Characterizing the Variance Risk Premium in Consumption-Based Models*

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Abstract

We show that in a general consumption-based asset pricing model with Epstein-Zin preferences, the market variance risk premium is linearly related to the leverage effect, the conditional covariance between the market return and its variance. This relation is due to the pricing of volatility risk. Empirically, we document a positive and statistically significant relationship between the variance risk premium and the leverage effect, using 1990-2016 data for the S&P 500 index. We exploit the relation between the variance risk premium and the leverage effect to characterize the historical behavior of the market variance risk premium dating back to 1926.

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1 Introduction

The variance of returns is one of the building blocks of modern finance, and fortunately it is possible to characterize and measure it very precisely, even over very short intervals. We now also have estimates of the variance risk premium, the difference between physical and risk-neutral expectations of future market return variance, following the growth in derivatives markets over the past few decades. The measurement and characterization of the variance risk premium has generated a lot of attention from academics as well as practitioners because it reveals how variance risk is perceived and incorporated into prices by investors. The existing literature finds that the variance risk premium is large, negative, and statistically significant. It is a strong predictor of short-horizon future market returns.[1] Several models featuring time-varying economic uncertainty have been proposed to explain these stylized facts in the data.

This paper shows that in a general consumption-based asset pricing framework, the variance risk premium is linearly related to leverage effect, which we define as the conditional covariance between the market return and its variance. While there is an extensive literature on the leverage effect starting with Black (1976), this relation to the variance risk premium has not yet been documented.

The relation between the variance risk premium and the leverage effect is driven by the pricing of volatility risk. If volatility risk is priced, shocks to the market variance result in shocks to the pricing kernel, and therefore in shocks to the market return. The variance risk premium measures how the market variance co-moves with the pricing kernel, while the leverage effect measures how the market variance co-moves with the market return. In other words, the variance risk premium and the leverage effect are generated through the same channel.

[1] Bollerslev, Tauchen, and Zhou (2009) document that the variance risk premium is a strong predictor of short-term U.S. equity market returns. Bollerslev, Marrone, Xu, and Zhou (2014) provide additional evidence based on international data. Kilic and Shaliastovich (2018) decompose the total variance risk premium into good and bad components and investigate the predictive power of each component.

We empirically confirm the theoretical relation between the variance risk premium and the leverage effect using monthly data on the S&P 500 for 1990-2016. We document a strong and statistically significant positive linear relationship between the two quantities, consistent with the theory. This finding is robust to alternative measurement of the variance risk premium and the leverage effect and obtains for various sample periods. These findings suggest that investors are willing to pay a higher premium for hedging against volatility risk when they expect stronger (i.e. more negative) co-movement between returns and volatility.

The relation between the variance risk premium and the leverage effect can be used to empirically study and document the properties of the variance risk premium. As an example, we provide estimates of the variance risk premium dating back to 1926, which is possible because the leverage effect can be estimated provided that daily market returns are available. Existing estimates of the variance risk premium typically start in 1990 because sufficient index option data are not available prior to this time. The resulting time series of the variance risk premium are therefore far more limited compared to available return time series as well as most other financial and economic variables. We find that the extended time series of the variance risk premium we construct has plausible properties. The variance risk premium is very large and displays extreme fluctuations during the late 1920s and throughout the 1930s, reflecting the adverse economic conditions during the Great Depression.

Our paper contributes to the expanding empirical literature on the variance risk premium. While a long position in the market index itself results in a positive expected return, it is well known that a long position in the second moment of the index on average generates significant losses. For instance, Coval and Shumway (2001) and Bakshi and Kapadia (2003) show that option-based strategies that are long index variance, such as delta-hedged portfolios and zero-beta straddles, earn negative average returns.[2] Recent evidence from volatility claims (e.g. variance swaps and VIX futures), which are designed to offer investors direct exposure to volatility risk, also documents a large negative variance risk premium (Carr and Wu, 2009;

[2]Another strand of literature investigates the pricing of volatility risk in the cross-section of stock returns. See, for example, Ang, Hodrick, Xing, and Zhang (2006).

Aït-Sahalia, Karaman, and Mancini, 2015; Eraker and Wu, 2017). Related, Todorov (2010) investigates the time series dynamics of the variance risk premium.

Several existing studies examine the variance risk premium in the context of consumption-based models with Epstein-Zin preferences. To explain the variance risk premium, the consumption dynamics originally proposed in the long-run risk model of Bansal and Yaron (2004) have been extended in several ways. For example, Bollerslev, Tauchen, and Zhou (2009) incorporate stochastic volatility-of-volatility into consumption growth, and Zhou and Zhu (2014) consider long-run and short-run components of consumption growth. Drechsler and Yaron (2011) augment the long-run risk model by introducing jumps into the model, emphasizing their importance in explaining the variance risk premium.[3] Separately, Seo and Wachter (2018) show that a model with stochastic disaster risk can also account for the variance risk premium. While these papers relate the variance risk premium to the dynamics of economic fundamentals like consumption growth, we focus on the relation between the variance risk premium and the leverage effect within a general theoretical setup.[4]

We also contribute to the literature on the leverage effect and asymmetric volatility.[5] Existing studies show that the leverage effect for the index is robustly negative. The negative sign implies that volatility responds asymmetrically to positive and negative returns. Black (1976) attributes this phenomenon to firm leverage, which is the origin of the terminology "leverage effect." When the stock price is lower, financial/operating leverage goes up, which makes the stock riskier and more volatile.[6] Bekaert and Wu (2000) empirically reject the leverage channel and instead support an alternative hypothesis based on time-varying risk premia, the volatility feedback effect. If volatility is negatively priced, a positive shock to

[3]Eraker (2008) also considers a similar model where the consumption variance process is subject to a Poisson jump.

[4]Our benchmark model in Section 2 is the continuous-time version of the long-run risk model of Bansal and Yaron (2004). This setup is extended in Section 6, which makes it possible to nest a wider range of models including the generalized long-run risk model (Drechsler and Yaron, 2011) and rare disaster models (Barro, 2006; Wachter, 2013).

[5]Alternatively, in the existing literature the leverage effect may refer to the the correlation between return and variance.

[6]This leverage hypothesis is also studied by Christie (1982), Schwert (1989), and Cheung and Ng (1992).

volatility leads to an immediate increase in investors' marginal utility and a decrease in the stock price, producing asymmetric volatility.[7] Note that we do not necessarily take a stance on which channel is more empirically plausible, but the model we use in this paper, like most other dynamic consumption-based models, generates the leverage effect through the volatility feedback channel.

The rest of the paper is organized as follows. Section 2 derives a theoretical relation between the variance risk premium and the leverage effect using the benchmark model. Section 3 discusses various aspects of this relation. Section 4 presents the main empirical analyses. Section 5 provides robustness results. Section 6 discusses a more general model. Section 7 concludes.

2 The Model

In this section, we present our main result. We deliberately focus on a simple version of the model to provide intuition for this result. In Section 6, we show that this result also obtains in a model with jumps and/or rare events, which is more general and is capable of matching additional stylized facts in equity index markets.

2.1 Model Setup

We assume that aggregate consumption C_t follows an affine diffusion process

$$\frac{dC_t}{C_{t^-}} = (\mu_C + X_t)dt + \sqrt{V_t}dB_{C,t},$$

where $B_{C,t}$ is a standard Brownian motion. The drift of consumption growth has a time-varying component, X_t, which mean-reverts back to zero. The variance of consumption

[7]Other studies on the volatility feedback effect include French, Schwert, and Stambaugh (1987) and Campbell and Hentschel (1992). Bandi and Reno (2012) and Yu (2012) report evidence for a time-variation in the leverage effect.

growth V_t is stochastic as well and follows the square root process of Cox, Ingersoll, and Ross (1985). This gives:

$$\begin{aligned} dX_t &= -\kappa_X X_t dt + \sigma_X \sqrt{V_t} dB_{X,t} \\ dV_t &= \kappa_V(\bar{V} - V_t)dt + \sigma_V \sqrt{V_t} dB_{V,t}, \end{aligned} \tag{1}$$

where $B_{X,t}$ and $B_{V,t}$ are also standard Brownian motions that are independent of each other and of $B_{C,t}$.

We assume complete markets with an infinitely-lived representative agent who has recursive preferences as in Epstein and Zin (1989) and Weil (1989). The utility function implied by Epstein-Zin preferences generalizes time-additive power utility by allowing the separation of relative risk aversion and the elasticity of intertemporal substitution (EIS). We adopt the continuous-time formulation of this utility function derived in Duffie and Epstein (1992):

$$J_t = \mathbb{E}^P \left[\int_t^\infty f(C_s, J_s) ds \right] \tag{2}$$

where

$$f\left(C, J\right) = \frac{\delta}{1 - \frac{1}{\psi}} (1-\gamma) J \left[\left(C \left[(1-\gamma) \, J \right]^{-\frac{1}{1-\gamma}} \right)^{1-\frac{1}{\psi}} - 1 \right]. \tag{3}$$

The parameters δ, γ, and ψ represent the rate of time preference, relative risk aversion, and the EIS, respectively. While the expression for the normalized aggregator f is defined only when the EIS is not equal to one, the limit of the EIS approaching one is well-defined.[8]

In this paper, we are agnostic about different choices of model parameters as our focus is to make a general statement about the relationship between the variance risk premium

[8] That is, in case of a unit EIS, the normalized aggregator is defined as

$$\lim_{\psi \to 1} f(C, J) = \delta(1-\gamma) J \left[\log C - \frac{1}{1-\gamma} \log \left[(1-\gamma) \, J \right] \right].$$

and the leverage effect. However, it is still worth mentioning that our setup reduces to (the continuous-time analogue of) the long-run risk model of Bansal and Yaron (2004) given a calibration in which the mean and volatility components of consumption growth are highly persistent and the representative agent prefers early resolution of uncertainty ($\psi > \frac{1}{\gamma} > 1$). In Section 6, we extend the current setup to incorporate the risk of jumps, in an effort to also nest other models (e.g. Drechsler and Yaron, 2011; Wachter, 2013) in our setup.

The value function J is a function of the three state variables (C_t, X_t, and V_t). Since the functional form of the normalized aggregator implies that the value function is homogeneous of degree $(1-\gamma)$ in consumption, it follows that

$$J_t = \frac{C_t^{1-\gamma}}{1-\gamma} I(X_t, V_t), \tag{4}$$

where $I(\cdot)$ needs to be determined. In general, dynamic Epstein-Zin models do not allow an exact closed-form solution for I unless the EIS equals one. Appendix A shows that we obtain an exponentially linear function

$$I(X, V) = e^{a + b_X X + b_V V} \tag{5}$$

for an arbitrary EIS, where a, b_X and b_V are defined in the Appendix. This result uses an approximation around the mean consumption-wealth ratio proposed by Campbell and Viceira (1999). In case of a unit EIS, this approximation is exact.

Based on equation (4) and the results from Duffie and Skiadas (1994), Appendix B derives the stochastic differential equation for the state-price density π_t as

$$\frac{d\pi_t}{\pi_{t^-}} = -r_{f,t}dt - \gamma\sqrt{V_t}dB_{C,t} + \left(1 - \frac{1}{\theta}\right)\left[b_X \sigma_X \sqrt{V_t} dB_{X,t} + b_V \sigma_V \sqrt{V_t} dB_{V,t}\right], \tag{6}$$

where $\theta = (1-\gamma)/(1-1/\psi)$ and $r_{f,t}$ is the instantaneous risk-free rate. For details, see Appendix B.

2.2 Stock Price Dynamics

We define the aggregate stock as the claim to the aggregate dividend D_t, which evolves according to

$$\frac{dD_t}{D_{t^-}} = \mu_{D,t}dt + \phi\sqrt{V_t}dB_{C,t},$$

where $\mu_{D,t} = \mu_{D0} + \mu_{DX}X_t + \mu_{DV}V_t$.[9]

The pricing relation implies that the price of this dividend claim, S_t, equals the expected future cash flows discounted by the state-price density. In Appendix C, we derive the following expression for S_t:

$$S_t = \mathbb{E}_t^P\left[\int_t^\infty \frac{\pi_s}{\pi_t}D_s ds\right] = D_t\int_0^\infty \exp\Big(a_\phi(\tau) + b_{\phi X}(\tau)X_t + b_{\phi V}(\tau)V_t\Big)\,d\tau,$$

where $a_\phi(\tau)$, $b_{\phi X}(\tau)$, and $b_{\phi V}(\tau)$ solve a system of ordinary differential equations provided in the Appendix. Note that the price-dividend ratio, $G(X_t, V_t) = S_t/D_t$, is not exactly a log-linear function of X_t and V_t, although it is fairly close because it is an integral of a log-linear function. Following Seo and Wachter (2017), we log-linearize the price-dividend ratio as:[10]

$$G(X, V) \;\; \simeq \;\; \exp\left(a_{\bar\phi} + b_{\bar\phi X}X + b_{\bar\phi V}V\right), \tag{7}$$

where $a_{\bar\phi}$, $b_{\bar\phi X}$, and $b_{\bar\phi V}$ are defined in the Appendix. By applying Ito's lemma to $S_t = D_tG_t$, the dynamics of the stock price are given by

$$\frac{dS_t}{S_{t^-}} = \mu_{S,t}dt + \phi\sqrt{V_t}dB_{C,t} + b_{\bar\phi X}\sigma_X\sqrt{V_t}dB_{X,t} + b_{\bar\phi V}\sigma_V\sqrt{V_t}dB_{V,t}. \tag{8}$$

[9] This general specification of the dividend process nests the standard assumption that the aggregate dividend is a levered consumption claim (i.e. $D_t = C_t^\phi$), in which case $\mu_{D,t} = \phi\mu_C + \phi X_t + \frac{1}{2}\phi(\phi-1)V_t$. Since dividends are more volatile than consumption, in this specification ϕ is assumed to be larger than 1.

[10] It is convenient to log-linearize this function; without this, the dynamics of the stock price do not exactly follow an affine-jump diffusion process.

For additional details, see Appendix C.

We also derive the risk-neutral dynamics of the stock price in the model. Based on the Radon-Nikodym derivative process $L_t = \pi_t \left(\int_0^t r_{f,s} ds \right)$ implied by equation (6), Girsanov's theorem suggests

$$\frac{dS_t}{S_{t^-}} = \mu_{S,t}^{Q} dt + \phi \sqrt{V_t} dB_{C,t}^{Q} + b_{\bar{\phi}X} \sigma_X \sqrt{V_t} dB_{X,t}^{Q} + b_{\bar{\phi}V} \sigma_V \sqrt{V_t} dB_{V,t}^{Q}, \tag{9}$$

where $B_{C,t}^{Q}$, $B_{X,t}^{Q}$, and $B_{V,t}^{Q}$ are standard Brownian motions under the risk-neutral measure defined in Appendix B. Under the risk-neutral measure, the drift of the stock dynamics becomes the risk-free rate less the dividend yield (i.e. $\mu_{S,t}^{Q} = r_{f,t} - G_t^{-1}$).

2.3 The Variance Risk Premium

The variance risk premium is defined as the difference between the physical and risk-neutral expectations of future realized variance.[11] Empirically, realized variance is measured as the sum of squared high-frequency log returns over an interval. This quantity converges to the quadratic variation of the log stock price as the sampling interval becomes smaller. Thus, in the model, we calculate realized variance using the quadratic variation

$$QV_{t,t+\tau} = \int_t^{t+\tau} d\left[\log S, \log S\right]_s = \int_t^{t+\tau} \xi V_s ds, \tag{10}$$

where $\xi = \left(\phi^2 + b_{\bar{\phi}X}^2 \sigma_X^2 + b_{\bar{\phi}V}^2 \sigma_V^2 \right)$. We define $V_t^S \equiv \xi V_t$ as the time-t instantaneous stock variance. The use of quadratic variance makes it easy to analytically calculate expectations of realized variance and the variance risk premium in our model.

[11] Existing studies report a sizable variance risk premium at the index level (e.g., Bakshi and Kapadia, 2003; Carr and Wu, 2009). Additional evidence for the existence of the index variance risk premium is provided in the option pricing literature (e.g., Bates, 2000; Pan, 2002; Eraker, 2004). The variance risk premium is a strong predictor of short term stock index returns (Bollerslev, Tauchen, and Zhou, 2009). Zhou (2018) provides a comprehensive survey on the predictive power of the variance risk premium.

