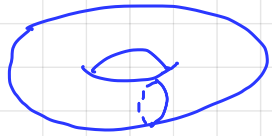
1.1.1 Show that if M^m, N^n are smooth manifolds, then $M^m \times N^n$ is also a $(m+n)$ dimensional smooth manifold. Hence, the n-dimensional torus or simply n-torus

$$T^n = \underbrace{S^1 \times \cdots \times S^1}_{n}$$

$T^2 =$  2-dim manifold

is a smooth manifold.

proof) We want to show :

[1] $M^m \times N^n$ is manifold [2] It is smooth manifold

[1] : ① Hausdorff

Since M^m, N^n are smooth manifold, these are Hausdorff. Then, for any U_M, V_M in M^m and U_N, V_N in N^n, let $U_M \times U_N = U \subset M^m \times N^n$ and $V_M \times V_N \subset N^n$, then $U_M \times U_N \cap V_M \times V_N = \phi$. ⋯ (*)

(∵)

(1) : $(U_M \times U_N) \cap V_M = V_M \cap U_M \times V_M \cap U_N = \phi$

because $V_M \cap U_M = \phi$ since M^m is Hausdorff.

(2) : $(U_M \times U_N) \cap V_N = V_N \cap U_M \times V_N \cap U_N = \phi$

because $V_N \cap U_N = \phi$ since N^n is Hausdorff.

(*) $= (1) \times (2) = \phi$.

Thus, $M^m \times N^n$ is Hausdorff.

② second countable

By the assumption, M^m, N^n have a countable basis $\mathcal{B}_M, \mathcal{B}_N$. Then trivially $\beta_M \times \beta_N \subset M^m \times N^n$ and we can pick

$\beta_M \times \beta_N$ is a countable basis for $M^m \times N^n$.

(∵) $x \in \beta_M$ & $y \in \beta_N \Rightarrow (x, y) \in \beta_M \times \beta_N \subset M^m \times N^n$

and β_M, β_N : open $\Rightarrow \beta_M \times \beta_N$: open.

$\beta_M \times \beta_N$: countable since β_M, β_N are countable.

pf) Let $\beta_M \times \beta_N$: finite $\rightarrow$ trivial.

We assume β_M, β_N : countably infinite.

$$
\begin{array}{cccc}
(\beta_M^0, \beta_N^0) & (\beta_M^0, \beta_N^1) & (\beta_M^0, \beta_N^2) & \cdots \\
(\beta_M^1, \beta_N^0) & (\beta_M^1, \beta_N^1) & (\beta_M^1, \beta_N^2) & \cdots \\
(\beta_M^2, \beta_N^0) & (\beta_M^2, \beta_N^1) & (\beta_M^2, \beta_N^2) & \cdots \\
\vdots & \vdots & \vdots & \ddots
\end{array}
$$

First, we pick (β_M^0, β_N^0), then we pick

$(\beta_M^0, \beta_N^1), (\beta_M^1, \beta_N^0)$, then we pick

$(\beta_M^0, \beta_N^2), (\beta_M^1, \beta_N^1)\ (\beta_M^2, \beta_N^0) \cdots\cdots$

Continue to this processes, then we can

define the one-to-one correspondence

between $\beta_M \times \beta_N \rightarrow \mathbb{N}$ (the set of natural #).

Thus, by definition of countable,

the assertion is proved.

③ Homeomorphism

Let $\varphi_M : U \rightarrow \mathbb{R}^m$ & $p \in \varphi_M(U)$ and

$\varphi_N : V \rightarrow \mathbb{R}^n$ & $q \in \varphi_N(V)$, then we can define

$\varphi_{MN}(r) = (\varphi_M \times \varphi_N)(p, q) = (\varphi_M(p), \varphi_N(q))$

if $\varphi_{MN} : U \times V \to \mathbb{R}^{m+n}$.

(i) injective

$\varphi_{MN}(r_1) = \varphi_{MN}(r_2)$

$\Rightarrow (\varphi_M(p_1), \varphi_N(q_1)) = (\varphi_M(p_2), \varphi_N(q_2))$

$\Rightarrow \varphi_M(p_1) = \varphi_M(p_2)$ & $\varphi_N(q_1) = \varphi_N(q_2)$

$\Rightarrow p_1 = p_2$ & $q_1 = q_2$ since φ_M, φ_N : injective.

(ii) surjective

For $\forall y = \varphi_{MN}(\bar{r}) \in \mathbb{R}^{m+n}$, $\exists (\bar{p}, \bar{q}) \in U \times V$ s.t.

$y = \varphi_{MN}(\bar{r}) = (\varphi_M(\bar{p}), \varphi_N(\bar{q}))$ since

φ_M & φ_N are surjective.

By (i), (ii), φ_{MN} : bijection on $U \times V \subset M^m \times N^n$.

Thus, $\exists \varphi_{MN}^{-1}$: inverse of φ_{MN}.

In case of continuity, for any θ, β be open in $\mathbb{R}^m, \mathbb{R}^n$,

$\varphi_M^{-1}(\theta), \varphi_N^{-1}(\beta)$ are open by the assumption.

Since $\theta \times \beta$: open and its pre image

$\varphi_{MN}^{-1}(\theta \times \beta)$ is open.

$\therefore$ φ_{MN} is continuous.

($\because$) if $\theta = \varphi_M(\alpha)$, $\beta = \varphi_N(\beta)$ for any open sets α, β in U, V, then

$\varphi_{MN}^{-1}(\varphi_{MN}(\theta \times \beta)) = \theta \times \beta = \varphi_M(\alpha) \times \varphi_N(\beta)$: open.

For φ_{MN}^{-1} : inverse of φ_{MN}, $\varphi_{MN}^{-1}(\mathcal{O}\times\beta)$ is open

$\Rightarrow \varphi_{MN}(\varphi_{MN}^{-1}(\mathcal{O}\times\beta)) = \mathcal{O}\times\beta$ is open

$\therefore$ φ_{MN}^{-1} is continuous.

$\therefore$ φ_{MN} is Homeomorphism.

Therefore, $M^m \times N^n$ is a manifold.

[2] : Since M^m, N^n are smooth manifold, they have a C^∞-structure, so that the coordinate charts (U, φ_M), (V, φ_N) is C^∞-compatible with all charts in the atlas of M^m, N^n, respectively.

By [1], we defined the homeomorphism φ_{MN}, hence we can write the coordinate chart of $M^m \times N^n$ that $(U\times V, \varphi_{MN})$. Consider another chart $(U'\times V', \varphi_{MN}^*)$, then

$$\varphi_{MN} \circ \varphi_{MN}^* = (\varphi_M \times \varphi_N)\circ(\varphi_M^* \times \varphi_N^*)^{-1}$$

$$= \varphi_M \circ \varphi_M^{*-1} \times \varphi_N \circ \varphi_N^{*-1}.$$

Since φ_M, φ_N, φ_M^{*-1}, φ_N^{*-1} are C^∞,

$\varphi_{MN} \circ \varphi_{MN}^*$ is C^∞.

$\therefore$ $M^m \times N^n$ is a smooth manifold.

Thus, By the proof above, T^n is smooth manifold. $\square$

1.1.2 Let $U \subset \mathbb{R}^n$ be open and $f : U \to \mathbb{R}^m$ be continuous. Show that the graph of f

$$\Gamma_f = \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^m : x \in U \text{ and } y = f(x)\}$$

is an n-dimensional manifold.

proof) By the example 1.7. (i) of the lecture note of professor Han, $\mathbb{R}^n$, $\mathbb{R}^m$ are n, m dimensional smooth manifold and hence $\mathbb{R}^n \times \mathbb{R}^m$ is smooth manifold by the exercise 1.1.1.

Thus, the graph of f Γ_f is the subspace topology of $\mathbb{R}^n \times \mathbb{R}^m$. Hence, Γ_f is Hausdorff and 2nd-countable space.

So, we want to show that Γ_f has the locally Euclidean property only.

Let $\pi_x : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is the projection onto x, and let $\varphi : \Gamma_f \to U$ be the restriction of π_x to Γ_f that $\varphi(x, y) = x$, $(x, y) \in \Gamma_f$.

Since π_x is continuous (clearly),

the restriction of π_x φ is continuous, and bijective also. Thus $\exists\, \varphi^{-1}$: inverse of φ and since $\varphi^{-1}(x) = (x, f(x))$, φ^{-1} is continuous.

$\therefore$ φ : Homeomorphism.

$\therefore$ Γ_f is n-dimensional manifold. $\square$

1.2.1 Complete the proof of proposition 1.14 :
Suppose that $\pi : M \to M/\sim$ is an open map. Then
(ii) $M/\sim$ is Hausdorff $\Rightarrow$ $R = \{(p,q) : p \sim q\}$ is closed in $M \times M$.

proof) Note that :

$$[x]_\sim = \{x \in M : x \sim \alpha,\ \alpha \in M\}.$$

$$M/\sim = \{[x]_\sim : x \in M\}$$

$\mathcal{O} \subset M/\sim$ is open $\Leftrightarrow$ $\pi^{-1}(\mathcal{O}) = \{x : \pi(x) = [x] \in \mathcal{O}\}$ is open in M.

Assume that $M/\sim$ is Hausdorff.

Claim : $R \subset M \times M$ is closed

$\Leftrightarrow$ $M \times M - R$ is open.

Let $(p,q) \in M \times M - R$, then $\pi(p) \neq \pi(q)$ $\Rightarrow$ $(p,q) \notin R$. Thus we can take the disjoint open sets $\pi(p) \in U_1$, $\pi(q) \in U_2$ since $M/\sim$ is Hausdorff.

Let $V_1 = \pi^{-1}(U_1)$ & $V_2 = \pi^{-1}(U_2)$.

If $(V_1 \times V_2) \cap R \neq \phi$, then $^{\exists}(v_1, v_2) \in V_1 \times V_2$ such that $\pi(v_1) = \pi(v_2)$, $\pi(v_1) \in U_1$, $\pi(v_2) \in U_2$.

But, $U_1 \cap U_2 = \phi$, that is contradiction.

$\therefore$ R is closed in $M \times M$. $\square$

1.2.2 Let $f : S^n \to S^n$ be the antipodal map defined by $f(x) = -x$. Define an relation $\sim$ on S^n by $x \sim y$ iff $y = x$ or $y = f(x)$. Show that $\sim$ is an equivalence relation and $S^n/\sim \; = \mathbb{R}P^n$.

proof) ① Equivalence relation

「 $x \sim y \iff y = x \iff y - x = 0$ 」

(i) $x \sim x$ since $x - x = 0$

(ii) if $x \sim y$, then

$$y = x \iff y - x = 0 \iff -(x - y) = 0$$
$$\iff x - y = 0 \iff x = y$$
$$\iff y \sim x.$$

$$y = -x \iff -y = x \iff x = f(y)$$
$$\iff y \sim x$$

(iii) if $x \sim y$ & $y \sim z$, then

$y = x$ and $z = y$ and so

$$z = y = x \iff x \sim z.$$

$y = -x$ & $z = -y$, then

$$z = -y = -(-x) = x \iff x \sim z.$$

② $S^n/\sim \; = \mathbb{R}P^n$

$$[x]_M = \bigcup_i [x_i]_{S^n}. \quad (\because [x]_{S^n} \subset [x]_M)$$

$$\Rightarrow \bigcup_j \left(\bigcup_i [x_i]_{S^n} \right)_j = M/\sim \; = \mathbb{R}P^n$$

For arbitrary $r \in \mathbb{R}^{n+1} - \{0\}$, we can let

$S^n = \{ \vec{x} : \|\vec{x}\| = r \}$, and thus

$S^n/\sim \; = \mathbb{R}P^n$.

