Chapter 4

Sang Tran

September 12, 2024

RUDIN Chapter 4 problems 1, 2, 3, 4, 5 (first question), 6, 8, 14, 15, 16, 17, 18, 19. For 6, say f maps X to Y. Use the metric $d((x_1, y_1), (x_2, y_2)) = d_X(x_1, x_2) + d_Y(y_1, y_2)$.

1.

No. For example take
$$f(x) = \begin{cases} 0 & \text{at } x = 0 \\ 1 & \text{otherwise} \end{cases}$$
then f satisfies the condition trivially for all x but it is not continuous at 0 (take $\epsilon = \frac{1}{2}$ then $\forall \delta > 0$ and $\forall x \in \mathbb{R},\ 0 < d(x, 0) < \delta \implies d(f(x), f(0)) = 1 > \epsilon$).

2.

Sol1:
$\forall y \in f(\overline{E})$, $\exists x \in \overline{E}$ such that $f(x) = y$. If $x \in E$, then $y = f(x) \in f(E) \subset \overline{f(E)}$, hence the containment holds. If $x \in E'$, by theorem 4.2 and 4.6, f is continuous at x if and only if for all sequences $p_n \to x$ where $p_n \in E$ and $p_n \neq x$, we have $f(p_n) \to f(x)$. In other words, $y = f(x) \in f(E)' \subset \overline{f(E)}$. Thus $f(\overline{E}) \subset \overline{f(E)}$.

Sol2:
$\forall y \in f(\overline{E})$, $\exists x \in \overline{E}$ such that $f(x) = y$.
If $x \in E$, then $y = f(x) \in f(E) \subset \overline{f(E)}$.
If $x \in E'$, since f is continuous then given any $\epsilon > 0$, $\exists \delta > 0$, $\forall p \in E$, $d(x, p) < \delta \implies d(y, f(p)) < \epsilon$. There always exists some $x \neq p \in E$ within δ distance from x since $x \in E'$, which means $y \in f(E)' \subset \overline{f(E)}$.
Hence, $f(\overline{E}) \subset \overline{f(E)}$.

Consider $f : \mathbb{R} \to \mathbb{R}$ where $f(x) = \frac{1}{1+x}$. Let $E = \overline{E} = [0, \infty)$, then $f(\overline{E}) = f(E) = (0, 1]$, but $\overline{f(E)} = [0, 1]$.

3.

Since f is continuous and $Z(f) = f^{-1}(\{0\})$ where $\{0\}$ is closed, by the corollary of theorem 4.8, $Z(f)$ is closed.

4.

Since E is dense in X, $E \cup E' = X$. Thus, $f(E)$ is dense in $f(X)$ if and only if $f(E) \cup f(E)' = f(X) = f(E \cup E')$, or $\overline{f(E)} = f(\overline{E})$.
Suffice to show if $y \in f(X)$ but $y \notin f(E)$, then $y \in f(E)'$.
$y \in f(X)$ but $y \notin f(E) \implies \exists q \in X$ and $q \notin E$ such that $f(q) = y$. Since E is dense in X, $q \in E'$. Repeat the same argument as in problem 2, we have $y \in f(E)'$. Hence $f(E)$ is dense in $f(X)$.

Define $s(p) = g(p) - f(p)$ for all $p \in X$. Since g and f are continuous, z is continuous by theorem 4.9. By assumption $s(p) = 0$ for all $p \in E$ and $E \subset X$, thus $E \subset Z(s)$, where $Z(s)$ is defined as in problem 3. On the other hand, also by problem 3, we know $Z(s)$ is closed.
$\implies \overline{E} \subset Z(s)$ by theorem 2.27.
But $X = \overline{E}$ since E is dense in X.
$\implies X \subset Z(s)$.
It is obvious that $Z(s) \subset X$. Therefore $X = Z(s)$, i.e. $g(p) = f(p)$ for all $p \in X$.

5.

Following the hint, let $g(x) = 1$ on each of the segments which constitute E^c, which is open. From problem 29 of chapter 2, every open set in $\mathbb{R}$ is the union of an at most countable collection of disjoint segments. Since g is continuous on any disjoint segment (any $\delta > 0$ works in the definition of continuity of a function since $\epsilon > 0$), g is continuous on R^1 such that $g(x) = f(x)$ for all $x \in E$.

