Chapter 8

Sang Tran

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Chapter 8 problems 1, 4, 5, 6, 7, 8, 9(a), 10, 11, 12, 13, 14, 22 (for problem 22 only prove Newton's binomial theorem), 23, 25

8.1

Let $y = \frac{1}{x}$ and $x \neq 0$, then $y \to \infty$ as $x \to 0$.

$$\begin{aligned} f'(0) &= \lim_{x \to 0} \frac{f(x) - f(0)}{x} \\ &= \lim_{x \to 0} \frac{e^{-1/x^2}}{x} \\ &= \lim_{y \to \infty} y e^{-y^2} = 0, \end{aligned}$$

by Theorem 8.6f.

Suppose it is true for k, i.e. $f^{(k)}(0) = 0$. Also note that $f^{(k)}(x) = e^{-y^2} P_k(y)$, where $P_k(y)$ is some polynomial of y with order k. We have

$$\begin{aligned} f^{(k+1)}(0) &= \lim_{x \to 0} \frac{f^{(k)}(x) - f^{(k)}(0)}{x} \\ &= \lim_{y \to \infty} y P_k(y) e^{-y^2} \\ &= \lim_{y \to \infty} P_{k+1}(y) e^{-y^2} = 0, \end{aligned}$$

by Theorem 8.6f.

By induction, we're done. Therefore, $f^{(n)}(0) = 0$, for $n = 1, 2, 3, ...$

8.4

(a)

Using the fact that $b^x = E(xL(b))$ for $b > 0$ and then apply L'Hospital's Rule, we have

$$\begin{aligned}\lim_{x\to 0}\frac{b^x-1}{x} &= \lim_{x\to 0}\frac{e^{x\log b}-1}{x}\\ &= \lim_{x\to 0} e^{x\log b}\log b\\ &= \log b.\end{aligned}$$

(b)
Apply L'Hospital's Rule and use the fact that $L'(y)=\frac{1}{y}$ for $y>0$, we have

$$\lim_{x\to 0}\frac{\log(x+1)}{x} = \lim_{x\to 0}\frac{1}{x+1} = 1$$

(c) For small enough x, or large enough y, consider

$$\begin{aligned}\log\left(\lim_{x\to 0}(1+x)^{1/x}\right) &= \lim_{y\to\infty} y\log\left(1+\frac{1}{y}\right) && (1)\\ &= \lim_{y\to\infty}\frac{\log\left(1+\frac{1}{y}\right)}{\frac{1}{y}} && (2)\\ &= \lim_{y\to\infty}\frac{-\frac{1}{y^2}\cdot\frac{1}{1+\frac{1}{y}}}{-\frac{1}{y^2}}\\ &= \lim_{y\to\infty}\frac{1}{1+\frac{1}{y}}\\ &= 1,\end{aligned}$$

where we use the fact that *log* is continuous for (1) with $|y|>1$, and then apply L'Hospital's Rule for (2).
Thus, if we take exponential on both sides we obtain the desired result, which is $\lim_{x\to 0}(1+x)^{1/x}=e$.

(d) If $x=0$, the equality holds trivially. Suppose $x\neq 0$, then for large enough n, consider

$$\begin{aligned}
\log\left(\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n\right) &= \lim_{n\to\infty} n\log\left(1+\frac{x}{n}\right) && (1)\\
&= \lim_{n\to\infty}\frac{\log\left(1+\frac{x}{n}\right)}{\frac{1}{n}} && (2)\\
&= \lim_{n\to\infty}\frac{-\frac{x}{n^2}\cdot\frac{1}{1+\frac{x}{n}}}{-\frac{1}{n^2}}\\
&= \lim_{n\to\infty}\frac{x}{1+\frac{x}{n}}\\
&= x,
\end{aligned}$$

where we use the fact that *log* is continuous for (1) and then apply L'Hospital's Rule for (2).
Thus, if we take exponential on both sides we obtain the desired result, which is $\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n = e^x$.

