Chapter 6

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Ch 6 problems 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13(abd), 15, 17, 19

6.1

Fix $\epsilon > 0$,
Since α is continuous at x_0, this means

$$\exists\delta > 0, \forall x \in [a, b], |x - x_0| < \delta \implies |\alpha(x) - \alpha(x_0)| < \frac{\epsilon}{2}$$

Let P be the partition that $P = \{z_0, z_1, z_2, z_3\}$, where $z_0 = a$, $z_1 = x_0 - \delta$, $z_2 = x_0 + \delta$, and $z_3 = b$.

$$\begin{aligned}
|U(P, f, \alpha) - L(P, f, \alpha)| &= |\sum_i^3 M_i \Delta\alpha_i - \sum_i^3 m_i \Delta\alpha_i| \\
&= |M_2 \Delta\alpha_2 - m_2 \Delta\alpha_2| \\
&= |\Delta\alpha_2| \\
&= |\alpha(x_0 + \delta) - \alpha(x_0 - \delta)| \\
&= |\alpha(x_0 + \delta) - \alpha(x_0) + \alpha(x_0) - \alpha(x_0 - \delta)| \\
&\leq |\alpha(x_0 + \delta) - \alpha(x_0)| + |\alpha(x_0) - \alpha(x_0 - \delta)| \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.
\end{aligned}$$

Hence $f \in \mathcal{R}(\alpha)$.
$\int f = \underline{\int} f = sup(L(P, f, \alpha)) = sup\{0\} = 0.$

6.2

Then suppose $f(z) \neq 0$ for some $z \in [a, b]$, then let $f(z) = M \in \mathbb{R}$ then $\exists r$ such that $0 < r < M$.

f is continuous on $[a,b]$ then $\exists \delta > 0, \forall t \in [a,b]$, $|t - z| < \delta \implies |f(t) - f(z)| < M - r$.
In other words, $f(t) > r - M + f(z) = r > 0$.
Then let P be a partition that $\Delta x_i < \delta$ for all i, then $z \in [x_{j-1}, x_j]$ for some j.
$\implies 0 < m_j \Delta x_j \leq L(P,f) \leq sup(L(P,f)) = \underline{\int_a^b} f(x)dx = \int_a^b f(x)dx = 0$, contradiction.

6.3
(a)
$(\impliedby)$ Suppose $f(0+) = f(0)$. Let $P = \{x_0, x_1, x_2, x_3\}$ where $x_0 = -1$ and $x_1 = 0 < x_2 < x_3 = 1$.
Then $U(P,f,\beta_1) - L(P,f,\beta_1) = M_2 - m_2$. As $x_2 \to 0$, we have $f(x_2) \to f(0)$ i.e. $M_2 \to f(0)$ and $m_2 \to f(0)$.
$\implies U(P,f,\beta_1) - L(P,f,\beta_1) \to 0$. Hence $f \in \mathcal{R}(\beta_1)$.
$\implies \int f d\beta_1 = sup(L(P,f,\beta_1)) = sup(\sum_i m_i \Delta\beta_i) = f(0)$.

$(\implies)$ Suppose $f \in \mathcal{R}(\beta_1)$, for any $\epsilon > 0$, $\exists$ a partition P of $[-1,1]$ such that $U(P,f,\beta_1) - L(P,f,\beta_1) < \epsilon$.
Suppose $0 \in [x_{j-1}, x_j]$ for some j where either $x_{j-1} = 0$ or $x_{j-1} < 0 < x_j$, then $U(P,f,\beta_1) - L(P,f,\beta_1) = M_j - m_j < \epsilon$. For any $\delta > 0$ such that $[0,\delta] \subseteq [x_{j-1}, x_j]$.
We have $\forall x \in [0,\delta] \implies |f(x) - f(0)| \leq M_j - m_j < \epsilon$.
$\implies f(0+) = \lim_{x\to 0^+} f(x) = f(0)$.

(b)
$(\impliedby)$ Suppose $f(0-) = f(0)$. Let $P = \{x_0, x_1, x_2, x_3\}$ where $x_0 = -1$ and $x_1 < x_2 = 0$, and $x_3 = 1$.
Then $U(P,f,\beta_2) - L(P,f,\beta_2) = M_2 - m_2$. As $x_1 \to 0$, we have $f(x_2) \to f(0)$ i.e. $M_2 \to f(0)$ and $m_2 \to f(0)$.
$\implies U(P,f,\beta_2) - L(P,f,\beta_2) \to 0$. Hence $f \in \mathcal{R}(\beta_2)$.
$\implies \int f d\beta_2 = sup(L(P,f,\beta_2)) = sup(\sum_i m_i \Delta\beta_i) = f(0)$.

$(\implies)$ Suppose $f \in \mathcal{R}(\beta_2)$, for any $\epsilon > 0$, $\exists$ a partition P of $[-1,1]$ such that $U(P,f,\beta_2) - L(P,f,\beta_2) < \epsilon$.
Suppose $0 \in [x_{j-1}, x_j]$ for some j where either $x_{j-1} = 0$ or $x_{j-1} < 0 < x_j$, then $U(P,f,\beta_2) - L(P,f,\beta_2) = M_j - m_j < \epsilon$. For any $\delta > 0$ such that $[-\delta, 0] \subseteq [x_{j-1}, x_j]$.
We have $\forall x \in [-\delta, 0] \implies |f(x) - f(0)| \leq M_j - m_j < \epsilon$.
$\implies f(0+) = \lim_{x\to 0^+} f(x) = f(0)$.

