**1.1.1** Show that if $M^m, N^n$ are smooth manifolds, then $M^m \times N^n$ is also a $(m+n)$ dimensional smooth manifold. Hence, the $n$-dimensional torus or simply $n$-torus

$$\mathbb{T}^n = \underbrace{S^1 \times \cdots \times S^1}_{n}$$

$\mathbb{T}^2 =$ 



2-dim manifold

is a smooth manifold.

**proof)** We want to show :

[1] $M^m \times N^n$ is manifold [2] It is smooth manifold

[1] : ① Hausdorff

Since $M^m, N^n$ are smooth manifold, these are Hausdorff. Then, for any $U_M, V_M$ in $M^m$ and $U_N, V_N$ in $N^n$, let $U_M \times U_N = U \subset M^m \times N^n$ and $V_M \times V_N \subset N^n$, then $U_M \times U_N \cap V_M \times V_N = \phi$. $\cdots (*)$

(∵)

(1) : $(U_M \times U_N) \cap V_M = V_M \cap U_M \times V_N \cap U_N = \phi$

because $V_M \cap U_M = \phi$ since $M^m$ is Hausdorff.

(2) : $(U_M \times U_N) \cap V_N = V_N \cap U_M \times V_N \cap U_N = \phi$

because $V_N \cap U_N = \phi$ since $N^n$ is Hausdorff.

$(*) = (1) \times (2) = \phi$.

Thus, $M^m \times N^n$ is Hausdorff.

② second countable

By the assumption, $M^m, N^n$ have a countable basis $\beta_M, \beta_N$. Then

trivially $\beta_M \times \beta_N \subset M^m \times N^n$ and we can pick

$\beta_M \times \beta_N$ is a countable basis for $M^m \times N^n$.

($\because$) $x \in \beta_M$ & $y \in \beta_N \Rightarrow (x,y) \in \beta_M \times \beta_N \subset M^m \times N^n$

and $\beta_M$, $\beta_N$ : open $\Rightarrow \beta_M \times \beta_N$ : open.

$\beta_M \times \beta_N$ : countable since $\beta_M$, $\beta_N$ are countable.

pf) Let $\beta_M \times \beta_N$ : finite $\rightarrow$ trivial.

We assume $\beta_M$, $\beta_N$ : countably infinite.

$(\beta_M^0, \beta_N^0)$ $(\beta_M^0, \beta_N^1)$ $(\beta_M^0, \beta_N^2)$ $\cdots$

$(\beta_M^1, \beta_N^0)$ $(\beta_M^1, \beta_N^1)$ $(\beta_M^1, \beta_N^2)$ $\cdots$

$(\beta_M^2, \beta_N^0)$ $(\beta_M^2, \beta_N^1)$ $(\beta_M^2, \beta_N^2)$ $\cdots$

$\vdots$ $\quad$ $\vdots$ $\quad$ $\vdots$

First, we pick $(\beta_M^0, \beta_N^0)$, then we pick

$(\beta_M^0, \beta_N^1)$, $(\beta_M^1, \beta_N^0)$, then we pick

$(\beta_M^0, \beta_N^2)$, $(\beta_M^1, \beta_N^1)$ $(\beta_M^2, \beta_N^0)$ $\cdots$

Continue to this processes, then we can

define the one-to-one correspondence

between $\beta_M \times \beta_N \rightarrow \mathbb{N}$ (the set of natural #).

Thus, by definition of countable,

the assertion is proved.


③ Homeomorphism

Let $\varphi_M : U \rightarrow \mathbb{R}^m$ & $p \in \varphi_M(U)$ and

$\varphi_N : V \rightarrow \mathbb{R}^n$ & $q \in \varphi_N(V)$, then we can define

$$\varphi_{MN}(r) = (\varphi_M \times \varphi_N)(p, g) = (\varphi_M(p), \varphi_N(g))$$

if $\varphi_{MN} : U \times V \longrightarrow \mathbb{R}^{m+n}$.

