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Who Should Buy Portfolio Insurance?

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I. Introduction

THE EXISTENCE OF OPTIONS markets can generate new opportunities for portfolio management. As Ross [1976] has shown, a complete set of options markets on a reference stock or portfolio will enable investors to achieve any desired pattern of returns conditional on the terminal value of the reference asset. While "buy-and-hold" equity strategies allow investors to achieve returns *proportional* to the terminal value of a reference portfolio, buy-and-hold option strategies permit *nonproportional* returns to be achieved.

A nonproportional return of particular interest to some investors is that which provides portfolio insurance. Equivalent to a put option on the reference portfolio, portfolio insurance enables an investor to avoid losses, but capture gains, at the cost of a fixed "premium." Unfortunately, options markets do not currently exist for portfolios of securities, and a portfolio of options is *not* equivalent to an option on a portfolio.

Even when options markets do not exist, however, investors may be able to achieve nonproportional returns on terminal asset values by following *dynamic* investment strategies. If security returns are lognormally distributed at any future time, and continuous trading is possible, Black and Scholes [1973] show that the returns to any option on an asset can be duplicated by an appropriate trading strategy involving the asset and a riskless security. This implies that, in a Black-Scholes world, there exists a dynamic investment strategy which can generate insured portfolio values. The investment strategy involves trading only in the securities of the portfolio and in the riskless asset; no options need exist to achieve insured values.

While the theory of option pricing suggests how to value options, and therefore how to value portfolio insurance, it does not suggest the nature of investors who would benefit from purchasing options or insurance. Unlike traditional insurance, in which everyone can benefit from a pooling of independent risks, portfolio insurance involves hedging against a common (market) risk. For every investor buying portfolio insurance, some other investor(s) must be selling it, either by writing the appropriate put option, or by following the inverse dynamic trading strategy. Who should buy, and who should sell?

In this paper, we provide a characterization of investors who will benefit from purchasing portfolio insurance. Indeed, our results are considerably more general: we characterize investors who demand arbitrary nonproportional patterns of returns on a reference portfolio, and thereby characterize the nature of investors

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581

who wish to buy (or sell) options. Since in the Black-Scholes environment there is a correspondence between options and dynamic strategies, our results also characterize the nature of investors who would wish to follow dyamnic trading strategies. Some "rules of thumb," such as "run with your winners, cut your losses," and "sell at a new high, buy at a new low" will be shown to approximate the optimal dynamic trading strategies for certain types of investors.

The question we address is closely related to the theory of optimal risk sharing and insurance considered by Borch [1960], Wilson [1968], Ross [1973], and Leland [1978]. In an important recent paper, Brennan and Solanki [1979] have posed the question that we consider here. These studies, however, have all assumed identical expectations on the part of market participants. Besides extending results for this case, we consider the optimal behavior of investors whose expectations *differ* from those of the average investor. Since much of the demand for options is attributed to investors who are either more bullish or more bearish on the expected return of the underlying stocks, it seems important to include differing expectations as a possible source of demand for options or for portfolio insurance.

Our principal conclusions are:

1) Investors who have average expectations, but whose risk tolerance increases with wealth more rapidly than average, will wish to obtain portfolio insurance.

2) Investors who have average risk tolerance, but whose expectations of returns are more optimistic than average, will wish to obtain portfolio insurance.

Institutional investors falling in class (1) might include pension or endowment funds which at all costs must exceed a minimum value, but thereafter can accept reasonable risks. "Safety-first" investors would find portfolio insurance attractive on this basis.

Institutional investors falling in class (2) would include well-diversified funds which believe themselves to have positive "α's"—i.e., funds which expect on average to achieve excess returns by superior stock selection. In order to exploit these excess returns to equities, but at the same time keep risk within tolerable levels, insured-type strategies are optimal.

Given the connection between options and portfolio insurance, which is developed in detail in section II, classes (1) and (2) above also characterize the types of investors who would wish to buy call options on a reference portfolio. Since the dynamic trading strategy which yields call option returns (or insured returns) involves buying into the portfolio as its value goes up, but selling out as its value goes down, our results also suggest that investors in classes (1) and (2) would benefit from a "run with your winners, cut your losses" kind of dynamic strategy,[1] rather than a simply "buy and hold" policy. Investors with opposite characteristics would prefer the "buy low, sell high" strategy which is equivalent to writing a call (or selling insurance).

