Chapter 3

Sang Tran

September 12, 2024

RUDIN Chapter 3 problems 2, 4, 5, 6, 7, 8, 9, 10, 11(abc), 12, 13, 14(ab), 16(a), 17(abc), 19, 20, 21, 22, 23

2.

$$\begin{aligned}\lim_{n\to\infty}(\sqrt{n^2+n}-n) &= \lim_{n\to\infty}\frac{n}{\sqrt{n^2+n}+n}\\ &= \lim_{n\to\infty}\frac{1}{\sqrt{1+\frac{1}{n}}+1}\\ &= \frac{1}{2}.\end{aligned}$$

4.
Observe that $\{s_n\} = \{0, 0, \frac{1}{2}, \frac{1}{4}, \frac{3}{4}, \frac{3}{8}, \frac{7}{8}, ...\}$
Claim: $s_{2m+2} = \frac{1}{2} - \frac{1}{2^{m+1}}$; $s_{2m+1} = 1 - \frac{1}{2^m}$ for $m \geq 0$.
Since either one is determined by the other in the definition, it suffices to prove only one. The base case holds trivially. Suppose both hold for $m \leq r$. Then

$$s_{2(r+1)+2} = \frac{s_{2(r+1)+1}}{2} = \frac{1}{2} - \frac{1}{2^{(r+1)+1}}.$$

Thus by induction we are done.
The subsequence with odd terms converges to 1 and since $s_n \in [0, 1)$, and all terms of the original sequence are less than 1 and therefore, by theorem 3.17, $\limsup_{n\to\infty} s_n = 1$.
The subsequence with even terms converges to $\frac{1}{2}$ and for any $0 < x < \frac{1}{2}$, by A.P. $\exists k \in \mathbb{N}$ such that $\frac{1}{2} - x > \frac{1}{k}$. But for any such k, $\exists m \in \mathbb{N}$ such that $2^{m+1} > k$, or $\frac{1}{k} > \frac{1}{2^{m+1}}$ (i.e. take $m = k$). Thus

$$\frac{1}{2} - x > \frac{1}{k} > \frac{1}{2^{m+1}} \to s_{2m+2} > x \; \forall m \geq k.$$

It is trivial for $x \leq 0$ since after the first term, all terms in the subsequence are greater 0. Hence, by theorem 3.17, $\liminf_{n\to\infty} s_n = \frac{1}{2}$.

5.
Let $r \in \mathbb{R}$.
If the RHS is of the form $\infty + \infty$, $\infty \pm r$, or $r + \infty$, then the RHS $= \infty$ and the inequality holds trivially.
If the RHS is of the form $r - \infty$ or $-\infty + r$, then the RHS $= -\infty$ and W.L.O.G. suppose $\limsup_{n\to\infty} a_n = r$ and $\limsup_{n\to\infty} b_n = -\infty$. Then by definition of the upper limit, $\lim_{n\to\infty} b_n = -\infty$. On the other hand, by theorem 3.17 part (b), $\exists M > r$ such that $a_n < M$, $\forall n$. Thus, by theorem 3.19,

$$\limsup_{n\to\infty}(a_n + b_n) \leq \limsup_{n\to\infty}(M + b_n) = -\infty,$$

which implies $\limsup_{n\to\infty}(a_n + b_n) = -\infty$. So the inequality holds.
If the RHS $\in \mathbb{R}$, or $\limsup_{n\to\infty}(a_n) = A$ and $\limsup_{n\to\infty}(b_n) = B$. Then let $\limsup_{n\to\infty}(a_n + b_n) = C$. Suppose on the contrary that $A + B < C \to A < C - B$. Thus $\exists S$ such that $A < S < C - B$.
By theorem 3.17 part (b), $\exists N$ such that $\forall n \geq N$, $a_n < S$. Therefore, when $n \geq N$, $a_n + b_n < S + b_n$ and by theorem 3.19,

$$C = \limsup_{n\to\infty}(a_n + b_n) < \limsup_{n\to\infty}(S + b_n) = S + \limsup_{n\to\infty}(b_n) = S + B < C,$$

a contradiction.

6

(a)

$$\begin{aligned}\sum_{n=1}^{m} a_n &= (\sqrt{2} - \sqrt{1}) + (\sqrt{3} - \sqrt{2}) + ... + \sqrt{m+1} - \sqrt{m} \\ &= \sqrt{m+1} - 1\end{aligned}$$

$$\implies \sum a_n = \lim_{m\to\infty}(\sqrt{m+1} - 1) = \infty.$$

Hence, the series diveges.

(b)

$$\begin{aligned}0 \leq a_n &= \frac{\sqrt{n+1} - \sqrt{n}}{n} \\ &= \frac{1}{n(\sqrt{n+1} + \sqrt{n})} < \frac{1}{2n\sqrt{n}} = \frac{1}{2n^{\frac{3}{2}}}.\end{aligned}$$

The p-series $\sum \frac{1}{n^{\frac{3}{2}}}$ converges with $p = \frac{3}{2} > 1$, and thus $\sum \frac{1}{2n^{\frac{3}{2}}}$ also converges. Since the above relation holds for all n, by comparison test, we have $\sum a_n$ converges.

