Statistical and Mathematical Methods for Data Analysis

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Textbook

- **Probability & Statistics for Engineers & Scientists**, Ninth Edition, Ronald E. Walpole, Raymond H. Myer

References

Readings for these lecture notes:

- ❑Probability & Statistics for Engineers & Scientists, Ninth edition, Ronald E. Walpole, Raymond H. Myer

- ❑Elementary Statistics, Tenth Edition, Mario F. Triola

These notes contain material from the above resources.

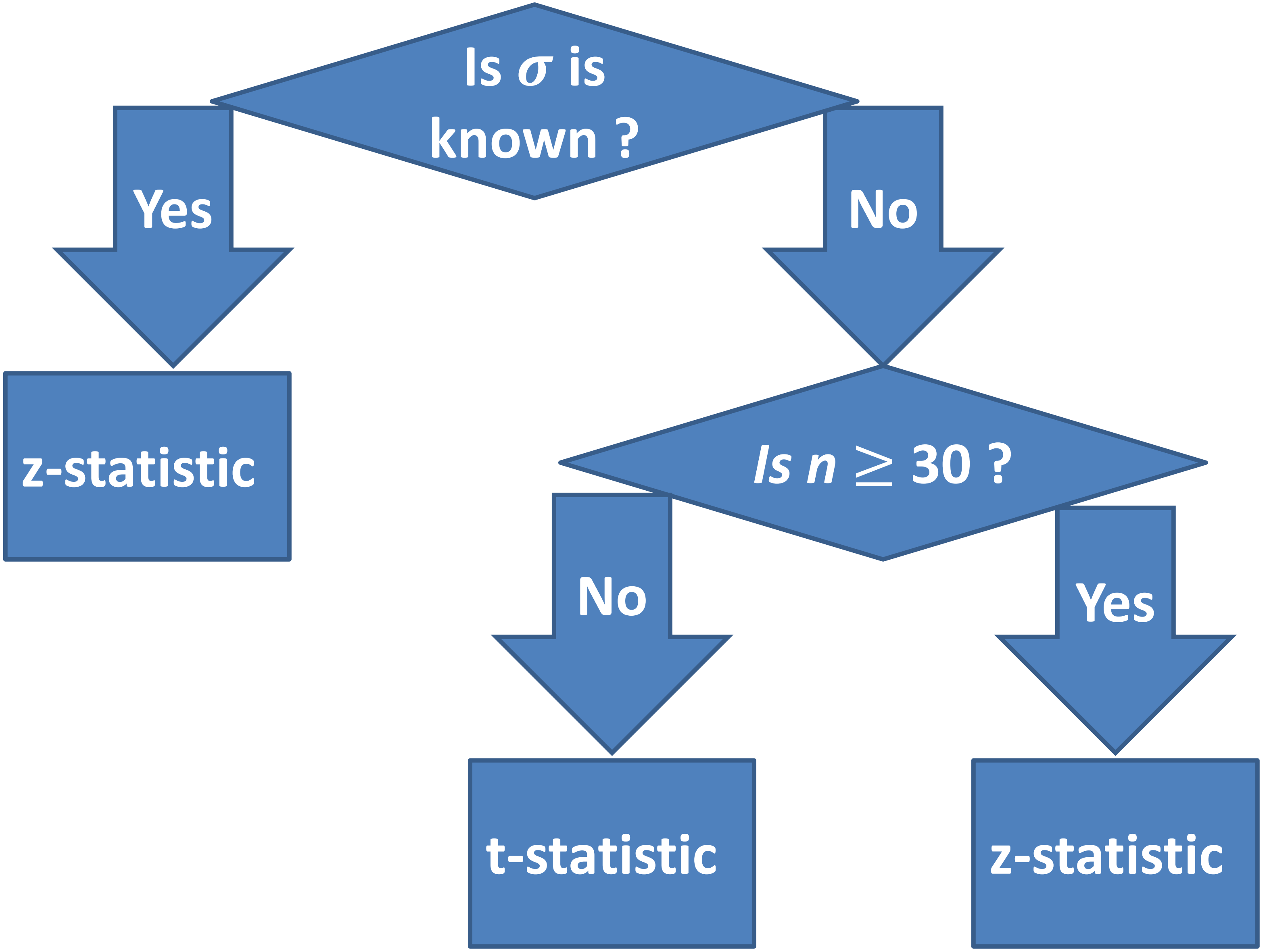
Is σ is known ?

Yes

If either the population is normally distributed or $n \geq 30$, then use the use the standard normal distribution or Z-test

No

If either the population is normally distributed or $n \geq 30$, then use the t-distribution or t-test


Is σ is known ?
Yes
No
z-statistic
Is n ≥ 30 ?
No
Yes
t-statistic
z-statistic

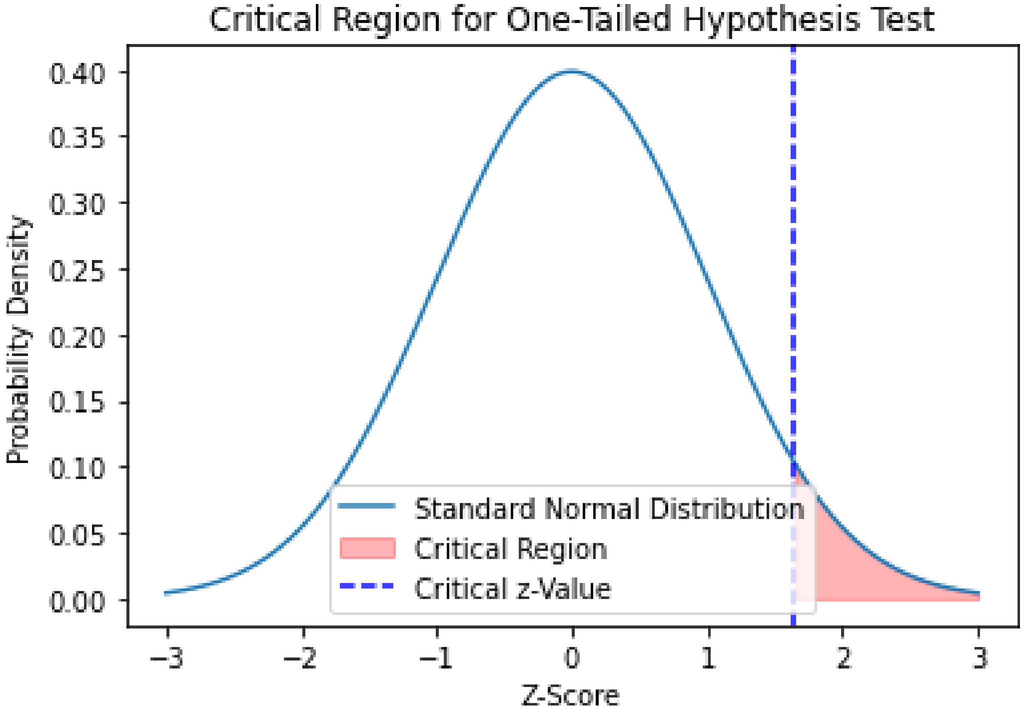
When both ***n* < 30** and the **population is *not* normally distributed**, we *cannot* use the **standard normal distribution** or the ***t*-distribution**.

