Chapter 2

Sang Tran

September 12, 2024

RUDIN Chapter 2 problems: 2, 6, 7, 8, 9, 10 (without the compactness question), 11, 12, 13, 14, 16, 17, 18, 19, 20, 21, 22, 23, 24, 29.

2.
Let S be the set of all algebraic complex numbers.
Then $S = \{s \in \mathbb{C} | \exists f \in F : f(s) = 0\} = \bigcup_{f \in F} \{s \in \mathbb{C} | f(s) = 0\}$, where F is the set of all integer coefficient polynomials. If we can show F is countable, we are done, since for any $f \in F$, the set $\{s \in \mathbb{C} | f(s) = 0\}$ is finite, or its cardinality is at most degree of f by fundamental theorem of algebra and thus the results follows from theorem 2.8.
Define:

$$h : F \longrightarrow A = \{(n, a_0, a_1, ..., a_n) \ : \ n \in \mathbb{Z}_+, \ a_i \in \mathbb{Z} \text{ for } 0 \leq i \leq n\}$$

It follows from the definition of F that h is bijective. Thus, we want to show A is countable.

We can rewrite A as

$$\begin{aligned} A &= \bigcup_{N \in \mathbb{Z}_+} \{(n, a_0, a_1, ..., a_n) \ : \ n + |a_0| + |a_1| + ... + |a_n| = N\} \\ &= \bigcup_{N \in \mathbb{Z}_+} A_N \end{aligned}$$

Consider the set A_N for any fixed N,

- $1 \leq n \leq N$
- $-N \leq a_i \leq N$ for $0 \leq i \leq n$

There are N choices for n, and for any chosen n, there are $n+1$ entires, for each of which we have $2n+1$ choices. Therefore,

$$|A_N| \leq N \cdot (N+1)^{2N+1},$$

and this implies A_N is finite and thus countable. By theorem 2.12, it follows that A is countable.

6.

Prove that E' is closed.

If E' is empty, we are done.

Suppose E' is not empty, for any $x \in E''$, by definition,

$$\forall r_1 > 0\ ,\ B_{r_1}(x)\backslash\{x\} \bigcap E' \neq \emptyset.$$

Let $p \in B_{r_1}(x)\backslash\{x\} \bigcap E'$.

$p \in B_{r_1}(x)\backslash\{x\} \Rightarrow \exists r_2 > 0$ such that $B_{r_2}(p) \subset B_{r_1}(x)\backslash\{x\}$.

$p \in E' \Rightarrow B_{r_2}(p)\backslash\{p\} \bigcap E \neq \emptyset$.

Thus,

$$\forall r_1 > 0,\ B_{r_1}(x)\backslash\{x\} \bigcap E \neq \emptyset \Rightarrow x \in E'.$$

Since x is arbitrary E' contains all its limit points, which means E' is closed.

Prove that E and $\overline{E}$ have the same limit points.

For any $x \in E'$, we have

$$\forall r_0 > 0,\ \emptyset \neq B_{r_0}(x)\backslash\{x\} \bigcap E \subseteq B_{r_0}(x)\backslash\{x\} \bigcap \overline{E},$$

$\Rightarrow x \in \overline{E}'$.

For any $x \in \overline{E}'$, we have

$$\forall r_1 > 0,\ B_{r_1}(x)\backslash\{x\} \bigcap \overline{E} \neq \emptyset.$$

Let $p \in B_{r_1}(x)\backslash\{x\} \bigcap \overline{E}$. Since $\overline{E} = E \bigcup E'$,

if $p \in E$, then $x \in E'$, and

if $p \in E'$, then $\exists r_2 > 0$ such that $B_{r_2}(p) \subset B_{r_1}(x)\backslash\{x\}$ and $B_{r_2}(p)\backslash\{p\} \bigcap E \neq \emptyset$, which means $x \in E'$.

$\Rightarrow x \in E'$.

Do E and E' always have the same limit points?

No. Take $E = \{\frac{1}{n} : n \in \mathbb{N}\}$, then $E' = \{0\}$, but $E'' = \emptyset$.

7.

(a)

For any $b \in \overline{B}_n,\ b \in B_n \vee b \in B_n'$.

If $b \in B_n$, then for some i in $1 \leq i \leq n$, $b \in A_i \subset \overline{A}_i \subset \bigcup_{i=1}^{n} \overline{A}_i$.

If $b \in B_n'$, then

$$\forall r > 0,\ (B_r(b)\backslash\{b\}) \bigcap \left(\bigcup_{i=1}^{n} A_i \right) \neq \emptyset$$

$$\Rightarrow \forall r > 0,\ (B_r(b) \backslash \{b\}) \bigcap \left(\bigcup_{i=1}^{n} \overline{A}_i \right) \neq \emptyset$$

So, $b \in \left(\bigcup_{i=1}^{n} \overline{A}_i\right)' \subset \bigcup_{i=1}^{n} \overline{A}_i$, since finite union of closed sets is closed.
$\Rightarrow \overline{B}_n \subset \bigcup_{i=1}^{n} \overline{A}_i$.

