# CS 228 : Logic in Computer Science

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# Basic Rules So Far

- $\wedge$i, $\wedge e_1$, $\wedge e_2$ (and introduction and elimination)
- $\neg\neg e$, $\neg\neg i$ (double negation elimination and introduction)
- MP (Modus Ponens)
- $\rightarrow i$ (Implies Introduction : remember opening boxes)
- $\vee i_1, \vee i_2, \vee e$ (Or introduction and elimination)

# Associativity Using Or Elimination

- $(p \lor q) \lor r \vdash p \lor (q \lor r)$

| | | |
|---|---|---|
| 1. | $(p \lor q) \lor r$ | premise |
| 2. | | |

# Associativity Using Or Elimination

- $(p \vee q) \vee r \vdash p \vee (q \vee r)$

| | | |
|---|---|---|
| 1. | $(p \vee q) \vee r$ | premise |
| 2. | $p \vee q$ | assumption |
| 3. | | |

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- $(p \vee q) \vee r \vdash p \vee (q \vee r)$

| | | |
|---|---|---|
| 1. | $(p \vee q) \vee r$ | premise |
| 2. | $p \vee q$ | assumption |
| 3. | $p$ | assumption |
| 4. | | |

# Associativity Using Or Elimination

- $(p \vee q) \vee r \vdash p \vee (q \vee r)$

| | | |
|---|---|---|
| 1. | $(p \vee q) \vee r$ | premise |
| 2. | $p \vee q$ | assumption |
| 3. | $p$ | assumption |
| 4. | $p \vee (q \vee r)$ | $\vee i_1$ 3 |
| 5. | | |

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- $(p \lor q) \lor r \vdash p \lor (q \lor r)$

| | | |
|---|---|---|
| 1. | $(p \lor q) \lor r$ | premise |
| 2. | $p \lor q$ | assumption |
| 3. | $p$ | assumption |
| 4. | $p \lor (q \lor r)$ | $\lor i_1$ 3 |
| 5. | $q$ | assumption |
| 6. | | |

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- $(p \vee q) \vee r \vdash p \vee (q \vee r)$

| | | |
|---|---|---|
| 1. | $(p \vee q) \vee r$ | premise |
| 2. | $p \vee q$ | assumption |
| 3. | $p$ | assumption |
| 4. | $p \vee (q \vee r)$ | $\vee i_1$ 3 |
| 5. | $q$ | assumption |
| 6. | $q \vee r$ | $\vee i_1$ 5 |
| 7. | | |

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| 2. | $p \vee q$ | assumption |
| 3. | $p$ | assumption |
| 4. | $p \vee (q \vee r)$ | $\vee i_1$ 3 |
| 5. | $q$ | assumption |
| 6. | $q \vee r$ | $\vee i_1$ 5 |
| 7. | $p \vee (q \vee r)$ | $\vee i_2$ 6 |
| 8. | | |

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| | | |
|---|---|---|
| 1. | $(p \vee q) \vee r$ | premise |
| 2. | $p \vee q$ | assumption |
| 3. | $p$ | assumption |
| 4. | $p \vee (q \vee r)$ | $\vee i_1$ 3 |
| 5. | $q$ | assumption |
| 6. | $q \vee r$ | $\vee i_1$ 5 |
| 7. | $p \vee (q \vee r)$ | $\vee i_2$ 6 |
| 8. | $p \vee (q \vee r)$ | $\vee e$ 2, 3-4, 5-7 |
| 9. | | |

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| 1. | $(p \vee q) \vee r$ | premise |
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| 6. | $q \vee r$ | $\vee i_1$ 5 |
| 7. | $p \vee (q \vee r)$ | $\vee i_2$ 6 |
| 8. | $p \vee (q \vee r)$ | $\vee e$ 2, 3-4, 5-7 |
| 9. | $r$ | assumption |
| 10. | | |

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| 2. | $p \vee q$ | assumption |
| 3. | $p$ | assumption |
| 4. | $p \vee (q \vee r)$ | $\vee i_1$ 3 |
| 5. | $q$ | assumption |
| 6. | $q \vee r$ | $\vee i_1$ 5 |
| 7. | $p \vee (q \vee r)$ | $\vee i_2$ 6 |
| 8. | $p \vee (q \vee r)$ | $\vee e$ 2, 3-4, 5-7 |
| 9. | $r$ | assumption |
| 10. | $q \vee r$ | $\vee i_2$ 9 |
| 11. | | |

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|---|---|---|
| 1. | $(p \vee q) \vee r$ | premise |
| 2. | $p \vee q$ | assumption |
| 3. | $p$ | assumption |
| 4. | $p \vee (q \vee r)$ | $\vee i_1$ 3 |
| 5. | $q$ | assumption |
| 6. | $q \vee r$ | $\vee i_1$ 5 |
| 7. | $p \vee (q \vee r)$ | $\vee i_2$ 6 |
| 8. | $p \vee (q \vee r)$ | $\vee e$ 2, 3-4, 5-7 |
| 9. | $r$ | assumption |
| 10. | $q \vee r$ | $\vee i_2$ 9 |
| 11. | $p \vee (q \vee r)$ | $\vee i_2$ 10 |

