Chapter 5

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RUDIN Chapter 5 problems 2, 4, 6, 8, 9, 11, 15, 17, 22, 26.

5.2

$f'(x) < 0 \ \forall x \in (a,b) \implies f$ is differentiable and hence continuous on (a,b).
Then for any $a < x_1 < x_2 < b$, by theorem 5.10, $\exists x \in (a,b)$ such that

$$f(x_2) - f(x_1) = (x_2 - x_1)f'(x),$$

but $f'(x) > 0 \ \forall x \in (a,b)$ and $x_2 - x_1 > 0$ by assumption
$\implies f(x_2) - f(x_1) > 0 \implies f(x_2) > f(x_1)$. Hence f is strictly increasing on (a,b).

Fix $y_0 = f(x_0)$ for some $x_0 \in (a,b)$, and let $y = f(x)$ for $x \in (a,b)$.
W.T.S. g is differentiable at y_0 i.e.

$$\forall \epsilon > 0, \ \exists \delta > 0, \ \forall y \in f(a,b), \ 0 < |y - y_0| < \delta \implies \left| \frac{g(y) - g(y_0)}{y - y_0} - \frac{1}{f'(x_0)} \right| < \epsilon. \tag{$\star$}$$

Fix $\epsilon > 0$,
We know that $\lim_{x \to x_0} \frac{1}{\frac{f(x) - f(x_0)}{x - x_0}} = \frac{1}{f'(x_0)}$ so

$$\exists \delta_1 > 0, \ \forall x \in (a,b), \ 0 < |x - x_0| < \delta_1 \implies \left| \frac{1}{\frac{f(x)-f(x_0)}{x-x_0}} - \frac{1}{f'(x_0} \right| < \epsilon. \tag{1}$$

If g is continuous at y_0 (proof below), then

$$\exists \delta_2 > 0, \ \forall y \in f(a,b), \ |y - y_0| < \delta_2 \implies |g(y) - g(y_0)| = |x - x_0| < \delta_1. \tag{2}$$

Therefore, if we choose $\delta = \delta_2$, and since $\frac{g(y) - g(y_0)}{y - y_0} = \frac{1}{\frac{f(x)-f(x_0)}{x - x_0}}$, by (1) and (2), we get ($\star$).
<u>Claim</u>: g is continuous
Suppose for a contradiction that g is not continuous at y_0 then

$$\exists \epsilon_0 > 0, \ \forall \delta > 0, \ \exists y \in f(a,b) \text{ s.t. } |y - y_0| < \delta \wedge |x - x_0| \geq \epsilon_0. \tag{3}$$

Consider $\epsilon = min\{\epsilon_0, \frac{1}{2}|x_0-a|, \frac{1}{2}|x_0-b|\}$. Then let $A = [x_0-\epsilon, x_0+\epsilon] \subset (a,b)$ is compact and since f is continuous, $f(A)$ is compact. Thus $\exists$ a maximum y_M and a minimum $y_m \in f(A)$. O.T.O.H, f is strictly increasing thus $y_m = f(x_0-\epsilon)$ and $y_M = f(x_0+\epsilon)$.
If we pick $\delta = min\{|y_0 - y_m|, |y_0 - y_M|\}$ in (3), then $y \in (y_0 - \delta, y_0 + \delta) \implies y_m < y < y_M$.
$\implies \exists\, x_0 - \epsilon < x < x_0 + \epsilon$ (apply intermediate values theorem 4.23).
$\implies |x - x_0| < \epsilon \leq \epsilon_0$.
$\implies |x - x_0| < \epsilon_0$ contradicting (3).

5.4

Let $f(x) = C_0 x + \frac{C_1}{2}x^2 + ... + \frac{C_{n-1}}{n}x^n + \frac{C_n}{n+1}x^{n+1}$.
Let $g(x) = C_0 + C_1 x + ... + C_{n-1}x^{n-1} + C_n x^n$.
Then we know $f(0) = f(1) = 0$ by assumption and $f'(x) = g(x)$.
Since f is a polynomial with real coefficients, f is continuous and differentiable on $(0,1)$.
Thus, by theorem 5.10, $\exists x_0 \in (0,1)$ such that $0 = f(1) - f(0) = (1-0)f'(x_0) = g(x_0)$.
In other words, there exists some $x_0 \in (0,1)$ such that $g(x_0) = 0$, which is the required conclusion.

