# Machine Learning

## Logistic Regression (Classification Algorithm)

Dr. Muhammad Kamran Malik

# Logistic Regression

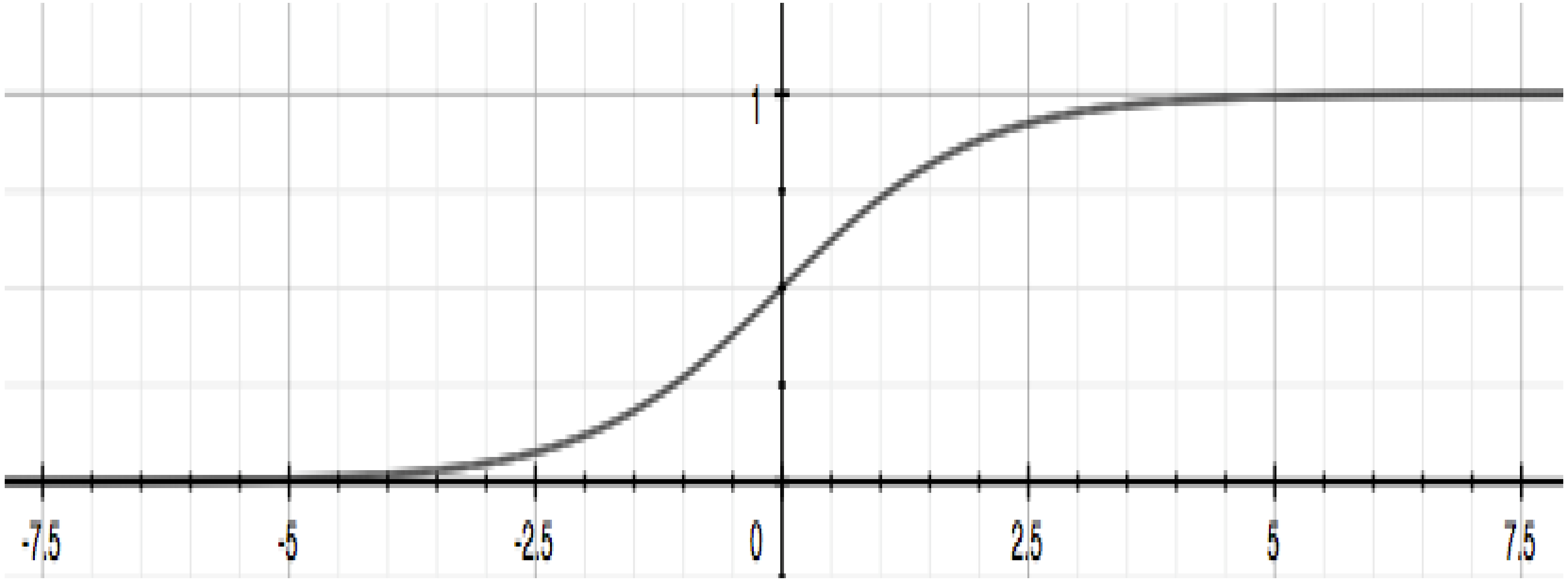
Source: https://www.coursera.org/learn/machine-learning

# Logistic Regression

- Supervised Machine Learning algorithm
- Don't be confused by the name "Logistic Regression“.
- It is named that way for historical reasons and is actually an approach to classification problems, not regression problems
- Binary Classification Algorithm
- Instead of our output vector y being a continuous range of values, it will only be 0 or 1. $y \in \{0,1\}$
- Where 0 is usually taken as the "negative class" and 1 as the "positive class", but you are free to assign any representation to it.
- One method is to use linear regression and map all predictions greater than 0.5 as a 1 and all less than 0.5 as a 0.
- This method doesn't work well because classification is not actually a linear function.

# Logistic Regression

- Our hypothesis should satisfy:
  - $0 \leq h\vartheta(x) \leq 1$
- Our new form uses the "Sigmoid Function" also called the "Logistic Function”
  - $h\vartheta(x)=g(\vartheta^T x)$
  - $z=\vartheta^T x$
  - $g(z)=1/\ 1+e^{-z}$




| z | sig(z) |
|---|---|
| -2 | 0.12 |
| -1.5 | 0.18 |
| -1 | 0.27 |
| -0.5 | 0.38 |
| 0 | 0.50 |
| 0.5 | 0.62 |
| 1 | 0.73 |
| 1.5 | 0.82 |
| 2 | 0.88 |
| 2.5 | 0.92 |

# Logistic Regression

- We start with our old hypothesis (linear regression), except that we want to restrict the range to 0 and 1.
- This is accomplished by plugging $\vartheta^T x$ into the Logistic Function.
- $h\vartheta$ will give us the **probability** that our output is 1.
- For example, $h\vartheta(x)=0.7$ gives us the probability of 70% that our output is 1.
  - $h\vartheta(x) = P(y=1|x;\vartheta) = 1 - P(y=0|x;\vartheta)$
  - $P(y=0|x;\vartheta) + P(y=1|x;\vartheta) = 1$
- Our probability that our prediction is 0 is just the complement of our probability that it is 1 (e.g. if probability that it is 1 is 70%, then the probability that it is 0 is 30%).