For any $a \in \bigcup_{i=1}^{n} \overline{A}_i$, $a \in \overline{A}_i$ for some i in $1 \leq i \leq n$.
If $a \in A_i$, $a \in B_n \subset \overline{B}_n$.
If $a \in A_i'$, then

$$\begin{aligned} \forall r > 0,\ \emptyset &\neq (B_r(a) \backslash \{a\}) \bigcap A_i \\ &\subset (B_r(a) \backslash \{a\}) \bigcap B_n \end{aligned}$$

$\Rightarrow a \in B_n' \subset \overline{B}_n$.
$\Rightarrow \bigcup_{i=1}^{n} \overline{A}_i \subset \overline{B}_n$.
Hence, $\overline{B}_n = \bigcup_{i=1}^{n} \overline{A}_i$.

(b)
For any $a \in \bigcup_{i=1}^{\infty} \overline{A}_i$, $a \in \overline{A}_i$ for some $i \geq 1$.
If $a \in A_i$, $a \in B \subset \overline{B}$.
If $a \in A_i'$, then

$$\begin{aligned} \forall r > 0,\ \emptyset &\neq (B_r(a) \backslash \{a\}) \bigcap A_i \\ &\subset (B_r(a) \backslash \{a\}) \bigcap B \end{aligned}$$

$\Rightarrow a \in B' \subset \overline{B}$.
$\Rightarrow \bigcup_{i=1}^{\infty} \overline{A}_i \subset \overline{B}$.
An example, that this inclusion can be proper is to take $A_i = \{\frac{1}{i}\}$, for $i \in \mathbb{N}$.
Then $A_i = \overline{A}_i$ and if $B = \bigcup_{i=1}^{\infty} A_i$, we have

$$\overline{B} = \left\{ \frac{1}{n} \middle| n \in \mathbb{N} \right\} \bigcup \{0\} \supset \left\{ \frac{1}{n} \middle| n \in \mathbb{N} \right\} = \bigcup_{i=1}^{\infty} \overline{A}_i.$$

8.
<u>Is every point of every open set $E \subset \mathbb{R}^2$ a limit point of E?</u>
Yes. For any $x \in E$, $\exists r > 0$ such that $B_r(x) \subset E$.
$\Rightarrow B_r(x) \backslash \{x\} \bigcap E \neq \emptyset$.
For any $r' \geq r > 0$, $B_r(x) \subseteq B_{r'}(x) \Rightarrow B_{r'}(x) \backslash \{x\} \bigcap E \neq \emptyset$.
For $0 < r' < r$, $B_{r'}(x) \subset B_r(x)$ and thus

$$\emptyset \neq \left(B_{r'}(x) \backslash \{x\} \bigcap B_r(x) \backslash \{x\} \right) = \left(B_{r'}(x) \backslash \{x\} \bigcap E \right),$$

$\Rightarrow x \in E'$.

Answer the same question for closed sets in $\mathbb{R}^2$.
No. Take $E = \{0\}$, then E is closed, but $E' = \emptyset$.

10. (ignore the compactness question)
Since $d(p,q) = 1 > 0 \ \forall p \neq q$ and $d(p,q) = 0$ if $p = q$, this function satisfies (a).
Since $d(p,q) = 1 = d(q,p) \ \forall p \neq q$ and $d(p,q) = 0 = d(q,p)$ if $p = q$, this function satisfies (b).
If $p = q$, $d(p,q) = 0 \leq d(r,p) + d(r,q) \ \forall r \in X$ holds trivially. If $p \neq q$, then $(r = p \wedge r \neq q) \vee (r = q \wedge r \neq p) \vee (r \neq p \wedge r \neq q)$, and each of these cases holds since either $1 \leq 1$ or $1 \leq 2$. This function satisfies (c)
$\Rightarrow$ this function is a metric.
Every subset of X is both open and closed. It is trivial for the $\emptyset$. For $S \subseteq X$, take $r = \frac{1}{2}$, then $B_r(s) \subseteq S$ for any $s \in S$ hense S is open, which implies every subset of X is open. The result follows by taking complement of S.

