Assignment-7

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Q5) $\int_0^1 \frac{f(x)}{x^{0.9}} dx \approx \sum_{i=1}^{n} w_i f(a_i)$ [exact for polynomials of degree $\leq 2n-1$

For $n=1$

$f(x) = 1 \rightarrow \int_0^1 \frac{1\,dx}{x^{0.9}} = w_1 f(a_1) \Rightarrow w_1 = \frac{1}{0.1} = 10$

$\Rightarrow$ so for $n=1$ $\int_0^1 \frac{f(x)\,dx}{x^{0.9}} \simeq 10 f(a_1)$

For $n=2$

$f(x) = 1 \rightarrow \int_0^1 \frac{1\,dx}{x^{0.9}} = 10 = w_1 + w_2 \rightarrow (1)$

$f(x) = x \rightarrow \int_0^1 x^{0.1} dx = \left[\frac{x^{1.1}}{1.1}\right]_0^1 = w_1 f(a_1) + w_2 f(a_2)$

$\Rightarrow \frac{10}{11} = w_1 a_1 + w_2 a_2 \rightarrow (2)$

$\rightarrow (2) - a_1 \times \rightarrow (1)$

$\Rightarrow (a_2 - a_1) w_2 = 10\left(\frac{1}{11} - a_1\right)$

$\Rightarrow w_2 = \frac{(1-11a_1)10}{11(a_2-a_1)}$ & $w_1 = 10 - w_2$

$= 10\left[1 - \frac{1-11a_1}{11a_2 - 11a_1}\right]$

$= \frac{(11a_2 - 1)10}{11(a_2-a_1)}$

$\Rightarrow$ so, for $n=2$

$$\int_0^1 \frac{f(x)\,dx}{x^{0.9}} \simeq \frac{10(11a_2-1)}{11(a_2-a_1)} f(a_1) + \frac{10(1-11a_1)}{11(a_2-a_1)} f(a_2)$$

Q6) $S_1 = \sum_{i=1}^{n} i^5 = \frac{n^2(n+1)^2(2n^2+2n-1)}{12}$

$i^5 = i^{(5)} + a_4 i^{(4)} + a_3 i^{(3)} + a_2 i^{(2)} + a_1 i^{(1)}$

$i^{(n)} = i(i-1)(i-2)\ldots\ldots(i-n+1)$

so,

$i^{(5)} = i(i-1)\ldots(i-4)$

$i^{(4)} = i(i-1)\ldots(i-3)$

$i^{(3)} = i(i-1)(i-2)$

$i^{(2)} = i(i-1)$

$i=1$

$1 = a_1 i^{(1)} \Rightarrow a_1 = 1$

$i=2$

$32 = a_2 2^{(2)} + a_1 2^{(1)}$

$= a_2 \times 2 \times 1 + a_1 \times 2$

$\Rightarrow a_1 + a_2 = 16 \Rightarrow a_2 = 15$

$i=3$

$243 = a_3 3^{(3)} + 15 \times 3^{(2)} + 3^{(1)} = 6a_3 + 90 + 3 \Rightarrow a_3 = 25$

$i=4$

$1024 = a_4 4^{(4)} + 25 \times 4^{(3)} + 15 \times 4^{(2)} + 4^{(1)}$

$= 24a_4 + 600 + 180 + 4 \Rightarrow a_4 = 10$

so, $i^5 = i^{(5)} + 10i^{(4)} + 25i^{(3)} + 15i^{(2)} + i$

$\sum_{i=1}^{n} i^{(5)} = \frac{(n+1)^{(6)} - 1^{(6)}}{6} = \frac{1}{6}(n+1) \times n \times \ldots\ldots (n-4)$

$10 \times \sum_{i=1}^{n} i^{(4)} = \frac{(n+1)^{(5)}}{5} = \frac{1}{5}(n+1)n \times \ldots (n-3) \times 10$

$+$

$25 \times \sum_{i=1}^{n} i^{(3)} = \frac{(n+1)^{(4)}}{4} = \frac{1}{4}(n+1)n \times \ldots (n-2) \times 25$

