Name - Sangram keshari patro

(4) We have already derived in class that

$$E_n(x) = \frac{P_n(x)}{(n+1)!} f^{(n+1)}(\xi_x) \leq \frac{1}{(n+1)!} \max |P_n(x)| \max |f^{(n+1)}(x)|$$

on $[a,b]$

(maximum possible error)

(i) linear $(n=1)$

$$E_1(x) \leq \frac{1}{2!} \max |(x-x_0)(x-x_1)| \max |e^x| \quad \begin{bmatrix} f(x) = e^x \\ f''(x) = e^x \end{bmatrix}$$

on $[0,1]$

$$E_1(x) \leq \frac{1}{2!} \left| \frac{(x_0-x_1)^2}{4} \right| \times e \quad \begin{bmatrix} h(x) = (x-x_0)(x-x_1) \\ h'(x) = 0 = 2x - (x_0+x_1) \\ \Rightarrow x = \frac{x_0+x_1}{2} \end{bmatrix}$$

$$\Rightarrow E_1(x) \leq \frac{h^2}{8} \times e$$

so, $\frac{h^2 e}{8} < 10^{-7} \Rightarrow h < \sqrt{\frac{8 \times 10^{-7}}{e}} \simeq 0.00054$

So it is the maximum spacing for linear interpolation.

(ii) quadratic $(n=2)$

$$E_2(x) \leq \frac{1}{3!} \max |(x-x_0)(x-x_1)(x-x_2)| \max |f'''(x)|$$

on $[0,1]$ $[f'''(x) = e^x]$

$$E_2(x) \leq \frac{1}{3!} \times \frac{2h^3}{3\sqrt{3}} \times e$$

$$\leq \frac{h^3}{9\sqrt{3}} e$$

$x \leftarrow t \rightarrow$

$x_0 \leftarrow h \rightarrow x_1 \leftarrow h \rightarrow x_2$

$x - x_1 = t$, $x - x_0 = t - h$, $x + x_2 = t + h$

$h(x) = (x-x_0)(x-x_1)(x-x_2)$

$= t(t-h)(t+h) = (t^2 - h^2)t$

$h'(x) = 3t^2 - h^2 = 0 \Rightarrow t = \pm\frac{h}{\sqrt{3}}$

$\Rightarrow h(t) = \pm \frac{2h^3}{3\sqrt{3}}$ at $t = \pm\frac{h}{\sqrt{3}}$

so, $\frac{h^3 e}{9\sqrt{3}} < 10^{-7} \Rightarrow h < \sqrt[3]{\frac{10^{-7} \times 9\sqrt{3}}{e}}$

$\simeq 0.008308$

so, the maximum spacing for quadratic interpolation is 0.008308.

(iii) cubic $(n=3)$

$$E_3(n) \leq \frac{1}{4!} \max |(n-n_0)(n-n_1)(n-n_2)(n-n_3)| \max_{\text{on } [0,1]} |f^{(iv)}(n)|$$

$$\leq \frac{1}{4!} |-1 \times h^4| \times e$$

$$E_3(n) \leq \frac{h^4 e}{24}$$

So, $\frac{h^4 e}{24} \leq 10^{-7}$

$$\Rightarrow h < \sqrt[4]{\frac{24 \times 10^{-7}}{e}} \simeq 0.03065$$

So, the maximum spacing for cubic interpolation is 0.03065.

$h(n) = (n-n_0)(n-n_1)(n-n_2)(n-n_3)$

$n-n_1 = t,\ n-n_2 = t+h,\ n-n_3 = t+2h$

$n-n_0 = t-h.$

$n_0 \quad n_1 \quad n_2 \quad n_3$

$h(t) = t(t^2-h^2)(t+2h)$

$= (t^3 - th^2)(t+2h)$

$= h^4(k^3-k)(k+2) \quad [t = kh]$

$= h^4(k^4 - k^2 + 2k^3 - 2k)$

$h'(t) = h^4(4k^3 - 2k + 6k^2 - 2)$

$h'(t) = 0 = 2k^3 - k + 3k^2 - 1$

$\Rightarrow k = -\frac{1}{2}, \frac{-1 \pm \sqrt{5}}{2}$

for $k = -\frac{1}{2}$ $f(k) = \frac{9}{16}$

for $k = \frac{-1 \pm \sqrt{5}}{2}$, $f(k) = -1$

We can clearly see that for linear the max spacing is less but it increases as order of interpolation increases.

