MATH7003-00: Assignment #8

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Problem.

1. Consider the example on slide 10 and implement the Gauss-Jacobi method to obtain the Table 8.3. Discuss about the results. [1].

Table 8.3 Numerical results for the Gauss–Jacobi method

m	$x_1^{(m)}$	$x_2^{(m)}$	$x_3^{(m)}$	$\|e^{(m)}\|_\infty$	Ratio
0	0	0	0	1.0	
1	1.4	.5	1.4	.5	.5
2	1.11	1.20	1.11	.2	.4
3	.929	1.055	.929	.071	.36
4	.9906	.9645	.9906	.0355	.50
5	1.01159	.9953	1.01159	.01159	.33
6	1.000251	1.005795	1.000251	.005795	.50

2. Consider the example on slide 15 and implement the Gauss-Seidel method to obtain the Table 8.4. Discuss about the results. [1].

Table 8.4 Numerical results for the Gauss–Seidel method

m	$x_1^{(m)}$	$x_2^{(m)}$	$x_3^{(m)}$	$\|e^{(m)}\|_\infty$	Ratio
0	0	0	0	1	
1	1.4	.78	1.026	.4	.4
2	1.063400	1.020480	.987516	6.34E − 2	.16
3	.995104	.995276	1.001907	4.90E − 3	.077
4	1.001227	1.000817	.999632	1.23E − 3	.25
5	.999792	.999848	1.000066	2.08E − 4	.17
6	1.000039	1.000028	.999988	3.90E − 5	.19

3. Show the following inequalities (use the norm definition on slide 6);

3-(1) $\| Av \|_1 \leq \| A \|_1 \| v \|_1$

3-(2) $\| Av \|_\infty \leq \| A \|_\infty \| v \|_\infty$

Solution. (problem #1 and #2)

For the problem $Ax = b$ with $A = \begin{bmatrix} 10 & 3 & 1 \\ 2 & -10 & 3 \\ 1 & 3 & 10 \end{bmatrix}$, $b = \begin{bmatrix} 14 \\ -5 \\ 14 \end{bmatrix}$, and initial guess $x^{(0)} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$, the result of the Gauss-Jacobi method and Gauss-Seidel method is as follows.

명령 창

Table 8.3 Numerical results for the Gauss-Jacobi method

m	{x_{1}}^{(m)}	{x_{2}}^{(m)}	{x_{3}}^{(m)}	||e^{(m)}||_{inf}	Ratio
0	0.000000	0.000000	0.000000	1.000000	
1	1.400000	0.500000	1.400000	0.500000	0.5
2	1.110000	1.200000	1.110000	0.200000	0.4
3	0.929000	1.055000	0.929000	0.071000	0.35
4	0.990600	0.964500	0.990600	0.035500	0.5
5	1.011590	0.995300	1.011590	0.011590	0.33
6	1.000251	1.005795	1.000251	0.005795	0.5

Table 8.4 Numerical results for the Gauss-Seidel method

m	{x_{1}}^{(m)}	{x_{2}}^{(m)}	{x_{3}}^{(m)}	||e^{(m)}||_{inf}	Ratio
0	0.000000	0.000000	0.000000	1.000000	
1	1.400000	0.780000	1.026000	4.00e-01	0.4
2	1.063400	1.020480	0.987516	6.34e-02	0.16
3	0.995104	0.995276	1.001907	4.90e-03	0.077
4	1.001227	1.000817	0.999632	1.23e-03	0.25
5	0.999792	0.999848	1.000066	2.08e-04	0.17
6	1.000039	1.000028	0.999988	3.90e-05	0.19

fx >>

Figure 1. The results of the problem 1 and 2 using MATLAB. Note that Ratio $= \| e^{(m)} \|_{\infty} / \| e^{(m-1)} \|_{\infty}$.

