \in G$.

Then, μ_g, μ_h are also smooth since

μ is smooth. $(\because \mu = \mu_h \cdot \mu_g)$

Hence, we can define the inverse $inv(\alpha)$

by $inv : G \to G$, $inv(\alpha) = \alpha^{-1}$ as

$\alpha := h^{-1}$, since G is a group.

$(\because)$ Since G is a group, for each $h^{-1} \in G$,

there exists the inverse $h = (h^{-1})^{-1}$ such that

$h(h^{-1}) = e = (h^{-1})h$, e : identity of G.

Therefore, by definition 4.1,

G is a Lie group. $\square$

4.1.2. Prove that if G_1 and G_2 are Lie groups, then $G_1 \times G_2$ is a Lie group. Hence, T^n is a Lie group.

proof) Suppose that G_1, G_2 are Lie groups.
Since G_1, G_2 are groups, it can be written by the direct product of two groups as

$G_1 \times G_2$ such that $(g_1, h_1)(g_2, h_2) = (g_1 h_1, g_2 h_2)$ for $g_1, h_1 \in G_1$, $g_2, h_2 \in G_2$.

Then, $G_1 \times G_2$ is a group is immediate.
On the other hand, by problem 1.1 (Han's note), we know that $G_1 \times G_2$ is a smooth manifold.

Hence, now we only have to show that $G_1 \times G_2$ is a Lie group.

Define $\mu : (G_1 \times G_2) \times (G_1 \times G_2) \to G_1 \times G_2$ by
$\mu((g_1, g_2), (h_1, h_2)) = (g_1, g_2)(h_1, h_2)$, $g_i, h_i \in G_i$.
Since $G_1 \times G_2$ is the direct product of G_1, G_2,
$\mu((g_1, g_2), (h_1, h_2)) = (g_1, g_2)(h_1, h_2) = (g_1 h_1, g_2 h_2)$.
(in this, $g_1 h_1 \in G_1$, $g_2 h_2 \in G_2$ since G_1, G_2 are Lie groups by the assumption.)

Since G_1, G_2 are Lie groups again, there exist x_1^{-1}, x_2^{-1} : inverse of G_1, G_2, respectively so that

we define $inv : G_1 \times G_2 \to G_1 \times G_2$ by
$inv(x_1, x_2) = (x_1, x_2)^{-1}$, then
$inv(x_1, x_2) = (x_1, x_2)^{-1} = (x_1^{-1}, x_2^{-1})$.
Then, $x_i^{-1} \in G_i$ since G_i are Lie group.

Note that the multivariable function is smooth if the components are smooth.
Then, μ, inv are smooth clearly.

($\because$ consider the each component functions as μ_i, inv_i of G_1, G_2. Then these are smooth since G_1, G_2 are Lie group. We just find the proper multiplication & inverse map.)

Therefore, by definition 4.1 in Han's note, $G_1 \times G_2$ is a Lie group.

By the proof above, we can extend the fact that T^n is a Lie group since the direct product is defined componentwise as represented by a tuple. $\square$

4.1.3. Verify Example 4.13.

proof) (i) The complex special linear group

$SL(n,\mathbb{C}) = \{A \in gl(n,\mathbb{C}) : \det A = 1\}$

is $(2n^2-2)$ dimensional Lie subgroup of $GL(n,\mathbb{C})$.

pf) ① Subgroup

For any $A, B \in SL(n,\mathbb{C}) = \det^{-1}(1) \subseteq GL(n,\mathbb{C})$,

we can pick B^{-1} : inverse of B since

$1+i\cdot 0 \rightarrow$ $\det(B) = 1 \neq 0 \iff B$ is invertible.

By the property of determinent,

$\det(B^{-1}) = \det(B)^{-1} = 1 \Rightarrow B^{-1} \in SL(n,\mathbb{C})$.

Then,

$$\begin{aligned}\det(AB^{-1}) &= \det(A)\det(B^{-1}) \\ &= \det(A)\det(B)^{-1} = 1.\end{aligned}$$

$\therefore AB^{-1} \in SL(n,\mathbb{C})$

$\therefore SL(n,\mathbb{C})$ is a subgroup of $GL(n,\mathbb{C})$.

② Submanifold.

Define $\det_{*A} : T_A GL(n,\mathbb{C}) = gl(n,\mathbb{C}) \to \mathbb{C}$.

For $A \in GL(n,\mathbb{C})$ & $B \in gl(n,\mathbb{C})$,

$\gamma(s) = A + sB$, $s \in (-\varepsilon, \varepsilon)$ for small ε.

Using the formula in example 4.11,

we obtain this in the same way as $\mathbb{C}$.

$\rightsquigarrow \det_{*A}(B) = (\det A)\,tr(A^{-1}B)$.

Note that $\dim GL(n,\mathbb{C}) = 2n^2$, $\dim \mathbb{C} = 2$.

$\det_{*A}(A) = \det A \operatorname{tr}(I) = \det(A)(n) \neq 0$.

Hence, $\det_{*A}$ is a submersion for $\forall A \in GL(n,\mathbb{C})$.

Moreover, $SL(n,\mathbb{C}) = \det^{-1}(1)$ is a regular submanifold of $GL(n,\mathbb{C})$ by Thm 3.14 and its dimension is $2n^2-2$.

Thus, by Theorem 4.8, $SL(n,\mathbb{C})$ is a closed Lie subgroup. $\square$

(ii) The unitary group $U(n) = \{A \in gl(n,\mathbb{C}) : A^*A = I_n\}$ is n^2 dimensional Lie subgroup $GL(n,\mathbb{C})$.

pf) ① Subgroup

$$(AB)^*(AB) = (\overline{B^T A^T})(AB) = (\overline{B^T})(\overline{A^T})(AB)$$
$$= B^*A^*AB = B^*I_nB = B^*B = I_n.$$

$\therefore U(n) \leq GL(n,\mathbb{C})$.

② Submanifold.

Let $S(n,\mathbb{C}) = \{A \in gl(n,\mathbb{C}) : A^* = A\}$.

{ <u>Note</u> : $A \in gl(n,\mathbb{R})$ is Hermitan $\Leftrightarrow$ A : symmetric.

↝ symmetric real matrix is the special case of Hermitan. }

Then, $\dim S(n,\mathbb{C}) = n^2$

($\because$ Consider the basis for 2x2 complex matrix)

↳ in linear combination.

