Riskiness, endogenous productivity dispersion and business cycles

CrossMark

Can Tian

Shanghai University of Finance and Economics, School of Economics, China

ARTICLE INFO

ABSTRACT

In the data, cross-sectional productivity dispersion is countercyclical at both the plant level and the firm level, see e.g. Bloom (2009). I incorporate a firm's choice of risk level into a model of firm dynamics with real business cycle features to explain this empirical finding both qualitatively and quantitatively. In the model, in every period, each firm chooses the investment amount and the risk level associated with a production project every period. All projects available to each firm have the same expected flow return, determined by the aggregate and idiosyncratic shocks to the firm's productivity, and differ from one another only in their risk. The endogenous option of exiting the market and the limited funding for new investment jointly play an important role in motivating firms' risk-taking behavior. The model predicts that, in each period, relatively small firms are more likely to take risk and hence exhibit a higher exit rate, and that the cross-sectional productivity dispersion, measured as the standard deviation of the realized individual component of productivity, is larger in recessions.

1. Introduction

Cross-sectional productivity dispersion increases in bad times, as does volatility. This is the case for productivity at the plant, firm, and industry levels (Fig. 1). Recently, this phenomenon has attracted growing attention.[1] Macroeconomists are divided over the explanation for this pattern. The majority of the literature views an economic downturn as a result of an exogenous increase in uncertainty combined with various frictions. Meanwhile, other researchers advocate the hypothesis that increased dispersion and volatility is a consequence of recessions. The goal of this paper is to complement existing theories on what causes the negative correlation between business cycles and cross-sectional productivity dispersion.

This paper studies a mechanism through which limited liability and the option of exiting the market jointly induce more risk-taking behavior in recessions and increase the realized productivity dispersion as a result.[2] The main intuition behind the mechanism is that firms will take more risk when there is little to lose. Imagine a firm with limited capital. The firm can always exit and take with it the value from liquidating its capital. Or the firm may choose to continue its production. If the firm is on the edge of exiting, that is, the additional benefit from continuing is not very large, it may find some extra risk

---

[1] Examples are Higson et al. (2002, 2004), Bloom (2009), Bloom et al. (2014), Bachmann and Bayer (2013), Arellano et al. (2010), Bachmann et al. (2013), Kehrig (2011), to name a few.

[2] In what follows, the difference between a firm and a plant is not distinguished. The optimal number of plants or establishments a firm should have, although an interesting and important consideration, is not the focus here.

very appealing. With such risk, the worst case scenario (in which the project fails, yielding no payoff, and the firm may have to exit) occurs with positive probability. This does not seem too bad given that the firm has very little to lose to begin with. Or, also with positive probability, the project may succeed and result in exceptionally high output, which can significantly improve the firm's situation and pull it away from the edge of exiting. A bad shock pushes more firms to the edge and, as a result, they all choose to take more risk, which in turn leads to a more dispersed distribution of realized individual productivity.

The model employed is in line with the standard industry dynamics model with firm entry and exit built in the seminal work of Hopenhayn (1992), with aggregate technology shocks as the driving force of business cycles.[3] Specifically, the model features the following elements: Firms are heterogeneous in size and idiosyncratic shocks and can choose the level of risk to which their production is exposed. Firms cannot save or borrow, and therefore the investment cannot exceed the revenue at hand net of the operating costs. In each period, each firm has the option to exit the market and take the revenue from selling all of its capital. The choice of risk level is the major new twist to an otherwise standard model in order to capture the willingly chosen lack of diversification and additional exiting hazard, both of which can be size-dependent and can affect a firm's contemporaneous payoff.[4] For each firm, the productivity of a riskier project is a mean-preserving spread of the productivity of a less risky one. Although firms are risk-neutral and riskier projects do not give a higher expected flow payoff, there is a positive fraction of firms that strictly prefer to take on risky projects. This is because the option of exit provides a lower bound for a firm's continuation value as a function of working capital and creates a local convexity called the *risky region*. This convexity gives the firms an incentive to randomize over their future values by choosing riskier projects, and when the uncertain productivity is realized, dispersion arises. Due to the assumption that preowned capital has a discounted selling price, the risky region in the quantitative model also lies on the lower end of the capital axis. Therefore, cross-sectionally, smaller firms tend to take higher risk and exhibit higher volatility. In bad times, this risky region gets larger and the fraction of firms willing to take risks rises. In fact, if the productivity shock is bad enough, aggregate or idiosyncratic, even firms of the largest sizes will fall into the risky region and willingly take on extra risk. Consequently, the average riskiness in firms' production increases following a bad productivity shock, and so does the realized productivity dispersion.

The nature of the mechanism leads to its prediction on firm dynamics and the cyclicality of productivity dispersion by firm size groups, in addition to the model's direct implication on the cyclical feature of the whole economy. Note that the by-size predictions are twofold: (i) on the cross-sectional differences in firm dynamics and (ii) on the time-series differences in the correlation with cyclical indicators.

First, the model predicts that, on average, smaller firms bear higher risks than bigger ones and, in addition, smaller firms are more volatile over time. This claim is supported by the previous findings.[5] A well-known fact is that the survival rate increases in firm size while average growth rate decreases in size. If we look at the entry and exit behavior, Fig. 2 shows that, as plant size grows, not only do the entry and exit rates drop but their volatility decreases too. Combined with the pro- and countercyclicality in entry and exit rates, respectively, this suggests that smaller businesses contribute more to the entry and exit volatility by facing even higher risks in recessions. Fort et al. (2013), Haltiwanger et al. (2013), and Haltiwanger (2012) document the "up or out" dynamics of young businesses and the difference in cyclical sensitivity by size (young and small businesses are more vulnerable to business cycle shocks and their net growth rates fall more in contractions). In fact, Gertler and Gilchristm (1994), Davis and Haltiwanger (2001), Chari et al. (2013), and Fort et al. (2013) point out that small firms are more responsive to contractions, especially those associated with tighter credit markets. This also shows up in the cyclicality of stock returns; see, for example, Perez-Quiros and Timmermann (2000). Evidence also shows a lack of diversification in the higher risks faced by small businesses, according to findings in the entrepreneurship literature, suggesting an inverse relationship between the size of a business and its level of risk. Examples are Hamilton (2000), Moskowitz and Vissing-Jørgensen (2002), and Herranz et al. (2009).[6]

Meanwhile, the result of the quantitative exercise shows that the productivity dispersion within each size group exhibits similar degrees of countercyclicality, measured as the correlation between the standard deviation in the realized individual productivity and the aggregate cyclical indicators. This does not come as a surprise. The theoretical model predicts that smaller firms will consistently show larger productivity dispersion. However, it is the *changes* in the dispersion in response to aggregate productivity shocks that determine the cyclicality. Moreover, in the model, it is with positive probability that both the aggregate and idiosyncratic shocks can be so bad that even some of the largest firms will prefer riskier production projects. Similarly, the smaller ones sometimes choose not to take any risk. Hence, admittedly, the parameter values largely determine which size group has the larger degree of countercyclicality of productivity dispersion. As it turns out, the calibrated model predicts consistent countercyclicality across size groups with the correlation coefficients on similar scales.

---

[3] However, I do not consider the general equilibrium in this paper. Instead I focus on the aggregation of the firms' individual decisions.

[4] One can think of this as a change of sales strategy that has short-term impact on a firm's revenue, which is similar to the idea of pricing experiments studied by Bachmann and Moscarini (2012). However, the choice of risk level is not intended to capture the R&D expenditure, which is largely procyclical and may only pay off in the long run.

[5] Moscarini and Postel-Vinay (2012) document that large firms contribute disproportionately to the changes in the unemployment rate. However, unemployment rate and other cyclical indicators are not perfectly lined up, hence their paper does not contradict other findings.

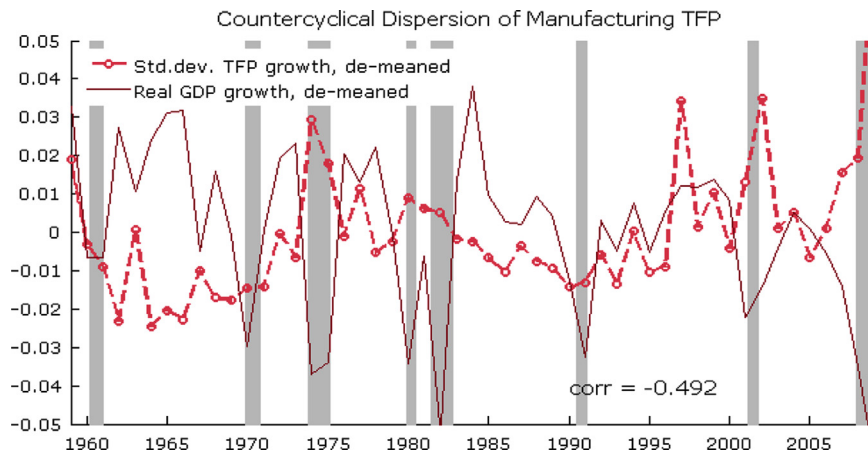[6] See Quadrini (2009) for a detailed review.

Fig. 1. Countercyclical dispersion in productivity at industry level. Productivity dispersion is measured as the standard deviation in the industry-level total factor productivity (TFP) growth, plotted with annual growth in real GDP. The shaded bars indicate official NBER recessions. Real GDP data are from Federal Reserve Economic Data at the Federal Reserve Bank of St. Louis (FRED); TFP data are from the NBER Manufacturing Industry Productivity Database.
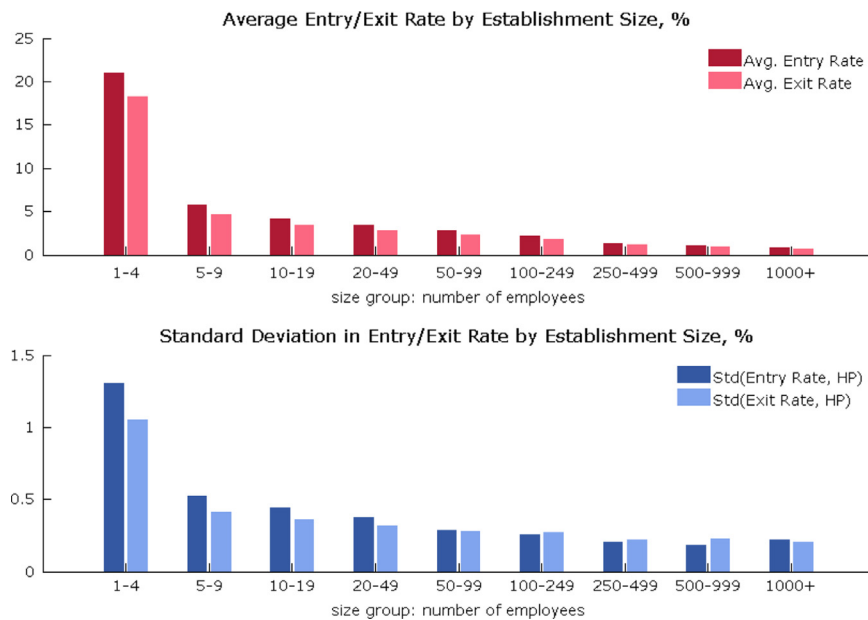

Fig. 2. Establishment Entry and Exit by Size in U.S. The top panel plots the time-series averages of entry and exit rates grouped by number of employees. The bottom panel plots the time-series standard deviations in Hodrick–Prescott (HP)-filtered entry and exit rates by size group. Data are from Business Dynamics Statistics, annual frequency from 1977 to 2012.

1.1. Literature review

The real option literature that aims to explain such countercyclicality suggests that the causality goes from increased uncertainty to a decline in aggregate economic activity. An influential paper is Bloom (2009), which was later generalized by Bloom et al. (2014). Bloom shows that increased uncertainty, through the channel of adjustment costs to capital and labor, leads to a larger option value of waiting and a pause in investment and employment. A sizable drop in aggregate economic activity occurs because of this "wait-and-see" effect. However, findings by Bachmann and Bayer (2013) and Bachmann et al. (2013) suggest that other mechanisms may be in play. Also, Baker and Bloom (2013) study the causality between first- and second-moment shocks and find that both play significant roles in explaining GDP growth. A recent work by Komaromi (2014) provides evidence supporting the first-to-second-moment direction.