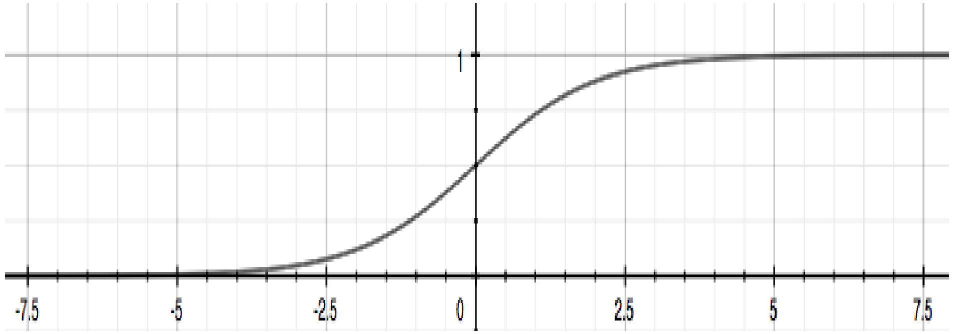
# Q & A

# Logistic Regression (Decision Boundary)

Source: https://www.coursera.org/learn/machine-learning

# Logistic Regression (Decision Boundary)

- In order to get our discrete 0 or 1 classification, we can translate the output of the hypothesis function as follows:
  - $h\vartheta(x) \geq 0.5 \rightarrow y = 1$
  - $h\vartheta(x) < 0.5 \rightarrow y = 0$
- The way our logistic function g behaves is that when its input is greater than or equal to zero, its output is greater than or equal to 0.5:
  - $g(z) \geq 0.5 \text{ when } z \geq 0$
- So if our input to g is $\vartheta^T X$, then that means:
  - $h\vartheta(x)=g(\vartheta^T x) \geq 0.5 \text{ when } \vartheta^T x \geq 0$
- From these statements we can now say:
  - $\vartheta^T x \geq 0 \Rightarrow y = 1$
  - $\vartheta^T x < 0 \Rightarrow y = 0$




- The **decision boundary** is the line that separates the area where y = 0 and where y = 1. It is created by our hypothesis function.

# Logistic Regression (Decision Boundary)

- $\Theta = \begin{bmatrix} 5 \\ -1 \\ 0 \end{bmatrix}$

$y$=1 *if* $5 + (-1) * x_1 + 0 * x_2 \geq 0$

$5 - x_1 \geq 0$

$-x_1 \geq -5$

$x_1 \leq 5$

- In this case, our decision boundary is a straight vertical line placed on the graph where $x_1 = 5$, and everything to the left of that denotes y = 1, while everything to the right denotes y = 0.
- Again, the input to the sigmoid function g(z) (e.g. $\vartheta^{T}X$) doesn't need to be linear, and could be a function that describes a circle (e.g. $z = \vartheta_0 + \vartheta_1 x_1^2 + \vartheta_2 x_2^2$) or any shape to fit our data.

# Q & A

# Logistic Regression (Cost Function)

Source: https://www.coursera.org/learn/machine-learning

# Logistic Regression (Cost Function)

- The more our hypothesis is off from y, the larger the cost function output. If our hypothesis is equal to y, then our cost is 0:

$$J(\theta) = \frac{1}{m} \sum_{i=1}^{m} \mathrm{Cost}(h_\theta(x^{(i)}), y^{(i)})$$

$$\mathrm{Cost}(h_\theta(x), y) = -\log(h_\theta(x)) \qquad \text{if y} = 1$$

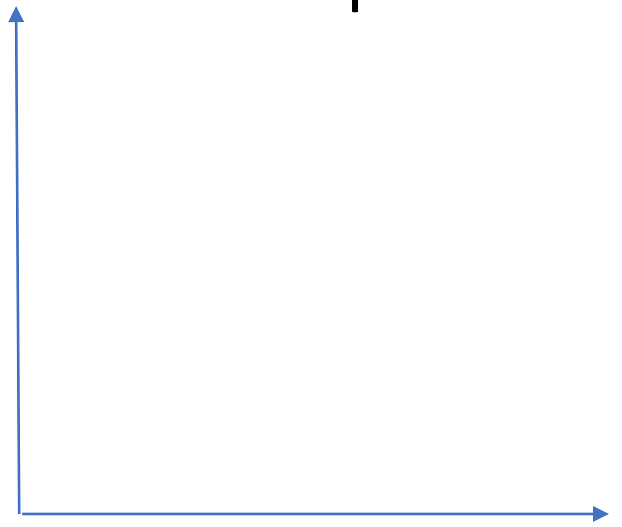
$$\mathrm{Cost}(h_\theta(x), y) = -\log(1 - h_\theta(x)) \qquad \text{if y} = 0$$