Discussion. (problem #1 and #2)

For the case of the problem 1 in our textbook, we already obtained the inequality for the error that

$$\| e^{(m+1)} \|_{\infty} \le \mu \| e^{(m)} \|_{\infty} = 0.5 \| e^{(m)} \|_{\infty},$$

And also we have the inequality for the error in the case of the problem 2 that

$$\| e^{(m+1)} \|_{\infty} \le \eta \| e^{(m)} \|_{\infty} \quad \text{with} \quad \eta = \max_{1 \le i \le n} \frac{\beta_i}{1-\alpha_i}, \ \alpha_i = \sum_{j=1}^{i-1} \left| \frac{a_{ij}}{a_{ii}} \right|, \ \beta_i = \sum_{j=i+1}^{n} \left| \frac{a_{ij}}{a_{ii}} \right| \ \text{ for } i = 1, \ldots, n.$$

Then, by the calculation of above, we have $\eta = 0.4 < \mu = 0.5 < 1$. This shows the convergence of $e^{(m)} \to 0$ as $m \to \infty$, and also the speed of convergence for Gauss-Seidel method is significantly better than for the case of the Gauss-Jacobi method. The values of Ratio appear to converge to about 0.18.

Solution. (problem #3)

3-(1) proof:

Note that $\| A \|_1 = \underset{j}{\mathrm{Max}} \sum_i |a_{ij}|$ and $\| v \|_1 = \sum_i |v_i|$ by definition in Table 7.1. Then,

$$\| Av \|_1 = \sum_i |(Av)_i| \le \sum_i \sum_j |a_{ij}| |v_j| = \sum_j \left(\sum_i |a_{ij}| \right) |v_j|$$

$$\le \sum_j \left(\underset{k}{\mathrm{Max}} \sum_i |a_{ik}| \right) |v_j| = \| A \|_1 \| v \|_1$$

Thus, $\| Av \|_1 \le \| A \|_1 \| v \|_1$.

■

3-(2) proof:

Note that $\| A \|_\infty = \underset{i}{\mathrm{Max}} \sum_j |a_{ij}|$ and $\| v \|_\infty = \underset{i}{\mathrm{Max}} |v_i|$ by definition in Table 7.1. Then,

$$\| Av \|_\infty = \underset{i}{\mathrm{Max}} \sum_i |(Av)_i| \le \underset{i}{\mathrm{Max}} \sum_j |a_{ij}| |v_j|$$

$$\le \underset{i}{\mathrm{Max}} \left(\sum_j |a_{ij}| \right) \underset{j}{\mathrm{Max}} |v_j| = \| A \|_\infty \| v \|_\infty$$

Thus, $\| Av \|_\infty \le \| A \|_\infty \| v \|_\infty$.

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Table_8_3_and_Table_8_4_HW8.m

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%% MATH7003-00: Assignment #8, 2019310290 Sangman Jung.
clear,clc
% the problem for a linear equation Ax=b
m_iter = 7; % set the iteration numbers
A = [10 3 1; 2 -10 3; 1 3 10]; % matrix A
b = [14 -5 14]'; % vector b
true_x = [1 1 1]'; % solution of Ax=b
x(:,1) = [0 0 0]'; % initial guess for Gauss-Jacobi (GB)
x_sei(:,1) = x(:,1); % initial guess for Gauss-Seidal (GS)
e(:,1) = true_x - x(:,1); % error for (GB)
e_sei(:,1) = true_x - x_sei(:,1); % error for (GS)
norm_e(:,1) = norm(e(:,1),inf); % infinite norm of error 'e' (GB)
norm_e_sei(:,1) = norm(e_sei(:,1),inf); % infinite norm of error 'e' (GS)
Ratio = zeros(m_iter-1,1); % ratio == ||e^{(m)}||_{inf} / ||e^{(m-1)}||_{inf}
Ratio_sei = Ratio;