Define $F : GL(n,\mathbb{C}) \to S(n,\mathbb{C})$ by $F(A) = A^*A$.
Then $U(n) = F^{-1}(I_n)$. For $A \in U(n)$, we see that

$$F_{*A} : T_A GL(n,\mathbb{C}) = \mathfrak{gl}(n,\mathbb{C}) \to T_{F(A)} S(n,\mathbb{C}).$$

Given $B \in S(n,\mathbb{C})$, the curve $\overset{\text{(smooth)}}{\gamma(t)} = A + tB$ is well defined for all $t \in (-\varepsilon, \varepsilon)$, $\varepsilon > 0$ is small.
Thus,

$$\begin{aligned}
F_{*A}(B) &= \frac{d}{dt}\Big|_{t=0} F \circ \gamma(t) \\
&= \frac{d}{dt}\Big|_{t=0} F(A+tB) \\
&= \frac{d}{dt}\Big|_0 (A+tB)^*(A+tB) \\
&= \frac{d}{dt}\Big|_0 (A^*+tB^*)(A+tB) \quad (\because (A+B)^* = A^*+B^*). \\
&= \underbrace{B^*(A+tB) + (A^*+tB^*)B \Big|_{t=0}}_{(\because)\ dA^* = d(A^*) = (dA)^*} \\
&= B^*A + A^*B \in S(n,\mathbb{C}).
\end{aligned}$$

Hence, $T_{F(A)} S(n,\mathbb{C}) \subset S(n,\mathbb{C})$. For given $C \in S(n,\mathbb{C})$,

$$\begin{aligned}
F_{*A}\left(\tfrac{1}{2}AC\right) &= \tfrac{1}{2}(AC)^*A + \tfrac{1}{2}A^*(AC) \\
&= \tfrac{1}{2}(C^*A^*)A + \tfrac{1}{2} I_n C \quad (\because A \in U(n)) \\
&= \tfrac{1}{2}C^* + \tfrac{1}{2}C = C \quad (\because C \in S(n,\mathbb{C}))
\end{aligned}$$

Thus, $T_{F(A)} S(n, \mathbb{C}) = S(n, \mathbb{C})$ and

F_{*A} is surjective for any $A \in U(n)$ and

thus I_n is a regular value of F.

Then, by Theorem 3.14, $U(n)$ is a regular

submanifold of $GL(n, \mathbb{C})$ with dimension n^2.

Therefore, Theorem 4.8, $U(n)$ is a Lie subgroup.

$\square$

(iii) The special unitary group

$$SU(n) = SL(n, \mathbb{C}) \cap U(n)$$

is (n^2-1) dimensional Lie subgroup $GL(n, \mathbb{C})$.

pf) $SL(n, \mathbb{C}) = \det^{-1}(1)$ is closed in $GL(n, \mathbb{C})$.

For any $A \in U(n)$, $\det(A) = \pm 1$.

$(\because)$ $1 = \det(A)\det(A)^{-1} = \det(AA^{-1})$

$= \det(AA^*) = \det(A)\det(A^*) = \det(A)\det(A)^*$.

$\Rightarrow |\det A| = 1$.

Thus, $SU(n)$ is an open submanifold of $U(n)$.

$(\because)$ $U^+(n) \cup U^-(n) = U(n)$. $(\dim U(n) = 1)$.

$U^+(n) = \det^{-1}(1) \cap U(n)$

$\Rightarrow U^{\pm}(n) \equiv$ closed in $U(n)$

$\equiv$ open in $U(n)$ $(\because$ disjoint$)$

Hence, $SU(n)$ is a regular submanifold.

Thus, by Theorem 4.8, $SU(n)$ is a Lie subgroup of

$GL(n, \mathbb{C})$ with $\dim = n^2 - 1$.

$\square$

4.1.4. (i) Prove that

$$SU(2) = \left\{ \begin{pmatrix} z & -\bar{w} \\ w & \bar{z} \end{pmatrix} \in gl(2,\mathbb{C}) : z\bar{z} + w\bar{w} = 1 \right\}$$

(ii) Show that $SU(2)$ is diffeomorphic to S^3.

proof) (i) Special unitary group.

In Example 4.13 (iii), $SU(n) = SL(n,\mathbb{C}) \cap U(n)$.

That is, $SU(n) = \{A \in gl(n,\mathbb{C}) : \det A = 1 \;\&\; A^*A = I_n\}$,

$A^* = (\bar{A})^T$ is the Hermitian conjugate of A.

Let $A = \begin{pmatrix} z & -\bar{w} \\ w & \bar{z} \end{pmatrix} \in gl(2,\mathbb{C})$, for any $z, w \in \mathbb{C}$.

$$\begin{cases} \det A = z\bar{z} - (-w\bar{w}) = z\bar{z} + w\bar{w} \\ A^* = \overline{A^T} = \begin{pmatrix} \bar{z} & \bar{w} \\ -w & z \end{pmatrix} \end{cases}$$

$$A^*A = \begin{pmatrix} \bar{z} & \bar{w} \\ -w & z \end{pmatrix} \begin{pmatrix} z & -\bar{w} \\ w & \bar{z} \end{pmatrix} = \begin{pmatrix} \bar{z}z + \bar{w}w & \bar{w}\bar{z} - \bar{z}\bar{w} \\ -wz + zw & w\bar{w} + z\bar{z} \end{pmatrix}$$

$$= \begin{pmatrix} z\bar{z} + w\bar{w} & 0 \\ 0 & z\bar{z} + w\bar{w} \end{pmatrix}$$

Since $\bar{\alpha}\alpha = \alpha\bar{\alpha}$, $z_1 z_2 = z_2 z_1$ and $\bar{\bar{\alpha}} = \alpha$.

Thus, if $A \in SU(2)$, then $z\bar{z} + w\bar{w}$ must be 1,

so that $A^*A = I_2$ and $\det A = 1$.

(ii) $SU(2) \cong S^3$ (diffeomorphic)

Since $\mathbb{R}^4 \cong \mathbb{C}^2$, $S^3 = \{(z,w) \in \mathbb{C}^2 : |z|^2 + |w|^2 = 1\}$.

($\because$ $z\bar{z} = \bar{z}z = |z|^2$).

Define a map $f : S^3 \to SU(2)$ by $f(z,w) = \begin{pmatrix} z & -\bar{w} \\ w & \bar{z} \end{pmatrix}$.