(Additional Information i thought)

$$\left\{ \begin{array}{l} S^n = \{ (x_1, \cdots, x_{n+1}) : x_1^2 + \cdots + x_{n+1}^2 = r^2 \}. \\ (0, 0, \cdots, 0) \notin S^n \subset M = \mathbb{R}^{n+1} - \{0\}. \\ [x]_M = \{ x \in M : x \sim y \iff y = tx \text{ for some } t \neq 0 \} \\ [x]_{S^n} = \{ x \in S^n : x \sim y \iff y = \pm x \} \\ \Rightarrow \quad [x]_{S^n} \subset [x]_M. \quad (\because [x]_{S^n} = \{-x, x\}) \\ \Rightarrow [x]_{S^n} \in M/\sim \; = \mathbb{R}P^n. \\ \therefore \quad S^n/\sim \; \subset M/\sim \; = \mathbb{R}P^n. \end{array} \right.$$

$\square$

1.2.3. The complex projective space $\mathbb{C}P^n$ is the set of all line through the origin in $\mathbb{C}^{n+1}$, i.e., the set of 1-dimensional subspaces of $\mathbb{C}^{n+1}$. If we define an equivalence relation on $M = \mathbb{C}^{n+1} - \{0\}$ by $z \sim w \Leftrightarrow w = \lambda z$ for some $\lambda \in \mathbb{C}^*$, then $\mathbb{C}P^n = M/\sim$. Show that $\mathbb{C}P^n$ is a 2n-dimensional smooth manifold.

proof) ① 2nd-countable

Since M is 2nd-countable, the quotient set of M is 2nd-countable.

② Hausdorff.

$[z_1], [z_2] \in U_j$ for some j

$\Rightarrow [z_1]$ and $[z_2]$ are disjoint open set,

$(\because)$ $\varphi_j(z_1), \varphi_j(z_2) \in \mathbb{C}^n$.

<u>Claim</u>: $\nexists U_j$ containing $[z_1]$ & $[z_2]$.

Given $j \neq k$, let

$A_{j,k} = \{[z] : |z^j| > |z^k|\} \subset \mathbb{C}P^n$.

Then $A_{j,k}$ is open since

$\pi^{-1}(A_{j,k})$ is open in $\mathbb{C}^{n+1} - \{0\}$.

By the assumption, $\exists j \neq k$ s.t.

$[z_1] \in U_j$ and $[z_2] \in U_k$, but

$z_1^j = z_2^k = 0$.

$\therefore$ $z_1 \in A_{j,k}$, $z_2 \in A_{k,j}$.

$\therefore A_{j,k} \cap A_{k,j} = \phi$.

③ local Euclidean

For $z = (z^0, \cdots, z^n) \in \mathbb{C}^{n+1}$, define

$U_i = \{[z] : z^i \neq 0\} \subset \mathbb{CP}^n$, then we can

define $\varphi_i : U_i \to \mathbb{C}^n$ by

$$\varphi_i([z]) = \left(\frac{z^0}{z^i}, \cdots, \frac{z^{i-1}}{z^i}, \frac{z^{i+1}}{z^i}, \cdots, \frac{z^n}{z^i}\right).$$

continuous
$\left\{\begin{array}{l} \text{For a projection } \pi : M \to M/\sim \text{ by } \pi(z) = [z], \\ \varphi_i^{-1}(V) \text{ is open for any open } V \subset \mathbb{C}^n \\ \Leftrightarrow \pi^{-1} \circ \varphi_i^{-1}(V) = (\varphi_i \circ \pi)^{-1}(V) \text{ is open in } \mathbb{C}^{n+1}. \\ \text{Since } \varphi_i \circ \pi \text{ is clearly continuous, } (\because z^i \neq 0) \\ (\varphi_i \circ \pi)^{-1}(V) \text{ is open in } \mathbb{C}^{n+1}. \end{array}\right.$

① injective

$$\varphi_i([z_1]) = \varphi_i([z_2]) \Rightarrow \frac{z_1^j}{z_2^j} = \frac{z_1^i}{z_2^i}, \quad j \neq i$$

$$\Rightarrow [z_1] = [z_2]. \quad (\because z_1^j = t z_2^j \text{ for } \forall j)$$

②
$v = (v^1, \cdots, v^n) \in \mathbb{C}^n$, then

$$\varphi_i^{-1}(v) = \pi(v^1, \cdots, v^{i-1}, 1, v^i, \cdots, v^n).$$

Since π is continuous, φ_i^{-1} is continuous.

$(\because)$ $\varphi_i(\pi(v^1, \cdots, v^{i-1}, 1, v^i, \cdots, v^n))$

$$= \left(\frac{v^1}{1}, \cdots, \frac{v^{i-1}}{1}, \frac{v^{i+1}}{1}, \cdots, \frac{v^n}{1}\right) = v$$

By ①, ②, φ_i is a homeomorphism.

③ assuming w.r.o.g, $i < j$, the transition maps

$$\varphi_j \circ \varphi_i^{-1} : \varphi(U_i \cap U_j) = \{ z = (z^1, \cdots, z^n) \in \mathbb{C}^n : z^j \neq 0 \}$$

$$\to \varphi(U_i \cap U_j)$$

$$\varphi_j \circ \varphi_i^{-1}(z^1, \cdots, z^n) = \varphi_j([(z^1, \cdots, z^{i-1}, 1, z^{i+1}, \cdots, z^n)])$$

$$= \left(\frac{z^1}{z^j}, \cdots, \frac{z^i}{z^j}, \frac{1}{z^i}, \frac{z^{i+1}}{z^j}, \cdots, \frac{z^{j-1}}{z^j}, \frac{z^{j+1}}{z^j}, \cdots, \frac{z^n}{z^j} \right)$$

is smooth,

Thus, for $\{(U_i, \varphi_i) \mid i = 1, \cdots, n+1\} = \mathcal{A}$,

$\mathcal{A}$ is C^∞ atlas.

$\therefore$ $\mathbb{CP}^n$ is a $2n$-dimensional smooth manifold.

$\square$

1.3.1. Let $N = M = \mathbb{R}P^1$ and write a point in $\mathbb{R}P^1$ as $[(x,y)]$ for $(x,y) \in \mathbb{R}^2$. Show that the map $F: N \to M$ given by $F([(x,y)]) = [(x^2, y^2)]$ is smooth.

proof) For $\{(U, \varphi)\} = \mathcal{A}_N$ and $\{(V, \psi)\} = \mathcal{A}_M$,

let $\varphi = \psi = \pi^{-1}$, i.e.

$\varphi: N \to \mathbb{R}^2$ by $\varphi([[(x,y)]]) = (x,y)$

$\psi: M \to \mathbb{R}^2$ by $\psi([x,y]) = (x,y)$

Then, These can be a homeomorphism.

Thus,

$$f \equiv \psi \circ F \circ \varphi^{-1} = \psi(F([(x,y)]))$$

$$\mathbb{R}^2 \to \mathbb{R}^2 \qquad = \psi([(x^2,y^2)]) = (x^2, y^2).$$

since the components of f is smooth,

$f = \psi \circ F \circ \varphi^{-1}$ is smooth, and therefore,

$F: N \to M$ is smooth mapping.

$\square$

1.3.2 Prove that (ii) of Theorem 1.28.

proof) For each $q \in A$, $\exists (\varphi, U)$ near q
that $U_q \subset U$ & $\varphi(U_q) \subset V$,
$V = \{$ Contains the open ball $B_3(0)$ $\}$.
$B_3(0) \equiv$ open ball of radius 3 centered at 0.

Let $\tilde{U}_q = \varphi^{-1}(B_1(0))$ and
define the function called "Bump function" that

$$g(t) = \begin{cases} 1 & \text{for } t \le 1 \\ 0 & \text{for } t \ge 2 \end{cases},$$

and let

$$f(p) = \begin{cases} g(\varphi(p)) & p \in U_q \\ 0 & p \notin U_q \end{cases}.$$

Then, $f \in C^\infty(M)$ such that $0 \le f \le 1$,
$f \equiv 1$ on $\tilde{U}_q \subset A$ and $\mathrm{supp}(f) \subset U_p \subset U$. ■

2.1.1 For a smooth map $F : N^n \to M^m$, the push forward $F_{*p} : T_pN \to T_{F(p)}M$ at $p \in N$ was defined in terms of derivations. This can be also defined by the equivalence class of curves as

$$F_{*p}([\gamma]) = [F \circ \gamma].$$

Show that this definition is well defined. In other words, $\gamma_1 \sim \gamma_2$ implies that $F \circ \gamma_1 \sim F \circ \gamma_2$.

proof) For a coordinate chart (U, φ) at p,

$$\gamma_1 \sim \gamma_2 \iff (\varphi \circ \gamma_1)'(0) = (\varphi \circ \gamma_2)'(0).$$

$F \circ \gamma = F \circ \varphi^{-1} \circ \varphi \circ \gamma$ since φ : homeomorphism.

$$\begin{aligned}
\Rightarrow (F \circ \gamma_1)'(0) &= (F \circ \varphi^{-1} \circ \varphi \circ \gamma_1)'(0) \\
&= (F \circ \varphi^{-1})'(\varphi \circ \gamma_1)(0). \\
(\because \gamma_1 \sim \gamma_2) \quad &= (F \circ \varphi^{-1})'(\varphi \circ \gamma_2)(0) \\
&= (F \circ \varphi^{-1} \circ \varphi \circ \gamma_2)'(0) \\
&= (F \circ \gamma_2)'(0)
\end{aligned}$$

$\iff F \circ \gamma_1 \sim F \circ \gamma_2$ $\square$

2.1.2. Let $F : N^n \to M^m$ be a smooth map.
Prove that if N is connected and $F_{*p} = 0$ for any $p \in N$, then F is constant map.

proof) Let $f \in C^\infty(M)$ and let $X_p \in T_pN$.
By the assumption, $F_{*p}[f] = X_p(f \circ F) = 0$.

Let (U, φ) : smooth chart containing p. Then

$$X_p = \sum_i X_p^i \frac{\partial}{\partial x^i}\Big|_p = \sum_i X^i (\varphi^{-1})_{*p} \frac{\partial}{\partial x^i}\Big|_{\varphi(p)}$$

$$\Rightarrow \left(\sum_i X^i (\varphi^{-1})_{*p} \frac{\partial}{\partial x^i}\Big|_{\varphi(p)}\right)(f \circ F) = \sum_i X^i \frac{\partial}{\partial x^i}\Big|_{\varphi(p)} (f \circ F \circ \varphi^{-1}) = 0$$

$\Rightarrow$ F is constant on U.

