Chapter 4: Linear Transformations

1. Let A be an $m \times n$ matrix with real entries.

2. Then A "acts" on the n-dimensional space $\mathbb{R}^n$ by left multiplication : If $v \in \mathbb{R}^n$ then $Av \in \mathbb{R}^m$.

3. In other words, A defines a function

$$T_A : \mathbb{R}^n \longrightarrow \mathbb{R}^m, \qquad T_A(v) = Av.$$

4. By properties of matrix multiplication, T_A satisfies the following conditions:

   i. $T_A(v + w) = T_A(v) + T_A(w)$

   ii. $T_A(cv) = cT_A(v)$

   where $c \in \mathbb{R}$ and $v, w \in \mathbb{R}^n$.

5. We say that T_A respects the two operations in the vector space $\mathbb{R}^n$.

6. In this lecture we study such maps between the vector spaces.

Linear Transformations

Definition

Let V, W be vector spaces over $\mathbb{F}$. A linear transformation $T : V \longrightarrow W$ is a function satisfying

$$T(v + w) = T(v) + T(w) \text{ and } T(cv) = cT(v)$$

where $v, w \in V$ and $c \in \mathbb{F}$.

1. If $T : V \to W$ is a linear transformation, then $T(0) = 0$ (why?)
2. Examples:

i. For any pair of vector spaces V, W over $\mathbb{F}$, the "zero map" $T_0 : V \to W$ defined as $T_0(v) = 0$ for all $v \in V$, is clearly a linear transformation.

3. Can you now think of another linear map from a vector space V to itself?

ii. The identity map $I : V \to V$ defined as $I(v) = v$ for all $v \in V$, is clearly a linear map.

Linear Transformations: Examples

iii. Let $c \in \mathbb{R}, V = W = \mathbb{R}^2$. Define $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ by

$$T \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} c & 0 \\ 0 & c \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} cx \\ cy \end{bmatrix} = c \begin{bmatrix} x \\ y \end{bmatrix}.$$

It follows that T is a linear transformation (why?) since

$$\begin{aligned} &T(v+w) = c(v+w) = cv + cw = T(v) + T(w) \\ &T(dv) = c(dv) = d(cv) = dT(v), \text{ for } v, w \in \mathbb{R}^2, d \in \mathbb{R}. \end{aligned}$$

iv. Rotation: Fix θ and define $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ by

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix} \right) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x\cos\theta - y\sin\theta \\ x\sin\theta + y\cos\theta \end{bmatrix}.$$

1. Then $T(e_1) = (\cos\theta, \sin\theta)^t$ and $T(e_2) = (-\sin\theta, \cos\theta)^t$.
2. Thus T rotates the whole space by θ. (Draw a picture to convince yourself of this. Another way is to identify the vector $(x, y)^t$ with the complex number $z = x + iy$. Then we can write $T(z) = ze^{i\theta}$).

Linear Transformations: Examples

v. Let $\mathcal{D}$ be the vector space of differentiable functions $f : \mathbb{R} \longrightarrow \mathbb{R}$ such that $f^{(n)}$ exists for all n. Define $D : \mathcal{D} \longrightarrow \mathcal{D}$ by

$$D(f) = f^{'}.$$

1. Then D is a linear transformation (why?) since $D(af + bg) = af^{'} + bg^{'} = aD(f) + bD(g)$, for $f, g \in \mathcal{D}$ and $a, b \in \mathbb{R}$.

vi. Define $\mathcal{I} : \mathcal{D} \longrightarrow \mathcal{D}$ by

$$\mathcal{I}(f)(x) = \int_0^x f(t)\, dt.$$

2. By properties of integration, $\mathcal{I}$ is a linear transformation.

vii. The map $T : \mathbb{R} \to \mathbb{R}$ given by $T(x) = x^2$ is not linear (why?).

viii. Let $V = M_{n\times n}(\mathbb{F})$ be the vector space of all $n \times n$ matrices over $\mathbb{F}$. Fix $A \in V$. The map $T : V \to V$ given by $T(N) = AN$ is linear (why?).

