Fifth SI Edition

CHAPTER

3

MECHANICS OF MATERIALS

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Torsion

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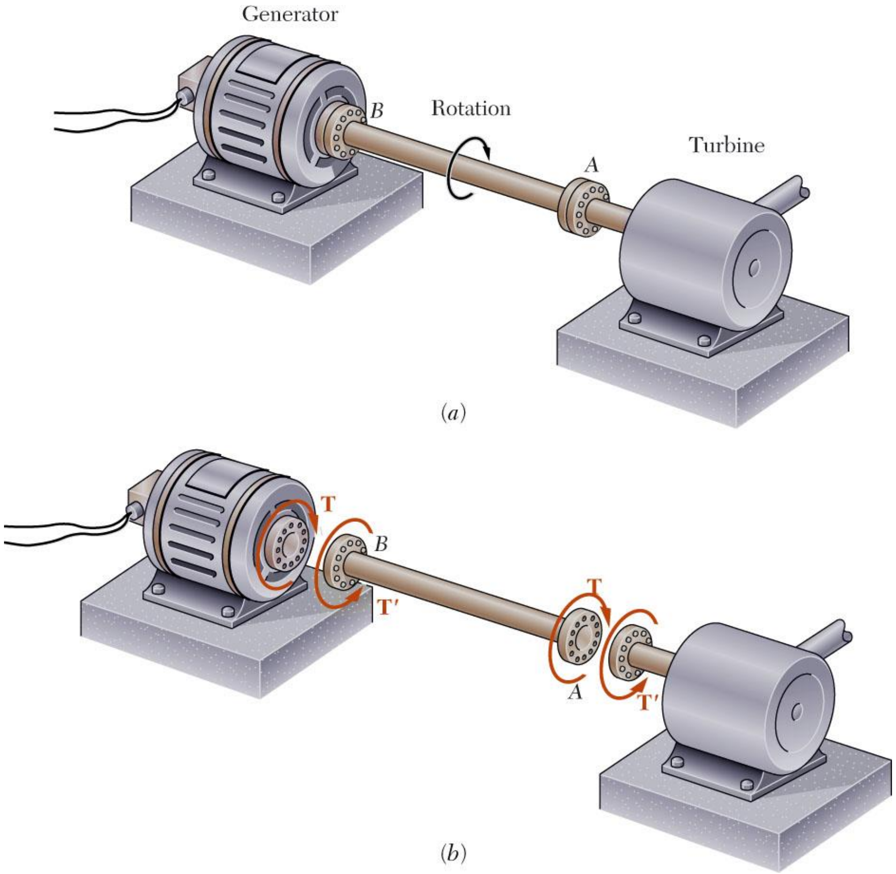
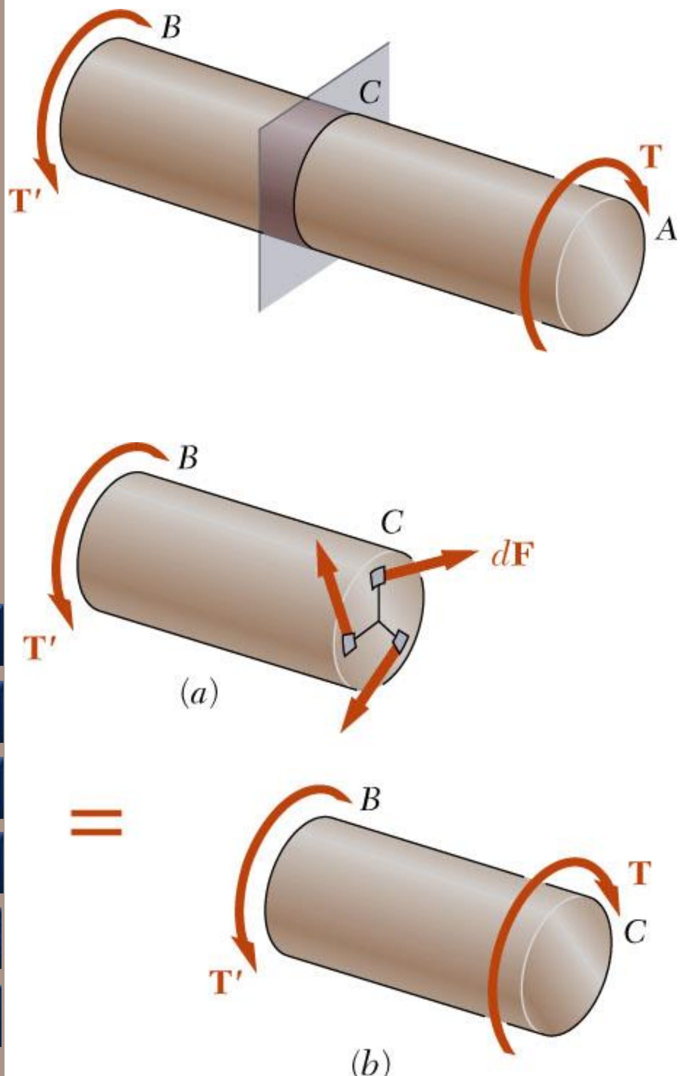
Torsional Loads on Circular Shafts



