# NM - ASSIGNMENT-3

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(4) Take, ~~the~~ $L^{-1} = [y_1\ y_2 \cdots y_n]$ where $y_K$ is a column vector of $n \times 1$.

Now we have,

$$LL^{-1} = I = [\underbrace{e_1\ e_2\ e_3 \cdots e_n}_{\text{basis vectors}}]$$

$$\Rightarrow LL^{-1} = L[y_1 \cdots y_n] = [Ly_1\ Ly_2 \cdots Ly_n]$$

$$\Rightarrow Ly_K = e_K \quad (1 \le K \le n)$$

$e_K \rightarrow 1$ in the $K^{th}$ row otherwise 0.

$$\Rightarrow \begin{bmatrix} l_{11} & 0 & \cdots & 0 \\ l_{21} & l_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ l_{n1} & & & l_{nn} \end{bmatrix} \begin{bmatrix} y_{1K} \\ y_{2K} \\ \vdots \\ y_{nK} \end{bmatrix} = \begin{bmatrix} \\ \cdot \\ \\ \end{bmatrix} \longrightarrow (1)$$

by putting $k = 2, 3, 4 \cdots n$ in →(1) we can clearly see that $y_K$ has 0s only above the $K^{th}$ row i.e, $L^{-1}_{ij} = y_{ij}$ is a lower triangular matrix.

So, $L \times L^{-1} = I$ can be written as,

$$\begin{bmatrix} l_{11} & 0 & \cdots & 0 \\ l_{21} & l_{22} & 0 \cdots & \vdots \\ \vdots & \vdots & \ddots & 0 \\ l_{n1} & \cdots & & l_{nn} \end{bmatrix} \begin{bmatrix} y_{11} & 0 & \cdots & \vdots \\ y_{21} & y_{22} & & \vdots \\ \vdots & & \ddots & 0 \\ y_{n1} & \cdots & & y_{nn} \end{bmatrix} = \begin{bmatrix} 1 & & & 0 \\ & 1 & & \\ & & 1 \ddots & \\ 0 & & & 1 \end{bmatrix}$$

By multiplying $j^{th}$ row of $L$ matrix with $j^{th}$ column of $L^{-1}$ matrix we get.

$$l_{11} y_{11} = 1,\ l_{22} y_{22} = 1 \cdots\cdots l_{jj} y_{jj} = 1 \quad (1 \le j \le n)$$

$$\Rightarrow y_{jj} = l^{-1}_{jj} = \frac{1}{l_{jj}}$$

$(j=1)$

Multiplying $L$ with $1^{st}$ column of $L^{-1}$ we get, ★

$i=2 \rightarrow$ $l_{21}y_{11} + l_{22}y_{21} = 0 \Rightarrow y_{21} = \underline{\frac{-1}{l_{22}}\left(\sum_{k=1}^{1} l_{2k}y_{k1}\right)}$

$j=1$

$= \frac{-1}{l_{22}}(l_{21}y_{11})$

$i=3$ $l_{31}y_{11} + l_{32}y_{21} + l_{33}y_{31} = 0 \Rightarrow y_{31} = \frac{-1}{l_{33}}(l_{31}y_{11} + l_{32}y_{21})$

$j=1$

$= \frac{-1}{l_{33}}\left(\sum_{k=1}^{2} l_{3k}y_{k1}\right)$ ★

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$i \rightarrow$ $\sum_{k=1}^{i} l_{ik}y_{k1} = 0 \Rightarrow y_{i1} = \frac{-1}{l_{ii}}\left(\sum_{k=1}^{i-1} l_{ik}y_{k1}\right)$

$j=1$

Similarly for $j=2$ & $i = 3, 4 \ldots n$

$\sum_{k=2}^{i} l_{ik}y_{k2} = 0 \Rightarrow y_{i2} = \frac{-1}{l_{ii}}\left(\sum_{k=1}^{i-1} l_{ik}y_{k2}\right)$

