# ASSIGNMENT

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Q3) (i) $e^x = \cos x$

$\Rightarrow x = \ln(\cos x)$

$\Rightarrow f(x) = \ln(\cos x)$

$|f'(x)| < 1 \Rightarrow |-\tan x| < 1$ (✗)

It doesn't converge. (except for $x=0$)

or $x = \cos^{-1}(e^x)$

so $f(x) = \cos^{-1}(e^x)$

$\Rightarrow |f'(x)| < 1$

$\Rightarrow \left|\dfrac{-e^x}{\sqrt{1-e^{2x}}}\right| < 1$

$\Rightarrow e^x < \sqrt{1-e^{2x}}$

$\Rightarrow e^{2x} < \frac{1}{2}$

$\Rightarrow 2x < -\ln 2$

$\Rightarrow x < -\frac{\ln 2}{2}$ (except the trivial root $x=0$)

($\cos^{-1}x$ gives result in the range $[0, \pi]$)

$\Rightarrow x = \cos^{-1}(\cos x) = \cos^{-1}(e^x)$

general solution is

$x = -2n\pi \pm \cos^{-1}(e^x)$

The roots are:-

$0, -1.293, -4.721, -7.854$

------ etc.

(ii) $\ln(1+x) = x^5$

$\Rightarrow 1+x = e^{x^5}$ so $x = e^{x^5} - 1$

$|f'(x)| < 1 \Rightarrow |5x^4 e^{x^5}| < 1$ ✗

It doesn't converge (except for $x=0$)

or $x = [\ln(1+x)]^{1/5}$

$f'(x) = \left|\frac{1}{5} \frac{1}{[\ln(1+x)]^{4/5}} \times \frac{1}{(1+x)}\right| < 1$ ✓

for $x = 0.918$

So, possible forms are. $\begin{cases} \ln(1+x)^{1/5}, & \text{for } x = 0.918 \\ e^{x^5} - 1, & \text{for } x = 0 \end{cases}$

(iii) $e^x = 4x^6$

$\Rightarrow x = \ln(4x^6)$ or $x = \left(\frac{e^x}{4}\right)^{1/6}$

$\Rightarrow |f'(x)| < 1$ $\qquad f(x) = \frac{e^{x/6}}{4^{1/6}}$

$\Rightarrow \left|\frac{24x^3}{4x^6}\right| < 1 \Rightarrow \left|\frac{1}{x^3}\right| < \frac{1}{6}$ $\qquad f'(x) = \left|\frac{1}{6}\frac{e^{x/6}}{4^{1/6}}\right| < 1$

for $x = 19.0773$ $\qquad$ for $x = 0.92618, -0.70653$

The roots are :- $0.92618, -0.70563, 19.0773$

so different forms are.

$$\begin{cases} \ln(4x^6) & \text{for } x = 19.0773 \\ \dfrac{e^{x/6}}{4^{1/6}} & \text{for } x = 0.92618, -0.70563 \end{cases}$$

(5) $x_{i+\frac{1}{2}} = x_i - \dfrac{f(x_i)}{f'(x_i)} \rightarrow (1)$ and $x_{i+1} = x_{i+\frac{1}{2}} - \dfrac{f(x_{i+\frac{1}{2}})}{f'(x_i)} \rightarrow (2)$

and we also have, $\epsilon_{i+\frac{1}{2}} = x_{i+\frac{1}{2}} - \alpha$, $\epsilon_i = x_i - \alpha$
and $\epsilon_{i+1} = x_{i+1} - \alpha$ $\longrightarrow (3)$

from → (1) & (3),

$$\epsilon_{i+\frac{1}{2}} + \cancel{\alpha} = \epsilon_i + \cancel{\alpha} - \frac{f(\epsilon_i+\alpha)}{f'(\epsilon_i+\alpha)} = \epsilon_i - \frac{\left[f(\alpha) + \epsilon_i f'(\alpha) + \frac{\epsilon_i^2}{2} f''(\alpha) + \dots\right]}{\left[f'(\alpha) + \epsilon_i f''(\alpha) + \dots\right]}$$

$$\Rightarrow \epsilon_{i+\frac{1}{2}} = \epsilon_i - \frac{1}{f'(\alpha)}\left(\epsilon_i f'(\alpha) + \frac{\epsilon_i^2}{2} f''(\alpha)\right) \times \left[1 + \epsilon_i \frac{f''(\alpha)}{f'(\alpha)}\right]^{-1}$$