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[1] This is, of course, only a rough approximation of the exact trading strategy which reproduces insured returns. And it should be noted that "run with your winners" typically describes policies towards individual stocks in a portfolio, rather than the portfolio as a whole which is our focus here.

II. Portfolio Insurance, Puts, and Calls: A Review

In this section, we show that purchasing portfolio insurance is equivalent to either:

1) holding the reference portfolio, and buying a put option on the portfolio with striking price equal to the initial portfolio value;

2) buying a call option on the reference portfolio with striking price equal to the initial portfolio value, plus holding cash equal to the initial portfolio value discounted by the riskless interest rate over the insured period.

These results have been derived elsewhere (see Brennan and Schwartz [1976], for example), but are reviewed here for completeness.

Let W_0 and W_T represent the initial and terminal values, respectively, of a reference portfolio of stocks. If an investor obtains full portfolio insurance, he is assured of an end-of-period value Y given by

$$Y(W_T; W_0) = \text{Max}[W_T, W_0].$$

That is, the insured investor gets the larger of the reference portfolio's initial or terminal value. Of course, he must pay a premium to obtain insured values; the cost of insurance will be discussed later.

Now consider an investor who owns the reference portfolio, and who can buy a *put option* on the portfolio with striking price W_0. Such an option has end-period returns given by

$$P(W_T; W_0) = \text{Max}[W_0 - W_T, 0].$$

Holding the reference portfolio plus the put option will give terminal values

$$W_T + P(W_T; W_0) = W_T + \text{Max}[W_0 - W_T, 0]$$

$$= \text{Max}[W_T, W_0]$$

$$= Y[W_T; W_0].$$

Therefore, holding the reference portfolio plus purchasing a put option with striking price W_0 gives insured returns. We now see that the price of insurance must equal the price of the put option with striking price W_0.

Finally, consider a portfolio consisting of a *call option* on the reference portfolio with striking price W_0, plus initial cash equal to $W_0/(1 + r)$, where r is the rate of interest paid on cash over the period of insurance. The call option will have terminal value

$$C[W_T; W_0] = \text{Max}[W_T - W_0, 0],$$

while the cash will have terminal value W_0. Together, the call option plus cash will have terminal value

$$W_0 + C[W_T; W_0] = W_0 + \text{Max}[W_T - W_0, 0]$$

$$= \text{Max}[W_T, W_0]$$

$$= Y[W_T; W_0].$$

Thus we see a second way to obtain insured returns: buying a call option on the reference portfolio, plus holding cash. This implies that the dynamic trading strategy which creates portfolio insurance will be identical to the dynamic strategy which creates the equivalent call option. Rubinstein and Cox [1980] provide a full analysis of the strategy's properties. For our purpose, it is important to note that the dynamic trading strategy that creates insured portfolio values requires higher investment in the reference portfolio as its value rises, and higher amounts in cash as its value falls. Thus anyone obtaining portfolio insurance via a dynamic trading strategy should follow a rule loosely described as "run with your winners, cut your losses." This is, of course, a crude approximation of the exact trading strategy.

III. Generalized Insurance Contracts and Convex Payoff Functions

In the previous section, we considered full portfolio insurance with zero deductible. But, as suggested by Brennan and Solanki [1979], in general it may not be optimal for an investor to have 100 percent protection of value below a certain level, and none above. Investors may simply wish increasing amounts of protection as the level of potential loss increases. Insurance policies with this property are termed *general insurance policies*.

General insurance policies can be created by insuring different fractions of the reference portfolio at different levels of deductible. For example, consider the terminal values of a portfolio with one-fifth of its value insured at zero deductible, and successive fifths insured at 5, 10, 15, and 20 percent deductible. These returns can be duplicated by holding the reference portfolio, plus buying a *portfolio* of put options on the reference fund with striking prices W_0, $.95W_0$, $.90W_0$, $.85W_0$, and $.80W_0$. Note that each put option is held in only one-fifth the quantity that was required for the full insurance discussed in the previous section.

The insured portfolio payoff schedule

$$Y(W_T; \cdot)$$

is a convex function of the reference portfolio's terminal value. Indeed, *we characterize general insurance policies as those that provide strictly convex payoff functions*, since convexity implies greater protection from loss at lower values of the reference portfolio.

