

Numerical Partial Differential Equations

Final Exam Presentation

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Finite-Difference Schemes for Reaction–Diffusion Equations Modeling Predator–Prey Interactions in MATLAB

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Abstract We present two finite-difference algorithms for studying the dynamics of spatially extended predator–prey interactions with the Holling type II functional response and logistic growth of the prey. The algorithms are stable and convergent provided the time step is below a (non-restrictive) critical value. This is advantageous as it is well-known that the dynamics of approximations of differential equations (DEs) can differ significantly from that of the underlying DEs themselves. This is particularly important for the spatially extended systems that are studied in this paper as they display a wide spectrum of ecologically relevant behavior, including chaos. Furthermore, there are implementational advantages of the methods. For example, due to the structure of the resulting linear systems, standard direct, and iterative solvers are guaranteed to converge. We also present the results of numerical experiments in one and two space dimensions and illustrate the simplicity of the numerical methods with short programs in MATLAB. Users can download, edit, and run the codes from <http://www.uoguelph.ca/~mgarvie/>, to investigate the key dynamical properties of spatially extended predator–prey interactions.

Keywords Reaction-diffusion system · Predator-prey interaction · Finite difference method · MATLAB

1. Introduction

1.1. Model equations

In this paper, we study the numerical solutions of 2-component reaction–diffusion systems with the following general form (cf. May, 1974, p. 84; Murray, 1993, p. 71;

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$$\text{Model : } \begin{cases} \frac{\partial u}{\partial t} = \Delta u + u(1-u) - vh(au) \\ \frac{\partial v}{\partial t} = \delta \Delta v + bv h(au) - cv \end{cases}, \quad \Rightarrow \quad \begin{cases} \frac{\partial u}{\partial t} = \Delta u + f(u, v) \\ \frac{\partial v}{\partial t} = \delta \Delta v + g(u, v) \end{cases}$$

2

$$\text{Reaction term : } f(u, v) = \begin{cases} u(1-u) - \frac{uv}{u+\alpha} & (\text{Kinetics 1}) \\ u(1-u) - v(1-e^{-\gamma u}) & (\text{Kinetics 2}) \end{cases}$$

$$g(u, v) = \begin{cases} \frac{\beta uv}{u+\alpha} - \gamma v & (\text{Kinetics 1}) \\ \beta v(\alpha - 1 - \alpha e^{-\gamma u}) & (\text{Kinetics 2}) \end{cases}$$

$$\text{Parameters : } \begin{cases} u^* = \frac{\alpha\gamma}{\beta-\gamma}, \quad v^* = (1-u^*)(u^* + \alpha) \\ u^* = \frac{1}{\gamma} \ln\left(\frac{\alpha-1}{\alpha}\right), \quad v^* = \frac{u^*(1-u^*)}{1-e^{-\gamma u^*}} \end{cases}$$

Zero-Flux Boundary Condition

One-Dimensional PDE

$$U_{-1}^n := U_1^n, \quad U_{J+1}^n := U_{J-1}^n, \quad V_{-1}^n := V_1^n, \quad V_{J+1}^n := V_{J-1}^n.$$

Two-Dimensional PDE

$$\begin{aligned} U_{i,-1}^n &:= U_{i,1}^n, & U_{i,J+1}^n &:= U_{i,J-1}^n, \\ U_{J+1,j}^n &:= U_{J-1,j}^n, & U_{-1,j}^n &:= U_{1,j}^n, \end{aligned}$$

$$\begin{aligned} U_{0,-1}^n &= (U_{0,0}^n + U_{0,1}^n)/2, & U_{-1,0}^n &= (U_{0,0}^n + U_{1,0}^n)/2, & (\text{SW corner}) \\ U_{J,J+1}^n &= (U_{J,J}^n + U_{J,J-1}^n)/2, & U_{J+1,J}^n &= (U_{J,J}^n + U_{J-1,J}^n)/2, & (\text{NE corner}) \\ U_{J,-1}^n &= 2U_{J,1}^n - U_{J,0}^n, & U_{J+1,0}^n &= 2U_{J-1,0}^n - U_{J,0}^n, & (\text{SE corner}) \\ U_{0,J+1}^n &= 2U_{0,J-1}^n - U_{0,J}^n, & U_{-1,J}^n &= 2U_{1,J}^n - U_{0,J}^n, & (\text{NW corner}) \end{aligned}$$

We use the explicit scheme (FTCS) in contrast the implicit scheme in the paper.

Note that for the 2-dimensional PDE,

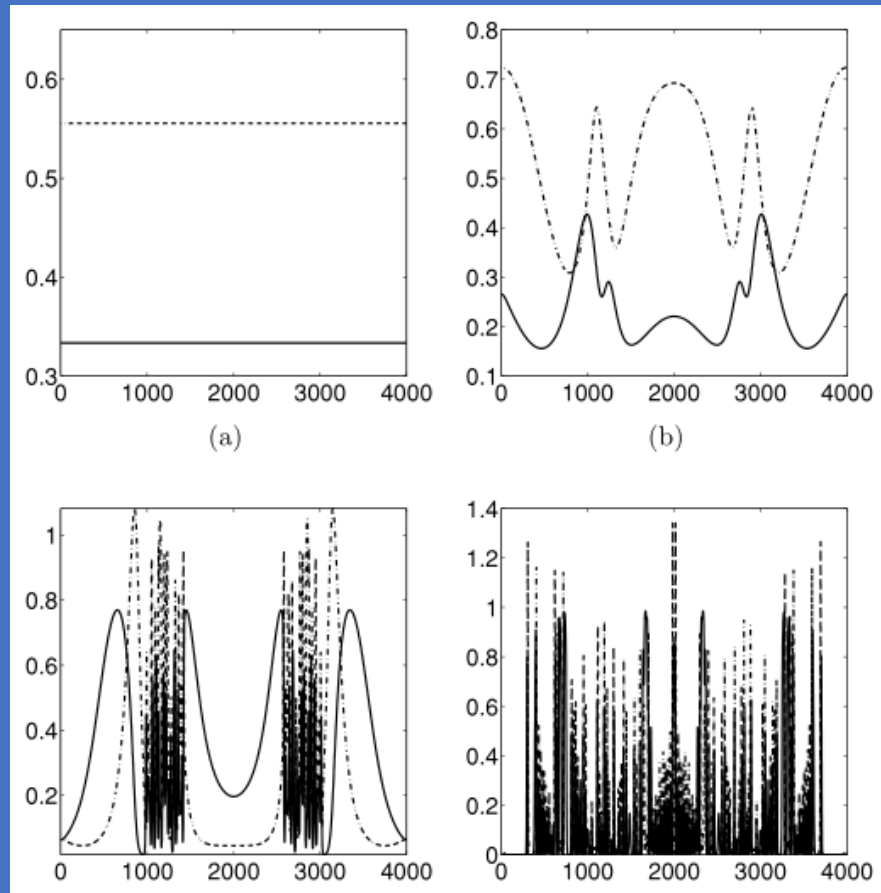
$$\frac{\partial u}{\partial t} \approx \frac{u_{l,m}^{n+1} - u_{l,m}^n}{k} \qquad \frac{\partial^2 u}{\partial x^2} \approx \frac{u_{l+1,m}^n - 2u_{l,m}^n + u_{l-1,m}^n}{h^2}$$

Laplace operator $\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$ for $u(x, y, t)$

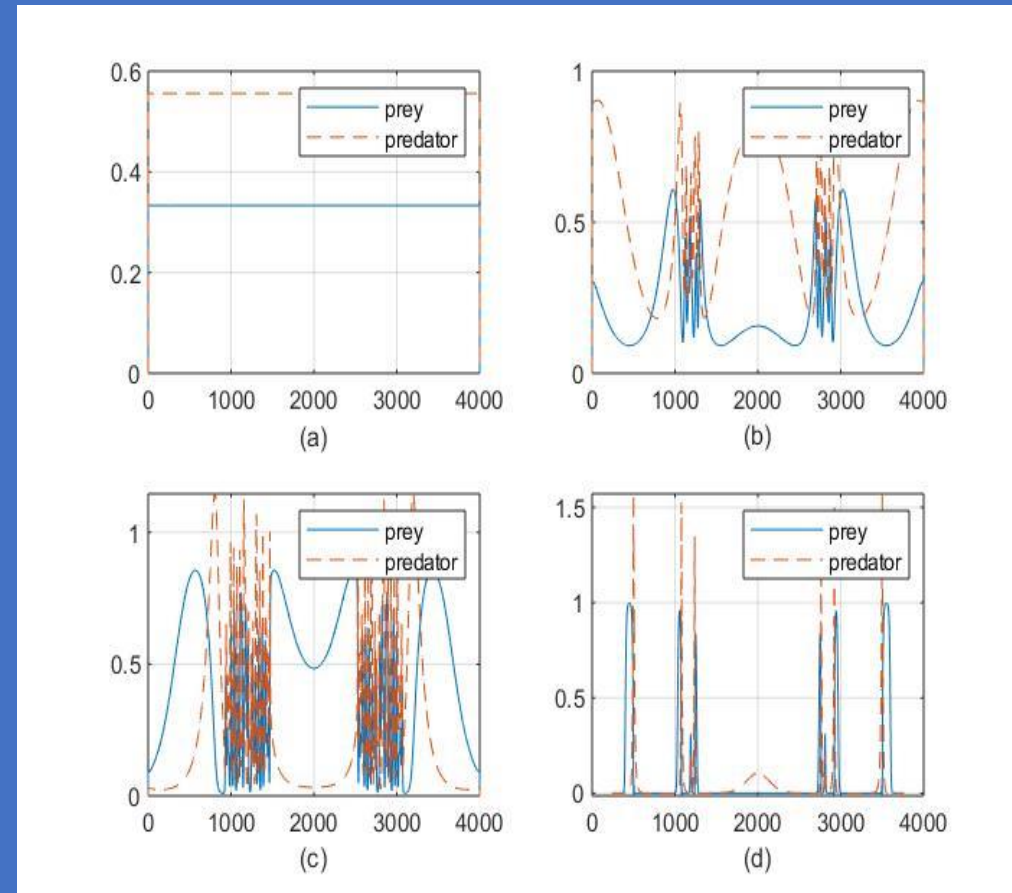
Stability condition : step size $\Delta t < \frac{1}{4}$. Thus we take $\Delta t = \frac{1}{5}$.