- Interested in stresses and strains of circular shafts subjected to twisting couples or *torques*
- Turbine exerts torque *T* on the shaft
- Shaft transmits the torque to the generator
- Generator creates an equal and opposite torque *T*’

End

Net Torque Due to Internal Stresses

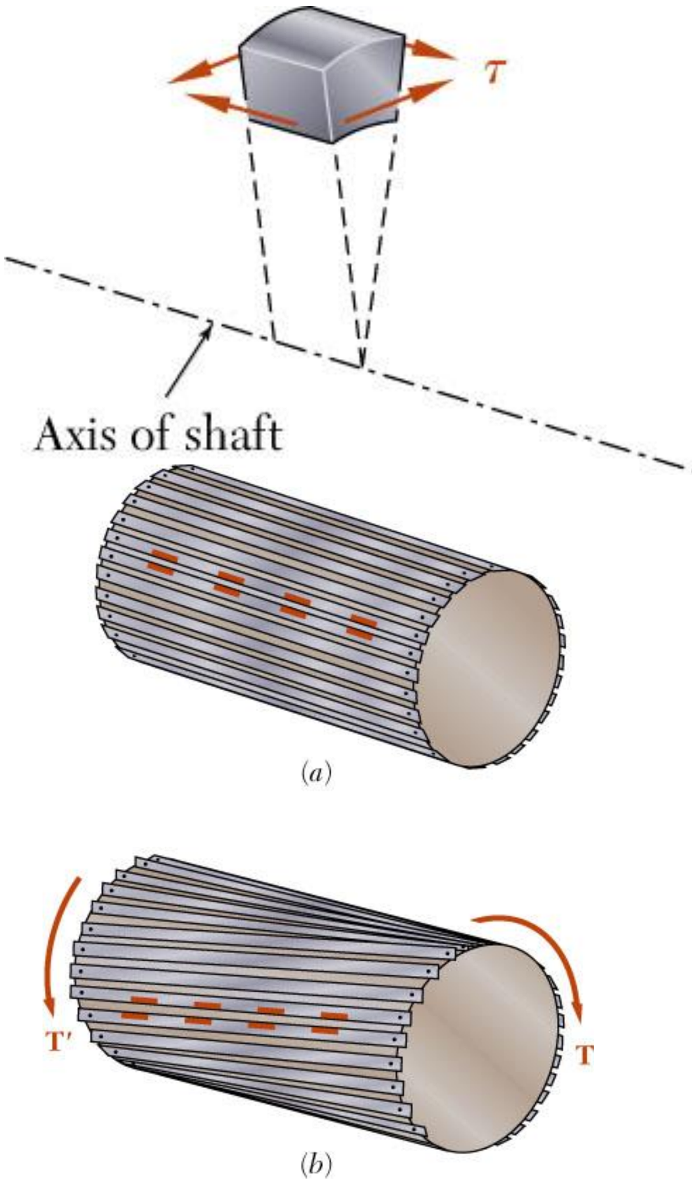


- Net of the internal shearing stresses is an internal torque, equal and opposite to the applied torque,

$$T = \int \rho \, dF = \int \rho (\tau \, dA)$$

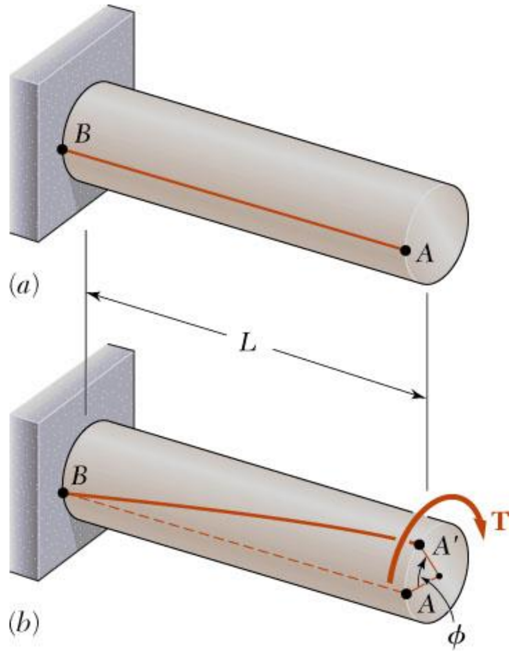
- Although the net torque due to the shearing stresses is known, the distribution of the stresses is not.
- Distribution of shearing stresses is statically indeterminate – must consider shaft deformations.
- Unlike the normal stress due to axial loads, the distribution of shearing stresses due to torsional loads can not be assumed uniform.