By observing the similar pattern we can say

for $j=j, i = j+1, 2, \ldots n$

$\sum_{k=j}^{i} l_{ik}y_{kj} = 0 \Rightarrow y_{ij} = \boxed{l^{-1}_{ij} = \frac{-1}{l_{ii}}\left(\sum_{k=1}^{i-1} l_{ik}l^{-1}_{kj}\right)}$

(5) Consider a Square matrix $A^{*}_{(n\times n)}$ which is a diagonal dominant matrix.

$$A^{*} = \begin{bmatrix} a_{11} - a_{j1} - a_{n1} \\ a_{21} - a_{2j} - - \\ \vdots \quad \vdots \\ a_{j1} \quad a_{ji} \cdots \\ a_{n1} \quad a_{nj} \cdots \end{bmatrix}$$

So, after 1st step of gaussian elimination, we have

$$A^{(1)} = \begin{bmatrix} a_{11} \cdots a_{1j} \cdots \\ 0 \quad \cdots a_{2j} - \frac{a_{21}}{a_{11}} a_{1j} \cdots \\ \vdots \quad \cdots \quad \vdots \quad \cdots \\ 0 \quad a_{nj} - \frac{a_{n1}}{a_{11}} a_{1j} \end{bmatrix}$$

Considering the diagonal dominance of 1st row of A.

$$\sum_{j=2}^{n} |a_{1j}| \leq |a_{11}| \longrightarrow (1) \Rightarrow \sum_{j=2}^{n} \left|\frac{a_{1j}}{a_{11}}\right| \leq 1 \ (\text{for } |a_{11}| \neq 0) \longrightarrow (2)$$

considering the non-diagonal elements of ith row of $A^{(1)}$.

$$\sum_{\substack{j=2\\ i\neq j}}^{n} \left| a_{ij} - a_{1j}\frac{a_{i1}}{a_{11}} \right| \leq \sum_{\substack{j=2\\ i\neq j}}^{n} |a_{ij}| + \sum_{\substack{j=2\\ i\neq j}}^{n} \left| a_{1j}\frac{a_{i1}}{a_{11}} \right|$$

$$= \left( \sum_{\substack{j=1\\ i\neq j}}^{n} |a_{ij}| - |a_{i1}| \right) + \left( \sum_{j=2}^{n} \left| a_{1j}\frac{a_{i1}}{a_{11}} \right| - \left| a_{1i}\frac{a_{i1}}{a_{11}} \right| \right)$$

$i=j$ case

Now (By using diagonal dominance of $r$th row of A we have $\sum_{j=1(i\neq j)}^{n} |a_{ij}| \leq |a_{ii}|$)

$$\leq |a_{ii}| - |a_{i1}| + |a_{i1}| \sum_{j=2}^{n} \left|\frac{a_{1j}}{a_{11}}\right| - \left| a_{1i}\frac{a_{i1}}{a_{11}} \right|$$

So, we get

$$\sum_{\substack{j=2 \\ i\neq j}}^{n} \left| a_{ij} - a_{1j}\frac{a_{i1}}{a_{11}} \right| \leq |a_{ii}| + |a_{i1}|\left(\sum_{j=2}^{n}\left|\frac{a_{1j}}{a_{11}}\right| - 1\right) - \left| a_{1i}\frac{a_{i1}}{a_{11}} \right|$$

Now from eq→② using $\sum_{j=2}^{n}\left|\frac{a_{1j}}{a_{11}}\right| - 1 \leq 0$ we have

$$\sum_{\substack{j=2 \\ i\neq j}}^{n} \left| a_{ij} - a_{1j}\frac{a_{i1}}{a_{11}} \right| \leq |a_{ii}| - \left| a_{1i}\frac{a_{i1}}{a_{11}} \right|$$

$$\leq \left| a_{ii} - a_{1i}\frac{a_{i1}}{a_{11}} \right|$$

which is the diagonal term of $i^{th}$ row of matrix $A^{(1)}$.