No. of entries in interpolation

→ linear → $\frac{1}{0.00054} \simeq 1851$

→ quadratic → $\frac{1}{0.008308} \simeq 120$

→ cubic → $\frac{1}{0.03065} \simeq 32$

(here we want maximum of $|P_4(n)|$ so we should take -1 because $\left|\frac{9}{16}\right| < |-1|$)

5) We have already derived ~~$E_1(n) \leq \frac{h^2}{8} \max |f''(\xi)|$~~

$E_1(n) \leq \frac{h^2}{8} |f''(\xi_c)|$ in 2n(4). So,

$$E_1(n) \leq \frac{h^2}{8} \times \frac{1}{4 \xi_c^{3/2}} = \frac{h^2}{32 \xi_c^{3/2}}$$

$$\left[f(n) = \sqrt{n} \quad f'(n) = \frac{1}{2\sqrt{n}} \quad f''(n) = \frac{1}{2} \times -\frac{1}{2 n^{3/2}} \right]$$

$h = 0.01 \Rightarrow \frac{0.01^2}{32 \xi_c^{3/2}} \leq 10^{-5}$ or 10^{-6} $\Rightarrow \frac{3.125 \times 10^{-6}}{\xi_c^{3/2}} < 10^{-5}$ or 10^{-6}

For 10^{-5}

$\xi_c^{3/2} > 0.3125 \Rightarrow \xi_c > 0.4605$

For 10^{-6}

$\xi_c^{3/2} > 3.125 \Rightarrow \xi_c > 2.137$ (excluding extrapolation)

but ξ_c should be between 0 & 1, so for a spacing of 0.01 this expression can't be $< 10^{-6}$.

(6) $f(x) = e^{ax}$, $x_j = a_j = jh$, $(j = 0, 1, \ldots)$

We know that,

$$f[a_j, a_{j+1} \ldots a_{j+k}] = \frac{1}{k!\,h^k} \Delta^k f(a_j) \rightarrow (1) \quad (k \geq 1)$$

Forward differences

$$\Delta f(x) = f(x+h) - f(x)$$

$$\Delta^2 f(x) = \Delta f(x+h) - \Delta f(x) = [f(x+2h) - f(x+h)] - [f(x+h) - f(x)]$$

$$= f(x+2h) - 2f(x+h) + f(x)$$

$$\Delta^3 f(x) = [f(x+3h) - f(x+2h)] - 2[f(x+2h) - f(x+h)] + [f(x+h) - f(x)]$$

$$= \underbrace{f(x+3h)}_{^3C_0} - \underbrace{3f(x+2h)}_{-^3C_1} + \underbrace{3f(x+h)}_{^3C_2} - \underbrace{f(x)}_{-^3C_3}$$

- - - - - - - - - - -

- - - - - - - - - - -

$$\Delta^k f(x) = {}^kC_0\, f(x+kh) - {}^kC_1\, f(x+(k-1)h) + \ldots \pm {}^kC_k\, f(x) \rightarrow (2)$$

for $x = x_0$ & $f(x) = e^{ax}$

$$\Delta^k f(x_0) = {}^kC_0\, e^{akh} - {}^kC_1\, e^{a(k-1)h} + \ldots \pm {}^kC_k \times 1$$

$$= (e^{ah} - 1)^k \rightarrow (3) \quad [f(x_0) = 1,\ f(x_1) = e^{ah},\ f(x_2) = e^{2ah} \ldots]$$

for $x = x_1$ & $f(x) = e^{ax}$

$$\Delta^k f(x_1) = {}^kC_0\, e^{a(k+1)h} - {}^kC_1\, e^{akh} + \ldots \pm {}^kC_k\, e^{ah}$$

$$= e^{ah} (e^{ah} - 1)^k \rightarrow (4)$$

To prove $eq^n \to (1)$ by induction.