(i) injective

$$\varphi_{MN}(r_1) = \varphi_{MN}(r_2)$$

$$\Rightarrow (\varphi_M(p_1), \varphi_N(g_1)) = (\varphi_M(p_2), \varphi_N(g_2))$$

$$\Rightarrow \varphi_M(p_1) = \varphi_M(p_2) \quad \& \quad \varphi_N(g_1) = \varphi_N(g_2)$$

$$\Rightarrow p_1 = p_2 \quad \& \quad g_1 = g_2 \quad \text{since } \varphi_M, \varphi_N : \text{injective.}$$

(ii) surjective

For $\forall \; y = \varphi_{MN}(\bar{r}) \in \mathbb{R}^{m+n}$, $\exists \; (\bar{p}, \bar{g}) \in U \times V$ s.t.

$$y = \varphi_{MN}(\bar{r}) = (\varphi_M(\bar{p}), \varphi_N(\bar{g})) \quad \text{since}$$

$\varphi_M \; \& \; \varphi_N$ are surjective.

By (i), (ii), $\varphi_{MN}$ : bijection on $U \times V \subset M^m \times N^n$.

Thus, $\exists \; \varphi_{MN}^{-1}$ : inverse of $\varphi_{MN}$.

In case of continuity, for any $\mathcal{O}, B$ $\overset{\text{be open}}{\frown}$ in $\mathbb{R}^m, \mathbb{R}^n$,

$\varphi_M^{-1}(\mathcal{O}), \varphi_N^{-1}(B)$ are open by the assumption.

Since $\mathcal{O} \times B$ : open and its preimage

$\varphi_{MN}^{-1}(\mathcal{O} \times B)$ is open.

$\therefore \; \varphi_{MN}$ is continuous.

($\because$) if $\mathcal{O} = \varphi_M(\alpha), B = \varphi_N(\beta)$ for any open sets

$\alpha, \beta$ in $U, V$, then

$$\varphi_{MN}^{-1}(\varphi_{MN}(\mathcal{O} \times \beta)) = \mathcal{O} \times \beta = \varphi_M(\alpha) \times \varphi_N(\beta) : \text{open.}$$

For $\varphi_{MN}^{-1}$ : inverse of $\varphi_{MN}$, $\varphi_{NN}^{-1}(\theta \times \beta)$ is open

$\Rightarrow \varphi_{MN}(\varphi_{MN}^{-1}(\theta \times \beta)) = \theta \times \beta$ is open

$\therefore \varphi_{MN}^{-1}$ is continuous.

$\therefore \varphi_{MN}$ is Homeomorphism.

Therefore, $M^m \times N^n$ is a manifold.

[2] : Since $M^m, N^n$ are smooth manifold, they have a $C^\infty$-structure, so that the coordinate charts $(U, \varphi_M)$, $(V, \varphi_N)$ is $C^\infty$-compatible with all charts in the atlas of $M^m, N^n$, respectively. By [1], we defined the homeomorphism $\varphi_{MN}$, hence we can write the coordinate chart of $M^m \times N^n$ that $(U \times V, \varphi_{MN})$. Consider another chart $(U' \times V', \varphi_{MN}^*)$, then

$$\varphi_{MN} \circ \varphi_{MN}^* = (\varphi_M \times \varphi_N) \circ (\varphi_M^* \times \varphi_N^*)^{-1}$$

$$= \varphi_M \circ \varphi_M^{*-1} \times \varphi_N \circ \varphi_N^{*-1}.$$

Since $\varphi_M, \varphi_N, \varphi_M^{*-1}, \varphi_N^{*-1}$ are $C^\infty$,

$\varphi_{MN} \circ \varphi_{MN}^*$ is $C^\infty$.

$\therefore M^m \times N^n$ is a smooth manifold.

Thus, By the proof above, $T^n$ is smooth manifold.

$\square$

**1.1.2** Let $U \subset \mathbb{R}^n$ be open and $f: U \to \mathbb{R}^m$ be continuous. Show that the graph of $f$

$$\Gamma_f = \{(x,y) \in \mathbb{R}^n \times \mathbb{R}^m : x \in U \text{ and } y = f(x)\}$$

is an $n$-dimensional manifold.

**proof)** By the example 1.7.(i) of the lecture note of professor Han, $\mathbb{R}^n$, $\mathbb{R}^m$ are $n, m$ dimensional smooth manifold and hence $\mathbb{R}^n \times \mathbb{R}^m$ is smooth manifold by the exercise 1.1.1.