H_0	Value of Test Statistic	H_1	Critical Region
$\mu = \mu_0$	$z = \dfrac{\bar{x} - \mu_0}{\sigma/\sqrt{n}}$; σ known	$\mu < \mu_0$ $\mu > \mu_0$ $\mu \neq \mu_0$	$z < -z_\alpha$ $z > z_\alpha$ $z < -z_{\alpha/2}$ or $z > z_{\alpha/2}$
$\mu = \mu_0$	$t = \dfrac{\bar{x} - \mu_0}{s/\sqrt{n}}$; $v = n - 1$, σ unknown	$\mu < \mu_0$ $\mu > \mu_0$ $\mu \neq \mu_0$	$t < -t_\alpha$ $t > t_\alpha$ $t < -t_{\alpha/2}$ or $t > t_{\alpha/2}$
$\mu_1 - \mu_2 = d_0$	$z = \dfrac{(\bar{x}_1 - \bar{x}_2) - d_0}{\sqrt{\sigma_1^2/n_1 + \sigma_2^2/n_2}}$; σ_1 and σ_2 known	$\mu_1 - \mu_2 < d_0$ $\mu_1 - \mu_2 > d_0$ $\mu_1 - \mu_2 \neq d_0$	$z < -z_\alpha$ $z > z_\alpha$ $z < -z_{\alpha/2}$ or $z > z_{\alpha/2}$
$\mu_1 - \mu_2 = d_0$	$t = \dfrac{(\bar{x}_1 - \bar{x}_2) - d_0}{s_p\sqrt{1/n_1 + 1/n_2}}$; $v = n_1 + n_2 - 2$, $\sigma_1 = \sigma_2$ but unknown, $s_p^2 = \dfrac{(n_1 - 1)s_1^2 + (n_2 - 1)s_2^2}{n_1 + n_2 - 2}$	$\mu_1 - \mu_2 < d_0$ $\mu_1 - \mu_2 > d_0$ $\mu_1 - \mu_2 \neq d_0$	$t < -t_\alpha$ $t > t_\alpha$ $t < -t_{\alpha/2}$ or $t > t_{\alpha/2}$
$\mu_1 - \mu_2 = d_0$	$t' = \dfrac{(\bar{x}_1 - \bar{x}_2) - d_0}{\sqrt{s_1^2/n_1 + s_2^2/n_2}}$; $v = \dfrac{(s_1^2/n_1 + s_2^2/n_2)^2}{\frac{(s_1^2/n_1)^2}{n_1 - 1} + \frac{(s_2^2/n_2)^2}{n_2 - 1}}$, $\sigma_1 \neq \sigma_2$ and unknown	$\mu_1 - \mu_2 < d_0$ $\mu_1 - \mu_2 > d_0$ $\mu_1 - \mu_2 \neq d_0$	$t' < -t_\alpha$ $t' > t_\alpha$ $t' < -t_{\alpha/2}$ or $t' > t_{\alpha/2}$
$\mu_D = d_0$ paired observations	$t = \dfrac{\bar{d} - d_0}{s_d/\sqrt{n}}$; $v = n - 1$	$\mu_D < d_0$ $\mu_D > d_0$ $\mu_D \neq d_0$	$t < -t_\alpha$ $t > t_\alpha$ $t < -t_{\alpha/2}$ or $t > t_{\alpha/2}$

Critical Region: Scenario 1

H_1: $\mu > \mu_0$ (One tailed test)

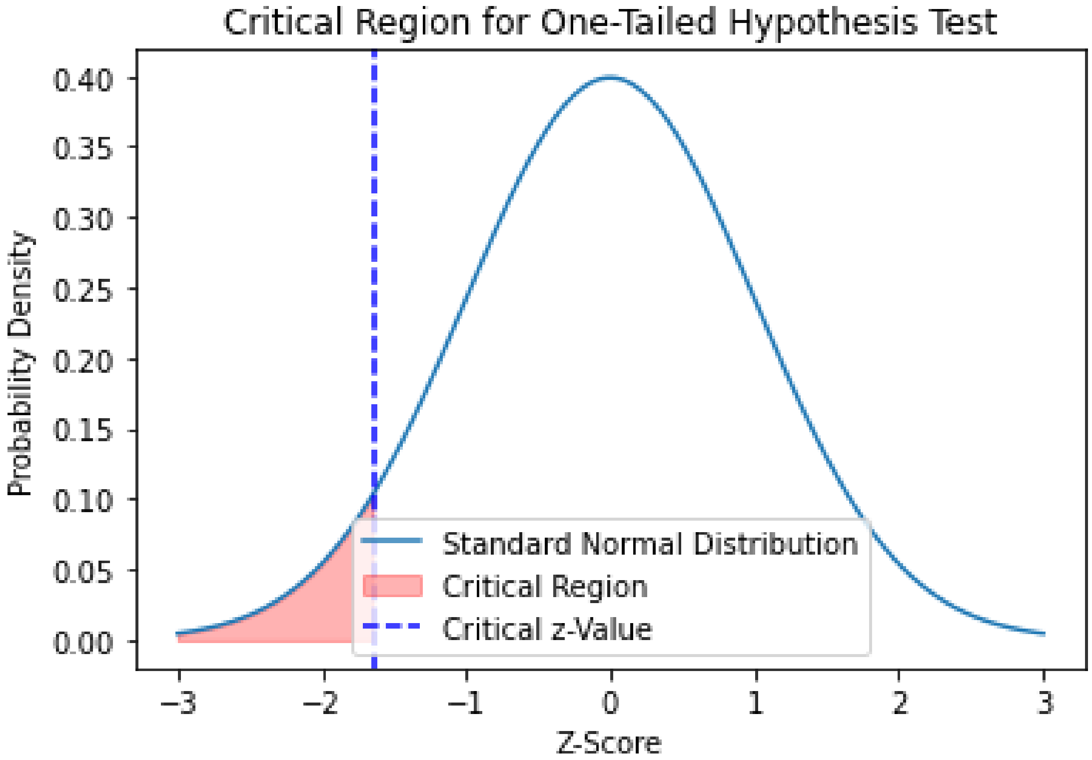
$Z_{cal} > Z_{tab}$, where $Z_{tab} = Z_\alpha$



Critical Region: Scenario 2

H_1: $\mu < \mu_0$ (One tailed test)

