HELMHOLTZ'S THEOREM

Is this solution unique?

YES, as long as the vector field $\boldsymbol{F}(\boldsymbol{r})$ itself goes to zero at infinity

If the divergence D(**r**) and the curl **C(r)** of a vector function **F(r)** are specified, and if they both go to zero faster than $1/r^2$ as r goes to infinity, and if **F(r)** itself goes to zero as r goes to infinity, then **F(r)** is uniquely given by

$$\boldsymbol{F} = -\nabla U + \nabla \times \boldsymbol{W}$$

MAXWELL'S EQUATIONS

$$\nabla . \vec{E} = \frac{\rho}{\epsilon_0}$$

$$\nabla . \vec{B} = 0$$

$$\nabla \times \vec{E} = \frac{-\partial \vec{B}}{\partial t}$$

$$\nabla \times \vec{B} = \mu_0 \vec{J} + \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t}$$

Maxwell's equations specify the divergence and curl of the electric and magnetic fields. Using Helmholtz's theorem, we can then determine the electric and magnetic fields from Maxwell's equations.

MAXWELL'S EQUATIONS - ELECTROSTATICS

$$\nabla . \boldsymbol{E} = \rho/\epsilon_0$$

$$\nabla \times \boldsymbol{E} = 0$$

$$\boldsymbol{E}(\boldsymbol{r}) = -\nabla\left(\frac{1}{4\pi\epsilon_0}\int\frac{\rho(\boldsymbol{r}')}{\mathfrak{R}}dV'\right)$$

$$= \frac{1}{4\pi\epsilon_0}\int\frac{\rho(\boldsymbol{r}')}{\mathfrak{R}^2}\hat{\mathfrak{R}}\,dV'$$

$$\mathfrak{R} = \boldsymbol{r} - \boldsymbol{r}'$$

Coulomb's Law!

For a continuous surface charge, $\boldsymbol{E}(\boldsymbol{r}) = \frac{1}{4\pi\epsilon_0}\int\frac{\sigma(\boldsymbol{r}')}{\mathfrak{R}^2}\hat{\mathfrak{R}}\,da'$

For a continuous line charge, $\boldsymbol{E}(\boldsymbol{r}) = \frac{1}{4\pi\epsilon_0}\int\frac{\lambda(\boldsymbol{r}')}{\mathfrak{R}^2}\hat{\mathfrak{R}}\,dl'$

For a collection of discrete charges, $\boldsymbol{E}(\boldsymbol{r}) = \frac{1}{4\pi\epsilon_0}\sum_i\frac{q_i}{\mathfrak{R}_i^2}\hat{\mathfrak{R}}_{\boldsymbol{i}}$

ELECTROSTATICS

ELECTROSTATICS

$$\nabla . \boldsymbol{E} = \rho/\epsilon_0$$
$$\nabla \times \boldsymbol{E} = 0$$

Maxwell's Equations
for Electrostatics

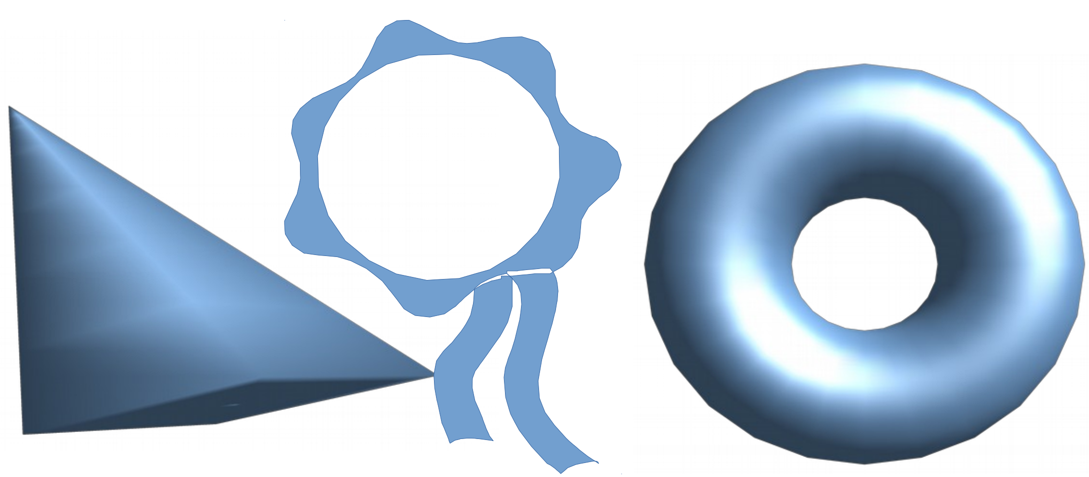
Helmholtz Theorem $\longrightarrow$

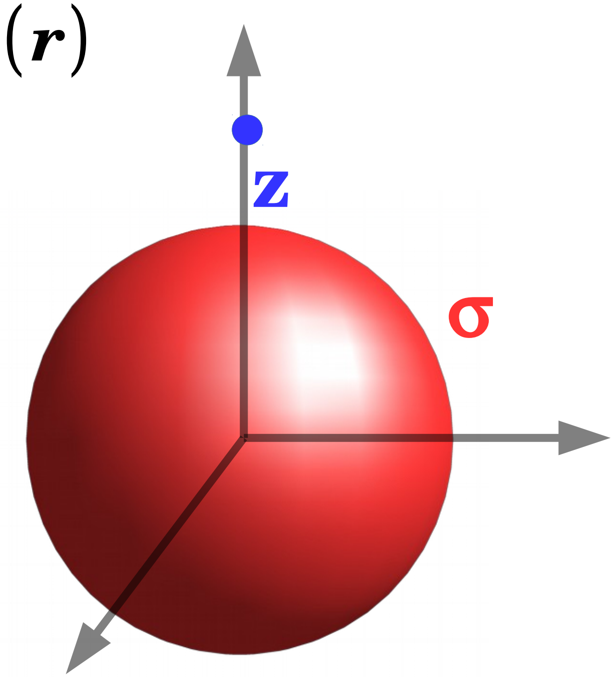
$$\boldsymbol{E}(\boldsymbol{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\boldsymbol{r}')}{\Re^2}\hat{\boldsymbol{\Re}}\, dV'$$