(c)
$(\impliedby)$ Suppose f is continuous at 0. Let $P = \{x_0, x_1, x_2, x_3\}$ where $x_0 = -1$ and $x_1 = -\delta$, $x_2 = \delta$, and $x_3 = 1$, where $0 < \delta < 1$.
Then $U(P,f,\beta_3) - L(P,f,\beta_3) = M_2 - m_2$. As $\delta \to 0$, we have $M_2 \to f(0)$ and $m_2 \to f(0)$.
$\implies U(P,f,\beta_3) - L(P,f,\beta_3) \to 0$. Hence $f \in \mathcal{R}(\beta_3)$.

$\implies \int f d\beta_3 = sup(L(P,f,\beta_3)) = sup(\sum_i m_i \Delta\beta_i) = f(0)$.

($\implies$) Suppose $f \in \mathcal{R}(\beta_3)$, for any $\epsilon > 0$, $\exists$ a partition P of $[-1,1]$ such that $U(P,f,\beta_3) - L(P,f,\beta_3) < \epsilon$.
Suppose $0 \in (x_{j-1}, x_j)$ for some j where either $x_{j-1} = 0$ or $x_{j-1} < 0 < x_j$, then $U(P,f,\beta_3) - L(P,f,\beta_3) = M_j - m_j < \epsilon$. Pick any $\delta > 0$ such that $[-\delta,\delta] \subseteq [x_{j-1}, x_j]$.
We have $\forall x \in [-\delta,\delta] \implies |f(x) - f(0)| \leq M_j - m_j < \epsilon$.
$\implies f(0) = \lim_{x\to 0} f(x) = f(0)$, i.e. f is continuous at 0.

Suppose $0 = x_j$ for some j, then $U(P,f,\beta_3) - L(P,f,\beta_3) = \frac{1}{2}(M_j - m_j) + \frac{1}{2}(M_{j+1} - m_{j+1}) < \epsilon$.
Then $max\{M_j, M_{j+1}\} - min\{m_j, m_{j-1}\} < 2\epsilon$.
Pick any $\delta > 0$ such that $[-\delta,\delta] \subseteq [x_{j-1}, x_{j+1}]$.
We have $\forall x \in [-\delta,\delta] \implies |f(x) - f(0)| < 2\epsilon$.
Since $\epsilon > 0$ is arbitrary,
$\implies f(0) = \lim_{x\to 0} f(x) = f(0)$, i.e. f is continuous at 0.
(d)
If f is continuous then $f(0) = f(0-) = f(0+)$. By (a)-(c), we have the desired results.

6.4
Suppose it is integrable, then for any ϵ, $\exists$ a partition P such that $U(P,f) - L(P,f) < \epsilon$.
Let $\epsilon = b - a - 1$ and since $\mathbb{Q}$ and $\mathbb{R}\backslash\mathbb{Q}$ are dense in $\mathbb{R}$, we have
$U(P,f) - L(P,f) = \sum_{i=1}^n M_i \Delta x_i - \sum_{i=1}^n m_i \Delta x_i = \sum_{i=1}^n \Delta x_i = b - a > \epsilon$ for any partition P.
Hence $f \notin \mathcal{R}$.

6.5
No. $f^2 \in \mathcal{R}$ does not implies $f \in \mathcal{R}$. Counter-example, $f = \begin{cases} -1 & x \in \mathbb{Q} \\ 1 & x \in \mathbb{R}\backslash\mathbb{Q} \end{cases}$
Then $f^2(x) = 1$, which is bounded and continuous on $\mathbb{R}$, hence $f^2 \in \mathcal{R}$. The proof for $f \notin \mathcal{R}$, is similar to excercise 6.4.
Yes. $f^3 \in \mathcal{R}$ implies $f \in \mathcal{R}$.
Fix $\epsilon > 0$, by assumption, $\exists P$ such that $U(P,f^3) - L(P,f^3) < \epsilon$.
Then, $U(P,f) - L(P,f) = \sum_i (M_i - m_i)\triangle x_i \leq \sum_i (M_i^3 - m_i^3)\triangle x_i = U(P,f^3) - L(P,f^3) < \epsilon$.
Since $M_i^2 \geq M_i m_i \implies M_i^2 + M_i m_i > 0$,
and $0 < (M_i - m_i) < (M_i - m_i)(M_i^2 + M_i m_i + m_i^2) = M_i^3 - m_i^3$.

