Affine Option Pricing Model in Discrete Time

Stanislav Khrapov* — New Economic School
Eric Renault† — Brown University

April 18, 2016

Abstract

We propose an extension with leverage effect of the discrete time stochastic volatility model of Darolles et al. (2006). This extension is shown to be the natural discrete time analog of the Heston (1993) option pricing model. It shares with Heston (1993) the advantage of structure preserving change of measure: with an exponentially affine stochastic discount factor, the historical and the risk neutral models belong to the same family of joint probability distributions for return and volatility processes. This allows computing option prices in semi-closed form through Fourier transform. The discrete time approach has several advantages. First, it allows relaxing the constraints on higher order moments implied by the specification of a diffusion process. Second, it makes more transparent the role of various parameters: leverage versus volatility feedback effect, connection with daily realized volatility measure on high-frequency intraday returns, closed-form formulas for affine dynamics of the first two moments of return and volatility that are robust to temporal aggregation, impact of leverage on the volatility smile, etc. This sheds some new light on the identification issue of the various risk premium parameters. An empirical illustration is provided.

Keywords: stochastic volatility; realized variance; leverage; option pricing; equity risk premium; volatility risk premium
JEL Classification: C58, G13

*Address: New Economic School, 100A Novaya Street, Skolkovo, Moscow, 143026 Russia. Phone: +7 (495) 956 9508. E-mail: khrapovs@gmail.com

†Address: Brown University, Robinson Hall, Providence, RI 02912, USA. Phone: +1 (401) 863 3519. Email: Eric_Renault@brown.edu

1 Introduction

Affine Jump-Diffusion models have been put forward by Duffie et al. (2000) as a convenient model for state variable to get closed- or nearly-closed form expressions for derivative asset prices. Their model nests in particular the popular Cox et al. (1985) model for interest rates as well as Heston (1993) stochastic volatility model for currency and equity prices for the purpose of option pricing. Duffie et al. (2000) synthesized this strand of literature to show that generally speaking the use of Fourier transform allows not only to define the affine model through conditional moment restrictions but also to derive nearly-closed form expressions for option prices.

Since then, Affine Jump-Diffusion models have often been criticized for their poor empirical fit. The key intuition is that they maintain an assumption of local conditional normality, up to jumps. Jumps are to some extent the only degree of freedom to reproduce the pattern of time-varying skewness and excess kurtosis commonly observed in asset returns. As a response to this criticism, at least two strands of literature have promoted specifications of discrete time models that remain true as much as possible to the affine structure. The goal is to use the additional degree of freedom provided by discrete time modeling to get a better empirical fit of higher order moments while keeping closed- or nearly-closed form expressions for securities prices. While Duan (1995), Heston & Nandi (2000) have initiated a strand of literature on closed-form GARCH option pricing (see Christoffersen et al. (2010a, 2012), and references therein for the most recent contributions), the paper by Darolles et al. (2006) has been seminal to provide a class of discrete time affine stochastic volatility models as general as the class of Affine Jump-Diffusion models for continuous time arbitrage pricing.

The stochastic volatility model provides a versatile framework to capture asymmetric volatility dynamics with possibly different parameters for historical and risk-neutral dynamics. While a similar exercise has been performed by Barone-Adesi et al. (2008) in a GARCH framework (thanks to calibration of option prices data), Meddahi & Renault (2004) have shown that affine discrete-time volatility dynamics may be seen as a relevant weakening of the GARCH restrictions. This weakening restores robustness to temporal aggregation, at least for the affine specification of the first two moments. However, Meddahi & Renault (2004) approach is only semi-parametric while a complete specification of the conditional probability distributions is warranted for option pricing. Compound AutoRegressive (Car) models of Darolles et al. (2006) provide exactly the relevant framework for doing so. However, the focus is only on volatility dynamics and there is no attempt to specify a joint model for volatility and return process, incorporating the leverage effect as in particular in Heston (1993) model. Bertholon et al. (2008) move in the direction of joint return and volatility modeling within Car-type framework. As an example, they develop the model with asymmetric GARCH volatility to produce the leverage effect. They also note the theoretical possibility to introduce instantaneous correlation between returns and volatility by considering Car framework.

The main contribution of this paper is to extend the framework of Darolles et al. (2006) to a bivariate model of return and volatility that allows for leverage effect and volatility feedback as well. This provides a convenient large class of affine models for option pricing, nesting Heston (1993) model as a particular continuous time limit. Moreover, by contrast with the debates about the right way to define continuous time limits of GARCH models, our limit arguments are underpinned by temporal aggregation formulas and as such, are immune to the criticism of ad hoc specification. The challenge to provide a versatile discrete time extension of Heston (1993) option pricing with stochastic volatility and leverage effect is twofold:

First, the discrete time approach complicates the separate identification of Granger causality and instantaneous causality (see e.g. Renault et al., 1998). This is especially important in the context of stochastic volatility models since, as documented by Bollerslev et al. (2006), the only way to disentangle leverage effect (as defined by Black, 1976) from volatility feedback due to risk premium, is to assess the direction of causality between volatility and return. While Bollerslev et al. (2006) enhanced the usefulness of high frequency data to do so, our parametric modeling must carefully leave room for a mixture of these two effects in discrete time. Note that, on the other hand, we maintain the assumption that returns do not Granger cause volatility. This assumption is key (see Renault, 1997) to get option pricing formulas which, like Black and Scholes are homogeneous of degree one with respect to underlying stock price and strike price and as a result, allow us to see the volatility smile as a function of moneyness. The lack of such homogeneity property is another weakness of GARCH option pricing (see Garcia & Renault, 1998).

Second, we want to keep in discrete time the main features of Heston (1993), namely volatility dynamics that are affine for both the historical and the risk-neutral distribution, while keeping the same leverage effect. To the best of our knowledge, the only attempt to do so in the extant literature has been recently proposed by Feunou & Tedongap (2012). However, we show that their affine specification with leverage effect cannot work simultaneously for the historical and the risk neutral distribution, More precisely, a general exponentially affine pricing kernel is not structure preserving in their context. They can use their model either for risk neutral distribution or for the historical one, but not both. Our specification is structure preserving (while keeping the same leverage effect) with a general exponential affine stochastic discount factor. While the shape of volatility smile without leverage effect is well-known (see Renault & Touzi, 1996) our closed form expressions allow us to give new insights on distortions of volatility smiles produced by leverage. Moreover, these formulas also provide conditional moment restrictions for econometric inference as a discrete time extension of the work of Pan (2002). Finally, we are also able to characterize the information content of option price data, in particular their crucial role in statistical identification of risk neutral dynamics. This paves the way for an extension of Gagliardini et al. (2011) to general discrete time affine models with leverage effect.

The rest of the paper is organized as follows. Section 2 sets up a general model. Section 3 introduces stochastic discount factor and shows how to construct risk neutral distributions of return and volatility. Section 4 derives the generalized Black-Scholes option price formula, and analyzes the effect of leverage on implied volatilities. Section 5 shows how to estimate the joint model of returns and volatility with leverage effect using the moment restrictions based on Laplace transform. Section 6 concludes.

2 Affine Historical Probability Measure

2.1 General framework

Let S_t be the price for a stock at time t. The observed time series of interest will be the continuously computed rate of return in excess of the risk free rate:

$$r_{t+1} = \log\left(S_{t+1}/S_t\right) - r_{f,t},$$

where $r_{f,t}$ stands for the risk-free rate over period $[t, t+1]$.

We will assume throughout that the return volatility is driven by one factor, that is a stochastic process denoted by σ_t. We do not need to be more specific at this stage regarding the exact connection between the process σ_t and the volatility of the return process r_t. Let us just say that the $(t+1)$'th "observation" $\left(r_{t+1}, \sigma^2_{t+1}\right)$ has a joint conditional distribution given past information characterized by its Laplace transform:

$$L\left(u, v \left| I_t \right.\right) \equiv E\left[\exp\left(-u\sigma^2_{t+1} - v r_{t+1}\right)\Big| I_t\right],$$

where u and v are complex arguments and I_t stands for the natural filtration of the state variables:

$$I_t = \{r_s, r_{f,s}, \sigma_s;\ s \leq t\}$$

Note that this setting is fully general since the Laplace transform can always be used to characterize the probability distribution, for instance by letting the arguments u and v be purely imaginary numbers, so that we get the characteristic function of the distribution.

However, we want to remain true to a well-founded tradition in option pricing (see Renault, 1997; Garcia & Renault, 1998 and references therein for a discussion) to see option prices as homogeneous functions of degree one with respect to the pair (S_t, K) of the underlying stock price and the strike price. Since Black and Scholes pricing formula fulfills this homogeneity property, assuming that our option pricing model is also homogeneous amounts to see Black-Scholes implied volatilities as depending only on the moneyness S_t/K, in line with a common tradition of representation of the volatility smile. As shown in Renault (1997), this homogeneity assumption is tantamount to the conjunction of two assumptions:

- First, we preclude any kind of Granger causality from return to volatility. The conditional probability distribution of σ^2_{t+1} given I_t depends on conditioning information only through the past of the volatility factor. Adding a common Markov assumption, we will simply write:

$$L\left(u, 0 \left| I_t \right.\right) = L_\sigma\left(u \left| \sigma^2_t \right.\right) = E\left[\exp\left(-u\sigma^2_{t+1}\right)\Big| \sigma^2_t\right].$$

  Note that the Markov assumption could be relaxed by considering several volatility factors. The non-causality assumption from return to volatility, albeit common in the stochastic volatility literature, is at odds with some popular models of GARCH option pricing, as initiated by Duan (1995). For this reason, GARCH option pricing is not homogeneous (see Garcia & Renault, 1998).

- Second, we assume that stock returns are conditionally serially independent given the volatility path. The conditional probability distribution of return r_{t+1} given I_t and σ_{t+1}^2 depends on conditioning information only through past and current values σ_t^2 and σ_{t+1}^2 of the volatility factor. Adding a joint Markov assumption for the process $(r_{t+1}, \sigma_{t+1}^2)$, we will simply write:

$$L\left(u, v \left| I_t \right.\right) = E\left[\exp\left(-u\sigma_{t+1}^2\right) L_r\left(v \left| \sigma_t^2, \sigma_{t+1}^2 \right.\right) \middle| \sigma_t^2\right],$$

where:

$$L_r\left(v \left| \sigma_t^2, \sigma_{t+1}^2 \right.\right) \equiv E\left[\exp\left(-vr_{t+1}\right) \middle| I_t^\sigma\right],$$

and I_t^σ stands for the augmented filtration:

$$I_t^\sigma = \left\{r_s, r_{f,s}, \sigma_{s+1}; s \leq t\right\}.$$

The focus of our interest is precisely the possible difference between two conditional distributions of returns:

$$L_r\left(v \left| \sigma_t^2, \sigma_{t+1}^2 \right.\right) \neq L\left(0, v \left| I_t \right.\right) = E\left[L_r\left(v \left| \sigma_t^2, \sigma_{t+1}^2 \right.\right) \middle| \sigma_t^2\right].$$

This difference, that could be dubbed instantaneous causality between return and volatility factor, is known to have several economic interpretations like "leverage effect" or "volatility feedback".

Of course, the above maintained assumptions imply in particular the following restriction on the joint Laplace transform:

$$L\left(u, v \left| I_t \right.\right) = L_{\sigma,r}\left(u, v \left| \sigma_t^2 \right.\right).$$

Following Darolles et al. (2006), we will assume throughout that this Laplace transform defines a bivariate compound autoregressive process of order one (Car(1)), that is:

$$L_{\sigma,r}\left(u, v \left| \sigma_t^2 \right.\right) = \exp\left\{-l\left(u, v\right)\sigma_t^2 - g\left(u, v\right)\right\}. \tag{2.1}$$

It implies in particular a univariate Car(1) model for the volatility factor:

$$L_\sigma\left(u \left| \sigma_t^2 \right.\right) = \exp\left\{-a\left(u\right)\sigma_t^2 - b\left(u\right)\right\}, \tag{2.2}$$

while we also assume a similar structure for the conditional distribution of r_{t+1} given I_t and σ_{t+1}^2:

$$L_r\left(v \left| \sigma_t^2, \sigma_{t+1}^2 \right.\right) = \exp\left\{-\alpha\left(v\right)\sigma_{t+1}^2 - \beta\left(v\right)\sigma_t^2 - \gamma\left(v\right)\right\}. \tag{2.3}$$

In summary, our model is specified through the definition of five functions $a(\cdot)$, $b(\cdot)$, $\alpha(\cdot)$, $\beta(\cdot)$, and $\gamma(\cdot)$ in definitions of Laplace transforms (2.2) and (2.3). The functions $a(\cdot)$, $b(\cdot)$, $\alpha(\cdot)$, $\beta(\cdot)$, and $\gamma(\cdot)$ are all defined on some neighborhood of zero in the complex plan, and we have by definition:

$$a(0) = b(0) = \alpha(0) = \beta(0) = \gamma(0) = 0.$$

We will assume throughout the existence of the first and second derivatives at zero of these five functions.

Note that our joint model for return and volatility factor, as defined by (2.2) and (2.3) is akin to impose the following constraint on the bivariate Car(1) defined by (2.1):

$$\begin{aligned} l\left(u,v\right) &= a\left[u+\alpha\left(v\right)\right]+\beta\left(v\right), \\ g\left(u,v\right) &= b\left[u+\alpha\left(v\right)\right]+\gamma\left(v\right). \end{aligned} \tag{2.4}$$

2.2 Return volatility

From the conditional Laplace transform of return given I_t^σ:

$$E\left[\exp\left(-vr_{t+1}\right)\middle|\, I_t^\sigma\right] = \exp\left\{-\alpha\left(v\right)\sigma_{t+1}^2-\beta\left(v\right)\sigma_t^2-\gamma\left(v\right)\right\}$$

we deduce the first two conditional moments:

$$\begin{aligned} E\left[r_{t+1}\middle|\, I_t^\sigma\right] &= \alpha'\left(0\right)\sigma_{t+1}^2+\beta'\left(0\right)\sigma_t^2+\gamma'\left(0\right), \\ V\left[r_{t+1}\middle|\, I_t^\sigma\right] &= -\alpha''\left(0\right)\sigma_{t+1}^2-\beta''\left(0\right)\sigma_t^2-\gamma''\left(0\right). \end{aligned}$$

This suggests that the missing link between the volatility factor σ_{t+1}^2 and the return (squared volatility) $V\left[r_{t+1}\middle|\, I_t\right]$ goes through the modified volatility factor:

$$\tilde{\sigma}_{t+1}^2 = \sigma_{t+1}^2+\frac{\beta''\left(0\right)\sigma_t^2+\gamma''\left(0\right)}{\alpha''\left(0\right)}. \tag{2.5}$$

More precisely, the variance decomposition allows us to write:

$$V\left[r_{t+1}\middle|\, I_t\right] = \left[\alpha'\left(0\right)\right]^2 V\left[\tilde{\sigma}_{t+1}^2\middle|\, I_t\right]-\alpha''\left(0\right)E\left[\tilde{\sigma}_{t+1}^2\middle|\, I_t\right].$$

In other words, volatility dynamics is created by the conjunction of two effects:

- First a time varying risk premium, leading to a variance of the expected return equal to $\left[\alpha'\left(0\right)\right]^2 V\left[\tilde{\sigma}_{t+1}^2\middle|\, I_t\right]$,
- Second the expected integrated (squared) volatility, dampened by leverage effect.

We mean that we interpret the modified volatility factor $\tilde{\sigma}_{t+1}^2$ as integrated variance (in continuous time):

$$\tilde{\sigma}_{t+1}^2 = \int_t^{t+1}\sigma^2\left(u\right)du,$$

while its effect on return volatility is dampened by leverage effect:

$$V\left[r_{t+1}-E\left[r_{t+1}\middle|\, I_t^\sigma\right]\middle|\, I_t\right] = -\alpha''\left(0\right)E\left[\tilde{\sigma}_{t+1}^2\middle|\, I_t\right]\leq E\left[\tilde{\sigma}_{t+1}^2\middle|\, I_t\right].$$

Hence the parameterization:

$$\begin{aligned} \alpha'(0) &= \psi, \\ -\alpha''(0) &= 1-\phi^2, \quad |\phi|<1, \end{aligned}$$

so that:

$$V\left[r_{t+1}|\, I_t\right] = \psi^2 V\left[\tilde{\sigma}^2_{t+1}\middle|\, I_t\right] + \left(1-\phi^2\right) E\left[\tilde{\sigma}^2_{t+1}\middle|\, I_t\right]. \tag{2.6}$$

It is worth noting that, by contrast for instance with Feunou & Tedongap (2012), we do not want to maintain a decomposition:

$$r_{t+1} = E\left[r_{t+1}|\, I_t^{\sigma}\right] + \tilde{\sigma}_{t+1}\varepsilon_{t+1}, \quad V\left[\varepsilon_{t+1}|\, I_t^{\sigma}\right] = 1,$$

because it would impose $\phi = 0$, meaning that all possible leverage effect is confused with volatility feedback inside expected return. By contrast, both parameters ψ and ϕ play a role in our measure of leverage effect since the conditional correlation between return and volatility is:

$$Corr\left[r_{t+1}, \tilde{\sigma}^2_{t+1}\middle|\, I_t\right] = \frac{Corr\left[\psi\tilde{\sigma}^2_{t+1}, \tilde{\sigma}^2_{t+1}|\, I_t\right]}{\sqrt{V\left[\tilde{\sigma}^2_{t+1}|\, I_t\right]}\sqrt{\psi^2 V\left[\tilde{\sigma}^2_{t+1}|\, I_t\right] + \left(1-\phi^2\right) E\left[\tilde{\sigma}^2_{t+1}|\, I_t\right]}},$$

that can be written:

$$Corr\left[r_{t+1}, \tilde{\sigma}^2_{t+1}\middle|\, I_t\right] = \frac{\phi}{\sqrt{\phi^2 + \left(1-\phi^2\right)\frac{k_t^2}{\tilde{k}^2}}},$$

where:

$$\psi = \tilde{k}\phi, \quad k_t^2 = \frac{E\left[\tilde{\sigma}^2_{t+1}|\, I_t\right]}{V\left[\tilde{\sigma}^2_{t+1}|\, I_t\right]}.$$

For sake of subsequent discussion, it will be worth recalling that, if for whatever reason, the random variable k_t^2 takes values quite concentrated around $\tilde{k}^2$, then we have:

$$\begin{aligned} Corr\left[r_{t+1}, \tilde{\sigma}^2_{t+1}\middle|\, I_t\right] &\approx \phi, \\ V\left[r_{t+1}|\, I_t\right] &\approx E\left[\tilde{\sigma}^2_{t+1}\middle|\, I_t\right]. \end{aligned}$$

We will actually discuss the parameterization further with these approximations in mind: we want to see the return conditional variance $V\left[r_{t+1}|\, I_t\right]$ as the conditional expectation of the "integrated variance" $\tilde{\sigma}^2_{t+1}$ and the conditional correlation coefficient $Corr\left[r_{t+1}, \tilde{\sigma}^2_{t+1}|\, I_t\right]$ that measures the leverage effect as almost equal to the parameter ϕ. In particular, we will always see this parameter as lying in the interval $(-1, 0]$.

2.3 Affine Volatility Dynamics

2.3.1 Model specification

As already discussed in Darolles et al. (2006), the Car modeling of volatility nicely fits into the affine framework, as of popular use in financial econometrics (see e.g. Duffie et al., 2000, 2003; Meddahi & Renault, 2004). To see that, note that we immediately deduce from (2.2) that:

$$\begin{aligned} E\left[\sigma_{t+1}^2 \middle| I_t\right] &= a'(0)\,\sigma_t^2 + b'(0)\,, \\ V\left[\sigma_{t+1}^2 \middle| I_t\right] &= -a''(0)\,\sigma_t^2 - b''(0)\,. \end{aligned} \tag{2.7}$$

We can then without loss of generality characterize the conditional mean and variance of the volatility factor through four parameters ρ, c, δ, and ω defined as follows:

$$\begin{aligned} \rho &= a'(0) \in (0,1)\,, \\ c &= -\frac{a''(0)}{2a'(0)} > 0, \\ \delta &= -2\frac{a'(0)\,b'(0)}{a''(0)} > 0, \\ \omega &= -4\frac{b''(0)\,[a'(0)]^2}{[a''(0)]^2} > 0, \end{aligned}$$

such that the affine volatility dynamics are parameterized as follows:

$$\begin{aligned} E\left[\sigma_{t+1}^2 \middle| I_t\right] &= \rho\sigma_t^2 + \delta c, \\ V\left[\sigma_{t+1}^2 \middle| I_t\right] &= 2c\rho\sigma_t^2 + \omega c^2. \end{aligned} \tag{2.8}$$

Note that the positive parameter c is nothing but a scale parameter for the volatility factor σ_{t+1}^2. In particular, this volatility factor σ_{t+1}^2 is endowed with AR(1) dynamics characterized by the correlation coefficient $\rho \in [0,1)$. Note that in general the modified volatility factor, that we interpret as integrated variance:

$$\tilde{\sigma}_{t+1}^2 = \sigma_{t+1}^2 + \frac{\beta''(0)\,\sigma_t^2 + \gamma''(0)}{\alpha''(0)}$$

will be endowed with ARMA(1,1) dynamics.

More precisely, $\tilde{\sigma}_{t+1}^2$ is ARMA(1,1) (resp. AR(1)) when we have a non-zero (resp zero) $\beta''(0)$, for instance depending whether the function $\beta(\cdot)$ is linear or not. If one realizes that integrated variance is almost "observed" through realized variance computed with high frequency data, this model specification issue is germane with considering squared returns that are AR(1) (as in the ARCH(1) model) or simply ARMA(1,1) (as in the GARCH(1,1) model).

2.3.2 Leading example: ARG(1) volatility process

Following Gourieroux & Jasiak (2006), we will pay a special attention to the so-called autoregressive gamma process ARG(1) which is defined by:

$$a(u) = \frac{\rho u}{1 + cu}, \quad b(u) = \delta \log(1 + cu), \tag{2.9}$$

with

$$\rho \in [0, 1), \quad c > 0, \quad \delta > 0.$$

Obviously, this model fits within the general class of affine volatility models (2.8) discussed above, while imposing the constraint $\delta = \omega$. For $c = 1$, this model can be interpreted by introducing a latent variable with integer values Z_t such that:

- The conditional distribution of Z_t given σ_t^2 is Poisson with parameter $\rho\sigma_t^2$,
- The conditional distribution of σ_{t+1}^2 given $\left(\sigma_t^2, Z_t\right)$ is gamma with degree of freedom $(\delta + Z_t)$.

Note, in particular, that we have in this case:

$$E\left[\sigma_{t+1}^2 \middle| \sigma_t^2\right] = E\left[\delta + Z_t \middle| \sigma_t^2\right] = \delta + \rho\sigma_t^2,$$

such that the condition $\rho \in [0, 1)$ is devised to ensure both persistence and mean reversion for the process of the volatility factor.