The time-t variance risk premium between t and $t+\tau$ is given by:

$$VRP_{t,t+\tau} = \mathbb{E}_t^P[QV_{t,t+\tau}] - \mathbb{E}_t^Q[QV_{t,t+\tau}] = \xi \left(\mathbb{E}_t^P \left[\mathbb{V}_{t,t+\tau} \right] - \mathbb{E}_t^Q \left[\mathbb{V}_{t,t+\tau} \right] \right), \tag{11}$$

where $\mathbb{V}_{t,t+\tau} = \left(\int_t^{t+\tau} V_s ds \right)$ is the integrated consumption variance between t and $t+\tau$. $\mathbb{E}^P[\cdot]$ and $\mathbb{E}^Q[\cdot]$ represent expectations under the physical and risk-neutral measure, respectively. We calculate the closed-form expression for equation (11) in Appendix D. Under both measures, the time-t conditional expectation of $\mathbb{V}_{t,t+\tau}$ is linear in V_t with coefficients that are functions of τ. Consequently, the time-t variance risk premium in the model is also obtained as a linear function of V_t.

Equation (11) indicates that the model-implied variance risk premium arises from the stochastic volatility in equations (8) and (9). In a dynamic Epstein-Zin model, shocks to V_t (i.e. $dB_{V,t}$) are priced, which makes the risk-neutral dynamics of V_t different from its physical counterpart. Specifically, under the risk neutral measure,

$$dV_t = \kappa_V^Q (\bar{V}^Q - V_t) dt + \sigma_V \sqrt{V_t} dB_{V,t}^Q$$

where $\kappa_V^Q = \kappa_V - \left(1 - \frac{1}{\theta}\right) b_V \sigma_V^2$ and $\bar{V}^Q = \bar{V} \kappa_V / \kappa_V^Q$. See Appendix B for details. Since V_t and, by extension, $\mathbb{V}_{t,t+\tau}$ have different conditional means under the two measures, the model-implied variance risk premium is nonzero.[12]

[12]The literature has documented the important role of jump risk in explaining the variance risk premium in the data. Drechsler and Yaron (2011) find that the long-run risk model enhanced by jumps is able to explain the behavior of the variance risk premium. The results in Todorov (2010), Bollerslev and Todorov (2011), and Bollerslev, Todorov, and Xu (2015) also suggest that the compensation for jump risk accounts for a large fraction of the variance risk premium. To illustrate that our findings can accommodate these stylized facts, Section 6 presents an extension of the model where the variance risk premium is also driven by jump risk.

2.4 The Leverage Effect and the Variance Risk Premium

In many empirical applications, the variance risk premium is measured over a one-month time horizon τ. For such short horizons, the integrated variance $\mathbb{V}_{t,t+\tau} = \left(\int_t^{t+\tau} V_s ds\right)$ is well approximated by $\tau V_{t+\tau}$. Thus, in this case, it follows that

$$VRP_{t,t+\tau} \simeq \xi \left(\mathbb{E}_t^P \left[\tau V_{t+\tau}\right] - \mathbb{E}_t^Q \left[\tau V_{t+\tau}\right] \right) = \mathbb{E}_t^P \left[\tau V_{t+\tau}^S\right] - \mathbb{E}_t^Q \left[\tau V_{t+\tau}^S\right], \tag{12}$$

where $\tau V_{t+\tau}^S$ represents the stock variance over horizon τ. This approximation provides intuition by simplifying the expression for the variance risk premium, eliminating integrals over short horizons. However, Appendix D and Appendix E indicate that this approximation is not required to derive our main result, the linear relationship between the variance risk premium and the leverage effect. Note also that if the continuous-time model is implemented in discrete time using one-month intervals, this approximation is exact.

Since the Radon-Nikodym derivative L_t is a martingale, we know that $\mathbb{E}_t^P \left[L_{t+1}/L_t\right] = 1$. Furthermore, it follows from the Radon-Nikodym theorem that $\mathbb{E}_t^Q[V_{t+\tau}^S] = \mathbb{E}_t^P \left[\frac{L_{t+\tau}}{L_t} V_{t+\tau}^S\right]$. These two relations imply that equation (12) can be expressed as

$$VRP_{t,t+\tau} = \left(\mathbb{E}_t^P \left[\frac{L_{t+\tau}}{L_t}\right] \mathbb{E}_t^P \left[\tau V_{t+\tau}^S\right] - \mathbb{E}_t^P \left[\frac{L_{t+\tau}}{L_t} \tau V_{t+\tau}^S\right] \right) = -\mathrm{Cov}_t \left(\frac{L_{t+\tau}}{L_t}, \tau V_{t+\tau}^S \right). \tag{13}$$

The expression for $(L_{t+\tau} - L_t)/L_t$ is given by:

$$\frac{L_{t+\tau} - L_t}{L_t} = -\gamma \sqrt{V_t} \Delta B_C + \left(1 - \frac{1}{\theta}\right) \left[b_X \sigma_X \sqrt{V_t} \Delta B_X + b_V \sigma_V \sqrt{V_t} \Delta B_V \right]. \tag{14}$$

where ΔB_C, ΔB_X, and ΔB_V are Brownian increments that are normally distributed with mean 0 and variance τ. It can be seen from equation (6) that this is the innovation to the pricing kernel.[13] Equation (13) therefore states that the time-t variance risk premium is the time-t covariance between the innovation to the stock variance ($\tau V_{t+\tau}^S - \mathbb{E}_t^P \left[\tau V_{t+\tau}^S\right]$) and the

[13] This result also follows from the fact that L_t is a martingale with $dL_t/L_{t^-} = d\pi_t/\pi_{t^-} + r_{f,t} dt$.

innovation to the pricing kernel.

This expression is analogous to the economic intuition behind the equity premium: the equity premium originates from the covariance between the innovations to the stock return and the pricing kernel. Equation (13) addresses the second moment, while the equity premium addresses the first moment.

The pricing kernel contains several shocks, but only ΔB_V is relevant to the variance premium. While the ΔB_C and ΔB_X shocks are uncorrelated with the innovation to the stock variance, ΔB_V is, by definition, the shock to the consumption variance V_t and is perfectly correlated with the shock to the stock variance V_t^S. Since the loading on this shock in equation (14) is $\left(1-\frac{1}{\theta}\right) b_V \sigma_V \sqrt{V_t}$, equation (13) becomes

$$VRP_{t,t+\tau} \quad = \quad -\left(1-\frac{1}{\theta}\right)\frac{b_V}{b_{\bar{\phi}V}}\mathrm{Cov}_t\left(b_{\bar{\phi}V}\sigma_V\sqrt{V_t}\Delta B_V, \tau V_{t+\tau}^S\right). \tag{15}$$

As can be seen from equation (8), the term $b_{\bar{\phi}V}\sigma_V\sqrt{V_t}\Delta B_V$ is the only part of the log stock price dynamics that is correlated with the shock to the stock variance. Thus, equation (15) can also be expressed as

$$VRP_{t,t+\tau} \quad = \quad -\left(1-\frac{1}{\theta}\right)\frac{b_V}{b_{\bar{\phi}V}}\mathrm{Cov}_t\left(\log S_{t+\tau}, \tau V_{t+\tau}^S\right), \tag{16}$$

where $\mathrm{Cov}_t\left(\log S_{t+\tau}, \tau V_{t+\tau}^S\right) = \mathrm{Cov}_t\left(\log\frac{S_{t+\tau}}{S_t}, \tau V_{t+\tau}^S\right)$ is the conditional covariance between the log stock return and stock variance, which we refer to as the leverage effect in the stock market. We denote this quantity as $LE_{t,t+\tau}$. Equation (16) shows that the variance risk premium is affine, i.e. linear with zero intercept, in the leverage effect.

Recall that this derivation assumes a short horizon τ and the absence of jumps. Section 6 and Appendix D show that even in a setup with jumps, we obtain a similar result for arbitrary τ; however, in this case the variance risk premium is linear in the leverage effect with a nonzero intercept

$$VRP_{t,t+\tau} \;=\; \alpha + \beta LE_{t,t+\tau}. \tag{17}$$

Equation (17) is the main relation analyzed in our empirical work.

3 Discussion

The model generates a nontrivial variance risk premium because shocks to volatility are priced. In Section 3.1, we emphasize the critical role of Epstein-Zin preferences, which allow for volatility risk to be priced in the model. In the resulting economy, there is an economically intuitive relation between the leverage effect and the variance risk premium due to the pricing of volatility risk. We explain the intuition behind this relation in Section 3.2. In Section 3.3, we discuss the relation between the variance risk premium, the leverage effect, and the variance of variance.

3.1 The Role of Recursive Preferences

The model economy uses the continuous-time formulation of Epstein-Zin recursive utility in Duffie and Epstein (1992). We adopt a continuous-time framework for analytical convenience. However, the use of non-time-separable recursive utility is critical because it generalizes time-additive power utility and allows for preferences over the timing of uncertainty resolution. The agent prefers early (late) resolution of uncertainty when ψ is greater (smaller) than $1/\gamma$.

To appreciate the importance of this model feature, consider the case of power utility, which means that the timing of uncertainty resolution is irrelevant. This special case can be obtained by setting $\psi = \frac{1}{\gamma}$. The state-price density in equation (6) then becomes

$$\frac{d\pi_t}{\pi_{t^-}} = -r_{f,t}dt - \gamma\sqrt{V_t}dB_{C,t}. \tag{18}$$

That is, shocks to the mean and the variance of consumption growth (i.e. $dB_{X,t}$ and $dB_{V,t}$) disappear because $\left(1-\frac{1}{\theta}\right)$ equals zero.[14]

Comparing equation (18) with equation (6) highlights the implications of the utility function for the variance risk premium. Shocks to the variance no longer enter the state-price density, and therefore also do not enter the pricing kernel in equation (14). In other words, the innovation to the pricing kernel is no longer correlated with the innovation to the variance, and the variance risk premium is zero.

Alternatively, this implication can be directly verified from the expression for the variance risk premium in equation (15): the right-hand side of equation (15) simply becomes zero under power utility because the coefficient $\left(1-\frac{1}{\theta}\right)$ is zero.

3.2 Why is the Relation Linear?

The definitions of the variance risk premium and the leverage effect provided in Section 2.4 provide intuition for why these two quantities are so closely connected. We have:

$$
\begin{aligned}
VRP_{t,t+\tau} &= -\mathrm{Cov}_t\left(\tau V^S_{t+\tau}, M_{t+\tau}\right) && (19)\\
LE_{t,t+\tau} &= \mathrm{Cov}_t\left(\tau V^S_{t+\tau}, r_{S,t+\tau}\right), && (20)
\end{aligned}
$$

where $M_{t+\tau} \equiv \frac{L_{t+\tau}}{L_t}$ is the pricing kernel and $r_{S,t+\tau} \equiv \log \frac{S_{t+\tau}}{S_t}$ is the log return on the aggregate market.

Equation (19) states that the variance risk premium is (the negative of) the covariance between the stock variance and the pricing kernel. In the model, the stock variance is a constant multiple of the consumption variance: $V^S_{t+\tau} = \xi V_{t+\tau}$. Thus, the innovation to the stock variance originates from the innovation to the consumption variance (i.e. $\sigma_V\sqrt{V_t}\Delta B_V$). By assuming Epstein-Zin preferences, such shocks are priced and thus appear in the pricing kernel with a loading of $\left(1-\frac{1}{\theta}\right)b_V$ as shown in equation (6). Typically, this loading is

[14] Since γ is equal to the inverse of ψ, the parameter $\theta = (1-\gamma)/(1-1/\psi)$ is one under power utility.

assumed to be positive so that the marginal utility of the representative agent rises when there is a positive volatility shock. This positive co-movement between the stock variance and the pricing kernel generates a negative variance risk premium.

Furthermore, the pricing of volatility risk generates the so-called volatility feedback effect, as documented, for example, by French, Schwert, and Stambaugh (1987), Campbell and Hentschel (1992), and Bekaert and Wu (2000). A positive shock to the pricing kernel originated from a positive volatility shock would lead to a higher required return on the aggregate market, which in turn causes a drop in the stock price. The stock variance therefore exhibits a negative co-movement with the stock return through this volatility feedback channel. In the model, equation (8) clarifies this relation. The loading of the stock return on the innovation to the variance is $b_{\bar{\phi}V}$, and this coefficient is typically calibrated to be negative (so the price-dividend ratio is low when V_t is high). This negative coefficient generates a negative leverage effect according to equation (20).

We conclude that the variance risk premium and the leverage effect arise from the same source: the pricing of volatility risk. The two quantities measure how the stock variance comoves with others; the former is with the pricing kernel while the latter is with the stock return. Since the pricing kernel and the stock return are both affected by shocks to variance under an economy where volatility risk is priced, it naturally follows that the variance risk premium and the leverage effect are closely associated.

In our model, these two quantities are linearly related. Note that the loadings of the pricing kernel and the stock return on the innovation to variance are $\left(1-\frac{1}{\theta}\right)b_V$ and $b_{\bar{\phi}V}$, respectively. Thus, the linear coefficient between the variance risk premium and the leverage effect becomes (the negative of) the ratio of these two loadings, namely $\left[-\left(1-\frac{1}{\theta}\right)\frac{b_V}{b_{\bar{\phi}V}}\right]$, as can be seen from equation (16).

It is worth noting the role of the log-linear approximation in demonstrating this linear relation. Under Epstein-Zin utility, a dynamic model does not permit an exact closed-form solution unless the EIS is one. Hence, in Section 2, we follow Campbell and Viceira (1999) and approximate the value function in closed form.[15] However, this does not imply that the relation between the variance risk premium and the leverage effect is an artifact of our model specification and solution method. The intuition from expressions (19) and (20) is not model-specific. The pricing of volatility risk ensures that when a volatility shock arrives, the pricing kernel in equation (19) and the aggregate market return in equation (20) comove in different directions. In the model, the relation between the variance risk premium and the leverage effect is linear under the log-linearization, which means that the leverage effect fully characterizes the variance risk premium. More generally, this relation may not be linear but could instead include some higher-order effects. For example, Kraus and Litzenberger (1976) and Dittmar (2002) examine nonlinear pricing kernels that depend on second- and third-degree polynomials of the market return.[16] In these more general setups, the linear relationship between the variance risk premium and the leverage effect continues to capture the first-order effect of the volatility feedback channel. We conclude that in general there is a strong relation between the variance risk premium and the leverage effect.

3.3 The Variance Risk Premium and the Variance of Variance

Economic intuition suggests that variance-of-variance risk is a major determinant of the variation in the variance risk premium, because the variance risk premium measures the compensation for the risk associated with changes in variance.[17] Is this intuition consistent with our findings regarding the variance risk premium and the leverage effect?

In Section 2, the consumption variance V_t follows a square root process. This assumption is made for parsimony because it ensures that the instantaneous variance of consumption variance, denoted as W_t, is completely determined by the consumption variance V_t itself.

[15] When the model is written in discrete-time, the log-linear approximation of Campbell and Shiller (1988) is necessary.

[16] Other examples include Barone-Adesi (1985) and Harvey and Siddique (2000).

[17] Another potential determinant of the time variation in the variance risk premium is time-varying jump risk (see, for example, Todorov, 2010). Section 6 extends the benchmark model in Section 2 by incorporating jump risk.

From equation (1), $W_t = \sigma_V^2 V_t$, which implies that variance risk and variance-of-variance risk are perfectly correlated in the model. To further explore the relation among variance-of-variance risk, the variance risk premium, and the leverage effect, we therefore use the extension of the volatility dynamics, following Bollerslev, Tauchen, and Zhou (2009), Bollerslev, Sizova, and Tauchen (2012), and Tauchen (2012):

$$\begin{aligned} dV_t &= \kappa_V(\bar{V} - V_t)dt + \sqrt{W_t}dB_{V,t} \\ dW_t &= \kappa_W(\bar{W} - W_t)dt + \sigma_W\sqrt{W_t}dB_{W,t}, \end{aligned}$$

where $B_{W,t}$ is an independent standard Brownian motion.

