A STUDY OF SPATIAL DYNAMICS UNDER THE PRISONER'S DILEMMA, HAWK-DOVE AND SNOWDRIFT GAME

Sangman Jung and Sunmi Lee

Department of Applied Mathematics, Kyung Hee University, Yongin-si, South Korea

Background

- We investigate the effects of three games on spatial dynamics: Prisoner's Dilemma, Hawk-Dove and Snowdrift game.
- We employ three different approaches including a lattice network model, replicator equations, and partial differential equations.
- We study spatial dynamics under various conditions: different payoff, and initial configurations.
- In game theory, the payoff matrix is the result of the model of strategic interaction between rational decision-makers.
- This model implies that the players obtained specific payoff when they play the game with each other, having their own's strategies.

We investigate the 2×2 symmetric games represented by the payoff matrix in game theory as follows.

PD	C	D
C	R	S
D	T	P

Prisoner's Dilemma

HD	D	C
D	$\frac{T-q}{2}$	T
C	0	$\frac{T}{2}$

Hawk-Dove

SD	C	D
C	$\frac{(T-q)}{2}$	$\frac{T}{2} - q$
D	$\frac{T}{2}$	0

Snowdrift

Table 1. The payoff matrix of three different games

T : temptation, R : reward, P : punishment, S : sucker's payoff, q : cost.

In this table 1, C : Cooperator, D : Defector. In Hawk-Dove game, D is a 'Hawk', C is a 'Dove', respectively.

These parameters are satisfied the inequality as

$$T > R > P \geq S$$

We set the values of the base parameters $T = 1.9$, $q = 1$, $R = 1$, $P = S = 0$ and vary the parameters T and c.

Three different models

1. Replicator equations

(Ordinary Differential Equations)

In evolutionary game theory, the replicator equation is the deterministic dynamics of a population. Let x_1, x_2 are the proportion of C and D such that $x_1 + x_2 = 1$, then the system of the equations as follows.

- **Prisoner's Dilemma**

$$\dot{x}_1 = x_1^2(1 - x_1 - T x_2)$$
$$\dot{x}_2 = x_1 x_2 (T(1 - x_2) - x_1)$$

- **Hawk-Dove game**

$$\dot{x}_1 = x_1\left(\frac{T-q}{2}x_1(1-x_1) + T x_2\left(1 - x_1 - \frac{x_2}{2}\right)\right)$$
$$\dot{x}_2 = x_2\left(\frac{T}{2}x_2 - \frac{T-q}{2}x_1^2 - T x_1 x_2 - \frac{T}{2}x_2^2\right)$$

- **Snowdrift game**

$$\dot{x}_1 = x_1\left(\frac{T-q}{2}x_1(1-x_1) + (q-T)x_1 x_2 + \left(\frac{T}{2} - q\right)x_2\right)$$
$$\dot{x}_2 = x_2\left(\frac{T}{2}x_1(1-x_1) + \frac{q}{2}x_1^2 + (q-T)x_1 x_2\right)$$

2. Modified replicator equations

(Partial Differential Equations)

We obtain the system of partial differential equations by adding the Laplace operator $\Delta u = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}$ when $u(x,y,t)$ in the replicator equations as follows.

- **Modified Prisoner's Dilemma**

$$\frac{\partial u_1}{\partial t} = \Delta u_1 + u_1^2(1 - u_1 - T u_2)$$
$$\frac{\partial u_2}{\partial t} = \Delta u_2 + u_1 u_2 (T(1 - u_2) - u_1)$$

