Chapter 2: Determinants

1. Axioms for determinant function.
2. Properties of determinant function.
3. Existence and uniqueness of determinant function.
4. $\det(AB) = \det A \det B$.
5. Invertibility of a matrix in terms of determinant.
6. Computation of determinant by Gauss-Jordan Method.
7. Inverse of a matrix in terms of the cofactor matrix.
8. Cramer's Rule for solving n linear equations in n unknowns.

Axiomatic approach for the Determinant Function

1. Recall the formula for determinants of $k \times k$ matrices, for $k = 1, 2, 3$.

$$\det[a] = a, \quad \det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc$$

$$\text{and } \det \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = a(ei - hf) - b(di - gf) + c(dh - eg).$$

2. Our approach to determinants of $n \times n$ matrices is via their properties (rather than via an explicit formula as above).

3. Let d be a function that associates a scalar $d(A) \in \mathbb{F}$ with every $n \times n$ matrix A over $\mathbb{F}$. Here $\mathbb{F}$ is either $\mathbb{R}$ or $\mathbb{C}$.

Axioms for the determinant function

1. If the columns of A are $A_1, A_2, \ldots, A_n$, we write $d(A) = d(A_1, A_2, \ldots, A_n)$.
2. (i) d is called **multilinear** if for each $k = 1, 2, \ldots, n$; scalars α, β and $n \times 1$ column vectors $A_1, \ldots, A_{k-1}, A_{k+1}, \ldots, A_n, B, C$

$$\begin{aligned} d(A_1, \ldots, A_{k-1}, \alpha B + \beta C, A_{k+1}, \ldots, A_n) &= \\ \alpha\, d(A_1, \ldots, A_{k-1}, B, A_{k+1}, \ldots, A_n) &+ \beta\, d(A_1, \ldots, A_{k-1}, C, A_{k+1}, \ldots, A_n). \end{aligned}$$

3. (ii) d is called **alternating** if $d(A_1, A_2, \ldots, A_n) = 0$ if $A_i = A_j$ for some $i \neq j$.
4. (iii) d is called **normalized** if $d(I) = d(e_1, e_2, \ldots, e_n) = 1$, where e_i is the i^{th} standard column vector with 1 in the i^{th} coordinate and 0's elsewhere.
5. A normalized, alternating, and multillinear function d on $n \times n$ matrices is called a **determinant function** of order n.

Properties of determinant function

1. Our immediate objective is to show that there is only one determinant function of order n.

2. **Lemma:** Suppose that $d(A_1, A_2, \ldots, A_n)$ is a multilinear alternating function on columns of $n \times n$ matrices. Then
(a) If some $A_k = 0$ then $d(A_1, A_2, \ldots, A_n) = 0$.
(b) $d(A_1, A_2, \ldots, A_k, A_{k+1}, \ldots A_n) = -d(A_1, A_2, \ldots, A_{k+1}, A_k, \ldots, A_n)$.
(c) $d(A_1, A_2, \ldots, A_i, \ldots, A_j, \ldots, A_n) = -d(A_1, A_2, \ldots, A_j, \ldots, A_i, \ldots, A_n)$.

3. **Proof:** (a) If $A_k = 0$ then by multilinearity
$$d(A_1, A_2, \ldots, 0A_k, \ldots, A_n) = 0\ d(A_1, A_2, \ldots, A_k, \ldots, A_n) = 0.$$

4. (b) Put $A_k = B, A_{k+1} = C$. By the alternating property
$$\begin{aligned} 0 &= d(A_1, A_2, \ldots, B + C, B + C, \ldots, A_n) \\ &= d(A_1, A_2, \ldots, B, B + C, \ldots, A_n) + d(A_1, A_2, \ldots, C, B + C, \ldots, A_n) \\ &= d(A_1, A_2, \ldots, B, C, \ldots, A_n) + d(A_1, A_2, \ldots, C, B, \ldots, A_n) \end{aligned}$$
$$\implies d(A_1, A_2, \ldots, B, C, \ldots, A_n) = -d(A_1, A_2, \ldots, C, B, \ldots, A_n).$$

5. (c) Follows from (b). $\square$

Formula for the determinant of a 2×2 matrix

1. Suppose $d(A_1, A_2)$ is an alternating multilinear normalized function on 2×2 matrices $A = (A_1, A_2)$. Then

$$d \begin{bmatrix} x & y \\ z & u \end{bmatrix} = xu - yz.$$

2. To derive this formula, write the first column as $A_1 = xe_1 + ze_2$ and the second column as $A_2 = ye_1 + ue_2$.

3. Then using the axioms for determinate function we get

$$\begin{aligned} d(A_1, A_2) &= d(xe_1 + ze_2, ye_1 + ue_2) \\ &= d(xe_1 + ze_2, ye_1) + d(xe_1 + ze_2, ue_2) \\ &= d(xe_1, ye_1) + d(ze_2, ye_1) \\ &\quad + d(xe_1, ue_2) + d(ze_2, ue_2) \\ &= yzd(e_2, e_1) + xud(e_1, e_2) \\ &= (xu - yz)d(e_1, e_2) = xu - yz. \end{aligned}$$

Uniqueness of the determinant function

1. **Lemma:** Suppose f is a multilinear alternating function on $n \times n$ matrices and $f(e_1, e_2, \ldots, e_n) = 0$. Then f is identically zero.

