Chapter 6: Eigenvalues and Eigenvectors

1. **Definition.** Let $A \in \mathbb{C}^{n \times n}$ and $\lambda \in \mathbb{C}$ and $v \in \mathbb{C}^n$ be a nonzero vector so that $Av = \lambda v$. Then we say that λ is an eigenvalue of A and v is an eigenvector for the eigenvalue λ.
2. **Solving $Av = \lambda v$:** We can write the equation $Av = \lambda v$ as
$$(A - \lambda I)v = 0.$$
3. Hence $\lambda \in \mathbb{C}$ is an eigenvalue for A if and only if $\mathcal{N}(A - \lambda I)$ contains a **nonzero** vector v, that is, $\det(A - \lambda I) = 0$.
4. Therefore λ is an eigenvalue of A if and only if $\det(A - \lambda I) = 0$.
5. Notice that $\det(A - \lambda I) = 0 \Leftrightarrow \det(\lambda I - A) = 0$. Also, $p_A(x) = \det(xI - A)$ is a polynomial in x which is called the **characteristic polynomial** of A.
6. **Exercise**. Show that $p_A(x) = \det(xI - A)$ is a monic polynomial of degree n, i.e., the coefficient of the term x^n is 1. (Hint: Use induction on n, the order of the square matrix A).
7. It follows that λ is an eigenvalue of A iff it is a root of the polynomial $p_A(x)$.

Eigenvalues and eigenvectors of linear operators

1. Recall that $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$. We can define eigenvalues and eigenvectors for linear operators too.
2. **Definition.** Let V be a vector space over $\mathbb{F}$ and let $T : V \to V$ be a linear operator. A scalar $\lambda \in \mathbb{F}$ is said to be an **eigenvalue** of T if there is a nonzero vector $v \in V$ such that $T(v) = \lambda v$.
3. We say that v is an **eigenvector** of T with eigenvalue λ.
4. Let A be an $n \times n$ matrix over $\mathbb{F}$.
5. Notice that an eigenvalue and eigenvector of A are an eigenvalue and eigenvector of the linear map $T_A : \mathbb{F}^n \to \mathbb{F}^n$ defined by $T_A(v) = Av, \ v \in \mathbb{F}^n$, i.e., $\lambda \in \mathbb{F}$ is an eigenvalue of A if there exists a nonzero (column) vector $v \in \mathbb{F}^n$ with $T_A(v) = Av = \lambda v$.
6. **Example.** Let V be the real vector space of all smooth real valued functions on $\mathbb{R}$. Let $D = \frac{d}{dx} : V \to V$ be the derivative map. The function $f(x) = e^{\lambda x}$ is an eigenvector with eigenvalue λ since $D(e^{\lambda x}) = \lambda e^{\lambda x}$.

Eigenvalues and eigenvectors of linear operators

1. **Example.** Let A be a diagonal matrix with scalars $\mu_1, \ldots, \mu_n$ on the diagonal. We write this as $A = \text{diag}(\mu_1, \ldots, \mu_n)$.
2. Then $Ae_i = \mu_i e_i$ for $1 \le i \le n$ and so $e_1, \ldots, e_n$ are eigenvectors of A with the corresponding eigenvalues $\mu_1, \ldots, \mu_n$.
3. Let $T : V \to V$ be linear and let $\lambda \in \mathbb{F}$. It can be checked that

$$V_\lambda = \{v \in V : T(v) = \lambda v\}$$

   is a subspace of V. If $V_\lambda \neq \{0\}$, then λ is an eigenvalue of T.
4. Any nonzero vector in V_λ is an eigenvector (of T) with eigenvalue λ.
5. In this case we say that V_λ is the **eigenspace** of the eigenvalue λ.
6. **Theorem.** Let $T : V \to V$ be a linear operator. Let $\lambda_1, \ldots, \lambda_k \in \mathbb{F}$ be k distinct eigenvalues of T and let $v_1, \ldots, v_k$ be corresponding eigenvectors. Then $v_1, v_2, \ldots, v_k$ are linearly independent.

Eigenvalues and eigenvectors of linear operators

1. **Proof.** Use induction on k, the case $k = 1$ being clear.
2. Let $k > 1$. Let $c_1, c_2, \ldots, c_k \in \mathbb{F}$ such that

$$c_1 v_1 + c_2 v_2 + \cdots + c_k v_k = 0 \cdots\cdots\cdots (1)$$

3. Apply T to equation (1) to get

$$c_1 \lambda_1 v_1 + c_2 \lambda_2 v_2 + \cdots + c_k \lambda_k v_k = 0 \cdots\cdots\cdots (2)$$

4. Now, $(2) - \lambda_1 \times (1)$ implies

$$c_2(\lambda_2 - \lambda_1) v_2 + \cdots + c_k(\lambda_k - \lambda_1) v_k = 0.$$

5. Since $\lambda_1, \lambda_2, \ldots, \lambda_k$ are distinct, we get by induction, that $c_2 = \cdots = c_k = 0$ and by substituting these values in (1) we get $c_1 = 0$ too. $\square$
6. **Example.** The functions $e^{\lambda_1 x}, \ldots, e^{\lambda_k x}$ where $\lambda_1, \ldots, \lambda_k$ are distinct real numbers, are linearly independent.