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- $(p \vee q) \vee r \vdash p \vee (q \vee r)$

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| 1. | $(p \vee q) \vee r$ | premise |
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| 3. | $p$ | assumption |
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| 5. | $q$ | assumption |
| 6. | $q \vee r$ | $\vee i_1$ 5 |
| 7. | $p \vee (q \vee r)$ | $\vee i_2$ 6 |
| 8. | $p \vee (q \vee r)$ | $\vee e$ 2, 3-4, 5-7 |
| 9. | $r$ | assumption |
| 10. | $q \vee r$ | $\vee i_2$ 9 |
| 11. | $p \vee (q \vee r)$ | $\vee i_2$ 10 |
| 12. | $p \vee (q \vee r)$ | $\vee e$ 1, 2-8, 9-11 |

# The Copy Rule

- $\vdash p \rightarrow (q \rightarrow p)$

| | | |
|---|---|---|
| 1. | *true* | premise |
| 2. | | |

# The Copy Rule

- $\vdash p \to (q \to p)$

| | | |
|---|---|---|
| 1. | *true* | premise |
| 2. | $p$ | assumption |
| 3. | | |

# The Copy Rule

- $\vdash p \rightarrow (q \rightarrow p)$

| | | |
|---|---|---|
| 1. | *true* | premise |
| 2. | $p$ | assumption |
| 3. | $q$ | assumption |
| 4. | | |

# The Copy Rule

- $\vdash p \rightarrow (q \rightarrow p)$

| | | |
|---|---|---|
| 1. | *true* | premise |
| 2. | $p$ | assumption |
| 3. | $q$ | assumption |
| 4. | $p$ | copy 2 |
| 5. | | |

# The Copy Rule

- $\vdash p \to (q \to p)$

| | | |
|---|---|---|
| 1. | *true* | premise |
| 2. | $p$ | assumption |
| 3. | $q$ | assumption |
| 4. | $p$ | copy 2 |
| 5. | $q \to p$ | $\to i$ 3-4 |
| 6. | | |

# The Copy Rule

- $\vdash p \rightarrow (q \rightarrow p)$

| | | |
|---|---|---|
| 1. | *true* | premise |
| 2. | $p$ | assumption |
| 3. | $q$ | assumption |
| 4. | $p$ | copy 2 |
| 5. | $q \rightarrow p$ | $\rightarrow i$ 3-4 |
| 6. | $p \rightarrow (q \rightarrow p)$ | $\rightarrow i$ 2-5 |

# The Rules of Single Negation

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- Any two contradictions are equivalent : $p \wedge \neg p$ is equivalent to $\neg r \wedge r$. Contradictions denoted by $\perp$.
- $\perp \rightarrow \varphi$ for any formula $\varphi$.

# Rules with $\perp$

## The $\perp$ elimination rule $\perp e$

$$\frac{\perp}{\psi}$$

## The $\perp$ introduction rule $\perp i$

$$\frac{\varphi \qquad \neg\varphi}{\perp}$$

# An Example

- $\neg p \vee q \vdash p \rightarrow q$

| | | |
|---|---|---|
| 1. | $\neg p \vee q$ | premise |
| 2. | | |

# An Example

- $\neg p \vee q \vdash p \rightarrow q$

| | | |
|---|---|---|
| 1. | $\neg p \vee q$ | premise |
| 2. | $\neg p$ | premise |
| 3. | | |

# An Example

- $\neg p \vee q \vdash p \rightarrow q$

| | | |
|---|---|---|
| 1. | $\neg p \vee q$ | premise |
| 2. | $\neg p$ | premise |
| 3. | $p$ | assumption |
| 4. | | |

# An Example

- $\neg p \vee q \vdash p \rightarrow q$

| | | |
|---|---|---|
| 1. | $\neg p \vee q$ | premise |
| 2. | $\neg p$ | premise |
| 3. | $p$ | assumption |
| 4. | $\perp$ | $\perp i$ 2,3 |
| 5. | | |

# An Example

- $\neg p \vee q \vdash p \rightarrow q$

| | | |
|---|---|---|
| 1. | $\neg p \vee q$ | premise |
| 2. | $\neg p$ | premise |
| 3. | $p$ | assumption |
| 4. | $\perp$ | $\perp i$ 2,3 |
| 5. | $q$ | $\perp e$ 4 |
| 6. | | |

# An Example

- $\neg p \lor q \vdash p \to q$

| | | |
|---|---|---|
| 1. | $\neg p \lor q$ | premise |
| 2. | $\neg p$ | premise |
| 3. | $p$ | assumption |
| 4. | $\bot$ | $\bot i$ 2,3 |
| 5. | $q$ | $\bot e$ 4 |
| 6. | $p \to q$ | $\to i$ 3-5 |
| 7. | $q$ | premise |
| 8. | | |

