

1.1.1 Show that if M^m, N^n are smooth manifolds, then $M^m \times N^n$ is also a $(m+n)$ dimensional smooth manifold. Hence, the n -dimensional torus or simply n -torus

$$\mathbb{T}^n = \underbrace{S^1 \times \cdots \times S^1}_n \quad \mathbb{T}^2 = \text{Diagram of a torus} \quad \begin{matrix} 2\text{-dim} \\ \text{manifold} \end{matrix}$$

is a smooth manifold.

proof) We want to show :

[1] $M^m \times N^n$ is manifold [2] It is smooth manifold

[1] : ① Hausdorff

Since M^m, N^n are smooth manifold, these are Hausdorff. Then, for any U_M, V_M in M^m and U_N, V_N in N^n , let $U_M \times U_N = U \subset M^m \times N^n$ and $V_M \times V_N \subset N^n$, then $U_M \times U_N \cap V_M \times V_N = \emptyset$. $\cdots (*)$

(\because)

(1) : $(U_M \times U_N) \cap V_M = V_M \cap U_M \times V_M \cap U_N = \emptyset$
because $V_M \cap U_M = \emptyset$ since M^m is Hausdorff.

(2) : $(U_M \times U_N) \cap V_N = V_N \cap U_M \times V_N \cap U_N = \emptyset$

because $V_N \cap U_N = \emptyset$ since N^n is Hausdorff.

$(*) = (1) \times (2) = \emptyset$.

Thus, $M^m \times N^n$ is Hausdorff.

② second countable

By the assumption, M^m, N^n have a countable basis β_M, β_N . Then

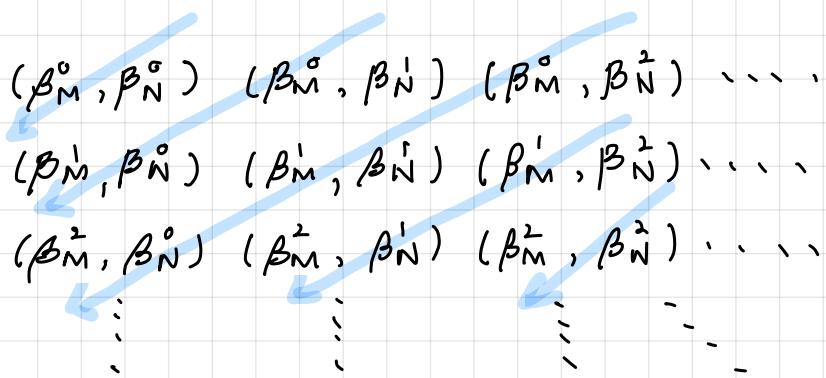
trivially $\beta_M \times \beta_N \subset M^m \times N^n$ and we can pick
 $\beta_M \times \beta_N$ is a countable basis for $M^m \times N^n$.

($\circ\circ$) $x \in \beta_M \text{ & } y \in \beta_N \Rightarrow (x, y) \in \beta_M \times \beta_N \subset M^m \times N^n$
 and $\beta_M, \beta_N : \text{open} \Rightarrow \beta_M \times \beta_N : \text{open}$.

$\beta_M \times \beta_N : \text{countable}$ since β_M, β_N are countable.

pf) Let $\beta_M \times \beta_N : \text{finite} \rightarrow \text{trivial}$.

We assume $\beta_M, \beta_N : \text{countably infinite}$.



First, we pick (β_M^0, β_N^0) , then we pick

$(\beta_M^0, \beta_N^1), (\beta_M^1, \beta_N^0)$, then we pick

$(\beta_M^0, \beta_N^2), (\beta_M^1, \beta_N^1), (\beta_M^2, \beta_N^0) \dots \dots$

Continue to this processes, then we can define the one-to-one correspondence

between $\beta_M \times \beta_N \rightarrow \mathbb{N}$ (the set of natural #).

Thus, by definition of countable,

the assertion is proved.

③ Homeomorphism

Let $\varphi_M : U \rightarrow \mathbb{R}^m$ & $p \in \varphi_M(U)$ and

$\varphi_N : V \rightarrow \mathbb{R}^n$ & $q \in \varphi_N(V)$, then we can define

$$\varphi_{MN}(r) = (\varphi_M \times \varphi_N)(p, q) = (\varphi_M(p), \varphi_N(q))$$

if $\varphi_{MN} : U \times V \rightarrow \mathbb{R}^{m+n}$.

(i) injective

$$\varphi_{MN}(r_1) = \varphi_{MN}(r_2)$$

$$\Rightarrow (\varphi_M(p_1), \varphi_N(q_1)) = (\varphi_M(p_2), \varphi_N(q_2))$$

$$\Rightarrow \varphi_M(p_1) = \varphi_M(p_2) \quad \& \quad \varphi_N(q_1) = \varphi_N(q_2)$$

$$\Rightarrow p_1 = p_2 \quad \& \quad q_1 = q_2 \quad \text{since } \varphi_M, \varphi_N \text{ : injective.}$$

(ii) surjective

For $\forall y = \varphi_{MN}(r) \in \mathbb{R}^{m+n}$, $\exists (\bar{p}, \bar{q}) \in U \times V$ s.t.

$$y = \varphi_{MN}(\bar{r}) = (\varphi_M(\bar{p}), \varphi_N(\bar{q})) \text{ since}$$

φ_M & φ_N are surjective.

By (i), (ii), φ_{MN} : bijection on $U \times V \subset M^m \times N^n$.

Thus, $\exists \varphi_{MN}^{-1}$: inverse of φ_{MN} .

be open

In case of continuity, for any Ω, β in $\mathbb{R}^m, \mathbb{R}^n$,

$\varphi_M^{-1}(\Omega), \varphi_N^{-1}(\beta)$ are open by the assumption.

Since $\Omega \times \beta$: open and its preimage

$\varphi_{MN}^{-1}(\Omega \times \beta)$ is open.

$\therefore \varphi_{MN}$ is continuous -

(\because) if $\Omega = \varphi_M(\alpha), \beta = \varphi_N(\beta)$ for any open sets α, β in U, V , then

$$\varphi_{MN}^{-1}(\varphi_{MN}(\Omega \times \beta)) = \Omega \times \beta = \varphi_M(\alpha) \times \varphi_N(\beta) \text{ : open.}$$

For φ_{MN}^{-1} : inverse of φ_{MN} , $\varphi_{MN}^{-1}(\theta \times \beta)$ is open

$\Rightarrow \varphi_{MN}(\varphi_{MN}^{-1}(\theta \times \beta)) = \theta \times \beta$ is open

$\therefore \varphi_{MN}^{-1}$ is continuous.

\circ φ_{MN} is Homeomorphism -

Therefore, $M^m \times N^n$ is a manifold.

[2] : Since M^m, N^n are smooth manifold, they have a C^∞ -structure, so that the coordinate charts $(U, \varphi_M), (V, \varphi_N)$ is C^∞ -compatible with all charts in the atlas of M^m, N^n , respectively.

By [1], we defined the homeomorphism φ_{MN} , hence we can write the coordinate chart of $M^m \times N^n$ that $(U \times V, \varphi_{MN})$. Consider another chart $(U' \times V', \varphi_{MN}^*)$, then

$$\begin{aligned}\varphi_{MN} \circ \varphi_{MN}^* &= (\varphi_M \times \varphi_N) \circ (\varphi_M^* \times \varphi_N^*)^{-1} \\ &= \varphi_M \circ \varphi_M^{*-1} \times \varphi_N \circ \varphi_N^{*-1}.\end{aligned}$$

Since $\varphi_M, \varphi_N, \varphi_M^{*-1}, \varphi_N^{*-1}$ are C^∞ ,

$\varphi_{MN} \circ \varphi_{MN}^*$ is C^∞ .

$\therefore M^m \times N^n$ is a smooth manifold.

Thus, By the proof above, \mathbb{T}^n is smooth manifold.

□

1.1.2 Let $U \subset \mathbb{R}^n$ be open and $f: U \rightarrow \mathbb{R}^m$ be continuous. Show that the graph of f

$$\Gamma_f = \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^m : x \in U \text{ and } y = f(x)\}$$

is an n -dimensional manifold.

(proof) By the example 1.1.1(i) of the lecture note of professor Han, \mathbb{R}^n , \mathbb{R}^m are n , m dimensional smooth manifold and hence $\mathbb{R}^n \times \mathbb{R}^m$ is smooth manifold by the exercise 1.1.1.

Thus, the graph of f Γ_f is the subspace topology of $\mathbb{R}^n \times \mathbb{R}^m$. Hence, Γ_f is Hausdorff and 2nd-countable space.

So, we want to show that Γ_f has the locally Euclidean property only.

Let $\pi_{\mathcal{X}}: \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^n$ is the projection onto \mathcal{X} , and let $\varphi: \Gamma_f \rightarrow U$ be the restriction of $\pi_{\mathcal{X}}$ to Γ_f that $\varphi(x, y) = x$, $(x, y) \in \Gamma_f$.

Since $\pi_{\mathcal{X}}$ is continuous (clearly),

the restriction of $\pi_{\mathcal{X}}$ φ is continuous, and bijective also. Thus $\exists \varphi^{-1}$: inverse of φ and since $\varphi^{-1}(x) = (x, f(x))$, φ^{-1} is continuous.