Diagonalizable matrices and linear operators

1. **Definition.** Let V be a finite dimensional vector space over $\mathbb{F}$ and let $T : V \to V$ be a linear operator. We say that T is **diagonalizable** if there exists a basis of V consisting of eigenvectors of T.
2. If $B = (v_1, \ldots, v_n)$ is an ordered basis with $T(v_i) = \lambda_i v_i,\ \lambda_i \in \mathbb{F}$, then
$$M_B^B(T) = \operatorname{diag}(\lambda_1, \ldots, \lambda_n).$$
3. **Definition.** An $n \times n$ matrix A over $\mathbb{F}$ is said to be diagonalizable if $T_A : \mathbb{F}^n \to \mathbb{F}^n$, given by $T_A(v) = Av,\ v \in \mathbb{F}^n$, is diagonalizable.
4. **Proposition.** An $n \times n$ matrix A over $\mathbb{F}$ is diagonalizable if and only if $P^{-1}AP$ is a diagonal matrix, for some invertible matrix P over $\mathbb{F}$.
5. In that case, the columns of P are eigenvectors of A and the ith diagonal entry of $P^{-1}AP$ is the eigenvalue associated with the ith column of P.
6. **Proof.** Let A be diagonalizable and let $(v_1, \ldots, v_n)$ be an ordered basis of $\mathbb{F}^n$ with $T_A(v_i) = Av_i = \lambda_i v_i$.

Similar matrices have same characteristic polynomial

1. Let $P = [v_1 \ v_2 \cdots v_n]$ and $D = \operatorname{diag}(\lambda_1, \ldots, \lambda_n)$. Then
$$AP = A[v_1 \ v_2 \cdots v_n] = [Av_1 \ Av_2 \cdots Av_n] = [\lambda_1 v_1 \ \lambda_2 v_2 \cdots \lambda_n v_n] = PD.$$
2. Since P is invertible (why?), we have $P^{-1}AP = D$.
3. Conversely, suppose $P^{-1}AP = D$, where $D = \operatorname{diag}(\lambda_1, \ldots, \lambda_n)$. Then for $P = [u_1 \ u_2 \cdots u_n]$, $P^{-1}AP = D \implies AP = PD$
$$\implies A[u_1 \ u_2 \cdots u_n] = [Au_1 \ Au_2 \cdots Au_n] = [\lambda_1 u_1 \ \lambda_2 u_2 \cdots \lambda_n u_n].$$
4. Hence the ith column vector u_i of P is an eigenvector with eigenvalue λ_i. $\square$
5. **Proposition.** If $A = PBP^{-1}$ then $p_A(x) = p_B(x)$.
6. **Proof.** We have
$$\begin{aligned} p_A(x) &= \det(xI - PBP^{-1}) \\ &= \det(P(xI - B)P^{-1}) \\ &= \det(P)\det(xI - B)\det(P^{-1}) = p_B(x). \quad \square \end{aligned}$$

Roots of the characteristic polynomials

1. **Proposition.** (1) Eigenvalues of a square matrix $A \in \mathbb{F}^{n \times n}$ are the roots of $p_A(x)$ lying in $\mathbb{F}$. (2) For a scalar $\lambda \in \mathbb{F}$, V_λ = nullspace of $A - \lambda I$.
2. **Proof.** (1) $\lambda \in \mathbb{F}$ is an eigenvalue of $A \iff Av = \lambda v$ for some nonzero v

   $\iff (A - \lambda I)v = 0$ for some nonzero v

   $\iff$ the nullity of $(A - \lambda I) > 0$

   $\iff \operatorname{rank}(A - \lambda I) < n \iff \det(A - \lambda I) = 0 \iff p_A(\lambda) = 0.$
3. (2) $V_\lambda = \{v \mid Av = \lambda v\} = \{v \mid (A - \lambda I)v = 0\} = \mathcal{N}(A - \lambda I).$ $\square$
4. **Example.** Let $A = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$. To find the eigenvalues of A we solve the equation

$$p_A(x) = \det(xI - A) = \det \begin{bmatrix} x-1 & -2 \\ 0 & x-3 \end{bmatrix} = (x-1)(x-3) = 0.$$

Examples

1. Hence the eigenvalues of A are 1 and 3.
2. Let us calculate the eigenspaces V_1 and V_3. By definition
$$V_1 = \{v \mid (A - I)v = 0\} \text{ and } V_3 = \{v \mid (A - 3I)v = 0\}.$$
3. For $A - I = \begin{bmatrix} 0 & 2 \\ 0 & 2 \end{bmatrix}$ and $v = \begin{bmatrix} x \\ y \end{bmatrix}$, solve $\begin{bmatrix} 0 & 2 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.
4. It follows that $V_1 = L\{[1, 0]^t\}$. Now,
for $A - 3I = \begin{bmatrix} -2 & 2 \\ 0 & 0 \end{bmatrix}$ and $v = \begin{bmatrix} x \\ y \end{bmatrix}$, solve $\begin{bmatrix} -2 & 2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.
5. It follows that $V_3 = L(\{[1, 1]^t\})$. Is there a P such that $P^{-1}AP = \mathrm{diag}(1, 3)$?
6. Notice that any two eigenvectors corresponding to the distinct eigenvalues 1 and 3 are linearly independent.

Examples

1. **Example.** We use the notation $i = \sqrt{-1}$.
2. Let $A = \begin{bmatrix} \cos\ \theta & -\sin\ \theta \\ \sin\ \theta & \cos\ \theta \end{bmatrix}$, where $\theta \neq n\pi,$ for any $n \in \mathbb{Z}$. Now

$$\begin{aligned} p_A(x) = \det(xI - A) &= \det \begin{bmatrix} x - \cos\ \theta & \sin\ \theta \\ -\sin\ \theta & x - \cos\ \theta \end{bmatrix} \\ &= (x - \cos\ \theta)^2 + \sin^2\ \theta = x^2 - 2\cos\theta x + 1 \\ &= (x - e^{i\theta})(x - e^{-i\theta}). \end{aligned}$$

3. So, the real matrix A has no eigenvalues and thus no eigenvectors.
4. Recall that A represents the rotation by θ.
5. But as a complex matrix A has two distinct eigenvalues $e^{i\theta}$ and $e^{-i\theta}$.
6. An eigenvector corresponding to $e^{i\theta}$ is $[1, -i]^{\mathrm{t}}$ and an eigenvector corresponding to $e^{-i\theta}$ is $[-i, 1]^{\mathrm{t}}$.