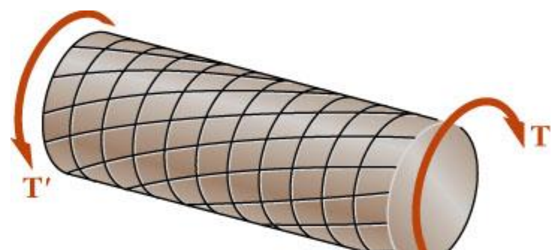
Axial Shear Components

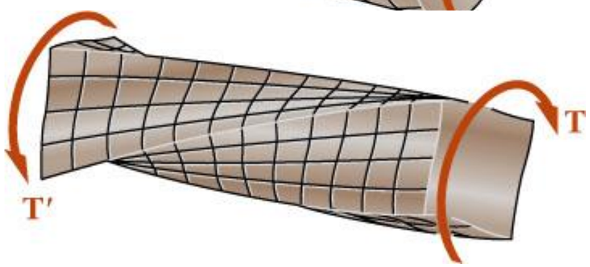


- Torque applied to shaft produces shearing stresses on the faces perpendicular to the axis.
- Conditions of equilibrium require the existence of equal stresses on the faces of the two planes containing the axis of the shaft.
- The existence of the axial shear components is demonstrated by considering a shaft made up of axial slats.
- The slats slide with respect to each other when equal and opposite torques are applied to the ends of the shaft.

Shaft Deformations





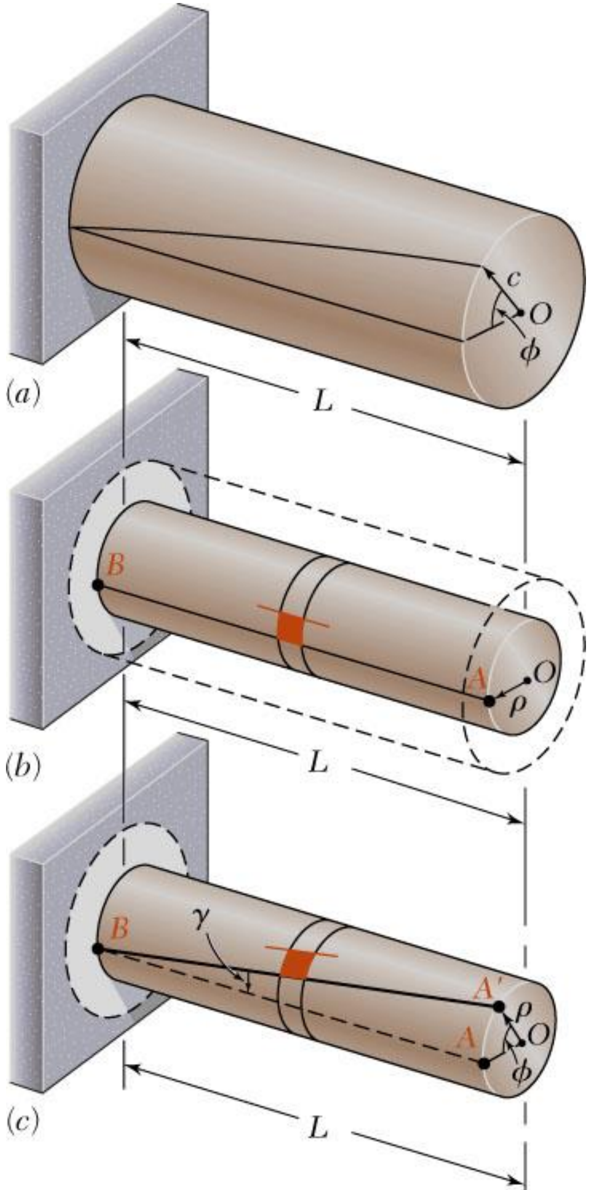


- From observation, the angle of twist of the shaft is proportional to the applied torque and to the shaft length.

$$\phi \propto T$$
$$\phi \propto L$$

- When subjected to torsion, every cross-section of a circular shaft remains plane and undistorted.
- Cross-sections for hollow and solid circular shafts remain plain and undistorted because a circular shaft is axisymmetric.
- Cross-sections of noncircular (non-axisymmetric) shafts are distorted when subjected to torsion.

Shearing Strain



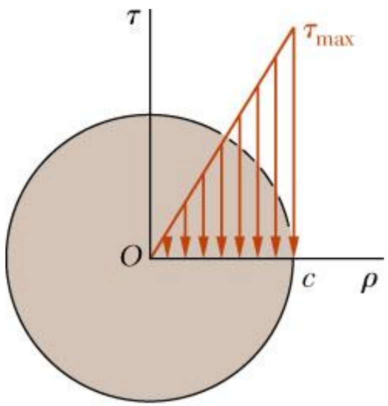
- Consider an interior section of the shaft. As a torsional load is applied, an element on the interior cylinder deforms into a rhombus.
- Since the ends of the element remain planar, the shear strain is equal to angle of twist.
- It follows that

$$L\gamma = \rho\phi \quad \text{or} \quad \gamma = \frac{\rho\phi}{L}$$

- Shear strain is proportional to twist and radius

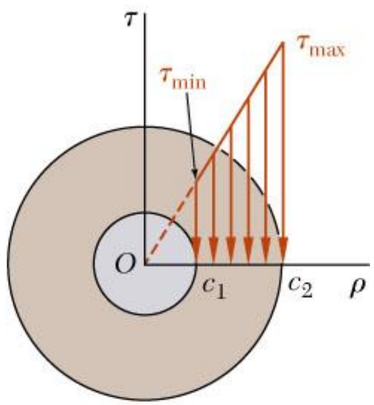
$$\gamma_{\max} = \frac{c\phi}{L} \quad \text{and} \quad \gamma = \frac{\rho}{c}\gamma_{\max}$$