If the word "closed" is ommited, take $E = (0, 1)$ and function $f(x) = \frac{1}{x}$. Clearly f is continuous on E, we cannot extend f to a new function which is continuous on $\mathbb{R}$, since $f(0+) = \infty$. In other words, f will have a discontinuity at 0.

6. Say f maps X to Y. Use the metric $d((x_1, y_1), (x_2, y_2)) = d_X(x_1, x_2) + d_Y(y_1, y_2)$

" $\implies$ " Let $A =$ the set of points $(x, f(x))$ for $x \in E$. A is compact iff any infinite sequence in A has a subsequential limit.
Let $\{(x_{n_k}, f(x_{n_k})\}$ be an arbitrary sequence in A. Since E is compact $\{x_{n_k}\} \to x' \in E$. And since f is continuous on E, by theorem 4.2 and 4.6, $f(x_{n_k}) \to f(x')$. Then given $\epsilon > 0$, $\exists N_1$ and N_2 such that $\forall n_k \geq N_1 \implies d(x_{n_k}, x') < \epsilon$ and $\forall n_k \geq N_2 \implies d(f(x_{n_k}), f(x')) < \epsilon$.

Pick $N = max\{N_1, N_2\}$. Then $\forall n_k \geq N$, we have

$$d((x_{n_k}, f(x_{n_k})), (x', f(x'))) = d_X(x_{n_k}, x') + d_Y(f(x_{n_k}), f(x')) < 2\epsilon$$

and since ϵ is arbitrary, we have $\{(x_{n_k}, f(x_{n_k})\} \to (x', f(x')) \in A$.
Since the infinite sequence was arbitrary, A is compact.

" $\Longleftarrow$ " Suppose E and A are compact. By theorem 4.2 and theorem 4.6, assume f is not continuous, then $\exists$ a converging sequence $\{x_n\} \to x'$ in E such that $f(x_n) \not\to f(x')$.
Consider the sequence $\{(x_n, f(x_n))\}$ and since A is compact, the sequence has a subsequential limit say (x_0, y_0), or $\{(x_{n_k}, f(x_{n_k}))\} \to (x_0, y_0) \in A$.
<u>Claim</u>: $x_{n_k} \to x_0$ and $f(x_{n_k}) \to y_0$.
Given $\epsilon > 0$,
$\exists N_1$ such that for all $n_k \geq N_1$, $d((x_{n_k}, f(x_{n_k})), (x_0, y_0)) < \epsilon$.
So $d_X(x_{n_k}, x_0) < \epsilon - d_Y(f(x_{n_k}), f(y_0)) < \epsilon$.
And $d_Y(f(x_{n_k}), f(y_0)) < \epsilon - d_X(x_{n_k}, x_0) < \epsilon$.
Hence $x_{n_k} \to x_0$ and $f(x_{n_k}) \to y_0$.
Since $\{x_n\} \to x'$, and limit of a sequence is unique, $x_0 = x_1$.
So $\{(x_{n_k}, f(x_{n_k}))\} \to (x', y_0) \in A$. If $y_0 \neq f(x')$, then $(x', y_0) \notin A$, a contradiction. And hence $y_0 = f(x')$, which means $f(x_{n_k}) \to f(x')$, and similarly since a limit of a sequence is unique, $f(x_n) \to f(x')$, contradicting f is not continuous at x'.
Thus, f is continuous.

8.

Sol1:

WLOG Suppose f is not bounded above on E, then for any $M > 0$, there is an $x \in E$ such that $f(x) > M$. Then we can obtain a sequence of $\{x_n\} \in E$ such that $f(x_n) > n$.
We have $f(x_n) \to \infty$ as $n \to \infty$. Since $\{x_n\}$ is an infinite sequence in $E \subset \mathbb{R}$, and E is bounded then by theorem 3.6 (b), there exists a subsequence $\{x_{n_k}\}$ that converges. And since the sequence converges in $\mathbb{R}$, it is Cauchy.
f is uniformly continuous then for any $\epsilon > 0$, $\exists \delta > 0$ such that $\forall p, q \in E$ $d(p, q) < \delta \implies d(f(p), f(q)) < \epsilon$.
Pick such δ then $\exists N$ such that for all $n_{k_2} > n_{k_1} \geq N$ we have $d(x_{n_{k_1}}, x_{n_{k_2}}) < \delta \implies d(f(x_{n_{k_1}}), f(x_{n_{k_2}})) < \epsilon$, which does not hold if we fix n_{k_1} and let $n_{k_2} \to \infty$ then we must have $d(f(x_{n_{k_1}}), f(x_{n_{k_2}})) \to \infty > \epsilon$ as $f(x_{n_{k_2}}) \to \infty$ by construction.
Therefore, f must be bounded on E.