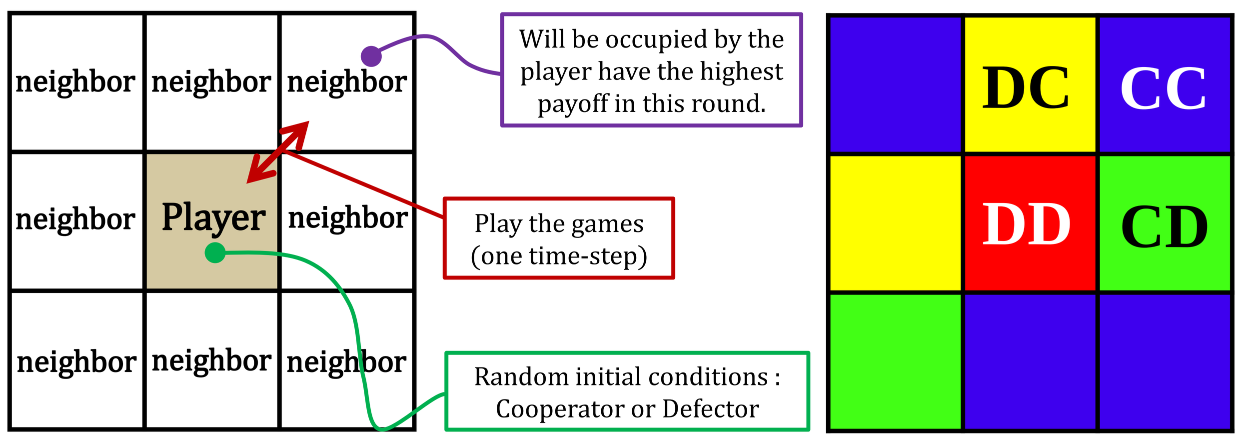
- **Modified Hawk-Dove game**

$$\frac{\partial u_1}{\partial t} = \Delta u_1 + u_1\left(\frac{T-q}{2}u_1(1-u_1) + T u_2\left(1 - u_1 - \frac{u_2}{2}\right)\right)$$
$$\frac{\partial u_2}{\partial t} = \Delta u_2 + u_2\left(\frac{T}{2}u_2 - \frac{T-q}{2}u_1^2 - T u_1 u_2 - \frac{T}{2}u_2^2\right)$$

- **Modified Snowdrift game**

$$\frac{\partial u_1}{\partial t} = \Delta u_1 + u_1\left(\frac{T-q}{2}u_1(1-u_1) + (q-T)u_1 u_2 + \left(\frac{T}{2} - q\right)u_2\right)$$
$$\frac{\partial u_2}{\partial t} = \Delta u_2 + u_2\left(\frac{T}{2}u_1(1-u_1) + \frac{q}{2}u_1^2 + (q-T)u_1 u_2\right)$$

3. Spatial structure : Lattice network model



CC : Cooperator → Cooperator **DC** : Defector → Cooperator

CD : Cooperator → Defector **DD** : Defector → Defector

Figure 1. The rule of the game and color map when the strategy changing on $n \times n$ lattice network model.