(c)

$$\begin{aligned}\alpha = \limsup \sqrt[n]{|a_n|} &= \limsup \left(\sqrt[n]{n} - 1\right) \\ &= \left(\limsup \sqrt[n]{n}\right) - 1 \\ &= 1 - 1 \\ &= 0,\end{aligned}$$

where the equality follows from theorem 3.20 (c) i.e. $\lim \sqrt[n]{n} = 1 \implies \limsup \sqrt[n]{n} = 1$.
Since $\alpha = 0 < 1$, by the root test, $\sum a_n$ converges.

(d)
If $|z| \leq 1$, then $|1 + z^n| \leq 1 + |z^n| \leq 1 + |z|^n \leq 2$

$$|a_n| = \left|\frac{1}{1+z^n}\right| \geq \frac{1}{2}\ \forall n$$

which means a_n does not tend to 0 and hence $\sum a_n$ does not converge.
If $|z| > 1$, then since $\forall n \geq 1$, $|1 + z^n| + 1 \geq |1 + z^n - 1| = |z^n|$, or
$|1 + z^n| \geq |z|^n - 1 > 0 \implies \frac{1}{|1+z^n|} \leq \frac{1}{|z|^n - 1}$ and
$-|z|^{n-1} \leq -1 \implies |z|^n - |z|^{n-1} \leq |z|^n - 1$
Therefore, we have

$$\begin{aligned}|a_n| = \left|\frac{1}{1+z^n}\right| &\leq \frac{1}{|z|^n - 1} \\ &\leq \frac{1}{|z|^n - |z|^{n-1}} \\ &= \frac{1}{1 - \frac{1}{|z|}} \cdot \frac{1}{|z|^n} \\ &= \frac{|z|}{|z| - 1} \cdot \frac{1}{|z|^n},\end{aligned}$$

and by assumption $\frac{1}{|z|} < 1$ so $\sum \frac{1}{|z|^n}$ converges, hence $\frac{|z|}{|z|-1} \sum \frac{1}{|z|^n}$ converges.
By the comparison test, $\sum a_n$ converges.

7

Sol1:
We have

$$\left(\sqrt{a_n} - \frac{1}{n}\right)^2 \geq 0 \implies \frac{\sqrt{a_n}}{n} \leq \frac{1}{2}\left(a_n + \frac{1}{n^2}\right)$$

Since $\sum a_n$ and $\sum \frac{1}{n^2}$ converges, where the latter is p-series with $p = 2 > 1$, hence $\frac{1}{2} \sum \left(a_n + \frac{1}{n^2}\right)$ converges. By the comparison test, $\sum \frac{\sqrt{a_n}}{n}$ converges.

Sol2:
By the Cauchy-Schwarz inequality

$$\left|\sum_{k=n}^{m} \frac{\sqrt{a_k}}{k}\right| \le \left(\sum_{k=n}^{m} a_k\right)^{\frac{1}{2}} \left(\sum_{k=n}^{m} \frac{1}{k^2}\right)^{\frac{1}{2}}$$

Let $\epsilon > 0$, then $\exists N_1, N_2 \in \mathbb{N}$ such that $\forall m_1 \ge n_1 \ge N_1$ and $m_2 \ge n_2 \ge N_2$ implies

$$0 \le \sum_{k=n_1}^{m_1} a_k \le \epsilon \text{ and } 0 \le \sum_{k=n_2}^{m_2} \frac{1}{k^2} \le \epsilon$$

where $\sum \frac{1}{n^2}$ is p-series with $p = 2 > 1$.
Let $N = max\{N_1, N_2\}$, then

$$\left|\sum_{k=n}^{m} \frac{\sqrt{a_k}}{k}\right| \le \sqrt{\epsilon \cdot \epsilon} = \epsilon.$$

8

Sol 1:
Since $\sum a_n$ converges, it is bounded and hence the partial sums A_n form a bounded sequence. Since $\{b_n\}$ is monotonic and bounded, it converges to, say b.
If $\{b_n\}$ is monotonically decreasing, let $c_n = b_n - b$, then $\{c_n\}$ is monotonically decreasing and $\lim c_n = 0$. By theorem 3.42, $\sum a_n c_n$ converges. On the other hand,

$$\sum a_n b_n = \sum a_n c_n + \sum a_n b = \sum a_n c_n + b \sum a_n$$

The RHS converges from theorem 3.47 and above and thus, $\sum a_n b_n$ converges.
If $\{b_n\}$ is monotonically increasing, let $c_n = b - b_n$, then by similar reasoning as above we also conclude $\sum a_n b_n$ converges.

Sol2:
Let $s_n = \sum_{i=1}^{n} a_i$, and $s_0 = 0$, then $a_k = s_k - s_{k-1}$.
Since $\sum a_n$ converges,

$$(\forall \epsilon > 0)(\exists N \in \mathbb{N})\ m \ge n \ge N \implies |s_m - s_n| = \left|\sum_{i=n}^{m} a_i\right| < \epsilon.$$

In other words $\{s_n\} \to s$, and hence $\{s_n\}$ is bounded. We also have $\{b_n\} \to b$ since $\{b_n\}$ is monotonic and bounded. Therefore, $\{b_n s_n\} \to bs$.