Stresses in Elastic Range



(a)

$$J = \frac{1}{2}\pi c^4$$



(b)

$$J = \frac{1}{2}\pi\left(c_2^4 - c_1^4\right)$$

- Multiplying the previous equation by the shear modulus,

$$G\gamma = \frac{\rho}{c} G\gamma_{\max}$$

From Hooke's Law, $\tau = G\gamma$, so

$$\tau = \frac{\rho}{c}\tau_{\max}$$

The shearing stress varies linearly with the radial position in the section.

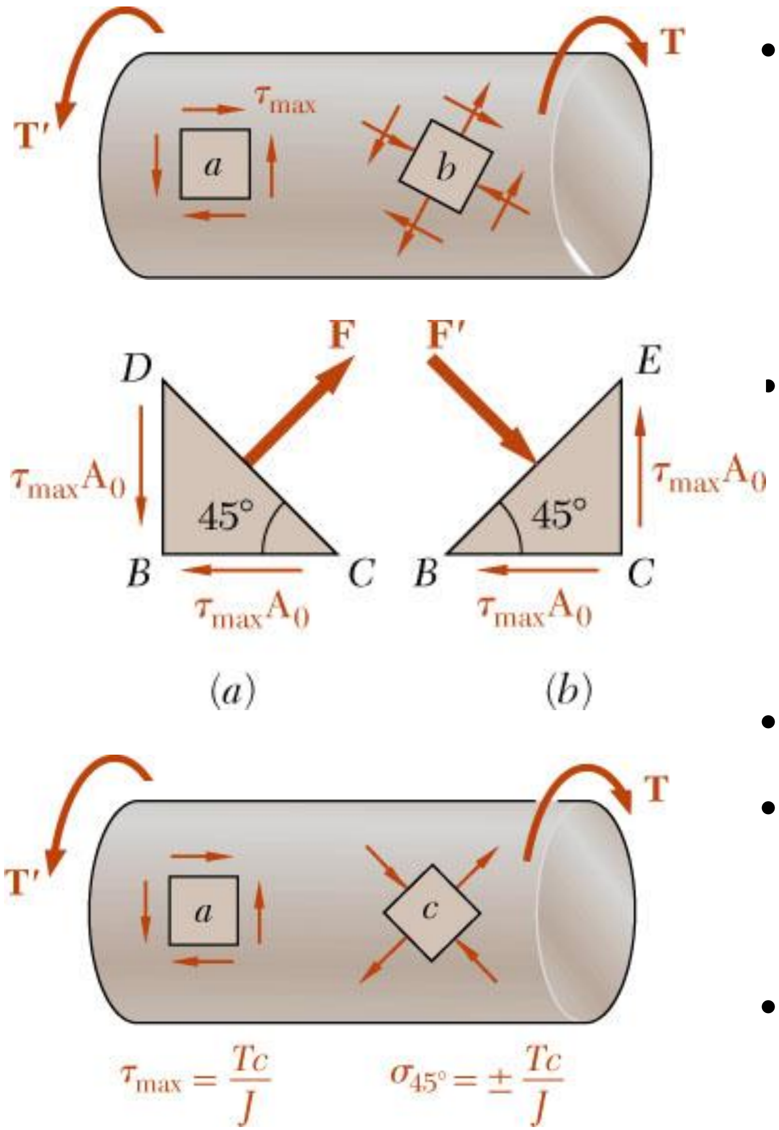
- Recall that the sum of the moments from the internal stress distribution is equal to the torque on the shaft at the section,

$$T = \int \rho\tau \, dA = \frac{\tau_{\max}}{c}\int \rho^2 \, dA = \frac{\tau_{\max}}{c} J$$

- The results are known as the *elastic torsion formulas*,

$$\tau_{\max} = \frac{Tc}{J} \quad \text{and} \quad \tau = \frac{T\rho}{J}$$

Normal Stresses



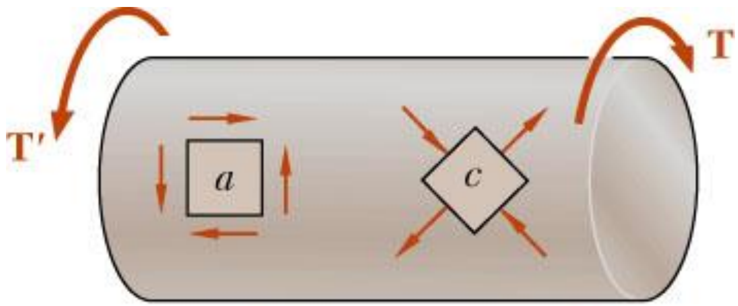
- Elements with faces parallel and perpendicular to the shaft axis are subjected to shear stresses only. Normal stresses, shearing stresses or a combination of both may be found for other orientations.
- Consider an element at 45° to the shaft axis,