$Z_{cal} < Z_{tab}$, where $Z_{tab} = -Z_{\alpha}$



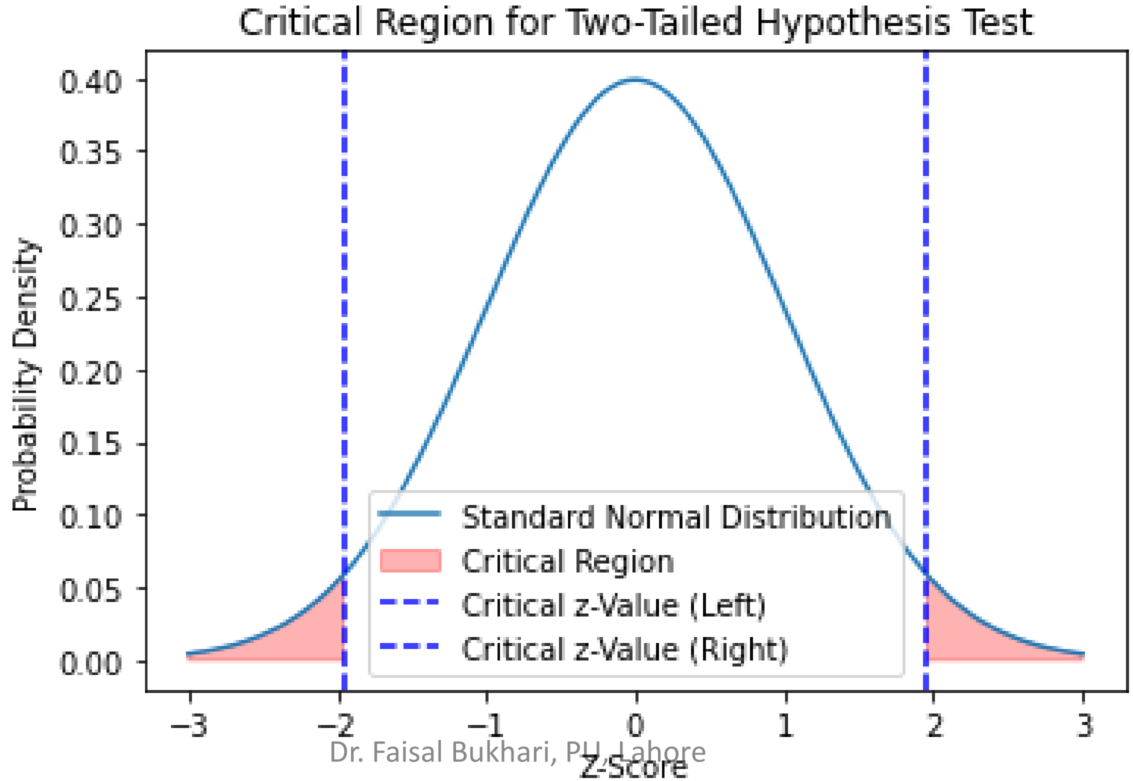
Critical Region: Scenario 3

$H_1: \mu \neq \mu_0$ **(Two tailed test)**

$Z_{cal} < Z_{tab}$, where $Z_{tab} = -Z_{\alpha/2}$ **or** $Z_{cal} > Z_{tab}$, where $Z_{tab} = Z_{\alpha/2}$

OR

$|Z_{cal}| > Z_{tab}$, where $Z_{tab} = Z_{\alpha/2}$



Approach to Hypothesis Testing with Fixed Probability of Type I Error

1. State the null and alternative hypotheses.
2. Choose a fixed significance level α.
3. Test statistic to be used it
4. Calculations
5. Critical region
6. Conclusion

Area under the Normal Curve [1]

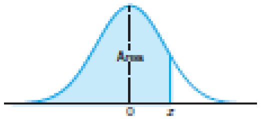


Table A.3 Areas under the Normal Curve

z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
−3.4	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0002
−3.3	0.0005	0.0005	0.0005	0.0004	0.0004	0.0004	0.0004	0.0004	0.0004	0.0003
−3.2	0.0007	0.0007	0.0006	0.0006	0.0006	0.0006	0.0006	0.0005	0.0005	0.0005
−3.1	0.0010	0.0009	0.0009	0.0009	0.0008	0.0008	0.0008	0.0008	0.0007	0.0007
−3.0	0.0013	0.0013	0.0013	0.0012	0.0012	0.0011	0.0011	0.0011	0.0010	0.0010
−2.9	0.0019	0.0018	0.0018	0.0017	0.0016	0.0016	0.0015	0.0015	0.0014	0.0014
−2.8	0.0026	0.0025	0.0024	0.0023	0.0023	0.0022	0.0021	0.0021	0.0020	0.0019
−2.7	0.0035	0.0034	0.0033	0.0032	0.0031	0.0030	0.0029	0.0028	0.0027	0.0026
−2.6	0.0047	0.0045	0.0044	0.0043	0.0041	0.0040	0.0039	0.0038	0.0037	0.0036
−2.5	0.0062	0.0060	0.0059	0.0057	0.0055	0.0054	0.0052	0.0051	0.0049	0.0048
−2.4	0.0082	0.0080	0.0078	0.0075	0.0073	0.0071	0.0069	0.0068	0.0066	0.0064
−2.3	0.0107	0.0104	0.0102	0.0099	0.0096	0.0094	0.0091	0.0089	0.0087	0.0084
−2.2	0.0139	0.0136	0.0132	0.0129	0.0125	0.0122	0.0119	0.0116	0.0113	0.0110
−2.1	0.0179	0.0174	0.0170	0.0166	0.0162	0.0158	0.0154	0.0150	0.0146	0.0143
−2.0	0.0228	0.0222	0.0217	0.0212	0.0207	0.0202	0.0197	0.0192	0.0188	0.0183
−1.9	0.0287	0.0281	0.0274	0.0268	0.0262	0.0256	0.0250	0.0244	0.0239	0.0233
−1.8	0.0359	0.0351	0.0344	0.0336	0.0329	0.0322	0.0314	0.0307	0.0301	0.0294
−1.7	0.0446	0.0436	0.0427	0.0418	0.0409	0.0401	0.0392	0.0384	0.0375	0.0367
−1.6	0.0548	0.0537	0.0526	0.0516	0.0505	0.0495	0.0485	0.0475	0.0465	0.0455
−1.5	0.0668	0.0655	0.0643	0.0630	0.0618	0.0606	0.0594	0.0582	0.0571	0.0559
−1.4	0.0808	0.0793	0.0778	0.0764	0.0749	0.0735	0.0721	0.0708	0.0694	0.0681
−1.3	0.0968	0.0951	0.0934	0.0918	0.0901	0.0885	0.0869	0.0853	0.0838	0.0823
−1.2	0.1151	0.1131	0.1112	0.1093	0.1075	0.1056	0.1038	0.1020	0.1003	0.0985
−1.1	0.1357	0.1335	0.1314	0.1292	0.1271	0.1251	0.1230	0.1210	0.1190	0.1170
−1.0	0.1587	0.1562	0.1539	0.1515	0.1492	0.1469	0.1446	0.1423	0.1401	0.1379
−0.9	0.1841	0.1814	0.1788	0.1762	0.1736	0.1711	0.1685	0.1660	0.1635	0.1611
−0.8	0.2119	0.2090	0.2061	0.2033	0.2005	0.1977	0.1949	0.1922	0.1894	0.1867
−0.7	0.2420	0.2389	0.2358	0.2327	0.2296	0.2266	0.2236	0.2206	0.2177	0.2148
−0.6	0.2743	0.2709	0.2676	0.2643	0.2611	0.2578	0.2546	0.2514	0.2483	0.2451
−0.5	0.3085	0.3050	0.3015	0.2981	0.2946	0.2912	0.2877	0.2843	0.2810	0.2776
−0.4	0.3446	0.3409	0.3372	0.3336	0.3300	0.3264	0.3228	0.3192	0.3156	0.3121
−0.3	0.3821	0.3783	0.3745	0.3707	0.3669	0.3632	0.3594	0.3557	0.3520	0.3483
−0.2	0.4207	0.4168	0.4129	0.4090	0.4052	0.4013	0.3974	0.3936	0.3897	0.3859
−0.1	0.4602	0.4562	0.4522	0.4483	0.4443	0.4404	0.4364	0.4325	0.4286	0.4247
−0.0	0.5000	0.4960	0.4920	0.4880	0.4840	0.4801	0.4761	0.4721	0.4681	0.4641