$\therefore \varphi$: Homeomorphism.

$\therefore \Gamma_f$ is n -dimensional manifold.



1.2.1 Complete the proof of proposition 1.14 :

Suppose that $\pi : M \rightarrow M/\sim$ is an open map. Then

(ii) M/\sim is Hausdorff $\Rightarrow R = \{(p, q) : p \sim q\}$ is closed in $M \times M$.

proof) Note that :

$$[x]_{\sim} = \{x \in M : x \sim d, x \in M\}.$$

$$M/\sim = \{[x]_{\sim} : x \in M\}$$

$O \subset M/\sim$ is open $\Leftrightarrow \pi^{-1}(O) = \{x : \pi(x) = [x] \in O\}$
is open in M .

Assume that M/\sim is Hausdorff.

Claim : $R \subset M \times M$ is closed

$\Leftrightarrow M \times M - R$ is open,

Let $(p, q) \in M \times M - R$, then $\pi(p) \neq \pi(q)$

$\Rightarrow (p, q) \notin R$. Thus we can take the

disjoint open sets $\pi(p) \in U_1$, $\pi(q) \in U_2$

since M/\sim is Hausdorff.

Let $V_1 = \pi^{-1}(U_1)$ & $V_2 = \pi^{-1}(U_2)$.

If $(V_1 \times V_2) \cap R \neq \emptyset$, then $\exists (v_1, v_2) \in V_1 \times V_2$

such that $\pi(v_1) = \pi(v_2)$, $\pi(v_1) \in U_1$, $\pi(v_2) \in U_2$.

But, $U_1 \cap U_2 = \emptyset$, that is contradiction.

$\therefore R$ is closed in $M \times M$.

□

1.2.2 Let $f : S^n \rightarrow S^n$ be the antipodal map defined by $f(x) = -x$. Define an relation \sim on S^n by $x \sim y$ iff $y = x$ or $y = f(x)$. Show that \sim is an equivalence relation and $S^n/\sim = \mathbb{RP}^n$.

proof) ① Equivalence relation

$$\boxed{x \sim y \iff y = x \iff y - x = 0}$$

$$(i) x \sim x \text{ since } x - x = 0$$

$$(ii) \text{ if } x \sim y, \text{ then}$$

$$\begin{aligned} y = x &\iff y - x = 0 \iff -(x - y) = 0 \\ &\iff x - y = 0 \iff x = y \\ &\iff y \sim x. \end{aligned}$$

$$\begin{aligned} y = -x &\iff -y = x \iff x = f(y) \\ &\iff y \sim x \end{aligned}$$

$$(iii) \text{ if } x \sim y \text{ \& } y \sim z, \text{ then}$$

$$y = x \text{ and } z = y \text{ and so}$$

$$z = y = x \iff x \sim z.$$

$$y = -x \text{ \& } z = -y, \text{ then}$$

$$z = -y = -(-x) = x \iff x \sim z.$$

$$\textcircled{2} \quad S^n / \sim = \mathbb{R}P^n$$

$$[x]_M = \bigcup_i [x_i]_{S^n}. \quad (\because [x]_{S^n} \subset [x]_M)$$

$$\Rightarrow \bigcup_j \left(\bigcup_i [x_i]_{S^n} \right)_j = M / \sim = \mathbb{R}P^n$$

For arbitrary $r \in \mathbb{R}^{n+1} - \{0\}$, we can let

$$S^n = \{ \vec{x} : \|\vec{x}\| = r \}, \text{ and thus}$$

$$S^n / \sim = \mathbb{R}P^n.$$

(Additional Information i thought)

$$S^n = \{ (x_1, \dots, x_{n+1}) : x_1^2 + \dots + x_{n+1}^2 = r^2 \}.$$

$$(0, 0, \dots, 0) \notin S^n \subset M = \mathbb{R}^{n+1} - \{0\}.$$

$$[x]_M = \{ x \in M : x \sim y \iff y = tx \text{ for some } t \neq 0 \}$$

$$[x]_{S^n} = \{ x \in S^n : x \sim y \iff y = \pm x \}$$

$$\Rightarrow [x]_{S^n} \subset [x]_M. \quad (\because [x]_{S^n} = \{-x, x\})$$

$$\Rightarrow [x]_{S^n} \in M / \sim = \mathbb{R}P^n.$$

$$\therefore S^n / \sim \subset M / \sim = \mathbb{R}P^n.$$

□

1.2.3. The complex projective space $\mathbb{C}P^n$ is the set of all line through the origin in \mathbb{C}^{n+1} , i.e., the set of 1-dimensional subspaces of \mathbb{C}^{n+1} . If we define an equivalence relation on $M = \mathbb{C}^{n+1} - \{0\}$ by $z \sim w \Leftrightarrow w = \lambda z$ for some $\lambda \in \mathbb{C}^*$, then $\mathbb{C}P^n = M/\sim$. Show that $\mathbb{C}P^n$ is a $2n$ -dimensional smooth manifold.

proof) ① 2nd-countable

Since M is 2nd-countable, the quotient set of M is 2nd-countable.

② Hausdorff.

$[z_1], [z_2] \in U_j$ for some j

$\Rightarrow [z_1]$ and $[z_2]$ are disjoint open set,

(\because) $\varphi_j(z_1), \varphi_j(z_2) \in \mathbb{C}^n$.

Claim: $\exists U_k$ containing $[z_1] \& [z_2]$.

Given $j \neq k$, let

$A_{j,k} = \{[z] : |z^j| > |z^k|\} \subset \mathbb{C}P^n$.

Then $A_{j,k}$ is open since

$\pi^{-1}(A_{j,k})$ is open in $\mathbb{C}^{n+1} - \{0\}$.

By the assumption, $\exists j \neq k$ s.t.

$[z_1] \in U_j$ and $[z_2] \in U_k$, but

$$z_1^j = z_2^k = 0.$$

$\therefore z_1 \in A_{j,k}, z_2 \in A_{k,j}$.

$$\therefore A_{j,k} \cap A_{k,j} = \emptyset.$$

③ local Euclidean

For $\underline{z} = (z^0, \dots, z^n) \in \mathbb{C}^{n+1}$, define

$U_i = \{[\underline{z}] : z^i \neq 0\} \subset \mathbb{C}\mathbb{P}^n$, then we can define $\varphi_i : U_i \rightarrow \mathbb{C}^n$ by

$$\varphi_i([\underline{z}]) = \left(\frac{z^0}{z^i}, \dots, \frac{z^{i-1}}{z^i}, \frac{z^{i+1}}{z^i}, \dots, \frac{z^n}{z^i} \right).$$

continuous

For a projection $\pi : M \rightarrow M/\sim$ by $\pi(\underline{z}) = [\underline{z}]$,

$\varphi_i^{-1}(V)$ is open for any open $V \subset \mathbb{C}^n$

$\Leftrightarrow \pi^{-1} \circ \varphi_i^{-1}(V) = (\varphi_i \circ \pi)^{-1}(V)$ is open in \mathbb{C}^{n+1} .

Since $\varphi_i \circ \pi$ is clearly continuous, ($\because z^i \neq 0$)
 $(\varphi_i \circ \pi)^{-1}(V)$ is open in \mathbb{C}^{n+1} .

① injective

$$\varphi_i([\underline{z}_1]) = \varphi_i([\underline{z}_2]) \Rightarrow \frac{z_1^j}{z_2^j} = \frac{z_1^i}{z_2^i}, j \neq i$$

$$\Rightarrow [z_1] = [z_2]. (\because z_1^j = \lambda z_2^j \text{ for } \forall j)$$

②

$v = (v^1, \dots, v^n) \in \mathbb{C}^n$, then

$$\varphi_i^{-1}(v) = \pi(v^1, \dots, v^{i-1}, 1, v^i, \dots, v^n).$$

Since π is continuous, φ_i^{-1} is continuous.

$$\begin{aligned} (\because) \quad & \varphi_i(\pi(v^1, \dots, v^{i-1}, 1, v^i, \dots, v^n)) \\ &= \left(\frac{v^1}{1}, \dots, \frac{v^{i-1}}{1}, \frac{v^i}{1}, \dots, \frac{v^n}{1} \right) = v \end{aligned}$$

By ①, ②, φ_i is a homeomorphism.

② assuming w.r.o.g., $i < j$, the transition maps

$$\varphi_j \circ \varphi_i^{-1} : \varphi(U_i \cap U_j) = \{ z = (z^1, \dots, z^n) \in \mathbb{C}^n : z^i \neq 0 \} \rightarrow \varphi(U_i \cap U_j)$$

$$\begin{aligned} \varphi_j \circ \varphi_i^{-1}(z^1, \dots, z^n) &= \varphi_j([(z^1, \dots, z^{i-1}, 1, z^{i+1}, \dots, z^n)]) \\ &= \left(\frac{z^1}{z^i}, \dots, \frac{z^i}{z^i}, \frac{1}{z^i}, \frac{z^{i+1}}{z^i}, \dots, \frac{z^{j-1}}{z^i}, \frac{z^{j+1}}{z^i}, \dots, \frac{z^n}{z^i} \right) \end{aligned}$$

is smooth.