Linear Transformations: Rank and Nullity

1. Let $T : V \to W$ be a linear transformation of vector spaces. There are two important subspaces associated with T.
   - Nullspace of T $= \mathcal{N}(T) = \{v \in V \mid T(v) = 0\}$.
   - Image of T $= \mathrm{Im}\,(T) = \{T(v) \mid v \in V\}$.
2. Let V be a finite dimensional vector space. Suppose that α, β are scalars. If $v, w \in \mathcal{N}(T)$ then $T(\alpha v + \beta w) = \alpha T(v) + \beta T(w) = 0$. Hence $\alpha v + \beta w \in \mathcal{N}(T)$.
3. Thus $\mathcal{N}(T)$ is a subspace of V. The dimension of $\mathcal{N}(T)$ is called the nullity of T and it is denoted by nullity (T).
4. Suppose that $v, w \in V$. Then
$$\alpha T(v) + \beta T(w) = T(\alpha v + \beta w) \in \mathrm{Im}\,(T).$$
5. Thus $\mathrm{Im}\,(T)$ is a subspace of W. The dimension of $\mathrm{Im}\,(T)$, denoted by rank(T), is called the rank of T.

Linear Transformations: Rank and Nullity

Proposition

Let $T : V \to W$ be a linear map of vector spaces. Then T is 1-1 if and only if $\mathcal{N}(T) = \{0\}$.

1. Proof: $(\Longleftarrow)$ $T(u) = T(v) \implies T(u - v) = 0 \implies u = v$.

   $(\Longrightarrow)$ $v \in \mathcal{N}(T) \implies T(v) = 0 = T(0) \implies v = 0$. $\square$

Proposition

Let V, W be vector spaces. Assume V is finite dimensional with $\{v_1, \ldots, v_n\}$ as a basis. Let $(w_1, \ldots, w_n)$ (these w_j's need not be distinct) be an arbitrary sequence of vectors in W. Then there is a unique linear map $T : V \to W$ with $T(v_i) = w_i$, for all $i = 1, \ldots, n$.

2. Proof: (uniqueness) Given $v \in V$ we can write (uniquely) $v = a_1 v_1 + \cdots + a_n v_n$, for scalars a_i. Then $T(v) = a_1 T(v_1) + \cdots + a_n T(v_n) = a_1 w_1 + \cdots + a_n w_n$. So T is determined by $(w_1, \ldots, w_n)$.

Linear Transformations: Rank and Nullity

1. (existence) Define T as follows.

2. Given $v \in V$ write (uniquely) $v = a_1 v_1 + \cdots + a_n v_n$, for scalars a_i and then define $T(v) = a_1 w_1 + \cdots + a_n w_n$.

3. Show that T is linear (exercise). □

Theorem (The Rank-Nullity Theorem)

Let $T : V \to W$ be a linear transformation of vector spaces where V is finite dimensional. Then

$$\operatorname{rank}(T) + \textit{nullity}\,(T) = \dim V.$$

4. Proof: Suppose $\dim V = n$. Let $B = \{v_1, v_2, \ldots, v_k\}$ be a basis of $\mathcal{N}(T)$. We can extend B to a basis $C = \{v_1, v_2, \ldots, v_k, w_1, w_2, \ldots, w_{n-k}\}$ of V.

5. We show that

$$D = \{T(w_1), T(w_2), \ldots, T(w_{n-k})\}$$

is a basis of $\mathrm{Im}\,(T)$.

Rank-Nullity Theorem: Proof continues...

1. Note that any $v \in V$ can be expressed uniquely as
$$v = \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_k v_k + \beta_1 w_1 + \cdots + \beta_{n-k} w_{n-k}.$$

2. This implies that
$$\begin{aligned} T(v) &= \alpha_1 T(v_1) + \cdots + \alpha_k T(v_k) + \beta_1 T(w_1) + \cdots + \beta_{n-k} T(w_{n-k}) \\ &= \beta_1 T(w_1) + \cdots + \beta_{n-k} T(w_{n-k}). \end{aligned}$$

3. Hence D spans $\mathrm{Im}\, T$.

4. We now show that D is linearly independent. For, suppose
$$\beta_1 T(w_1) + \cdots + \beta_{n-k} T(w_{n-k}) = T(\beta_1 w_1 + \cdots + \beta_{n-k} w_{n-k}) = 0.$$

5. Then $\beta_1 w_1 + \cdots + \beta_{n-k} w_{n-k} \in \mathcal{N}(T)$ and hence there are scalars $\alpha_1, \alpha_2, \ldots, \alpha_k$ such that
$$\begin{gathered} \beta_1 w_1 + \beta_2 w_2 + \cdots + \beta_{n-k} w_{n-k} = \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_k v_k \\ \implies \beta_1 = \beta_2 = \cdots = \beta_{n-k} = 0. \end{gathered}$$

Rank-Nullity Theorem: Proof continues...