Let $|M|$ be the upper bound for both $\{b_n\}$ and $\{s_n\}$. Then

$$\begin{aligned}
\left|\sum_{k=n+1}^{m} a_k b_k\right| &= \left|\sum_{k=n+1}^{m} (s_k - s_{k-1}) b_k\right| \\
&= \left|b_m s_m - b_n s_n + \sum_{k=n}^{m-1} (b_k - b_{k+1}) s_k\right| \\
&\le |b_m s_m - b_n s_n| + \left|\sum_{k=n}^{m-1} (b_k - b_{k+1}) s_k\right| \\
&\le |b_m s_m - b_n s_n| + |M| \sum_{k=n}^{m-1} |(b_k - b_{k+1})| \\
&= |b_m s_m - b_n s_n| + |M| \left|\sum_{k=n}^{m-1} (b_k - b_{k+1})\right| \\
&\text{(since } \{b_n\} \text{ is monotonic)} \\
&= |b_m s_m - b_n s_n| + |M| \, |b_n - b_m|
\end{aligned}$$

Since $\{b_n\}$, and $\{b_n s_n\}$ are convergent sequences, which means they are Cauchy. Given $\epsilon > 0$, pick $N = max\{N_1, N_2\}$ such that

$$|b_m s_m - b_n s_n| < \frac{2\epsilon}{3} \ \forall m \ge n \ge N_1.$$

$$|b_n - b_m| < \frac{\epsilon}{3|M|} \ \forall m \ge n \ge N_2.$$

Then

$$\left|\sum_{k=n+1}^{m} a_k b_k\right| < \epsilon.$$

9

(a)

$$\limsup \left|\frac{(n+1)^3}{n^3}\right| = \limsup \frac{(1+\frac{1}{n})^3}{1} = 1$$

$\implies R = 1.$

(b)

$$\limsup \left|\frac{2^{n+1} n!}{(n+1)! 2^n}\right| = \limsup \frac{2}{n+1} = 0$$

$\implies R = \infty.$

(c)

$$\limsup \left| \frac{2^{n+1}n^2}{2^n(n+1)^2} \right| = \limsup \frac{2}{(\frac{1}{n}+1)^2} = 2$$

$\implies R = \frac{1}{2}$.

(d)

$$\limsup \left| \frac{(n+1)^3 3^n}{n^3 3^{n+1}} \right| = \limsup \frac{(1+\frac{1}{n})^3}{3} = \frac{1}{3}$$

$\implies R = 3$.

10

Let $\alpha = \limsup \sqrt[n]{|a_n|}$. Suppose $R > 1$, then $\alpha < 1$. Choose β and integer N (by theorem 3.17 (b)) so that $\alpha < \beta < 1$, and

$$\sqrt[n]{|a_n|} < \beta \implies |a_n| < \beta^n < 1 \; \forall n \geq N$$

$\implies a_n = 0 \; \forall n \geq N$ and thus for $1 \leq i \leq N$, there could only be finite number of non-zero integer a_i's, which contradicts the fact that $\{a_n\}$ contains infinitely non-zero integers.

Hence, $R \leq 1$.

16(a).

$$x_n - x_{n+1} = \frac{1}{2}\left(x_n - \frac{\alpha}{x_n}\right) \implies x_n - x_{n+1} = \frac{1}{2}\left(\frac{x_n^2 - \alpha}{x_n}\right)$$

$$\implies x_n - x_{n+1} = \frac{1}{2}\left(\frac{(x_n - \sqrt{\alpha})(x_n + \sqrt{\alpha})}{x_n}\right).$$

We know $x_1 > 0$ and from the recursion formula, it is obvious that $x_2, x_3, ... > 0$. $x_1 - \sqrt{\alpha} > 0$. For $n > 1$,

$$\begin{aligned} x_n - \sqrt{\alpha} &= \frac{1}{2}\left(x_{n-1} + \frac{\alpha}{x_{n-1}}\right) - \sqrt{\alpha} \\ &= \frac{1}{2}\left(\frac{x_{n-1}^2 + \alpha - 2x_{n-1}\sqrt{\alpha}}{x_{n-1}}\right) \\ &= \frac{1}{2}\left(\frac{(x_{n-1} - \alpha)^2}{x_{n-1}}\right) \geq 0 \end{aligned}$$

If $x_{n-1} - \alpha = 0 \implies x_{n-1} = \alpha$ and $x_n = \sqrt{\alpha}$. But from the recursion formula $x_n = \frac{1}{2}\left(\alpha - \frac{\alpha}{\alpha}\right) = \frac{1}{2}(\alpha - 1) \neq \sqrt{\alpha}$. Thus $x_n - \sqrt{\alpha} > 0$.

And hence, $x_n - x_{n+1} > 0$, which means the sequence decreases monotonically and is bounded below by $\sqrt{\alpha}$.

By theorem 3.14, $\{x_n\}$ converges to some $p \in \mathbb{R}$.

$$0 < p = \lim x_n = \frac{1}{2}\left(\lim x_{n-1} + \frac{\alpha}{\lim x_{n-1}}\right) = \frac{1}{2}\left(p + \frac{\alpha}{p}\right) \implies p^2 = \alpha,$$

or $p = \sqrt{\alpha}$ since $p \geq \sqrt{\alpha} > 0$. If $p < \sqrt{\alpha}$, $\exists x_l \in B_{\sqrt{\alpha}-p}(p)\setminus\{p\}$, so $x_l < \sqrt{\alpha}$, contradicting $x_n > \sqrt{\alpha}$ for all n. Thus, $\lim x_n = \sqrt{\alpha}$.

11(abc).