One alternative that has been extensively studied is the interaction between uncertainty and financial frictions. Arellano et al. (2010) show that a sizable aggregate economic downturn can occur when risk is higher because firms reduce their investment projects to avoid default. Gilchrist et al. (2014) argue that financial frictions are important in the transmission of uncertainty shocks to idiosyncratic productivity. In fact, their model features both costly reversible investment and financial frictions in the debt and equity markets. They show that the financial frictions channel greatly enhances the "wait-and-see"
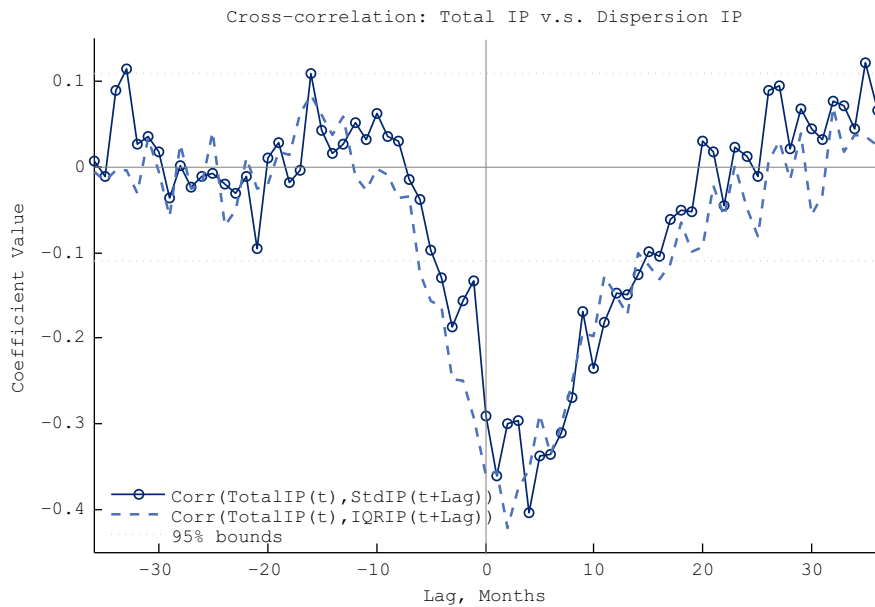
Fig. 3. Cross-correlation between total IP growth and dispersion of IP Growth at Industry Level. The 1986–2013 monthly data on Industrial Production (IP) indices by industry are from the Federal Reserve Board of Governors G17 series. The two measures of dispersion are (i) the standard deviation in IP growth (solid line with marks) and (ii) the inter-quartile range in IP growth (dashed line). The cyclical indicator is the growth of total IP.

effects of idiosyncratic uncertainty shocks and the quantitative magnitude is plausible. Di Tella (2012) shows that uncertainty shocks can drive balance sheet recessions due to the moral hazard problem, even when agents can write complete contracts on all observable variables. Christiano et al. (2014) also conclude that risk shocks to idiosyncratic productivity account for a large share of macroeconomic fluctuations in the U.S. in the presence of financial frictions. However, Chugh (2013) obtains a much lower estimate of the scale of the risk shocks, which results in a lower explanatory power of such shocks, even with financial frictions.

Note that the importance of the uncertainty shock is not denied in this paper, and the reverse causality may still be at work. But there is a measurement issue of uncertainty, which relates to the lead–lag relationship between uncertainty and business cycles. Time-series volatility of major business condition indicators is often interpreted as uncertainty. In addition, a parallel family of uncertainty measures concerns the realized cross-sectional dispersion in micro-level performance, which includes, among other things, the cross-sectional standard deviations in measured firm-level TFP level or growth rate, and sales growth rate. However, realized cross-sectional dispersion is only a proxy for uncertainty and increased micro-level cross-sectional dispersion tends to lag recessions. The cross-correlation between the total industrial production index (IP) and the industry-level IP dispersion is plotted in Fig. 3. The negative correlation reaches the highest absolute value when the dispersion measures lag the total IP by two to four months. This suggests that the aggregate economic state may cause the changes in measured uncertainty, in particular, it may cause the changes in cross-sectional dispersion in productivities.

The idea that countercyclical dispersion of productivity, sales, and prices can occur endogenously is not a new one. Van Nieuwerburgh and Veldkamp (2006) provide a model to address sudden recessions and gradual booms due to procyclical learning. The model also predicts countercyclical output dispersion as information is noisier in recessions. There are other papers that entertain the endogenous dispersion hypothesis. Bachmann and Moscarini (2012) build a model in which firms need to run costly pricing experimentations to learn about their own demand. Firms face uncertainty in demand but learn gradually from sales. As a result of lower experimentation costs, the dispersion of productivity measured in sales is larger during recessions due to the increase in experiments conducted. My model shares the idea that bad times incentivize risky behavior at the firm level. The major differences are that, in my model, (i) firm size is crucial in determining how much risk a firm is willing to take as well as in determining the option value of exiting and (ii) endogenous entry and exit are explicitly modeled and calibrated. In addition, my model is driven by standard first-moment productivity shocks, which allows for close comparison with the framework studied by Bloom (2009) and other conventional real business cycle models. Another alternative is to explore the more standard risk diversification channel. Decker et al. (2014) provide a theory in which positive aggregate TFP shocks enable firms to pay a cost to participate in more markets and thus reduce firm-level volatility through a standard diversification mechanism. Notably, in the model, low-productivity firms are more volatile and they do not react to the cycle. Empirical evidence from several firm-level data sets confirms the procyclical market exposure.

The role of entry and exit in a model with aggregate fluctuations is also largely examined. Several papers study how entry and exit may enhance the effects of aggregate productivity shocks in competitive economies. Campbell (1998) examines a vintage capital model in which new technology is available to entrants only and shows that entry and exit behaviors exhibit plausible cyclical patterns. Samaniego (2008) finds entry and exit insensitive to aggregate productivity shocks in a
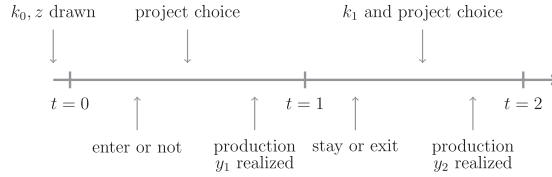
Fig. 4. Timing of the simple model.

frictionless model with a balanced growth path. Lee and Mukoyama (2015) and Clementi and Palazzo (2013) are the most relevant to my model. Lee and Mukoyama (2015) document plants' patterns of entry and exit in U.S. manufacturing and construct a model to match the data. Clementi and Palazzo (2013) show that entry and exit are important in shaping the aggregate dynamics by amplifying and propagating the effects of aggregate productivity shocks. In what follows, I will compare my model's prediction to these two papers and examine the role of choice of risk level.

The idea of allowing for choice of risk-taking resembles the model of occupational choice presented by Vereshchagina and Hopenhayn (2009). In their model, a poor entrepreneur may choose a riskier investment project because he can always quit and receive a fixed wage as a worker in case of business failure. While they focus on each individual's occupational choice and the resulting cross-sectional differences among entrepreneurs and workers, my paper examines the cyclical change of variables (especially firm-level productivity dispersion) over time; it also examines the entry and exit dynamics of firms.

The rest of the paper is organized as follows. Section 2 contains a simple three-period model that illustrates the mechanism and shows preliminary results. Section 3 takes the simple model to an infinite horizon and Section 4 examines the quantitative performance. Section 5 concludes. Omitted proofs are in the appendix.

2. A simple model

To demonstrate the mechanism, I start with a simplified and tractable three-period version of the model. The simple individual decision problem permits an analytical solution to each firm's decision-making rules for entry, exit, and the choice of risk levels. The comparative statics study shows the potential of the model in explaining countercyclical productivity dispersion while capturing procyclical entry and countercyclical exit of firms.

2.1. Setup

There are three periods, $t = 0, 1, 2$. There is a single type of good, the demand for which has infinite elasticity. The price is normalized to be 1 in each period. The good can be used as capital for future production. The capital fully depreciates in each period. A firm is defined as a production plant.

Each firm can choose from two projects, a risky one and a safe one, indexed by $\{p, 1\}$, respectively, with $p \in (0, 1)$. The index not only represents the probability of success of a project but it also determines the extra return conditional on success. Specifically, if a firm with the individual productivity component z chooses capital level k_t and project $p_t \in \{p, 1\}$, then the production outcome is a random variable that takes the following form:

$$y_{t+1} = AzB(p_t)k_t^\alpha \quad \text{where } B(p_t) = \begin{cases} 1/p_t, & w \cdot p \cdot p_t, \\ 0, & w \cdot p \cdot 1 - p_t. \end{cases} \tag{1}$$

The amount of capital k_t is determined at time t, and so is the choice of project p_t. The output level y_{t+1} is to be taken into the following period. Clearly, a firm that chooses the safe project $p_t = 1$ gets the output Azk_t^α for certain. The binary-outcome random variable $B(p)$ determines the outcome of the risky project. Note that,

$$\mathbb{E}(B(p)) = 1,$$
$$Var(B(p)) = (1-p)/p,$$

so the expected output for each project, safe or risky, is always Azk_t^α, given A, z, and k_t. The riskiness of the risky project can be measured as the standard deviation of $B(p)$, that is, $\sqrt{(1-p)/p}$. The riskiness of the safe project is zero. Hence, the risky project is in fact a mean-preserving spread of the safe one. Under this setup, A corresponds to the average firm-level productivity. The riskiness of each project represents the micro-level risk, while the realized cross-sectional variation in productivity measures the dispersion.

In period 0, a continuum of risk-neutral potential entrant firms decides whether to enter the market. Each entrant independently draws the initial capital $k_0 \geq 0$ and the individual component of its productivity $z > 0$ from the common distributions $G^k(k_0)$ on support $[0, \bar{k}]$ and $G^z(z)$ on $[\underline{z}, \bar{z}]$ with $\underline{z} > 0$, respectively. If the potential entrant chooses not to enter the market, it takes away a fixed value $V^0 > 0$. Otherwise, it enters as a firm and chooses a production project that requires all of its initial capital k_0. Assume that each production project requires the full capacity of a firm and the full attention of the firm owner, therefore a firm can undertake only one production project in each period. The period-0 flow payoff to each entering firm is zero.
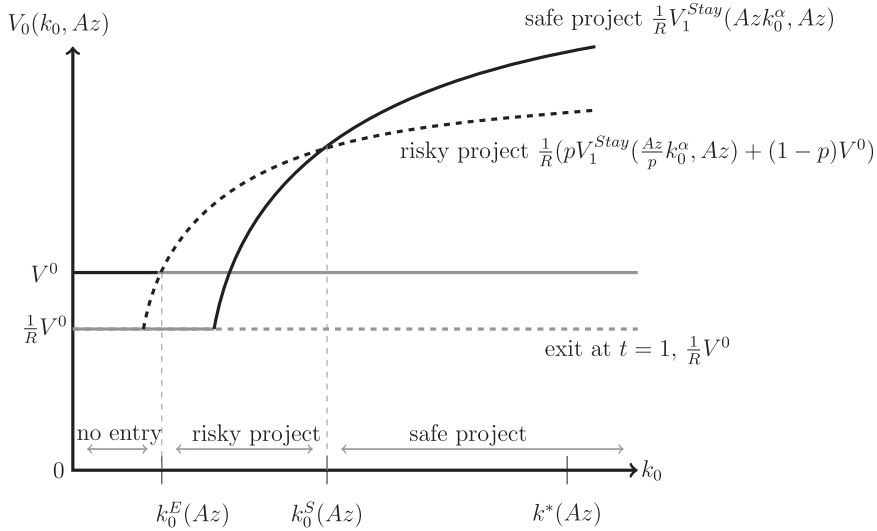
Fig. 5. Value function $V_0(k_0, Az)$ with fixed A, z, p, and a low V^0. The solid curve shows the continuation value of the safe project as a function of k_0; the dashed curve is that of the risky one; the horizontal V^0 line is the fixed value of no entry; the lower $(1/R)V^0$ line is the fixed continuation value of exiting.

In period 1, a surviving firm with output y_1 decides whether to exit the market given the value of exiting V^0. If the firm stays, then it decides on the investment level k_1 and the riskiness of the project, similar to the last period. The flow payoff to a surviving firm is $y_1 - k_1$. Assume that the flow payoff cannot be negative, hence $k_1 \in [0, y_1]$. This assumption is intended to capture the difficulty in financing; it constrains the maximum investment a firm can make. In period 2, each surviving firm takes away the realized output y_2. Each firm discounts the future flow of profit at rate $1/R$, $R > 1$. The timing is illustrated in Fig. 4.

2.2. Analysis

Let $V_t(y_t, Az)$ be the time t value for a firm with realized output y_t, aggregate productivity A, and individual productivity z. To solve the problem, it is convenient to work backwards. In the last period, $t=2$, each surviving firm gets its output y_2:

$$V_2(y_2, Az) = y_2.$$

At time $t=1$, each firm is indifferent between operating a safe project and a risky one that has the same expected productivity due to risk neutrality. Assume that all firms choose the safe project in this case, which is consistent with the choice under risk aversion. The period 1 value for a staying firm will be

$$V_1^{Stay}(y_1, Az) = \max_{0 \le k_1 \le y_1} \left\{ (y_1 - k_1) + \frac{1}{R} Az k_1^\alpha \right\}. \tag{2}$$

Let $k^*(Az)$ be the optimal unconstrained capital choice for this firm and $\pi^*(Az)$ the discounted output from choosing $k^*(Az)$; then

$$k^*(Az) = \left(\frac{\alpha Az}{R} \right)^{1/(1-\alpha)} \quad \text{and} \quad \pi^*(Az) = \frac{Az}{R}[k^*(Az)]^\alpha = \frac{k^*(Az)}{\alpha}. \tag{3}$$

The value of a firm with (y_1, Az) at the beginning of period 1 will be given by

$$V_1(y_1, Az) = \max\left\{ V^0, V_1^{Stay}(y_1, Az) \right\}. \tag{4}$$

Let $y_1^*(Az)$ be such that $V^0 = V_1^{Stay}(y_1^*, Az)$. Note that there is a kink at y_1^* and $V_1(y_1, Az)$ is convex in the neighborhood of $y_1^*(Az)$. This gives a firm with relatively low level of capital an incentive to choose a risky project before it enters period 1.