$$\boldsymbol{E}(\boldsymbol{r}) = \frac{1}{4\pi\epsilon_0}\sum_i \frac{q_i}{\Re_i^2}\hat{\boldsymbol{\Re}}_{\boldsymbol{i}}$$

Coulomb's Law

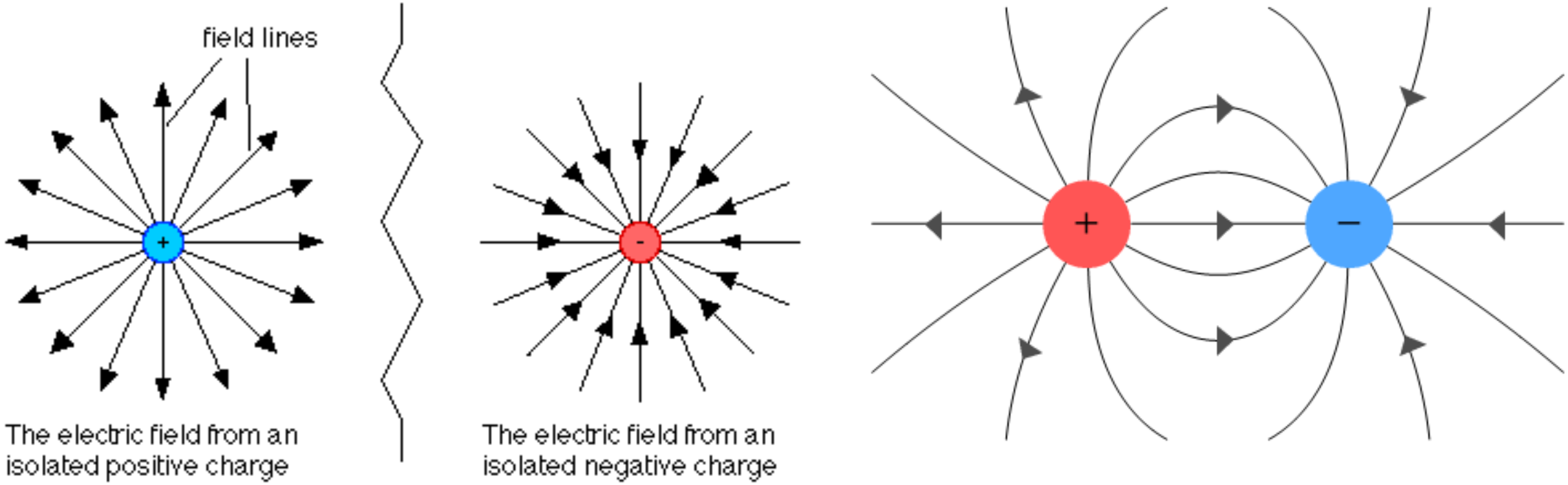
Force on a test charge Q, $\boldsymbol{F}(\boldsymbol{r}) = Q\,\boldsymbol{E}(\boldsymbol{r})$





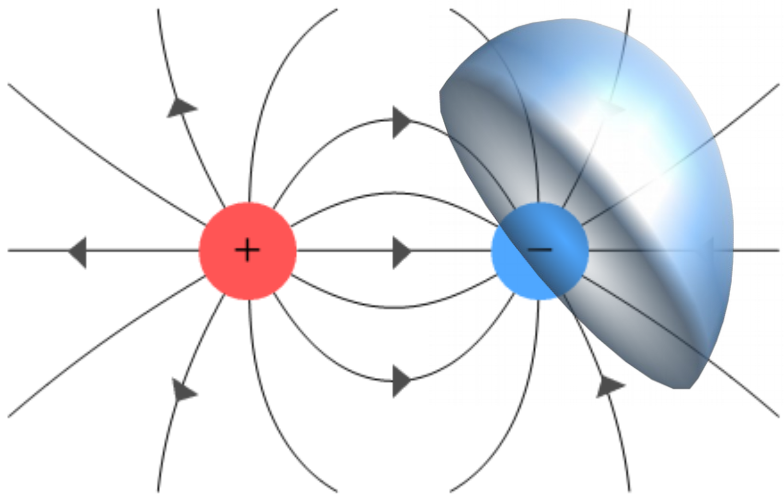
In principle, we're done with Electrostatics!

