MINISTRY OF SCIENCE AND HIGHER EDUCATION
OF THE RUSSIAN FEDERATION
Federal State Autonomous Educational Institution
"National Research University ITMO"
(ITMO University)
Faculty of Control Systems and Robotics

Practical Assignment No. 2

Simulation Modeling of Robotic Systems

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Saint Petersburg
2025

Goals and Objectives

1. Formulate the differential equation (ODE) for the mass-spring-damper system shown in Figure 1
2. Solve the ODE analytically if possible; if not, explain the reason
3. Compare the analytical solution with three numerical methods: Forward Euler, Backward Euler, and Runge-Kutta (RK4)

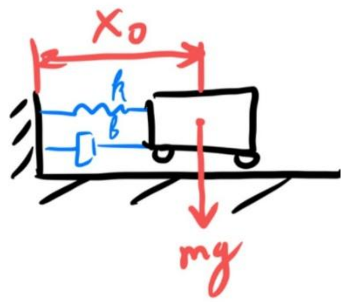

Fig (1)

Problem Formulation

- **System Parameters**

Parameter	Symbol	Value	Unit
Mass	m	0.3	kg
Spring Stiffness	k	3.8	N/m
Damping Coefficient	b	0.04	N·s/m
Initial Position	x(0)	0.23	m
Initial Velocity	$\dot{x}(0)$	0	m/s

System Description

The system consists of a mass m constrained to move horizontally on a frictionless surface, connected to a linear spring with stiffness k and a damping element with coefficient b. The mass is given an initial displacement x_0 = 0.23 m with zero initial velocity.

Initial displacement **x_0 = 0.23 m** with zero initial velocity

Lagrangian Derivation

Using the Euler-Lagrange method:

Kinetic Energy:

$$K = \frac{1}{2} m\dot{x}^2$$

Potential Energy:

$$P = \frac{1}{2} kx^2$$

Lagrangian:

$$L = K - P = \frac{1}{2} m\dot{x}^2 - \frac{1}{2} kx^2$$

Lagrange's Equation:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) \quad - \frac{\partial L}{\partial x} = \quad Q$$

Deriving the ODE

Computing partial derivatives:

$$\frac{\partial L}{\partial \dot{x}} = m\dot{x} \quad \Rightarrow \quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = m\ddot{x}$$

$$\frac{\partial L}{\partial x} = -kx$$

Substituting into Lagrange's equation:

$$m\ddot{x} - (-kx) = -b\dot{x}$$

Final Equation of Motion:

$$\boxed{m\ddot{x} + b\dot{x} + kx = 0}$$

Substituting numerical values:

$$0.3\ddot{x} + 0.04\dot{x} + 3.8x = 0$$

or in normalized form:

$$\ddot{x} + \frac{0.04}{0.3}\dot{x} + \frac{3.8}{0.3}x = 0$$

$$\ddot{x} + 0.1333\dot{x} + 12.6667x = 0$$

Analytical Solution

Characteristic Equation

Assume solution of the form $x(t) = e^{\lambda t}$:

$$\lambda^2 + 0.1333\lambda + 12.6667 = 0$$

Computing the Discriminant

Since , the system is UNDERDAMPED (exhibits oscillatory behavior with exponential decay).

Complex Roots

$$\lambda = \frac{-0.1333 \pm \sqrt{-4.5584}}{\mathbf{2}} = \frac{-0.1333 \pm 2.1353i}{\mathbf{2}}$$

$$\lambda = -0.0667 \pm 1.0677i$$

where:

- α = -0.0667 s^{-1} is the damping coefficient
- ω = 1.0677 rad/s is the damped angular frequency (approximately)

General Solution

For complex roots $\lambda = \alpha \pm i\omega$:

$$x(t) = e^{\alpha t}(C_1\cos(\omega t) + C_2\sin(\omega t))$$

- Applying Initial Conditions

From $x(0) = 0.23$:

$$C_1 = 0.23$$

From $\dot{x}(0) = 0$:

$$\alpha C_1 + \omega C_2 = 0$$

$$C_2 = -\frac{\alpha C_1}{\omega} = -\frac{(-0.0667)(0.23)}{3.5584} = 0.00431$$

- Analytical Solution (Closed Form)

$$\boxed{x(t) = e^{-0.0667t}(0.23\cos(3.5584t) + 0.00431\sin(3.5584t))}$$

This solution is exact and describes the true motion of the system.