Hence we get if $A$ is diagonally dominant, $A^{(1)}$ is also diagonally dominant. If this procedure is used ~~by~~ inductively then we can get any submatrix in any stage of the gauss-elimination is diagonally dominant by rows, which means pivoting is never necessary for such matrices. And the pivot element $a_{kk}^{(k-1)}$ is non-zero as it is ~~[illegible]~~ in denominator part.

(6) Writing the system of eq$^n$s in matrix form we have,

$$\left(\begin{array}{cccc|c} 2 & 5 & 8 & 9 & 10 \\ 2 & 3 & 5 & 7 & 6 \\ 1 & 5 & 8 & 7 & 10 \\ 1 & 2 & 3 & 4 & 4 \end{array}\right)$$

By using $R_1 \to \frac{R_1}{2}$, ~~$R_2 \to 2R_1 \to R_2$~~ $R_2 \to R_2 - 2R_1$

$R_3 \to R_3 - R_1$

$R_4 \to R_4 - R_1$

$$\left(\begin{array}{cccc|c} 1 & \frac{5}{2} & 4 & \frac{9}{2} & 5 \\ 0 & -2 & -3 & -2 & -4 \\ 0 & \frac{5}{2} & 4 & \frac{5}{2} & 5 \\ 0 & -\frac{1}{2} & -1 & -\frac{1}{2} & -1 \end{array}\right)$$

By using, $R_2 \to -\frac{R_2}{2}$, $R_1 \to R_1 - \frac{5}{2}R_2$

$R_3 \to R_3 - \frac{5}{2}R_2$

$R_4 \to R_4 + \frac{1}{2}R_2$

$$\left(\begin{array}{cccc|c} 1 & 0 & \frac{10}{4} & 2 & 0 \\ 0 & 1 & \frac{3}{2} & 1 & 2 \\ 0 & 0 & \frac{1}{4} & 0 & 0 \\ 0 & 0 & -\frac{1}{4} & 0 & 0 \end{array}\right)$$

using $R_3 \to 4R_3$, $R_1 \to R_1 - \frac{R_3}{4}$

$R_2 \to R_2 - \frac{3}{2}R_3$

$R_4 \to R_4 + \frac{R_3}{4}$

$$\left(\begin{array}{cccc|c} 1 & 0 & 0 & 2 & 0 \\ 0 & 1 & 0 & 1 & 2 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$\Rightarrow x_1 + 2x_4 = 0$, $x_2 + x_4 = 2$, $x_3 = 0$

so, the solution of $Ax = b \Rightarrow$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} K \\ 2+\frac{K}{2} \\ 0 \\ -\frac{K}{2} \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 0 \\ 0 \end{bmatrix} + K \begin{bmatrix} 1 \\ \frac{1}{2} \\ 0 \\ -\frac{1}{2} \end{bmatrix}$$

[putting $x_1 = K$, $x_4 = -\frac{K}{2}$, $x_2 = 2 + \frac{K}{2}$, $x_3 = 0$]

it is a particular solution

It is the general solution of the eq$^n$ $Ax = 0$

So, the null space of the matrix is →

$$N(A) = \text{span} \begin{bmatrix} 1 \\ \frac{1}{2} \\ 0 \\ -\frac{1}{2} \end{bmatrix}$$

From the reduced row echelon form of A we can clearly see that first 3 columns of A are linearly independent of each other & fourth column can be constructed by linear combination of 1st & 2nd column. Hence the 1st 3 columns of A form a basis for its range space:

$$\left\{ \begin{bmatrix} 2 \\ 2 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 5 \\ 3 \\ 5 \\ 2 \end{bmatrix}, \begin{bmatrix} 8 \\ 5 \\ 8 \\ 3 \end{bmatrix} \right\}$$

We can also check the Null space by calculating.

$$Ax = \begin{bmatrix} 2 & 5 & 8 & 9 \\ 2 & 3 & 5 & 7 \\ 1 & 5 & 8 & 7 \\ 1 & 2 & 3 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 1/2 \\ 0 \\ -1/2 \end{bmatrix} = 0 \checkmark$$