August 20, 2018

1 Robust Inference for Risk Price

$\varsigma = 1 - \phi^2$

δ

The reduced-form parameters are $\omega = (\rho, c, \gamma, \beta, \psi, \phi^2)$. Using the conditional mean and conditional variance derived in the paper, we estimate $\omega_1 = (\rho, c)$ by the GMM estimator, estimate $\omega_2 = (\psi, \beta, \gamma)$ by the GLS estimator, and estimate $\omega_3 = \phi^2$ by the method of moments estimator for the variance. We can show that the estimator $\widehat{\omega}$ satisfies

$$n^{1/2}(\widehat{\omega} - \omega_0) \rightarrow_d \xi_\omega \sim N(0, V). \tag{1.1}$$

See the next section for details. Note that these estimators do not involve the structural parameters θ and π. We do not plug in β, γ, ψ as functions of θ and π. Instead, we treat β, γ, ψ just as linear coefficients and estimate them by GLS.

We estimate the structural parameters θ and π using $\widehat{\omega}$ and the link functions specified below. First, we know that

$-\sqrt{\max\{1-\varsigma, 0\}}$ $\quad\varsigma\quad\varsigma$

$$\psi_0 = \phi_0 \left(c_0 \left(1 + \rho_0\right)\right)^{-1/2} + \left(1 - \phi_0^2\right)/2 - (1 - \phi_0^2)\theta_0 \tag{1.2}$$

when all parameters are evaluated at the true values. This equation strongly identifies θ_0 because ϕ_0^2 is assumed to be bounded away from 1. It follows from (1.2) that

$$\theta_0 = L(\omega_0) = (1 - \phi_0^2)^{-1} \left[\psi_0 - \phi_0 \left(c_0 \left(1 + \rho_0\right)\right)^{-1/2} - \left(1 - \phi_0^2\right)/2\right]. \tag{1.3}$$

Thus, we estimate θ_0 by

$$\widehat{\theta} = L(\widehat{\omega}). \tag{1.4}$$

By the delta method, we know that

$$n^{1/2}(\widehat{\theta} - \theta_0) \rightarrow_d L_\omega(\omega_0)'\xi_\omega, \tag{1.5}$$

where $L_\omega(\omega) \in R^{d_\omega}$denote the derivative of $L(\omega)$ wrt to ω. The inference for θ is standard. A confidence interval for θ can be obtained by inverting the t statistic with a critical value obtained from the standard normal distribution.

Next, we consider inference for the structural parameter π. This is a non-standard problem because π is potentially weakly identified. Define

$$g(\pi, \omega) = \begin{pmatrix} \gamma - \left[B\left(\pi + C\left(\overset{L(w)}{\cancel{\theta_L}} - 1\right)\right) - B\left(\pi + C\left(\overset{L(w)}{\cancel{\theta_L}}\right)\right)\right] \\ \beta - \left[A\left(\pi + C\left(\underset{L(w)}{\theta_L} - 1\right)\right) - A\left(\pi + C\left(\underset{L(w)}{\cancel{\theta_L}}\right)\right)\right] \end{pmatrix}, \text{ where } \cancel{\theta_L = L(\omega).} \tag{1.6}$$

non-smooth

We know that

$$g(\pi_0, \omega_0) = 0 \in R^2. \tag{1.7}$$

Inference on π is based on the function $g(\pi, \widehat{\omega})$ because $\widehat{\omega}$ is a consistent estimator of ω_0.

By the consistency of $\widehat{\omega}$,

$$T^{1/2}\left[g(\pi, \widehat{\omega}) - g(\pi, \omega_0)\right] \Rightarrow \xi(\pi) = G(\pi, \omega_0)'\xi_\omega, \tag{1.8}$$

where $G(\pi, \omega)$ denote the derivative of $g(\pi, \omega)$ wrt to ω. The Gaussian process $\xi(\pi)$ has covariance kernel

$$\Sigma(\pi_1, \pi_2) = G(\pi_1, \omega_0)'VG(\pi_2, \omega_0). \tag{1.9}$$

We can estimate $\Sigma(\pi_1, \pi_2)$ by

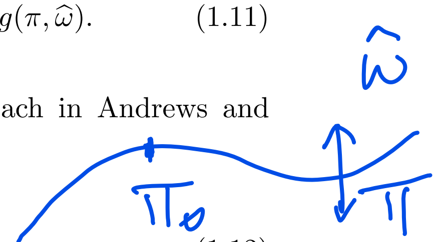
$$\widehat{\Sigma}(\pi_1, \pi_2) = G(\pi_1, \widehat{\omega})'\widehat{V}G(\pi_2, \widehat{\omega}), \tag{1.10}$$

where $\widehat{V}$ is a consistent estimator of V.

