

Assume F is sat.

\exists an assignment α s.t. $\alpha \models F$.

If $a = T$ in α , then

$$\alpha \models (\neg a \vee C_{k+1}) \wedge (\neg a \vee C_{k+2}) \wedge \dots \wedge B_1 \wedge \dots \wedge B_m$$

$$\Rightarrow \alpha \models C_{k+1} \wedge C_{k+2} \wedge \dots \wedge C_n \wedge B_1 \wedge \dots \wedge B_m$$

$$\Rightarrow \alpha \models R(F)$$

else if $a = \perp$ in α , then

$$\alpha \models (a \vee C_1) \wedge (a \vee C_2) \wedge \dots \wedge (a \vee C_k) \wedge B_1 \wedge \dots \wedge B_m$$

$$\Rightarrow \alpha \models C_1 \wedge \dots \wedge C_k \wedge B_1 \wedge \dots \wedge B_m$$

$$\Rightarrow \alpha \models R(F).$$

Conversely, let $R(F)$ be sat.

Then $\exists \alpha$ s.t. $\alpha \models R(F)$.

$$\therefore \alpha \models B_1 \wedge \dots \wedge B_m.$$

① $\alpha \models C_1 \wedge \dots \wedge C_k$. Then \exists an assignment β which agrees with α , and assigns $a = \perp$ s.t.

$$\beta \models B_1 \wedge \dots \wedge B_m \text{ and}$$

$$\beta \models (a \vee C_1) \wedge (a \vee C_2) \wedge \dots \wedge (a \vee C_k)$$

$$\text{and } \beta \models (\neg a \vee C_{k+1}) \wedge (\neg a \vee C_{k+2}) \wedge \dots \wedge (\neg a \vee C_n)$$

$$\therefore \beta \models F$$

② $\exists j \in [1, k]$ s.t. $\alpha \not\models C_j$.

$$\text{But } \alpha \models (C_j \vee C_{k+1}) \wedge (C_j \vee C_{k+2}) \wedge \dots \wedge (C_j \vee C_n)$$

$$\Rightarrow \alpha \models C_{k+1} \wedge C_{k+2} \wedge \dots \wedge C_n$$

As in ①, let β agree with α and assign $a = T$. Then

$$\beta \models B_1 \wedge \dots \wedge B_m \text{ and}$$

$$\beta \models (a \vee C_1) \wedge \dots \wedge (a \vee C_k)$$

Thus $\beta \models F$.

Thus, it is enough to maintain clauses in $R(F)$.

$F \rightarrow F'$ and $F \rightarrow \#$ gives

$$F \rightarrow \#.$$

Thus we have $F \rightarrow \#$ and $\# \rightarrow G$.

4. $S_1 = \{p \wedge \neg p\}$ no proper subset

$$S_2 = \{p, \neg p\}$$

$$S_3 = \{p_1, p_2, \neg(p_1 \wedge p_2)\}$$

$$S_4 = \{p_1, p_2, p_3, \neg(p_1 \wedge p_2 \wedge p_3)\}$$

5. (a) Routine.

(b) The idea is to remove the parents of a resolvent and ask if it will work.

Resolution: Resolving F to $R(F)$ if \exists clauses a, c and a literal a' s.t.
 $a \in c$ and $\neg a \in c$

F can be written as a conjunction of clauses

$$(a \vee c_1), (a \vee c_2), \dots, (a \vee c_k), (\neg a \vee c_{k+1}), \dots, (\neg a \vee c_n), b_1, \dots, b_m$$

s.t. B has neither a , $\neg a$, and
 $\neg c$, $a \notin c$, $\neg a \notin c$

Define $R(F)$ as a conjunction of clauses

$$(a \vee c_{k+1}), (c_2 \vee c_{k+1}), \dots, (c_k \vee c_{k+1}), \\ (c_1 \vee c_{k+2}), (c_2 \vee c_{k+2}), \dots, (c_k \vee c_{k+2}), \\ \vdots \\ (c_1 \vee c_n), (c_2 \vee c_n), \dots, (c_k \vee c_n), \\ b_1, \dots, b_m.$$

Indeed, $R(F)$ contains neither a nor $\neg a$.

P.T. F is sat $\iff R(F)$ is sat.

1) Assume $\mathcal{F} \equiv G$.

Eg $\mathcal{F} = \{p, q\}$ $G = \{p \wedge q\}$.

(2) false $p \neq p \wedge q, q \neq p \wedge q$.

Now consider $\mathcal{F} = \{p, q, p \vee \neg p\}$

$G = \{p \vee q, q \vee \neg q\}$.

Then $\forall F \in \mathcal{F}, F \in G$

$\forall G \in G, G \in \mathcal{F}$

However $p \wedge q \wedge (p \vee \neg p) \neq (p \vee q) \wedge (q \vee \neg q)$

2) \mathcal{F} is of the form

$\{F_1, F_2, \dots, F_n, F_1 \wedge F_2, F_1 \wedge F_3, \dots, F_1 \wedge \dots \wedge F_n\}$, and is

inconsistent. That is $\bigwedge_{F \in \mathcal{F}} F \vdash \perp$.

Consider any $F \in \mathcal{F}$.

Since $\bigwedge_{F \in \mathcal{F}} F \vdash \perp$, and $\bigwedge_{F \in \mathcal{F}} F \in \mathcal{F}$

Choose $G = \bigwedge_{F \in \mathcal{F}} F$. Indeed $G \vdash \perp$, hence $G \vdash \neg F$.

3) $\not\models F \rightarrow G$, F not a contradiction and G not a tautology.

$\text{Prop}(F) = \text{Prop variables in } F$

$\text{Prop}(G) = \text{Prop variables in } G$

Induct on $|\text{Prop}(F) - \text{Prop}(G)|$.

Base $|\text{Prop}(F) - \text{Prop}(G)| = 0$ then choose

$H = F$. Indeed, $\models F \rightarrow F$ and $\models F \rightarrow G$,

and $\text{Prop}(F) \subseteq \text{Prop}(F) \cap \text{Prop}(G)$

(since $|\text{Prop}(F) - \text{Prop}(G)| = 0$).

Inductive hypothesis. Assume the result whenever

$|\text{Prop}(F) - \text{Prop}(G)| \leq n$. let

$|\text{Prop}(F) - \text{Prop}(G)| = n+1$.

Let $q \in \text{Prop}(F) - \text{Prop}(G)$.

Define $F' = F[q \mapsto \top] \vee F[q \mapsto \perp]$.

Then $\models F' \rightarrow G$ (why?), $\models F \rightarrow F'$.

Indeed, $|\text{Prop}(F') - \text{Prop}(G)| = n$ and the inductive hypothesis applies.

Hence, $\exists H$ st. $\models F' \rightarrow H, \models H \rightarrow G$.