Then, f is well-defined since

$(z,w) \in S^3 \Rightarrow f(z,w) \in SU(2)$ clearly.

First, we now show that f is bijective.

① injective.

Clearly, $f(z_1, w_1) = f(z_2, w_2) \Rightarrow (z_1, w_1) = (z_2, w_2)$.

② surjective.

$f(S^3) = f(\{|z|^2 + |w|^2 = 1\}) = SU(2)$.

$\therefore$ f is bijective $\Rightarrow \exists$ inverse f^{-1}.

Note that $SU(2) \subseteq M(2, \mathbb{C}) \cong \mathbb{R}^8$.

Then, $SU(2)$ is a submanifold of $M(2,\mathbb{C}) \cong \mathbb{R}^8$.

Thus, if we define $F : \mathbb{R}^4 \to \mathbb{R}^8$,

f is just a restriction of F and

f, f^{-1} are smooth since F, F^{-1} are smooth,

and S^3, $SU(2)$ are submanifolds.

therefore, definition 1.16, f is diffeomorphism. $\blacksquare$

4.1.5. Prove that $SO(2)$ is diffeomorphic to S^1.

proof) $SO(n) = SL(n,\mathbb{R}) \cap O(n)$

$\Leftrightarrow \{A \in gl(n,\mathbb{R}) : \det A = 1 \text{ and } A^TA = I_n\}$.

In $SO(2)$, by calculation of $\det A$ and A^TA,

$$SO(2) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a^2+c^2=1,\ b^2+d^2=1,\ ad-bc=1,\ ab+cd=0 \right\}.$$

Let $a=x$, $b=-y$, $c=y$, $d=x$, then

all of the condition of $SO(2)$ is satisfied so that

we obtain

$$SO(2) = \left\{ \overbrace{\begin{pmatrix} x & -y \\ y & x \end{pmatrix}}^{A} \in gl(2,\mathbb{R}) : \det A = 1 \ \&\ A^TA = I_2 \right\}.$$

Define $f : S^1 \to SO(2)$ by $f(x,y) = \begin{pmatrix} x & -y \\ y & x \end{pmatrix}$.

Then, clearly, $(x,y) \in S^1 \Rightarrow f(x,y) \in SO(2)$ so

f is well-defined.

Trivially, f is bijective, thus $\exists$ inverse f^{-1} of f.

Since S^1 is a submanifold of $\mathbb{R}^2$ and $SO(2)$ is a

submanifold of $GL(2,\mathbb{R})$ by example 4.12,

Consider $F : \mathbb{R}^2 \to GL(2,\mathbb{R}) \cong \mathbb{R}^4$ be a diffeomorphism,

then f is just a restriction of F and so

f, f^{-1} is smooth.

Therefore, by definition 1.16, f is a diffeomorphism.

$\blacksquare$

4.2.1. Verify Example 4.27.

proof) ① $\mathfrak{sl}(n,\mathbb{C}) = \{X \in \mathfrak{gl}(n,\mathbb{C}) : \operatorname{tr}(X) = 0\}$

pf) Let $X \in \mathfrak{sl}(n,\mathbb{C})$. Set $\gamma(s) = e^{sX}$, $s \in \mathbb{R}$.

Then, $\det(\gamma(s)) = \det(e^{sX}) = e^{\operatorname{tr}(sX)} = e^0 = 1$.

$\Rightarrow$ $\gamma(s)$ is a curve on $SL(n,\mathbb{C})$.

Since $\gamma(0) = I_n$ & $\gamma'(0) = X$, we have

$\gamma'(0) = X \in T_{I_n} SL(n,\mathbb{C}) \subset \operatorname{Lie} SL(n,\mathbb{C})$.

Hence, $\mathfrak{sl}(n,\mathbb{C}) \subset \operatorname{Lie} SL(n,\mathbb{C})$.

Since $\dim \operatorname{Lie} SL(n,\mathbb{C}) = \dim \mathfrak{sl}(n,\mathbb{C}) = 2n^2-2$,

we conclude that $\mathfrak{sl}(n,\mathbb{C}) = \operatorname{Lie} SL(n,\mathbb{C})$. △

② $\mathfrak{u}(n) = \{X \in \mathfrak{gl}(n,\mathbb{C}) : X^* + X = 0\}$

similar to ①, $\dim \mathfrak{u}(n) = n^2$.

pf) For $X \in \operatorname{Lie} U(n) \subset \operatorname{Lie} \mathfrak{gl}(n,\mathbb{C}) = T\,GL(n,\mathbb{C}) = \mathfrak{gl}(n,\mathbb{C})$,

let $\gamma : I \to U(n)$ be a curve such that

$\gamma(0) = I_n$, $\gamma'(0) = X$. Then $\gamma(s)^* \gamma(s) = I_n$ such that

$\gamma(s) \in U(s)$. Hence,

$X^* + X = \gamma'(0)^* \cdot \gamma(0) + \gamma(0)^* \cdot \gamma'(0) = 0$.

So, $X \in \mathfrak{u}(n)$.

Conversely, suppose $X \in \mathfrak{u}(n)$. Set $\beta(s) = e^{sX}$ for $s \in \mathbb{R}$.

Then, $\beta(s)^* \beta(s) = \overline{(e^{sX})^T}\, e^{sX}$

$= \overline{e^{sX^T}}\, e^{sX} = e^{sX^*} e^{sX}$

$= e^{s(X^*+X)}$ ($\because X^*X = I_n = XX^*$)

$= I_n$ ($\because X^* + X = 0$ in $\mathfrak{u}(n)$)

$\Rightarrow$ $\beta(s)$ is a curve on $U(n)$. Since $\beta(0) = I_n$ and

$\beta'(0) = X$, we have $X \in T_{I_n} U(n) = \operatorname{Lie} U(n)$. △

③ $su(n) = \{X \in gl(n, \mathbb{C}) : X^* + X = 0 \ \& \ tr(X) = 0\}$.

pf) Let $X \in su(n)$, set $\gamma(s) = e^{sX}$ for $s \in \mathbb{R}$.

By ①, ②, we can check that $\gamma : I \to su(n)$,

$\gamma(s) \in SL(n, \mathbb{C}) \cap U(n)$ such that $\gamma(0) = I_n$, $\gamma'(0) = X$.