Numerical simulations

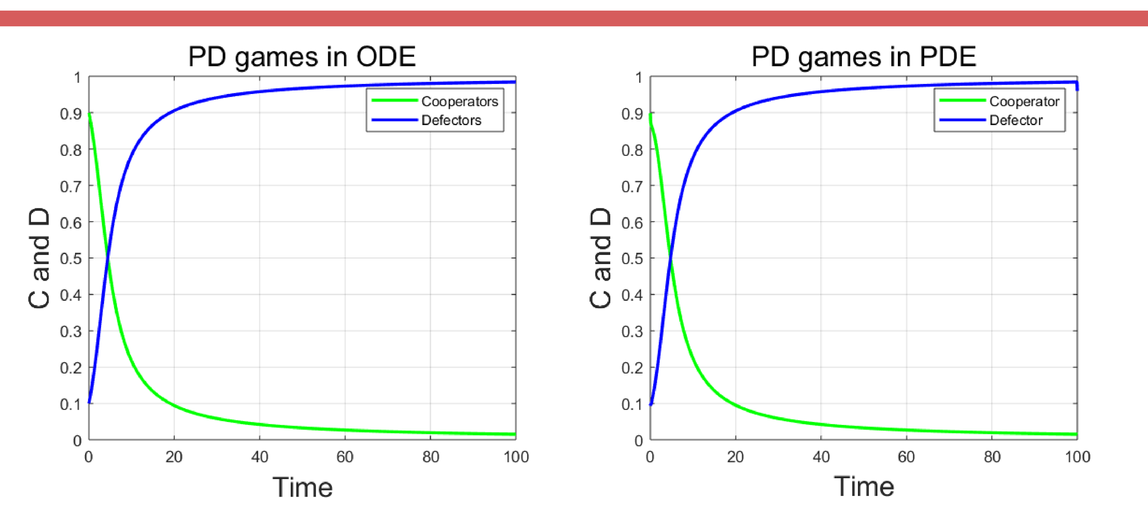
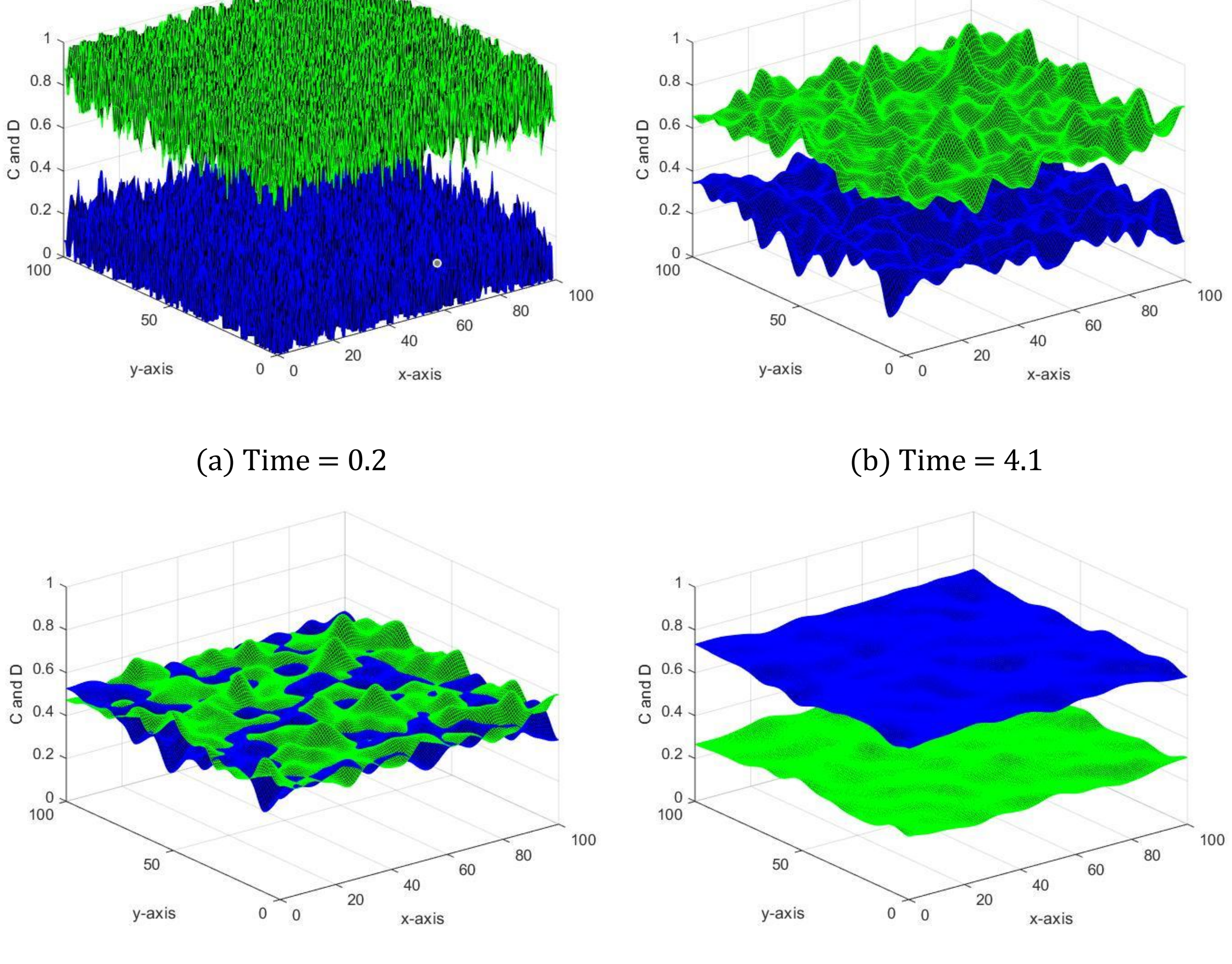


Figure 2. Comparison between ODEs and PDEs in the Prisoner's Dilemma (PD). 'C' : cooperator, 'D' : defector. The initial condition for C is 0.9, D is 0.1. The space domain $(x,y) \in [0,100] \times [0,100]$, T=1.9, random initial condition.



(a) Time = 0.2 (b) Time = 4.1

(c) Time = 4.8 (d) Time = 8.9

Figure 3. Spatial Prisoner's Dilemma (PDEs): mesh graph simulation of each time in figure 2.