% Problem 1
fprintf('Table 8.3  Numerical results for the Gauss-Jacobi method\n');
fprintf('---------------------------------------------------------------------------------------------------\n');
fprintf('\tm\t\t{x_{1}}^{(m)}\t\t{x_{2}}^{(m)}\t\t{x_{3}}^{(m)}\t\t||e^{(m)}||_{inf}\t\tRatio\t\n');
fprintf('---------------------------------------------------------------------------------------------------\n');
fprintf('\t%d\t\t\t%1.6f\t\t\t%1.6f\t\t\t%1.6f\t\t\t%1.6f\t\t\t%1.2f\t\n',0,x(1,1),x(2,1),x(3,1),norm_e(:,1),'');
for m = 1:m_iter-1 % m iteration
    % search (i,j)-entry for the matrix A
    for i = 1:size(A,1) % position i
        % allocations
        sum_ax = 0;
        seisum1 = 0;
        seisum2 = 0;
        for j = 1:size(A,1) % position j
            if i ~= j
                sum_ax = sum_ax + A(i,j)*x(j,m); % inner product for the rows of A and the vector x
            end
        end
        x(i,m+1) = (b(i)-sum_ax)/A(i,i); % compute 'x' using Gauss-Jacobi method

        if i-1 == 0
            seisum1 = 0; % consideration for the end is zero
        else
            for j = 1:i-1
                seisum1 = seisum1 + A(i,j)*x_sei(j,m+1); % the first summation in GS
            end
        end
        for j = i+1:size(A,1)
            seisum2 = seisum2 + A(i,j)*x_sei(j,m); % the second summation in GS
        end
        x_sei(i,m+1) = (b(i)-seisum1-seisum2)/A(i,i); % compute 'x' using Gauss-Seidel method
    end
    % error update
    e(:,m+1) = true_x - x(:,m+1);
    e_sei(:,m+1) = true_x - x_sei(:,m+1);
    % infinite norm of error update
    norm_e(:,m+1) = norm(e(:,m+1),inf);
    norm_e_sei(:,m+1) = norm(e_sei(:,m+1),inf);
    % ratio update
    Ratio(m) = norm_e(:,m+1)/norm_e(:,m);
    Ratio_sei(m) = norm_e_sei(:,m+1)/norm_e_sei(:,m);
    fprintf('\t%d\t\t\t%1.6f\t\t\t%1.6f\t\t\t%1.6f\t\t\t%1.6f\t\t\t%1.2g\t\n',...
        m,x(1,m+1),x(2,m+1),x(3,m+1),norm_e(:,m+1),Ratio(m)); % print the values
end
fprintf('---------------------------------------------------------------------------------------------------\n');

% Problem 2
fprintf('\n\nTable 8.4  Numerical results for the Gauss-Seidel method\n');
fprintf('---------------------------------------------------------------------------------------------------\n');
fprintf('\tm\t\t{x_{1}}^{(m)}\t\t{x_{2}}^{(m)}\t\t{x_{3}}^{(m)}\t\t||e^{(m)}||_{inf}\t\tRatio\t\n');
fprintf('---------------------------------------------------------------------------------------------------\n');
fprintf('\t%d\t\t\t%1.6f\t\t\t%1.6f\t\t\t%1.6f\t\t\t%1.6f\t\t\t%1.2f\t\n',0,x(1,1),x(2,1),x(3,1),norm_e(:,1),'');
for m = 1:m_iter-1
    fprintf('\t%d\t\t\t%1.6f\t\t\t%1.6f\t\t\t%1.6f\t\t\t%1.2e\t\t\t%1.2g\t\n',...
        m,x_sei(1,m+1),x_sei(2,m+1),x_sei(3,m+1),norm_e_sei(:,m+1),Ratio_sei(m)); % print the values
end
fprintf('---------------------------------------------------------------------------------------------------\n');
```

References.

[1] Atkinson, K. E. (2008). An introduction to numerical analysis. John wiley & sons.

[2] Atkinson, K., Han, W., & Stewart, D. E. (2011). Numerical solution of ordinary differential equations (Vol. 108). John Wiley & Sons.

[3] Atkinson, K. E., & Han, W. (1985). Elementary numerical analysis (p. 17). New York et al.: Wiley.