$\Rightarrow X \in T_{I_n} SL(n, \mathbb{C}) \cap U(n) \subset \mathrm{Lie}\, SU(n)$.

Hence, $su(n) \subset \mathrm{Lie}\, SU(n)$.

Conversely, if $X \in \mathrm{Lie}\, SU(n)$, let $\beta : I \to SU(n)$

that $\beta(0) = I_n$, $\beta'(0) = X$. Then

$\beta(s)^* \beta(s) = I_n$ and $\det(\beta(s)) = 1$, $\beta(s) \in SU(n)$.

Then, $X^* + X = \beta'(0)^* \beta(0) + \beta(0)^* \beta'(0) = 0$ and

$$tr(X) = tr(\beta'(0)) = \det(\beta(0))\, tr(\beta(0)^* \beta'(0))$$

$$= \left.\frac{d}{ds}\right|_{s=0} \det \circ \beta(s) = 0.$$

($\because \beta'(s) \in T_{\beta(s)} SU(n) = SU(n) \Rightarrow \det(\beta'(s)) = 1$).

$\Rightarrow X \in su(n)$ and thus, $su(n) \supset \mathrm{Lie}\, SU(n)$.

$\therefore su(n) = \mathrm{Lie}\, SU(n)$. △

□

4.2.2. Prove that the Pauli matrices

$$E_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad E_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

are generators of $su(2)$.

proof) $A \in su(2) \Rightarrow A^* + A = 0$ and $tr(A) = 0$.

Let $A = \begin{bmatrix} x_1 + iy_1 & x_2 + iy_2 \\ x_3 + iy_3 & x_4 + iy_4 \end{bmatrix}$. Then

$$A^* = \begin{bmatrix} x_1 - iy_1 & x_3 - iy_3 \\ x_2 - iy_2 & x_4 - iy_4 \end{bmatrix}$$ and $A^* + A$ is

$$\begin{bmatrix} 2x_1 & x_2 + x_3 + i(y_2 - y_3) \\ x_2 + x_3 + i(y_3 - y_2) & 2x_4 \end{bmatrix} = 0, \quad x_1 = x_4 = 0.$$

$x_2 + x_3 = 0 \Rightarrow x_2 = -x_3$, $y_2 = y_3$.

$tr(A) = x_1 + iy_1 + x_4 + iy_4 = +i(y_1 + y_4) = 0$

$\Rightarrow y_1 = -y_4$. Thus,

$$A = \begin{bmatrix} iy_1 & x_2 + iy_2 \\ -x_2 + iy_2 & -iy_1 \end{bmatrix} \xrightarrow{\text{general}} \begin{bmatrix} ix & y + iz \\ -y + iz & -ix \end{bmatrix}, \quad x, y, z \in \mathbb{R}.$$

$$\Rightarrow A = z \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix} + y \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} + x \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$$

$:= z\sigma_1 + y\sigma_2 + x\sigma_3$ for any $x, y, z \in \mathbb{R}$.

Then, we see that $\sigma_1 = iE_1$, $\sigma_2 = iE_2$, $\sigma_3 = iE_3$.

Clearly, $\{\sigma_1, \sigma_2, \sigma_3\}$: Linearly independent.

$\therefore$ $\{\sigma_1, \sigma_2, \sigma_3\}$ is a basis for $su(2)$

$\therefore$ E_1, E_2, E_3 are generator of $su(2)$. $\square$

4.3.1. Prove Example 4.31.

Let G and H be a group and $\psi: G \to H$ be a homomorphism. Prove that $\theta: G \times H \to H$ is defined by $\theta(g,h) = \psi(g)h$ for $g \in G$ and $h \in H$, then θ is a left action.

proof) Let G, H be a group have a binary operation $*$.

(i) For the identity element of $G \equiv e_g$,

$$\theta(e_g, h) = \psi(e_g) * h$$
$$= e_h * h \quad (\because \psi : \text{homo}.\ \psi(G) \subseteq H)$$
$$= h \text{ for all } h \in H.$$

($\because$) Let e_g, e_h be the identity of G, H, respectively.

ψ : homomorphism $\Rightarrow \psi(e_g) = e_h$.

pf) $\psi(e_g) = \psi(e_g * e_g)$

$= \psi(e_g) * \psi(e_g)$ since ψ : homo.

$\Rightarrow \psi(e_g) * \psi^{-1}(e_g)$

$= \psi(e_g) * \psi(e_g) * \psi^{-1}(e_g)$

since $\psi(e_g) \in H$, H : group, $\exists$ inverse ψ^{-1}.

$\Rightarrow e_h = \psi(e_g)$. $\triangle$

(ii) $\theta(g_1, \theta(g_2, h)) = \theta(g_1, \psi(g_2) * h)$

$= \psi(g_1) * \psi(g_2) * h$

$= \psi(g_1 * g_2) * h$ since ψ : homo.

$= \theta(g_1 * g_2, h)$ for $g_1, g_2 \in G$.

$\therefore$ θ : left action on H. $\square$

4.3.2. Prove that Example 4.40.

$S^{2n-1} \cong U(n)/U(n-1) \cong SU(n)/SU(n-1)$.

In particular, $S^3 \cong SU(2)/SU(1) = SU(2)$.

Recall Problem 4.1.4.

proof) We want to apply the theorem 4.38. So,

① well-defined action

Recall that Example 4.39 (i).

$GL(n,\mathbb{C}) \times \mathbb{C}^n \longrightarrow \mathbb{C}^n$, $(A,v) \longmapsto Av \in \mathbb{C}^n$.

If $A \in U(n)$, then $A \in U(n)$.

$\langle Av, Av \rangle \underset{(*)}{=\!=} \langle v, A^*Av \rangle = \langle v, v \rangle$.

Hence, the action $U(n) \times S^{2n-1} \to S^{2n-1}$ is well-defined as the restriction of the above.

(*) : Let $T \in \mathcal{L}(\mathbb{C}^n, \mathbb{C}^n)$ is a linear transformation.
The adjoint linear transformation of T denoted by T^* is a linear transformation s.t. for any $v \in \mathbb{C}^n, w \in \mathbb{C}^n$, $\langle Tv, w \rangle = \langle v, T^*w \rangle$.
If we assume the finite dimensional inner product space, then we obtain the following.
$\langle Ax, y \rangle = \langle x, A^*y \rangle$ for $x, y \in \mathbb{C}^n$, $A \in \mathbb{C}^{n \times n}$.