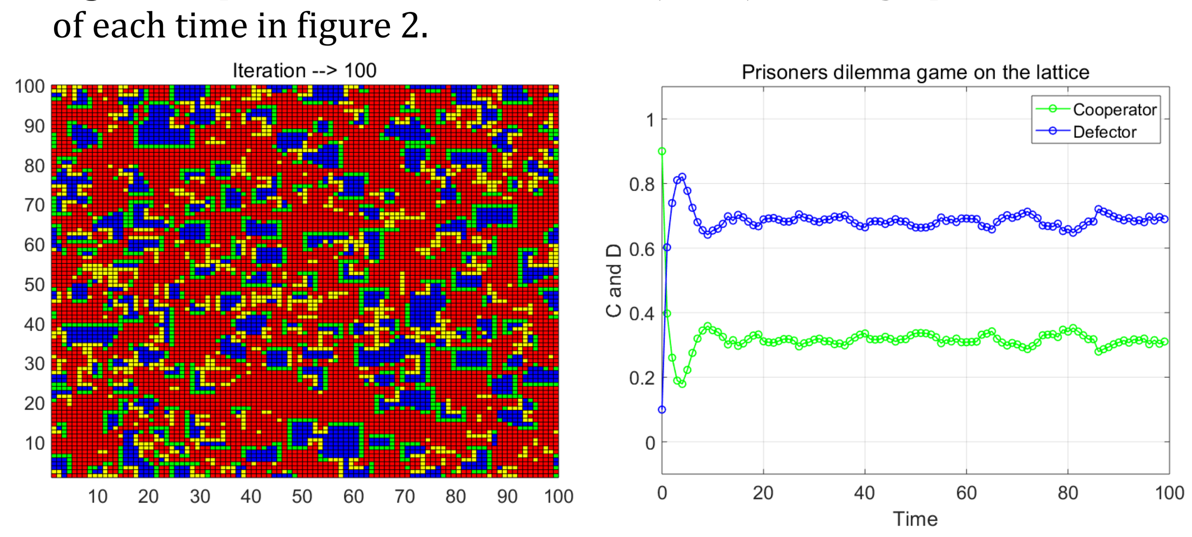


Figure 4. Prisoner's Dilemma game on the lattice network model. Lattice size : 100×100, T=1.9, random initial condition.

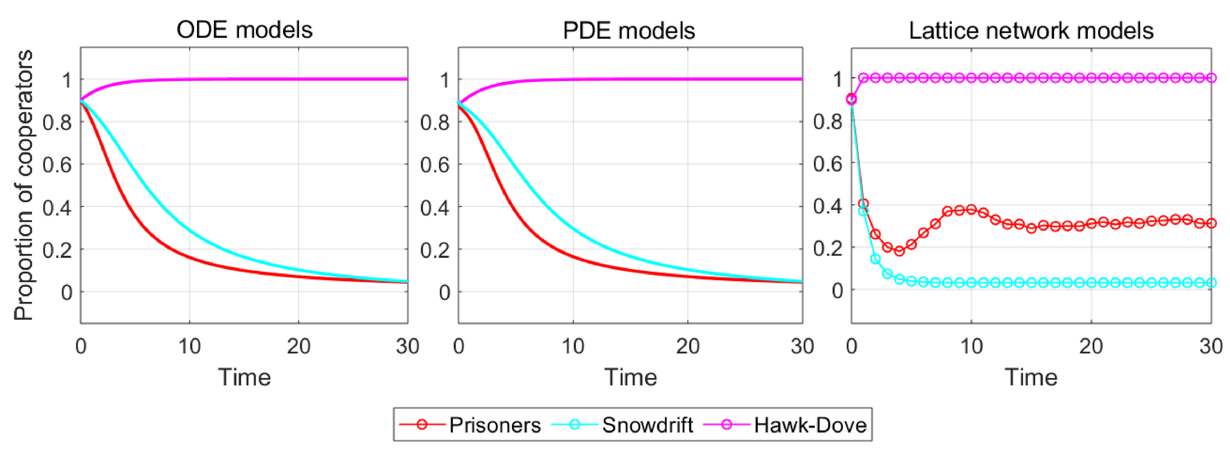
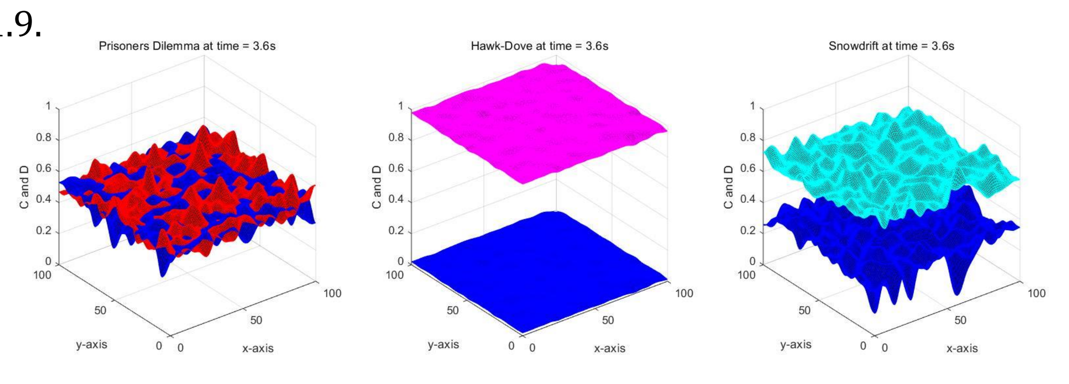