8.5

(a) From excercise 8.4 part (c) we can apply L'Hospital Rule we have

$$\lim_{x\to 0}\frac{e-(1+x)^{1/x}}{x} = -\lim_{x\to 0} y', \qquad (1)$$

where $y=(1+x)^{1/x}$.
Consider $\log y = \log(1+x)^{1/x}$, then $x\log y = \log(1+x)$ and

$$\log y + \frac{x}{y}y' = \frac{1}{1+x}$$

$$\implies y' = (1+x)^{1/x}\left[\frac{1}{x(1+x)} - \frac{\frac{1}{x}\log(1+x)}{x}\right]$$

$$\implies y' = (1+x)^{1/x}\left[\frac{1}{x} - \frac{1}{1+x} - \frac{1}{x} + \frac{1}{x} - \frac{\log(1+x)}{x^2}\right]$$

$$\implies y' = (1+x)^{1/x}\left[-\frac{1}{1+x} + \frac{x-\log(1+x)}{x^2}\right]$$

but $\lim_{x\to 0}\frac{x-log(1+x)}{x^2} = \lim_{x\to 0}\frac{1-\frac{1}{1+x}}{2x} = \lim_{x\to 0}\frac{1}{2(1+x)^2} = \frac{1}{2}$.
Thus, using excercise 8.4 part (c), we have $\lim_{x\to 0} y' = e\left(-1+\frac{1}{2}\right) = -\frac{e}{2}$.
From (1), we have $\lim_{x\to 0}\frac{e-(1+x)^{1/x}}{x} = \frac{e}{2}$.

(b)

$$\lim_{n\to\infty}\frac{n}{\log n}[n^{1/n}-1] = \lim_{n\to\infty}\frac{n^{1/n}-1}{\log n/n} \qquad (1)$$

Let $y = n^{1/n}$, then $n \log y = \log n$ and therefore

$$\log y + \frac{n}{y} y' = \frac{1}{n}$$

$$\implies y' = n^{1/n} \left(\frac{1}{n^2} - \frac{\log n}{n^2} \right) = n^{1/n} \left(\frac{1 - \log n}{n^2} \right)$$

O.T.O.H., $\left(\frac{\log n}{n}\right)' = \frac{1-\log n}{n^2}$

Since $\lim_{n\to\infty} n^{1/n} = 1$ and $\lim_{n\to\infty} \frac{\log n}{n} = 0$, we apply L'Hospital Rule for (1) then we have

$$\lim_{n\to\infty} \frac{y'}{(1-\log n)/n^2} = \lim_{n\to\infty} n^{1/n} = 1.$$

(c) Since $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$, $\cos x = \frac{1}{2}(e^{ix} + e^{-ix})$ and $\sin x = \frac{1}{2i}(e^{ix} - e^{-ix})$, we have

$$\cos x = \frac{1}{2} \left(\sum_{n=0}^{\infty} \frac{(ix)^n}{n!} + \sum_{n=0}^{\infty} \frac{(-ix)^n}{n!} \right) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots$$

$$\sin x = \frac{1}{2i} \left(\sum_{n=0}^{\infty} \frac{(ix)^n}{n!} - \sum_{n=0}^{\infty} \frac{(-ix)^n}{n!} \right) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$$

Then apply L'Hospital Rule we have

$$\lim_{x\to 0} \frac{\tan x - x}{x(1-\cos x)} = \lim_{x\to 0} \frac{\tan x - x}{x^3/2! - x^5/4! + \cdots} = \lim_{x\to 0} \frac{1/\cos^2 x - 1}{3x^2/2! - 5x^4/4! + \cdots}$$

$$= \lim_{x\to 0} \frac{\sin^2 x}{\cos^2 x(3x^2/2! - 5x^4/4! + \cdots}) = \lim_{x\to 0} \frac{\frac{\sin^2 x}{x^2}}{\cos^2 x(3/2 - 5x^2/4! + \cdots)} = \frac{2}{3}.$$

Since $\lim_{x\to 0} \frac{\sin x}{x} = \lim_{x\to 0} \frac{x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots}{x} = \lim_{x\to 0} 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \cdots = 1$.

(d) From part (c) above and by L'Hospital Rule we have

$$\begin{aligned}
\lim_{x\to 0} \frac{1-\cos x}{1/\cos^2 x - 1} &= \lim_{x\to 0} \frac{\cos^2 x(1-\cos x)}{\sin^2 x} \\
&= \lim_{x\to 0} \frac{\cos^2 x(\frac{1}{x^2} - \frac{\cos x}{x^2})}{\frac{\sin^2 x}{x^2}} \\
&= \lim_{x\to 0} \frac{\cos^2 x(\frac{1}{2!} - \frac{x^2}{4!} + \frac{x^4}{6!} + \cdots)}{\frac{\sin^2 x}{x^2}} \\
&= \frac{1}{2}.
\end{aligned}$$