(a)

Sol 1:
Suppose not, i.e. the series converges. Then $\lim \frac{a_n}{1+a_n} = \lim \frac{1}{\frac{1}{a_n}+1} = 0$.
$\implies \frac{1}{a_n} \to \infty$
$\implies a_n \to 0$.
$\implies \exists N$ such that for all $n \geq N$, $a_n < 1$. But then $1 + a_n < 2 \implies \frac{a_n}{2} < \frac{a_n}{1+a_n}$ since $a_n > 0$.
Since $\sum a_n$ diverges, then $\frac{1}{2}\sum a_n$ diverges and therefore by the comparison test $\sum \frac{a_n}{1+a_n}$ diverges, contradiction.

Sol2:
If $a_n \to \infty$ as $n \to \infty$ then $\lim \frac{a_n}{1+a_n} = \lim(1 - \frac{1}{1+a_n}) = 1 \neq 0$, so the series diverges.
If a_n remains bounded on $(0, M)$ for some $M \in \mathbb{R}_+$, then
$1 + a_n < 1 + M \implies \frac{1}{1+M} < \frac{1}{1+a_n} \implies \frac{a_n}{1+M} < \frac{a_n}{1+a_n}$
Since $\sum a_n$ diverges, then $\frac{1}{1+M}\sum a_n$ diverges and therefore by the comparison test $\sum \frac{a_n}{1+a_n}$ diverges.

(b)

$$\begin{aligned}\frac{a_{N+1}}{s_{N+1}} + ... + \frac{a_{N+k}}{s_{N+k}} &\geq 1 - \frac{s_N}{s_{N+k}} \\ &= \frac{a_{N+1}}{s_{N+k}} + ... + \frac{a_{N+k}}{s_{N+k}}\end{aligned}$$

Since $a_n > 0$, thus $s_i > s_j$ for all $i > j$ since the partial sums form an increasing sequence, which implies $\frac{a_j}{s_j} > \frac{a_j}{s_i}$. After cancelling the last term on the both hand sides, the inequality holds.
Suppose the series converges, then given $\epsilon > 0$, there exists N such that for all $N + k \geq N$, we have

$$\frac{a_{N+1}}{s_{N+1}} + ... + \frac{a_{N+k}}{s_{N+k}} < \epsilon \implies 1 - \frac{s_N}{s_{N+k}} < \epsilon$$

which is a contradiction as N is fixed and $\lim_{k\to\infty} s_{N+k} = \infty$ i.e. pick $0 < \epsilon < 1$, the latter inequality above does not hold for sufficiently large k.

(c)

Given $a_n > 0$, we have

$$\begin{aligned}
&\frac{a_n}{s_n^2} \leq \frac{1}{s_{n-1}} - \frac{1}{s_n} \\
\iff &\frac{a_n}{s_n^2} \leq \frac{s_n - s_{n-1}}{s_n s_{n-1}} \\
\iff &\frac{a_n}{s_n} \leq \frac{a_n}{s_{n-1}} \\
\iff &\frac{1}{s_n} \leq \frac{1}{s_{n-1}} \iff s_{n-1} \leq s_n.
\end{aligned}$$

The last inequality holds since $\{s_n\}$ is an increasing sequence.
Suppose the series diverges, then by comparison test, $\sum \left(\frac{1}{s_{n-1}} - \frac{1}{s_n}\right)$ also diverges. Then
$\exists\, \epsilon > 0, \forall N \in \mathbb{N},\ m \geq n \geq N \implies \left|\sum_{k=n}^{m} \left(\frac{1}{s_{k-1}} - \frac{1}{s_k}\right)\right| = \sum_{k=n}^{m} \left(\frac{1}{s_{k-1}} - \frac{1}{s_k}\right) = \frac{1}{s_{n-1}} - \frac{1}{s_m} \geq \epsilon.$
which is not true since $s_n \to \infty$, thus $\frac{1}{s_{n-1}} - \frac{1}{s_m} \to 0 - 0 = 0 < \epsilon$ as $n \to \infty$, i.e. the inequality will not hold for sufficiently large N.

Alternatively, since $s_1 = a_1$ and $\frac{a_n}{s_n^2} > 0$
$\sum_{k=1}^{n} \frac{a_k}{s_k^2} = \frac{a_1}{a_1^2} + \sum_{k=2}^{n} \frac{a_k}{s_k^2} = \frac{1}{a_1} + \sum_{k=2}^{n} \frac{a_k}{s_k^2} \leq \frac{1}{s_1} + \sum_{k=2}^{n} \left(\frac{1}{s_{k-1}} - \frac{1}{s_k}\right)$
$\implies \sum_{k=1}^{n} \frac{a_k}{s_k^2} \leq \frac{1}{s_1} + \frac{1}{s_1} - \frac{1}{s_n} = \frac{2}{s_1} - \frac{1}{s_n} < \frac{2}{s_1}. \implies$ the partial sums form a bounded sequence.
On the other hand, $\sum_{k=1}^{n+1} \frac{a_k}{s_k^2} - \sum_{k=1}^{n} \frac{a_k}{s_k^2} = \frac{a_{n+1}}{s_{n+1}^2} > 0$, the partial sums form an increasing bounded sequence $\implies$ it converges.