Thus, the graph of $f$ $\Gamma_f$ is the subspace topology of $\mathbb{R}^n \times \mathbb{R}^m$. Hence, $\Gamma_f$ is Hausdorff and 2nd-countable space.

So, we want to show that $\Gamma_f$ has the locally Euclidean property only.

Let $\pi_x : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is the projection onto $x$, and let $\varphi : \Gamma_f \to U$ be the restriction of $\pi_x$ to $\Gamma_f$ that $\varphi(x,y) = x$, $(x,y) \in \Gamma_f$.

Since $\pi_x$ is continuous (clearly), the restriction of $\pi_x$ $\varphi$ is continuous, and bijective also. Thus $\exists \varphi^{-1}$: inverse of $\varphi$ and since $\varphi^{-1}(x) = (x, f(x))$, $\varphi^{-1}$ is continuous.

$\therefore$ $\varphi$ : Homeomorphism.

$\therefore$ $\Gamma_f$ is $n$-dimensional manifold. $\qquad \blacksquare$

**1.2.1** Complete the proof of proposition 1.14 :

Suppose that $\pi : M \to M/\sim$ is an open map. Then

(ii) $M/\sim$ is Hausdorff $\Rightarrow R = \{(p,q) : p \sim q\}$ is closed in $M \times M$.

**proof)** Note that :

$$[x]_\sim = \{x \in M : x \sim \alpha, \ \alpha \in M\}.$$

$$M/\sim = \{[x]_\sim : x \in M\}$$

$$\Theta \subset M/\sim \text{ is open} \Leftrightarrow \pi^{-1}(\Theta) = \{x : \pi(x) = [x] \in \Theta\}$$
$$\text{is open in } M.$$

Assume that $M/\sim$ is Hausdorff.

<u>Claim</u> : $R \subset M \times M$ is closed

$\Leftrightarrow M \times M - R$ is open.

Let $(p,q) \in M \times M - R$, then $\pi(p) \neq \pi(q)$

$\Rightarrow (p,q) \notin R$. Thus we can take the

disjoint open sets $\pi(p) \in U_1$, $\pi(q) \in U_2$

since $M/\sim$ is Hausdorff.

Let $V_1 = \pi^{-1}(U_1)$ & $V_2 = \pi^{-1}(U_2)$.

If $(V_1 \times V_2) \cap R \neq \emptyset$, then $\exists (v_1, v_2) \in V_1 \times V_2$

such that $\pi(v_1) = \pi(v_2)$, $\pi(v_1) \in U_1$, $\pi(v_2) \in U_2$.

But, $U_1 \cap U_2 = \emptyset$, that is contradiction.

$\therefore R$ is closed in $M \times M$. $\qquad \square$

1.2.2 Let $f: S^n \to S^n$ be the antipodal map defined by $f(x) = -x$. Define an relation $\sim$ on $S^n$ by $x \sim y$ iff $y = x$ or $y = f(x)$. Show that $\sim$ is an equivalence relation and $S^n/\sim = \mathbb{R}P^n$.

proof) ① Equivalence relation

$$\boxed{x \sim y \iff y = x \iff y - x = 0}$$

(i) $x \sim x$ since $x - x = 0$

(ii) if $x \sim y$, then

$$y = x \iff y - x = 0 \iff -(x - y) = 0$$
$$\iff x - y = 0 \iff x = y$$
$$\iff y \sim x.$$

$$y = -x \iff -y = x \iff x = f(y)$$
$$\iff y \sim x$$

(iii) if $x \sim y$ & $y \sim z$, then

$$y = x \quad \text{and} \quad z = y \quad \text{and so}$$
$$z = y = x \iff x \sim z.$$

$$y = -x \quad \& \quad z = -y, \quad \text{then}$$
$$z = -y = -(-x) = x \iff x \sim z.$$

② $S^n/_\sim = \mathbb{R}P^n$

$$[x]_M = \bigcup_i [x_i]_{S^n} . \quad (\because [x]_{S^n} \subset [x]_M )$$

$$\Rightarrow \bigcup_j \left( \bigcup_i [x_i]_{S^n} \right)_j = M/_\sim = \mathbb{R}P^n$$

For arbitrary $r \in \mathbb{R}^{n+1} - \{0\}$, we can let

$S^n = \{ \vec{x} : \|\vec{x}\| = r \}$, and thus

$$S^n/_\sim = \mathbb{R}P^n .$$

(Additional Information i thought)

$S^n = \{ (x_1, \cdots, x_{n+1}) : x_1^2 + \cdots + x_{n+1}^2 = r^2 \}$.