Area under the Normal Curve [2]

Table A.3 (continued) Areas under the Normal Curve

z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990
3.1	0.9990	0.9991	0.9991	0.9991	0.9992	0.9992	0.9992	0.9992	0.9993	0.9993
3.2	0.9993	0.9993	0.9994	0.9994	0.9994	0.9994	0.9994	0.9995	0.9995	0.9995
3.3	0.9995	0.9995	0.9995	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9997
3.4	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9998

Two Samples: Tests on Two Means

Example: An experiment was performed to compare the abrasive wear of two different laminated materials. **Twelve** pieces of material 1 were tested by exposing each piece to a machine measuring wear. **Ten** pieces of material 2 were similarly tested. In each case, the depth of wear was observed. The **samples** of material 1 gave an **average** (coded) wear of **85** units with a **sample standard deviation** of **4**, while the samples of material **2** gave an **average of 81** with a **sample standard** deviation of **5**. Can we conclude at the **0.05** level of significance that the abrasive wear of material 1 exceeds that of material 2 by more than 2 units? Assume the populations to be approximately normal with **equal variances**.

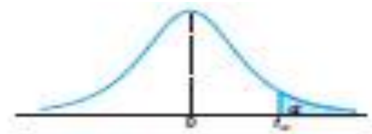


Table A.4 Critical Values of the t-Distribution

	α						
v	0.40	0.30	0.20	0.15	0.10	0.05	0.025
1	0.325	0.727	1.376	1.963	3.078	6.314	12.706
2	0.289	0.617	1.061	1.386	1.886	2.920	4.303
3	0.277	0.584	0.978	1.250	1.638	2.353	3.182
4	0.271	0.569	0.941	1.190	1.533	2.132	2.776
5	0.267	0.559	0.920	1.156	1.476	2.015	2.571
6	0.265	0.553	0.906	1.134	1.440	1.943	2.447
7	0.263	0.549	0.896	1.119	1.415	1.895	2.365
8	0.262	0.546	0.889	1.108	1.397	1.860	2.306
9	0.261	0.543	0.883	1.100	1.383	1.833	2.262
10	0.260	0.542	0.879	1.093	1.372	1.812	2.228
11	0.260	0.540	0.876	1.088	1.363	1.796	2.201
12	0.259	0.539	0.873	1.083	1.356	1.782	2.179
13	0.259	0.538	0.870	1.079	1.350	1.771	2.160
14	0.258	0.537	0.868	1.076	1.345	1.761	2.145
15	0.258	0.536	0.866	1.074	1.341	1.753	2.131
16	0.258	0.535	0.865	1.071	1.337	1.746	2.120
17	0.257	0.534	0.863	1.069	1.333	1.740	2.110
18	0.257	0.534	0.862	1.067	1.330	1.734	2.101
19	0.257	0.533	0.861	1.066	1.328	1.729	2.093
20	0.257	0.533	0.860	1.064	1.325	1.725	2.086
21	0.257	0.532	0.859	1.063	1.323	1.721	2.080
22	0.256	0.532	0.858	1.061	1.321	1.717	2.074
23	0.256	0.532	0.858	1.060	1.319	1.714	2.069
24	0.256	0.531	0.857	1.059	1.318	1.711	2.064
25	0.256	0.531	0.856	1.058	1.316	1.708	2.060
26	0.256	0.531	0.856	1.058	1.315	1.706	2.056
27	0.256	0.531	0.855	1.057	1.314	1.703	2.052
28	0.256	0.530	0.855	1.056	1.313	1.701	2.048
29	0.256	0.530	0.854	1.055	1.311	1.699	2.045
30	0.256	0.530	0.854	1.055	1.310	1.697	2.042
40	0.255	0.529	0.851	1.050	1.303	1.684	2.021
60	0.254	0.527	0.848	1.045	1.296	1.671	2.000
120	0.254	0.526	0.845	1.041	1.289	1.658	1.980
∞	0.253	0.524	0.842	1.036	1.282	1.645	1.960

	α						
v	0.02	0.015	0.01	0.0075	0.005	0.0025	0.0005
1	15.894	21.205	31.821	42.433	63.656	127.321	636.578
2	4.849	5.643	6.965	8.073	9.925	14.089	31.600
3	3.482	3.896	4.541	5.047	5.841	7.453	12.924
4	2.999	3.298	3.747	4.088	4.604	5.598	8.610
5	2.757	3.003	3.365	3.634	4.032	4.773	6.869
6	2.612	2.829	3.143	3.372	3.707	4.317	5.959
7	2.517	2.715	2.998	3.203	3.499	4.029	5.408
8	2.449	2.634	2.896	3.085	3.355	3.833	5.041
9	2.398	2.574	2.821	2.998	3.250	3.690	4.781
10	2.359	2.527	2.764	2.932	3.169	3.581	4.587
11	2.328	2.491	2.718	2.879	3.106	3.497	4.437
12	2.303	2.461	2.681	2.836	3.055	3.428	4.318
13	2.282	2.436	2.650	2.801	3.012	3.372	4.221
14	2.264	2.415	2.624	2.771	2.977	3.326	4.140
15	2.249	2.397	2.602	2.746	2.947	3.286	4.073
16	2.235	2.382	2.583	2.724	2.921	3.252	4.015
17	2.224	2.368	2.567	2.706	2.898	3.222	3.965
18	2.214	2.356	2.552	2.689	2.878	3.197	3.922
19	2.205	2.346	2.539	2.674	2.861	3.174	3.883
20	2.197	2.336	2.528	2.661	2.845	3.153	3.850
21	2.189	2.328	2.518	2.649	2.831	3.135	3.819
22	2.183	2.320	2.508	2.639	2.819	3.119	3.792
23	2.177	2.313	2.500	2.629	2.807	3.104	3.768
24	2.172	2.307	2.492	2.620	2.797	3.091	3.745
25	2.167	2.301	2.485	2.612	2.787	3.078	3.725
26	2.162	2.296	2.479	2.605	2.779	3.067	3.707
27	2.158	2.291	2.473	2.598	2.771	3.057	3.689
28	2.154	2.286	2.467	2.592	2.763	3.047	3.674
29	2.150	2.282	2.462	2.586	2.756	3.038	3.660
30	2.147	2.278	2.457	2.581	2.750	3.030	3.646
40	2.123	2.250	2.423	2.542	2.704	2.971	3.551
60	2.099	2.223	2.390	2.504	2.660	2.915	3.460
120	2.076	2.196	2.358	2.468	2.617	2.860	3.373
∞	2.054	2.170	2.326	2.432	2.576	2.807	3.290