If we consider more generally that the above Poisson-mixture of gamma distributions actually characterizes the conditional probability distribution of $\left(\sigma_{t+1}^2/c\right)$ given $\left(\sigma_t^2/c\right)$, we end up with a conditional distribution of σ_{t+1}^2 given σ_t^2 corresponding to the Laplace transform defined by (2.9). Again, c is nothing but a scaling volatility factor.

2.4 Temporal Aggregation and Continuous Time Limit

2.4.1 Temporal aggregation of the volatility factor as a Markov state

Model (2.7) specifies the joint process $\left(\sigma_t^2, \sigma_t^4\right)$ as a vector auto-regressive VAR(1) process. This setting is robust to temporal aggregation.

Proposition 1 (Volatility factor aggregation)**.** *The affine volatility dynamics defined in* (2.8) *implies that*

$$\begin{aligned} E\left[\sigma_{t+1}^2 \middle| I_t\right] &= \rho\sigma_t^2 + c\delta, \\ E\left[\sigma_{t+1}^4 \middle| I_t\right] &= \rho^2\sigma_t^4 + a\sigma_t^2 + b, \end{aligned}$$

with:

$$\begin{aligned} a &= 2c\rho(1 + \delta), \\ b &= c^2\left(\omega + \delta^2\right), \end{aligned}$$

and then, for any integer $H \geq 2$:

$$E\left[\sigma_{t+H}^2\middle| I_t\right] = \rho^H \sigma_t^2 + c\delta\left(H\right),$$
$$E\left[\sigma_{t+H}^4\middle| I_t\right] = \rho^{2H}\sigma_t^4 + a\rho^{H-1}\frac{1-\rho^H}{1-\rho}\sigma_t^2 + d\left(H\right),$$

where

$$\delta\left(H\right) = \delta\frac{1-\rho^H}{1-\rho},$$
$$d\left(H\right) = ac\frac{1-\rho^{H-1}}{1-\rho^2}\delta\left(H\right) + b\frac{1-\rho^{2H}}{1-\rho^2}.$$

Proof. See Appendix C.1. □

A byproduct of temporal aggregation formulas is to display without ambiguity what should be the disaggregation formulas, that is the formulas for small time intervals of length $\Delta < 1$ that are subsets of the unit time interval considered so far. Therefore, we can revisit for instance Le et al. (2010) when (see p. 2203) they "put the model parameters in connection with the time interval Δ." By contrast with them, we do not need to resort to any approximation by Euler discretization and just write:

Proposition 2 (Volatility factor disaggregation)**.** *For any integer $N \geq 2$:*

$$E\left[\sigma_{t+1/N}^2\middle| I_t\right] = \rho^{1/N}\sigma_t^2 + \delta_N c,$$
$$E\left[\sigma_{t+1/N}^4\middle| I_t\right] = \rho^{2/N}\sigma_t^4 + a_N\sigma_t^2 + b_N,$$

where

$$\delta_N = \delta\frac{1-\rho^{1/N}}{1-\rho},$$
$$a_N = a\frac{1-\rho^{1/N}}{1-\rho}\rho^{-1+1/N},$$
$$b_N = b\frac{1-\rho^{2/N}}{1-\rho^2} - ac\delta\frac{\left(1-\rho^{1/N}\right)\left(\rho^{-1+1/N}-1\right)}{\left(1-\rho\right)\left(1-\rho^2\right)},$$

Proof. This proposition is proven (see Appendix C.2) by checking that, when aggregating these formulas over N consecutive periods (just applying Proposition 1 with $H = N$ and parameters $\left[\rho^{1/N}, \delta_N, a_N, b_N\right]$ instead of $[\rho, \delta, a, b]$), one gets the initial formulas (2.8). □

2.4.2 Temporal aggregation of integrated volatility

As far as option pricing or hedging is concerned (see e.g. Mykland, 2000), the key challenge is to take a stab at predicting the cumulative volatility. In other words, for option pricing at maturity H, the object of interest should be:

$$\sigma_{t,H}^{2}\left(N\right)=\frac{1}{HN}\sum_{n=1}^{HN}\sigma_{t+n/N}^{2}, \tag{2.10}$$

where N is the number of subintervals in a unit interval. Our normalization by the factor HN allows up to keep the interpretation of each $\sigma_{t,H}^{2}(N)$ as a volatility factor on a given (the smallest possible) unit of time.

It is well known (see e.g. Bollerslev & Zhou, 2002; Meddahi, 2003; Meddahi & Renault, 2004) that with a spot volatility process that is autoregressive AR(1), the cumulative volatility process (also called integrated variance) will be ARMA(1,1). In other words, the expectation equations (2.8) should be only understood for a minimum period of time while, for any longer period, additional errors would be MA(1) instead of martingale difference sequences. The price to pay for discrete time modeling is some arbitrariness on what we call the "minimum period of time". Proposition 4 below will show that we could actually consider an infinitesimal period, in a continuous time framework. In order to define non-ambiguously a continuous time limit of our model, it is worth checking first its robustness to temporal aggregation, at least as regards the ARMA(1,1) specification of the dynamics of cumulative volatility $\sigma_{t}^{2}(H;N)$ as well as its square, as implied by the two equations in (2.8).

Proposition 3 (Volatility factor as ARMA process)**.** *The aggregated volatility in (2.10) satisfies two ARMA(1,1)-type conditional moment restrictions:*

$$\begin{aligned} E\left[\sigma_{t+H,H}^{2}\left(N\right)-\rho^{H}\sigma_{t,H}^{2}\left(N\right)-c\delta\left(H\right)\middle|\tilde{\mathcal{F}}_{t}\right]&=0,\\ E\left[\sigma_{t+H,H}^{4}\left(N\right)-\rho^{2H}\sigma_{t,H}^{4}\left(N\right)-a\left(H;N\right)\sigma_{t,H}^{2}\left(N\right)-b\left(H;N\right)\middle|\tilde{\mathcal{F}}_{t}\right]&=0, \end{aligned} \tag{2.11}$$

for deterministic coefficients $\delta(H)$, $a(H;N)$, and $b(H;N)$, are given in (C.2), (C.13), and (C.14), respectively, for any $H,N=1,2,\ldots$, and information set $\tilde{\mathcal{F}}_{t}=\left\{\sigma_{t-kH,H}^{2}(N),\ k\geq1\right\}$.

Proof. See Appendix C.3. □

2.4.3 Continuous time limit

A byproduct of the temporal (dis)aggregation formulas given in the former subsection is to provide an unambiguous definition of the continuous time limit of our model. It is obviously about the limit of the above formulas when N goes to infinity. However, for the sake of getting the instantaneous analog of $\sigma_{t,H}^{2}(N)$, we will also consider that the horizon H may go to zero, while always assuming $HN\geq1$ and (for sake of notational simplicity) maintaining the assumption that HN is an integer.

The following lemma, directly deduced from Proposition 2, will be useful to get the continuous time limit of our model:

Lemma 1. *It is true that*

$$\lim_{H\to0}E\left[\sigma_{t,H}^{2}\left(N\right)\middle|I_{t}\right]=\sigma_{t}^{2}, \tag{2.12}$$

$$\lim_{H\to0}\frac{1}{H}V\left[\sigma_{t,H}^{2}\left(N\right)\middle|I_{t}\right]=\frac{1}{2}\lim_{H\to0}\frac{1}{H}V\left[\sigma_{t+H,H}^{2}\left(N\right)\middle|I_{t}\right]. \tag{2.13}$$

Proof. See Appendix C.4. □

We can then prove:

Proposition 4 (Continuous-time limit)**.** *The ARMA(1,1)-type model* (2.11) *for all integer N and $H \in [1/N, +\infty)$, jointly with the regularity conditions* (2.12) *and* (2.13) *implies that:*

$$\begin{aligned} \lim_{H\to 0} \frac{1}{H} E\left[\sigma^2_{t+H,H}(N) - \sigma^2_{t,H}(N) \middle| \tilde{\mathcal{F}}_t \right] &= -\log(\rho)\left(\frac{c\delta}{1-\rho} - \sigma^2_t \right), \\ \lim_{H\to 0} \frac{1}{H} V\left[\sigma^2_{t,H}(N) \middle| \tilde{\mathcal{F}}_t \right] &= -2c\frac{\log(\rho)}{1-\rho}\left(\sigma^2_t + c\frac{\delta-\omega}{1+\rho} \right). \end{aligned}$$

Proof. See Appendix C.5. □

In other words, in terms of local behavior of the first two moments, the spot volatility factor σ^2_t can be seen as a Brownian diffusion process driven by the stochastic differential equation:

$$d\sigma^2_t = \kappa\left(\bar{\sigma}^2 - \sigma^2_t\right) dt + \sqrt{\nu + \eta\sigma^2_t}\, dW_t$$

for some Wiener process W_t and:

$$\begin{aligned} \kappa &= -\log(\rho) > 0, \\ \bar{\sigma}^2 &= \frac{c\delta}{1-\rho} = E\left[\sigma^2_t\right] > 0, \\ \eta &= \frac{2\kappa}{\delta}\bar{\sigma}^2, \\ \nu &= \eta c\frac{\delta-\omega}{1+\rho} \geq 0, \quad \text{if} \quad \delta \geq \omega. \end{aligned}$$

Therefore, if $\omega = \delta$ we get for σ^2_t a square root process of Feller (1951), as used for interest rate by Cox et al. (1985) and for volatility by Heston (1993). The three parameters $(\kappa, \bar{\sigma}^2, \eta)$ are unconstrained (up to standard inequality constraints) as one-to-one functions of the three initial parameters ρ, δ, and c. Therefore, as far as the first two moments are concerned, any square root process can be seen as a continuous time limit of our volatility factor model. In particular, this volatility factor will never hit the zero barrier when $\delta(1-\rho) \geq -2c\log(\rho)$, since it is tantamount to the classical necessary and sufficient condition $2\kappa\bar{\sigma}^2 \geq \eta^2$. More generally, any affine process in continuous time (Duffie et al., 2000) can be seen as the continuous time limit of our discrete time model thanks to the degree of freedom $\omega \neq \delta$. Recall that the ARG(1) model of Gourieroux & Jasiak (2006) maintains the constraint $\omega = \delta$, the reason why they only get the square root process as continuous time limit. Even in the particular case $\omega = \delta$, our result is more general since it is semi-parametric in nature; we do not maintain the ARG (1) parametric specification. This actually means that our continuous time limit must be interpreted only as regards the first two moments. The advantage of the discrete time specification is that, by contrast with Brownian diffusions, the specification of the first two conditional moments does not constrain us regarding higher order moments. This may allow us in particular to accommodate stylized facts that take jumps both in returns and in volatility (see e.g. Bandi & Reno, 2016) to be captured by a continuous time model.

3 Affine Risk-neutral Probability Measure

3.1 Structure-preserving change of measure

A pricing kernel $M_{t,t+1}(\theta)$, function of some preference parameters θ, would allow to characterize the price C_t at time t of any payoff at time $(t+1)$ function $H(r_{t+1}, \sigma^2_{t+1}, I_t)$ of the state variables as:

$$C_t = E\left[M_{t,t+1}(\theta) H\left(r_{t+1}, \sigma^2_{t+1}, I_t\right)\Big| I_t\right]. \tag{3.1}$$

Then, a risk-neutral probability measure should allow to compute these prices as discounted expected payoffs:

$$C_t = \exp(-r_{f,t}) E^*\left[H\left(r_{t+1}, \sigma^2_{t+1}, I_t\right)\Big| I_t\right] \tag{3.2}$$

where $E^*[\cdot|I_t]$ stands for the conditional expectation operator with respect to the risk-neutral probability distribution. This change of measure between historical and risk-neutral will be dubbed "structure-preserving" if the specification of risk-neutral dynamics goes through the specification of some functions $a^*(\cdot)$, $b^*(\cdot)$, $\alpha^*(\cdot)$, $\beta^*(\cdot)$, and $\gamma^*(\cdot)$ in a way similar to (2.2) and (2.3):

$$\begin{aligned}
E^*\left[\exp\left(-u\sigma^2_{t+1}\right)\Big|\sigma^2_t\right] &= \exp\left\{-a^*(u)\sigma^2_t - b^*(u)\right\}, \\
E^*\left[\exp(-vr_{t+1})\Big|\sigma^2_t, \sigma^2_{t+1}\right] &= \exp\left\{-\alpha^*(v)\sigma^2_{t+1} - \beta^*(v)\sigma^2_t - \gamma^*(v)\right\}.
\end{aligned}$$

By comparing (3.1) and (3.2), the structure preserving property is tantamount to the following conditions for any complex numbers (u, v) in a convenient neighborhood of zero:

$$\begin{aligned}
\exp(r_{f,t}) E\left[M_{t,t+1}(\theta)\exp\left(-u\sigma^2_{t+1} - vr_{t+1}\right)\Big| I_t\right] &= \exp\left\{-l^*(u,v)\sigma^2_t - g^*(u,v)\right\}, \\
l^*(u,v) &= a^*(u + \alpha^*(v)) + \beta^*(v), \\
g^*(u,v) &= b^*(u + \alpha^*(v)) + \gamma^*(v),
\end{aligned}$$

in order to preserve the structure (2.1) with (2.4) described in Section 2.

This existence of functions $l^*(u,v)$ and $g^*(u,v)$ to deduce a risk-neutral bivariate Car(1) from the historical one is obviously achieved by an exponential affine pricing kernel:

$$M_{t,t+1}(\theta) = \exp(-r_{f,t})\exp\left\{m_0(\theta) + m_1(\theta)\sigma^2_t - \theta_1\sigma^2_{t+1} - \theta_2 r_{t+1}\right\}, \tag{3.3}$$

where:

- θ_1 and θ_2 are the two preference parameters corresponding to the two sources of risk. Parameter θ_1, expected to be non-positive, characterizes the price of volatility risk, while θ_2, expected to be nonnegative, characterizes the price of equity risk.

- The functions $m_0(\theta)$ and $m_1(\theta)$, with $\theta = (\theta_1, \theta_2)$, are defined in order to match the exogenously specified dynamics of interest rate, which is akin to the following restriction:

$$E\left[\exp\left\{m_0(\theta)+m_1(\theta)\sigma_t^2-\theta_1\sigma_{t+1}^2-\theta_2 r_{t+1}\right\}\Big| I_t\right]=1,$$

or equivalently, by the law of iterated expectations:

$$E\left[\exp\left\{m_0(\theta)+m_1(\theta)\sigma_t^2-\theta_1\sigma_{t+1}^2\right\}\exp\left\{-\alpha(\theta_2)\sigma_{t+1}^2-\beta(\theta_2)\sigma_t^2-\gamma(\theta_2)\right\}\Big| I_t\right]=1,$$

meaning that:

$$\begin{aligned}m_0(\theta)&=\gamma(\theta_2)+b\left[\alpha(\theta_2)+\theta_1\right],\\ m_1(\theta)&=\beta(\theta_2)+a\left[\alpha(\theta_2)+\theta_1\right],\end{aligned}\tag{3.4}$$

that is, from (2.4):

$$m_0(\theta)=g(\theta),\quad m_1(\theta)=l(\theta).$$

The characterization of functions $a^*(\cdot)$, $b^*(\cdot)$, $\alpha^*(\cdot)$, $\beta^*(\cdot)$, and $\gamma^*(\cdot)$ to achieve the right decompositions of functions $l^*(u,v)$ and $g^*(u,v)$ will be discussed in Section 3.2 below.

3.2 Risk-neutral parameters

As usual, the risk-neutral dynamics as defined by functions $a^*(\cdot)$, $b^*(\cdot)$, $\alpha^*(\cdot)$, $\beta^*(\cdot)$, and $\gamma^*(\cdot)$ are fully known when we know both the historical dynamics, that is the functions $a(\cdot)$, $b(\cdot)$, $\alpha(\cdot)$, $\beta(\cdot)$, and $\gamma(\cdot)$, and the price of risk parameters θ_1 and θ_2. More precisely we can prove:

Proposition 5. *Assume that historical dynamics are defined by a constrained bivariate Car(1) as in Section 2 (equations 2.1 and 2.4), and we have an exponential affine pricing kernel as described in Section 3.1 (equation 3.3). Then, risk-neutral dynamics are defined by the following constrained bivariate Car(1):*

$$\begin{aligned}E^*\left[\exp\left(-u\sigma_{t+1}^2-vr_{t+1}\right)\Big| I_t\right]&=\exp\left\{-l^*(u,v)\sigma_t^2-g^*(u,v)\right\},\\ l^*(u,v)&=a^*\left[u+\alpha^*(v)\right]+\beta^*(v),\\ g^*(u,v)&=b^*\left[u+\alpha^*(v)\right]+\gamma^*(v),\end{aligned}$$

where:

$$\begin{aligned}\alpha^*(v)&=\alpha(\theta_2+v)-\alpha(\theta_2),\\ \beta^*(v)&=\beta(\theta_2+v)-\beta(\theta_2),\\ \gamma^*(v)&=\gamma(\theta_2+v)-\gamma(\theta_2),\end{aligned}\tag{3.5}$$

and:

$$\begin{aligned}a^*(u)&=a\left[u+\theta_1+\alpha(\theta_2)\right]-a\left[\theta_1+\alpha(\theta_2)\right],\\ b^*(u)&=b\left[u+\theta_1+\alpha(\theta_2)\right]-b\left[\theta_1+\alpha(\theta_2)\right].\end{aligned}\tag{3.6}$$

Proof. See Section C.6 of the Appendix. □

We note that the wedge between historical (resp. risk neutral) return dynamics (given the volatility path) as described by the difference between functions $\alpha(\cdot)$, $\beta(\cdot)$, and $\gamma(\cdot)$ (resp. $\alpha^*(\cdot)$, $\beta^*(\cdot)$, and $\gamma^*(\cdot)$) depends only on the (arguably positive) price θ_2 of the equity risk but not on the (arguably negative) price θ_1 of the volatility risk. However, it would be premature to conclude that the historical return dynamics is not informative about the price θ_1 of the volatility risk. The key remark is that return dynamics entail volatility dynamics and for the latter, the wedge between risk neutral and historical dynamics depends in general on both risk premium parameters through the function $\xi(\theta) = \theta_1 + \alpha(\theta_2)$. It is only when the leverage function is identically nil that the role of the two parameters is clearly disentangled: price of equity risk in return dynamics, price of volatility risk in volatility dynamics.

Finally, it is worth noting that Proposition 5 shows that the functions $\alpha(\cdot)$, $\beta(\cdot)$, $\gamma(\cdot)$, $a(\cdot)$, and $b(\cdot)$ and their risk neutral analog are tightly related by a shape restriction: their derivatives should coincide up to a translation of the variables. This means that the change of measure can be "structure preserving", that is allowing the econometrician to use the same parametric model for historical and risk-neutral dynamics, only when functions $\alpha(\cdot)$, $\beta(\cdot)$, $\gamma(\cdot)$, $a(\cdot)$, and $b(\cdot)$ are defined by polynomials, and/or ratio and logarithms of these polynomials. In this respect, the inverse Gaussian distribution whose Laplace transform involves a square root function and that may be well suited to accommodate return skewness does not fulfill the restriction of structure preserving change of measure. This leads Feunou & Tedongap (2012) to use it separately for historical and risk-neutral modeling, while the two models cannot be consistent together.

3.3 Identification of prices of risk

The stock pricing equation encapsulates the identification information brought by the observation of underlying asset return data:

$$E\left[M_{t,t+1}(\theta)\exp(r_{t+1})\middle| I_t\right] = 1,$$

that can be rewritten:

$$E\left[\exp\left\{-\theta_1\sigma_{t+1}^2 - (\theta_2 - 1)r_{t+1}\right\}\middle| I_t\right] = \exp\left\{-m_0(\theta) - m_1(\theta)\sigma_t^2\right\},$$

with $m_0(\theta)$ and $m_1(\theta)$ defined by (3.4). By the law of iterative expectations again, these equations are equivalent to the two following equations about the two unknown parameters θ_1 and θ_2:

$$\begin{aligned}\gamma(\theta_2 - 1) + b[\alpha(\theta_2 - 1) + \theta_1] &= \gamma(\theta_2) + b[\alpha(\theta_2) + \theta_1],\\ \beta(\theta_2 - 1) + a[\alpha(\theta_2 - 1) + \theta_1] &= \beta(\theta_2) + a[\alpha(\theta_2) + \theta_1].\end{aligned} \tag{3.7}$$

It is worth realizing that the two equations (3.7) may lead to two very different identification schemes:

- One can find a price θ_2 of equity risk which solves the three equations:

$$\alpha(\theta_2) = \alpha(\theta_2 - 1), \quad \beta(\theta_2) = \beta(\theta_2 - 1), \quad \gamma(\theta_2) = \gamma(\theta_2 - 1), \tag{3.8}$$

Then, the two equations (3.7) are fulfilled for this value of θ_2, irrespective of the value of the price θ_1 of volatility risk. This price is not constrained at all by the observed value of the stock price. This lack of identification result is actually consistent with a common belief. It takes derivative asset prices to identify the price of volatility risk. This price is not identified by the stock price alone.

- However, it is worth keeping in mind that there is another possible identification scheme where the solution (θ_1, θ_2) of (3.7) does not fulfill (3.8), that is θ_1 is also constrained by (3.7).

In order to elicit the right identification scheme, it helps to get started by answering the following question: "Is the solution θ_2 of the equation $\alpha(\theta_2) = \alpha(\theta_2 - 1)$ a reasonable value for the price of equity risk?" Our answer to this question will be yes if and only if there is no leverage effect ($\phi = 0$). To see that, we can investigate the case where the function $\alpha(\cdot)$ is well approximated by its quadratic Taylor expansion in the neighborhood of zero (true at least in the conditionally Gaussian case). Then:

$$\begin{aligned}\alpha(\theta_2) - \alpha(\theta_2 - 1) \approx& \alpha'(0) + \alpha''(0)\left(\theta_2 - \frac{1}{2}\right) \\ =& \psi - \left(1 - \phi^2\right)\left(\theta_2 - \frac{1}{2}\right).\end{aligned}$$

Therefore, the question asked amounts to wonder whether it sounds reasonable to assume that:

$$\psi = \left(1 - \phi^2\right)\left(\theta_2 - \frac{1}{2}\right). \tag{3.9}$$

We will now argue that the answer must be negative when there is genuine leverage effect, that is a non-zero parameter ϕ. To see that, it is worth recalling that the parameter $\psi = \alpha'(0)$ has been devised to encapsulate the role of current volatility in the return forecast equation. It follows directly from this equation that:

$$E\left[r_{t+1} \middle| I_t^\sigma\right] - E\left[r_{t+1} \middle| I_t\right] = \psi\left\{\sigma_{t+1}^2 - E\left[\sigma_{t+1}^2 \middle| I_t\right]\right\}.$$

In other words, when forecasting at horizon 1 from date t, the knowledge of future volatility σ_{t+1}^2 may help to improve the forecast of the future return r_{t+1} in due proportion of the parameter ψ. As well reminded by Bollerslev et al. (2006), this forecast improvement stems from two different economic phenomenons that are impossible to disentangle in discrete time:

- First, volatility feedback,
- Second, leverage effect.