9.
(a)
Let $p \in E^\circ$, p is an interior point of E.

$$\Rightarrow \exists r > 0,\ B_r(p) \subset E.$$

If r is such that $B_r(p) \subset E^\circ$, then we are done.
On the other hand, suppose $q \in B_r(p) \bigcap (E^\circ)^c \neq \emptyset$, then

$$B_{r-d(p,q)}(q) \subset B_r(p) \subset E,$$

which means q is an in interior point of E since for any $x \in B_{r-d(p,q)}(q)$:

$$d(x,p) \leq d(x,q) + d(q,p) < r - d(p,q) + d(p,q) = r,$$

and $q \in B_r(p) \Rightarrow r - d(p,q) > 0$. This means q is an interior point of E, and thus $q \in E^\circ$, contradicting $q \in (E^\circ)^c$.
Hence, $B_r(p) \subset E^\circ \Rightarrow E^\circ$ is open.

(b)
($\Rightarrow$) Suppose E is open.
Let $x \in E^\circ$, then x is an interior point of E, so

$$\exists r > 0, x \in B_r(x) \subset E \Rightarrow x \in E,$$

$\Rightarrow E^\circ \subset E$.
Let $x \in E$, since E is open, x is an interior point of E, so $x \in E^\circ$.
$\Rightarrow E \subset E^\circ$.
Therefore, $E^\circ = E$.

($\Leftarrow$) follows from definition 2.18 (f): E is open if every point of E is an interior point of E.

(c)
Let $g \in G$, since G is open

$$\exists r > 0,\ B_r(g) \subset G \subset E,$$

thus $g \in E^\circ$, which means $G \subset E^\circ$.

(d) <u>Prove $(E^\circ)^c = \overline{E^c}$</u>

Let $x \in (E^\circ)^c$, then $x \notin E^\circ$, or

$$\forall r > 0, B_r(x) \not\subset E,$$

$$\Rightarrow \forall r > 0, B_r(x) \bigcap E^c \neq \emptyset.$$

$\Rightarrow x \in (E^c)' \subset \overline{E^c}$
$\Rightarrow (E^\circ)^c \subset \overline{E^c}$.

Let $x \in \overline{E^c}$.
If $x \in E^c$, then $x \notin E$, or $\forall r > 0,\ B_r(x) \not\subset E$ and therefore $x \notin E^\circ$, which means $x \in (E^\circ)^c$.
If $x \in (E^c)'$, then

$$\forall r > 0,\ B_r(x) \backslash \{x\} \bigcap E^c \neq \emptyset$$

$$\Rightarrow \forall r > 0,\ B_r(x) \not\subset E$$

thus $x \notin E^\circ$, which means $x \in (E^\circ)^c$.
$\Rightarrow \overline{E^c} \subset (E^\circ)^c$.
Thus, $(E^\circ)^c = \overline{E^c}$.

(e)
No. Let the whole space be $\mathbb{R}$ and $E = \mathbb{Q}$, then $\overline{E} = \mathbb{R}$. It is obvious that $E^\circ = \emptyset$ and $\overline{E}^\circ = \mathbb{R}$.

(f)
No. Let the whole space be $\mathbb{R}$ and $E = \mathbb{Q}$, then $\overline{E} = \mathbb{R}$ but since $E^\circ = \emptyset$, $\overline{E^\circ} = \emptyset$.

11.
d_1 is not a metric, since $d_1(1,0) > d_1(1,2) + d_1(2,0)$.

d_2 is a metric. $|x-y| = |x-z+z-y| \leq |x-z| + |z-y|$. Taking square root both side, we have

$$\sqrt{|x-y|} \leq \sqrt{|x-z|+|z-y|} \leq \sqrt{|x-z|} + \sqrt{|z-y|},$$

where the last inequality comes from taking square root of

$$\left(\sqrt{|x-z|}+\sqrt{|z-y|}\right)^2 = |x-z|+|z-y|+2\sqrt{|x-z|+|z-y|}$$
$$\geq |x-z|+|z-y|$$

and d_2 satisfies other properties of a metric trivially.

d_3 is not a metric since $d_3(-1,1)=0$.

d_4 is not since $d_4(1,1)\neq 0$.

Let $a=|x-y|, b=|x-z|, c=|z-y|$, where $a,b,c\geq 0$ and $a\leq b+c$. Then

$$\frac{b}{1+b}+\frac{c}{1+c}=\frac{b+c+2bc}{1+bc+b+c},$$

$$(1+a)(b+c+2bc)=b+c+2bc+ab+ac+2abc,$$

and

$$a(1+bc+b+c)=a+abc+ab+ac.$$

So $a(1+bc+b+c)\leq(1+a)(b+c+2bc)\Rightarrow \frac{a}{1+a}\leq\frac{b+c+2bc}{1+bc+b+c}=\frac{b}{1+b}+\frac{c}{1+c}$, which means the triangle inequality holds. The other two also holds trivially so d_5 is a metric.