Thus, for $\{(U_i, \varphi_i) \mid i=1, \dots, n+1\} = \mathcal{A}$,

\mathcal{A} is C^∞ atlas.

$\sigma_0 \mathbb{CP}^n$ is a $2n$ -dimensional smooth manifold.

□

1.3.1. Let $N = M = \mathbb{R}P^1$ and write a point in $\mathbb{R}P^1$ as $[(x, y)]$ for $(x, y) \in \mathbb{R}^2$. Show that the map $F : N \rightarrow M$ given by $F([(x, y)]) = [(x^2, y^2)]$ is smooth.

proof) For $f(U, \varphi) \cap f(V, \psi) = \emptyset$ and $f(V, \psi) \cap f(W, \omega) = \emptyset$, let $\varphi = \psi = \pi^{-1}$, i.e.

$$\varphi : N \rightarrow \mathbb{R}^2 \text{ by } \varphi([(x, y)]) = (x, y)$$

$$\psi : M \rightarrow \mathbb{R}^2 \text{ by } \psi([(x, y)]) = (x, y)$$

Then, these can be a homeomorphism.

Thus,

$$f = \psi \circ F \circ \varphi^{-1} = \psi(F([(x, y)]))$$

$$\mathbb{R}^2 \rightarrow \mathbb{R}^2 = \psi([(x^2, y^2)]) = (x^2, y^2).$$

Since the components of f is smooth,

$f = \psi \circ F \circ \varphi^{-1}$ is smooth, and therefore,

$F : N \rightarrow M$ is smooth mapping.

□

1.3.2 Prove that (ii) of Theorem 1.28.

proof) For each $g \in A$, $\exists (\varphi, U)$ near g such $U_g \subset U$ & $\varphi(U_g) \subset V$,
 $V = \{ \text{Contains the open ball } B_3(0) \}$.
 $B_3(0) \equiv \text{open ball of radius 3 centered at } 0$.

Let $\tilde{U}_g = \varphi^{-1}(B_1(0))$ and

define the function called "Bump function" that

$$g(t) = \begin{cases} 1 & \text{for } t \leq 1 \\ 0 & \text{for } t \geq 2 \end{cases},$$

and let

$$f(p) = \begin{cases} g(\varphi(p)) & p \in U_g \\ 0 & p \notin U_g \end{cases}.$$

Then, $f \in C^\infty(M)$ such that $0 \leq f \leq 1$,
 $f \equiv 1$ on $\tilde{U}_g \subset A$ and $\text{supp}(f) \subset U_p \subset U$. \square

2.1.1 For a smooth map $F: N^n \rightarrow M^m$, the push forward $F_* p : T_p N \rightarrow T_{F(p)} M$ at $p \in N$ was defined in terms of derivations. This can be also defined by the equivalence class of curves as

$$F_* p([\gamma]) = [F \circ \gamma].$$

Show that this definition is well defined. In other words, $\gamma_1 \sim \gamma_2$ implies that $F \circ \gamma_1 \sim F \circ \gamma_2$.

proof) For a coordinate chart (U, φ) at p ,

$$\gamma_1 \sim \gamma_2 \Leftrightarrow (\varphi \circ \gamma_1)'(0) = (\varphi \circ \gamma_2)'(0).$$

$F \circ \gamma = F \circ \varphi^{-1} \circ \varphi \circ \gamma$ since φ is homeomorphism.

$$\begin{aligned} \Rightarrow (F \circ \gamma_1)'(0) &= (F \circ \varphi^{-1} \circ \varphi \circ \gamma_1)'(0) \\ &= (F \circ \varphi^{-1})'(\varphi \circ \gamma_1)(0). \end{aligned}$$

$$\begin{aligned} (\because \gamma_1 \sim \gamma_2) \quad &= (F \circ \varphi^{-1})'(\varphi \circ \gamma_2)(0) \\ &= (F \circ \varphi^{-1} \circ \varphi \circ \gamma_2)'(0) \\ &= (F \circ \gamma_2)'(0) \end{aligned}$$

$$\Leftrightarrow F \circ \gamma_1 \sim F \circ \gamma_2$$

□

2. 1. 2. Let $F : N^n \rightarrow M^m$ be a smooth map.

Prove that if N is connected and $F_{*p} = 0$ for any $p \in N$, then F is constant map.

Proof) Let $f \in C^\infty(M)$ and let $X_p \in T_p N$.

By the assumption, $F_{*p}[f] = X_p(f \circ F) = 0$.

Let (U, φ) : smooth chart containing p . Then

$$X_p = \sum_i X_p^i \frac{\partial}{\partial x^i} \Big|_p = \sum_i X^i(\varphi^{-1})_* p \frac{\partial}{\partial x^i} \Big|_{\varphi(p)}$$

$$\Rightarrow \left(\sum_i X^i(\varphi^{-1})_* p \frac{\partial}{\partial x^i} \Big|_{\varphi(p)} \right) (f \circ F) = \sum_i X^i \frac{\partial}{\partial x^i} \Big|_{\varphi(p)} (f \circ F \circ \varphi^{-1}) = 0$$

$\Rightarrow F$ is constant on U .

Since N is connected, N : path connected.

Let $q \in N$ & let $\gamma : [0, 1] \rightarrow N$ be a path connecting p & q .

Since F is constant on each smooth chart $(U_{\gamma(x)}, \varphi_{\gamma(x)})$ containing $\gamma(x)$ for every $x \in [0, 1]$, $F \equiv c$ on N since $F(p) = c$ & γ is continuous. □

2.1.3. Prove that for any $p \in S^n$,

$$T_p S^n = \{X \in \mathbb{R}^{n+1} : \langle p, X \rangle = 0\}.$$

proof) Let $p \in S^n \subset \mathbb{R}^{n+1} - \{0\}$ and let $X \in \mathbb{R}^{n+1}$.

By the example 2.2, $T_p \mathbb{R}^m = \mathbb{R}^m$.

$$m = n+1 \Rightarrow T_p \mathbb{R}^{n+1} = \mathbb{R}^{n+1} \supset S^n.$$

$$[X]_{S^n} = \{X \in \mathbb{R}^{n+1} : X \sim p \Leftrightarrow p = tX \text{ for some } t \neq 0\}.$$

$$\langle p, X \rangle = 0 \Rightarrow p = -X.$$

Define

$$\gamma_{p,X}(t) = \begin{cases} p+tX & t \neq 1 \\ p & t=1 \end{cases}.$$

$$\text{Then, } \frac{d}{dt} \gamma_{p,X}(t) \Big|_{t=0} = p+X, \gamma_{p,X}(0) = p = -X$$

$$\Rightarrow [\gamma_{p,X}] \in T_p S^n.$$

$$\therefore T_p S^n = \{X \in \mathbb{R}^{n+1} : \langle p, X \rangle = 0\}. \quad \square$$

2.3.1. Let $F : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ such that

$F(x, y) = (x^2 - 2y, 4x^3y^2)$. For $X = 4x \frac{\partial}{\partial x} + 3y^2 \frac{\partial}{\partial y}$, compute $F_* X$.

proof) For the vector field $X = 4x \frac{\partial}{\partial x} + 3y^2 \frac{\partial}{\partial y}$,

let $x' = x^2 - 2y$ & $y' = 4x^3y^2$, then

$$F_* \left(\frac{\partial}{\partial x} \right) = \frac{\partial x'}{\partial x} \frac{\partial}{\partial x} + \frac{\partial y'}{\partial x} \frac{\partial}{\partial y} = (2x^2y^2) \frac{\partial}{\partial y}$$

$$F_* \left(\frac{\partial}{\partial y} \right) = \frac{\partial x'}{\partial y} \frac{\partial}{\partial x} + \frac{\partial y'}{\partial y} \frac{\partial}{\partial y} = -2 \frac{\partial}{\partial x} + 8x^3y \frac{\partial}{\partial y}$$

$$F_* X = 4x F_* \left(\frac{\partial}{\partial x} \right) + 3y^2 F_* \left(\frac{\partial}{\partial y} \right)$$

$$= -6y^2 \frac{\partial}{\partial x} + x^3y^2 (24y + 48) \frac{\partial}{\partial y}$$

□

2.3.2. Express the following planar vector fields in polar coordinates.

$$X = x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y}, \quad Y = -y \frac{\partial}{\partial x} + x \frac{\partial}{\partial y}.$$

What is $[X, Y]$?

Proof) Let $x = r \cos \theta$, $y = r \sin \theta$, then

$$\begin{aligned}\frac{\partial}{\partial r} &= \frac{\partial x}{\partial r} \frac{\partial}{\partial x} + \frac{\partial y}{\partial r} \frac{\partial}{\partial y} = \cos \theta \frac{\partial}{\partial x} + \sin \theta \frac{\partial}{\partial y} \\ &= \frac{x}{r} \frac{\partial}{\partial x} + \frac{y}{r} \frac{\partial}{\partial y} = \frac{1}{r} X\end{aligned}$$

$$\begin{aligned}\frac{\partial}{\partial \theta} &= \frac{\partial x}{\partial \theta} \frac{\partial}{\partial x} + \frac{\partial y}{\partial \theta} \frac{\partial}{\partial y} = -r \sin \theta \frac{\partial}{\partial x} + r \cos \theta \frac{\partial}{\partial y} \\ &= -y \frac{\partial}{\partial x} + x \frac{\partial}{\partial y} = Y\end{aligned}$$

$$\therefore X = r \frac{\partial}{\partial r}, \quad Y = \frac{\partial}{\partial \theta}.$$

By definition 2.21.(iii), $[X, Y] = \left[r \frac{\partial}{\partial r}, \frac{\partial}{\partial \theta} \right] = 0$.