Solution:

$n_1 = 12$ (Sample size of material 1)

$\overline{x_1} = 85$ (Sample mean of material 1)

$s_1 = 4$ (sample standard deviation of material 1)

$n_2 = 10$ (Sample size of material 2)

$\overline{x_2} = 81$ (Sample mean of material 2)

$s_2 = 5$ (sample standard deviation of material 1)

$\alpha = 0.05$ (Level of significance

Given assumption the populations to be approximately normal with **equal variances**

$\Rightarrow$ $\boldsymbol{\sigma_1^2 = \sigma_1^2}$ **but unknown**

1. **We state our hypothesis as**:

$H_0: \mu_1 - \mu_2 = 2$

$H_1: \mu_1 - \mu_2 > 2$

(One tailed test)

2. **The level of significance is set** $\alpha = 0.05$.

3. **Test statistic to be used is**

$$t_{cal} = \frac{(\overline{x}_1 - \overline{x}_2) - (\mu_1 - \mu_2)}{s_p \sqrt{1/n_1 + 1/n_2}}$$

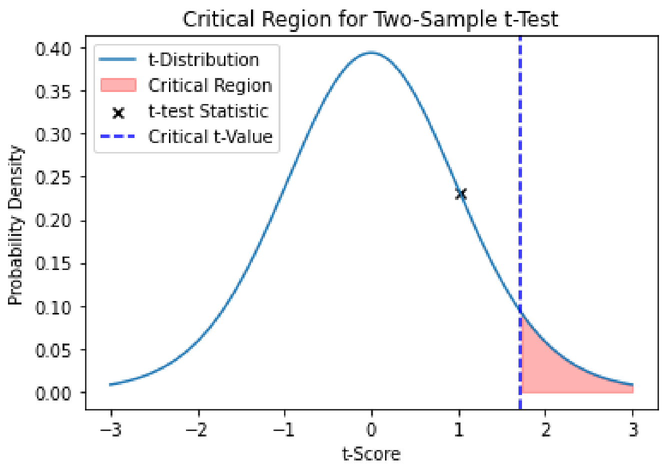
where, $sp = \sqrt{\dfrac{(n_1 - 1)s_1^2 + (n_2 - 1)s_2^2}{n_1 + n_2 - 2}}$

$$s_1^2 = \frac{\sum(x_1 - \overline{x}_1)^2}{n_1 - 1} \quad \text{and} \quad s_2^2 = \frac{\sum(x_2 - \overline{x}_2)^2}{n_2 - 1}$$

4. **Calculations:** $s_p = \sqrt{\dfrac{(12-1)4^2 + (10-1)5^2}{12+10-2}} = 4.478$

$$t_{cal} = \frac{(85-81)-2}{4.478\sqrt{1/12 + 1/10}} = \mathbf{1.04}$$

5. **Critical region:**



$t_{cal} > t_{tab}$

Where $t_{tab} = t_{(\alpha,\, n1+n2-2)} = t_{(0.05,\, 20)} = \mathbf{1.725}$

1.04 > **1.725** (False)

6. **Conclusion:** Since calculated value of t_{cal} is less than the tabulate value of t, so we accept H_O

```python
import numpy as np
from scipy import stats

# Given data for material 1
mean_material1 = 85
std_dev_material1 = 4
sample_size_material1 = 12

# Given data for material 2
mean_material2 = 81
std_dev_material2 = 5
sample_size_material2 = 10

# Set the significance level
alpha = 0.05

# Calculate the pooled standard deviation
pooled_std_dev = np.sqrt(((sample_size_material1 - 1) * std_dev_material1**2 + (sample_size_material2 - 1) * std_dev_material2**2) / (sample_size_material1 + sample_size_material2 - 2))
```

```python
# Calculate the standard error of the difference between means
standard_error_diff = pooled_std_dev * np.sqrt((1/sample_size_material1) + (1/sample_size_material2))


# Calculate the t-statistic
t_statistic = (mean_material1 - mean_material2 - 2) / standard_error_diff

# Degrees of freedom
degrees_of_freedom = sample_size_material1 + sample_size_material2 - 2

# Calculate the critical t-value for a two-tailed test
critical_t_value = stats.t.ppf(1 - alpha/2, degrees_of_freedom)

# Round the results to four decimal places
t_statistic = round(t_statistic, 4)
critical_t_value = round(critical_t_value, 4)
```

```python
# Print the results
print(f"t-statistic: {t_statistic}")
print(f"Critical t-value: {critical_t_value}")

# Compare the t-statistic with the critical t-value
if t_statistic > critical_t_value:
    print("Reject the null hypothesis. There is enough evidence to suggest that the abrasive wear of material 1 exceeds that of material 2 by more than 2 units.")
else:
    print("Fail to reject the null hypothesis. There is not enough evidence to suggest that the abrasive wear of material 1 exceeds that of material 2 by more than 2 units.")
```

Output:

```
t-statistic: 1.0432
Critical t-value: 2.086
Fail to reject the null hypothesis. There is not enough evidence
to suggest that the abrasive wear of material 1 exceeds that of
material 2 by more than 2 units.
```

Two Samples: Tests on Two Means

Example: An experiment was performed to compare the abrasive wear of two different laminated materials. **Twelve** pieces of material 1 were tested by exposing each piece to a machine measuring wear. **Ten** pieces of material 2 were similarly tested. In each case, the depth of wear was observed. The **samples** of material 1 gave an **average** (coded) wear of **85** units with a **sample standard deviation** of **4**, while the samples of material **2** gave an **average of 81** with a **sample standard** deviation of **5**.

Cont.

Can we conclude at the **0.05** level of significance that the abrasive wear of material 1 exceeds that of material 2 by more than 2 units? Assume the populations to be approximately normal with **unequal variances**.

