NOISE ENHANCED DETECTION IN RESTRICTED NEYMAN-PEARSON FRAMEWORK

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ABSTRACT

Noise enhanced detection is studied for binary composite hypothesis-testing problems in the presence of prior information uncertainty. The restricted Neyman-Pearson (NP) framework is considered, and a formulation is obtained for the optimal additive noise that maximizes the average detection probability under constraints on worst-case detection and false-alarm probabilities. In addition, sufficient conditions are provided to specify when the use of additive noise can or cannot improve performance of a given detector according to the restricted NP criterion. A numerical example is presented to illustrate the improvements obtained via additive noise.

***Index Terms*–** Binary hypothesis-testing, noise enhanced detection, Neyman-Pearson, spectrum sensing.

1. INTRODUCTION AND MOTIVATION

In recent studies, performance improvements that can be obtained via *noise* have been investigated for various problems ([1] and references therein). Although increasing noise levels or injecting additive noise to a system commonly results in degraded performance, it can also lead to performance enhancements in some cases. Enhancements obtained via noise can, for example, be in the form of increased signal-to-noise ratio (SNR), mutual information or detection probability, or in the form of reduced average probability of error [1]-[9].

In hypothesis-testing (detection) problems, additive noise can be employed in order to improve performance of a suboptimal detector according to Bayesian, minimax, or Neyman-Pearson (NP) criterion. In [4], the Bayesian criterion is considered under uniform cost assignment, and it is proved that the optimal noise that minimizes the probability of decision error has a constant value. The study in [6] investigates optimal additive noise for suboptimal variable detectors in the Bayesian and minimax frameworks based on the results in [4] and [8]. In [5], noise enhanced detection is studied in the restricted Bayesian framework, which generalizes the Bayesian and minimax criteria and cover them as special cases.

In the NP framework, additive noise can be used to increase detection probability of a suboptimal detector under a constraint on false-alarm probability [7]-[10]. In [7], an example is presented to illustrate the improvements in detection probability via additive noise for the problem of detecting a constant signal in Gaussian mixture noise. A theoretical framework is established in [8] and sufficient conditions are presented for the improvability and nonimprovability of detection probability via additive noise. It is shown that optimal additive noise can be represented by a randomization

between at most two different signal levels. An optimization theoretic framework is provided in [9] for the same problem, which also proves the two mass point structure of the optimal additive noise probability distribution.

Noise enhanced detection is studied also for *composite* hypothesis-testing problems, in which there can be multiple possible distributions, hence, multiple parameter values, under each hypothesis [11]. Such problems are encountered in various cases such as noncoherent communications receivers, radar systems, and spectrum sensing in cognitive radio networks [11]-[13]. Noise enhanced detection is investigated for composite hypothesis-testing problems according to the Bayesian and NP criteria in [5, 14, 15]. However, no studies have considered the noise enhanced detection problem in the restricted NP framework, which focuses on composite hypothesis-testing problems in the presence of uncertainty in the prior probability distribution under the alternative hypothesis. In the restricted NP framework, the aim is to maximize the average detection probability under the constraints on the worst-case detection and false-alarm probabilities [16, 17].

In this paper, noise enhanced detection is studied for composite hypothesis-testing problems according to the restricted NP criterion. A formulation is obtained for calculating the optimal additive noise in the restricted NP framework. In addition, sufficient conditions are provided to specify when the use of additive noise can or cannot improve performance of a given detector according to the restricted NP criterion. Also, a numerical example is presented to illustrate the improvements obtained via additive noise. Since this study considers a generic composite hypothesis-testing problem with prior distribution uncertainty, it generalizes the previous studies in the literature and covers them as special cases [8, 9, 15].

The results in this study can also have some implications for binary detection problems encountered in wireless communications systems. An important example is the spectrum sensing problem in cognitive radio systems [18], where the aim is to detect the presence of a primary user (cf. (25)). This problem can be formulated as a binary composite hypothesis-testing problem as stated in [13]. In practice, there exists prior information on the unknown parameters of the primary user; however, this information is never perfect and should be modeled to include certain uncertainty. Therefore, the restricted NP framework is well-suited for this problem as it takes prior information uncertainty into account. Hence, investigation of noise enhancements according to the restricted NP criterion carries certain importance for spectrum sensing problems.