Is there a relationship between convex payoff functions and portfolios that include the reference portfolio plus a further portfolio of put options on the reference portfolio? The answer is yes: a portfolio consisting of the reference portfolio plus put options will always provide a convex payoff function. Conversely, a (twice continuously differentiable) convex payoff function can always be generated by holding the reference portfolio and cash, plus a suitable portfolio of put options on the reference portfolio.[2]

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[2] We are focusing on the twice continuously differentiable case here. However, our results could be extended to consider piecewise linear functions. Note also that Ross [1976] and Breeden and Litzenberger [1978] show any payoff function can be provided through a portfolio of options. The content of our result is that a convex function can be provided with a portfolio which contains only *long* positions in options.

To show this, consider an investor who holds a fraction α of some reference portfolio. In addition, he purchases put options at alternative striking prices. Let $\beta(K) \geqq 0$ denote the number of put options bought with striking price K. Then his insured returns will be given by

$$Y[W_T; \beta(\cdot)] = \alpha W_T + \int_0^\infty \beta(K) P[W_T; K] \, dK$$

$$= \alpha W_T + \int_9^\infty \beta(K) \text{Max}[K - W_T, 0] \, dK$$

$$= \alpha W_T + \int_{W_T}^\infty \beta(K)[K - W_T] \, dK,$$

from whence it follows that

$$Y''[W_T; \beta(\cdot)] = \beta(W_T) \geqq 0.$$

Thus, portfolios including the reference portfolio plus put options on that portfolio generate convex returns. Similarly, it can be seen that any twice-differentiable pattern of returns can be obtained by holding put options, a fraction of the reference portfolio, plus cash—a holding that includes portfolio insurance. As can be inferred from the preceding section, this pattern of returns also could be obtained by a suitable portfolio of call options, or by following the equivalent dynamic trading strategy.

Most of our future results will be aimed at characterizing investors who wish strictly convex payoff functions over the terminal value of the reference portfolio. The previous discussion indicates that these investors would demand a general portfolio insurance policy.

IV. The Model

Consider a portfolio whose end-period value is given by W_T, a random variable, and whose initial value W_0 is normalized to one. Let

$p(W_T)$ denote the pricing function at the initial time period for \$1 delivered at the terminal time period, contingent on the value of the reference portfolio.

This pricing function, which reflects the market's expectations and attitudes towards risk, may be imputed from a number of possible environments:

   a) A complete set of options markets on the reference portfolio exists, permitting computation of the $p(W_T)$ function.

  b) The conditions for a risk-neutral valuation relationship[3] are satisfied.
  c) The market behaves as if it were composed of representative or "average" investors, with utility function $V(W_{MT})$ over terminal period market wealth per capita, and with probability function $h(W_{MT}, W_T)$ giving the joint density of end-period market and portfolio values.[4]

These scenarios need not be mutually exclusive. We shall further assume that the prices $p(W_T)$ are competitive; that is, they are unaffected by the portfolio decision of any individual investor.

Now consider an individual whose terminal wealth is dependent upon the value W_T of a reference portfolio. While the reference portfolio is itself a choice variable, we shall not focus on this choice, but rather take it as given. Let

$$Y(W_T) \text{ denote the individual's wealth given } W_T.$$

The individual investor will choose an optimal $Y(\cdot)$ schedule to maximize his expected utility subject to his budget constraint:

$$\underset{Y(\cdot)}{\text{Maximize}} \int_{-\infty}^{\infty} U[Y(W_T)]f(W_T)\, dW_T \tag{1}$$

subject to

$$\int_{-\infty}^{\infty} Y(W_T)p(W_T)\, dW_T = I,$$

where

$$f(W_T) \text{ denotes the investor's probability density function over}$$
$$\text{terminal portfolio values,}$$

and

$$I \text{ denotes the investor's initial wealth.}$$

This problem is similar to that posed by the optimal risk-sharing literature, except that fixed market prices $p(W_T)$ substitute for the (variable) implicit contingency claim prices of the second party.