At $t=0$, a newborn firm with (k_0, z) chooses a project:

$$V_0^{Enter}(k_0, Az) = \frac{1}{R} \max\left\{ V_1(Az k_0^\alpha, Az), pV_1\left(\frac{Az}{p} k_0^\alpha, Az \right) + (1-p)V_1(0, Az) \right\}.$$

Lastly, at the beginning of period 0, each potential entrant with initial draw (k_0, z) decides whether to enter the market and become a new firm:

$$V_0(k_0, Az) = \max\{ V_0^{Enter}(k_0, Az), V^0 \}. \tag{5}$$

The following proposition describes the solution to each firm's problem, which is then illustrated in Fig. 5.

Proposition 1. *Consider the economy described by a triplet (A, p, V^0). Assume $R < \frac{1-p}{1-p^{1-\alpha}}$. At $t=0$, the decision of a potential entrant with initial draw (k_0, z) follows a cutoff rule characterized by thresholds $k_0^E(Az)$ and $k_0^S(Az)$ with $0 < k_0^E(Az) < k_0^S(Az)$ when $k_0^S(Az)$ exists, such that the firm will not enter if $k_0 < k_0^E(Az)$ and it will enter otherwise. An entering firm chooses the safe project if $k_0 \geq k_0^S(Az)$ and it chooses the risky one otherwise. Specifically,*

$$k_0^E(Az) = \left(\frac{(R+p-1)RV^0}{p^{1-\alpha}(Az)^{1+\alpha}} \right)^{1/\alpha^2};$$
(6)

also, for higher realizations of A and/or z such that $\pi^*(Az) \geq \frac{1-p}{p^\alpha - p} V^0$,

$$k_0^S(Az) = \left(\frac{(1-p)RV^0}{(1-p^{1-\alpha})(Az)^{1+\alpha}} \right)^{1/\alpha^2};$$
(7)

for realizations of A and/or z such that $V^0/(1-\alpha) \leq \pi^*(Az) < \frac{1-p}{p^\alpha - p} V^0$, $k_0^S(Az) \leq k^*(Az)$ uniquely solves the following:

$$\frac{Az}{R} \left(Az \left[k_0^S(Az) \right]^\alpha \right)^\alpha - Az \left[k_0^S(Az) \right]^\alpha = p(1-\alpha)\pi^*(Az) + (1-p)V^0;$$
(8)

and for low realizations of A and/or z such that $\pi^*(Az) < V^0/(1-\alpha)$, $k_0^S(Az)$ does not exist, in which case all entering firms choose the risky project. At $t=1$, the decision of an active firm with output y_1 follows a cutoff rule characterized by $y_1^*(Az)$ such that the firm exits if $y_1 < y_1^*(Az)$; otherwise, it stays and invests $\min\{y_1, k^*(Az)\}$ in the safe project:

$$y_1^*(Az) = \begin{cases} \left(\dfrac{RV^0}{Az} \right)^{1/\alpha} & \text{if } \pi^*(Az) \geq V^0, \\ V^0 - (1-\alpha)\pi^*(Az) & \text{otherwise.} \end{cases}$$
(9)

The proof can be found in Appendix A.1. The assumption is to ensure that the risky project is always more appealing to some firms. Intuitively, the discount rate must be high enough so that some firms with initial draw (k_0, z) close to the safe-project threshold $k_0^S(Az)$ find entering with a risky project preferable to not entering at all. In addition, for some (larger) firms to choose the safe project, i.e., for the existence of $k_0^S(Az)$ for some z, the option value of exiting V^0 cannot be too high. Specifically, the unconstrained optimal net return from investing $k^*(Az)$ must exceed V^0 for some sufficiently large realization of z. Fig. 5 illustrates the agent's decision rule specified in Proposition 1 when this is the case. The potential entrants with the lowest draws of initial capital k_0 do not enter the market. Conditional on entry, the continuation value function of the safe project is strictly concave if exiting is not allowed. The option value of exiting not only forms a lower bound in a firm's continuation value but also creates a *risky region* on the lower end of capital holdings, $[k_0^E(Az), k_0^S(Az)]$, in which the value function is convex, resulting in voluntary risk-taking. Hence, smaller firms find the risky project more appealing while larger ones stay on the safe side due to the concavity in the value function. If, on the other hand, for some (lower) realization of z, the unconstrained optimal return from investing $k^*(Az)$ is lower than the value of exiting V^0, the value function of the safe project bounded below by V^0 is convex for any initial capital k_0, and hence all entering firms are willing to take the risk. So, to summarize, with some fixed A, for each realization of z, either (i) all entering firms with z choose the risky project or (ii) the smaller entering firms with (k_0, z) such that $k_0 \in [k_0^E(Az), k_0^S(Az)]$ prefer the risky project while the larger ones choose the safe project. The fraction of risk-taking firms is determined by the distributions $G^k(k_0)$ and $G^z(z)$, the common component of productivity A, the probability of success p and the option value of exiting V^0 as well. The explicit solution to each firm's decision rule permits the intuitive comparative statics exercise in the following section, which is intended to capture some of the flavor of the business cycle analysis in the quantitative model.

2.3. Comparative statics

This section shows that the simple model has the potential to explain the countercyclical dispersion in productivity and is able to capture other cyclical features.

Proposition 2. *The thresholds $k_0^E(Az)$ and $y_1^*(Az)$ are decreasing in Az, and so is $k_0^S(Az)$ when it exists. In addition, $k_0^S(Az) - k_0^E(Az)$ is also decreasing in Az when it exists.*

The conclusion is proven in Appendix A.2. Observe that, for each firm, the aggregate and individual productivity components A and z enter its production function symmetrically and hence changes of the same magnitude have the same effect. With the aggregate productivity component A fixed, we first examine the cross-sectional differences in firms' decision rules. A potential entrant with a low productivity draw enters the market only if it is relatively large in size. Conditional on entry, low productivity firms are prone to take more risk and face a higher ex ante exiting rate. Combined, these are consistent with the fact that smaller firms exhibit "up-or-out" dynamics: they tend to have a higher exit rate, but those that stay show higher productivity.

The change in A affects the whole economy and has an aggregate effect. A decrease in A increases the entry size threshold for each z, resulting in a smaller volume of entrants with fixed $G^k(k_0)$ and $G^z(z)$. Meanwhile, the increased exiting threshold can lead to the exit of more and larger firms. The direction and the magnitude of the changes in cross-sectional productivity dispersion and exit rate following the decrease in A depend on the shape of the distributions $G^k(k_0)$ and $G^z(z)$. However, the enlarged risky region $\left[k_0^E(Az), k_0^S(Az)\right]$ shows that the model can potentially generate countercyclicality in productivity dispersion through the firms' risk-taking. In fact, when the realization of A is low enough, all entering firms are willing to take risk, regardless of their size.

Therefore, cross-sectionally, smaller firms tend to bear higher risk and show more entry and exit activity, while, when the aggregate productivity worsens, even the largest firms may show risk-taking behavior in response. Actually, if firms are allowed to choose from a continuum of projects, each of which is associated with a riskiness level indexed by the probability $p \in [\underline{p}, 1]$, there will be more risk-taking larger firms as each of them can choose a higher probability of success, i.e., a project with low risk. In this case, when the aggregate state changes, firms respond by choosing different risk levels. This is captured in the quantitative model.

3. Quantitative model

The simple three-period model illustrates the main mechanism in a tractable setting. This section builds a richer, more dynamic version of the simple model allowing for capital accumulation with friction, time-varying productivities, state-dependent exit value, and a larger range of risky projects. The model is then confronted by the data to examine the quantitative importance of the mechanism.

3.1. Setup

Time is discrete with an infinite horizon. The common discount rate is $1/R$, $R > 1$. All firms produce a single type of good every period, the demand for which has infinite elasticity. The price of the good is normalized to be 1. The good can also be used as capital for future production. All capital depreciates at a constant rate $\delta > 0$, and the capital accumulates or decumulates based on the sign of investment level i_t, so that

$$k_t = (1-\delta)k_{t-1} + i_t. \tag{10}$$

In each period t, all firms decide k_t and choose from a continuum of projects indexed $p_t \in [\underline{p}, 1]$, $\underline{p} > 0$. Projects differ from one another in riskiness. Each firm can choose only one project in a period. Given the aggregate productivity component A_t, for a firm with idiosyncratic productivity z_t and capital k_t, the production outcome takes the form

$$y_t = A_t z_t B(p_t) k_t^\alpha, \quad 0 < \alpha < 1, \tag{11}$$

where y_t is the end-of-period revenue that is carried into next period, and the binary random variable $B(p_t)$ characterizes the project p_t (which may result in a good outcome or a bad one) and determines the realized productivity and level of output, such that

$$B(p_t) = \begin{cases} 1/p_t, & w \cdot p \cdot p_t, \\ 0, & w \cdot p \cdot 1 - p_t. \end{cases} \tag{12}$$

Hence, a larger p_t indicates (i) higher probability of a good outcome and (ii) lower realized productivity conditional on that good outcome. Note that, for each $p_t \in [\underline{p}, 1]$,

$$\mathbb{E}(B(p_t)) = 1,$$
$$Var(B(p_t)) = (1-p_t)/p_t. \tag{14}$$

Hence, all projects deliver the same expected output and higher p_t means lower riskiness, measured as $\sqrt{(1-p_t)/p_t}$. Also, for $p_t < p_t'$ such that $p_t, p_t' \in [\underline{p}, 1]$, $B(p_t)$ is a mean-preserving spread of $B(p_t')$. The lower bound $\underline{p}$ sets the highest possible productivity and the highest possible riskiness to be finite.

The aggregate productivity A_t evolves as an AR(1) process, such that

$$\ln A_t = \rho_A \ln A_{t-1} + \sigma_A u_t^A \quad \text{where } u_t^A \sim N\left(-\frac{1-\rho_A}{2}\sigma_A, 1\right), \tag{15}$$

and therefore the unconditional expectation of A_t is normalized to be one. The firms also face idiosyncratic productivity shocks, and the individual productivity z_t evolves independently in cross-section and independent of the aggregate process:

$$\ln z_t = \rho_z \ln z_{t-1} + \sigma_z u_t^z \quad \text{where } u_t^z \sim N\left(-\frac{1-\rho_z}{2}\sigma_z, 1\right), \tag{16}$$

hence the unconditional expectation of z_t is also normalized to be one. Note that σ_z measures the time-series volatility of the idiosyncratic shocks, which is assumed to be constant in this model. However, it is not a valid measure of the time-varying cross-sectional productivity dispersion. In addition to σ_z, the productivity dispersion also depends on the joint distribution of firm size and productivity, which is in turn shaped by the entry and exit dynamics and the aggregate A_t.

Following conventional real business cycle models, I assume time-invariant volatility in A_t and z_t, in terms of constant σ_A and σ_z. This implies that this modeled economy is driven by the traditional "technology shocks," that is, the change in the first moment. This is different from Bloom (2009) and Bloom et al. (2014), who use time-varying higher moments as the pure source of aggregate fluctuation. Meanwhile, this is also distinct from, for example, Bachmann and Bayer (2013) and Chugh (2013), who allow time-varying higher moments in addition to the usual first-moment movement to account for the countercyclical dispersion observed in the data.

Production is costly. In each period, a staying firm needs to pay a fixed operating cost c_f, and the adjustment of capital level is subject to partial irreversibility, measured as the gap between the buying price of new capital and the selling price of old capital. When a firm makes a positive investment and grows, the price paid for every unit of new capital is normalized to be one. However, if a firm wants to reduce in scale, the selling price for each unit of old capital is $\theta < 1$, reflecting capital specificity and a lemons problem. Assume, as in the simple model, that additional financing is infinitely costly, so the firm does not borrow. In fact, in each period t, a surviving firm must pay the operating cost and any positive investment out of its revenue inherited from the previous period. The remaining non-negative profit is the flow payoff to the firm. The constraint of non-negative profit is intended to mimic the fund insufficiency faced by smaller firms who lack sufficient resources to invest the optimal amount.[7, 8].

If a firm exits, the payoff is the sale revenue of all the capital stock at price θ. The firm cannot re-open for business again in the future after exiting.