The implications of this model extension for the aggregate stock index are straightforward. First, the price-dividend ratio becomes a function of W_t as well as X_t and V_t. Reflecting this additional source of aggregate market volatility, the stock price dynamics follow

$$\frac{dS_t}{S_{t^-}} = \mu_{S,t}dt + \phi\sqrt{V_t}dB_{C,t} + b_{\bar{\phi}X}\sigma_X\sqrt{V_t}dB_{X,t} + b_{\bar{\phi}V}\sqrt{W_t}dB_{V,t} + b_{\bar{\phi}W}\sigma_W\sqrt{W_t}dB_{W,t}.$$

The coefficient $b_{\bar{\phi}W}$ refers to the loading of the log price-dividend ratio on the variance-of-variance process. Consequently, the instantaneous stock market variance is expressed as $V_t^S = \xi_V V_t + \xi_W W_t$, where $\xi_V = \left(\phi^2 + b_{\bar{\phi}X}^2\sigma_X^2\right)$ and $\xi_W = \left(b_{\bar{\phi}V}^2 + b_{\bar{\phi}W}^2\sigma_W^2\right)$. It follows from the definition of the (annualized) leverage effect that

$$\frac{1}{\tau}\left[LE_{t,t+\tau}\right] = \mathrm{Cov}_t\left(r_{S,t+\tau}, V_{t+\tau}^S\right) = \tau\left[b_{\bar{\phi}V}\xi_V + b_{\bar{\phi}W}\xi_W\sigma_W^2\right]W_t. \tag{21}$$

Under Epstein-Zin preferences, shocks to the new state variable W_t are also priced. The pricing of variance-of-variance risk yields an additional term in the expression of the state-price density:

$$\frac{d\pi_t}{\pi_{t^-}} = -r_{f,t}dt - \gamma\sqrt{V_t}dB_{C,t} + \left(1 - \frac{1}{\theta}\right)\left[b_X\sigma_X\sqrt{V_t}dB_{X,t} + b_V\sqrt{W_t}dB_{V,t} + b_W\sigma_W\sqrt{W_t}dB_{W,t}\right],$$

where b_W determines the market price of variance-of-variance risk. From equation (13), the (annualized) variance risk premium is derived as

$$\frac{1}{\tau}\left[VRP_{t,t+\tau}\right] = -\mathrm{Cov}_t\left(M_{t+\tau}, V_{t+\tau}^S\right) = -\tau\left(1-\frac{1}{\theta}\right)\left[b_V\xi_V + b_W\xi_W\sigma_W^2\right]W_t. \tag{22}$$

Expressions (21) and (22) highlight the added intuition from the extended model. The time variation in the variance risk premium and the leverage effect are entirely driven by variance-of-variance risk (W_t), not variance risk (V_t). Note that in Section 2, variance-of-variance risk is perfectly correlated with variance risk; by replacing the W_t terms in equations (21) and (22) with $\sigma_V^2 V_t$, we indeed obtain the results for the benchmark model.[18]

We can also express the variance risk premium and the leverage effect in terms of the variance of variance over horizon τ, which is empirically more relevant than the instantaneous variable W_t. The τ-horizon variance of market variance is given by $\mathrm{Var}_t\left(V_{t+\tau}^S\right) = \tau\left(\xi_V^2 + \xi_W^2\sigma_W^2\right)W_t$, which implies that equations (21) and (22) can be written as

$$\begin{aligned}\frac{1}{\tau}\left[LE_{t,t+\tau}\right] &= \left[\frac{b_{\bar{\phi}V}\xi_V + b_{\bar{\phi}W}\xi_W\sigma_W^2}{\xi_V^2+\xi_W^2\sigma_W^2}\right]\mathrm{Var}_t\left(V_{t+\tau}^S\right)\\ \frac{1}{\tau}\left[VRP_{t,t+\tau}\right] &= -\left(1-\frac{1}{\theta}\right)\left[\frac{b_V\xi_V + b_W\xi_W\sigma_W^2}{\xi_V^2+\xi_W^2\sigma_W^2}\right]\mathrm{Var}_t\left(V_{t+\tau}^S\right).\end{aligned} \tag{23}$$

The coefficients $b_{\bar{\phi}V}$ and $\left[-\left(1-\frac{1}{\theta}\right)b_V\right]$ are typically assumed to be negative, as discussed in Section 3.2. Similarly, $b_{\bar{\phi}W}$ and $\left[-\left(1-\frac{1}{\theta}\right)b_W\right]$ are usually calibrated to be negative so the price-dividend ratio falls and the marginal utility rises when variance-of-variance risk goes up. Since ξ_V and ξ_W are positive, the variance risk premium and the leverage effect are both negatively associated with the variance of market variance: when we expect a higher variance of market variance, the variance risk premium and the leverage effect become more negative.

Equation (23) indicates that the variance risk premium is not only linear in the leverage

[18] In the benchmark model, ξ_V corresponds to ξ while ξ_W, $b_{\bar{\phi}W}$, and b_W are all zero.

effect, but also in the variance of market variance. This suggests an alternative way to characterize the variance risk premium based on the variance of market variance. In fact, the variance risk premium, the leverage effect, and the variance of market variance are all perfectly correlated with one another in the model. Hence, we can equivalently think of the variance of market variance as generating the variance risk premium.

However, potential distinctions between the leverage effect and the variance of variance surface when the variance dynamics are subject to shocks that are uncorrelated with the pricing kernel. Estimates of the variance may be noisy, and this noise is by definition uninformative about investors' marginal utility. Furthermore, models such as Johnson (2004) and Johnson and Lee (2014) emphasize the role of unpriced uncertainty shocks in explaining various aspects of market data.[19]

In such circumstances, the leverage effect may be more informative about the variance risk premium than the variance of market variance. Recall that both the variance risk premium and the leverage effect are generated through the pricing of volatility risk. This channel remains in the presence of unpriced volatility shocks. On the other hand, the variance risk premium and the variance of market variance may differ if some volatility shocks are not priced. Variance of variance captures every shock to the variance process regardless of its implications for the pricing kernel. In summary, the leverage effect may characterize the variance risk premium in a more robust way because both the quantities reflect the pricing of volatility shocks.

4 Empirical Results

In this section, we construct time series for the variance risk premium and the leverage effect for the period January 1990 through December 2016, and investigate the empirical relation

[19]Meanwhile, Dew-Becker, Giglio, Le, and Rodriguez (2017), Dew-Becker, Giglio, and Kelly (2017), and Berger, Dew-Becker, and Giglio (2018) consistently find from various derivatives data that shocks to realized volatility are significantly priced whereas shocks to expected future volatility are not.

between the two time series. Both time series are available at the monthly frequency; for notational convenience, we use the time subscript $t+1$ to denote one month from time t.

The variance risk premium is typically measured over one month. We denote it by $VRP_{t,t+1}$ for month t. We also calculate the leverage effect over a one-month period, from t to $t+1$, $LE_{t,t+1}$. Since there is no ambiguity with respect to the horizon, we simply denote these two quantities as VRP_t and LE_t in this section.

4.1 Measuring the Variance Risk Premium and the Leverage Effect

To calculate the variance risk premium, we need both the physical and risk-neutral expectations of the future variance, as indicated in equation (11). For the risk-neutral component, for each month t we take the closing value of the VIX index squared on the last trading day of the month.[20] The squared VIX, which represents the risk-neutral expectation of the future variance over the subsequent 30 days, is calculated in a model-free way using out-of-the-money call and put options.[21] Since the VIX index is expressed in annualized percentage volatility terms, we divide the squared VIX by 12 to express it in conventional units as squared monthly percentages. Denoting this scaled quantity as VIX_t^2, the variance risk premium can be expressed as

$$VRP_t = \mathbb{E}_t^P\left[RV_{t,t+1}\right] - VIX_t^2, \tag{24}$$

where $RV_{t,t+1}$ is the realized variance between time t and $t+1$.

As demonstrated in the literature, estimating the realized variance based on high-frequency

[20]The time series of the VIX is obtained from the Chicago Board Options Exchange (CBOE) website.

[21]The CBOE developed the first-ever volatility index in 1993, which was based on Black-Scholes implied volatilities from at-the-money S&P 100 index options. In 2003, the CBOE modified the methodology and started publishing a new VIX index, calculated in a model-free manner based on prices of S&P 500 index options. In this paper, we use the new VIX. For more details on the model-free approach, see Dupire (1994), Neuberger (1994), Britten-Jones and Neuberger (2000), and Jiang and Tian (2005).

data provides accurate ex-post measures.[22] We obtain the high-frequency S&P 500 index time series from TICKDATA. While it is possible to further increase our sampling frequency, we follow the standard approach and sample every five minutes to avoid potential microstructure effects. Liu, Patton, and Sheppard (2015) argue that it is difficult to outperform five-minute realized variance even with more sophisticated sampling techniques.

The calculation of the physical expectation proceeds in two steps. First, we calculate the realized variance $RV_{t,t+1}$ by summing up the squared five-minute log returns on the S&P 500 index over the month $t+1$.[23] We multiply the outcome by 10,000 in order to express realized variance in monthly percentage squared terms. Panel A of Figure 1 presents the resulting time series from January 1990 to December 2016.

In the second step, we estimate the time-t ex-ante physical variance in equation (24) by projecting the realized variance between t and $t+1$ onto the predictor variables at time t. Specifically, following Drechsler and Yaron (2011), we consider the following linear projection:

$$\log RV_{t,t+1} = \delta_0 + \delta_1 \log RV_{t-1,t} + \delta_2 \log VIX_t^2 + \epsilon. \tag{25}$$

We use the logarithm of the variance to ensure that variance forecasts always remain positive.[24] Table 1 shows the results from this regression. While the setup is parsimonious, its predictive power is similar to that of other variance forecasting models, generating an adjusted R^2 of 68%. We consider robustness with respect to the variance forecasting model in Section 5.1.

The proxy for the time series of $\mathbb{E}_t^P\left[RV_{t,t+1}\right]$, which we refer to as ERV_t, is obtained using

[22]See Andersen, Bollerslev, Diebold, and Ebens (2001), Andersen, Bollerslev, Diebold, and Labys (2003), and Barndorff-Nielsen and Shephard (2002).

[23]For any two consecutive trading days, we calculate the overnight/weekend return and treat it as an additional five-minute interval of the latter trading day.

[24]An additional reason to use logarithms is that the distribution of the variance variables is much closer to a log-normal distribution than a normal distribution.

the fitted values of the regression:

$$ERV_t = \mathbb{E}_t^P \left[RV_{t,t+1} \right] \simeq \exp \left(\delta_0 + \delta_1 \log RV_{t-1,t} + \delta_2 \log VIX_t^2 + \frac{1}{2}\sigma_\epsilon^2 \right). \tag{26}$$

Panel B of Figure 1 shows the time series of ex-ante physical variance estimates together with that of the squared VIX. Following equation (24), the variance risk premium is measured as the difference between these two time series, depicted in Panel C.

Now we turn to the leverage effect. We define $COV_{t,t+1}$ as the realized covariance between the return on the S&P 500 index and its conditional variance over the month $t+1$. For each month, we estimate $COV_{t,t+1}$ using a sample covariance between daily log returns and daily changes in the conditional variance.[25] We estimate the daily time series of conditional variances using the EGARCH(1,1) model of Nelson (1991), which incorporates the asymmetric response of volatility to positive and negative returns. In Section 5.2 below, we investigate if our empirical results are robust to different computations of the leverage effect.

The leverage effect at time t, which we denote as LE_t, is calculated as the time-t physical expectation of $COV_{t,t+1}$. We estimate LE_t using the following linear projection:

$$COV_{t,t+1} = \delta_0 + \delta_1 RV_{t-1,t} + \delta_2 COV_{t-1,t} + \epsilon. \tag{27}$$

In contrast to equation (25), we do not take logarithms of the covariances because realized covariance can take on negative as well as positive values. The leverage effect is computed as the fitted value of the regression:

$$LE_t = \mathbb{E}_t^P \left[COV_{t,t+1} \right] \simeq \delta_0 + \delta_1 RV_{t-1,t} + \delta_2 COV_{t-1,t}.$$

Panel B of Figure 2 shows the resulting time series of the leverage effect. Note again that

[25] In general, calculating realized covariance using high-frequency data is challenging due to asynchronicity (Epps, 1979; Sheppard, 2006). Furthermore, Aït-Sahalia, Fan, and Li (2013) show that leverage effect estimates based on high-frequency data are biased towards zero.

the leverage effect is defined in terms of covariance, not correlation.

Panel A of Table 2 reports summary statistics for the variables used in the empirical analysis: RV (realized variance), ERV (ex-ante physical variance), VIX^2 (squared VIX), VRP (variance risk premium), COV (realized covariance), and LE (leverage effect). We also include the monthly returns on the S&P 500 index, which we denote as SP.

The sample mean of RV is 19.64. The sample distribution of RV is far from a normal distribution: it is highly skewed to the right, which makes the median value (10.89) much lower than the mean. Furthermore, the distribution is extremely fat-tailed, exhibiting a sample kurtosis of 95.58. Lastly, RV has an AR(1) coefficient of 0.65, implying that it is a relatively quickly mean-reverting process.

ERV, the ex-ante physical expectation of future realized variance, can be viewed as a smoothed version of RV because it is obtained by projecting RV of each month onto predictor variables. ERV is much more persistent, having a monthly AR(1) coefficient of 0.77, and has a skewness (5.58) and kurtosis (47.22) that are much smaller than those of RV. While the general magnitude of ERV is similar to RV, the mean is slightly smaller and the median is slightly larger, because the distribution is less skewed.

The squared VIX, which represents the ex-ante risk-neutral expectation of future realized variance, is on average higher than its physical counterpart (37.01 vs 18.62). This implies that the variance risk premium is negative on average (-18.39). In fact, the squared VIX is always larger than the ex-ante physical variance in our sample, which makes the variance risk premium consistently negative, as evident from Panel C of Figure 1.

Similarly, COV is consistently negative in the sample: the mean value is -16.94 and it is always negative as illustrated in Figure 2. This is intuitive given that it is a proxy for the realized covariance between log returns and conditional return variances. LE, the leverage effect, is obtained as a smoothed version of COV. As we can see in Figure 2, the peak of LE in October 2008 is much smaller compared to that of COV. While the standard deviation of LE is smaller compared to COV, other sample statistics are fairly close. For example, the AR(1) coefficients of both variables are 0.77 and 0.78 respectively.

Panel B of Table 2 reports the correlation coefficients among these variables. Evidently, RV and ERV as well as COV and LE are highly correlated (0.97 and 0.99 respectively). ERV and VIX^2 are also highly positively correlated (0.95) because they are both ex-ante expectations of future realized variance under two different measures. The fact that VRP is negatively correlated with both ERV and VIX^2 implies that the variance risk premium becomes larger in magnitude (i.e. more negative) during "bad times" when the volatility of the market is high. This is consistent with a positive correlation between VRP and SP, which implies that the variance risk premium becomes smaller in magnitude (i.e. less negative) during "good times" when the market returns are high.

Importantly, the variance risk premium is significantly and positively correlated with the leverage effect: the correlation coefficient is 0.46 and in several subsamples this correlation is even higher. Motivated by this correlation (and our theoretical framework), we investigate the linear relationship between these two variables more closely in the next section.

4.2 Testing the Linear Relation Between the VRP and the Leverage Effect

Based on the time series of the variance risk premium and leverage effect as discussed in Section 4.1, we empirically examine the theoretical relation derived in equation (17). Specifically, we run the following regression:

$$VRP_t = \alpha + \beta LE_t + \epsilon_t. \tag{28}$$

Table 3 displays the outcomes of this regression for various sample periods. As illustrated in the first panel, the variance risk premium and the leverage effect are statistically significantly positively related in the January 1990 through December 2016 sample. The estimated slope coefficient β is 0.15 with a Newey-West (1987) t-statistic of 3.27.[26] The leverage effect explains a sizable portion of the variation in the variance risk premium, generating an adjusted R^2 of 21%. The positive relationship implies that times when the variance risk premium is high (so-called "bad times") tend to coincide with periods with a high covariance between the market return and the conditional market variance.

The positive linear relationship between the variance risk premium and leverage effect is robust to the exclusion of the Great Recession period (December 2007 to June 2009). The slope coefficient increases (0.43) and is still statistically significant with a Newey-West t-statistic of 2.65, as can be seen in the second row. The last two rows of Table 3 show the results from the regression with the two subperiods that are of the same length: the first half of the sample is from January 1990 to June 2003 and the second half is from July 2003 to December 2016. While both samples yield significantly positive slope coefficients, the estimates are somewhat different: the estimate for the first period is 0.58, which is larger than the 0.13 estimate for the second period. The main reason for this difference seems to be the Great Recession. Table 3 indicates that the estimates for the first half of the sample and the sample without the Great Recession are similar, presumably because neither sample contains the Great Recession period. Moreover, the second half of the sample, which includes the Great Recession period, gives an estimate that is closer to the full sample result. The leverage effect exhibits an extremely high peak in October 2008, which leads to a smaller estimate for the slope coefficient. The explanatory power of the leverage effect increases after splitting the sample into two. Specifically, the values for the adjusted R^2 in the first and second subsamples are 37% and 28% respectively.