(Recall that the stability condition for the diffusion term has the inequality $b\mu < \frac{1}{2}$ for 1-dimensional PDE, $b = 1$.)

In the paper

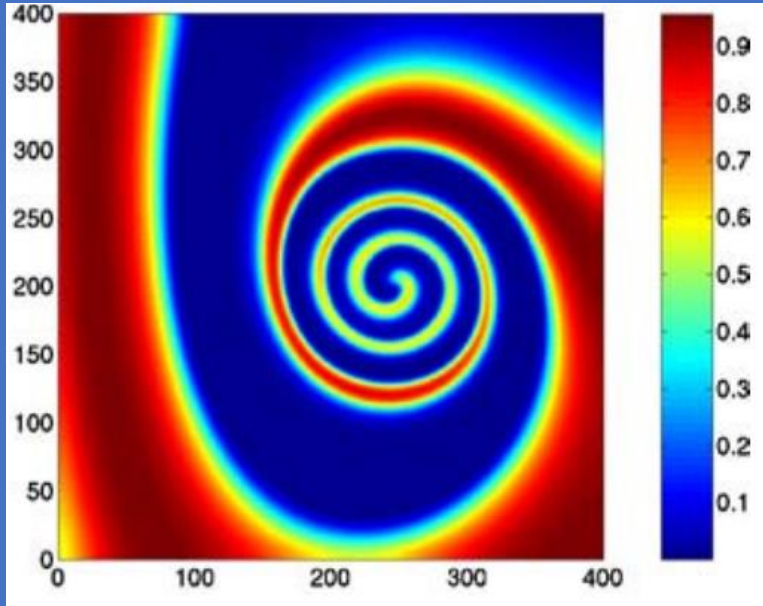


Simulation

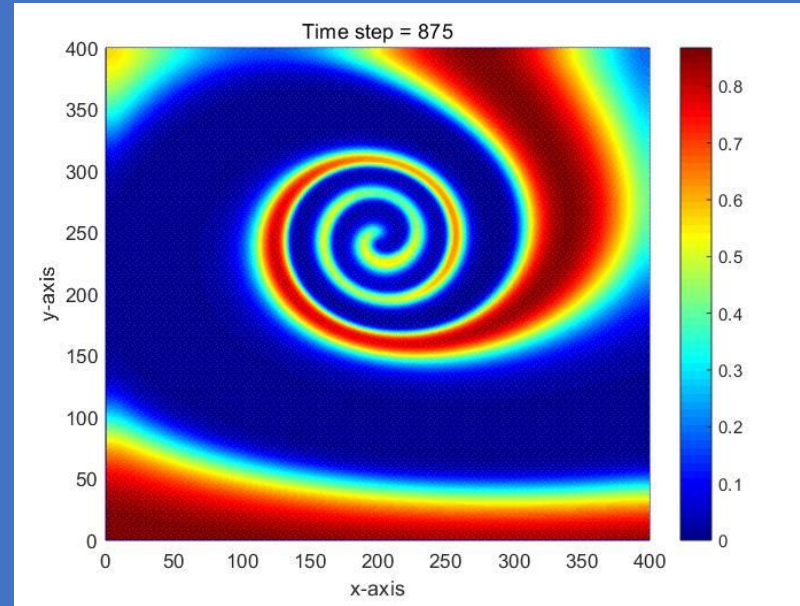


One-dimensional numerical solutions of the equations, $\Delta t=1/10$.

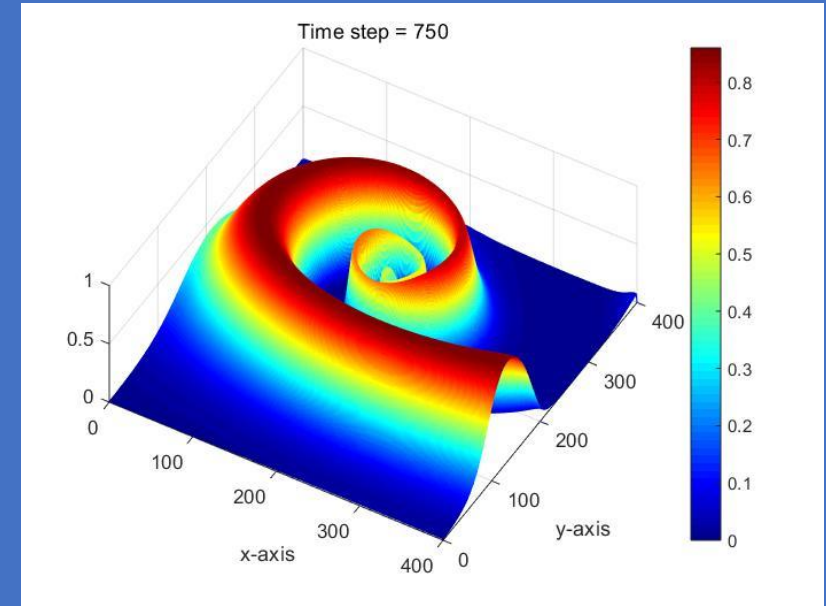
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Simulation 1



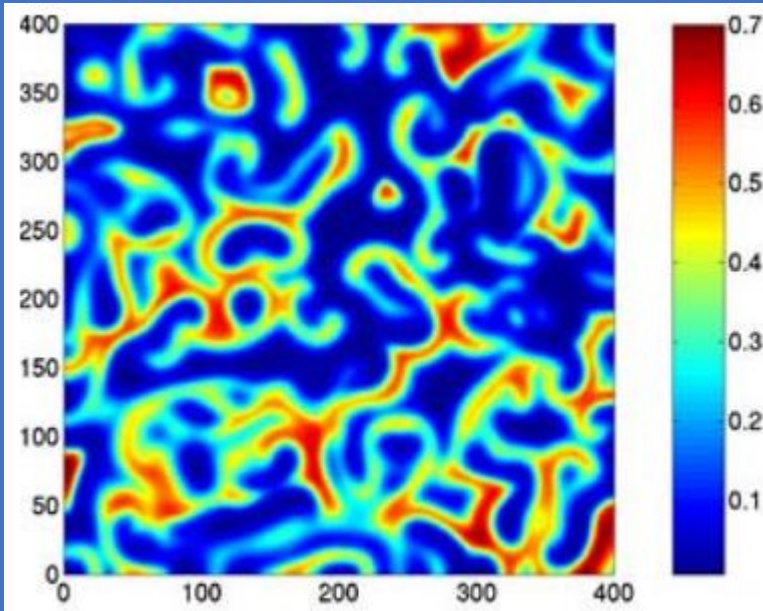
Simulation 2



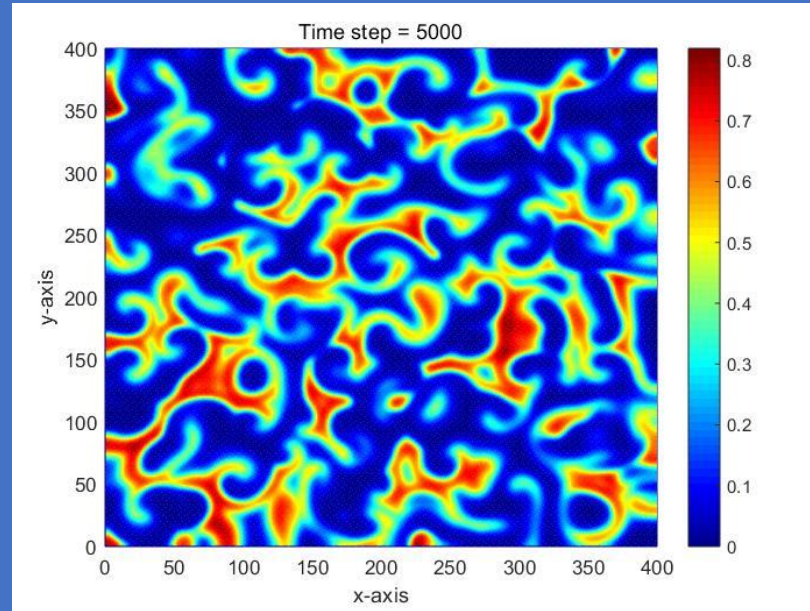
Two-dimensional numerical solutions of the equations Kinetics 1.