There is a constant mass $M > 0$ of potential entrant firms in every period. Each potential entrant draws its initial capital k_0 and its initial individual productivity component z_t from time-invariant distributions $G^k(k_0)$ and $G^z(z_t)$, respectively, where $G^z(z_t)$ is the stationary distribution of z_t, so

$$\ln z_t \sim N\left(-\frac{1}{2}\frac{\sigma_z^2}{1-\rho_z^2}, \frac{\sigma_z^2}{1-\rho_z^2}\right), \tag{17}$$

and $G^k(k_0)$ is assumed to be a Pareto distribution with exponent ξ. $G^k(k_0)$ and $G^z(z_t)$ jointly determine the size and productivity distribution of the entering firms in each period. Each entering firm pays the initial setup cost $c_e > 0$. The setup cost c_e also serves as the lower bound of the Pareto distribution $G^k(k_0)$. Once entered, an entrant acts as an incumbent thereafter as long as it stays. Under the non-zero profit constraint with k_0 given, a potential entrant's problem is similar to an incumbent's: $k_0 - c_e$ bounds from above an entering firm's investment amount in its first period in the same way that the previous-period revenue may constrain a surviving firm. Additionally, the shape of the Pareto distribution also has the desired property that the majority of entering firms are relatively small.[9]

Fig. 6 illustrates the timing of the quantitative model. Each time period has several stages, resembling the simple model:

- *Stage* 1: *Observation of state variables.* Aggregate productivity A_t and individual productivity z_t are realized. An incumbent firm observes (A_t, z_t) and enters this period with remaining capital after depreciation $(1-\delta)k_{t-1}$, together with last period's realized output $y_{t-1} = Z_{t-1}k_{t-1}^\alpha$ where $Z_{t-1} = A_{t-1}z_{t-1}B(p_{t-1})$ is the firm's realized productivity. A potential entrant draws (k_0, z_t) and observes A_t.
- *Stage* 2: *Entry and exit.* Each potential entrant decides whether to enter given (k_0, A_t, z_t). Similarly, each incumbent decides whether to exit given $(Z_{t-1}, k_{t-1}, A_t, z_t)$.
- *Stage* 3: *Investment and project decision.* Staying firms and new entrants decide on investment i_t and project choice associated with a distinct riskiness indexed by p_t. At the same time, the operating cost and capital adjustment cost is paid.
- *Stage* 4: *Production.* The level of output depends on capital k_t and the realized productivity $Z_t = A_t z_t B(p_t)$.

3.2. Individual decision

Incumbent's problem: At the beginning of each period t, an incumbent firm is characterized by $(Z_{t-1}, k_{t-1}, A_t, z_t)$.[10] The realized productivity $Z_{t-1} \in \{A_{t-1}z_{t-1}/p_{t-1}, 0\}$, where $p_{t-1} \in [\underline{p}, 1]$ represents the riskiness of the project the firm chose in the previous period. Let $V(Z_{t-1}, k_{t-1}, A_t, z_t)$ denote the value function of this firm at time t. It first decides whether to exit the

---

[7] This is essentially a zero-borrowing constraint and it binds firms with low levels of working capital and/or firms with bad productivity realizations. It can be conjectured that it is still these firms that face a binding (non-zero) borrowing constraint if allowed. However, allowing for a standard borrowing constraint will greatly complicate the quantitative analysis without adding much insight.

[8] The timing may seem unconventional. However, if we interpret the realization of Z_{t-1}, where $Z_{t-1} = A_{t-1}z_{t-1}B(p_{t-1})$, as current productivity, k_{t-1} as current capital stock, then the timing becomes a standard one for real business cycle models in which firms own their capital and face a non-negative-profit constraint. The difference, though, from the standard model is that firms perfectly foresee $A_t z_t$, the expectation of next-period productivity, and have to choose the standard deviation by selecting p_t while deciding on the level of investment

[9] Alternatively, one can assume that $G^z(z_t)$ is a Pareto distribution concentrating on the low-productivity end and the initial capital is chosen optimally based on the initial productivity, similar to the case studied by Clementi and Palazzo (2013). This alternative assumption also leads to a similar distribution of entrants.

[10] To avoid computational complexity, I do not consider the price feedback effect in this model. Therefore, the distribution of firms is not a state variable in this model because agents do not need to forecast future prices using information on distribution.
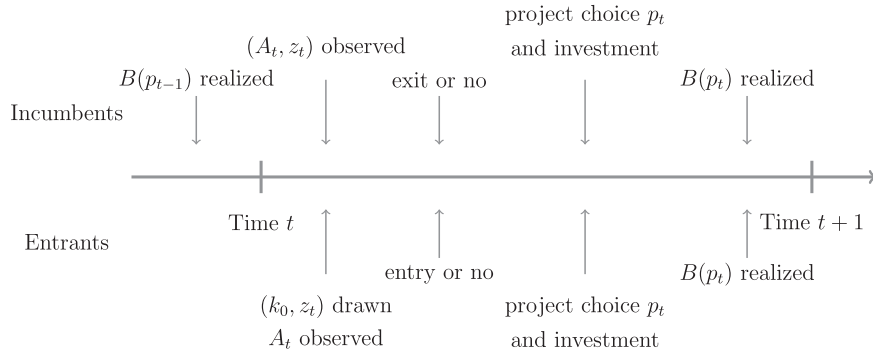
Fig. 6. Timing of the quantitative model.

market:

$$V(Z_{t-1}, k_{t-1}, A_t, z_t) = \max\left\{V^{Stay}(Z_{t-1}, k_{t-1}, A_t, z_t), V^X(Z_{t-1}, k_{t-1})\right\}, \tag{18}$$

where $V^{Stay}(Z_{t-1}, k_{t-1}, A_t, z_t)$ is the firm's continuation value if staying and $V^X(Z_{t-1}, k_{t-1})$ is the value of the exiting option. Specifically, the exiting value depends only on remaining capital and previous output:

$$V^X(Z_{t-1}, k_{t-1}) = \theta(Z_{t-1}k_{t-1}^\alpha + (1-\delta)k_{t-1}). \tag{19}$$

An exiting firm gets to take with it the revenue from selling all of its capital. In addition, it can also take the same θ fraction of its last-period output.

If this firm chooses to stay, it must then decide on investment i_t and project choice p_t. The objective of a staying firm is to maximize the discounted expected value of its future flow payoff, hence the continuation value has the following form:

$$V^{Stay}(Z_{t-1}, k_{t-1}, A_t, z_t)$$
$$= \max_{p_t, i_t}\left\{D(i_t; Z_{t-1}, k_{t-1}) + \frac{1}{R}\mathbb{E}^{p_t}\mathbb{E}^{A_{t+1}, z_{t+1}}\left[V(Z_t(p_t), k_t, A_{t+1}, z_{t+1})|A_t, z_t\right]\right\}, \tag{20}$$

subject to the rule of capital evolvement (10) and the constraint of no external financing:

$$D(i_t; Z_{t-1}, k_{t-1}) \geq 0, \tag{21}$$

as well as

$$p_t \in [\underline{p}, 1], \quad k_t \geq 0. \tag{22}$$

The lowest probability of choice $\underline{p}$ is bounded away from zero which ensures that, for each (A_t, z_t), the realized productivity of any project is always bounded.

The time t operating profit $D(i_t; Z_{t-1}, k_{t-1})$ depends on the amount of investment or disinvestment due to the friction in capital adjustment, specifically,

$$D(i_t; Z_{t-1}, k_{t-1}) = \begin{cases} Z_{t-1}k_{t-1}^\alpha - \theta i_t - c_f & \text{if } i_t < 0, \\ Z_{t-1}k_{t-1}^\alpha - i_t - c_f & \text{if } i_t \geq 0. \end{cases} \tag{23}$$

The firm has intertemporal concerns when choosing investment and project riskiness, simply because they directly affect both the level and the distribution of continuation values realized in the future:

$$\mathbb{E}^{p_t}\mathbb{E}^{A_{t+1}, z_{t+1}}\left[V(Z_t(p_t), k_t, A_{t+1}, z_{t+1})|A_t, z_t\right]$$
$$= \mathbb{E}^{A_{t+1}, z_{t+1}}\left[p_t V\left(\frac{A_t z_t}{p_t}, k_t, A_{t+1}, z_{t+1}\right) + (1-p_t)V(0, k_t, A_{t+1}, z_{t+1})\Big|A_t, z_t\right]. \tag{24}$$

Similar to the case in the simple model, the option value of exiting creates a convex portion in a firm's value function that motivates voluntary risk-taking. Firms in this region strictly prefer some $p_t < 1$ to $p_t = 1$. Compared to firms that choose a safe project, risk-taking firms are smaller on average. Due to the riskiness of their chosen projects, these smaller firms face (1) a higher probability of significant growth in size in the following period resulting from a good outcome and (2) a higher exiting hazard due to a bad outcome.

Potential entrant's problem: In the beginning of each period t, a potential entrant draws initial capital holding k_0 and z_t from their invariant distributions, observing the current-period A_t. The value of staying outside the market is independent of the productivity:

$$V_0^{Out}(k_0) = \theta k_0. \tag{25}$$

An entering firm must pay a startup cost c_e from its initial capital, and thereafter the firm acts as an incumbent identified by

Table 1
Parameter values.

Notation	Description	Value
α	Capital scale	0.592
R	Interest rate	1.04
δ	Depreciation rate	0.1
θ	Capital resale price	0.795
c_f	Fixed operating cost	4.0
c_e	Entry cost	10.8
ξ	Pareto exponent	3.0
M	Mass of potential entrants	100
$\underline{p}$	Riskiness bound	0.5
ρ_z	Persistence of idiosyncratic shock	0.885
σ_z	Volatility of idiosyncratic shock	0.177
ρ_A	Persistence of aggregate shock	0.95
σ_A	Volatility of aggregate shock	0.0156

state $(0, (k_0 - c_e)/(1-\delta), A_t, z_t)$. Hence, the payoff from opening a firm will be

$$V_0^{Enter}(k_0, A_t, z_t) = V^{Stay}\left(0, (k_0 - c_e)/(1-\delta), A_t, z_t\right). \tag{26}$$

A new firm with initial draw (k_0, z_t) enters if

$$V_0^{Enter}(k_0, A_t, z_t) > V_0^{Out}(k_0). \tag{27}$$

3.3. Calibration

The calibration strategy is to use the stationary case of the model with constant A_t set to be $\mathbb{E}A_t = 1$ as the benchmark, and then to add the aggregate fluctuations later on. The parameter values are listed in Table 1.

The duration of one period is chosen to be one year. Consistent with the majority of macroeconomic studies, I assume that $R = 1.04$ and $\delta = 0.1$. The production function, $F(Z, k) = Zk^{\alpha}$, is the same as the profit function used by Cooper and Haltiwanger (2006), so I follow their estimation and set α to be 0.592 and the capital resale price $\theta = 0.795$. Borrowing from the same work, I set the persistence in the idiosyncratic component of the productivity as $\rho_z = 0.885$.[11] Different values of the mass M of potential entrants in each period yield stationary distributions and flows of firms identical to each other except for the total volume. The scale of the model is not a relevant aspect and I set $M = 100$. The lowest feasible probability of a good outcome for each risky project is set as $\underline{p} = 0.5$.[12] This ensures that, for each pair of realized shocks (A_t, z_t), the highest realized productivity is $2A_t z_t$ and the associated highest riskiness measured as the standard deviation of $B(\underline{p})$ is 1.

The remaining parameters are chosen such that a number of statistics computed from the simulated data are close to their empirical counterparts. Table 2 lists the simulated statistics from the stationary model together with their empirical counterparts. Table 3 contains the targets for the model with aggregate fluctuations.

Due to endogenous entry/exit and choice of risk level, the model is non-linear and hence the parameters cannot be matched directly to the target moments. Nevertheless, the mechanism indicates that each parameter contributes differently to the shaping of the model statistics.

The targets that describe the entry and exit dynamics are the average entry and exit rates and the average relative sizes of entering and exiting firms obtained by Lee and Mukoyama (2015). In the absence of aggregate fluctuation and starting from an empty market, the simulated economy will eventually become one with the stationary volume of firms and the outflow of exiting firms will equal the inflow of entering ones. The simulated entry and exit rates are the same and are jointly determined by the shape of $G^k(k_0)$ with Pareto exponent ξ, the operating cost c_f, and the idiosyncratic volatility σ_z. The latter two parameters also contribute to the average relative size of exiting firms. The relative size of the entrants then pins down the entry cost c_e.

Once the partial irreversibility of capital (measured by the resale price θ) is fixed, the mean and the dispersion of the investment rate are mainly determined by the volatility σ_z of the idiosyncratic shock process. The targets are estimated by Cooper and Haltiwanger (2006). The simulated standard deviation in investment rate is a bit different from the target because σ_z is not a free parameter.

For the case with aggregate fluctuation, two additional parameters ρ_A and σ_A need to be calibrated. The target for ρ_A is the autocorrelation of the residual of the HP-filtered real U.S. GDP from 1977 to 2013. The volatility is chosen to match that of the real GDP growth in the same period.

---

[11] Other studies provide different estimates of ρ_z. For example, Lee and Mukoyama (2015) estimate it to be 0.97, Foster et al. (2008)'s estimate is 0.8, and Castro et al. (2015) give the sector-specific estimates for manufacturing industries at the three-digit SIC level ranging from 0.3 to 0.7.
[12] There is no clear guideline for the choice of $\underline{p}$. See the appendix for a robustness check with alternative values of $\underline{p}$.

Table 2
Calibration targets: stationary.

Statistic	Model	Data
Average investment rate	0.135	0.122
Standard deviation of investment rate	0.232	0.337
Entry rate	0.061	0.062
Exit rate	0.061	0.055
Relative size, entering	0.586	0.60
Relative size, exiting	0.463	0.49

Table 3
Calibration targets: aggregate fluctuation.