Since N is connected, N : path connected.
Let $q \in N$ & let $\gamma : [0,1] \to N$ be a path connecting p & q.
Since F is constant on each smooth chart $(U_{\gamma(x)}, \varphi_{\gamma(x)})$ containing $\gamma(x)$ for every $x \in [0,1]$, $F \equiv c$ on N since $F(p) = c$ & γ is continuous. $\square$

2.1.3. Prove that for any $p \in S^n$,

$$T_p S^n = \{ X \in \mathbb{R}^{n+1} : \langle p, X \rangle = 0 \}.$$

proof) Let $p \in S^n \subset \mathbb{R}^{n+1} - \{0\}$ and let $X \in \mathbb{R}^{n+1}$.

By the example 2.2, $T_p \mathbb{R}^m = \mathbb{R}^m$.

$m = n+1 \Rightarrow T_p \mathbb{R}^{n+1} = \mathbb{R}^{n+1} \supset S^n$.

$[X]_{S^n} = \{ X \in \mathbb{R}^{n+1} : X \sim p \Leftrightarrow p = tX \text{ for some } t \neq 0 \}$.

$\langle p, X \rangle = 0 \Rightarrow p = -X$.

Define

$$\gamma_{p,X}(t) = \begin{cases} p + tX & t \neq 1 \\ p & t = 1 \end{cases}.$$

Then, $\frac{d}{dt} \gamma_{p,X}(t) \Big|_{t=0} = p + X$, $\gamma_{p,X}(0) = p = -X$

$\Rightarrow [\gamma_{p,X}] \in T_p S^n$.

$\therefore T_p S^n = \{ X \in \mathbb{R}^{n+1} : \langle p, X \rangle = 0 \}$. ■

2.3.1. Let $F : \mathbb{R}^2 \to \mathbb{R}^2$ such that $F(x, y) = (a^2 - 2y,\ 4x^3y^2)$. For $X = 4x\frac{\partial}{\partial x} + 3y^2\frac{\partial}{\partial y}$, compute $F_* X$.

proof) For the vector field $X = 4x\frac{\partial}{\partial x} + 3y^2\frac{\partial}{\partial y}$,

let $x' = a^2 - 2y$ & $y' = 4x^3y^2$, then

$$F_*\left(\frac{\partial}{\partial x}\right) = \frac{\partial x'}{\partial x}\frac{\partial}{\partial x} + \frac{\partial y'}{\partial x}\frac{\partial}{\partial y} = 12x^2y^2\frac{\partial}{\partial y}$$

$$F_*\left(\frac{\partial}{\partial y}\right) = \frac{\partial x'}{\partial y}\frac{\partial}{\partial x} + \frac{\partial y'}{\partial y}\frac{\partial}{\partial y} = -2\frac{\partial}{\partial x} + 8x^3y\frac{\partial}{\partial y}$$

$$\begin{aligned} F_* X &= 4x\, F_*\left(\frac{\partial}{\partial x}\right) + 3y^2\, F_*\left(\frac{\partial}{\partial y}\right) \\ &= -6y^2\frac{\partial}{\partial x} + x^3y^2(24y + 48)\frac{\partial}{\partial y} \end{aligned}$$

$\square$

2.3.2. Express the following planar vector fields in polar coordinates.

$$X = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}, \quad Y = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}.$$

What is $[X, Y]$?

proof) Let $x = r\cos\theta$, $y = r\sin\theta$, then

$$\frac{\partial}{\partial r} = \frac{\partial x}{\partial r}\frac{\partial}{\partial x} + \frac{\partial y}{\partial r}\frac{\partial}{\partial y} = \cos\theta\frac{\partial}{\partial x} + \sin\theta\frac{\partial}{\partial y}$$

$$= \frac{x}{r}\frac{\partial}{\partial x} + \frac{y}{r}\frac{\partial}{\partial y} = \frac{1}{r}X$$

$$\frac{\partial}{\partial \theta} = \frac{\partial x}{\partial \theta}\frac{\partial}{\partial x} + \frac{\partial y}{\partial \theta}\frac{\partial}{\partial y} = -r\sin\theta\frac{\partial}{\partial x} + r\cos\theta\frac{\partial}{\partial y}$$

$$= -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y} = Y$$

$$\therefore X = r\frac{\partial}{\partial r}, \quad Y = \frac{\partial}{\partial \theta}.$$

By definition 2.21.(iii), $[X, Y] = \left[r\frac{\partial}{\partial r}, \frac{\partial}{\partial \theta}\right] = 0$.

professor Han's note

$\square$

2.3.3. In $\mathbb{R}^3$, let

$$X = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x} \quad \text{and} \quad Y = y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}.$$

Compute $[X, Y]$.

proof) By definition 2.22, we have

$$\begin{aligned}[X, Y] &= \left[x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x},\ y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right] \\ &= \left[x\frac{\partial}{\partial y},\ y\frac{\partial}{\partial z}\right]^{①} + \left[x\frac{\partial}{\partial y},\ -z\frac{\partial}{\partial y}\right]^{②} + \\ &\quad \left[-y\frac{\partial}{\partial x},\ y\frac{\partial}{\partial z}\right]^{③} + \left[-y\frac{\partial}{\partial x},\ -z\frac{\partial}{\partial y}\right]^{④}\end{aligned}$$

By definition 2.21 (iii), we have

① $= x \cdot 0 \frac{\partial}{\partial z} + y \cdot 0 \frac{\partial}{\partial y} = 0$

② $= x \cdot 0 \frac{\partial}{\partial y} + (-z) \cdot 0 \frac{\partial}{\partial y} = 0$

③ $= -y \cdot 0 \frac{\partial}{\partial z} + y \cdot 0 \frac{\partial}{\partial x} = 0$

④ $= -y \cdot 0 \frac{\partial}{\partial y} - z \cdot 0 \frac{\partial}{\partial x} = 0$

$\therefore [X, Y] = 0$ $\square$

2.3.4. Verify Example 2.23.

proof) (i) $\mathbb{R}^n$ is a Lie algebra.

Since $[a,b] = (a+b)-(b+a) = 0$,

bilinear & skew summetric satisfied.

Check Jacobi identity condition. For $c \in \mathbb{R}^n$,

$$[a,[b,c]]+[b,[c,a]]+[c,[a,b]]$$

$$= a+[b,c]-([b,c]+a) + b + [c,a]-([c,a]+b)$$
$$+ c+[a,b]-([a,b]+c)$$

$$= a+0-(0+a)+b+0-(0+b)+c+0-(0+c)$$

$$= a-a+b-b+c-c = 0.$$

$\therefore$ $\mathbb{R}^n$ is a Lie algebra.

(ii) $GL(n,\mathbb{R})$ is a Lie algebra.

Check only the condition for Jacobi.

$[A,B] = AB-BA$ for $A, B \in GL(n,\mathbb{R})$,

pick $C \in GL(n,\mathbb{R})$, then

$$[A,[B,C]]+[B,[C,A]]+[C,[A,B]]$$

$$= A([B,C])-([B,C]A) + B([C,A])-([C,A]B)$$
$$+ C([A,B])-([A,B]C)$$

$$= A(BC-CB)-(BC-CB)A+B(CA-AC)-(CA-AC)B$$
$$+ C(AB-BA)-(AB-BA)C$$

$$= ABC-ACB-BCA+CBA+BCA-BAC-CAB+ACB$$
$$+CAB-CBA-ABC+BAC$$

$= 0$. $\therefore$ $GL(n,\mathbb{R})$ is a Lie algebra.

(iii) $\mathbb{R}^3$ is a Lie algebra with $[u,v] = u\times v$, $u,v\in\mathbb{R}^3$

Cross product satisfy the skew symmetric condition

($\because$ $u\times v = -v\times u$). and bilinear condition also.

($\because$ Let $u,v,w\in\mathbb{R}^3$, i,j,k : standard basis of $\mathbb{R}^3$.

$u = u_1 i + u_2 j + u_3 k$, $v = v_1 i + v_2 j + v_3 k$,

$w = w_1 i + w_2 j + w_3 k$. Then, for $c\in\mathbb{R}$,

$(cu+v)\times w = c(u\times w) + v\times w$.

pf)

$$(cu+v)\times w = \begin{vmatrix} i & j & k \\ cu_1+v_1 & cu_2+v_2 & cu_3+v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$

$$= \begin{vmatrix} i & j & k \\ cu_1 & cu_2 & cu_3 \\ w_1 & w_2 & w_3 \end{vmatrix} + \begin{vmatrix} i & j & k \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$

$$= c\begin{vmatrix} i & j & k \\ u_1 & u_2 & u_3 \\ w_1 & w_2 & w_3 \end{vmatrix} + \begin{vmatrix} i & j & k \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$

$= c(u\times w) + v\times w$. Q.E.D)

Check Jacobi identity condition. For $u,v,w\in\mathbb{R}^3$,

$[u,[v,w]] + [v,[w,u]] + [w,[u,v]]$

$= u\times(v\times w) + v\times(w\times u) + w\times(u\times v)$

$= (u\cdot w)v - (u\cdot v)w + (v\cdot u)w - (v\cdot w)u$

$\quad + (w\cdot v)u - (w\cdot u)v$

$= 0$.

$\therefore$ $\mathbb{R}^3$ is a Lie algebra with $[u,v] = u\times v$ for $u,v\in\mathbb{R}^3$.

(iv) $\mathfrak{G}, \mathfrak{h}$ are Lie algebras $\Rightarrow \mathfrak{G} \times \mathfrak{h}$ is also a Lie algebra under the bracket

$$[(X_1, Y_1), (X_2, Y_2)] = ([X_1, Y_1], [X_2, Y_2]).$$

① Bilinearity

Clearly we obtain the property after complicate calculation.

Note that $[X_1, Y_1], [X_2, Y_2]$ are satisfy the bilinearity in $\mathfrak{G}, \mathfrak{h}$, respectively.

② skew - symmetric.

For the simplicity, denote $X_1, Y_1 \equiv x_1, y_1$.

$$\begin{aligned} [(x_1, y_1), (x_2, y_2)] &= ([x_1, y_1], [x_2, y_2]) \\ \mathfrak{G}, \mathfrak{h} : \text{Lie algebra} \longrightarrow &= (-[y_1, x_1], -[y_2, x_2]) \\ &= -([y_1, x_1], [y_2, x_2]) \\ &= -[(y_1, x_1), (y_2, x_2)] \end{aligned}$$

③ Jacobi identity

$$\begin{aligned} &[(x_1, y_1), [(x_2, y_2), (x_3, y_3)]] \\ &+ [(x_2, y_2), [(x_3, y_3), (x_1, y_1)]] \\ &+ [(x_3, y_3), [(x_1, y_1), (x_2, y_2)]] = 0 \end{aligned}$$

using the previous results we proved and our Lie bracket. $\square$

2.3.5. Prove Theorem 2.24.

proof) Check the Jacobi identity.