Examples

1. **Example.** Find A^8, where $A = \begin{bmatrix} 4 & -3 \\ 2 & -1 \end{bmatrix}$. The eigenvalues of A are $2, 1$.

   The corresponding eigenvectors are $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

2. Set $P = \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}$ and $D = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$.

3. Then $P^{-1} = \begin{bmatrix} 1 & -1 \\ -2 & 3 \end{bmatrix}$ and $A = PDP^{-1}$.

4. We find A^8 using the eigenvalues.

$$\begin{aligned} A^8 &= (PDP^{-1})^8 = (PDP^{-1}) \cdots (PDP^{-1}) = PD^8P^{-1} \\ &= \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 2^8 & 0 \\ 0 & 1^8 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -2 & 3 \end{bmatrix} \\ &= \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 256 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -2 & 3 \end{bmatrix} = \begin{bmatrix} 766 & -765 \\ 510 & -509 \end{bmatrix}. \end{aligned}$$

Algebraic and geometric multiplicity of eigenvalues

1. Let $T : V \to V$ be a linear transformation of a finite dimensional vector space over $\mathbb{F}$.
2. We define the **characteristic polynomial** $p_T(x)$ of T to be $p_A(x)$, where $A = M_B^B(T)$ with respect to an ordered basis B of V.
3. Since $M_B^B(T) = (M_C^B)^{-1} M_C^C(T) M_C^B$ where C is another ordered basis of V, $p_T(x)$ is well defined (as for $P = M_C^B$, $M_B^B(T) = P^{-1} M_C^C(T) P$).
4. Let $f(x)$ be a polynomial with coefficients in $\mathbb{F}$.
5. Let $\mu \in \mathbb{F}$ be a root of $f(x)$. Then $(x - \mu)$ divides $f(x)$.
6. The **multiplicity** of the root μ is the **largest positive integer** k such that $(x - \mu)^k$ divides $f(x)$.
7. Let V be a fdvs over $\mathbb{F}$ and let $T : V \to V$ be a linear operator.
8. Let μ be an eigenvalue of T. Then the **geometric multiplicity** of μ is $\dim V_\mu$ and the **algebraic multiplicity** of μ is the multiplicity of μ as a root of the characteristic polynomial $p_T(x)$.

Geometric multiplicity $\leq$ algebraic multiplicity

1. **Theorem.** Let V be a finite dimensional vector space over $\mathbb{F}$. Then the geometric multiplicity of an eigenvalue $\mu \in \mathbb{F}$ of T is less than or equal to the algebraic multiplicity of μ.
2. **Proof.** Suppose that the algebraic multiplicity of μ is k and the geometric multiplicity of μ is ℓ.
3. Hence V_μ has a basis of ℓ eigenvectors $v_1, v_2, \ldots, v_\ell$.
4. We can extend this basis of V_μ to an ordered basis of V, say $B = (v_1, v_2, \ldots, v_\ell, \ldots, v_n)$.
5. Now

$$M_B^B(T) = \left[\begin{array}{c|c} \mu I_\ell & D \\ \hline 0 & C \end{array}\right]$$

where D is an $\ell \times (n-\ell)$ matrix and C is an $(n-\ell) \times (n-\ell)$ matrix.
6. It is now clear from the form of $M_B^B(T)$ that $(x-\mu)^\ell$ divides $p_T(x) = \det(xI - M_B^B(T))$. Thus $\ell \leq k$. $\square$

Criterion for diagonalizability

1. **Theorem.** Let $T : V \to V$ be a linear operator, where V is an n-dimensional vector space over $\mathbb{F}$. Then
   1. T is diagonalizable $\iff \sum_\lambda \dim V_\lambda = \dim V$.
   2. Assume $\mathbb{F} = \mathbb{C}$. Then T is diagonalizable iff the algebraic and geometric multiplicities are equal for each eigenvalue of T.
2. **Proof.** (1) Suppose that T is diagonalizable. Let $\lambda_1, \ldots, \lambda_k$ be the distinct eigenvalues of T. Let B_i be a basis of V_{λ_i} for $i = 1, 2, \ldots, k$.
3. Note that $V_\lambda \cap V_\mu = \{0\}$ for $\lambda \neq \mu$ and hence $B_1 \cup B_2$ is L.I. and it follows by induction that $B_1 \cup B_2 \cup \cdots \cup B_k$ is L.I. and, being a spanning set for V, it is a basis of V having eigenvectors of T.
4. Since $B_i \cap B_j = \varnothing$ for $i \neq j$ (why?), $\sum_\lambda \dim V_\lambda = \dim V$.
5. The converse assumption ensures that V has a basis containing eigenvectors of T and hence T is diagonalizable.

Proof of the second part

1. (2) Let $\mathbb{F} = \mathbb{C}$. Let $\lambda_1, \ldots, \lambda_k$ be the distinct eigenvalues of T.
2. By the Fundamental theorem of Algebra, $p_T(x) = \prod_{i=1}^{k}(x - \lambda_i)^{m_i}$, where m_i is the algebraic multiplicity of λ_i.
3. Since $\sum_i m_i = n$, if T is diagonalizable it follows from the first part that $m_i = \dim V_{\lambda_i}$=geometric multiplicity of λ_i (why?).
4. Conversely, if the algebraic and geometric multiplicities are equal for each eigenvalue of T, it follows from the fact $\sum_i m_i = n$ that $\sum_\lambda \dim V_\lambda = \dim V$ and hence T is diagonalizable by the first part. $\square$

Examples

1. **Example.** (1) $A = \begin{bmatrix} 3 & 0 & 0 \\ -2 & 4 & 2 \\ -2 & 1 & 5 \end{bmatrix}$, $\det(xI - A) = (x-3)^2(x-6)$.
2. Hence eigenvalues of A are 3 and 6. The eigenvalue $\lambda = 3$ has algebraic multiplicity 2 and the algebraic multiplcity of 6 is one.
3. Let us find the eigenspaces V_3 and V_6.

$$\text{For } \lambda = 3 : A - 3I = \begin{bmatrix} 0 & 0 & 0 \\ -2 & 1 & 2 \\ -2 & 1 & 2 \end{bmatrix}. \quad \text{Hence } \operatorname{rank}(A - 3I) = 1.$$

4. Therefore nullity $(A - 3I) = 2$. So the geometric multiplicity of the eigenvalue 3 is 2 which is also the algebraic multiplicity of the eigenvalue 3.
5. By solving the linear system $(A - 3I)v = 0$, we find that $\mathcal{N}(A - 3I) = V_3 = L(\{[1, 0, 1]^t, [1, 2, 0]^t\})$.