Statistic	Model	Data
Autocorrelation of HP-filtered real GDP	0.639	0.631
Standard deviation of real GDP growth	0.0194	0.0196

4. Quantitative results

This section explores the simulated data and shows the quantitative performance of the model with aggregate fluctuation. In the examination of cyclical behavior, the indicators for the business cycle are chosen to be the aggregate productivity level and the residual of the HP-filtered total output.

Section 4.1 shows that the calibrated benchmark model is capable of generating countercyclical productivity dispersion, with a correlation comparable to the data. The cyclicality of variables related to entry and exit is also in line with the facts. The model's implication on the cyclicality and firm dynamics by size group is then discussed in 4.2. Then, how important is the choice of risk level in this model? Section 4.3 answers this question by comparing the quantitative result of the benchmark model to the case without such choices. The alternative model is identical to the benchmark model, except that in the alternative model, firms do not have the option to choose the level of risk to which they expose their production activity. As it turns out, the choice of risk level is critical in shaping the endogenous countercyclicality of the productivity dispersion; it also increases the dispersion of investment rate. On a related note, Section 4.4 discusses the cyclicality of investment dispersion in light of recent research.

4.1. Countercyclical productivity dispersion

Now I look at how aggregate productivity shocks affect the cross-sectional moments. In particular, I compute the correlation coefficients between the cyclical indicators and the contemporaneous productivity dispersion, together with entry and exit rates. The measure for productivity dispersion is the cross-sectional standard deviation of the realized *idiosyncratic* productivity level at the end of each simulation period t, that is, $SD(z_{jt}B(p_{jt}))$. Although I do not include productivity growth in the model, I choose to include the HP-filtered logarithm of the dispersion as an additional measure, interpreted as the percentage deviation of the productivity dispersion from its trend. The smoothing parameter for the HP-filter is 100. In terms of the cyclical indicators, since there is no long-term growing trend in the model, I select the total output (or simply the GDP) in each simulation period together with the driving force of the model, the exogenous aggregate component of productivity, A_t. In addition, for compatibility with the data, I also include the cyclical component gdp_t^{HP} of the HP-filtered logarithm of GDP with smoothing parameter set at 100 and the growth rate of GDP, denoted Δgdp_t, as well.

Table 4 shows a negative relationship between idiosyncratic productivity dispersion and various cyclical indicators. Although previous empirical findings have established the countercyclicality of the dispersion in both the *levels* and the *growth rates* of productivity, I will focus on the dispersion in levels when comparing the simulated statistics and data.[13] Kehrig (2011) measures the productivity dispersion as the median of within-industry standard deviation in the level of plant productivity and finds that the correlation between non-durable good output and its dispersion is -0.293, and -0.502 for

---

[13] The modeling assumption that production failure leads to zero output prohibits a well-defined productivity growth rate for every firm in every period. However, conditional on project *success*, I can calculate the cross-sectional standard deviation in productivity growth rate for all surviving incumbents. In fact, this measure of growth rate dispersion is still negatively correlated with all cyclical indicators (at 1% significance level) except Δgdp_t (not statistically significant, *p*-value=0.76).

Table 4
Generated cyclicality.

Correlations with cyclical indicators				
Variables of interest	Cyclical indicators			
	A_t	GDP_t	gdp_t^{HP}	Δgdp_t
Dispersion productivity, $SD(z_{jt}B(p_{jt}))$	−0.712	−0.923	−0.388	−0.167
Dispersion productivity, HP-filtered logarithm	−0.500	−0.311	−0.665	−0.354
Entry rate	0.660	0.067	0.551	0.583
Exit rate	−0.885	−0.729	−0.682	−0.657

durables for U.S. manufacturing. The correlations are −0.336 and −0.537, respectively, when the business cycle indicator is HP-filtered GDP, and −0.250 and −0.413 when using GDP growth rate. The model-simulated correlation coefficients are in line with the data when gdp_t^{HP} and Δgdp_t serve as the corresponding cyclical indicators. In fact, if we look at the *level* measures of both the dispersion and the cyclical indicators, the negative correlation becomes even more pronounced. As expected, the change in the realized productivity dispersion is completely driven by the change in firms' risk-seeking behavior over business cycles. In bad times, a larger fraction of firms choose even riskier projects, which results in larger realized productivity dispersion.

Also consistent with the facts, the entry rate is procyclical and the exit rate is countercyclical. The volume of entrants increases when the aggregate productivity increases; so does the volume of continuing firms (due to fewer exits), and the entry volume increases disproportionately more than the continuing volume does, leading to the net result of procyclical entry rates. The explanation for the countercyclical exit rate is twofold. First, a low aggregate productivity directly pushes out more firms as they expect no additional future payoff from continuing. Second, a low aggregate productivity induces more firms to take greater risk and a fraction of the risk-taking firms face a bad outcome, resulting in more exits in the following period.

4.2. Firm sizes and business cycles

In the model, the size of a firm, measured as its total amount of capital, largely shapes its optimal choices. Therefore, I turn to the predictions of the model regarding the differences across size groups.

In the model, it is expected that smaller firms always take higher risk and exhibit more movement on the entry and exit margins over time. However, it is the *change* in the realized within-group productivity dispersion in response to the aggregate productivity shock that determines the cyclicality in each group, measured as the coefficient of correlation between within-group productivity dispersion and the aggregate cyclical indicators. The parameters determine which group shows a higher level of such correlation.

Table 5 reports the simulated relationship between statistics of entry, exit and firm size. In the model, the size of a firm is measured as its amount of capital. The first quartile contains the smallest firms in each simulation period while the fourth contains the largest. For each size group, the time-series average and the standard deviation in entry and exit rates are calculated and listed. Consistent with the evidence shown in Fig. 2, smaller firms show not only higher average rates of entry and exit, but also higher time-series volatility in the rates, measured as time-series standard deviation. While the difference in the movement on the entry side comes mainly from the Pareto size distribution of potential entrants, that on the exit side stems from the features of the model. The direct contributor to the exiting volume is the cutoff rule for exiting and the indirect one is the probability of project failure due to choice of risk level. The smallest firms show high averages and volatility in their exit rates due to the combination of the two effects, while the largest firms are only subject to the indirect one – and it has a lower probability of affecting them.

Table 6 lists the time-series change in the within-group productivity dispersion and compares the correlation coefficients between within-group productivity dispersion and cyclical indicators across size groups. The measure for dispersion is still the standard deviation of the realized individual productivity. The division into groups is done the same as in Table 5. The measure for change in dispersion is the time-series coefficient of variation for the cross-sectional standard deviation. The theoretical model predicts that smaller firms will consistently show larger productivity dispersion. However, in the model, it is with positive probability that both the aggregate and idiosyncratic shocks can be so bad that even some of the largest firms will prefer riskier production projects. Similarly, the smaller ones sometimes choose not to take any risk. Hence, admittedly, the resulting within-group countercyclicality of productivity dispersion depends on the parameter values. As it turns out, the calibrated model predicts comparable volatility in the dispersion and consistent countercyclicality across size groups, with the correlation coefficients on similar scales.

Table 5
Entry and exit by size quartile.

Size quartile	Entry rate (%)		Exit rate (%)	
	Avg.	Vol.	Avg.	Vol.
First quartile	19.52	0.72	14.71	0.96
Second quartile	3.65	0.58	5.37	0.44
Third quartile	0.15	0.06	3.11	0.23
Fourth quartile	0.00	0.00	0.39	0.11

Table 6
Cyclicality of productivity dispersion by size group.

Size quartile	Coef. Var. (%)	Correlation with			
		A_t	GDP_t	gdp_t^{HP}	Δgdp_t
First quartile	0.92	− 0.613	− 0.923	− 0.326	− 0.076
Second quartile	1.91	− 0.426	− 0.573	− 0.219	− 0.044
Third quartile	3.33	− 0.400	− 0.289	− 0.301	− 0.193
Fourth quartile	1.13	− 0.424	− 0.254	− 0.291	− 0.133

Table 7
Stationary targets without choice of risk.

Statistic	No choice of risk	Baseline model
Productivity dispersion	0.406	0.563
Average investment rate	0.124	0.135
Standard deviation of investment rate	0.163	0.232
Entry rate	0.050	0.061
Exit rate	0.050	0.061
Relative size, entering	0.688	0.586
Relative size, exiting	0.541	0.463

4.3. The role of choice of risk level

At the core of the mechanism is the ability of continuing firms to choose the risk level associated with each production project. Together with the option value of exiting, the choice of risk level creates the countercyclical productivity dispersion.

To see the role of risk level choice quantitatively, I simulate a version of the model without this feature. First, it is worthwhile to see how choice of risk level affects the key cross-sectional moments without aggregate fluctuation. Table 7 compares the statistics that are used for calibration purposes together with realized productivity dispersion, simulated from the two models using the same parameter values. Compared to the baseline model, the realized productivity dispersion measured as the cross-sectional standard deviation drops by more than a quarter. Clearly, the absence of risky projects results in fewer extreme realizations in productivity. Consequently, the dispersion in investment rate drops from 23.2% to 16.3%. This drop comes largely from the decrease in the measure of firms with higher investment rates, due to infinitely costly financing. A good productivity realization makes the constraint less binding and a higher investment rate becomes feasible. Without the choice of risk level, firms only get high productivity realization from positive idiosyncratic shocks, compared to the baseline case where high productivity can also be the outcome of risky projects. Therefore, the investment rate distribution is less dispersed without the choice of risk level. The exit rate becomes lower as the firms cannot voluntarily take risk and randomize between exiting and higher productivity.

It is important, then, to see the role of risk level choice in shaping cyclicality. Table 8 lists the results when the aggregate productivity evolves as the calibrated process. The productivity dispersion, measured as the cross-sectional standard deviation of the idiosyncratic productivity $SD(z_{jt})$, is strongly procyclical without the risk level choice, $B(p_{jt})$. Such a result highlights the "cleansing effect" of recessions. Given the cyclical behavior of firm entry and exit rates, in bad times, firms with low productivity exit and entrants have relatively high productivity, so the productivity distribution is more concentrated at the higher end and hence less dispersed. The opposite happens in good times. Admittedly, the resulting procyclicality in the absence of the choice of risk may depend on the parameter values. In fact, since the entrants in recessions must have relatively high idiosyncratic productivity, the standard deviation may increase under certain parameter values. Even when this is actually the case, the choice of risk can increase the productivity dispersion even further, thus enhancing the countercyclicality.

Table 8
Cyclicality without risk level choice.

Correlations with cyclical indicators				
Variables of interest	No choice of risk		Baseline model	
	A_t	gdp_t^{HP}	A_t	gdp_t^{HP}
Dispersion productivity	0.926	0.816	−0.712	−0.388
Entry rate	0.806	0.629	0.660	0.551
Exit rate	−0.828	−0.612	−0.816	−0.682

The cyclicality of entry and exit dynamics do not show significant change without the risk level choice. The selection of productivity at entry still exists and is highly cyclical. As many have shown, entry and exit are important in shaping aggregate fluctuations.[14] This section confirms that the entry and exit dynamics are robust in a model with or without the choice of risk level. However, this exercise also shows that, under current specification of parameter values, entry and exit alone have difficulty in matching the countercyclical productivity dispersion as they seem to drive the cross-sectional standard deviation of idiosyncratic productivity to move procyclically. The choice of risk level adds an opposite force and corrects the cyclicality while preserving the desired feature of the entry and exit dynamics.

4.4. Investment dispersion

Another interesting implication of the model is the cyclicality of investment rate distribution. This model has two corresponding features: partial irreversibility of capital and unavailability of external financing. Table 9 reports the quantitative performance of the model along this dimension. First, both the aggregate investment rate and the cross-sectional average investment rate are strongly procyclical. Next, the dispersion in investment rate measured as cross-sectional standard deviation is also procyclical. Then, the fraction of firms undergoing investment spikes is procyclical while the fraction of firms undergoing investment inaction is countercyclical.

The literature has documented several related empirical facts. Doms and Dunne (1998) use establishment-level data on U.S. manufacturing investment and show that both the frequency of investment spikes and the Herfinahl index of investment co-move with aggregate investment. Exploring plan-level data from Chile and the U.S., Gourio and Kashyap (2007) find that not only do investment spikes move procyclically but the extensive margin of spikes also accounts for the majority of variation in aggregate investment over time. In particular, they document the correlation between investment spikes and the aggregate investment rate at 0.87 for the U.S. and the correlation between the near-zero investments and the aggregate investment rate at −0.94 for the U.S. Kehrig and Vincent (2013) document procyclical investment dispersion at the plant level in U.S. manufacturing. They find the contemporaneous correlation coefficient between dispersion and aggregate investment rate to be 0.73. Bachmann and Bayer (2014) document that the dispersion in firm-level investment rate moves significantly and robustly procyclically in Germany, the U.S., and the U.K. In particular, using COMPUSTAT, they find the correlation in the U.S. to be 0.6. Their model, which uses non-convex capital adjustment cost and uncertainty shocks, is able to capture this fact quantitatively.