- $T = 150, \alpha = 0.4, \beta = 2.0, \gamma = 0.6, \delta = 1, h = 1, \Delta t = 1/5, u^* = 6/35, v^* = 116/245$
- $U_{i,j}^0 = u^* - 2 \times 10^{-7} (x_i - 0.1y_j - 225)(x_i - 0.1y_j - 675),$
- $V_{i,j}^0 = v^* - 3 \times 10^{-5} (x_i - 450) - 1.2 \times 10^{-4} (y_j - 150).$

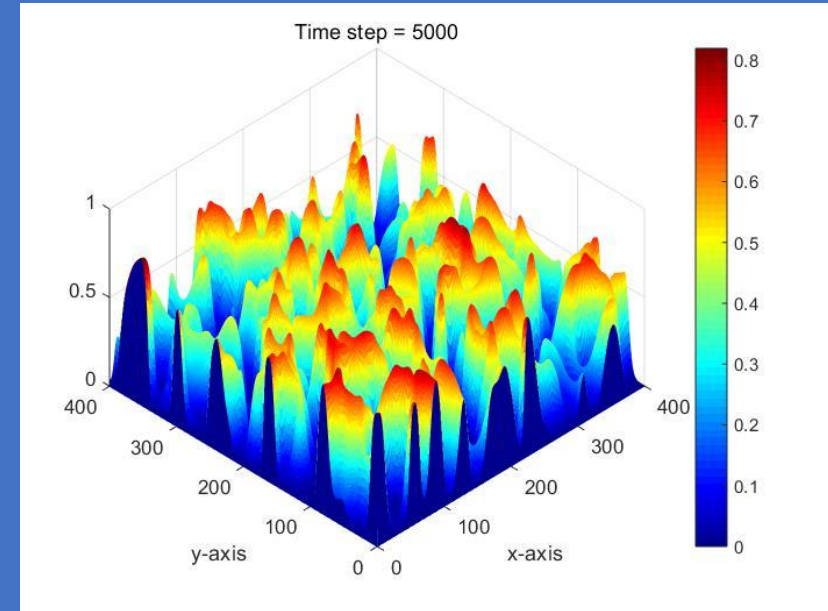
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Simulation 1



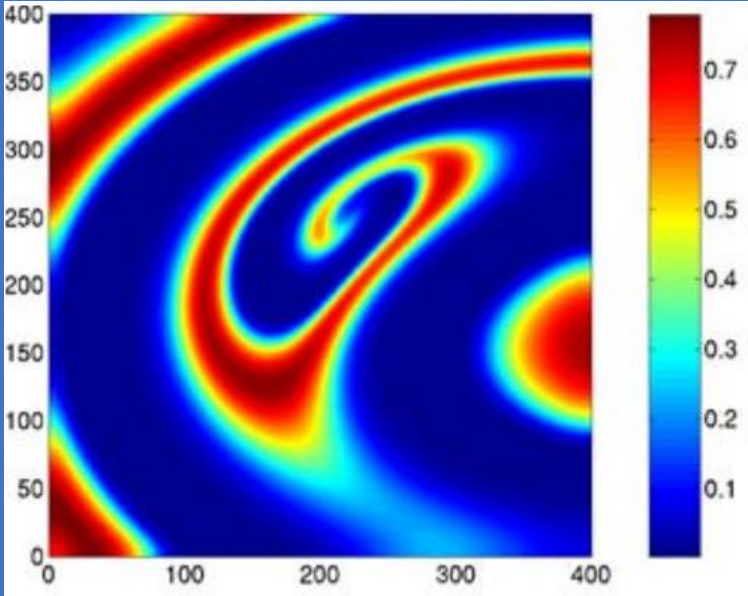
Simulation 2



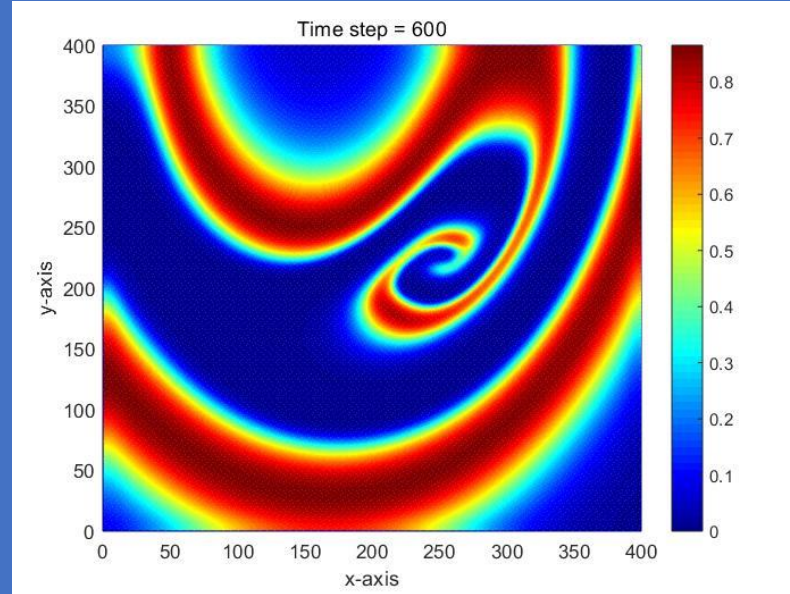
Two-dimensional numerical solutions of the equations Kinetics 1.

- $T = 1000, \alpha = 0.4, \beta = 2.0, \gamma = 0.6, \delta = 1, h = 1, \Delta t = 1/5, u^* = 6/35, v^* = 116/245$
- $U_{i,j}^0 = u^* - 2 \times 10^{-7} (x_i - 0.1y_j - 225)(x_i - 0.1y_j - 675),$
- $V_{i,j}^0 = v^* - 3 \times 10^{-5} (x_i - 450) - 1.2 \times 10^{-4} (y_j - 150).$

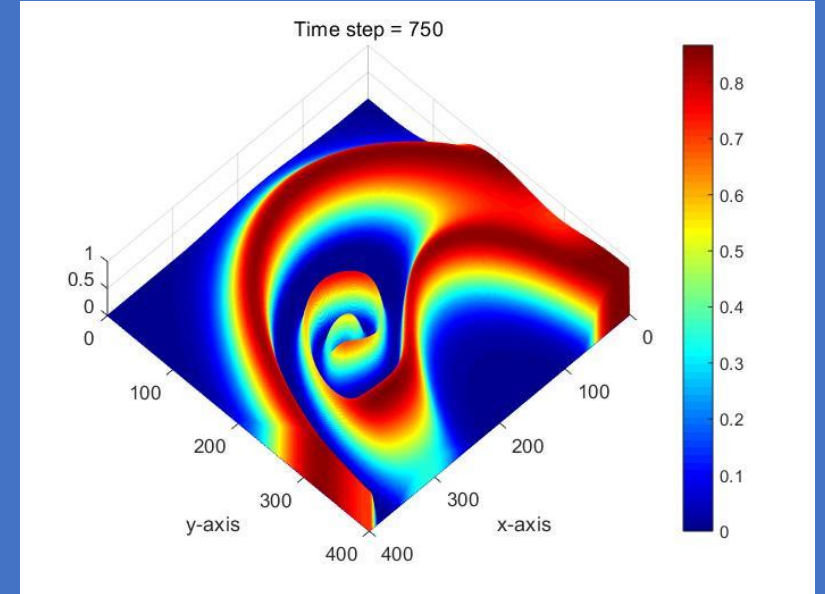
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Simulation 1



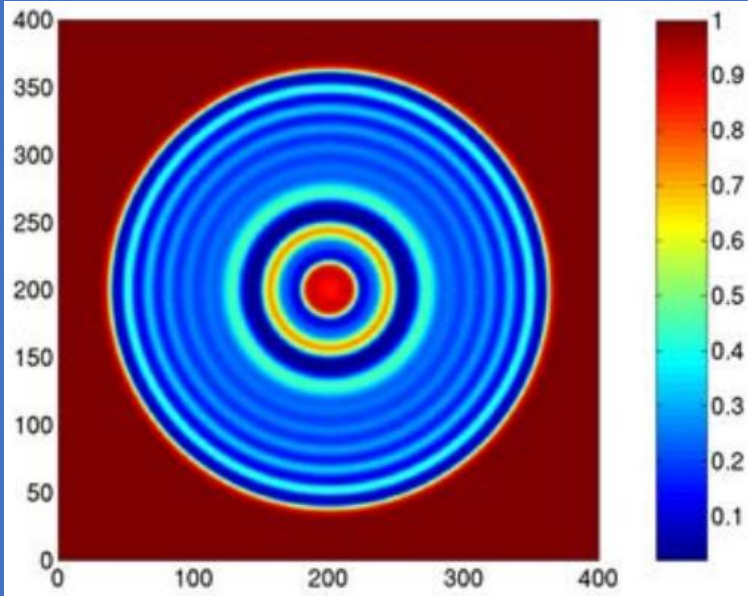
Simulation 2



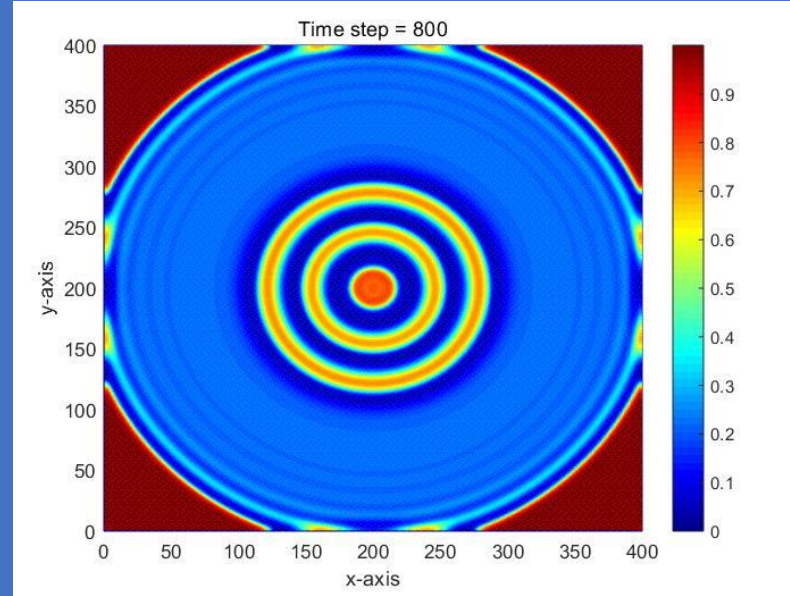
Two-dimensional numerical solutions of the equations Kinetics 1.