~~Sup~~ For $k=1$

$$f[a_0, a_1] = \frac{f(a_1)-f(a_0)}{h} = \frac{e^{ah}-1}{h} \quad (a_1 - a_0 = h)$$

$$= \frac{(e^{ah}-1)^1}{1!\,h^1}$$

Assume that it is true for $k=t$

$$f[a_0, a_1, \ldots a_t] = \frac{(e^{ah}-1)^t}{t!\,h^t}$$

so, for $k = t+1$

$$f[a_0, \ldots a_{t+1}] = \frac{f[a_1, \ldots a_{t+1}] - f[a_0, \ldots a_t]}{a_{t+1} - a_0}$$

$$= \frac{\frac{\Delta^t f(a_1)}{t!\,h^t} - \frac{\Delta^t f(a_0)}{t!\,h^t}}{(t+1)h} \quad [\text{from } eq \to 1]$$

$$= \frac{\Delta^t[f(a_1)] - \nabla^t[f(a_0)]}{(t+1)!\,h^{t+1}}$$

$$= \frac{e^{ah}(e^{ah}-1)^t - (e^{ah}-1)^t}{(t+1)!\,h^{t+1}} \quad [\text{from } (3) \,\&\, (4)]$$

$$= \frac{(e^{ah}-1)^{t+1}}{(t+1)!\,h^{t+1}}$$

Hence we proved, that

$$f[a_0, \ldots\ldots a_n] = \frac{(e^{ah}-1)^n}{n!\,h^n} \longrightarrow (5)$$

so, $f(n) = f[a_0] + (n-a_0) f[a_0, a_1] + (n-a_0)(n-a_1) f[a_0, a_1, a_2] + \ldots P_n(n) f[a_0, \ldots a_n] + P_{n+1}(n) f[a_0, \ldots a_{n+1}] + \ldots$

For $n \to \infty$

$$\frac{n+1^{th}\text{ term}}{n^{th}\text{ term}} = \frac{(n-a_n) f[a_0, \ldots a_{n+1}]}{f[a_0, \ldots a_n]} = \frac{(e^{ah}-1)(n-nh)}{h(n+1)}$$

(by using $eq \to (5)$) $\quad (a_n = nh)$

so, $L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$

$= \lim_{n\to\infty} \left| (e^{ah}-1) \frac{(x-nh)}{h(n+1)} \right|$ (as $e^{ah}-1>0$)

$= e^{ah}-1$ (for x not a positive integral multiple of h)

So for the series to converge $L<1$

$\Rightarrow e^{ah}-1<1 \Rightarrow e^{ah}<2$

and for the series to diverge $L>1$

$\Rightarrow e^{ah}>2.$

For $e^{ah}=2$

n^{th} term $= a_n = P_n(x) f[a_0, \dots a_n]$

$$= \left(\prod_{j=0}^{n-1} (x-a_j) \right) \times \frac{(e^{ah}-1)^n}{n!\, h^n}$$

$$a_n = \left(\prod_{j=0}^{n-1} (Kh - jh) \right) \times \frac{1}{n!\, h^n} \qquad \left[\begin{array}{l} a_j = nh \\ x = Kh \end{array} \right]$$

$$a_n = \prod_{j=1}^{n} \left[\frac{(Kh-(j-1)h)}{hj} \right] = (-1)^n \prod_{j=1}^{n} \left(1 - \frac{K+1}{j}\right)$$

$$= \cancel{(-1)} \prod_{j=1}^{n} \left(\frac{K+1}{j} - 1 \right)$$

~~for $n\to\infty$, the j^{th} term~~

apply log both sides

$$\ln(|a_n|) = \sum_{j=1}^{n} \ln\left(1 - \frac{K+1}{j}\right)$$

For $n\to\infty$, the j^{th} term of this series tends to $\frac{K+1}{j}$. so, if $K \neq -1$ the series diverges.