A. Prove that the set of all injections from the set of natural numbers to itself is uncountable.

Let I denote such a set and suppose it were countable, then I consist of $\phi_1,\phi_2,\phi_3,....$ Construct ϕ as follows. Define $\phi:\mathbb{N}\longrightarrow\mathbb{N}$ such that

$$\phi(n)=\begin{cases} n \text{ if } \phi_n(n)\neq n \\ \text{remove } \{n\} \text{ from domain otherwise}\end{cases}$$

It is clear that ϕ is injective, or $\phi\in I$ and by construction, ϕ is different from ϕ_n at least at n, which means $\phi\notin I$, a contradiction.

12.
Let $\mathbb{R}\supset K=\{\frac{1}{n}|n\in\mathbb{N}\}\bigcup\{0\}$, and $\mathbb{V}=\bigcup_{\alpha\in I}V_\alpha$ be an open cover for K.

Claim: 0 is a limit point of K.
Let $\epsilon>0$, by A.P. $\exists N\in\mathbb{N}$ such that $N\epsilon>1$. Thus, take $N+1>N$, then $\epsilon>\frac{1}{N}>\frac{1}{N+1}\in K$. Hence, $B_\epsilon(0)\backslash\{0\}\bigcap K\neq\emptyset$.

Claim: $\frac{1}{n}$ is a not a limit point.
Given any $\frac{1}{n}$, take $r=min\{\frac{1}{n-1},\frac{1}{n+1}\}$, then $B_r(\frac{1}{n})\bigcap K=\{\frac{1}{n}\}$.

Fixing any open ball around 0 will leave only finite number of other elements of K left to cover. Since $0 \in K$, $0 \in$ some V_α, say V_{α_0}. Since V_{α_0} is open, $\exists r > 0$ such that $B_r(0) \in V_{\alpha_0}$. Then, by A.P. $\exists N \in \mathbb{N}$ such that $Nr > 1$. Thus, for $n > N$, $r > \frac{1}{N} > \frac{1}{n} \in V_{\alpha_0}$. Let V_{α_i} contain $\frac{1}{i}$ for $i \leq N$, then we have

$$K \subset \bigcup_{i=0}^{N} V_{\alpha_i}.$$

13
Take $A_k = \{\frac{1}{k} - \frac{1}{nk} : n \in \mathbb{N}\} \cap (\frac{1}{k+1}, \frac{1}{k})$ where $k = 1, 2, 3, ...$ Then

$A_1 = \{1 - \frac{1}{n} : n \in \mathbb{N}\} \cap (\frac{1}{2}, 1)$,
$A_2 = \{\frac{1}{2} - \frac{1}{2n} : n \in \mathbb{N}\} \cap (\frac{1}{3}, \frac{1}{2})$,
$A_3 = \{\frac{1}{3} - \frac{1}{3n} : n \in \mathbb{N}\} \cap (\frac{1}{4}, \frac{1}{3})$,
...

Let $A = (\bigcup_{k=1}^{\infty} A_k) \bigcup \{\frac{1}{n} : n \in \mathbb{N}\}$, then the set $\{0\} \cup \{\frac{1}{n} : n \in \mathbb{N}\}$ is the countable set of all limit points of A that is contained in A and hence A is closed. A is compact since it is closed and bounded on $[0, 1]$.

Claim: For any $x \in A$ but $x \notin \{\frac{1}{n} : n \in \mathbb{N}\}$, x is not a limit point.

If $x \in A$, then $x \in A_k$, and $\exists l \in \mathbb{N}$ such that $\frac{1}{k+1} < 1 - \frac{1}{l+1} < x < 1 - \frac{1}{l-1} < \frac{1}{k}$, so take $r = min\{|x - 1 + \frac{1}{l-1}|, |x - 1 + \frac{1}{l+1}|\}$ then $B_r(x) \backslash \{x\} \cap A_k = \emptyset$, and thus x is not a limit point.

14.
Take $\mathbb{O} = (1 - \frac{1}{2}, 1 + \frac{1}{2}) \cup (\frac{1}{2} - \frac{1}{4}, \frac{1}{2} + \frac{1}{4}) \cup ... = \bigcup_{i \in \mathbb{N}} B_{r_i}(\frac{1}{i})$, where $r_i = \frac{1}{2i}$, to be open cover of segment $(0, 1)$. $\mathbb{O}$ covers the segment since for any $0 < x < 1$, by A.P., $\exists n \in \mathbb{N}$ such that $x > \frac{1}{n}$, which means $x \in (\frac{1}{n}, 1) \subset (\frac{1}{n} - \frac{1}{2n}, 1 + \frac{1}{2}) = \bigcup_{i=1}^{n} B_{r_i}(\frac{1}{i})$.
Suppose there were finite subcover $\bigcup_{i=1}^{k} B_{r_i}(\frac{1}{i})$ of $\mathbb{O}$ that covers the segment, but

$$\bigcup_{i=1}^{k} B_{r_i}\left(\frac{1}{i}\right) = \left(\frac{1}{k} - \frac{1}{2k}, 1 + \frac{1}{2}\right) \supset \left(\frac{1}{2k}, 1\right) \not\supset (0, 1).$$

16

$E \neq \emptyset$ since $\frac{3}{2} \in E$.