For the simplicity, denote $X, Y, \cdots \equiv x, y, \cdots$.

$[x,[y,z]] + [y,[z,x]] + [z,[x,y]]$

$= x(yz - zy) - (yz - zy)x + y(zx - xz) - (zx - xz)y$

$\quad + z(xy - yx) - (xy - yx)z$

$= xyz - xzy - yzx + zyx + yzx - yxz - zxy + xzy$

$\quad + zxy - zyx - xyz + yxz$

$= 0.$ $\square$

2.3.6. Prove Theorem 2.26.

proof) By the assumption, X_i, Y_i are F-related.

i.e. $F_*(X_i) = Y_i$ by definition 2.25.

<u>Claim</u> : $F_*([X_1, X_2]) = [Y_1, Y_2]$.

Choose $g \in C^\infty(M)$ and $x \in N$, then

$$(Y_i g)(F(x)) = (F_*)_x (X_i)(g) = X_i(g \circ F)$$

Thus, $(Y_i g) \circ F = X_i(g \circ F)$ ⋯ (*)

Let $f \in C^\infty(M)$ be arbitrary. Using (*)

$\Rightarrow Y_1(Y_2 f) \circ F = X_1((Y_2 f) \circ F)$. ⋯ (**)

By (*), we also obtain

$$(Y_2 f) \circ F = X_2(f \circ F) \text{ and thus}$$

$(**) = Y_1(Y_2 f) \circ F = X_1(X_2(f \circ F))$.

Likewise, we get

$$Y_2(Y_1 f) \circ F = X_2(X_1(f \circ F)).$$

Hence, $([Y_1, Y_2] f) \circ F = [X_1, X_2](f \circ F)$.

Therefore, $[Y_1, Y_2]$ is F-related to $[X_1, X_2]$ $\square$

2.3.7. Let $F: N \longrightarrow M$ be a diffeomorphism. Prove that for any $Y \in \mathfrak{X}(M)$, there is a unique $X \in \mathfrak{X}(N)$ such that X is F-related to Y.

proof) Assume that X is F-related to Y.

i.e. $X_{F(p)} = F_{*p}(Y_p)$.

If F is a diffeomorphism, we define X by

$$X_q = F_{*F^{-1}(q)}(Y_{F^{-1}(q)})$$

Then, it is clear that X is the unique vector field such that F-related to Y.

「 Note that

$X : N \to TN$, N : manifold, TN : tangent bundle.

Then X is the composition that

$$N \xrightarrow{F^{-1}} M \xrightarrow{X} TM \xrightarrow{dF} TN$$

$\Rightarrow$ X is smooth. 」

$\square$

2.3.8. Express the planar 1-form $\omega = x\,dx + y\,dy$ in polar coordinates.

proof) Let $x = r\cos\theta$ and $y = r\sin\theta$.

By Chain rule,

$$dx = \frac{\partial x}{\partial r} dr + \frac{\partial x}{\partial \theta} d\theta = \cos\theta\, dr - r\sin\theta\, d\theta$$

$$dy = \frac{\partial y}{\partial r} dr + \frac{\partial y}{\partial \theta} d\theta = \sin\theta\, dr + r\cos\theta\, d\theta$$

The differential 1-form ω is expressed by

$$\omega = \left(\frac{\partial x}{\partial r} x + \frac{\partial y}{\partial r} y\right) dr + \left(\frac{\partial x}{\partial \theta} x + \frac{\partial y}{\partial \theta} y\right) d\theta$$

$$= \left(r\cos^2\theta + r\sin^2\theta\right) dr + \left(-r^2\sin\theta\cos\theta + r^2\cos\theta\sin\theta\right) d\theta$$

$$= r\,dr + 0\,d\theta$$

$$= r\,dr.$$

$\blacksquare$

3.2.1. Let $M = \{(x,y,z) \in \mathbb{R}^3 : x^3+y^3+z^3-3xyz = 1\}$.
Prove that M is a 2-dimensional regular submanifold of $\mathbb{R}^3$. What is T_pM at $p=(0,0,1)$?

proof) Let $F : \mathbb{R}^3 \to \mathbb{R}$ be defined by

$$F(x,y,z) = x^3+y^3+z^3-3xyz-1.$$

Note that :

<u>Definition</u>

The rank of a smooth map $f : N \to M$ between two manifolds at a point $p \in N$ is the rank of the derivative of f at p.

For $q = 0 \in \mathbb{R}$, $F^{-1}(q) = M$ and Jacobian is

$$DF = F_* = \left[3x^2-3yz \quad 3y^2-3xz \quad 3z^2-3xy\right], \vec{0} \notin M.$$

Then, $(0,0,0)$ is the only critical point of F.
Thus, q is a regular value of F.

By Theorem 3.14 (ii) in Han's lecture note,
$F^{-1}(q) = M$ is a 2-dimensional regular submanifold.

T_pM at $p=(0,0,1)$ is :
We have the equation of T_pM that
$0(x-0)+0(y-0)+3(z-1) = 3(z-1) = 0$.
$\therefore T_pM = \{(x,y,z) \in \mathbb{R}^3 : 3(z-1) = 0\}$. $\square$

3.2.2. Show that $F : \mathbb{R}P^2 \to \mathbb{R}^4$ be defined by

$$F[(x,y,z)] = \frac{1}{x^2+y^2+z^2}(x^2-y^2, xy, xz, yz)$$

is a smooth embedding.

proof) Consider the quotient projection $\pi : p \mapsto [p]$.
We know that π is local diffeomorphism and $\mathbb{R}P^2$ has the quotient topology of S^2 via π.

$$\begin{tikzcd} S^2 \arrow[r, "f"] \arrow[d, "\pi"'] & \mathbb{R}^4 \\ \mathbb{R}P^2 \arrow[ur, "F"'] & \end{tikzcd}$$

Define f by

$$f : S^2 \to \mathbb{R}^4, \quad f(x,y,z) = (x^2-y^2, xy, xz, yz).$$

Then, $f = F \circ \pi$ since

$$\pi(p_1) = \pi(p_2) \Rightarrow p_1 = \pm p_2 \Rightarrow f(p_1) = f(p_2).$$

$$f(p) = f(-p).$$

Calculate the Jacobian of f:

$$Df = \begin{bmatrix} 2x & -2y & 0 \\ y & x & 0 \\ z & 0 & x \\ 0 & z & y \end{bmatrix}$$

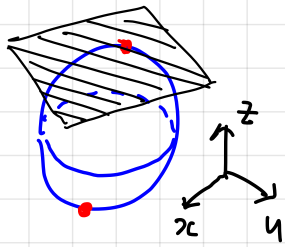
If $x \neq 0, y \neq 0$, then $\operatorname{rank}(Df) = 3$.

$\Rightarrow$ it is injective linear map $: \mathbb{R}^3 \to \mathbb{R}^4$.

(in this, restricted to the tangent plane of S^2 at $p \in S^2$ is still injective.

$\Rightarrow$ the map has rank $= 2$ at p $\to$ a map of S^2.)

If $x = y = 0$, then $p = (0, 0, \pm 1)$



$\Rightarrow$ $T_p S^2$ is xy-plane in $\mathbb{R}^3$.

$\Rightarrow$ $T_p S^2$ is mapped injectively into $\mathbb{R}^4$

$\Rightarrow$ $\operatorname{rank}(f) = 2$.

Thus, $F = f \circ \pi^{-1}$, $\operatorname{rank}(F) = 2$ locally.

Now, we claim $f(p) = f(q)$ for $p, q \in S^2$, then $p = \pm q$.

Consider $\varphi : \mathbb{R}^2 \to \mathbb{R}^2$ by $(x, y) \mapsto (x^2 - y^2, xy)$.

Let $x^2 - y^2 = a$, $xy = b$. Then

$$a^2 + 4b^2 = x^4 + 2x^2y^2 + y^4 = (x^2+y^2)^2$$

$$x^2 + y^2 = \sqrt{a^2+4b^2}$$

$$x^2 = \frac{1}{2}(a + \sqrt{a^2+4b^2})$$

$$y^2 = \frac{1}{2}(-a + \sqrt{a^2+4b^2})$$

$\Rightarrow$ $\pm x, \pm y$: uniquely determined by a, b

xy already determined by b.

$\Rightarrow$ only $\pm(x, y)$ are mapped to (a, b).

Now, (x_1, y_1, z_1), (x_2, y_2, z_2) are mapped to the same point, then we know that

$(x_1, y_1) = \pm(x_2, y_2)$, so $z_1 = \pm z_2$.

$\Rightarrow (x_1, y_1, z_1) = \pm(x_2, y_2, z_2)$ in S^2

$\rightsquigarrow$ the two points are antipodal.

Thus, F is injective, so F is injective immersion.

Since $\mathbb{RP}^2$ is compact because

it is the image of S^2 by the continuous map π.

Therefore, by Theorem 3.6.(iii),

F is a smooth embedding. $\blacksquare$

3.2.3. Let $F: \mathbb{R}^3 \to \mathbb{R}^2$ be defined by $F(x,y,z) = (x^2+y,\ x^2+y^2+z^2+y)$. Show that $q = (0,1)$ is regular value of F and $F^{-1}(q)$ is diffeomorphic to S^1.

proof) Let $f_1 = x^2+y$, $f_2 = x^2+y^2+z^2+y$.

$f_1 = 0 \Rightarrow y = -x^2$ and $f_2 = 1 \Rightarrow y^2+z^2 = 1$.

$$\underbrace{F^{-1}(0,1)}_{\text{Level set.}} = \{(x,y,z) \in \mathbb{R}^3 : y = -x^2 \text{ and } y^2+z^2 = 1\}$$

$$DF = \begin{bmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} & \frac{\partial f_1}{\partial z} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} & \frac{\partial f_2}{\partial z} \end{bmatrix} = \begin{bmatrix} 2x & 1 & 0 \\ 2x & 2y+1 & 2z \end{bmatrix}$$

consider $x = y = 0$ or $x \neq 0, y \neq 0$

$\Rightarrow$ rank $DF = 2$ for all $(x,y,z) \in F^{-1}(0,1)$.

By the definition 3.13, $q = (1,0)$ is regular value.

Thus, by theorem 3.14 (ii), $F^{-1}(q)$ is an 1-dimensional regular submanifold of $\mathbb{R}^3$.

Now, we want to show that $F^{-1}(q)$ is diffeomorphic to S^1.

Recall the definition of diffeomorphic.

「 Two manifolds M, N are diffeomorphic if there is a diffeomorphism f from M to N 」

(f : C^∞, homeo, f^{-1} : C^∞)

Define $\varphi : F^{-1}(q) \to S^1$ by $\varphi(x,y,z) = (y,z)$.

Then, clearly, φ is smooth and bijective.

$(\because)$ φ : surjective $\to$ trivial.

$\varphi(x_1, y_1, z_1) = \varphi(x_2, y_2, z_2)$

$\Rightarrow (y_1, z_1) = (y_2, z_2) \Rightarrow y_1 = y_2$ & $z_1 = z_2$

Since $y = -x^2$, $x_1 = x_2$.