2. NOISE ENHANCED DETECTION ACCORDING TO RESTRICTED NP CRITERION

Consider the following binary composite hypothesis-testing problem:

$$\mathcal{H}_0 : p_\theta^{\mathbf{X}}(\mathbf{x}) , \ \theta \in \Lambda_0 , \quad \mathcal{H}_1 : p_\theta^{\mathbf{X}}(\mathbf{x}) , \ \theta \in \Lambda_1 \quad (1)$$

where $p_\theta^{\mathbf{X}}(\cdot)$ represents the probability density function (PDF) of observation $\mathbf{X}$ for a given value of parameter, $\Theta = \theta$, and Λ_i is the set of possible parameter values under hypothesis $\mathcal{H}_i$ for $i = 0, 1$ [11]. Parameter sets Λ_0 and Λ_1 are disjoint, and their union forms the parameter space, $\Lambda = \Lambda_0 \cup \Lambda_1$. The observation (measurement), $\mathbf{x}$, is a vector with K components; i.e., $\mathbf{x} \in \mathbb{R}^K$. The probability distributions of parameter θ under $\mathcal{H}_0$ and $\mathcal{H}_1$ are represented by $w_0(\theta)$ and $w_1(\theta)$, respectively, which are known *with some uncertainty*. For instance, these distributions can be obtained as PDF estimates based on previous decisions (experience). Then, uncertainty is related to estimation errors, and higher amount of uncertainty is observed as the estimation error increases [17].

At the receiver, observation $\mathbf{x}$ is processed by a detector (decision rule) in order to decide between the two hypotheses. For the theoretical analysis, a generic detector is considered at the receiver, which is expressed as

$$\phi(\mathbf{x}) = i , \ \text{if} \ \mathbf{x} \in \Gamma_i \quad (2)$$

for $i = 0, 1$, where Γ_0 and Γ_1 are the decision regions for $\mathcal{H}_0$ and $\mathcal{H}_1$, respectively, and form a partition of the observation space Γ.

In some situations, addition of independent "noise" to observations can improve the detection performance of a suboptimal detector [1], [5]-[9]. By adding noise $\mathbf{n}$ to original observation $\mathbf{x}$, the noise modified observation is formed as $\mathbf{y} = \mathbf{x} + \mathbf{n}$, where $\mathbf{n}$ has a PDF denoted by $p_{\mathbf{N}}(\cdot)$. As in [5, 8, 9], it is assumed that the detector in (2) is fixed, and that the only means for enhancing the performance of the detector is to optimize the additive noise $\mathbf{n}$. Then, the aim of this study is to find the optimal $p_{\mathbf{N}}(\cdot)$ according to the restricted NP criterion [16, 17]. In other words, the optimal additive noise that maximizes the average detection probability under constraints on the worst-case detection and false-alarm probabilities are sought. This problem can be formulated as follows:

$$\max_{p_{\mathbf{N}}(\cdot)} \int_{\Lambda_1} P_D^{\mathbf{y}}(\phi; \theta) \, w_1(\theta) \, d\theta$$
$$\text{subject to } P_D^{\mathbf{y}}(\phi; \theta) \geq \beta , \ \forall \theta \in \Lambda_1$$
$$P_F^{\mathbf{y}}(\phi; \theta) \leq \alpha , \ \forall \theta \in \Lambda_0 \quad (3)$$

where α is the false-alarm constraint, β is the lower limit on the worst-case detection probability, and $P_D^{\mathbf{y}}(\phi; \theta)$ and $P_F^{\mathbf{y}}(\phi; \theta)$ denote, respectively, the detection and false-alarm probabilities of detector ϕ based on the noise modified observation $\mathbf{y}$ for a given parameter value θ. It is noted that $\int_{\Lambda_1} P_D^{\mathbf{y}}(\phi; \theta) \, w_1(\theta) \, d\theta = \mathrm{E}\{P_D^{\mathbf{y}}(\phi; \Theta)\} \triangleq P_D^{\mathbf{y}}(\phi)$ is the average detection probability, which is calculated based on the prior distribution $w_1(\theta)$.