2. **Proof:** Let $A = (a_{ij})$ be an $n \times n$ matrix with columns $A_1, \ldots, A_n$. Write A_j as

$$A_j = a_{1j}e_1 + a_{2j}e_2 + \cdots + a_{nj}e_n.$$

3. Since f is multilinear we have

$$f(A_1, \ldots, A_n) = \sum_h a_{h(1)1}a_{h(2)2}\cdots a_{h(n)n}\, f(e_{h(1)}, e_{h(2)}, \ldots, e_{h(n)}),$$

where the sum is over all functions $h : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$.

4. Since f is alternating we have

$$f(A_1, \ldots, A_n) = \sum_h a_{h(1)1}a_{h(2)2}\cdots a_{h(n)n}\, f(e_{h(1)}, e_{h(2)}, \ldots, e_{h(n)}),$$

where the sum is now over all bijections $h : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$.

Uniqueness of the determinant function

1. By using part (c) of the lemma above we see that we can write

$$f(A_1, \ldots, A_n) = \sum_h \pm a_{h(1)1} a_{h(2)2} \cdots a_{h(n)n} \, f(e_1, e_2, \ldots, e_n),$$

   where the sum is over all bijections $h : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$.

2. Therefore $f(A) = 0$. □

Theorem

Let f be an alternating multilinear function of order n and d a determinant function of order n. Then for all $n \times n$ matrices $A = (A_1, A_2, \ldots, A_n)$,

$$f(A_1, A_2, \ldots, A_n) = d(A_1, A_2, \ldots, A_n) f(e_1, e_2, \ldots, e_n).$$

In particular, if f is also a determinant function then

$$f(A_1, A_2, \ldots, A_n) = d(A_1, A_2, \ldots, A_n).$$

Proof of uniqueness of determinant function

1. **Proof:** Consider the function

$$g(A_1, A_2, \ldots, A_n) = f(A_1, A_2, \ldots, A_n) - d(A_1, A_2, \ldots, A_n) f(e_1, e_2, \ldots, e_n).$$

2. Since f, d are alternating and multilinear so is g. Since

$$g(e_1, e_2, \ldots, e_n) = 0 \qquad \text{(why?)}$$

   the result follows from the previous lemma. □

3. **Notation:** We shall denote the determinant of A by $\det A$ or $|A|$.
4. Setting $\det[a] = a$ shows the existence of the determinant function for $n = 1$.
5. Assume that we have shown the existence of the determinant function of order $(n-1) \times (n-1)$. The determinant of an $n \times n$ matrix A can be computed in terms of certain $(n-1) \times (n-1)$ determinants.
6. Let A_{ij} = the $(n-1) \times (n-1)$ submatrix obtained from A by deleting the $i\text{th}$ row and $j\text{th}$ column of A.

Existence of determinant function

Theorem

Let $A = (a_{ij})$ be an $n \times n$ matrix. Then the function

$$f(A) = a_{11} \det A_{11} - a_{12} \det A_{12} + \cdots + (-1)^{n+1} a_{1n} \det A_{1n}$$

is multilinear, alternating, and normalized on $n \times n$ matrices, hence is the determinant function.

1. **Proof:** Denote the function $f(A)$ by $f(A_1, A_2, \ldots, A_n)$. Suppose that the columns A_j and A_{j+1} of A are equal. Then A_{1i} have equal columns except when $i = j$ or $i = j + 1$. By induction $\det A_{1i} = 0$ for $i \neq j, j+1$. Thus

$$f(A) = a_{1j} \left[(-1)^{j+1} \det(A_{1j}) \right] + \left[(-1)^{j+2} a_{1j+1} \det(A_{1j+1}) \right].$$

2. Since $A_j = A_{j+1}, a_{1j} = a_{1j+1}$ and $A_{1j} = A_{1j+1}$. Thus $f(A) = 0$. Therefore $f(A_1, A_2, \ldots, A_n)$ is alternating.