In this model, it is also the capital adjustment cost that drives the cyclicality of investment rate dispersion. In terms of such cost, the model allows for a gap between the buying price of new capital and the selling price of old capital, together with infinitely costly external financing. The price gap distorts the marginal incentive to disinvest while the infinite cost of financing constrains the maximum amount of investment each firm can make in each period. A good realization of productivity loosens the maximum investment constraint and incentivizes higher investment. Therefore higher realization of A_t increases the fraction of investment spikes as well as the aggregate and average investment rate and it reduces the fraction of inaction. When the realized productivity is low, firms trade off between the incentive to disinvest and the extra cost of doing so due to the gap in capital price; meanwhile, more firms face tighter maximum investment constraints. Hence, more firms are concentrated in the region with very low level of adjustment in capital, which increases the fraction of inaction and, more importantly, reduces the investment rate dispersion. Admittedly, the countercyclical productivity dispersion is a force that drives the investment rate dispersion to move countercyclically, however, under the calibrated parameter values, this force is not strong enough to offset the effect of investment cost. Combined, the net result shows strongly procyclical investment dispersion, consistent with the empirical findings.

5. Conclusion

Productivity dispersion tends to be larger during recessions. The prevailing view is that increased uncertainty causes a decline in aggregate economic activity. However, although uncertainty matters, this story seems to contradict the

---

[14] Examples are Campbell (1998), Clementi and Palazzo (2013), Jaimovich and Floetotto (2008), and Lee and Mukoyama (2015).

Table 9
Cyclicality of investment rate distribution.

Correlation	Cyclical indicators		
	I_t/K_t	A_t	gdp_t^{HP}
Aggregate investment rate (I_t/K_t)	1	0.854	0.700
Average investment rate	0.682	0.801	0.569
Standard deviation investment rate	0.675	0.796	0.520
Fraction of investment spikes ($\geq 20\%$)	0.681	0.446	0.496
Fraction of inaction ($|i/k| \leq 1\%$)	−0.696	−0.840	−0.555

Table 10
Cyclicality with alternative $\underline{p}$.

Variables of interest	$\underline{p} = 0.1$		$\underline{p} = 0.3$		$\underline{p} = 0.7$	
	A_t	gdp_t^{HP}	A_t	gdp_t^{HP}	A_t	gdp_t^{HP}
Disp. prod.	−0.302	−0.230	−0.412	−0.266	−0.241	−0.124
Entry rate	0.487	0.430	0.649	0.587	0.757	0.665
Exit rate	−0.852	−0.591	−0.850	−0.656	−0.853	−0.656

Table 11
Cyclicality of investment rate dispersion.

Correlation	$\underline{p} = 0.1$		$\underline{p} = 0.3$		$\underline{p} = 0.7$	
	A_t	gdp_t^{HP}	A_t	gdp_t^{HP}	A_t	gdp_t^{HP}
Agg. inv. rate (I_t/K_t)	0.795	0.622	0.837	0.705	0.855	0.720
Avg. inv. rate	0.743	0.605	0.786	0.681	0.757	0.702
Std. dev. inv. rate	0.471	0.337	0.731	0.649	0.828	0.764
Frac. of spikes ($\geq 20\%$)	0.627	0.545	0.700	0.609	0.555	0.540
Frac. of inaction ($\leq 1\%$)	−0.835	−0.613	−0.583	−0.506	−0.815	0.691

observation that recessions lead to an increase in productivity dispersion. To complement existing theories, I explore a simple mechanism through which aggregate fluctuations due to standard "technology shocks" can endogenously generate countercyclical dispersion in the realized individual component of productivity. I alter the standard industry dynamics model with business cycle features by incorporating project choice as part of the individual decision problem. Because of this feature, a firm in this model can choose the risk level of its production. The model provides the following predictions: small firms are more likely to take risks and will have lower survival rates but, conditional on surviving, they exhibit higher productivity; a larger proportion of firms take risks in bad times, which also leads to higher exit rates; and realized micro-level productivity dispersion is larger in recessions. The next step would be to generalize the mechanism into a general-equilibrium framework. In particular, it would be interesting to see how the endogenous prices can affect the predictions both qualitatively and quantitatively. The model may also shed light on labor market variables if further enriched to include firms' employment decisions. I will leave this to future research.

Acknowledgments

Appendix A. Proofs

A.1. Proof of Proposition

Assumption 1. $R < \frac{1-p}{1-p^{1-\alpha}}$. With fixed A, for each $z \in [\underline{z}, \overline{z}]$, recall the unconstrained optimal capital choice for a staying firm in period 1:

$$k^*(Az) = \left(\frac{\alpha Az}{R}\right)^{1/(1-\alpha)}.$$

Then

$$V_1^{Stay}(y_1, Az) = \begin{cases} \frac{1}{R} Az y_1^\alpha & \text{if } y_1 < k^*(Az), \\ y_1 - k^*(Az) + \frac{1}{R} Az[k^*(Az)]^\alpha & \text{otherwise}. \end{cases}$$

Obviously, V_1^{Stay} is strictly increasing in y_1. It is convenient to define the following:

$$\pi^*(Az) = \frac{Az}{R}[k^*(Az)]^\alpha = \left(\frac{\alpha^\alpha Az}{R}\right)^{1/(1-\alpha)}.$$

Hence, $k^*(Az) = \alpha \pi^*(Az)$ and $-k^*(Az) + \frac{1}{R} Az[k^*(Az)]^\alpha = (1-\alpha)\pi^*(Az)$. Also, define the gain from choosing the risky project, ΔV, as

$$\Delta V(k_0, Az) = V^{Risky}(k_0, Az) - V^{Safe}(k_0, Az)$$
$$= pV_1\left(\frac{Az}{p}k_0^\alpha, Az\right) + (1-p)V^0 - V_1\left(Azk_0^\alpha, Az\right).$$

Abusing the notations a bit and letting $y_1 = Azk_0^\alpha$ lead to

$$\Delta V(y_1, Az) = pV_1\left(\frac{y_1}{p}, Az\right) + (1-p)V^0 - V_1(y_1, Az).$$

Case 1: Consider the case where

$$V^0 \geq \pi^*(Az).$$

The following proves that all firms for which this inequality holds prefer the risky project.

The exit threshold $y_1^*(Az)$ in period 1 must satisfy

$$V^0 = y_1^*(Az) - k^*(Az) + \frac{Az}{R}[k^*(Az)]^\alpha = y_1^*(Az) + (1-\alpha)\pi^*(Az),$$

namely

$$y_1^*(Az) = V^0 - (1-\alpha)\pi^*(Az) \geq \alpha\pi^*(Az) = k^*(Az).$$

Then, for large k_0 such that $y_1^*(Az) \leq y_1 = Azk_0^\alpha < y_1/p = \frac{Az}{p}k_0^\alpha$, we have

$$\Delta V(y_1, Az) = p\left(\frac{y_1}{p} + (1-\alpha)\pi^*(Az)\right) + (1-p)V^0 - (y_1 + (1-\alpha)\pi^*(Az))$$
$$= (1-p)\left[V^0 - (1-\alpha)\pi^*(Az)\right] = (1-p)y_1^*(Az) > 0.$$

Hence, the riskier project p is preferred by the largest firms associated with productivity realization Az. For k_0 such that $py_1^*(Az) \leq y_1 = Azk_0^\alpha < y_1^*(Az)$, we have

$$\Delta V(y_1, Az) = p\left(\frac{y_1}{p} + (1-\alpha)\pi^*(Az)\right) + (1-p)V^0 - V^0$$
$$= y_1 - p[V^0 - (1-\alpha)\pi^*(Az)],$$

which is increasing in y_1, hence in k_0 on the interval, and reaches the minimum at $y_1 = py_1^*(Az)$, $\Delta V(y_1, Az) \geq \Delta V(py_1^*(Az), Az) = 0$. Therefore, the riskier project p is still strictly preferred. And lastly, firms with low levels of k_0 such that $y_1 < py_1^*(Az)$ will choose not to enter in period 0 and get V^0 immediately. Therefore, the safe-project threshold $k_0^S(Az)$ does not exist in this case. And hence,

$$RV_0^{Enter}(y_1, Az) = V^{Risky} = \begin{cases} V^0 & \text{if } y_1 < py_1^*(Az), \\ y_1 + p(1-\alpha)\pi^*(Az) + (1-p)V^0 & \text{otherwise}, \end{cases}$$

which increases in y_1. Setting $V^0 = V_0^{Enter}$ yields

$$y_1^E(Az) = (R-1)V^0 + py_1^*(Az) > py_1^*(Az),$$

or equivalently,

$$k_0^E(Az) = \left[\frac{(R-1)V^0 + p[V^0 - (1-\alpha)\pi^*(Az)]}{Az}\right]^{1/\alpha}.$$

Case 2: Consider the case where

$$V^0 \leq p^\alpha \pi^*(Az).$$

Now

$$y_1^*(Az) = \left(\frac{RV^0}{Az}\right)^{1/\alpha}$$

and

$$py_1^*(Az) < y_1^*(Az) \leq pk^*(Az) < k^*(Az).$$

The function $\Delta V(y_1, Az)$ is piecewise defined and continuous.
When $y_1 < py_1^*(Az)$, $\Delta V = V^0 - V^0 = 0$. When $y_1 \in [py_1^*(Az), y_1^*(Az)]$,

$$\Delta V(y_1, Az) = p^{1-\alpha}\left(\frac{Az}{R}y_1^\alpha\right) - pV^0 \geq \Delta V(py_1^*(Az), Az) = p\frac{Az}{R}(y_1^*(Az))^\alpha - pV^0 = 0,$$

which increases in y_1. Hence, the risky project is preferred in this interval. When $y_1 \in [y_1^*(Az), pk^*(Az)]$,

$$\Delta V(y_1, Az) = p^{1-\alpha}\frac{Az}{R}y_1^\alpha + (1-p)V^0 - \frac{Az}{R}y_1^\alpha = (1-p)V^0 - \left(1 - p^{1-\alpha}\right)\frac{Az}{R}y_1^\alpha$$

$$\geq \Delta V(pk^*(Az), Az) = (1-p)\left(V^0 - \frac{p^\alpha - p}{1-p}\pi^*(Az)\right),$$

which decreases in y_1. When $y_1 \in [pk^*(Az), k^*(Az)]$,

$$\Delta V(y_1, Az) = y_1 - \frac{Az}{R}y_1^\alpha + p(1-\alpha)\pi^*(Az) + (1-p)V^0$$

$$\geq \Delta V(k^*(Az), Az) = (1-p)[V^0 - (1-\alpha)\pi^*(Az)],$$

which still decreases in y_1. Lastly, when $y_1 \geq k^*(Az)$,

$$\Delta V(y_1, Az) = (1-p)[V^0 - (1-\alpha)\pi^*(Az)].$$

- $V^0 \leq \frac{p^\alpha - p}{1-p}\pi^*(Az)$: $\exists y_1^S = Az[k_1^S(Az)]^\alpha \in (y_1^*(Az), pk^*(Az)]$ such that $\Delta V(y_1^S, Az) = 0$, and

$$k_0^S(Az) = \left(\frac{(1-p)RV^0}{(1 - p^{1-\alpha})(Az)^{1+\alpha}}\right)^{1/\alpha^2}.$$

And the cutoff rule holds. Also, $V_0^{Enter}(y_1, Az)$ is continuous and increasing in y_1, and

$$RV_0^{Enter}(y_1, Az) = \begin{cases} V^0 & \text{if } y_1 < py_1^*(Az), \\ p^{1-\alpha}\frac{Az}{R}y_1^\alpha + (1-p)V^0 & \text{if } y_1 \in [py_1^*(Az), y_1^S) \\ \frac{Az}{R}y_1^\alpha & \text{if } y_1 \in [y_1^S, k^*(Az)) \\ y_1 + (1-\alpha)\pi^*(Az) & \text{if } y_1 \geq k^*(Az). \end{cases}$$

Hence, $\exists y_1^E = Az[k_1^E(Az)]^\alpha$ such that $V^0 = V_0^{Enter}$. To ensure $k_1^E < k_1^S$, we need $Az(y_1^S)^\alpha/R > RV^0$, which leads to $R < \frac{1-p}{1 - p^{1-\alpha}}$, and

$$k_0^E(Az) = \left(\frac{(R+p-1)RV^0}{p^{1-\alpha}(Az)^{1+\alpha}}\right)^{1/\alpha^2}.$$

- $V^0 \in \left(\frac{p^\alpha - p}{1-p}\pi^*(Az), (1-\alpha)\pi^*(Az)\right] \cap \left(\frac{p^\alpha - p}{1-p}\pi^*(Az), p^\alpha\pi^*(Az)\right]$: $\exists y_1^S = Az[k_1^S(Az)]^\alpha \in (pk^*(Az), k^*(Az)]$ such that $\Delta V(y_1^S, Az) = 0$, i.e.,

$$\frac{Az}{R}(y_1^S)^\alpha - y_1^S = p(1-\alpha)\pi^*(Az) + (1-p)V^0,$$

which does not have an explicit solution. The cutoff rule still holds. $V_0^{Enter}(y_1, Az)$ can still be written piecewise and is

continuous and increasing in y_1:

$$RV_0^{Enter}(y_1, Az) = \begin{cases} V^0 & \text{if } y_1 < py_1^*(Az), \\ p^{1-\alpha}\dfrac{Az}{R}y_1^\alpha + (1-p)V^0 & \text{if } y_1 \in [py_1^*(Az), pk^*(Az)), \\ y_1 + p(1-\alpha)\pi^*(Az) + (1-p)V^0 & \text{if } y_1 \in [pk^*(Az), y_1^S), \\ \dfrac{Az}{R}y_1^\alpha & \text{if } y_1 \in [y_1^S, k^*(Az)), \\ y_1 + (1-\alpha)\pi^*(Az) & \text{if } y_1 \geq k^*(Az). \end{cases}$$

