# MATH7003-00: Assignment #3

**Sangman Jung**
e-mail: sangmanjung@khu.ac.kr
Kyung Hee University —April 1, 2020

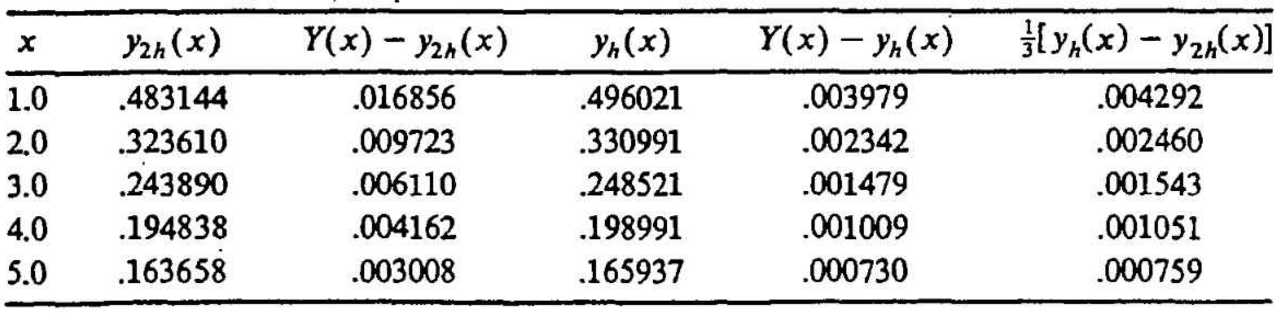
**Problem.** For the example on slide 20 (in week3-1), generate the same Table 6.8 using the Richardson error estimation. Discuss about the result. [1].

***Example.*** Consider the problem

$$y' = -y^2 \qquad\qquad y(0) = 1$$

which has the solution $Y(x) = 1/(1+x)$. The results in Table 6.8 are for stepsizes $h = 0.25$ and $2h = 0.5$. The last column is the error estimate (6.5.27), and it is an accurate estimator of the true error $Y(x) - y_h(x)$.

Table 6.8 Trapezoidal method and Richardson error estimation



| $x$ | $y_{2h}(x)$ | $Y(x) - y_{2h}(x)$ | $y_h(x)$ | $Y(x) - y_h(x)$ | $\frac{1}{3}[y_h(x) - y_{2h}(x)]$ |
|---|---|---|---|---|---|
| 1.0 | .483144 | .016856 | .496021 | .003979 | .004292 |
| 2.0 | .323610 | .009723 | .330991 | .002342 | .002460 |
| 3.0 | .243890 | .006110 | .248521 | .001479 | .001543 |
| 4.0 | .194838 | .004162 | .198991 | .001009 | .001051 |
| 5.0 | .163658 | .003008 | .165937 | .000730 | .000759 |

**Solution.** First, in order to estimate the numerical solution of the given differential equation, we have to find the numerical scheme of the equation. Using trapezoidal method, we obtain the numerical scheme of our equation as follows.

$$y_{n+1} = y_n - \frac{h}{2}\left(y_n^2 + y_{n+1}^2\right), \quad h > 0, \ n \geq 0.$$

We can see that the scheme is implicit method. There are many methods and formulas to compute the implicit scheme, we use the method as called **predictor-corrector method** in this problem[2][3]. According to [1], since the trapezoidal scheme is a nonlinear equation with root $y_{n+1}$, we adopt the root finding method of Chapter 2 in [1]. Hence, let $y_{n+1}^{(0)}$ be a good initial guess of the solution $y_{n+1}$, and define

$$y_{n+1}^{(j+1)} = y_n - \frac{h}{2}\left[y_n^2 + \left(y_{n+1}^{(j)}\right)^2\right], \quad j = 0,1,2,\cdots.$$

The initial guess is usually obtained using an explicit method such as Euler's method or midpoint method. For the successive iteration of this scheme, we have to investigate two conditions that

**(1)** choose the initial guess $y_{n+1}^{(0)} \simeq y_{n+1}$, **(2)** consider the iteration $j$ such that $\left|y_{n+1} - y_{n+1}^{(j)}\right| = O(h^4)$.

In the textbook [1] of our lecture, the case of initial guess using the Euler's method has $O(h^2)$ and the case of midpoint method has $O(h^3)$ because of :

$$y_{n+1} - y_{n+1}^{(0)} = y_{n+1} - u_n(x_{n+1}) + u_n(x_{n+1}) - y_{n+1}^{(1)} = O(h^2) \text{ in Euler's method case,}$$
$$\because \; y_{n+1} - u_n(x_{n+1}) = O(h^3),\; u_n - y_{n+1}^{(1)} = O(h^2).$$

By the Lipschitz condition,

$$\left|y_{n+1} - y_{n+1}^{(1)}\right| \le \frac{hK}{2}\left|y_{n+1} - y_{n+1}^{(0)}\right| \le O(h^3)$$

in Euler's method case and furthermore, we can obtain

$$\left|y_{n+1} - y_{n+1}^{(2)}\right| \le \frac{hK}{2}\left|y_{n+1} - y_{n+1}^{(1)}\right| \le O(h^4).$$