$$F = 2(\tau_{max}A_0)\cos 45° = \tau_{max}A_0\sqrt{2}$$

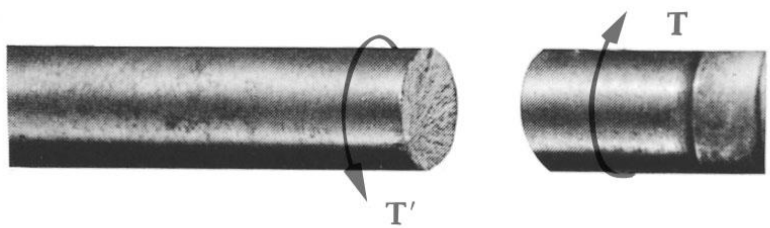
$$\sigma_{45°} = \frac{F}{A} = \frac{\tau_{max}A_0\sqrt{2}}{A_0\sqrt{2}} = \tau_{max}$$

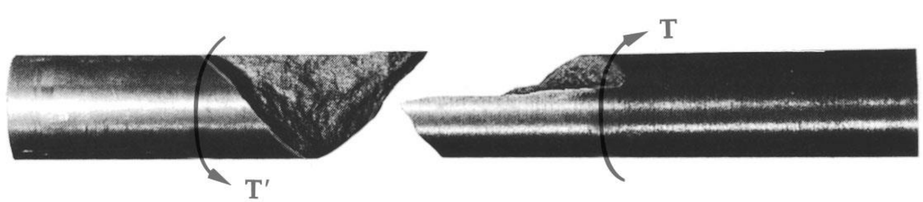
- Element *a* is in pure shear.
- Element *c* is subjected to a tensile stress on two faces and compressive stress on the other two.
- Note that all stresses for elements *a* and *c* have the same magnitude

Torsional Failure Modes



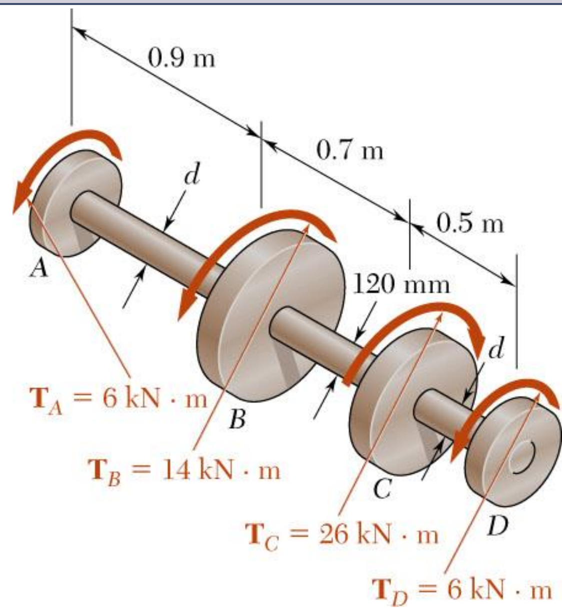
$$\tau_{\max} = \frac{Tc}{J} \qquad \sigma_{45^\circ} = \pm\frac{Tc}{J}$$





- Ductile materials generally fail in shear. Brittle materials are weaker in tension than shear.
- When subjected to torsion, a ductile specimen breaks along a plane of maximum shear, i.e., a plane perpendicular to the shaft axis.
- When subjected to torsion, a brittle specimen breaks along planes perpendicular to the direction in which tension is a maximum, i.e., along surfaces at 45^{o} to the shaft axis.

Sample Problem 3.1



Shaft *BC* is hollow with inner and outer diameters of 90 mm and 120 mm, respectively. Shafts *AB* and *CD* are solid of diameter *d*. For the loading shown, determine (*a*) the minimum and maximum shearing stress in shaft *BC*, (*b*) the required diameter *d* of shafts *AB* and *CD* if the allowable shearing stress in these shafts is 65 MPa.

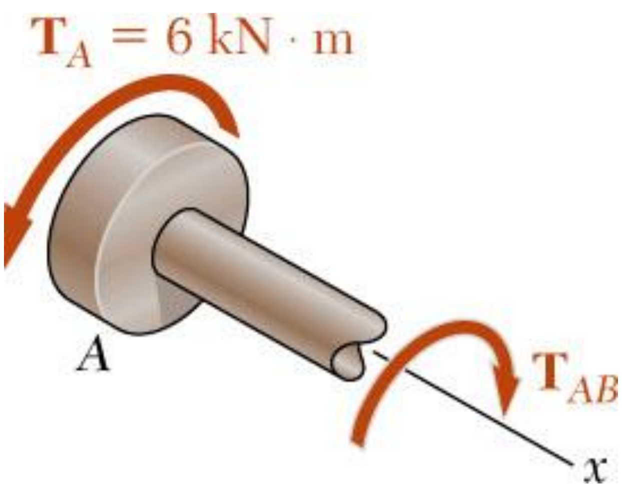
SOLUTION:

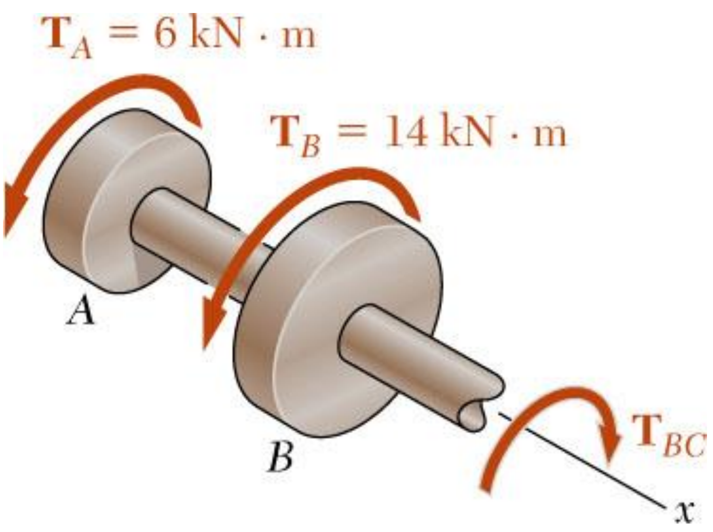
- Cut sections through shafts *AB* and *BC* and perform static equilibrium analyses to find torque loadings.
- Apply elastic torsion formulas to find minimum and maximum stress on shaft *BC*.
- Given allowable shearing stress and applied torque, invert the elastic torsion formula to find the required diameter.

Sample Problem 3.1

SOLUTION:

- Cut sections through shafts *AB* and *BC* and perform static equilibrium analysis to find torque loadings.



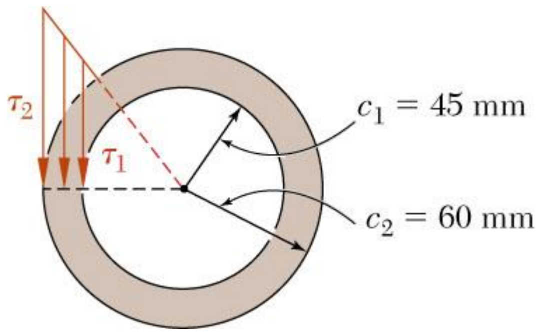


$$\sum M_x = 0 = (6\,\mathrm{kN \cdot m}) - T_{AB}$$
$$T_{AB} = 6\,\mathrm{kN \cdot m} = T_{CD}$$

$$\sum M_x = 0 = (6\,\mathrm{kN \cdot m}) + (14\,\mathrm{kN \cdot m}) - T_{BC}$$
$$T_{BC} = 20\,\mathrm{kN \cdot m}$$

Sample Problem 3.1

- Apply elastic torsion formulas to find minimum and maximum stress on shaft *BC*.



$$J = \frac{\pi}{2}\left(c_2^4 - c_1^4\right) = \frac{\pi}{2}\left[(0.060)^4 - (0.045)^4\right]$$
$$= 13.92 \times 10^{-6}\,\mathrm{m}^4$$

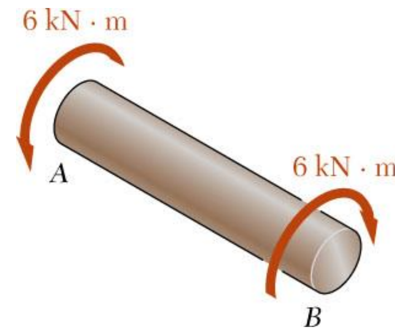
$$\tau_{max} = \tau_2 = \frac{T_{BC}c_2}{J} = \frac{(20\,\mathrm{kN\cdot m})(0.060\,\mathrm{m})}{13.92\times 10^{-6}\,\mathrm{m}^4}$$
$$= 86.2\,\mathrm{MPa}$$

$$\frac{\tau_{min}}{\tau_{max}} = \frac{c_1}{c_2} \qquad \frac{\tau_{min}}{86.2\,\mathrm{MPa}} = \frac{45\,\mathrm{mm}}{60\,\mathrm{mm}}$$

$$\tau_{min} = 64.7\,\mathrm{MPa}$$

$$\boxed{\begin{aligned}\tau_{max} &= 86.2\,\mathrm{MPa}\\ \tau_{min} &= 64.7\,\mathrm{MPa}\end{aligned}}$$

- Given allowable shearing stress and applied torque, invert the elastic torsion formula to find the required diameter.