6.6
Let $\epsilon > 0$ and put $M = sup|f(x)|$.

According to the construction of the Cantor set $\mathbb{P}$ in sec. 2.44, then $\exists n$ such that $|E_n| = 2^n \cdot \frac{1}{3^n} = \left(\frac{2}{3}\right)^n < \epsilon$.
Remove these 2^n segments from $[0, 1]$ then we have the remaining set K is compact and since f is continuous on this set by assumption, we have f is uniformly continuous. Thus, $\exists \delta > 0$ such that $|f(s) - f(t)| < \epsilon$ if $s, t \in K$ and $|s - t| < \delta$.
Construct a partition $P = \{x_0, ..., x_z\}$ as follows.
$-$ $x_0 = 0$ and $x_z = 1$. $-$ Every endpoint of intervals in E_n belongs to P, a total of 2^{n+1} points. $-$ No points of 2^n segments are in P. $-$ $\forall i \in \{1, ..., z-1)$, if x_i is not an endpoint of intervals in E_n, $\triangle x_{i+1} < \delta$.
Note that $M_i - m_i \leq 2M$ for every i, and $M_i - m_i < \epsilon$ if x_i is not an endpoint of intervals in E_n.
Then $U(P, f) - L(P, f) = \sum_{i=1}^{z} (M_i - m_i) \triangle x_i < 2M\epsilon + \epsilon(1 - 0) = (2M + 1)\epsilon$.
Since ϵ is arbitrary, by theorem 6.6. $f \in \mathcal{R}$.

6.7
(a)
We need to show that $\forall \epsilon > 0$, $\exists r > 0$ such that $0 < c < r \leq 1 \implies |\int_0^1 fd - \int_c^1 fd| < \epsilon$.
Claim: f is bounded. Proof: Since $f \in \mathcal{R}$, suppose f is not bounded, then $\forall P$, $\exists [x_{i-1}, x_i]$ where f is not bounded. Then $\exists x, y \in [x_{i-1}, x_i]$such that $|f(x) - f(y)| > \frac{\epsilon}{\delta}$ where $\delta = x_i - x_{i-1}$, which implies $U(P, f) - L(P, f) > \epsilon$, a contradiction.
Let $M = sup|f(x)|$ on $[0, 1]$ and fix $\epsilon > 0$,
Let P_1 be any partition on $[0, 1]$ that contains c, for any fixed $c \in (0, 1]$, then we have

$$U(P_1, f) - U_c(P_c, f) \leq M(c - 0) = Mc.$$

If we let $r = \frac{\epsilon}{M}$, then $\forall 0 < c < r$, $U(P_1, f) - U_c(P_1, f) < \epsilon$.
Since $f \in \mathcal{R}$ on $[0, 1]$ and $[c, 1]$, we have $\int_0^1 fdx = infU(P_{[0,1]}, f)$ and $\int_c^1 fdx = infU(P_{[c,1]}, f)$ then
$\exists P_2$ over $[0, 1]$ that contains c such that $U(P_2, f) - \int_0^1 fdx < \epsilon$, since we can refine any $P_{[0,1]}$ to get P' where $U(P', f) \leq U(P_{[0,1]}, f)$.
$\exists P_3$ over $[c, 1]$ such that $U(P_3, f) - \int_c^1 fdx < \epsilon$.
Let $P^* = P_1 \cup P_2 \cup P_3$. Then $\forall 0 < c < r$, then

$$\begin{aligned} &\left| \int_0^1 fdx - \int_c^1 fdx \right| \\ \leq &\left| \int_0^1 fdx - U(P^*, f) \right| + |U(P^*, f) - U(P_c^*, f)| + \left| U(P_c^*, f) - \int_c^1 fdx \right|. \end{aligned}$$

$\implies \left| \int_0^1 fdx - \int_c^1 fdx \right| < 3\epsilon$.
Since $\epsilon > 0$ is arbitrary, the result follows.

(b)
Let $f(x) = (-1)^n(n+1)$ for $x \in \left[\frac{1}{n+1}, \frac{1}{n}\right]$ for $n \in \mathbb{N}$. Then if $c \in \left[\frac{1}{N+1}, \frac{1}{N}\right]$ then

$$\int_c^1 f dx = (-1)^N(N+1)\left(\frac{1}{N} - c\right) + \sum_{k=1}^{N-1} \frac{(-1)^k}{k}$$

As $c \to 0$, we have $N \to \infty$, which means $(-1)^N(N+1)\left(\frac{1}{N} - c\right) \to 0$ and by theorem 3.43, $\sum \frac{(-1)^k}{k}$ converges, hence the limit exists.
O.T.O.H.

$$\int_c^1 |f| dx = (N+1)\left(\frac{1}{N} - c\right) + \sum_{k=1}^{N-1} \frac{1}{k}$$

As $c \to 0$, we have $N \to \infty$, which means $(N+1)\left(\frac{1}{N} - c\right) \to 0$ but the series $\sum \frac{(-1)^k}{k}$ is the harmonic series, which diverges.