ELECTRIC FIELD LINES



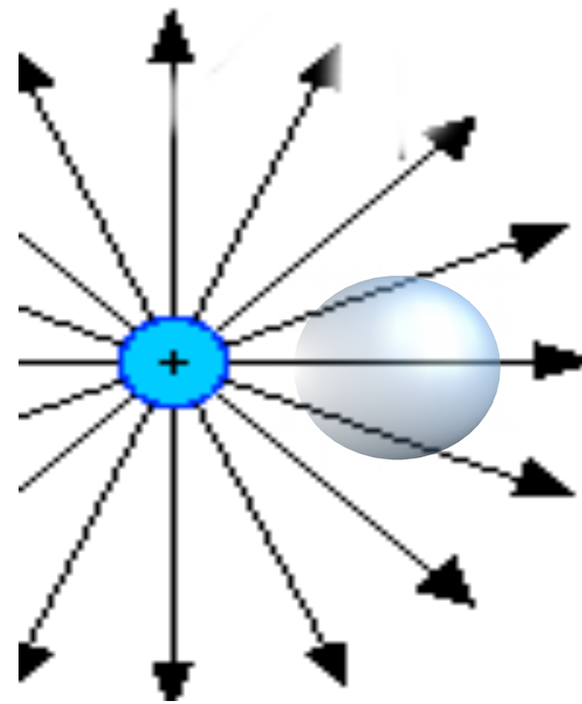
- Field lines begin on positive charges
- Field lines end on negative charges, or they extend upto infinity
- The strength of the field is indicated by the density of the field lines
- Field lines can never cross

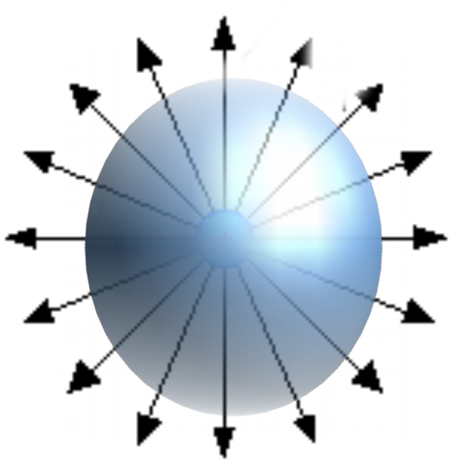
ELECTRIC FIELD LINES - CLOSED SURFACE

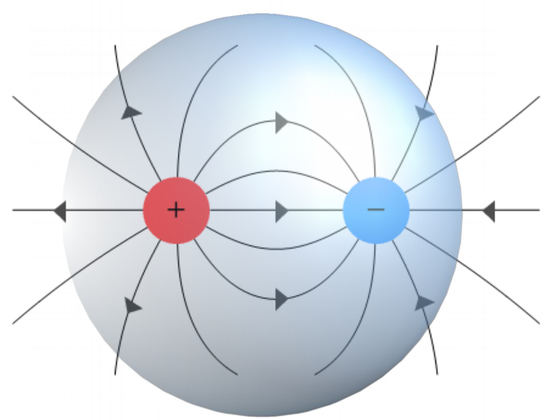


Flux $\Phi_E = \oint_S \boldsymbol{E} . d\boldsymbol{S}$

Flux $\propto$ Number of field lines

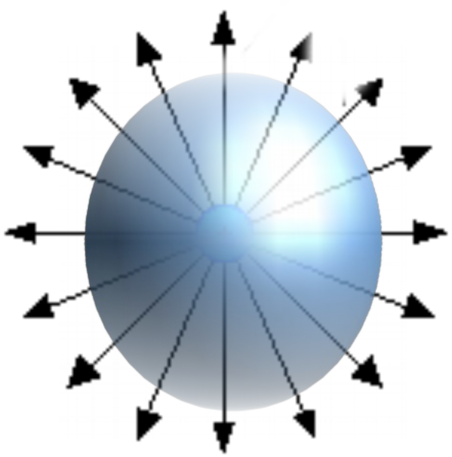






Flux through a closed surface is a measure of the total charge inside the surface – Gauss's Law

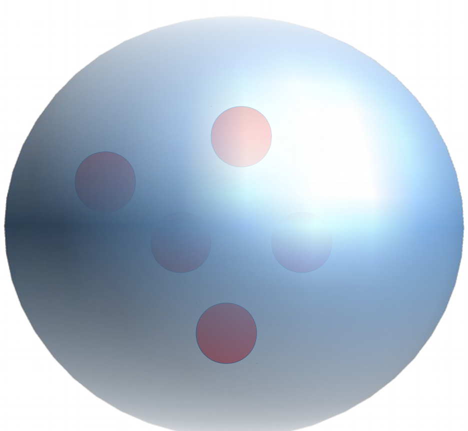
GAUSS'S LAW



Point charge q at origin

$$\oint \boldsymbol{E}.d\boldsymbol{a} = \int \frac{1}{4\pi\epsilon_0}\left(\frac{q}{r^2}\hat{\boldsymbol{r}}\right)(r^2 \sin\theta\, d\theta\, d\phi\, \hat{\boldsymbol{r}}) = \frac{q}{\epsilon_0}$$

Principle of superposition $\boldsymbol{E} = \sum_{i=1}^{N} \boldsymbol{E}_i$

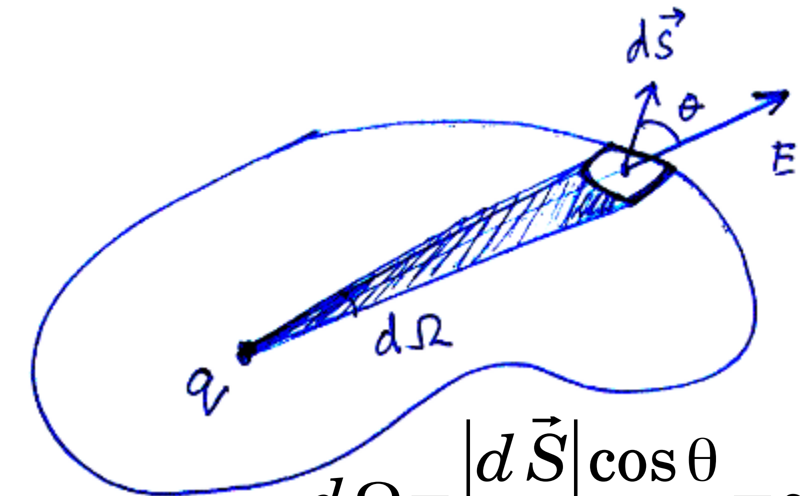


Collection of point charges q_i

$$\oint \boldsymbol{E}.d\boldsymbol{a} = \sum_{i=1}^{N}\left(\oint \boldsymbol{E}.d\boldsymbol{a}\right) = \sum_{i=1}^{N}\left(\frac{q_i}{\epsilon_0}\right)$$