While the importance of the former effect is directly drawn by the size of the equity risk price θ_2, the latter should on the contrary be proportional to the leverage effect parameter ϕ. This leads us to consider that the value of ψ should be an aggregate computed as follows:

$$\psi = H\left[\left(1 - \phi^2\right)\left(\theta_2 - \frac{1}{2}\right), k\phi\right] \tag{3.10}$$

In particular, when $\phi \neq 0$, this aggregation formula implies that the constraint (3.9) will be violated. Note that we assume that this aggregate depends on the two key parameters only through their rescaled impact (3.9) and $k\phi$. For the former, the dampening by $(1-\phi^2)$ due to the effect of conditioning by the volatility path and the Jensen effect ($\theta_2 - 1/2$ instead of θ_2) are taken into account. For the latter, the impact of leverage must be rescaled by a parameter k as explained in Section 2. Typically, the constraint (3.10) will be implemented with the specification:

$$k^2 = \frac{E\left[\tilde{\sigma}^2_{t+1}\right]}{E\left[V\left[\tilde{\sigma}^2_{t+1}\middle| I_t\right]\right]} \tag{3.11}$$

in order for the component $k\phi$ of ψ to match as well as possible the natural impact $k_t\phi$ of leverage on coefficient ψ with, as discussed in Section 2:

$$k_t^2 = \frac{E\left[\tilde{\sigma}^2_{t+1}\middle| I_t\right]}{V\left[\tilde{\sigma}^2_{t+1}\middle| I_t\right]}.$$

Irrespective of this specific parameterization, the bottom line of this subsection is that:

- On the one hand, when there is no genuine leverage effect ($\phi = 0$), the stock return data are uninformative about the volatility risk price θ_1. The only identification content of stock return data about prices of risk is encapsulated in the three identities:

$$\varpi\left(\theta_2\right) = \varpi\left(\theta_2 - 1\right), \quad \varpi\left(\cdot\right) \in \left\{\alpha\left(\cdot\right), \beta\left(\cdot\right), \gamma\left(\cdot\right)\right\}. \tag{3.12}$$

  In particular, if the function $\alpha\left(\cdot\right)$ is well approximated by its quadratic expansion around zero:

$$\psi \approx \theta_2 - \frac{1}{2}.$$

- On the other hand, when there is leverage effect ($\phi \neq 0$), the constraints (3.12) are likely to be violated. On the contrary the two prices of risk (θ_1, θ_2) are likely to be jointly determined as solutions of the two equations (3.8) stemming from stock pricing. In particular, we don't expect the parameter ψ to be conformable to its approximation (3.9) implied by (3.12), but rather to be an aggregate of two effects according to (3.10).

While Gagliardini et al. (2011) had drawn the implications in terms of identification of the volatility risk price from option data in the former case, Bandi & Reno (2016) have more recently provided some evidence that in the latter case, due to leverage effect, the volatility risk price can be identified from return data only. However, they stress that this identification is fragile and obtained by a sophisticated inference strategy from high-frequency return data as well as a careful study of possible jumps in both return and volatility process. While the value of the leverage effect parameter is known to be hard to estimate (the so-called "leverage effect puzzle", see Ait-Sahalia et al., 2013) it is not surprising that identification of volatility risk price that relies completely upon this effect (through the impact of in the second component of the aggregation (3.10)) can never be very strong.

3.4 Parameterization for conditionally Gaussian returns

Conditionally Gaussian returns are characterized by a Laplace transforms with functions $\alpha(\cdot)$, $\beta(\cdot)$, $\gamma(\cdot)$ that are all quadratic. It is worth stressing that this does not imply thin tails and/or zero skewness for the marginal distribution of returns. Even the conditional distribution of return r_{t+1} given past information I_t (including stochastic volatility) may be skewed and leptokurtic after integrating out the current volatility. Since risk-neutral Laplace transforms are related to historical ones by formulas:

$$\varpi^*(v) = \varpi(\theta_2 + v), \quad \varpi(\cdot) \in \{\alpha(\cdot), \beta(\cdot), \gamma(\cdot)\},$$

we must have:

$$\begin{aligned}
\varpi^{*\prime}(0) &= \varpi'(\theta_2) = \varpi'(0) + \theta_2 \varpi''(0), \\
\varpi^{*\prime\prime}(0) &= \varpi''(\theta_2), \\
\varpi(\cdot) &\in \{\alpha(\cdot), \beta(\cdot), \gamma(\cdot)\},
\end{aligned}$$

and in particular:

$$\begin{aligned}
\psi^* =& \psi - \left(1 - \phi^2\right)\theta_2, \\
1 - \phi^{*2} =& 1 - \phi^2.
\end{aligned} \tag{3.13}$$

Two conclusions can be drawn from formulas (3.13):

- First, as it is the case in common continuous time models, the amount of genuine leverage effect is the same for historical and risk-neutral dynamics:

$$\phi^* = \phi.$$

- Second, the consistency condition imposing that the risk neutral volatility feedback parameter ψ^* does not depend on equity risk price θ_2 implies that the aggregator (3.10) is additive with respect to this risk price:

$$H\left[\left(1 - \phi^2\right)\left(\theta_2 - \frac{1}{2}\right), k\phi\right] = \left(1 - \phi^2\right)\left(\theta_2 - \frac{1}{2}\right) + \tilde{H}(k\phi).$$

Since we also want to see the volatility feedback parameter directly drawn by $k\phi$ with k defined by (3.11), we will elicit the following specification:

$$\begin{aligned}
\psi &= k\phi + \left(1 - \phi^2\right)\left(\theta_2 - \frac{1}{2}\right), \\
\psi^* &= k\phi - \frac{1}{2}\left(1 - \phi^2\right), \\
k^2 &= \frac{E\left[\tilde{\sigma}_{t+1}^2\right]}{E\left[V\left[\tilde{\sigma}_{t+1}^2 \middle| I_t\right]\right]}.
\end{aligned}$$

It is worth making explicit how this volatility feedback specification impacts the expected return:

$$E\left[\exp\left(r_{t+1}\right)\middle| I_t^\sigma\right] = \exp\left\{-\alpha\left(-1\right)\sigma_{t+1}^2 - \beta\left(-1\right)\sigma_t^2 - \gamma\left(-1\right)\right\},$$

and thus:

$$E\left[\exp\left(r_{t+1}\right)\middle| I_t\right] = \exp\left\{-\beta\left(-1\right)\sigma_t^2 - \gamma\left(-1\right)\right\}\exp\left\{-a\left(\alpha\left(-1\right)\right)\sigma_t^2 - b\left(\alpha\left(-1\right)\right)\right\},$$

with:

$$\alpha\left(-1\right) = -\psi - \frac{1}{2}\left(1-\phi^2\right) = -k\phi - \theta_2\left(1-\phi^2\right).$$

Therefore, the net return $\exp\left(r_{t+1}\right)$ will have a unit conditional expectation when the equity risk price θ_2 is nil if and only if the two following conditions are fulfilled:

$$\begin{aligned}\beta^*\left(-1\right) &= -\,a^*\left(-k\phi\right),\\ \gamma^*\left(-1\right) &= -\,b^*\left(-k\phi\right).\end{aligned} \tag{3.14}$$

In other words, the equity risk compensation introduces a wedge between the specification of the return dynamics and the risk neutral volatility dynamics. The restriction (3.14) will be maintained throughout. Note that the value of k will be implied by the specification of the volatility dynamics as described below.

3.5 Risk Neutral Parameters for ARG(1) dynamics:

Besides conditionally Gaussian returns, we set a special focus on ARG(1) volatility dynamics defined in Section 2.3.2. Even though our continuous-time limit results show that our most general model is consistent with continuous time affine diffusions, it is worth considering discrete time distributions that exactly fit what is implied by the continuous diffusion. Moreover, with ARG(1) volatility dynamics jointly with conditionally Gaussian returns we have an appealing example where the constraint of structure preserving change of measure is fulfilled.

Since risk-neutral Laplace transforms are related to historical ones by formulas:

$$\begin{aligned}\varpi^*\left(u\right) &= \varpi\left(\xi\left(\theta\right)+u\right) - \varpi\left(\xi\left(\theta\right)\right), \quad \varpi\left(\cdot\right)\in\left\{a\left(\cdot\right), b\left(\cdot\right)\right\},\\ \xi\left(\theta\right) &= \theta_1 + \alpha\left(\theta_2\right),\end{aligned}$$

we must have:

$$\varpi^{*\prime}\left(0\right) = \varpi'\left(\xi\left(\theta\right)\right), \quad \varpi\left(\cdot\right)\in\left\{a\left(\cdot\right), b\left(\cdot\right)\right\}. \tag{3.15}$$

We have:

$$a'\left(u\right) = \frac{\rho}{\left(1+cu\right)^2}, \quad b'\left(u\right) = \frac{\delta c}{1+c}.$$

Therefore, the change of measure will be based on the quantity:

$$\chi\left(\theta\right) = 1 + c\xi\left(\theta\right) = 1 + c\left[\theta_1 + \alpha\left(\theta_2\right)\right].$$

We deduce directly from (3.15) and the formulas above for $a'(\cdot)$ and $b'(\cdot)$ that risk-neutral volatility dynamics are also ARG(1) with parameters:

$$\rho^* = \frac{\rho}{\chi^2(\theta)}, \quad c^* = \frac{c}{\chi(\theta)}, \quad \delta^* = \delta.$$

4 Option pricing

We maintain in this section the assumption of conditional log-normality of return, as in Section 3.4. Note that by virtue of Proposition 5, (conditional) log-normality of return for historical dynamics implies (conditional) log-normality of returns for risk-neutral dynamics: when the functions $\alpha(\cdot)$, $\beta(\cdot)$ and $\gamma(\cdot)$ are quadratic, the risk neutral functions $\alpha^*(\cdot)$, $\beta^*(\cdot)$ and $\gamma^*(\cdot)$ are quadratic as well.

4.1 Generalized Black and Scholes

We want to compute the price C_t at time t of an European Call with strike price K and maturity date $t+1$. Then, using the risk-neutral expectation operator E^*, we have:

$$\begin{aligned} C_t(x_t) &= \exp(-r_{f,t}) E^* [\max\{0, S_{t+1} - K\} | I_t] \\ &= S_t E^* [\max\{0, \exp(r_{t+1}) - \exp(x_t)\} | I_t], \end{aligned}$$

where $x_t = \log(K/S_t) - r_{f,t}$ stands for the log-moneyness of the option (with K properly compared to the forward stock price $S_t \exp(r_{f,t})$).

By the law of iterated expectations:

$$C_t(x_t) = S_t E^* [E^* [\max\{0, \exp(r_{t+1}) - \exp(x_t)\} | I_t^\sigma] | I_t].$$

The trick of the so-called Generalized Black and Scholes (GBS) formula, as put forward by Garcia et al. (2010) is to realize that the inside conditional expectation, based on conditional log-normality of return, can be computed as a Black and Scholes formula, but with two adjustments of the traditional inputs of this formula:

- The actual stock price should be replaced by:

$$\tilde{S}_t = \exp(-r_{f,t}) E^* [S_{t+1} | I_t^\sigma] = S_t E^* [\exp(r_{t+1}) | I_t^\sigma] = S_t \xi_{t,t+1},$$

with results of Section 3.4:

$$\begin{aligned} \xi_{t,t+1} &= E^* [\exp(r_{t+1}) | I_t^\sigma] \\ &= \exp\left\{-\alpha^*(-1)\sigma_{t+1}^2 - \beta^*(-1)\sigma_t^2 - \gamma^*(-1)\right\} \end{aligned} \tag{4.1}$$

where

$$\alpha^*(-1) = -k\phi, \quad \beta^*(-1) = -a^*(-k\phi), \quad \gamma^*(-1) = -b^*(-k\phi).$$

- The squared volatility parameter is:

$$V^*\left[\exp\left(r_{t+1}\right)\middle| I_t^{\sigma}\right] = \left(1-\phi^2\right)\tilde{\sigma}_{t+1}^2.$$

Thus, we can state:

Proposition 6 (Option price with leverage)**.** *The price at time t of a one-period call option with log-moneyness $x_t = \log\left(K/S_t\right) - r_{f,t}$ is given by:*

$$C_t\left(x_t\right) = E^*\left[BS\left(S_t\xi_{t,t+1}, \left(1-\phi^2\right)\tilde{\sigma}_{t+1}^2, K, r_{f,t}\right)\middle| I_t\right],$$

with:

$$\begin{aligned}
BS\left(S,V,K,r\right) &= S\Phi\left(d_1\left(S,V,K,r\right)\right) - Ke^{-r}\Phi\left(d_2\left(S,V,K,r\right)\right),\\
d_1\left(S,V,K,r\right) &= \frac{\log\left(S/K\right)+r}{\sqrt{V}} + \frac{1}{2}\sqrt{V},\\
d_2\left(S,V,K,r\right) &= \frac{\log\left(S/K\right)+r}{\sqrt{V}} - \frac{1}{2}\sqrt{V},
\end{aligned}$$

and $\xi_{t,t+1}$ given by (4.1), $\Phi\left(\cdot\right)$ being the cumulative distribution function of the standard normal.

Note that the above strategy of application of GBS formula would be easily extended to a call option with time to maturity $t+h$, $h>1$. Since given the volatility path, consecutive returns are serially independent, the multi-period horizon is easily tackled by the law of iterated expectations, ending up with a formula where the volatility factor is replaced by its sum over the unit periods until maturity.

4.2 Volatility smile

The GBS formula gives the option price as an expectation of the BS price, where the expectation is computed over the (risk neutral) probability distribution of future volatility $\tilde{\sigma}_{t+1}$ given information I_t. The random variable $\tilde{\sigma}_{t+1}$ has actually a double occurrence inside the GBS formula:

- First, the (squared) volatility parameter V is $\left(1-\phi^2\right)\tilde{\sigma}_{t+1}^2$.
- Second, the stock price S_t is replaced by $S_t\xi_{t,t+1}$.

It has been shown (first in Renault & Touzi (1996) and second in Renault (1997) for a simplified proof) that with the first occurrence only (with $\xi_{t,t+1}\equiv 1$), the GBS formula would lead to a volatility smile (giving BS Implied Volatility as function of the log-moneyness x) that is a U-shape symmetric curve, with a minimum at the money ($x=0$). However, Renault (1997) had also numerically checked that a deterministic factor $\xi_{t,t+1}$, even slightly different from 1, will give rise to huge distortions of the volatility smile, that basically amount to adding a monotonic curve (decreasing or increasing depending whether $\xi_{t,t+1}>1$ or $\xi_{t,t+1}<1$, respectively).

The situation here is different. We do find (see formula (4.1)) that $\xi_{t,t+1}\equiv 1$ when $\phi=0$, leading as expected to a symmetric volatility smile when there is no leverage effect. In the general case, it is worth noting that $\xi_{t,t+1}$ is a random variable of expectation one since by definition of the risk neutral distribution:

$$E^*\left[\exp\left(r_{t+1}\right)\right| I_t] = 1.$$

In other words, the possible asymmetries of the volatility smile will be due to some Jensen effect when $\xi_{t,t+1}$ is random (because $\phi \neq 0$), due to the fact that the BS price is not a linear function of the underlying stock price. In order to assess the impact of the leverage parameter ϕ on the smile asymmetry, it is worth disentangling two kinds of occurrences of ϕ within the GBS formula. With simplified notations:

$$C_t\left(x_t\right) = C\left[\phi, \phi\right],$$

with:

$$C\left[\phi_1, \phi_2\right] = E^*\left[BS\left(S_t \xi_{t,t+1}\left(\phi_1\right), \left(1-\phi_2^2\right)\tilde{\sigma}_{t+1}^2\left(\phi_2\right), K, r_{f,t}\right)\middle| I_t\right],$$

where:

$$\begin{aligned}
\xi_{t,t+1}\left(\phi\right) &= \exp\left\{k\phi\sigma_{t+1}^2 + a^*\left(-k\phi\right)\sigma_t^2 + b^*\left(-k\phi\right)\right\}, \\
\tilde{\sigma}_{t+1}^2\left(\phi\right) &= \sigma_{t+1}^2 + \frac{\beta^{*\prime\prime}\left(0\right)\sigma_t^2 + \gamma^{*\prime\prime}\left(0\right)}{\alpha^{*\prime\prime}\left(0\right)}.
\end{aligned}$$

It is worth noting that $\tilde{\sigma}_{t+1}^2\left(\phi\right)$ does not depend on ϕ when it coincides with σ_{t+1}^2 (because we would assume that functions $\beta\left(\cdot\right)$ and $\gamma\left(\cdot\right)$ are linear). We have:

$$\frac{\partial C\left[\phi_1, \phi_2\right]}{\partial \phi_1} = S_t E^*\left[\Phi\left(\tilde{d}_1\left(\phi_1, \phi_2\right)\right)\frac{\partial \xi_{t,t+1}}{\partial \phi_1}\middle| I_t\right],$$

with:

$$\begin{aligned}
\tilde{d}_1\left(\phi_1, \phi_2\right) &= d_1\left(S_t \xi_{t,t+1}\left(\phi_1\right), \left(1-\phi_2^2\right)\tilde{\sigma}_{t+1}^2\left(\phi_2\right), K, r_{f,t}\right), \\
\frac{\partial \xi_{t,t+1}}{\partial \phi_1} &= k\xi_{t,t+1}\left[\sigma_{t+1}^2 - a^{*\prime}\left(-k\phi_1\right)\sigma_t^2 - b^{*\prime}\left(-k\phi_1\right)\right].
\end{aligned}$$

The sign of the sensitivity of the option price to parameter ϕ_1 is ambiguous in the general case. Then, it is worthwhile to consider this sensitivity in the neighborhood of zero leverage because the above formula can then be simplified as:

$$\frac{\partial \xi_{t,t+1}}{\partial \phi_1}\left(\phi_1 = 0\right) = k\left\{\sigma_{t+1}^2 - E^*\left[\sigma_{t+1}^2\middle| I_t\right]\right\}.$$

Hence, we get the following result:

Proposition 7 (Option price around zero leverage)**.**

$$\frac{\partial C\left[\phi_1, \phi_2\right]}{\partial \phi_1}\left(\phi_1 = 0\right) = kS_t Cov_t^*\left\{\Phi\left(d_1\left(S_t, \left(1-\phi_2^2\right)\tilde{\sigma}_{t+1}^2\left(\phi_2\right), K, r_{f,t}\right)\right), \tilde{\sigma}_{t+1}^2\left(\phi_2\right)\right\},$$

where $Cov_t^*\left\{\cdot\right\}$ stands for the covariance operator computed with the conditional risk neutral probability *distribution given information I_t.*

Note that the formula of Proposition 7 can be interpreted in two different ways:

- Either one sticks to the case $\tilde{\sigma}^2_{t+1}(\phi_2) = \sigma^2_{t+1}$ (linear functions $\beta(\cdot)$ and $\gamma(\cdot)$) and then Proposition 7 actually provides the global sensitivity of the option price to the leverage parameter in the neighborhood of zero leverage:

$$\begin{aligned}\frac{\partial C[\phi_1, 0]}{\partial \phi_1}(\phi_1 = 0) &= \frac{\partial C_t(x_t)}{\partial \phi}(\phi = 0) \\ &= kS_t Cov^*_t\left\{\Phi\left(d_1\left(S_t, \sigma^2_{t+1}, K, r_{f,t}\right)\right), \sigma^2_{t+1}\right\}.\end{aligned}$$

  To see that, note that when $\tilde{\sigma}^2_{t+1}(\phi_2) = \sigma^2_{t+1}$:

$$\frac{\partial C[\phi_1, \phi_2]}{\partial \phi_2}(\phi_2 = 0) = 0,$$

  since it is proportional to the derivative of $\left(1 - \phi_2^2\right)$.

- Or, in the general case, Proposition 7 sets the focus on the specific effect of leverage on smile asymmetry through its impact around the symmetric smile that would show up for $\phi_1 = 0$ (irrespective of the value of ϕ_2).

In order to assess the sign of the covariance term in Proposition 7, first note that:

$$\frac{\partial d_1(S, V, K, r)}{\partial V} = -\frac{1}{2V\sqrt{V}}\left[\log\left(\frac{S}{K}\right) + r\right] + \frac{1}{4\sqrt{V}} > 0,$$

if and only if

$$\frac{1}{V}\left[\log\left(\frac{S}{K}\right) + r\right] < \frac{1}{2}.$$

This inequality is automatically fulfilled out of the money (that is when $\log(S/K) + r < 0$) and even in the money, as long as the log-moneyness is at least $(-V/2)$.

Note in addition that:

$$\frac{\partial}{\partial K}\left\{\frac{\partial d_1(S, V, K, r)}{\partial V}\right\} = \frac{1}{2KV\sqrt{V}} > 0.$$

In other words, when log-moneyness is at least $(-V/2)$, the larger the log-moneyness, the faster $d_1(S, V, K, r)$ is increasing with V. As a result, we can claim that the covariance in Proposition 7 is positive for a sufficiently large log-moneyness (at least at and out of the money) and even more so that the log-moneyness is higher. In particular, at the money we have:

$$\frac{\partial C[\phi_1, \phi_2]}{\partial \phi_1}(\phi_1 = 0) = kS_t Cov^*_t\left\{\Phi\left(\sqrt{1 - \phi_2^2}\frac{\tilde{\sigma}_{t+1}(\phi_2)}{2}\right), \tilde{\sigma}^2_{t+1}(\phi_2)\right\} > 0.$$

Therefore, leverage effect (meaning ϕ_1 negative) will dampen the option prices, more and more when log-moneyness increases. In other words, we are adding a monotonically decreasing curve to the benchmark symmetric smile (corresponding to zero leverage) as if $\xi_{t,t+1}$ were a deterministic number larger than one. Then, we expect that the larger the absolute value of negative ϕ, the more the increasing part of the smile on the right side of the moneyness is dampened, possibly even becoming decreasing. It is actually a kind of smirk often documented as a stylized fact and it is confirmed by our numerical work in appendix (see Figure 4 on page 39). We find in particular that for $\phi < -0.5$, the smile is always a monotonically decreasing curve.

5 Empirical results

In this section we illustrate estimation and option pricing performance of the simple ARG(1)-Normal model.[1] In Section 5.1 we start by stating the exact model that we work with in this empirical exercise. Next, we outline two estimation methods applicable for the model parameters. In particular, we use both full information Maximum Likelihood (Section 5.2) and Spectral Generalized Method of Moments (Section 5.3). Data and estimation results are described in Section 5.4. Finally, we discuss the option pricing performance of the model in Section 5.5.