Figure 5. Comparison of ODE, PDE and Lattice network models in three games. The red color is the Prisoner's Dilemma, cyan and magenta are Snowdrift, Hawk-Dove, respectively. The curve represents the distribution of the cooperator in population. The parameters are q=1, T=1.9.



(a) Prisoner's Dilemma (b) Hawk-Dove (c) Snowdrift

Figure 6. Mesh graphs of time=3.6s. : the blue is the distribution of defectors and the red, magenta and cyan color are of cooperators.

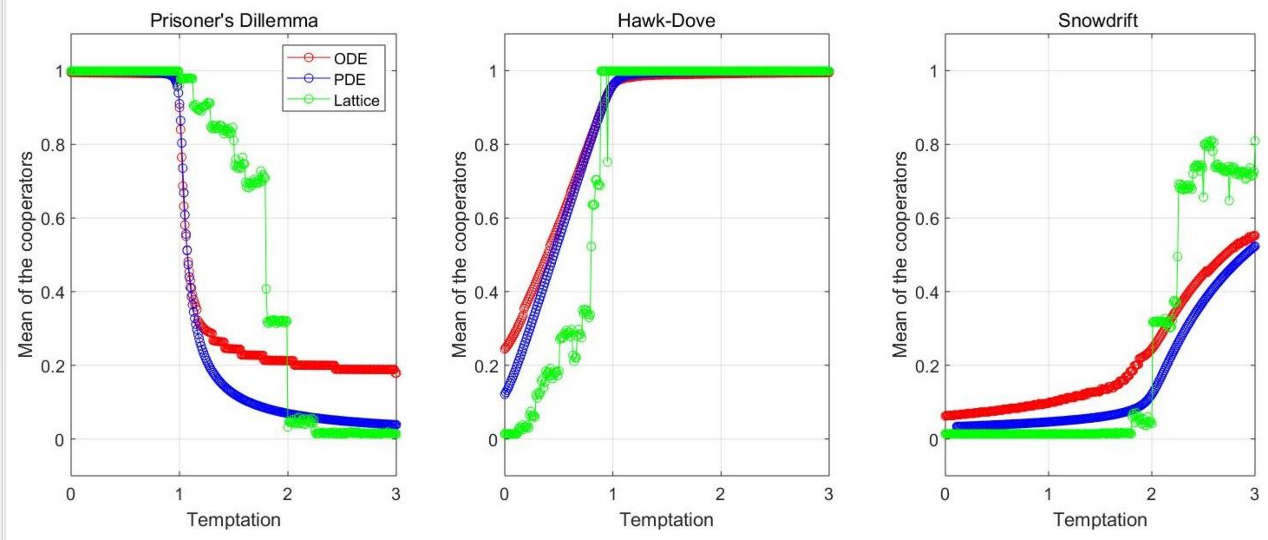