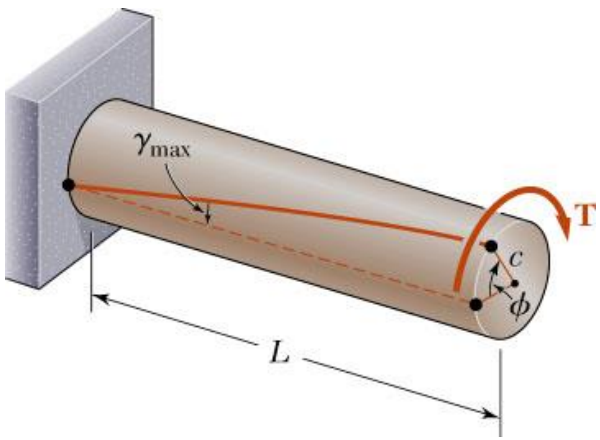


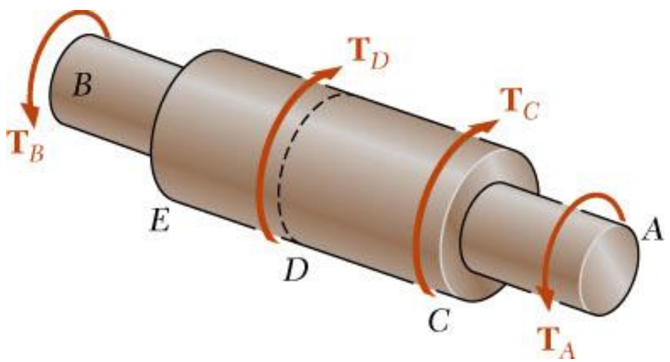
$$\tau_{max} = \frac{Tc}{J} = \frac{Tc}{\frac{\pi}{2}c^4} \qquad 65\,MPa = \frac{6\,\mathrm{kN\cdot m}}{\frac{\pi}{2}c^3}$$

$$c = 38.9\times 10^{-3}\,\mathrm{m}$$

$$\boxed{d = 2c = 77.8\,\mathrm{mm}}$$

Angle of Twist in Elastic Range





- Recall that the angle of twist and maximum shearing strain are related,

$$\gamma_{max} = \frac{c\phi}{L}$$

- In the elastic range, the shearing strain and shear are related by Hooke's Law,

$$\gamma_{max} = \frac{\tau_{max}}{G} = \frac{Tc}{JG}$$

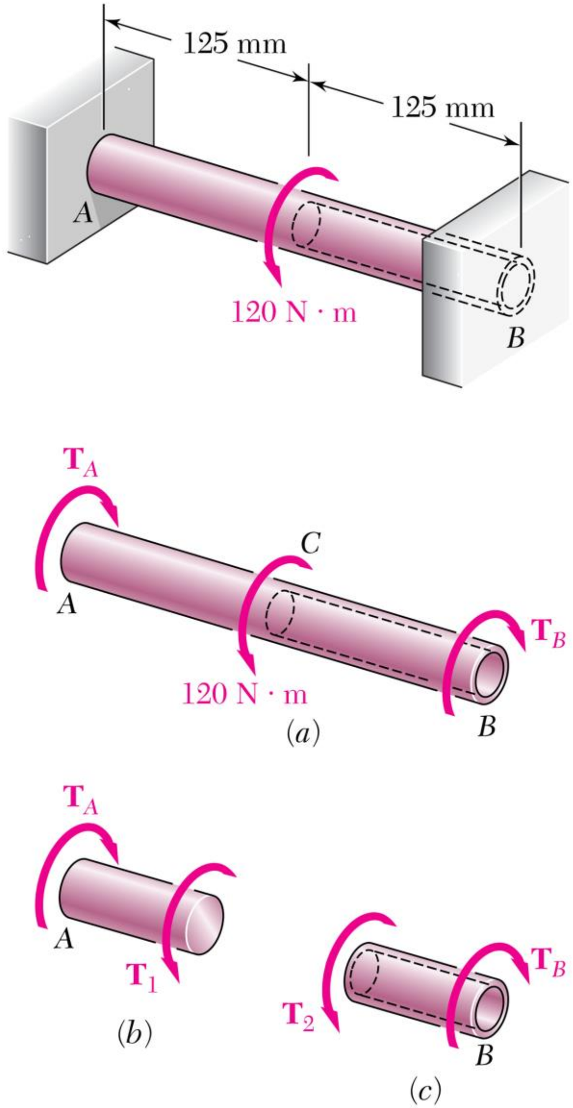
- Equating the expressions for shearing strain and solving for the angle of twist,

$$\phi = \frac{TL}{JG}$$

- If the torsional loading or shaft cross-section changes along the length, the angle of rotation is found as the sum of segment rotations

$$\phi = \sum_i \frac{T_i L_i}{J_i G_i}$$

Statically Indeterminate Shafts



- Given the shaft dimensions and the applied torque, we would like to find the torque reactions at *A* and *B*.
- From a free-body analysis of the shaft,

$$T_A + T_B = 120\,\mathrm{N\cdot m}$$

which is not sufficient to find the end torques. The problem is statically indeterminate.

