exercise 1.1.1.

Consider the initial value problem for the equation

$$u_t + a u_x = f(t, x)$$

with $u(0, x) = 0$ and

$$f(t, x) = \begin{cases} 1 & x \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

Assume that a is positive. Show that the solution is

$$u(t, x) = \begin{cases} 0 & x \leq 0 \\ x/a & x \geq 0 \text{ \& } x - at \leq 0 \\ t & x \geq 0 \text{ \& } x - at \geq 0 \end{cases}$$

proof) Change of variables that

$$\begin{cases} t = \tau \\ \xi = x - at \end{cases} \Rightarrow \begin{cases} u(t, x) = \tilde{u}(\tau, \xi) \\ u(0, x) = \tilde{u}(0, \xi) = 0 \end{cases}$$

$$\frac{\partial \tilde{u}}{\partial \tau} = \frac{\partial u}{\partial t}\frac{\partial t}{\partial \tau} + \frac{\partial u}{\partial x}\frac{\partial x}{\partial t} = u_t(1) + u_x(a) = f(t, x)$$

$$\Rightarrow \frac{\partial \tilde{u}}{\partial \tau} = f(\tau, \xi + a\tau)$$

$$\Rightarrow \int_0^\tau \frac{\partial \tilde{u}}{\partial \sigma} d\sigma = \int_0^\tau f(\sigma, \xi + a\sigma) d\sigma$$

$$\Rightarrow \tilde{u}(\tau, \xi) - \tilde{u}(0, \xi) = \int_0^\tau f(\sigma, \xi + a\sigma) d\sigma$$

$$\therefore \tilde{u}(\tau, \xi) = \int_0^\tau f(\sigma, \xi + a\sigma) d\sigma$$

$$\Rightarrow u(t, x) = \int_0^t f(s, x - at + as) ds$$

We know that $a, s \geq 0$ & $0 < s < t$.

① $x \leq 0$: $f = 0 \Rightarrow u = 0$.

② $x \geq 0$ & $x - at \leq 0$:

Let $x^* = x - at + as$, & assume that $x^* \leq 0$. Then

$$x^* \leq 0 \iff 0 \leq s \leq t - \frac{x}{a}. \quad (f(s, x^*) = 0)$$

If $x^* \geq 0$, then

$$t - \frac{x}{a} \leq s \leq t. \quad (f(s, x^*) = 1)$$

$$u(t, x) = \int_0^t f(s, x^*)\, ds = \int_0^{t - \frac{x}{a}} f\, ds + \int_{t-\frac{x}{a}}^{t} f\, ds$$

$$= 0 + t - t + \frac{x}{a} = \frac{x}{a}.$$

③ $x \geq 0$ & $x - at \geq 0$:

$$x - at \geq 0 \Rightarrow x - at + as = x^* \geq 0 \quad (\because as \geq 0)$$

$$u(t, x) = \int_0^t f(s, x^*)\, ds = \int_0^t 1\, ds = t.$$

By ①, ②, ③,

$$u(t, x) = \begin{cases} 0 & x \leq 0 \\ x/a & x \geq 0 \ \& \ x - at \leq 0 \\ t & x \geq 0 \ \& \ x - at \geq 0 \end{cases}$$

$\square$

exercise 1.2.1.

Consider the system (*) :

$$\begin{bmatrix} u^1 \\ u^2 \end{bmatrix}_t + \begin{bmatrix} a & b \\ b & a \end{bmatrix}\begin{bmatrix} u^1 \\ u^2 \end{bmatrix}_x = \vec{0}, \quad a = 0 \ \& \ b = 1 \text{ on } [0,1]$$

with B.C : $u^1(t,0) = 0$, $u^1(t,1) = 1$

with I.C : $u^1(0,x) = x$, $u^2(0,x) = 1$.

Then, the solution is $u^1(t,x) = x$ and $u^2(t,x) = 1-t$ for all (t,x) with $0 \le x \le 1$, $0 \le t$.

proof) Since the system is hyperbolic form,

$A := \begin{bmatrix} a & b \\ b & a \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ is diagonalizable by definition.

$\det(A - \lambda I) = \lambda^2 - 1 = 0 \Rightarrow \lambda = \pm 1$.

