ASSIGNMENT

→ Sangram Keshari Patro

Q.2)
$$\int_a^{a+3h} f(x)\,dx = a_1 f(a) + a_2 f(a+h) + a_3 f(a+2h) + a_4 f(a+3h) + e f^{(4)}(\eta)$$

which should be exact for $f(x) = 1, x, x^2, x^3$

[choose $a = -2h$]

$f(x) = 1 \rightarrow \quad 3h = a_1 + a_2 + a_3 + a_4 \rightarrow (1)$

$f(x) = x \rightarrow \quad \frac{1}{2}[h^2 - 4h^2] = -\frac{3h^2}{2} = -2ha_1 - ha_2 + ha_4$

$\Rightarrow \frac{3h}{2} = 2a_1 + a_2 - a_4 \rightarrow (2)$

$f(x) = x^2 \rightarrow \quad \frac{1}{3}[h^3 + 8h^3] = 3h^3 = 4h^2 a_1 + a_2 h^2 + h^2 a_4$

$\Rightarrow 3h = 4a_1 + a_2 + a_4 \rightarrow (3)$

$f(x) = x^3 \rightarrow \quad \frac{1}{4}[h^4 - 16h^4] = -\frac{15h^4}{4} = -8h^3 a_1 - h^3 a_2 + h^3 a_4$

$\Rightarrow \frac{15h}{4} = 8a_1 + a_2 - a_4 \rightarrow (4)$

$\rightarrow (4) - (2) = 6a_1 = \frac{9h}{4} \Rightarrow a_1 = \frac{3h}{8}$

so, eq → (2) & (3) becomes

$\frac{3h}{4} = a_2 - a_4$ & $\frac{3h}{2} = a_2 + a_4$

$\Rightarrow a_2 = \frac{9h}{8}$ & $a_4 = \frac{3h}{8}$

substituting all these in → (1) we get

$a_3 = 3h - \frac{9h}{8} - \frac{3h}{8} - \frac{3h}{8} = \frac{9h}{8}$

To find e put $f(x) = x^4$ so,

$\frac{1}{5}[h^5 + 32h^5] = \frac{33h^5}{5} = \frac{3h}{8}[\cancel{h} + 16h^4 + 3 \times h^4 + 0 + h^4] + 24e$

$\cancel{\Rightarrow \frac{33h^5}{5} = \frac{3h^6}{8} \times 24}$

$\frac{33h^5}{5} = \frac{3h}{8}[16h^4 + 3h^4 + h^4] + 24e \Rightarrow -\frac{9h^5}{10 \times 24} = e \Rightarrow e = -\frac{3}{80}h^5$

so, we get,

$$\int_a^{a+3h} f(x)\,dx = \frac{3h}{8}\left[f(a) + 3f(a+h) + 3f(a+2h) + f(a+3h)\right] - \frac{3}{80}h^5 f^{(4)}(\eta)$$

Q3) $\int_0^1 \frac{1}{\sqrt{x}} f(x)\,dx \simeq \sum_{i=0}^{n} w_i f(i/n)$ [Exact for polynomials of degree $\leq n$

For $n=2$

$f(x)=1 \rightarrow \int_0^1 \frac{1}{\sqrt{x}} f(x)\,dx = w_0 f(0) + w_1 f\left(\frac{1}{2}\right) + w_2 f(0)$

$\Rightarrow w_0 + w_1 + w_2 = 2 \longrightarrow (1)$

$f(x) = x$ $\int_0^1 \sqrt{x}\,dx = w_1 \times \frac{1}{2} + w_2 = \frac{2}{3} = w_1 + 2w_2 = \frac{4}{3} \rightarrow (2)$

$f(x) = x^2$ $\int_0^1 x\sqrt{x}\,dx = \frac{2}{5} = \frac{w_1}{4} + w_2 \Rightarrow w_1 + 4w_2 = \frac{8}{5} \rightarrow (3)$

$(3) - (2) = 2w_2 = \frac{4}{15} \Rightarrow w_2 = \frac{2}{15}$

so, we get by solving $\rightarrow$ $\boxed{w_1 = \frac{16}{15},\ w_2 = \frac{2}{15},\ w_0 = \frac{4}{5}}$

so, for $n=2$, we have,

$\int_0^1 f(x) \times \frac{1}{\sqrt{x}}\,dx = \frac{4}{5} f(0) + \frac{16}{15} f\left(\frac{1}{2}\right) + \frac{2}{15} f(1)$

For $n=3$

$f(x) = 1 \rightarrow \int_0^1 \frac{dx}{\sqrt{x}} = 2 = w_0 f(0) + w_1 f\left(\frac{1}{3}\right) + w_2 f\left(\frac{2}{3}\right) + w_3 f(1)$