Examples

1. For $\lambda = 6$: $A - 6I = \begin{bmatrix} -3 & 0 & 0 \\ -2 & -2 & 2 \\ -2 & 1 & -1 \end{bmatrix}$. Hence $\text{rank}(A - 6I) = 2$.
2. Therefore $\dim V_6 = 1$. We can show that $\{[0, 1, 1]^t\}$ is a basis of V_6.
3. Therefore the algebraic and geometric multiplicities of the eigenvalue 6 are 1.
4. Let $P = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 1 & 0 & 1 \end{bmatrix}$, then $P^{-1}AP = \text{diag}(3, 3, 6)$. $\square$
5. **Example.** (2) Let $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$.
6. Then $\det(xI - A) = (x - 1)^2$ and hence A has only one eigenvalue 1 with the algebraic multiplicity 2.
7. $A - I = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. Hence nullity $(A - I) = 1$ and $V_1 = L(\{e_1\})$.
8. Therefore the geometric multiplicity $<$ algebraic multiplicity of the eigenvalue 1 and hence A is not a diagonalizable matrix. $\square$

Revisiting orthogonal and unitary matrices

1. **Proposition.** The set $\{v_1, v_2, \ldots, v_n\}$ is an orthonormal basis of $\mathbb{R}^n$ (resp. $\mathbb{C}^n$) if and only if the matrix $P = [v_1 \;\; v_2 \cdots v_n]$ is an orthogonal (resp. unitary) matrix, that is, $P^t P = I$ (resp. $P^* P = I$).

2. **Proof.** Note that $P^t P = [v_1 \;\; v_2 \cdots v_n]^t [v_1 \;\; v_2 \cdots v_n] = \begin{bmatrix} v_1^t \\ v_2^t \\ \vdots \\ v_n^t \end{bmatrix} [v_1 \;\; v_2 \cdots v_n]$

$$= \begin{bmatrix} v_1^t v_1 & v_1^t v_2 & \cdots & v_1^t v_n \\ v_2^t v_1 & v_2^t v_2 & \cdots & v_2^t v_n \\ \vdots & \vdots & \vdots & \vdots \\ v_n^t v_1 & v_n^t v_2 & \cdots & v_n^t v_n \end{bmatrix} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$

3. if and only if the set $\{v_1, v_2, \ldots, v_n\}$ is orthonormal in $\mathbb{R}^n$ (and hence is an orthonormal basis of $\mathbb{R}^n$). The respective case in $\mathbb{C}^n$ is proved similarly. $\square$

Orthogonally and unitarily diagonalizable matrices

1. Recall that a complex $n \times n$ matrix A is **diagonalizable** if there is an invertible matrix $P \in \mathbb{C}^{n \times n}$ so that $P^{-1}AP = \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$. Recall also that the column vectors of P are eigenvectors of A corresponding to the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ of A.
2. **Definition.** A matrix $A \in \mathbb{C}^{n \times n}$ is called **unitarily diagonalizable** if there is an orthonormal basis of $\mathbb{C}^n$ consisting of eigenvectors of A, and this is equivalent of saying that there exists a unitary matrix U such that $U^{-1}AU = U^*AU = \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$.
3. **Definition.** A real $n \times n$ matrix A is called **orthogonally diagonalizable** if there is an orthonormal basis of $\mathbb{R}^n$ consisting of eigenvectors of A, and this is equivalent of saying that there exists an orthogonal matrix P such that $P^{-1}AP = P^tAP = \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$.
4. **Theorem (Spectral Theorem for real matrices).** $A \in \mathbb{R}^{n \times n}$ is a symmetric matrix if and only if A is orthogonally diagonalizable.

Normal and symmetric matrices

1. **Theorem.** (a) $A \in \mathbb{R}^{n \times n}$ is orthogonally diagonalizable $\implies A^t = A$.
2. (b) $A \in \mathbb{C}^{n \times n}$ is unitarily diagonalizable $\implies A^*A = AA^*$.
3. **Proof.** (a) Let A be a real $n \times n$ orthogonally diagonalizable matrix.
4. Let $\{v_1, v_2, \dots, v_n\}$ be an orthonormal basis of $\mathbb{R}^n$ with $Av_i = \lambda_i v_i$, $\lambda_i \in \mathbb{R}$ and let $D = \mathrm{diag}(\lambda_1, \dots, \lambda_n)$.
5. Let $P = [v_1 \ v_2 \cdots v_n]$. Then

$$AP = [Av_1 \ Av_2 \cdots Av_n] = [\lambda_1 v_1 \ \lambda_2 v_2 \cdots \lambda_n v_n] = PD.$$