8.6

(a) Since f is differentiable and not zero, $f^{(k)}(x) < \infty$, $\forall k$, and $f(x) \neq 0\ \forall x$. Take $x = y = 0$, then we have $f(0)f(0) = f(0) \neq 0 \implies f(0) = 1$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = cf(x),$$

where let $c = \lim_{h\to 0} \frac{f(h)-1}{h} = \lim_{h\to 0} \frac{f(h)-f(0)}{h} = f'(0) < \infty$ and since $f(x+h) = f(x)f(h)$. Suppose it is indeed true for n such that $f^{(n)}(x) = c^n f(x)$, then

$$f^{(n+1)}(x) = \lim_{h\to 0} \frac{c^n f(x+h) - c^n f(x)}{h} = c^{n+1} f(x).$$

By induction, we have $f^{(k)}(x) = c^k f(x)\ \forall k$.
Consider the function, $g(x) = e^{-cx} f(x)$.
Then $g'(x) = e^{-cx} f'(x) - ce^{-cx} f(x) = ce^{-cx} f(x) - ce^{-cx} f(x) = 0$, which means g is a constant function.
Plug $x = 0$, we have $g(0) = 1$. Thus, $e^{-cx} f(x) = 1$, in other words, $f(x) = e^{cx}$.

(b)
Since $f(x+y) = f(x)f(y)$, if f has a root, then f is the zero function. By the assumption that f is continuous and from (a) we know $f(0) = 1$ so f cannot be the zero function and also f is always positive since it cannot change sign.
Define $g(x) := \log(f(x))$. Then $g(x+y) = \log(f(x+y)) = \log(f(x)) + \log(f(y)) = g(x) + g(y)$, and since f and log are continuous, g is continuous.
Want to show: $g(x) = cx$ for all x, where $c = g(1)$.
Let $A = \{x : g(x) = cx\}$. Note that $A \neq \emptyset$ since $0, 1 \in A$, for $g(0) = \log 1 = 0$ and $g(1) = \log(f(1))$.
We know that $g(0) = \log(f(0)) = 0$. Also, $g(-x) = \log(f(-x)) = \log(f(x)^{-1}) = -g(x)$ since $1 = f(0) = f(x-x) = f(x)f(-x)$. Therefore, it is trivial to show by induction that $g(nx) = ng(x)$ for all $n \in \mathbb{Z}$.
Now, if $y \in A$, then $ny \in A$ for all $n \in \mathbb{Z}$ since

$$g(ny) = \log(f(ny)) = \log(nf(y)) = n\log(f(y)) = ng(y) = cny.$$

Also, if $y \in A$, then $\frac{y}{n} \in A$ for all $n \in \mathbb{Z}$ since $g(y) = g(n \cdot (y/n)) = n\log(f(y/n)) = ng(y/n)$. Thus

$$g(y/n) = \frac{1}{n} g(y) = \frac{cy}{n}.$$

We have shown so far that $g(x) = cx$ for all $x \in A = \mathbb{Q}$ where $c = g(1)$.
Since g and cx are continuous, $\mathbb{Q}$ is dense in $\mathbb{R}$ and $g(x) = cx$ for all $x \in \mathbb{Q}$, we conclude that $g(x) = cx$ for all $x \in \mathbb{R}$. Hence, $f(x) = e^{cx}$ where c is some constant.

8.7

Consider $f(x) = \sin x - x$, then $f'(x) = \cos x - 1 < 0$, $\forall x \in (0, \frac{\pi}{2})$. Thus, f is monotonically decreasing, or in particular, $f(x) < f(0) = \sin 0 - 0 = 0$, and hence, $\sin x < x$, or

$$\frac{\sin x}{x} < 1 \tag{1}$$

Consider $g(x) = \tan x - x$, then $g'(x) = \frac{1-\cos^2 x}{\cos^2 x} = \frac{\sin^2 x}{\cos^2 x} > 0$, $\forall x \in (0, \frac{\pi}{2})$. Thus, g is monotonically increasing, or in particular, $g(x) > g(0) = 0$, and hence, $\tan x > x$, or $\sin x - x\cos x > 0$.
But then $\left(\frac{\sin x}{x}\right)' = \frac{x\cos x - \sin x}{\cos^2 x} < 0$, i.e. $\frac{\sin x}{x}$ is monotonically decreasing on $(0, \frac{\pi}{2})$.

$$\implies \frac{\sin x}{x} > \frac{\sin\frac{\pi}{2}}{\frac{\pi}{2}} = \frac{2}{\pi} \tag{2}$$

From (1) and (2), we obtained the desired inequality.