$\therefore$ φ : injective.

Thus, there exists an inverse of $\varphi \equiv \varphi^{-1}$.

Now, we only check that φ^{-1} is smooth.

Consider the inclusion map $i : F^{-1}(q) \to \mathbb{R}^3$ of submanifold $F^{-1}(q)$ and the projection $\pi : \mathbb{R}^3 \to \mathbb{R}^2$. Then i is smooth and π is also smooth. Thus, $\varphi = \pi \circ i$.

Hence, $\varphi^{-1} = (\pi \circ i)^{-1} = i^{-1} \circ \pi^{-1}$ is smooth.

$\therefore$ φ is a diffeomorphism.

$\therefore$ the level set $F^{-1}(q)$ is diffeomorphic to S^1. $\blacksquare$

3.2.4. Let $F : N \to M$ be a smooth map of constant rank. Prove that if F is injective, then it is an immersion.

proof) Let $\dim M = m$ and $\dim N = n$ and suppose that F has constant rank r.

Suppose that F is not an immersion, i.e. $r < n$. By the rank theorem, for each $p \in N$, $\exists$ (U, φ) for N centered at p & (V, ψ) for M centered at $F(p)$ such that F has the coordinate representation

$$\psi \circ F \circ \varphi^{-1}(x^1, \cdots, x^n) = (x^1, \cdots, x^r, 0, \cdots, 0).$$

It follows that $F(0, \cdots, 0, \varepsilon) = F(0, \cdots, 0, 0)$ for any sufficiently small ε.

$\therefore$ F is not injective. $\blacksquare$

~~3.2.2.~~ Show that $F : \mathbb{RP}^2 \to \mathbb{R}^4$ be defined by

$$F[(x,y,z)] = \frac{1}{x^2+y^2+z^2}(x^2-y^2, xy, xz, yz)$$

is a smooth embedding.

~~proof)~~ Note that $\mathbb{RP}^2 = S^2/\{\pm 1\}$ of S^2 which is obtained by identifying antipodal points.

reference 3 생김 Then, F naturally reduced to the map

$$f : S^2 \to \mathbb{R}^4, \quad f(x,y,z) = (x^2-y^2, xy, xz, yz).$$

In order to apply Theorem 3.6 (Han's note), we have to check two conditions that

① injective immersion ② $\mathbb{RP}^2$ is compact.

<u>Claim</u> : ① → (1) : f is immersion,

Γ f is immersion $\Longleftrightarrow$ $Df : T_pM \to T_{f(p)}M$ is injective.

$(f : M \to N)$

pf) We know that $\dim M = \dim T_pM$ and the fact $\dim V = \operatorname{rank} A + \dim \ker A$, for any linear map A on vector space V.

Since Df is linear, by definition, we obtain $\dim T_pM = \operatorname{rank} Df + \dim \ker Df$.

Now, by the definition of immersion (3.2, (i)), we have :

$$f : \text{immersion} \Leftrightarrow \dim M = \operatorname{rank} Df$$
$$\Leftrightarrow \dim \ker Df = 0$$
$$\Leftrightarrow \ker Df = \{0\}$$
$$\Leftrightarrow Df \text{ is injective}. \quad \lrcorner$$

Clearly, $\dim \mathbb{RP}^2 = 2$.

We know that $\dim M = \dim T_pM$ and the fact $\dim V = \operatorname{rank} A + \dim \ker A$, for any linear map A on vector space V.

Hence, f : injective $\Rightarrow Df$: injective.

Since Df is linear, by definition, we obtain $\dim T_pM = \operatorname{rank} Df + \dim \ker Df$.

$\left\{ (\because) \; T_pM \text{ is a vector space spanned by a basis } \left\{ \frac{\partial}{\partial x^i}\Big|_p : i \in \{1, \ldots, n\} \right\}, n \in \mathbb{N}. \right\}$

Now, by the definition of immersion (3.2, (i)), we have :

$$Df \text{ is injective} \Leftrightarrow \ker Df = \{0\}$$
$$\Leftrightarrow \dim \ker Df = 0$$
$$\Leftrightarrow \dim M = \operatorname{rank} Df$$
$$\Leftrightarrow f : \text{immersion}$$

3.2.5. Give an example of an immersion $\iota : N \hookrightarrow M$ and $\omega \in \Gamma(M)$ such that $\iota^* \omega \equiv 0$ on N although $\omega \neq 0$ everywhere on M.

proof) Let $M = \mathbb{R}^2$, $\omega := dy \in \Gamma(M)$.

Consider S be the x-axis $\equiv N$.

(Note that S : embedded submanifold of $\mathbb{R}^2$.)

Then, as a covector field on M,

ω is nonzero everywhere since one of its component functions is always 1.

But, for the restriction $\iota^* \omega$ where $\iota : N \hookrightarrow M$ be an immersion,

$$\iota^* \omega = \iota^* dy = d(y \circ \iota) = 0$$

proposition 2.19 (iii) in Han's note.

y vanishes identically on S.

$\blacksquare$

4.1.1. Let G be a manifold with a group structure. Prove that if the map $G \times G \to G$ defined by $(g,h) \mapsto gh^{-1}$ is smooth, then G is a Lie group.

proof) Let $\mu : G \times G \to G$, $\mu(g,h) = gh^{-1}$.
By the assumption, μ is smooth for all (g,h).
Thus, consider the restriction map defined by
$\mu_g = \mu(e,h) = h^{-1}$ and $\mu_h = \mu(g,e) = g$
for the identity $e \in G$.

Then, μ_g, μ_h are also smooth since
μ is smooth. $(\because \mu = \mu_h \cdot \mu_g)$
Hence, we can define the inverse $inv(\alpha)$
by $inv : G \to G$, $inv(\alpha) = \alpha^{-1}$ as
$\alpha := h^{-1}$, since G is a group.

$(\because)$ Since G is a group, for each $h^{-1} \in G$,
there exists the inverse $h = (h^{-1})^{-1}$ such that
$h(h^{-1}) = e = (h^{-1})h$, e : identity of G.

Therefore, by definition 4.1,

G is a Lie group. $\square$

4.1.2. Prove that if G_1 and G_2 are Lie groups, then $G_1 \times G_2$ is a Lie group. Hence, T^n is a Lie group.

proof) Suppose that G_1, G_2 are Lie groups.
Since G_1, G_2 are groups, it can be written by the direct product of two groups as

$G_1 \times G_2$ such that $(g_1, h_1)(g_2, h_2) = (g_1h_1, g_2h_2)$
for $g_1, h_1 \in G_1$, $g_2, h_2 \in G_2$.

Then, $G_1 \times G_2$ is a group is immediate.
On the other hand, by problem 1.1 (Han's note), we know that $G_1 \times G_2$ is a smooth manifold.

Hence, now we only have to show that $G_1 \times G_2$ is a Lie group.

Define $\mu : (G_1 \times G_2) \times (G_1 \times G_2) \to G_1 \times G_2$ by
$\mu((g_1, g_2), (h_1, h_2)) = (g_1, g_2)(h_1, h_2)$, $g_i, h_i \in G_i$.
Since $G_1 \times G_2$ is the direct product of G_1, G_2,
$\mu((g_1, g_2), (h_1, h_2)) = (g_1, g_2)(h_1, h_2) = (g_1h_1, g_2h_2)$.
(in this, $g_1h_1 \in G_1$, $g_2h_2 \in G_2$ since G_1, G_2 are Lie groups by the assumption.)

Since G_1, G_2 are Lie groups again, there exist x_1^{-1}, x_2^{-1} : inverse of G_1, G_2, respectively so that

we define $inv : G_1 \times G_2 \to G_1 \times G_2$ by
$inv(x_1, x_2) = (x_1, x_2)^{-1}$, then
$inv(x_1, x_2) = (x_1, x_2)^{-1} = (x_1^{-1}, x_2^{-1})$.
Then, $x_i^{-1} \in G_i$ since G_i are Lie group.

Note that the multivariable function is smooth if the components are smooth.
Then, μ, inv are smooth clearly.

($\because$ consider the each component functions as μ_i, inv_i of G_1, G_2. Then these are smooth since G_1, G_2 are Lie group. We just find the proper multiplication & inverse map.)

Therefore, by definition 4.1 in Han's note, $G_1 \times G_2$ is a Lie group.

By the proof above, we can extend the fact that T^n is a Lie group since the direct product is defined componentwise as represented by a tuple. $\blacksquare$

4.1.3. Verify Example 4.13.

proof) (i) The complex special linear group

$SL(n,\mathbb{C}) = \{A \in gl(n,\mathbb{C}) : \det A = 1\}$

is $(2n^2-2)$ dimensional Lie subgroup of $GL(n,\mathbb{C})$.

pf) ① Subgroup

For any $A, B \in SL(n,\mathbb{C}) = \det^{-1}(1) \subseteq GL(n,\mathbb{C})$,

we can pick B^{-1} : inverse of B since

$1+i\cdot 0 \longrightarrow$ $\det(B) = 1 \neq 0 \iff B$ is invertible.

By the property of determinent,

$\det(B^{-1}) = \det(B)^{-1} = 1 \Rightarrow B^{-1} \in SL(n,\mathbb{C})$.

Then,

$$\det(AB^{-1}) = \det(A)\det(B^{-1})$$
$$= \det(A)\det(B)^{-1} = 1.$$

$\therefore AB^{-1} \in SL(n,\mathbb{C})$

$\therefore SL(n,\mathbb{C})$ is a subgroup of $GL(n,\mathbb{C})$.

② Submanifold.

Define $\det_{*A} : T_A GL(n,\mathbb{C}) = gl(n,\mathbb{C}) \to \mathbb{C}$.

For $A \in GL(n,\mathbb{C})$ & $B \in gl(n,\mathbb{C})$,

$\gamma(s) = A + sB$, $s \in (-\varepsilon, \varepsilon)$ for small ε.

Using the formula in example 4.11,

we obtain this in the same way as $\mathbb{C}$.

$\rightsquigarrow \det_{*A}(B) = (\det A)\,tr(A^{-1}B)$.

Note that $\dim GL(n, \mathbb{C}) = 2n^2$, $\dim \mathbb{C} = 2$.

$\det_{*A}(A) = \det A \operatorname{tr}(I) = \det(A)(n) \neq 0$.

Hence, $\det_{*A}$ is a submersion for $\forall A \in GL(n, \mathbb{C})$.

Moreover, $SL(n, \mathbb{C}) = \det^{-1}(1)$ is a regular submanifold of $GL(n, \mathbb{C})$ by Thm 3.14 and its dimension is $2n^2 - 2$.

Thus, by Theorem 4.8, $SL(n, \mathbb{C})$ is a closed Lie subgroup. $\square$

(ii) The unitary group

$U(n) = \{A \in gl(n, \mathbb{C}) : A^* A = I_n\}$ is n^2 dimensional Lie subgroup $GL(n, \mathbb{C})$.

pf) ① Subgroup

$$(AB)^*(AB) = \overline{(B^T A^T)}(AB) = (\overline{B^T})(\overline{A^T})(AB)$$
$$= B^* A^* A B = B^* I_n B = B^* B = I_n.$$

$\therefore U(n) \leq GL(n, \mathbb{C})$.