Solution:

$n_1 = 12$ (Sample size of material 1)

$\overline{x_1} = 85$ (Sample mean of material 1)

$s_1 = 4$ (sample standard deviation of material 1)

$n_2 = 10$ (Sample size of material 2)

$\overline{x_2} = 81$ (Sample mean of material 2)

$s_2 = 5$ (sample standard deviation of material 1)

$\alpha = 0.05$ (Level of significance

Given assumption the populations to be approximately normal with **unequal variances**

$\Rightarrow$ $\boldsymbol{\sigma_1^2 \neq \sigma_1^2}$ **but unknown**

1. **We state our hypothesis as**:

H_0: $\mu_1 - \mu_2 = 2$

H_1: $\mu_1 - \mu_2 > 2$

(One tailed test)

2. **The level of significance is set** $\alpha = 0.05$.

3. **Test statistic to be used is**

$$t_{cal} = \frac{(\overline{x}_1 - \overline{x}_2) - (\mu_1 - \mu_2)}{\sqrt{\frac{s_1{}^2}{n_1} + \frac{s_2{}^2}{n_2}}}$$

where $s_1^2 = \frac{\sum(x_1 - \overline{x}_1)^2}{n_1 - 1}$ and $s_2^2 = \frac{\sum(x_2 - \overline{x}_2)^2}{n_2 - 1}$

4. **Calculations:** $t_{cal} = \frac{(85 - 81) - 2}{\sqrt{4^2/12 + 5^2/10}} = 1.0215$

5. **Critical region:**

$t_{cal} > t_{tab}$

where $t_{tab} = t_{(\alpha, v)}$

$$v = \frac{(s_1^2/n_1 + s_2^2/n_2)^2}{[(s_1^2/n_1)^2/(n_1-1) + (s_2^2/n_2)^2/(n_2-1)]}$$

$$v = \frac{(4^2/12 + 5^2/10)^2}{[(4^2/12)^2/11 + (5^2/10)^2/9]}$$

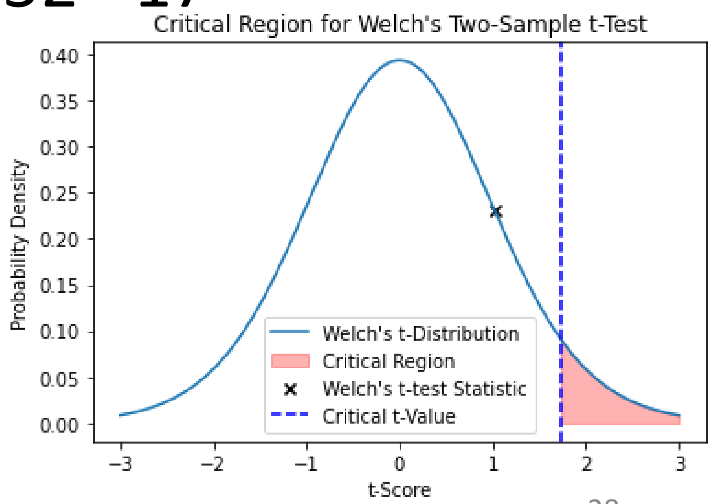
$v = 16.6944/(0.1616+0.6944) = 17.1652 = 17$

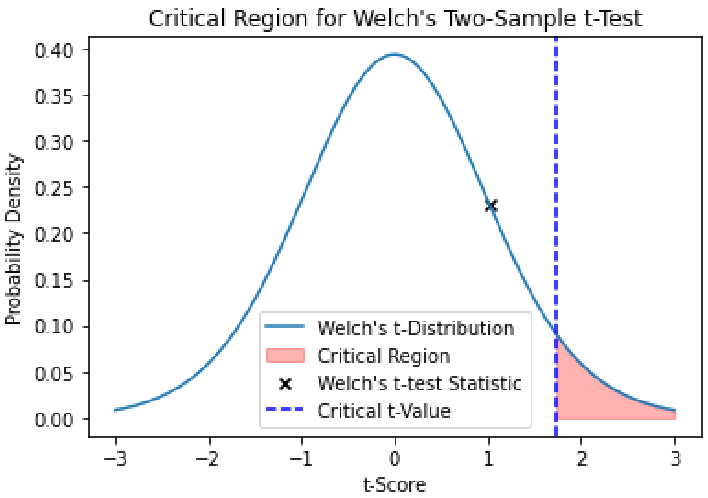
$\mathbf{t_{(\alpha, v)} = t_{(0.05, 17)} = 1.740}$

$\mathbf{t_{cal} = 1.0691}$

$t_{cal} > t_{tab}$

1.0251 > **1.740 (False)**





t_{cal} = 1.0691

$t_{cal} < t_{tab}$

1.0691 < 1.725 (False)

6. **Conclusion:** Since calculated value of t_{cal} is less than the tabulate value of t, so we accept H_O

```python
import numpy as np
from scipy import stats

# Given data for material 1
mean_material1 = 85
std_dev_material1 = 4
sample_size_material1 = 12

# Given data for material 2
mean_material2 = 81
std_dev_material2 = 5
sample_size_material2 = 10

# Calculate degrees of freedom for t-test
degrees_of_freedom = ((std_dev_material1**2 / sample_size_material1 + std_dev_material2**2 / sample_size_material2)**2) / \
                      ((std_dev_material1**2 / sample_size_material1)**2 / (sample_size_material1 - 1) + (std_dev_material2**2 / sample_size_material2)**2 / (sample_size_material2 - 1))
```

```python
# Calculate t-statistic for t-test
t_statistic = (mean_material1 - mean_material2 - 2) / 
np.sqrt((std_dev_material1**2 / sample_size_material1) + 
(std_dev_material2**2 / sample_size_material2))

# Set the significance level
alpha = 0.05

# Calculate the critical t-value
critical_t_value = stats.t.ppf(1 - alpha/2, degrees_of_freedom)
# Print the results
print(f"Degrees of Freedom: {round(degrees_of_freedom,4)}")
print(f"t-statistic: {round(t_statistic, 4)}")
print(f"Critical t-value: {round(critical_t_value, 4)}")
# Compare the t-statistic with the critical t-value
if abs(t_statistic) > critical_t_value:
    print("Reject the null hypothesis. There is enough evidence to 
suggest that the abrasive wear of material 1 exceeds that of 
material 2 by more than 2 units.")
else:
    print("Fail to reject the null hypothesis. There is not enough 
evidence to suggest that the abrasive wear of material 1 exceeds 
that of material 2 by more than 2 units.")
```

1. When $\alpha = 0.05$ or $\alpha = 5\%$

$z_{\alpha/2} = z_{0.0250} = 1.96$ $\quad \because 1 - \alpha/2 = 1 - 0.250 = 0.9750$

2. When $\alpha = 0.01$ or $\alpha = 1\%$

$z_{\alpha/2} = z_{0.005} = 2.575$ $\quad \because 1 - \alpha/2 = 1 – 0.005 = 0.9950$

3. When $\alpha = 0.10$ or $\alpha = 10\%$

$z_{\alpha/2} = z_{0.05} = 1.645$ $\quad \because 1 - \alpha/2 = 1 – 0.05 = 0.9500$

Testing a Proportion

1. H_0: $p = p_0$.

One of the alternatives H_1: $p < p_0$, $p>p_0$, or $p \neq p_0$.