$\lambda = 1 \Rightarrow \vec{v}_1 = (1,1)^T$, $\lambda = -1 \Rightarrow \vec{v}_2 = (-1,1)^T$.

$\therefore P = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$, $P^{-1} = \begin{bmatrix} 1/2 & 1/2 \\ -1/2 & 1/2 \end{bmatrix}$ such that

$\Lambda := PAP^{-1}$: diagonal matrix.

Note that $\begin{cases} \lambda_i \equiv \text{diagonal elements of } \Lambda \\ \quad \equiv \text{characteristic speed of } u^i \end{cases}$

$(*) \Rightarrow P\vec{u}_t + PA\vec{u}_x = \vec{0} \Rightarrow P\vec{u}_t + \Lambda P \vec{u}_x = \vec{0}$

$$\Rightarrow \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} u^1 \\ u^2 \end{bmatrix}_t + \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1/2 & 1/2 \\ -1/2 & 1/2 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} u^1 \\ u^2 \end{bmatrix}_x = \vec{0}$$

$$\Rightarrow \begin{bmatrix} u^1 - u^2 \\ u^1 + u^2 \end{bmatrix}_t + \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} u^1 - u^2 \\ u^1 + u^2 \end{bmatrix}_x = \vec{0}$$

Let $w^1 = u^1 - u^2$ & $w^2 = u^1 + u^2$. Since λ_i are the characteristic speed of w^i, we can define

$$\xi_1 = x + t \quad \& \quad \xi_2 = x - t .$$

Now, the system is

$$w^1_t - w^2_x = 0 \quad \& \quad w^1_t + w^2_x = 0$$

with I.C : $w^1(0,x) = u^1(0,x) - u^2(0,x) = x - 1$

$w^2(0,x) = u^1(0,x) + u^2(0,x) = x + 1$

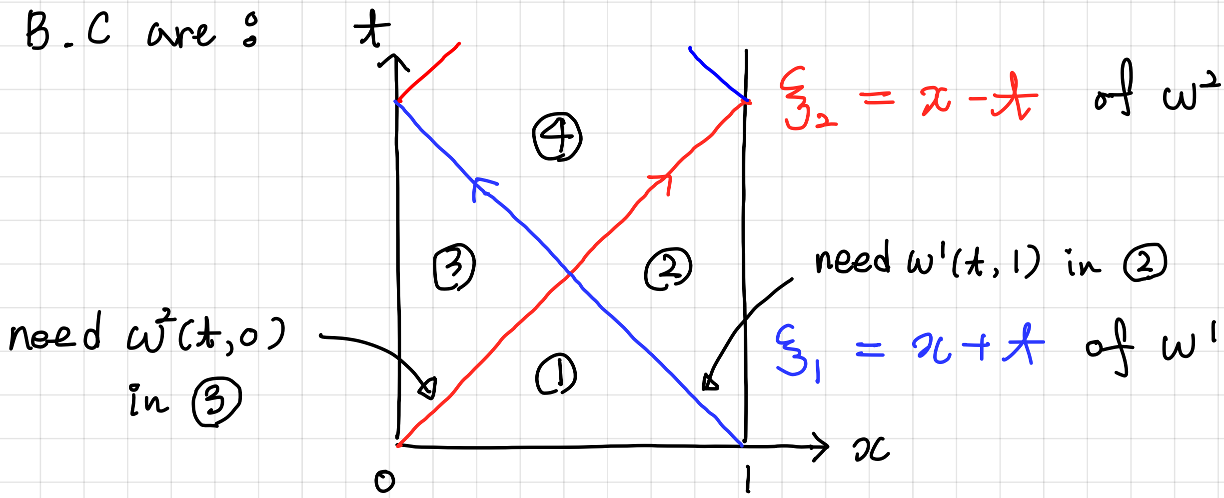
with B.C : $w^2(t,0) = u^1 + u^2 = u^1 + u^1 - w^1$

$= 2u^1(t,0) - w^1(t,0) = - w^1(t,0)$

$w^1(t,1) = u^1 - u^2 = u^1 - w^2 + u^1$

$= 2u^1 - w^2 = 2 - w^2(t,1)$

(∵ B.C are :



$$\begin{cases} \xi_2 \text{ need boundary : } w^1(t,1) \leftrightarrow w^2(t,1) \\ \xi_1 \text{ need boundary : } w^2(t,0) \leftrightarrow w^1(t,0) \end{cases}$$

→ 영역을 먼저 고려한 후 필요한 B.C.에 대한 계산.