$$\oint_S \boldsymbol{E}.d\boldsymbol{a} = \frac{1}{\epsilon_0} Q_{enc}$$

GAUSS'S LAW



$$\int_{surface} \boldsymbol{E}.d\boldsymbol{S} = \frac{q}{4\pi\epsilon_0}\int_{surface}\frac{\hat{\boldsymbol{r}}.d\boldsymbol{S}}{r^2}$$

$$= \frac{q}{4\pi\epsilon_0}\int_{surface}\frac{|d\boldsymbol{S}|\cos\theta}{r^2}$$

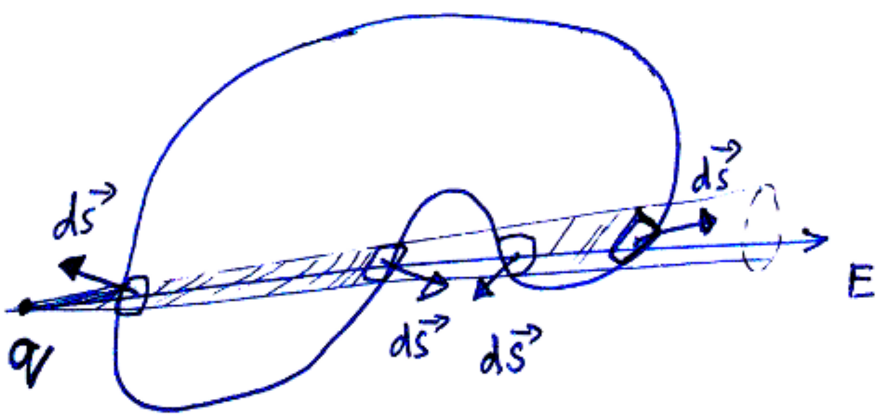
$$d\Omega = \frac{|d\vec{S}|\cos\theta}{r^2} = \sin\theta\, d\theta\, d\phi$$

$$= \frac{q}{4\pi\epsilon_0}\int_{surface} d\Omega$$

$$= \frac{q}{4\pi\epsilon_0}4\pi = \frac{q}{\epsilon_0}$$

The solid angle subtended by a surface S is defined as the surface area of a unit sphere covered by the surface's projection onto the sphere.

A measure of how large an object appears to an observer looking from that point



If the point is located outside then the contributions exactly cancel

Use superposition principle ---> Add contribution from each charge

GAUSS'S LAW - DIFFERENTIAL FORM

$$\int_{surface} \boldsymbol{E} . d\boldsymbol{S} = \int_{vol} \nabla . \boldsymbol{E}\, dV$$

$$\frac{1}{\epsilon_0} Q_{enc} = \int_{vol} \frac{\rho(\boldsymbol{r})}{\epsilon_0} dV$$

$$\oint_S \boldsymbol{E} . d\boldsymbol{a} = \frac{1}{\epsilon_0} Q_{enc} \Rightarrow \nabla . \boldsymbol{E} = \frac{\rho(\boldsymbol{r})}{\epsilon_0}$$

Helmholtz Theorem

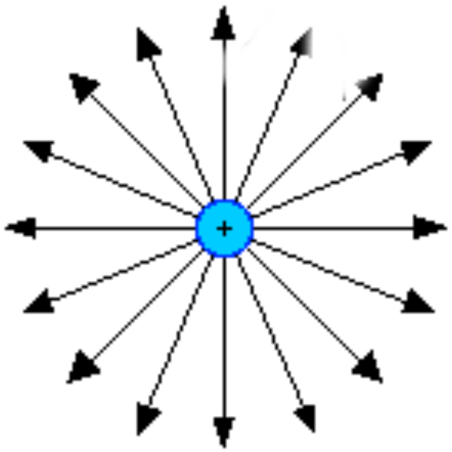
$$\nabla . \boldsymbol{E} = \frac{\rho(\boldsymbol{r})}{\epsilon_0} \qquad \boldsymbol{E} = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{\boldsymbol{r}}$$

Valid for moving charges!

Only for static charges

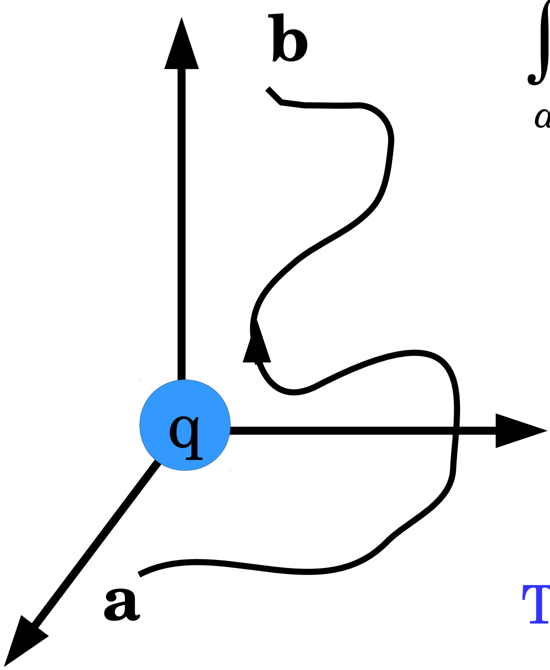
Gauss's Law

CURL OF THE ELECTRIC FIELD



$$\boldsymbol{E} = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{\boldsymbol{r}}$$