The positive relationship between the variance risk premium and the leverage effect is expected through the lens of an Epstein-Zin asset pricing model. Equation (16) indicates that the variance risk premium is positively associated with the leverage effect when the coefficient $\left[-\left(1-\frac{1}{\theta}\right)\frac{b_V}{b_{\phi V}}\right]$ is positive. This sign is consistent with economic intuition: a

[26]The lag selection follows Newey and West (1994).

positive $\left(1-\frac{1}{\theta}\right) b_V$ means that the marginal utility of the agent rises when V_t is hit by a positive shock, i.e. during “bad times”, as evident from equation (6). Moreover, $b_{\phi V}$ is typically determined to be negative because the price-dividend ratio falls when V_t goes up. See equation (C.2). Thus, the negative of the ratio between $\left(1-\frac{1}{\theta}\right) b_V$ and $b_{\phi V}$ is expected to be positive in this model.

4.3 The Variance Risk Premium 1926-2016

The construction of the variance risk premium requires an estimate of the expected future market variance under the risk-neutral measure. To obtain this risk-neutral expectation, a large cross section of option prices is required: the squared VIX, which we use as a proxy for this very quantity for a one-month horizon, is calculated using a range of out-of-the-money call and put option prices. The available time series of the variance risk premium is therefore rather limited due to the limited availability of options data. Existing studies of the variance risk premium focus on a relatively short sample starting in 1990.

Our analysis suggests an alternative approach for constructing a longer time series of the variance risk premium. Section 2 demonstrates that in a fairly general setup, the variance risk premium is a linear function of the leverage effect, and this linear relationship is confirmed in Section 4 using various samples for which both variables are available. Note that the computation of the leverage effect does not require options data. Our estimate of the leverage effect can be constructed when daily market returns are available. Thus, we can construct extended time series of the leverage effect, which in turn can be used to characterize the variance risk premium over a much longer sample period, based on an estimate of the relation between the leverage effect and the variance risk premium.

We proceed by estimating the leverage effect starting in January 1926. Daily return data on the S&P 500 index are available starting in July 1962, but we use the CRSP value-weighted index as a proxy for the S&P 500 index prior to that. Using this extended time series of returns, we follow the procedure described in Section 4.1: we run the EGARCH(1,1) model to estimate the daily time series of conditional market variances and calculate the realized covariance (COV) for each month as the sample covariance between daily log returns and daily changes in conditional return variances. We then obtain the leverage effect (LE) by projecting realized covariance onto lagged realized covariance and realized variance.

We also combine two data sources to estimate the realized variance. High-frequency index data are available starting in October 1985, and we compute the realized variance for subsequent months as the summation of the squared five-minute index returns over each month.[27] Prior to October 1985, we compute the realized variance using the French, Schwert, and Stambaugh (1987) measure, which is the sum of the squared daily log returns over each month, adjusted for autocorrelations.

Figure 3 shows the extended time series of the variance risk premium, extrapolated based on the leverage effect. For each month, the variance risk premium VRP_t is calculated as $\alpha + \beta LE_t$, where α and β are estimated from the sample that excludes the Great Recession in the second panel of Table 3. We use these estimates rather than those from the full sample because accurately estimating the variance risk premium and the leverage effect during the Great Recession is more challenging. Estimates of the slope coefficient are sensitive to extremely high values of the leverage effect during this period, as discussed in Section 4.2. Figure 3 indicates that the variance risk premium is consistently negative and highly time-varying in the long sample. Among occasional downward spikes, a few are noticeable: October 1929 (Black Tuesday or the so-called Great Crash), July 1933 (the depth of the Great Depression), October 1987 (Black Monday), and October 2008 (the Great Recession). The pattern of the variance risk premium in the 1930s is very distinct: not only is the level extremely high, it also shows a lot of variation, reflecting the bad economic climate during the Great Depression. This period is followed by a few decades of economic recovery after the Second World War where the variance risk premium remains at a low level and

[27] TICKDATA provides high-frequency index data starting in February 1982, but we found that the data prior to October 1985 have different trading hours and occasionally contain zero prices. We therefore use the data starting in October 1985 to avoid biases in the computation.

exhibits limited variation. In the extended period, the variance risk premium has a sample mean of -17.55 and a standard deviation of 8.70.

5 Robustness

We investigate the robustness of the empirical findings in Section 4. We show that the positive relationship between the variance risk premium and the leverage effect is robust to alternative measurements of these two variables.

5.1 Alternative Measures of the Variance Risk Premium

Recall that the variance risk premium is defined as the difference between the physical and risk-neutral expectations of future market variance. While the risk-neutral expectation can be simply captured by the squared VIX, estimating the corresponding physical counterpart is generally less straightforward. In our benchmark analysis, we follow Drechsler and Yaron (2011) and calculate this physical expectation by projecting realized variance onto lagged realized variance and the squared VIX. In this section, we consider three alternative projections.

The variance risk premium in Panel A of Figure 4 is constructed under the assumption that the realized variance process follows a random walk. For each month t, the ex-ante physical variance is simply $\mathbb{E}_t^P [RV_{t,t+1}] = RV_{t-1,t}$. This approach, first adopted by Bollerslev, Tauchen, and Zhou (2009), is popular in the literature partly due to its simplicity. However, this assumption is somewhat tenuous and it may bias the estimate of the variance risk premium, especially during highly volatile periods. In fact, Panel A shows that the variance risk premium has a massive positive spike in October 2008 during the Great Recession. This is puzzling because we generally expect the variance risk premium to be significantly negative during bad economic times. Although realized variance was extremely high during October 2008, it is possible that the market did not expect high volatility in the subsequent month.

This suggests that at the end of October 2008, the random walk model might overestimate the expected physical variance, and therefore, the overall variance risk premium.

Panel B of Figure 4 follows Zhou (2018) and calculates the expected physical variance by assuming that realized variance follows an autoregressive model with 12 lags (AR(12)). The autoregressive coefficients are estimated based on the full sample period, January 1990 to December 2016.[28] Although the time series in Panel B still exhibits some positive spikes during the Great Recession, their magnitudes are moderate compared to those in Panel A. Furthermore, in contrast to Panel A, the variance risk premium remains negative most of the time.

Panel C of Figure 4 considers the variance forecasting model of Bekaert and Hoerova (2014) using lagged realized variances over different horizons together with the squared VIX as the predictor variables:

$$\log RV_{t,t+1} = \delta_0 + \delta_1 \log RV_{t-1,t} + \delta_2 \log RV_t^W + \delta_3 \log RV_t^D + \delta_4 \log VIX_t^2 + \epsilon$$

where $RV_{t-1,t}$ is the monthly realized variance over the month, and RV_t^D and RV_t^W denote the realized variances over the last day and last week, respectively. The expectation of the future realized variance is calculated as

$$\mathbb{E}_t^P \left[RV_{t,t+1}\right] = \exp\left(\delta_0 + \delta_1 \log RV_{t-1,t} + \delta_2 \log RV_t^W + \delta_3 \log RV_t^D + \delta_4 \log VIX_t^2 + \frac{1}{2}\sigma_\epsilon^2\right).$$

The resulting time series of the variance risk premium is consistently negative in the sample and appears to have a low-frequency pattern that is similar to our benchmark measure.

Panel A of Table 4 examines whether the positive relationship between the variance risk premium and leverage effect is sensitive to the choice of variance risk premium measures. Rows (1), (2), and (3) correspond to the models of Bollerslev, Tauchen, and Zhou (2009), Zhou (2018), and Bekaert and Hoerova (2014), respectively. We report results for the full

[28] The time series of the resulting variance risk premium can be downloaded from Hao Zhou's website.

sample period as well as for the sample excluding the Great Recession. When using the first alternative measure with the random walk assumption, the slope coefficient is estimated to be negative but it is statistically insignificant. The reason is that this variance risk premium measure exhibits an extremely large positive spike during the Great Recession. When we run the same regression using the sample without the Great Recession, the estimate of the slope coefficient is positive. Consistent with our benchmark results, the other two measures based on Zhou (2018) and Bekaert and Hoerova (2014) always yield significantly positive slope coefficients regardless of the sample period, demonstrating the robustness of the positive relationship between the variance risk premium and the leverage effect.

5.2 Alternative Measures of the Leverage Effect

In this section, we construct three alternative measures of the leverage effect. These three measures as well as our benchmark measure in Section 4.1 differ with respect to the computation of the realized covariance COV. We use the same projection regression with lagged realized variance and realized covariance in equation (27) for all four measures. Unlike the expected physical variance, the leverage effect is relatively less sensitive to different projection specifications. Recall that our benchmark measure estimates COV using the sample covariance between daily log returns and daily changes in conditional variances estimated from the EGARCH(1,1) model.

The computation of COV in the first alternative measure is similar to our benchmark implementation in that we compute realized covariance between returns and changes in conditional variance. However, instead of using EGARCH(1,1), here we obtain estimates of daily conditional variance by applying the variance forecasting model in equation (25) at the daily frequency. The resulting time series of the leverage effect is displayed in Panel A of Figure 5.

The second alternative measure simply uses changes in daily realized variances (rather than changes in the conditional expectations) when computing the sample covariance with daily log returns. For each month, the realized covariance COV is estimated by the sample covariance between daily log returns and changes in daily realized variances over the month. Panel B of Figure 5 shows the time series of the leverage effect based on this definition.

Lastly, the third alternative measure considers the sample covariance between daily log returns and daily variance innovations. For each trading day, we compute the variance innovation as the realized variance for that day minus the expected one-day ahead realized variance (i.e. the one-day ahead conditional variance) at the end of the previous trading day. The leverage effect based on this alternative approach is plotted in Panel C of Figure 5.

Panel B of Table 4 reports the results from regressing the benchmark measure of the variance risk premium on the three alternative measures of the leverage effect, not only for the full sample but also for the sample without the Great Recession. Regardless of how we measure the leverage effect, the estimated slope coefficients are consistently positive and statistically significant. These results demonstrate that the positive relationship between the variance risk premium and the leverage effect is robust to various measurements of the leverage effect.

6 Generalizing the Model

In Section 2, we use a stylized model in order to highlight the mechanics underlying the relation between the variance risk premium and the leverage effect. However, this model is not sufficiently rich to explain several important stylized facts of the variance risk premium (see, for example, Drechsler and Yaron, 2011). We now discuss a more general model that allows for jump risk in the level of aggregate consumption as well as in its mean and variance. The model in this section nests the model in Section 2. We demonstrate that this extension does not change the paper's main conclusion, the linear relationship between the variance risk premium and the leverage effect, and that our empirical analysis accounts for the presence of jumps.

We now assume that aggregate consumption follows an affine jump diffusion process

$$\frac{dC_t}{C_{t^-}} = (\mu_C + X_t)dt + \sqrt{V_t}dB_{C,t} + (e^{Z_{C,t}} - 1)dN_t,$$

where N_t is a Poisson process with time-varying intensity λ_t. The drift and the variance of consumption growth are also subject to jumps:

$$\begin{aligned} dX_t &= -\kappa_X X_t dt + \sigma_X \sqrt{V_t} dB_{X,t} + Z_{X,t} dN_t \\ dV_t &= \kappa_V(\bar{V} - V_t)dt + \sigma_V \sqrt{V_t} dB_{V,t} + Z_{V,t} dN_t. \end{aligned}$$

For notational convenience, we define the vector of jump size distributions as $Z_t = [Z_{C,t}, Z_{X,t}, Z_{V,t}]^\top$ whose time-invariant distribution is independent of N_t and the Brownian motions.[29] This distribution is characterized by the moment generating function of Z_t, which we denote as $\Phi_Z(u) = \mathbb{E}^P\left[e^{u^\top Z}\right]$.

This more general setup includes a wide range of existing consumption-based asset pricing models that aim to explain the high equity premium and high stock market volatility in the post-war data. First, the benchmark model in Section 2 is obtained by setting $\lambda_t \equiv 0$. In this case, with highly persistent mean and volatility components and a representative agent who prefers early resolution of uncertainty, our setup reduces to the long-run risk model of Bansal and Yaron (2004). If we allow for jumps in the state variables (except for consumption growth), we obtain a model that is close to Drechsler and Yaron (2011). Second, note that different calibrations and parameter values can result in very different models: assuming large but rare jumps in consumption results in rare disaster models such as Barro (2006) and Rietz (1988) (who assume constant jump intensity) as well as Wachter (2013) (who assumes time-varying jump intensity). Our objective is to make a general statement about the relationship between the variance risk premium and the leverage effect; therefore we do not impose any restrictions on the model parameters and jump size distributions.

[29] Since Z_t has a time-invariant distribution, we omit the subscript t when it is not necessary.

Based on the empirical observation that periods with more frequent jumps usually coincide with higher volatility, it is reasonable to model time-varying jump risk λ_t as a function of the stochastic variance V_t. For parsimony, we assume $\lambda_t = \rho_0 + \rho_V V_t$. This simple approach is fairly standard in the option pricing literature because it has been demonstrated that separately identifying jump risk and volatility risk can be challenging.[30]

In Appendix A, we show that the value function J_t has the same functional form as equation (4). However, the stochastic differential equation for the state-price density π_t now takes an additional jump term:

$$\begin{aligned}\frac{d\pi_t}{\pi_{t^-}} = &\Big[-r_{f,t} - \lambda_t\left(\Phi_Z(\eta_\pi) - 1\right)\Big]dt - \gamma\sqrt{V_t}dB_{C,t} \\ &+ \left(1-\frac{1}{\theta}\right)\Big[b_X\sigma_X\sqrt{V_t}dB_{X,t} + b_V\sigma_V\sqrt{V_t}dB_{V,t}\Big] + \left(e^{\eta_\pi^\top Z_t} - 1\right)dN_t, \quad (29)\end{aligned}$$

where $\eta_\pi = [-\gamma, \left(1-\frac{1}{\theta}\right)b_X, \left(1-\frac{1}{\theta}\right)b_V]^\top$. Appendix B provides the details.