④ : ①, ②, ③ 에서 구한 값으로 일반화 가능하므로 제외.)

① : Both (solutions) are determined I.C.

$$w^1(t,x) = w^1(0,x+t) = x+t-1$$
$$w^2(t,x) = w^2(0,x-t) = x-t+1$$

$$\Rightarrow \left.\begin{aligned} u^1(t,x) &= \frac{w^1(t,x)+w^2(t,x)}{2} = x \\ u^2(t,x) &= -\frac{w^1(t,x)-w^2(t,x)}{2} = 1-t \end{aligned}\right\} \text{inverse relation.}$$

② : w^2 determined by ①. But w^1 need B.C at $x=1$.

$$w^1(t,1) = 2 - w^2(t,1) = t \quad (\because w^2(t,x) = x-t+1 \text{ by ①})$$

$$\therefore w^1(t,x) = w^1(\underset{(1)}{x+t-1}, 1) = x+t-1$$

$$\Rightarrow u^1(t,x) = \frac{w^1(t,x)+w^2(t,x)}{2} = x$$
$$u^2(t,x) = -\frac{w^1(t,x)-w^2(t,x)}{2} = 1-t$$

③ : w^1 determined by ①. But w^2 need B.C at $x=0$.

$$w^2(t,0) = -w^1(t,0) = 1-t \quad (\because w^1(t,x) = x+t-1 \text{ by ①})$$

$$\therefore w^2(t,x) = w^2(\underset{(2)}{t-x}, 0) = 1-t+x.$$

$$\Rightarrow u^1(t,x) = \frac{w^1(t,x)+w^2(t,x)}{2} = x$$
$$u^2(t,x) = -\frac{w^1(t,x)-w^2(t,x)}{2} = 1-t$$

By ①, ②, ③, we obtain the solution of u^1, u^2 :

$$u^1(t,x) = x, \quad u^2(t,x) = 1-t.$$

$\square$

In exercise (1.1.1)

$u_t + a u_x + bu = f(t,x)$: general hyperbolic eq.

with initial condition $u(0,x) = u_0(x)$, a, b : constant.

$\rightsquigarrow$ PDE to ODE using a change of variable.

Let $\tau = t$, $\xi = x - at \iff t = \tau$, $x = \xi + a\tau$.

($\because$) the solution at (t,x) depends only on the value

of $\xi = x - at$: "characteristic curve"

Then, $u(t,x) = \tilde{u}(t,x)$ ($\because$ $\sim$: need to distinguish)

Since $\dfrac{\partial \tilde{u}}{\partial \tau} = \dfrac{\partial u}{\partial t}\dfrac{\partial t}{\partial \tau} + \dfrac{\partial u}{\partial x}\dfrac{\partial x}{\partial \tau} = u_t + a u_x$

$= -bu + f(\tau, \xi + a\tau)$,

$\therefore \dfrac{\partial \tilde{u}}{\partial \tau} = -b\tilde{u} + f(\tau, \xi + a\tau) \iff \tilde{u}' + b\tilde{u} = f$

integrating factor $\lambda = e^{\int b \, d\tau} = e^{b\tau}$.

$\Rightarrow \dfrac{\partial \tilde{u}}{\partial \tau} e^{b\tau} + b\tilde{u} e^{b\tau} = f(\tau, \xi + a\tau) e^{b\tau}$

$\Rightarrow \dfrac{\partial}{\partial \tau}\{e^{b\tau}\tilde{u}\} = f(\tau, \xi + a\tau) e^{b\tau}$ ($\because$ 곱의 미분법)