# An Example

- $\neg p \lor q \vdash p \to q$

| | | |
|---|---|---|
| 1. | $\neg p \lor q$ | premise |
| 2. | $\neg p$ | premise |
| 3. | $p$ | assumption |
| 4. | $\bot$ | $\bot i$ 2,3 |
| 5. | $q$ | $\bot e$ 4 |
| 6. | $p \to q$ | $\to i$ 3-5 |
| 7. | $q$ | premise |
| 8. | $p$ | assumption |
| 9. | | |

# An Example

- $\neg p \vee q \vdash p \rightarrow q$

| | | |
|---|---|---|
| 1. | $\neg p \vee q$ | premise |
| 2. | $\neg p$ | premise |
| 3. | $p$ | assumption |
| 4. | $\perp$ | $\perp i$ 2,3 |
| 5. | $q$ | $\perp e$ 4 |
| 6. | $p \rightarrow q$ | $\rightarrow i$ 3-5 |
| 7. | $q$ | premise |
| 8. | $p$ | assumption |
| 9. | $q$ | copy 7 |
| 10. | $p \rightarrow q$ | $\rightarrow i$ 8-9 |
| 11. | $p \rightarrow q$ | $\vee e$ 1, 2-6, 7-10 |

# Introducing Negations (PBC)

- In the course of a proof, if you assume $\varphi$ (by opening a box) and obtain $\perp$ in the box, then we conclude $\neg\varphi$
- This rule is denoted $\neg i$ and is read as $\neg$ introduction.
- Also known as Proof By Contradiction

# An Example

- $p \to \neg p \vdash \neg p$

| | | |
|---|---|---|
| 1. | $p \to \neg p$ | premise |
| 2. | | |

# An Example

- $p \rightarrow \neg p \vdash \neg p$

| | | |
|---|---|---|
| 1. | $p \rightarrow \neg p$ | premise |
| 2. | $p$ | assumption |
| 3. | | |

# An Example

- $p \to \neg p \vdash \neg p$

| | | |
|---|---|---|
| 1. | $p \to \neg p$ | premise |
| 2. | $p$ | assumption |
| 3. | $\neg p$ | MP 1,2 |
| 4. | | |

# An Example

- $p \to \neg p \vdash \neg p$

| | | |
|---|---|---|
| 1. | $p \to \neg p$ | premise |
| 2. | $p$ | assumption |
| 3. | $\neg p$ | MP 1,2 |
| 4. | $\bot$ | $\bot i$ 2,3 |
| 5. | $\neg p$ | $\neg i$ 2-4 |

# The Last One

**Law of the Excluded Middle (LEM)**

$$\frac{}{\varphi \vee \neg\varphi}$$

# Summary of Basic Rules

- $\wedge i$, $\wedge e_1$, $\wedge e_2$,
- $\neg\neg e$
- MP
- $\rightarrow i$
- $\vee i_1, \vee i_2, \vee e$
- Copy, $\neg i$ or PBC
- $\bot e$, $\bot i$

# Derived Rules

- MT (derive using MP, $\bot i$ and $\neg i$)
- $\neg\neg i$ (derive using $\bot i$ and $\neg i$)
- LEM (derive using $\vee i_1, \bot i, \neg i, \vee i_2, \neg\neg e$)

# The Proofs So Far

- So far, the “proof” we have seen is purely syntactic, no true/false meanings were attached

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# The Proofs So Far

- So far, the "proof" we have seen is purely syntactic, no true/false meanings were attached
- Intuitively, $p \to q \vdash \neg p \vee q$ makes sense because you think semantically. However, we never used any semantics so far.
- Now we show that whatever can be proved makes sense semantically too.

# Semantics

- Each propositional variable is assigned values true/false. Truth tables for each of the operators $\vee, \wedge, \neg, \rightarrow$ to determine truth values of complex formulae.

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- $\varphi_1, \ldots, \varphi_n \models \psi$ iff whenever $\varphi_1, \ldots, \varphi_n$ evaluate to true, so does $\psi$. $\models$ is read as semantically entails
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  - Recall $\vdash$, and compare with $\models$
- Formulae $\varphi$ and $\psi$ are provably equivalent iff $\varphi \vdash \psi$ and $\psi \vdash \varphi$
- Formulae $\varphi$ and $\psi$ are semantically equivalent iff $\varphi \models \psi$ and $\psi \models \varphi$

# Soundness of Propositional Logic

$\varphi_1, \ldots, \varphi_n \vdash \psi \Rightarrow \varphi_1, \ldots, \varphi_n \models \psi$

Whenever $\psi$ can be proved from $\varphi_1, \ldots, \varphi_n$, then $\psi$ evaluates to true whenever $\varphi_1, \ldots, \varphi_n$ evauate to true