(d)

Take $a_n = \frac{1}{n}$, then $\sum a_n$ diverges and $\sum \frac{a_n}{1+na_n} = \sum \frac{1}{2n}$ diverges.
Take $a_n = 1$ for $n = 2^k$ where $k \in \mathbb{Z}_+$, and $a_n = \frac{1}{n^2}$ otherwise then $\sum a_n$ will be at least $\sum 1$ which diverges. However, $\frac{a_n}{1+na_n} = \frac{1}{1+2^k} < \frac{1}{2^k}$ for $n = 2^k$ where $k \in \mathbb{Z}_+$ and $\frac{a_n}{1+a_n} = \frac{1}{n^2+n} < \frac{1}{n^2}$ otherwise. By the comparison test, the two subsequences of partial sums converge (geometric series and p-series). Thus $\sum \frac{a_n}{1+na_n}$ converge.
Hence, $\sum \frac{a_n}{1+na_n}$ may converge or diverge.

$\frac{a_n}{1+n^2a_n} = \frac{1}{\frac{1}{a_n}+n^2} < \frac{1}{n^2}$ since $a_n > 0$. So $\sum \frac{a_n}{1+n^2a_n}$ converges using the compar-

ison test and the fact that $\sum \frac{1}{n^2}$ converges (p-series with $p = 2$).
Hence, $\sum \frac{a_n}{1+n^2 a_n}$ converges.

12.

(a)
Given $a_n > 0$,
$m < n \implies r_m > r_n$. In other words $r_m > r_{m+k} \forall k \in \mathbb{N}$.
Thus, $\frac{a_m}{r_{m+k}} < \frac{a_m}{r_m}$

$$\begin{aligned}\frac{a_m}{r_m} + ... + \frac{a_n}{r_n} &> \frac{a_m + ... + a_n}{r_m} \\ &= \frac{r_m - r_n}{r_m} + \frac{a_n}{r_m} \\ &> \frac{r_m - r_n}{r_m} \\ &= 1 - \frac{r_n}{r_m}.\end{aligned}$$

Suppose the series converges, then given $\epsilon > 0$, $\exists N$ such that for all $\forall n \geq m \geq N$, we have
$1 - \frac{r_n}{r_m} < \sum_{k=m}^{n} \frac{a_k}{r_k} = \left|\sum_{k=m}^{n} \frac{a_k}{r_k}\right| < \epsilon$
$\implies 1 - \frac{r_n}{r_m} < \epsilon$ for all $n \geq m \geq N'$.
$\implies r_m - r_n < r_m \epsilon$ for all $n \geq m \geq N'$.
$\implies r_n > (1 - \epsilon) r_m$ for all $n \geq m \geq N'$.
Fixing some $m \geq N'$, and since $\sum a_n$ converges, $\exists N''$ such that $\forall n \geq N''$, we have
$r_{n+1} = |\sum_{k=1}^{n} a_k - \sum a_n| < (1 - \epsilon) r_m$.
Then pick $N = max\{m, N''\}$, and for $n > N$ we have a contradiction.

(b)

$$\begin{aligned}\frac{a_n}{\sqrt{r_n}} &< 2(\sqrt{r_n} - \sqrt{r_{n+1}}) \\ \iff a_n - r_n &< r_n - 2\sqrt{r_n r_{n+1}} \\ &= \left(\sqrt{r_n} - \sqrt{r_{n+1}}\right)^2 - r_{n+1} \\ \iff a_n - (r_n - r_{n+1}) &< \left(\sqrt{r_n} - \sqrt{r_{n+1}}\right)^2 \\ \iff 0 = a_n - a_n &< \left(\sqrt{r_n} - \sqrt{r_{n+1}}\right)^2\end{aligned}$$

The inequality holds since $\sqrt{r_n} \neq \sqrt{r_{n+1}}$, because $a_n > 0$.
Let $L = \sum a_n$, then

$\sum_{k=1}^{n} \frac{a_k}{\sqrt{r_k}} < 2\sum_{k=1}^{n}(\sqrt{r_k} - \sqrt{r_{k+1}}) = 2(\sqrt{r_1} - \sqrt{r_{n+1}}) < 2\sqrt{r_1} = 2\sqrt{L}$.
Hence the partial sums of the series form a bounded sequence.
On the other hand, $\sum_{k=1}^{n+1} \frac{a_k}{\sqrt{r_k}} - \sum_{k=1}^{n} \frac{a_k}{\sqrt{r_k}} = \frac{a_{n+1}}{\sqrt{r_{n+1}}} > 0$
$\implies$ the partial sums of the series form an increasing bounded sequence
$\implies \sum \frac{a_n}{\sqrt{r_n}}$ converges.

13.
Let $A = \sum |a_n|$, and $B = \sum |b_n|$ be the two absolutely convergent sequence.

$$\begin{aligned}
C_m = \sum_{n=0}^{m} |c_m| &= \sum_{n=0}^{m} \left| \sum_{k=0}^{n} a_k b_{n-k} \right| \\
&\le \sum_{n=0}^{m} \sum_{k=0}^{n} |a_k b_{n-k}| \\
&\le \sum_{n=0}^{m} \sum_{k=0}^{n} |a_k|\,|b_{n-k}| \\
&= \sum_{k=0}^{n} |a_k| \sum_{n=0}^{m} |b_{n-k}| \\
&= \sum_{k=0}^{n} |a_k| \sum_{n=0}^{m-k} |b_n| \\
&\le \sum |a_n| \sum |b_n| \\
&\le AB
\end{aligned}$$

$\implies \{C_m\}$ is bounded and also $\{C_m\}$ increases monotonically.
$\implies \{C_m\}$ converges.