1. Hence D is a basis of $\mathrm{Im}\, T$. Thus
$$\mathrm{rank}(T) = n - k = \dim V - \dim \mathcal{N}(T). \quad \square$$

2. Coordinate Vectors:

3. Let V be a finite dimensional vector space (fdvs) of dimension n over $\mathbb{F}$. By an ordered basis of V we mean a sequence $(v_1, v_2, \ldots, v_n)$ of distinct vectors of V such that the set $\{v_1, \ldots, v_n\}$ is linearly independent.

4. Let $u \in V$. Write uniquely (why?)
$$u = a_1 v_1 + a_2 v_2 + \cdots + a_n v_n,\ a_i \in \mathbb{F}.$$

5. Define the coordinate vector of u with respect to (wrt) the ordered basis B by
$$[u]_B = \begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix}^t.$$

6. Note that (why?) for vectors $u, v \in V$ and scalar $a \in \mathbb{F}$, we have
$$[u+v]_B = [u]_B + [v]_B,\ \ [av]_B = a[v]_B.$$

Coordinate Vectors: Change of Basis

1. Suppose $C = (u_1, \ldots, u_n)$ is another ordered basis of V.
2. Given $u \in V$, what is the relation between $[u]_B$ and $[u]_C$?
3. Define M_B^C, the transition matrix from C to B, to be the $n \times n$ matrix whose jth column is $[u_j]_B$:

$$M_B^C = [[u_1]_B \ [u_2]_B \cdots [u_n]_B] .$$

Proposition

Set $M = M_B^C$. Then, for all $u \in V$, we have

$$[u]_B = M[u]_C .$$

4. Proof: Let

$$[u]_C = \begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix}^t .$$

Coordinate Vectors: Change of Basis

1. Then $u = a_1u_1 + a_2u_2 + \cdots + a_nu_n$ and we have

$$\begin{aligned}
[u]_B &= [a_1u_1 + \cdots + a_nu_n]_B \\
&= a_1[u_1]_B + \cdots + a_n[u_n]_B \\
&= [[u_1]_B \; [u_2]_B \cdots [u_n]_B] \begin{bmatrix} a_1 \\ a_2 \\ \cdot \\ \cdot \\ a_n \end{bmatrix} \\
&= M[u]_C.
\end{aligned}$$

□

Coordinate Vectors: Change of Basis

1. Example: Let $V = \mathbb{R}^3$ and let

$$v_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix},\ v_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix},\ v_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},\ u_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix},\ u_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix},\ u_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

2. Consider the ordered bases $B = (v_1, v_2, v_3)$ and $C = (u_1, u_2, u_3)$. We have (why?)

$$M = M_B^C = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}.$$

3. Let $u = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$. So (why?) $[u]_C = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$.

4. Then

$$[u]_B = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}.$$

Coordinate Vectors: Change of Basis

1. Check that
$$\begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} = 2\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

Proposition

Let V be a finite dimensional vector space and B and C be two ordered bases. Then
$$M_B^C = (M_C^B)^{-1}.$$

2. Proof: Put $M = M_C^B$ and $N = M_B^C$. We need to show that $MN = NM = I$.
3. We have, for all $u \in V$, $[u]_B = N[u]_C$, $[u]_C = M[u]_B$.
4. It follows that, for all $u \in V$,
$$\begin{aligned} [u]_B &= N[u]_C = NM[u]_B \\ [u]_C &= M[u]_B = MN[u]_C. \end{aligned}$$
5. Thus (why?) $MN = NM = I$. $\square$

Coordinate Vectors: Change of Basis

1. Example: Let M be the $(n+1) \times (n+1)$ matrix, with rows and columns indexed by $\{0, 1, \ldots, n\}$, and with entry in row i and column j, $0 \leq i, j \leq n$, given by $\binom{j}{i}$. We show that M is invertible and find the inverse explicitly.
2. Consider the vector space $\mathcal{P}_n(\mathbb{R})$ of real polynomials of degree $\leq n$. Then $B = (1, x, x^2, \ldots, x^n)$ and $C = (1, x-1, (x-1)^2, \ldots, (x-1)^n)$ are both ordered bases (why?).
3. We claim that $M = M_C^B$. To see this note the following computation. For $0 \leq j \leq n$ we have

$$\begin{aligned} x^j &= (1 + (x-1))^j \\ &= \sum_{i=0}^{j} \binom{j}{i} (x-1)^i \\ &= \sum_{i=0}^{n} \binom{j}{i} (x-1)^i, \end{aligned}$$

where in the last step we have used the fact that $\binom{j}{i} = 0$ for $i > j$.