τ에 대한 f 적분을 얻기위해 정적분 $0 \sim \tau$ 을 취하면,

$$\int_0^\tau \frac{\partial}{\partial \sigma}\{e^{b\sigma}\tilde{u}\}\, d\sigma = e^{b\tau}\tilde{u}(\tau,\xi) - \tilde{u}(0,\xi)$$

$$= e^{b\tau}\tilde{u}(\tau,\xi) - u_0(\xi)$$

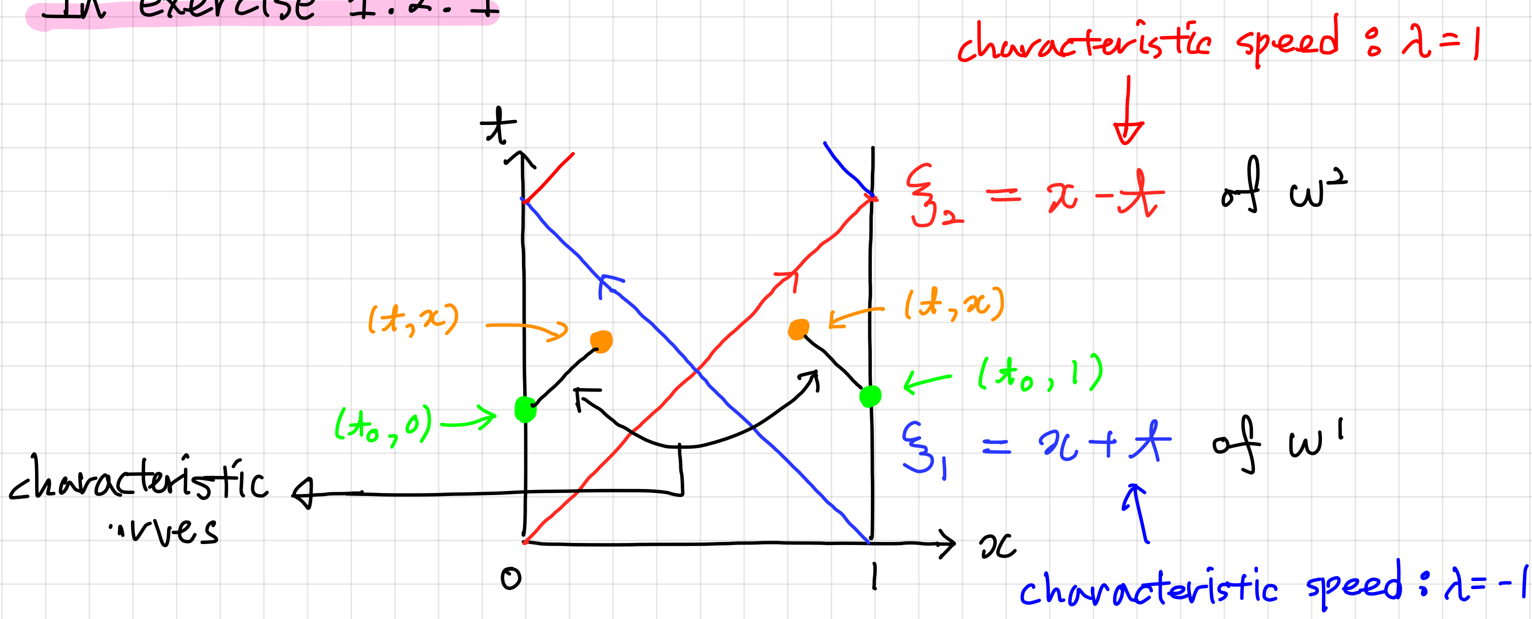
$$= \int_0^\tau f(\sigma, \xi + a\sigma) e^{b\sigma}\, d\sigma$$

$\Rightarrow \tilde{u}(\tau,\xi) = u_0(\xi)e^{-b\tau} + e^{-b\tau}\int_0^\tau f(\sigma, \xi + a\sigma) e^{b\sigma}\, d\sigma$

put it

$\therefore u(t,x) = u_0(x)e^{-bt} + \int_0^t f(s, x - at + as) e^{b(s-t)}\, ds$.

In exercise 1.2.1



② : $w^1(t, x) = w^1(\underbrace{x+t-1}_{(1)}, 1) = x + t - 1$

③ : $w^2(t, x) = w^2(\underbrace{t - x}_{(2)}, 0) = 1 - t + x$

"Considering extending to the interior along the characteristics."

(1) in ②, the relationship of $\lambda = -1$ is :

$$\frac{x - 1}{t - t_0} = -1 \iff t_0 = x + t - 1$$

(2) in ③, the relationship of $\lambda = 1$ is :

$$\frac{x - 0}{t - t_0} = 1 \iff t_0 = t - x$$

$*$ speed $= \dfrac{\text{distance}}{\text{time}} = \dfrac{dx}{dt} \equiv a$.

exercise 1.4.1.