17(abc).
It is trivial to see that $x_n > 0$ for all n by the recursion formula.
<u>Claim</u>: $x_{2n} < \sqrt{\alpha}$
Since $\alpha > 1 \implies \sqrt{\alpha} > 1$. For $n = 1$, we know $x_1 > \sqrt{\alpha}$.

$$\begin{aligned}
x_2 - \sqrt{\alpha} &= \frac{\alpha + x_1 - \sqrt{\alpha} - \sqrt{\alpha}x_1}{1 + x_1} \\
&= \frac{\sqrt{\alpha}(\sqrt{\alpha} - 1) - x_1(\sqrt{\alpha} - 1)}{1 + x_1} \\
&= \frac{(\sqrt{\alpha} - 1)(\sqrt{\alpha} - x_1)}{1 + x_1} < 0
\end{aligned}$$

Suppose it is true for $n = k$ i.e. $x_{2k} < \sqrt{\alpha}$. Consider $n = k + 1$

$$x_{2k+1} - \sqrt{\alpha} = \frac{(\sqrt{\alpha} - 1)(\sqrt{\alpha} - x_{2k})}{1 + x_1} < 0$$

By induction, we are done.

<u>Claim</u>: $x_{2n-1} > \sqrt{\alpha}$.

$$\begin{aligned}
x_3 - \sqrt{\alpha} &= \frac{\alpha + x_2 - \sqrt{\alpha} - \sqrt{\alpha} x_2}{1 + x_2} \\
&= \frac{\sqrt{\alpha}(\sqrt{\alpha} - 1) - x_2(\sqrt{\alpha} - 1)}{1 + x_2} \\
&= \frac{(\sqrt{\alpha} - 1)(\sqrt{\alpha} - x_2)}{1 + x_2} > 0,
\end{aligned}$$

since $\sqrt{\alpha} - x_2 > 0$.
By similar argument as above, claim is proven by induction.
So every even terms are less than every odd terms.

(a) (b)
For $n \geq 1$

$$\begin{aligned}
x_{2n+1} - x_{2n-1} &= \frac{\alpha + x_{2n}}{1 + x_{2n}} - x_{2n-1} \\
&= \frac{\alpha + \frac{\alpha + x_{2n-1}}{1 + x_{2n-1}}}{1 + \frac{\alpha + x_{2n-1}}{1 + x_{2n-1}}} - x_{2n-1} \\
&= \frac{2\alpha + (\alpha + 1)x_{2n-1}}{1 + \alpha + 2x_{2n-1}} - x_{2n-1} \\
&= \frac{2\alpha - 2x_{2n-1}^2}{1 + \alpha + 2x_{2n-1}} \\
&= \frac{2(\alpha - x_{2n-1}^2)}{1 + \alpha + 2x_{2n-1}} < 0
\end{aligned}$$

since $x_{2n-1} > \sqrt{\alpha}$.
(c)
For all $k, l \geq 1$, this relation holds $0 < x_{2k} < \sqrt{\alpha} < x_{2l-1}$ from above.
$\{x_{2n-1}\}$ decreases monotonically and is bounded below by $\sqrt{\alpha}$.
By theorem 3.14, $\{x_{2n-1}\}$ converges to some $p \in \mathbb{R}$.

$$\begin{aligned}
p = \lim_{l \to \infty} x_{2l-1} &= \lim_{l \to \infty} \frac{2\alpha + (\alpha + 1)x_{2l-3}}{1 + \alpha + 2x_{2l-3}} \\
&= \frac{2\alpha + (\alpha + 1)p}{1 + \alpha + 2p}
\end{aligned}$$

$$\implies p^2 = \alpha$$

or $p = \sqrt{\alpha}$ since $p \geq \sqrt{\alpha} > 0$. If $p < \sqrt{\alpha}$, $\exists x_{2l-1} \in B_{\sqrt{\alpha}-p}(p)\backslash\{p\}$, so $x_{2l-1} < \sqrt{\alpha}$, contradicting $x_{2l-1} > \sqrt{\alpha}$ for all l. Thus, $\lim_{l\to\infty} x_{2l-1} = \sqrt{\alpha}$.
Similarly repeat the same argument above we conclude $\lim_{k\to\infty} x_{2k} = \sqrt{\alpha}$.
So for any $\epsilon > 0$, $\exists N_1, N_2$ such that $\forall l \geq N_1,$ and $\forall k \geq N_2$

$$\implies d(x_{2k}, \sqrt{\alpha}) < \epsilon \text{ and } d(x_{2l-1}, \sqrt{\alpha}) < \epsilon$$

