Chapter 1

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RUDIN Chapter 1 problems 1, 4, 5, 7(abc), 6, 7(defg), 8, 9, 10, 13, 14, 15, 17.

1.
Suppose $r + x$ is rational then

$$x = x + (r - r) = (x + r) - r \in \mathbb{Q}$$

since $\mathbb{Q}$ is closed under addition, which contradicts x is irrational.

Suppose rx is rational, since r is rational and $r \neq 0$ we have

$$\frac{1}{r} \in \mathbb{Q} \text{ and } x = x\frac{r}{r} = (rx)\frac{1}{r} \in \mathbb{Q}$$

since $\mathbb{Q}$ is closed under multiplication, which contradicts x is irrational.

4.
Since E is non-empty subset of an ordered set S, take $x \in E$ and thus by our assumption, we have $\alpha, \beta \in S$ where $\alpha \leq x$ and $x \leq \beta$. By transitivity property of an ordered set, we conclude $\alpha \leq \beta$.

5.
$A \neq \emptyset$, $A \subseteq \mathbb{R}$, and bounded below. Hence, $\beta = inf A$ exists.
Since $A \neq \emptyset$, $-A \neq \emptyset$. Suffice to prove $sup(-A) = -\beta$.

We know that $\forall x \in A$, $x \geq \beta$, or equivalently $\forall -x \in -A$, $-x \leq -\beta$, thus $-\beta$ is an upper bound of $-A$.
On the other hand, since $\beta = inf A$, thus $\beta + e$ for any $e > 0$ is not a lower bound of A, or in other words, $\exists x' \in A$ such that $x' < \beta + e$, but this means $\exists -x' \in -A$ such that $-x' > -\beta - e$ and $-\beta - e < -\beta$. Thus, $-\beta - e$ for any $e > 0$ is not an upper bound of $-A$.

7(abc). Fix $b > 1$ and $y > 0$,

(a)

Base case for $n = 1$ is true trivially. Suppose true for n, want to show

$$b^{n+1} - 1 \geq (n+1)(b-1)$$

We have

$$b^{n+1} - 1 = b^{n+1} - b + b - 1 = b(b^n - 1) + b - 1 \geq nb(b-1) + b - 1$$

But since $b > 1$, so

$$b^{n+1} - 1 \geq n(b-1) + b - 1 = (n+1)(b-1)$$

By induction, we're done.

(b)
Substitute b with $b^{\frac{1}{n}} > 1$ to (a) we have (b).

(c)
Since $t > 1$, $t - 1 > 0$. Then $n(t-1) > b - 1 \geq n(b^{\frac{1}{n}} - 1)$, and thus, $t > b^{\frac{1}{n}}$.

6. For (c) change the definition of B(x) to require $t < x$ (instead of $t \leq x$).

Fix $b > 1$.

(a) Let $k = mq = np$.
We have $((b^m)^{\frac{1}{n}})^{nq} = (b^m)^q = b^{mq}$ and $((b^p)^{\frac{1}{q}})^{nq} = (b^p)^n = b^{np}$. By theorem 1.21, there exists only one $y > 0$ such that $y^{nq} = b^k$. Hence, $(b^m)^{\frac{1}{n}}) = (b^p)^{\frac{1}{q}})$. Thus $b^r = (b^m)^{\frac{1}{n}}$ is well-defined, since any two representations of r yield the same value.

(b)
Let $r = \frac{m}{n}$, $s = \frac{p}{q}$ where $n, q \neq 0$. Then we have

$$b^{r+s} = b^{\frac{mq+np}{nq}} = (b^{mq+np})^{\frac{1}{nq}} = (b^{mq}b^{np})^{\frac{1}{nq}}$$

where the last equality comes from the rules for integer exponents. By the corrolary of theorem 1.21, we have

$$b^{r+s} = (b^{mq})^{\frac{1}{nq}}(b^{np})^{\frac{1}{nq}} = b^{\frac{m}{n}}b^{\frac{p}{q}} = b^r b^s$$

where the second to last equality is from (a).

(c)
Define $B(x) = \{b^t | t \in \mathbb{Q}, x \in \mathbb{R}, t < x\}$.
Since $t < r$, we have $r - t > 0$. Let $r - t = \frac{m}{n}$, $m, n > 0$.