6. Since the set $\{v_1, v_2, \dots, v_n\}$ is orthonormal we have $P^t P = I = PP^t$.
7. Therefore

$$A = PDP^t \text{ and } A^t = PD^t P^t.$$

8. Since D is a diagonal matrix we have $D = D^t$ and hence $A^t = A$.

Normal and symmetric matrices

1. (b) Let A be a complex $n \times n$ unitarily diagonalizable matrix.
2. Let $\{v_1, v_2, \ldots, v_n\}$ be an orthonormal basis of $\mathbb{C}^n$ with $Av_i = \lambda_i v_i$, $\lambda_i \in \mathbb{C}$ and let $D = \text{diag}(\lambda_1, \ldots, \lambda_n)$.
3. Let $P = [v_1 \ v_2 \cdots v_n]$. Then
$$AP = [Av_1 \ Av_2 \cdots Av_n] = [\lambda_1 v_1 \ \lambda_2 v_2 \cdots \lambda_n v_n] = PD.$$
4. Since the set $\{v_1, v_2, \ldots, v_n\}$ is orthonormal we have $P^*P = I = PP^*$.
5. Thus
$$A = PDP^* \text{ and } A^* = PD^*P^*.$$
6. Therefore $AA^* = (PDP^*)(PD^*P^*) = PDD^*P^*$ and $A^*A = PD^*DP^*$.
7. Since D is a diagonal matrix, $D^*D = DD^*$ and hence $AA^* = A^*A$. □
8. **Definition.** A square complex matrix A is called **normal** if $A^*A = AA^*$.
9. A square complex matrix A is called **Hermitian** or **self-adjoint** (resp. skew-Hermitian) if $A^* = A$ (resp. $A^* = -A$). Notice that a real symmetric matrix is Hermitian and Hermitian matrices are normal.

Statement of the Spectral Theorems

1. **Theorem (Spectral Theorem for real symmetric matrices).** Any real symmetric $n \times n$ matrix A is orthogonally diagonalizable. In other words, there is an orthonormal basis of $\mathbb{R}^n$ consisting of eigenvectors of A.
2. **Theorem (Spectral Theorem for normal matrices).** Let A be an $n \times n$ complex normal matrix. Then there is an orthonormal basis of $\mathbb{C}^n$ consisting of eigenvectors of A. In other words, A is unitarily diagonalizable.
3. **Note.** We shall prove the Spectral Theorem for Hermitian matrices first and then deduce the one for normal matrices.
4. **Theorem.** The eigenvalues of a Hermitian matrix are real.
5. **Proof.** Let A be a Hermitian matrix. Then for any $v \in \mathbb{C}^n$

$$(v^*Av)^* = v^*A^*v = v^*Av.$$

6. Therefore v^*Av is a real number. Let λ be an eigenvalue of A with eigenvector v. Then $v^*Av = v^*(\lambda v) = \lambda(v^*v) = \lambda\|v\|^2 \implies \lambda \in \mathbb{R}$. $\square$

Self-adjoint operators on inner product spaces

1. Though a proof of the spectral theorem for self-adjoint matrices can be given working only with matrices, a coordinate free approach is more intuitive and more memorable.
2. Therefore, we first develop a coordinate free version of the concept of a self-adjoint matrix. The following definition covers both the real and complex cases.
3. **Definition.** Let V be a finite dimensional inner product space over $\mathbb{F}$. A linear operator $T : V \to V$ is said to be **self-adjoint** if

$$\langle x,\ T(y)\rangle = \langle T(x),\ y\rangle,\ \ x, y \in V.$$

4. **Example.** Let $A \in \mathbb{R}^{n\times n}$ be symmetric and $T_A : \mathbb{R}^n \to \mathbb{R}^n$ be the associated linear operator. Then T_A is self-adjoint. To see this, let $x, y \in \mathbb{R}^n$. Then

$$\langle x,\ T_A(y)\rangle = \langle x,\ Ay\rangle = x^t Ay = x^t A^t y = \langle Ax,\ y\rangle = \langle T_A(x),\ y\rangle.$$

$\square$

Characterization of self-adjoint operators

1. **Theorem.** Let V be a finite dimensional inner product space over $\mathbb{F}$ and let $T : V \to V$ be a linear operator. Then T is self-adjoint iff $M_B^B(T)$ is self-adjoint for every **ordered orthonormal basis** B of V.
2. **Proof.** Let $B = (v_1, \ldots, v_n)$ be an ordered orthonormal basis of V.
3. Suppose that T is self-adjoint and $A = (a_{ij}) = M_B^B(T)$.
4. Then $T(v_j) = \sum_{k=1}^n a_{kj} v_k$. So $\langle T(v_j),\, v_i\rangle = \langle \sum_{k=1}^n a_{kj} v_k,\; v_i\rangle = \overline{a_{ij}}$.
5. Therefore $\overline{a_{ij}} = \langle T(v_j),\, v_i\rangle = \langle v_j,\; T(v_i)\rangle = \langle v_j,\, \sum_{k=1}^n a_{ki} v_k\rangle = a_{ji}$.
6. Conversely suppose that $A = (a_{ij}) = M_B^B(T)$ is self-adjoint, i.e., $\overline{a_{ij}} = a_{ji}$.
7. Then $\langle T(v_j),\, v_i\rangle = \langle v_j,\; T(v_i)\rangle$. Let $x = \sum_{j=1}^n a_j v_j$ and $y = \sum_{i=1}^n b_i v_i$. Then

$$\begin{aligned}
\langle x,\; T(y)\rangle &= \langle \sum_j a_j v_j, \sum_i b_i T(v_i)\rangle = \sum_{j,i} \overline{a_j} b_i \langle v_j,\; T(v_i)\rangle, \\
\langle T(x),\; y\rangle &= \langle \sum_j a_j T(v_j), \sum_i b_i v_i\rangle = \sum_{j,i} \overline{a_j} b_i \langle T(v_j),\; v_i\rangle.
\end{aligned}$$