In order to provide an alternative formulation of (3), $P_D^{\mathbf{y}}(\phi; \theta)$ and $P_F^{\mathbf{y}}(\phi; \theta)$ can be expressed as

$$P_D^{\mathbf{y}}(\phi; \theta) = \mathrm{E}\{\phi(\mathbf{Y}) \mid \Theta = \theta\} = \int_\Gamma \phi(\mathbf{y}) \, p_\theta^{\mathbf{Y}}(\mathbf{y}) \, d\mathbf{y} \quad (4)$$

for $\theta \in \Lambda_1$, and

$$P_F^{\mathbf{y}}(\phi; \theta) = \mathrm{E}\{\phi(\mathbf{Y}) \mid \Theta = \theta\} = \int_\Gamma \phi(\mathbf{y}) \, p_\theta^{\mathbf{Y}}(\mathbf{y}) \, d\mathbf{y} \quad (5)$$

for $\theta \in \Lambda_0$, where $p_\theta^{\mathbf{Y}}(\cdot)$ is the PDF of the noise modified observation for a given value of $\Theta = \theta$. Then, $P_D^{\mathbf{y}}(\phi; \theta)$ in (4) can further be manipulated as follows:[1]

$$P_D^{\mathbf{y}}(\phi; \theta) = \int_\Gamma \int_{\mathbb{R}^K} \phi(\mathbf{y}) \, p_\theta^{\mathbf{X}}(\mathbf{y} - \mathbf{n}) \, p_{\mathbf{N}}(\mathbf{n}) \, d\mathbf{n} \, d\mathbf{y} \quad (6)$$

$$= \int_{\mathbb{R}^K} p_{\mathbf{N}}(\mathbf{n}) \left[\int_\Gamma \phi(\mathbf{y}) p_\theta^{\mathbf{X}}(\mathbf{y} - \mathbf{n}) \, d\mathbf{y} \right] d\mathbf{n} \quad (7)$$

$$= \int_{\mathbb{R}^K} p_{\mathbf{N}}(\mathbf{n}) \, F_\theta(\mathbf{n}) \, d\mathbf{n} \quad (8)$$

$$= \mathrm{E}\{F_\theta(\mathbf{N})\} \quad (9)$$

where
$$F_\theta(\mathbf{n}) \triangleq \int_\Gamma \phi(\mathbf{y}) \, p_\theta^{\mathbf{X}}(\mathbf{y} - \mathbf{n}) \, d\mathbf{y} . \quad (10)$$

Note that $F_\theta(\mathbf{n})$ defines the detection probability given $\theta \in \Lambda_1$ for a constant value of additive noise, $\mathbf{N} = \mathbf{n}$. Therefore, for $\mathbf{n} = \mathbf{0}$, $F_\theta(\mathbf{0}) = P_D^{\mathbf{x}}(\phi; \theta)$ is obtained; that is, $F_\theta(\mathbf{0})$ is equal to the detection probability of the detector given $\theta \in \Lambda_1$ for the original observation $\mathbf{x}$.

Via similar manipulations, $P_F^{\mathbf{y}}(\phi; \theta)$ in (5) is expressed as

$$P_F^{\mathbf{y}}(\phi; \theta) = \mathrm{E}\{G_\theta(\mathbf{N})\} \quad (11)$$

where
$$G_\theta(\mathbf{n}) \triangleq \int_\Gamma \phi(\mathbf{y}) \, p_\theta^{\mathbf{X}}(\mathbf{y} - \mathbf{n}) \, d\mathbf{y} . \quad (12)$$

Note that $G_\theta(\mathbf{n})$ defines the false alarm probability for given $\theta \in \Lambda_0$ for a constant value of additive noise, $\mathbf{N} = \mathbf{n}$. Therefore, for $\mathbf{n} = \mathbf{0}$, $G_\theta(\mathbf{0}) = P_F^{\mathbf{x}}(\phi; \theta)$ is obtained; that is, $G_\theta(\mathbf{0})$ is equal to the false alarm probability of the detector given $\theta \in \Lambda_0$ for the original observation $\mathbf{x}$.