5.1 ARG(1)-Normal model

The model for the excess log return r_t is characterized by the conditional Laplace transform:

$$E\left[\exp\left(-vr_{t+1}\right)\middle|\sigma_t^2,\sigma_{t+1}^2\right]=\exp\left\{-\alpha\left(v\right)\sigma_{t+1}^2-\beta\left(v\right)\sigma_t^2-\gamma\left(v\right)\right\}, \tag{5.1}$$

with:

$$\alpha\left(v\right)=\psi v-\frac{1}{2}v^2\left(1-\phi^2\right),\quad \beta\left(v\right)=va^*\left(-\phi k\right),\quad \gamma\left(v\right)=vb^*\left(-\phi k\right),$$

This specification for the functions $\beta\left(\cdot\right)$ and $\gamma\left(\cdot\right)$ follows the following rationale:

- A linear specification of these functions allows us to equalize the volatility factor σ_t^2 with the factor $\tilde{\sigma}_t^2$ defined in (2.5). The validity of this simplification is an empirical question discussed below.
- When functions $\beta\left(\cdot\right)$ and $\gamma\left(\cdot\right)$ are linear, they coincide by (3.5) with their risk neutral counterparts, so that:

$$\begin{aligned}\beta(v) &= -\beta\left(-1\right)v=-\beta^*\left(-1\right)v=va^*\left(-k\phi\right),\\ \beta(v) &= -\beta\left(-1\right)v=-\beta^*\left(-1\right)v=va^*\left(-k\phi\right),\end{aligned}$$

  where the last equalities are given by (3.14).
- By (2.6), we can then consider as an empirical approximation that:

$$V\left[r_{t+1}\middle|I_t\right]\approx E\left[\sigma_{t+1}^2\middle|I_t\right]=E\left[\tilde{\sigma}_{t+1}^2\middle|I_t\right], \tag{5.2}$$

[1]For this paper we used Python and scientific libraries SciPy/NumPy for numerical implementation of our estimation methods. The code is partially available here: https://github.com/khrapovs/argamma.

if we can assume that:

$$\left[\frac{\psi}{\phi}\right]^2 \approx \frac{E\left[\left.\sigma_{t+1}^2\right| I_t\right]}{V\left[\left.r_{t+1}\right| I_t\right]}$$

target that we hope to meet by taking:

$$\psi = k\phi + \left(1-\phi^2\right)\left(\theta_2 - \frac{1}{2}\right)$$

with:

$$k^2 = \frac{E\left[\sigma_{t+1}^2\right]}{E\left[V\left[\left.\sigma_{t+1}^2\right| I_t\right]\right]}.$$

The advantage of approximation (5.2) will be to allow us to consider the volatility factor σ_{t+1}^2 as "observed", through daily realized volatility, computed from high-frequency intraday data. Then, our model is about the observed bivariate process $\left(r_{t+1}, \sigma_{t+1}^2\right)$, whose joint probability distribution is defined by the specification of conditional return dynamics given volatility in (5.1) as well as an ARG(1) model for the volatility factor. Recall that this process is a convenient simplification of the general framework (2.8) in which we impose in particular $\omega = \delta$:

$$\begin{aligned} E\left[\left.\sigma_{t+1}^2\right| I_t\right] &= \rho\sigma_t^2 + \delta c, \\ V\left[\left.\sigma_{t+1}^2\right| I_t\right] &= 2c\rho\sigma_t^2 + \delta c^2, \end{aligned}$$

so that:

$$\begin{aligned} E\left[\sigma_t^2\right] &= \frac{\delta c}{1-\rho}, \\ E\left[V\left[\left.\sigma_{t+1}^2\right| I_t\right]\right] &= 2c\rho\frac{\delta c}{1-\rho} + \delta c^2. \end{aligned}$$

Thus:

$$\frac{1}{k^2} = 2c\rho + \delta c^2 \frac{1-\rho}{\delta c} = c\left(1+\rho\right).$$

To summarize, we end up with a parametric model corresponding to the observation of a time series $\left(r_{t+1}, \sigma_{t+1}^2\right)$, whose joint probability distribution is specified by the true unknown value of the parameters:

$$c, \rho, \delta, \phi, \theta_1, \theta_2.$$

The value of these parameters determine the functions $a\left(\cdot\right)$, $b\left(\cdot\right)$, $\alpha\left(\cdot\right)$, $\beta\left(\cdot\right)$, and $\gamma\left(\cdot\right)$ as:

$$\begin{aligned} a\left(u\right) &= \frac{\rho u}{1+cu}, \\ b\left(u\right) &= \delta\log\left(1+cu\right), \end{aligned}$$

and:

$$\begin{aligned}
\alpha(v) &= \psi v - \frac{1}{2}v^2\left(1-\phi^2\right), \\
\beta(v) &= va^*(-k\phi), \\
\gamma(v) &= vb^*(-k\phi),
\end{aligned}$$

where k and ψ are given by:

$$\begin{aligned}
k &= k(c,\rho) = \frac{1}{\sqrt{c[1+\rho]}}, \\
\psi &= \psi(c,\rho,\phi,\theta_2) = k(c,\rho)\phi + \left(1-\phi^2\right)\left(\theta_2 - \frac{1}{2}\right),
\end{aligned}$$

while:

$$\begin{aligned}
a^*(u) &= \frac{\rho^* u}{1+c^* u}, \\
b^*(u) &= \delta\log(1+c^* u), \\
\rho^* &= \frac{\rho}{\chi^2(\theta)}, \\
c^* &= \frac{c}{\chi(\theta)}, \\
\chi(\theta) &= 1 + c[\theta_1 + \alpha(\theta_2)].
\end{aligned}$$

5.2 Maximum Likelihood

For the purpose of conditional MLE given an initial observation of σ_0^2, the joint conditional density of $(r_{t+1}, \sigma_{t+1}^2)$ given past observations is written as:

$$f\left(r_{t+1}, \sigma_{t+1}^2 \middle| \sigma_t^2; c,\rho,\delta,\phi,\theta_1,\theta_2\right) = f\left(r_{t+1} \middle| \sigma_{t+1}^2, \sigma_t^2; c,\rho,\delta,\phi,\theta_1,\theta_2\right) f\left(\sigma_{t+1}^2 \middle| \sigma_t^2; c,\rho,\delta\right).$$

The density function $f\left(\sigma_{t+1}^2 \middle| \sigma_t^2; 1,\rho,\delta\right)$ is computed as a mixture of densities of gamma with degree of freedom $(\delta + Z_t)$ where Z_t is drawn in the Poisson distribution with parameter $\rho\sigma_t^2$. For $c \neq 1$, we use instead the Poisson distribution with parameter $(\rho\sigma_t^2/c)$ to drive mixing variable Z_t of gamma of degree of freedom $(\delta + Z_t)$ to compute the probability density function of (σ_{t+1}^2/c). Then we deduce $f\left(\sigma_{t+1}^2 \middle| \sigma_t^2; 1,\rho,\delta\right)$ from the Jacobian formula.

The density function $f\left(r_{t+1} \middle| \sigma_{t+1}^2, \sigma_t^2; c,\rho,\delta,\phi,\theta_1,\theta_2\right)$ is the density of the normal distribution with mean and variance given by:

$$\begin{aligned}
E\left[r_{t+1} \middle| \sigma_{t+1}^2, \sigma_t^2\right] &= \psi(c,\rho,\phi,\theta_2)\sigma_{t+1}^2 - \frac{\rho^* k(c,\rho)\phi}{1 - c^* k(c,\rho)\phi}\sigma_t^2 + \delta\log(1 - c^* k(c,\rho)\phi), \\
V\left[r_{t+1} \middle| \sigma_{t+1}^2, \sigma_t^2\right] &= \left(1-\phi^2\right)\sigma_{t+1}^2,
\end{aligned}$$

where:

$$\begin{aligned}
c^* &= \frac{c}{\chi(\theta)}, \\
\rho^* &= \frac{\rho}{\chi^2(\theta)}, \\
\chi(\theta) &= 1 + c\left[\theta_1 + \alpha(\theta_2)\right], \\
\alpha(\theta_2) &= \psi(c, \rho, \phi, \theta_2)\,\theta_2 - \frac{1}{2}\theta_2^2\left(1 - \phi^2\right).
\end{aligned}$$

As obvious from the discussion in Section 3.3, the identification of θ_1 from observations on $\left(r_{t+1}, \sigma_{t+1}^2\right)$ only is likely to be fragile. It comes only from the occurrence of θ_1 in the function $\chi(\theta)$ while the other risk premium parameter θ_2 has clearly a much more pervasive effect. This is the reason why we choose not to estimate θ_1 from these data but instead to perform maximum likelihood for a given value of θ_1. This value of the price of the volatility risk will be obtained from calibration of option prices. Ideally an iterative procedure should be applied: estimate $[c, \rho, \delta, \phi, \theta_1, \theta_2]$ given θ_1, then calibrate θ_1 given $[c, \rho, \delta, \phi, \theta_1, \theta_2]$ and repeat until convergence. In our empirical exercise, we varied θ_1 inside sufficiently wide bounds and did not notice any effect on the estimates of other parameters. So, as expected, the price $\theta_1$of volatility risk is almost exclusively identified from option prices, and the iterative procedure converges very fast.

As far as maximum likelihood given θ_1 is concerned, we first note that the log-likelihood admits an additive decomposition:

$$\begin{aligned}
\log f\left(r_{t+1}, \sigma_{t+1}^2 \middle| \sigma_t^2; c, \rho, \delta, \phi, \theta_1, \theta_2\right) &= A_t(c, \rho, \delta) + B_t(c, \rho, \delta, \phi, \theta_2), \\
A_t(c, \rho, \delta) &= \log f\left(\sigma_{t+1}^2 \middle| \sigma_t^2; c, \rho, \delta\right), \\
B_t(c, \rho, \delta, \phi, \theta_2) &= \log f\left(r_{t+1} \middle| \sigma_{t+1}^2, \sigma_t^2; c, \rho, \delta, \phi, \theta_1, \theta_2\right).
\end{aligned}$$

By maximizing $\sum_{t=1}^{T} A_t(c, \rho, \delta)$, we get an estimator $\left(\hat{c}, \hat{\rho}, \hat{\delta}\right)$ that would be efficient if we had only observed the stochastic process σ_t^2. This estimator will not be efficient in general when we have the joint observation of $\left(r_{t+1}, \sigma_{t+1}^2\right)$ since the parameters (c, ρ, δ) also show up in the second part of the log-likelihood. However, we will get a consistent estimator $\left(\hat{\phi}, \hat{\theta}_2\right)$ of other parameters (ϕ, θ_2) as maximizer of the second part of the log-likelihood where the first step estimators have been plugged in:

$$\left(\hat{\phi}, \hat{\theta}_2\right) = \arg\max_{\phi, \theta_2} \sum_{t=1}^{T} \log f\left(r_{t+1} \middle| \sigma_{t+1}^2, \sigma_t^2; \hat{c}, \hat{\rho}, \hat{\delta}, \phi, \theta_2\right).$$

However, this estimator is not efficient since it does not take into account the joint maximization of the two additive parts of the log-likelihood. We also perform joint estimation. Note that maximization by parts of Fan et al. (2015) is a way to get this joint maximization without additional computational burden.

5.3 Spectral GMM

As an alternative to full information ML, we employ spectral GMM proposed by Singleton (2001) and Chacko & Viceira (2003). The main idea of this method is to utilize conditional characteristic function as a moment restriction for a certain grid of its arguments. Although our model was formulated in terms of Laplace transforms, they are easily converted into characteristic functions by a simple replacement of a real argument with a complex one. Since a characteristic function is a on-to-one mapping with the density, with the number of grid points going to infinity the efficiency of the estimator reaches that of ML. On the other hand, the standard GMM methodology does not allow for this limit since the efficient weighting matrix becomes singular with neighboring moments becoming perfectly correlated. Although the solution to this problem is well known (Carrasco & Florens, 2000; Carrasco et al., 2007) we do not chase efficiency in this empirical exercise but rather point estimates used later for option pricing. The main selling point of the spectral GMM is that it works best in situations where characteristic function is known while the likelihood function is not or hard to obtain in closed form, for example in the context of affine jump diffusion models.

The model stated in Section 5.1 implies the following set of moment functions

$$g_t(u,\theta) = Z_t \otimes \begin{bmatrix} \exp\left\{-u\sigma_{t+1}^2\right\} - \exp\left\{-a(u)\,\sigma_t^2 - b(u)\right\} \\ \exp\left\{-ur_{t+1}\right\} - \exp\left\{-\alpha(u)\,\sigma_{t+1}^2 - \beta(u)\,\sigma_t^2 - \gamma(u)\right\} \end{bmatrix},$$

with instruments given by

$$Z_t = \left[1, \exp\left(-i\sigma_t^2\right), \ldots, \exp\left(-i\sigma_{t-l}^2\right)\right],$$

for lag order $l = 1, 2$, complex unity $i = \sqrt{-1}$, and $u \in \mathbb{C}$. The unconditional moment restrictions are

$$E\begin{bmatrix} \operatorname{Re}\left\{g_t(u,\theta)\right\} \\ \operatorname{Im}\left\{g_t(u,\theta)\right\} \end{bmatrix} = 0,$$

The complex grid, where characteristic function is evaluated, is comprised of equally spaced 5 points between $[i, 10i]$. This grid was chosen after multiple experiments with both simulated and real data. These experiments have shown that the corresponding moments are sufficiently apart from each other and sufficiently informative for the estimation algorithm to be robust to initial parameter values.

Real and complex parts of the unconditional moments, three instruments, five grid points, all give us in total 60 unconditional moment restrictions. These moments jointly identify the vector of 5 parameters $\theta = (c, \rho, \delta, \phi, \theta_2)$ given the price of volatility risk, θ_1. Same as in the ML estimation strategy, this parameter will be identified from option price data as described in Section 5.5. Further on we estimate model parameters in two steps similarly to MLE. Each step would use only half of components of function g.

5.4 Data and results

The data for the estimation (S&P500 index, realized volatility) is obtained from Oxford-Man Institute[2]. The descriptive statistics of daily log returns, r_t, and realized volatility, σ_t^2, both annualized, are given in Table 1 on page 37. The data is plotted in Figure 1 on page 36.

Estimation results both for MLE and Spectral GMM are given in Table 2 on page 37. We show parameter estimates, corresponding standard errors, and t-statistics. Besides that the table contains some statistics which we will come back to in Section 5.5 when we discuss option pricing performance of our model. For a rough visual check of the model fit we plot actual and expected volatility and returns in Figure 2 on page 38.

Most importantly, notice that the point estimate of the leverage parameter is -0.17 for MLE and -0.3 for GMM. What we see in the data and also noted by others on a similar data set (i.e. Bollerslev et al., 2012), the sample correlation between the returns and next day volatility is actually not far from -0.2. This parameter value is very similar to the result of Garcia et al. (2011) who estimate Heston (1993) model on observed returns and realized volatility. On the other hand, studies that use option data to estimate the leverage parameter (i.e. Christoffersen et al., 2010b) estimate it much closer to the range from -0.6 to as low as -0.75. Hence, we also estimate the model under the restriction that the leverage parameter is no larger than -0.7. Corresponding GMM estimation results are reported in Table 2 on page 37. The necessity for this restriction is coming from the need to reproduce asymmetric implied volatility smile observed in the data. As we can see in Figure 4 on page 39, the model is perfectly capable of reproducing the smirk and the term structure of implied volatility for some parameters. So, later in Section 5.5 we will use the restricted set of parameter estimates to reproduce the asymmetry of implied volatility more accurately.

Another important parameter is the persistence of the volatility factor, ρ. In our model it coincides with the first order autocorrelation. From the descriptive statistics in Table 1 on page 37 we see that the sample first order autocorrelation is 0.772. In contrast, our ML estimates are much closer to 0.69. This outcome is not surprising given the likelihood which is based on first order Markov assumption. Now if we take the power 4 of autocorrelation 0.69 it produces 0.23 which is well below sample autocorrelation of the fourth order, 0.608. For 90 days, 0.69^{90} is hardly distinguishable from zero while realized volatility autocorrelation at this horizon is still above 0.15. Hence, a more flexible estimation method is better suited to reproduce long range persistence of the observed volatility.

For spectral GMM estimation, besides parameter estimates, standard errors, and their ratios, we also report J-statistics with a corresponding p-value of asymptotic Chi-squared distribution. The estimation is performed in two steps similarly to separable MLE, hence the actual number of moments is not 60, as it would have been in joint estimation, but 30. On the first step we fit volatility model and obtain estimated of $\theta_\sigma = (c, \rho, \delta)$. On the second step we fit the return model to estimate $\theta_r = (\phi, \theta_2)$ using the first stage estimates of θ_σ as given. The price of volatility risk, θ_1, is set at -7 which is close to the optimum in calibration exercise.

Unfortunately, the model is rejected by the data, but this is not too surprising given the extreme

[2]Oxford-Man Institute's "realized library", http://realized.oxford-man.ox.ac.uk

simplicity of our combination of ARG(1) and conditional log-normal distribution of returns. Besides, the large number of moments contributes to the bias in the computation of J-statistic. At the same time the standard errors are very small and allow us to interpret precisely the parameter estimates of the model. In particular, note that parameter ρ matches the long-term autocorrelation of the realized daily volatility much better than the estimate of MLE. Parameter δ points to the marginal overdispersion of the volatility process (see Proposition 4 in Gourieroux & Jasiak, 2006). The central parameter of this study, the leverage parameter ϕ, is negative and statistically significant. Equity risk price has the positive sign which is natural from theoretical standpoint.

In order to assess the degree of misspecification of our model for the realized volatility we perform analysis in terms of persistence. In particular, in Figure 3 on page 39 we plot autocorrelation function (ACF) for the data, and theoretically implied by the following models: AR(1), ARMA(1,1), and our preferred ARG(1). The first two models are estimated using QML, and the last one is estimated using spectral GMM with moment conditions given in the previous Section 5.3. The results are as follows.

ACF for the realized volatility drops to 0.6 very quickly (only a few days) and then continues its descend very slowly. The persistence parameter in AR(1) model is estimated to be 0.77. The theoretical ACF is simply 0.77^h as a function of horizon h. It decays exponentially. On the Figure 3 on page 39 this is the lowest line. Clearly, it is inadequate for the data at hand. Persistence parameter of ARMA(1,1) model is 0.94 and MA parameter is -0.47. The corresponding theoretical ACF drops after the first lag and then decays exponentially. On the figure it is the second line from below. In comparison to RV it gets the first few correlations right, but then dies out too quickly. Persistence parameter in ARG(1) model estimated via spectral GMM is equal to 0.95. The model itself implies first order autoregressive conditional mean for the volatility factor. Hence, the ACF is expected to be close to 0.95^h, still exponential rate of decay. This is exactly what we see on Figure 3 on page 39. The theoretical ACF is above all other theoretically implied and crosses its RV counterpart around two weeks mark.

To conclude, we see that the simplest AR(1) model is furthest away from the data if judged only based on its persistence properties. ARG(1) model estimated using GMM seems to be the closest. ARMA(1,1) model takes the second place in this comparison. Finally, we must acknowledge that our results confirm the already well documented evidence that the persistence of integrated variance is better captured by an ARMA(1,1) model than by a simple auto-regression. This should lead us to question our convenient approximation of the latent volatility factor σ_{t+1}^2 by the integrated variance $\tilde{\sigma}_{t+1}^2$ as discussed in section Section 2.3.1.

Finally, we report the estimates of expected equity risk premium (see ERP row in Table 2 on page 37).

5.5 Option pricing performance

Our option pricing exercise is similar in spirit, for example, to Christoffersen et al. (2008), Feunou & Tedongap (2012), Corsi et al. (2013). The data is obtained from OptionMetrics database. It covers the period from Jan 1996 to Aug 2013. The descriptive statistics including number of observations, mean option premium, and mean Black-Scholes implied volatility may be found in Table 5 on page 41 together with the corresponding data implied by our model.

In Section C.7 of the Appendix we show that the risk-neutral Laplace transform (or characteristic function) of the cumulative log return

$$r_{t,n} \equiv \log\left(S_{t+n}/S_t\right) = \sum_{i=0}^{n} r_{t+i}$$

is

$$E^{*}\left[\exp\left(-v r_{t,n}\right)\middle|\,\sigma_t^2\right] = \exp\left\{-\Psi_n\left(v\right)\sigma_t^2 - \Upsilon_n\left(v\right)\right\}. \tag{5.3}$$

This formula gives us a direct way to compute model implied option prices using the inverse Fourier transform[3] given the set of model parameters $\theta = (c, \rho, \delta, \phi, \theta_2, \theta_1)$, and the series of daily volatility factor. Just for an illustrative example we plot implied volatility smiles for some specific set of parameter values in Figure 4 on page 39. From the left plot we can see that the leverage parameter ϕ is responsible for the shape of the smile. Specifically, for more negative ϕ the smile becomes more skewed and shifted to the right and down. On the right side of the panel we see that the smile flattens with an increase of an option maturity.

The first five parameters in parameter vector θ are estimated, while the price of volatility risk θ_1 was held constant. This parameter is chosen by minimizing implied volatility root mean squared error (IVRMSE) put forward by Renault (1997) and computed as

$$IVRMSE = \sqrt{\frac{1}{N}\sum_{j=1}^{N}\left(IV_j^{Market} - IV_j^{Model}\left(\theta\right)\right)^2},$$

with j being the index of an option contract available in the data. Each implied volatility IV_j is obtained by simple inversion of Black-Scholes option pricing formula given a corresponding option premium. As it was described in the end of Section 5.5, the estimation and calibration can be recursively repeated until convergence. But in practice, we did not notice any substantial impact of calibrated parameter θ_1 on the estimation step, hence collapsing the procedure to one iteration only.

The description below is based on parameter estimates obtained from restricted Spectral GMM estimation. Calibration of θ_1 given this set of parameters produces the optimal choice of approximately -7 as seen on the plot of IVRMSE for different risk prices, Figure 5 on page 40. The resulting minimal value of IVRMSE is equal to 4.54%. We did the same calibration for each set of parameter estimates and reported the resulting pair of volatility risk price and IVRMSE in corresponding rows of Table 2 on page 37. There we see that the set of parameters produced by restricted GMM ($\phi \leq -0.7$) produces the best results in terms of implied volatility fit. The resulting IVRMSE is a good result given the simplicity of our model especially in comparison to more sophisticated models.[4]

[3] For numerical implementation we use methodology proposed in Fang & Oosterlee (2009). The code written in Python is available at https://github.com/khrapovs/fangoosterlee

[4] Feunou & Tedongap (2012) compares a multitude of discrete option pricing model such as Heston & Nandi (2000), Christoffersen et al. (2006, 2008), as well as continuous-time models by Pan (2002), Andersen et al. (2002), Chernov et al. (2003), and Bates (2006). They show that the best IVRMSE among them is in vicinity of 2.5%. Corsi et al. (2013) with the model of long memory in volatility produce 3.8%. Christoffersen et al. (2009) are close to 2% with several volatility factors.