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[3] A risk neutral valuation relationship exists if the relationship between the value of the payoff function and the value of the reference portfolio is the same as would exist if all market participants are risk neutral. Black and Scholes [1979] show that sufficient conditions for a risk neutral valuation relationship with continuous trading are that the value of the reference portfolio follow a Gauss-Wiener process and a riskless asset exists. Rubinstein [1976] and Brennan [1978] show that a necessary and sufficient condition for a risk neutral valuation relationship to exist in a discrete time model with lognormal returns is that the average investor exhibit constant relative risk aversion. Breeden and Litzenberger [1978] suggest this latter condition is also necessary with continuous trading when the value of the market portfolio is lognormally distributed.

[4] See Rubinstein [1974] for conditions sufficient for a representative or "average" investor to exist. These conditions are closely related to those consistent with risk-neutral valuation relationships existing for the market portfolio.

The solution to (1) is straightforward. For every level of wealth W_T,

$$f(W_T) U'[Y(W_T)] = \lambda p(W_T), \tag{2}$$

where primes denote derivatives, and λ is a positive constant whose level is determined through the budget constraint. Differentiating (2) with respect to W_T, solving for $Y'(W_T)$, and substituting from (2) for λ gives

$$Y' = -\frac{U'}{U''} \left(\frac{f'}{f} - \frac{p'}{p} \right) \tag{3}$$

for all W_T, where functions' arguments have been suppressed. Differentiating again with respect to W_T and simplifying gives

$$Y'' = \left[\frac{d}{dY}\left(\frac{-U'}{U''} \right) + \frac{d}{dW_T}\left(\frac{f'}{f} - \frac{p'}{p} \right) \middle/ \left(\frac{f'}{f} - \frac{p'}{p} \right) \right] \left[\left(\frac{-U'}{U''} \right)\left(\frac{f'}{f} - \frac{p'}{p} \right)^2 \right]. \tag{4}$$

Without further specification of the price, probability, and utility functions, little can be said at this stage. One rather weak result is, however, suggested by (4):

PROPOSITION I: *Other things being equal, investors whose risk tolerance increases with wealth more rapidly are more likely to desire portfolio insurance.*

This can be verified by noting that the term in the right bracket of (4) is always positive. If the f and p functions are fixed, Y'' will more likely be positive if $\frac{d}{dY}\left(\frac{-U'}{U''} \right)$ is large—i.e., if the investor's risk tolerance is increasing rapidly with wealth. Of course $Y'' > 0$ implies the function is (locally) convex; if it is convex at all Y, generalized insurance is desirable.

Brennan and Solanki [1979] derive further results by assuming $f(\cdot)$ and $p(\cdot)$ are consistent with a risk-neutral valuation relationship. We pursue a somewhat different route; our results are compared in the conclusion.

V. Contingency Claim Prices Reflecting an Aggregate Investor

We now consider an environment where the contingency claim price function $p(W_T)$ reflects the preferences of an average (or aggregate) investor.[5] Then there exists an aggregate or market utility function $V(W_T)$ and a market probability density function $h(W_T)$, such that

$$p(W_T) = h(W_T) V'(W_T), \tag{5}$$

where $V'(W_T)$ is the (normalized) marginal utility of a dollar when the value of

[5] Again, see Rubinstein [1974]. Note that the individual investor we consider will in general differ in a significant way from the aggregate investor—either because of differing risk tolerance, differing expectations, or both. This could raise questions as to the consistency of our modelling unless the set of investors differing from the average is "small."

the reference portfolio is W_T.[6]

Given (5), we may now express (3) as

$$Y' = \frac{-U'}{U''}\left(\frac{-V''}{V'} + \frac{f'}{f} - \frac{h'}{H}\right). \tag{6}$$

a: Investor Has Same Expectations as the Market

We first consider the case examined by the optimal risk-sharing literature and by Brennan and Solanki [1979], where $f(W_T) \equiv h(W_T)$, i.e., the investor shares the market's expectations.