Take $N = max\{N_1, N_2\}$ then for any $n \geq N$

$$d(x_n, \sqrt{\alpha}) < \epsilon.$$

19.
Let $a = \{\alpha_n\}$ where $a_n \in \{0, 2\}$ and $x(a) = \sum \frac{\alpha_n}{3^n}$.
Recall that the Cantor Set was defined as $C = \bigcap C_n$ where $C_n = \left(\frac{1}{3}C_{n-1}\right) \cup \left(\frac{1}{3}C_{n-1} + \frac{2}{3}\right)$ and $C_0 = [0, 1]$.
Observe that under normal ternary expasion in base three i.e. $a_i \in \{0, 1, 2\}$, there could be two sequences representing the same point in $[0, 1]$, namely 0.10200000... and 0.10122222... Other than these cases, ternary expansion of $x \in [0, 1]$ is unique.
Need to show the map $\{0, 2\}^{\mathbb{N}} \to C$ is bijective.
$x \in C_1 \iff \alpha_1 = 0 \vee \alpha_1 = 2$ in the ternary expasion in base three of x since it is of the form $\frac{1}{3}y + \frac{2}{3}$ for some $y \in [0, 1]$. Inductively, $x \in C_n \iff \alpha_k = 0 \vee \alpha_k = 2$ in the ternary expasion in base three of x for $1 \leq k \leq n$.
$\implies$ ternary expansions in base three where every digit is either 0 or 2 belong to C_n for every n, hence give an element of C. $\implies$ under the ternary expansion, the map is onto.
If x has two different ternary expansions , let n be the first digit they differ then either $a_k = 0\ \forall k > n$ or $a_k = 2\ \forall k > n$. One of the two ternary expansion must have $a_n = 1$, the other must have $a_n = 0$ or 2. For example, 0.20$\underline{2}$000... and 0.20$\underline{1}$222... or 0.20$\underline{1}$000... and 0.20$\underline{0}$2222... But then in either case there is only one expansion that has $a_k \neq 1\ \forall k$, thus the map is injective.

20.
Sol1:
Suppose not, then
Case 1: the set of subsequential limit is empty, which is not true by assumption.
Case 2: the set of subsequential limit contains at least one other point than p.
In other words, $\exists p_{n_k} \to p' \neq p$. Then given $\epsilon > 0$,
$\exists N_1$ such that $\forall n_i \geq N_1,\ d(p_{n_i}, p) < \frac{\epsilon}{2}$ and
$\exists N_2$ such that $\forall n_k \geq N_2,\ d(p_{n_k}, p') < \frac{\epsilon}{2}$.

Take $N = max\{N_1, N_2\}$ then $\forall n_i, n_k \geq N$ we have:

$$\begin{aligned} d(p_{n_i}, p_{n_k}) &\leq d(p_{n_i}, p) + d(p, p') + d(p_{n_k}, p') \\ &< \epsilon + d(p, p') \end{aligned}$$

Since ϵ is arbitrary and $0 \neq d(p, p')$ is some constant, we can conclude that $d(p_{n_i}, p_{n_k}) \to d(p, p')$.
But then if we let $\epsilon = \frac{d(p,p')}{2}$, then $\nexists N'$ such that $\forall m, n \geq N'$, $d(p_n, p_m) < \epsilon$, which means $\{p_n\}$ is not Cauchy, contradicting our assumption.
Sol2:
$\{p_n\}$ is Cauchy then $\forall \epsilon > 0$, $\exists N_1 \in \mathbb{N}$ such that $m, n \geq N_1 \implies d(p_n, p_m) < \frac{\epsilon}{2}$.
On the other hand, since $\{p_{n_i}\} \to p$, $\exists N_2 \in \mathbb{N}$ such that $\forall n_i \geq N_2 \implies d(p_{n_i}, p) < \frac{\epsilon}{2}$.
Take $N = max\{N_1, N_2\}$ then when $n, n_i \geq N$

$$d(p_n, p) \leq d(p_n, p_{n_i}) + d(p_{n_i}, p) < \epsilon.$$

Therefore, $p_n \to p$.

21

Since each $E_n \neq \emptyset$, define $\{p_n\}$ in X as choosing $p_n \in E_n$ for every n.
Given $\epsilon > 0$, $\lim diam\, E_n = 0 \implies \exists N \in \mathbb{N}$ such that for $n \geq N$, $diam\, E_n < \epsilon$.
But then for $m \geq n \geq N$, $p_m \in E_m \subset E_N$ and $p_n \in E_n \subset E_N$, so $d(p_m, p_n) \leq diam\, E_N < \epsilon$. Hence $\{p_n\}$ is Cauchy.
X is complete $\implies$ $\{p_n\}$ converges to some p in X. Since $E_n \supset E_{n+1}$, the sequence $\{p_n\} \in E_n$ and therefore $p \in E_n$, since E_n is closed.
Suppose $p \notin \bigcap E_n$, $\exists N' \in \mathbb{N}$ such that $p \notin E_{N'}$, $p \in E^c_{N'}$, which is open since $E_{N'}$ is closed. Then
$\exists r > 0$, such that $B_r(p) \backslash \{p\} \cap E_{N'} = \emptyset$, which means $\exists r > 0$, $\forall n \geq N' \implies d(p, p_n) \geq r$ since $p_n \in E_n \subset E_{N'}$ for all $n \geq N'$, or there could only possibly be finite number of p_n's $\in B_r(p) \backslash \{p\}$ for $n < N'$, contradicting the fact that $\{p_n\} \to p$.
So $\bigcap E_n \neq \emptyset$.
Suppose there are at least two points in $\bigcap E_n$, then $\lim diam \bigcap E_n > 0$ and the two points are also in E_n for any n. We have $\bigcap E_n \subset E_n$ for all n.
$\implies diam \bigcap E_n \leq diam\, E_n$. But $\lim diam\, E_n = 0 \implies diam \bigcap E_n = 0$, contradiction.

22

Since $X \neq \emptyset$, fix a point $x_0 \in X$. We know G_1 is dense in X, and making use of the fact that intersection of finite number of open sets is open, we have $B_1(x_0) \cap G_1$ is non-empty and open.

