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Q4)

arbitary vector

$$C_3(\lambda)=\begin{bmatrix}\lambda&1&0\\0&\lambda&1\\0&0&\lambda\end{bmatrix}\Rightarrow C_3(\lambda)\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}\lambda&1&0\\0&\lambda&1\\0&0&\lambda\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\lambda\begin{bmatrix}x\\y\\z\end{bmatrix}+\begin{bmatrix}y\\z\\0\end{bmatrix}=\begin{bmatrix}\lambda x+y\\\lambda y+z\\\lambda z\end{bmatrix}$$

$$C_3(\lambda)\begin{bmatrix}\lambda x+y\\\lambda y+z\\\lambda z\end{bmatrix}=\begin{bmatrix}\lambda&1&0\\0&\lambda&1\\0&0&\lambda\end{bmatrix}\begin{bmatrix}\lambda x+y\\\lambda y+z\\\lambda z\end{bmatrix}=\begin{bmatrix}\lambda^2x+2\lambda y+z\\\lambda^2y+2\lambda z+z\\\lambda^2z\end{bmatrix}=\lambda^2\begin{bmatrix}x\\y\\z\end{bmatrix}+2\lambda\begin{bmatrix}y\\z\\0\end{bmatrix}+z\begin{bmatrix}1\\1\\0\end{bmatrix}$$

$$C_3(\lambda)\begin{bmatrix}\lambda^2x+2\lambda y+z\\\lambda^2y+2\lambda z+z\\\lambda^2z\end{bmatrix}=\begin{bmatrix}\lambda^3x+3\lambda^2y+3\lambda z+z\\\lambda^3y+3\lambda^2z+\lambda z\\\lambda^3z\end{bmatrix}=\lambda^3\begin{bmatrix}x\\y\\z\end{bmatrix}+3\lambda^2\begin{bmatrix}y\\z\\0\end{bmatrix}+3\lambda\begin{bmatrix}z\\\frac{z}{3}\\0\end{bmatrix}$$

$$C_3^s u_j \simeq \lambda^s u_j+\binom{s}{1}\lambda^{s-1}u_{j-1}+\binom{s}{2}\lambda^{s-2}u_{j-2}+\cdots\binom{s}{j-1}\lambda^{s-j+1}u_1+\begin{bmatrix}z\\0\\0\end{bmatrix}\quad (s>j)$$

for $s\to\infty$, u_1 becomes dominant

$$C_3^s u_j\simeq\binom{s}{j-1}\lambda^{s-j+1}\left(u_1+\frac{j-1}{s-j+2}\lambda_1 u_2+\cdots\right)$$

As the matrix has a linear division of matrix M, the components of other eigenvalue except λ will fall so we consider eigenvectors corresponding to λ.

If A is any general matrix $A=VJV^{-1}$ (where 1st column of V is eigenvector of A corresponding to dominant eigenvalue λ_1)

so, $x_0=c_1v_1+c_2v_2+\cdots+c_nv_n$ (A is a $n\times n$ matrix)

$$x_k=\frac{A^k x_0}{\|A^k x_0\|}=\frac{(VJV^{-1})^k x_0}{\|(VJV^{-1})^s x_0\|}=\frac{VJ^kV^{-1}x_0}{\|VJ^kV^{-1}x_0\|}$$

$$=\frac{VJ^kV^{-1}(c_1v_1+c_2v_2+\cdots c_nv_n)}{\|VJ^kV^{-1}(c_1v_1+\cdots+c_nv_n)\|}=\frac{VJ^k(c_1e_1+\cdots+c_ne_n)}{\|VJ^k(c_1e_1+c_2e_2+\cdots+c_nv_n)\|}$$

$$=\left(\frac{\lambda_1}{|\lambda_1|}\right)^k\frac{c_1}{|c_1|}\frac{\left(v_1+\frac{V}{c_1}\left(\frac{1}{\lambda_1}J\right)^k(c_2e_2+\cdots+c_ne_n)\right)}{\|v_1+\frac{V}{c_1}\left(\frac{1}{\lambda_1}J\right)^k(c_2e_2+\cdots+c_ne_n)\|}$$

$$\simeq\left(\frac{\lambda_1}{|\lambda_1|}\right)^k\frac{c_1}{|c_1|}\frac{v_1}{\|v_1\|}\quad\left(\text{as }\left(\frac{1}{\lambda_1}J_i\right)^k\to 0\text{ as }k\to\infty\right)$$