② Submanifold.

Let $S(n, \mathbb{C}) = \{A \in gl(n, \mathbb{C}) : A^* = A\}$.

{ <u>Note</u> : $A \in gl(n, \mathbb{R})$ is Hermitian $\Leftrightarrow$ A : symmetric.

$\rightsquigarrow$ symmetric real matrix is the special case of Hermitian. }

Then, $\dim S(n, \mathbb{C}) = n^2$

($\because$ Consider the basis for 2x2 complex matrix)

↳ in linear combination.

Define $F : GL(n,\mathbb{C}) \to S(n,\mathbb{C})$ by $F(A) = A^*A$.
Then $U(n) = F^{-1}(I_n)$. For $A \in U(n)$, we see that

$$F_{*A} : T_A GL(n,\mathbb{C}) = \mathfrak{gl}(n,\mathbb{C}) \to T_{F(A)} S(n,\mathbb{C}).$$

Given $B \in S(n,\mathbb{C})$, the curve $\overset{C^\infty \text{ (smooth)}}{\gamma(t)} = A + tB$ is well defined for all $t \in (-\varepsilon, \varepsilon)$, $\varepsilon > 0$ is small.
Thus,

$$\begin{aligned}
F_{*A}(B) &= \frac{d}{dt}\Big|_{t=0} F \circ \gamma(t) \\
&= \frac{d}{dt}\Big|_{t=0} F(A+tB) \\
&= \frac{d}{dt}\Big|_{0} (A+tB)^*(A+tB) \\
&= \frac{d}{dt}\Big|_{0} (A^*+tB^*)(A+tB) \quad (\because (A+B)^* = A^*+B^*). \\
&= \underbrace{B^*(A+tB) + (A^*+tB^*)B}_{(\because)\ dA^* = d(A^*) = (dA)^*}\Big|_{t=0} \\
&= B^*A + A^*B \in S(n,\mathbb{C}).
\end{aligned}$$

Hence, $T_{F(A)} S(n,\mathbb{C}) \subset S(n,\mathbb{C})$. For given $C \in S(n,\mathbb{C})$,

$$\begin{aligned}
F_{*A}(\tfrac{1}{2}AC) &= \tfrac{1}{2}(AC)^*A + \tfrac{1}{2}A^*(AC) \\
&= \tfrac{1}{2}(C^*A^*)A + \tfrac{1}{2} I_n C \quad (\because A \in U(n)) \\
&= \tfrac{1}{2}C^* + \tfrac{1}{2}C = C \quad (\because C \in S(n,\mathbb{C}))
\end{aligned}$$

Thus, $T_{F(A)} S(n, \mathbb{C}) = S(n, \mathbb{C})$ and F_{*A} is surjective for any $A \in U(n)$ and thus I_n is a regular value of F.
Then, by Theorem 3.14, $U(n)$ is a regular submanifold of $GL(n, \mathbb{C})$ with dimension n^2.

Therefore, Theorem 4.8, $U(n)$ is a Lie subgroup.

$\square$

(iii) The special unitary group

$$SU(n) = SL(n, \mathbb{C}) \cap U(n)$$

is (n^2-1) dimensional Lie subgroup $GL(n, \mathbb{C})$.

pf) $SL(n, \mathbb{C}) = \det^{-1}(1)$ is closed in $GL(n, \mathbb{C})$.

For any $A \in U(n)$, $\det(A) = \pm 1$.

$(\because)$ $1 = \det(A)\det(A)^{-1} = \det(AA^{-1})$

$= \det(AA^*) = \det(A)\det(A^*) = \det(A)\det(A)^*$.

$\Rightarrow |\det A| = 1$.

Thus, $SU(n)$ is an open submanifold of $U(n)$.

$(\because)$ $U^+(n) \cup U^-(n) = U(n)$. $(\dim U(n) = 1)$.

$U^+(n) = \det^{-1}(1) \cap U(n)$

$\Rightarrow U^{\pm}(n) \equiv$ closed in $U(n)$

$\equiv$ open in $U(n)$ $(\because$ disjoint$)$

Hence, $SU(n)$ is a regular submanifold.
Thus, by Theorem 4.8, $SU(n)$ is a Lie subgroup of $GL(n, \mathbb{C})$ with $\dim = n^2 - 1$.

$\blacksquare$

4.1.4. (i) Prove that

$$SU(2) = \left\{ \begin{pmatrix} z & -\bar{w} \\ w & \bar{z} \end{pmatrix} \in gl(2,\mathbb{C}) : z\bar{z} + w\bar{w} = 1 \right\}$$

(ii) Show that $SU(2)$ is diffeomorphic to S^3.

proof) (i) Special unitary group.

In Example 4.13 (iii), $SU(n) = SL(n,\mathbb{C}) \cap U(n)$.

That is, $SU(n) = \{A \in gl(n,\mathbb{C}) : \det A = 1 \ \& \ A^*A = I_n\}$,

$A^* = (\bar{A})^T$ is the Hermitian conjugate of A.

Let $A = \begin{pmatrix} z & -\bar{w} \\ w & \bar{z} \end{pmatrix} \in gl(2,\mathbb{C})$, for any $z, w \in \mathbb{C}$.

$$\begin{cases} \det A = z\bar{z} - (-w\bar{w}) = z\bar{z} + w\bar{w} \\ A^* = \overline{A^T} = \begin{pmatrix} \bar{z} & \bar{w} \\ -w & z \end{pmatrix} \end{cases}$$

$$A^*A = \begin{pmatrix} \bar{z} & \bar{w} \\ -w & z \end{pmatrix}\begin{pmatrix} z & -\bar{w} \\ w & \bar{z} \end{pmatrix} = \begin{pmatrix} \bar{z}z + \bar{w}w & \bar{w}\bar{z} - \bar{z}\bar{w} \\ -wz + zw & w\bar{w} + z\bar{z} \end{pmatrix}$$

$$= \begin{pmatrix} z\bar{z} + w\bar{w} & 0 \\ 0 & z\bar{z} + w\bar{w} \end{pmatrix}$$

Since $\bar{\alpha}\alpha = \alpha\bar{\alpha}$, $z_1 z_2 = z_2 z_1$ and $\bar{\bar{\alpha}} = \alpha$.

Thus, if $A \in SU(2)$, then $z\bar{z} + w\bar{w}$ must be 1,

so that $A^*A = I_2$ and $\det A = 1$.

(ii) $SU(2) \cong S^3$ (diffeomorphic)

Since $\mathbb{R}^4 \cong \mathbb{C}^2$, $S^3 = \{(z,w) \in \mathbb{C}^2 : |z|^2 + |w|^2 = 1\}$.

($\because$ $z\bar{z} = \bar{z}z = |z|^2$).

Define a map $f : S^3 \to SU(2)$ by $f(z,w) = \begin{pmatrix} z & -\bar{w} \\ w & \bar{z} \end{pmatrix}$.

Then, f is well-defined since

$(z,w) \in S^3 \Rightarrow f(z,w) \in SU(2)$ clearly.

First, we now show that f is bijective.

① injective.

Clearly, $f(z_1,w_1) = f(z_2,w_2) \Rightarrow (z_1,w_1) = (z_2,w_2)$.

② surjective.

$f(S^3) = f(\{|z|^2+|w|^2=1\}) = SU(2)$.

$\therefore$ f is bijective $\Rightarrow \exists$ inverse f^{-1}.

Note that $SU(2) \subseteq M(2,\mathbb{C}) \cong \mathbb{R}^8$.

Then, $SU(2)$ is a submanifold of $M(2,\mathbb{C}) \cong \mathbb{R}^8$.

Thus, if we define $F : \mathbb{R}^4 \to \mathbb{R}^8$,

f is just a restriction of F and

f, f^{-1} are smooth since F, F^{-1} are smooth,

and S^3, $SU(2)$ are submanifolds.

Therefore, definition 1.16, f is diffeomorphism. $\blacksquare$

4.1.5. Prove that $SO(2)$ is diffeomorphic to S^1.

proof) $SO(n) = SL(n, \mathbb{R}) \cap O(n)$

$\Leftrightarrow \{A \in gl(n, \mathbb{R}) : \det A = 1 \text{ and } A^T A = I_n\}$.

In $SO(2)$, by calculation of $\det A$ and A^TA,

$$SO(2) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a^2+c^2=1,\ b^2+d^2=1,\ ad-bc=1,\ ab+cd=0 \right\}.$$

Let $a = x$, $b = -y$, $c = y$, $d = x$, then

all of the condition of $SO(2)$ is satisfied so that

we obtain

$$SO(2) = \left\{ \underbrace{\begin{pmatrix} x & -y \\ y & x \end{pmatrix}}_{A} \in gl(2, \mathbb{R}) : \det A = 1 \ \& \ A^TA = I_2 \right\}.$$

Define $f : S^1 \to SO(2)$ by $f(x, y) = \begin{pmatrix} x & -y \\ y & x \end{pmatrix}$.

Then, clearly, $(x, y) \in S^1 \Rightarrow f(x, y) \in SO(2)$ so

f is well-defined.

Trivially, f is bijective, thus $\exists$ inverse f^{-1} of f.

Since S^1 is a submanifold of $\mathbb{R}^2$ and $SO(2)$ is a

submanifold of $GL(2, \mathbb{R})$ by example 4.12,

Consider $F : \mathbb{R}^2 \to GL(2, \mathbb{R}) \cong \mathbb{R}^4$ be a diffeomorphism,

then f is just a restriction of F and so

f, f^{-1} is smooth.

Therefore, by definition 1.16, f is a diffeomorphism.

$\blacksquare$

4.2.1. Verify Example 4.27.

proof) ① $\mathfrak{sl}(n,\mathbb{C}) = \{X \in \mathfrak{gl}(n,\mathbb{C}) : \mathrm{tr}(X) = 0\}$

pf) Let $X \in \mathfrak{sl}(n,\mathbb{C})$. Set $\gamma(s) = e^{sX}$, $s \in \mathbb{R}$.

Then, $\det(\gamma(s)) = \det(e^{sX}) = e^{\mathrm{tr}(sX)} = e^0 = 1$.

$\Rightarrow$ $\gamma(s)$ is a curve on $SL(n,\mathbb{C})$.

Since $\gamma(0) = I_n$ & $\gamma'(0) = X$, we have

$\gamma'(0) = X \in T_{I_n} SL(n,\mathbb{C}) \subset \mathrm{Lie}\, SL(n,\mathbb{C})$.

Hence, $\mathfrak{sl}(n,\mathbb{C}) \subset \mathrm{Lie}\, SL(n,\mathbb{C})$.

