Q : A person is swimming across several rivers. Speeds of those rivers are different: v1, v2, …, vn. To simplify this problem, we only consider the speed in vertical direction. The person's speed is v. It's a constant, no way to change that. And the angle of the person's velocity to horizontal line is a1, a2, …, an. The total time for swimming is T. And, the person must pass those rivers.

Your task is: Find out an equation to determine by choosing what angles (a1, a2, …, an) the person can get maximum distance in vertical direction (That is to say, please maximize dh by determining a1, a2, …, an) under the total time T. 【You are not required to give out concrete angle numbers, a "cost function" that can be derived from is enough】

Tips: For this question, a mathematical tool you may need is called "Lagrangian Multiplier". Which means, when you provide a formula, say E, which still need to satisfy some more conditions, say a > 1, for the convenience of calculating, we can write those 2 parts (formula E and condition a > 1) together as one new formula. Here the new formula will be: E – λ(a - 1).

已知: $S = (s_1, s_2, \ldots, s_n)$

$V = (v_1, v_2, \ldots, v_n)$

$v_p : speed\ of\ person$

T

求: $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ **使得垂直方向的距离最大**

垂直方向的距离如下

$$\begin{cases} dH = \sum_{i=1}^{n} (v_p \times \sin(\alpha_i) + v_i) \times (\frac{s_i}{v_p \times \cos(\alpha_i)}) \\ \qquad s.b. \quad \sum_{i=1}^{n} \frac{s_i}{v_p \times \cos(\alpha_i)} \end{cases}$$

1. **求**dH**的最大值即求** $(-dH)$**的最小值，根据拉格朗日乘子法，可以构造如下的拉格朗日函数**

$$L(\alpha, \lambda) = -\sum_{i=1}^{n} \left\{ (v_p \times \sin(\alpha_i) + v_i) \times (\frac{s_i}{v_p \times \cos(\alpha_i)}) \right\} + \lambda * \sum_{i=1}^{n} \frac{s_i}{v_p \times \cos(\alpha_i)}$$

2. **将上述公式转为矩阵表达如下**

$$
\begin{aligned}
L(\alpha,\lambda) &= -\sum_{i=1}^{n}\left\{(v_p \times \sin(\alpha_i) + v_i) \times (\frac{s_i}{v_p \ \times \ \cos(\alpha_i)})\right\} + \lambda \times \sum_{i=1}^{n} \frac{s_i}{v_p \ \times \ \cos(\alpha_i)} \\
&= -\sum_{i=1}^{n}\left\{s_i \times tan(\alpha_i) + \frac{v_i \times s_i}{v_p \ \times \ \cos(\alpha_i)}\right\} + \frac{\lambda}{v_p} \times \sum_{i=1}^{n} \frac{s_i}{\cos(\alpha_i)} \\
&= -\left\{S * tan(\alpha)^T + \frac{V \cdot S}{v_p} * \left[\frac{1}{\cos(\alpha)}\right]^T\right\} + \frac{\lambda}{v_p} \times S * \left[\frac{1}{\cos(\alpha)}\right]^T
\end{aligned}
$$

3. 对变量 α, λ 分别求导，可求 $L(\alpha, \lambda)$ 的最小值

$$
\begin{cases}
\dfrac{\partial L(\alpha,\lambda)}{\partial \alpha} = 0 \\
\dfrac{\partial L(\alpha,\lambda)}{\partial \lambda} = 0
\end{cases}
$$

In [ ]:

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