All of the option pricing results can be found in Section 5.5. Implied prices and volatilities grouped only by moneyness, maturity, and current VIX level are given in Table 3 on page 40, Table 4 on page 41, and Table 5 on page 41, respectively. For a more in-depth understanding of where the model performs worse we compute aggregate statistics for option pricing errors inside several groups of two dimensions (log-moneyness and maturity) and present the results in Table 6 on page 42 and Figure 6 on page 43. There, in addition to the implied volatility error $\left(IV_j^{Market} - IV_j^{Model}\right)$, we compute price error normalized by the spot price, $\left(P_j^{Market} - P_j^{Model}\right)/S$. Two kinds of statistics are given: bias which is the average of the pricing errors, and Root Mean Squared Error (RMSE).

In the Figure 7 on page 44 we can observe the average premium and implied volatility both in the data and in the model grouped by log-forward-moneyness and maturity. Here we see that the implied volatility smile is almost exactly matched on average by our model for the short term options. At the same time, the smile flattens out too fast to match the data at long horizons. This result is not surprising given one factor volatility model such as ours (see e.g. Christoffersen et al., 2009 for multifactor option pricing model).

Time series of average implied volatilities at-the-money seen in the data and the model are shown in Figure 8 on page 45. One can see that the overall dynamics coming from realized volatility is carried over to the dynamics of implied volatility.

Figure 9 on page 46 reflects the terms structure of implied volatilities for three different current levels of VIX. Notice that the speed of convergence of the implied volatility to its long-run level is somewhat faster for the model against that of the data. This observation confirms our previous conclusion that one factor volatility model is insufficient to match quantitatively the term structure of implied volatility.

Given all of the above estimates, including the risk prices, we can estimate volatility risk premium of -0.2% by evaluating $\left(\hat{E}_t^P\left[\sigma_{t+1}^2\right]\right)^{1/2} - \left(\hat{E}_t^Q\left[\sigma_{t+1}^2\right]\right)^{1/2}$ and scaling it appropriately to the annual interval. This number can be found in Table 2 on page 37 together with analogous statistics for other estimation methods in the row called VRP.

6 Conclusion

Variance-dependent pricing kernels are very popular nowadays (see e.g. Christoffersen et al., 2013) to accommodate a specific variance risk premium, which allows in particular to understand that the projection of the pricing kernel on the stock return alone may not be a monotonic function. We have been able in this paper to devise a stochastic volatility model whose latent volatility factor shows up in the pricing kernel. Moreover, an exponentially affine pricing kernel allows us to preserve an affine structure for both historical and risk-neutral dynamics. The affine structure is convenient for historical volatility dynamics that are robust to temporal and cross-sectional aggregation while risk-neutral affine dynamics are convenient for closed form formula in option pricing. Moreover, the discrete time setting makes the affine model much less restrictive than the corresponding continuous time model in order to reproduce stylized facts about nonlinear features of return dynamics.

We stress that the main challenge of discrete time affine volatility modeling is to accommodate the leverage effect and to disentangle it from volatility feedback. We are able to derive a new parameterization which fulfills these requirements. We can show that the continuous time limit of our model, computed in a non-ambiguous way thanks to our temporal aggregation formulas, coincides with the generalization of the Heston model where the squared diffusion term of the volatility process is a general affine function of the volatility process. Our model allows us not only to reproduce observed asymmetry of the volatility smile but also to understand the mechanism: how leverage effect amounts to add to a benchmark symmetric volatility smile (corresponding to zero leverage, see Renault & Touzi, 1996) a monotonically decreasing curve. Moreover, our general setting sheds some light on the reason why, as recently documented by Bandi & Reno (2016), leverage effect may allow the researcher to identify the volatility risk premium from underlying asset return data alone, but that this identification remains quite fragile as long as option price data are not available.

As already documented by Pan (2002) in continuous time, the affine structure is also convenient for GMM estimation based on the Fourier transform. Moreover we manage to interpret the volatility factor in terms of realized volatility, so that intraday data can be used to compute an observable proxy of the volatility factor. We estimate the simplest version of our model in which the Markovian volatility factor coincides with realized volatility. With this crude approximation, we get meaningful estimates of the two risk premium parameters, equity risk premium and variance risk premium. However, we acknowledge that our simplifying assumption is obviously a bit stretched since it is well known that, both for theoretical and empirical reasons, realized volatility dynamics are better captured by an ARMA(1,1) model. Although we are able to describe precisely what should be done to disentangle the Markovian latent volatility factor and the observed realized volatility, we do not try to estimate in this paper the parameter dependent relationship between volatility factor in the pricing kernel and observed realized volatility. This is left for future research.

References

Ait-Sahalia, Y., Fan, J., & Li, Y. (2013). The leverage effect puzzle: Disentangling sources of bias at high frequency. *Journal of Financial Economics*, 109(1), 224–249.

Andersen, T. G., Benzoni, L., & Lund, J. (2002). An Empirical Investigation of Continuous-Time Equity Return Models. *Journal of Finance*, 57(3), 1239–1284.

Bandi, F. M. & Reno, R. (2016). Price and volatility co-jumps. *Journal of Financial Economics*, 119(1), 107–146.

Barone-Adesi, G., Engle, R. F., & Mancini, L. (2008). A GARCH option pricing model with filtered historical simulation. *Review of Financial Studies*, 21(3), 1223–1258.

Bates, D. S. (2006). Maximum Likelihood Estimation of Latent Affine Processes. *Review of Financial Studies*, 19(3), 909–965.

Bertholon, H., Monfort, A., & Pegoraro, F. (2008). Econometric asset pricing modelling. *Journal of Financial Econometrics*, 6(4), 407–458.

Black, F. (1976). Studies of Stock Price Volatility Changes. *Proceedings of the 1976 Meetings of the American Statistical Association, Business and Economics Statistics Section*, (pp. 177–181).

Bollerslev, T., Litvinova, J., & Tauchen, G. (2006). Leverage and Volatility Feedback Effects in High-Frequency Data. *Journal of Financial Econometrics*, 4(3), 353–384.

Bollerslev, T., Sizova, N., & Tauchen, G. (2012). Volatility in Equilibrium: Asymmetries and Dynamic Dependencies. *Review of Finance*, 16(1), 31–80.

Bollerslev, T. & Zhou, H. (2002). Estimating stochastic volatility diffusion using conditional moments of integrated volatility. *Journal of Econometrics*, 109, 33 – 65.

Carrasco, M., Chernov, M., Florens, J.-P., & Ghysels, E. (2007). Efficient estimation of general dynamic models with a continuum of moment conditions. *Journal of Econometrics*, 140(2), 529–573.

Carrasco, M. & Florens, J.-P. (2000). Generalization of GMM to a Continuum of Moment Conditions. *Econometric Theory*, 16(6), 797–834.

Chacko, G. & Viceira, L. M. (2003). Spectral GMM estimation of continuous-time processes. *Journal of Econometrics*, 116(1-2), 259–292.

Chernov, M., Gallant, R. A., Ghysels, E., & Tauchen, G. (2003). Alternative models for stock price dynamics. *Journal of Econometrics*, 116(1-2), 225–257.

Christoffersen, P., Elkamhi, R., Feunou, B., & Jacobs, K. (2010a). Option Valuation with Conditional Heteroskedasticity and Nonnormality. *Review of Financial Studies*, 23(5), 2139–2183.

Christoffersen, P., Heston, S. L., & Jacobs, K. (2006). Option valuation with conditional skewness. *Journal of Econometrics*, 131(1-2), 253–284.

Christoffersen, P., Heston, S. L., & Jacobs, K. (2009). The Shape and Term Structure of the Index Option Smirk: Why Multifactor Stochastic Volatility Models Work So Well. *Management Science*, 55(12), 1914–1932.

Christoffersen, P., Heston, S. L., & Jacobs, K. (2013). Capturing Option Anomalies with a Variance-Dependent Pricing Kernel. *Review of Financial Studies*, 26(8), 1963–2006.

Christoffersen, P., Jacobs, K., & Mimouni, K. (2010b). Volatility Dynamics for the S&P500: Evidence from Realized Volatility, Daily Returns, and Option Prices. *Review of Financial Studies*, 23(8), 3141–3189.

Christoffersen, P., Jacobs, K., & Ornthanalai, C. (2012). GARCH Option Valuation: Theory and Evidence. CREATES Research Paper No. 2012-50.

Christoffersen, P., Jacobs, K., Ornthanalai, C., & Wang, Y. (2008). Option valuation with long-run and short-run. *Journal of Financial Economics*, 90(3), 272–297.

Corsi, F., Fusari, N., & La Vecchia, D. (2013). Realizing smiles: Options pricing with realized volatility. *Journal of Financial Economics*, 107(2), 284–304.

Cox, J. C., Ingersoll, J. E., & Ross, S. A. (1985). A Theory of the Term Structure of Interest Rates. *Econometrica*, 53(2), 385–407.

Darolles, S., Gourieroux, C., & Jasiak, J. (2006). Structural Laplace Transform and Compound Autoregressive Models. *Journal of Time Series Analysis*, 27(4), 477–503.

Duan, J.-C. (1995). The GARCH option pricing model. *Mathematical Finance*, 5(1), 13–32.

Duffie, D., Filipovic, D., & Schachermayer, W. (2003). Affine Processes and Applications in Finance. *Annals of Applied Probability*, 13(3), 984–1053.

Duffie, D., Pan, J., & Singleton, K. J. (2000). Transform Analysis and Asset Pricing for Affine Jump-Diffusions. *Econometrica*, 68(6), 1343–1376.

Fan, Y., Pastorello, S., & Renault, E. (2015). Maximization by parts in extremum estimation. *Econometrics Journal*, 18(2), 147–171.

Fang, F. & Oosterlee, C. W. (2009). A Novel Pricing Method for European Options Based on Fourier-Cosine Series Expansions. *SIAM Journal on Scientific Computing*, 31(2), 826.

Feller, W. (1951). Two Singular Diffusion Problems. *Annals of Mathematics*, 54(1), 173–182.

Feunou, B. & Tedongap, R. (2012). A Stochastic Volatility Model With Conditional Skewness. *Journal of Business and Economic Statistics*, 30(4), 576–591.

Gagliardini, P., Gourieroux, C., & Renault, E. (2011). Efficient Derivative Pricing by the Extended Method of Moments. *Econometrica*, 79(4), 1181–1232.

Garcia, R., Ghysels, E., & Renault, E. (2010). The Econometrics of Option Pricing. In Y. Ait-sahalia & L. P. Hansen (Eds.), *Handbook of Financial Econometrics: Tools and Techniques*, volume 1 chapter 9, (pp. 479–552). Elsevier B.V.

Garcia, R., Lewis, M.-A., Pastorello, S., & Renault, E. (2011). Estimation of objective and risk-neutral distributions based on moments of integrated volatility. *Journal of Econometrics*, 160(1), 22–32.

Garcia, R. & Renault, E. (1998). A Note on Hedging in ARCH and Stochastic Volatility Option Pricing Models. *Mathematical Finance*, 8(2), 153–161.

Gourieroux, C. & Jasiak, J. (2006). Autoregressive gamma processes. *Journal of Forecasting*, 25(2), 129–152.

Heston, S. L. (1993). A Closed-Form Solution for Options With Stochastic Volatility With Applications to Bond and Currency Options. *Review of Financial Studies*, 6, 327–343.

Heston, S. L. & Nandi, S. (2000). A closed-form GARCH option valuation model. *Review of Financial Studies*, 13(3), 585–625.

Le, A., Singleton, K. J., & Dai, Q. (2010). Discrete-Time AffineQ Term Structure Models with Generalized Market Prices of Risk. *Review of Financial Studies*, 23(5), 2184–2227.

Meddahi, N. (2003). ARMA representation of integrated and realized variances. *Econometrics Journal*, 6(2), 335–356.

Meddahi, N. & Renault, E. (2004). Temporal aggregation of volatility models. *Journal of Econometrics*, 119(2), 355–379.

Mykland, P. A. (2000). Conservative Delta Hedging. *Annals of Applied Probability*, 10(2), 664–683.

Pan, J. (2002). The jump-risk premia implicit in options: evidence from an integrated time-series study. *Journal of Financial Economics*, 63(1), 3–50.

Renault, E. (1997). Econometric Models of Option Pricing Errors. In D. M. Kreps & K. F. Wallis (Eds.), *Advances in Economics and Econometrics: Theory and Applications*, volume 3 chapter 8, (pp. 223–278). Cambridge University Press.

Renault, E., Sekkat, K., & Szafarz, A. (1998). Testing for spurious causality in exchange rates. *Journal of Empirical Finance*, 5(1), 47–66.

Renault, E. & Touzi, N. (1996). Option Hedging and Implied Volatilities in a Stochastic Volatility Model. *Mathematical Finance*, 6(3), 279–302.

Singleton, K. J. (2001). Estimation of affine asset pricing models using the empirical characteristic function. *Journal of Econometrics*, 102(1), 111–141.

Appendix

A Estimation results

Figure 1: S&P500 index, log returns, realized volatility, and VIX

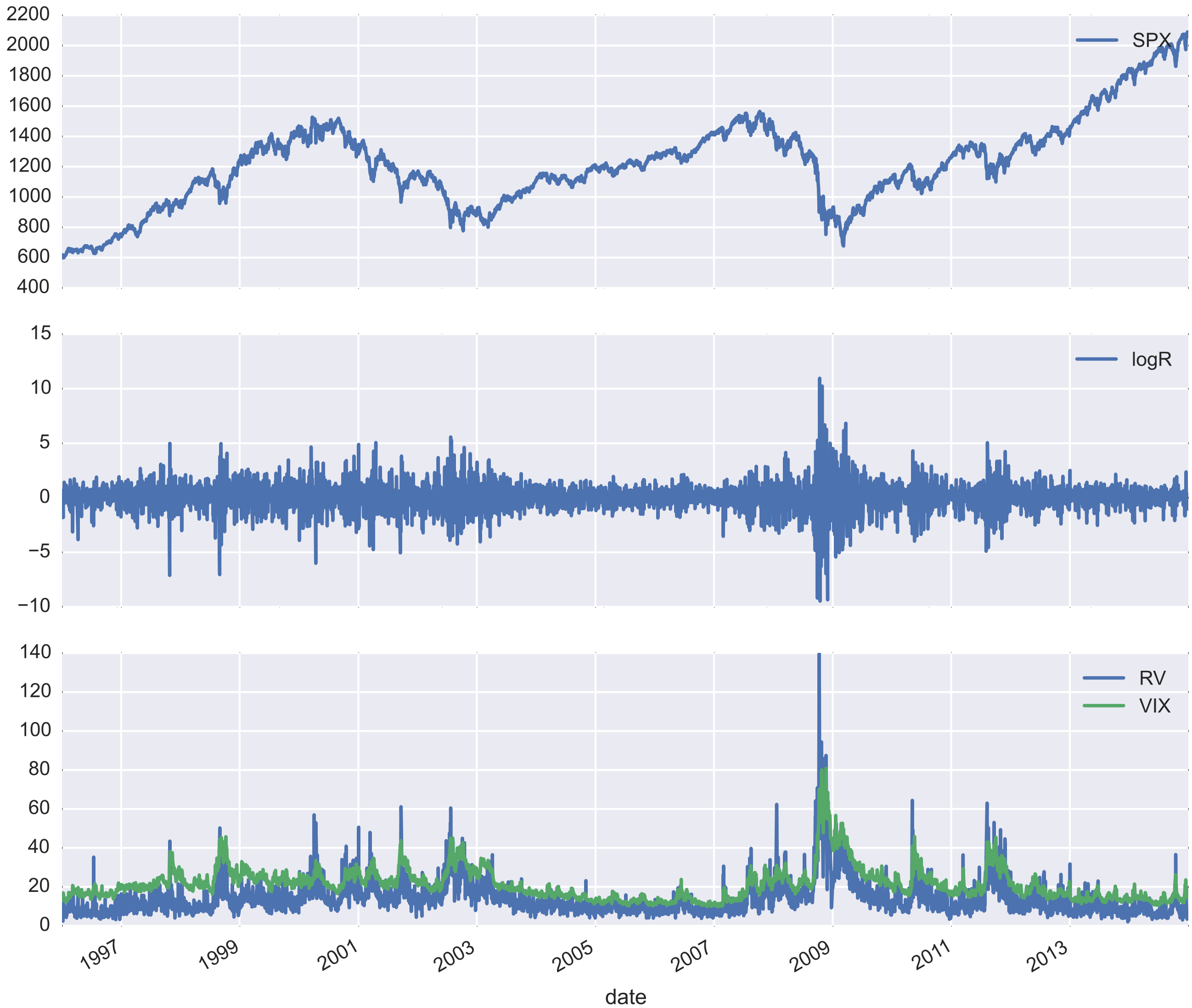


Table 1: Descriptive statistics returns and realized volatility

Daily log return (annualized) and realized volatility (annualized). Data sample is from January 3, 1996 to August 30, 2013.

	N	Mean	Std	Min	Max	γ_1	γ_2	γ_3	γ_4	...	γ_{90}
$r_t \times 100$	4314	0.04	20.59	-150.63	165.15						
$\sigma_t^2 \times 100$	4314	2.97	5.55	0.10	89.08	0.77	0.69	0.63	0.61	...	0.15

Table 2: Estimated parameters of ARG(1)-Normal model

Data sample is from January 3, 1996 to August 30, 2013. Standard errors are given in parenthesis, the ratio of the parameter and its standard error is given brackets.

	Unrestricted				$\phi \leq -0.7$	
	MLE step	MLE joint	GMM step		GMM step	
			Vol	Ret	Vol	Ret
$E\left[\sigma_t^2\right]$	**0.03**	**0.03**	**0.02**		**0.02**	
	(0.00)	(0.00)	(0.00)		(0.00)	
	[31.02]	[31.55]	[243.55]		[243.55]	
ρ	**0.69**	**0.69**	**0.95**		**0.95**	
	(0.01)	(0.01)	(0.01)		(0.01)	
	[60.38]	[60.63]	[150.97]		[150.97]	
δ	**1.22**	**1.24**	**1.25**		**1.25**	
	(0.03)	(0.03)	(0.07)		(0.07)	
	[43.47]	[44.29]	[18.98]		[18.98]	
ϕ	**-0.17**	**-0.17**		**-0.30**		**-0.70**
	(0.01)	(0.01)		(0.01)		(0.02)
	[-15.44]	[-15.41]		[-31.03]		[-45.76]
θ_2	**0.50**	**0.50**		**0.53**		**0.54**
	(0.09)	(0.09)		(0.08)		(0.11)
	[5.75]	[5.65]		[6.45]		[5.01]
$\log L$	-3.34	-3.34				
J			299	179	299	241
df			27	28	27	28
p-value			0.0	0.0	0.0	0.0
θ_1	-7.	-7.		-7.		-7.
ERP	1.71	1.70		2.24		2.11
VRP	-0.78	-0.78		-0.15		-0.20
$IVRMSE$	6.26	6.27		5.15		4.54

Figure 2: Returns and volatility: data and model.
Realized volatility and returns with their model predicted values in time series representation (left panel), and scatter representation (right panel). Data sample is from January 3, 1996 to August 30, 2013.

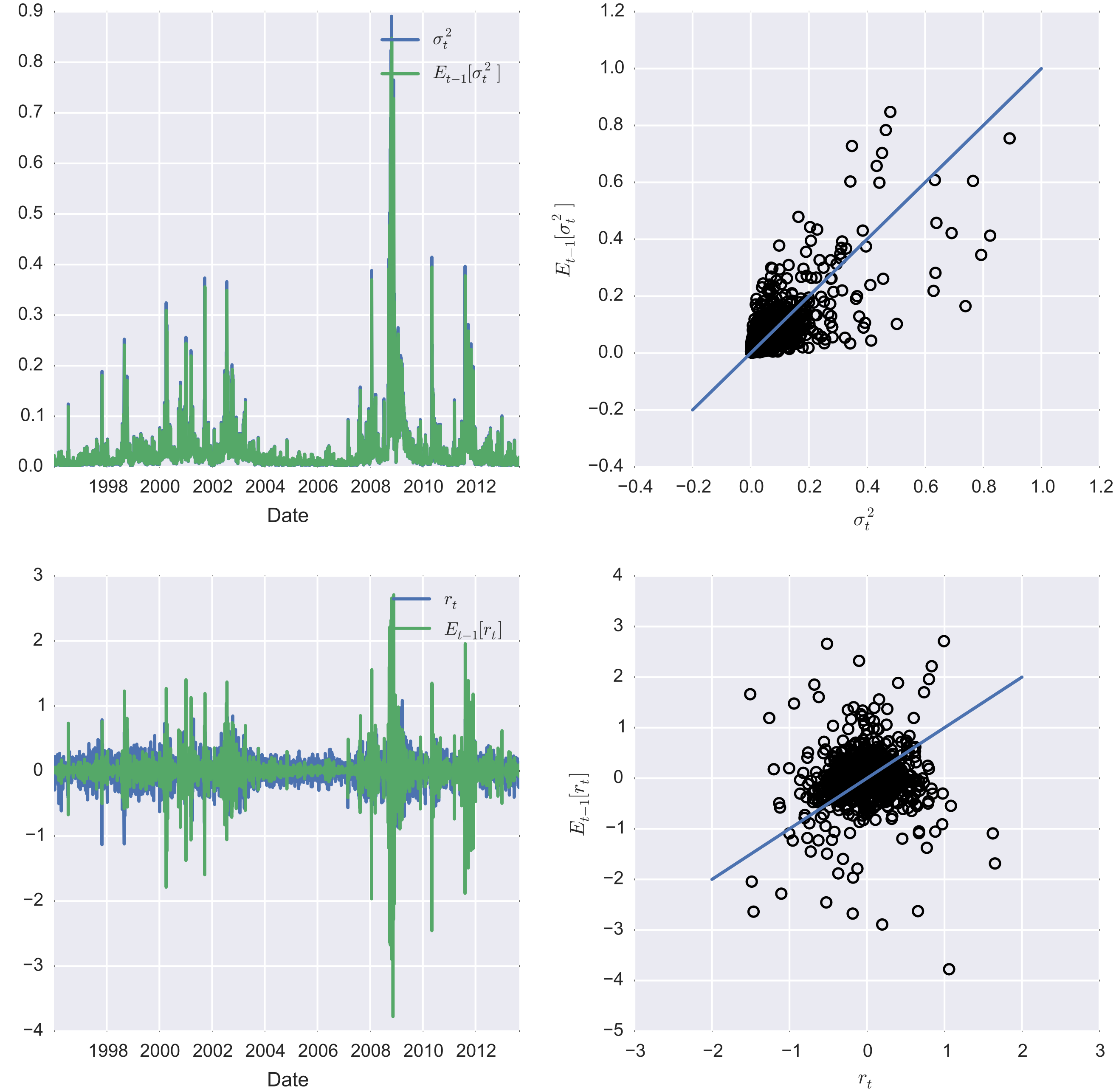
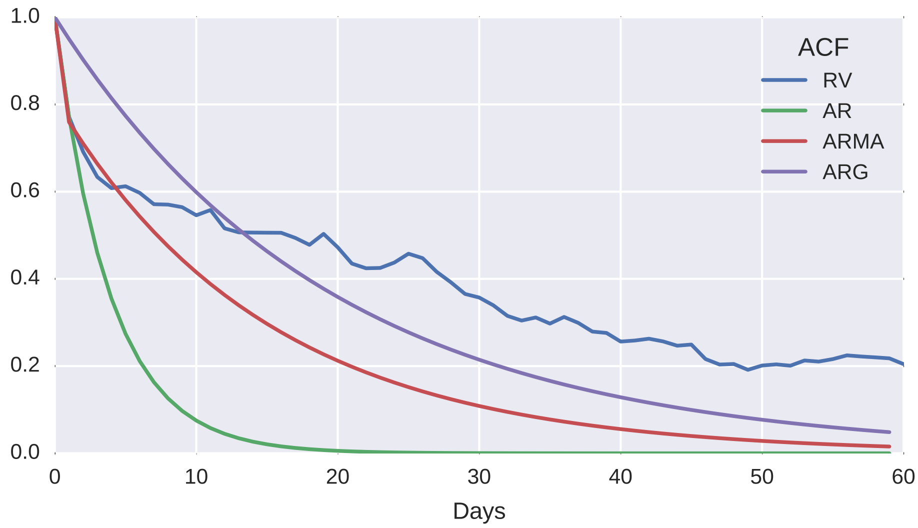


Figure 3: ACF for realized volatility (RV) and model implied values.
The models are AR(1) with persistence 0.77; ARMA(1,1) with 0.94 and −0.47 for AR and MA part, respectively; and our preferred ARG with persistence 0.95.