Sol2:

If $E = \emptyset$, there is nothing to prove. If $E \neq \emptyset$, since E is a bounded subset of $\mathbb{R}$, then let $M_1 = infE$ and $M_2 = supE$.

f is uniformly continuous on E, we can fix $\delta > 0$ such that $\forall p, q \in E$, $d(p,q) < \delta \implies d(f(p), f(q)) < 1$.
Let $N > \lceil \frac{M_2 - M_1}{\delta} \rceil$, then there are N partitions of the interval $[M_1, M_2]$ each of length $\frac{M_2-M_1}{N} < \delta$.
Consider the intervals $I_k = \left[M_1 + (k-1)\frac{M_2-M_1}{N}, M_1 + k\frac{M_2-M_1}{N}\right]$ where $k = 1, 2, ...N$. It is clear that $E \subset \bigcup_{k=1,2,...,N} I_k$, and thus $E \cap I_k$ cannot be empty for all k.
Let $x_k = I_k \cap E \neq \emptyset$ for each k, then the set $\{f(x_k)\}$ is non-empty and finite.
Consider $x \in E$, then $x \in I_k \cap E$ for some k, which means $d(x, x_k) < \delta$. Thus $d(f(x), f(x_k)) < 1$.
However,

$$|f(x)| = d(0, f(x)) \leq d(0, f(x_k)) + d(f(x_k), f(x)) < max\{d(0, f(x_k))|k = 1, 2, ..., N\} + 1$$

$\implies |f(x)| < max\{|f(x_k))|_{k=1,2,...,N}\} + 1$
$\implies f$ is bounded on E.

If E is not bounded, take $E = \mathbb{R}$ and the function $f(x) = x$. Given any $\epsilon > 0$, take $\delta = \epsilon$ then $\forall x, y \in E$, $d(x,y) < \delta \implies d(f(x), f(y) = d(x,y) < \delta = \epsilon$, which means f is uniformly continuous on E. However, f is not bounded.

14.

Sol1:
If $f(0) = 0 \vee f(1) = 1$, we are done.
Suppose $f(0) \neq 0 \wedge f(1) \neq 1$. Since $f : I \to I$, $f(0) > 0 \wedge f(1) < 1$.

Suppose not, i.e. $\forall x \in I$, $f(x) \neq x \ \forall x \in I$. But if $f(x) > x \ \forall x \in I$, then $f(1) > 1$, which is a contradiction since $f(1) < 1$. Similarly, if $f(x) < x \ \forall x \in I$, then $f(0) < 0$, which is a contradiction.

Sol2: (using theorem 4.23)
Let $h(x) = f(x) - x$, which is continuous on I. Suppose not, i.e. $h(x) \neq 0 \ \forall x \in I$. Since $f(x) \subseteq I$, $h(0) = f(0) \neq 0 \implies h(0) > 0$, and $h(1) = f(1) - 1 < 0$ (since $f(1) \neq 1$).
Now, since h is continuous on I and $h(1) < 0 < h(0)$, by theorem 4.23, $\exists x_0$ such that $h(x_0) = 0$, contradicting our assumption.

15.

Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous open mapping. Suppose f is not monotonic, then there exists $x < y < z$ such that $f(x) < f(y)$, and $f(y) > f(z)$.

Let $I = (x, z)$.
$I \subset [x, z]$, which is compact then by theorem 4.16, there exists $p, q \in I$, such that $f(q) = sup_{s \in [x,z]} f(s)$ and $f(p) = inf_{s \in [x,z]} f(s)$.

Now, if $x \neq p \vee x \neq q$, then $p \in (x, z) \vee q \in (x, z)$. In other words, $f(I)$ contains the maximum or the minimum but $f(I) = (c, d)$ is open in $\mathbb{R}$ since f is open. Clearly, if $f(I)$ contains either c or d, neither $(c, d]$ or $[c, d)$ is open. Thus we must have $x = p$ or $x = q$. Similarly, $z = p$ or $z = q$. Thus $f(x)$ and $f(z)$ must be the maximum and minimum over $[x, z]$. Since $x < y < z$,

If $f(x) < f(y) < f(z)$, then it contradicts our assumption that $f(y) > f(z)$.
If $f(z) < f(y) < f(x)$, then it contradicts our assumption that $f(x) < f(y)$.