8.8

W.T.S. $\sin(a+b) = \sin a\cos b + \cos a \sin b$.

$$\begin{aligned}
\sin(a+b) &= \frac{e^{i(a+b)} - e^{-i(a+b)}}{2i} \\
&= \frac{e^{ia}e^{ib} - e^{ia}e^{-ib} + e^{ia}e^{-ib} - e^{-ia}e^{-ib}}{2i} \\
&= \frac{e^{ia}(e^{ib} - e^{-ib}) + e^{-ib}(e^{ia} - e^{-ia})}{2i} \\
&= e^{ia}\sin b - e^{-ib}\sin a \\
&= (\cos a + i\sin a)\sin b + (\cos(-b) + i\sin(-b))\sin a \\
&= (\cos a + i\sin a)\sin b + (\cos b - i\sin b)\sin a \\
&= \cos a\sin b + \cos b \sin a.
\end{aligned}$$

Base cases are true trivially for $n = 0, 1$. Suppose it is true for $n = k > 1$ that is $\left|\sin kx\right| \leq k|\sin x|$ then

$$\begin{aligned}
\left|\sin(k+1)x\right| &= \left|\sin(kx + x)\right| \\
&= \left|\sin(kx)\cos x + \cos(kx)\sin x\right| \\
&\leq |\sin(kx)||\cos x| + |\cos(kx)||\sin x| \\
&\leq k|\sin x| + |\sin x| \\
&= (k+1)|\sin x|.
\end{aligned}$$

By induction, we are done.

8.9(a)

Consider the sequence $\{g(n)\}_{n\geq 1}$ where $g(n) = s_n - \log n$.
$g(n+1) - g(n) = s_{n+1} - s_n - [\log(n+1) - \log n] = \frac{1}{n+1} - \int_n^{n+1} \frac{1}{t} dt < 0$, since $\int_n^{n+1} \frac{1}{t} dt > L(\frac{1}{x}, P) = \frac{1}{n+1}$ where $P = \{x_0 = n, x_1 = n+1\}$ and $\frac{1}{x}$ is decreasing for $x > 0$.
$\Longrightarrow$ the sequence is decreasing.
Also, $\log n = \int_1^n \frac{1}{t} dt < U(\frac{1}{x}, P) = 1 + \frac{1}{2} + ... + \frac{1}{n-1} < s_n$ where $P = \{x_0 = 1, x_1 = 2, ..., x_n = n\}$ and since $\frac{1}{x}$ is decreasing for $x > 0$.
$\Longrightarrow$ the sequence in positive.
$\Longrightarrow$ the sequence must converge to a non-negative number, i.e. the limit exists.

8.10

Following the hint: Given N, let $p_1, \ldots, p_k$ be those primes that divide at least one integer and less than N. Then

$$\sum_{i=1}^{N} \frac{1}{n} \leq \prod_{j=1}^{k} \left(1 + \frac{1}{p_j} + \frac{1}{p_j^2} \cdots\right) = \prod_{j=1}^{k} \left(1 - \frac{1}{p_j}\right)^{-1} \tag{1}$$

Let $f(x) = 2x + \log(1-x)$, then $f'(x) = \frac{2-2x}{1-x} > 0$, $\forall x \in [0, \frac{1}{2}]$ so f is monotonically increasing.
$\Longrightarrow$ $0 = f(0) \leq f(x)$.
$\Longrightarrow$ $\log(1-x)^{-1} \leq 2x$.
$\Longrightarrow$ $(1-x)^{-1} \leq e^{2x}$.
Since p_j's are at least 2, apply this to (1) we have

$$\prod_{j=1}^{k} \left(1 - \frac{1}{p_j}\right)^{-1} \leq \prod_{j=1}^{k} e^{\frac{2}{p_j}} = exp\Big(2 \sum_{j=1}^{k} \frac{1}{p_j}\Big).$$

Since the harmonic series diverges, the series $\sum \frac{1}{p}$ also diverges.