Claim: if $b > 1$, then $b^{\frac{1}{n}} > 1$ for $n \in \mathbb{N}$.

$$0 < b - 1 = (b^{\frac{1}{n}})^n - 1^n = (b^{\frac{1}{n}} - 1)((b^{\frac{1}{n}})^{n-1} + (b^{\frac{1}{n}})^{n-2} + ... + 1)$$

then both terms must have the same sign, and the sign must be positive since if it is negative then

$$(b^{\frac{1}{n}})^{n-1} + (b^{\frac{1}{n}})^{n-2} + ... + 1 < 0$$

but each of these terms $(b^{\frac{1}{n}})^{n-i}$, for $1 \leq i \leq n$, is an integer power of a positive number (by theorem 1.21), which is positive. Hence $b^{\frac{1}{n}} > 1$.

Thus, $b^{r-t} = b^{\frac{m}{n}} = (b^m)^{\frac{1}{n}} > 1$. By theorem 1.21, we have $b^t > 0$,

$$b^{r-t} > 1 \Rightarrow b^t b^{r-t} > b^t \Rightarrow b^r > b^t.$$

Thus, b^r is an upper bound to $B(r)$.

Suffice to show next that if $y \in \mathbb{R}$ and $y < b^r$ or $y^{-1}b^r > 1$, then y is not an upper bound to $B(r)$.
Since $b^r > 0$, $y > 0$. Apply 7(c), with $t = y^{-1}b^r > 1$, and $n > \frac{b-1}{y^{-1}b^r - 1}$, then

$$b^{\frac{1}{n}} < y^{-1}b^r \Leftrightarrow y < b^{r-\frac{1}{n}} < b^r$$

where the last inequality follows from that fact that $b^{\frac{1}{n}} > 1$. So $\forall y < b^r$, for sufficiently large n, by the Archimedian property, there exists $b^t \in B(r)$ such that $y < b^t < b^r$. Therefore,

$$b^r = supB(r).$$

Hence it makes sense to define $b^x = supB(x)$, $\forall x \in \mathbb{R}$ because by the above argument the equation holds $\forall x \in \mathbb{Q}$.

(d)
For any $r \in \mathbb{Q}$ such that $r < x + y$, we can write as $r = s + t$ where $s, t \in \mathbb{Q}$ and $s < x$ and $t < y$. Since take any s such that $r - y < s < x$ ($\mathbb{Q}$ is dense in $\mathbb{R}$). Then we can take $t = r - s$, since $r - y < s < x \Rightarrow -x < -s < y - r \Rightarrow r - x < r - s < y$. And conversely, for any $s, t \in \mathbb{Q}$, such that $s < x$, and $t < y$, the sum gives a rational $t = s + t < x + y$.

Thus by definition, we can rewrite $B(x + y) = \{b^{s+r} | s, r \in \mathbb{Q} \wedge s < x, r < y\}$.
Hence $b^s b^r = b^{s+r} \leq b^{x+y} = supB(x + y)$.

So $b^s \leq \frac{b^{x+y}}{b^r}$ for fixed $r < y$. But since b^x is $sup\{b^s | s \in \mathbb{Q},\ s < x\}$. Then $b^x \leq \frac{b^{x+y}}{b^r} \Leftrightarrow b^r \leq \frac{b^{x+y}}{b^x}$. But since b^y is $sup\{b^r | r \in \mathbb{Q},\ r < y\}$, then $b^y \leq \frac{b^{x+y}}{b^x}$.
Hence, $b^x b^y \leq b^{x+y}$.

For any $z \in B(x + y)$, we have $z = b^{s+r} = b^s b^r \leq b^x b^r \leq b^x b^y$ since $b^x = sup\{b^s | s \in \mathbb{Q},\ s < x\}$ and $b^y = sup\{b^r | r \in \mathbb{Q},\ r < y\}$. So $b^{x+y} \leq b^x b^y$.
Thus, $b^{x+y} = b^x b^y$.

7(defg).