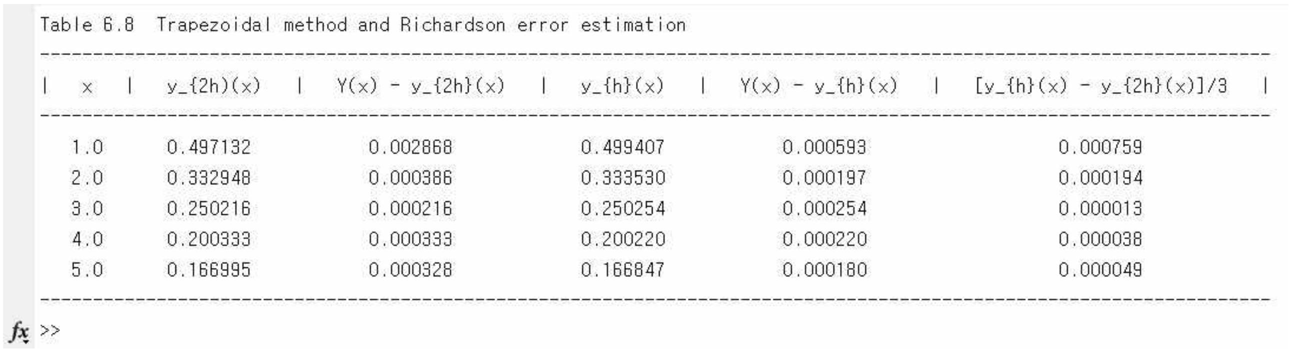
Thus, this means that we just iterate a twice times for initial guess, using Euler's method. In the same way to Euler's case, we can catch the number of iteration by

$$\left|y_{n+1} - y_{n+1}^{(1)}\right| \le \frac{hK}{2}\left|y_{n+1} - y_{n+1}^{(0)}\right| = O(h^3) \le O(h^4) \text{ in midpoint method case.}$$

Now, more precisely, we let the initial guess $y_{n+1}^{(0)} = y_{pred}^{(0)} = y_{n-1} - 2hy_n^2$ using midpoint method. $y_n$ is computed by using Euler's method as $y_n = y_{n-1} - hy_{n-1}^2$. Note that this initial guess is called **predictor**. From our trapezoidal scheme, now we can set the scheme as

$$y_{n+1}^{(j+1)} = y_n - \frac{h}{2}\left\{y_n^2 + \left[y_{n+1}^{(j)}\right]^2\right\} = y_n - \frac{h}{2}\left\{y_n^2 + \left[y_{pred}^{(j)}\right]^2\right\},\; j = 0,1,2,\cdots,\; n \ge 0.$$

The **corrector** term is just the trapezoidal method of the scheme. Hence, finally we update $y_{n+1} = y_{correct}^{(j+1)}$ in the iteration, we can get the numerical solution of the equation as follows.



```
Table 6.8  Trapezoidal method and Richardson error estimation
------------------------------------------------------------------------------------------------------------
|   x   |  y_{2h)(x)   |  Y(x) - y_{2h}(x)   |  y_{h}(x)   |  Y(x) - y_{h}(x)   |  [y_{h}(x) - y_{2h}(x)]/3   |
------------------------------------------------------------------------------------------------------------
   1.0     0.497132        0.002868            0.499407       0.000593               0.000759
   2.0     0.332948        0.000386            0.333530       0.000197               0.000194
   3.0     0.250216        0.000216            0.250254       0.000254               0.000013
   4.0     0.200333        0.000333            0.200220       0.000220               0.000038
   5.0     0.166995        0.000328            0.166847       0.000180               0.000049
------------------------------------------------------------------------------------------------------------
fx >>
```

**Figure 1.** Numerical computation of Table 6.8 using MATLAB.

The error $[y_h(x) - y_{2h}(x)]/3$ is called **Richardson error** and it was used to predict the error and to obtatin a more rapidly convergent numerical integration method. This is a practical procedure for estimating the global error. More briefly, the Richardson error is the approximation of the true error if we don't know the exact solution of $y'$. Note that $Y(x_n) - y_h(x_n) \doteq [y_h(x) - y_{2h}(x)]/3$. In **Figure 1.**, we can see that the Richardson error is an accurate estimator of the true error $Y(x) - y_h(x)$.