8.11

Fix $t > 0$ and $\epsilon > 0$, then since $e^{-tx} > 0$ and by assumption $\exists M > 0$ s.t. $\forall x > M \implies 1 - \epsilon < f(x) < 1 + \epsilon$.

$$\begin{aligned}
\Longrightarrow\ t\int_M^\infty e^{-tx}(1-\epsilon)dx < t\int_M^\infty e^{-tx} f(x)dx < t\int_M^\infty e^{-tx}(1+\epsilon)dx \\
\Longrightarrow\ (1-\epsilon)e^{-tM} < t\int_M^\infty e^{-tx} f(x)dx < (1+\epsilon)e^{-tM} \\
\Longrightarrow\ 1-\epsilon < \lim_{t\to 0} t\int_M^\infty e^{-tx} f(x)dx < 1+\epsilon.
\end{aligned}$$

Since ϵ is arbitrary, $\lim_{t\to 0} t\int_M^\infty e^{-tx}f(x)dx = 1$.
Since $f \in \Re$ on $[0,M]$, $|f(x)| \in \Re$ on $[0,M]$ by Theorem 6.13(b) and also, $|e^{-tx}| = e^{-tx} < 1$ for $t > 0$. Thus

$$\Big|\int_0^M e^{-tx}f(x)dx\Big| \le \int_0^M |e^{-tx}||f(x)|dx \le \int_0^M |f(x)|dx = K$$

$\implies 0 \le \left|t\int_0^M e^{-tx}f(x)dx\right| < tK.$
$\implies \lim_{t\to 0} |t\int_0^M e^{-tx}f(x)dx| = 0.$
$\implies \lim_{t\to 0} t\int_0^M e^{-tx}f(x)dx = 0.$
Hence,

$$\begin{aligned}\lim_{t\to 0} t\int_0^\infty e^{-tx}f(x)dx &= \lim_{t\to 0} t\int_0^M e^{-tx}f(x)dx + \lim_{t\to 0} t\int_M^\infty e^{-tx}f(x)dx \\ &= 0 + 1 = 1.\end{aligned}$$

8.12

(a)

$$c_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{2\pi}\int_{-\delta}^{\delta} dx = \frac{\delta}{\pi}.$$

and

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}\,dx = \frac{1}{2\pi}\int_{-\delta}^{\delta} e^{-inx}\,dx = \frac{1}{2\pi}\frac{e^{in\delta}-e^{-in\delta}}{in} = \frac{\sin(n\delta)}{n\pi},$$

for $n \neq 0$.

(b)
Let $y = \sum_{n=1}^\infty c_n$. Since sin is odd and therefore $c_{-n} = c_n$, we have

$$\sum_{n=-\infty}^{-1} c_n = \sum_{n=1}^{\infty} c_n = y,$$

and hence,

$$2y + c_0 = \sum_{n=-\infty}^{\infty} c_n = f(0) \quad (\text{since } f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}).$$

This gives $2y + \frac{\delta}{\pi} = 1$, and thus $y = \frac{1-\delta/\pi}{2}$.

$$\implies \sum_{n=1}^{\infty}\frac{\sin(n\delta)}{n} = \sum_{n=1}^{\infty}\pi c_n = \pi y = \frac{\pi-\delta}{2}.$$

(c)
Let $y = \sum_{n=1}^{\infty} |c_n|^2$. Since $c_{-n} = c_n$, we have

$$\sum_{n=-\infty}^{-1} |c_n|^2 = \sum_{n=1}^{\infty} |c_n|^2 = y,$$

and therefore,

$$2y + |c_0|^2 = \sum_{n=-\infty}^{\infty} |c_n|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x)|^2 \, dx = \frac{1}{2\pi} \int_{-\delta}^{\delta} 1 \, dx = \frac{\delta}{\pi},$$

by Parseval's theorem. So

$$2y + \frac{\delta^2}{\pi^2} = \frac{\delta}{\pi} \implies y = \frac{\pi\delta - \delta^2}{2\pi^2}.$$

Hence,

$$\sum_{n=1}^{\infty} \frac{\sin^2(n\delta)}{n^2\delta} = \frac{\pi^2}{\delta} \sum_{n=1}^{\infty} |c_n|^2 = \frac{\pi^2}{\delta} \cdot \frac{\pi\delta - \delta^2}{2\pi^2} = \frac{\pi - \delta}{2}.$$

(d)
Let $A > 0$, and let $f(x) = \left(\frac{\sin x}{x}\right)^2$. Define

$$g(x) = \begin{cases} f(x) & \text{if } x > 0 \\ 1 & \text{if } x = 0 \end{cases}$$