In this case, (6) reduces to

$$Y' = \frac{-U'}{U''}\bigg/\frac{-V'}{V''}. \tag{7}$$

Note that $Y' > 0$ given risk aversion. We can now prove the following proposition, which is found in slightly different form in Leland [1978]:

PROPOSITION II: *If the investor shares the market's expectations, the optimal schedule $Y(\cdot)$ is strictly convex if and only if the risk tolerance of the individual is increasing more rapidly with Y than the aggregate investor's is increasing with W_T.*

Proof: Differentiating (7) with respect to W_T gives

$$Y'' = \left(\frac{-V'}{V''}\left[\frac{d}{dY}\left(\frac{-U'}{U''}\right)Y'\right] + \frac{U'}{U''}\left[\frac{d}{dW_T}\left(\frac{-V'}{V''}\right)\right]\right)\bigg/K, \tag{8}$$

where

$$K = (-V'/V'')^2 > 0.$$

Substituting for Y' from (7) yields

$$\text{sign } Y'' = \text{sign}\left(\frac{-U'}{U''}\right)\left[\frac{d}{dY}\left(\frac{-U'}{U''}\right) - \frac{d}{dW_T}\left(\frac{-V'}{V''}\right)\right]. \tag{9}$$

Since risk tolerance $(-U'/U'') > 0$, Y'' will be positive iff

$$\frac{d}{dY}\left(\frac{-U'}{U''}\right) > \frac{d}{dW_T}\left(\frac{-V'}{V''}\right),$$

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[6] This equation holds as is when $W_T = W_{TM}$, i.e., the reference portfolio is the market portfolio, and $V(\cdot)$ is utility dependent upon the terminal value of the market portfolio. When $W_T \equiv W_{TM}$, we can *define*

$$V'(W_T) = \int V'(W_{MT})h(W_{MT}/W_T)\, dW_{MT},$$

where $h(W_{MT}/W_T)$ is the conditional probability function of W_{MT} given W_T. Equation (5) then holds with this newly defined $V'(W_T)$ function, interpreting $h(W_T)$ as the marginal density of W_T derived from the joint density function $h(W_{TM}, W_T)$. If $V(W_{MT})$ exhibits constant relative risk aversion, and W_{MT} and W_T are jointly lognormally distributed, then $V(W_T)$ will also exhibit constant relative risk aversion, although generally of a different degree.

i.e., the risk tolerance of the investor at $Y(W_T)$ is increasing more rapidly than that of the market at W_T, for all W_T.

Remarks

Proposition II does not assume any specific form for investor and market utility functions or expectations. However, if investors are not identical, Rubinstein [1974] has shown that sufficient conditions on market participants for the market to reflect an aggregate or average investor are

(a) identical expectations
(b) linear risk tolerance functions with identical slopes,[7] implying an aggregate or market utility function

$$-V'/V'' = a_M + b_M W_{TM},$$

where W_{TM} is the end-period value of the market portfolio.

Thus, if we restrict our attention to the case where the reference or insured portfolio is the market portfolio, and where the investor also has a linear risk tolerance function

$$-U'/U'' = a_i + b_i Y,$$

then $Y(W_{TM})$ will be a strictly convex function if and only if

$$b_i > b_M. \tag{10}$$

Note that Proposition II does *not* imply that more risk averse investors necessarily demand portfolio insurance. Indeed, if $a_M = a_i = 0$, then $b_i > b_M$ implies that at equal levels of wealth, the investor demanding portfolio insurance is *less* risk averse than average. He will demand a more levered or "higher beta" portfolio, but take out insurance on his levered position.

If the reference portfolio is not the market portfolio, the technique suggested in footnote 6 could be used to compute an appropriate "b" for the market. The work of Brennan and Solanki [1979] suggests that if returns are lognormally distributed, the appropriate b_M will be given by $\sigma^2/\hat{a}$, where σ^2 is the variance of the logarithm of the return, and $\hat{a}$ is the "risk premium" commanded by the portfolio in the market.[8]

b. Investors with Differing Expectations

We now examine cases where the investor's expectations differ from those of the market average. To derive specific results, we shall assume

(i) the individual investor has a linear risk tolerance

$$-U'/U'' = a_i + b_i Y;$$

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[7] Linear risk tolerance is, of course, an alternative terminology for "HARA class" utility functions. These functions include the quadratic, logarithmic, exponential, and power utility functions.

[8] $\hat{a}$ is the difference between the instantaneous rate of return ($\mu_M + 0.5\,\sigma_M^2$) and the risk-free rate r.

(ii) the market utility function exhibits constant relative risk aversion, or proportional risk tolerance

$$-V'/V'' = b_M W_{TM};$$

(iii) the end-of-period market value W_{TM} is lognormally distributed, with

$$E[\ln W_{TM}] = \mu_M \quad \text{for the market}, \quad \mu_i \quad \text{for the investor};$$

$$\text{Var}[\ln W_{TM}] = \sigma_M^2 \quad \text{for the market}, \quad \sigma_i^2 \quad \text{for the investor}.$$

These assumptions are similar to those of Brennan and Solanki [1979].[9] However, those authors assumed identical expectations. As before, we shall concentrate attention on where the individual investor's reference portfolio is the market portfolio, i.e., $W_T = W_{TM}$.