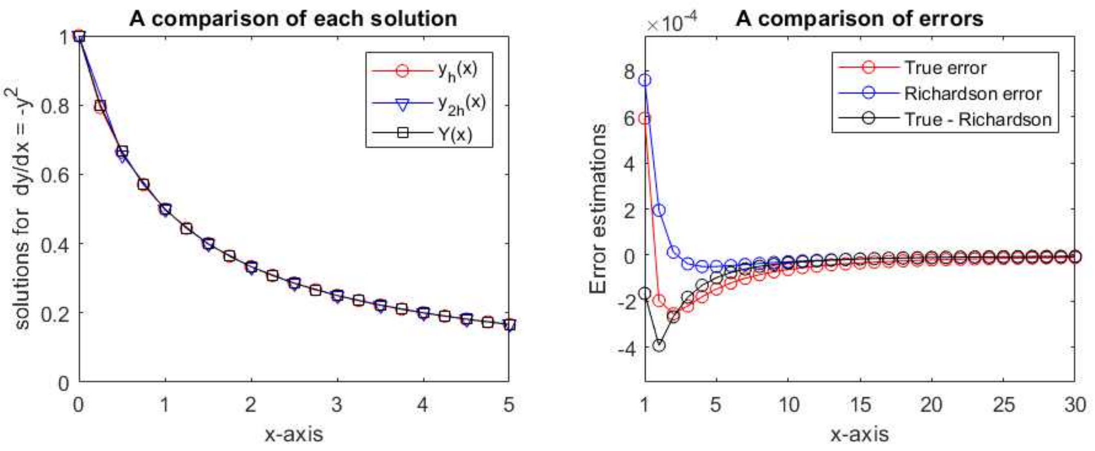


**Figure 2.** Two comparison graphs of the equation $y' = -y^2$. On the left side of this figure, we can see the comparison of three solutions : $y_h(x)$ (red color) and $y_{2h}(x)$ (blue color) for $h = 0.25$ and the exact solution, $Y(x)$ (black color), $x \in [0, 5]$. On the right side, we computed three error estimates : true error $e_T = Y(x) - y_h(x)$ at $h = 0.25$ (red color), Richardson error $e_{Richardson} = [y_h(x) - y_{2h}(x)]/3$ (blue color) and their difference $e_T - e_{Richardson}$ (black color), $x \in [0, 30]$.

In **Figure 2.**, the numerical scheme is nicely approximated to the exact solution of $y'$. It is sufficient to draw out the error comparison graph as shown above.

## Trapezoidal_method_HW3-(1).m (except Figure 2. code)

```
%% MATH7003-00: Assignment #3-(1), 2019310290 Sangman Jung
clear,clc

% Initial parameters, initial conditions
h = [0.25 0.5]; % step size
y(1) = 1; % initial value of y
Y = @(x) 1./(1+x)'; % exact solution of y'

% Trapezoidal method iteration to solve y'=-y^2
% Using predictor-corrector method, we can obtain the scheme as follows.
for h_iter = 1:2 % the loop of h and 2h
    x = 0:h(h_iter):5; % apply a different step size
    y = ones(size(x)); % reset the computed results of h=0.25 before
    y(2) = y(1)-h(h_iter)*y(1)^2; % define the term for midpoint, using Euler's method
    y_p(1)=y(1)-2*h(h_iter)*y(2)^2; % initial guess of y_p (midpoint method)
    for j =1:length(x)-1 % Trapezoidal method loop
        y_p(j+1)=y(j)-h(h_iter)*(y(j)^2+y_p(j)^2)/2; % predictor
        y(j+1)=y(j)-h(h_iter)*(y(j)^2+y_p(j+1)^2)/2; % corrector (In this code, numerical sol. of y')
    end
    fval(h_iter) = {y'}; % save the values of y have different size for y_h and y_2h
    xval(h_iter) = {x'}; % save the values of x have different size for y_h and y_2h
end

% Print out setting
x_h = cell2mat(xval(1)); % for the case of h = 0.25
x_2h = cell2mat(xval(2)); % for the case of h = 0.5
y_h = cell2mat(fval(1)); % numerical y for h = 0.25
y_2h = cell2mat(fval(2)); % numerical y for h = 0.5
y_ht = y_h(5:4:end);
y_2ht = y_2h(3:2:end);
t = 1:5;

% Print the Table 6.8
fprintf("Table 6.8  Trapezoidal method and Richardson error estimation\n");
fprintf("--------------------------------------------------------------------------------------------
------------------\n");
fprintf("|   x   |   y_{2h}(x)   |   Y(x) - y_{2h}(x)   |   y_{h}(x)   |   Y(x) - y_{h}(x)   |   [y_{h}(x) -
y_{2h}(x)]/3   |\n");
fprintf("--------------------------------------------------------------------------------------------
------------------\n");
for i = 1:length(t)
    fprintf('  %1.1f     %1.6f         %1.6f          %1.6f         %1.6f               %1.6f\n',...
        abs([t(i) y_2ht(i) Y(t(i))-y_2ht(i) y_ht(i) Y(t(i))-y_ht(i) (y_ht(i)-y_2ht(i))/3]));
end
fprintf("--------------------------------------------------------------------------------------------
------------------\n");
```

## References.

[1] Atkinson, K. E. (2008). An introduction to numerical analysis. John wiley & sons.

[2] Atkinson, K., Han, W., & Stewart, D. E. (2011). Numerical solution of ordinary differential equations (Vol. 108). John Wiley & Sons.

[3] Atkinson, K. E., & Han, W. (1985). Elementary numerical analysis (p. 17). New York et al.: Wiley.