Ninth Edition

CHAPTER

5

VECTOR MECHANICS FOR ENGINEERS: STATICS

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Lecture Notes:

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Distributed Forces: Centroids and Centers of Gravity

Contents

Introduction

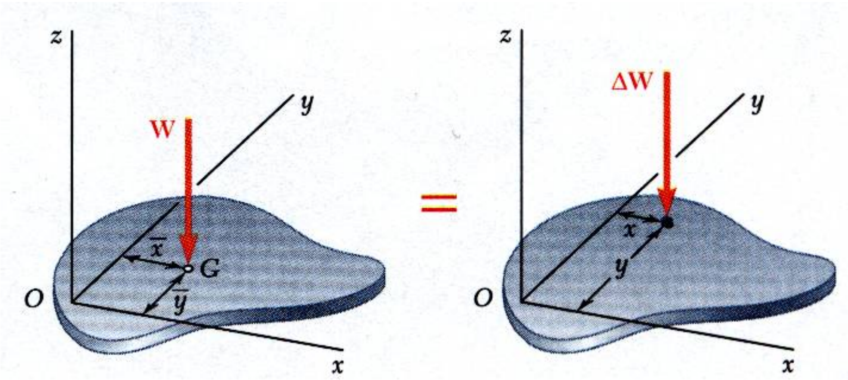
- The earth exerts a gravitational force on each of the particles forming a body. These forces can be replace by a single equivalent force equal to the weight of the body and applied at the *center of gravity* for the body.

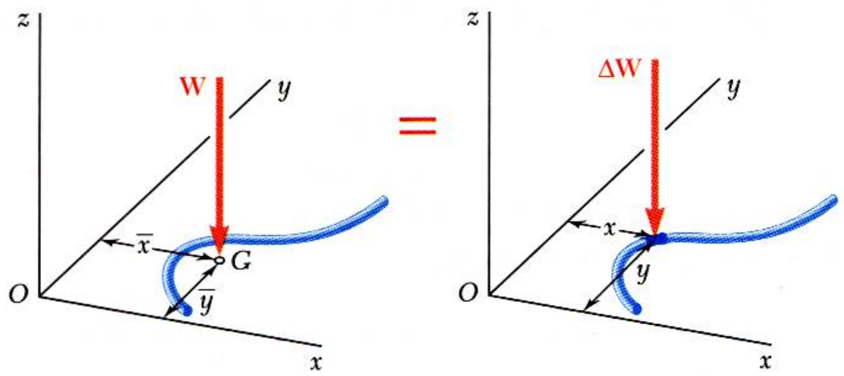
- The *centroid of an area* is analogous to the center of gravity of a body. The concept of the *first moment of an area* is used to locate the centroid.

- Determination of the area of a *surface of revolution* and the volume of a *body of revolution* are accomplished with the *Theorems of Pappus-Guldinus*.

Center of Gravity of a 2D Body

- Center of gravity of a plate
- Center of gravity of a wire

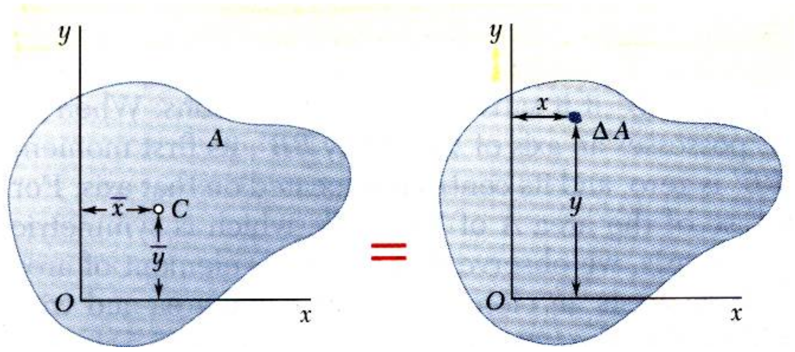




$$\sum M_y \quad \bar{x}W = \sum x\Delta W$$
$$= \int x\, dW$$
$$\sum M_y \quad \bar{y}W = \sum y\Delta W$$
$$= \int y\, dW$$