$\Rightarrow w_0 + w_1 + w_2 + w_3 = 2 \rightarrow (1)$

$f(x) = x \rightarrow \int_0^1 \sqrt{x}\,dx = \frac{2}{3} = \frac{w_1}{3} + \frac{2w_2}{3} + w_3 \rightarrow (2)$

$f(x) = x^2 \rightarrow \int_0^1 x\sqrt{x}\,dx = \frac{2}{5} = \frac{w_1}{9} + \frac{4w_2}{9} + w_3 \rightarrow (3)$

$f(x) = x^3 \rightarrow \int_0^1 x^2\sqrt{x}\,dx = \frac{2}{7} = \frac{w_1}{27} + \frac{8w_2}{27} + w_3 \rightarrow (4)$

Solving these 4 eq^ns we get, $w_0 = \frac{68}{105}, w_1 = \frac{6}{7}, w_2 = \frac{12}{35}$

$w_3 = \frac{16}{105}$

$\Rightarrow \int_0^1 \frac{1}{\sqrt{x}} f(x)\,dx = \frac{68}{105} f(0) + \frac{6}{7} f\left(\frac{1}{3}\right) + \frac{12}{35} f\left(\frac{2}{3}\right) + \frac{16}{105} f(1)$

Q4) $\int_0^{\pi} \cos(kx) f(x)\,dx \approx w_1(k) f(0) + w_2(k) f\left(\frac{\pi}{2}\right) + w_3(k) f(\pi)$

$f(x) = 1 \longrightarrow \int_0^{\pi} \cos(kx)\,dx = 0 = w_1 + w_2 + w_3$

$f(x) = x \longrightarrow \int_0^{\pi} x\cos(kx)\,dx = \frac{\pi}{2} w_2 + w_3 \pi = \left[x \frac{\sin kx}{k}\right]_0^{\pi} + \frac{1}{k^2}\left[\cos kx\right]_0^{\pi}$

$\Rightarrow$ $\underline{k \text{ even}}$: $\frac{w_2}{2} + w_3 = 0$ & $\underline{k \text{ odd}}$: $\frac{w_2}{2} + w_3 = \frac{-2}{\pi k^2}$

$f(x) = x^2 \longrightarrow \int_0^{\pi} x^2 \cos(kx)\,dx = \frac{\pi^2}{4} w_2 + \pi^2 w_3$

$[kx = t]$

$\int_0^{\pi} x^2 \cos(kx)\,dx = \int_0^{k\pi} \frac{1}{k^2}\left[t^2 \cos t\right] \frac{dt}{k} = \frac{1}{k^3}\left[\left[t^2 \sin t\right]_0^{k\pi} - \int_0^{k\pi} 2t \sin t\,dt\right]$

$= \frac{-2}{k^3}\left[\left[-t\cos t\right]_0^{k\pi} + \left[\sin t\right]_0^{k\pi}\right] = \frac{2}{k^3} \times k\pi (-1)^k = \frac{2\pi}{k^2}(-1)^k$

$\underline{k \text{ even}}$: $\frac{\pi^2}{4} w_2 + \pi^2 w_3 = \frac{2\pi \cancel{k}}{k^2}$

$\underline{k \text{ odd}}$: $\frac{\pi^2}{4} w_2 + \pi^2 w_3 = -\frac{2\pi \cancel{k}}{k^2}$

For k even

$w_1 + w_2 + w_3 = 0$

$w_2 + 2w_3 = 0$

$\pi w_2 + 4\pi w_3 = \frac{8\cancel{\pi}}{k^2}$

$\Rightarrow w_1 = \frac{4}{\pi k^2},\ w_2 = -\frac{8}{\pi k^2},\ w_3 = \frac{4}{\pi k^2}$

For k odd

$w_1 + w_2 + w_3 = 0$

$w_2 + 2w_3 = \frac{-4}{\pi k^2}$

$\pi w_2 + 4\pi w_3 = -\frac{8}{k^2}$

$\Rightarrow w_1 = \frac{2}{\pi k^2},\ w_2 = 0,\ w_3 = \frac{-2}{\pi k^2}$

so, for any k, from these 2 we can write as,

$w_1 = \frac{(3 + (-1)^k)}{\pi k^2}$, $w_2 = -\frac{4(1 + (-1)^k)}{\pi k^2}$, $\cancel{w_3 = 3(1+(-1)^k)}$

$w_3 = \frac{(3(-1)^k + 1)}{\pi k^2}$

so, we get

$$\int_0^{\pi} \cos(kx) f(x)\,dx \approx \frac{1}{\pi k^2}\left[\left[3 + (-1)^k\right] f(0) - \left[4(1 + (-1)^k)\right] f\left(\frac{\pi}{2}\right) + \left[3(-1)^k + 1\right] f(\pi)\right]$$