Recalling that we have set $W_0 = 1$, the lognormality assumption implies

$$h(W_T) = \frac{1}{\sqrt{2\pi}\sigma_m W_T} \exp\left[\frac{-1}{2\sigma_M^2}(\ln W_T - \mu_M)^2\right],$$

whereas the individual's density function is given by

$$f(W_T) = \frac{1}{\sqrt{2\pi}\sigma_i W_T} \exp\left[\frac{-1}{2\sigma_i^2}(\ln W_T - \mu_i)^2\right].$$

Substituting these relationships into (6) gives

$$Y' = (a_i + b_i Y)\left[\frac{1}{B_M W_T} + \frac{1}{W_T}\left[\frac{\sigma_i^2(\ln W_T - \mu_M) - \sigma_M^2(\ln W_T - \mu_i)}{\sigma_M^2\sigma_i^2}\right]\right] \quad (11)$$

which has solution

$$Y = c_0 W_T^{[(b_i/b_M) + b_i k_1 + b_i k_2 \ln W_T]} - a_i/b_i, \quad (12)$$

where

c_0 is a constant of integration determined by the budget constraint;

$$k_i = \frac{\mu_i\sigma_M^2 - \mu_M\sigma_i^2}{\sigma_i^2\sigma_M^2},$$

$$k_2 = \frac{\sigma_i^2 - \sigma_M^2}{2\sigma_i^2\sigma_M^2}.$$

While (12) is a general solution to the case with differing expectations, we focus on the environment where the individual and market estimates of variance of ln

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[9] A fuller discussion of the relationship between the assumptions and Brennan/Solanki is contained in the final section.

W_T are identical, but the estimates of the mean return of ln W_T differ.[10] In this environment, $k_1 > 0$ if $\mu_i > \mu_M$, and $k_2 = 0$. From (12), the following proposition is immediate:

PROPOSITION III: *If the market and individual have identical risk tolerance behavior* ($b_i = b_M$) *and identical estimates of riskiness* ($\sigma_i = \sigma_M$), *the optimal schedule* $Y(W_T)$ *will be strictly convex if and only if* $\mu_i > \mu_M$, *i.e., if the investor has more optimistic expectations than the market.*

Remarks

Proposition III implies that an investor with "average" risk tolerance behavior will demand portfolio insurance if he has a more optimistic expected return than the market. From the results of section II, such an investor could obtain insured returns either by (i) holding the reference portfolio (perhaps levered) and buying put options on the portfolio, (ii) investing in call options on the reference portfolio, or (iii) following an appropriate dynamic trading strategy. An investor with less optimistic expectations would be willing to provide insurance, or equivalently, to write put or call options, or follow the inverse dynamic trading strategy.

Portfolios managed with the expectation of excess returns, or "positive α's," will benefit from portfolio insurance. This may seem counterintuitive: why should an investor who is willing to seek out excess returns also wish to protect himself through insurance? The answer is that excess expected returns can more thoroughly be exploited, but risks controlled, through insured strategies. The investor can more fully exploit positive α situations through greater levels of risky investment. At the same time, risks can be kept within manageable bounds by the use of portfolio insurance.

Proposition III also indicates that the lessons of static portfolio theory must be broadened in the context where complete options markets do not exist, but continuous trading is possible. Static theory suggests that higher expected returns will lead to greater levels of investment in the risky portfolio. Our results indicate that not only is greater investment desirable, but that dynamic strategies may enable achievement of even higher levels of expected utility. That is, *dynamic strategies should be an intrinsic part of any portfolio optimization.*

It should be stressed that the dynamic strategies considered here are not "market timing" strategies. They are not predicated on the idea that excess returns can be achieved by buying and selling at the "right" time. Rather, the dynamic strategies are used to create desirable nonproportional end-of-period values. Buying and selling are triggered only by changes in the value of the reference portfolio, according to the "informationless" hedging rules that recreate option returns. The resulting convexity of the end-of-period returns yield greater expected utility than any buy-and-hold strategy involving the reference portfolio,

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[10] If the random terminal value W_T is created by a (continuous) logarithmic random walk over the period, and continuous trading is possible, the individual investor must have the same instantaneous estimate of the variance as the market has. Otherwise, a riskless arbitrage opportunity would be perceived, and equilibrium could not exist. Of course if continuous trading is not possible, equilibrium could support differences in variances.

even though the trading strategy does not increase the α associated with that portfolio.

