Problem Set 3

1. Let $\mathcal{F}$ and $\mathcal{G}$ be two sets of formulae. We say $\mathcal{F} \equiv \mathcal{G}$ iff for any assignment α, $\alpha \models \mathcal{F}$ iff $\alpha \models \mathcal{G}$ ($\alpha \models \mathcal{F}$ iff $\alpha \models F_i$ for every $F_i \in \mathcal{F}$). Prove or disprove: For any $\mathcal{F}$ and $\mathcal{G}$, $\mathcal{F} \equiv \mathcal{G}$ iff
   (1) For each $G \in \mathcal{G}$, there exists $F \in \mathcal{F}$ such that $G \models F$, and
   (2) For each $F \in \mathcal{F}$, there exists $G \in \mathcal{G}$ such that $F \models G$,

2. A set of sentences $\mathcal{F}$ is said to be closed under conjunction if for any F and G in $\mathcal{F}$, $F \wedge G$ is also in $\mathcal{F}$. Suppose $\mathcal{F}$ is closed under conjunction and is inconsistent. Prove that for any $G \in \mathcal{F}$, there exists $F \in \mathcal{F}$ such that $\{F\} \vdash \neg G$.

3. Suppose $\models (F \rightarrow G)$ and F is not a contradiction and G is not a tautology. Show that there exists a formula H such that the atomic propositions in H are in both F and G and $\models F \rightarrow H$ and $\models H \rightarrow G$.

4. Call a set of formulae minimal unsatisfiable iff it is unsatisfiable, but every proper subset is satisfiable. Show that there exist minimal unsatisfiable sets of formulae of size n for each $n \geq 1$.

5. Let $\psi = (a \vee \neg b) \wedge (b \vee \neg c) \wedge (c \vee \neg a) \wedge (\neg a \vee \neg b \vee \neg c) \wedge (a \vee b \vee c)$. Check if ψ is satisfiable using
   (a) Resolution as discussed in class, that is, maintain all clauses with you, and check if $\emptyset$ is reached. Why is this correct?
   (b) Can we optimize resolution? Instead of propagating all clauses (resolvents, and the parent clauses) obtained so far to the next step, can we only keep the new clauses obtained as a result of resolution in each step, and drop the parent clauses? Will this result in a correct resolution?