- $T = 120, \alpha = 0.4, \beta = 2.0, \gamma = 0.6, \delta = 1, h = 1, \Delta t = 1/5, u^* = 6/35, v^* = 116/245$
- $U_{i,j}^0 = u^* - 2 \times 10^{-7}(x_i - 180)(x_i - 720) - 6 \times 10^{-7}(y_j - 90)(y_j - 210),$
- $V_{i,j}^0 = v^* - 3 \times 10^{-5}(x_i - 450) - 6 \times 10^{-5}(y_j - 135).$

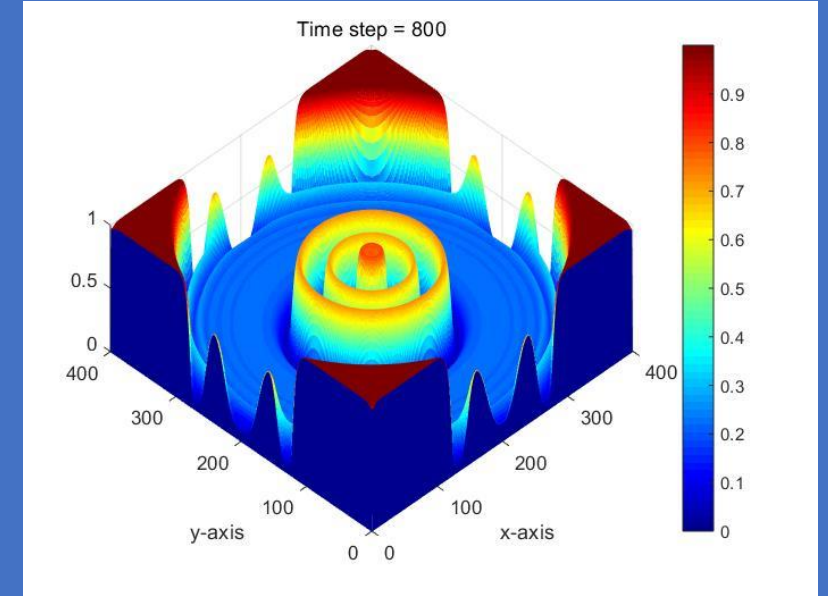
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Simulation 1



Simulation 2



Two-dimensional numerical solutions of the equations **Kinetics 2**.

- $T = 150, \alpha = 1.5, \beta = 1.0, \gamma = 5.0, \delta = 1, h = 1, \Delta t = 1/5$
- $U_{i,j}^0 = 1.0$ if $(x_i - 200)^2 + (y_j - 200)^2 < 400$ and $U_{i,j}^0 = 0$ otherwise
- $V_{i,j}^0 = 0.2$ if $(x_i - 200)^2 + (y_j - 200)^2 < 400$ and $V_{i,j}^0 = 0$ otherwise

OPEN

Emergence of unusual coexistence states in cyclic game systems

Junpyo Park¹, Younghae Do², Bongsoo Jang² & Ying-Cheng Lai³

Evolutionary games of cyclic competitions have been extensively studied to gain insights into one of the most fundamental phenomena in nature: biodiversity that seems to be excluded by the principle of natural selection. The Rock-Paper-Scissors (RPS) game of three species and its extensions [e.g., the Rock-Paper-Scissors-Lizard-Spock (RPSLS) game] are paradigmatic models in this field. In all previous studies, the intrinsic symmetry associated with cyclic competitions imposes a limitation on the resulting coexistence states, leading to only selective types of such states. We investigate the effect of nonuniform intraspecific competitions on coexistence and find that a wider spectrum of coexistence states can emerge and persist. This surprising finding is substantiated using three classes of cyclic game models through stability analysis, Monte Carlo simulations and continuous spatiotemporal dynamical evolution from partial differential equations. Our finding indicates that intraspecific competitions or alternative symmetry-breaking mechanisms can promote biodiversity to a broader extent than previously thought.

Fundamental to species coexistence and biodiversity are competitions. In ecosystems there are two types of competitions: interspecific (competitions among individuals from different species) and intraspecific (competitions among individuals in the same species), where both types can either promote or hinder species coexistence^{1,2}. The purpose of this paper is to demonstrate, through a systematic study of several models of cyclic evolutionary game, that intraspecific competitions can induce unusual states of coexistence that have not been reported previously. Intraspecific competitions may thus be more fundamental to biodiversity than previously thought.

A natural and typical mechanism for interspecific competitions is predator-prey interaction, while intraspecific competitions occur because individuals in the same species compete for essential life-sustaining resources such as food, water, light, and opposite sex. A well known type of intraspecific competitions is cannibalism or intraspecific predation^{3–10}, which can occur with high likelihood especially when there is lack of sufficient resources. Such competitions can also occur when individuals fight each other for mating opportunities, which were observed for side-blotched lizards in California¹¹. In the past decade there were studies of the effect of intraspecific competitions on biodiversity^{12–16}, with results such as the experimental finding that the competitions tend to drive disruptive selection¹², enhanced host survival through intraspecific competition between co-infecting parasite strains¹³, and directional selection of certain fish species^{15,16}.

To understand coexistence and biodiversity, the approach of mathematical modeling has proven to be useful, providing fundamental insights into the various mechanisms underlying species coexistence at both the macroscopic, population^{17–19} and the microscopic, individual competition levels^{20–23}. Historically, the theoretical approach began with mathematically modeling growth and competitions through dynamical equations at the population level^{17–19}. In the past fifteen years or so, microscopic models at the level of individual interactions were extensively studied based on the mathematical paradigm of evolutionary games^{24–26}. A milestone result^{25,26} is the elucidation of the role of species mobility in coexistence, which traditionally had been regarded as detrimental to coexistence. In particular, utilizing the framework of three cyclic competing species, the rock-paper-scissors (RPS) model, the authors^{22,26} demonstrated robust coexistence in the weak mobility regime, providing a resolution to the paradox that macroscopic models exclude coexistence of mobile species but, in realistic ecological processes ranging from bacteria run and tumble to animal migration, coexistence is ubiquitous. The basic dynamical structure supporting the coexistence of mobile species was identified to be spiral wave patterns that emerge and evolve with time in the physical space²², which are robust against noise²⁸. Other issues that have