We construct a confidence interval for π by inverting tests $H_0 : \pi = \pi_0$ vs $H_0 : \pi \neq \pi_0$. The test statistic is the QLR statistic

$$QLR = Tg(\pi_0, \widehat{\omega})'\widehat{\Sigma}(\pi_0, \pi_0)^{-1}g(\pi_0, \widehat{\omega}) - \min_{\pi \in \Pi} Tg(\pi, \widehat{\omega})'\widehat{\Sigma}(\pi, \pi)^{-1}g(\pi, \widehat{\omega}). \tag{1.11}$$

To obtain the critical value, we follow the conditional inference approach in Andrews and Mikusheva (2016). To this end, first construct a projection residual process



$$h(\pi, \widehat{\omega}) = g(\pi, \widehat{\omega}) - \widehat{\Sigma}(\pi, \pi_0)\widehat{\Sigma}(\pi_0, \pi_0)^{-1}g(\pi_0, \widehat{\omega}). \tag{1.12}$$

$Y - \beta \cdot X$

By construction, $h(\pi, \widehat{\omega})$ and $g(\pi_0, \widehat{\omega})$ are independent asymptotically. Conditional on $h(\pi, \widehat{\omega})$, we obtain the $1 - \alpha$ quantile of the QLR statistic, denoted by $c_\alpha(h)$, by sampling from the asymptotic distributions of $g(\pi_0, \widehat{\omega})$ under the null. Specifically, we take independent draws $\xi^* \sim N(0, \widehat{\Sigma}(\pi_0, \pi_0))$ and produce simulated process

realization in sample, compute once

$$g^*(\pi, \widehat{\omega}) = h(\pi, \widehat{\omega}) + \widehat{\Sigma}(\pi, \pi_0)\widehat{\Sigma}(\pi_0, \pi_0)^{-1}\xi^*. \tag{1.13}$$

d

$\underline{g(\pi_0, \hat{w})} \perp h(\pi, \hat{w})$ $\quad$ $g(\pi_0, \hat{w})$ distn

We then calculate

$$QLR^* = Tg^*(\pi_0, \widehat{\omega})' \widehat{\Sigma}(\pi_0, \pi_0)^{-1} g^*(\pi_0, \widehat{\omega}) - \min_{\pi \in \Pi} Tg^*(\pi, \widehat{\omega})' \widehat{\Sigma}(\pi, \pi)^{-1} g^*(\pi, \widehat{\omega}), \tag{1.14}$$

which is a random drawn from the conditional distribution of the QLR statistic given $h_T(\pi, \widehat{\omega})$, when $g(\pi_0, \widehat{\omega})$ is drawn from its asymptotic distribution. In practice, we repeat this process for a large number of times and obtain $c_\alpha(h)$ by simulation.

We reject the null $H_0 : \pi = \pi_0$ if $QLR \geq c_\alpha(h)$. The confidence interval for π is the collection of null values that are not rejected as the null value. Note that the construction of this CI does not involve estimation of π.

2 Asymptotic Distribution of the Reduced-Form Parameter

This section gives the asymptotic distribution of the reduced-form parameter. This will be given before the CLR session in the paper.

Write $\omega = (\omega_1, \omega_2, \omega_3)$, where $\omega_1 = (\rho, c)$ δ, $\omega_2 = (\gamma, \beta, \psi)$, and $\omega_3 = \phi^2$. Below we describe the estimator $\widehat{\omega}_1, \widehat{\omega}_2, \widehat{\omega}_3$ and provide the asymptotic distribution of $\widehat{\omega} = (\widehat{\omega}_1, \widehat{\omega}_2, \widehat{\omega}_3)$. We estimate these parameters separately because ω_1 only shows up in the conditional mean and variance of r_{t+1}; ω_2 only shows up in the conditional mean of σ_{t+1}^2; and ϕ only shows up in the conditional variance of σ_{t+1}^2.

We estimate ω_1 by GMM based on the moment condition

$$\begin{aligned}
\mathbb{E}[h_t(\omega_{1,0}) &= 0, \text{ where} \\
h_t(\omega_1) &= \begin{pmatrix}
\sigma_{t+1}^2 - \left(c\delta + \rho\sigma_t^2\right) \\
\sigma_t^2 \left(\sigma_{t+1}^2 - \left(c\delta + \rho\sigma_t^2\right)\right) \\
\sigma_{t+1}^4 - \left(c^2\delta + 2c\rho\sigma_t^2 + \left(c\delta + \sigma_{t+1}^2 - \left(c\delta + \rho\sigma_t^2\right)^2\right)\right) \\
\sigma_t^2 \left(\sigma_{t+1}^4 - \left(c^2\delta + 2c\rho\sigma_t^2 + \left(c\delta + \sigma_{t+1}^2 - \left(c\delta + \rho\sigma_t^2\right)^2\right)\right)\right) \\
\sigma_t^4 \left(\sigma_{t+1}^4 - \left(c^2\delta + 2c\rho\sigma_t^2 + \left(c\delta + \sigma_{t+1}^2 - \left(c\delta + \rho\sigma_t^2\right)^2\right)\right)\right)
\end{pmatrix}.
\end{aligned} \tag{2.1}$$