From (9) and (11), the optimization problem in (3) can be formulated as follows:

$$\max_{p_{\mathbf{N}}(\cdot)} \int_{\Lambda_1} \mathrm{E}\{F_\theta(\mathbf{N})\} w_1(\theta) \, d\theta$$
$$\text{subject to } \min_{\theta \in \Lambda_1} \mathrm{E}\{F_\theta(\mathbf{N})\} \geq \beta$$
$$\max_{\theta \in \Lambda_0} \mathrm{E}\{G_\theta(\mathbf{N})\} \leq \alpha . \quad (13)$$

Defining a new function $F(\mathbf{n})$ as

$$F(\mathbf{n}) \triangleq \int_{\Lambda_1} F_\theta(\mathbf{n}) w_1(\theta) \, d\theta , \quad (14)$$

the optimization problem in (13) can be reformulated in the following more compact form:

$$\max_{p_{\mathbf{N}}(\cdot)} \mathrm{E}\{F(\mathbf{N})\} ,$$
$$\text{subject to } \min_{\theta \in \Lambda_1} \mathrm{E}\{F_\theta(\mathbf{N})\} \geq \beta$$
$$\max_{\theta \in \Lambda_0} \mathrm{E}\{G_\theta(\mathbf{N})\} \leq \alpha . \quad (15)$$

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[1]Note that the independence of $\mathbf{X}$ and $\mathbf{N}$ is used to obtain (6) from (4).

From (10) and (14), it is noted that $F(\mathbf{0}) = P_D^{\mathbf{x}}(\phi)$. Namely, $F(\mathbf{0})$ is equal to the average detection probability for the original observation $\mathbf{x}$; that is, the average detection probability in the absence of additive noise.

The exact solution of the optimization problem in (15) is quite difficult to obtain in general since a search over all possible additive noise PDFs is required. Therefore, an approximate solution can be obtained based on the Parzen window density estimation technique [5, 19]. Namely, the additive noise PDF can be parameterized as

$$p_{\mathbf{N}}(\mathbf{n}) \approx \sum_{l=1}^{L} \mu_l \, \varphi_l(\mathbf{n}) \qquad (16)$$

where $\mu_l \geq 0$, $\sum_{l=1}^{L} \mu_l = 1$, and $\varphi_l(\cdot)$ is a window function that satisfies $\varphi_l(\mathbf{x}) \geq 0 \; \forall \mathbf{x}$ and $\int \varphi_l(\mathbf{x})d\mathbf{x} = 1$, for $l = 1, \ldots, L$. A common window function is the Gaussian window, for which $\varphi_l(\mathbf{n})$ corresponds to the PDF of a Gaussian random vector with a certain mean vector and a covariance matrix. In that case, the optimization problem in (15) can be solved over a number of parameters instead of PDFs. Even in that case, the problem is nonconvex in general. Therefore, global optimization algorithms such as particle swarm optimization (PSO) need to be employed [5, 20].

3. IMPROVABILITY AND NONIMPROVABILITY CONDITIONS

Since it is quite complex to obtain a solution of the optimization problem in (15), it is useful to determine, without solving the optimization problem, whether additive noise can improve the performance of the original system. In the restricted NP framework, a detector is called *improvable*, if there exists a noise PDF such that $\mathrm{E}\{F(\mathbf{N})\} > F(\mathbf{0}) = P_D^{\mathbf{x}}(\phi)$, $\min_{\theta \in \Lambda_1} P_D^{\mathbf{y}}(\phi; \theta) = \min_{\theta \in \Lambda_1} \mathrm{E}\{F_\theta(\mathbf{N})\} \geq \min_{\theta \in \Lambda_1} P_D^{\mathbf{x}}(\phi; \theta) \geq \beta$, and $\max_{\theta \in \Lambda_0} P_F^{\mathbf{y}}(\phi; \theta) = \max_{\theta \in \Lambda_0} \mathrm{E}\{G_\theta(\mathbf{N})\} \leq \max_{\theta \in \Lambda_0} P_F^{\mathbf{x}}(\phi; \theta) \leq \alpha$ (cf. (3) and (15)). Otherwise, the detector is called *nonimprovable*.

First, a nonimprovability condition is obtained based on the properties of F_θ in (10), G_θ in (12), and F in (14).