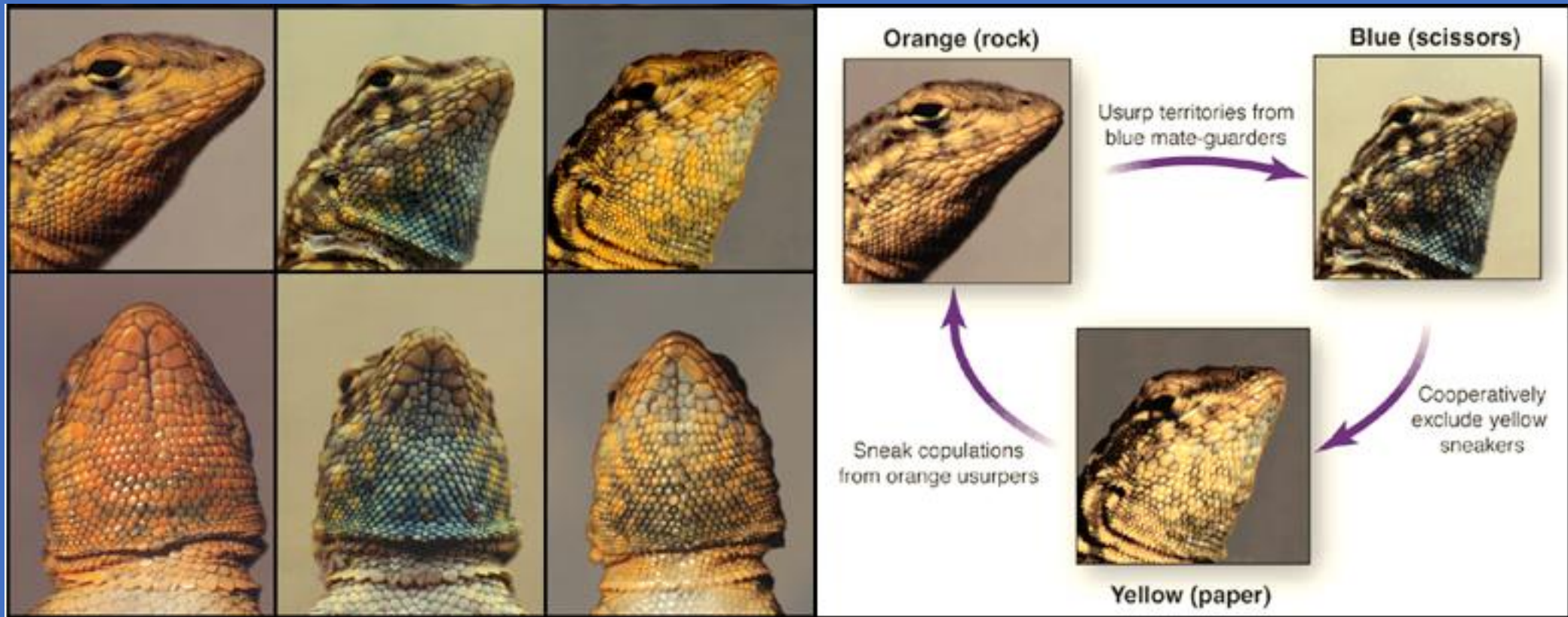
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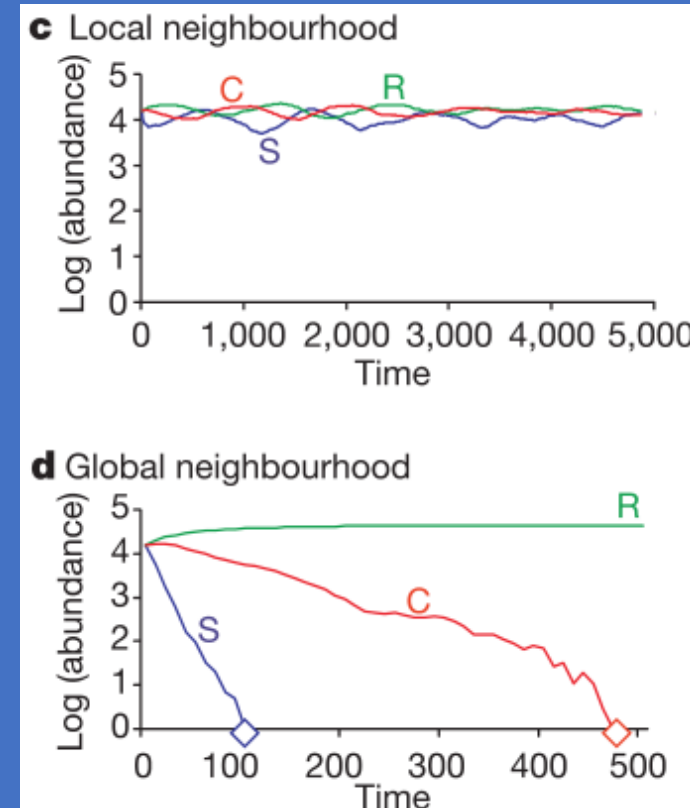
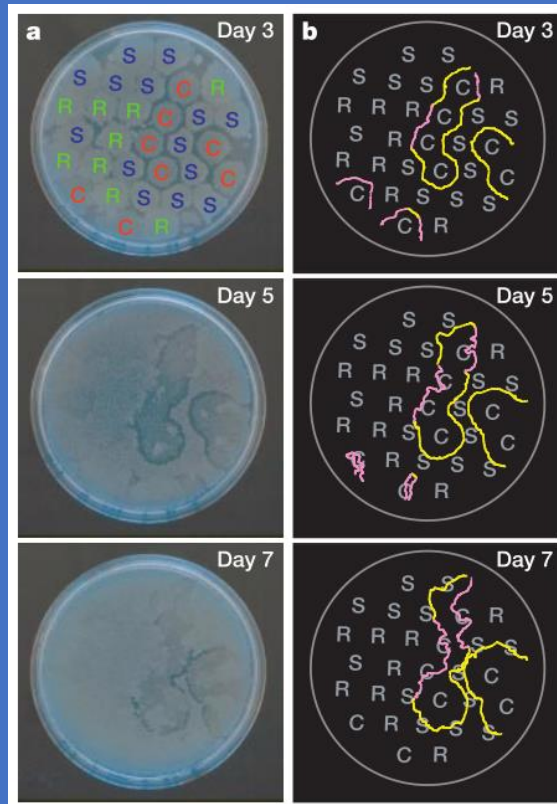
Nature, 2017

What is the cyclic games? (in ecology)



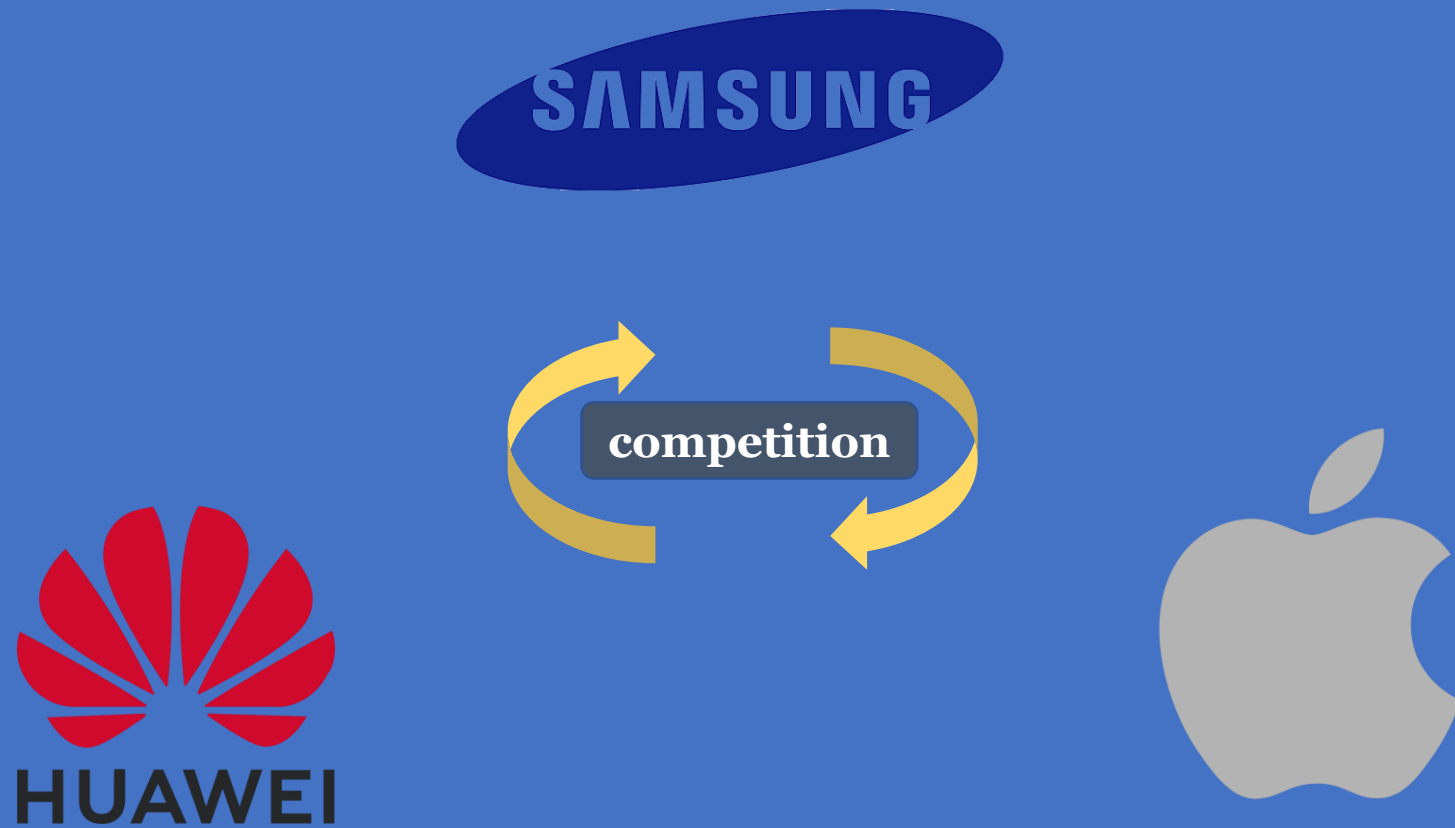
B. Sinervo & M. Lively (Nature, 1996)

What is the cyclic games? (in biology)

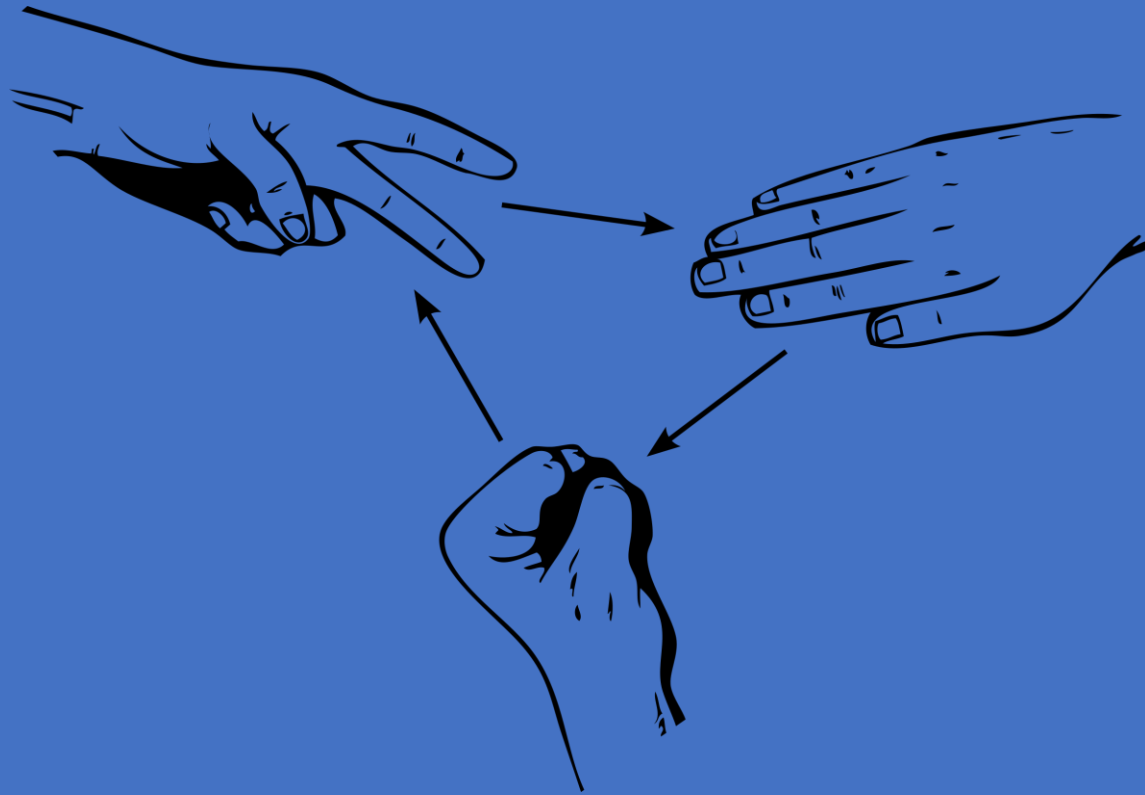


**Benjamin Kerr, Margaret A. Riley, Marcus W. Feldman
& Brendan J. M. Bohannan, (Nature, 2002)**

What is the cyclic games? (in economics)