6.8
For any n, define $\alpha_1(x) = f(n)$ and $\alpha_2(x) = f(n+1)$ for $x \in (n, n+1)$. Since f is monotonically decreasing, we have $\alpha_2(x) \leq f(x) \leq \alpha_1(x)$ for all $x \in [1, \infty]$.
Note that $\int_n^{n+1} \alpha_1(x)dx = f(n)$, and $\int_n^{n+1} \alpha_2(x)dx = f(n+1)$.
Thus, by theorem 6.12 part (b) and (c), for any $N \in \mathbb{N}$, we have

$$\begin{aligned}
\sum_1^N f(n) &= f(1) + \int_1^N \alpha_2(x)dx \\
&\leq f(1) + \int_1^N f(x)dx \\
&\leq f(1) + \int_1^N \alpha_1(x)dx \\
&= f(1) + \sum_1^{N-1} f(n).
\end{aligned}$$

$\implies \sum_1^N f(n) - f(1) \leq \int_1^N f(x)dx \leq \sum_1^{N-1} f(n)$.
By assumption $f(x) \geq 0$, therefore
$\sum f(n)$ is an increasing sequence, if $\int_1^\infty f dx$ converges, then $\sum f(n)$ is bounded above, hence converges.
Similarly, since $\int_1^t f dx$ is increasing as $t \to \infty$, if $\sum f(n)$ converges, so is $\int_1^\infty f dx$.

6.10
(a)
$p = \frac{q}{q-1} > 0 \implies q - 1 > 0$ since $p, q > 0$.

Fix u, then let $f(v) = \frac{u^p}{p} - \frac{v^q}{q} - uv$. Then $f'(v) = v^{q-1} - u$ and $f''(v) = (q-1)v^{q-2} \geq 0$ for $v \geq 0$. So f achieves a minimum at $v = u^{1/(q-1)}$.
So,

$$\begin{aligned}
\frac{u^p}{p} + \frac{v^q}{q} - uv &\geq \frac{u^p}{p} + \frac{u^{q/(q-1)}}{q} - u^{1+1/(q-1)} \\
&= \frac{u^p}{p} + \frac{u^p}{q} - u^p \\
&= \left(\frac{1}{p} + \frac{1}{q} - 1\right) u^p \\
&= 0.
\end{aligned}$$

So $uv \leq \frac{u^p}{p} + \frac{v^q}{q}$.
Equality holds when $v = u^{1/(q-1)} \implies v^q = u^{q/(q-1)} = u^p$.

(b)
From part (a), we have

$$\begin{aligned}
\int_a^b fg d\alpha &\leq \int_a^b \left(\frac{f^p}{p} + \frac{g^q}{q}\right) d\alpha \\
&= \frac{1}{p}\int_a^b f^p d\alpha + \frac{1}{q}\int_a^b g^q d\alpha \\
&= \frac{1}{p} + \frac{1}{q} \\
&= 1.
\end{aligned}$$

(c)
Suppose the $RHS \neq 0$, then we have that

$$\begin{aligned}
\frac{\left|\int_a^b fg d\alpha\right|}{\left\{\int_a^b |f|^p d\alpha\right\}^{1/p}\left\{\int_a^b |g|^q d\alpha\right\}^{1/q}} &\leq \frac{\int_a^b |fg| d\alpha}{\left\{\int_a^b |f|^p d\alpha\right\}^{1/p}\left\{\int_a^b |g|^q d\alpha\right\}^{1/q}} \\
&= \int_a^b \left(\frac{|f|^p}{\int_a^b |f|^p d\alpha}\right)^{1/p}\left(\frac{|g|^q}{\int_a^b |g|^q d\alpha}\right)^{1/q} d\alpha \\
&\leq \int_a^b \left(\frac{|f|^p}{p\int_a^b |f|^p d\alpha}\right)\left(\frac{|g|^q}{q\int_a^b |g|^q d\alpha}\right) d\alpha \text{ by (a)} \\
&= \frac{1}{p}\left(\frac{\int_a^b |f|^p d\alpha}{\int_a^b |f|^p d\alpha}\right) + \frac{1}{q}\left(\frac{\int_a^b |g|^q d\alpha}{\int_a^b |g|^q d\alpha}\right) \\
&= 1.
\end{aligned}$$

Hence, $\left|\int_a^b fgd\alpha\right| \leq \left\{\int_a^b |f|^p d\alpha\right\}^{1/p} \left\{\int_a^b |g|^q d\alpha\right\}^{1/q}$.

(d)
Apply (c), for excercise 6.7, where $f, g \in \mathcal{R}$ on $[c, 1]$ and for every $c > 0$, $\int_c^1 |f|^p dx$ and $\int_c^1 |g|^q dx$ exists. We have
$\left|\int_c^1 fgdx\right| \leq \left\{\int_c^1 |f|^p dx\right\}^{1/p} \left\{\int_c^1 |g|^q dx\right\}^{1/q}$.
Take the limit on both sides we have

$$\begin{aligned}\lim_{c\to 0}\left|\int_c^1 fgdx\right| &\leq \lim_{c\to 0}\left\{\int_c^1 |f|^p dx\right\}^{1/p} \left\{\int_c^1 |g|^q dx\right\}^{1/q} \\ &= \lim_{c\to 0}\left\{\int_c^1 |f|^p dx\right\}^{1/p} \lim_{c\to 0}\left\{\int_c^1 |g|^q dx\right\}^{1/q}\end{aligned}$$

$$\implies \left|\int_0^1 fgdx\right| \leq \left\{\int_0^1 |f|^p dx\right\}^{1/p} \left\{\int_0^1 |g|^q dx\right\}^{1/q}.$$