16.

$[x]$ has a simple discontinuity at any integer since $f(x-) = [x]-1 \neq [x] = f(x+)$.

(x) has a simple discontinuity at any integer since $f(x-) = 1 \neq 0 = f(x+)$.

17.

Let f be a real function defined on (a, b).

Following the hint, let E be the set on which $f(-x) < f(x+)$. With each point $x \in E$, associate a triple (p, q, r) of rational numbers such that

(a) $f(x-) < p < f(x+)$,
(b) $a < q < t < x \implies f(t) < p$,
(c) $x < t < r < b \implies f(t) > p$.
WTS each triple is associated with at most one point of E.
Suppose not, i.e. $\exists x, y \in E$ such that $x \neq y$ and both is associated with the same triple (p, q, r).
W.L.O.G if $x < y$, then by (b) and (c), we have $x < t < y \implies f(t) > p \wedge f(t) < p$, contradiction.

Therefore, E is at most countable.

Let F be the set on which $f(-x) > f(x+)$. Repeating the same argument as above, we obtain F is at most countable.

Let G be the set on which $f(x-) = f(x+) < f(x)$. With each point $x \in G$, associate a triple (m, l, n) of rational numbers such that

(a) $f(x-) = f(x+) < l < f(x)$,
(b) $a < m < t < x \implies f(t) < l$,
(c) $x < t < n < b \implies f(t) < l$.

WTS each triple is associated with at most one point of G.
Suppose not, i.e. $\exists x, y \in E$ such that $x \neq y$ and both is associated with the same triple (m, l, n).
W.L.O.G consider $x < y$, then from (b) and (c), we have $f(x) < l \vee f(y) < l$, either of which is a contradiction. Thus, each triple is associated with at most one point in G, hence G is at most countable.

Let H be the set on which $f(x-) = f(x+) > f(x)$. Repeating the same argument as above, we obtain H is at most countable.
Therefore, $E \cup F \cup G \cup H$ is at most countable.

18.

Prove that f is continuous at every irrational point.
Let $x \in \mathbb{R} \backslash \mathbb{Q}$, and fix $\epsilon > 0$, by A.P. $\exists N \in \mathbb{N}$ such that $\frac{1}{N} < \epsilon$ and $\exists m \in \mathbb{Z}$ such that $m < x < m + 1$.
Choose $0 < \delta = min\{|x - m - \frac{k}{n}|\}$ where $k \leq n$ and $n < N$, which exists since it is the set of reduced fractions in $(0, 1)$ with denominators less than N, and it is finite.

$\forall y \in \mathbb{R}$ such that $|x - y| < \delta$,

if $y \in \mathbb{R} \backslash \mathbb{Q}$, then $|f(x) - f(y)| = 0 < \epsilon$,

else if $y \in \mathbb{Q}$, then $y \in B_\delta(x)$ such that y has a denominator $n \geq N$. Therefore, $|f(x) - f(y)| = |f(y)| = \frac{1}{n} \leq \frac{1}{N} < \epsilon$.
Since $x \in \mathbb{R} \backslash \mathbb{Q}$ is arbitrary, f is continuous at every irrational point.

Prove that f has a simple discontinuity at every rational point.
Let $x \in \mathbb{Q}$ and $x = \frac{s}{t}$ where $gcd(s, t) = 1$ and $t > 0$.
W.L.O.G. Suppose $x > 0$. Consider any sequence $\{\frac{s_n}{t_n}\} = \{p_n\} \to x$ and fix $\epsilon > 0$.
$\implies \exists N$ such that $\frac{1}{N} < \epsilon$.
Choose δ as above then $\exists N^*$ such that $\forall n \geq N^*$, then if $p_n \in \mathbb{Q}$
$\implies f(p_n) = \frac{1}{t_n} \leq \frac{1}{N} < \epsilon$ if $p_n \in \mathbb{Q}$ and $f(p_n) = 0 < \epsilon$ if $p_n \in \mathbb{R} \backslash \mathbb{Q}$.
$\implies f(x-) = f(x+) = 0 \neq \frac{1}{t} = f(x)$.

Hence, f has a simple discontinuity at every rational point.

19.