The result in Proposition III also applies to portfolios other than the market portfolio, with the recognition that the relevant b_M must be adjusted by the technique in footnote 6. For portfolios whose returns are highly correlated with the market's, as are many institutional portfolios, virtually no adjustment will be required.

c. Investors with Differing Expectations and Risk Tolerance Behavior

The following proposition also is an immediate consequence of (12) and generalizes Proposition III to the case where $b_i \neq b_M$:

PROPOSITION IV: *If the market and individual have identical estimates of the variance* $(\sigma_i = \sigma_M)$, *the optimal schedule* Y *will be strictly convex if and only if*

$$\frac{\mu_i - \mu_M}{\sigma_M^2} > \frac{1}{b_i} - \frac{1}{b_M}.$$

This follows from the convexity requirement that $\dfrac{b_i}{b_M} + b_i k_1 > 1$.

Thus, even if he has risk tolerance growing more slowly than the market average $(b_i < b_M)$, the individual investor still may want a general insurance contract if his expectations are sufficiently positive. And note that the effects of more rapidly growing risk tolerance $(b_i > b_M)$ and more optimistic expectations $(\mu_i > \mu_M)$ on the demand for insurance are cumulative.

As either the risk tolerance of the investor increases more rapidly with Y, or his expectations become more optimistic, the optimal schedule $Y(\cdot)$ becomes increasingly steep and convex. Relative curvature (Y''/Y') also increases. The average steepness of the function can be associated with the exposure to risk—greater steepness implies a more levered position in the reference portfolio. Convexity is associated with the amount of insurance purchased. Relative curvature is associated with the ratio of insurance purchased to exposure—loosely speaking, it measures the "fraction" of portfolio held that is insured. Thus, changes in either the risk tolerance parameter b_i or in expected returns μ_i (or both) lead to greater insurance coverage—or equivalently, greater purchases of call options on the portfolio. If insurance is provided by a dynamic trading strategy, greater insurance coverage implies more extensive trading as the value of the portfolio fluctuates.

VI. Conclusion and Relationship to Previous Work

Our objective has been to characterize the nature of investors who would benefit from portfolio insurance. Our conclusions, summarized by Propositions I to IV, indicate that investors (i) whose risk tolerance increases with wealth more rapidly than that of the average investor, and/or (ii) whose expectations are more optimistic than average, would benefit from portfolio insurance. Investors be-

longing to class (i) would include safety-first investors. Class (ii) investors would include institutions with portfolios managed with the expectation of above-average returns, or positive α.

We indicated that portfolio insurance can be obtained in a number of ways. If option markets exist on the reference portfolio, insurance can be achieved by purchasing appropriate put options in addition to holding the refrence portfolio. It could also be achieved by purchasing appropriate call options, plus holding cash.

If (as is usually the case) options do not exist on the reference portfolio, insured returns can nonetheless be achieved when continuous trading is possible and portfolio returns are lognormally distributed. The dynamic trading strategy that reproduces the appropriate options' returns will provide the optimal portfolio insurance. This result has an important bearing on portfolio optimization strategies. Static portfolio theory suggests that more optimistic expectations should lead to greater investment in the risky portfolio. But our analysis suggests that in addition to greater investment, dynamic strategies should also be followed to produce the optimal pattern of returns. While current applied portfolio optimization techniques focus on static portfolios that minimize risk for a given level of return, further improvements in risk/return ratios can be achieved by following appropriate dynamic strategies. We noted that the dynamic strategy associated with portfolio insurance requires buying more of the reference portfolio as its value rises, and selling out as its value falls. The opposite strategy ("sell at a high, buy at a low") is appropriate for an investor wishing a concave payoff schedule.