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□

2.3.3. In \mathbb{R}^3 , let

$$X = x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x} \text{ and } Y = y \frac{\partial}{\partial z} - z \frac{\partial}{\partial y} .$$

Compute $[X, Y]$.

proof) By definition 2.22, we have

$$\begin{aligned}[X, Y] &= \left[x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x}, y \frac{\partial}{\partial z} - z \frac{\partial}{\partial y} \right] \\ &= \left[x \frac{\partial}{\partial y}, y \frac{\partial}{\partial z} \right] + \left[x \frac{\partial}{\partial y}, -z \frac{\partial}{\partial y} \right] + \\ &\quad \left[-y \frac{\partial}{\partial x}, y \frac{\partial}{\partial z} \right] + \left[-y \frac{\partial}{\partial x}, -z \frac{\partial}{\partial y} \right]\end{aligned}$$

By definition 2.21 (iii), we have

$$\textcircled{1} = x \cdot 0 \frac{\partial}{\partial z} + y \cdot 0 \frac{\partial}{\partial y} = 0$$

$$\textcircled{2} = x \cdot 0 \frac{\partial}{\partial y} + (-z) \cdot 0 \frac{\partial}{\partial y} = 0$$

$$\textcircled{3} = -y \cdot 0 \frac{\partial}{\partial z} + y \cdot 0 \frac{\partial}{\partial x} = 0$$

$$\textcircled{4} = -y \cdot 0 \frac{\partial}{\partial y} - z \cdot 0 \frac{\partial}{\partial x} = 0$$

$$\therefore [X, Y] = 0$$

□

2.3.4. Verify Example 2.23.

proof) (i) \mathbb{R}^n is a Lie algebra.

Since $[a, b] = (a+b) - (b+a) = 0$,

bilinear & skew symmetric satisfied.

Check Jacobi identity condition. For $c \in \mathbb{R}^n$,

$$[a, [b, c]] + [b, [c, a]] + [c, [a, b]]$$

$$= a + [b, c] - ([b, c] + a) + b + [c, a] - ([c, a] + b)$$

$$+ c + [a, b] - ([a, b] + c)$$

$$= a + 0 - (0 + a) + b + 0 - (0 + b) + c + 0 - (0 + c)$$

$$= a - a + b - b + c - c = 0.$$

$\therefore \mathbb{R}^n$ is a Lie algebra.

(ii) $GL(n, \mathbb{R})$ is a Lie algebra.

Check only the condition for Jacobi.

$$[A, B] = AB - BA \text{ for } A, B \in GL(n, \mathbb{R}),$$

pick $C \in GL(n, \mathbb{R})$, then

$$[A, [B, C]] + [B, [C, A]] + [C, [A, B]]$$

$$= A([B, C]) - ([B, C]A) + B([C, A]) - ([C, A]B)$$

$$+ C([A, B]) - ([A, B]C)$$

$$= A(BC - CB) - (BC - CB)A + B(CA - AC) - (CA - AC)B$$

$$+ C(AB - BA) - (AB - BA)C$$

$$= ABC - ACB - BCA + CBA + BCA - BAC - CAB + ACB$$

$$+ CAB - CBA - ABC + BAC$$

$$= 0. \quad \therefore GL(n, \mathbb{R}) \text{ is a Lie algebra.}$$

(iii) \mathbb{R}^3 is a Lie algebra with $[u, v] = u \times v$, $u, v \in \mathbb{R}^3$

Cross product satisfy the skew symmetric condition

($\because u \times v = -v \times u$). and bilinear condition also.

(\therefore Let $u, v, w \in \mathbb{R}^3$, i, j, k : standard basis of \mathbb{R}^3 .

$$u = u_1 i + u_2 j + u_3 k, \quad v = v_1 i + v_2 j + v_3 k,$$

$$w = w_1 i + w_2 j + w_3 k. \text{ Then, for } c \in \mathbb{R},$$

$$(cu + v) \times w = c(u \times w) + v \times w.$$

$$\text{pf)} \quad (cu + v) \times w = \begin{vmatrix} i & j & k \\ cu_1 + v_1 & cu_2 + v_2 & cu_3 + v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$

$$= \begin{vmatrix} i & j & k \\ cu_1 & cu_2 & cu_3 \\ w_1 & w_2 & w_3 \end{vmatrix} + \begin{vmatrix} i & j & k \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$

$$= c \begin{vmatrix} i & j & k \\ u_1 & u_2 & u_3 \\ w_1 & w_2 & w_3 \end{vmatrix} + \begin{vmatrix} i & j & k \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$

$$= c(u \times w) + v \times w. \quad \text{Q.E.D.)}$$

Check Jacobi identity condition. For $u, v, w \in \mathbb{R}^3$,

$$[u, [v, w]] + [v, [w, u]] + [w, [u, v]]$$

$$= u \times (v \times w) + v \times (w \times u) + w \times (u \times v)$$

$$= (u \cdot w)v - (u \cdot v)w + (v \cdot u)w - (v \cdot w)u$$

$$+ (w \cdot v)u - (w \cdot u)v$$

$$= 0.$$

$\therefore \mathbb{R}^3$ is a Lie algebra with $[u, v] = u \times v$ for $u, v \in \mathbb{R}^3$.

(iv) G , \mathfrak{H} are Lie algebras $\Rightarrow G \times \mathfrak{H}$ is also a Lie algebra under the bracket

$$[(X_1, Y_1), (X_2, Y_2)] = ([X_1, Y_1], [X_2, Y_2]).$$

① Bilinearity

Clearly we obtain the property after complicate calculation.

Note that $[X_1, Y_1]$, $[X_2, Y_2]$ are satisfy the bilinearity in G , \mathfrak{H} , respectively.

② skew - symmetric .

For the simplicity, denote $X_1, Y_1 \equiv x_1, y_1$.

$$[(x_1, y_1), (x_2, y_2)] = ([x_1, y_1], [x_2, y_2])$$

$$\begin{aligned} G, \mathfrak{H} : \text{Lie algebra} &\rightarrow = (-[y_1, x_1], -[y_2, x_2]) \\ &= -([y_1, x_1], [y_2, x_2]) \\ &= -[(y_1, x_1), (y_2, x_2)] \end{aligned}$$

③ Jacobi identity

$$\begin{aligned} &[(x_1, y_1), [(x_2, y_2), (x_3, y_3)]] \\ &+ [(x_2, y_2), [(x_3, y_3), (x_1, y_1)]] \\ &+ [(x_3, y_3), [(x_1, y_1), (x_2, y_2)]] = 0 \end{aligned}$$

using the previous results we proved
and our Lie bracket. □

2. 3. 5. Prove Theorem 2.24.

proof) Check the Jacobi identity.

For the simplicity, denote $X, Y, \dots \equiv x, y, \dots$.

$$[x, [y, z]] + [y, [z, x]] + [z, [x, y]]$$

$$= x(yz - zy) - (yz - zy)x + y(zx - xz) - (zx - xz)y$$

$$+ z(xy - yx) - (xy - yx)z$$

$$= xyz - xzy - yzx + zyx + yxz - yzx - zx y + xzy$$

$$+ zxy - zyx - xzy + yxz$$

$$= 0.$$

□

2.3.6. Prove Theorem 2.26.

proof) By the assumption, X_i, Y_i are F -related.

i.e. $F_*(X_i) = Y_i$ by definition 2.25.

Claim: $F_*([X_1, X_2]) = [Y_1, Y_2]$.

Choose $g \in C^\infty(M)$ and $\alpha \in N$, then

$$(Y_1 g)(F(\alpha)) = (F_*)_\alpha(X_i)(g) = X_i(g \circ F)$$

$$\text{Thus, } (Y_1 g) \circ F = X_i(g \circ F) \quad \cdots (*)$$

Let $f \in C^\infty(N)$ be arbitrary. Using $(*)$

$$\Rightarrow Y_1(Y_2 f) \circ F = X_i((Y_2 f) \circ F). \quad \cdots (**)$$

By $(*)$, we also obtain

$$(Y_2 f) \circ F = X_2(f \circ F) \text{ and thus}$$

$$(**) = Y_1(Y_2 f) \circ F = X_i(X_2(f \circ F)).$$

Likewise, we get

$$Y_2(Y_1 f) \circ F = X_2(X_1(f \circ F)).$$

$$\text{Hence, } ([Y_1, Y_2]f) \circ F = [X_1, X_2](f \circ F).$$

Therefore, $[Y_1, Y_2]$ is F -related to $[X_1, X_2]$

□

2.3.1. Let $F: N \rightarrow M$ be a diffeomorphism. Prove that for any $Y \in \mathcal{X}(M)$, there is a unique $X \in \mathcal{X}(N)$ such that X is F -related to Y .

proof) Assume that X is F -related to Y .

$$\text{i.e. } X_{F(p)} = F_* p(Y_p).$$

If F is a diffeomorphism, we define X by

$$X_g = F_*_{F^{-1}(g)}(Y_{F^{-1}(g)})$$

Then, it is clear that X is the unique vector field such that F -related to Y .