$$\nabla\times\boldsymbol{E} = 0$$



$$\int_a^b \boldsymbol{E}.d\boldsymbol{l} = \int_a^b \left(\frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{\boldsymbol{r}}\right).(dr\,\hat{\boldsymbol{r}} + r\,d\theta\,\hat{\boldsymbol{\theta}} + r\sin\theta\,d\phi\,\hat{\boldsymbol{\phi}})$$

$$\int_a^b \boldsymbol{E}.d\boldsymbol{l} = \frac{1}{4\pi\epsilon_0}\int_a^b \frac{q}{r^2}dr = \frac{q}{4\pi\epsilon_0}\left(\frac{1}{r_a} - \frac{1}{r_b}\right)$$

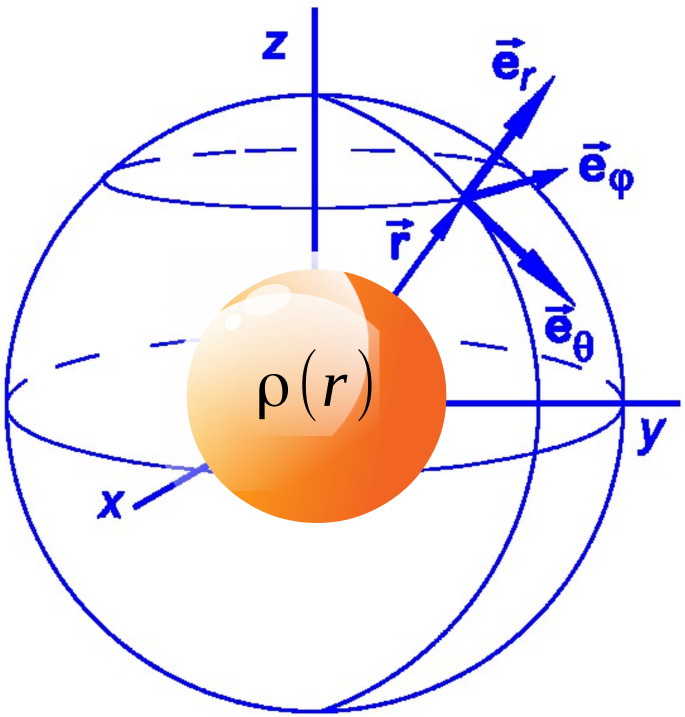
$$\oint \boldsymbol{E}.d\boldsymbol{l} = 0 \quad \overset{\text{Stokes Theorem}}{\Rightarrow} \quad \nabla\times\boldsymbol{E} = 0$$

True for any charge configuration due to superposition!

Valid only for static charges

GAUSS'S LAW + SYMMETRY - SPHERE

Consider a spherically symmetric charge distribution $\rho(\boldsymbol{r})$



$E_\phi = 0$ Why?

Rotate about the z-axis

$$\oint \boldsymbol{E}.d\boldsymbol{l} = 0 \quad \Rightarrow \quad E_\phi = 0$$

$E_\theta = 0$ Why?

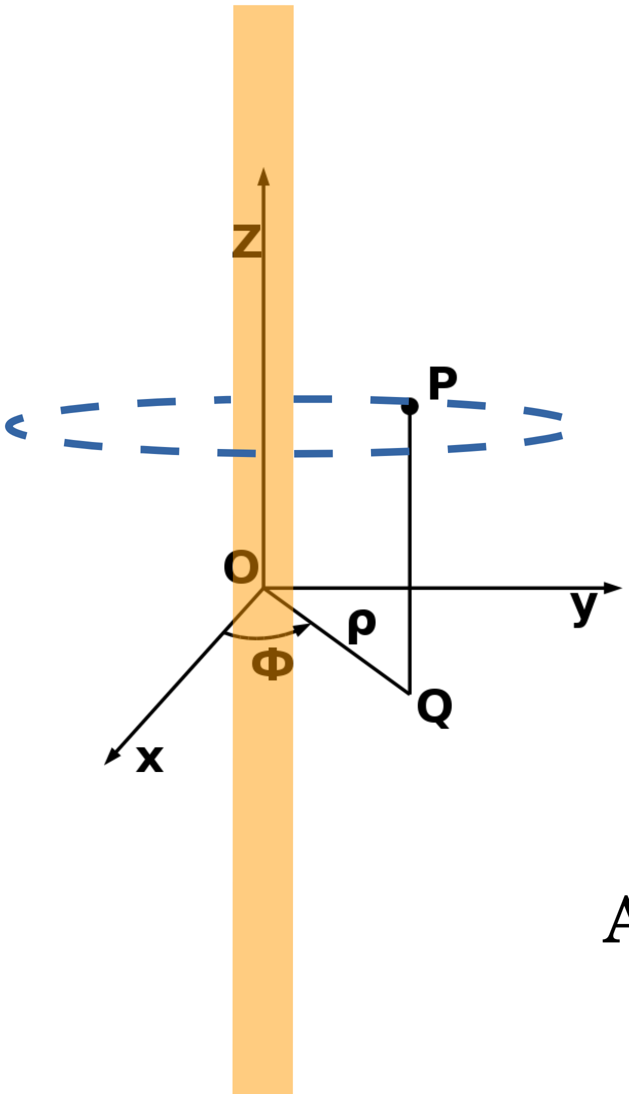
Rotate about the x-axis

$$\oint \boldsymbol{E}.d\boldsymbol{l} = 0 \quad \Rightarrow \quad E_\theta = 0$$

Apply Gauss's Law: $$E_r.4\pi R^2 = \frac{1}{\epsilon_0}\int_0^R \rho(r)4\pi r^2.dr$$

GAUSS'S LAW + SYMMETRY - CYLINDER

Consider a long, narrow wire with a charge per unit length λ



$E_\phi = 0$ Why?