B Option pricing performance

Figure 4: Implied volatility smile of ARG(1)-Normal model
Default parameter values are $\phi = 0$, $T = 30/365$, $E\left[\sigma_t^2\right] = c\delta/\left(1-\rho\right) = 0.2^2/365$, $\delta = 1.1$ and $\rho = 0.9$. Risk prices are set at $\theta_1 = -1$ and $\theta_2 = 0.5$.

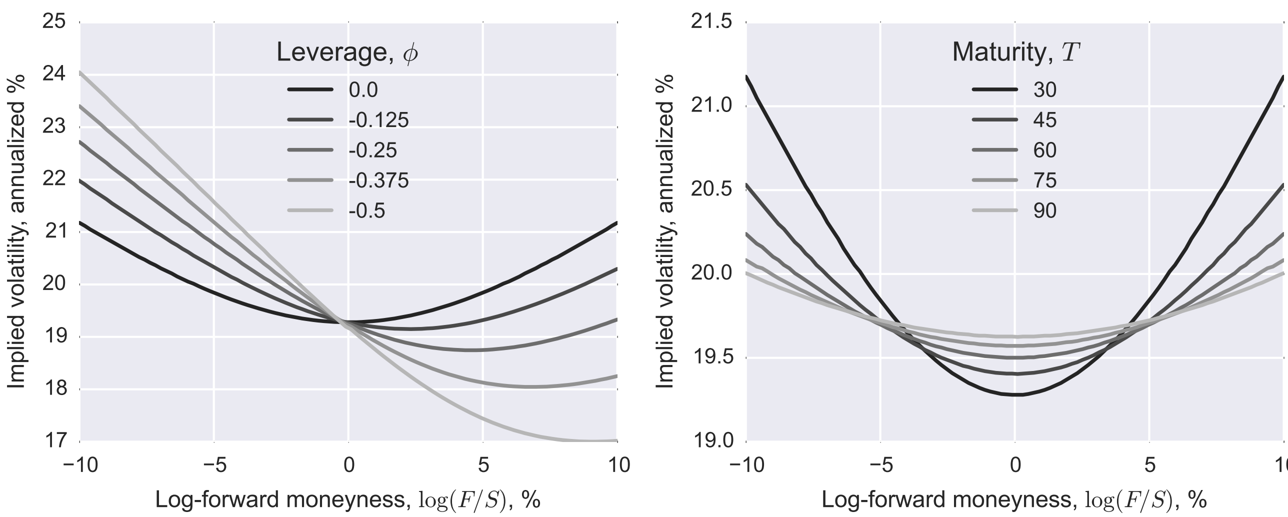


Figure 5: Calibration of volatility risk price

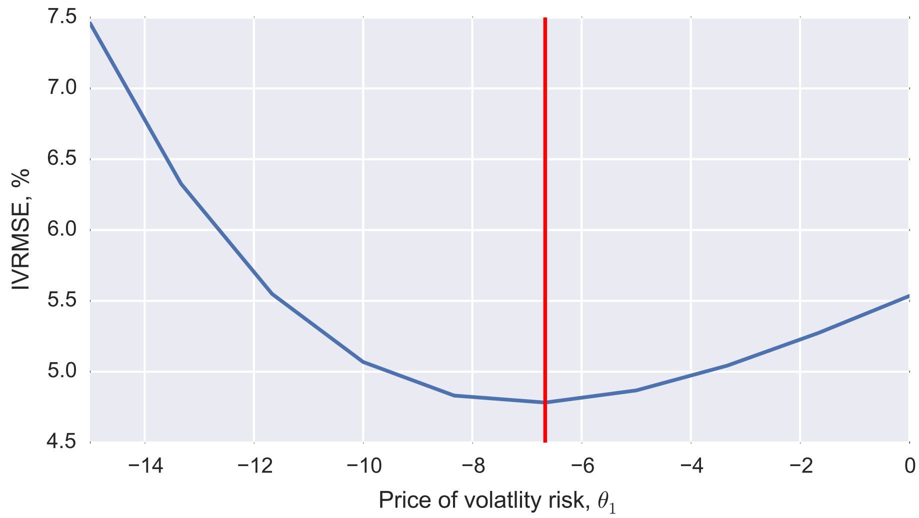


Table 3: Option pricing performance by moneyness

	Log-Moneyness, $\log(K/F)$, %					
	<-4	(-4, -2]	(-2, 2]	(2, 4]	>4	All
Number of obs.	16633	10174	24218	12347	20252	83624
Premium (data)	**2.85**	**3.37**	**3.30**	**2.60**	**2.38**	**2.89**
std	1.69	2.19	1.96	1.87	1.62	1.89
Premium (model)	**2.72**	**3.31**	**3.54**	**2.95**	**2.70**	**3.06**
std	1.49	1.95	1.93	1.89	1.58	1.80
IV (data)	**22.69**	**20.44**	**17.67**	**17.17**	**18.35**	**19.10**
std	6.32	6.18	5.67	5.59	5.58	6.17
IV (model)	**21.65**	**19.95**	**18.47**	**18.42**	**19.25**	**19.46**
std	4.51	4.68	4.39	4.40	4.42	4.62
IV bias	**-1.04**	**-0.49**	**0.79**	**1.25**	**0.90**	**0.37**
std	4.51	4.36	4.22	4.44	4.69	4.53
IVRMSE	**4.62**	**4.39**	**4.30**	**4.61**	**4.78**	**4.54**
std	6.75	6.62	6.34	6.51	7.17	6.71

Table 4: **Option pricing performance by maturity**

	Maturity, days					
	≤30	(30, 90]	(90, 180]	(180, 270]	>270	All
Number of obs.	12140	11944	33272	10072	16196	83624
Premium (data)	**1.29**	**1.85**	**2.70**	**3.48**	**4.90**	**2.89**
std	0.73	1.01	1.41	1.66	2.10	1.89
Premium (model)	**1.23**	**1.82**	**2.80**	**3.72**	**5.45**	**3.06**
std	0.80	0.97	1.19	1.23	1.40	1.80
IV (data)	**19.35**	**19.29**	**19.04**	**18.96**	**18.96**	**19.10**
std	7.70	6.83	5.92	5.49	5.21	6.17
IV (model)	**18.57**	**19.00**	**19.44**	**19.80**	**20.31**	**19.46**
std	7.72	5.78	3.85	2.81	1.99	4.62
IV bias	**-0.78**	**-0.30**	**0.40**	**0.84**	**1.34**	**0.37**
std	4.99	4.64	4.34	4.25	4.34	4.53
IVRMSE	**5.05**	**4.65**	**4.36**	**4.34**	**4.55**	**4.54**
std	8.65	7.34	5.98	5.42	5.50	6.71

Table 5: **Option pricing performance by current VIX**

	VIX level					
	≤15	(15, 20]	(20, 25]	(25, 30]	>30	All
Number of obs.	21653	23943	21881	8705	7442	83624
Premium (data)	**1.97**	**2.57**	**3.17**	**3.69**	**4.86**	**2.89**
std	1.19	1.52	1.78	1.96	2.66	1.89
Premium (model)	**3.03**	**2.88**	**2.92**	**3.27**	**3.87**	**3.06**
std	1.66	1.71	1.74	1.91	2.25	1.80
IV (data)	**13.14**	**17.05**	**21.00**	**24.38**	**31.23**	**19.10**
std	2.15	2.66	2.73	2.94	7.28	6.17
IV (model)	**17.41**	**18.02**	**19.53**	**22.29**	**26.53**	**19.46**
std	2.41	2.39	2.66	5.26	8.54	4.62
IV bias	**4.27**	**0.97**	**-1.47**	**-2.09**	**-4.69**	**0.37**
std	2.26	2.69	2.81	5.33	6.88	4.53
IVRMSE	**4.84**	**2.86**	**3.17**	**5.73**	**8.33**	**4.54**
std	4.75	3.45	3.96	8.53	10.12	6.71

Table 6: Option pricing performance by moneyness and maturity

	Maturity				
Log-Moneyness	(., 30]	(30, 90]	(90, 180]	(180, 270]	(270, .)
			$Bias_P$		
$(-\infty, -0.04]$	-0.27	-0.25	-0.18	-0.12	0.04
$(-0.04, -0.02]$	-0.11	-0.07	-0.07	-0.07	0.02
$(-0.02, +0.02]$	-0.01	0.09	0.27	0.39	0.71
$(+0.02, +0.04]$	-0.06	0.12	0.38	0.58	0.95
$(+0.04, +\infty)$	-0.04	-0.14	0.10	0.38	0.85
			$RMSE$		
$(-\infty, -0.04]$	0.59	0.68	0.86	1.00	1.36
$(-0.04, -0.02]$	0.48	0.68	0.93	1.18	1.45
$(-0.02, +0.02]$	0.48	0.67	0.97	1.21	1.66
$(+0.02, +0.04]$	0.48	0.65	1.01	1.29	1.85
$(+0.04, +\infty)$	0.75	0.72	0.88	1.11	1.71
			$Bias_{IV}$		
$(-\infty, -0.04]$	-3.45	-2.10	-0.98	-0.53	-0.08
$(-0.04, -0.02]$	-1.11	-0.45	-0.38	-0.36	-0.20
$(-0.02, +0.02]$	-0.08	0.52	1.10	1.23	1.66
$(+0.02, +0.04]$	-0.71	0.86	1.72	2.01	2.35
$(+0.04, +\infty)$	-1.13	-1.31	0.43	1.49	2.35
			$IVRMSE$		
$(-\infty, -0.04]$	6.80	5.22	4.38	4.10	4.09
$(-0.04, -0.02]$	4.68	4.53	4.32	4.39	4.13
$(-0.02, +0.02]$	4.31	4.23	4.26	4.30	4.45
$(+0.02, +0.04]$	4.66	4.33	4.57	4.66	4.92
$(+0.04, +\infty)$	7.89	5.29	4.32	4.38	4.90

Figure 6: Option pricing performance

The plot is a graphical representation of the bias and RMSE given in Table 6 on page 42.

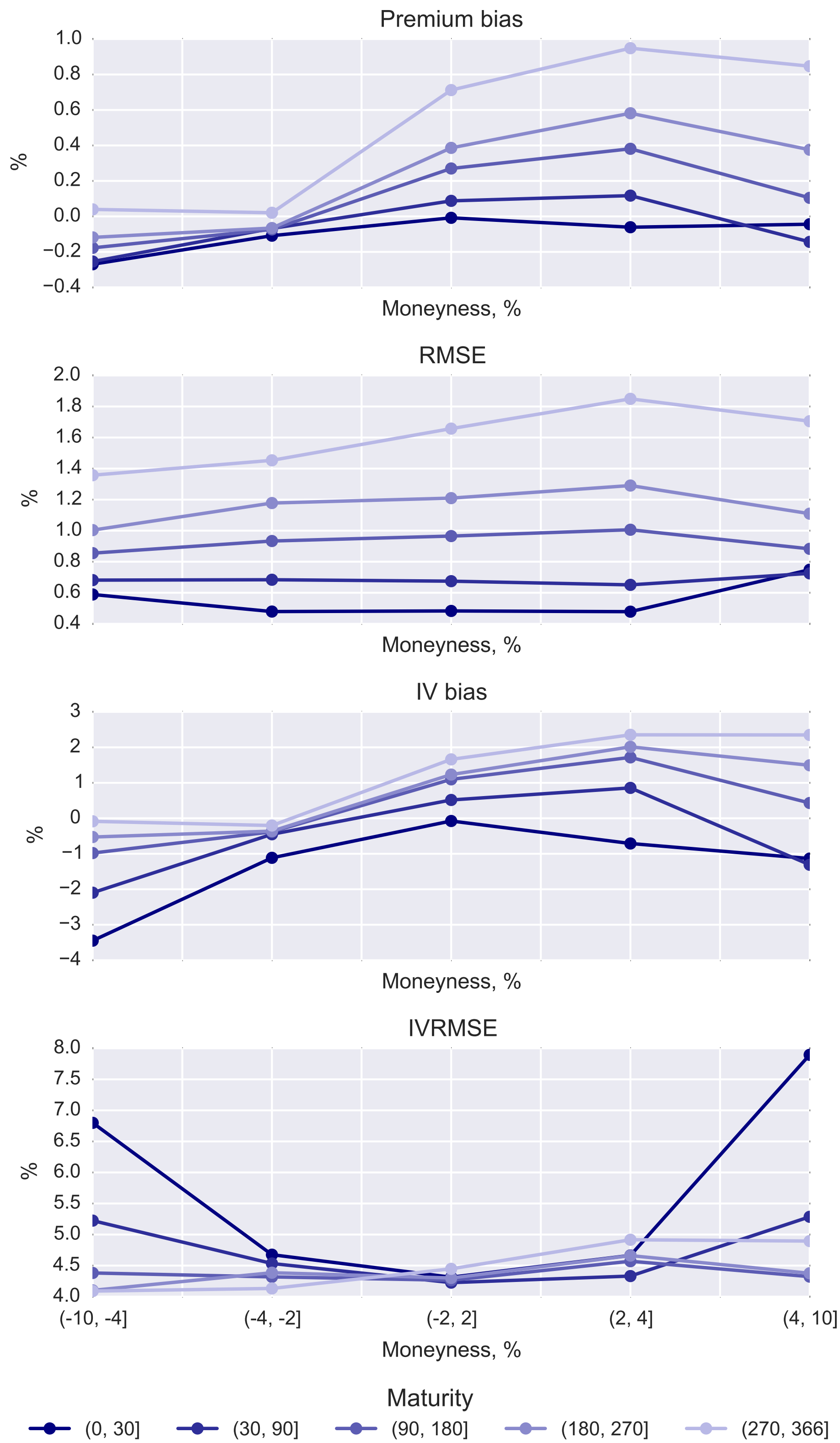


Figure 7: Premium and implied volatility in the data and in the model

The plot is a graphical representation of the average out-of-the-money premium normalized by asset price (left column) and implied volatility (right column) both in the data (blue lines) and derived from the model (green lines) sorted by maturity (rows) and moneyness (horizontal axis).

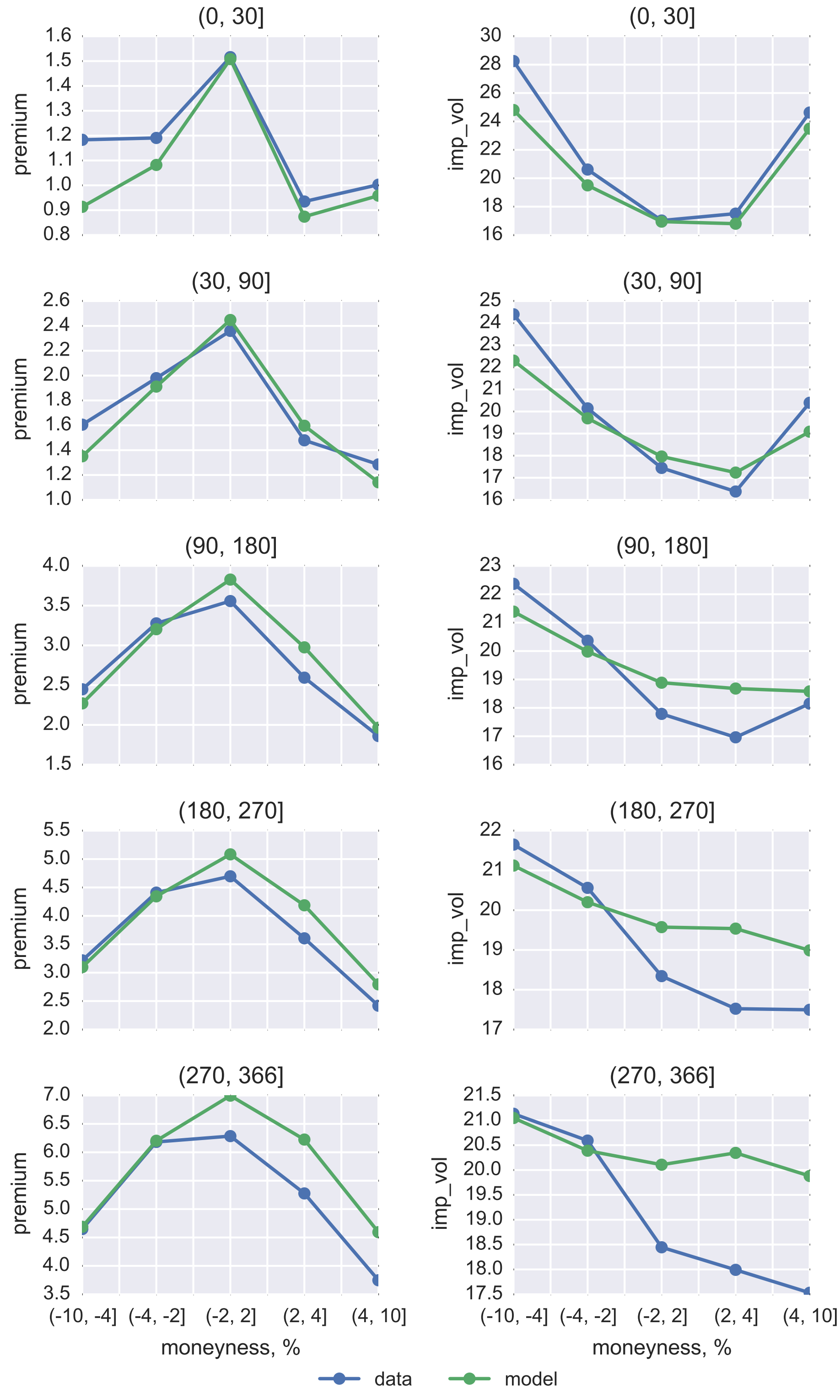


Figure 8: Data and model implied volatility

Implied volatilities are computed at the money, that is averages of all implied volatilities with log-moneyness between -2% and 2%.

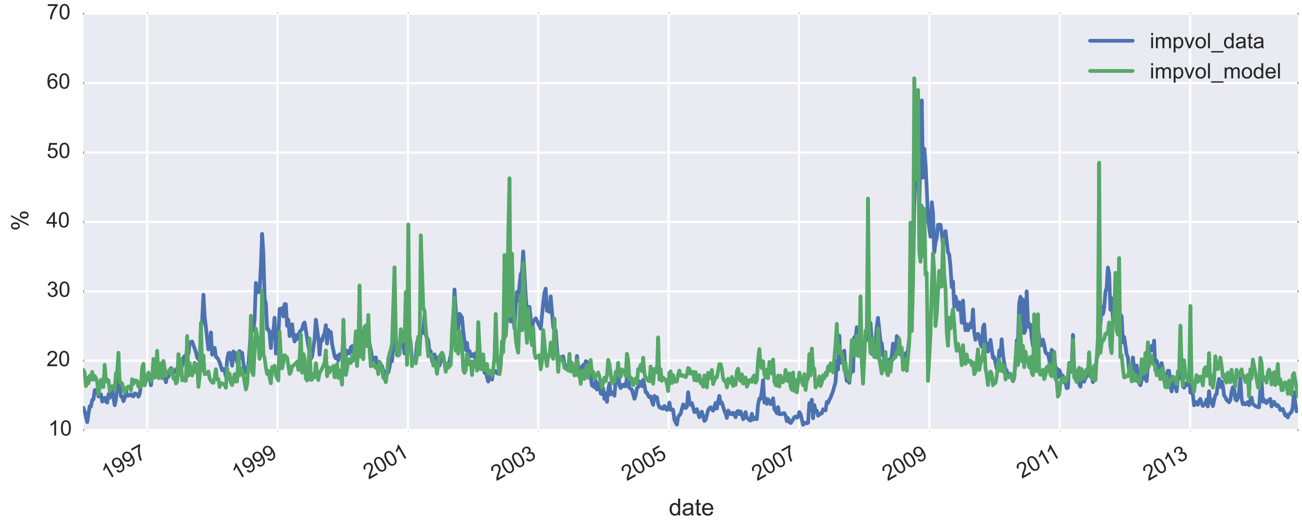
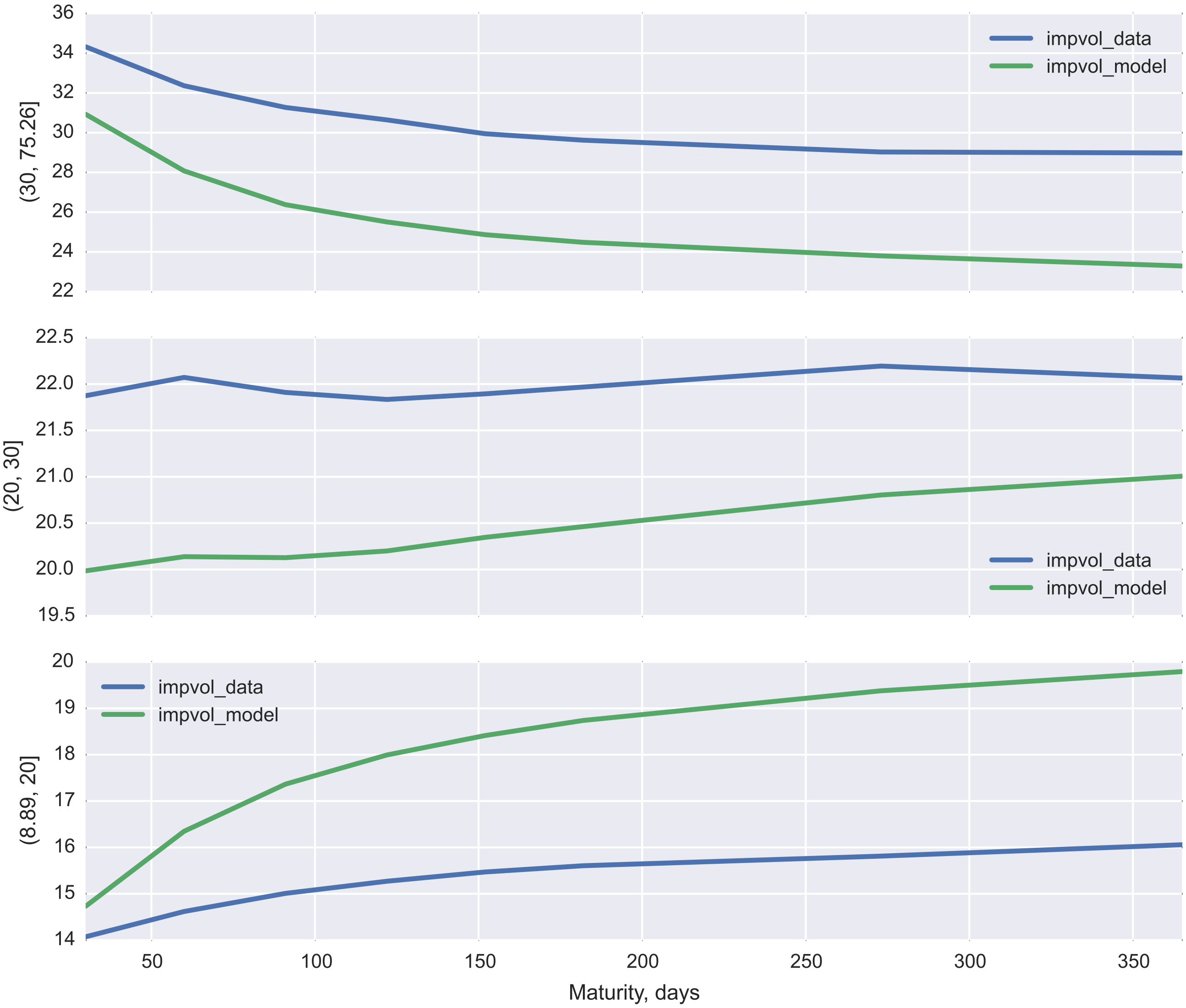


Figure 9: Implied volatility term structure

Three levels of the current VIX. Top panel refers to the high level of volatility, bottom - low level.