□

Note that

$X: N \rightarrow TN$, N : manifold, TN : tangent bundle.

Then X is the composition that

$$N \xrightarrow{F^{-1}} M \xrightarrow{X} TM \xrightarrow{dF} TN$$

$\Rightarrow X$ is smooth.

□

□

2.3.8. Express the planar 1-form $\omega = xdx + ydy$ in polar coordinates.

proof) Let $x = r\cos\theta$ and $y = r\sin\theta$.

By Chain rule,

$$dx = \frac{\partial x}{\partial r} dr + \frac{\partial x}{\partial \theta} d\theta = \cos\theta dr - r\sin\theta d\theta$$

$$dy = \frac{\partial y}{\partial r} dr + \frac{\partial y}{\partial \theta} d\theta = \sin\theta dr + r\cos\theta d\theta$$

The differential 1-form ω is expressed by

$$\omega = \left(\frac{\partial x}{\partial r} x + \frac{\partial y}{\partial r} y \right) dr + \left(\frac{\partial x}{\partial \theta} x + \frac{\partial y}{\partial \theta} y \right) d\theta$$

$$= (r\cos^2\theta + r\sin^2\theta) dr + (-r^2\sin\theta\cos\theta + r^2\cos\theta\sin\theta) d\theta$$

$$= r dr + 0 d\theta$$

$$= r dr.$$

□

3.2.1. Let $M = \{(x, y, z) \in \mathbb{R}^3 : x^3 + y^3 + z^3 - 3xyz = 1\}$.

Prove that M is a 2-dimensional regular submanifold of \mathbb{R}^3 . What is $T_p M$ at $p = (0, 0, 1)$?

proof) Let $F : \mathbb{R}^3 \rightarrow \mathbb{R}$ be defined by

$$F(x, y, z) = x^3 + y^3 + z^3 - 3xyz - 1.$$

Note that :

Definition

The rank of a smooth map $f : N \rightarrow M$ between two manifolds at a point $p \in N$ is the rank of the derivative of f at p .

For $g = 0 \in \mathbb{R}$, $F^{-1}(g) = M$ and Jacobian is

$$DF = F_* = \begin{bmatrix} 3x^2 - 3yz & 3y^2 - 3xz & 3z^2 - 3xy \end{bmatrix}, \vec{0} \notin M.$$

Then, $(0, 0, 0)$ is the only critical point of F .

Thus, g is a regular value of F .

By Theorem 3.14 (ii) in Han's lecture note, $F^{-1}(g) = M$ is a 2-dimensional regular submanifold.

$T_p M$ at $p = (0, 0, 1)$ is :

We have the equation of $T_p M$ that

$$0(x-0) + 0(y-0) + 3(z-1) = 3(z-1) = 0.$$

$$\therefore T_p M = \{(x, y, z) \in \mathbb{R}^3 : 3(z-1) = 0\}.$$

□

3.2.2. Show that $F : \mathbb{RP}^2 \rightarrow \mathbb{R}^4$ be defined by

$$F(x, y, z) = \frac{1}{x^2 + y^2 + z^2} (x^2 - y^2, xy, xz, yz)$$

is a smooth embedding.

proof)

□

Consider the quotient projection $\pi : P \mapsto [P]$.

We know that π is local diffeomorphism and \mathbb{RP}^2 has the quotient topology of S^2 via π .

$$\begin{array}{ccc} S^2 & \xrightarrow{f} & \mathbb{R}^4 \\ \pi \downarrow & & \nearrow F \\ \mathbb{RP}^2 & & \end{array}$$

□

Define f by

$$f : S^2 \rightarrow \mathbb{R}^4, f(x, y, z) = (x^2 - y^2, xy, xz, yz).$$

Then, $f = F \circ \pi$ since

$$\pi(p_1) = \pi(p_2) \Rightarrow p_1 = \pm p_2 \Rightarrow \underline{f(p_1) = f(p_2)}.$$

$$f(p) = f(-p).$$

Calculate the Jacobian of f :

$$Df = \begin{bmatrix} 2x & -2y & 0 \\ y & x & 0 \\ z & 0 & x \\ 0 & z & y \end{bmatrix}$$

If $x \neq 0, y \neq 0$, then $\text{rank}(Df) = 3$.

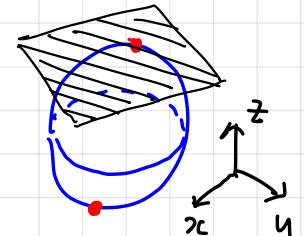
\Rightarrow it is injective linear map : $\mathbb{R}^3 \rightarrow \mathbb{R}^4$.

(in this, restricted to the tangent plane of S^2 at $p \in S^2$ is still injective.)

\Rightarrow the map has rank = 2 at $p \rightarrow$ a map of S^2 .)

If $x = y = 0$, then $p = (0, 0, \pm 1)$

$\Rightarrow T_p S^2$ is xy -plane in \mathbb{R}^3 .



$\Rightarrow T_p S^2$ is mapped injectively into \mathbb{R}^4

$\Rightarrow \text{rank}(f) = 2$.

Thus, $F = f \circ \pi^{-1}$, $\text{rank}(F) = 2$ locally.

Now, we claim $f(p) = f(q)$ for $p, q \in S^2$,
then $p = \pm q$.

Consider $\varphi : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ by $(x, y) \mapsto (x^2 - y^2, xy)$.

Let $x^2 - y^2 = a$, $xy = b$. Then

$$a^2 + 4b^2 = x^4 + 2x^2y^2 + y^4 = (x^2 + y^2)^2$$

$$x^2 + y^2 = \sqrt{a^2 + 4b^2}$$

$$x^2 = \frac{1}{2}(a + \sqrt{a^2 + 4b^2})$$

$$y^2 = \frac{1}{2}(-a + \sqrt{a^2 + 4b^2})$$

$\Rightarrow \pm x, \pm y$: uniquely determined by a, b

xy already determined by b .

\Rightarrow only $\pm(x, y)$ are mapped to (a, b) .

Now, (x_1, y_1, z_1) , (x_2, y_2, z_2) are mapped to the same point, then we know that $(x_1, y_1) = \pm(x_2, y_2)$, so $z_1 = \pm z_2$.

$\Rightarrow (x_1, y_1, z_1) = \pm(x_2, y_2, z_2)$ in S^2
~D the two points are antipodal.

Thus, F is injective, so F is injective immersion.

Since \mathbb{RP}^2 is compact because it is the image of S^2 by the continuous map Π .

Therefore, by Theorem 3.6. (iii),

F is a smooth embedding. □

3.2.3. Let $F : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ be defined by
 $F(x, y, z) = (x^2 + y, x^2 + y^2 + z^2 + y)$. Show that
 $g = (0, 1)$ is regular value of F and $F^{-1}(g)$ is
diffeomorphic to S^1 .

proof) Let $f_1 = x^2 + y$, $f_2 = x^2 + y^2 + z^2 + y$.
 $f_1 = 0 \Rightarrow y = -x^2$ and $f_2 = 1 \Rightarrow y^2 + z^2 = 1$.

$F^{-1}(0, 1) = \{ (x, y, z) \in \mathbb{R}^3 : y = -x^2 \text{ and } y^2 + z^2 = 1 \}$
 Level set.

$$DF = \begin{bmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} & \frac{\partial f_1}{\partial z} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} & \frac{\partial f_2}{\partial z} \end{bmatrix} = \begin{bmatrix} 2x & 1 & 0 \\ 2x & 2y+1 & 2z \end{bmatrix}$$

→ $\Rightarrow \text{rank } DF = 2 \text{ for all } (x, y, z) \in F^{-1}(0, 1)$.

consider
 $x = y = 0$
or
 $x \neq 0, y \neq 0$

By the definition 3.13, $g = (1, 0)$ is regular value.

Thus, by theorem 3.14 (ii), $F^{-1}(g)$ is an 1-dimensional regular submanifold of \mathbb{R}^3 .

Now, we want to show that $F^{-1}(g)$ is diffeomorphic to S^1 .

Recall the definition of diffeomorphic.

F Two manifolds M, N are diffeomorphic if there is a diffeomorphism f from M to N
(f: C[∞], homeo, f⁻¹: C[∞])

Define $\varphi : F^{-1}(g) \rightarrow S^1$ by $\varphi(x, y, z) = (y, z)$.

Then, clearly, φ is smooth and bijective.

($\circ\circ$) φ is surjective \rightarrow trivial.

$$\varphi(x_1, y_1, z_1) = \varphi(x_2, y_2, z_2)$$

$$\Rightarrow (y_1, z_1) = (y_2, z_2) \Rightarrow y_1 = y_2 \text{ & } z_1 = z_2$$

$$\text{Since } y = -x^2, \quad x_1 = x_2.$$

$\therefore \varphi$ is injective.

Thus, there exists an inverse of $\varphi \equiv \varphi^{-1}$.

Now, we only check that φ^{-1} is smooth.