5.6

By (a) and (b) and since x is continuous and differentiable on $(0,\infty)$, g is continuous and differentiable on $(0,\infty)$ by theorem 4.9 and theorem 5.3.
By (d) and theorem 5.11, $f'(x) \geq 0\ \forall x \in (0,\infty)$.
Consider $0 < x_1 < x_2$,
Since f, g are continuous and differentiable on $(0,\infty)$, then by theorem 5.3 we have:

$$g'(x) = \frac{xf'(x) - f(x)}{x^2}$$

WTS $g'(x) \geq 0$ for $x > 0$, or equivalently, $xf'(x) - f(x) \geq 0$.
$\iff f'(x) \geq \frac{f(x)}{x} = \frac{f(x)-f(0)}{x-0} = f'(x_0)$ for some $x_0 \in (0,1)$, where the last equality holds by theorem 5.10.
$\iff f'(x) \geq f'(x_0)$ which holds by (d) and $0 < x_0 < x$.
Hence, $g'(x) \geq 0$ for $x > 0$.

5.8
Since f' is continuous on $[a,b]$, which is compact is $\mathbb{R}$, f' is uniformly continuous. Then fix $\epsilon > 0$, $\exists \delta > 0$, such that $\forall x, t \in [a,b]$ and $0 < |t - x| < \delta \implies |f'(t) - f'(x)| < \epsilon$.

Since f' exists on $[a,b]$, f is continuous on $[a,b]$. Apply mean value theorem on the interval $[t,x]$ or $[x,t]$, then W.L.O.G $\exists y \in (t,x)$ such that $0 < |x-y| < |t-x| < \delta$, where $f'(y) = \frac{f(t)-f(x)}{t-x}$ and thus since $0 < |x-y| < \delta$, we have $|\frac{f(t)-f(x)}{t-x} - f'(x)| < \epsilon$.

This still holds for vector-valued functions.
Define $\mathbf{f} := \mathbb{R}^1 \to \mathbb{R}^k$ where $k \geq 2$ and $\mathbf{f} = (f_1, f_2, ..., f_k)$. Similarly, since $\mathbf{f'}$ is continuous on $[a,b]$ we have $\mathbf{f'}$ is uniformly continuous.
Fix $\epsilon > 0$,

$$\exists \delta > 0,\ \forall x,t \in [a,b],\ |x-t| < \delta \implies |\mathbf{f'}(t) - \mathbf{f'}(x)| < \frac{\epsilon}{\sqrt{k}}. \tag{4}$$

Since $\mathbf{f'}$ exists on $[a,b]$, by definition,

$$f_1', f_2', ..., f_k' \text{ exists on } [a,b]. \tag{5}$$

Note that

$$\frac{\mathbf{f}(t) - \mathbf{f}(x)}{t-x} = \left(\frac{f_1(t) - f_1(x)}{t-x}, \frac{f_2(t) - f_2(x)}{t-x}, ..., \frac{f_k(t) - f_k(x)}{t-x} \right).$$

W.L.O.G. Suppose $t < x$. We apply similar technique as above for $1 \leq i \leq k$, we have

$$\frac{f_i(t) - f_i(x)}{t-x} = f_i'(\theta_i) \text{ where } \theta_i \in (t,x).$$

And , thus $0 < |x - \theta_i| < \delta$, and by (4) and (5), we have

$$|f_i'(\theta_i) - f_i'(x)| \leq |\mathbf{f'}(\theta_i) - \mathbf{f'}(x)| < \frac{\epsilon}{\sqrt{k}}.$$