3. If $A = (e_1, e_2, \ldots, e_n)$ then $f(A) = 1 \det(A_{11}) = 1$, by induction. (Exercise) Show the multilinear property of $f(A_1, \ldots, A_n)$. □

Determinant of elementary and upper triangular matrices

1. **Theorem:** (i) Let U be an upper triangular or a lower triangular matrix. Then $\det U =$ product of diagonal entries of U.

2. (ii) Let E be an elementary matrix of the type $I + \alpha e_{ij}$, for some $i \neq j$. Then $\det E = 1$.

3. (iii) Let E be an elementary matrix of the type $I + e_{ij} + e_{ji} - e_{ii} - e_{jj}$, for some $i \neq j$. Then $\det E = \ -1$.

4. (iv) Let E be an elementary matrix of the type $I + (\alpha - 1)e_{ii},\ \alpha \neq 0$. Then $\det E = \alpha$.

5. **Proof:** (i) Let $U = (u_{ij})$ be upper triangular. Arguing as in the proof of one of the last lemma we see that

$$\det U = \sum_h u_{h(1)1} u_{h(2)2} \cdots u_{h(n)n} \det(e_{h(1)}, \ldots, e_{h(n)})$$

where the sum is over all bijections $h : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$.

Determinant and invertibility

1. Since U is upper triangular, the only choice of h yielding a nonzero term is the identity function and hence
$$\det U = u_{11}u_{22}\cdots u_{nn}\det(e_1,\ldots,e_n) = u_{11}u_{22}\cdots u_{nn}.$$
2. (ii) Follows from part (i).
3. (iii) Observe that E is obtained from the identity matrix by interchanging columns i and j. The result follows since determinant is an alternating function.
4. (iv) Follows form part (i). □
5. **Theorem:** Let A, B be two $n \times n$ matrices. Then
$$\det(AB) = \det A \det B.$$
6. **Proof:** Let D_i denote the $i\mathrm{th}$ column of a matrix D. Then $(AB)_i = AB_i$.
7. Therefore we need to prove that
$$\det(AB_1, AB_2\ldots, AB_n) \quad = \quad \det(A_1, A_2, \ldots, A_n)\det(B_1,\ldots,B_n).$$

$\det(AB) = \det A \det B$

1. Keep A fixed and define $f(B_1, B_2, \ldots, B_n) = \det(AB_1, AB_2, \ldots, AB_n)$.
2. We show that f is alternating and multilinear.
3. Let C be an $n \times 1$ column vector. Then

$$\begin{aligned}
f(B_1, \ldots, B_i, \ldots, B_i, \ldots, B_n) &= \det(AB_1, \ldots, AB_i, \ldots, AB_i, \ldots, AB_n) = 0 \\
f(B_1, \ldots, B_k + \alpha C, \ldots, B_n) &= \det(AB_1, \ldots, A(B_k + \alpha C), \ldots, AB_n) \\
&= \det(AB_1, \ldots, AB_k + \alpha AC, \ldots, AB_n) \\
&= \det(AB_1, \ldots, AB_k, \ldots, AB_n) \\
&+ \det(AB_1, \ldots, \alpha AC, \ldots, AB_n) \\
&= f(B_1, \ldots, B_n) + \alpha f(B_1, \ldots, C, \ldots, B_n).
\end{aligned}$$

4. Therefore $f(B_1, B_2, \ldots, B_n) = \det(B_1, \ldots, B_n) f(e_1, e_2, \ldots, e_n)$.
5. Now note that
$f(e_1, e_2, \ldots, e_n) = \det(Ae_1, \ldots, Ae_n) = \det(A_1, \ldots, A_n) = \det A$.
6. Hence $\det(AB) = \det A \det B$. $\square$

Determinant and invertibility

1. **Lemma:** (i) If A is an invertible matrix then $\det A \neq 0$ and
$$\det A^{-1} = \frac{1}{\det A}.$$
2. (ii) $\det A \neq 0$ implies A is invertible.
3. (iii) Suppose A, B are square matrices with $AB = I$. Then A is invertible and $B = A^{-1}$.
4. **Proof:** (i) Since $AA^{-1} = I, \det A^{-1} \det A = \det I = 1$.
5. (ii) Suppose A is not invertible. Then it follows that there is a nontrivial column vector x such that $Ax = 0$.
6. So some column of A is a linear combination of other columns (i.e., excluding itself) of A.
7. It now follows from multilinearity and alternating properties that $\det A = 0$.
8. (iii) Taking determinants we have $\det A \det B = 1$. So $\det A \neq 0$ and A is invertible. Now $B = (A^{-1}A)B = A^{-1}(AB) = A^{-1}$. □