Proposition 1: *Assume that there exits $\theta^* \in \Lambda_0$ ($\theta^* \in \Lambda_1$) such that $G_{\theta^*}(\mathbf{n}) \leq \alpha$ ($F_{\theta^*}(\mathbf{n}) \geq \beta$) implies $F(\mathbf{n}) \leq F(\mathbf{0})$ for all $\mathbf{n} \in \mathcal{S}_{\mathrm{n}}$, where $\mathcal{S}_{\mathrm{n}}$ is a convex set[2] consisting of all possible values of additive noise $\mathbf{n}$. If $G_{\theta^*}(\mathbf{n})$ is a convex function ($F_{\theta^*}(\mathbf{n})$ is a concave function), and $F(\mathbf{n})$ is a concave function over $\mathcal{S}_{\mathrm{n}}$, then the detector is nonimprovable.*

Proof: The proof employs an approach that is similar to that in the proof of Proposition 1 in [10]. Due to the convexity of $G_{\theta^*}(\cdot)$, the false alarm probability in (9) can be bounded from below, via Jensen's inequality, as

$$P_F^{\mathbf{y}}(\phi; \theta^*) = \mathrm{E}\{G_{\theta^*}(\mathbf{N})\} \geq G_{\theta^*}\left(\mathrm{E}\{\mathbf{N}\}\right). \qquad (17)$$

Because $P_F^{\mathbf{y}}(\phi; \theta^*) \leq \alpha$ is a necessary condition for improvability, (17) implies that $G_{\theta^*}\left(\mathrm{E}\{\mathbf{N}\}\right) \leq \alpha$ must be satisfied. Since $\mathrm{E}\{\mathbf{N}\} \in \mathcal{S}_{\mathrm{n}}$, $G_{\theta^*}\left(\mathrm{E}\{\mathbf{N}\}\right) \leq \alpha$ means $F\left(\mathrm{E}\{\mathbf{N}\}\right) \leq F(\mathbf{0})$ due to the assumption in the proposition. Therefore,

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[2]$\mathcal{S}_{\mathrm{n}}$ can be modeled as convex because convex combination of individual noise components can be generated via randomization [21].

$$P_D^{\mathbf{y}}(\phi) = \mathrm{E}\{F(\mathbf{N})\} \leq F\left(\mathrm{E}\{\mathbf{N}\}\right) \leq F(\mathbf{0}) \qquad (18)$$

where the first inequality follows from the concavity of F. Then, from (17) and (18), it is concluded that $P_F^{\mathbf{y}}(\phi; \theta^*) \leq \alpha$ implies $P_D^{\mathbf{y}}(\phi; \theta^*) \leq F(\mathbf{0}) = P_D^{\mathbf{x}}(\phi)$. Therefore, the detector is nonimprovable. (The alternative nonimprovability conditions stated in the parentheses in the proposition can be proved in a similar fashion.) □

The conditions in Proposition 1 can be employed to specify when the detector performance cannot be improved via additive noise. In this way, unnecessary efforts for trying to solve the optimization problem in (15) can be prevented. However, it should also be noted that Proposition 1 provides only sufficient conditions; hence, the detector can still be nonimprovable although the conditions in the proposition are not satisfied.

Next, sufficient conditions under which the detector performance can be improved via additive noise are derived. To that aim, it is first assumed that $F(\mathbf{x})$, $F_\theta(\mathbf{x}) \; \forall \theta \in \Lambda_1$, and $G_\theta(\mathbf{x}) \; \forall \theta \in \Lambda_0$ are second-order continuously differentiable around $\mathbf{x} = \mathbf{0}$. In addition, the following functions are defined for notational convenience:

$$g_\theta^{(1)}(\mathbf{x}, \mathbf{z}) \triangleq \mathbf{z}^T \nabla G_\theta(\mathbf{x}) \qquad (19)$$

$$f_\theta^{(1)}(\mathbf{x}, \mathbf{z}) \triangleq \mathbf{z}^T \nabla F_\theta(\mathbf{x}) \qquad (20)$$

$$f^{(1)}(\mathbf{x}, \mathbf{z}) \triangleq \mathbf{z}^T \nabla F(\mathbf{x}) \qquad (21)$$

$$g_\theta^{(2)}(\mathbf{x}, \mathbf{z}) \triangleq \mathbf{z}^T H(G_\theta(\mathbf{x})) \, \mathbf{z} \qquad (22)$$

$$f_\theta^{(2)}(\mathbf{x}, \mathbf{z}) \triangleq \mathbf{z}^T H(F_\theta(\mathbf{x})) \, \mathbf{z} \qquad (23)$$