Now, if $n=3$ and $V=I$ i.e. $A_{3\times3}=I\,J_{3\times3}\,I^{-1}=C_3$

Q5)

$$y_{j+1} = a_0 y_j + a_1 y_{j-1} + h(b_{-1} y'_{j+1} + b_0 y'_j + b_1 y'_{j-1})$$

maximum order is 4 which can be achieved by simpson's rule whose stability is already analysed in class.

For this 3^{rd} order 2step numerical integration, the formula is exact for $y = 1, t, t^2, t^3$ $(y_j = t_j^i)$ $(t_k = (k-j)h)$

For $i=0$ $(y=1)$

$1 = a_0 + a_1 \rightarrow (1)$

For $i=1$ $(y=t)$

$h = -a_1 h + h(b_{-1} + b_0 + b_1) \Rightarrow 1 = -a_1 + b_{-1} + b_0 + b_1 \rightarrow (2)$

For $i=2$ $(y=t^2)$

$h^2 = a_1 h^2 + h^2(2b_{-1} - 2b_1) \Rightarrow 1 = a_1 + 2b_{-1} - 2b_1 \rightarrow (3)$

For $i=3$ $(y=t^3)$

$h^3 = -a_1 h^3 + 3h^3[b_{-1} + b_1] \Rightarrow 1 = -a_1 + 3(b_{-1} + b_1) \rightarrow (4)$

$\rightarrow (3) + (4) \Rightarrow 2 = 5b_{-1} + b_1 \Rightarrow \boxed{b_1 = 2 - 5b_{-1}} \rightarrow (5)$

$\rightarrow 3 \times (2) - (4) \Rightarrow 2 = -2a_1 + 3b_0 \Rightarrow \boxed{b_0 = \frac{2}{3}(1 + a_1)} \rightarrow (6)$

$(2) + (3) \Rightarrow 2 = 3b_{-1} + b_0 - b_1$

$\Rightarrow 2 = 3b_{-1} + \frac{2}{3}(1 + a_1) - 2 + 5b_1 \Rightarrow \frac{10}{3} = 8b_{-1} + \frac{2}{3}a_1$

$\Rightarrow \boxed{a_1 = 5 - 12b_{-1}} \rightarrow (7)$

from (6) $b_0 = \frac{2}{3}(1 + 5 - 12b_{-1}) = 4 - 8b_{-1} \rightarrow (8)$

from (1) $a_0 = 1 - a_1 = -4 + 12b_{-1} \rightarrow (9)$

using $y' = \lambda y$ we get the equation;

$(1 - h\lambda b_{-1}) y_{j+1} = (a_0 + h\lambda b_0) y_j + (a_1 + h\lambda b_1) y_{j-1}$

look for solutions of the form $y_j = c\alpha^j$

$\Rightarrow (1 - h\lambda b_{-1})\alpha^2 - (a_0 + h\lambda b_0)\alpha - (a_1 + h\lambda b_1) = 0$

method is stable near $h=0$ so put $h=0$, we get,

$\alpha^2 - a_0\alpha - a_1 = 0 \Rightarrow \alpha^2 - (12b_{-1} - 4)\alpha + (12b_{-1} - 5) = 0$

$\Rightarrow |\alpha_i| = 1 \rightarrow$ for $\alpha = 1 \rightarrow$ (trivial result)

for $\alpha = -1 \Rightarrow 1 + 24b_{-1} - 9 = 0 \Rightarrow b_{-1} = \frac{1}{3}$

$|\alpha_i|^2 = \alpha_1\alpha_2 = 12b_{-1} - 5 = 1 \Rightarrow b_{-1} = \frac{1}{2}$

$\Delta = B^2 - 4AC = (12b_{-1} - 4)^2 - 4(12b_{-1} - 5) = 36(2b_{-1} - 1)^2 > 0$

The formula is relatively stable for $\frac{1}{3} \leq b_{-1} < \frac{1}{2}$