Similar, apply (c) to excercise 6.8, then suppose all the assumptions hold and for every $b > a$, we have
$\left|\int_a^b fgdx\right| \leq \left\{\int_a^b |f|^p dx\right\}^{1/p} \left\{\int_a^b |g|^q dx\right\}^{1/q}$
Take the limit on both sides we have

$$\begin{aligned}\lim_{b\to\infty}\left|\int_a^b fgdx\right| &\leq \lim_{b\to\infty}\left\{\int_a^b |f|^p dx\right\}^{1/p} \left\{\int_a^b |g|^q dx\right\}^{1/q} \\ &= \lim_{b\to\infty}\left\{\int_a^b |f|^p dx\right\}^{1/p} \lim_{b\to\infty}\left\{\int_a^b |g|^q dx\right\}^{1/q}\end{aligned}$$

$$\implies \left|\int_a^\infty fgdx\right| \leq \left\{\int_a^\infty |f|^p dx\right\}^{1/p} \left\{\int_a^\infty |g|^q dx\right\}^{1/q}.$$

6.9

(i) Theorem: If F, G are differentiable on $[a, \infty)$, $F' = f \in \mathcal{R}$ and $G' = g \in \mathcal{R}$ on $[a, b]$ for every $b > a$. Then

$$\int_a^\infty Fgdx = \lim_{b\to\infty}\{F(b)G(b)\} - F(a)G(a) - \int_a^\infty fGdx,$$

if $\lim_{b\to\infty}\{F(b)G(b)\}$ exists and $\int_a^\infty fGdx$ converges.

Proof. Let $H(x) = F(x)G(x)$, then $H' \in \mathcal{R}$ by theorem 6.13. Apply theorem 6.21 to H and its derivative, then for any $b > a$, we have $\int_a^b H' dx = H(b) - H(a)$.

$$\int_a^b Fg dx = \{F(b)G(b)\} - F(a)G(a) - \int_a^b fG dx,$$

By assumptions, limit on the RHS as $b \to \infty$ exists. Lets call it M.
Suffice to show $\lim_{b\to\infty} \int_a^b Fg dx = M$.
Fix $\epsilon > 0$,
$\exists B$ such that $\forall b > B$, $|\{F(b)G(b)\} - F(a)G(a) - \int_a^b fG dx - M| < \epsilon$.
Equivalently, $|\int_a^b Fg dx - M| < \epsilon$. Hence, limit on the LHS as $b \to \infty$ is M. $\square$

(ii) By excercise 8, $\int_0^\infty \left|\frac{sinx}{(1+x)^2}\right| dx$ converges iff $\sum \left|\frac{sinn}{(1+n)^2}\right|$ converges.
Since $\left|\frac{sinn}{(1+n)^2}\right| \le \frac{1}{(1+n)^2}$ for all n, and $\sum \frac{1}{(1+n)^2}$ converges,
we have $\int_0^\infty \left|\frac{sinx}{(1+x)^2}\right| dx$ converges. Thus, $\int_0^\infty \frac{sinx}{(1+x)^2} dx$ converges absolutely.
Hence, apply the theorem and we have the LHS of the following converges.
$\int_0^\infty \frac{cosx}{1+x} dx = \lim_{b\to\infty} \frac{sin(b)}{1+b} - \frac{sin(0)}{1+0} - \int_0^\infty \frac{-sinx}{(1+x)^2} dx = \int_0^\infty \frac{sinx}{(1+x)^2}$.
O.T.O.H. we have

$$\begin{aligned}
\lim_{b\to\infty} \int_0^b \frac{|cosx|}{1+x} dx &= \sum_{k=0}^{\infty} \int_{2\pi k}^{2\pi(k+1)} \frac{|cosx|}{1+x} dx \\
&\ge \sum_{k=0}^{\infty} \frac{1}{2\pi(k+1)+1} \int_{2\pi k}^{2\pi(k+1)} |cosx| dx \\
&\ge \sum_{k=0}^{\infty} \frac{1}{2\pi(k+1)+2\pi} \cdot 4 \\
&\ge \frac{2}{\pi} \sum_{k=0}^{\infty} \frac{1}{k+2} = -\frac{2}{\pi} + \frac{2}{\pi} \sum_{k=1}^{\infty} \frac{1}{k}.
\end{aligned}$$

where the last term diverges. Hence, $\int_0^\infty \frac{cosx}{1+x} dx$ does not converge absolutely.