The process for the aggregate dividend D_t is also extended to incorporate jump risk:

$$\frac{dD_t}{D_{t^-}} = \mu_{D,t}dt + \phi\sqrt{V_t}dB_{C,t} + \left(e^{\phi Z_{C,t}} - 1\right)dN_t.$$

Based on the new dividend process, Appendix C derives the dynamics for the stock price:

$$\frac{dS_t}{S_{t^-}} = \mu_{S,t}dt + \phi\sqrt{V_t}dB_{C,t} + b_{\bar{\phi}X}\sigma_X\sqrt{V_t}dB_{X,t} + b_{\bar{\phi}V}\sigma_V\sqrt{V_t}dB_{V,t} + \left(e^{\eta_{\bar{\phi}}^\top Z_t} - 1\right)dN_t,$$

where $\eta_{\bar{\phi}} = [\phi, b_{\bar{\phi}X}, b_{\bar{\phi}V}]^\top$. Similarly, the risk-neutral dynamics of the stock price is given by

$$\frac{dS_t}{S_{t^-}} = \mu^Q_{S,t}dt + \phi\sqrt{V_t}dB^Q_{C,t} + b_{\bar{\phi}X}\sigma_X\sqrt{V_t}dB^Q_{X,t} + b_{\bar{\phi}V}\sigma_V\sqrt{V_t}dB^Q_{V,t} + \left(e^{\eta_{\bar{\phi}}^\top Z_t} - 1\right)dN_t,$$

where $\mu^Q_{S,t} = r_{f,t} - G_t^{-1} - \lambda_t^Q\mathbb{E}^Q\left[e^{\eta_{\bar{\phi}}^\top Z_t} - 1\right]$. Under the risk-neutral measure, λ_t is no longer

[30] See, for example, Bates (2000), Pan (2002), Eraker, Johannes, and Polson (2003), Eraker (2004), Broadie, Chernov, and Johannes (2007), Christoffersen, Jacobs, and Ornthanalai (2012), and Andersen, Fusari, and Todorov (2015).

the intensity process for N_t; instead, we have $\lambda_t^Q = \lambda_t \Phi_Z(\eta_\pi)$. The jump intensity remains linear in V_t under the risk-neutral measure

$$\lambda_t^Q = \rho_0^Q + \rho_V^Q V_t \tag{30}$$

with different coefficients $\rho_0^Q = \rho_0 \Phi_Z(\eta_\pi)$ and $\rho_V^Q = \rho_V \Phi_Z(\eta_\pi)$. Furthermore, the jump distribution changes as well. The moment generating function of Z_t under the risk-neutral measure is given by

$$\mathbb{E}^Q \left[e^{u^\top Z} \right] = \Phi_Z^Q(u) = \frac{\Phi_Z(u + \eta_\pi)}{\Phi_Z(\eta_\pi)}. \tag{31}$$

In the extended model, the variance risk premium is not only affected by the diffusive component of the log stock return, but also by the jump component. For notational convenience, let $Z_{S,t} = \eta_\phi^\top Z_t$ denote the jump size distribution for the log stock price. Since the quadratic variation is expressed as

$$QV_{t,t+\tau} = \int_t^{t+\tau} d\left[\log S, \log S\right]_s = \xi \int_t^{t+\tau} V_s ds + \int_t^{t+\tau} Z_{S,s}^2 dN_s, \tag{32}$$

the variance risk premium consists of two distinctive parts:

$$\begin{aligned} VRP_{t,t+\tau} = \xi & \left(\mathbb{E}_t^P \left[\mathbb{V}_{t,t+\tau} \right] - \mathbb{E}_t^Q \left[\mathbb{V}_{t,t+\tau} \right] \right) \\ & + \left(\mathbb{E}^P \left[Z_S^2 \right] \mathbb{E}_t^P \left[\rho_0 + \rho_V \mathbb{V}_{t,t+\tau} \right] - \mathbb{E}^Q \left[Z_S^2 \right] \mathbb{E}_t^Q \left[\rho_0^Q + \rho_V^Q \mathbb{V}_{t,t+\tau} \right] \right). \end{aligned} \tag{33}$$

Specifically, the second line of equation (33) shows how distinct jump intensities and size distributions under the two measures affect the variance risk premium. In Appendix D, we show that the variance risk premium is linear in V_t because the conditional expectation of $\mathbb{V}_{t,t+\tau}$ is linear in V_t regardless of the measure.

The leverage effect is calculated as the time-t conditional covariance between the log stock return and stock variance:

$$LE_{t,t+\tau} = \mathrm{Cov}_t\left(\log\frac{S_{t+\tau}}{S_t}, \tau V^S_{t+\tau}\right) = \mathbb{E}^P_t\left[\int_t^{t+\tau} d\left[\log S, \tau V^S\right]_s\right].$$

Appendix E demonstrates that this quantity is linear in V_t as well.

In sum, for any arbitrary τ, both the variance risk premium and the leverage effect are linear in V_t. This implies that there exist coefficients α and β such that $VRP_{t,t+\tau} = \alpha + \beta LE_{t,t+\tau}$. In other words, the variance risk premium and the leverage effect exhibit a linear relationship in the model.

In our theoretical analysis, the relation between the variance risk premium and the leverage effect is affine with a zero intercept if the state variables do not exhibit jumps. Table 3 shows that the estimates of the intercept α are between -12 and -16, and are statistically significant in various sample periods. This finding is consistent with the stylized fact that jump components are crucial to explaining the dynamics of the variance risk premium.

7 Conclusion

A risk factor pays a premium when it comoves with the pricing kernel. The variance risk premium captures the conditional covariance between the market variance and the pricing kernel. A wealth of empirical evidence suggests that the variance risk premium is economically and statistically significant, which implies that the market variance is correlated with the pricing kernel and that variance risk is priced.

This paper emphasizes the relation between the variance risk premium and the leverage effect in economies with priced volatility risk. We demonstrate this relation in a general consumption-based framework that nests various models of the variance risk premium. Epstein-Zin preferences play a key role in the pricing of volatility risk. When the market variance is impacted by a shock, this also results in a shock to the pricing kernel and therefore a shock to the market return. The variance risk premium, which is the covariance between the market variance and the pricing kernel, and the leverage effect, which is the covariance between the market variance and the market return, therefore result from the same source.

We find that this theoretical relation is supported by the data. Using data for 1990-2016, we find a strong and positive linear relation between the variance risk premium and the leverage effect. This result is confirmed in various subsamples and using alternative measures for the two quantities.

This relation between the variance risk premium and the leverage effect can be used to empirically study and document the properties of the variance risk premium. As an example, we provide estimates of the variance risk premium dating back to 1926, which is possible because the leverage effect can be estimated provided that daily market returns are available. The relation between the variance risk premium and the leverage effect therefore provides additional evidence and insight on the variance risk premium, for which currently available estimates cover a very limited time horizon.

Appendix

This appendix provides detailed derivations for the generalized model in Section 6. The corresponding results for the benchmark model in Section 2 are obtained as a special case where jump risk is ignored. This model reduces to the benchmark model by setting $\rho_0 = \rho_V = 0$ (i.e. $\lambda_t \equiv 0$) or by setting $Z_t \equiv 0$.

A The Value Function

Adding $\int_0^t f(C_s, J_s)ds$ to both sides of equation (2), we obtain

$$J_t + \int_0^t f(C_s, J_s)ds = \mathbb{E}_t^P \left[\int_0^\infty f(C_s, J_s)ds \right],$$

where the right-hand side is a martingale by the law of iterated expectations. Hence, the left-hand side should also be a martingale. Based on the functional form of J_t in equation (4), it follows from Ito's lemma that

$$
\begin{aligned}
\frac{dJ_t}{J_{t^-}} + \frac{f(C_t, V_t)}{J_t} dt \;=\; & \Bigg[(1-\gamma)\left(\mu_C + X_t\right) - \frac{I_X}{I}\kappa_X X_t + \frac{I_V}{I}\kappa_V(\bar{V} - V_t) - \frac{1}{2}\gamma(1-\gamma)V_t \\
& + \frac{1}{2}\frac{I_{XX}}{I}\sigma_X^2 V_t + \frac{1}{2}\frac{I_{VV}}{I}\sigma_V^2 V_t + \lambda_t \mathbb{E}^P\left[e^{(1-\gamma)Z_{C,t}}\frac{I_t}{I_{t^-}} - 1\right] + \frac{f(C_t, V_t)}{J_t}\Bigg] dt \\
& + (1-\gamma)\sqrt{V_t}dB_{C,t} + \frac{I_X}{I}\sigma_X\sqrt{V_t}dB_{X,t} + \frac{I_V}{I}\sigma_V\sqrt{V_t}dB_{V,t} \\
& + \left(e^{(1-\gamma)Z_{C,t}}\frac{I_t}{I_{t^-}} - 1\right) dN_t - \lambda_t \mathbb{E}^P\left[e^{(1-\gamma)Z_{C,t}}\frac{I_t}{I_{t^-}} - 1\right] dt.
\end{aligned}
$$

The martingale condition implies that

$$
\begin{aligned}
0 = (1-\gamma)\left(\mu_C + X_t\right) - \frac{I_X}{I}\kappa_X X_t + \frac{I_V}{I}\kappa_V(\bar{V} - V_t) - \frac{1}{2}\gamma(1-\gamma)V_t \\
+ \frac{1}{2}\frac{I_{XX}}{I}\sigma_X^2 V_t + \frac{1}{2}\frac{I_{VV}}{I}\sigma_V^2 V_t + \lambda_t \mathbb{E}^P\left[e^{(1-\gamma)Z_{C,t}}\frac{I_t}{I_{t^-}} - 1\right] + \frac{f(C_t, V_t)}{J_t},
\end{aligned}
\quad (A.1)
$$

and we can show by combining equations (3) and (4) that

$$
\frac{f(C_t, V_t)}{J_t} = \begin{cases} -\delta \log I & \text{if } \psi = 1 \\ \delta\theta\left(I_t^{-\frac{1}{\theta}} - 1\right) & \text{if } \psi \neq 1. \end{cases}
$$

If $\psi = 1$, it is straightforward that equation (5) satisfies equation (A.1), which implies that I is an exponentially linear function in X and V. However, when $\psi \neq 1$, equation (A.1) does not allow a closed-form solution due to a non-linear term $I_t^{-\frac{1}{\theta}}$. Following Campbell and Viceira (1999), we log-linearize this term.[31] Since $\delta I_t^{-\frac{1}{\theta}}$ represents the consumption-wealth

[31] See also Campbell and Viceira (2002), Campbell, Chacko, Rodriguez, and Viceira (2004), Nowotny (2011), and Tsai and Wachter (2018).

ratio in the model, we can show that

$$\delta I^{-\frac{1}{\theta}} = \frac{C}{W} = \exp\left(\log\left(\frac{C}{W}\right)\right) \simeq i_0 + i_1\left(\log\delta - \frac{1}{\theta}\log I_t\right),$$

where $i_1 = e^{\mathbb{E}^P\left[\log\frac{C}{W}\right]}$ and $i_0 = i_1(1 - i_1)$. This is essentially the first-order Taylor expansion of an exponential function around the mean log consumption-wealth ratio. Under this approximation, it follows that $\frac{f(C_t,J_t)}{J_t} \simeq -\delta\theta + \theta i_0 + \theta i_1 \log\delta - i_1 \log I_t$, and this implies that equation (5) becomes a solution for equation (A.1) as in the case with $\psi = 1$.[32] Specifically, the martingale condition in equation (A.1) becomes

$$\begin{aligned}0 = &\Big[(1-\gamma)\mu_C + b_V\kappa_V\bar{V} + \theta\left(i_0 + i_1\log\delta - \delta\right) - i_1 a + \rho_0\left(\Phi_Z(\eta_J) - 1\right)\Big] \\ &+ \Big[(1-\gamma) - b_X\kappa_X - i_1 b_X\Big] X_t \\ &+ \left[-b_V\kappa_V - \frac{1}{2}\gamma(1-\gamma) + \frac{1}{2}b_X^2\sigma_X^2 + \frac{1}{2}b_V^2\sigma_V^2 - i_1 b_V + \rho_V\left(\Phi_Z(\eta_J) - 1\right)\right] V_t,\end{aligned} \quad \text{(A.2)}$$

where $\eta_J = [1-\gamma, b_X, b_V]^\top$. Since equation (A.2) should hold for arbitrary X_t and V_t, it follows that

$$\begin{aligned}0 &= (1-\gamma)\mu_C + b_V\kappa_V\bar{V} + \theta\left(i_0 + i_1\log\delta - \delta\right) - i_1 a + \rho_0\left(\Phi_Z(\eta_J) - 1\right) \\ 0 &= (1-\gamma) - b_X\kappa_X - i_1 b_X \\ 0 &= -b_V\kappa_V - \frac{1}{2}\gamma(1-\gamma) + \frac{1}{2}b_X^2\sigma_X^2 + \frac{1}{2}b_V^2\sigma_V^2 - i_1 b_V + \rho_V\left(\Phi_Z(\eta_J) - 1\right).\end{aligned}$$

From this system of equations, we can determine the values of a, b_X, and b_V.[33]

[32] Furthermore, the exact result for $\psi = 1$ can be obtained as the limit case with $\psi \to 1$ because

$$\lim_{\psi\to 1}\frac{f(C_t,J_t)}{J_t} \simeq \lim_{\psi\to 1} -\delta\theta + \theta i_0 + \theta i_1\log\delta - i_1\log I_t = -\delta\log I_t.$$

[33] After determining b_X from the second equation, the third equation becomes a quadratic equation in b_V, which generates two roots. While both roots satisfy the equation, only the negative root remains if we consider the limit case of the model where σ_V approaches 0, as discussed in Tauchen (2012).

B The State-Price Density and the Radon-Nikodym Derivative

According to Duffie and Skiadas (1994), the state price density π_t follows

$$\pi_t = \exp\left(\int_0^\infty f_J(C_s, J_s)ds\right) f_C(C_t, J_t),$$

which implies, by Ito's lemma, that

$$\frac{d\pi_t}{\pi_{t^-}} = f_J(C_t, J_t)dt + \frac{df_C(C_t, J_t)}{f_C(C_t, J_t)}.$$

We can show that

$$\begin{aligned}\frac{d\pi_t}{\pi_{t^-}} = \Bigg[&-\delta - \frac{1}{\psi}(\mu_C + X_t) + \left\{\frac{\gamma}{2}\left(1+\frac{1}{\psi}\right) - \frac{1}{2\theta}\left(1-\frac{1}{\theta}\right)\left(b_X^2\sigma_X^2 + b_V^2\sigma_V^2\right)\right\} V_t \\ &- \left(1-\frac{1}{\theta}\right)\Big(\Phi_Z(\eta_J) - 1\Big)\lambda_t\Bigg] dt - \gamma\sqrt{V_t}dB_{C,t} \\ &+ \left(1-\frac{1}{\theta}\right) b_X\sigma_X\sqrt{V_t}dB_{X,t} + \left(1-\frac{1}{\theta}\right) b_V\sigma_V\sqrt{V_t}dB_{V,t} + \left(e^{\eta_\pi^\top Z_t} - 1\right) dN_t.\end{aligned}$$

In equilibrium, the infinitesimal generator (i.e the sum of the drift and jump compensator) of the state-price density growth equals the negative of the risk-free rate:

$$\begin{aligned}r_{f,t} = \delta + \frac{1}{\psi}(\mu_C + X_t) - \left[\frac{\gamma}{2}\left(1+\frac{1}{\psi}\right) - \frac{1}{2\theta}\left(1-\frac{1}{\theta}\right)\left(b_X^2\sigma_X^2 + b_V^2\sigma_V^2\right)\right] V_t \\ + \left[\left(1-\frac{1}{\theta}\right)\Big(\Phi_Z(\eta_J) - 1\Big) - \Big(\Phi_Z(\eta_\pi) - 1\Big)\right]\lambda_t.\end{aligned}$$

Let Q denote the equivalent martingale measure associated with the bank account numeraire. This measure is the so-called risk-neutral measure. In our model, its existence and uniqueness are guaranteed because markets are complete. Therefore, there exists a one-to-one correspondence between the state-price density π_t and L_t, the Radon-Nikodym derivative process of Q with respect to the physical measure. It follows that

$$L_t = \pi_t \left(\int_0^t r_{f,s} ds \right). \tag{B.1}$$

Applying Ito's lemma to equation (B.1) yields

$$\frac{dL_t}{L_{t^-}} = \frac{d\pi_t}{\pi_{t^-}} + r_{f,t}dt,$$

which subsequently implies

$$\begin{aligned}\frac{dL_t}{L_{t^-}} = -\gamma\sqrt{V_t}dB_{C,t} + \left(1 - \frac{1}{\theta}\right) b_X \sigma_X \sqrt{V_t}dB_{X,t} + \left(1 - \frac{1}{\theta}\right) b_V \sigma_V \sqrt{V_t}dB_{V,t} \\ + \underbrace{\left(e^{\eta_\pi^\top Z_t} - 1\right) dN_t - \left(\Phi_Z(\eta_\pi) - 1\right) \lambda_t dt}_{\text{compensated jump}}.\end{aligned} \tag{B.2}$$

Equation (B.2) reaffirms that the Radon-Nikodym derivative process is a martingale. Specifically, in our model, L_t consists of four martingales, three of which are Brownian components, and one of which is a compensated jump.