(d)
If w is such that $b^w < y$. Since $b^{-w} > 0$, then $yb^{-w} > 1$. Let $t = yb^{-w}$. By the Archimedian property, $\exists n \in \mathbb{N}$ such that $n > \frac{b-1}{t-1}$, thus we can apply part (c), and since $b^{-w} > 0$ we have

$$b^{\frac{1}{n}} < yb^{-w} \Leftrightarrow b^{w+\frac{1}{n}} = b^w b^{\frac{1}{n}} < y.$$

(e)
If w is such that $b^w > y$. Since $y > 0$, then $y^{-1}b^w > 1$. Let $t = y^{-1}b^w$. By the Archimedian property, $\exists n \in \mathbb{N}$ such that $n > \frac{b-1}{t-1}$, thus we can apply part (c), and since $b^{-\frac{1}{n}}, y > 0$ we have

$$b^{\frac{1}{n}} < y^{-1}b^w \Leftrightarrow b^{w-\frac{1}{n}} = b^w b^{-\frac{1}{n}} > y.$$

(f)
Let $A = \{w | b^w < y\}$. If $x = supA$, by the trichotomy law, either $b^x < y$, $b^x > y$, or $b^x = y$.

If $b^x < y$, then by (d), we can have for sufficiently large n that $b^{x+\frac{1}{n}} < y$, which implies $x + \frac{1}{n} \in A$. But $x + \frac{1}{n} > x = supA$, a contradiction.

If $b^x > y$, then by (e), we can have for sufficiently large n that $b^{x-\frac{1}{n}} > y$, which implies $x - \frac{1}{n} \notin A$. If we can show $b^w < y < b^{x-\frac{1}{n}} \Rightarrow w < x - \frac{1}{n}$ $\forall w \in A$, we will thus reach a contradiction since $x - \frac{1}{n} < x = supA$ is then an upper bound of A.
And hence, $b^x = y$.

Claim: if $b > 1$ and $\alpha, \beta \in \mathbb{R}$, then $b^\alpha < b^\beta \Rightarrow \alpha < \beta$.
$b^\alpha < b^\beta \Leftrightarrow b^{\alpha-\beta} < 1$ since $b^{-\beta} > 0$. If $\alpha - \beta = 0$, then $b^{\alpha-\beta} = 1$ is not greater than 1. If $\alpha - \beta > 0$, by the following claim, then $b^{\alpha-\beta} > 1$, contradiction. Thus $\alpha - \beta < 0 \Rightarrow \alpha < \beta$.

Claim: if $b > 1$ and $0 < x \in \mathbb{R}$, $b^x > 1$.
Since $\mathbb{Q}$ is dense in $\mathbb{R}$, $\exists 0 < q < x$. By the proven claim in (c), $b^q > 1$. And by definition of $B(x)$, we have $b^x \geq b^q$ $\forall q \in \mathbb{Q}$ $q < x$. Thus, $b^x > 1$.

(g)
Suppose $\exists x_1, x_2 \in \mathbb{R}$ such that $b^{x_1} = b^{x_2} = y$.
If $x_1 < x_2$, then $x_2 - x_1 > 0$. By the previous proven claim in (f) and since $b^{x_1} > 0$, we have

$$b^{x_2-x_1} > 1 \Rightarrow b^{x_2} > b^{x_1},$$

a contradiction.

Similarly, if $x_1 > x_2$, then $x_1 - x_2 > 0$. By the previous proven claim in (f) and since $b^{x_2} > 0$, we have

$$b^{x_1-x_2} > 1 \Rightarrow b^{x_1} > b^{x_2},$$

a contradiction.

8.
Suppose $\mathbb{C}$ is an ordered field by definition 1.17, we have then by proposition 1.18, if $x \neq 0$ then $x^2 > 0$. Take $i \in \mathbb{C}$, then it follows that $-1 = i^2 > 0$ and also $1 = 1^2 > 0$. But then by (i) of definition 1.17,

$$-1 + 1 > 0 + 1 \Rightarrow 0 > 1,$$

contradiction.