6.11

Let $u, v \in \mathcal{R}$ on $[a, b]$, then lets show $||u + v||_2 \le ||u||_2 + ||v||_2$.

$$
\begin{aligned}
||u+v||_2^2 &= \int_a^b |u+v|^2 d\alpha \\
&\leq \int_a^b |u+v|^2 d\alpha \\
&\leq \int_a^b (|u|+|v|)^2 d\alpha \\
&\text{(by Schwarz Inequality)} \\
&= \int_a^b |u|^2 d\alpha + \int_a^b |v|^2 d\alpha + 2\int_a^b |u||v| d\alpha \\
&\leq \int_a^b |u|^2 d\alpha + \int_a^b |v|^2 d\alpha + 2\left\{\int_a^b |u|^2 d\alpha\right\}^{1/2} \left\{\int_a^b |v|^2 d\alpha\right\}^{1/2} \\
&\text{(by Holder's inequality)} \\
&= \left(\left\{\int_a^b |u|^2 d\alpha\right\}^{1/2} + \left\{\int_a^b |v|^2 d\alpha\right\}^{1/2}\right)^2 \\
&= (||u||_2 + ||v||_2)^2.
\end{aligned}
$$

Take square root on both sides we get what we wanted to show. And then replace $u = f - g$ and $v = g - h$, we get the desired result.

6.12

Fix $\epsilon > 0$.
Since $f \in \mathcal{R}(\alpha)$ on $[a,b]$, f is bounded. Let $M = sup|f(x)|$. Choose a partition $P = \{x_0, ..., x_n\}$ such that $U(P,f,\alpha) - L(P,f,\alpha) < \frac{\epsilon^2}{2M}$.
Define: $g(t) = \frac{x_i - t}{\triangle x_i} f(x_{i-1}) + \frac{t - x_{x_{i-1}}}{\triangle x_i} f(x_i)$ for all $x_{i-1} \leq t \leq x_i$.
g is continuous since for any $\forall t \in [a,b]$, say $t \in [x_{i-1}, x_i]$ for some i, then pick $\delta = min\{\frac{\epsilon \triangle x_j}{2M}, |t - x_j|\}$ if t is not an endpoint, and pick $\delta = min\{\frac{\epsilon \triangle x_j}{2M}, |t - x_j|\}$ for $j \in \{0, ..., n\} \backslash \{i\}$ if $t = x_i$ for some i and thus $\forall s \in [a,b]$, such that $|t - s| < \delta$,

then

$$\begin{aligned}|g(s)-g(t)| &= \left|\frac{x_i-t}{\triangle x_i}f(x_{i-1})+\frac{t-x_{i-1}}{\triangle x_i}f(x_i)-\frac{x_i-s}{\triangle x_i}f(x_{i-1})-\frac{s-x_{i-1}}{\triangle x_i}f(x_i)\right|\\ &= \left|\frac{s-t}{\triangle x_i}f(x_{i-1})+\frac{t-s}{\triangle x_i}f(x_i)\right|\\ &\le \left|\frac{s-t}{\triangle x_i}\right|(|f(x_{i-1})|+|f(x_i)|) \qquad \text{(by triangle inequality)}\\ &< \frac{\delta}{\triangle x_i}2M < \epsilon.\end{aligned}$$

O.T.O.H. we have $\forall x_{i-1} \le t \le x_i$,

$$\begin{aligned}|f(t)-g(t)| &= \left|\frac{x_i-t}{\triangle x_i}(f(t)-f(x_{i-1}))+\frac{t-x_{i-1}}{\triangle x_i}(f(t)-f(x_i))\right|\\ &\le \left|\frac{x_i-t}{\triangle x_i}\right||f(t)-f(x_{i-1})|+\left|\frac{t-x_{i-1}}{\triangle x_i}\right||(f(t)-f(x_i))|\\ &\le M_i-m_i.\end{aligned}$$

Thus,

$$\begin{aligned}||f-g||_2^2 &= \int_a^b |f-g|^2 d\alpha\\ &= \sum_{i=1}^n \int_{x_{i-1}}^{x_i} |f-g|^2 d\alpha\\ &\le \sum_{i=1}^n \int_{x_{i-1}}^{x_i} (M_i-m_i)^2 d\alpha\\ &\le 2M\sum_{i=1}^n \int_{x_{i-1}}^{x_i} (M_i-m_i) d\alpha\\ &= 2M\sum_{i=1}^n (M_i-m_i)\triangle\alpha_i\\ &= 2M[U(P,f,\alpha)-L(P,f,\alpha)]\\ &< 2M\frac{\epsilon^2}{2M} = \epsilon^2.\end{aligned}$$

$\implies ||f-g||_2 < \epsilon.$

6.13(abd)

(a)
Suppose $x > 0$, let $u = t^2$, then $du = 2tdt$ and $t = \sqrt{u}$. By change of variables, we have

$$f(x) = \int_{x^2}^{(x+1)^2} \frac{sin(u)}{2\sqrt{u}} du$$

Let $F = \frac{1}{2}u^{-1/2}$ and $G = -cos(u)$, then F, G are differentiable and $G' = sin(u)$ and $F' = -\frac{1}{4}u^{-3/2} \in \mathcal{R}$, and $u \neq 0$. Apply integration by parts we have

$$f(x) = -\frac{cos[(x+1)^2]}{2(x+1)} + \frac{cos(x^2)}{2x} - \int_{x^2}^{(x+1)^2} \frac{cos(u)}{4u^{3/2}} du$$