Rotate about the z-axis

$$\oint \boldsymbol{E}.d\boldsymbol{l} = 0 \quad \Rightarrow \quad E_\phi = 0$$

$E_z = 0$ Why?

Flip about z-axis

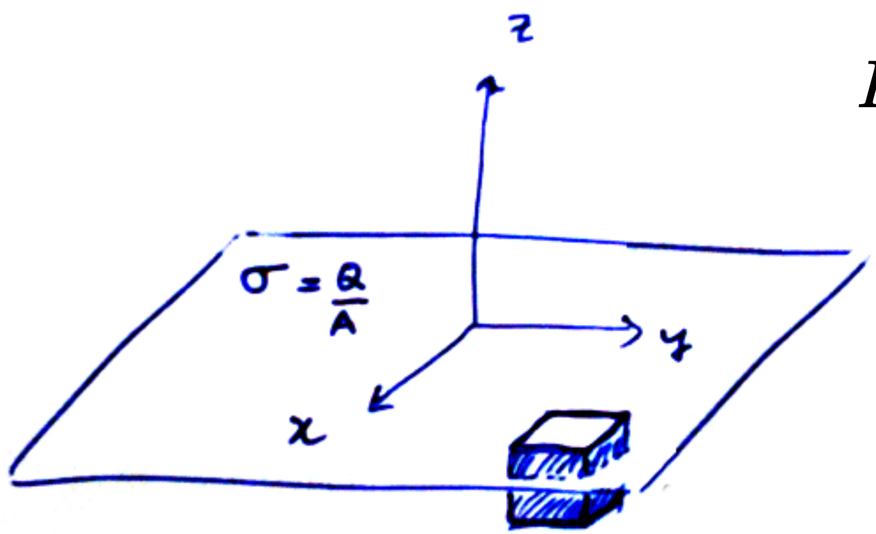
Nothing distinguishes z from -z

$$\Rightarrow \quad E_z = 0$$

Apply Gauss's Law: $E_\rho . 2\pi\rho = \dfrac{1}{\epsilon_0}\lambda$

GAUSS'S LAW + SYMMETRY - SURFACE

Consider an infinite sheet of charge with a surface charge density σ



$E_{//}\ \ (E_x, E_y) = 0$ Why?

Rotate the sheet about any point

Translate by any in-plane vector

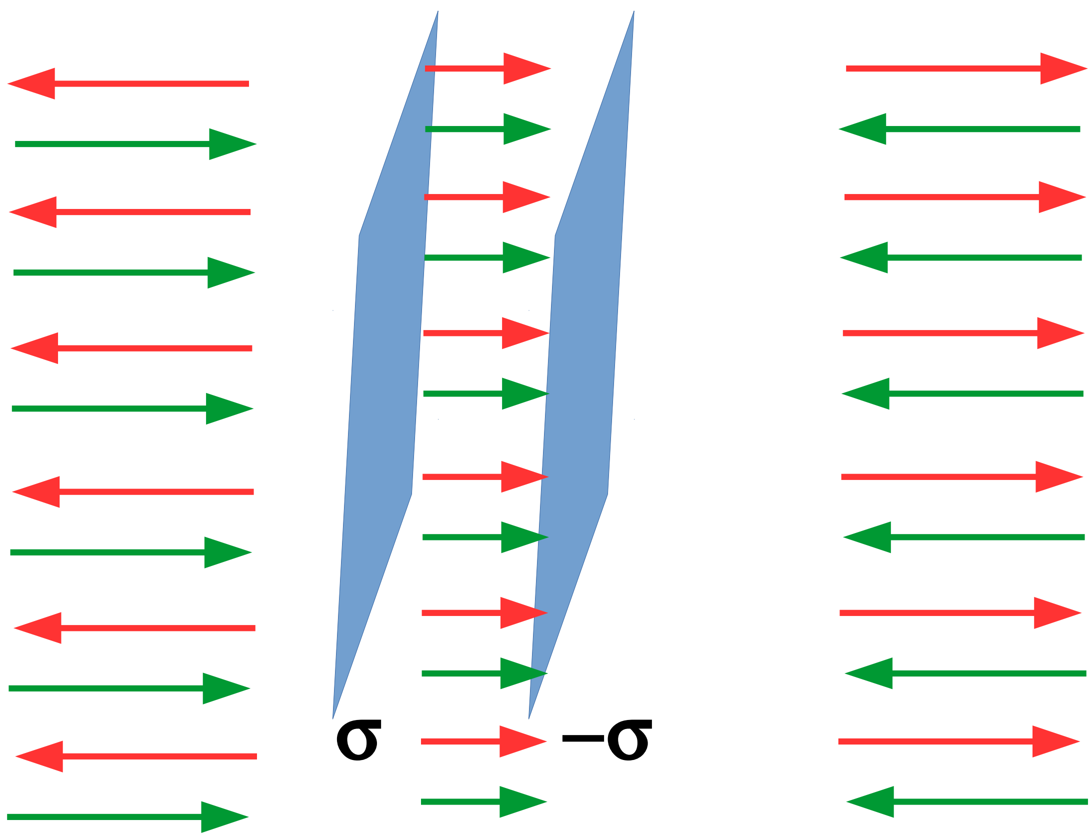
Field cannot change $\Rightarrow \quad E_{//} = 0$