Since

$$\lim_{x \to 0} g(x) = \lim_{x \to 0} f(x) = 1 = g(0),$$

since if we apply L'Hospital we get $\lim_{x \to 0} \frac{\sin x}{x} = 1$, and the square function is continuous on $[0, A]$.
So $g(x)$ is continuous on every $[0, A]$ and hence $g \in \Re$ on $[0, A]$. Since

$$\int_0^A f(x) \, dx = \lim_{c \to 0} \int_c^A f(x) \, dx = \lim_{c \to 0} \int_c^A g(x) \, dx = \int_0^A g(x) \, dx,$$

$f \in \Re$ on $[0, A]$.
Let $P^* = \{x_0 = 0, x_1 = \delta, \ldots, x_m = A\}$ (suppose $(m-1)\delta < A < m\delta$) be a partition on $[0, A]$. Suppose $P = \{y_0 = 0, y_1, \ldots, y_k = A\}$ is any other partition on $[0, A]$. Choose $\delta > 0$ small enough so that $x_{n_i - 1} = (n_i - 1)\delta \leq y_i < x_{n_i} = n_i\delta$, for $1 \leq i < k$ and denote S to be the set that contains such $x'_{n_i}s$. We then have

$$\left|\sum_{n_i}[f(n_i\delta)\Delta x_{n_i} - M_{1i}(y_i - x_{n_i-1}) - M_{2i}(x_{n_i} - y_i)]\right|$$

$$\leq \sum_{n_i}[|f(n_i\delta)|\Delta x_{n_i} + M_{1i}(y_i - x_{n_i-1}) + M_{2i}(x_{n_i} - y_i)]$$

$$\leq \sum_{n_i}[M\Delta x_{n_i} + M(y_i - x_{n_i-1}) + M(x_{n_i} - y_i)] = 2M\sum_{n_i}\Delta x_{n_i} = 2Mk\delta,$$

where $M = \sup_{x\in[0,A]}|f(x)|$, $M_{1i} = \sup_{x\in[y_{i-1},y_i]}|f(x)|$,
and $M_{2i} = \sup_{x\in[y_i,y_{i+1}]}|f(x)|$. Therefore,

$$\sum_{n_i} f(n_i\delta)\Delta x_{n_i} \leq \sum_{n_i}[M_{1i}(y_i - x_{n_i-1}) + M_{2i}(x_{n_i} - y_i)] + 2Mk\delta,$$

$$\begin{aligned}
\Longrightarrow \sum_{n=1}^{m} f(x_n)\Delta x_n &= \sum_{n=1}^{m} f(n\delta)\delta \\
&= \sum_{n_j\notin S} f(n_j\delta)\Delta x_{n_j} + \sum_{n_i} f(n_i\delta)\Delta x_{n_i} \\
&\leq \sum_{n_j\notin S} f(n_j\delta)\Delta x_{n_j} + \sum_{n_i}[M_{1i}(y_i - x_{n_i-1}) + M_{2i}(x_{n_i} - y_i)] + 2Mk\delta \\
&= U(P\cup P^*, f) + 2Mk\delta \\
&\leq U(P, f) + 2Mk\delta.
\end{aligned}$$

and thus

$$\lim_{\delta\to 0}\sum_{n=1}^{m} f(x_n)\Delta x_n \leq U(P, f).$$

Since P is arbitrary, we have

$$\lim_{\delta\to 0}\sum_{n=1}^{m} f(x_n)\Delta x_n = \lim_{\delta\to 0}\sum_{n=1}^{m} f(\delta n)\delta = \lim_{\delta\to 0}\sum_{n=1}^{m}\frac{\sin^2(n\delta)}{n^2\delta} \leq \overline{\int_0^A} f(x)\,dx.$$

Similarly, we can also show

$$\lim_{\delta\to 0}\sum_{n=1}^{m} f(x_n)\Delta x_n \geq L(P, f),$$

and conclude that

$$\lim_{\delta\to 0}\sum_{n=1}^{m} f(x_n)\Delta x_n = \lim_{\delta\to 0}\sum_{n=1}^{m} f(\delta n)\delta = \lim_{\delta\to 0}\sum_{n=1}^{m}\frac{\sin^2(n\delta)}{n^2\delta} \geq \underline{\int_0^A} f(x)\,dx.$$

So

$$\underline{\int_0^A} f(x)\,dx \le \lim_{\delta\to 0}\sum_{n=1}^{m}\frac{\sin^2(n\delta)}{n^2\delta} \le \overline{\int_0^A} f(x)\,dx.$$