The optimal GMM estimator is

$$
\begin{aligned}
\widehat{\omega}_1 &= \arg\min_{\omega_1 \in \Lambda_1} \overline{h}_T(\omega_1)' W_T \overline{h}_T(\omega_1), \text{ where} \\
\overline{h}_T(\omega_1) &= T^{-1} \sum_{t=1}^{T} h_t(\omega_1), \\
W_T &= T^{-1} \sum_{t=1}^{T} h_t(\widetilde{\omega}_1) h_t(\widetilde{\omega}_1)' - \overline{h}_T(\widetilde{\omega}_1) \overline{h}_T(\widetilde{\omega}_1)', \qquad (2.2)
\end{aligned}
$$

where $\widetilde{\omega}_1$ is the preliminary GMM estimator based on the identify covariance matrix.

We estimate ω_2 by the GLS estimator because γ, β, ψ are the intercept and linear coefficients of the conditional mean function and the conditional variance is proportional to σ_{t+1}^2. Define $x_t = \sigma_{t+1}^{-1}(1, \sigma_t^2, \sigma_{t+1}^2)$ and $y_t = \sigma_{t+1}^{-1} r_{t+1}$. The GLS estimator of ω_2 is

$$
\widehat{\omega}_2 = \left(\sum_{t=1}^{T} x_t x_t' \right)^{-1} \sum_{t=1}^{T} x_t y_t. \qquad (2.3)
$$

$\hat{Var} = 1 - \hat{\phi}^2$

We estimate ω_3 by the sample variance estimator. Define

$$
\widehat{y}_t = x_t \widehat{\omega}_2 = \sigma_{t+1}^{-1}(\widehat{\gamma} + \widehat{\beta}\sigma_t^2 + \widehat{\psi}\sigma_{t+1}^2). \qquad (2.4)
$$

$\phi^2 \in [0,1]$

The estimator of ω_3 is

$$
\widehat{\omega}_3 = \max\{1 - T^{-1} \sum_{t=1}^{T} (y_t - \widehat{y}_t)^2, 0\}. \qquad (2.5)
$$

[**XC. In practice, do we need to impose the estimator is positive?]

The next lemma provides the asymptotic distribution of the estimator $\widehat{\omega}$. Let $h_{\omega,t}(\omega_1) \in R^{5\times 2}$ denote the derivative of $h_t(\omega_1)$ wrt ω_1. Define

$$
\begin{aligned}
\Omega_1 &= \left\{ \mathbb{E}\left[h_{\omega,t}(\omega_{1,0})\right]' \mathbb{E}[h_t(\omega_{1,0}) h_t(\omega_{1,0})']^{-1} \mathbb{E}\left[h_{\omega,t}(\omega_{1,0})\right] \right\}^{-1}, \\
\Omega_2 &= \mathbb{E}\left[x_t x_t'\right]^{-1} \mathbb{E}[(y_t - x_t'\omega_{2,0})^2], \\
\Omega_3 &= \mathbb{V}[\left(y_t - x_t'\omega_{2,0}\right)^2] \quad \cdot C_{\phi^2} \qquad (2.6)
\end{aligned}
$$

Lemma 2.1 *Suppose Assumptions *** hold. Then,*

$$
T^{1/2} \begin{pmatrix} \widehat{\omega}_1 - \omega_{1,0} \\ \widehat{\omega}_2 - \omega_{2,0} \\ \widehat{\omega}_3 - \omega_{3,0} \end{pmatrix} \to_d \xi_\omega = \begin{pmatrix} \xi_{\omega 1} \\ \xi_{\omega 2} \\ \xi_{\omega 3} \end{pmatrix} \sim N \left(0, \begin{pmatrix} \Omega_1 & 0 & 0 \\ 0 & \Omega_2 & 0 \\ 0 & 0 & \Omega_3 \end{pmatrix} \right).
$$

$- \sqrt{\phi^2}$

$1 - \frac{1}{2}(\phi^2)^{-\frac{1}{2}} \cdot \xi_{\phi^2}$

C_ϕ

Proof. Will be added later. ■