2. Choose a level of significance equal to α.

3. Test statistic: Binomial variable X with $p = p_0$.

$$Z_{cal} = \frac{x - np_0}{\sqrt{np_0q_0}} \quad \text{or } Z_{cal} = \frac{\hat{p} - p_0}{\sqrt{p_0q_{0\,/\,n}}}$$

Where $\mathbf{p_0}$ is the **population proportion** and $\hat{\mathbf{p}}$ is the **sample proportion**

4. Computations: Find x, the number of successes, and compute the appropriate P-value.

5. Decision: Draw appropriate conclusions based on the *P*-value.

Example: A **builder claims** that heat pumps are installed in **70%** of all homes being constructed today in the city of Richmond, Virginia. Would you **agree with this claim** if a **random survey** of new homes in this city showed that **8 out of 15** had heat pumps installed? Use a **0.10 level of significance**.

Solution:

p or $p_0 = 0.70$ **(population proportion)**

$\hat{p} = x/n = 8/15 = 0.53$ **(sample proportion)**

$\hat{q} = 1 - 0.53 = 0.47$

$n=15$

1. **We state our hypothesis as**

H_0: p = 0.7.

H_1: p ≠ 0.7 (two tailed test)

2. **The level of significance is set at α** = 0.10.

3. **Test statistic to be used is**:

$$Z_{cal} = \frac{\hat{p} - p_0}{\sqrt{p_0 q_0 / n}}$$

4. **Calculations:**

$$Z_{cal} = \frac{0.53 - 0.70}{\sqrt{0.70(0.30)/15}}$$

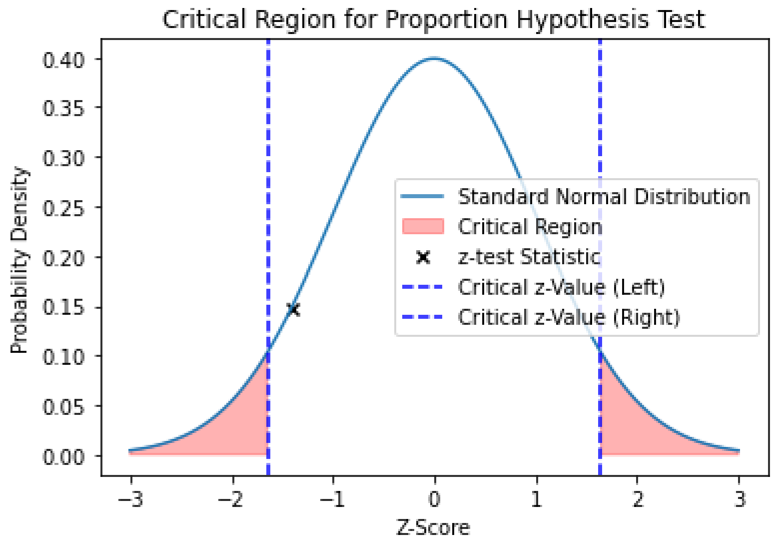
$$= -0.17/0.1183 = \mathbf{-1.4371}$$

5. **Critical region:**

$|Z_{cal}| > Z_{tab}$

Where $Z_{tab} = Z_{\alpha/2} = Z_{0.05} = 1.645$

$1.4371 > 1.645$ (False)



6. **Conclusion:** Since calculated value of Z is less than the tabulated value of Z, so we accept H_o and conclude that there is insufficient reason to doubt the builder's claim.

```python
from scipy.stats import norm

# Number of homes with heat pumps installed in the survey
x = 8

# Total number of homes surveyed
n = 15

# Assumed proportion under the null hypothesis
p_null = 0.70

# Sample proportion
p_sample = x / n

# Standard error of the proportion
se = ((p_null * (1 - p_null)) / n) ** 0.5
```

```python
# Calculate the z-score
z_score = (p_sample - p_null) / se

# Significance level
alpha = 0.10

# Find the critical value
critical_value = norm.ppf(1 - alpha / 2)

# Compare the z-score to the critical value
if abs(z_score) > critical_value:
    print(f"Reject the null hypothesis. There is enough evidence to suggest that the true proportion is different from {p_null}.")
else:
    print(f"Fail to reject the null hypothesis. There is not enough evidence to suggest that the true proportion is different from {p_null}.")
```

Example: A commonly prescribed drug for relieving nervous tension is believed to be only **60% effective**. Experimental results with a new drug administered to a random sample of **100 adults** who were suffering from nervous tension show that **70 received relief**. Is this sufficient evidence to conclude that the new drug is **superior** to the one **commonly prescribed**? Use a 0.05 level of significance.

Solution:

p or $p_0 = 0.60$ **(population proportion)**

$\hat{p} = x/n = 70/100 = 0.70$ **(sample proportion)**

$\hat{q} = 1 - 0.70 = 0.30$

$n = 100$ **(sample size)**

1. We state our hypothesis as

H_0: p = 0.60

H_1: p > 0.60 (one tailed test)

2. **The level of significance is set at α = 0.05**.

3. **Test statistic to be used is**:

$$Z_{cal} = \frac{\hat{p} - p_0}{\sqrt{p_0 q_0 / n}}$$

4. **Calculations:**

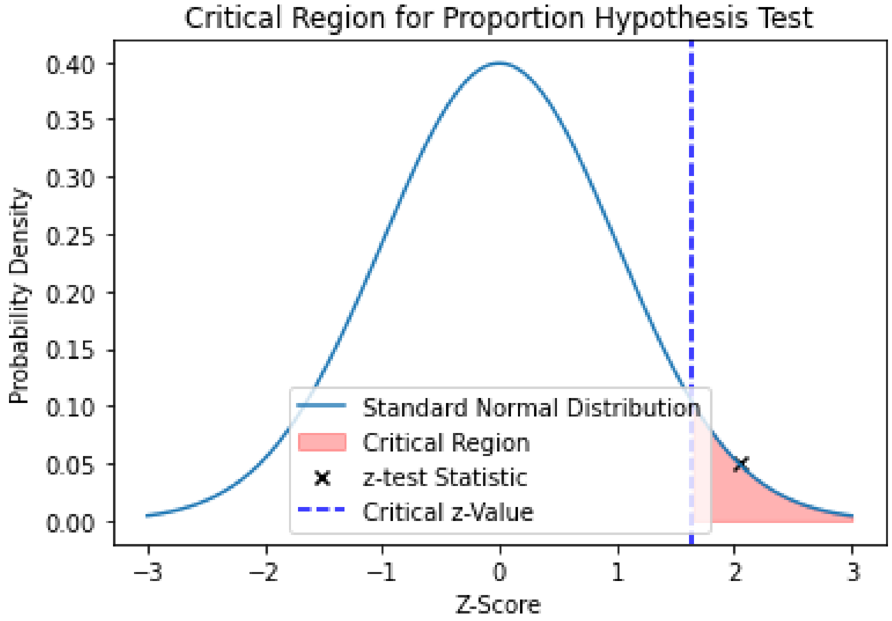
$$Z_{cal} = \frac{0.70 - 0.60}{\sqrt{(0.60)(0.40)/100}} = 2.04$$