Consider the inclusion map $i : F^{-1}(g) \rightarrow \mathbb{R}^3$

of submanifold $F^{-1}(g)$ and the projection

$\pi : \mathbb{R}^3 \rightarrow \mathbb{R}^2$. Then i is smooth and

π is also smooth. Thus, $\varphi = \pi \circ i$.

Hence, $\varphi^{-1} = (\pi \circ i)^{-1} = i^{-1} \circ \pi^{-1}$ is smooth.

$\circ\circ$ φ is a diffeomorphism.

$\circ\circ$ the level set $F^{-1}(g)$ is diffeomorphic to S^1 .



3.2.4. Let $F: N \rightarrow M$ be a smooth map of constant rank. Prove that if F is injective, then it is an immersion.

proof) Let $\dim M = m$ and $\dim N = n$ and suppose that F has constant rank r .

Suppose that F is not an immersion, i.e. $r < n$. By the rank theorem, for each $p \in N$, $\exists (U, \varphi)$ for N centered at p & (V, ψ) for M centered at $F(p)$ such that F has the coordinate representation

$$\psi \circ F \circ \varphi^{-1}(x^1, \dots, x^n) = (x^1, \dots, x^r, 0, \dots, 0).$$

It follows that $F(0, \dots, 0, \varepsilon) = F(0, \dots, 0, 0)$ for any sufficiently small ε .

$\therefore F$ is not injective. □

3.2.2. Show that $F : \mathbb{RP}^2 \rightarrow \mathbb{R}^4$ be defined by

$$F[(x, y, z)] = \frac{1}{x^2 + y^2 + z^2} (x^2 - y^2, xy, xz, yz)$$

is a smooth embedding.

proof) Note that $\mathbb{RP}^2 = S^2 / \{\pm 1\}$ of S^2 which is obtained by identifying antipodal points.

reference Then, F naturally reduced to the map

3.4.2. $f : S^2 \rightarrow \mathbb{R}^4$, $f(x, y, z) = (x^2 - y^2, xy, xz, yz)$.

In order to apply Theorem 3.6 (Han's note), we have to check two conditions that

① injective immersion ② \mathbb{RP}^2 is compact.

Claim : ① \rightarrow (1) : f is immersion.

P f is immersion $\Leftrightarrow Df : T_p M \rightarrow T_{f(p)} N$
 $(f : M \rightarrow N)$ is injective.

pf) We know that $\dim M = \dim T_p M$ and the fact $\dim V = \text{rank } A + \dim \ker A$, for any linear map A on vector space V .

Since Df is linear, by definition, we obtain $\dim T_p M = \text{rank } Df + \dim \ker Df$.

Now, by the definition of immersion (3.2, (i)), we have :

$$\begin{aligned}
 f : \text{immersion} &\Leftrightarrow \dim M = \text{rank } Df \\
 &\Leftrightarrow \dim \ker Df = 0 \\
 &\Leftrightarrow \ker Df = \{0\} \\
 &\Leftrightarrow Df \text{ is injective.}
 \end{aligned}$$

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Clearly, $\dim \mathbb{R}\mathbb{P}^2 = 2$.

We know that $\dim M = \dim T_p M$ and the fact $\dim V = \text{rank } A + \dim \ker A$, for any linear map A on vector space V .

Hence, $f : \text{injective} \Rightarrow Df : \text{injective}$.

Since Df is linear, by definition, we obtain

$$\dim T_p M = \text{rank } Df + \dim \ker Df.$$

($\circ\circ$) $T_p M$ is a vector space spanned by

$$\text{a basis } \left\{ \frac{\partial}{\partial x^i} \Big|_p : i \in \{1, \dots, n\} \right\}, n \in \mathbb{N}.$$

Now, by the definition of immersion (3.2, (i)), we have :

$$\begin{aligned}
 Df \text{ is injective} &\Leftrightarrow \ker Df = \{0\} \\
 &\Leftrightarrow \dim \ker Df = 0 \\
 &\Leftrightarrow \dim M = \text{rank } Df \\
 &\Leftrightarrow f : \text{immersion}
 \end{aligned}$$

3.2.5. Give an example of an immersion $\iota : N \hookrightarrow M$ and $\omega \in \Gamma(M)$ such that $\iota^*\omega = 0$ on N although $\omega \neq 0$ everywhere on M .

proof) Let $M = \mathbb{R}^2$, $\omega := dy \in \Gamma(M)$.

Consider S be the x -axis $\equiv N$.

(Note that S : embedded submanifold of \mathbb{R}^2 .)

Then, as a covector field on M ,

ω is nonzero everywhere since one of its component functions is always 1.

But, for the restriction $\iota^*\omega$ where

$\iota : N \hookrightarrow M$ be an immersion,

$$\iota^*\omega = \iota^*dy = d(y \circ \iota) = 0$$

\uparrow \uparrow
proposition 2.11 (iii) in Ham's note.

|

y vanishes identically on S .

□

4.1.1. Let G be a manifold with a group structure.
 Prove that if the map $G \times G \rightarrow G$ defined by $(g, h) \mapsto gh^{-1}$
 is smooth, then G is a Lie group.

proof) Let $\mu : G \times G \rightarrow G$, $\mu(g, h) = gh^{-1}$.

By the assumption, μ is smooth for all (g, h) .

Thus, consider the restriction map defined by

$\mu_g = \mu(e, h) = h^{-1}$ and $\mu_h = \mu(g, e) = g$
 for the identity $e \in G$.

Then, μ_g, μ_h are also smooth since
 μ is smooth. ($\because \mu = \mu_h \cdot \mu_g$)

Hence, we can define the inverse $\text{inv}(\alpha)$
 by $\text{inv} : G \rightarrow G$, $\text{inv}(\alpha) = \alpha^{-1}$ as
 $\alpha := h^{-1}$, since G is a group.

($\circ\circ$) Since G is a group, for each $h^{-1} \in G$,
 there exists the inverse $h = (h^{-1})^{-1}$ such that
 $h(h^{-1}) = e = (h^{-1})h$, e : identity of G .

Therefore, by definition 4.1,

G is a Lie group. □

4.1.2. Prove that if G_1 and G_2 are Lie groups, then $G_1 \times G_2$ is a Lie group. Hence, \mathbb{F}^n is a Lie group.

proof) Suppose that G_1, G_2 are Lie groups.

Since G_1, G_2 are groups, it can be written by the direct product of two groups as

$G_1 \times G_2$ such that $(g_1, h_1)(g_2, h_2) = (g_1h_1, g_2h_2)$ for $g_1, h_1 \in G_1, g_2, h_2 \in G_2$.

Then, $G_1 \times G_2$ is a group is immediate.

On the other hand, by problem 1.1 (Han's note), we know that $G_1 \times G_2$ is a smooth manifold.

Hence, now we only have to show that $G_1 \times G_2$ is a Lie group.

Define $\mu : (G_1 \times G_2) \times (G_1 \times G_2) \rightarrow G_1 \times G_2$ by

$$\mu((g_1, g_2), (h_1, h_2)) = (g_1, g_2)(h_1, h_2), \quad g_i, h_i \in G_i.$$

Since $G_1 \times G_2$ is the direct product of G_1, G_2 ,

$$\mu((g_1, g_2), (h_1, h_2)) = (g_1, g_2)(h_1, h_2) = (g_1h_1, g_2h_2).$$

(in this, $g_1h_1 \in G_1, g_2h_2 \in G_2$ since G_1, G_2 are Lie groups by the assumption.)

Since G_1, G_2 are Lie groups again,

there exist $\gamma_1^{-1}, \gamma_2^{-1}$: inverse of G_1, G_2 , respectively so that

we define $\text{inv} : G_1 \times G_2 \rightarrow G_1 \times G_2$ by

$$\text{inv}(x_1, x_2) = (x_1, x_2)^{-1}, \text{ then}$$

$$\text{inv}(x_1, x_2) = (x_1, x_2)^{-1} = (x_1^{-1}, x_2^{-1}).$$

Then, $x_i^{-1} \in G_i$ since G_i are Lie group.

Note that the multivariable function is smooth if the components are smooth.

Then, μ , inv are smooth clearly.

(\because consider the each component functions as μ_i , inv_i of G_1, G_2 . Then these are smooth since G_1, G_2 are Lie group.

We just find the proper multiplication & inverse map.)

Therefore, by definition 4.1 in Han's note, $G_1 \times G_2$ is a Lie group.

By the proof above, we can extend the fact that \mathbb{T}^n is a Lie group since the direct product is defined componentwise as represented by a tuple. □

4.1.3. Verify Example 4.13.

proof) (i) The complex special linear group

$$SL(n, \mathbb{C}) = \{A \in gl(n, \mathbb{C}) : \det A = 1\}$$

is $(2n^2 - 2)$ dimensional Lie subgroup of $GL(n, \mathbb{C})$.

pf) ① Subgroup

For any $A, B \in SL(n, \mathbb{C}) = \det^{-1}(1) \subseteq GL(n, \mathbb{C})$,

we can pick B^{-1} : inverse of B since

$$1+i \cdot 0 \rightarrow \det(B) = 1 \neq 0 \Leftrightarrow B \text{ is invertible}.$$

By the property of determinant,

$$\det(B^{-1}) = \det(B)^{-1} = 1 \Rightarrow B^{-1} \in SL(n, \mathbb{C}).$$

Then,

$$\begin{aligned} \det(AB^{-1}) &= \det(A)\det(B^{-1}) \\ &= \det(A)\det(B)^{-1} = 1. \end{aligned}$$

$$\therefore AB^{-1} \in SL(n, \mathbb{C})$$

$\therefore SL(n, \mathbb{C})$ is a subgroup of $GL(n, \mathbb{C})$.