8. Therefore T is self-adjoint. □

Spectral Theorem for self-adjoint operators

1. **Theorem (Spectral Theorem for Self-Adjoint Operators).** Let V be a finite dimensional inner product space over $\mathbb{F}$ and let $T : V \to V$ be a self-adjoint linear operator. Then there exists an orthonormal basis of V consisting of eigenvectors of T.
2. **Proof.** By the fundamental theorem of algebra and the fact that Hermitian matrices have only real eigenvalues, there exists $\lambda \in \mathbb{R}$ and a unit vector $v \in V$ with $T(v) = \lambda v$. Put $W = L(\{v\})^{\perp}$.
3. **Claim.** (a). $w \in W$ implies $T(w) \in W$ (b). $T : W \to W$ is self-adjoint.
4. **Proof.** (a). $\langle T(w), v\rangle = \langle w, T(v)\rangle = \langle w, \lambda v\rangle = \lambda\langle w, v\rangle = 0$, since $w \in W$. Therefore $T(w) \in W$. (b). This is clear (as T is self-adjoint on V).
5. By induction on dimension, there is an orthonormal basis B of W consisting of eigenvectors of $T : W \to W$.
6. Now $\{v\} \cup B$ is the required orthonormal basis of V. $\square$

Eigenspaces of self-adjoint matrices are mutually $\perp$

1. **Proposition.** Let T be a self-adjoint operator on a finite-dimensional inner product space V. Let u, v be eigenvectors of T with distinct eigenvalues λ and μ respectively. Then $u \perp v$.
2. **Proof.** As T is self-adjoint, $\lambda, \mu \in \mathbb{R}$. Therefore,

$$\begin{aligned}(\lambda - \mu)\langle u, v\rangle = \lambda\langle u, v\rangle - \mu\langle u, v\rangle &= \langle \lambda u, v\rangle - \langle u, \mu v\rangle \\ &= \langle Tu, v\rangle - \langle u, Tv\rangle \\ &= \langle u, Tv\rangle - \langle u, Tv\rangle = 0.\end{aligned}$$

3. Since $\lambda \neq \mu$, $\langle u, v\rangle = 0$ and hence u and v are mutually perpendicular. $\square$
4. **Theorem.** Let T be a self-adjoint linear operator on a finite dimensional inner product space V. Let $\lambda_1, \dots, \lambda_k$ be the distinct eigenvalues of T. Then
$$V = V_{\lambda_1} \oplus V_{\lambda_2} \oplus \cdots \oplus V_{\lambda_k} \text{ and } \dim V = \textstyle\sum_{i=1}^{k} \dim V_{\lambda_i}.$$
5. **Proof.** Exercise (Hint: T is diagonalizable).

Spectral Theorem for real symmetric matrices

1. **Theorem (Spectral Theorem for Real Symmetric matrices).** Let A be an $n \times n$ real symmetric matrix with (real) eigenvalues $\lambda_1, \ldots, \lambda_n$. Set $D = \operatorname{diag}(\lambda_1, \ldots, \lambda_n)$. Then there exists an $n \times n$ real orthogonal matrix S such that $S^t AS = D$.
2. **Theorem.** Let A be an $n \times n$ Hermitian matrix with eigenvalues $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$. Set $D = \operatorname{diag}(\lambda_1, \ldots, \lambda_n)$. Then there exists an $n \times n$ unitary matrix U such that $U^* AU = D$.
3. **Proof of the theorems.** Recall that for given $n \times n$ real symmetric (resp. Hermitian) matrix A as in theorems, T_A is a self-adjoint operator on $\mathbb{R}^n$ (resp. on $\mathbb{C}^n$).
4. Now it follows from the spectral theorem for self-adjoint operators that there is an orthonormal basis B of $\mathbb{R}^n$ (resp. $\mathbb{C}^n$) such that
$$M_B^B(T_A) = \operatorname{diag}(\lambda_1, \ldots, \lambda_n) = D.$$
5. Recall that if E is the standard basis of $\mathbb{R}^n$ (resp. $\mathbb{C}^n$), then $M_E^E(T_A) = A$.

An algorithm for diagonalizing a self-adjoint matrix

1. Hence it follows from the relation $M_B^B(T_A) = M_B^E M_E^E(T_A) M_E^B$ that $M_B^B(T_A) = P^{-1}AP$ for $P = M_E^B$.
2. Since B is an orthonormal basis of $\mathbb{R}^n$ (resp. $\mathbb{C}^n$), the matrix $P = M_E^B$ is orthogonal (resp. unitary) in real (resp. complex) case and hence $P^{-1} = P^t$ (resp. $P^{-1} = P^*$). Thus $P^tAP = M_B^B(T_A) = D$ (resp. $P^*AP = D$).
3. Notice that this P is S of the first theorem and U of the second theorem. $\square$
4. **An algorithm for diagonalizing an $n \times n$ self-adjoint matrix:**
5. Find the distinct eigenvalues $\mu_1, \ldots, \mu_k$ of A. These are all real.
6. For each μ_i construct a basis of V_{μ_i} using Gauss elimination. Convert it to an orthonormal basis $B(\mu_i)$, for all $i = 1, 2, \ldots, k$, using Gram-Schmidt process.
7. Suppose this basis has d_i vectors. Then $d_1 + \cdots + d_k = n$. (why?)
8. Form an $n \times n$ matrix as follows: the first d_1 columns are the vectors in $B(\mu_1)$ the next d_2 columns are the vectors in $B(\mu_2)$ and so on. This is the matrix U in the complex case or S in the real case and $D = \mathrm{diag}(\mu_1, \ldots, \mu_1, \ldots, \mu_k)$.