$+$

$15 \times \sum_{i=1}^{n} i^{(2)} = \frac{(n+1)^{(3)}}{3} = \frac{1}{3}(n+1) \times n \times (n-1) \times 15$

$+$

$\sum_{i=1}^{n} i^{(1)} = \frac{(n+1)^{(2)}}{2} = \frac{1}{2} n(n+1)$

$S_1 = \sum_{i=1}^{n} i^5 = \frac{n(n+1)}{2}\Big[1 + 10(n-1) + \frac{25}{2}(n-2)(n-1) + 4(n-3)(n-2)(n-1) + \frac{1}{3}(n-4)(n-3)(n-2)(n-1)\Big]$

$= \frac{n(n+1)}{2}\left[\frac{n}{6}(2n^3+4n^2+n-1)\right] = \frac{n(n+1) \times n}{2 \times 6} \times \left[2n^3+2n^2-n+2n^2+2n-1\right]$

$= \frac{n^2(n+1)}{12} \times \left[(n+1)(2n^2+2n-1)\right] = \frac{n^2(n+1)^2(2n^2+2n-1)}{12}$

$$S_2 = \sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$$

$$i^3 = i^{(3)} + a_2 i^{(2)} + a_1 i^{(1)}$$

$$\underline{i=1} \qquad \underline{i=2}$$

$$1 = a_1 \qquad 8 = 2a_2 + 2 \Rightarrow a_2 = 3$$

$$i^3 = i^{(3)} + 3 i^{(2)} + i^{(1)}$$

$$\sum_{i=1}^{n} i^{(3)} = \frac{(n+1)^{(4)}}{4} = \frac{n(n+1)(n-1)(n-2)}{4}$$

$$3 \times \sum_{i=1}^{n} i^{(2)} = \frac{(n+1)^{(3)}}{3} = \frac{n(n+1)(n-1)}{3} \times 3$$

$$\sum_{i=1}^{n} i^{(1)} = \frac{(n+1)^{(2)}}{2} = \frac{n(n+1)}{2}$$

$$S_2 = \sum_{i=1}^{n} i^3 = \frac{n(n+1)}{2}\left[1 + 2(n-1) + \frac{1}{2}(n^2-3n+2)\right]$$

$$= \frac{n(n+1)}{2}\left[1 + 2n - 2 + \frac{n^2}{2} + \frac{n}{2} + 1\right]$$

$$= \frac{n(n+1)}{4}(n^2+n)$$

$$\Rightarrow S_2 = \frac{n^2(n+1)^2}{4}$$

$$S_4 = \sum_{n=1}^{N} \frac{1}{n(n+1)}$$

$$= \sum_{n=1}^{N}\left(\frac{1}{n} - \frac{1}{n+1}\right) = \left(1 - \cancel{\frac{1}{2}}\right) + \left(\cancel{\frac{1}{2}} - \cancel{\frac{1}{3}}\right) + \left(\cancel{\frac{1}{3}} - \frac{1}{4}\right) + \cdots + \left(\cancel{\frac{1}{N}} - \frac{1}{N+1}\right)$$

$$= 1 - \frac{1}{N+1} = \frac{N}{N+1}$$

or, $$S_4 = \sum_{n=1}^{N} \frac{1}{(n+1)^{(2)}} = \sum_{n=1}^{N} (n-1)^{(-2)}$$

$$= \sum_{n-1=0}^{N-1} (n-1)^{(-2)} = \frac{N^{(-1)} - 0^{(-1)}}{-1} = 1 - \frac{1}{N+1}$$

$$= \frac{N}{N+1}$$

$$S_3 = \sum_{i=2}^{n} \frac{i-4}{i(i^2-1)} = \sum_{i-2=0}^{n-2} \frac{\cancel{i-2}}{}(i-2)$$

$$\frac{i-4}{i(i+1)(i-1)} = \frac{i-4}{(i+1)^{(3)}} \qquad (i-2)^{(-3)}$$

$$= \frac{1}{(i+1)(i-1)} = \frac{1}{2}\left[\frac{1}{i-1} - \frac{1}{i+1}\right.$$