② Submanifold.

Define $\det_{*A} : T_A GL(n, \mathbb{C}) = gl(n, \mathbb{C}) \rightarrow \mathbb{C}$.

For $A \in GL(n, \mathbb{C})$ & $B \in gl(n, \mathbb{C})$,

$$\gamma(s) = A + sB, \quad s \in (-\varepsilon, \varepsilon) \text{ for small } \varepsilon.$$

Using the formula in example 4.11,

we obtain this in the same way as C.

$$\approx \det_{*A}(B) = (\det A) + r(A^{-1}B).$$

Note that $\dim GL(n, \mathbb{C}) = 2n^2$, $\dim \mathbb{C} = 2$.

$\det_{*A}(A) = \det A + \text{tr}(I) = \det(A)(n) \neq 0$.

Hence, \det_{*A} is a submersion for $\forall A \in GL(n, \mathbb{C})$.

Moreover, $SL(n, \mathbb{C}) = \det^{-1}(1)$ is a regular submanifold of $GL(n, \mathbb{C})$ by Thm 3.14 and its dimension is $2n^2 - 2$.

Thus, by Theorem 4.8,

$SL(n, \mathbb{C})$ is a closed Lie subgroup. \square

(ii) The unitary group

$U(n) = \{A \in gl(n, \mathbb{C}) : A^*A = I_n\}$ is n^2 dimensional Lie subgroup $GL(n, \mathbb{C})$.

pf) ① Subgroup

$$\begin{aligned}(AB)^*(AB) &= \overline{(B^T A^T)}(AB) = \overline{(B^T)} \overline{(A^T)}(AB) \\ &= B^* A^* AB = B^* I_n B = B^* B = I_n.\end{aligned}$$

$\therefore U(n) \leq GL(n, \mathbb{C})$.

② Submanifold.

Let $S(n, \mathbb{C}) = \{A \in gl(n, \mathbb{C}) : A^* = A\}$.

Note : $A \in gl(n, \mathbb{R})$ is Hermitian $\Leftrightarrow A$: symmetric.

\rightsquigarrow symmetric real matrix is the special case of Hermitian.

Then, $\dim S(n, \mathbb{C}) = n^2$

(\because Consider the basis for 2×2 complex matrix)

\hookrightarrow in linear combination.

Define $F : GL(n, \mathbb{C}) \rightarrow S(n, \mathbb{C})$ by $F(A) = A^*A$.

Then $U(n) = F^{-1}(I_n)$. For $A \in U(n)$, we see that

$$F_{*A} : T_A GL(n, \mathbb{C}) = gl(n, \mathbb{C}) \rightarrow T_{F(A)} S(n, \mathbb{C}).$$

Given $B \in S(n, \mathbb{C})$, the curve $\gamma(t) = A + tB$ is well defined for all $t \in (-\varepsilon, \varepsilon)$, $\varepsilon > 0$ is small.

Thus,

$$\begin{aligned} F_{*A}(B) &= \frac{d}{dt} \Big|_{t=0} F \circ \gamma(t) \\ &= \frac{d}{dt} \Big|_{t=0} F(A + tB) \\ &= \frac{d}{dt} \Big|_0 (A + tB)^*(A + tB) \\ &= \frac{d}{dt} \Big|_0 (A^* + tB^*)(A + tB) \quad (\because (A + B)^* = A^* + B^*) \\ &= B^*(A + tB) + (A^* + tB^*)B \Big|_{t=0} \\ &\quad \underbrace{\qquad\qquad\qquad}_{(\because dA^* = d(A^*) = (dA)^*)} \\ &= B^*A + A^*B \in S(n, \mathbb{C}). \end{aligned}$$

Hence, $T_{F(A)} S(n, \mathbb{C}) \subset S(n, \mathbb{C})$. For given $C \in S(n, \mathbb{C})$,

$$\begin{aligned} F_{*A}\left(\frac{1}{2}AC\right) &= \frac{1}{2}(AC)^*A + \frac{1}{2}A^*(AC) \\ &= \frac{1}{2}(C^*A^*)A + \frac{1}{2}I_n C \quad (\because A \in U(n)) \\ &= \frac{1}{2}C^* + \frac{1}{2}C = C \quad (\because C \in S(n, \mathbb{C})) \end{aligned}$$

Thus, $T_{F(A)} S(n, \mathbb{C}) = S(n, \mathbb{C})$ and
 F_{*A} is surjective for any $A \in U(n)$ and
thus I_n is a regular value of F .

Then, by Theorem 3.14, $U(n)$ is a regular
submanifold of $GL(n, \mathbb{C})$ with dimension n^2 .

Therefore, Theorem 4.8, $U(n)$ is a Lie subgroup.

□

(iii) The special unitary group

$$SU(n) = SL(n, \mathbb{C}) \cap U(n)$$

is $(n^2 - 1)$ dimensional Lie subgroup $GL(n, \mathbb{C})$.

pf) $SL(n, \mathbb{C}) = \det^{-1}(1)$ is closed in $GL(n, \mathbb{C})$.

For any $A \in U(n)$, $\det(A) = \pm 1$.

$$\begin{aligned} (\because) \quad 1 &= \det(A) \det(A)^{-1} = \det(AA^{-1}) \\ &= \det(AA^*) = \det(A) \det(A^*) = \det(A) \det(A)^*. \\ \Rightarrow |\det A| &= 1. \end{aligned}$$

Thus, $SU(n)$ is an open submanifold of $U(n)$.

($\circ\circ$) $U^+(n) \cup U^-(n) = U(n)$. ($\dim U(n) = 1$).

$$U^+(n) = \det^{-1}(1) \cap U(n)$$

$\Rightarrow U^\pm(n) \equiv$ closed in $U(n)$

\equiv open in $U(n)$ (\because disjoint)

Hence, $SU(n)$ is a regular submanifold.

Thus, by Theorem 4.8, $SU(n)$ is a Lie subgroup of
 $GL(n, \mathbb{C})$ with $\dim = n^2 - 1$.

□

4.1.4. (i) Prove that

$$SU(2) = \left\{ \begin{pmatrix} z & -\bar{w} \\ w & \bar{z} \end{pmatrix} \in gl(2, \mathbb{C}) : z\bar{z} + w\bar{w} = 1 \right\}$$

(ii) Show that $SU(2)$ is diffeomorphic to S^3 .

proof) (i) Special unitary group.

In Example 4.13 (iii), $SU(n) = SL(n, \mathbb{C}) \cap U(n)$.

That is, $SU(n) = \{A \in gl(n, \mathbb{C}) : \det A = 1 \text{ & } A^*A = I_n\}$,

$A^* = (\bar{A})^T$ is the Hermitian conjugate of A .

Let $A = \begin{pmatrix} z & -\bar{w} \\ w & \bar{z} \end{pmatrix} \in gl(2, \mathbb{C})$, for any $z, w \in \mathbb{C}$.

$$\begin{cases} \det A = z\bar{z} - (-w\bar{w}) = z\bar{z} + w\bar{w} \\ A^* = \bar{A}^T = \begin{pmatrix} \bar{z} & \bar{w} \\ -\omega & \bar{z} \end{pmatrix} \end{cases}$$

$$A^*A = \begin{pmatrix} \bar{z} & \bar{w} \\ -\omega & \bar{z} \end{pmatrix} \begin{pmatrix} z & -\bar{w} \\ w & \bar{z} \end{pmatrix} = \begin{pmatrix} \bar{z}z + \bar{w}\omega & \bar{w}\bar{z} - \bar{z}\bar{w} \\ -w\bar{z} + z\bar{w} & w\bar{w} + \bar{z}\bar{z} \end{pmatrix}$$

$$= \begin{pmatrix} z\bar{z} + w\bar{w} & 0 \\ 0 & z\bar{z} + w\bar{w} \end{pmatrix}$$

Since $\bar{\alpha}\alpha = \alpha\bar{\alpha}$, $z_1\bar{z}_2 = \bar{z}_2z_1$ and $\bar{\bar{\alpha}} = \alpha$.

Thus, if $A \in SU(2)$, then $z\bar{z} + w\bar{w}$ must be 1, so that $A^*A = I_2$ and $\det A = 1$.

(ii) $SU(2) \cong S^3$ (diffeomorphic)

Since $\mathbb{R}^4 \cong \mathbb{C}^2$, $S^3 = \{(z, w) \in \mathbb{C}^2 : |z|^2 + |w|^2 = 1\}$.
 $(\because z\bar{z} = \bar{z}z = |z|^2)$.

Define a map $f : S^3 \rightarrow SU(2)$ by $f(z, w) = \begin{pmatrix} z & -\bar{w} \\ w & \bar{z} \end{pmatrix}$.

Then, f is well-defined since

$(z, w) \in S^3 \Rightarrow f(z, w) \in SU(2)$ clearly.

First, we now show that f is bijective.

① injective.

Clearly, $f(z_1, w_1) = f(z_2, w_2) \Rightarrow (z_1, w_1) = (z_2, w_2)$.