Apply Gauss's Law: $\int \boldsymbol{E} . d\boldsymbol{a} \; = \; 2A|\boldsymbol{E}| = \frac{1}{\epsilon_0}\sigma A$

$$\boldsymbol{E} = \frac{\sigma}{2\epsilon_0}\hat{\boldsymbol{n}}$$

GAUSS'S LAW + SYMMETRY - SURFACE

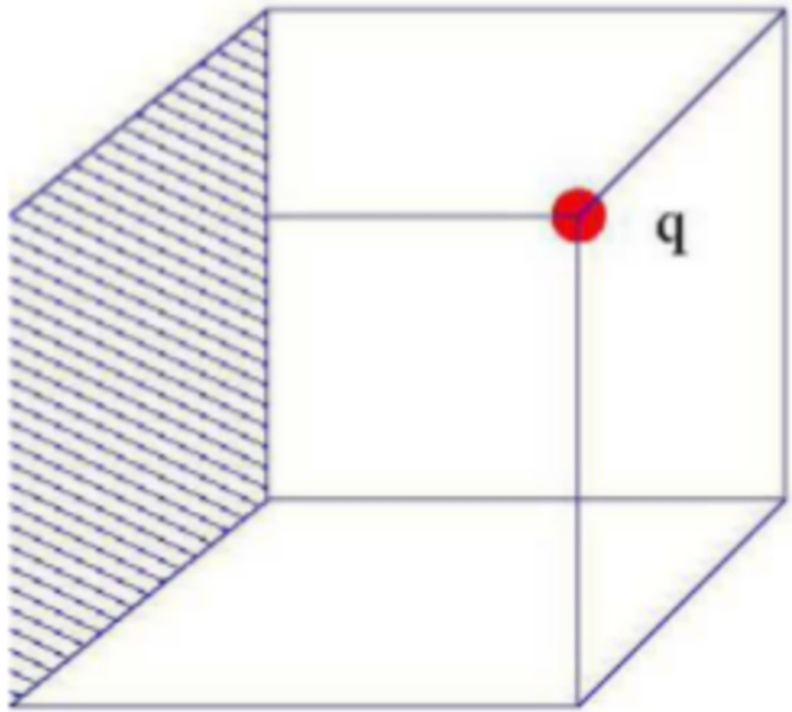
Consider two parallel plates with equal and opposite charge densities $\pm\sigma$. What is the electric field?

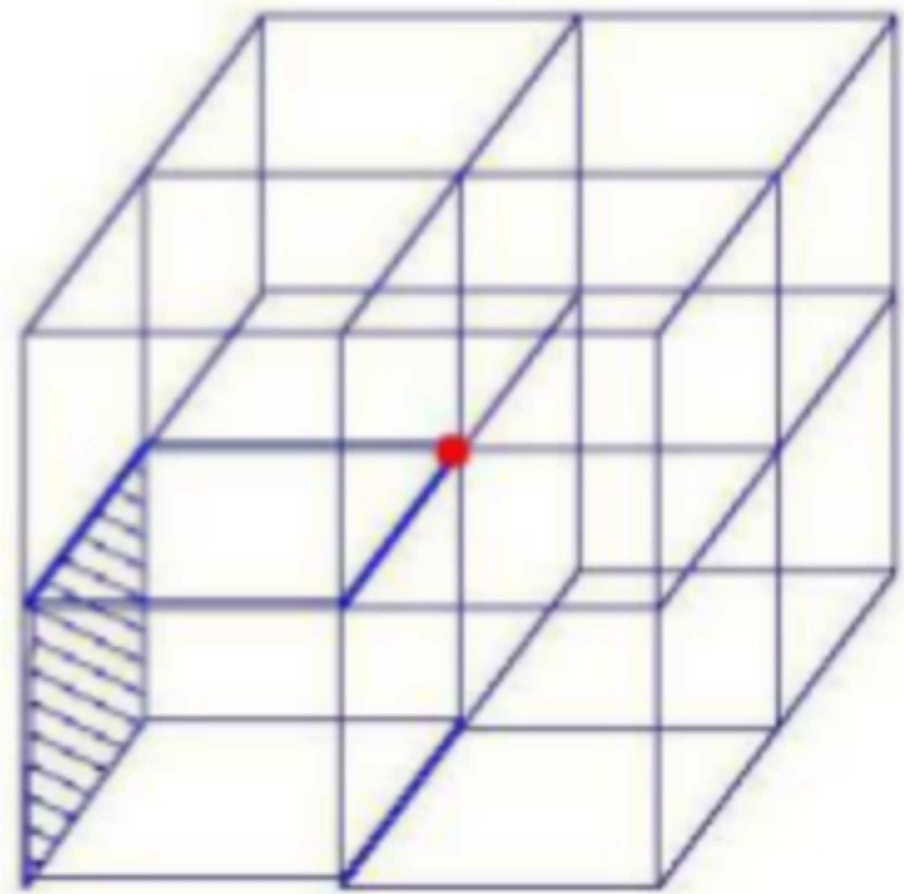


$$C = \frac{\epsilon_0 A}{d}$$

$$\boldsymbol{E} = \begin{cases} \dfrac{\sigma}{2\epsilon_0}\hat{\boldsymbol{n}} + \dfrac{\sigma}{2\epsilon_0}\hat{\boldsymbol{n}} = \dfrac{\sigma}{\epsilon_0}\hat{\boldsymbol{n}} & \text{between the plates} \\ 0 & \text{everywhere else} \end{cases}$$

GAUSS'S LAW + SYMMETRY





What is the flux of the electric field through the shaded face?

$$\frac{q}{24\,\epsilon_0}$$

THE ELECTRIC POTENTIAL

Gauss's Law is always true.