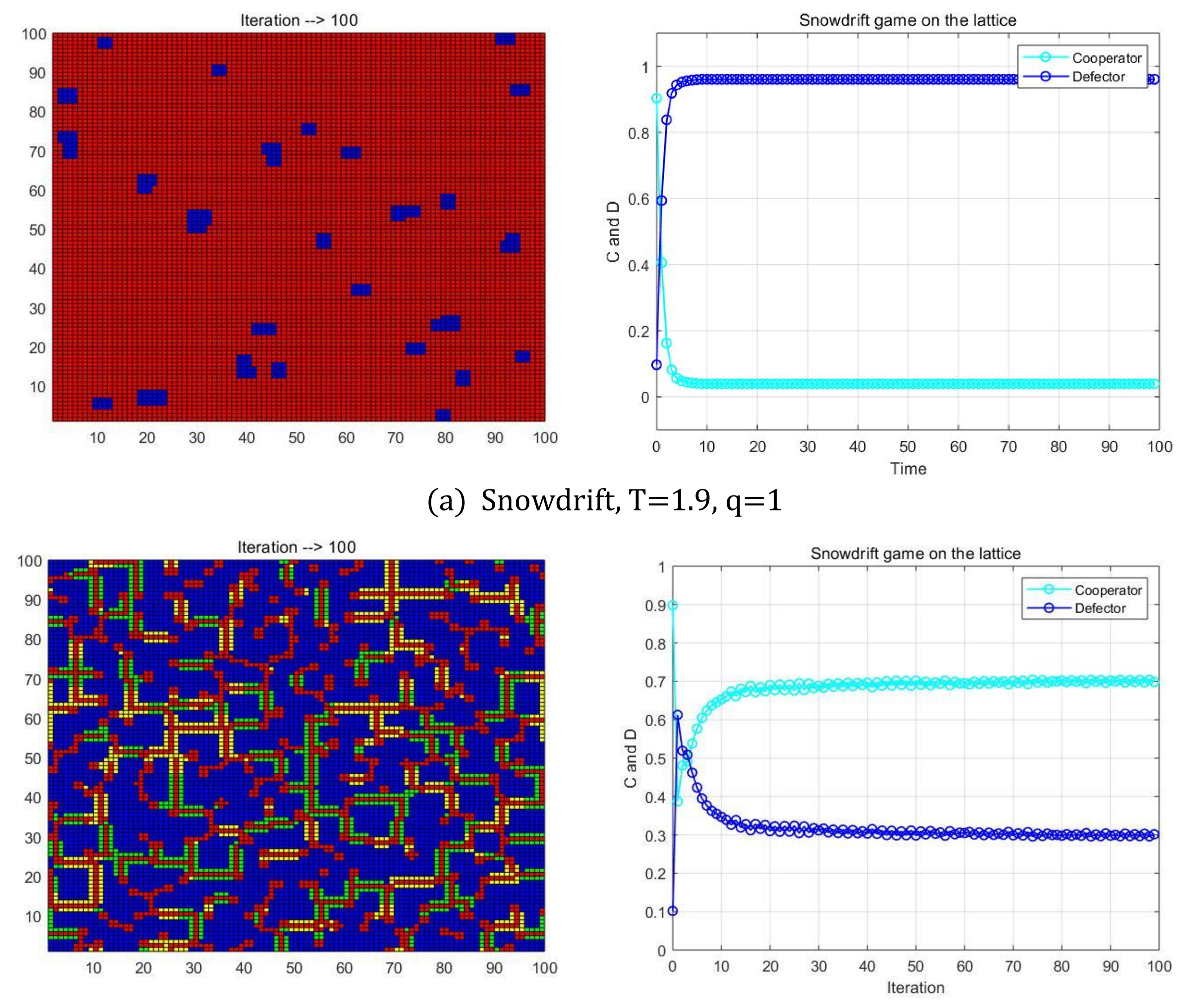


Figure 7. Bifurcation graph of the temptation parameter $0 \leq T \leq 3$ in three models. 'Cost' is fixed by q = 1. All models employ the conditions : $(x,y) \in [0,100] \times [0,100]$, 100 generations. Red, blue and green colors are ODE, PDE, Lattice network model, respectively.



(a) Snowdrift, T=1.9, q=1

(b) Snowdrift, T=1.9, q=0.8

Figure 8. Snowdrift game on the lattice network model. In this simulation, lattice size is 100×100, random initial condition.