② surjective.

$f(S^3) = f(\{ |z|^2 + |w|^2 = 1 \}) = SU(2)$.

$\therefore f$ is bijective $\Rightarrow \exists$ inverse f^{-1} .

Note that $SU(2) \subseteq M(2, \mathbb{C}) \cong \mathbb{R}^8$.

Then, $SU(2)$ is a submanifold of $M(2, \mathbb{C}) \cong \mathbb{R}^8$.

Thus, if we define $F : \mathbb{R}^4 \rightarrow \mathbb{R}^8$,

f is just a restriction of F and

f, f^{-1} are smooth since F, F^{-1} are smooth,

and $S^3, SU(2)$ are submanifolds.

Therefore, definition 1.46, f is diffeomorphism. □

4.1.5. Prove that $SO(2)$ is diffeomorphic to S^1 .

proof) $SO(n) = SL(n, \mathbb{R}) \cap O(n)$

$$\Leftrightarrow \{ A \in gl(n, \mathbb{R}) : \det A = 1 \text{ and } A^T A = I_n \}.$$

In $SO(2)$, by calculation of $\det A$ and $A^T A$,

$$SO(2) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a^2 + c^2 = 1, b^2 + d^2 = 1, ad - bc = 1, ab + cd = 0 \right\}.$$

Let $a = x$, $b = -y$, $c = y$, $d = x$, then

all of the condition of $SO(2)$ is satisfied so that

we obtain \boxed{A}

$$SO(2) = \left\{ \begin{pmatrix} x & -y \\ y & x \end{pmatrix} \in gl(2, \mathbb{R}) : \det A = 1 \& A^T A = I_2 \right\}.$$

Define $f : S^1 \rightarrow SO(2)$ by $f(x, y) = \begin{pmatrix} x & -y \\ y & x \end{pmatrix}$.

Then, clearly, $(x, y) \in S^1 \Rightarrow f(x, y) \in SO(2)$ so
 f is well-defined.

Trivially, f is bijective, thus \exists inverse f^{-1} of f .

Since S^1 is a submanifold of \mathbb{R}^2 and $SO(2)$ is a
submanifold of $GL(2, \mathbb{R})$ by example 4.12,

Consider $F : \mathbb{R}^2 \rightarrow GL(2, \mathbb{R}) \cong \mathbb{R}^4$ be a diffeomorphism,
then f is just a restriction of F and so
 f, f^{-1} is smooth.

Therefore, by definition 1.16, f is a diffeomorphism.



4.2.1. Verify Example 4.27.

proof) ① $sl(n, \mathbb{C}) = \{X \in gl(n, \mathbb{C}) : \text{tr}(X) = 0\}$

pf) Let $X \in sl(n, \mathbb{C})$. Set $\gamma(s) = e^{sX}$, $s \in \mathbb{R}$.

Then, $\det(\gamma(s)) = \det(e^{sX}) = e^{\text{tr}(sX)} = e^0 = 1$.

$\Rightarrow \gamma(s)$ is a curve on $SL(n, \mathbb{C})$.

Since $\gamma(0) = I_n$ & $\gamma'(0) = X$, we have

$\gamma'(0) = X \in T_{I_n} SL(n, \mathbb{C}) \subset \text{Lie } SL(n, \mathbb{C})$.

Hence, $sl(n, \mathbb{C}) \subset \text{Lie } SL(n, \mathbb{C})$.

Since $\dim \text{Lie } SL(n, \mathbb{C}) = \dim sl(n, \mathbb{C}) = 2n^2 - 2$,

we conclude that $sl(n, \mathbb{C}) = \text{Lie } SL(n, \mathbb{C})$. Δ

② $U(n) = \{X \in gl(n, \mathbb{C}) : X^* + X = 0\}$

pf) For $X \in \text{Lie } U(n) \subset \text{Lie } gl(n, \mathbb{C}) = T GL(n, \mathbb{C}) = gl(n, \mathbb{C})$,

similar to let $\gamma : I \rightarrow U(n)$ be a curve such that

①,

$\dim U(n) = n^2$.

$\gamma(0) = I_n$, $\gamma'(0) = X$. Then $\gamma(s)^* \gamma(s) = I_n$ such that
 $\gamma(s) \in U(s)$. Hence,

$$X^* + X = \gamma'(0)^* \cdot \gamma(0) + \gamma(0)^* \cdot \gamma'(0) = 0.$$

So, $X \in U(n)$.

Conversely, suppose $X \in U(n)$. Set $\beta(s) = e^{sX}$ for $s \in \mathbb{R}$.

$$\text{Then, } \beta(s)^* \beta(s) = \overline{(e^{sX})^T} e^{sX}$$

$$= \overline{e^{sXT}} e^{sX} = e^{sX^*} e^{sX}$$

$$= e^{s(X^* + X)} \quad (\because X^* X = I_n = X X^*)$$

$$= I_n \quad (\because X^* + X = 0 \text{ in } U(n))$$

$\Rightarrow \beta(s)$ is a curve on $U(n)$. Since $\beta(0) = I_n$ and

$\beta'(0) = X$, we have $X \in T_{I_n} U(n) = \text{Lie } U(n)$. Δ

$$\textcircled{3} \quad \text{su}(n) = \{X \in \mathfrak{gl}(n, \mathbb{C}) : X^* + X = 0 \text{ & } \text{tr}(X) = 0\}.$$

if) Let $X \in \text{su}(n)$, set $\gamma(s) = e^{sX}$ for $s \in \mathbb{R}$.

By \textcircled{1}, \textcircled{2}, we can check that $\gamma : \mathbb{I} \rightarrow \text{su}(n)$,
 $\gamma(s) \in \text{SL}(n, \mathbb{C}) \cap \text{U}(n)$ such that $\gamma(0) = I_n$, $\gamma'(0) = X$.

$$\Rightarrow X \in T_{I_n} \text{SL}(n, \mathbb{C}) \cap \text{U}(n) \subset \text{Lie SU}(n).$$

Hence, $\text{su}(n) \subset \text{Lie SU}(n)$.

Conversely, if $X \in \text{Lie SU}(n)$, let $\beta : \mathbb{I} \rightarrow \text{SU}(n)$
such that $\beta(0) = I_n$, $\beta'(0) = X$. Then

$$\beta(s)^* \beta(s) = I_n \text{ and } \det(\beta(s)) = 1, \quad \beta(s) \in \text{SU}(n).$$

Then, $X^* + X = \beta'(0)^* \beta(0) + \beta(0)^* \beta'(0) = 0$ and
 $\text{tr}(X) = \text{tr}(\beta'(0)) = \det(\beta(0)) \text{tr}(\beta(0)^* \beta'(0))$

$$= \frac{d}{ds} \Big|_{s=0} \det \circ \beta'(s) = 0.$$

$$(\because \beta'(s) \in T_{\beta(s)} \text{SU}(n) = \text{SU}(n) \Rightarrow \det(\beta'(s)) = 1).$$

$\Rightarrow X \in \text{su}(n)$ and thus, $\text{su}(n) \supset \text{Lie SU}(n)$.

$\therefore \text{su}(n) = \text{Lie SU}(n)$.

Δ

\square

4.2.2. Prove that the Pauli matrices

$$E_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, E_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, E_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

are generators of $\text{su}(2)$.

proof) $A \in \text{su}(2) \Rightarrow A^* + A = 0$ and $\text{tr}(A) = 0$.

Let $A = \begin{bmatrix} x_1 + iy_1 & x_2 + iy_2 \\ x_3 + iy_3 & x_4 + iy_4 \end{bmatrix}$. Then

$$A^* = \begin{bmatrix} x_1 - iy_1 & x_3 - iy_3 \\ x_2 - iy_2 & x_4 - iy_4 \end{bmatrix} \text{ and } A^* + A \text{ is}$$

$$\begin{bmatrix} 2x_1 & x_2 + x_3 + i(y_2 - y_3) \\ x_2 + x_3 + i(y_3 - y_2) & 2x_4 \end{bmatrix} = 0, x_1 = x_4 = 0.$$

$$x_2 + x_3 = 0 \Rightarrow x_2 = -x_3, y_2 = y_3.$$

$$\text{tr}(A) = x_1 + iy_1 + x_4 + iy_4 = +i(y_1 + y_4) = 0$$

$$\Rightarrow y_1 = -y_4. \text{ Thus,}$$

$$A = \begin{bmatrix} iy_1 & x_2 + iy_2 \\ -x_2 + iy_2 & -iy_1 \end{bmatrix} \xrightarrow{\text{general}} \begin{bmatrix} ix & y + iz \\ -y + iz & -ix \end{bmatrix}, x, y, z \in \mathbb{R}.$$

$$\Rightarrow A = z \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix} + y \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} + x \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$$

$$:= z G_1 + y G_2 + x G_3 \text{ for any } x, y, z \in \mathbb{R}.$$

Then, we see that $G_1 = iE_1, G_2 = iE_2, G_3 = iE_3$.

Clearly, $\{G_1, G_2, G_3\}$ is linearly independent.

∴ $\{G_1, G_2, G_3\}$ is a basis for $\text{su}(2)$

∴ E_1, E_2, E_3 are generator of $\text{su}(2)$. □