Show that the FTCS scheme :

$$\frac{v_m^{n+1} - v_m^n}{k} + a\frac{v_{m+1}^n - v_{m-1}^n}{2h} = 0$$

is consistent with equation $u_t + a u_x = 0$.

proof) Let $P = \frac{\partial}{\partial t} + a\frac{\partial}{\partial x}$, then $u_t + a u_x = Pu = 0$.

Define $\phi(t,x)$ is smooth function such that $P\phi = \phi_t + a\phi_x$. Then, the difference operator $P_{k,h}$ is given as follows.

$$P_{k,h}\phi = \frac{\phi_m^{n+1} - \phi_m^n}{k} + a\frac{\phi_{m+1}^n - \phi_{m-1}^n}{2h}$$

Using the Taylor series, we obtain

$$\phi_m^{n+1} = \phi_m^n + \phi_t k + \frac{\phi_{tt}}{2}k^2 + O(k^3)$$
$$\phi_{m\pm1}^n = \phi_m^n \pm \phi_x h + \frac{\phi_{xx}}{2}h^2 \pm O(h^3)$$

$$P_{k,h}\phi = \phi_t + \frac{k}{2}\phi_{tt} + O(k^3) + a/2h\,(2h\phi_x + O(h^3))$$
$$= \phi_t + a\phi_x + \frac{k}{2}\phi_{tt} + O(k^3 + h^3)$$

$$\Rightarrow P\phi - P_{k,h}\phi = \frac{k}{2}\phi_{tt} + O(k^3 + h^3)$$

$$P\phi - P_{k,h}\phi \to 0 \quad \text{as} \quad (k,h) \to 0$$

$\therefore$ This scheme is consistent. $\square$

exercise 1.4.2.

Show that the leapfrog scheme :

$$\frac{v_m^{n+1} - v_m^{n-1}}{2k} + a\frac{v_{m+1}^n - v_{m-1}^n}{2h} = 0$$

is consistent with equation $u_t + au_x = 0$

proof) $u_t + au_x = 0 \Leftrightarrow Pu = 0$, $P = \frac{\partial}{\partial t} + a\frac{\partial}{\partial x}$.

Define ϕ : smooth fn s.t. $P\phi = \phi_t + a\phi_x$.

$$P_{k,h}\phi = \frac{\phi_m^{n+1} - \phi_m^{n-1}}{2k} + a\frac{\phi_{m+1}^n - \phi_{m-1}^n}{2h} = 0$$

Using Taylor series of ϕ, we obtain

$$\phi_m^{n\pm1} = \phi_m^n \pm k\phi_t + \frac{k^2}{2}\phi_{tt} \pm \frac{k^3}{3!}\phi_{ttt} + O(k^4)$$

$$\phi_{m\pm1}^n = \phi_m^n \pm h\phi_x + \frac{h^2}{2}\phi_{xx} \pm \frac{h^3}{3!}\phi_{xxx} + O(h^4)$$

$$\Rightarrow P_{k,h}\phi = \phi_t + \frac{k^2}{6}\phi_{ttt} + O(k^4) + a\phi_x + \frac{h^2}{6}\phi_{xxx} + O(h^4)$$

$$= \phi_t + a\phi_x + \frac{1}{6}(k^2\phi_{ttt} + h^2\phi_{xxx}) + O(k^4 + h^4)$$

$$P\phi - P_{k,h}\phi = \frac{1}{6}(k^2\phi_{ttt} + h^2\phi_{xxx}) + O(k^4 + h^4)$$

$P\phi - P_{k,h}\phi \to 0$ as $(k,h) \to 0$.

$\therefore$ This scheme is consistent. $\square$

exercise 2.1.4.