$$\begin{aligned}
\implies |f(x)| &\leq \left|\frac{cos[(x+1)^2]}{2(x+1)}\right| + \left|\frac{cos(x^2)}{2x}\right| + \left|\int_{x^2}^{(x+1)^2} \frac{cos(u)}{4u^{3/2}} du\right| \\
&\leq \frac{|cos[(x+1)^2]|}{2(x+1)} + \frac{|cos(x^2)|}{2x} + \int_{x^2}^{(x+1)^2} \frac{|cos(u)|}{4u^{3/2}} du \\
&< \frac{1}{2(x+1)} + \frac{1}{2x} + \int_{x^2}^{(x+1)^2} \frac{1}{4u^{3/2}} du \\
&= \frac{1}{2(x+1)} + \frac{1}{2x} + \frac{1}{2}\left(\frac{1}{x} - \frac{1}{x+1}\right) \\
&= \frac{1}{x},
\end{aligned}$$

where the strict inequality comes from the fact that $|cos(u)|$ will not be constantly 1 when integrating over $[x^2, (x+1)^2]$.
(b)
From part (a), we see that
$f(x) < -\frac{cos[(x+1)^2]}{2(x+1)} + \frac{cos(x^2)}{2x} + \frac{1}{2x(x+1)}$
$\implies 2xf(x) < cos(x^2) - \frac{xcos[(x+1)^2]}{x+1} + \frac{1}{x+1} = cos(x^2) + \frac{cos[(x+1)^2]}{x+1} - cos[(x+1)^2] + \frac{1}{x+1}$.
Then set $r(x) = 2xf(x) - cos(x^2) + cos[(x+1)^2] < \frac{cos[(x+1)^2]}{x+1} + \frac{1}{x+1}$.
$\implies |r(x)| < \frac{2}{x+1} < \frac{2}{x}$.
(d)
Fix $N \in \mathbb{N}$,

$$\int_0^N sin(t^2)dt = \sum_{k=0}^{N} f(k)$$
$$= f(0) + \sum_{k=1}^{N} \frac{r(k)}{2k} + \sum_{k=1}^{N} \frac{cos(k^2) - cos[(k+1)^2]}{2k}$$
$$= f(0) + \sum_{k=1}^{N} \frac{r(k)}{2k} + \left[\frac{cos1}{2} - \frac{cos4}{2} + \frac{cos4}{4} - \frac{cos9}{4} + ... - \frac{cos[(N+1)^2]}{2N}\right]$$
$$= f(0) + \sum_{k=1}^{N} \frac{r(k)}{2k} + \left[\frac{cos1}{2} - \frac{cos[(N+1)^2]}{2N}\right] - \frac{1}{2}\sum_{k=2}^{N} \frac{cos(k^2)}{k(k-1)}$$

Since $\left|\sum \frac{r(k)}{2k}\right| < \sum \frac{|r(k)|}{2k} \leq \sum \frac{c}{2k^2}$, where the last sum converges. Therefore, the first sum oh RHS above converges.
Consider $\left|\sum_{k=2}^{\infty} \frac{cos(k^2)}{k(k-1)}\right| \leq \sum_{k=2}^{\infty} \frac{|cos(k^2)|}{k(k-1)} \leq \sum_{k=2}^{\infty} \frac{1}{k(k-1)} = \sum_{k=2}^{\infty} \left(\frac{1}{k-1} - \frac{1}{k}\right)$, and the last sum converges (fix N,then $\sum_{k=2}^{N} \left(\frac{1}{k-1} - \frac{1}{k}\right) = 1 - \frac{1}{N}$, and the limit as $N \to \infty$ is 1).
The other terms on the RHS above clearly converges.
Suffice to show $\lim_{x\to\infty} \int_{\lfloor x \rfloor}^{x} sin(t^2)dt = 0$, where $\lfloor x \rfloor \leq x < \lfloor x \rfloor + 1$ and $\lfloor x \rfloor \in \mathbb{Z}$.
From (a), we apply integration by parts we get

$$\int_{\lfloor x \rfloor}^{x} sin(t^2)dt = -\frac{cos(x^2)}{2x} + \frac{cos(\lfloor x \rfloor^2)}{2\lfloor x \rfloor} - \int_{\lfloor x \rfloor^2}^{x^2} \frac{cos(u)}{4u^{3/2}}du,$$

the limit approaches 0 as $x \to \infty$.

6.15

Since x and $f(x)$ are differentiable functions, $xf(x)$ is differentiable. Also, $f'(x)$ is continuous, which means $f'(x) \in \mathcal{R}$ and $[xf(x)]' = f(x) + xf'(x) \in \mathcal{R}$.
Apply integration by parts we have

$$\int_a^b xf(x)f'(x)dx = bf^2(b) - af^2(a) - \int_a^b [f(x) + xf'(x)]f(x)dx$$
$$= 0 - 0 - \int_a^b f^2(x)dx - \int_a^b xf(x)f'(x)dx$$

$$\iff 2\int_a^b xf(x)f'(x)dx = -1$$
$$\iff \int_a^b xf(x)f'(x)dx = -\frac{1}{2}.$$