Determinant of transpose of a matrix

1. **Theorem:** For any $n \times n$ matrix A,
$$\det A = \det A^t.$$
2. **Proof:** If A is not invertible then A^t is also not invertible (why?) and both determinants are 0. So we may assume that A is invertible. Then we write
$$A = E_1 E_2 \cdots E_k$$
where $E_1, \ldots, E_k$ are elementary matrices.
3. Now transpose of an elementary matrix is also an elementary matrix (of the same type) and has the same determinant.
4. The result follows by multiplicativity of determinant. $\square$
5. **Corollary:** Let $A = (a_{ij})$ be an $n \times n$ matrix and let $1 \leq k \leq n$. Then
$$\det A = \sum_{i=1}^{n} (-1)^{k+i} a_{ik} \det A_{ik}.$$
6. **Proof**: Exercise. (Use the theorem and properties of the det function.) $\square$

Computation of determinant by Gauss-Jordan elimination

1. Let A be an $n \times n$ matrix.
2. Suppose $E =$ the $n \times n$ elementary matrix for the row operation $R_i \to R_i + cR_j$
3. $F =$ the $n \times n$ elementary matrix for the row operation $R_i \leftrightarrow R_j$
4. $G =$ the $n \times n$ elementary matrix for the row operation $R_i \to cR_i$.
5. Suppose that U is the RCF of A. If $c_1, c_2, \ldots, c_p$ are the multipliers used for the row operations $R_i \to cR_i$ and r row exchanges have been used to get U from A then for any alternating multilinear function d,
$$d(U) = (-1)^r c_1 c_2 \ldots c_p \, d(A).$$
6. To see this, we simply note that (recall that $\det A^t = \det A$)
$$d(FA) = -d(A),\ d(EA) = d(A) \text{ and } d(GA) = cd(A). \qquad \text{(why?)}$$
7. If $u_{11}, u_{22}, \ldots, u_{nn}$ are the diagonal entries of U, then
$$d(A) = (-1)^r (c_1 c_2, \ldots c_p)^{-1} u_{11} u_{22} \cdots u_{nn}.$$

The cofactor matrix

1. **Definition:** Let $A = (a_{ij})$ be an $n \times n$ matrix. The **cofactor** of a_{ij}, denoted by $\operatorname{cof} a_{ij}$ is defined as

$$\operatorname{cof} a_{ij} = (-1)^{i+j} \det A_{ij}.$$

2. The **cofactor matrix** of A denoted by $\operatorname{cof} A$ is the matrix

$$\operatorname{cof} A = (\operatorname{cof} a_{ij}).$$

Theorem

For any $n \times n$ matrix A,

$$A(\operatorname{cof} A)^t = (\det A)I = (\operatorname{cof} A)^t A.$$

In particular, if $\det A$ is nonzero then $A^{-1} = \frac{1}{\det A}(\operatorname{cof} A)^t$, hence A is invertible.

Matrix inverse and the cofactor matrix

1. **Proof:** The $(i,j)^{\mathrm{th}}$ entry of $(\operatorname{cof} A)^t A$ is

$$\sum_{k=1}^{n} (\operatorname{cof} A)^t_{ik} a_{kj} = \sum_{k=1}^{n} (\operatorname{cof} a_{ki}) a_{kj} = a_{1j} \operatorname{cof} a_{1i} + a_{2j} \operatorname{cof} a_{2i} + \cdots + a_{nj} \operatorname{cof} a_{ni}.$$

2. When $i = j$, it is easy to see that it is $\det A$.
3. When $i \neq j$, consider the matrix B obtained by replacing j^{th} column of A by i^{th} column of A.
4. So B has a repeated column and then the $(i,j)^{\mathrm{th}}$ entry of $(\operatorname{cof} A)^t A$ actually represents $\det B$ (since $a_{kj} = a_{ki}$ for each $1 \leq k \leq n$) and hence is equal to 0.
5. The other equation $A(\operatorname{cof} A)^t = (\det A)I$ is proved similarly. $\square$
6. **Cramer's Rule:** Suppose

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}.$$

Cramer's Rule

1. Suppose the coefficient matrix $A = (a_{ij})$ is invertible.
2. Let C_j be the matrix obtained from A by replacing j^{th} column of A by $b = [b_1, b_2, \ldots, b_n]^t$.
3. Then for $j = 1, 2, \ldots, n$,
$$x_j = \frac{\det C_j}{\det A}.$$
4. **Proof:** Let $A_1, \ldots, A_n$ be the columns of A.
5. Write $b = x_1 A_1 + x_2 A_2 + \cdots + x_n A_n$.
6. Then $\det(b, A_2, A_3, \ldots, A_n) = x_1 \det A$ and hence $x_1 = \frac{\det C_1}{\det A}$.
7. By a similar computation we obtain $x_j = \frac{\det C_j}{\det A}$ for all $j = 1, 2, \ldots, n$. $\square$