MATH7003-00: Assignment #1

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Problem. Apply Euler's method for the following example and generate Table 6.5 [1].

Example Solve the pendulum equation,

$$\theta''(t) = -\sin(\theta(t)), \quad \theta(0) = \frac{\pi}{2}, \quad \theta'(0) = 0.$$

Convert this to a system by letting $y_1 = \theta$, $y_2 = \theta'$, and replace the variable t by x. Then

$$\begin{aligned} y_1' &= y_2 & y_1(0) &= \frac{\pi}{2} \\ y_2' &= -\sin(y_1) & y_2(0) &= 0 \end{aligned}$$

The numerical results are given in Table 6.5. Note that the error decreases by about half when h is halved.

Table 6.5 Euler's method for example (6.2.57)

h	x_n	$y_{1,n}$	$Y_1(x_n)$	Error	$y_{2,n}$	$Y_2(x_n)$	Error
0.2	.2	1.5708	1.5508	−.0200	−.20000	−.199999	.000001
	.6	1.4508	1.3910	−.0598	−.59984	−.59806	.00178
	1.0	1.1711	1.0749	−.0962	−.99267	−.97550	.01717
0.1	.2	1.5608	1.5508	−.0100	−.20000	−.199999	.000001
	.6	1.4208	1.3910	−.0298	−.59927	−.59806	.00121
	1.0	1.1223	1.0749	−.0474	−.98568	−.97550	.01018

Solution. Prior to compute the equation using Euler method in MATLAB, we have to calculate the exact solution of Y_1 and Y_2. According to the paper [2], our equation of the problem is generally written by

$$\frac{d^2\theta}{dt^2} + w_0^2 \sin\theta = 0, \qquad \theta(0) = \theta_0, \qquad \left(\frac{d\theta}{dt}\right)_{t=0} = 0,$$

Then the exact solution of the pendulum equation is as follows.

$$\theta(t) = 2\arcsin\left\{\sin\frac{\theta_0}{2}\, sn\left[K\left(\sin^2\frac{\theta_0}{2}\right) - w_0 t\, ;\, \sin^2\frac{\theta_0}{2}\right]\right\},$$

where sn is the **Jacobi elliptic function**, K is the **complete elliptical integral of the first kind**. The formal definitions of those are well documented in the paper. Let $\theta_0 = \pi/2$, $w_0 = 1$, then the equation above is the same as our equation, and we obtain the solution as follows.

$$\theta(t) = 2\arcsin\left\{\frac{\sqrt{2}}{2}\, sn\left[K\left(\frac{1}{2}\right) - t\, ;\, \frac{1}{2}\right]\right\}$$

The derivative of $\theta(t)$ is

$$\frac{d\theta(t)}{dt} = -\frac{2cn\left(t-K\left(\frac{1}{2}\right);\frac{1}{2}\right)dn\left(t-K\left(\frac{1}{2}\right);\frac{1}{2}\right)}{\sqrt{2-sn\left(t-K\left(\frac{1}{2}\right);\frac{1}{2}\right)^2}},$$

where cn, dn are other Jacobi elliptic function. We can find the value of $K(1/2)$ is $K(1/2)=8\pi^{3/2}/\Gamma(-1/4)^2$, and this term is computed $1.8540746\ldots$, using ***WolframAlpha***. So we finally obtain $K(1/2)\approx 1.8540746$. This can obtain '**`ellipke`**' in MATLAB also.

The Jacobi elliptic functions sn, cn, dn are easily computed by using Built-in function, '**`ellipj`**' in MATLAB. Thus we can catch the exact solution $y_1=\theta$ and $y_2=\theta'$.