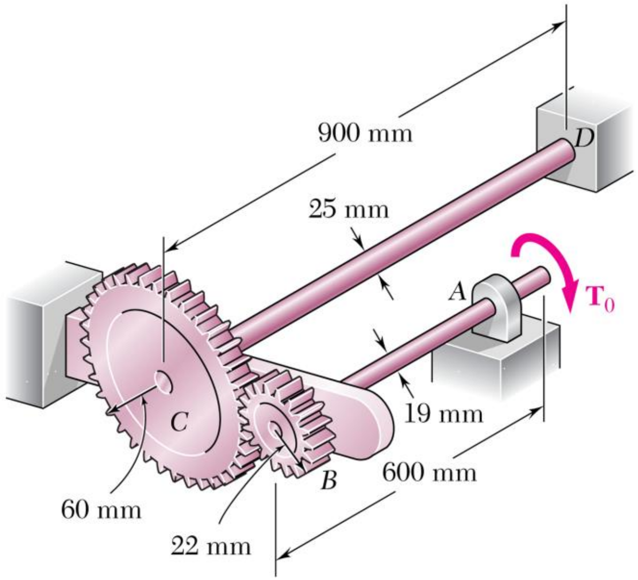
- Divide the shaft into two components which must have compatible deformations,

$$\phi = \phi_1 + \phi_2 = \frac{T_A L_1}{J_1 G} - \frac{T_B L_2}{J_2 G} = 0 \qquad T_B = \frac{L_1 J_2}{L_2 J_1} T_A$$

- Substitute into the original equilibrium equation,

$$T_A + \frac{L_1 J_2}{L_2 J_1} T_A = 120\,\mathrm{N\cdot m}$$

Sample Problem 3.4



Two solid steel shafts are connected by gears. Knowing that for each shaft G = 77 GPa and that the allowable shearing stress is 55 MPa, determine (*a*) the largest torque T_0 that may be applied to the end of shaft *AB*, (*b*) the corresponding angle through which end *A* of shaft *AB* rotates.

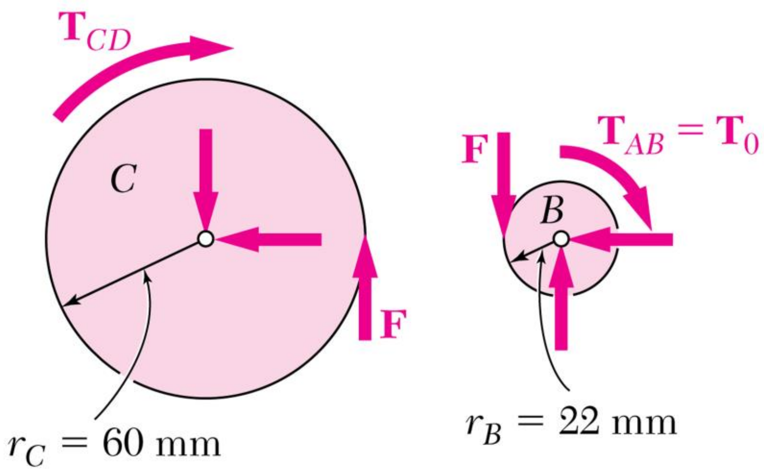
SOLUTION:

- Apply a static equilibrium analysis on the two shafts to find a relationship between T_{CD} and T_0 .
- Apply a kinematic analysis to relate the angular rotations of the gears.
- Find the maximum allowable torque on each shaft – choose the smallest.
- Find the corresponding angle of twist for each shaft and the net angular rotation of end *A*.

Sample Problem 3.4

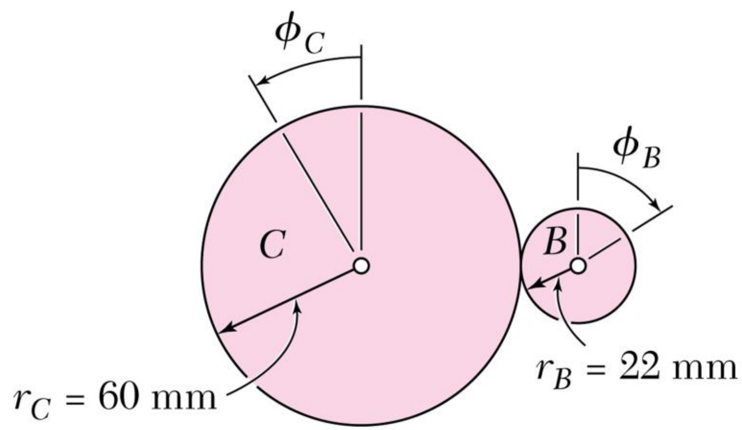
SOLUTION:

- Apply a static equilibrium analysis on the two shafts to find a relationship between T_{CD} and T_0 .



$$\sum M_B = 0 = F(22\,\text{mm}) - T_0$$
$$\sum M_C = 0 = F(60\,\text{mm}) - T_{CD}$$
$$T_{CD} = 2.73T_0$$

- Apply a kinematic analysis to relate the angular rotations of the gears.



$$r_B\phi_B = r_C\phi_C$$
$$\phi_B = \frac{r_C}{r_B}\phi_C = \frac{60\,\text{mm}}{20\,\text{mm}}\phi_C$$
$$\phi_B = 2.73\phi_C$$