Centroids and First Moments of Areas and Lines

- Centroid of an area



$$\bar{x}W = \int x\,dW$$

$$\bar{x}(\gamma A t) = \int x(\gamma t)dA$$

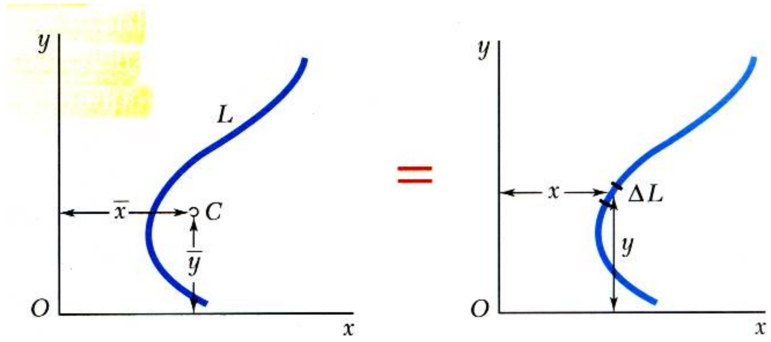
$$\bar{x}A = \int x\,dA = Q_y$$

= first moment with respect to y

$$\bar{y}A = \int y\,dA = Q_x$$

= first moment with respect to x

- Centroid of a line



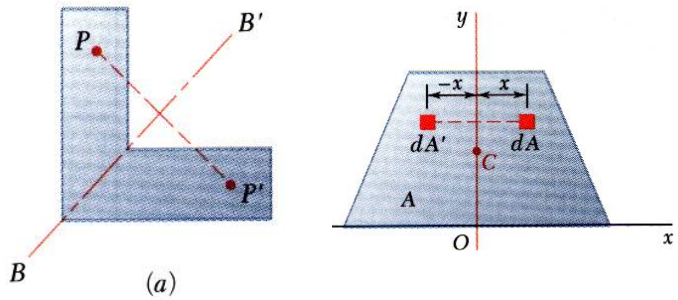
$$\bar{x}W = \int x\,dW$$

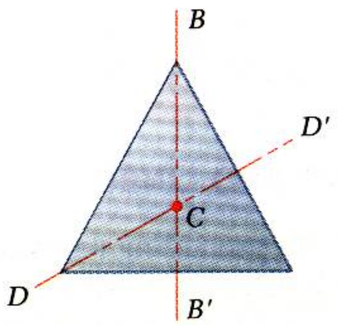
$$\bar{x}(\gamma L a) = \int x(\gamma a)dL$$

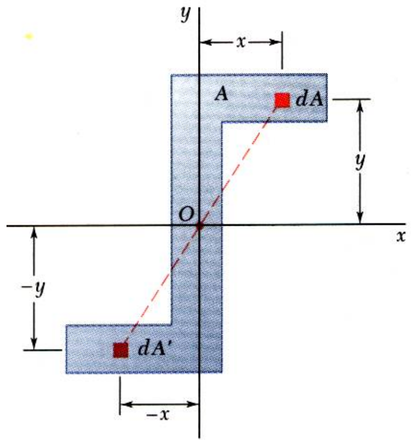
$$\bar{x}L = \int x\,dL$$

$$\bar{y}L = \int y\,dL$$

First Moments of Areas and Lines

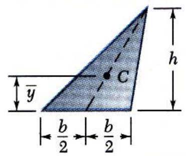
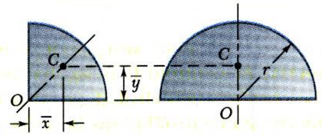
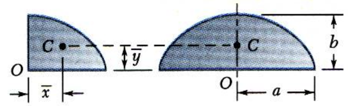
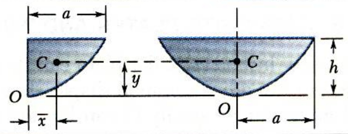
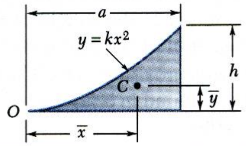
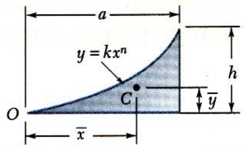
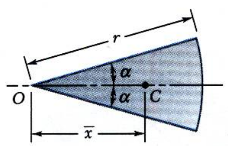




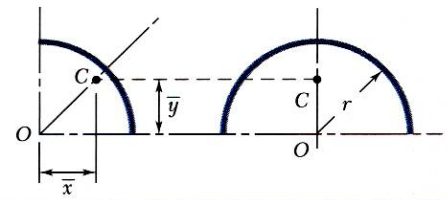
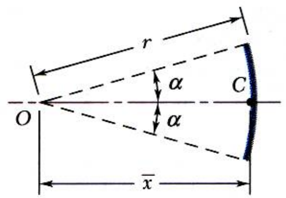