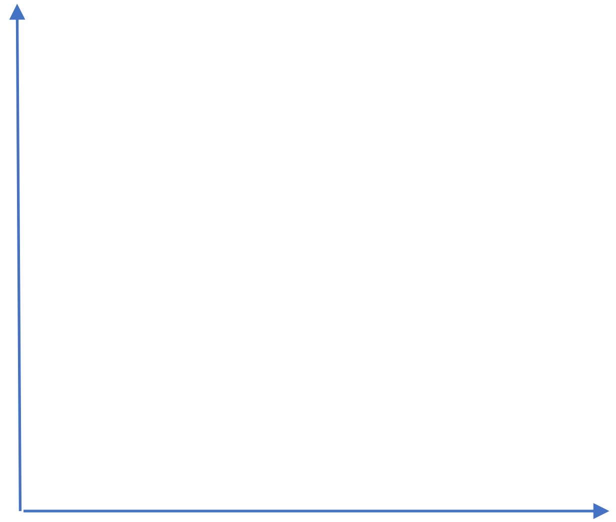
$$\mathrm{Cost}(h_\theta(x), y) = 0 \text{ if } h_\theta(x) = y$$

$$\mathrm{Cost}(h_\theta(x), y) \to \infty \text{ if } y = 0 \text{ and } h_\theta(x) \to 1$$

$$\mathrm{Cost}(h_\theta(x), y) \to \infty \text{ if } y = 1 \text{ and } h_\theta(x) \to 0$$

| x | $-\log(h_\vartheta(x))$ | $-\log(1-h_\vartheta(x))$ |
|---|---|---|
| 0.1 | 1.00 | 0.05 |
| 0.2 | 0.70 | 0.10 |
| 0.3 | 0.52 | 0.15 |
| 0.4 | 0.40 | 0.22 |
| 0.5 | 0.30 | 0.30 |
| 0.6 | 0.22 | 0.40 |
| 0.7 | 0.15 | 0.52 |
| 0.8 | 0.10 | 0.70 |
| 0.9 | 0.05 | 1.00 |





# Logistic Regression (Simplified Cost Function)

- We can compress our cost function's two conditional cases into one case:
  - Cost $(h\vartheta(x), y) = -\, y \log (h\vartheta\,(x)) - (1-y) \log (1 - h\vartheta\,(x))$
- Notice that when y is equal to 1, then the second term $-(1-y) \log (1 - h\vartheta(x))$ will be zero and will not affect the result. If y is equal to 0, then the first term $-\, y \log (h\vartheta\,(x))$ will be zero and will not affect the result.
- We can fully write out our entire cost function as follows:
  - $J(\theta) = \frac{-1}{m} \sum_{i=1}^{m} [y^i \log(y'^i) + (1 - \ y^i) \log(1 - y'^i)]$

# Q & A

# Logistic Regression (Gradient Descent)

Source: https://www.coursera.org/learn/machine-learning

# Logistic Regression (Gradient Descent)

- *Repeat*{

$$\vartheta_j := \vartheta_j - \alpha \frac{\partial}{\partial \vartheta_j} J(\vartheta)$$

}

$$y'^{(i)} = h_\vartheta(x^{(i)}) = \sigma(\vartheta_0 + \vartheta_1 x^{(i)})$$

$$\sigma(z) = \frac{1}{1 + e^{-z}}$$

$$J(\theta) = \frac{-1}{m} \sum_{i=1}^{m} [y^i \log(y'^i) + (1 - y^i) \log(1 - y'^i)]$$

$$\sigma(z)' = \sigma(z)\left(1 - \sigma(z)\right)$$

$$Repeat\ \{$$

$$\theta_j := \theta_j - \frac{\alpha}{m} \sum_{i=1}^{m} (h_\theta(x^{(i)}) - y^{(i)}) x_j^{(i)}$$

$$\}$$

# Logistic Regression (Vectorized Implementation)

- $h = g(X\vartheta)$
- $J(\vartheta) = 1/m * (-y^T \log(h) - (1-y)^T \log(1-h))$

- Weight Update
- $\vartheta := \vartheta - \alpha / m * X^T (g(X\vartheta) - y)$

$$\nabla J(\theta) = \frac{1}{m} \cdot X^T \cdot \left(g\left(X \cdot \theta\right) - \vec{y}\right)$$