Show that the initial value problem for the equation $u_t = u_{xxx}$ is well-posed.

proof) Note that Fourier transform of $u(x)$ is

$$\hat{u}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\omega x} u(x)\, dx \quad (*)$$

Fourier inversion formula of $\hat{u}(\omega)$ is

$$u(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\omega x} \hat{u}(\omega)\, d\omega \quad (**)$$

$$\Rightarrow \frac{\partial u}{\partial x} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\omega x} i\omega \hat{u}(\omega)\, d\omega \quad \text{by } (**)$$

$$\Rightarrow \frac{\partial \hat{u}}{\partial x} = i\omega \hat{u}(\omega) \quad \text{by } (*)$$

$$\Rightarrow u_{xxx} = (i\omega)^3 \hat{u}(\omega) = -i\omega^3 \hat{u}(\omega) = \hat{u}_t$$

By Parseval's relation, (PR)

$$\int_{-\infty}^{\infty} |u(t,x)|^2 dx = \int_{-\infty}^{\infty} |\hat{u}(t,\omega)|^2 d\omega \leftarrow \text{PR}$$

$\hat{u}(t,\omega) = e^{-ia\omega t}\hat{u}_0(\omega) \rightarrow$ $$= \int_{-\infty}^{\infty} |e^{-i\omega^3 t} \hat{u}_0(\omega)| d\omega$$

$|e^{i\theta}| = 1 \rightarrow$ $$= \int_{-\infty}^{\infty} |\hat{u}_0(\omega)|^2 d\omega$$

$$= \int_{-\infty}^{\infty} |u_0(x)|^2 dx \leftarrow \text{PR}$$

Let $C_T = 1$, then, the equation $u_t = u_{xxx}$ is well-posed. ∎

exercise 2.1.5.

Show that the initial value problem for the equation $u_t + u_x + bu = 0$ is well-posed.

proof) Using Fourier transform of u & its inversion formula, we obtain

$$\frac{\partial \hat{u}}{\partial x} = i\omega \hat{u}(\omega) \Rightarrow \hat{u}_t = -i\omega \hat{u}(\omega) - b\hat{u}(\omega)$$
$$= -(i\omega + b)\hat{u}(\omega).$$

Then, the solution is,

$$\hat{u}(t, \omega) = e^{-(i\omega+b)t}\hat{u}_0(\omega).$$

By Parseval's relation, then

$$\int_{-\infty}^{\infty} |u(t,x)|^2 dx = \int_{-\infty}^{\infty} |\hat{u}(t,\omega)|^2 d\omega$$
$$= e^{-bt}\int_{-\infty}^{\infty} |e^{-i\omega t}\hat{u}_0(\omega)|^2 d\omega$$
$$= e^{-bt}\int_{-\infty}^{\infty} |\hat{u}_0(\omega)|^2 d\omega$$
$$= e^{-bt}\int_{-\infty}^{\infty} |u_0(x)|^2 dx$$

Let $C_T = e^{-bt}$, then, the equation is well-posed. ∎

exercise 2.2.1.

Show that the BTCS scheme

$$\frac{v_m^{n+1} - v_m^n}{k} + a\frac{v_{m+1}^{n+1} - v_{m-1}^{n+1}}{2h} = 0$$

is consistent with $u_t + a u_x = 0$ & is unconditionally stable.

proof) ① Consistency

Let ϕ be a smooth function that

$P\phi = \phi_t + a\phi_x$ where $P = \frac{\partial}{\partial t} + a\frac{\partial}{\partial x}$, and

the difference operator $P_{k,h}$ that

$$P_{k,h}\phi = \frac{\phi_m^{n+1} - \phi_m^n}{k} + a\frac{\phi_{m+1}^{n+1} - \phi_{m-1}^{n+1}}{2h}.$$

Then, using Taylor series of ϕ,

$$\phi_m^{n+1} = \phi_m^n + k\phi_t + \frac{k^2}{2}\phi_{tt} + \frac{k^3}{6}\phi_{ttt} + O(k^4)$$

$$\phi_{m\pm1}^{n+1} = \phi_m^n \pm h\phi_x + \frac{h^2}{2}\phi_{xx} \pm \frac{h^3}{6}\phi_{xxx} + O(h^4)$$

$$\Rightarrow P_{k,h}\phi = \phi_t + \frac{k}{2}\phi_{tt} + \frac{k^2}{6}\phi_{ttt} + O(k^4)$$
$$+ a\phi_x + \frac{h^2}{6}\phi_{xxx} + O(h^4)$$

$$\Rightarrow P\phi - P_{k,h}\phi = \frac{k}{2}\phi_{tt} + \frac{1}{6}(k^2\phi_{ttt} + h^2\phi_{xxx}) + O(k^4+h^4)$$

$P\phi - P_{k,h}\phi \to 0$ as $k, h \to 0$

$\therefore$ this scheme is consistent.