$$S_3 = \sum_{i=2}^{n} \frac{i-4}{i(i^2-1)} = \sum_{i=2}^{n} \frac{\cancel{i}}{\cancel{i}(i+1)(i-1)} - 4\sum_{i-2=0}^{n-2} \frac{1}{(i-1)i(i+1)}$$

$$= A + (-4)\times B$$

$$A = \frac{1}{2}\sum_{i=2}^{n}\left(\frac{1}{i-1} - \frac{1}{i+1}\right) = \frac{1}{2}\left[\left(1-\cancel{\frac{1}{3}}\right) + \left(\frac{1}{2} - \cancel{\frac{1}{4}}\right) + \left(\cancel{\frac{1}{3}} - \cancel{\frac{1}{5}}\right) + \left(\cancel{\frac{1}{4}} - \cancel{\frac{1}{6}}\right) + \cdots \left(\cancel{\frac{1}{n-2}} - \frac{1}{n}\right) + \left(\cancel{\frac{1}{n-1}} - \frac{1}{n+1}\right)\right]$$

$$= \frac{1}{2}\left\{\frac{3}{2} - \left[\frac{1}{n} + \frac{1}{n+1}\right]\right\}$$

$$= \frac{1}{2}\left[\frac{3}{2} - \frac{2n+1}{n(n+1)}\right]$$

$$= \frac{3n(n+1) - 4n - 2}{4n(n+1)} = \frac{3n^2 - n - 2}{4n(n+1)}$$

$$B = \left[\sum_{i-2=0}^{n-2} \frac{1}{(i+1)^{(-3)}}\right] = -4 \times \sum_{i-2=0}^{n-2} (i-2)^{(-3)}$$

$$\left(\text{using } \sum_{n=M}^{N} n^{(n)} = \frac{(N+1)^{(n+1)} - M^{(n+1)}}{n+1}\right) = \frac{(n-2+1)^{(-2)} - 0^{(-2)}}{-2}$$

$$= \left(\frac{1}{(n+1)^{(2)}} - \frac{1}{2^{(2)}}\right) \times -\frac{1}{2}$$

$$= \left(\frac{1}{n(n+1)} - \frac{1}{2}\right) \times -\frac{1}{2} = \frac{n^2 + n - 2}{4n(n+1)}$$

$$\Rightarrow S_3 = A - 4B = \frac{3n^2 - n - 2}{4n(n+1)} - \frac{4n^2 + 4n - 8}{4n(n+1)} = -\frac{n^2 + 5n - 6}{4n(n+1)}$$

Q4) $I_1 = \int_0^1 e^x dx = e - 1$

expecte order of error $\rightarrow \frac{\sigma}{\sqrt{n}}$

where variance $\sigma^2 = \int_0^1 e^{2x} dx - \left(\int_0^1 e^x dx\right)^2$

$= \left[\frac{e^{2x}}{2}\right]_0^1 - (e-1)^2$

$= \frac{1}{2}(e^2-1) - (e-1)^2 \approx 0.24204$

$I_2 = \int_0^1 (e^x - 1 - x) dx = e - \frac{5}{2}$

error $\rightarrow \frac{\sigma}{\sqrt{n}}$ where $\sigma^2 = \int_0^1 (e^x - 1 - x)^2 dx - \left(\int_0^1 (e^x - 1 - x) dx\right)^2$

$\Rightarrow \sigma^2 = -\int (e^{2x} - 2xe^x - 2e^x + x^2 + 2x + 1) dx + I_2$

$= -\left[\frac{e^{2x}}{2} - 2\left[xe^x - \int e^x dx\right] - 2e^x + \frac{x^3}{3} + x^2 + x + I_2\right]$

$= -\left[\frac{e^{2x}}{2} - 2xe^x + \frac{x^3}{3} + x^2 + x\right]_0^1 + I_2$

$= -\left[\frac{e^2}{2} - 2e + \frac{11}{6}\right] + \left(e - \frac{5}{2}\right)$

$= 3e - \frac{e^2}{2} - \frac{53}{12} \approx 0.044$

so, the no. of function evaluation to achieve the same accuracy is going to reduce by a factor of $\frac{0.044}{.24204} \approx 0.182$