To show E is closed, suffice to show E^c is open. $E^c = \{p \in \mathbb{Q} : p^2 \leq 2 \vee p^2 \geq 3\}$. Since $\sqrt{3}$ and $\sqrt{2} \notin \mathbb{Q}$ so
$E^c = \{p \in \mathbb{Q} : p^2 < 2 \vee p^2 > 3\} = \{p \in \mathbb{Q} : p^2 < 2\} \cup \{p \in \mathbb{Q} : p^2 > 3\} = E_1 \cup E_2$.
W.L.O.G., suffice to prove E_1 is open. From chapter 1, we know E_1 contains

no largest number so for any $x \in E_1$, $\exists \epsilon_1, \epsilon_2 > 0$ such that $(x+\epsilon_1)^2 < 2$ and $(x-\epsilon_1)^2 < 2$, so take $r = min\{\epsilon_1, \epsilon_2\}$ then $B_r(x) \subset E_1$. Since union of open sets is open, thus E^c is open and thus E is closed.

E is bounded. Take $p = 2 \Rightarrow 4 > 3$, and $p = -2 \Rightarrow 4 > 3$.

E is not compact, since if it is compact in $\mathbb{Q}$ then it must be compact in $\mathbb{R}$ by theorem 2.33., which is not true since E is not closed in $\mathbb{R}$ and by Heine-Borel theorem.

E is open since for any $x \in E$, $r = min\{\frac{|x|-\sqrt{2}}{2}, \frac{\sqrt{3}-|x|}{2}\}$ then $B_r(x) \subset E$.

17
Denote $E = \{0.x_1x_2x_3... : x_i \in \{4,7\}, i \in \mathbb{N}\}$

E is not countable since the set of all sequences whose elements are the digits 4 and 7, which is uncountable by theorem 2.14. Thus, E is uncountable.

E is not dense in $[0,1]$ since E is bounded below by 0.4 and above by 0.8, and 0.3 is obviously not in E or is a limit point of E.

E is closed. Need to show $p \in E'$, then $p \in E$. By contrapositive, suppose $p \notin E$. If $p \notin [0,1]$, then $p \notin E'$. So consider such $p \in [0,1]$, then there is the smallest N such that $p_N \notin \{4,7\}$. If $p_N \in \{5,6\}$, then let $\alpha = sup\{0.p_1p_2...p_{N-1}4...\}$, which exists since the set is non-empty and bounded above by $0.p_1p_2...p_{N-1}5$. On the other hand, $(p, 0.p_1p_2...p_{N-2}7) \cap E = \emptyset$. Also, $|p-\alpha| \neq 0$, since $\alpha \neq p$ (both have different value at the N^{th} decimal place. Thus, take

$$r = min\{|p-\alpha|, |p - 0.p_1p_2...p_{N-2}7|\},$$

then $B_r(p) \cap E = \emptyset$. The same algorithm can be applied to the cases where $p_N \in \{1,2,3,8,9\}$. Therefore, we have shown $p \notin E'$.

E is not perfect since $p = 0.4 \in E$ but $B_{0.01}(p)\backslash\{p\} \cap E = \emptyset$ and thus $p \notin E'$.

18
Construct the set as follows:

Let $E_0 = [\sqrt{2}, 1+\sqrt{2}]$. Since $\mathbb{Q}$ is countable, let $Q = \{r_1, r_2, ...\}$ be the enumeration of the rational numbers in E_0. Take $\epsilon_1 = \frac{\sqrt{2}}{m_1} > 0$ for some $m_1 \in \mathbb{N}$ so that $(r_1 - \epsilon_1, r_1 + \epsilon_1) \subset E_0\backslash[r_2, 1+\sqrt{2}]$.

Then let $E_1 = E_0\backslash(r_1 - \epsilon_1, r_1 + \epsilon_1)$.

Continuing in this fashion we have each E_n is disjoint union of intervals. E_{n+1} is obtained by removing segments, where each segment is centered around its associated rational number with an irrational radius in each of the intervals in E_n and additionally if r_h is the associated rational number of the segment, then the segment is the subset of $[r_{h-1}, r_{h+1}]$.

So the set we want is $A = \bigcap_{n\in\mathbb{N}} E_n$.

- A is closed since each E_n is closed, so the intersection is also closed.
- $A \neq \emptyset$ due to the corolary of finite intersection property.
- A contains endpoints of E_n $\forall n \in \mathbb{N}$, which are irrational numbers.

Suffice to prove A is perfect.