C Proofs

C.1 Proof of Proposition 1 (Volatility factor aggregation)

C.1.1 Proof for the first moment of volatility

Everywhere below we use the following notation:

$$E_t[X] = E[X \,|I_t].$$

The first equation in the theorem statement is trivially

$$E_t\left[\sigma_{t+1}^2\right] = \rho E_t\left[\sigma_t^2\right] + c\delta.$$

Taking conditional expectation analogously:

$$E_t\left[\sigma_{t+2}^2\right] = \rho E_t\left[\sigma_{t+1}^2\right] + c\delta.$$

Hence, by the law of iterated expectations and plugging the former formula in the latter:

$$\begin{aligned} E_t\left[\sigma_{t+2}^2\right] &= \rho E_t\left[\sigma_{t+1}^2\right] + c\delta \\ &= \rho^2 E_t\left[\sigma_t^2\right] + c\delta\left(1+\rho\right). \end{aligned}$$

By iterating H times the same argument, we get:

$$\begin{aligned} E_t\left[\sigma_{t+H}^2\right] &= \rho^H E_t\left[\sigma_t^2\right] + c\delta\left(1+\rho+...+\rho^{H-1}\right) \\ &= \rho^H E_t\left[\sigma_t^2\right] + c\delta\frac{1-\rho^H}{1-\rho} \\ &= \rho^H E_t\left[\sigma_t^2\right] + c\delta\left(H\right), \end{aligned} \tag{C.1}$$

with

$$\delta\left(H\right) = \delta\frac{1-\rho^H}{1-\rho}. \tag{C.2}$$

C.1.2 Proof for the second moment of volatility

Since (2.11) is assumed to be valid at least for $H = 1$, we have:

$$E_t\left[\sigma_{t+1}^4\right] = \rho^2 E_t\left[\sigma_t^4\right] + aE_t\left[\sigma_t^2\right] + b.$$

But, for the same reason,

$$E_{t+1}\left[\sigma_{t+2}^4\right] = \rho^2 E_{t+1}\left[\sigma_{t+1}^4\right] + aE_{t+1}\left[\sigma_{t+1}^2\right] + b.$$

Hence, by the law of iterated expectations and plugging the former formula in the latter:

$$E_t\left[\sigma_{t+2}^4\right]=\rho^4 E_t\left[\sigma_t^4\right]+a\left(E_t\left[\sigma_{t+1}^2\right]+\rho^2 E_t\left[\sigma_t^2\right]\right)+b\left(1+\rho^2\right).$$

By iterating H times the same argument, we get:

$$E_t\left[\sigma_{t+H}^4\right]=\rho^{2H} E_t\left[\sigma_t^4\right]+a\sum_{h=0}^{H-1}\rho^{2(H-1-h)}E_t\left[\sigma_{t+h}^2\right]+b\sum_{h=0}^{H-1}\rho^{2h}.$$

By applying (C.1) to the second term in the above equation separately, we get:

$$\begin{aligned}\sum_{h=0}^{H-1}\rho^{2(H-1-h)}E_t\left[\sigma_{t+h}^2\right] &= \sum_{h=0}^{H-1}\rho^{2(H-1-h)}\left(\rho^h E_t\left[\sigma_t^2\right]+c\delta\left(h\right)\right)\\ &= \rho^{H-1}\frac{1-\rho^H}{1-\rho}E_t\left[\sigma_t^2\right]+C_1,\end{aligned}$$

where

$$C_1=c\delta\sum_{h=0}^{H-1}\rho^{2(H-1-h)}\frac{1-\rho^h}{1-\rho}=c\delta\frac{\left(1-\rho^{H-1}\right)\left(1-\rho^H\right)}{\left(1-\rho\right)\left(1-\rho\right)^2}=c\delta\left(H\right)\frac{1-\rho^{H-1}}{1-\rho}. \tag{C.3}$$

Hence:

$$E_t\left[\sigma_{t+H}^4\right]=\rho^{2H}E_t\left[\sigma_t^4\right]+a\rho^{H-1}\frac{1-\rho^H}{1-\rho}E_t\left[\sigma_t^2\right]+d\left(H\right),$$

where

$$\begin{aligned}d\left(H\right)=&aC_1+b\sum_{h=0}^{H-1}\rho^{2h}\\ =&ac\delta\left(H\right)\frac{1-\rho^{H-1}}{1-\rho}+b\frac{1-\rho^{2H}}{1-\rho^2}.\end{aligned} \tag{C.4}$$

C.2 Proof of Proposition 2 (Volatility factor disaggregation)

Assume that model (2.8) is given with parameters:

$$\left(\tilde\rho,\tilde\delta,\tilde a,\tilde b\right)=\left(\rho^{1/N},\delta_N,a_N,b_N\right).$$

Then, by Proposition 1, we can compute the parameters of this model aggregated over $H=N$ periods. We get:

$$\begin{aligned}\rho\left(H\right) &= \tilde\rho^H=\rho,\\ \delta\left(H\right) &= \delta_N\frac{1-\tilde\rho^H}{1-\tilde\rho}=\delta\frac{1-\rho^{1/N}}{1-\rho}\frac{1-\rho^{H/N}}{1-\rho^{1/N}}=\delta,\end{aligned}$$

and:

$$\begin{aligned} a(H) &= \tilde{a}\tilde{\rho}^{H-1}\frac{1-\tilde{\rho}^{H}}{1-\tilde{\rho}} \\ &= a\frac{1-\rho^{1/N}}{1-\rho}\rho^{-1+1/N}\rho^{(H-1)/N}\frac{1-\rho^{H/N}}{1-\rho^{1/N}} = a, \end{aligned}$$

as well as:

$$\begin{aligned} d(H) &= \tilde{a}c\frac{1-\tilde{\rho}^{H-1}}{1-\tilde{\rho}^{2}}\delta(H) + \tilde{b}\frac{1-\tilde{\rho}^{2H}}{1-\tilde{\rho}^{2}} \\ &= ac\delta\frac{1-\rho^{1/N}}{1-\rho}\rho^{-1+1/N}\frac{1-\rho^{(H-1)/N}}{1-\rho^{2/N}} \\ &+ \left[b\frac{1-\rho^{2/N}}{1-\rho^{2}} - ac\delta\frac{\left(1-\rho^{1/N}\right)\left(\rho^{-1+1/N}-1\right)}{(1-\rho)\left(1-\rho^{2}\right)}\right]\frac{1-\rho^{2H/N}}{1-\rho^{2/N}} \\ &= b. \end{aligned}$$

C.3 Proof of Proposition 3 (Volatility factor as ARMA process)

C.3.1 Proof for the first moment of volatility

Continuing from equation (C.1) in Section C.1.1, for any real $h \geq 0$:

$$E_{t+h}\left[\sigma^2_{t+H+h}\right] = \rho^H E_{t+h}\left[\sigma^2_{t+h}\right] + c\delta(H).$$

and, by the law of iterated expectations:

$$E_{t}\left[\sigma^2_{t+H+h}\right] = \rho^H E_{t}\left[\sigma^2_{t+h}\right] + c\delta(H).$$

Adding all above equations for $h = \frac{1}{N}, \frac{2}{N}, \ldots, HN-1, HN$, and dividing by HN, we get:

$$E_{t}\left[\sigma^2_{t+H,H}(N)\right] = \rho^H E_{t}\left[\sigma^2_{t,H}(N)\right] + c\delta(H),$$

or

$$E_{t}\left[\sigma^2_{t+H,H}(N) - \rho^H\sigma^2_{t,H}(N) - c\delta(H)\right] = 0,$$

or

$$E_{t}\left[\left(1-\rho^H L\right)\sigma^2_{t+H,H}(N)\right] = c\delta(H),$$

with:

C.3.2 Proof for the second moment of volatility

Continuing from where we left off in Section C.1.2, for any $h \geq 0$ (with an additional application of the law of iterated expectations) we have:

$$E_t\left[\sigma_{t+H+h}^4\right] = \rho^{2H} E_t\left[\sigma_{t+h}^4\right] + a\rho^{H-1}\frac{1-\rho^H}{1-\rho} E_t\left[\sigma_{t+h}^2\right] + d\left(H\right).$$

This equation can be rewritten using lag operator L as

$$E_t\left[\left(1-\rho^{2H}L^H\right)\sigma_{t+H+h}^4\right] = a\rho^{H-1}\frac{1-\rho^H}{1-\rho} E_t\left[\sigma_{t+h}^2\right] + d\left(H\right). \tag{C.5}$$

Adding all of the above equations for $h = \frac{1}{N}, \frac{2}{N}, \ldots, H - \frac{1}{N}, H$, and dividing by N we get:

$$E_t\left[\left(1-\rho^{2H}L^H\right)\frac{1}{HN}\sum_{n=1}^{HN}\sigma_{t+H+n/N}^4\right] = a\rho^{H-1}\frac{1-\rho^H}{1-\rho} E_t\left[\sigma_{t,H}^2\left(N\right)\right] + d\left(H\right). \tag{C.6}$$

We are actually interested in computing $E_t\left[\sigma_{t+H,H}^4\left(N\right)\right]$, where:

$$\begin{aligned}\sigma_{t,H}^4\left(N\right) &= \left[\frac{1}{HN}\sum_{n=1}^{HN}\sigma_{t+n/N}^2\right]^2 \\ &= \frac{1}{H^2N^2}\sum_{n=1}^{HN}\sigma_{t+n/N}^4 + \frac{2}{H^2N^2}\sum_{j=1}^{HN-1}\sum_{n=1}^{HN-j}\sigma_{t+n/N}^2\sigma_{t+(n+j)/N}^2.\end{aligned}$$

This means, after multiplying by HN and shifting time by H, that

$$\frac{1}{HN}\sum_{n=1}^{HN}\sigma_{t+H+n/N}^4 = HN\sigma_{t+H,H}^4\left(N\right) - \frac{2}{HN}\sum_{j=1}^{HN-1}\sum_{n=1}^{HN-j}\sigma_{t+H+n/N}^2\sigma_{t+H+(n+j)/N}^2.$$

Making the corresponding substitution in (C.6), and dividing by HN, we can write

$$\begin{aligned}E_t\left[\left(1-\rho^{2H}L^H\right)\sigma_{t+H,H}^4\left(N\right)\right] =& E_t\left[\frac{2}{H^2N^2}\sum_{j=1}^{HN-1}\sum_{n=1}^{HN-j}\left(1-\rho^{2H}L^H\right)\sigma_{t+H+n/N}^2\sigma_{t+H+(n+j)/N}^2\right] \\ &+ a_0\left(H;N\right)E_t\left[\sigma_{t,H}^2\left(N\right)\right] + \frac{1}{HN}d\left(H\right),\end{aligned}$$

where

$$a_0\left(H;N\right) = \frac{1}{HN}a\rho^{H-1}\frac{1-\rho^H}{1-\rho}. \tag{C.7}$$

Note that (C.1) can be rewritten in terms of $\delta\left(H\right)$ defined in (C.2), and H can be fractional:

$$E_t\left[\sigma_{t+H}^2\right] = \rho^H E_t\left[\sigma_t^2\right] + c\delta\left(H\right). \tag{C.8}$$

By the law of iterated expectations and (C.8) the expectation of cross-term is:

$$\begin{aligned}E_t\left[\sigma^2_{t+H+n/N}\sigma^2_{t+H+(n+j)/N}\right] =&E_t\left[\sigma^2_{t+H+n/N}E_{t+H+n/N}\left[\sigma^2_{t+H+(n+j)/N}\right]\right]\\ =&\rho^{j/N}E_t\left[\sigma^4_{t+H+n/N}\right]+c\delta\left(j/N\right)E_t\left[\sigma^2_{t+H+n/N}\right].\end{aligned}$$

For $h=n/N$ equation (C.5) is

$$E_t\left[\left(1-\rho^{2H}L^H\right)\sigma^4_{t+H+n/N}\right]=a\rho^{H-1}\frac{1-\rho^H}{1-\rho}E_t\left[\sigma^2_{t+n/N}\right]+d\left(H\right).$$

Applying (C.8) to $E_t\left[\sigma^2_{t+H+n/N}\right]$ gives us

$$\begin{aligned}E_t\left[\left(1-\rho^{2H}L^H\right)\sigma^2_{t+H+n/N}\right]=&E_t\left[\sigma^2_{t+H+n/N}\right]-\rho^{2H}E_t\left[\sigma^2_{t+n/N}\right]\\ =&\rho^H\left(1-\rho^H\right)E_t\left[\sigma^2_{t+n/N}\right]+c\delta\left(H\right).\end{aligned}$$

Hence, the expectation of the cross-term multiplied by $\left(1-\rho^{2H}L^H\right)$ is

$$\begin{aligned}&E_t\left[\left(1-\rho^{2H}L^H\right)\sigma^2_{t+H+n/N}\sigma^2_{t+H+(n+j)/N}\right]=\\ &=\rho^{j/N}E_t\left[\left(1-\rho^{2H}L^H\right)\sigma^4_{t+H+n/N}\right]+c\delta\left(j/N\right)E_t\left[\left(1-\rho^{2H}L^H\right)\sigma^2_{t+H+n/N}\right]\\ &=\left[a\rho^{j/N}\rho^{H-1}\frac{1-\rho^H}{1-\rho}+c\delta\left(j/N\right)\rho^H\left(1-\rho^H\right)\right]E_t\left[\sigma^2_{t+n/N}\right]+C_3\left(j\right),\end{aligned}\tag{C.9}$$

where we denoted

$$\begin{aligned}C_3\left(j\right)=&\rho^{j/N}d\left(H\right)+c^2\delta\left(j/N\right)\delta\left(H\right)\\ =&\rho^{j/N}\left[ac\delta\left(H\right)\frac{1-\rho^{H-1}}{1-\rho}+b\frac{1-\rho^{2H}}{1-\rho^2}\right]+c^2\delta^2\frac{1-\rho^{j/N}}{1-\rho}\frac{1-\rho^H}{1-\rho}.\end{aligned}\tag{C.10}$$

Next we need to express $E_t\left[\sigma^2_{t+n/N}\right]$ in terms of $E_t\left[\sigma^2_{t,H}\left(N\right)\right]$. For that purpose apply (C.8) again:

$$E_t\left[\sigma^2_{t+n/N}\right]=\rho^{n/N}E_t\left[\sigma^2_t\right]+c\delta\left(n/N\right).$$

From the definition of the aggregated variance we easily find that

$$\begin{aligned}E_t\left[\sigma^2_{t,H}(N)\right] =& \frac{1}{HN}\sum_{n=1}^{HN} E_t\left[\sigma^2_{t+n/N}\right]\\ =& \frac{1}{HN}\sum_{n=1}^{HN}\left(\rho^{n/N}E_t\left[\sigma^2_t\right] + c\delta(n/N)\right)\\ =& \frac{1}{HN}\sum_{n=1}^{HN}\rho^{n/N}E_t\left[\sigma^2_t\right] + \frac{1}{HN}\sum_{n=1}^{HN}c\delta(n/N)\\ =& \frac{\rho^{1/N}}{HN}\frac{1-\rho^H}{1-\rho^{1/N}}E_t\left[\sigma^2_t\right] + C_4,\end{aligned}$$

where

$$C_4 = \frac{1}{HN}\sum_{n=1}^{HN}c\delta(n/N) = \frac{c\delta}{HN}\frac{\left(1-\rho^{1/N}\right)-\rho^{1/N}\left(1-\rho^H\right)}{(1-\rho)\left(1-\rho^{1/N}\right)}. \tag{C.11}$$

Solving for $E_t\left[\sigma^2_t\right]$ we have

$$E_t\left[\sigma^2_t\right] = \frac{HN}{\rho^{1/N}}\frac{1-\rho^{1/N}}{1-\rho^H}\left(E_t\left[\sigma^2_{t,H}(N)\right] - C_4\right).$$

At the same time we also have

$$\begin{aligned}E_t\left[\sigma^2_{t+n/N}\right] =& \rho^{n/N}E_t\left[\sigma^2_t\right] + c\delta(n/N)\\ =& HN\rho^{(n-1)/N}\frac{1-\rho^{1/N}}{1-\rho^H}\left(E_t\left[\sigma^2_{t,H}(N)\right] - C_4\right) + c\delta(n/N).\end{aligned}$$

Substituting this result to the expression (C.9) for the cross-terms we obtain

$$\begin{aligned}&E_t\left[\left(1-\rho^{2H}L^H\right)\sigma^2_{t+H+n/N}\sigma^2_{t+H+(n+j)/N}\right] =\\ &= \left(a\rho^{j/N}\rho^{H-1}\frac{1-\rho^H}{1-\rho} + c\delta(j/N)\rho^H\left(1-\rho^H\right)\right)\\ &\quad\times\left(HN\rho^{(n-1)/N}\frac{1-\rho^{1/N}}{1-\rho^H}\left(E_t\left[\sigma^2_{t,H}(N)\right] - C_4\right) + c\delta(n/N)\right) + C_3(j)\\ &= \left(a\rho^{j/N}\rho^{H-1}\frac{1-\rho^H}{1-\rho} + c\delta(j/N)\rho^H\left(1-\rho^H\right)\right)\rho^{(n-1)/N}\frac{1-\rho^{1/N}}{1-\rho^H}HNE_t\left[\sigma^2_{t,H}(N)\right] + C_5(j,n),\end{aligned}$$

where

$$\begin{aligned}C_5(j,n) =& \left(a\rho^{j/N}\rho^{H-1}\frac{1-\rho^H}{1-\rho} + c\delta(j/N)\rho^H\left(1-\rho^H\right)\right)\\ &\times\left(c\delta(n/N) - HN\rho^{(n-1)/N}\frac{1-\rho^{1/N}}{1-\rho^H}C_4\right) + C_3(j).\end{aligned} \tag{C.12}$$

Collecting the terms we find that the coefficient in (2.11) is

$$
\begin{aligned}
a\left(H;N\right) =& a_{0}\left(H;N\right) \\
&+\frac{2}{HN}\frac{\rho^{H}}{1-\rho^{H}}\left(1-\rho^{1/N}\right)\sum_{j=1}^{HN-1}\sum_{n=1}^{HN-j}\left(\rho^{j/N}\frac{a}{\rho}\frac{1-\rho^{H}}{1-\rho}+c\delta\left(H\right)\left(1-\rho^{n/N}\right)\right)\rho^{(n-1)/N},
\end{aligned}
\tag{C.13}
$$

and

$$
b\left(H;N\right)=\frac{2}{H^{2}N^{2}}\sum_{j=1}^{HN-1}\sum_{n=1}^{HN-j}C_{5}\left(j,n\right),
\tag{C.14}
$$

with $a_0\left(H;N\right)$ defined above in (C.7), $\delta\left(H\right)$ defined in (C.2), coefficients C_1 through C_5 defined in (C.3), (C.4), (C.10), (C.11), and (C.12).