$- - - - - - - - - - - - - - - - - - - -$ Note that $f'(x), xf(x) \in \mathcal{R}(\alpha)$. Apply Holder's inequality we have

$$\begin{aligned}
\int_a^b [f'(x)]^2 dx \cdot \int_a^b x^2 f^2(x) dx &= \left(\left\{ \int_a^b [f'(x)]^2 dx \right\}^{1/2} \cdot \left\{ \int_a^b x^2 f^2(x) dx \right\}^{1/2} \right)^2 \\
&\geq \left(\left| \int_a^b x f'(x) f(x) dx \right| \right)^2 \\
&= \frac{1}{4}.
\end{aligned}$$

6.17

Take g real, W.L.O.G. Given $P = \{x_0, x_1, ..., x_n\}$, choose $t_i \in \{x_{i-1}, x_i\}$ so that $g(t_i)\triangle x_i = G(x_i) - G(x_{i-1})$, by mean value theorem. Then we have

$$\begin{aligned}
\sum_{i=1}^n \alpha(x_i) g(t_i) \triangle x_i &= \sum_{i=1}^n \alpha(x_i)(G(x_i) - G(x_{i-1}) \\
&= \sum_{i=1}^n [\alpha(x_i)G(x_i) - \alpha(x_{i-1})G(x_{i-1}) + \alpha(x_{i-1})G(x_{i-1}) - \alpha(x_i)G(x_{i-1})] \\
&= \sum_{i=1}^n [\alpha(x_i)G(x_i) - \alpha(x_{i-1})G(x_{i-1})] - \sum_{i=1}^n G_{i-1} \triangle \alpha_i \quad (\alpha \text{ increases monotonically}) \\
&= G(b)\alpha(b) - G(a)\alpha(a) - \sum_{i=1}^n G_{i-1} \triangle \alpha_i.
\end{aligned}$$

g is continuous and since α is monotonic on $[a, b]$, α will have at most finitely many jump discontinuities. Hence $g\alpha$ will have at most finitely many jump discontinuities. By theorem 6.10. $g\alpha \in \mathcal{R}$.
G' exists thus G is continuous and by theorem 6.8., $G \in \mathcal{R}$.
Since

$$L(P, g\alpha) \leq \sum_{i=1}^n \alpha(x_i) g(t_i) \triangle x_i \leq U(P, g\alpha)$$

and

$$L(P, G, \alpha) \leq \sum_{i=1}^n G_{i-1} \triangle \alpha_i \leq U(P, G, \alpha)$$

$$\implies L(P, g\alpha) + L(P, G, \alpha) \leq G(b)\alpha(b) - G(a)\alpha(a) \leq U(P, g\alpha) + U(P, G, \alpha)$$

Since P is arbitrary and $g\alpha, G \in \mathcal{R}$, thus we can refine P such that
$\int_a^b \alpha(x)g(x)dx + \int_a^b G(x)d\alpha \leq G(b)\alpha(b) - G(a)\alpha(a)$
$\leq \int_a^b \alpha(x)g(x)dx + \int_a^b G(x)d\alpha$.
In other words, $\int_a^b \alpha(x)g(x)dx + \int_a^b G(x)d\alpha = G(b)\alpha(b) - G(a)\alpha(a)$,
or $\int_a^b \alpha(x)g(x)dx = G(b)\alpha(b) - G(a)\alpha(a) - \int_a^b G(x)d\alpha$.

6.19

Since $\gamma_2(s) = \gamma_1(\phi(s))$, and ϕ is one-to-one, γ_1 is one-to-one iff γ_2 is one-to-one.
Hence γ_1 is an arc iff γ_2 is an arc.
Since ϕ is continuous and bijective, ϕ is strictly increasing. Suppose not, then $\exists a < x < y < b$ such that $\phi(x) > \phi(y) > c = \phi(a)$. By IVT, $\exists a < z < x$ such that $\phi(z) = \phi(y)$, contradicting the fact that ϕ is one-to-one. Hence, $\phi(d) = b$.
Thus,

$$\gamma_2(d) = \gamma_1(\phi(d)) = \gamma_1(b) = \gamma_1(a) = \gamma_2(\phi(c)) = \gamma_2(c),$$

In other words, γ_1 is a closed curve iff γ_2.
W.L.O.G. suppose γ_1 is rectifiable. Let $P = \{x_0 = a, .., x_n = c\}$ be a partition of $[a, b]$. Then $\phi^{-1}(P)$ is a partition of $[c, d]$.

$$\begin{aligned}
\Lambda(\gamma_2) &= sup\Lambda(\phi^{-1}(P), \gamma_2) \\
&= sup\sum_{i=1}^{n} \left|\gamma_2(\phi^{-1}(x_i)) - \gamma_2(\phi^{-1}(x_{i-1}))\right| \\
&= sup\sum_{i=1}^{n} |\gamma_1(x_i) - \gamma_1(x_{i-1})| \\
&= sup\Lambda(P, \gamma_1) = \Lambda(\gamma_1).
\end{aligned}$$

$\implies$ γ_2 is rectifiable.
$\implies$ γ_1 is rectifiable iff γ_2 is rectifiable.
And if one of them is rectifiable, from the above process, it is clear they must have the same length.