(ignoring higher order terms & $f(\alpha) = 0$)

$$\Rightarrow \epsilon_{i+\frac{1}{2}} = \cancel{\epsilon_i} - \cancel{\epsilon_i} - \frac{\epsilon_i^2}{2}\frac{f''(\alpha)}{f'(\alpha)} + \epsilon_i^2 \frac{f''(\alpha)}{f'(\alpha)} + O(\epsilon_i^3) + \underbrace{HOT}_{\text{higher order terms}}$$

$$\Rightarrow \epsilon_{i+\frac{1}{2}} = \frac{\epsilon_i^2}{2}\frac{f''(\alpha)}{f'(\alpha)} \text{ (ignoring HOT)} \rightarrow (4)$$

from → (2) & (3),

$$\epsilon_{i+1} + \cancel{\alpha} = \epsilon_{i+\frac{1}{2}} + \cancel{\alpha} - \frac{f(\alpha + \epsilon_{i+\frac{1}{2}})}{f'(\epsilon_i + \alpha)} = \epsilon_{i+\frac{1}{2}} - \frac{\left[0 + \epsilon_{i+\frac{1}{2}} f'(\alpha) + \frac{\epsilon_{i+\frac{1}{2}}^2}{2} f''(\alpha) + HOT\right]}{\left[f'(\alpha) + \epsilon_i f''(\alpha) + HOT\right]}$$

$$\Rightarrow \epsilon_{i+1} = \epsilon_{i+\frac{1}{2}} - \frac{1}{f'(\alpha)}\left(\epsilon_{i+\frac{1}{2}} f'(\alpha) + \frac{\epsilon_{i+\frac{1}{2}}^2}{2} f''(\alpha)\right)\left[1 + \epsilon_i \frac{f''(\alpha)}{f'(\alpha)}\right]^{-1}$$

$$\Rightarrow \epsilon_{i+1} = \cancel{\epsilon_{i+\frac{1}{2}}} - \cancel{\epsilon_{i+\frac{1}{2}}} - \frac{\epsilon_{i+\frac{1}{2}}^2}{2}\frac{f''(\alpha)}{f'(\alpha)} + \epsilon_i \epsilon_{i+\frac{1}{2}} \frac{f''(\alpha)}{f'(\alpha)} + HOT$$

$$\Rightarrow \epsilon_{i+1} = -\frac{\epsilon_{i+\frac{1}{2}}^2}{2}\frac{f''(\alpha)}{f'(\alpha)} + \epsilon_i \epsilon_{i+\frac{1}{2}} \frac{f''(\alpha)}{f'(\alpha)} \rightarrow (5) \text{ (ignoring HOT)}$$

From (4) & (5)

$$\epsilon_{i+1} = O(\epsilon_i^4) + \frac{\epsilon_i^3}{2}\left(\frac{f''(\alpha)}{f'(\alpha)}\right)^2 + HOT$$

Teacher's Signature

Ignoring $o(\epsilon_i^4)$ & HOT.

$$\epsilon_{i+1} = \frac{\epsilon_i^3}{2}\left(\frac{f''(\alpha)}{f'(\alpha)}\right)^2 \quad (P=3, \text{ the convergence is cubic})$$

$$\Rightarrow \lim_{i\to\infty} \frac{x_{i+1}-\alpha}{(x_i-\alpha)^3} = \frac{1}{2}\left(\frac{f''(\alpha)}{f(\alpha)}\right)^2$$

(b) Computational efficiency $= E = P^{1/\theta} = 3^{1/3}$ (assuming cost of computing of $f(x) = f'(x) = 1$)

$= 1.442$

If cost of computing $f(x) = 1$, $f'(x) = \theta$

(i) This method is more efficient than NR method.

In this method efficiency $(E) = 3^{\frac{1}{2+\theta}}$ ← 2 times $f(\,)$ is calculated

In NR method $E = 2^{\frac{1}{1+\theta}}$

so, $3^{\frac{1}{2+\theta}} > 2^{\frac{1}{1+\theta}}$

$$\Rightarrow \frac{\ln 3}{2+\theta} > \frac{\ln 2}{1+\theta} \Rightarrow \theta > \frac{\ln 2}{\ln 3 - \ln 2} - 1 = 0.7095 \simeq 0.71$$

(ii) In secant method, $E = 1.618$

so, $3^{\frac{1}{2+\theta}} > 1.618$

$$\Rightarrow \frac{\ln 3}{\ln(1.618)} > 2+\theta \Rightarrow \theta < \frac{\ln 3}{\ln(1.618)} - 2 \simeq 0.2831$$

Hence, This method will be more efficient than the Newton-Raphson method if $\theta > 0.71$ while for $\theta < 0.2831$, it is more efficient than Secant method.