We have $A' \subset A$, since A is closed. Consider $x \in A$, and $l > 0$, WLOG choose a rational number r_k such that $x < r_k < x + l$ then $r_k \in [r_i - \epsilon_k, r_i + \epsilon_k]$ for some i. Since $x \in E$ we must have $x < r_i - \epsilon_k$, which means $r_i - \epsilon_k \in (x, x + \epsilon)$ since $x < r_i - \epsilon_k < r_k < x + l$. But $r_i - \epsilon_k \in A$, so $B_l(x) \backslash \{x\} \cap A \neq \emptyset$, or $x \in A'$. Hence, A is perfect.

19
(a)
Since A and B are disjoint closed sets, $A' \subset A$ and $B' \subset B$. Thus $\overline{A} \cap \overline{B} = \emptyset$, and this follows that $\overline{A} \cap B = \emptyset = A \cap \overline{B}$, which means A and B are separated.

(b)
Since A and B are disjoint, $A \cap B = \emptyset$. If $x \in A'$, and suppose $x \in B$, then $\exists r > 0$, $B_r(x) \subset B$ since B is open, but let $q \in B_r(x) \backslash \{x\} \cap A \neq \emptyset$, then $q \in A \cap B \neq \emptyset$, a contradiction. Thus, $\overline{A} \cap B = \emptyset$. By similar argument, we can prove $A \cap \overline{B} = \emptyset$. Thus A and B are separated.

(c)
Let $p \in X$, and $\delta > 0$.
Define $A := \{q \in X : d(p, q) < \delta\}$, and $B := \{q \in X : d(p, q) > \delta\}$.

It is obvious that A and B are disjoint and by definition A is open. To prove they are separated, it suffices to show B is open, or B^c is closed.

$B^c = \{q \in X : d(p, q) \leq \delta\}$. Let $x \in (B^c)'$, then we have

$$\forall r > 0,\ x' \in B_r(x) \backslash \{x\} \cap B^c \neq \emptyset,$$

But $d(x, p) \leq d(x, x') + d(x', p) < r + \delta$, $\forall r > 0$ which implies $d(x, p) \leq \delta$, so $x \in B^c$. Thus, B^c is closed.

(d)
Let X be a connected metric space with at least two points and suppose on the contrary that X is countable.
Fix $p \in X$ then since X contains at least two points,
Let $D = \{d(p, q) : p \neq q \wedge q \in X\} \neq \emptyset$, and D is at most countable and is a subset of $(0, \infty)$. By corolary of theorem 2.43, $(0, \infty)$ is uncountable so $\exists\ 0 < \delta \notin D$. Then if we define the sets A and B as in (c), we get $X = A \cup B$ and A and B are separated according to (c), hence X is not connected, a contradiction.

20

(a) Are closures of connected sets always connected? YES

Let X be a connected set. If $\overline{X} = \emptyset$, we are done. Suppose not, and suppose for a contradiction that $\overline{X}$ is not connected, then $\exists$ non-empty A and B that $\overline{A} \cap B = \emptyset = A \cap \overline{B}$ and $\overline{X} = A \cup B$. Since $X \subset \overline{X}$, thus $X \subset A \cup B$. Let $A_X = X \cap A$ and $B_X = X \cap B$, then $X = A_X \cup B_X$.

$A_X \neq \emptyset$, since if so, $X = B_X \subset B$ and thus $\overline{X} \subset \overline{B}$. But $\overline{B} \cap A = \emptyset$, hence $A = (A \cap \overline{X}) \subset (A \cap \overline{B}) = \emptyset$ contradicting our assunptions. Similarly, $B_X \neq \emptyset$ for the same reason.

$A_X = (X \cap A) \Rightarrow \overline{A_X} \subset \overline{A}$. So $(\overline{A_X} \cap B_X) \subset (\overline{A} \cap B_X)$. But $B_X \subset B$, therefore

$$(\overline{A_X} \cap B_X) \subset (\overline{A} \cap B) = \emptyset,$$

similarly, we can show $A_X \cap \overline{B_X} = \emptyset$ for the same reasoning. Hence, X is not connected, which is a contradiction.

Note: the converse is not true in $\mathbb{R}$. Take $(-1,1)\backslash\{0\}$.

(b) Are interiors of connected sets always connected? NO

Counter-example: Consider two closed disk in $\mathbb{R}^2$: $\overline{B_1((1,0))}$ and $\overline{B_1((-1,0))}$. Their union will make a connected set but the interior part will be two separated open balls.

Note: It is true in $\mathbb{R}$.