② transitivity of the action.

It is similar to (ii) in Example 4.39.

For the standard basis $\{e_1, \cdots, e_n\}$ of $\mathbb{C}^n$ and

$\{v_1, \cdots, v_n\}$ is an orthonormal basis of $\mathbb{C}^n$,
$v_i = A_i^j e_j$. Then,

$$\delta_{ij} = \langle v_i, v_j \rangle = \langle A_i^k e_k, A_j^l e_l \rangle$$

orthonormal

using $(*)$ $\longrightarrow$ $= \langle e_k, \overline{A_k^i} A_j^l e_l \rangle$

sesquilinear of inner pdt. $\longrightarrow$ $= A_i^k \overline{A_l^j} \langle e_k, e_l \rangle$

$k \neq l \Rightarrow \langle e_k, e_l \rangle = 0$ $\longrightarrow$ $= A_i^k \overline{A_k^j}$

$$= (AA^*)_i^j$$

Hence, $A \in U(n)$.

Now, given $v, w \in S^{2n-1}$, we can choose $A, B \in U(n)$ such that

$$Ae_1 = v \quad \text{and} \quad Be_1 = w.$$

Thus, $w = BA^* v = (BA^*) \cdot v$.

By definition 4.32, the action is transitive.

③ $K \equiv$ isotropy subgroup of e_n $= H$.

Let

$$H = \left\{ A \in U(n) : A = \begin{bmatrix} \tilde{A} & 0 \\ 0 & 1 \end{bmatrix}, \tilde{A} \in U(n-1) \right\}.$$

Then, $H \cong U(n-1)$ is trivial.

$(\because)$ $A^*A = \begin{bmatrix} \tilde{A}^* & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \tilde{A} & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \tilde{A}^*\tilde{A} & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} I_n & 0 \\ 0 & 1 \end{bmatrix}.$

Clearly, every element of $U(n-1)$ leaves e_n.

Conversely, suppose that $Ae_n = e_n$ for some $A = (A^1, \cdots, A^n) \in U(n)$. Then $A^n = e_n$ s.t. $A_i^n = 0$ for $i < n$ and $A_n^n = 1$.

$$\left(Ae_n = \begin{bmatrix} \tilde{A} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} = e_n . \right)$$

Since $A^*A = I_n$, we have

$$1 = \sum_{i=1}^{n} (A_n^i)^2 = (A_n^n) + \sum_{i=1}^{n-1} (A_n^i)^2 .$$

So, $A_n^i = 0$ for $i < n \Rightarrow A \in H$.

By Theorem 4.38, $S^{2n-1} \cong U(n) / U(n-1)$.

By a similar argument, $S^{2n-1} \cong SU(n) / SU(n-1)$.

We can show $S^3 \cong SU(2) / SU(1) = SU(2) / \{1\}$ $= SU(2)$ using the above arguments. $\square$

4.3.3. Prove Example 4.42.

$\mathbb{C}P^{n-1} \cong SU(n)/U(n-1)$. In particular, by Problem 1.3.2 and Example 4.40

$$S^2 \cong \mathbb{C}P^1 \cong SU(2)/U(1) \cong S^3/S^1.$$

proof) Define an $SU(n) \times \mathbb{C}P^{n-1} \to \mathbb{C}P^{n-1}$ by

$(A, [x]) \mapsto A \cdot [x] := [Ax]$.

If $A \in SU(n)$ & $[x] = [y]$, then

$y = \lambda x$ for some $\lambda \in \mathbb{C}^*$ s.t. $Ay = \lambda Ax$.

$\therefore$ the action is well defined.

For $[x], [y] \in \mathbb{C}P^{n-1}$, we can choose $A, B \in SU(n)$ such that

$$A[z_1] = [x], \ A[z_1] = [y].$$

(We can find the basis $[z_1]$, by the same argument of 4.3.2.)

$\therefore$ the action is transitive.

Note that $A \in SU(n)$ fixes $[z_n]$

$\Leftrightarrow Az_n = z_n$ or $Az_n = -z_n$

$\Leftrightarrow A = \begin{pmatrix} \tilde{A} & 0 \\ 0 & 1 \end{pmatrix}$ or $A = \begin{pmatrix} \tilde{A} & 0 \\ 0 & -1 \end{pmatrix}$ where $\tilde{A} \in U(n-1)$

$\Leftrightarrow A = \begin{pmatrix} \tilde{A} & 0 \\ 0 & \det\tilde{A} \end{pmatrix}$ where $\tilde{A} \in U(n-1)$

So, the isotropy group of z_n is $U(n-1)$.

By Theorem 4.38, we get the desired result.

Now, we want to show that

$$S^2 \underset{①}{\cong} \mathbb{C}P^1 \underset{②}{\cong} SU(2)/U(1) \underset{③}{\cong} S^3/S^1$$

② : Using the problem 4.3.3 we proved, this follows for $n=2$.

③ : Consider the quotient map $f : \mathbb{C}^2 - \{0\} \to \mathbb{C}P^1$ and the restriction $\pi : S^3 \to \mathbb{C}P^1$.

① : Define a charts on S^2

$\phi_{\pm} : S^2 - \{(0,0,\pm 1)\} \to \mathbb{R}^2$ by $(x_0, x_1, x_2) \mapsto \left(\frac{x_0}{1 \mp x_2}, \frac{x_1}{1 \mp x_2}\right)$

and define

$$F : \mathbb{C}P^1 \to S^2, \quad F([z_0, z_1]) := \begin{cases} \phi_N^{-1}\left(\frac{z_0}{z_1}\right), & z_1 \neq 0 \\ (0,0,1), & z_1 = 0 \end{cases}.$$

Then F is diffeomorphism. $\blacksquare$

5.1.1. Find the integral curve of

$$X = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} \quad \text{on } \mathbb{R}^2.$$

proof) Let $\gamma(t) = (\gamma_1(t), \gamma_2(t))$ be an integral curve such that $\gamma(t)$ is smooth, and $\gamma'(t) = X_{\gamma(t)}$, X is a vector field on $\mathbb{R}^2$. Then,

$$\gamma'(t) = X_{\gamma(t)} = \gamma_1(t)\frac{\partial}{\partial \gamma_1} + \gamma_2(t)\frac{\partial}{\partial \gamma_2}.$$

Since $\gamma'(t) = (\gamma_1'(t), \gamma_2'(t))$, we obtain ODEs:

$$\gamma_1' = \gamma_1,\ \gamma_2' = \gamma_2 \Rightarrow \gamma_1 = C_1 e^t,\ \gamma_2 = C_2 e^t$$

solutions, $C_1, C_2 \in \mathbb{R}$.