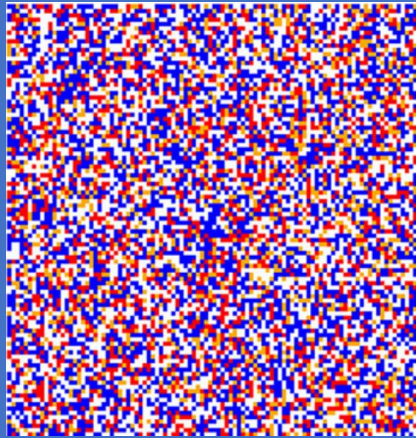


What is the cyclic games? (in general)

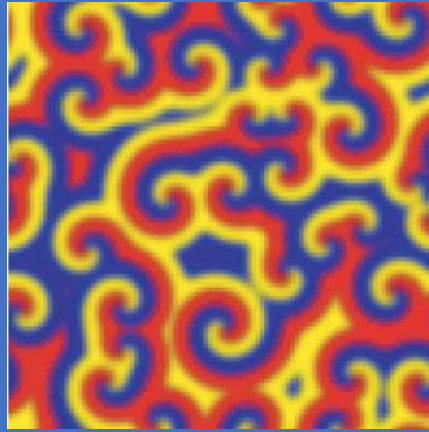


Rock-Paper-Scissors Game (RPS game)

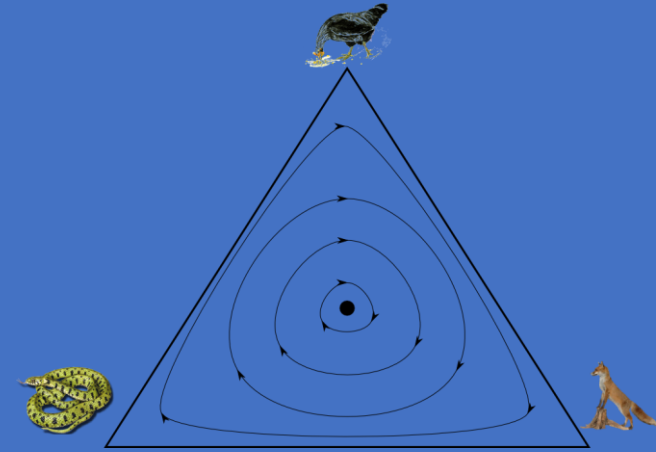
In this paper, they consider the population dynamics as :



Lattice
(micro-scale)



PDEs



ODEs
(macro-scale)

Extinction



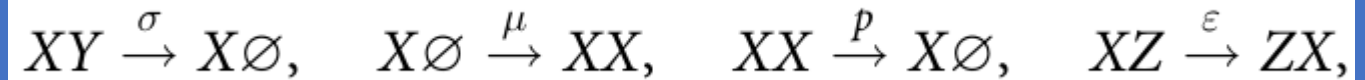
Coexistence

**we consider only :
ODEs and PDEs in terms of the numerical simulations.**

$$\begin{aligned}\frac{da}{dt} &= a \left[\mu(1 - \rho) - \sigma c - \frac{p_a}{2} a \right], \\ \frac{db}{dt} &= b \left[\mu(1 - \rho) - \sigma a - \frac{p_b}{2} b \right], \\ \frac{dc}{dt} &= c \left[\mu(1 - \rho) - \sigma b - \frac{p_c}{2} c \right],\end{aligned}$$

ODEs

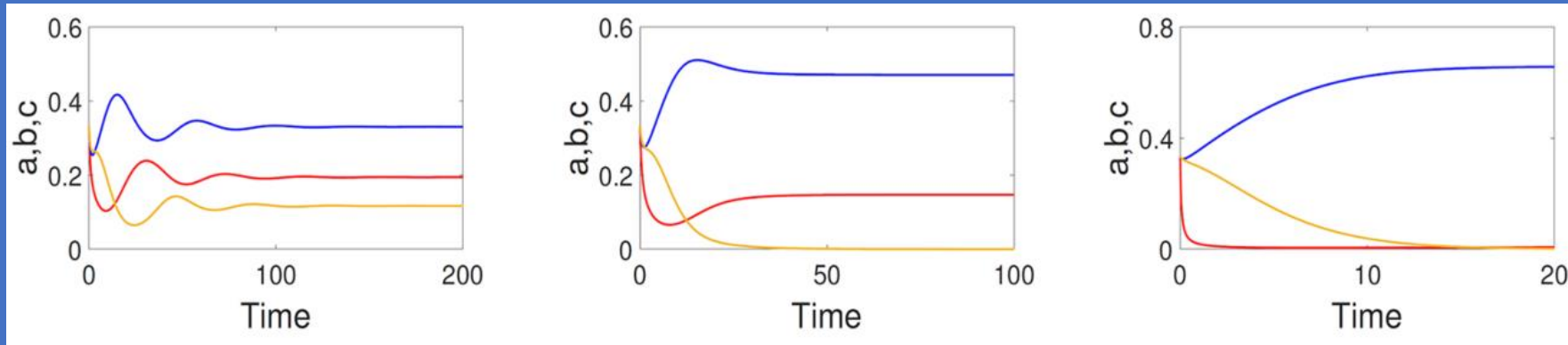
$a(x, y, t), b(x, y, t), c(x, y, t)$: density of the species
 $\rho(x, y, t) = a + b + c$: total density of these species
 $M = 10^{-3}$ is individual mobility parameter



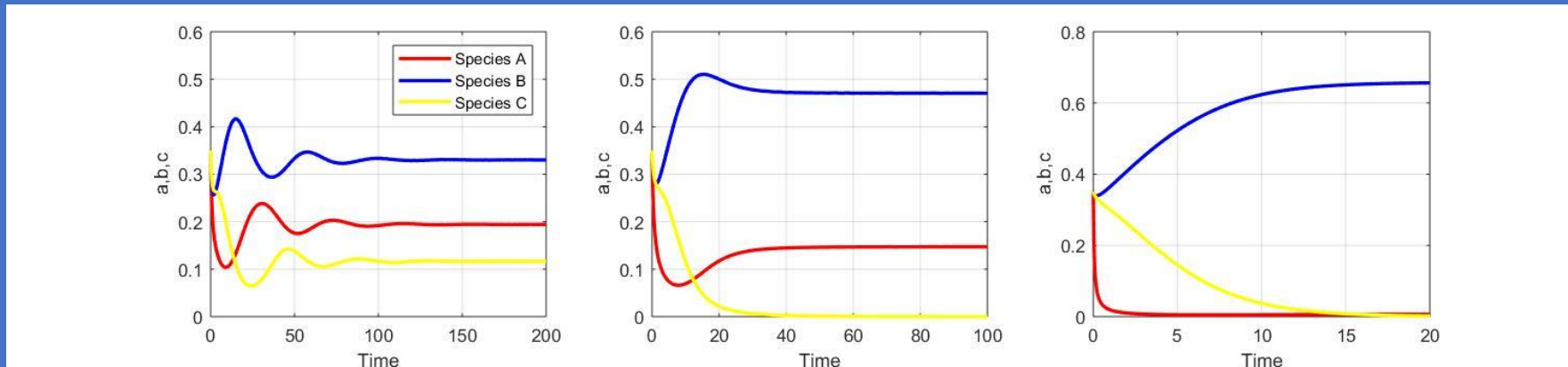
$$\begin{aligned}\frac{\partial a(\mathbf{x}, t)}{\partial t} &= M \Delta a(\mathbf{x}, t) + \mu a(\mathbf{x}, t) [1 - \rho(\mathbf{x}, t)] - \sigma a(\mathbf{x}, t) c(\mathbf{x}, t) - \frac{p_a}{2} a(\mathbf{x}, t) a(\mathbf{x}, t), \\ \frac{\partial b(\mathbf{x}, t)}{\partial t} &= M \Delta b(\mathbf{x}, t) + \mu b(\mathbf{x}, t) [1 - \rho(\mathbf{x}, t)] - \sigma b(\mathbf{x}, t) a(\mathbf{x}, t) - \frac{p_b}{2} b(\mathbf{x}, t) b(\mathbf{x}, t), \\ \frac{\partial c(\mathbf{x}, t)}{\partial t} &= M \Delta c(\mathbf{x}, t) + \mu c(\mathbf{x}, t) [1 - \rho(\mathbf{x}, t)] - \sigma b(\mathbf{x}, t) c(\mathbf{x}, t) - \frac{p_c}{2} c(\mathbf{x}, t) c(\mathbf{x}, t).\end{aligned}$$

PDEs

σ : interspecific competition rate, movement rate $\varepsilon \equiv 2MN$,
 μ : species reproduction rate, N : total number of individual,
 p : intraspecific competition rate.