# Sigmoid Derivation

$$\sigma(x) = \frac{1}{1 + e^{-x}}$$

$$\frac{d\sigma(x)}{dx} = \sigma(x)'$$

$$\frac{d\sigma(x)}{dx} = \frac{d}{dx}\left(\frac{1}{1 + e^{-x}}\right)$$

$$= \frac{d}{dx}(1 + e^{-x})^{-1}$$

$$= -1 * (1 + e^{-x})^{-2} \frac{d}{dx}(1 + e^{-x})$$

$$= \frac{-1}{(1 + e^{-x})^2} \frac{d}{dx}(1 + e^{-x})$$

$$= \frac{-1}{(1 + e^{-x})^2} \left(\frac{d}{dx}(1) + \frac{d}{dx}(e^{-x})\right)$$

$$= \frac{-1}{(1 + e^{-x})^2} \left(0 + e^{-x} \frac{d}{dx}(-x)\right)$$

$$= \frac{-1}{(1 + e^{-x})^2} (0 + -e^{-x})$$

$$= \frac{e^{-x}}{(1 + e^{-x})^2}$$

$$= \left(\frac{1}{1 + e^{-x}}\right)\left(\frac{e^{-x}}{1 + e^{-x}}\right)$$

$$= \sigma(x) \left(\frac{e^{-x}}{1 + e^{-x}}\right)$$

$$= \sigma(x) \left(\frac{1 - 1 + e^{-x}}{1 + e^{-x}}\right)$$

$$= \sigma(x) \left(\frac{1 + e^{-x} - 1}{1 + e^{-x}}\right)$$

$$= \sigma(x) \left(\frac{1 + e^{-x}}{1 + e^{-x}} - \frac{1}{1 + e^{-x}}\right)$$

$$= \sigma(x) \left(1 - \sigma(x)\right)$$

# Binary Cross Entropy

Cost function:

$$J(\theta) = \frac{-1}{m}\sum_{i=1}^{m}[y^i \log(y'^i) + (1- y^i)\log(1-y'^i)]$$

*Objective:*

$$\min_{\vartheta_0,\vartheta_1} J(\vartheta_0,\vartheta_1)$$

Hypothesis Function:

$$y'^{(i)} = h_\vartheta(x^{(i)}) = \sigma\,(\vartheta_0+\vartheta_1 x^{(i)})$$

$$\frac{\partial J(\theta)}{\partial \theta_j} = \frac{\partial}{\partial \theta_j}\frac{-1}{m}\sum_{i=1}^{m}[y^i \log(y'^i) + (1- y^i)\log(1-y'^i)]$$

$$= \frac{-1}{m}\sum_{i=1}^{m}[y^i \frac{\partial}{\partial \theta_j}\log(y'^i) + (1- y^i)\frac{\partial}{\partial \theta_j}\log(1-y'^i)]$$

$$= \frac{-1}{m}\sum_{i=1}^{m}[\frac{y^i \frac{\partial}{\partial \theta_j} y'^i}{y'^i} + \frac{(1-y^i)\frac{\partial}{\partial \theta_j}(1-y'^i)}{1-y'^i}]$$

$$= \frac{-1}{m}\sum_{i=1}^{m}[\frac{y^i \frac{\partial}{\partial \theta_j} y'^i}{y'^i} + \frac{(1-y^i)\frac{\partial}{\partial \theta_j}(1-y'^i)}{1-y'^i}]$$

$$= \frac{-1}{m}\sum_{i=1}^{m}[\frac{y^i\,(y'^i)(1-y'^i)\frac{\partial}{\partial \theta_j}\theta^T x}{y'^i} + \frac{-(1-y^i)\,(y'^i)(1-y'^i)\frac{\partial}{\partial \theta_j}\theta^T x}{1-y'^i}]$$

$$= \frac{-1}{m}\sum_{i=1}^{m}[\frac{y^i\,(y'^i)(1-y'^i)\frac{\partial}{\partial \theta_j}\theta^T x}{y'^i} - \frac{(1-y^i)\,(y'^i)(1-y'^i)\frac{\partial}{\partial \theta_j}\theta^T x}{1-y'^i}]$$

$$= \frac{-1}{m}\sum_{i=1}^{m}[y^i(1-y'^i)x_j^i - (1- y^i)y'^i x_j^i]$$

$$= \frac{-1}{m}\sum_{i=1}^{m}[y^i(1-y'^i) - (1- y^i)y'^i]x_j^i$$

$$= \frac{-1}{m}\sum_{i=1}^{m}[y^i - y^i y'^i - y'^i + y^i y'^i]x_j^i$$

$$= \frac{-1}{m}\sum_{i=1}^{m}[y^i - y'^i]x_j^i$$

$$= \frac{1}{m}\sum_{i=1}^{m}[y'^i - y^i]x_j^i$$

# Q & A