Then

$$\begin{aligned}
\left| \frac{\mathbf{f}(t) - \mathbf{f}(x)}{t-x} - \mathbf{f'}(x) \right| &= \left| (f_1'(\theta_1), ..., f_k'(\theta_k)) - \mathbf{f'}(x) \right| \\
&= \sqrt{[f_1'(\theta_1) - f_1'(x)]^2 + ... + [f_k'(\theta_k) - f_k'(x)]^2} \\
&< \sqrt{k \cdot \frac{\epsilon^2}{k}} \\
&= \epsilon.
\end{aligned}$$

5.9
By definition, $f'(0) = \lim_{t \to 0^-} \frac{f(t)-f(0)}{t-0} = \lim_{t \to 0^+} \frac{f(t)-f(0)}{t}$.
If $t \to 0^-$, since f is continuous on $\mathbb{R}$, and differentiable everywhere except 0, for each t_i in the sequence $\{t_n\}$ approaching 0^-, by the mean value theorem $\exists$

an associated $x_i \in (t_i, 0)$ such that $f(t_i) - f(0) = (t_i - 0)f'(x_i)$. Observe that $x_n \to 0$ as $t_n \to 0$, therefore

$$\lim_{t \to 0^-} \frac{f(t) - f(0)}{t} = \lim_{x \to 0^-} f'(x_t) = 3.$$

Similarly, if $t \to 0^+$, we have

$$\lim_{t \to 0^+} \frac{f(t) - f(0)}{t} = \lim_{x \to 0^+} f'(x_t) = 3.$$

Hence, $f'(0)$ exists.

5.11

Following the hint, using theorem 5.13 and since$f$ is defined around x and $f''(x)$ exists

$$\begin{aligned}\lim_{h \to 0} \frac{f(x+h) + f(x-h) - 2f(x)}{h^2} &= \lim_{h \to 0} \frac{f'(x+h) + f'(x-h)}{2h} \\ &= \frac{1}{2} \lim_{h \to 0} \left(\frac{f'(x+h) - f'(x)}{h} + \frac{f'(x) - f'(x-h)}{h} \right) \\ &= \frac{1}{2}(f''(x) + f''(x)) \\ &= f''(x).\end{aligned}$$

Take $f(x) = sgn(x)$, then consider at $x = 0$, we have $\lim_{h \to 0} \frac{f(x+h)+f(x-h)-2f(x)}{h^2} = \lim_{h \to 0} \frac{f(x+h)+f(x-h)}{h^2} = 0$ but $f''(0)$ does not exist.

5.15

It is trivial to show the inequality holds for $M_0 = \infty$ or $M_2 = \infty$, thus suppose M_0, M_2 are both finite.

If $h > 0$, Taylor's theorem shows that

$$f(x + 2h) = f(x) + 2hf'(x) + \frac{(2h)^2}{2!} f''(\xi)$$

for some $\xi \in (x, x + 2h)$.

$\implies f'(x) = \frac{1}{2h}[f(x + 2h) - f(x)] + hf''(\xi)$

$\implies |f'(x)| \leq \frac{1}{2h}(M_0 + M_0) + hM_2 = \frac{M_0}{h} + hM_2$

$\implies sup(|f'(x)|) = M_1 \leq \frac{M_0}{h} + hM_2$

Let $g(h) = \frac{M_0}{h} + hM_2$ which is continuous on $(0, \infty)$.

$g'(h) = -\frac{M_0}{h^2} + M_2$ and $g'(h) = 0 \iff M_2h^2 - M_0 = 0 \implies h = \pm\sqrt{\frac{M_0}{M_2}}$.

Since $h > 0$, $h = \sqrt{\frac{M_0}{M_2}}$. For $0 < h < \sqrt{\frac{M_0}{M_2}}$, then $h^2 < \frac{M_0}{M_2}$ and thus $g'(h) = M_2 - \frac{M_0}{h^2} < 0$. Similarly, for $h > \sqrt{\frac{M_0}{M_2}}$, $g'(h) > 0$. Hence, $g(h)$ attains minimum at $h = \sqrt{\frac{M_0}{M_2}}$.
$\implies M_1 \leq inf(\frac{M_0}{h} + hM_2)$
$\implies M_1 \leq 2\sqrt{M_0 M_2}$
$\implies M_1^2 < 4M_0M_2$.
Note that if $M_0 = 0$, there is nothing to prove. If $M_2 = 0$, then f' is a some constant and f is some linear function. Since we assume $M_0 < \infty$, f must be a constant. It suffices to assume $0 < M_0 < \infty$ and $0 < M_2 < \infty$.