5. **Critical region:**

$Z_{cal} > Z_{tab}$

Where $\mathbf{Z_{tab} = Z_{\alpha} = Z_{0.05} = 1.645}$

2.04 > 1.645 (True)



6. **Conclusion:** Since calculated value of Z is greater than the tabulated value of Z, so we reject H_o.

```python
import scipy.stats as stats

# Number of homes with heat pumps installed
x = 8

# Total number of homes surveyed
n = 15

# Assumed proportion under the null hypothesis
p_null = 0.70

# Calculate the sample proportion
p_sample = x / n

# Calculate the standard error of the proportion
se = ((p_null * (1 - p_null)) / n) ** 0.5

# Calculate the z-score
z_score = (p_sample - p_null) / se

# Significance level
alpha = 0.10
```

```python
# Find the critical region boundaries
critical_region_left = stats.norm.ppf(alpha / 2)
critical_region_right = stats.norm.ppf(1 - alpha / 2)

# Compare the z-score to the critical region boundaries
if z_score < critical_region_left or z_score > critical_region_right:
    print(f"Reject the null hypothesis. The true proportion is likely different from {p_null}.")
else:
    print(f"Fail to reject the null hypothesis. There is not enough evidence to suggest that the true proportion is different from {p_null}.")
```

Two Samples: Tests on Two Proportions

$$Z_{cal} = \frac{(\hat{p}_1 - \hat{p}_2) - (p_1 - p_2)}{\sqrt{P_c q_c \left(\frac{1}{n_1} + \frac{1}{n_2}\right)}}$$

where $p_c = \frac{x_1 + x_2}{n_1 + n_2}$ and $q_c = 1 - p_c$

Where $\mathbf{p_1}$ and $\mathbf{p_2}$ are the **population proportion from population 1 and population 2** and $\widehat{\mathbf{p_1}}$ and $\widehat{\mathbf{p_2}}$ are the corresponding respective **sample proportions.**

Example: A vote is to be taken among the residents of a town and the surrounding county to determine whether a proposed chemical plant should be constructed. The construction site is within the town limits, and for this reason many voters in the county believe that the proposal will pass because of the large proportion of town voters who favor the construction. To determine if there is a significant difference in the proportions of town voters and county voters favoring the proposal, a poll is taken. If **120 of 200** town voters favor the proposal and **240 of 500** county residents favor it, would you agree **that the proportion of town voters favoring the proposal is higher than the proportion of county voters**? Use an $\alpha = 0.05$ level of significance.

Solution:

$\widehat{p_1}$ = x_1/n_2 = 120/200 = 0.60 **(sample proportion of town voters, favoring it)**

$\widehat{q_1}$ = 1 – 0.60 = 0.40

$\mathbf{n_1}$ = 100 (sample size of town voters)

$\widehat{p_2}$ = x_2/n_2 = 240/500 = 0.60 **(sample proportion of county residents, favoring it)**

$\widehat{q_2}$ = 1 – 0.60 = 0.40

$\mathbf{n_2}$ = 100 (sample size of county residents)

$\mathbf{p_c = \frac{x_1+x2}{n_1+n2}}$ = (120 + 240)/(200 + 500) = 0.51

and $\mathbf{q_c}$ = 1 - p_c = 0.49

1. We state our hypothesis as

$H_0: p_1 = p_2$

$H_1: p_1 > p_2$ (one tailed test)

2. The level of significance is set at $\alpha = 0.05$.

3. Test statistic to be used is:

$$Z_{cal} = \frac{(\hat{p}_1 - \hat{p}_2) - (p_1 - p_2)}{\sqrt{P_c q_c \left(\frac{1}{n_1} + \frac{1}{n_2}\right)}}$$

where $p_c = \frac{x_1 + x_2}{n_1 + n_2}$, and $q_c = 1 - p_c$

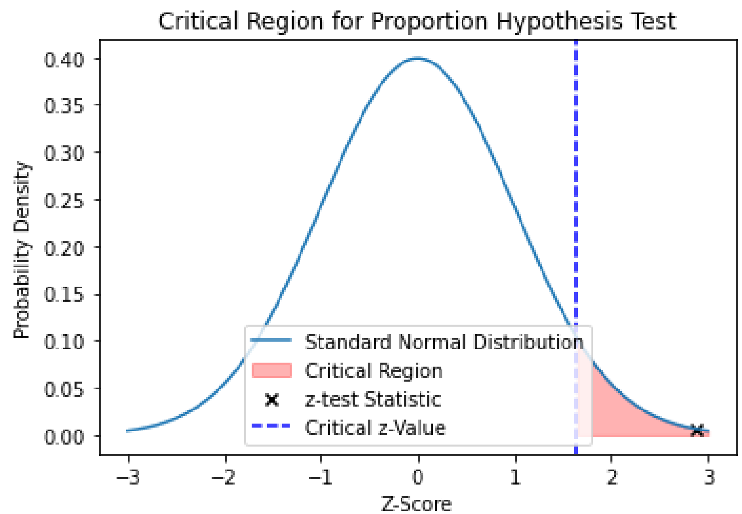
4. **Calculations:**

$$Z_{cal} = \frac{(0.60 - 0.48) - 0}{\sqrt{(0.51)(0.49)\left(\frac{1}{200} + \frac{1}{500}\right)}} = \mathbf{2.9}$$

5. **Critical region:**

$Z_{cal} > Z_{tab}$, where $Z_{tab} = Z_{\alpha} = Z_{0.05} = 1.645$

2.9 > **1.645 (True)**



6. **Conclusion:** Since calculated value of Z is greater than the tabulated value of Z, so we are unable to accept H_o agree that the proportion of town voters favoring the proposal is higher than the proportion of county voters.

```python
import math

# Town voters
n1 = 200
x1 = 120

# County voters
n2 = 500
x2 = 240

# Calculate sample proportions
p_hat1 = x1 / n1
p_hat2 = x2 / n2

# Calculate pooled sample proportion and qc
p_c = (x1 + x2) / (n1 + n2)
qc = 1 - p_c

# Calculate the standard error of the difference between proportions
se = math.sqrt(p_c * qc * (1/n1 + 1/n2))
```

```python
# Calculate the z-calculated value
z_cal = (p_hat1 - p_hat2) / se

# Significance level
alpha = 0.05

# Two-sided test, so we need to compare against a critical z-value
critical_value = stats.norm.ppf(1 - alpha / 2)

# Compare the z-calculated value to the critical value
if z_cal > critical_value:
    print("Reject the null hypothesis. The proportion of town voters favoring the proposal is higher than the proportion of county voters.")
else:
    print("Fail to reject the null hypothesis. There is not enough evidence to suggest that the proportion of town voters favoring the proposal is higher than the proportion of county voters.")
```