Since $f \in \Re$ on $[0, A]$,

$$\lim_{\delta\to 0}\sum_{n=1}^{m}\frac{\sin^2(n\delta)}{n^2\delta} = \int_0^A f(x)\ dx,$$

which holds for all $A > 0$.

$$\begin{aligned}\implies \int_0^\infty f(x)\ dx &= \lim_{A\to\infty}\int_0^A f(x)\ dx \\ &= \lim_{\delta\to 0}\sum_{n=1}^{\infty}\frac{\sin^2(n\delta)}{n^2\delta} \\ &= \lim_{\delta\to 0}\frac{\pi-\delta}{2} = \frac{\pi}{2}.\end{aligned}$$

(d)
Since $\sin^2(n\pi/2) = 1$ if n is odd and 0 otherwise then

$$\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2} = \frac{\pi}{2}\cdot\frac{\pi-\pi/2}{2} = \frac{\pi^2}{8}.$$

8.13

Computing the Fourier coefficients of f, we get

$$c_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} x\,dx = 0,$$

and

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} xe^{-inx}\ dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} xe^{-inx}\ dx = \frac{(-1)^{n+1}}{in},$$

where $n \neq 0$. Thus $|c_n|^2 = \frac{1}{n^2}$. Let $y = \sum_{n=1}^{\infty}|c_n|^2$, then

$$\sum_{i=-\infty}^{-1}|c_{-n}|^2 = \sum_{n=1}^{\infty}|c_n|^2 = y,$$

which gives

$$2y + |c_0|^2 = \sum_{n=-\infty}^{\infty} |c_n|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x)|^2 \, dx = \frac{1}{2\pi} \int_{-\pi}^{\pi} x^2 \, dx.$$

Evaluating the integral, we have

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} x^2 \, dx = \frac{\pi^2}{3}.$$

Hence,

$$2y = \frac{\pi^2}{3}, \quad \text{i.e.,} \quad y = \frac{\pi^2}{6}.$$

Therefore,

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

8.14

Computing the Fourier coefficients of f, we get

$$c_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) \, dx = \frac{1}{2\pi} \int_{-\pi}^{\pi} (\pi - |x|)^2 \, dx = \frac{\pi^2}{3},$$

and

$$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} (\pi - |x|)^2 e^{-inx} \, dx = \frac{2}{n^2} \quad (n \neq 0).$$

Hence,

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx} = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{2}{n^2} e^{inx} + \sum_{n=-\infty}^{-1} \frac{2}{n^2} e^{inx}.$$

$$
\begin{aligned}
&= \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{2}{n^2} e^{inx} + \sum_{n=1}^{\infty} \frac{2}{n^2} e^{-inx} \\
&= \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{2}{n^2} (e^{inx} + e^{-inx}) \\
&= \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{2}{n^2} (\cos nx + i \sin nx + \cos nx - i \sin nx) \\
&= \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{2}{n^2} (2 \cos nx) \\
&= \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4}{n^2} \cos nx.
\end{aligned}
$$

Put $x = 0$, we get

$$
f(0) = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{2}{n^2} \cdot 2 = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4}{n^2} = \pi^2,
$$

which gives

$$
\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.
$$

Since $c_n = c_{-n}$, let $y = \sum_{n=1}^{\infty} |c_n|^2$, we have

$$
2y + |c_0|^2 = \sum_{n=-\infty}^{\infty} |c_n|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x)|^2 \, dx.
$$

Hence,

$$
2y + \left(\frac{\pi^2}{3} \right)^2 = \sum_{n=-\infty}^{\infty} |c_n|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} (\pi - |x|)^4 \, dx.
$$

which gives

$$
2y + \frac{\pi^4}{9} = \frac{\pi^4}{5} \implies y = \frac{2\pi^4}{45}.
$$

$\implies \sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}.$

8.22 (only prove Newton's binomial theorem).
Following the hint, denote the RHS to be $f(x)$.

$$
\lim_{n \to \infty} \left| \frac{\frac{\alpha(\alpha-1)\ldots(\alpha-n)}{(n+1)!}}{\frac{\alpha(\alpha-1)\ldots(\alpha-n+1)}{n!}} \right| = \lim_{n \to \infty} \left| \frac{\alpha + n}{n+1} \right| = 1.
$$

Then the power series convergers if $|x| < 1$.