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To show $M_1^2 = 4M_0M_2$, following the hint, take $a = -1$.
Define: $f(x) = \begin{cases} 2x^2 - 1 & (-1 < x < 0), \\ \frac{x^2-1}{x^2+1} & (0 \leq x < \infty). \end{cases}$
For $x \in (-1, 0)$, $0 < 2x^2 < 2 \implies -1 < 2x^2 - 1 < 1 \implies |f(x)| \leq 1$. For $x \in [0, \infty)$, since $\frac{x^2-1}{x^2+1} = 1 - \frac{2}{x^2+1}$, for any $x_1 < x_2$, we have $x_1^2 + 1 < x_2^2 + 1$ $\implies 1 - \frac{2}{x_2^2+1} > 1 - \frac{2}{x_1^2+1}$ so f is strictly increasing and as $x \to \infty$, $\frac{2}{x^2+1} \to 0$.
$\implies M_0 = 1$.

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Now, we have
$f'(x) = \begin{cases} 4x & (-1 < x < 0), \\ \frac{4x}{(x^2+1)^2} & (0 \leq x < \infty) \end{cases}$
and
$f''(x) = \begin{cases} 4 & (-1 < x < 0), \\ \frac{4-12x^2}{(x^2+1)^3} & (0 \leq x < \infty) \end{cases}$
Since f'' exists, f' is continuous and we have
$f'(0-) = \lim_{x \to 0-} \frac{f(x)-f(0)}{x} = \lim_{x \to 0^-} \frac{2x^2-1+1}{x} = \lim_{x \to 0^-} 2x = 0$. Similarly, we have
$f'(0+) = \lim_{x \to 0+} \frac{f(x)-f(0)}{x} = \lim_{x \to 0^+} \frac{x^2-1+x^2+1}{x(x^2+1)} = \lim_{x \to 0^+} \frac{2x}{x^2+1} = 0$.
Since f is continuous then by problem 9, this implies $f'(0) = 0$.
O.T.O.H. we have $\frac{4x}{(x^2+1)^2} = \frac{\frac{4}{x^3}}{(1+\frac{1}{x^2})^2} \to 0$ as $x \to \infty$ or $f'(x) \to 0$ as $x \to \infty$.
So f' is continuous on $[0, \infty)$ and $f'(x) > 0 \; \forall x \in [0, \infty)$.
There must exist at least one local maximum on $[0, \infty)$. And by theorem 5.8,

$$f''(x) = 0 \iff \frac{4 - 12x^2}{(x^2+1)^3} = 0 \implies x = \frac{1}{\sqrt{3}}$$

Plug in x, we have $f'(\frac{1}{\sqrt{3}}) = \frac{\frac{4}{\sqrt{3}}}{(\frac{1}{3}+1)^2} = \frac{3\sqrt{3}}{4}$.
So $M_1 = sup|f'(x)| = max\{4, \frac{3\sqrt{3}}{4}\}$=4.

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For $x \in [0, \infty)$, we have $\frac{4-12x^2}{(x^2+1)^3} = \frac{4(1-3x^2)}{(x^2+1)^3}$ then for any $x_1 < x_2$, $\frac{4-12x_1^2}{(x_1^2+1)^3} > \frac{4-12x_2^2}{(x_2^2+1)^3} \implies f''$ is strictly decreasing.
Since $\frac{4-12x^2}{(x^2+1)^3} \to 4$ as $x \to 0$, and f' is continuous, by problem 9, $f''(0) = 4$. Therefore, $M_2 = sup|f''(x)| = max\{4, f''(0)\}$=4.
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5.17
By assumption, we can follow the hint and apply theorem 5.15 with $\alpha = 0$ and $\beta = \pm 1$. Then,