$$f^{(2)}(\mathbf{x}, \mathbf{z}) \triangleq \mathbf{z}^T H(F(\mathbf{x})) \, \mathbf{z} \qquad (24)$$

where ∇ and H denote the gradient and Hessian, respectively. For example, $\nabla G_\theta(\mathbf{x})$ is a K-dimensional vector with its ith element being given by $\frac{\partial G_\theta(\mathbf{x})}{\partial x_i}$, where x_i represents the ith component of $\mathbf{x}$, and $H(G_\theta(\mathbf{x}))$ is a $K \times K$ matrix with its element in row l and column i being expressed as $\frac{\partial^2 G_\theta(\mathbf{x})}{\partial x_l \partial x_i}$.

Based on the definitions in (19)-(24), the following proposition provides sufficient conditions for improvability. For simplicity of expressions, we denote the values of $g_\theta^{(1)}(\mathbf{x}, \mathbf{z})$, $f_\theta^{(1)}(\mathbf{x}, \mathbf{z})$, $f^{(1)}(\mathbf{x}, \mathbf{z})$, $g_\theta^{(2)}(\mathbf{x}, \mathbf{z})$, $f_\theta^{(2)}(\mathbf{x}, \mathbf{z})$, and $f^{(2)}(\mathbf{x}, \mathbf{z})$ at $\mathbf{x} = \mathbf{0}$ as $g_\theta^{(1)}$, $f_\theta^{(1)}$, $f^{(1)}$, $g_\theta^{(2)}$, $f_\theta^{(2)}$, and $f^{(2)}$, respectively.

Proposition 2: *Let $\mathcal{L}_0$ and $\mathcal{L}_1$ denote the set of θ values that maximize $G_\theta(\mathbf{0})$ and minimize $F_\theta(\mathbf{0})$, respectively. Then, the detector is improvable if there exists a K-dimensional vector $\mathbf{z}$ such that one of the following conditions is satisfied at $\mathbf{x} = \mathbf{0}$ for all $\theta_0 \in \mathcal{L}_0$ and $\theta_1 \in \mathcal{L}_1$:*

- $f^{(1)} > 0$, $f_{\theta_1}^{(1)} > 0$, $g_{\theta_0}^{(1)} < 0$
- $f^{(1)} < 0$, $f_{\theta_1}^{(1)} < 0$, $g_{\theta_0}^{(1)} > 0$
- $f^{(2)} > 0$, $f_{\theta_1}^{(2)} > 0$, $g_{\theta_0}^{(2)} < 0$

Proof: The proof is omitted due to the space constraint. Mainly, the proof builds upon similar arguments to those in the proof of Theorem 2 in [5]. □

Proposition 2 states that under the stated conditions, it is possible to obtain a noise PDF that can increase the average detection probability under the constraints on the worst

case detection and false alarm probabilities. In other words, it guarantees the existence of additive noise that improves the detection performance according to the restricted NP criterion.

4. NUMERICAL RESULTS AND CONCLUSIONS

In this section, a binary hypothesis-testing problem [17] is studied in order to investigate noise benefits in the restricted NP framework. The hypotheses are defined as

$$\mathcal{H}_0 : X = V \ , \quad \mathcal{H}_1 : X = \Theta + V \quad (25)$$

where $X \in \mathbb{R}$, Θ is an unknown parameter, and V is symmetric Gaussian mixture noise represented by the following PDF $p_V(v) = \sum_{i=1}^{N_m} \omega_i \psi_i(v - m_i)$, with $\omega_i \geq 0$ for $i = 1, \ldots, N_m$, $\sum_{i=1}^{N_m} \omega_i = 1$, and $\psi_i(x) = 1/(\sqrt{2\pi}\,\sigma_i) \exp\big(-x^2/(2\,\sigma_i^2)\big)$ for $i = 1, \ldots, N_m$. Due to the symmetry assumption, $m_l = -m_{N_m-l+1}$, $\omega_l = \omega_{N_m-l+1}$ and $\sigma_l = \sigma_{N_m-l+1}$ for $l = 1, \ldots, \lfloor N_m/2 \rfloor$, where $\lfloor y \rfloor$ denotes the largest integer smaller than or equal to y.