Now, the result of this problem and MATLAB code are attached below.

```
Table 6.5  Euler's method for example (6.2.57)
-----------------------------------------------------------------------------
| h | x_{n} | y_{1,n} | Y_{1}(x_{n}) | Error | y_{2,n} | Y_{2}(x_{n}) | Error |
-----------------------------------------------------------------------------
0.2   0.0    1.5708       1.5708    0.0000   0.00000   -0.000000   -0.000000
0.2   0.2    1.5708       1.5508    -0.0200   -0.20000   -0.199992   0.000008
0.2   0.4    1.5308       1.4908    -0.0400   -0.40000   -0.399744   0.000256
0.2   0.6    1.4508       1.3910    -0.0598   -0.59984   -0.598061   0.001779
0.2   0.8    1.3308       1.2519    -0.0789   -0.79840   -0.791877   0.006524
0.2   1.0    1.1711       1.0749    -0.0962   -0.99267   -0.975510   0.017161
-----------------------------------------------------------------------------
0.1   0.0    1.5708       1.5708    0.0000   0.00000   -0.000000   -0.000000
0.1   0.1    1.5708       1.5658    -0.0050   -0.10000   -0.100000   0.000000
0.1   0.2    1.5608       1.5508    -0.0100   -0.20000   -0.199992   0.000008
0.1   0.3    1.5408       1.5258    -0.0150   -0.30000   -0.299939   0.000056
0.1   0.4    1.5108       1.4908    -0.0200   -0.39995   -0.399744   0.000206
0.1   0.5    1.4708       1.4459    -0.0249   -0.49977   -0.499220   0.000550
0.1   0.6    1.4208       1.3910    -0.0298   -0.59927   -0.598061   0.001209
0.1   0.7    1.3609       1.3263    -0.0346   -0.69815   -0.695819   0.002329
0.1   0.8    1.2911       1.2519    -0.0392   -0.79595   -0.791877   0.004076
0.1   0.9    1.2115       1.1680    -0.0435   -0.89207   -0.885437   0.006630
0.1   1.0    1.1223       1.0749    -0.0474   -0.98568   -0.975510   0.010171
-----------------------------------------------------------------------------
fx >>
```

Eulers_method_HW1.m

```
%% MATH7003-00: Assignment #1, 2019310290 Sangman Jung
clear,clc

% parameters, initial conditions
K = ellipke(1/2); % the complete elliptical integral of the first kind
h = [0.2 0.1]; % step size
y1(1) = pi/2; % initial value of y1
y2(1) = 0; % initial value of y2

% Euler method for the pendulum equation
fprintf("Table 6.5  Euler's method for example (6.2.57)\n");
fprintf("--------------------------------------------------------------------------------------\n");
fprintf("| h | x_{n} | y_{1,n} | Y_{1}(x_{n}) | Error | y_{2,n} | Y_{2}(x_{n}) | Error |\n");
fprintf("--------------------------------------------------------------------------------------\n");
for h_iter = 1:2
    x = 0:h(h_iter):1;
    for n = 1:length(x)
        y1(n+1) = y1(n)+h(h_iter)*y2(n);
        y2(n+1) = y2(n)-h(h_iter)*sin(y1(n));
        [SN(n),CN(n),DN(n)] = ellipj(K-x(n),1/2); % the Jacobi elliptic functions
        Y1(n) = 2*asin(sqrt(2)/2*SN(n)); % the exact solution of y1
        Y2(n) = -2*CN(n)*DN(n)/sqrt(2-SN(n)^2); % the exact solution of y2
        Error1(n) = Y1(n)-y1(n);
        Error2(n) = Y2(n)-y2(n);
        fprintf('%1.1f   %1.1f    %1.4f      %1.4f    %1.4f   %1.5f   %1.6f   %1.6f\n',...
            [h(h_iter) x(n) y1(n) Y1(n) Error1(n) y2(n) Y2(n) Error2(n)]);
    end
    fprintf("--------------------------------------------------------------------------------------\n");
end
```

References.

[1] Atkinson, K. E. (2008). An introduction to numerical analysis. John wiley & sons.

[2] Beléndez, A., Pascual, C., Méndez, D. I., Beléndez, T., & Neipp, C. (2007). Exact solution for the nonlinear pendulum. Revista brasileira de ensino de física, 29(4), 645-648.