Therefore, we can find x_1 and $0 < r_1 < 1$ such that $E_1 = \overline{B_{r_1}(x_1)} \subset (B_1(x_0) \cap G_1)$, i.e. we can take $r_1 = \frac{1}{2} min\{r, 1\}$ where $B_{r_1}(x_1) \subset (B_1(x_0) \cap G_1)$.
Having chosen x_{n-1} and r_{n-1}, we know G_n is dense in X, $B_{r_{n-1}}(x_{n-1}) \cap G_n$ is non-empty and open. We can find x_n and $0 < r_n < \frac{1}{n}$ such that $E_n = \overline{B_{r_n}(x_n)} \subset (B_{r_{n-1}}(x_{n-1}) \cap G_n)$.
Then by induction, this process generates a sequence $\{E_n\}$ where for each n, E_n is non-empty, closed (proof below) and bounded. Also by construction $E_n \supset E_{n+1}$ and thus $diam\, E_n \leq 2r_n < \frac{2}{n}$.
$\implies \lim diam\, E_n = 0$.
By Exercise 3.21, $\bigcap G_n \supset \bigcap E_n \neq \emptyset$.
In addition, if $\bigcap E_n = \{x\}$, then $x \in B_1(x_0)$.
In fact, since x_0 is arbitrary, either $x_0 \in \bigcap G_n$ or $x_0 \notin \bigcap G_n$. If the latter is the case, given any $\epsilon > 0$, we can generate a Cauchy sequence from$\{G_n\}$ such that it converges to some x within ϵ radius of x_0 and $x \in \bigcap G_n$ with the above procedure. In other words, $x_0 \in (\bigcap G_n)'$, i.e. $\bigcap G_n$ is dense in X.

<u>Claim</u>: A closed ball is closed in a metric space X.

Let $B = \{y \in X : d(x, y) \leq r, r > 0\}$. For any $p \in X \backslash B$, we have $p \notin B$, so $d(p, x) > r$. Let $r' = d(p, x) - r > 0$. Then for any $q \in B_{r'}(p)$,

$$d(x, p) \leq d(x, q) + d(q, p) \implies d(x, q) \geq d(x, p) - d(p, q) > r$$

Hence, $B_{r'}(p) \subset X \backslash B$, and $X \backslash B$ is open $\implies$ B is closed. 14(ab).
(a)
For any $\epsilon > 0$, $\exists N_1$ such that $\forall n \geq N_1$, $|s_n - s| < \epsilon$. On the other hand, by A.P. $\exists N_2$ such that

$$|s_0 + s_1 + ... + s_{N_1 - 1} - N_1 s| < N_2 \epsilon$$

So, take $n \geq max\{N_1, N_2\}$ we have $n + 1 > N_1 \wedge n + 1 > N_2$ and

$$\begin{aligned}
|\sigma_n - s| &= \frac{|s_0 + s_1 + ... + s_n - (n+1)s|}{n+1} \\
&\leq \frac{|s_0 + s_1 + ... + s_{N_1-1} - N_1 s|}{n+1} + \frac{|s_{N_1} - s + s_{N_1+1} - s + ... + s_n - s|}{n+1} \\
&< \frac{|s_0 + s_1 + ... + s_{N_1-1} - N_1 s|}{N_2} + \frac{|s_{N_1} - s + s_{N_1+1} - s + ... + s_n - s|}{n+1} \\
&< \epsilon + \frac{n - N_1 + 1}{n+1}\epsilon \\
&= \epsilon + \left(1 - \frac{N_1}{n+1}\right)\epsilon \\
&< 2\epsilon.
\end{aligned}$$

Since ϵ is arbitrary thus $\lim_{n\to\infty} \sigma_n = s$.

(b)

Let $s_n = (-1)^n$ for $n = 0, 1, 2, ...$ The subsequence with odd n converges to -1 and the subsequence with even n converges to 1 so $\{s_n\}$ does not converge.
$\{\sigma_n\} = \{0, -\frac{1}{2}, 0, -\frac{1}{4}, 0, -\frac{1}{6}, ...\}$ and $\sigma_n \to 0$.

23.
$\{p_n\}$ and $\{q_n\}$ are Cauchy sequences so given $\epsilon > 0$, $\exists N_1, N_2$ such that for all $m \geq n \geq N = max\{N_1, N_2\}$
$\implies d(p_n, p_m) < \frac{\epsilon}{2}$ and $d(q_n, q_m) < \frac{\epsilon}{2}$
Using triangle inequality,

$$\begin{aligned} d(p_n, q_n) &\leq d(p_n, p_m) + d(p_m, q_m) + d(q_m, q_n) \\ \implies d(p_n, q_n) - d(p_m, q_m) &\leq d(p_n, p_m) + d(q_m, q_n) < \epsilon \end{aligned}$$

Similarly,

$$\begin{aligned} d(p_m, q_m) &\leq d(p_m, p_n) + d(p_n, q_n) + d(q_n, q_m) \\ \implies d(p_m, q_m) - d(p_n, q_n) &\leq d(p_m, p_n) + d(q_n, q_m) < \epsilon \end{aligned}$$

So $|d(p_m, q_m) - d(p_n, q_n)| < \epsilon$ and thus, $\{d(p_n, q_n)\}$ is Cauchy in $\mathbb{R}^1$, which is complete. Hence $\{d(p_n, q_n)\}$ converges.