$$f'(x) = \sum_{n=1}^{\infty} \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!} nx^{n-1}$$

$$\begin{aligned}
(1+x)f'(x) &= (1+x)\sum_{n=1}^{\infty} \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!} nx^{n-1} \\
&= \sum_{n=1}^{\infty} \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!} nx^{n-1} + \sum_{n=1}^{\infty} \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!} nx^{n} \\
&= \alpha + \sum_{n=2}^{\infty} \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!} nx^{n-1} + \sum_{n=1}^{\infty} \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!} nx^{n} \\
&= \alpha + \sum_{n=1}^{\infty} \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)(\alpha-n)}{n!} x^{n} + \sum_{n=1}^{\infty} \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!} nx^{n} \\
&= \alpha + \sum_{n=1}^{\infty} \frac{\alpha(\alpha-1)\cdots(\alpha-n)}{n!} (\alpha-n+n)x^{n} \\
&= \alpha f(x).
\end{aligned}$$

$\frac{df(x)}{f(x)} = \alpha\frac{dx}{1+x} \implies \log f(x) = \alpha \log(1+x) + C$
$\implies f(x) = C'(1+x)^{\alpha}$.
Since $f(0) = 1$, so $C' = 1$. Thus, $f(x) = (1+x)^{\alpha}$.

8.23

Since γ is continuously differentiable, $\frac{\gamma'}{\gamma}$ is continuous on $[a,b]$. Define

$$\varphi(x) = \int_a^x \frac{\gamma'(t)}{\gamma(t)}\, dt, \quad x \in [a,b],$$

then $\varphi(a) = 0$ and by Theorem 6.20, $\varphi' = \frac{\gamma'}{\gamma}$.
Therefore,

$$\frac{d\varphi}{dx} = \frac{d\gamma}{\gamma dx} \implies d\varphi = \frac{d\gamma}{\gamma}.$$

Integrating both sides, we get

$$\varphi = \log\gamma + C, \quad \text{i.e.,} \quad \gamma = C'\exp(\varphi),$$

where $C' \neq 0$ is a constant.
Since $\gamma(a) = \gamma(b)$ (γ is closed), $\exp\varphi(b) = \exp\varphi(a) = 1$ (because $\varphi(a) = 0$).
Note that

$$\operatorname{Ind}(\gamma) = \frac{1}{2\pi i} \int_a^b \frac{\gamma'(t)}{\gamma(t)}\, dt = \frac{1}{2\pi i}(\varphi(b) - \varphi(a)) = \frac{\varphi(b)}{2\pi i},$$

and therefore, $\varphi(b) = 2\pi i \operatorname{Ind}(\gamma)$ but $\exp(2\pi i Ind(\gamma)) = \exp(\varphi(b)) = 1$, therefore $Ind(\gamma)$ has to be an integer.
When $\gamma(t) = e^{int}$, $a = 0$, $b = 2\pi$, we have

$$\operatorname{Ind}(\gamma) = \frac{1}{2\pi i} \int_0^{2\pi} \frac{(e^{int})'}{e^{int}}\, dt = \frac{1}{2\pi i} in \int_0^{2\pi} dt = n.$$

8.25

Put $\gamma = \frac{\gamma_2}{\gamma_1}$. Then $|1 - \gamma| < 1$, hence $\operatorname{Ind}(\gamma) = 0$, by Exercise 24. Also, since

$$\frac{\gamma'}{\gamma} = \frac{(\gamma_2/\gamma_1)'}{\gamma_2/\gamma_1} = \frac{(\gamma_2'\gamma_1 - \gamma_1'\gamma_2/\gamma_1^2)}{\gamma_2/\gamma_1} = \frac{\gamma_2'\gamma_1 - \gamma_1'\gamma_2}{\gamma_2\gamma_1} = \frac{\gamma_2'}{\gamma_2} - \frac{\gamma_1'}{\gamma_1}$$

we have

$$\operatorname{Ind}(\gamma) = \frac{1}{2\pi i} \int_a^b \frac{\gamma'(t)}{\gamma(t)}\, dt = \frac{1}{2\pi i} \int_a^b \left(\frac{\gamma_2'(t)}{\gamma_2(t)} - \frac{\gamma_1'(t)}{\gamma_1(t)} \right) dt = \operatorname{Ind}(\gamma_2) - \operatorname{Ind}(\gamma_1),$$

which gives $\operatorname{Ind}(\gamma_1) = \operatorname{Ind}(\gamma_2)$.