21

(a) Suppose $t \in \overline{A_0} \cap B_0 \neq \emptyset$, then if $t \in A_0 \cap B_0$, then $p(t) \in A \cap B$, contradicting to our assumption that $\overline{A} \cap B = \emptyset$. If $t \in A_0' \cap B_0$, then t is a limit point of A_0. Let $\epsilon > 0$, and $t' \in N_{\frac{\epsilon}{|a-b|}}(t)\backslash\{t\} \cap A_0 \neq \emptyset$ since $A \cap B = \emptyset \to a \neq b \to |a-b| > 0$, then we have

$$|p(t) - p(t')| = |-t(a-b) + t'(a-b)| = |t - t'||a-b| < \frac{\epsilon}{|a-b|}|a-b| = \epsilon,$$

and $p(t') \in A$, thus $p(t)$ is a limit point of A and $p(t) \in A' \cap B \subset \overline{A} \cap B$, contradicting again to our assumption that $\overline{A} \cap B = \emptyset$. Hence, $\overline{A_0} \cap B_0 = \emptyset$ and similarly we can also conclude $A_0 \cap \overline{B_0} = \emptyset$.

(b)

Suppose for a contrary that $\forall t_0 \in [0,1]$, $p(t_0) \in A \cup B$, then $t_0 \in A_0 \cup B_0$, or $t \in (A_0 \cap [0,1]) \cup (B_0 \cap [0,1])$. So $[0,1] \subset (A_0 \cap [0,1]) \cup (B_0 \cap [0,1])$ and the other direction is trivial so then $[0,1] = (A_0 \cap [0,1]) \cup (B_0 \cap [0,1])$. But by (a), $0 \in A_0 \cap [0,1] \neq \emptyset$ and $1 \in B_0 \cap [0,1] \neq \emptyset$ are separated since they are subsets of A_0 and B_0 respectively, which means $[0,1]$ is not connected, but by theorem 2.47, $[0,1]$ is connected, hence a contradiction. Thus, $\exists t_0 \in (0,1)$ such that $p(t_0) \notin A \cup B$.

(c)
$C \subset \mathbb{R}^k$ is convex if $p(t) \in C \ \forall t \in [0,1]$ where $p(0) = a \in C$ and $p(1) = b \in C$. By proving the contrapositive, if such C is not connected, then C is can be written as union of two non-empty separated sets, say A and B, and by the conditions given in the problem, part (a) and (b), we conclude C is not convex.

22.
Take $\mathbb{Q}^k = \{x = (x_1, x_2, ..., x_k) : x \in \mathbb{R}^k \wedge x_i \in \mathbb{Q}, 1 \leq i \leq k\}$, which is countable since $\mathbb{Q}$ is and $\mathbb{Q}^k$ is the product of countable sets. Suffice to show it is dense in $\mathbb{R}^k$. Let $y \in \mathbb{R}^k$, and $r > 0$. If $y \in \mathbb{Q}^k$, we are done. Suppose $y \notin \mathbb{Q}^k$, since $\mathbb{Q}$ is dense in $\mathbb{R}$, $\exists x_i \in \mathbb{Q}$ such that $|y_i - x_i| < \frac{r}{\sqrt{k}}$, for $1 \leq i \leq k$ and thus $x \in \mathbb{Q}^k$ and

$$|y - x| = \sqrt{\sum_{i=1}^{k} (y_i - x_i)^2} < \sqrt{k \frac{r^2}{k}} = r,$$

which implies $\forall r > 0, \exists x \in \mathbb{Q}^k$ such that $x \in B_r(y) \subset \mathbb{R}^k$, or y is a limit point of $\mathbb{Q}^k$.
Hence $\mathbb{R}^k$ is separable.

23.
Let X be a separable metric space and $X \supset D = \{x_1, x_2, ...\}$ be the countable dense subset. Following the hint, if we take the collection of open balls $B_r(x_i)$'s where $r \in \mathbb{Q}^+$, then we claim $V = \bigcup_{x_i \in D} \left(\bigcup_{r \in \mathbb{Q}^+} B_r(x_i) \right)$ to be the countable base of X.

Since given fixed $x_i \in D$, there are countably many balls in $\bigcup_{r \in \mathbb{Q}^+} B_r(x_i)$, thus V is the countable union of countable sets, which implies V is countable by the corolary of theorem 2.12. Denote $V = \{V_{x \in D}^{r \in \mathbb{Q}^+}\}$.

Let $x \in X$ and G be some open set such that $x \in G \subset X$.
$\Rightarrow \exists s > 0$ such that $B_s(x) \subset G$.
Since D is dense is X,
$\Rightarrow \exists x_j \in D$ such that $d(x, x_j) < \frac{s}{2}$.
Since $\mathbb{Q}$ is dense in $\mathbb{R}$, $\exists q \in \mathbb{Q}^+$ such that $d(x, x_j) < q < \frac{s}{2}$.

Then $x \in B_q(x_j) \subset B_s(x) \subset G$, where the containment comes from the fact that for any $x' \in B_q(x_j)$,

$$d(x', x) \leq d(x', x_j) + d(x_j, x) < q + \frac{s}{2} < s.$$

But $B_q(x_j) = V_{x_j}^q \subset V$, so $x \in V_{x_j}^q \subset G$.