$(0, 0, \cdots, 0) \notin S^n \subset M = \mathbb{R}^{n+1} - \{0\}$.

$[x]_M = \{ x \in M : x \sim y \Longleftrightarrow y = tx \text{ for some } t \neq 0 \}$

$[x]_{S^n} = \{ x \in S^n : x \sim y \Longleftrightarrow y = \pm x \}$

$\Rightarrow [x]_{S^n} \subset [x]_M . \quad (\because [x]_{S^n} = \{-x, x\} )$

$\Rightarrow [x]_{S^n} \in M/_\sim = \mathbb{R}P^n .$

$\therefore S^n/_\sim \subset M/_\sim = \mathbb{R}P^n .$

$\square$

**1.2.3.** The complex projective space $\mathbb{C}P^n$ is the set of all line through the origin in $\mathbb{C}^{n+1}$, i.e., the set of 1-dimensional subspaces of $\mathbb{C}^{n+1}$. If we define an equivalence relation on $M = \mathbb{C}^{n+1} - \{0\}$ by $z \sim w \iff w = \lambda z$ for some $\lambda \in \mathbb{C}^*$, then $\mathbb{C}P^n = M/\sim$. Show that $\mathbb{C}P^n$ is a $2n$-dimensional smooth manifold.

proof) ① 2nd - countable

Since $M$ is 2nd-countable, the quotient set of $M$ is 2nd-countable.

② Hausdorff.

$[z_1], [z_2] \in U_j$ for some $j$

$\Rightarrow [z_1]$ and $[z_2]$ are disjoint open set,

$(\because)$ $\varphi_j(z_1), \varphi_j(z_2) \in \mathbb{C}^n$.

<u>Claim</u> : $\nexists U_j$ containing $[z_1]$ & $[z_2]$.

Given $j \neq k$, let

$A_{j,k} = \{ [z] : |z^j| > |z^k| \} \subset \mathbb{C}P^n$.

Then $A_{j,k}$ is open since

$\pi^{-1}(A_{j,k})$ is open in $\mathbb{C}^{n+1} - \{0\}$.

By the assumption, $\exists j \neq k$ s.t.

$[z_1] \in U_j$ and $[z_2] \in U_k$, but

$z_1^j = z_2^k = 0$.

$\therefore z_1 \in A_{j,k}, z_2 \in A_{k,j}$.

$$\therefore A_{j,\kappa} \cap A_{\kappa,j} = \phi .$$

③ local Euclidean

For $\mathbb{Z} = (\mathbb{Z}^0, \cdots, \mathbb{Z}^n) \in \mathbb{C}^{n+1}$, define

$U_i = \{ [\mathbb{Z}] : \mathbb{Z}^i \neq 0 \} \subset \mathbb{C}P^n$, then we can

define $\varphi_i : U_i \rightarrow \mathbb{C}^n$ by

$$\varphi_i ([\mathbb{Z}]) = \left( \frac{\mathbb{Z}^0}{\mathbb{Z}^i}, \cdots, \frac{\mathbb{Z}^{i-1}}{\mathbb{Z}^i}, \frac{\mathbb{Z}^{i+1}}{\mathbb{Z}^i}, \cdots, \frac{\mathbb{Z}^n}{\mathbb{Z}^i} \right) .$$

con
tinu
ous

> For a projection $\pi : M \rightarrow M/_\sim$ by $\pi(\mathbb{Z}) = [\mathbb{Z}]$,
>
> $\varphi_i^{-1}(V)$ is open for any open $V \subset \mathbb{C}^n$
>
> $\Leftrightarrow \pi^{-1} \circ \varphi_i^{-1}(V) = (\varphi_i \circ \pi)^{-1}(V)$ is open in $\mathbb{C}^{n+1}$.
>
> Since $\varphi_i \circ \pi$ is clearly continuous, $(\because \mathbb{Z}^i \neq 0)$
>
> $(\varphi_i \circ \pi)^{-1}(V)$ is open in $\mathbb{C}^{n+1}$.