- $V^0 \in ((1-\alpha)\pi^*(Az), p^\alpha\pi^*(Az)]$ and $1-\alpha \leq p^\alpha$: In this case, $(1-p)[V^0 - (1-\alpha)\pi^*(Az)] > 0$, hence $\Delta V(y_1, Az) > 0$ for any $y_1 > py^*(Az)$. The safe-project threshold $k_0^S(Az)$ does not exist, and

$$RV_0^{Enter}(y_1, Az) = \begin{cases} V^0 & \text{if } y_1 < py_1^*(Az), \\ p^{1-\alpha}\dfrac{Az}{R}y_1^\alpha + (1-p)V^0 & \text{if } y_1 \in [py_1^*(Az), pk^*(Az)), \\ y_1 + p(1-\alpha)\pi^*(Az) + (1-p)V^0 & \text{if } y_1 \geq pk^*(Az). \end{cases}$$

Hence, when $V^0 > (p^\alpha - p)/(1-p)\pi^*(Az)$, $\exists y_1^E = Az[k_1^E(Az)]^\alpha$ such that $V^0 = V_0^{Enter}$. Conjecture that $y_1^E \in [py_1^*(Az), pk^*(Az))$, which requires that $pk^*(Az) + p(1-\alpha)\pi^*(Az) + (1-p)V^0 > RV^0$; meanwhile, we have $V^0 > (p^\alpha - p)/(1-p)\pi^*(Az)$, hence a sufficient condition is that $(p^\alpha - p)/(1-p) < p/(R+p-1)$, equivalent to $R < \frac{1-p}{1-p^{1-\alpha}}$. Therefore, once again we have

$$k_0^E(Az) = \left(\frac{(R+p-1)RV^0}{p^{1-\alpha}(Az)^{1+\alpha}}\right)^{1/\alpha^2}.$$

Case 3: Consider the case where

$$V^0 \in (p^\alpha\pi^*(Az), \pi^*(Az)).$$

Now

$$y_1^*(Az) = \left(\frac{RV^0}{Az}\right)^{1/\alpha}$$

and

$$py_1^*(Az) < pk^*(Az) < y_1^*(Az) < k^*(Az).$$

The function $\Delta V(y_1, Az)$ is again piecewise defined and continuous:

$$\Delta V(y_1, Az) = \begin{cases} V^0 - V^0 = 0 & \text{if } y_1 < py_1^*(Az), \\ p^{1-\alpha}\left(\dfrac{Az}{R}y_1^\alpha\right) - pV^0 \geq 0 & \text{if } y_1 \in [py_1^*(Az), pk^*(Az)), \\ y_1 + p(1-\alpha)\pi^*(Az) - pV^0 > 0 & \text{if } y_1 \in [pk^*(Az), y^*(Az)), \\ y_1 - \dfrac{Az}{R}y_1^\alpha + p(1-\alpha)\pi^*(Az) + (1-p)V^0 & \\ \qquad\qquad\qquad\qquad \text{if } y_1 \in [y^*(Az), k^*(Az)), \\ \qquad\quad > (1-p)[V^0 - (1-\alpha)\pi^*(Az)] & \\ (1-p)[V^0 - (1-\alpha)\pi^*(Az)] & \text{if } y_1 \geq k^*(Az). \end{cases}$$

- $V^0 \in (p^\alpha\pi^*(Az), (1-\alpha)\pi^*(Az)]$ and $1-\alpha \geq p^\alpha$: $\exists y_1^S = Az[k_1^S(Az)]^\alpha \in (y^*(Az), k^*(Az)]$ such that $\Delta V(y_1^S, Az) = 0$, i.e.,

$$\frac{Az}{R}(y_1^S)^\alpha - y_1^S = p(1-\alpha)\pi^*(Az) + (1-p)V^0,$$

which does not have an explicit solution.

$$RV_0^{Enter}(y_1, Az) = \begin{cases} V^0 & \text{if } y_1 < py_1^*(Az), \\ p^{1-\alpha}\dfrac{Az}{R}y_1^\alpha + (1-p)V^0 & \text{if } y_1 \in [py_1^*(Az), pk^*(Az)), \\ y_1 + p(1-\alpha)\pi^*(Az) + (1-p)V^0 & \text{if } y_1 \in [pk^*(Az), y_1^S), \\ \dfrac{Az}{R}y_1^\alpha & \text{if } y_1 \in [y_1^S, k^*(Az)), \\ y_1 + (1-\alpha)\pi^*(Az) & \text{if } y_1 \geq k^*(Az). \end{cases}$$

- $V^0 \in ((1-\alpha)\pi^*(Az), \pi^*(Az)] \cap (p^\alpha\pi^*(Az), \pi^*(Az)]$: Now, $(1-p)[V^0 - (1-\alpha)\pi^*(Az)] > 0$, hence $\Delta V(y_1, Az) > 0$ for any $y_1 > py^*(Az)$. The safe-project threshold $k_0^S(Az)$ does not exist, and

$$RV_0^{Enter}(y_1, Az) = \begin{cases} V^0 & \text{if } y_1 < py_1^*(Az), \\ p^{1-\alpha}\dfrac{Az}{R}y_1^\alpha + (1-p)V^0 & \text{if } y_1 \in [py_1^*(Az), pk^*(Az)) \\ y_1 + p(1-\alpha)\pi^*(Az) + (1-p)V^0 & \text{if } y_1 \geq pk^*(Az). \end{cases}$$

Hence, $\exists y_1^E = Az[k_0^E(Az)]^\alpha$ such that $V^0 = V_0^{Enter}$. And the expression for $k_0^E(Az)$ remains the same as in the previous case as long as $R < \frac{1-p}{1-p^{1-\alpha}}$.

To summarize, under Assumption 1, $k_0^E(Az)$ always exists and

$$k_0^E(Az) = \left(\frac{(R+p-1)RV^0}{p^{1-\alpha}(Az)^{1+\alpha}}\right)^{1/\alpha^2};$$

the safe-project threshold $k_0^S(Az)$ exists if $V^0 \leq (1-\alpha)\pi^*(Az)$. Therefore, if $V^0 > (1-\alpha)\pi^*(Az)$, all entering firms choose the risky project. When $V^0 \leq (1-\alpha)\pi^*(Az)$, the risky region and the safe region coexist if $k_0^E(Az) < k_0^S(Az)$ which is ensured by Assumption 1. In particular, when $V^0 \leq \frac{p^\alpha - p}{1-p}\pi^*(Az)$,

$$k_0^E(Az) < k_0^S(Az) = \left(\frac{(1-p)RV^0}{(1-p^{1-\alpha})(Az)^{1+\alpha}}\right)^{1/\alpha^2}.$$

When $V^0 \in \left(\frac{p^\alpha - p}{1-p}\pi^*(Az), (1-\alpha)\pi^*(Az)\right]$, we know that

$$Az[k_0^S(Az)]^\alpha = y_1^S \in \left(\max\left\{\left(\frac{RV^0}{Az}\right)^{1/\alpha}, pk^*(Az)\right\}, k^*(Az)\right]$$

and

$$\frac{Az}{R}(y_1^S)^\alpha - y_1^S = p(1-\alpha)\pi^*(Az) + (1-p)V^0$$

and Assumption 1 ensures that $y_1^E \in [py^*(Az), pk^*(Az))$, hence

$$y_1^E(Az) < pk^*(Az) < y_1^S(Az) \quad \text{and} \quad k_0^E(Az) < k_0^S(Az). \quad \square$$

A.2. Proof of Propositions

Assume that $R < \frac{1-p^{1-\alpha}}{1-p}$, then it is straightforward to show that

$$\frac{dk_0^E(Az)}{d(Az)} < 0.$$

Now we move on to show that $k_0^S(Az)$ also decreases in Az. Note that, $k^*(Az)$ and $\pi^*(Az)$ increase in Az. And as long as $\pi^*(Az) = \left(\frac{\alpha^\alpha Az}{R}\right)^{1/(1-\alpha)} \geq V^0$, $y_1^*(Az) = \left(\frac{RV^0}{Az}\right)^{1/\alpha}$, which decreases in Az.

When $V^0 \leq \frac{p^\alpha - p}{1-p}\pi^*(Az)$, the conclusions in Propositions 2 are reached by immediate algebra:

$$\frac{dk_0^S(Az)}{d(Az)} < 0 \quad \text{and} \quad \frac{d\left(k_0^S(Az) - k_0^E(Az)\right)}{d(Az)} < 0.$$

For Az such that $\frac{p^\alpha - p}{1-p}\pi^*(Az) < V^0 < (1-\alpha)\pi^*(Az)$, $y_1^S(Az) \in (pk^*(Az), k^*(Az))$, and

$$\frac{Az}{R}(y_1^S)^\alpha - y_1^S - p(1-\alpha)\pi^* - (1-p)V^0 = 0,$$

hence,

$$\frac{1}{R}\underbrace{\left(\frac{\alpha Az}{(y_1^S)^{1-\alpha}} - R\right)}_{> \alpha Az/(k^*(Az))^{1-\alpha} - R = 0}\, dy_1^S + \frac{1}{R}\underbrace{((y_1^S)^\alpha - p[k^*(Az)]^\alpha)}_{> (p^\alpha - p)[k^*(Az)]^\alpha > 0}\, d(Az) = 0.$$

Consequently,

$$\frac{dy_1^S(Az)}{d(Az)} < 0 \quad \text{and} \quad \frac{dk_0^S(Az)}{d(Az)} < 0.$$

When Az drops to $(1-\alpha)\pi^*(Az) = V^0$, $y_1^S(Az) = k^*(Az)$ and $k_0^S(Az) = (\alpha Az/R)^{1/(\alpha(1-\alpha))}$. When Az continues to drop, $y_1^S(Az)$ and

$k_0^S(Az)$ cease to exist. Now, to show that $\frac{d\left(k_0^S(Az) - k_0^E(Az)\right)}{d(Az)} < 0$, we start by showing $\frac{d(Azy_1^S(Az)/R - Azy_1^E(Az)/R)}{d(Az)} < 0$.

$$\frac{d}{d(Az)}\left(\frac{Az}{R}y_1^S(Az) - \frac{Az}{R}y_1^E(Az)\right) = \frac{d}{d(Az)}\left(\frac{(Az)^{1+\alpha}}{R}\left([k_0^S(Az)]^{\alpha^2} - [k_0^E(Az)]^{\alpha^2}\right)\right)$$

$$= \frac{d}{d(Az)}\left(y_1^S(Az) + p(1-\alpha)\pi^*(Az) + (1-p)V^0 - \frac{R+p-1}{p^{1-\alpha}}V^0\right)$$

$$= -\frac{((y_1^S)^\alpha - pk^*(Az)^\alpha)}{\left(\frac{\alpha Az}{(y_1^S)^{1-\alpha}} - R\right)} + \frac{pk^*(Az)^\alpha}{R} = \frac{-y_1^S + pk^*(Az)}{\alpha Az - R(y_1^S)^{1-\alpha}} < 0,$$

and hence,

$$\frac{d}{d(Az)}\left([k_0^S(Az)]^{\alpha^2} - [k_0^E(Az)]^{\alpha^2}\right) < 0 \quad \text{and} \quad \frac{d}{d(Az)}\left(k_0^S(Az) - k_0^E(Az)\right) < 0. \square$$

Appendix B. Numerical approximation

B.1. Value function iteration

1. Define grids for the state variables and control variables, namely A, z, p, and k. The grid for project choice p is equally spaced on $[\underline{p}, 1]$. The grid for capital stock k is placed on $[0, 1.5k_{SS}]$ such that the grid is finer on the lower end and coarser on the higher end, with k_{SS} being the unconstrained optimal capital at the steady state. The grids and transition probability matrices for A and z are constructed following Tauchen (1986). Given the grids for A, z and p, the grid for the last period's realized productivity Z is built.

