Chapter 7

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Chapter 7 problems 15, 18, 19, 20 (For 19, see page 151, just before Theorem 7.15, for the definition of "uniformly closed"), 21

7.15
Since f_n is equicontinuous on $[0,1]$ then fix $\epsilon > 0$, $\exists \delta > 0$, such that $|x-y| < \delta$ implies $|f_n(x) - f_n(y)| < \epsilon$, $x, y \in [0,1]$ and $n \in \mathbb{N}$. Or, equivalently, $|f(nx) - f(ny)| < \epsilon$, $x, y \in [0,1]$, $|x-y| < \delta$, and $n \in \mathbb{N}$.
If $n = 1$, we have $|f(x) - f(y)| < \epsilon$, $x, y \in [0,1]$, and $|x-y| < \delta$. This means, f is uniformly continuous on $[0,1]$.
If $n = 2$, we have $|f(2x) - f(2y)| < \epsilon$, $x, y \in [0,1]$, and $|x-y| < \delta$. Or, equivalenty, by change of variables, we have $|f(s) - f(t)| < \epsilon$, $s, t \in [0,2]$, and $|s-t| < 2\delta = \delta'$. This means, f is uniformly continuous on $[0,2]$.
Hence, in general, f is uniformly continuous on every interval $[0,n]$, and for each of these intervals corresponding to (n, f_n), we have $\delta' = n\delta$.

7.18
For each n, we have $f_n \in \Re$ on $[a,b]$ and thus by theorem 6.20, we have F_n is continuous on $[a,b]$.
Next, we want to show $\{F_n\}$ is equicontinuous. Since $\{f_n\}$ is uniformly bounded, we have $|f_n(x)| \leq M$, for all n and $x \in [a,b]$. Then fix $\epsilon > 0$, and let $0 < \delta < \frac{\epsilon}{M}$, we have

$$\begin{aligned} |F_n(x) - F_n(y)| &= \left| \int_a^x f_n(t)\,dt - \int_a^y f_n(t)\,dt \right| \\ &= \left| \int_y^x f_n(t)\,dt \right| \\ &\leq M|x-y| < M\delta < \epsilon, \end{aligned}$$

for all n and $x, y \in [a,b]$, $|x-y| < \delta$. Hence, F_n is equicontinuous.
Furthermore, we have

$$|F_n(x)| = \left| \int_a^x f_n(t)\,dt \right| \leq M \left| \int_a^x dt \right| = M|x-a| \leq M(b-a),$$

for every n and every $x \in [a,b]$. Thus, $\{F_n\}$ is uniformly bounded, which also implies that $\{F_n\}$ is pointwise bounded.
Since $[a,b]$ is compact and $F_n \in \mathscr{C}([a,b])$ for every n, by Theorem 7.25, $\{F_n\}$ contains a uniformly convergent subsequence on $[a,b]$.

7.19 (See page 151, just before Theorem 7.15, for the definition of "uniformly closed")
($\Longrightarrow$)
Suppose S is compact, then by Theorem 2.34, S is (uniformly) closed. Since f is bounded $\forall f \in \mathscr{C}(K)$, and $S \subset \mathscr{C}(K)$, S is pointwise bounded. Fix $\epsilon > 0$, for each $f \in S$, let $A(f,\epsilon)$ be the set of all functions $g \in S$ such that $d_{\mathscr{C}(K)}(f,g) = \|f-g\| < \epsilon$. Since S is compact, there are finitely many $f_i \in S$, $1 \le i \le n$, such that

$$S \subseteq A(f_1,\epsilon) \cup A(f_2,\epsilon) \cup \cdots \cup A(f_n,\epsilon).$$

Since each f_i, $1 \le i \le n$, is continuous, and K is compact, each f_i is uniformly continuous on K. Hence, there is a $\delta > 0$, such that $d_K(x,y) < \delta$, $x,y \in K$ $\Longrightarrow$ $|f_i(x) - f_i(y)| < \epsilon$, for each $1 \le i \le n$.
Now, for every $f \in S$, there is an f_s, $1 \le s \le n$, such that $f \in A(f_s,\epsilon)$, or, in other words, $\|f - f_s\| < \epsilon$. We then have

$$\begin{aligned}|f(x) - f(y)| &\le |f(x) - f_s(x)| + |f_s(x) - f_s(y)| + |f_s(y) - f(y)| \\ &\le \|f - f_s\| + |f_s(x) - f_s(y)| + \|f_s - f\| < 3\epsilon,\end{aligned}$$

where $x,y \in K$ and $d(x,y) < \delta$. Since ϵ is arbitrary, this gives that S is equicontinuous by definition.

($\Longleftarrow$)
Suppose S is uniformly closed, pointwise bounded, and equicontinuous. Let E be any infinite subset of S, and thus E is pointwise bounded and equicontinuous. By Theorem 7.25 part (b) of the conclusion, E contains a uniformly convergent subsequence on K. Suppose $\{f_n\} \to f$ uniformly on K, then $f \in E'$ and O.T.O.H. by Theorem 7.15, we know that $\mathscr{C}(K)$ is complete, so $f \in \mathscr{C}(K)$.
By Theorem 2.27, we have $f \in E' \subset \overline{E} \subset \overline{S} = S$, since S is uniformly closed. Therefore, $f \in S$.
We showed every infinite subset of S has a limit point in S, and thus by Theorem 2.41, we have that S is compact.

7.20
Since f is continuous on $[0,1]$, by Theorem 7.26 (Weierstrass's Theorem), $\exists$ $\{P_n\} \to f$ uniformly on $[0,1]$.
And also since $P_n \in \Re$ for all n, by Theorem 7.16, we have

$$\int_0^1 f^2(x)\,dx = \int_0^1 f(x)\left(\lim_{n\to\infty} P_n(x)\right)\,dx = \lim_{n\to\infty}\int_0^1 f(x)P_n(x)\,dx.$$

Since $\int_0^1 f(x)x^n\,dx = 0$ for all n, and by Theorem 6.12 part (a), we thus have that

$$\int_0^1 f(x)P_n(x)\,dx = 0, \text{ for all } n.$$

Therefore,

$$\int_0^1 f^2(x)\,dx = 0,$$

which implies $f(x) = 0$ on $[0,1]$.

7.21

Since $|z| = 1$, we can write $z = e^{i\theta}$ for some θ. Then functions in A then can be rewritten as

$$f(z) = \sum_{n=0}^{N} c_n z^n.$$

Clearly, A separates points on K and A vanishes at no point of K.
For every $f \in A$, we have

$$\int_0^{2\pi} f(e^{i\theta})e^{i\theta}\,d\theta = \int_0^{2\pi} \left(\sum_{n=0}^{N} c_n e^{i(n+1)\theta}\right) d\theta = \sum_{n=0}^{N} c_n \int_0^{2\pi} e^{i(n+1)\theta}\,d\theta = 0.$$

And for every g in the closure of A, we have $g = \lim_{n\to\infty} f_n$, and $f_n \to g$ uniformly, $f_n \in A$. We have

$$\int_0^{2\pi} g e^{i\theta}\,d\theta = \lim_{n\to\infty} \int_0^{2\pi} f_n e^{i\theta}\,d\theta = 0.$$

If we choose $\phi(e^{i\theta}) = e^{-i\theta}$, then ϕ is continuous on K but

$$\int_0^{2\pi} \phi(e^{i\theta})e^{i\theta}\,d\theta = \int_0^{2\pi} e^{-i\theta}e^{i\theta}\,d\theta = \int_0^{2\pi} 1\,d\theta = 2\pi \neq 0.$$

Thus, ϕ is not in the closure of A.