Parameter Θ in (25) is modeled as a random variable with a PDF in the form of

$$w_1(\theta) = \rho\,\delta(\theta - A) + (1 - \rho)\,\delta(\theta + A) \quad (26)$$

where A is a known positive constant, but ρ is known with some uncertainty.

For the problem formulation above, the parameter sets under $\mathcal{H}_0$ and $\mathcal{H}_1$ can be defined as $\Lambda_0 = \{0\}$ and $\Lambda_1 = \{-A, A\}$, respectively. In addition, the conditional PDF of original observation X for a given value of $\Theta = \theta$ is given by

$$p_\theta^{\mathbf{X}}(x) = \sum_{i=1}^{N_m} \frac{\omega_i}{\sqrt{2\pi}\,\sigma_i} \exp\left(\frac{-(x - \theta - m_i)^2}{2\,\sigma_i^2}\right) . \quad (27)$$

In addition, the detector is described by

$$\phi(y) = \begin{cases} 0 \, , & A/2 > y > -A/2 \\ 1 \, , & \text{otherwise} \end{cases} \quad (28)$$

where $y = x + n$, with n representing the additive independent noise term.

To obtain more compact expressions, we define $q_i(a) \triangleq Q\left((a - m_i - x)/\sigma_i\right)$ and $\tilde{q}_i(a) \triangleq Q\left((a + m_i + x)/\sigma_i\right)$, where $Q(x) = (1/\sqrt{2\pi}) \int_x^\infty e^{-t^2/2} dt$ denotes the Q-function. Then, $F_{\theta_{1i}}$ for $\theta_{11} = A$, and $\theta_{12} = -A$, $G_{\theta_{0i}}$ for $\theta_{01} = 0$ and F can be calculated from (10), (12) and (14) as follows:

$$F_A(x) = \sum_{i=1}^{N_m} w_i\Big(q_i(-A/2) + \tilde{q}_i(3A/2)\Big)$$

$$F_{-A}(x) = \sum_{i=1}^{N_m} w_i\Big(q_i(3A/2) + \tilde{q}_i(-A/2)\Big)$$

$$G_0(x) = \sum_{i=1}^{N_m} w_i\Big(q_i(A/2) + \tilde{q}_i(A/2)\Big)$$

$$F(x) = \rho F_A(x) + (1 - \rho)F_{-A}(x) \quad (29)$$

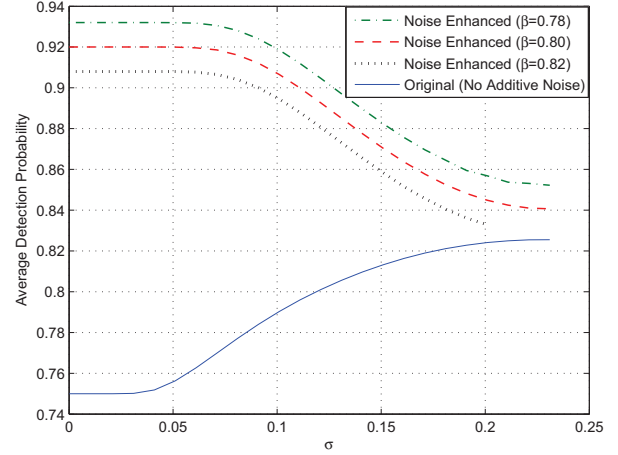In the numerical results, symmetric Gaussian mixture noise with $N_m = 4$ is considered, where the mean values of

Fig. 1. Average detection probability versus σ for various values of β, where $\alpha = 0.35$, $A = 1$ and $\rho = 0.8$.

Table 1. Values of $f^{(2)}$, $f_{\theta_1}^{(2)}$ (for $\theta_1 = A$ and $\theta_1 = -A$), and $g_{\theta_0}^{(2)}$ (for $\theta_0 = 0$) in Proposition 2 for various values of σ.

σ	$f^{(2)}$	$f_A^{(2)}$	$f_{-A}^{(2)}$	$g_0^{(2)}$
0.05	10.7982	10.7982	10.7982	-21.5964
0.10	6.0489	6.0489	6.0489	-12.0977
0.15	2.2500	2.2500	2.2500	-4.5000
0.20	0.5507	0.5507	0.5507	-1.1004

the Gaussian components in the mixture noise are specified as $[0.01 \ 0.6 \ -0.6 \ -0.01]$ with corresponding weights of $[0.25 \ 0.25 \ 0.25 \ 0.25]$. In addition, for all the cases, the variances of the Gaussian components in the mixture noise are assumed to be the same; i.e., $\sigma_i = \sigma$ for $i = 1, \ldots, N_m$.