$$\begin{aligned} 0 = f(-1) &= \frac{f^{(0)}(0)}{0!}(-1-0)^0 + \frac{f^{(1)}(0)}{1!}(-1-0)^1 + \frac{f^{(2)}(0)}{2!}(-1-0)^2 + \frac{f^{(3)}(0)}{3!}(-1)^3 \\ &= \frac{f^{(2)}(0)}{2!} - \frac{f^{(3)}(s)}{3!}, \end{aligned}$$

for some $s \in (-1, 0)$. $\implies f^{(3)}(s) = 3f^{(2)}(0)$.

$$\begin{aligned} 1 = f(1) &= \frac{f^{(0)}(0)}{0!}(1-0)^0 + \frac{f^{(1)}(0)}{1!}(1-0)^1 + \frac{f^{(2)}(0)}{2!}(1-0)^2 + \frac{f^{(3)}(0)}{3!}(1)^3 \\ &= \frac{f^{(2)}(0)}{2!} + \frac{f^{(3)}(t)}{3!}, \end{aligned}$$

for some $t \in (0, 1)$.
$\implies f^{(3)}(t) = 6 - 3f^{(2)}(0)$.
So, $f^{(3)}(s) + f^{(3)}(t) = 6$. If $f^{(3)}(s) < 3$ and $f^{(3)}(t) < 3$, their sum will be < 6. Thus, either $f^{(3)}(s) \geq 3$ or $f^{(3)}(t) \geq 3$, hence the required result.

5.22

(a)
Suppose not, then $\exists x \neq y$ such that $f(x) = x \wedge f(y) = y$.
Since f is differentiable, f is continuous. Consider $[x, y]$, by theorem 5.10, $\exists t \in (x, y)$ at which
$f(y) - f(x) = (y-x)f'(t) \implies f'(t) = \frac{f(y)-f(x)}{y-x} = 1$, contradiction.

(b)

$$
\begin{aligned}
\lim_{h\to 0} \frac{(1+e^{t+h})^{-1} - (1+e^t)^{-1}}{h} &= \lim_{h\to 0} \frac{e^t - e^{t+h}}{(1+e^t)(1+e^{t+h})h} \\
&= \frac{e^t}{1+e^t} \cdot \lim_{h\to 0} \frac{-e^h}{1+e^{t+h}+he^{t+h}} \text{ by L'Hopital's Rule} \\
&= \frac{e^t}{1+e^t} \cdot \frac{-1}{1+e^t} \\
&= \frac{-e^t}{(1+e^t)^2} = \frac{-e^t}{1+2e^t+e^{2t}}.
\end{aligned}
$$

So $f'(t) = 1 - \frac{e^t}{1+2e^t+e^{2t}}$. Since $0 < \frac{e^t}{1+2e^t+e^{2t}} < 1 \implies 0 < f'(t) < 1$.
Suppose there is a fixed point x_0, then $f(x_0) = x_0$ i.e.
$x_0 + (1+e^{x_0})^{-1} = x_0$
$\implies (1+e^{x_0})^{-1} = 0$
$\implies 1 = 0$, contradiction.

(c)
Start with $x_1 \in \mathbb{R}$, define $f(x_n) = x_{n+1}$. Since f' is differentiable, Consider the sequence $\{x_n\}$. Let $n, m \in N$ and $n > m$.
Then

$$
\begin{aligned}
|x_n - x_{n-1}| &= |f(x_{n-1}) - f(x_{n-2})| \\
&= |f'(\theta)(x_{n-1} - x_{n-2})| \text{ for some } \theta \text{ between } x_{n-1} \text{ and } x_{n-2} \\
&\leq A|f(x_{n-2}) - f(x_{n-3})| \\
&\leq \ ... \\
&\leq A^{n-2}|x_2 - x_1|.
\end{aligned}
$$