Diagonalization of a real symmetric matrix

1. **Example.** Consider the real symmetric matrix

$$A = \begin{bmatrix} 1 & 2 & -2 \\ 2 & 1 & 2 \\ -2 & 2 & 1 \end{bmatrix}.$$

2. Solve $\det(\lambda I - A) = 0$. Check that the eigenvalues of A are $3, 3, -3$.
3. The eigenvectors for $\lambda = 3$ are in $\mathcal{N}(A - 3I)$, the null space of $A - 3I$.
4. They are the nonzero solutions of

$$\begin{bmatrix} -2 & 2 & -2 \\ 2 & -2 & 2 \\ -2 & 2 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Leftrightarrow \begin{bmatrix} -2 & 2 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

5. It follows that $\dim \mathcal{N}(A - 3I) = 2$ and $\mathcal{N}(A - 3I) = L(\{[1, 1, 0]^t, [-1, 0, 1]^t\})$.

Diagonalization of a real symmetric matrix

1. Apply Gram-Schmidt process to get an orthonormal basis of $V_3 = L(\{[1,1,0]^t, [-1,0,1]^t\})$:

$$v_1 = \left[\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right]^t \quad \text{and} \quad v_2 = \left[\frac{-1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}\right]^t.$$

2. Similarly we find that $\left\{v_3 = \left[\frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right]^t\right\}$ is an orthonormal basis of V_{-3}.
3. Set $S = [v_1 \ v_2 \ v_3]$ and $D = \operatorname{diag}(3,3,-3)$. Then $S^tAS = D$. $\square$

Diagonalization of commuting self-adjoint operators

1. **Definition.** Two operators $A, B : V \to V$ are called commuting operators if $AB = BA$.
2. **Theorem.** Let V be an n-dimensional complex inner product space. Let A, B be two commuting self-adjoint operators on V. Then there exists an orthonormal basis $(v_1, \ldots, v_n)$ of V such that each v_i is an eigenvector of both A and B.
3. **Proof.** Let $V_1, \ldots, V_r$ be eigenspaces of A for distinct eigenvalues $\mu_1, \ldots, \mu_r$.
4. Let $v \in V_i$. Then we claim that $B(v) \in V_i$. For,

$$A(B(v)) = (AB)(v) = (BA)(v) = B(A(v)) = B(\mu_i v) = \mu_i B(v).$$

5. Therefore for all i, $B : V_i \to V_i$ is a self-adjoint operator.
6. Hence each V_i has an orthonormal basis of eigenvectors of B and all of these vectors are already eigenvectors of A. $\square$

Diagonalization of normal matrices

1. **Theorem (Spectral Theorem for Normal Matrices).** A complex normal matrix is unitarily diagonalizable.
2. **Proof.** Let N be a normal matrix. Write $N = \frac{N+N^*}{2} + \frac{N-N^*}{2}$.
3. Put $A = (N + N^*)/2$ and $B = (N - N^*)/2$.
4. Check that $A = A^*$ and $B^* = -B$ and $AB = BA$.
5. $C = iB$ is Hermitian as $C^* = -iB^* = iB = C$, and $AC = CA$.
6. Therefore there is a common orthnormal eigenbasis $\mathcal{B}$ of A and C.
7. As $B = -iC$, $\mathcal{B}$ is an orthonormal eigenbasis of B and $N = A + B$. □
8. **Proposition.** Let U be an $n \times n$ unitary matrix. Then U is unitarily diagonalizable and every eigenvalue λ of U satisfies $|\lambda| = 1$.
9. **Proof.** Since U is normal (why?), it is unitarily diagonalizable and for $x, y \in \mathbb{C}^n$, $Ux \cdot Uy = (Ux)^* Uy = x^* U^* Uy = x^* y = x \cdot y$. So $\|Ux\| = \|x\|$.
10. If $x \neq 0$ and $Ux = \lambda x$ then $\|x\| = \|Ux\| = \|\lambda x\| = |\lambda| \|x\| \implies |\lambda| = 1$. □

Applications of spectral theorem to geometry

1. **Definition.** Let $A = (a_{ij})$ be an $n \times n$ real symmetric matrix. The **quadratic form** associated with A is the map $Q : \mathbb{R}^n \to \mathbb{R}$ defined as follows. For $X = [x_1, x_2, \ldots, x_n]^t \in \mathbb{R}^n$,
$$Q(X) = X^t AX = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} x_i x_j = \sum_{i=1}^{n} a_{ii} x_i^2 + \sum_{1 \le i < j \le n} 2a_{ij} x_i x_j .$$
2. If $A = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$, then $Q(X) = \lambda_1 x_1^2 + \lambda_2 x_2^2 + \cdots + \lambda_n x_n^2$ is called a **diagonal form**.
3. **Example.** Let $A = \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix}$, $X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$. Then
$$Q(X) = X^t AX = [x_1, x_2] \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1^2 + 4x_1x_2 + 5x_2^2.$$
4. **Theorem.** Let U be an orthogonal matrix such that $U^t AU = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$. Then if we put $X = UY = U[y_1, y_2, \ldots, y_n]^t$, then $Q(X) = X^t AX = \lambda_1 y_1^2 + \lambda_2 y_2^2 + \cdots + \lambda_n y_n^2$.

Diagonalization of quadratic forms

1. **Proof.** Since $X = UY$, $X^tAX = (UY)^tA(UY) = Y^t(U^tAU)Y$.
2. Since $U^tAU = \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$, we get

$$\begin{aligned} X^tAX &= [y_1, y_2, \ldots, y_n] \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} \\ &= \lambda_1 y_1^2 + \lambda_2 y_2^2 + \cdots + \lambda_n y_n^2. \end{aligned}$$

3. **Example.** Let us determine the orthogonal matrix U which reduces the quadratic form $Q(X) = 2x_1^2 + 4x_1x_2 + 5x_2^2$ to a diagonal form.
4. For, we write $Q(X) = [x_1, x_2] \begin{bmatrix} 2 & 2 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = X^tAX$.

Diagonalization of quadratic forms

1. The symmetric matrix $A = \begin{bmatrix} 2 & 2 \\ 2 & 5 \end{bmatrix}$ can be diagonalized.
2. The eigenvalues of A are $\lambda_1 = 1$ and $\lambda_2 = 6$.
3. An orthonormal set of eigenvectors for λ_1 and λ_2 is
$$u_1 = \frac{1}{\sqrt{5}} \begin{bmatrix} 2 \\ -1 \end{bmatrix} \quad \text{and} \quad u_2 = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$
4. Hence $U = \frac{1}{\sqrt{5}} \begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix}$. Check that $U^t AU = \operatorname{diag}(1, 6)$. Now use $X = UY$.
5. The diagonal form is: $Y^t \begin{bmatrix} 1 & 0 \\ 0 & 6 \end{bmatrix} Y = y_1^2 + 6y_2^2$.