In Fig. 1 and Fig. 2, average detection probabilities are plotted with respect to σ for various values of β for the cases of $\alpha = 0.35$, and $\alpha = 0.45$ respectively, where $A = 1$ and $\rho = 0.8$ are used. It is observed that the use of additive noise improves the average detection probability, and significant improvements can be obtained via additive noise for low values of the standard deviation, σ. As the standard deviation increases, the amount of improvement in the average detection probability decreases. Another observation from the figures is that the average detection probabilities decrease as β increases. In other words, there is a tradeoff between β and the average detection probability, which is the essential characteristics of the restricted NP approach, as explained in [17]. In addition, after some values of σ, the constraints on the minimum detection probability or the false alarm probability cannot be satisfied; therefore, the restricted NP solution does not exist after certain values of σ. (Therefore, the curves are plotted up to those specific values.)

In order to check if any of the conditions in Proposition 2 are satisfied for the example above, the numerical values of $f^{(2)}$, $f_{\theta_1}^{(2)}$, and $g_{\theta_0}^{(2)}$ are calculated and tabulated in Table 1.[3]

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[3]Since scalar observations are considered, the signs of $f^{(2)}$, $f_{\theta_1}^{(2)}$, and $g_{\theta_0}^{(2)}$ in (22)-(24) do not depend on $\mathbf{z}$; hence, $\mathbf{z} = 1$ is used for Table 1.
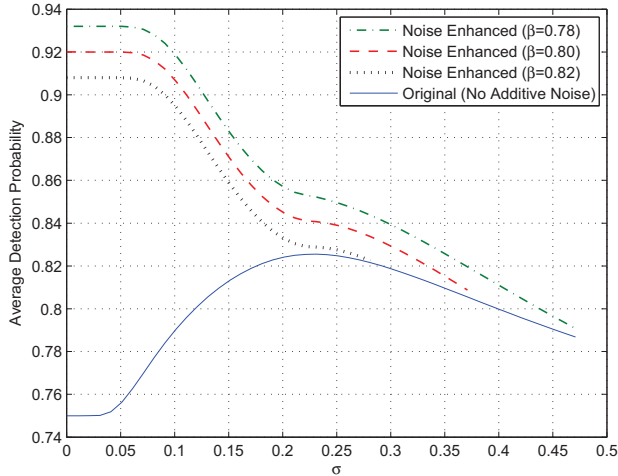
Fig. 2. Average detection probability versus σ for various values of β, where $\alpha = 0.45$, $A = 1$ and $\rho = 0.8$.

It is noted that, in this specific example, $F_{\theta_1}(\mathbf{0})$ has two minimizers; one is at $\theta_1 = -A$ and the other is at $\theta_1 = A$. Hence, sets $\mathcal{L}_1$ and $\mathcal{L}_0$ in Proposition 2 are defined as $\mathcal{L}_1 = \{-A, A\}$ and $\mathcal{L}_0 = \{0\}$, respectively. Therefore, the conditions in Proposition 2 must hold for two groups: $f^{(2)}, f_A^{(2)}, g_0^{(2)}$ and $f^{(2)}, f_{-A}^{(2)}, g_0^{(2)}$. From Table 1, it is observed that $f^{(2)}, f_A^{(2)}$ and $f_{-A}^{(2)}$ are always positive whereas $g_0^{(2)}$ is always negative for the given values of σ. Hence, the third condition in Proposition 2 is satisfied for both groups for those values of σ, meaning that the detector is improvable as a result of the proposition, which is also verified from Fig. 1 and Fig. 2.

In this study, noise enhanced detection has been investigated in the restricted NP framework, and a problem formulation has been provided for the PDF of optimal additive noise. Also, improvability and nonimprovability conditions have been obtained to specify whether additive noise can provide performance improvements according to the restricted NP criterion. A numerical example has been provided to illustrate the improvements via additive noise. The results can be applied in some wireless communications problems such as the spectrum sensing problem in cognitive radio systems.

5. REFERENCES

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