$\therefore$ integral curve of X is $\gamma(t) = e^t(C_1, C_2)$, $C_1, C_2 \in \mathbb{R}$.

$\square$

5.2.2. Complete the proof of Proposition 5.8.

proof) (ii) - 1 : $\mathcal{L}_X(fY) = (\mathcal{L}_X f)Y + f(\mathcal{L}_X Y)$

pf) Using the fact : $\mathcal{L}_X Y = [X, Y]$, $\mathcal{L}_X f = Xf$.

$$\begin{aligned} \mathcal{L}_X(fY) = [X, fY] &= X\cdot fY - fY\cdot X \\ &= Xf\cdot Y + fX\cdot Y - fY\cdot X \\ &= Xf\cdot Y - f\cdot[X,Y] \\ &= (\mathcal{L}_X f)Y + f(\mathcal{L}_X Y). \end{aligned}$$

Δ

(ii) - 2 : $\mathcal{L}_X[Y, Z] = [\mathcal{L}_X Y, Z] + [Y, \mathcal{L}_X Z]$
$= [\mathcal{L}_X Y, \mathcal{L}_X Z]$

pf) By the Jacobi identity, we obtain above :

$$[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0$$

prop. 5.8 (ii) → By the fact we proved : $\mathcal{L}_X Y = [X, Y]$ ⋯ (*)

$\mathcal{L}_X[Y,Z] = [X,[Y,Z]]$ and we know that

$$\begin{aligned}
&\xrightarrow{\text{Jacobi identity}} = -[Y,[Z,X]] - [Z,[X,Y]] \\
&\xrightarrow{\text{skew symmetric}} = [[X,Y],Z] - [Y,-[X,Z]] \\
&\xrightarrow{\text{bilinear}} = [[X,Y],Z] + [Y,[X,Z]] \\
&\xrightarrow{(*)} = [\mathcal{L}_X Y, Z] + [Y, \mathcal{L}_X Z].
\end{aligned}$$

Δ

(iii) - 1 : $\mathcal{L}_X(f\omega) = (\mathcal{L}_X f)\omega + f\mathcal{L}_X\omega$

pf) $\mathcal{L}_X(f\omega)_p = \lim_{t\to 0}\frac{1}{t}\left[\Phi_t^*(f\omega)(p) - (f\omega)(p)\right]$

$= \lim_{t\to 0}\frac{1}{t}\left[f(\Phi_t(p))\,\Phi_t^*(\omega(p)) - f(p)\omega(p)\right]$

$= \lim_{t\to 0}\frac{1}{t}\left[f(\Phi_t(p))\Phi_t^*(\omega(p)) - f(\Phi_t(p))(\omega(p)) +\right.$

$\left. f(\Phi_t(p))(\omega(p)) - f(p)\omega(p)\right]$

$= \lim_{t\to 0}\frac{1}{t}\left[\left[f(\Phi_t(p)) - f(p)\right]\cdot\omega(p)\right] +$

$\lim_{t\to 0}\frac{1}{t}\left[f(\Phi_t(p))\cdot\left[\Phi_t^*(\omega(p)) - \omega(p)\right]\right]$

$= (\mathcal{L}_X f)\omega + f\cdot\mathcal{L}_X\omega$ if $f(\Phi_t(p)) \to f(p)$ as $t\to 0$. Δ

(iii) - 2 : $\mathcal{L}_X(\omega Y) = (\mathcal{L}_X\omega)Y + \omega(\mathcal{L}_X Y)$

pf) $\mathcal{L}_X(\omega(Y)(p)) = \lim_{t\to 0}\frac{1}{t}\left[\omega(Y)(\Phi_t(p)) - \omega(Y)(p)\right]$

$= \lim_{t\to 0}\frac{1}{t}\left[\omega(\Phi_t(p))(Y_{\Phi_t(p)}) - \omega(p)(Y_p)\right]$

$= \lim_{t\to 0}\frac{1}{t}\left[(\Phi_t^*\omega)(p)(\Phi_{t*}Y_{\Phi_t(p)}) - \omega(p)(Y_p)\right]$

$= \lim_{t\to 0}\frac{1}{t}\left[(\Phi_t^*\omega)(p)(\Phi_{t*}Y)_p - \omega(p)(Y_p)\right]$

$= \lim_{t\to 0}\frac{1}{t}\left[(\Phi_t^*\omega)(p)(\Phi_{t*}Y)_p - \omega(p)(\Phi_{t*}Y)_p +\right.$

$\left.\omega(p)(\Phi_{t*}Y)_p - \omega(p)(Y_p)\right]$

$$= \lim_{t \to 0} \frac{1}{t} [(\Phi_t^* \omega)(p) - \omega(p)] \cdot (\Phi_{t*} Y)_p +$$

$$\lim_{t \to 0} \frac{1}{t} \omega(p) \cdot [Y_p - (\Phi_{t*} Y)_p]$$

$$= (\mathcal{L}_X \omega)(p)(Y_p) + \omega(p)(\mathcal{L}_X Y)(p) \text{ if}$$

$(\Phi_{t*} Y)_p \to Y_p$ as $t \to 0$. $\triangle$

$\square$

5.3.1. Let $A, B \in \mathfrak{gl}(2, \mathbb{R})$ be given by

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Compute the 1-parameter subgroup of $GL(2, \mathbb{R})$ generated by A and B.

proof) By Proposition 5.16 (ii) & Example 5.15, the 1-parameter subgroup of $GL(n, \mathbb{R})$ generated by $X \in \mathfrak{gl}(n, \mathbb{R}) \cong \mathrm{Lie}(GL(n, \mathbb{R}))$ is

$$\gamma(t) : t \in \mathbb{R} \longmapsto \exp(tX) = e^{tX} \in GL(n, \mathbb{R}).$$