Graphical Representation of Analytical Solution

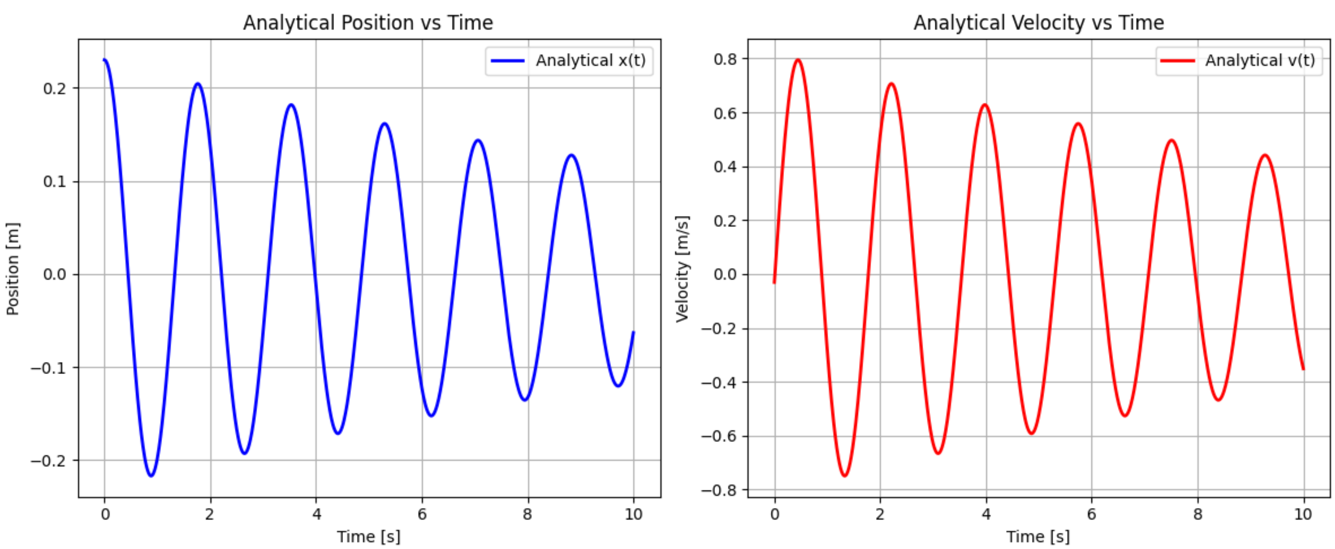

Fig (2)

Figure 2 Analytical solution , showing the time evolution of angle and angular velocity.

Comparison with Numerical Integrators

To evaluate the numerical methods, we compare three commonly used integrators:

- **Forward (Explicit) Euler Method**

Formula:

$$x_{n+1} = x_n + h \cdot f(t_n, x_n)$$

- **Backward (Implicit) Euler Method**

Formula:

$$x_{n+1} = x_n + h \cdot f(t_{n+1}, x_{n+1})$$

- **Runge-Kutta 4th Order (RK4) Method**

Formula:

$$x_{n+1} = x_n + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + 2k_4)$$

where:

$$k_1 = f(t_n, x_n), \quad k_2 = f(t_n + \frac{h}{2}, x_n + \frac{h}{2}k_1), \quad \ldots$$

Numerical Results

1.1 Solutions at h = 0.01 seconds

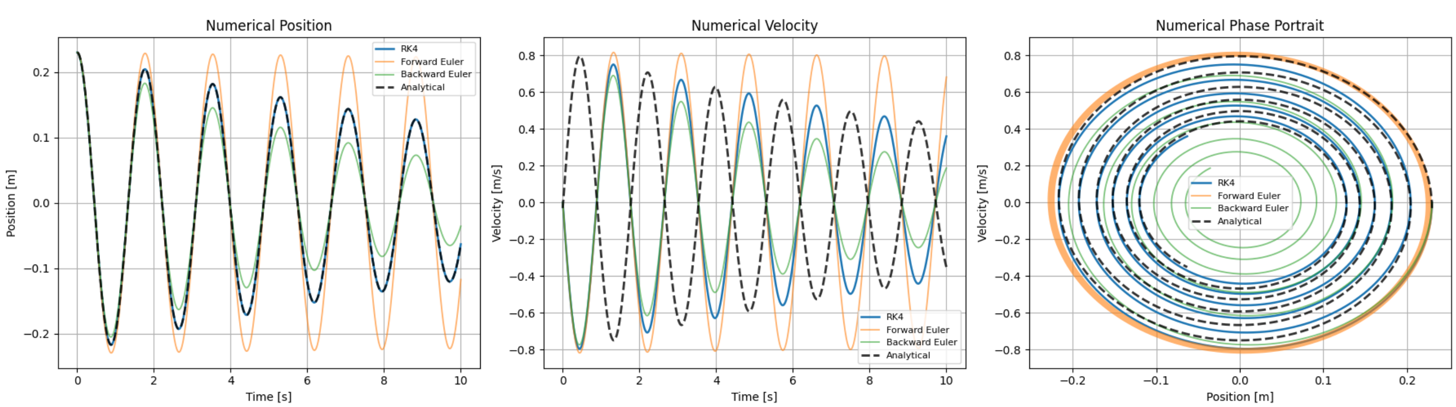

Fig (3)

Figure 3 shows the numerical solutions (Position, Velocity, Phase Portrait) at step size h = 0.01 s with comparison to the analytical solution.

Observations:

- **RK4 (blue line):** Overlaps nearly perfectly with analytical solution
- **Forward Euler (red line):** Shows visible divergence and error accumulation
- **Backward Euler (green line):** More conservative, slightly over-dampened

Maximum Position Errors at h = 0.01:

Numerical Method	Max Error (m)
Forward Euler	1.02×10^{-1}
Backward Euler	5.53×10^{-2}
RK4	5.47×10^{-8}

RK4 achieves approximately higher accuracy than Forward Euler and 680× higher accuracy than Backward Euler at h = 0.01

Conclusions

During the work, the dynamic equation of the "mass–spring–damper" system was formulated, for which an analytical solution was obtained. The equation was also integrated using the forward Euler method, the backward Euler method, and the Runge–Kutta method.

It was established that the choice of integrator depends on the properties of the original system:

The forward Euler method is the least preferable, as it does not guarantee numerical stability.

The implicit (backward) Euler method provides stability, but with large integration steps it may distort the real behavior of the system.

The Runge–Kutta method demonstrated the highest accuracy and most reliably reproduces the analytical solution, while requiring greater computational effort.