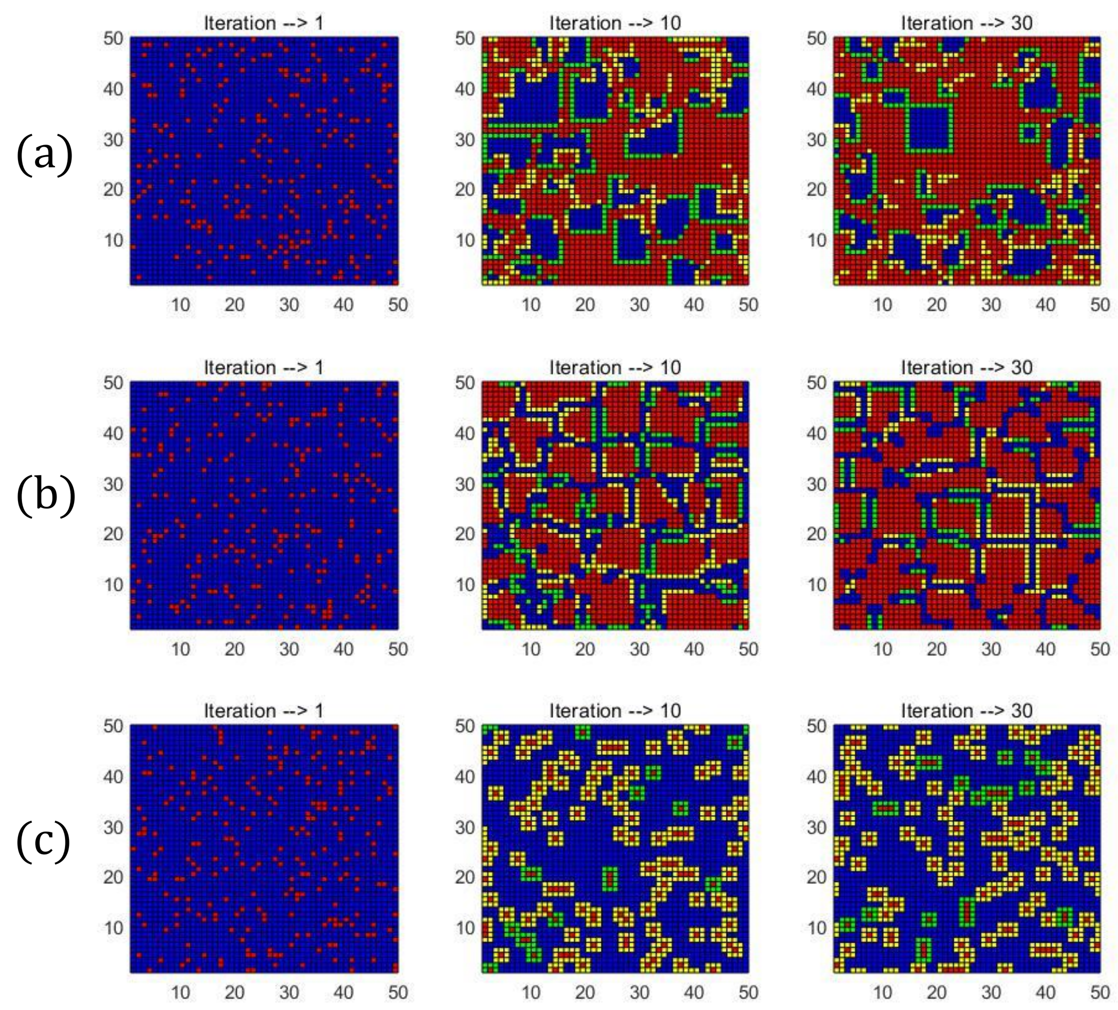


Figure 9. The time graph of three different payoff models fixed T=1.9 on the 50×50 lattice. (a) : Prisoner's, (b) : Hawk-Dove, q=2.5, (c) : Snowdrift, q=0.5.

Discussions

- We obtain the results that the spatial structure affects the proportion of the cooperators but the convergence behavior is similar between the models we suggest.
- In Hawk-Dove and Snowdrift game, both games get opposite results depending on the value of q, and opposite patterns on the spatial graph.
- Unlike other games, the proportion of C of the Prisoner's Dilemma is always converged to zero but does not converge to zero on the lattice.
- The results show that the spatial structure affects the proportion of the cooperators, however qualitative behavior is similar in all models.
- In Hawk-Dove and Snowdrift game, both games get opposite results depending on the value of q, and opposite patterns on spatial patterns.
- There is a discrepancy between the proportion of the cooperators of all three games (from ODE/PDE models) and the one from the lattice model.
- In fact, the difference in the proportion of the cooperators mentioned above is dependent on T and q.
- Prisoner's dilemma, Hawk-Dove, and Snowdrift game show different spatial patterns, hence, different the proportion of the cooperators.

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