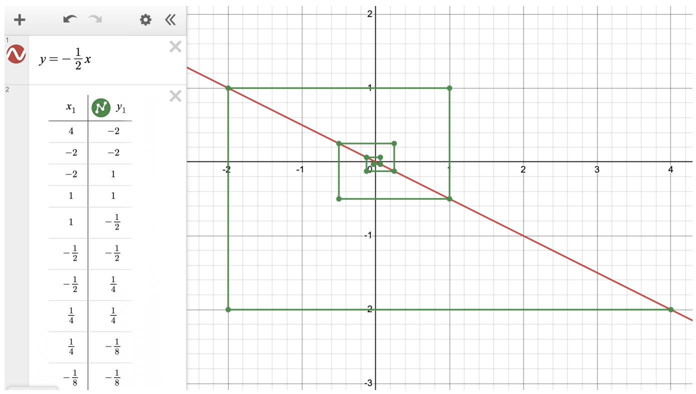
and

$$
\begin{aligned}
|x_n - x_m| &= |x_n - x_{n-1} + x_{n-1} - x_{n-2} + ... + x_{m+1} - x_m| \\
&\leq |x_n - x_{n-1}| + |x_{n-1} - x_{n-2}| + ... + |x_{m+1} - x_m| \\
&\leq A^{n-2}|x_2 - x_1| + A^{n-3}|x_2 - x_1| + ... + A^{m-1}|x_2 - x_1| \\
&= A^{m-1}(A^{n-m-1} + A + 1)|x_2 - x_1| \\
&< A^{m-1}\frac{|x_2 - x_1|}{1-A}
\end{aligned}
$$

Then given $\epsilon > 0$, $\exists N^*$ such that $\frac{1}{N^*}\frac{1-A}{|x_2-x_1|} < \epsilon$ by A.P. and we can find N large enough such that $A^N < \frac{1}{N^*}$ since $A < 1$, $A^{m-1} \to 0$ as $m \to \infty$.
Therefore, for all $n > m > N+1$, we have $|x_n - x_m| < A^{m-1}\frac{|x_2-x_1|}{1-A} < \epsilon$, and hence $\{x_n\}$ is Cauchy in $\mathbb{R}$, which is complete. Thus, $\{x_n\}$ converges to some $x \in \mathbb{R}$.

Since $x_n \to x$, we have $x = \lim_{n\to\infty} x_n = \lim_{n\to\infty} f(x_{n-1}) = f(x)$ since f is continuous.

(d)



5.26

Following the hint, if $A(x_0 - a) < 1$, and $M_0 \geq 0$ by definition of M_0. Suppose $M_0 > 0$, and from the assumption, we know

$$|f'(x)| \leq A|f(x)| < AM_0$$

for $x \in [a, b]$ and hence $M_1 \leq AM_0$.
For any $x \in [a, x_0]$, we have

$$|f(x)| = |f(x) - f(a)| = |f(x')|(x - a) \leq M_1(x_0 - a) \leq A(x_0 - a)M_0$$

for some $x' \in (a, x)$, which exists by the mean value theorem. Take the sup over $[a, x_0]$, we get

$$M_0 \leq A(x_0 - a)M_0 \implies A(x_0 - a) \geq 1$$

contradiction. Therefore, $M_0 = 0$ and $f(x) = 0$ on $[a, x_0]$.
Essentially, the above argument has shown that as long as we pick $x_0 \in (a, b]$ such that $A(x_0 - a) < 1$, we get the result we want in the restricted inverval $[a, x_0]$. For example, pick $x_0 = a + \frac{1}{2A}$. We can partition $[a, b]$ into intervals of length at most $a + \frac{1}{2A}$. If $b - a = k\left(a + \frac{1}{2A}\right)$ for some $k \in \mathbb{N}$, then just repeat the above argument k times by considering intervals $[a, x_0], ..., [x_{k-2}, b]$. If not, $\exists n \in \mathbb{N}$ such that $b - a = n\left(a + \frac{1}{2A}\right) + b - x_n$ and $b - x_n < x_0 - a < \frac{1}{A} \implies A(b - x_n) < 1$, then similarly repeat the above argument n times.