24.
Fix $\delta > 0$, following the hints, suppose for a contradiction that we could obtain an infinite set $\{x_i\}_\delta$, which by assumption must have a limit point, say l. Take $r = \frac{\delta}{2}$, then $B_{\frac{\delta}{2}}(l) \backslash \{l\} \cap \{x_i\}_\delta = \emptyset$ if $l \in \{x_i\}_\delta$, by construction $d(l, x_i) \geq \delta$,

if $l \notin \{x_i\}_\delta$, it is either l is a candidate for the infinite set during the process above or $d(l, x_i) < \delta$ for at least one $x_i \in \{x_i\}_\delta$. For the former case, continue the process until the set contains l, then similar conclusion will be reached as above. For the latter, consider $B_{\frac{\delta}{2}}(l)$ then $d(x_i, l) < \frac{\delta}{2}$ for some i. But

$$d(x_j, l) \geq d(x_j, x_i) - d(x_i, l) \geq \delta - \frac{\delta}{2} = \frac{\delta}{2},$$

for all j and $i \neq j$. Then $B_{\frac{\delta}{2}}(l) \cap \{x_i\}_\delta$ is finite, contradiction.

Following the hint, take $D = \bigcup_{n=1}^{\infty} E_n$, where $n \in \mathbb{N}$ and $E_n = \{x_i\}_{\delta=\frac{1}{n}}$. Then D is countable since it is a countable union of finite sets. Suffice to show it is dense in X.
Take $x \in X$, if $x \in D$, we are done. If $x \notin D$, then $\forall n \in \mathbb{N}$, $d(x, x^*) < \frac{1}{n}$ for some x^* in E_n. Then let $r > 0$, by A.P., $\exists N \in \mathbb{N}$ such that $\frac{1}{N} < r$ and $\exists y \in E_N$ such that $d(x, y) < \frac{1}{N} < r$, which means x is a limit point D.
Hence D is dense in X.

29.
Let $U \subset \mathbb{R}$ be open, then U is bounded.
<u>Claim</u>: U can be written as

$$U = \bigcup_{i \in I} (a_i, b_i) \text{ where } (a_i, b_i) \subset \mathbb{R},\ (a_i, b_i) \cap (a_j, b_j) = \emptyset \text{ for } i \neq j.$$

Fix $x \in U$,
Let $A = \{z < x;\ [z, x] \subset U\}$, and $B = \{z > x;\ [z, x] \subset U\}$. Since U is bounded, $x_A = infA$ and $x_B = supB$ exist. Then $(x_A, x_B) \subset U$.

We can write $U = \bigcup_{x \in U} (x_A, x_B)$ since
($\subset$) $\forall x \in U$, $x \in$ some $(x_A, x_B) \subset \bigcup_{x \in U} (x_A, x_B)$.
($\supset$) $\forall x \in \bigcup_{x \in U} (x_A, x_B)$, $x \in (x_A, x_B) \subset U$.

Suffice to show

$$\forall x, y \in U \text{ and } x \neq y,\ (x_A, x_B) = (y_A, y_B) \vee (x_A, x_B) \cap (y_A, y_B) = \emptyset.$$

Suppose $w \in (x_A, x_B) \cap (y_A, y_B) \neq \emptyset$, then $x_A < w < x_B$ and $y_A < w < y_B$.
On the other hand, we know $w \in U \Rightarrow w \in (w_A, w_B)$.
W.L.O.G., if we can prove $x_A = y_A = w_A$, we are done.

If $x_A < w_A$,

Since $w \in (x_A, x_B)$, whether $x \leq w$ or $x > w$,
then $(x_A, w) \subset U$, and any $c \in (x_A, w_A) \subset (x_A, w),\ c < w$
so $w_A \neq inf\{z < x;\ [z, w] \subset U\}$, a contradiction.

If $x_A > w_A$,
Since $w \in (x_A, x_B)$, whether $x \leq w$ or $x > w$,
then , $(w_A, x_A) \subset (w_A, w) \subset U$,
and any $c \in (w_A, x_A) \subset (w_A, x),\ c < x$ so $x_A \neq inf\{z < x;\ [z, x] \subset U\}$.

$\Rightarrow x_A = w_A$.
By similar reasoning as above, we can conclude $y_A = w_A$.

Claim: U can be written as countable union.
Fix any (a_i, b_i), $\exists q_i \in \mathbb{Q}$ such that $a_i < q_i < b_i$ since $\mathbb{Q}$ is dense in $\mathbb{R}$. Then for each $i \in I$, there is a corresponding $q_i \in \mathbb{Q}$ but this set of $q_i's$ is at most countable.
Hence, U can be written as a at most countable collection of segments.

The two claims above give the desired result.