Notes On Cantor Set

Cantor Set $C := \bigcap_{n \in N} E_n$

- $E_0 = [0, 1]$.
- each E_n is disjoint union of intervals each of length 3^{-n} $\forall n \in N$. E_{n+1} is obtained by removing the middle third of each intervals in E_n.
- C is closed since each E_n is closed.
- $[0, 1]$ is compact.
- $C \neq \emptyset$ due to the corolary of finite intersection property.
- if x is an endpoint of any E_n, then $x \in C$.
- C is perfect*

Remark: C is compact, perfect, and uncountable non-empty subset of R contains no interval, or is of length 0.
($\star$) By definition, C is perfect if C is closed and $C' = C$.

We have $C' \subset C$ since C is closed. Consider $x \in C$ and $r > 0$. Pick $n \in N$ such that $3^{-n} < r$, then let $I = E_n$ that contains x. The two endpoints of I are within radius r around x and the two endpoints are in C. Therefore, at least one of them is not x, which means $B_r(x) \backslash \{x\} \bigcap C \neq \emptyset$.

Suffice to show any non-empty perfect set is uncountable! (empty set is also a perfect set).

Lemma 1: If $p_n \in R^k,\ r_n > 0$ satisfying:

1. $B_{r_{n+1}}(p_{n+1}) \subset B_{r_n}(p_n)$
2. $B_{r_n}(p_n) \bigcap P \neq \emptyset$

Then $P \bigcap \left(\bigcap_{n \in N} \overline{B_{r_n}(p_n)} \right) \neq \emptyset$.

Proof. P is closed since P is perfect. Thus, $P \bigcap \overline{B_{r_n}(p_n)}$ is compact and non empty. And $P \bigcap \overline{B_{r_{n+1}}}(p_{n+1}) \subset P \bigcap \overline{B_{r_n}}(p_n)$. So by corolary to Theorem 2.36

$$P \bigcap \left(\bigcap_{n \in N} \overline{B_{r_n}(p_n)} \right) = \bigcap_{n \in N} \left(P \bigcap \overline{B_{r_n}(p_n)} \right) \neq \emptyset.$$

$\square$

Lemma 2: Let $p \neq x \in R^k$, and $r > 0$. If $q \in B_r(p) \backslash \{x\}$, then there is $s > 0$, such that $\overline{B_s(q)} \subset B_r(p) \backslash \{x\}$. (Excercise!)

Theorem: Non-empty Perfect Subset P In R^k Is Uncountable.

Proof. Consider any $x_1, x_2, ... \in P$. We will inductively choose $p_n \in P$ and $r_n > 0$ satisfying:

1. $x_n \notin \overline{B_{r_{n+1}}(p_{n+1})}$
2. $B_{r_{n+1}}(p_{n+1}) \subset B_{r_n}(p_n)$

Choose any $p_1 \in P$ and any $r_1 > 0$. Inductively assume that we have chosen n of them satisfying all the conditions.

Since $p_n \in P = P'$, we have that $B_{r_n}(p_n) \backslash \{p_n\} \bigcap P$ must be infinite by Theorem 2.20. Pick $p_{n+1} \in (B_{r_n}(p_n) \bigcap P) \backslash \{p_n, x_n\}$.

Choose $r_{n+1} > 0$ according to lemma 2. Thus we have obtained p_{n+1} satisfying all the conditions. By lemma 1, $P \bigcap \left(\bigcap_{n \in N} \overline{B_{r_n}(p_n)} \right) \neq \emptyset$.

By condition 1, $P \backslash \{x_1, x_2, ...\}$ contain the above set as a subset.

Hence $\{x_1, x_2, ...\}$ is a proper subset of P. Hence P is uncountable. $\square$

Reference:

Rudin W. Principles of Mathematical Analysis 3rd Ed.

The Cantor Set and the Cantor Function:
`https://wiki.math.ntnu.no/_media/tma4225/2015h/cantor_set_function.pdf`