2. For each pair (Z_{t-1}, k_{t-1}) on the grid, the exiting value is

$$V^X(Z_{t-1}, k_{t-1}) = \theta(Z_{t-1}k_{t-1}^\alpha + (1-\delta)k_{t-1}).$$

For each (Z_{t-1}, k_{t-1}) and each potential choice k_t on the grid, calculate the operating profit

$$D(k_t; Z_{t-1}, k_{t-1}) = \begin{cases} Z_{t-1}k_{t-1}^\alpha - \theta(k_t - (1-\delta)k_{t-1}) - c_f & \text{if } k_t < (1-\delta)k_{t-1}, \\ Z_{t-1}k_{t-1}^\alpha - (k_t - (1-\delta)k_{t-1}) - c_f & \text{if } k_t \geq (1-\delta)k_{t-1}. \end{cases}$$

For each $(Z_{t-1}, k_{t-1}, A_t, z_t)$ on the grid, guess an initial value for the value function, in particular, start with

$$V^0(Z_{t-1}, k_{t-1}, A_t, z_t) = V^X(Z_{t-1}, k_{t-1}).$$

3. Update the value function:

$$V^{Stay}(Z_{t-1}, k_{t-1}, A_t, z_t)$$

$$= \max_{p_t, k_t}\left\{ D(k_t; Z_{t-1}, k_{t-1}) + \frac{1}{R}\sum_{A_{t+1}}\sum_{z_{t+1}}\Pr\left(A_{t+1}|A_t\right)\Pr\left(z_{t+1}|z_t\right) \times \left[p_t V^0\left(\frac{A_t z_t}{p_t}, k_t, A_{t+1}, z_{t+1}\right)\right.\right.$$

$$\left.\left. + (1-p_t)V^0(0, k_t, A_{t+1}, z_{t+1})\right]\right\},$$

subject to

$$D(k_t; Z_{t-1}, k_{t-1}) \geq 0.$$

Then,

$$V^1(Z_{t-1}, k_{t-1}, A_t, z_t) = \max\{V^{Stay}(Z_{t-1}, k_{t-1}, A_t, z_t), V^X(Z_{t-1}, k_{t-1})\}.$$

4. Iterate until $\sum_{(Z_{t-1}, k_{t-1}, A_t, z_t)} \left| V^1(Z_{t-1}, k_{t-1}, A_t, z_t) - V^0(Z_{t-1}, k_{t-1}, A_t, z_t) \right| < 10^{-6}$.

5. A potential entrant with initial draw (k_0, A_t, z_t) does not enter if $k_0 < c_e$. Construct the initial capital grid such that $k_0 = (1-\delta)k_{t-1} + c_e$ for each k_{t-1} on the grid. Enter if

$$V_0^{Enter}(k_0, A_t, z_t) = V^{Stay}(0, (k_0 - c_e)/(1-\delta), A_t, z_t) > \theta k_0.$$

B.2. Simulation

Start with an initial distribution of firms Γ_0 over (Z, k, z). Simulate for T_{max} periods. Simulate the full history of aggregate shocks $\{A_t\}_{t=1}^{T_{max}}$ according to the transition matrix. Let the distribution of incumbent firms over (Z, k, z) at $t-1$ be Γ_{t-1}:

1. Construct the distribution for potential entrants Γ^0 over the grids of (k_0, z_0) such that $\Gamma^0(k_0^i, z_0^j) = \left(F^{Pareto}(k_0^i) - F^{Pareto}(k_0^{i-1}) \right) \Pr(z_0^j)$, where F^{Pareto} is the CDF for the Pareto distribution with parameters (c_e, ξ) and $\Pr(z_0^j)$ is calculated according to the transition matrix for the idiosyncratic shocks.
2. The potential entrants with mass $\Gamma^0(k_0, z)$ enter if $V^{Stay}(0, (k_0 - c_e)/(1-\delta), A_t, z) > \theta k_0$. The mass will be added to $\Gamma_t(0, (k_0 - c_e)/(1-\delta), z)$.
3. The firms $\Gamma_{t-1}(Z, k, z)$ exit if $V^{Stay}(Z, k, A_t, z) < V^X(Z, k)$.
4. The distribution of firms after the entry/exit stage becomes

$$\Gamma_t^{Active}(Z, k, z) = \Gamma^0((1-\delta)k + c_e, z)\mathbf{1}_{V^{Stay}(0, k, A_t, z) > \theta((1-\delta)k + c_e)}$$
$$+ \Gamma_{t-1}(Z, k, z)\mathbf{1}_{V^{Stay}(Z, k, A_t, z) \geq V^X(Z, k)},$$

where $\mathbf{1}_X$ is an indicator function such that $\mathbf{1}_X = 1$ if X is true and 0 otherwise.
5. Update the end-of-period distribution according to the policy functions:

$$\Gamma_t(0, k', z') = \sum_Z \sum_k \sum_z (1 - p^*(Z, k, A_t, z)) \Pr(z'|z) \Gamma_t^{Active}(Z, k, z)\mathbf{1}_{k' = k^*(Z, k, A_t, z)},$$

$$\Gamma_t(Z', k', z') = \sum_Z \sum_k \sum_z \frac{A_t z}{Z'} \Pr(z'|z) \Gamma_t^{Active}(Z, k, z)\mathbf{1}_{k' = k^*(Z, k, A_t, z)} \mathbf{1}_{p^*(Z, k, A_t, z) = A_t z/Z'},$$

where k^* and p^* are the policy functions for capital and project choice, respectively.
6. Compute size distribution of active firms, entrants and exiting firms accordingly.

The stationary simulation with $A_t = \mathbb{E}A_t = 1$, $\forall t$ starts with an empty market having no firms and ends when the distribution of firms does not change and the inflow and outflow of firms are the same. The initial distribution of firms for simulation with aggregate fluctuation is the converged stationary distribution. T_{max} is set to be 2000 and the first 1600 periods are ignored when calculating relevant time-series moments.

Appendix C. Robustness check: alternative parameter values

Since there is no clear guideline for calibrating the lower bound of probability (which determines the highest level of riskiness), the model is simulated with $\underline{p} = 0.1, 0.3, 0.7$. Tables 10 and 11 list the results. The qualitative results remain largely intact.

Appendix D. Supplementary data

Supplementary data associated with this paper can be found in the online version at http://dx.doi.org/10.1016/j.jedc.2015.05.015.

References

Arellano, Cristina, Bai, Yan, Kehoe, Patrick, 2010. Financial Markets and Fluctuations in Uncertainty. Federal Reserve Bank of Minneapolis Working Paper.
Bachmann, Rüdiger, Bayer, Christian, 2013. Wait-and-see business cycles? J. Monet. Econ. 60 (6), 704–719.
Bachmann, Rüdiger, Bayer, Christian, 2014. Investment dispersion and the business cycle. Am. Econ. Rev. 104 (4), 1392–1416.

Bachmann, Rüdiger, Moscarini, Giuseppe, 2012. Business Cycles and Endogenous Uncertainty. Working Paper.
Bachmann, Ruediger, Elstner, Steffen, Sims, Eric, 2013. Uncertainty and economic activity: evidence from business survey data. Am. Econ. J.: Macroecon. 5 (2), 217–249.
Baker, Scott R., Bloom, Nicholas, 2013. Does Uncertainty Reduce Growth? Using Disasters as Natural Experiments. Working Paper, National Bureau of Economic Research.
Bloom, Nicholas, 2009. The impact of uncertainty shocks. Econometrica 77 (3), 623–685.
Bloom, Nicholas, Floetotto, Max, Jaimovich, Nir, Eksten, Itay Saporta, Terry, Stephen, 2014. Really Uncertain Business cycles. US Census Bureau Center for Economic Studies Paper No. CES-WP-14-18.
Campbell, Jeffrey R., 1998. Entry, exit, embodied technology, and business cycles. Rev. Econ. Dyn. 1 (2), 371–408.
Castro, Rui, Clementi, Gian Luca, Lee, Yoonsoo, 2015. Cross-sectoral variation in the volatility of plant level idiosyncratic shocks. The Journal of Industrial Economics 63 (1), 1–29. http://dx.doi.org/10.1111/joie.12069.
Chari, V.V., Christiano, L.J., Kehoe, P., 2013. The Gertler–Gilchrist Evidence on Small and Large Firm Sales. Working Paper.
Christiano, Lawrence J., Motto, Roberto, Rostagno, Massimo, 2014. Risk shocks. Am. Econ. Rev. 104 (1), 27–65.
Chugh, Sanjay K., 2013. Firm Risk and Leverage Based Business Cycles. Working Paper, Boston College Department of Economics.
Clementi, Gian Luca, Palazzo, Berardino, 2013. Entry, Exit, Firm Dynamics, and Aggregate Fluctuations. Working Paper, National Bureau of Economic Research.
Cooper, Russell W., Haltiwanger, John C., 2006. On the nature of capital adjustment costs. Rev. Econ. Stud. 73 (3), 611–633.
Davis, Steven J., Haltiwanger, John, 2001. Sectoral job creation and destruction responses to oil price changes. J. Monet. Econ. 48 (3), 465–512.
Decker Ryan, D'Erasmo, Pablo, Moscoso Boedo, Hernan J., 2014. Market Exposure and Endogenous Firm Volatility Over the Business Cycle. American Economic Journal: Macroeconomics, forthcoming. URL: ⟨https://www.aeaweb.org/aej-macro/accepted_single.php?id=1827&jrnl=mac⟩.
Di Tella, Sebastian, 2012. Uncertainty Shocks and Balance Sheet Recessions. Working Paper.
Doms, Mark, Dunne, Timothy, 1998. Capital adjustment patterns in manufacturing plants. Rev. Econ. Dyn. 1 (2), 409–429.
Fort, Teresa C., Haltiwanger, John, Jarmin, Ron S., Miranda, Javier, 2013. How firms respond to business cycles: the role of firm age and firm size. IMF Econ. Rev. 61 (3), 520–559.
Foster, Lucia, Haltiwanger, John, Haltiwanger, Chad, 2008. Reallocation, firm turnover, and efficiency: selection on productivity or profitability? Am. Econ. Rev. 98 (1), 394–425.
Gertler, Mark, Gilchrist, Simon, 1994. Monetary policy, business cycles, and the behavior of small manufacturing firms. Q. J. Econ., 309–340.
Gilchrist, Simon, Sim, Jae W., Zakrajšek Egon, 2014. Uncertainty, Financial Frictions, and Investment Dynamics. Working Paper, National Bureau of Economic Research.
Gourio, François, Kashyap, Anil K., 2007. Investment spikes: new facts and a general equilibrium exploration. J. Monet. Econ. 54, 1–22.
Haltiwanger, John, 2012. Job creation and firm dynamics in the United States. Innovation Policy and the Economy, vol. 12. University of Chicago Press, pp. 17-38 URL: ⟨http://papers.nber.org/books/lern11-2⟩..
Haltiwanger, John, Jarmin, Ron S., Miranda, Javier, 2013. Who creates jobs? Small versus large versus young. Rev. Econ. Stat. 95 (2), 347–361.
Hamilton, B.H., 2000. Does entrepreneurship pay? An empirical analysis of the returns to self-employment. J. Polit. Econ. 108 (3), 604–631.
Herranz, Neus, Krasa, Stefan, Villamil, Anne P., 2009. Small firms in the SSBF. Ann. Finance 5 (3–4), 341–359.
Higson, Chris, Holly, Sean, Kattuman, Paul, 2002. The cross-sectional dynamics of the US business cycle: 1950–1999. J. Econ. Dyn. Control 26 (9), 1539–1555.
Higson, Chris, Holly, Sean, Kattuman, Paul, Platis, Stylianos, 2004. The business cycle, macroeconomic shocks and the cross-section: the growth of UK quoted companies. Economica 71 (282), 299–318.
Hopenhayn, Hugo A., 1992. Entry, exit, and firm dynamics in long run equilibrium. Econometrica: Journal of the Econometric Society, 1127–1150.
Jaimovich, Nir, Floetotto, Max, 2008. Firm dynamics, markup variations, and the business cycle. J. Monet. Econ. 55 (7), 1238–1252.
Kehrig, Matthias, 2011. The Cyclicality of Productivity Dispersion. US Census Bureau Center for Economic Studies Paper No. CES-WP-11-15.
Kehrig, Matthias, Vincent, Nicolas. Financial frictions and investment dynamics in multi-plant firms. US Census Bureau Center for Economic Studies Paper No. CES-WP-56, 2013.
Komaromi, Andras, 2014. Attention and Dispersion Over the Business Cycle. Working Paper, University of California, Berkeley.
Yoonsoo Lee, Toshihiko Mukoyama, 2015. Entry and exit of manufacturing plants over the business cycle. European Economic Review 77, 20–27. http://dx.doi.org/10.1016/j.euroecorev.2015.03.011.
Moscarini, Giuseppe, Postel-Vinay, Fabien, 2012. The contribution of large and small employers to job creation in times of high and low unemployment. Am. Econ. Rev. 102 (6), 2509–2539.
Moskowitz, Tobias J., Vissing-Jørgensen, Annette, 2002. The returns to entrepreneurial investment: a private equity premium puzzle?. Am. Econ. Rev. 92 (4), 745–778.
Perez-Quiros, Gabriel, Timmermann, Allan, 2000. Firm size and cyclical variations in stock returns. J. Finance 55 (3), 1229–1262.
Quadrini, Vincenzo, 2009. Entrepreneurship in macroeconomics. Ann. Finance 5 (3–4), 295–311.
Samaniego, Roberto M., 2008. Entry, exit and business cycles in a general equilibrium model. Rev. Econ. Dyn. 11 (3), 529–541. ISSN 1094-2025.
Tauchen, George, 1986. Finite state Markov-chain approximations to univariate and vector autoregressions. Econ. Lett. 20 (2), 177–181.
Van Nieuwerburgh, Stijn, Veldkamp, Laura, 2006. Learning asymmetries in real business cycles. J. Monet. Econ. 53 (4), 753–772.
Vereshchagina, Galina, Hopenhayn, Hugo A., 2009. Risk taking by entrepreneurs. Am. Econ. Rev. 99 (5), 1808–1830.