① injective

$$\varphi_i ([\mathbb{Z}_1]) = \varphi_i ([\mathbb{Z}_2]) \Rightarrow \frac{\mathbb{Z}_1^{0}}{\mathbb{Z}_2^{0}} = \frac{\mathbb{Z}_1^{i}}{\mathbb{Z}_2^{i}} , \quad j \neq i$$

$\Rightarrow [\mathbb{Z}_1] = [\mathbb{Z}_2] . \quad (\because \mathbb{Z}_1^j = t\mathbb{Z}_2^j$ for $\forall j )$

②

$v = (v^1, \cdots, v^n) \in \mathbb{C}^n$, then

$$\varphi_i^{-1}(v) = \pi (v^1, \cdots, v^{i-1}, 1, v^i, \cdots, v^n) .$$

Since $\pi$ is continuous, $\varphi_i^{-1}$ is continuous.

$(\because) \varphi_i (\pi(v^1, \cdots, v^{i-1}, 1, v^i, \cdots, v^n))$

$$= \left( \frac{v^1}{1}, \cdots, \frac{v^{i-1}}{1}, \frac{v^{i+1}}{1}, \cdots, \frac{v^n}{1} \right) = v$$

By ①, ②, $\varphi_i$ is a homeomorphism.

② assuming w.r.o.g, $i < j$, the transition maps

$$\varphi_j \circ \varphi_i^{-1} : \varphi(U_i \cap U_j) = \{ z = (z^1, \cdots, z^n) \in \mathbb{C}^n : z^j \neq 0 \}$$

$$\longrightarrow \varphi(U_i \cap U_j)$$

$$\varphi_j \circ \varphi_i^{-1} (z^1, \cdots, z^n) = \varphi_j([(z^1, \cdots, z^{i-1}, 1, z^{i+1}, \cdots, z^n)])$$

$$= \left( \frac{z^1}{z^j}, \cdots, \frac{z^i}{z^j}, \frac{1}{z^j}, \frac{z^{i+1}}{z^j}, \cdots, \frac{z^{j-1}}{z^j}, \frac{z^{j+1}}{z^j}, \cdots, \frac{z^n}{z^j} \right)$$

is smooth.

Thus, for $\{ (U_i, \varphi_i) \mid i = 1, \cdots, n+1 \} = \mathcal{A}$,

$\mathcal{A}$ is $C^\infty$ atlas.

$\therefore \mathbb{CP}^n$ is a $2n$-dimensional smooth manifold.

$\square$

**1.3.1.** Let $N = M = \mathbb{RP}^1$ and write a point in $\mathbb{RP}^1$ as $[(x,y)]$ for $(x,y) \in \mathbb{R}^2$. Show that the map $F: N \to M$ given by $F([(x,y)]) = [(x^2, y^2)]$ is smooth.

**proof)** For $\{(U, \varphi)\} = \mathcal{A}_N$ and $\{(V, \psi)\} = \mathcal{A}_M$,

let $\varphi = \psi = \pi^{-1}$, i.e.

$\varphi: N \to \mathbb{R}^2$ by $\varphi([(x,y)]) = (x,y)$

$\psi: M \to \mathbb{R}^2$ by $\psi([(x,y)]) = (x,y)$

Then, These can be a homeomorphism.

Thus,

$f = \psi \circ F \circ \varphi^{-1} = \psi(F([(x,y)]))$

$\mathbb{R}^2 \to \mathbb{R}^2 \quad = \psi([(x^2, y^2)]) = (x^2, y^2)$.

Since the components of $f$ is smooth,

$f = \psi \circ F \circ \varphi^{-1}$ is smooth, and therefore,

$F: N \to M$ is smooth mapping.

□

**1.3.2** Prove that (ii) of Theorem 1.23.

proof )