9.
Let $z = a + bi$, $w = c + di$, and $t = e + fi$. With the given ordering, the Trichotomy law holds since $a \neq b \vee c \neq d$ determines either $z < w$ or $z > w$, and therefore $\neg(a \neq b \vee c \neq d) \Leftrightarrow a = b \wedge c = d$ determines $z = w$. Thus it suffices to show the transitivity law holds.

If $z < w$ and $w < t$, want to show $z < t$.
Case 1: $z < w$ such that $a < c$.

- if $w < t$ such that $c < e$, then by transitivity of the reals $a < e$, which means $z < t$.
- if $w < t$ such that $c = e$ but $d < f$, then $a < e$, which means $z < t$.

Case 2: $z < w$ such that $a = c$ but $b < d$.

- if $w < t$ such that $c < e$, then $a < e$, which means $z < t$.
- if $w < t$ such that $c = e$ but $d < f$, then by transitivity of the reals $b < f$, which means $z < t$.

This ordered set does not have the least upper bound property. Counter example: take non-empty set $A = \{1 + bi | b \in \mathbb{R}\}$. This set bounded above by $2 + 0i$, but $\nexists\, supA$ since $b \in \mathbb{R}$ is unbounded.

10.

$$\begin{aligned}
z^2 &= (a + bi)^2 \\
&= a^2 - b^2 + 2abi \\
&= \frac{|w| + u}{2} - \frac{|w| - u}{2} + [(|w| - u)(|w| + u)]^{\frac{1}{2}} i \\
&= u + (|w|^2 - u^2)^{\frac{1}{2}} i \\
&= u + (u^2 + v^2 - u^2)^{\frac{1}{2}} i \\
&= u + |v| i.
\end{aligned}$$

And thus, similarly $\overline{z}^2 = u - |v|i$. Therefore,

- if $v > 0$, then $z^2 = w$.
- if $v < 0$, then $\overline{z}^2 = w$.
- if $v = 0$, then $z = \overline{z} = w$.

However, if $w = 0$, or $u = v = 0$, then $a = b = 0$, or $z = \overline{z} = 0$. This implies 0 is the unique square root of w. Thus, if $w \neq 0$, then by the above reasoning, we have z or $\overline{z}$ is the square root of w, i.e. $z^2 = w$ or $\overline{z}^2 = w$. Also by proposition 1.16 (d) of a field, we know then if z is the square root of w, $-z$ is also the square root of w, since $(-z)^2 = (-z)(-z) = zz = z^2 = w$ and similarly for $\overline{z}$.

13.
By applying theorem 1.33, we have

$$\begin{aligned} ||x| - |y||^2 &= |x|^2 + |y|^2 - 2|x||y| \\ &= |x|^2 + |y|^2 - 2|x||\overline{y}| \\ &= |x|^2 + |y|^2 - 2|x\overline{y}| \\ &\leq |x|^2 + |y|^2 - 2|Re(x\overline{y})| \\ &\leq |x|^2 + |y|^2 - 2Re(x\overline{y}) \\ &= |x - y|^2. \end{aligned}$$

From the uniqueness assertion of theorem 1.21, taking the square root on both sides of the above inequality yields the desired result.

14.

$$\begin{aligned} |1 + z|^2 + |1 - z|^2 &= (1 + z)(1 + \overline{z}) + (1 - z)(1 - \overline{z}) \\ &= 1 + z\overline{z} + 1 + z\overline{z} \\ &= 2 + 2(z\overline{z}) = 4. \end{aligned}$$

15.
In the proof, the equality holds when either $B = 0$ or $Ba_j - Cb_j = 0$. If $B \neq 0$, then $B > 0$, and $Ba_j = Cb_j$ or $a_j = \frac{C}{B}b_j$ for $1 \leq j \leq n$, the vector with entries $a_j's$ and the vector with entries $b_j's$ are not linearly independent.

17.

$$\begin{aligned} |x + y|^2 + |x - y|^2 &= (x + y) \cdot (x + y) + (x - y) \cdot (x - y) \\ &= |x|^2 + 2xy + |y|^2 + |x|^2 - 2xy + |y|^2 \\ &= 2|x|^2 + 2|y|^2. \end{aligned}$$

Geometrically, this means the sum of the squared norms of the diagonals of a parallelogram is the sum of all the squared norms of the sides.