It may not always be useful!

If we can take advantage of the symmetries of a problem, Gauss's Law can be a very powerful tool.

$$\nabla \times \boldsymbol{E} = 0$$

$$\boldsymbol{E} = -\nabla V$$

$$V(\boldsymbol{r}) \equiv \text{Electric Potential}$$

$$\nabla . \boldsymbol{E} = \rho / \epsilon_0 \quad \Rightarrow \quad \boxed{\nabla^2 V = -\rho / \epsilon_0} \quad \text{Poisson's Equation}$$

In regions of no charge $\boxed{\nabla^2 V = 0}$ Laplace Equation

THE ELECTRIC POTENTIAL

$$\nabla \times \boldsymbol{E} = 0 \quad \Leftrightarrow \quad \oint \boldsymbol{E} . d\boldsymbol{l} = 0$$

$$\boxed{V(\boldsymbol{r}) = -\int_{O}^{\boldsymbol{r}} \boldsymbol{E} . d\boldsymbol{l}}$$

O is some standard reference point

$$V(\boldsymbol{b}) - V(\boldsymbol{a}) = -\int_{O}^{\boldsymbol{b}} \boldsymbol{E} . d\boldsymbol{l} + \int_{O}^{\boldsymbol{a}} \boldsymbol{E} . d\boldsymbol{l} = -\int_{\boldsymbol{a}}^{\boldsymbol{b}} \boldsymbol{E} . d\boldsymbol{l}$$

$$\boldsymbol{E} = -\nabla V \qquad \nabla^2 V = -\rho / \epsilon_0$$

We have reduced a vector problem to a scalar one. How?

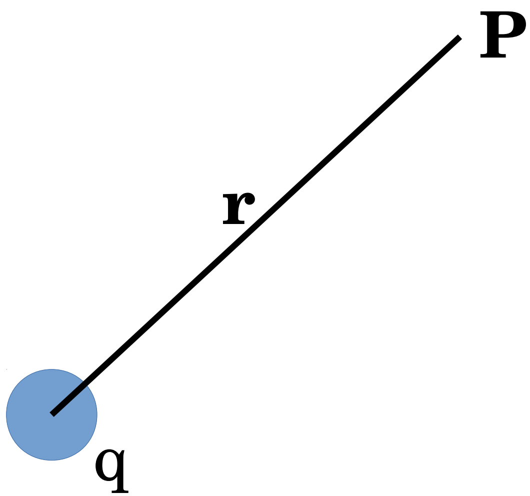
The choice of reference point is arbitrary. Changing the reference point does not change the potential difference or electric field between two point.

For most situations, $V(\infty) = 0$

Potential obeys the superposition principle.

Units: N-m/C, J/C, Volt

POTENTIAL FOR LOCALIZED CHARGE DISTRIBUTIONS

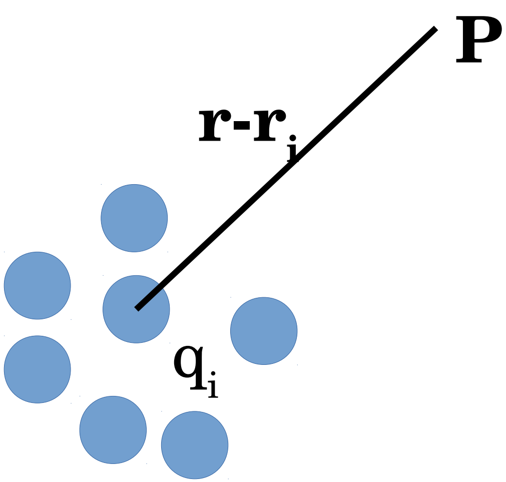


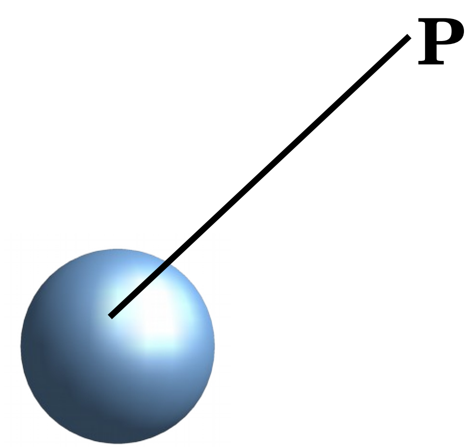
Point charge at origin

$$V(r) = -\frac{1}{4\pi\epsilon_0}\int_{\infty}^{r}\frac{q}{r'^2}dr'$$

$$= \frac{1}{4\pi\epsilon_0}\left[\frac{q}{r'}\right]_{\infty}^{r} = \frac{1}{4\pi\epsilon_0}\frac{q}{r}$$

$$V(\boldsymbol{r}) = \frac{1}{4\pi\epsilon_0}\frac{q}{|\boldsymbol{r}-\boldsymbol{r_s}|}$$





Superposition principle

$$V(\boldsymbol{r}) = \frac{1}{4\pi\epsilon_0}\sum_i \frac{q_i}{|\boldsymbol{r}-\boldsymbol{r_i}|}$$

$$V(\boldsymbol{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{q(\boldsymbol{r'})}{|\boldsymbol{r}-\boldsymbol{r'}|}dV'$$