C.4 Proof of Lemma 1

We simplify the proof by doing it only for the case $H = 1/N$. The general case follows by linear aggregation. We have:

$$
\sigma^{2}_{t,1/N}\left(N\right)=\sigma^{2}_{t+1/N},
$$

and thus:

$$
\lim_{N\to\infty}E\left[\left.\sigma^{2}_{t,1/N}\left(N\right)\right|I_{t}\right]=\lim_{N\to\infty}\rho^{1/N}\sigma^{2}_{t}+c\delta_{N}=\sigma^{2}_{t},
$$

since:

$$
\delta_{N}=\delta\frac{1-\rho^{1/N}}{1-\rho}\qquad\Rightarrow\qquad\lim_{N\to\infty}\delta_{N}=0.
$$

We deduce from Proposition 2 that:

$$
V\left[\left.\sigma^{2}_{t+1/N}\right|I_{t}\right]=\left[a_{N}-2c\delta_{N}\rho^{1/N}\right]\sigma^{2}_{t}+b_{N}-c^{2}\delta^{2}_{N},
$$

with:

$$
\lim_{N\to\infty}Na_{N}=\lim_{N\to\infty}Na\frac{1-\rho^{1/N}}{1-\rho}\rho^{-1+1/N}=-\frac{a\log\left(\rho\right)}{\rho\left(1-\rho\right)},
\tag{C.15}
$$

and,

$$
\lim_{N\to\infty}N\delta_{N}\rho^{1/N}=\lim_{N\to\infty}N\delta_{N}=\lim_{N\to\infty}N\delta\frac{1-\rho^{1/N}}{1-\rho}=-\frac{\delta\log\left(\rho\right)}{1-\rho},
$$

and,

$$
\lim_{N\to\infty}N\delta^{2}_{N}=-\lim_{N\to\infty}\delta_{N}\frac{\delta\log\left(\rho\right)}{1-\rho}=0,
$$

and,

$$\begin{aligned}
\lim_{N\to\infty} N b_N &= b \lim_{N\to\infty} N \frac{1-\rho^{2/N}}{1-\rho} \\
&\quad - \frac{ac\delta}{(1-\rho)(1-\rho^2)} \lim_{N\to\infty} N \left(1-\rho^{1/N}\right)\left(\rho^{-1+1/N}-1\right) \\
&= -\frac{2b\log(\rho)}{1-\rho} + \frac{ac\delta\log(\rho)}{\rho(1-\rho^2)}.
\end{aligned}$$

We deduce:

$$\frac{1-\rho}{\log(\rho)} \lim_{N\to\infty} V\left[\sigma^2_{t+1/N}\middle| I_t\right] = \left[-\frac{a}{\rho} + 2c\delta\right]\sigma_t^2 - 2b + \frac{ac\delta}{\rho(1+\rho)}.$$

Next,

$$\sigma^2_{t+H,H}(N) = \sigma^2_{t+2/N},$$

with:

$$\begin{aligned}
V\left[\sigma^2_{t+2/N}\middle| I_t\right] &= E\left[V\left[\sigma^2_{t+2/N}\middle| I_{t+1/N}\right]\middle| I_t\right] + V\left[E\left[\sigma^2_{t+2/N}\middle| I_{t+1/N}\right]\middle| I_t\right] \\
&= E\left[\left[a_N - 2c\delta_N\rho^{1/N}\right]\sigma^2_{t+1/N} + b_N - c^2\delta_N^2\middle| I_t\right] \\
&\quad + V\left[\rho^{1/N}\sigma^2_{t+1/N} + c\delta_N\middle| I_t\right] \\
&= \left[a_N - 2c\delta_N\rho^{1/N}\right]\left[\rho^{1/N}\sigma_t^2 + c\delta_N\right] + b_N - c^2\delta_N^2 \\
&\quad + \rho^{2/N}\left(\left[a_N - 2c\delta_N\rho^{1/N}\right]\sigma_t^2 + b_N - c^2\delta_N^2\right) \\
&= \left[a_N - 2c\delta_N\rho^{1/N}\right]\left[\rho^{1/N} + \rho^{2/N}\right]\sigma_t^2 + \left[b_N - c^2\delta_N^2\right]\left(1+\rho^{2/N}\right) \\
&\quad + c\delta_N\left[a_N - 2c\delta_N\rho^{1/N}\right].
\end{aligned}$$

We deduce from (C.15):

$$\lim_{N\to\infty} N\left[a_N - 2c\delta_N\rho^{1/N}\right]\left[\rho^{1/N} + \rho^{2/N}\right] = 2\left[-\frac{a\log(\rho)}{\rho(1-\rho)} + 2\frac{c\delta\log(\rho)}{1-\rho}\right],$$

and

$$\lim_{N\to\infty} N c\delta_N\left[a_N - 2c\delta_N\rho^{1/N}\right] = 0,$$

and

$$\lim_{N\to\infty} N\left[b_N - c^2\delta_N^2\right]\left(1+\rho^{2/N}\right) = 2\left[-\frac{2b\log(\rho)}{1-\rho} + \frac{ac\delta\log(\rho)}{\rho(1-\rho^2)}\right],$$

so that:

$$\begin{aligned}
\frac{1-\rho}{\log(\rho)} \lim_{N\to\infty} V\left[\sigma^2_{t+2/N}\middle| I_t\right] &= 2\left[-\frac{a}{\rho} + 2c\delta\right]\sigma_t^2 - 4b + \frac{2ac\delta}{\rho(1+\rho)} \\
&= 2\frac{1-\rho}{\log(\rho)} \lim_{N\to\infty} V\left[\sigma^2_{t+1/N}\middle| I_t\right].
\end{aligned}$$

C.5 Proof of Proposition 4 (Continuous-time limit for volatility)

We deduce from the first equation in (2.11) and (2.12) from Lemma 1 that:

$$\begin{aligned}\lim_{H\to 0}\frac{1}{H}E_t\left[\sigma^2_{t+H,H}(N)-\sigma^2_{t,H}(N)\right] &= \lim_{H\to 0}\frac{\rho^H-1}{H}E_t\left[\sigma^2_{t,H}(N)\right]+\lim_{H\to 0}\frac{c\delta(H)}{H}\\ &= \log(\rho)\left(\sigma^2(t)-\frac{c\delta}{1-\rho}\right),\end{aligned}$$

since:

$$\lim_{H\to 0}\frac{\delta(H)}{H}=\lim_{H\to 0}\frac{c\delta}{1-\rho}\frac{1-\rho^H}{H}=-c\delta\frac{\log(\rho)}{1-\rho}.$$

Straight from definition of the variance and application of (C.8) the equation (2.11) is the same as

$$\begin{aligned}V_t\left[\sigma^2_{t+H,H}(N)\right] &= -\left(E_t\left[\sigma^2_{t+H,H}(N)\right]\right)^2+\rho^{2H}E_t\left[\sigma^4_{t,H}(N)\right]+a(H;N)\,E_t\left[\sigma^2_{t,H}(N)\right]+b(H;N)\\ &= -\left(E_t\left[\rho^H\sigma^2_{t,H}(N)+c\delta(H)\right]\right)^2+\rho^{2H}E_t\left[\sigma^4_{t,H}(N)\right]\\ &\quad +a(H;N)\,E_t\left[\sigma^2_{t,H}(N)\right]+b(H;N)\\ &=\rho^{2H}V_t\left[\sigma^2_{t,H}(N)\right]+\left(a(H;N)-2\rho^Hc\delta(H)\right)E_t\left[\sigma^2_{t,H}(N)\right]+\left(b(H;N)-c^2\delta^2(H)\right).\end{aligned}$$

Next, divide this expression on both sides by H and take the limit ($N\to\infty$ implicitly since $\sigma^2_{t,H}(N)$ is only defined for $H\geq 1/N$):

$$\begin{aligned}\lim_{H\to 0}\frac{1}{H}V_t\left[\sigma^2_{t+H,H}(N)\right] &= \lim_{H\to 0}\rho^{2H}\frac{1}{H}V_t\left[\sigma^2_{t,H}(N)\right]+\lim_{H\to 0}\frac{a(H;N)-2\rho^Hc\delta(H)}{H}E_t\left[\sigma^2_{t,H}(N)\right]\\ &\quad +\lim_{H\to 0}\frac{b(H;N)-c^2\delta^2(H)}{H}.\end{aligned}$$

Using (2.13) from Lemma 1, we get that:

$$\begin{aligned}\lim_{H\to 0}\frac{1}{H}V_t\left[\sigma^2_{t+H,H}(N)\right]-\lim_{H\to 0}\rho^{2H}\frac{1}{H}V_t\left[\sigma^2_{t,H}(N)\right] &= \lim_{H\to 0}\frac{2}{H}V_t\left[\sigma^2_{t,H}(N)\right]-\lim_{H\to 0}\frac{\rho^{2H}}{H}V_t\left[\sigma^2_{t,H}(N)\right]\\ &= \lim_{H\to 0}\frac{1}{H}V_t\left[\sigma^2_{t,H}(N)\right]\left(2-\rho^{2H}\right)\\ &= \lim_{H\to 0}\frac{1}{H}V_t\left[\sigma^2_{t,H}(N)\right].\end{aligned}$$

and deduce that the above limit expression can be rewritten as:

$$\lim_{H\to 0}\frac{1}{H}V_t\left[\sigma^2_{t,H}(N)\right]=\lim_{H\to 0,N\to\infty}\frac{a(H;N)-2\rho^Hc\delta(H)}{H}E_t\left[\sigma^2_{t,H}(N)\right]+\lim_{H\to 0,N\to\infty}\frac{b(H;N)-c^2\delta^2(H)}{H}.$$

Simplifying $a\left(H;N\right)$. Before taking the limit with respect to $N\to\infty$ we need to simplify $a\left(H;N\right)$ by getting rid of summations in

$$\begin{aligned} a\left(H;N\right)-a_0\left(H;N\right)=&\frac{2}{HN}\frac{\rho^H}{1-\rho^H}\left(1-\rho^{1/N}\right)\\ &\times\sum_{j=1}^{HN-1}\left[\left(\rho^{j/N}\frac{a}{\rho}\frac{1-\rho^H}{1-\rho}+c\delta\left(H\right)\right)\sum_{n=1}^{HN-j}\rho^{(n-1)/N}-\rho^{1/N}c\delta\left(H\right)\sum_{n=1}^{HN-j}\rho^{2(n-1)/N}\right], \end{aligned}$$

with $a_0\left(H;N\right)$ defined in (C.7). Here the inner summations are reduced to

$$\sum_{n=1}^{HN-j}\rho^{(n-1)/N}=\sum_{n=0}^{HN-j-1}\rho^{n/N}=\frac{1-\rho^{H-j/N}}{1-\rho^{1/N}},\quad\text{and}\quad\sum_{n=1}^{HN-j}\rho^{2(n-1)/N}=\frac{1-\rho^{2H-2j/N}}{1-\rho^{2/N}}.$$

So the coefficient becomes

$$\begin{aligned} a\left(H;N\right)-a_0\left(H;N\right)=&\frac{2}{HN}\frac{\rho^H}{1-\rho^H}\left(1-\rho^{1/N}\right)\\ &\times\sum_{j=1}^{HN-1}\left[\left(\rho^{j/N}\frac{a}{\rho}\frac{1-\rho^H}{1-\rho}+c\delta\left(H\right)\right)\frac{1-\rho^{H-j/N}}{1-\rho^{1/N}}-\rho^{1/N}c\delta\left(H\right)\frac{1-\rho^{2H-2j/N}}{1-\rho^{2/N}}\right], \end{aligned}$$

or

$$\begin{aligned} a\left(H;N\right)-a_0\left(H;N\right)=&\frac{2}{HN}\frac{\rho^H}{1-\rho^H}\left(\frac{a}{\rho}\frac{1-\rho^H}{1-\rho}\sum_{j=1}^{HN-1}\left(\rho^{j/N}-\rho^H\right)+c\delta\left(H\right)\sum_{j=1}^{HN-1}\left(1-\rho^{H-j/N}\right)\right)\\ &-\frac{2}{N}\frac{\rho^H}{1-\rho^H}\frac{\rho^{1/N}}{1+\rho^{1/N}}c\delta\left(H\right)\sum_{j=1}^{HN-1}\left(1-\rho^{2H-2j/N}\right). \end{aligned}$$

In this expression we have three summations over j:

$$\sum_{j=1}^{HN-1}\rho^{H-j/N}=\sum_{j=1}^{HN-1}\rho^{j/N}=\frac{\rho^{1/N}-\rho^H}{1-\rho^{1/N}},\quad\text{and}\quad\sum_{j=1}^{HN-1}\rho^{2H-2j/N}=\frac{\rho^{2/N}-\rho^{2H}}{1-\rho^{2/N}}.$$

Substituting these we have

$$\begin{aligned} a\left(H;N\right)-a_0\left(H;N\right)=&-\frac{2}{HN}\frac{a}{\rho}\frac{\rho^H}{1-\rho}\left(\rho^H\left(HN-1\right)-\frac{\rho^{1/N}-\rho^H}{1-\rho^{1/N}}\right)\\ &+\frac{2}{HN}\frac{\rho^H}{1-\rho^H}c\delta\left(H\right)\left(\left(HN-1\right)-\frac{\rho^{1/N}-\rho^H}{1-\rho^{1/N}}\right)\\ &-\frac{2}{HN}\frac{\rho^H}{1-\rho^H}\frac{\rho^{1/N}}{1+\rho^{1/N}}c\delta\left(H\right)\left(\left(HN-1\right)-\frac{\rho^{2/N}-\rho^{2H}}{1-\rho^{2/N}}\right), \end{aligned}$$

with $a_0\left(H;N\right)$ defined in (C.7).

Taking the limit with $N \to \infty$. Taking the limit with respect to $N \to \infty$, the coefficient becomes

$$\begin{aligned}\lim_{N\to\infty} a\left(H;N\right) = & -\frac{2}{H}\frac{a}{\rho}\frac{\rho^H}{1-\rho}\left(\frac{1-\rho^H}{\log\left(\rho\right)}+\rho^H H\right) \\ & + 2\frac{\rho^H}{1-\rho^H}\frac{c\delta\left(H\right)}{H}\left(H+\frac{1-\rho^H}{\log\left(\rho\right)}\right) \\ & - \frac{\rho^H}{1-\rho^H}\frac{c\delta\left(H\right)}{H}\left(H+\frac{1-\rho^{2H}}{\log\left(\rho^2\right)}\right),\end{aligned} \tag{C.16}$$

while keeping in mind that

$$\lim_{N\to\infty} a_0\left(H;N\right) = \lim_{N\to\infty}\frac{1}{HN}a\rho^{H-1}\frac{1-\rho^H}{1-\rho} = 0.$$

Now divide (C.16) by H :

$$\begin{aligned}\frac{1}{H}\lim_{N\to\infty} a\left(H;N\right) = & -2\frac{a}{\rho}\frac{\rho^H}{1-\rho}\frac{1}{H}\left(\frac{1-\rho^H}{H\log\left(\rho\right)}+\rho^H\right) \\ & + 2\frac{\rho^H}{1-\rho^H}\frac{c\delta\left(H\right)}{H}\left(1+\frac{1-\rho^H}{H\log\left(\rho\right)}\right) \\ & - \frac{\rho^H}{1-\rho^H}\frac{c\delta\left(H\right)}{H}\left(1+\frac{1-\rho^{2H}}{H\log\left(\rho^2\right)}\right).\end{aligned}$$

Series expansion of this expression around $H = 0$ gives the following result:

$$\frac{1}{H}\lim_{N\to\infty} a\left(H;N\right) = -\frac{a}{\rho}\frac{\log\left(\rho\right)}{1-\rho} + O\left(H\right).$$

Hence,

$$\lim_{H\to 0, N\to\infty}\frac{a\left(H;N\right)}{H} = -\frac{a}{\rho}\frac{\log\left(\rho\right)}{1-\rho}.$$

Taking the limit of the constant we obtain[5]

$$\lim_{H\to 0, N\to\infty}\frac{b\left(H;N\right)}{H} = \left(\frac{a}{\rho}-\frac{2b}{c\delta}\right)\frac{c\delta\log\left(\rho\right)}{1-\rho^2}.$$

Finally,

$$\lim_{H\to 0}\frac{\delta^2\left(H\right)}{H} = \lim_{H\to 0}\left(\frac{\delta\left(H\right)}{H}\right)^2 H = 0.$$

[5] The analytical expression for $b\left(H;N\right)$ after taking all summations is several pages long. Taking the limit of this expression by hand does not seem feasible. These operations were performed in Mathematica software and available upon request.

This result concludes the proof and shows explicitly that

$$\begin{aligned}\lim_{H\to 0, N\to\infty}\frac{1}{H}V_t\left[\sigma_{t,H}^2(N)\right] &= \left(-\frac{a}{\rho}\frac{\log(\rho)}{1-\rho}+2c\delta\frac{\log(\rho)}{1-\rho}\right)\sigma_t^2+\left(\frac{a}{\rho}-\frac{2b}{c\delta}\right)\frac{c\delta\log(\rho)}{1-\rho^2}\\ &= -\frac{\log(\rho)}{1-\rho}\left[\left(\frac{a}{\rho}-2c\delta\right)\sigma_t^2+\left(\frac{a}{\rho}-\frac{2b}{c\delta}\right)\frac{c\delta}{1+\rho}\right].\end{aligned}$$

In case of affine first two moments as in (2.8), we have

$$a = 2c\rho(1+\delta), \quad b = c^2\left(\delta^2+\omega\right),$$

hence the limit becomes

$$\lim_{H\to 0, N\to\infty}\frac{1}{H}V_t\left[\sigma_{t,H}^2(N)\right] = -2c\frac{\log(\rho)}{1-\rho}\left(\sigma_t^2+c\frac{\delta-\omega}{1+\rho}\right).$$

For the ARG(1) case, where $\omega=\delta$, the same limit becomes

$$\lim_{H\to 0, N\to\infty}\frac{1}{H}V_t\left[\sigma_{t,H}^2(N)\right] = -2c\frac{\log(\rho)}{1-\rho}\sigma_t^2,$$

as expected from a particular case of Gourieroux & Jasiak (2006, p. 137).

C.6 Proof of Proposition 5

We have seen in Section 3.1 that:

$$\begin{aligned}&\exp\left(-l^*(u,v)\,\sigma_t^2-g^*(u,v)\right)=\\ &=\exp\left(r_{f,t}\right)E\left[M_{t,t+1}(\theta)\exp\left(-u\sigma_{t+1}^2-vr_{t+1}\right)\Big|\, I_t\right]\\ &=\exp\left(g(\theta)+l(\theta)\,\sigma_t^2\right)E\left[M_{t,t+1}(\theta)\exp\left(-(\theta_1+u)\,\sigma_{t+1}^2-(\theta_2+v)\,r_{t+1}\right)\Big|\, I_t\right]\\ &=\exp\left(g(\theta)-g(\theta_1+u,\theta_2+v)\right)\exp\left(\left[l(\theta)-l(\theta_1+u,\theta_2+v)\right]\sigma_t^2\right).\end{aligned}$$

Therefore:

$$\begin{aligned}g^*(u,v) &= g(\theta_1+u,\theta_2+v)-g(\theta_1,\theta_2),\\ l^*(u,v) &= l(\theta_1+u,\theta_2+v)-l(\theta_1,\theta_2).\end{aligned}$$

We want to check that these formulas coincide with the formulas for $g^*$and l^* given in Proposition 5. Let us then compute, while plugging in (3.5) and (3.6):

$$\begin{aligned}a^*\left[u+\alpha^*(v)\right]+\beta^*(v) &= a\left[u+\alpha^*(v)+\theta_1+\alpha(\theta_2)\right]-a\left[\theta_1+\alpha(\theta_2)\right]+\beta(\theta_2+v)-\beta(\theta_2)\\ &= a\left[u+\theta_1+\alpha(\theta_2+v)\right]-a\left[\theta_1+\alpha(\theta_2)\right]+\beta(\theta_2+v)-\beta(\theta_2)\\ &= \left(a\left[\theta_1+u+\alpha(\theta_2+v)\right]+\beta(\theta_2)+v\right)-\left(a\left[\theta_1+\alpha(\theta_2)\right]+\beta(\theta_2)\right)\\ &= l(\theta_1+u,\theta_2+v)-l(\theta_1,\theta_2)=l^*(u,v).\end{aligned}$$

A similar computation would obviously give:

$$b^*\left[u+\alpha\left(v\right)\right]+\gamma^*\left(v\right)=g^*\left(u,v\right).$$

C.7 Proof of equation (5.3) (Laplace transform for the aggregated return)

Recursive volatility characteristic function is

$$E\left[\exp\left\{-u\sigma_{t+n}^2\right\}\Big|\, I_t\right]=\exp\left\{-a_n\left(u\right)\sigma_t^2-b_n\left(u\right)\right\}.$$

One step further

$$E\left[\exp\left\{-u\sigma_{t+n+1}^2\right\}\Big|\, I_t\right]=\exp\left\{-a\left(a_n\left(u\right)\right)\sigma_t^2-b\left(a_n\left(u\right)\right)-b_n\left(u\right)\right\}.$$

Hence, it is clear that

$$\begin{aligned}
a_1\left(u\right) &= a\left(u\right),\\
b_1\left(u\right) &= b\left(u\right),\\
a_{n+1}\left(u\right) &= a\left(a_n\left(u\right)\right),\\
b_{n+1}\left(u\right) &= b\left(a_n\left(u\right)\right)+b_n\left(u\right).
\end{aligned}$$

Recursive return characteristic function is

$$E\left[\exp\left\{-vr_{t+1}\right\}|\, I_t\right]=\exp\left\{-h\left(v\right)\sigma_t^2-k\left(v\right)\right\}.$$

One step further

$$E\left[\exp\left\{-vr_{t+2}\right\}|\, I_t\right]=\exp\left\{-a\left(h\left(v\right)\right)\sigma_t^2-b\left(h\left(v\right)\right)-k\left(v\right)\right\}.$$

In general,

$$E\left[\exp\left\{-vr_{t+n}\right\}|\, I_t\right]=\exp\left\{-h_n\left(v\right)\sigma_t^2-k_n\left(v\right)\right\},$$

where

$$\begin{aligned}
h_1\left(v\right) &= l\left(0,v\right)\\
k_1\left(v\right) &= g\left(0,v\right)\\
h_{n+1}\left(v\right) &= a\left(h_n\left(v\right)\right)\\
k_{n+1}\left(v\right) &= b\left(h_n\left(v\right)\right)+k_n\left(v\right)
\end{aligned}$$

Recursive cumulative return characteristic function

$$
\begin{aligned}
E\left[\exp\left\{-vr_{t,t+n}\right\}\middle| I_t\right] =& E\left[\exp\left\{-v\sum_{j=1}^{n} r_{t+j}\right\}\middle| I_t\right] \\
=& E\left[\exp\left\{-v\sum_{j=1}^{n-1} r_{t+j}\right\} E\left[\exp\left\{-vr_{t+n}\right\}\middle| I_{t+n-1}\right]\middle| I_t\right] \\
=& E\left[\exp\left\{-v\sum_{j=1}^{n-1} r_{t+j}\right\} \exp\left\{-l\left(0,v\right)\sigma_{t+n-1}^2 - g\left(0,v\right)\right\}\middle| I_t\right] \\
=& E\left[\exp\left\{-v\sum_{j=1}^{n-1} r_{t+j}\right\} \exp\left\{-\Psi_1\left(v\right)\sigma_{t+n-1}^2 - \Upsilon_1\left(v\right)\right\}\middle| I_t\right] \\
=& E\left[\exp\left\{-v\sum_{j=1}^{n-2} r_{t+j}\right\} E\left[\exp\left\{-vr_{t+n-1} - \Psi_1\left(v\right)\sigma_{t+n-1}^2 - \Upsilon_1\left(v\right)\right\}\middle| I_{t+n-2}\right]\middle| I_t\right] \\
=& E\left[\exp\left\{-v\sum_{j=1}^{n-2} r_{t+j}\right\} \exp\left\{-l\left(\Psi_1\left(v\right),v\right)\sigma_{t+n-2}^2 - g\left(\Psi_1\left(v\right),v\right) - \Upsilon_1\left(v\right)\right\}\middle| I_t\right] \\
=& \exp\left\{-\Psi_n\left(v\right)\sigma_t^2 - \Upsilon_n\left(v\right)\right\},
\end{aligned}
$$

with

$$
\begin{aligned}
\Psi_1\left(v\right) &= l\left(0,v\right), \\
\Upsilon_1\left(v\right) &= g\left(0,v\right), \\
\Psi_{n+1}\left(v\right) &= l\left(\Psi_n\left(v\right),v\right), \\
\Upsilon_{n+1}\left(v\right) &= g\left(\Psi_n\left(v\right),v\right) + \Upsilon_n\left(v\right).
\end{aligned}
$$