Girsanov's theorem dictates how the dynamics of these four martingales change when the physical measure is switched to the risk-neutral measure. First of all, $B_{C,t}$, $B_{X,t}$, and $B_{V,t}$ are no longer standard Brownian motions under the risk-neutral measures. Instead, $B^Q_{C,t}$, $B^Q_{X,t}$, and $B^Q_{V,t}$ are:

$$\begin{aligned} B^Q_{C,t} &= B_{C,t} + \gamma\sqrt{V_t}dt \\ B^Q_{X,t} &= B_{X,t} - \left(1 - \frac{1}{\theta}\right) b_X \sigma_X \sqrt{V_t}dt \\ B^Q_{V,t} &= B_{V,t} - \left(1 - \frac{1}{\theta}\right) b_V \sigma_V \sqrt{V_t}dt. \end{aligned}$$

Accordingly, we can re-express the dynamics of consumption growth, dividend growth, and the two state variables in terms of Q-Brownian motions:

$$\begin{aligned}
\frac{dC_t}{C_{t^-}} &= (\mu_C + X_t - \gamma V_t)\, dt + \sqrt{V_t} dB^Q_{C,t} + (e^{Z_{C,t}} - 1) dN_t \\
\frac{dD_t}{D_{t^-}} &= (\mu_{D,t} - \gamma\phi V_t) dt + \phi\sqrt{V_t} dB^Q_{C,t} + \left(e^{\phi Z_{C,t}} - 1\right) dN_t \\
dX_t &= \left[-\kappa_X X_t + \left(1 - \frac{1}{\theta}\right) b_X \sigma_X^2 V_t\right] dt + \sigma_X \sqrt{V_t} dB^Q_{X,t} + Z_{X,t} dN_t \\
dV_t &= \left[\kappa_V - \left(1 - \frac{1}{\theta}\right) b_V \sigma_V^2\right] \left(\frac{\kappa_V}{\kappa_V - \left(1 - \frac{1}{\theta}\right) b_V \sigma_V^2} \bar{V} - V_t\right) dt + \sigma_V \sqrt{V_t} dB^Q_{V,t} + Z_{V,t} dN_t.
\end{aligned}$$

The change of measure also affects the jump process. Under the risk-neutral measure, the Poisson process N_t has a different intensity λ_t^Q, given by equation (30), and the jump size distribution Z_t follows a different distribution, given by equation (31).[34]

C The Price-Dividend Ratio

In equilibrium, the stock price (or the price of the dividend claim) is calculated as

$$S_t = \mathbb{E}_t^P \left[\int_t^\infty \frac{\pi_s}{\pi_t} D_s ds\right] = \int_t^\infty \mathbb{E}_t^P \left[\frac{\pi_s}{\pi_t} D_s\right] ds.$$

We define $H(D_t, X_t, V_t, s - t) = \mathbb{E}_t^P \left[\frac{\pi_s}{\pi_t} D_s\right]$ as the time-t price of the zero-coupon dividend claim at time s. We conjecture that

$$H_t = H(D_t, X_t, V_t, \tau) = D_t \exp\left(a_\phi(\tau) + b_{\phi X}(\tau) X_t + b_{\phi V}(\tau) V_t\right). \tag{C.1}$$

Using the fact that $\pi_t H_t = \mathbb{E}_t^P [\pi_s H_s]$ is a martingale (due to the law of iterated expectations), we verify this conjecture and derive the system of ordinary differential equations that a_ϕ,

[34] For more details, see Dai and Singleton (2003).

$b_{\phi X}$, and $b_{\phi V}$ satisfy. Note that Ito's lemma implies

$$
\begin{aligned}
\frac{d\left(\pi_t H_t\right)}{\pi_{t^-} H_{t^-}} = \Bigg[& -\delta - \frac{1}{\psi}\mu_C - \frac{1}{\psi}X_t + \left\{\frac{\gamma}{2}\left(1+\frac{1}{\psi}\right) - \frac{1}{2\theta}\left(1-\frac{1}{\theta}\right)\left(b_X^2\sigma_X^2 + b_V^2\sigma_V^2\right)\right\} V_t \\
& - \left(1-\frac{1}{\theta}\right)\Big(\Phi_Z(\eta_J) - 1\Big)\lambda_t + \mu_{D,t} - b_{\phi X}(\tau)\kappa_X X_t + \frac{1}{2}b_{\phi X}^2(\tau)\sigma_X^2 V_t \\
& + b_{\phi V}(\tau)\kappa_V(\bar{V} - V_t) + \frac{1}{2}b_{\phi V}^2(\tau)\sigma_V^2 V_t - \gamma\phi V_t + \left(1-\frac{1}{\theta}\right) b_X b_{\phi X}\sigma_X^2 V_t \\
& + \left(1-\frac{1}{\theta}\right) b_V b_{\phi V}\sigma_V^2 V_t - \Big(a_\phi'(\tau) + b_{\phi X}'(\tau)X_t + b_{\phi V}'(\tau)V_t\Big)\Bigg] dt \\
& + \Big[\phi - \gamma\Big]\sqrt{V_t}dB_{C,t} + \left[b_{\phi X}(\tau) + \left(1 - \frac{1}{\theta}b_X\right)\right]\sigma_X\sqrt{V_t}dB_{X,t} \\
& + \left[b_{\phi V}(\tau) + \left(1-\frac{1}{\theta}\right)b_V\right]\sigma_V\sqrt{V_t}dB_{V,t} + \left(e^{\left[\eta_\pi + \eta_\phi(\tau)\right]^\top Z_t} - 1\right)dN_t,
\end{aligned}
$$

where $\eta_\phi(\tau) = [\phi, b_{\phi X}(\tau), b_{\phi V}(\tau)]^\top$. The condition that the infinitesimal generator of this process equals zero for arbitrary values of X_t and V_t gives us the following system of ordinary-differential equations

$$
\begin{aligned}
a_\phi'(\tau) &= -\delta - \frac{1}{\psi}\mu_C + \mu_{D0} + b_{\phi V}(\tau)\kappa_V\bar{V} \\
&\quad +\rho_0\left[\Big(\Phi_Z\Big(\eta_\pi + \eta_\phi(\tau)\Big) - 1\Big) - \left(1-\frac{1}{\theta}\right)\Big(\Phi_Z(\eta_J) - 1\Big)\right] \\
b_{\phi X}'(\tau) &= -\frac{1}{\psi} + \mu_{DX} - b_{\phi X}(\tau)\kappa_X \\
b_{\phi V}'(\tau) &= \mu_{DV} + \frac{\gamma}{2}\left(1+\frac{1}{\psi}\right) - \frac{1}{2\theta}\left(1-\frac{1}{\theta}\right)\left(b_X^2\sigma_X^2 + b_V^2\sigma_V^2\right) - \gamma\phi - b_{\phi V}(\tau)\kappa_V \\
&\quad +\frac{1}{2}b_{\phi X}^2(\tau)\sigma_X^2 + \frac{1}{2}b_{\phi V}^2(\tau)\sigma_V^2 + \left(1-\frac{1}{\theta}\right)b_X b_{\phi X}\sigma_X^2 + \left(1-\frac{1}{\theta}\right)b_V b_{\phi V}\sigma_V^2 \\
&\quad +\rho_V\left[\Big(\Phi_Z\Big(\eta_\pi + \eta_\phi(\tau)\Big) - 1\Big) - \left(1-\frac{1}{\theta}\right)\Big(\Phi_Z(\eta_J) - 1\Big)\right].
\end{aligned}
$$

The boundary conditions for these differential equations are given by

$$a_\phi(0) = b_{\phi X}(0) = b_{\phi V}(0) = 0,$$

because $H(D_t, X_t, V_t, 0)$ should be equal to D_t by no-arbitrage.

From this, we can calculate the price-dividend ratio $G(X_t, V_t) = S_t/D_t$ as follows:

$$\begin{aligned} G(X_t, V_t) &= \int_t^\infty \exp\left(a_\phi(s-t) + b_{\phi X}(s-t)X_t + b_{\phi V}(s-t)V_t\right) ds \\ &= \int_0^\infty \exp\left(a_\phi(\tau) + b_{\phi X}(\tau)X_t + b_{\phi V}(\tau)V_t\right) d\tau. \end{aligned}$$

As discussed in Section 2.2, G is not a log-linear function of X and V so we use the approximation in Seo and Wachter (2017). By applying the first-order Taylor approximation to $g(X, V) = \log G(X, V)$ around the steady state level (X^*, V^*), we obtain

$$g(X, V) \simeq g(X^*, V^*) + g_X(X^*, V^*)(X - X^*) + g_V(X^*, V^*)(V - V^*).$$

This equation results in equation (7) when we define

$$\begin{aligned} a_{\bar\phi} &= g(X^*, V^*) - b_{\bar\phi X}X^* - b_{\bar\phi V}V^* \\ b_{\bar\phi X} &= g_X(X^*, V^*) = \frac{1}{G(X^*, V^*)} \int_0^\infty b_{\phi X}(\tau) \exp\left(a_\phi(\tau) + b_{\phi X}(\tau)X^* + b_{\phi V}(\tau)V^*\right) d\tau \\ b_{\bar\phi V} &= g_V(X^*, V^*) = \frac{1}{G(X^*, V^*)} \int_0^\infty b_{\phi V}(\tau) \exp\left(a_\phi(\tau) + b_{\phi X}(\tau)X^* + b_{\phi V}(\tau)V^*\right) d\tau. \end{aligned}$$

This approximation makes it simple to derive the dynamics for the price-dividend ratio:

$$\begin{aligned} \frac{dG_t}{G_t} = &\left[-b_{\bar\phi X}\kappa_X X_t + b_{\bar\phi V}\kappa_V(\bar V - V_t) + \frac{1}{2}b_{\bar\phi X}\sigma_X^2 V_t dt + \frac{1}{2}b_{\bar\phi V}\sigma_V^2 V_t dt\right] dt \\ &+ b_{\bar\phi X}\sigma_X\sqrt{V_t}dB_{X,t} + b_{\bar\phi V}\sigma_V\sqrt{V_t}dB_{V,t} + \left(e^{b_{\bar\phi X}Z_{X,t} + b_{\bar\phi V}Z_{V,t}} - 1\right) dN_t. \quad \text{(C.2)} \end{aligned}$$

D The Variance Risk Premium

The physical expectation of the quadratic variation in equation (32) equals

$$\begin{aligned}
\mathbb{E}_t^P \left[QV_{t,t+\tau} \right] &= \xi \mathbb{E}_t^P \left[\int_t^{t+\tau} V_s ds \right] + \mathbb{E}^P \left[Z_S^2 \right] \mathbb{E}_t^P \left[\int_t^{t+\tau} \lambda_s ds \right] \\
&= \xi \mathbb{E}_t^P \left[\mathbb{V}_{t,t+\tau} \right] + \mathbb{E}^P \left[Z_S^2 \right] \left(\rho_0 + \rho_V \mathbb{E}_t^P \left[\mathbb{V}_{t,t+\tau} \right] \right).
\end{aligned}$$

Therefore, we need to derive the expression for $\mathbb{E}_t^P \left[\mathbb{V}_{t,t+\tau} \right]$. The stochastic differential equation for V_t implies that

$$V_u - V_t = \kappa_V \bar{V} (u - t) - \kappa_V \int_t^u V_s ds + \sigma_V \int_t^u \sqrt{V_s} dB_{V,s} + \int_t^u Z_{V,s} dN_s,$$

for any $u \geq t$. We take time t-expectations on both sides and then differentiate both sides with respect to u, which results in the following ordinary differential equation:

$$\frac{d\mathbb{E}_t^P \left[V_u \right]}{du} = \left(\rho_V \mathbb{E}^P \left[Z_V \right] - \kappa_V \right) \mathbb{E}_t^P \left[V_u \right] + \kappa_V \bar{V} + \rho_0 \mathbb{E}^P \left[Z_V \right],$$

with the boundary condition $\mathbb{E}_t^P \left[V_t \right] = V_t$. Since this is a first-order linear differential equation, we can derive its solution as

$$\mathbb{E}_t^P \left[V_u \right] = \bar{V}' + e^{-\kappa_V' (u-t)} \left(V_t - \bar{V}' \right),$$

where $\kappa_V' = \kappa_V - \rho_V \mathbb{E}^P \left[Z_V \right]$ and $\bar{V}' = \bar{V} + \rho_0 \mathbb{E}^P \left[Z_V \right] / \kappa_V'$. By integrating this solution from t to $t + \tau$, we obtain

$$\mathbb{E}_t^P \left[\mathbb{V}_{t,t+\tau} \right] = \int_t^{t+\tau} \mathbb{E}_t^P \left[V_s \right] ds = \tau \bar{V}' + \frac{1}{\kappa_V'} \left(1 - e^{-\kappa_V' \tau} \right) \left(V_t - \bar{V}' \right). \tag{D.1}$$

Similarly, the risk-neutral expectation of the quadratic variation equals

$$\mathbb{E}_t^Q \left[QV_{t,t+\tau} \right] \quad = \quad \xi \mathbb{E}_t^Q \left[\mathbb{V}_{t,t+\tau} \right] + \mathbb{E}^Q \left[Z_S^2 \right] \mathbb{E}_t^Q \left[\rho_0^Q + \rho_V^Q \mathbb{V}_{t,t+\tau} \right], \tag{D.2}$$

and the conditional risk-neutral expectation of the integrated variance process is calculated as

$$\mathbb{E}_t^Q \left[\mathbb{V}_{t,t+\tau} \right] = \tau \bar{V}'^Q + \frac{1}{\kappa_V'^Q} \left(1 - e^{-\kappa_V'^Q \tau} \right) \left(V_t - \bar{V}'^Q \right),$$

where $\kappa_V'^Q = \kappa_V^Q - \rho_V^Q \mathbb{E}^Q \left[Z_V \right]$ and $\bar{V}'^Q = \bar{V}^Q + \rho_0^Q \mathbb{E}^Q \left[Z_V \right] / \kappa_V'^Q$. Given the expressions for the physical and risk-neutral expectations of quadratic variation, the variance risk premium is computed as their difference. Since both equations (D.1) and (D.2) are linear in V_t, the variance risk premium is also linear in V_t.

E The Leverage Effect

We define the time-t leverage effect as the time-t conditional covariance between the log stock return and stock variance:

$$LE_{t,t+\tau} \quad = \quad \mathrm{Cov}_t \left(\log \frac{S_{t+\tau}}{S_t}, \tau V_{t+\tau}^S \right) = \mathbb{E}_t^P \left[\int_t^{t+\tau} d \left[\log S, \tau V^S \right]_s \right],$$

where V_t^S is the time-t instantaneous stock variance. Note that when jumps are present, V_t^S is not simply equal to ξV_t, the instantaneous diffusive stock variance. Considering the effect of jumps, the stock variance can be calculated as $V_t^S = \lim_{\tau \to 0} \frac{1}{\tau} \mathbb{E}_t^P \left[QV_{t,t+\tau} \right]$. It follows that

$$V_t^S = \xi \lim_{\tau \to 0} \frac{1}{\tau} \mathbb{E}_t^P \left[\int_t^{t+\tau} V_s ds \right] + \mathbb{E}^P \left[Z_S^2 \right] \lim_{\tau \to 0} \frac{1}{\tau} \mathbb{E}_t^P \left[\int_t^{t+\tau} \lambda_s ds \right] = \xi V_t + \mathbb{E}^P \left[Z_S^2 \right] \lambda_t.$$

Since this implies that $dV_{S,t} = \left(\xi + \rho_V \mathbb{E}^P\left[Z_S^2\right]\right) dV_t$, the leverage effect is calculated as

$$LE_{t,t+\tau} = \tau\left(\xi + \rho_V \mathbb{E}^P\left[Z_S^2\right]\right)\left\{\rho_0 \mathbb{E}^P\left[Z_V Z_S\right] + \left(b_{\bar{\phi}V}\sigma_V^2 + \rho_V \mathbb{E}^P\left[Z_V Z_S\right]\right)\mathbb{E}_t^P\left[\mathbb{V}_{t,t+\tau}\right]\right\},$$

where the expression for $\mathbb{E}_t^P\left[\mathbb{V}_{t,t+\tau}\right]$ is given in equation (D.1).

References

Aït-Sahalia, Yacine, Jianqing Fan, and Yingying Li, 2013, The leverage effect puzzle: Disentangling sources of bias at high frequency, *Journal of Financial Economics* 109, 224–249.

Aït-Sahalia, Yacine, Mustafa Karaman, and Loriano Mancini, 2015, The term structure of variance swaps and risk premia, Working paper, Princeton University.

Andersen, Torben G., Tim Bollerslev, Francis X. Diebold, and Heiko Ebens, 2001, The distribution of realized stock return volatility, *Journal of Financial Economics* 61, 43–76.

Andersen, Torben G., Tim Bollerslev, Francis X. Diebold, and Paul Labys, 2003, Modeling and forecasting realized volatility, *Econometrica* 71, 579–625.

Andersen, Torben G., Nicola Fusari, and Viktor Todorov, 2015, The risk premia embedded in index options, *Journal of Financial Economics* 117, 558–584.

Ang, Andrew, Robert J Hodrick, Yuhang Xing, and Xiaoyan Zhang, 2006, The cross-section of volatility and expected returns, *Journal of Finance* 61, 259–299.

Bakshi, Gurdip, and Nikunj Kapadia, 2003, Delta-hedged gains and the negative market volatility risk premium, *Review of Financial Studies* 16, 527–566.