Identification of conic sections

1. A conic section is the locus in the Cartesian plane $\mathbb{R}^2$ of an equation
$$ax^2 + bxy + cy^2 + dx + ey + f = 0.$$
2. It can be proved that this equation represents one of the following:
3. (i) the empty set (ii) single point (iii) one or two straight lines
4. (iv) ellipse (v) hyperbola (vi) parabola.
5. We consider the second degree part $Q(x, y) = ax^2 + bxy + cy^2$.
6. This is a quadratic form. This determines the type of the conic.
7. We can write the matrix form after setting $x = x_1, y = x_2$:
$$[x_1, x_2] \begin{bmatrix} a & b/2 \\ b/2 & c \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + [d, e] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + f \quad = \quad 0$$

Identification of conic sections

1. Write $A = \begin{bmatrix} a & b/2 \\ b/2 & c \end{bmatrix}$. Let $U = [u_1, u_2]$ be an orthogonal matrix where u_1 and u_2 are eigenvectors of A with eigenvalues λ_1 and λ_2.
2. Apply the change of variables $X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = U \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$ to diagonalize the quadratic form $Q(x_1, x_2)$ to the diagonal form $\lambda_1 y_1^2 + \lambda_2 y_2^2$.
3. The orthonormal basis $\{u_1, u_2\}$ determines new coordinate axes.
4. The locus of the equation $X^t AX + BX + f = 0$, where $B = [d, e]$, is same as the locus of the equation

$$\begin{aligned} 0 &= Y^t \operatorname{diag}(\lambda_1, \lambda_2) Y + (BU)Y + f \\ &= \lambda_1 y_1^2 + \lambda_2 y_2^2 + [d, e][u_1, u_2] \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} + f. \end{aligned}$$

An Ellipse

1. **Example.** We shall identify the conic section represented by
$$2x_1^2 + 4x_1x_2 + 5x_2^2 + 4x_1 + 13x_2 - 1/4 = 0.$$
2. We have earlier diagonalized the quadratic form $2x_1^2 + 4x_1x_2 + 5x_2^2$.
3. The associated symmetric matrix, the eigenvectors and eigenvalues are displayed in the equation of diagonalization :
$$U^tAU = \frac{1}{\sqrt{5}}\begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 2 & 2 \\ 2 & 5 \end{bmatrix} \frac{1}{\sqrt{5}}\begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 6 \end{bmatrix}.$$
4. Set $t = 1/\sqrt{5}$. Then the new coordinates are defined by
$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2t & t \\ -t & 2t \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}.$$
5. This means $x_1 = t(2y_1 + y_2)$ and $x_2 = t(-y_1 + 2y_2)$.

An Ellipse

1. Substitute these into the original equation to get
$$y_1^2 + 6y_2^2 - \sqrt{5}y_1 + 6\sqrt{5}y_2 - \frac{1}{4} = 0.$$
2. Complete the square to write this as
$$(y_1 - \frac{1}{2}\sqrt{5})^2 + 6(y_2 + \frac{1}{2}\sqrt{5})^2 = 9.$$
3. This represents an ellipse with center $(\frac{1}{2}\sqrt{5}, -\frac{1}{2}\sqrt{5})$ in the y_1y_2-plane.
4. The y_1 and y_2 axes are determined by the eigenvectors u_1 and u_2.
5. **Example.** Let us identify the locus of the equation
$$2x_1^2 - 4x_1x_2 - x_2^2 - 4x_1 + 10x_2 - 13 = 0.$$
6. We write the equation in matrix form as
$$[x_1, x_2]\begin{bmatrix} 2 & -2 \\ -2 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + [-4, 10]\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} - 13 = 0.$$

A hyperbola

1. Let $t = 1/\sqrt{5}$. The eigenvalues of A are $\lambda_1 = 3, \lambda_2 = -2$.
2. An orthonormal set of eigenvectors is $\{u_1 = t(2,-1)^t, u_2 = t(1,2)^t\}$.
3. Now write $U = t\begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix}$ and $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = U\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$.
4. The transformed equation becomes

$$3y_1^2 - 2y_2^2 - 4t(2y_1 + y_2) + 10t(-y_1 + 2y_2) - 13 = 0$$

$$\implies 3y_1^2 - 2y_2^2 - 18ty_1 + 16ty_2 - 13 = 0.$$

5. Complete the square to get $3(y_1 - 3t)^2 - 2(y_2 - 4t)^2 = 12$. Therefore

$$\frac{(y_1 - 3t)^2}{4} - \frac{(y_2 - 4t)^2}{6} = 1.$$

6. This represents a hyperbola with center $(3t, 4t)$ in the y_1y_2-plane.
7. The vectors u_1 and u_2 are the directions of positive y_1 and y_2 axes.

A parabola

1. **Example.** Consider $9x_1^2 + 24x_1x_2 + 16x_2^2 - 20x_1 + 15x_2 = 0$.
2. The symmetric matrix for the quadratic part is $A = \begin{bmatrix} 9 & 12 \\ 12 & 16 \end{bmatrix}$.
3. The eigenvalues are $\lambda_1 = 25, \lambda_2 = 0$.
4. Put $a = 1/5$. An orthonormal set of eigenvectors is $\{u_1 = a(3,4)^t, u_2 = a(-4,3)^t\}$.
5. An orthogonal diagonalizing matrix is $U = a\begin{bmatrix} 3 & -4 \\ 4 & 3 \end{bmatrix}$.
6. The equations of change of coordinates are
$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = U \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \implies x_1 = a(3y_1 - 4y_2),\ x_2 = a(4y_1 + 3y_2).$$
7. The equation in y_1y_2-plane is $y_1^2 + y_2 = 0$.
8. This is an equation of parabola with its vertex at the origin.