Since $\dim \mathrm{Lie}\, SL(n,\mathbb{C}) = \dim \mathfrak{sl}(n,\mathbb{C}) = 2n^2-2$,

we conclude that $\mathfrak{sl}(n,\mathbb{C}) = \mathrm{Lie}\, SL(n,\mathbb{C})$. $\triangle$

② $\mathfrak{u}(n) = \{X \in \mathfrak{gl}(n,\mathbb{C}) : X^* + X = 0\}$

similar to ①, $\dim \mathfrak{u}(n) = n^2$.

pf) For $X \in \mathrm{Lie}\, U(n) \subset \mathrm{Lie}\, \mathfrak{gl}(n,\mathbb{C}) = T\,GL(n,\mathbb{C}) = \mathfrak{gl}(n,\mathbb{C})$,

let $\gamma : I \to U(n)$ be a curve such that

$\gamma(0) = I_n$, $\gamma'(0) = X$. Then $\gamma(s)^*\gamma(s) = I_n$ such that

$\gamma(s) \in U(s)$. Hence,

$$X^* + X = \gamma'(0)^* \cdot \gamma(0) + \gamma(0)^* \cdot \gamma'(0) = 0.$$

So, $X \in \mathfrak{u}(n)$.

Conversely, suppose $X \in \mathfrak{u}(n)$. Set $\beta(s) = e^{sX}$ for $s \in \mathbb{R}$.

Then,
$$\begin{aligned} \beta(s)^*\beta(s) &= \overline{(e^{sX})^T}\, e^{sX} \\ &= \overline{e^{sX^T}}\, e^{sX} = e^{sX^*} e^{sX} \\ &= e^{s(X^*+X)} \quad (\because X^*X = I_n = XX^*) \\ &= I_n \quad (\because X^*+X = 0 \text{ in } \mathfrak{u}(n)) \end{aligned}$$

$\Rightarrow$ $\beta(s)$ is a curve on $U(n)$. Since $\beta(0) = I_n$ and

$\beta'(0) = X$, we have $X \in T_{I_n} U(n) = \mathrm{Lie}\, U(n)$. $\triangle$

③ $su(n) = \{X \in gl(n, \mathbb{C}) : X^* + X = 0 \ \& \ tr(X) = 0\}$.

pf) Let $X \in su(n)$, set $\gamma(s) = e^{sX}$ for $s \in \mathbb{R}$.

By ①, ②, we can check that $\gamma : I \to su(n)$,

$\gamma(s) \in SL(n, \mathbb{C}) \cap U(n)$ such that $\gamma(0) = I_n$, $\gamma'(0) = X$.

$\Rightarrow X \in T_{I_n} SL(n, \mathbb{C}) \cap U(n) \subset \mathrm{Lie}\, SU(n)$.

Hence, $su(n) \subset \mathrm{Lie}\, SU(n)$.

Conversely, if $X \in \mathrm{Lie}\, SU(n)$, let $\beta : I \to SU(n)$

that $\beta(0) = I_n$, $\beta'(0) = X$. Then

$\beta(s)^* \beta(s) = I_n$ and $\det(\beta(s)) = 1$, $\beta(s) \in SU(n)$.

Then, $X^* + X = \beta'(0)^* \beta(0) + \beta(0)^* \beta'(0) = 0$ and

$$tr(X) = tr(\beta'(0)) = \det(\beta(0))\, tr(\beta(0)^* \beta'(0))$$

$$= \frac{d}{ds}\Big|_{s=0} \det \circ \beta(s) = 0 .$$

($\because \beta'(s) \in T_{\beta(s)} SU(n) = SU(n) \Rightarrow \det(\beta'(s)) = 1$).

$\Rightarrow X \in su(n)$ and thus, $su(n) \supset \mathrm{Lie}\, SU(n)$.

$\therefore su(n) = \mathrm{Lie}\, SU(n)$. △

□

4.2.2. Prove that the Pauli matrices

$$E_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad E_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

are generators of $su(2)$.

proof) $A \in su(2) \Rightarrow A^* + A = 0$ and $tr(A) = 0$.

Let $A = \begin{bmatrix} x_1 + iy_1 & x_2 + iy_2 \\ x_3 + iy_3 & x_4 + iy_4 \end{bmatrix}$. Then

$A^* = \begin{bmatrix} x_1 - iy_1 & x_3 - iy_3 \\ x_2 - iy_2 & x_4 - iy_4 \end{bmatrix}$ and $A^* + A$ is

$$\begin{bmatrix} 2x_1 & x_2 + x_3 + i(y_2 - y_3) \\ x_2 + x_3 + i(y_3 - y_2) & 2x_4 \end{bmatrix} = 0, \quad x_1 = x_4 = 0.$$

$x_2 + x_3 = 0 \Rightarrow x_2 = -x_3, \quad y_2 = y_3.$

$tr(A) = x_1 + iy_1 + x_4 + iy_4 = +i(y_1 + y_4) = 0$

$\Rightarrow y_1 = -y_4$. Thus,

$$A = \begin{bmatrix} iy_1 & x_2 + iy_2 \\ -x_2 + iy_2 & -iy_1 \end{bmatrix} \xrightarrow{\text{general}} \begin{bmatrix} ix & y + iz \\ -y + iz & -ix \end{bmatrix}, \quad x, y, z \in \mathbb{R}.$$

$$\Rightarrow A = z\begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix} + y\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} + x\begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$$

$:= z\sigma_1 + y\sigma_2 + x\sigma_3$ for any $x, y, z \in \mathbb{R}$.

Then, we see that $\sigma_1 = iE_1$, $\sigma_2 = iE_2$, $\sigma_3 = iE_3$.

Clearly, $\{\sigma_1, \sigma_2, \sigma_3\}$: Linearly independent.

$\therefore \{\sigma_1, \sigma_2, \sigma_3\}$ is a basis for $su(2)$

$\therefore$ E_1, E_2, E_3 are generator of $su(2)$. $\square$

4.3.1. Prove Example 4.31.

Let G and H be a group and $\psi : G \to H$ be a homomorphism. Prove that $\theta : G \times H \to H$ is defined by $\theta(g, h) = \psi(g)h$ for $g \in G$ and $h \in H$, then θ is a left action.

proof) Let G, H be a group have a binary operation $*$.

(i) For the identity element of $G \equiv e_g$,

$$\theta(e_g, h) = \psi(e_g) * h$$
$$= e_h * h \quad (\because \psi : \text{homo}.\ \psi(G) \subseteq H)$$
$$= h \text{ for all } h \in H.$$

($\because$) Let e_g, e_h be the identity of G, H, respectively.

ψ : homomorphism $\Rightarrow$ $\psi(e_g) = e_h$.

pf) $\psi(e_g) = \psi(e_g * e_g)$

$= \psi(e_g) * \psi(e_g)$ since ψ : homo.

$\Rightarrow \psi(e_g) * \psi^{-1}(e_g)$

$= \psi(e_g) * \psi(e_g) * \psi^{-1}(e_g)$

since $\psi(e_g) \in H$, H : group, $\exists$ inverse ψ^{-1}.

$\Rightarrow e_h = \psi(e_g)$. $\triangle$

(ii) $\theta(g_1, \theta(g_2, h)) = \theta(g_1, \psi(g_2) * h)$

$= \psi(g_1) * \psi(g_2) * h$

$= \psi(g_1 * g_2) * h$ since ψ : homo.

$= \theta(g_1 * g_2, h)$ for $g_1, g_2 \in G$.

$\therefore$ θ : left action on H. $\square$

4.3.2. Prove that Example 4.40.

$S^{2n-1} \cong U(n)/U(n-1) \cong SU(n)/SU(n-1)$.

In particular, $S^3 \cong SU(2)/SU(1) = SU(2)$.

Recall Problem 4.1.4.

proof) We want to apply the theorem 4.38. So,

① well-defined action

Recall that Example 4.39 (i).

$GL(n,\mathbb{C}) \times \mathbb{C}^n \longrightarrow \mathbb{C}^n$, $(A, v) \longmapsto Av \in \mathbb{C}^n$.

If $A \in U(n)$, then $\quad$ $A \in U(n)$.

$\langle Av, Av \rangle \underset{(*)}{=\!=} \langle v, A^*Av \rangle = \langle v, v \rangle$.

Hence, the action $U(n) \times S^{2n-1} \to S^{2n-1}$ is well-defined as the restriction of the above.

(*) : Let $T \in \mathcal{L}(\mathbb{C}^n, \mathbb{C}^n)$ is a linear transformation. The adjoint linear transformation of T denoted by T^* is a linear transformation s.t. for any $v \in \mathbb{C}^n, w \in \mathbb{C}^n$, $\langle Tv, w \rangle = \langle v, T^*w \rangle$. If we assume the finite dimensional inner product space, then we obtain the following. $\langle Ax, y \rangle = \langle x, A^*y \rangle$ for $x, y \in \mathbb{C}^n$, $A \in \mathbb{C}^{n\times n}$.

② transitivity of the action.

It is similar to (ii) in Example 4.39.

For the standard basis $\{e_1, \cdots, e_n\}$ of $\mathbb{C}^n$ and

$\{v_1, \cdots, v_n\}$ is an orthonormal basis of $\mathbb{C}^n$,

$v_i = A^j_i e_j$. Then,

$$\delta_{ij} = \langle v_i, v_j \rangle = \langle A^k_i e_k, A^l_j e_l \rangle$$

orthonormal

using $(*)$ $\longrightarrow$ $$= \langle e_k, \overline{A^i_k} A^l_j e_l \rangle$$

sesquilinear of inner pdt. $\longrightarrow$ $$= A^k_i \overline{A^j_l} \langle e_k, e_l \rangle$$

$k \neq l \Rightarrow \langle e_k, e_l \rangle = 0$ $\longrightarrow$ $$= A^k_i \overline{A^j_k}$$

$$= (AA^*)^j_i$$

Hence, $A \in U(n)$.

Now, given $v, w \in S^{2n-1}$, we can choose $A, B \in U(n)$ such that

$$Ae_1 = v \quad \text{and} \quad Be_1 = w.$$

Thus, $w = BA^*v = (BA^*) \cdot v$.

By definition 4.32, the action is transitive.

③ $K \equiv$ isotropy subgroup of e_n $= H$.

Let

$$H = \left\{ A \in U(n) : A = \begin{bmatrix} \tilde{A} & 0 \\ 0 & 1 \end{bmatrix}, \tilde{A} \in U(n-1) \right\}.$$

Then, $H \cong U(n-1)$ is trivial.

$$(\because) \quad A^*A = \begin{bmatrix} \tilde{A}^* & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \tilde{A} & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \tilde{A}^*\tilde{A} & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} I_n & 0 \\ 0 & 1 \end{bmatrix}.$$

Clearly, every element of $U(n-1)$ leaves e_n.

Conversely, suppose that $Ae_n = e_n$ for some $A = (A^1, \cdots, A^n) \in U(n)$. Then $A^n = e_n$ s.t. $A^n_i = 0$ for $i < n$ and $A^n_n = 1$.

$$\left(Ae_n = \begin{bmatrix} \tilde{A} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} = e_n . \right)$$

Since $A^*A = I_n$, we have

$$1 = \sum_{i=1}^{n} (A^i_n)^2 = (A^n_n) + \sum_{i=1}^{n-1} (A^i_n)^2 .$$

So, $A^i_n = 0$ for $i < n \Rightarrow A \in H$.