② unconditionally stable.

Let $v_m^n = g^n e^{im\theta}$, then the scheme is

$$\frac{g^{n+1}e^{im\theta} - g^n e^{im\theta}}{k} + a\frac{g^{n+1}e^{i(m+1)\theta} - g^{n+1}e^{i(m-1)\theta}}{2h} = 0$$

$$\Rightarrow g^n e^{im\theta} \left\{ \frac{g-1}{k} + a\,\frac{ge^{i\theta} - ge^{-i\theta}}{2h} \right\} = 0$$

$$\Rightarrow g - 1 + \frac{ak}{2h}\left(ge^{i\theta} - ge^{-i\theta}\right)$$

$$= g - 1 + \frac{ak}{2h} g\left(\cos\theta + i\sin\theta - \cos\theta + i\sin\theta\right)$$

$$= g\left(\frac{ak}{2h}(2i\sin\theta) + 1\right) - 1 = 0, \quad \lambda = \frac{k}{h},$$

$$\Rightarrow g(\theta) = \frac{1}{a\lambda i\sin\theta}. \quad \Rightarrow |g(\theta)| \le 1 \text{ clearly.}$$

$\therefore$ this scheme is stable. (unconditionally)

Note) Lax-equivalence theorem (linear PDE)

Consistency & Stability $\Leftrightarrow$ Convergence.

$\uparrow$ Taylor expension $\qquad$ $\uparrow$ Property of numerical scheme

$\blacksquare$

The Lax-Wendroff Scheme

→ second-order accuracy scheme.

For inhomogeneous equation $u_t + au_x = f$,

using Taylor series in time for $u(t+k, x)$, then

$$\left\{ u(t+k,x) = u(t,x) + k\,u_t(t,x) + \frac{k^2}{2}u_{tt}(t,x) + O(k^3) \right\}$$

Since $u_t = -au_x + f$, $u_{tt} = -a\,u_{xt} + f_t = -a\,u_{tx} + f_t$.

$$u_{tx} = (u_t)_x = (-au_x + f)_x = -au_{xx} + f_x.$$

$$\Rightarrow u_{tt} = a^2 u_{xx} - af_x + f_t$$ ← put in the Taylor series..

replacing $u_t \to u_x$ term, $u_{tt} \to u_{xx}$ term

$$u(t+k,x) = u(t,x) + k(-au_x + f) + \frac{k^2}{2}(a^2u_{xx} - af_x + f_t) + O(k^3)$$

∂x : 2nd-order difference, f_t : forward difference

$$u(t+k,x) = u(t,x) - ak\frac{u(t,x+h) - u(t,x-h)}{2h}$$

$$+ \frac{(ak)^2}{2}\left(\frac{u(t,x+h) - 2u(t,x) + u(t,x-h)}{h^2}\right)$$

$$+ \frac{k}{2}\left(f(t+k,x) - f(t,x)\right) - \frac{ak^2}{2}\left(\frac{f(t,x+h) - f(t,x-h)}{2h}\right)$$

$$+ O(kh^2) + O(k^3) \quad \cdots (*)$$

Replacing $(*)$ to difference operator v_m^n,

we obtain the Lax-Wendroff scheme. $\blacksquare$

The Crank-Nicolson Scheme

→ second-order accuracy scheme.

For the point $(t+k/2, x)$, use the formula that

$$u_t(t+\tfrac{1}{2}k, x) = \frac{u(t+k,x)-u(t,x)}{k} + O(k^2),$$ then

$$u_x(t+\tfrac{1}{2}k, x) = \frac{u_x(t+k,x)-u_x(t,x)}{2} + O(k^2)$$

$$= \frac{1}{2}\left[\frac{u(t+k,x+h)-u(t+k,x-h)}{2h} + \frac{u(t,x+h)-u(t,x-h)}{2h}\right] + O(k^2) + O(h^2).$$

$$(\ddot{\circ}) \Rightarrow \frac{v_m^{n+1}-v_m^n}{k} + a\frac{v_{m+1}^{n+1}-v_{m-1}^{n+1}+v_{m+1}^n-v_{m-1}^n}{4h} = \frac{f_m^{n+1}+f_m^n}{2}$$

$\square$