Thus, by definition 4.25,

$$\gamma(t) = e^{tA} = I + \begin{pmatrix} 0 & t \\ -t & 0 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 0 & t \\ -t & 0 \end{pmatrix}\begin{pmatrix} 0 & t \\ -t & 0 \end{pmatrix} + \cdots$$

$$= \begin{pmatrix} 1 - \frac{t^2}{2!} + \frac{t^4}{4!} + \cdots & t - \frac{t^3}{3!} + \cdots \\ -(t - \frac{t^3}{3!} + \cdots) & 1 - \frac{t^2}{2!} + \frac{t^4}{4!} + \cdots \end{pmatrix}$$

$$= \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}.$$

$$\gamma(t) = e^{tB} = I + \begin{pmatrix} 0 & t \\ 0 & 0 \end{pmatrix} + 0 + 0 + \cdots$$

$$= \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}.$$

$\square$

5.3.2. Prove Proposition 5.19.

proof) Note that $\mathrm{Ad}_g(e) = e$, and hence $\mathrm{Ad}_g : T_eG \to T_eG$.
Hence, $\exp \mathrm{Ad}_g A = g(\exp A) g^{-1}$ is well defined.

Since $\mathrm{Ad}_g : G \to G$ is a homomorphism for each $g \in G$, therefore,

$$\exp \mathrm{Ad}_g A = \mathrm{Ad}_g(\exp A) = g(\exp A) g^{-1}. \qquad \blacksquare$$

6.2.1. Verify Definition 6.13 (iii) and Definition 6.14 (ii).

proof) ① $T \in \mathcal{T}^r(V)$ is symmetric iff

$T(v_{\sigma(1)}, \cdots, v_{\sigma(r)}) = T(v_1, \cdots, v_r)$ for all

$v_1, \cdots, v_r \in V$ and $\sigma \in \mathcal{P}_r$.

pf) Let $\sigma : \{1, \cdots, r\} \longrightarrow \{1, \cdots, r\}$ be a bijection called permutation and $\sigma \in \mathcal{P}_r$ is a product of transposition by the assumption. Then, we can choose

$$\tau_{ij} = \begin{pmatrix} 1 & \cdots & i & \cdots & j & \cdots & r \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ 1 & \cdots & j & \cdots & i & \cdots & r \end{pmatrix} \in \mathcal{P}_r$$

that means that i and j are exchanged and all other elements are fixed.

a proper composition of transposition τ_{ij} → Hence, we obtain the identity by calculating a product of transposition $\sigma(i)$ and it can be symmetric. Thus, for $v_i \in V$ & $\sigma \in \mathcal{P}_r$, we get the symmetric representation

$$(v_1, \cdots, v_r) = (v_{\sigma(1)}, \cdots, v_{\sigma(r)})$$

and thus,

$$T(v_1, \cdots, v_r) = T(v_{\sigma(1)}, \cdots, v_{\sigma(r)})$$

is symmetric. △

③ Define sgn : $\mathcal{P}_r \to \{+1, -1\}$ by

$$\operatorname{sgn}(\sigma) = \begin{cases} +1 & \text{if } \sigma : \text{even \# of transposition} \\ -1 & \text{if } \sigma : \text{odd \# of transposition} \end{cases}$$

$$\sigma = \begin{pmatrix} 1 & \cdots & i & \cdots & j & \cdots & r \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \sigma(1) & \cdot & \sigma(i) & \cdots & \sigma(j) & \cdot & \sigma(r) \end{pmatrix} \in \mathcal{P}_r .$$

by the proof above, we can get the symmetric tensor and also we can set the permutation is compositions of odd number of transposition since the number of transpositions necessary to transform a permutation into the identity is not unique. Thus, we obtain $\operatorname{sgn}(\sigma) = -1$ and therefore

$$\begin{aligned} T(v_{\sigma(1)}, \cdots, v_{\sigma(r)}) &= \operatorname{sgn}(\sigma) T(v_1, \cdots, v_r) \\ &= -T(v_1, \cdots, v_r) . \end{aligned}$$

It is alternating.

$\triangle$

③ T : symmetric $\Longleftrightarrow$ $\mathrm{Sym}(T) = T$

By ①, $T(v_1, \cdots, v_r) = T(v_{\sigma(1)}, \cdots, v_{\sigma(r)})$

$$\Rightarrow \mathrm{Sym}(T) = \frac{1}{r!} \sum_{\sigma} T(v_{\sigma(1)}, \cdots, v_{\sigma(r)})$$

$$= \frac{1}{r!} \sum_{\sigma} T(v_1, \cdots, v_r)$$

Let $\sigma \in P_r$ in $(\Leftarrow)$.

$$= \frac{1}{r!} r! \, T(v_1, \cdots, v_r)$$

$$= T(v_1, \cdots, v_r).$$

△

④ T : alternating $\Longleftrightarrow$ $\mathrm{Alt}(T) = T$

$$\mathrm{Alt}(T) = \frac{1}{r!} \sum_{\sigma} (\mathrm{sgn}(\sigma)) T(v_{\sigma(1)}, \cdots, v_{\sigma(r)})$$

$$= T(v_1, \cdots, v_r) \text{ by using ②.}$$

$(\mathrm{sgn}(\sigma))^2 = 1$.

△

$\blacksquare$

6.2.2. Verify Example 2. In addition, compute $Sym(T_i)$ and $Alt(T_i)$ for $i=1,2,3$. ⟶ typo. example 6.15.

proof) It is easy to show that $T_i \in \mathcal{T}^2(\mathbb{R}^2)$. $\cdots$ (*)

For $\sigma \in \mathcal{P}_2 = \left\{ \sigma_1 = \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}, \sigma_2 = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \right\}$,

$$Sym(T_2) = \frac{1}{2}\left[\underset{\substack{\uparrow \\ \sigma_1}}{x_1 y_1} + \underset{\substack{\uparrow \\ \sigma_2}}{y_1 x_1} \right] \underset{\substack{\uparrow \\ x_1,\, y_1 \in \mathbb{R}}}{=} x_1 y_1 = T_2(x, y)$$

$$Alt(T_3) = \frac{1}{2}\,sgn(\sigma_1)\left[x_1 y_2 - x_2 y_1 \right] + \frac{1}{2}\,sgn(\sigma_2)\left[x_2 y_1 - x_1 y_2 \right]$$

Note that $\det(I_\sigma) = sgn(\sigma)$, I_σ = identity permutation matrix. So we can compute

$sgn(\sigma_1) = \det(I_{\sigma_1}) = 1$,

$$sgn(\sigma_2) = \det(I_{\sigma_2}) = \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = -1.$$

$$\Rightarrow Alt(T_3) = \frac{1}{2}(x_1 y_2 - x_2 y_1) - \frac{1}{2}(x_2 y_1 - x_1 y_2)$$

$$= x_1 y_2 - x_2 y_1 = T_3(x, y).$$

$\therefore$ T_2 is symmetric & T_3 is alternating.

