CS 228 : Logic in Computer Science

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Basic Rules So Far

- $ightharpoonup \land i, \land e_1, \land e_2$ (and introduction and elimination)
- $\rightarrow \neg \neg e, \neg \neg i$ (double negation elimination and introduction)
- ► MP (Modus Ponens)
- ightharpoonup
 ightharpoonup i (Implies Introduction : remember opening boxes)
- \lor $\lor i_1, \lor i_2, \lor e$ (Or introduction and elimination)

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► (p \lor q) \lor r \vdash p \lor (q \lor r)

1. (p \lor q) \lor r premise
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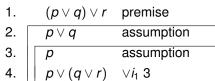
 $(p \lor q) \lor r \vdash p \lor (q \lor r)$

1.	$(p \lor q) \lor r$	premise
2.	$p \lor q$	assumption
3.		

$$(p \lor q) \lor r \vdash p \lor (q \lor r)$$

- 1. $(p \lor q) \lor r$ premise
- 2. $p \lor q$ assumption assumption
 - B. p assumption
 B. |

$$(p \lor q) \lor r \vdash p \lor (q \lor r)$$



 $(p \lor q) \lor r \vdash p \lor (q \lor r)$

1.	$(p \lor q) \lor r$	premise
2.	$p \lor q$	assumption
3.	р	assumption
4.	$p \lor (q \lor r)$	∨ <i>i</i> ₁ 3
5.	q	assumption
6.		

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2.	$p \lor q$	assumption
3.	р	assumption
4.	$p \lor (q \lor r)$	∨ <i>i</i> ₁ 3
5.	q	assumption
6.	$q \vee r$	∨ <i>i</i> ₁ 5
7.		

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2.	$p \lor q$	assumption
3.	p	assumption
4.	$p \lor (q \lor r)$	∨ <i>i</i> ₁ 3
5.	q	assumption
6.	$q \vee r$	∨ <i>i</i> ₁ 5
7.	$p \lor (q \lor r)$	∨ <i>i</i> ₂ 6
8.		

 $\blacktriangleright (p \lor q) \lor r \vdash p \lor (q \lor r)$

1.	$(p \lor q) \lor r$	premise
2.	$p \lor q$	assumption
3.	p	assumption
4.	$p \lor (q \lor r)$	∨ <i>i</i> ₁ 3
5.	q	assumption
6.	$ q \lor r$	<i>∨i</i> ₁ 5
7.	$p \lor (q \lor r)$	∨ <i>i</i> ₂ 6
8.	$p \lor (q \lor r)$	∨ <i>e</i> 2, 3-4, 5-7
a		

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7.	$p \lor (q \lor r)$	∨ <i>i</i> ₂ 6
8.	$p \lor (q \lor r)$	∨ <i>e</i> 2, 3-4, 5-7
9.	r	assumption
0.		

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2.	$p \lor q$	assumption
3.	p	assumption
4.	$p \lor (q \lor r)$	√ <i>i</i> ₁ 3
5.	q	assumption
6.	$ q \lor r$	∨ <i>i</i> ₁ 5
7.	$p \lor (q \lor r)$	∨ <i>i</i> ₂ 6
8.	$p \lor (q \lor r)$	∨ <i>e</i> 2, 3-4, 5-7
9.	r	assumption
0.	$q \lor r$	√ <i>i</i> ₂ 9
1.		

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2.	$p \lor q$	assumption
3.	p	assumption
4.	$p \lor (q \lor r)$	∨ <i>i</i> ₁ 3
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7.	$p \lor (q \lor r)$	∨ <i>i</i> ₂ 6
8.	$p \lor (q \lor r)$	∨ <i>e</i> 2, 3-4, 5-7
9.	r	assumption
10.	$q \vee r$	√ <i>i</i> ₂ 9
11.	$p \lor (q \lor r)$	√ <i>i</i> ₂ 10

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2.	$p \lor q$	assumption
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4.	$p \lor (q \lor r)$	∨ <i>i</i> ₁ 3
5.	q	assumption
6.	$ q \lor r$	∨ <i>i</i> ₁ 5
7.	$p \lor (q \lor r)$	∨ <i>i</i> ₂ 6
8.	$p \lor (q \lor r)$	∨ <i>e</i> 2, 3-4, 5-7
9.	r	assumption
١0.	$q \lor r$	∨ <i>i</i> ₂ 9
11.	$p \lor (q \lor r)$	√ <i>i</i> ₂ 10
12.	$p \lor (q \lor r)$	∨ <i>e</i> 1, 2-8, 9-11

$$\blacktriangleright \vdash p \rightarrow (q \rightarrow p)$$

1. true

premise

$$\blacktriangleright \vdash p \rightarrow (q \rightarrow p)$$

1.	true	premise
2.	р	assumption
3.		

▶
$$\vdash p \rightarrow (q \rightarrow p)$$

1.	true	premise
2.	р	assumption
3.	q	assumption
4.		