Coordinate Vectors: Change of Basis

1. Thus $M = \left[\binom{j}{i}\right] = M_C^B$ and hence it is invertible.
2. Since $M^{-1} = (M_C^B)^{-1} = M_B^C$, its entries are given by the following computation.
3. For $0 \leq j \leq n$, we have

$$\begin{aligned}(x-1)^j &= \sum_{i=0}^{j}(-1)^{j-i}\binom{j}{i}x^i \\ &= \sum_{i=0}^{n}(-1)^{j-i}\binom{j}{i}x^i.\end{aligned}$$

4. Thus the entry in row i and column j of M^{-1} is $(-1)^{j-i}\binom{j}{i}$, that is, $M^{-1} = \left[(-1)^{j-i}\binom{j}{i}\right]$. $\square$

Matrices and Linear Transformations

1. Let V and W be finite dimensional vector spaces with $\dim V = n$ and $\dim W = m$. Suppose $E = (v_1, v_2, \ldots, v_n)$ is an ordered basis for V and $F = (w_1, w_2, \ldots, w_m)$ is an ordered basis for W.

2. Let $T : V \longrightarrow W$ be a linear transformation.

3. We define $M_F^E(T)$, the matrix of T with respect to the ordered bases E and F, to be the $m \times n$ matrix whose jth column is $[T(v_j)]_F$:

$$M_F^E(T) \quad = \quad [[T(v_1)]_F \, [T(v_2)]_F \cdots [T(v_n)]_F] \, .$$

4. Please do the following important exercise.

5. Exercise: Let A be an $m \times n$ matrix over $\mathbb{F}$ and consider the linear map $T_A : \mathbb{F}^n \to \mathbb{F}^m$ given by $T_A(v) = Av$, for $v \in \mathbb{F}^n$ (we are considering column vectors here).

   Considering the ordered basis $E = (e_1, \ldots, e_n)$ and $F = (e_1, \ldots, e_m)$ of $\mathbb{F}^n$ and $\mathbb{F}^m$ respectively, show that $M_F^E(T_A) = A$.

Matrices and Linear Transformations

1. Let $\mathcal{L}(V, W)$ denote the set of all linear transformations from V to W. Suppose $S, T \in \mathcal{L}(V, W)$ and c is a scalar.
2. Define $S + T$ and cS as follows :

$$\begin{aligned} (S+T)(x) &= S(x) + T(x) \\ (cS)(x) &= cS(x) \end{aligned}$$

   for all $x \in V$.
3. It is easy to show that $\mathcal{L}(V, W)$ is a vector space under these operations.

Proposition

Fix ordered bases E and F of V and W respectively. For all $S, T \in \mathcal{L}(V, W)$ and scalar c we have

i. $M_F^E(S+T) = M_F^E(S) + M_F^E(T)$
ii. $M_F^E(cS) = cM_F^E(S)$
iii. $M_F^E(S) = M_F^E(T) \Leftrightarrow S = T$.

4. Proof: Exercise.

Matrices and Linear Transformations

Proposition

Suppose V, W are vector spaces of dimensions n, m respectively. Suppose $T : V \longrightarrow W$ is a linear transformation. Suppose $E = (v_1, \dots, v_n), F = (w_1, \dots, w_m)$ are ordered bases of V, W respectively. Then

$$[T(v)]_F = M_F^E(T)[v]_E, \quad v \in V.$$

1. Proof: Let
$$[v]_E = \begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix}^t.$$
2. Then $v = a_1v_1 + a_2v_2 + \cdots + a_nv_n$ and hence
$$T(v) = a_1T(v_1) + a_2T(v_2) + \cdots + a_nT(v_n).$$
3. We have
$$\begin{aligned} [T(v)]_F &= [a_1T(v_1) + \cdots + a_nT(v_n)]_F \\ &= a_1[T(v_1)]_F + \cdots + a_n[T(v_n)]_F \\ &= [[T(v_1)]_F \; [T(v_2)]_F \cdots [T(v_n)]_F] \begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix}^t \\ &= M_F^E(T)[v]_E. \quad \square \end{aligned}$$