Bandi, Federico M., and Roberto Reno, 2012, Time-varying leverage effects, *Journal of Econometrics* 169, 94–113.

Bansal, Ravi, and Amir Yaron, 2004, Risks for the long-run: A potential resolution of asset pricing puzzles, *Journal of Finance* 59, 1481–1509.

Barndorff-Nielsen, Ole E., and Neil Shephard, 2002, Econometric analysis of realized volatility and its use in estimating stochastic volatility models, *Journal of the Royal Statistical Society: Series B (Statistical Methodology)* 64, 253–280.

Barone-Adesi, Giovanni, 1985, Arbitrage equilibrium with skewed asset returns, *Journal of Financial and Quantitative Analysis* 20, 299–313.

Barro, Robert J., 2006, Rare disasters and asset markets in the twentieth century, *Quarterly Journal of Economics* 121, 823–866.

Bates, David S., 2000, Post-'87 crash fears in the S&P 500 futures option market, *Journal of Econometrics* 94, 181–238.

Bekaert, Geert, and Marie Hoerova, 2014, The VIX, the variance premium and stock market volatility, *Journal of Econometrics* 183, 181–192.

Bekaert, Geert, and Guojun Wu, 2000, Asymmetric volatility and risk in equity markets, *Review of Financial Studies* 13, 1–42.

Berger, David, Ian Dew-Becker, and Stefano Giglio, 2018, Uncertainty shocks as second-moment news shocks, Working paper, Northwestern University.

Black, Fischer, 1976, Studies of stock price volatility changes, *Proceedings of the 1976 Meetings of the American Statistical Association, Business and Economic Statistics Section* pp. 177–181.

Bollerslev, Tim, James Marrone, Lai Xu, and Hao Zhou, 2014, Stock return predictability and variance risk premia: Statistical inference and international evidence, *Journal of Financial and Quantitative Analysis* 49, 633–661.

Bollerslev, Tim, Natalia Sizova, and George Tauchen, 2012, Volatility in equilibrium: Asymmetries and dynamic dependencies, *Review of Finance* 16, 31–80.

Bollerslev, Tim, George Tauchen, and Hao Zhou, 2009, Expected stock returns and variance risk premia, *Review of Financial Studies* 22, 4463–4492.

Bollerslev, Tim, and Viktor Todorov, 2011, Tails, fears, and risk premia, *Journal of Finance* 66, 2165–2211.

Bollerslev, Tim, Viktor Todorov, and Lai Xu, 2015, Tail risk premia and return predictability, *Journal of Financial Economics* 118, 113–134.

Britten-Jones, Mark, and Anthony Neuberger, 2000, Option prices, implied price processes, and stochastic volatility, *Journal of Finance* 55, 839–866.

Broadie, Mark, Mikhail Chernov, and Michael Johannes, 2007, Model specification and risk premia: Evidence from futures options, *Journal of Finance* 62, 1453–1490.

Campbell, John Y, George Chacko, Jorge Rodriguez, and Luis M Viceira, 2004, Strategic asset allocation in a continuous-time VAR model, *Journal of Economic Dynamics and Control* 28, 2195–2214.

Campbell, John Y., and Ludger Hentschel, 1992, No news is good news: An asymmetric model of changing volatility in stock returns, *Journal of Financial Economics* 31, 281–318.

Campbell, John Y., and Robert J. Shiller, 1988, The dividend-price ratio and expectations of future dividends and discount factors, *Review of Financial Studies* 1, 195–228.

Campbell, John Y., and Luis M. Viceira, 1999, Consumption and portfolio decisions when expected returns are time-varying, *Quarterly Journal of Economics* 114, 433–495.

Campbell, John Y., and Luis M. Viceira, 2002, *Strategic Asset Allocation: Portfolio Choice for Long-Term Investors.* (Oxford University Press).

Carr, Peter, and Liuren Wu, 2009, Variance risk premiums, *Review of Financial Studies* 22, 1311–1341.

Cheung, Yin Wong, and Lilian K. Ng, 1992, Stock price dynamics and firm size: An empirical investigation, *Journal of Finance* 47, 1985–1997.

Christie, Andrew A., 1982, The stochastic behavior of common stock variances: Value, leverage and interest rate effects, *Journal of Financial Economics* 10, 407–432.

Christoffersen, Peter, Kris Jacobs, and Chayawat Ornthanalai, 2012, Dynamic jump intensities and risk premiums: Evidence from S&P500 returns and options, *Journal of Financial Economics* 106, 447–472.

Coval, Joshua D., and Tyler Shumway, 2001, Expected option returns, *Journal of Finance* 56, 983–1009.

Cox, John C., Jonathan C. Ingersoll, and Stephen A. Ross, 1985, A theory of the term structure of interest rates, *Econometrica* 53, 385–408.

Dai, Qiang, and Kenneth Singleton, 2003, Term structure dynamics in theory and reality, *Review of Financial Studies* 16, 631–678.

Dew-Becker, Ian, Stefano Giglio, and Bryan Kelly, 2017, How do investors perceive the risks from macroeconomic and financial uncertainty? Evidence from 19 option markets, Working paper, Northwestern University.

Dew-Becker, Ian, Stefano Giglio, Anh Le, and Marius Rodriguez, 2017, The price of variance risk, *Journal of Financial Economics* 123, 225–250.

Dittmar, Robert F., 2002, Nonlinear pricing kernels, kurtosis preference, and evidence from the cross section of equity returns, *Journal of Finance* 57, 369–403.

Drechsler, Itamar, and Amir Yaron, 2011, What's vol got to do with it, *Review of Financial Studies* 24, 1–45.

Duffie, Darrell, and Larry G Epstein, 1992, Asset pricing with stochastic differential utility, *Review of Financial Studies* 5, 411–436.

Duffie, Darrell, and Costis Skiadas, 1994, Continuous-time asset pricing: A utility gradient approach, *Journal of Mathematical Economics* 23, 107–132.

Dupire, Bruno, 1994, 'Pricing with a smile', Risk Magazine, *Incisive Media* pp. 327–343.

Epps, Thomas W., 1979, Comovements in stock prices in the very short run, *Journal of the American Statistical Association* 74, 291–298.

Epstein, Larry, and Stan Zin, 1989, Substitution, risk aversion and the temporal behavior of consumption and asset returns: A theoretical framework, *Econometrica* 57, 937–969.

Eraker, Bjørn, 2004, Do stock prices and volatility jump? Reconciling evidence from spot and option prices, *Journal of Finance* 59, 1367–1404.

Eraker, Bjørn, 2008, The volatility premium, Working paper, University of Wisconsin, Madison.

Eraker, Bjørn, Michael Johannes, and Nicholas Polson, 2003, The impact of jumps in volatility and returns, *Journal of Finance* 58, 1269–1300.

Eraker, Bjorn, and Yue Wu, 2017, Explaining the negative returns to VIX futures and ETNs: An equilibrium approach, Forthcoming, *Journal of Financial Economics.*

French, Kenneth R., G. William Schwert, and Robert F. Stambaugh, 1987, Expected stock returns and volatility, *Journal of Financial Economics* 19, 3–29.

Harvey, Campbell R., and Akhtar Siddique, 2000, Conditional skewness in asset pricing tests, *Journal of Finance* 55, 1263–1295.

Jiang, George J., and Yisong S. Tian, 2005, The model-free implied volatility and its information content, *Review of Financial Studies* 18, 1305–1342.

Johnson, Timothy C., 2004, Forecast dispersion and the cross section of expected returns, *Journal of Finance* 59, 1957–1978.

Johnson, Timothy C., and Jaehoon Lee, 2014, On the systematic volatility of unpriced earnings, *Journal of Financial Economics* 114, 84–104.

Kilic, Mete, and Ivan Shaliastovich, 2018, Good and bad variance premia and expected returns, Forthcoming, *Management Science*.

Kraus, Alan, and Robert H. Litzenberger, 1976, Skewness preference and the valuation of risk assets, *Journal of Finance* 31, 1085–1100.

Liu, Lily Y., Andrew J. Patton, and Kevin Sheppard, 2015, Does anything beat 5-minute RV? A comparison of realized measures across multiple asset classes, *Journal of Econometrics* 187, 293–311.

Nelson, Daniel B., 1991, Conditional heteroskedasticity in asset returns: A new approach, *Econometrica* 59, 347–370.

Neuberger, Anthony, 1994, The log contract, *Journal of Portfolio Management* 20, 74–80.

Newey, Whitney K., and Kenneth D. West, 1994, Automatic lag selection in covariance matrix estimation, *Review of Economic Studies* 61, 631–653.

Nowotny, Michael, 2011, Disaster begets crisis: The role of contagion in financial markets, Working paper, Boston University.

Pan, Jun, 2002, The jump-risk premia implicit in options: Evidence from an integrated time-series study, *Journal of Financial Economics* 63, 3–50.

Rietz, Thomas A., 1988, The equity risk premium: A solution, *Journal of Monetary Economics* 22, 117–131.

Schwert, William G., 1989, Why does stock market volatility change over time?, *Journal of Finance* 44, 1115–1153.

Seo, Sang Byung, and Jessica A. Wachter, 2017, Do rare events explain CDX tranche spreads?, Forthcoming, *Journal of Finance*.

Seo, Sang Byung, and Jessica A. Wachter, 2018, Option prices in a model with stochastic disaster risk, Forthcoming, *Management Science*.

Sheppard, Kevin, 2006, Realized covariance and scrambling, Working paper, University of Oxford.

Tauchen, George, 2012, Stochastic volatility in general equilibrium, *Quarterly Journal of Finance* 1, 707–731.

Todorov, Viktor, 2010, Variance risk-premium dynamics: The role of jumps, *Review of Financial Studies* 23, 345–383.

Tsai, Jerry, and Jessica A. Wachter, 2018, Pricing long-lived securities in dynamic endowment economies, Working paper, University of Oxford.

Wachter, Jessica A., 2013, Can time-varying risk of rare disasters explain aggregate stock market volatility?, *Journal of Finance* 68, 987–1035.

Weil, Philippe, 1989, The equity premium puzzle and the risk-free rate puzzle, *Journal of Monetary Economics* 24, 402–421.

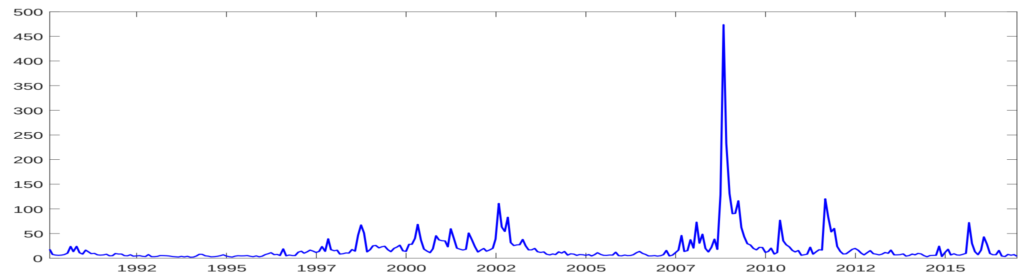
Yu, Jun, 2012, A semiparametric stochastic volatility model, *Journal of Econometrics* 167, 473–482.

Zhou, Guofu, and Yingzi Zhu, 2014, Macroeconomic volatilities and long-run risks of asset prices, *Management Science* 61, 413–430.

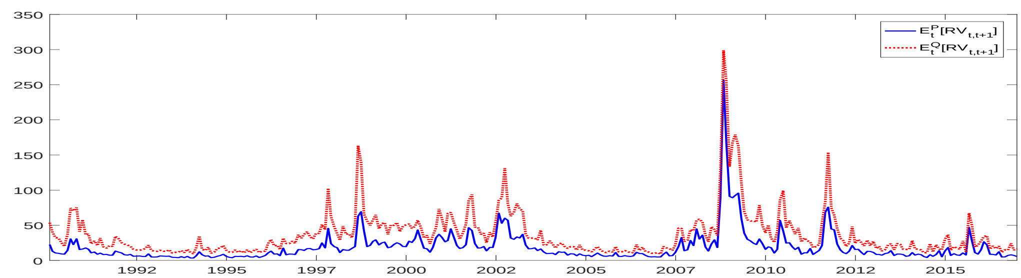
Zhou, Hao, 2018, Variance risk premia, asset predictability puzzles, and macroeconomic uncertainty, Forthcoming, *Annual Review of Financial Economics*.

Figure 1: S&P 500 Variance and Variance Risk Premium

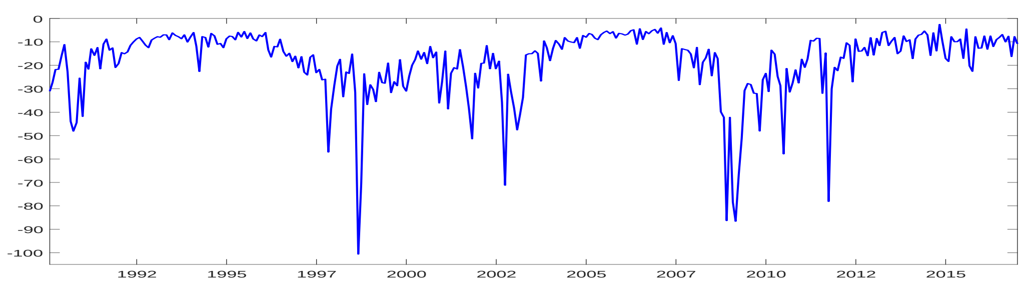
Panel A: Realized Variance



Panel B: Expected Physical and Risk-Neutral Variance



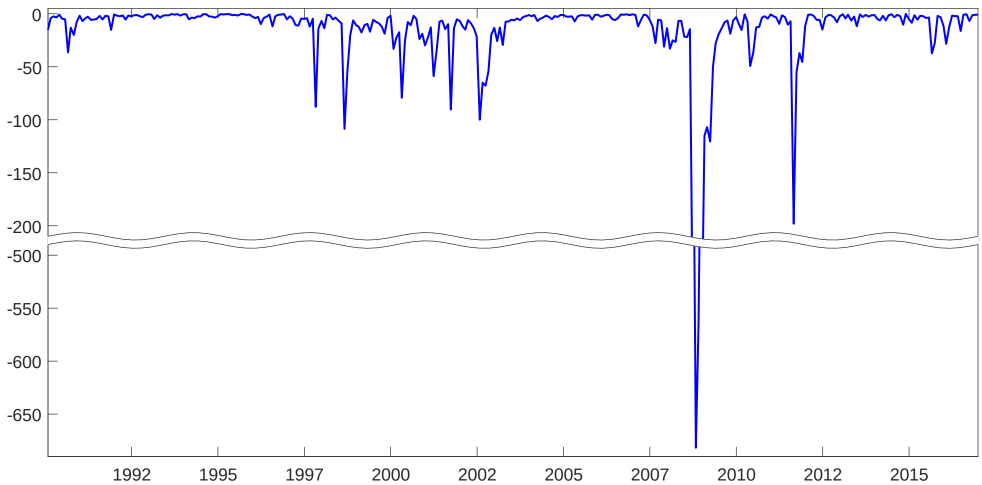
Panel C: The Variance Risk Premium



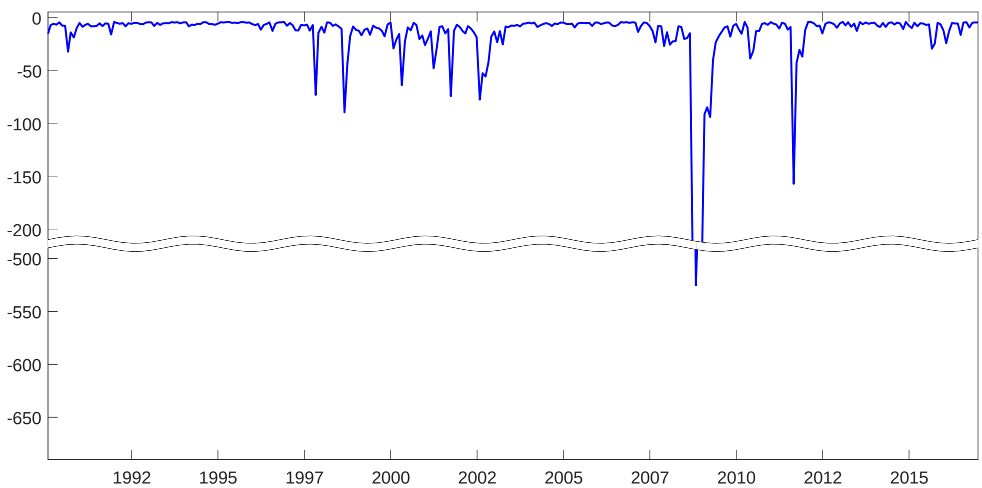
Notes: Panel A plots the time series of monthly realized variance (RV) calculated as the sum of the squared five-minute log returns on the S&P 500 index over the month. Panel B plots monthly physical and risk-neutral expectations of the one-month ahead variance for the S&P 500 index. We compute the physical expectation $\mathbb{E}_t^P(RV_{t,t+1})$ by projecting realized variance on lagged realized variance and the squared VIX index based on the full sample. The risk-neutral expectation $\mathbb{E}_t^Q(RV_{t,t+1})$ is proxied by the squared VIX index observed on the last trading day of each month. Panel C plots the monthly time series of the market variance risk premium, which is calculated as the difference between physical and risk neutral expectations listed above. Variance and variance risk premium are denoted in monthly percentage squared terms. The sample period is from January 1990 to December 2016.