Our results are related to a number of previous strands of research. Models of capital market equilibrium have focused on optimal portfolios and on valuation of risky assets. For analytical tractability, however, most of these models (including the CAPM) have assumed homogeneous expectations, and investors with linear risk tolerance utility functions with the same slope—precisely those assumptions that eliminate the demand for options! Thus these models are incapable of characterizing the nature of investors who demand options (or insured returns), since they make assumptions which in equilibrium preclude any investor from demanding them.

Our model is more closely related to the theory of insurance/agency/optimal risk sharing, which typically posits two parties sharing a random return, and examines the nature of the optimal sharing rule. By the assumption that an aggregate investor exists, we can examine the interaction between an individual investor and "the market." But a key difference between our study and the risk-sharing literature is that the market is perfectly competitive, and the portfolio choice of the individual investor does not affect the terms of trade. Nonetheless, Proposition II is almost identical to the result derived by Leland [1978] in the optimal risk-sharing context.

The closest work to ours is Brennan and Solanki [1979], who formulate the general problem in the same manner as we do. In deriving specific results, however, they focus on the case where prices $p(W_T)$ are derived from "risk-neutral valuation relationships," and the portfolio returns are lognormally distributed. They do not consider differences between investor and market expec-

tations. But it might appear that they avoid having to assume the existence of an aggregate investor, since a market utility function nowhere appears in their results. This is not the case, however. Brennan [1978] shows that risk-neutral valuation relationships and lognormality are consistent with a discrete time framework only if the market exhibits constant proportional risk aversion—i.e., has a utility function of the form $-V'/V'' = bW_T$, which is the special form assumed for our results in Propositions III and IV. Breeden and Litzenberger [1978] show that risk-neutral valuation and lognormality of market returns also requires constant proportional risk aversion for the market, when continuous trading is possible. Thus it appears that a market utility function with constant proportional risk aversion underlies the Brennan/Solanki results. Their key conclusion, that an individual investor will desire a convex payoff function if his index b_i of risk tolerance exceeds $\sigma^2/\hat{a}$, where $\hat{a}$ is the risk premium of the reference portfolio, is exactly the same as (10) when it is recognized that in equilibrium $\sigma^2/\hat{a} = b_M$. Note that (10) is a special case of Proposition II.

In sum, our results would seem to extend Brennan/Solanki's results in the case of identical investor-market expectations. And we have derived results for the case with differing expectations, which are often held to be the principal source of the demand for options.

REFERENCES

Black, F., and Scholes, M. 1973. "The Pricing of Options and Corporate Liabilities." *Journal of Political Economy* 81: 637–654 (May–June).

Borch, K. 1960. "The Safety Loading of Insurance Premiums." *Skandinansk Aktuarietidskrift* 43: 163–184.

Breeden, D., and Litzenberger, R. 1978. "Prices of State Contingent Claims Implicit in Option Prices." *Journal of Business* 51: 621–652 (October).

Brennan, M. 1979. "The Pricing of Contingent Claims in Discrete Time Models." *Journal of Finance* 34: 53–68 (March).

Brennan, M., and Solanki, R. 1979. "Optimal Portfolio Insurance." Working Paper No. 666. Faculty of Commerce and Business Administration, University of British Columbia.

Friend, I., and Blume, M. 1975. "The Demand for Risky Assets." *American Economic Review* 65: 900–922 (December).

Ibbotson, R., and Sinquefield, R. 1976. "Stocks, Bonds, Bills, and Inflation: Year by Year Historical Returns." *Journal of Business* 49: 11–47 (Jan).

Leland, H. 1978. "Optimal Risk Sharing and the Leasing of Natural Resources, with Application to Oil and Gas Leasing on the OCS." *Quarterly Journal of Economics*: 413–437 (August).

Ross, S. 1973. "The Economic Theory of Agency." *American Economic Review* 63: 134–139 (May).

———. 1976. "Options and Efficiency." *Quarterly Journal of Economics* 90: 75–89 (February).

Rubinstein, M. 1974. "An Aggregation Theorem for Securities Markets." *Journal of Financial Economics* 1: 225–244 (September).

———. 1976. "The Valuation of Uncertain Income Streams and the Pricing of Options." *Journal of Economics* 7: 407–425 (Autumn).

Rubinstein, M., and Cox, J. 1980. *Options Markets*. Prentice-Hall, forthcoming.

Wilson, R. 1968. "The theory of syndicates." *Econometrica* 36: 119–132 (Jan).