By Theorem 4.38, $S^{2n-1} \cong U(n) / U(n-1)$.

By a similar argument, $S^{2n-1} \cong SU(n) / SU(n-1)$.

We can show $S^3 \cong SU(2)/SU(1) = SU(2)/\{1\}$ $= SU(2)$ using the above arguments. $\square$

4.3.3. Prove Example 4.42.

$\mathbb{C}P^{n-1} \cong SU(n)/U(n-1)$. In particular, by Problem 1.3.2 and Example 4.40

$$S^2 \cong \mathbb{C}P^1 \cong SU(2)/U(1) \cong S^3/S^1.$$

proof) Define an $SU(n) \times \mathbb{C}P^{n-1} \to \mathbb{C}P^{n-1}$ by

$(A, [x]) \mapsto A \cdot [x] := [Ax]$.

If $A \in SU(n)$ & $[x] = [y]$, then

$y = \lambda x$ for some $\lambda \in \mathbb{C}^*$ s.t. $Ay = \lambda Ax$.

$\therefore$ the action is well defined.

For $[x], [y] \in \mathbb{C}P^{n-1}$, we can choose

$A, B \in SU(n)$ such that

$$A[z_1] = [x], \quad A[z_1] = [y].$$

(We can find the basis $[z_1]$, by the same argument of 4.3.2.)

$\therefore$ the action is transitive.

Note that $A \in SU(n)$ fixes $[z_n]$

$\Leftrightarrow Az_n = z_n$ or $Az_n = -z_n$

$$\Leftrightarrow A = \begin{pmatrix} \tilde{A} & 0 \\ 0 & 1 \end{pmatrix} \text{ or } A = \begin{pmatrix} \tilde{A} & 0 \\ 0 & -1 \end{pmatrix} \text{ where } \tilde{A} \in U(n-1)$$

$$\Leftrightarrow A = \begin{pmatrix} \tilde{A} & 0 \\ 0 & \det\tilde{A} \end{pmatrix} \text{ where } \tilde{A} \in U(n-1)$$

So, the isotropy group of z_n is $U(n-1)$.

By Theorem 4.38, we get the desired result.

Now, we want to show that

$$S^2 \overset{①}{\cong} \mathbb{CP}^1 \overset{②}{\cong} SU(2)/U(1) \overset{③}{\cong} S^3/S^1$$

② : Using the problem 4.3.3 we proved, this follows for $n=2$.

③ : Consider the quotient map $f : \mathbb{C}^2 - \{0\} \to \mathbb{CP}^1$ and the restriction $\pi : S^3 \to \mathbb{CP}^1$.

① : Define a charts on S^2

$\phi_\pm : S^2 - \{(0,0,\pm 1)\} \to \mathbb{R}^2$ by $(x_0, x_1, x_2) \mapsto \left(\frac{x_0}{1\mp x_2}, \frac{x_1}{1\mp x_2}\right)$

and define

$$F : \mathbb{CP}^1 \to S^2, \quad F([z_0, z_1]) := \begin{cases} \phi_N^{-1}\left(\frac{z_0}{z_1}\right), & z_1 \neq 0 \\ (0,0,1), & z_1 = 0 \end{cases}.$$

Then F is diffeomorphism. $\blacksquare$

5.1.1. Find the integral curve of

$$X = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} \quad \text{on } \mathbb{R}^2.$$

proof) Let $\gamma(t) = (\gamma_1(t), \gamma_2(t))$ be an integral curve such that $\gamma(t)$ is smooth, and $\gamma'(t) = X_{\gamma(t)}$, X is a vector field on $\mathbb{R}^2$. Then,

$$\gamma'(t) = X_{\gamma(t)} = \gamma_1(t)\frac{\partial}{\partial \gamma_1} + \gamma_2(t)\frac{\partial}{\partial \gamma_2}.$$

Since $\gamma'(t) = (\gamma_1'(t), \gamma_2'(t))$, we obtain ODEs:

$$\gamma_1' = \gamma_1, \ \gamma_2' = \gamma_2 \Rightarrow \gamma_1 = C_1 e^t, \ \gamma_2 = C_2 e^t$$

solutions, $C_1, C_2 \in \mathbb{R}$.

$\therefore$ integral curve of X is $\gamma(t) = e^t(C_1, C_2)$, $C_1, C_2 \in \mathbb{R}$.

$\square$

5.2.2. Complete the proof of Proposition 5.8.

proof) (ii) - 1 : $\mathcal{L}_X(fY) = (\mathcal{L}_X f)Y + f(\mathcal{L}_X Y)$

pf) Using the fact : $\mathcal{L}_X Y = [X, Y]$, $\mathcal{L}_X f = Xf$.

$$\begin{aligned}\mathcal{L}_X(fY) &= [X, fY] = X\cdot fY - fY\cdot X \\ &= Xf\cdot Y + fX\cdot Y - fY\cdot X \\ &= Xf\cdot Y - f\cdot [X, Y] \\ &= (\mathcal{L}_X f)Y + f(\mathcal{L}_X Y).\end{aligned}$$

$\triangle$

(ii) - 2 : $\mathcal{L}_X[Y, Z] = [\mathcal{L}_X Y, Z] + [Y, \mathcal{L}_X Z]$
$= [\mathcal{L}_X Y, \mathcal{L}_X Z]$

pf) By the Jacobi identity, we obtain above :

$$[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$$

prop. 5.8 (ii) → By the fact we proved : $\mathcal{L}_X Y = [X, Y]$ ⋯ (*)

$\mathcal{L}_X[Y, Z] = [X, [Y, Z]]$ and we know that

Jacobi identity ⟶ $= -[Y, [Z, X]] - [Z, [X, Y]]$

skew symmetric ⟶ $= [[X, Y], Z] - [Y, -[X, Z]]$

bilinear ⟶ $= [[X, Y], Z] + [Y, [X, Z]]$

(*) ⟶ $= [\mathcal{L}_X Y, Z] + [Y, \mathcal{L}_X Z]$.

$\triangle$

(iii) - 1 : $\mathcal{L}_X(f\omega) = (\mathcal{L}_X f)\omega + f\mathcal{L}_X\omega$

pf) $\mathcal{L}_X(f\omega)_p = \lim_{t\to 0}\frac{1}{t}[\Phi_t^*(f\omega)(p) - (f\omega)(p)]$

$= \lim_{t\to 0}\frac{1}{t}[f(\Phi_t(p))\Phi_t^*(\omega(p)) - f(p)\omega(p)]$

$= \lim_{t\to 0}\frac{1}{t}[f(\Phi_t(p))\Phi_t^*(\omega(p)) - f(\Phi_t(p))(\omega(p)) +$

$f(\Phi_t(p))(\omega(p)) - f(p)\omega(p)]$

$= \lim_{t\to 0}\frac{1}{t}[[f(\Phi_t(p)) - f(p)]\cdot\omega(p)] +$

$\lim_{t\to 0}\frac{1}{t}[f(\Phi_t(p))\cdot[\Phi_t^*(\omega(p)) - \omega(p)]]$

$= (\mathcal{L}_X f)\omega + f\cdot\mathcal{L}_X\omega$ if $f(\Phi_t(p)) \to f(p)$ as $t\to 0$.

Δ

(iii) - 2 : $\mathcal{L}_X(\omega Y) = (\mathcal{L}_X\omega)Y + \omega(\mathcal{L}_X Y)$

pf) $\mathcal{L}_X(\omega(Y)(p)) = \lim_{t\to 0}\frac{1}{t}[\omega(Y)(\Phi_t(p)) - \omega(Y)(p)]$

$= \lim_{t\to 0}\frac{1}{t}[\omega(\Phi_t(p))(Y_{\Phi_t(p)}) - \omega(p)(Y_p)]$

$= \lim_{t\to 0}\frac{1}{t}[(\Phi_t^*\omega)(p)(\Phi_{t*}Y_{\Phi_t(p)}) - \omega(p)(Y_p)]$

$= \lim_{t\to 0}\frac{1}{t}[(\Phi_t^*\omega)(p)(\Phi_{t*}Y)_p - \omega(p)(Y_p)]$

$= \lim_{t\to 0}\frac{1}{t}[(\Phi_t^*\omega)(p)(\Phi_{t*}Y)_p - \omega(p)(\Phi_{t*}Y)_p +$

$\omega(p)(\Phi_{t*}Y)_p - \omega(p)(Y_p)]$

$$= \lim_{t \to 0} \frac{1}{t} [(\Phi_t^* \omega)(p) - \omega(p)] \cdot (\Phi_{t*} Y)_p +$$

$$\lim_{t \to 0} \frac{1}{t} \omega(p) \cdot [Y_p - (\Phi_{t*} Y)_p]$$

$$= (\mathcal{L}_X \omega)(p)(Y_p) + \omega(p)(\mathcal{L}_X Y)(p) \text{ if}$$

$(\Phi_{t*} Y)_p \to Y_p$ as $t \to 0$.

$\triangle$

$\square$

5.3.1. Let $A, B \in \mathfrak{gl}(2,\mathbb{R})$ be given by

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Compute the 1-parameter subgroup of $GL(2,\mathbb{R})$ generated by A and B.

proof) By Proposition 5.16 (ii) & Example 5.15, the 1-parameter subgroup of $GL(n,\mathbb{R})$ generated by $X \in \mathfrak{gl}(n,\mathbb{R}) \cong \mathrm{Lie}(GL(n,\mathbb{R}))$ is

$$\gamma(t) : t \in \mathbb{R} \longmapsto \exp(tX) = e^{tX} \in GL(n,\mathbb{R}).$$

Thus, by definition 4.25,

$$\gamma(t) = e^{tA} = I + \begin{pmatrix} 0 & t \\ -t & 0 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 0 & t \\ -t & 0 \end{pmatrix}\begin{pmatrix} 0 & t \\ -t & 0 \end{pmatrix} + \cdots$$

$$= \begin{pmatrix} 1 - \frac{t^2}{2!} + \frac{t^4}{4!} + \cdots & t - \frac{t^3}{3!} + \cdots \\ -(t - \frac{t^3}{3!} + \cdots) & 1 - \frac{t^2}{2!} + \frac{t^4}{4!} + \cdots \end{pmatrix}$$

$$= \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}.$$

$$\gamma(t) = e^{tB} = I + \begin{pmatrix} 0 & t \\ 0 & 0 \end{pmatrix} + 0 + 0 + \cdots$$

$$= \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}. \qquad \square$$

5.3.2. Prove Proposition 5.19.

proof) Note that $\mathrm{Ad}_g(e) = e$, and hence $\mathrm{Ad}_g : T_eG \to T_eG$.
Hence, $\exp \mathrm{Ad}_g A = g(\exp A) g^{-1}$ is well defined.

Since $\mathrm{Ad}_g : G \to G$ is a homomorphism for each $g \in G$, therefore,

$$\exp \mathrm{Ad}_g A = \mathrm{Ad}_g(\exp A) = g(\exp A) g^{-1}. \qquad \blacksquare$$