Matrices and Linear Transformations

Proposition

Suppose U, V, W are vector spaces of dimension n, p, m respectively. Suppose $T : U \longrightarrow V$ and $S : V \longrightarrow W$ are linear transformations. Suppose E, F, G are ordered bases of U, V, W respectively. Then

$$M_G^E(SoT) = M_G^F(S)M_F^E(T).$$

1. Proof: Let $E = (u_1, u_2, \dots, u_n)$. Then, the jth column of $M_G^E(S \circ T)$ is
$$= [(S \circ T)(u_j)]_G = [S(T(u_j))]_G.$$

2. Now the jth column of $M_G^F(S)M_F^E(T)$ is
$$\begin{aligned} &= M_G^F(S)(j\text{th column of } M_F^E(T)) \\ &= M_G^F(S)[T(u_j)]_F \\ &= [S(T(u_j))]_G \quad (\text{since } [S(v)]_G = M_G^F(S)[v]_F). \end{aligned}$$

3. Hence $M_G^E(SoT) = M_G^F(S)M_F^E(T)$. □

Matrices and Linear Transformations

1. Let V be a finite dimensional vector space. A linear map $T : V \to V$ is said to be a linear operator on V. Let B, C be ordered bases of V.
2. The square matrix $M_B^B(T)$ is said to be the matrix of T with respect to the ordered basis B.
3. Note that the transition matrix M_B^C from C to B is the matrix $M_B^C(I)$ of the identity map with respect to the bases C and B.
4. Thus it follows that $M_B^C(I) = M_C^B(I)^{-1}$.

Matrices and Linear Transformations

Proposition

Let V be a finite dimensional vector space and B, C be a pair of two bases of V. Then, we have

$$M_B^B(T) = (M_C^B)^{-1} M_C^C(T) M_C^B.$$

1. Proof: Consider the sequence of linear operators where the bases used for computation of matrices of the linear transformations are specified:

$$(V, B) \xrightarrow{I} (V, C) \xrightarrow{T} (V, C) \xrightarrow{I} (V, B).$$

2. Note that the identity map I is just a map from V to V. It is not required that I maps B to C or C to B. This notation is used just to show that the mentioned bases are used for the computation of the matrices of the corresponding linear maps.
3. Then

$$\begin{aligned} T = I \circ T \circ I \quad &\implies \quad M_B^B(T) = M_B^C(I) M_C^C(T) M_C^B(I) \\ &\implies \quad M_B^B(T) = (M_C^B)^{-1} M_C^C(T) M_C^B. \end{aligned}$$

Matrices and Linear Transformations

1. Example: Consider the linear transformation
$$T : \mathbb{R}^2 \to \mathbb{R}^2,\ T(e_1) = e_1,\ \ T(e_2) = e_1 + e_2.$$

2. Let $C = (e_1, e_2)$ and $B = (e_1 + e_2,\ \ e_1 - e_2)$ be two ordered bases of $\mathbb{R}^2$.

3. Then
$$M_C^C(T) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix},\ M_C^B = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix},\ M_B^C = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{bmatrix}.$$

4. Since $M_B^B(T) = (M_C^B)^{-1} M_C^C(T) M_C^B = M_B^C M_C^C(T) M_C^B$, we get
$$M_B^B(T) \quad = \quad \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 3 & -1 \\ 1 & 1 \end{bmatrix}.$$

Sum of two subspaces and its dimension

1. Given subspaces V, W of a vector space U define the sum of V and W, denoted $V + W$, by
$$V + W = L(V \cup W).$$

Theorem

Let V, W be subspaces of a finite dimensional vector space U. Then
$$\dim(V + W) = \dim V + \dim W - \dim(V \cap W).$$

Proof of the formula for $\dim(V + W)$

1. Proof: We shall give a sketch of a proof leaving you to fill in the details.
2. Consider the set $V \times W = \{(v, w) : v \in V, w \in W\}$. This set is a vector space with component wise addition and scalar multiplication.
3. Check that the dimension of this space is $\dim V + \dim W$.
4. Define a linear map $T : V \times W \to V + W$ by $T((v, w)) = v - w$.
5. Check that T is onto and that the nullspace of T is $\{(v, v) : v \in V \cap W\}$.
6. The result now follows from the rank nullity theorem for linear maps. $\square$
7. Exercise:

i. Let V, W be finite dimensional vector spaces over $\mathbb{F}$ with dimensions n, m respectively. Fix ordered bases E, F for V, W respectively.

Consider the map $f : \mathcal{L}(V, W) \to M_{m \times n}(\mathbb{F})$ given by $f(T) = M_F^E(T)$, for $T \in \mathcal{L}(V, W)$. Show that f is linear, 1-1 and onto, which shows that $\dim \mathcal{L}(V, W) = mn$.