(*) : $T_1 : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $T_1(x, y) = x_1 y_2$.

$$T_1(ax + by, y) = (ax_1 + by_2) y_2$$

$$= ax_1 y_2 + by_2 y_2 = T_1(ax, y) + T_2(by, y).$$

$\square$

6.2.3. Complete the proof of Proposition 6.16.

proof) (i) For $T \in \mathcal{T}^r(V)$, $\mathrm{Alt}(T) \in \Lambda^r(V)$.

pf) If τ is transposition, then

$$\begin{aligned}
\mathrm{Alt}(T) \circ \tau &= \frac{1}{r!} \sum_{\sigma} (\mathrm{sgn}(\sigma))({}^{\sigma}T \circ \tau) \\
&= \frac{1}{r!} \sum_{\sigma} (\mathrm{sgn}(\sigma)) (T \circ \sigma \circ \tau) \\
&= \frac{1}{r!} (\mathrm{sgn}(\tau)) \sum_{\sigma} (\mathrm{sgn}(\sigma \circ \tau))(T \circ (\sigma \circ \tau))
\end{aligned}$$

$\sigma_0 = \sigma \circ \tau \longrightarrow$ $$= \frac{1}{r!} (\mathrm{sgn}(\tau)) \sum_{\sigma} (\mathrm{sgn}(\sigma_0))(T \circ \sigma_0)$$

$$= (\mathrm{sgn}(\tau)) \mathrm{Alt}(T) = -\mathrm{Alt}(T) \qquad \triangle$$

(ii) For $\omega \in \Lambda^r(V)$, $\mathrm{Alt}(\omega) = \omega$.

pf) Use the definition 6.14 (ii). $\triangle$

(iii) For $T \in \mathcal{T}^r(V)$, $\mathrm{Alt}^2(T) = \mathrm{Alt}(T)$.

pf) $\mathrm{Alt}^2 = \mathrm{Alt} \circ \mathrm{Alt}$.

By (i), $\mathrm{Alt}(T) \in \Lambda^r(V)$ and by (ii),

we obtain

$$\mathrm{Alt}(\mathrm{Alt}(T)) = \mathrm{Alt}(T) \qquad \triangle$$

$\blacksquare$

6.3.1. Prove Proposition 6.24.

proof; (i) If $\omega \in \Gamma^r(M)$ and $\eta \in \Gamma^s(M)$, then

$$F^*(\omega \wedge \eta) = (F^*\omega) \wedge (F^*\eta).$$

pf)
$$F^*(\omega \wedge \eta) = F^*\left(\frac{(r+s)!}{r!\,s!} \mathrm{Alt}(\omega \otimes \eta)\right)$$

$$\text{def.} \rightarrow \quad = \frac{(r+s)!}{r!\,s!} F^* \mathrm{Alt}(\omega \otimes \eta)$$

$$(*) \cdots \quad = \frac{(r+s)!}{r!\,s!} \mathrm{Alt}(F^*(\omega \otimes \eta))$$

$$(**) \cdots \quad = \frac{(r+s)!}{r!\,s!} \mathrm{Alt}(F^*\omega \otimes F^*\eta)$$

$$\text{def.} \rightarrow \quad = F^*\omega \wedge F^*\eta$$

$(*)$: $\mathrm{Alt}(F^*T)(v_1, \cdots, v_r) = \frac{1}{r!} \sum_{\sigma} \mathrm{sgn}(\sigma) F^*T(v_{\sigma(1)}, \cdots, v_{\sigma(r)})$

$$= \frac{1}{r!} F^*\left(\sum_{\sigma} \mathrm{sgn}(\sigma) T(v_{\sigma(1)}, \cdots, v_{\sigma(r)})\right)$$

$$= F^*\left(\frac{1}{r!} \sum_{\sigma} \mathrm{sgn}(\sigma) F^*T(v_{\sigma(1)}, \cdots, v_{\sigma(r)})\right)$$

$$= F^* \mathrm{Alt}(T)(v_1, \cdots, v_r)$$

$(**)$: $F^*(\omega \otimes \eta)(u,v) = \omega \otimes \eta(F_* u, F_* v)$ by Remark 6.22.

$$= \omega(F_* u)\eta(F_* v)$$

$$= F^*\omega(u) F^*\omega(v)$$

$$= F^*\omega \otimes F^*\eta\,(u,v).$$

△

(ii) $\omega \in \Gamma^n(M)$, $\omega = u\, dy^1 \wedge \cdots \wedge dy^n$ on a chart $(V, \psi = (y^1, \cdots, y^n))$ on M. If $(U, \varphi = (x^1, \cdots, x^n))$ is a chart on N, then

$$F^*\omega = (u \circ F) \det\left(\frac{\partial F^j}{\partial x^i}\right) dx^1 \wedge \cdots \wedge dx^n \text{ on } U \cap F^{-1}(V).$$

pf) By the assumption, $y^i = \hat{F}^i(x^1, \cdots, x^n)$, $i = 1, \cdots, n$,

$\hat{F} = y \circ F \circ x^{-1}$: local representation of F.

Then,

$$\begin{aligned}
F^*\omega &= F^*(u\, dy^1 \wedge \cdots \wedge dy^n) \\
&= F^*u\,(F^*(dy^1 \wedge \cdots \wedge dy^n)) \\
&= (u \circ F)(F^* dy^1) \wedge \cdots \wedge (F^* dy^n) \\
&= (u \circ F)\,(d(y^1 \circ F)) \wedge \cdots \wedge (d(y^n \circ F)) \\
&= (u \circ F)\,(d(\hat{F}^1 \circ x)) \wedge \cdots \wedge (d(\hat{F}^n \circ x)) \\
&= (u \circ F) \det\left(\frac{\partial F^j}{\partial x^i}\right) dx^1 \wedge \cdots \wedge dx^n .
\end{aligned}$$

△

□