Figure 2: Realized Covariance and the Leverage Effect

Panel A: Realized Covariance

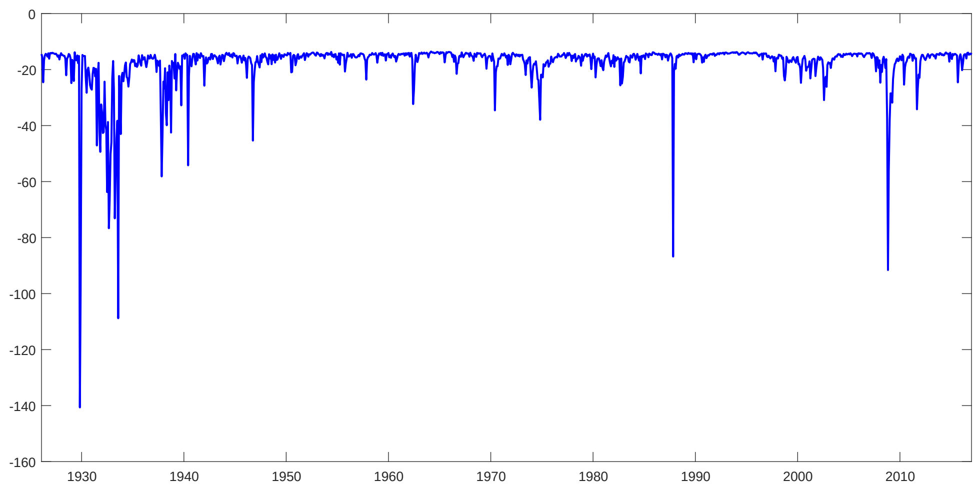


Panel B: The Leverage Effect



Notes: Panel A plots the time series of the monthly realized covariance between the return on the S&P 500 index and changes in its conditional variance, computed using daily data within each month. The time series of daily conditional variances is estimated based on the EGARCH (1,1) model using the full sample. Panel B plots the time series of the monthly leverage effect, which is defined as the expectation of one-month ahead realized covariance. We obtain this conditional expectation of the one-month ahead realized covariance by projecting realized covariance on lagged realized covariance and realized variance. The sample period is from January 1990 to December 2016.

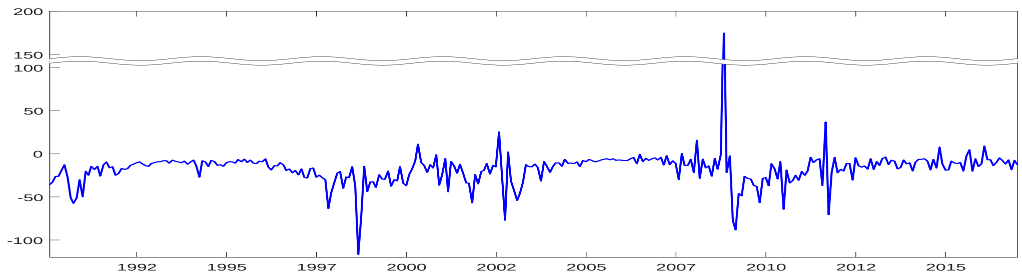
Figure 3: The Variance Risk Premium 1926-2016



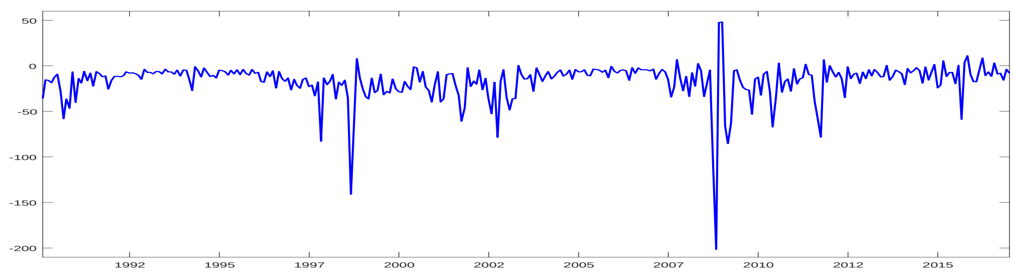
Notes: This figure plots the time series of the variance risk premium for the S&P 500 index in the extended sample starting in January 1926. Starting in July 1962, we use the daily return time series for the S&P 500 index. Before 1962, we use the CRSP value-weighted index as a proxy. The variance risk premium is extrapolated from estimates of the leverage effect for the January 1926 through December 2016 sample.

Figure 4: Robustness: Alternative Measures of the Variance Risk Premium

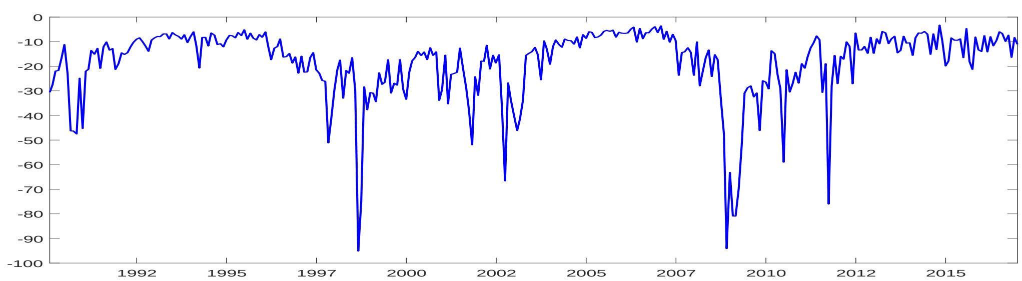
Panel A: Bollerslev, Tauchen, and Zhou (2009)



Panel B: Zhou (2018)

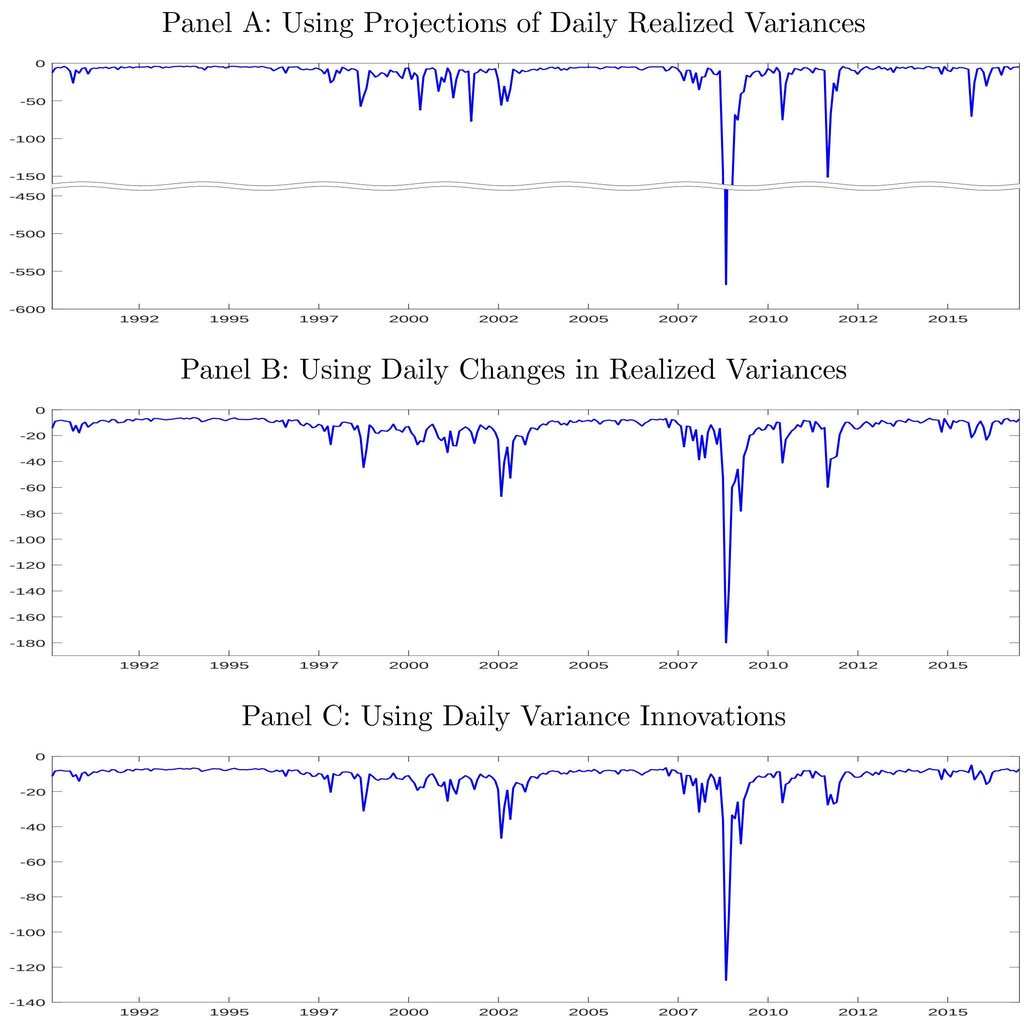


Panel C: Bekaert and Hoerova (2014)



Notes: This figure plots three alternative measures of the variance risk premium. We continue to use the squared VIX index as a proxy for the expected variance under the risk neutral measure, but we entertain different models to obtain different estimates of the expected variance under the physical measure. Panel A considers the random walk model as in Bollerslev, Tauchen, and Zhou (2009). Panel B follows Zhou (2018) and calculates the expected physical variance using an autoregressive model with 12 lags. Panel C obtains the time series of the expected variance by projecting realized variance on lagged VIX and realized variances over the past month, week, and day, as suggested by Bekaert and Hoerova (2014). The sample period is from January 1990 to December 2016.

Figure 5: Robustness: Alternative Measures of the Leverage Effect



Notes: This figure plots three alternative measures of the leverage effect for the S&P 500. These measures differ with respect to the computation of the realized covariance (COV). The first alternative measure calculates the sample covariance between daily log returns and daily changes in conditional variances, which are estimated based on daily realized variances and their projections. The second alternative measure computes the sample covariance between daily log returns and daily changes in realized variances. The third alternative measure calculates the sample covariance between daily log returns and daily variance innovations. The sample period is from January 1990 to December 2016.

Table 1: Forecasting Realized Variance and Covariance

Panel A: Realized Variance Projection				
	Const.	$\log RV_{t-1,t}$	$\log VIX_t^2$	Adj. R^2
$\log RV_{t,t+1}$	-0.44	0.47	0.52	68%
	(-2.45)	(6.92)	(5.71)	
Panel B: Realized Covariance Projection				
	Const.	$RV_{t-1,t}$	$COV_{t-1,t}$	Adj. R^2
$COV_{t,t+1}$	-4.57	0.08	0.82	60%
	(-1.70)	(0.47)	(8.37)	

Notes: This table reports the results of forecasting the one-month ahead realized variance (Panel A) and the one-month ahead realized covariance (Panel B). Realized variance is calculated as the sum of the squared five-minute log returns on the S&P 500 index over each month. Realized covariance is computed as the sample covariance between S&P 500 returns and changes in their conditional variance, using daily data within each month. The time series of daily conditional variances is estimated using the EGARCH(1,1) model. The sample period is from January 1990 to December 2016.

Table 2: Descriptive Statistics

Panel A: Summary Statistics							
	N	Mean	Median	Std.	Skew.	Kurt.	AR(1)
RV	324	19.64	10.89	34.69	8.27	95.58	0.65
ERV	324	18.62	11.91	22.26	5.58	47.22	0.77
VIX^2	324	37.01	27.03	33.76	3.54	18.47	0.80
VRP	324	-18.39	-14.06	14.33	-2.47	8.18	0.66
COV	324	-16.94	-4.68	57.51	-8.52	84.09	0.77
LE	324	-16.91	-7.40	44.53	-8.53	83.52	0.78
SP	324	0.66	1.03	4.16	-0.58	1.24	0.04
Panel B: Correlations							
	RV	ERV	VIX^2	VRP	COV	LE	SP
RV	1						
ERV	0.97	1					
VIX^2	0.84	0.95	1				
VRP	-0.49	-0.69	-0.88	1			
COV	-0.93	-0.90	-0.79	0.46	1		
LE	-0.92	-0.89	-0.78	0.46	0.99	1	
SP	-0.36	-0.39	-0.40	0.35	0.38	0.37	1

Notes: Panel A reports the mean, median, standard deviation, skewness , kurtosis, and first-order autocorrelation for the monthly realized variance (RV), the expected variance under the physical measure (ERV), the squared VIX index (VIX^2), the variance risk premium (VRP), the realized covariance (COV), the leverage effect (LE), and the S&P 500 return (SP). Panel B reports the correlations among these variables. The sample period is from January 1990 to December 2016.

Table 3: Regressing the Variance Risk Premium on the Leverage Effect

	α	β	Adj. R^2
Full Sample (Jan 1990 to Dec 2016)	-15.89 (-12.97)	0.15 (3.27)	21%
Excluding the Great Recession	-12.26 (-6.74)	0.43 (2.65)	24%
First Half of the Sample (Jan 1990 to Jun 2003)	-12.80 (-7.07)	0.58 (3.84)	37%
Second Half of the Sample (Jul 2003 to Dec 2016)	-13.82 (-9.81)	0.13 (3.93)	28%

Notes: This table reports estimation results for the following regression:

$$VRP_t = \alpha + \beta LE_t + \epsilon_t$$

for various sample periods. Reported in parentheses are Newey-West t-statistics that adjust for autocorrelation and heteroskedasticity. The first panel covers the full sample period from January 1990 to December 2016. The second panel excludes the Great Recession which corresponds to the period between December 2007 and June 2009. The bottom two panels provide the results for two sub-periods of equal length.

Table 4: Robustness

Panel A: Robustness to Different Measures of the Variance Risk premium						
	Jan 1990 to Dec 2016			Excluding the Great Recession		
	α	β	Adj. R^2	α	β	Adj. R^2
(1)	-19.50	-0.13	8%	-15.54	0.17	2%
	(-12.12)	(-1.72)		(-6.60)	(0.76)	
(2)	-13.08	0.21	19%	-7.37	0.72	42%
	(-9.29)	(2.33)		(-4.11)	(4.12)	
(3)	-15.71	0.16	24%	-12.31	0.42	23%
	(-12.78)	(3.49)		(-7.18)	(2.78)	

Panel B: Robustness to Different Measures of the Leverage Effect						
	Jan 1990 to Dec 2016			Excluding the Great Recession		
	α	β	Adj. R^2	α	β	Adj. R^2
(1)	-16.15	0.15	15%	-12.72	0.40	19%
	(-12.10)	(2.73)		(-7.86)	(2.78)	
(2)	-10.85	0.51	29%	-7.29	0.77	24%
	(-5.66)	(3.90)		(-3.64)	(4.63)	
(3)	-9.73	0.73	25%	-4.23	1.21	21%
	(-3.99)	(3.51)		(-1.47)	(4.29)	

Notes: This table reports estimation results for the regression

$$VRP_t = \alpha + \beta LE_t + \epsilon_t$$

using alternative measures of the variance risk premium (VRP) and the leverage effect (LE). Panel A considers different measures of the variance risk premium, and Panel B considers alternative measures of the leverage effect. We report results for the full sample period (January 1990 to December 2016) as well as the sample period without the Great Recession.