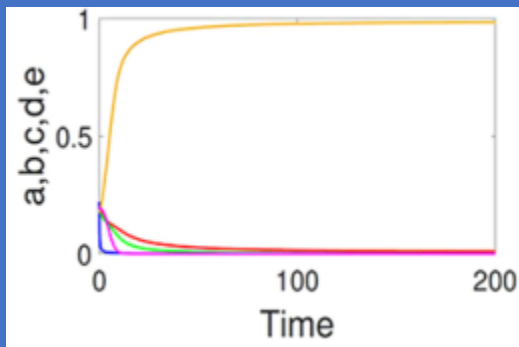
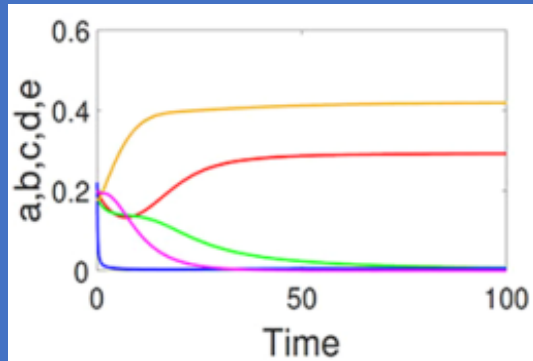
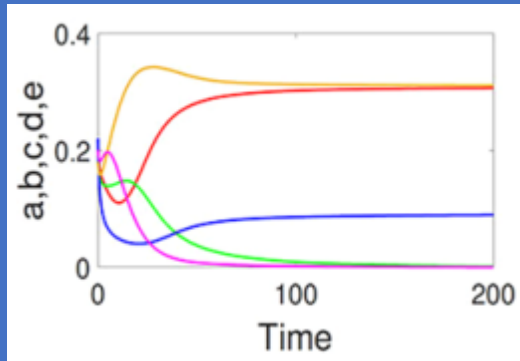
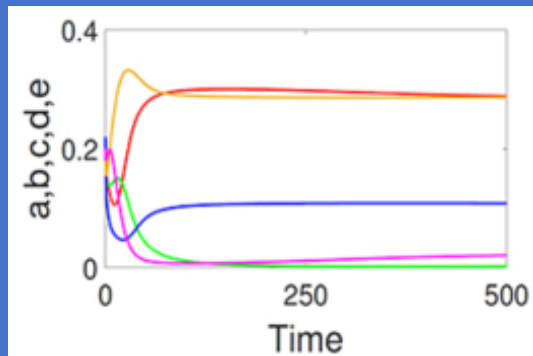
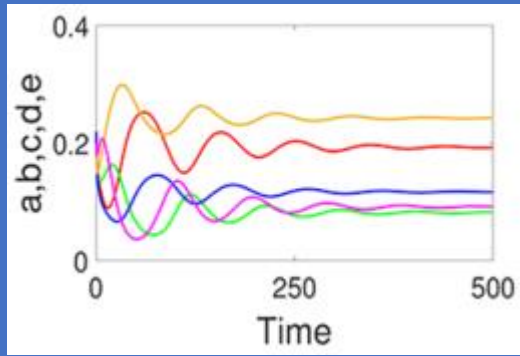


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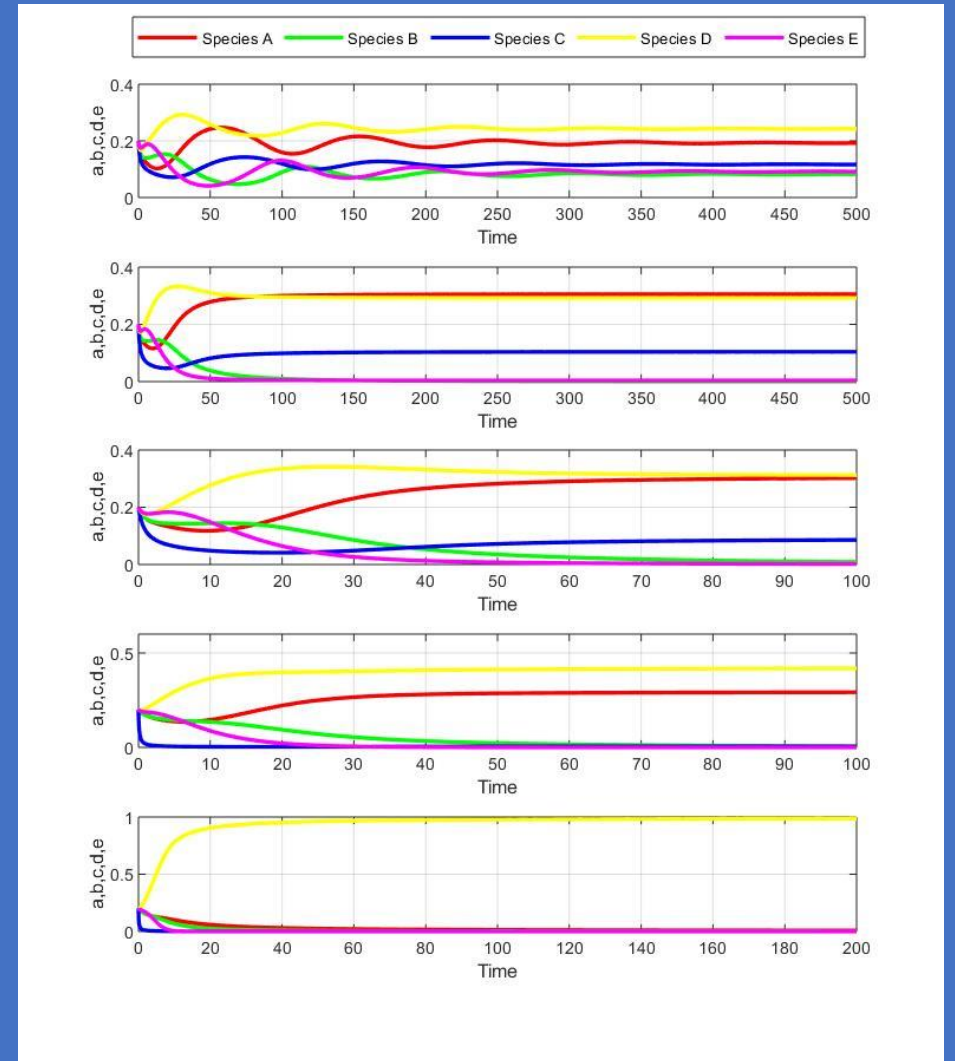


Simulation

Ordinary Differential Equation simulations (RPS game).
Use the random initial condition and finite difference scheme (FTCS)

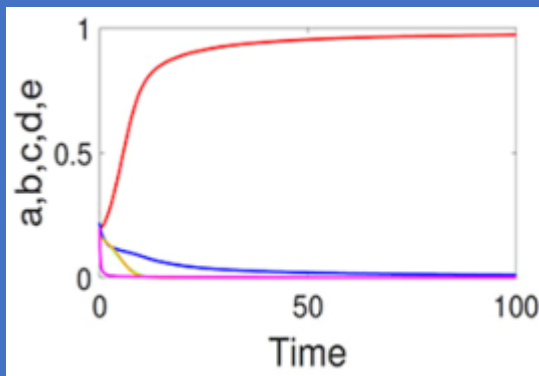
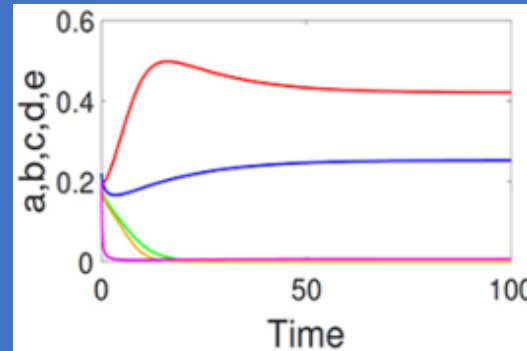
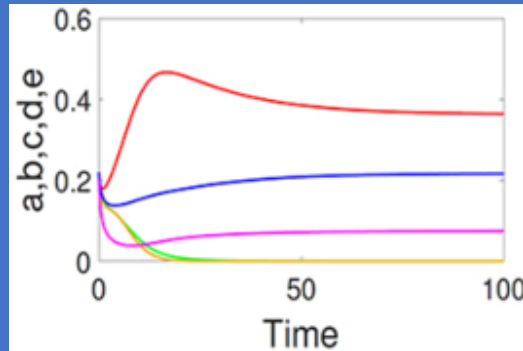
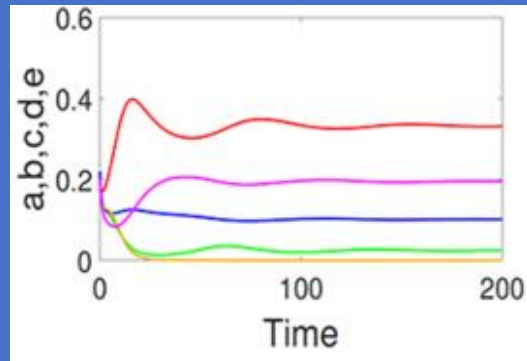
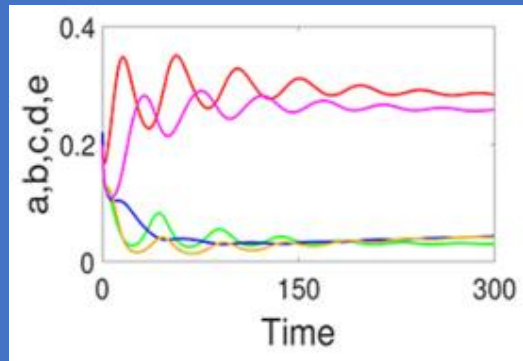