▶
$$\vdash p \rightarrow (q \rightarrow p)$$

	true	premise
2.	р	assumption
3.	q	assumption
ŀ.	р	copy 2
-		

▶
$$\vdash p \rightarrow (q \rightarrow p)$$

1.	true	premise
2.	р	assumption
3.	q	assumption
4.	p	copy 2
5.	$oldsymbol{q} ightarrow oldsymbol{p}$	→ <i>i</i> 3-4

6

▶
$$\vdash p \rightarrow (q \rightarrow p)$$

1.	true	premise
2.	р	assumption
3.	q	assumption
4.	р	copy 2
5.	$oldsymbol{q} ightarrow oldsymbol{p}$	→ <i>i</i> 3-4
6.	$p \rightarrow (q \rightarrow p)$	\rightarrow <i>i</i> 2-5

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- $ightharpoonup \perp \to \varphi$ for any formula φ .

Rules with \bot

The \perp elimination rule $\perp e$

$$\frac{\perp}{\psi}$$

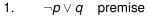
The \perp introduction rule $\perp i$

$$\frac{\varphi \qquad \neg \varphi}{\bot}$$

- ▶ $\neg p \lor q \vdash p \rightarrow q$
 - 1. $\neg p \lor q$ premise
 - 2.

▶
$$\neg p \lor q \vdash p \rightarrow q$$

- 1. $\neg p \lor q$ premise
- 2. $\neg p$ premise
- 3.

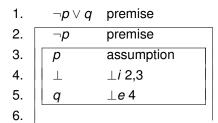


2. $\neg p$ premise

3. 4. p assumption

1.	$\neg p \lor q$	premise
2.	$\neg p$	premise
3.	р	assumption
4.		<i>⊥i</i> 2,3
5.		

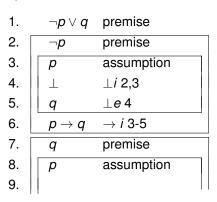
▶
$$\neg p \lor q \vdash p \rightarrow q$$



▶
$$\neg p \lor q \vdash p \rightarrow q$$

1.	$\neg p \lor q$	premise
2.	$\neg p$	premise
3.	р	assumption
4.		<i>⊥i</i> 2,3
5.	q	⊥ <i>e</i> 4
6.	p o q	→ <i>i</i> 3-5
7.	q	premise
8.		

▶
$$\neg p \lor q \vdash p \rightarrow q$$



▶
$$\neg p \lor q \vdash p \rightarrow q$$

1.	$\neg p \lor q$	premise	
2.	$\neg p$	premise	
3.	р	assumption	
4.		<i>⊥i</i> 2,3	
5.	q	⊥ <i>e</i> 4	
6.	p o q	→ <i>i</i> 3-5	
7.	q	premise	
8.	р	assumption	
9.	q	copy 7	
0.	p o q	<i>→ i</i> 8-9	
1.	p o q	∨ <i>e</i> 1, 2-6, 7-10	

Introducing Negations (PBC)

- In the course of a proof, if you assume φ (by opening a box) and obtain \bot in the box, then we conclude $\neg \varphi$
- ▶ This rule is denoted $\neg i$ and is read as \neg introduction.
- ► Also known as Proof By Contradiction

- 1. $p \rightarrow \neg p$ premise
- 2.

An Example

1.	p ightarrow eg p	premise
2.	р	assumption
3.		

An Example

1.	p ightarrow eg p	premise
2.	р	assumption
3.	$\neg p$	MP 1,2
4.		

An Example

1.	p ightarrow eg p	premise
2.	р	assumption
3.	$\neg p$	MP 1,2
4.		<i>⊥i</i> 2,3
5.	$\overline{\neg p}$	¬i 2-4

The Last One

Law of the Excluded Middle (LEM)



Summary of Basic Rules

- \blacktriangleright $\land i$, $\land e_1$, $\land e_2$,
- ¬¬e
- ► MP
- $\rightarrow i$
- $\triangleright \forall i_1, \forall i_2, \forall e$
- ▶ Copy, $\neg i$ or PBC
- ► *⊥e*, *⊥i*

Derived Rules

- ▶ MT (derive using MP, $\perp i$ and $\neg i$)
- $ightharpoonup \neg \neg i$ (derive using $\bot i$ and $\neg i$)
- ▶ LEM (derive using $\forall i_1, \bot i, \neg i, \forall i_2, \neg \neg e$)

The Proofs So Far

➤ So far, the "proof" we have seen is purely syntactic, no true/false meanings were attached

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The Proofs So Far

- So far, the "proof" we have seen is purely syntactic, no true/false meanings were attached
- ▶ Intuitively, $p \rightarrow q \vdash \neg p \lor q$ makes sense because you think semantically. However, we never used any semantics so far.
- Now we show that whatever can be proved makes sense semantically too.

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- $\varphi_1, \dots, \varphi_n \models \psi$ iff whenever $\varphi_1, \dots, \varphi_n$ evaluate to true, so does ψ . \models is read as semantically entails
 - ► Recall ⊢, and compare with ⊨

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- ▶ Formulae φ and ψ are provably equivalent iff $\varphi \vdash \psi$ and $\psi \vdash \varphi$

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Soundness of Propositional Logic

$$\varphi_1, \ldots, \varphi_n \vdash \psi \Rightarrow \varphi_1, \ldots, \varphi_n \models \psi$$

Whenever ψ can be proved from $\varphi_1, \dots, \varphi_n$, then ψ evaluates to true whenever $\varphi_1, \dots, \varphi_n$ evaluate to true