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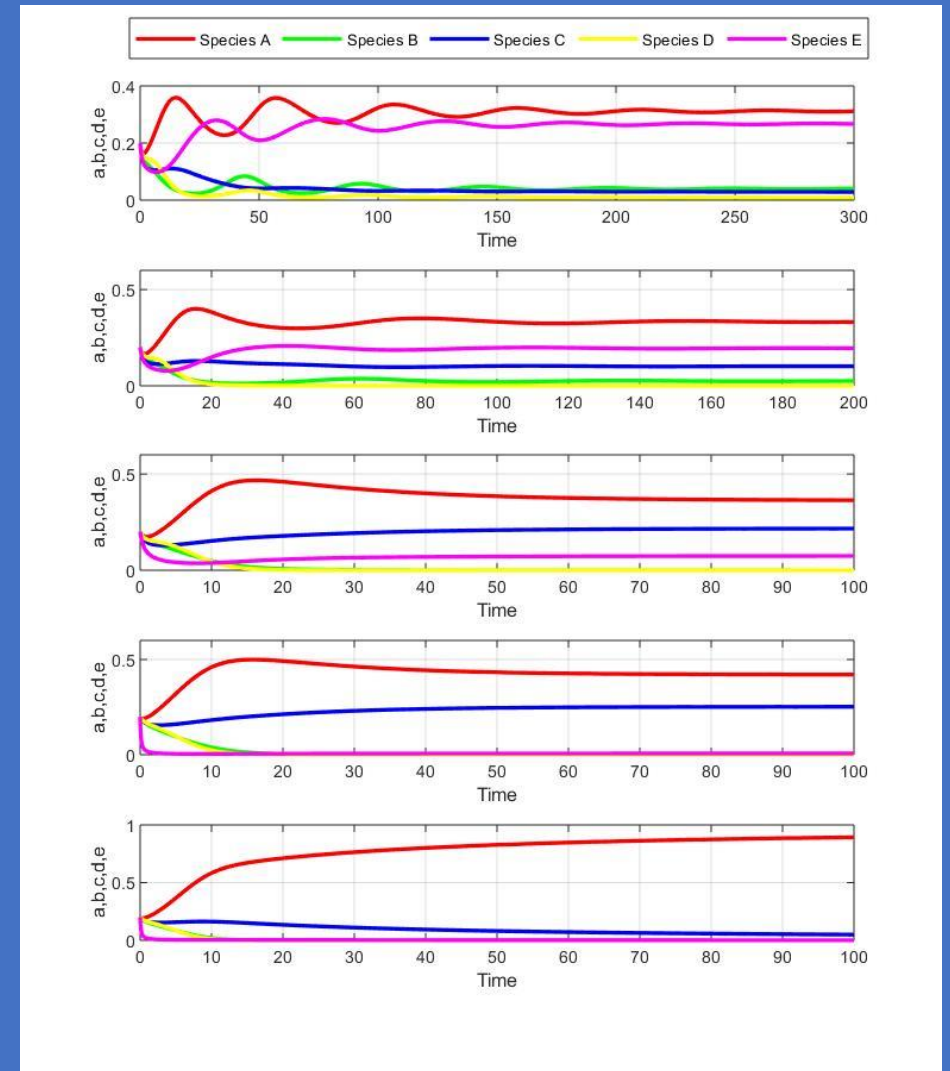


Simulation

Ordinary Differential Equation simulations. (Extended RPS game)
Use the random initial condition and finite difference scheme (FTCS)

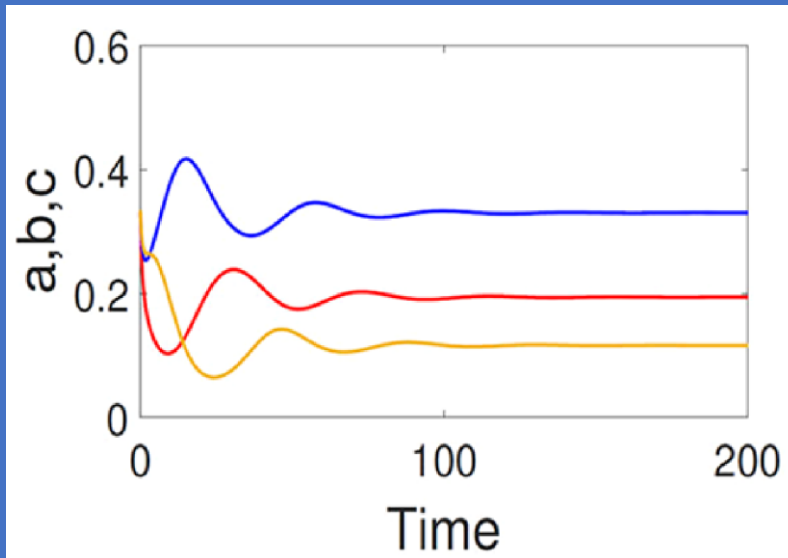
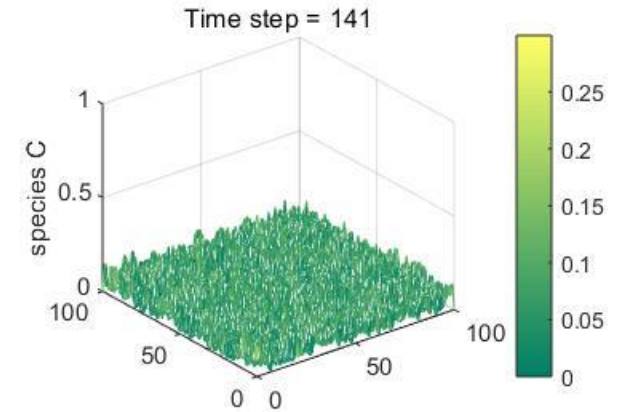
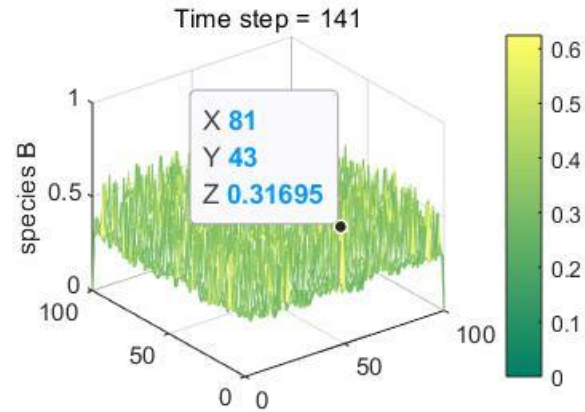
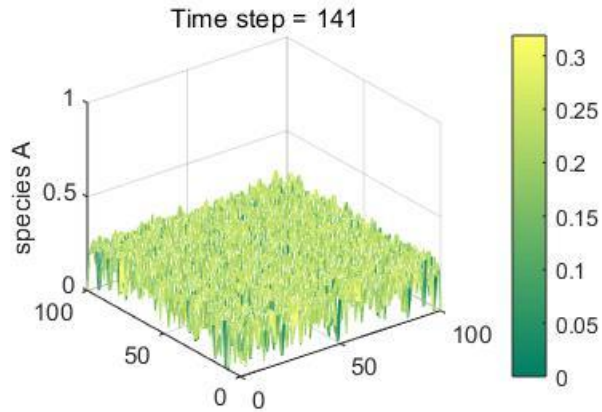


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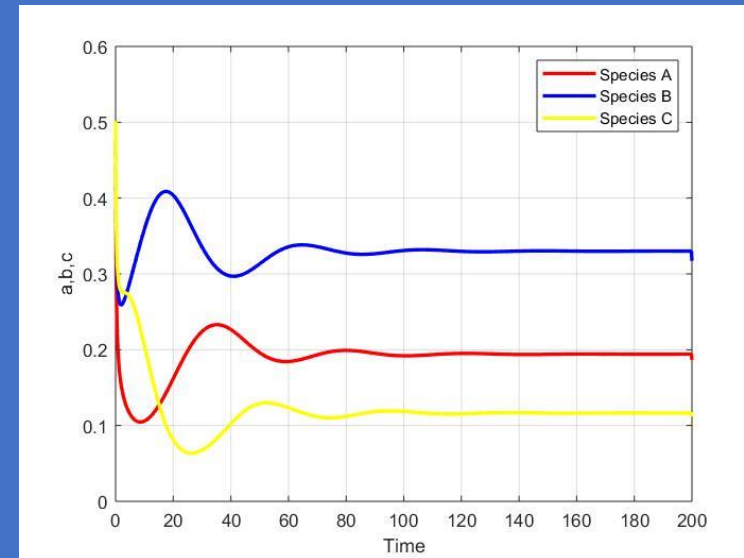


Simulation

Ordinary Differential Equation simulations. (RPSLS game)
Use the random initial condition and finite difference scheme (FTCS)



ODE simulation



PDE Simulation

Partial Differential Equation simulations. (RPS game)
Use the random initial condition and finite difference scheme (FTCS)

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Thank you!

Q & A

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