- An area is symmetric with respect to an axis *BB'* if for every point *P* there exists a point *P'* such that *PP'* is perpendicular to *BB'* and is divided into two equal parts by *BB'*.
- The first moment of an area with respect to a line of symmetry is zero.
- If an area possesses a line of symmetry, its centroid lies on that axis
- If an area possesses two lines of symmetry, its centroid lies at their intersection.
- An area is symmetric with respect to a center *O* if for every element *dA* at (*x*,*y*) there exists an area *dA'* of equal area at (-*x*,-*y*).
- The centroid of the area coincides with the center of symmetry.

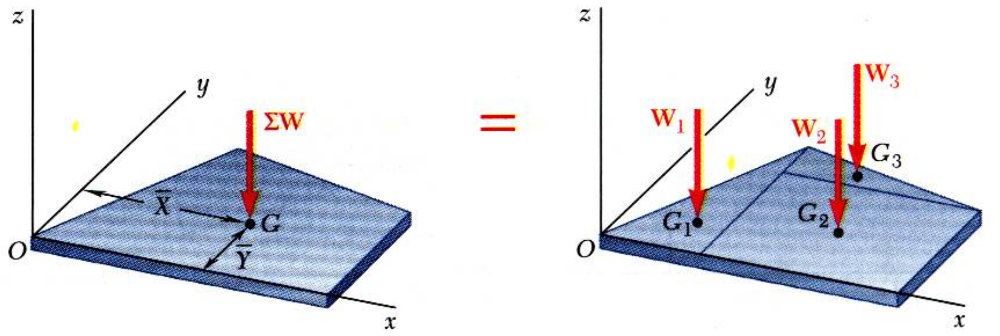
Centroids of Common Shapes of Areas

Shape		$\bar{x}$	$\bar{y}$	Area
Triangular area			$\frac{h}{3}$	$\frac{bh}{2}$
Quarter-circular area		$\frac{4r}{3\pi}$	$\frac{4r}{3\pi}$	$\frac{\pi r^2}{4}$
Semicircular area		0	$\frac{4r}{3\pi}$	$\frac{\pi r^2}{2}$
Quarter-elliptical area		$\frac{4a}{3\pi}$	$\frac{4b}{3\pi}$	$\frac{\pi ab}{4}$
Semielliptical area		0	$\frac{4b}{3\pi}$	$\frac{\pi ab}{2}$
Semiparabolic area		$\frac{3a}{8}$	$\frac{3h}{5}$	$\frac{2ah}{3}$
Parabolic area		0	$\frac{3h}{5}$	$\frac{4ah}{3}$
Parabolic spandrel		$\frac{3a}{4}$	$\frac{3h}{10}$	$\frac{ah}{3}$
General spandrel		$\frac{n+1}{n+2}a$	$\frac{n+1}{4n+2}h$	$\frac{ah}{n+1}$
Circular sector		$\frac{2r\sin\alpha}{3\alpha}$	0	αr^2

Centroids of Common Shapes of Lines

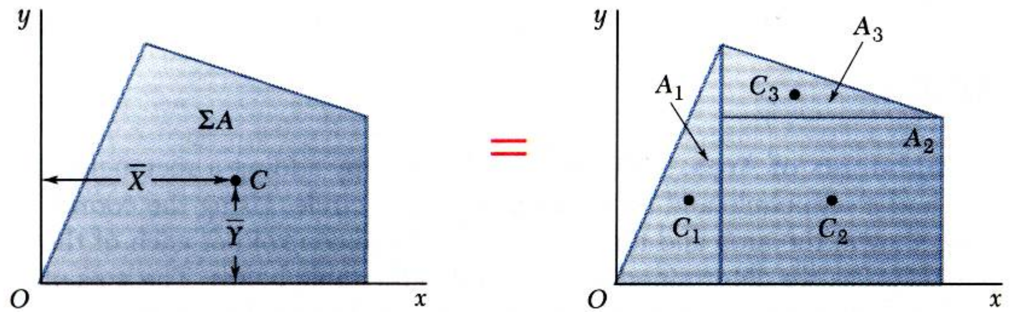
Shape		$\bar{x}$	$\bar{y}$	Length
Quarter-circular arc		$\frac{2r}{\pi}$	$\frac{2r}{\pi}$	$\frac{\pi r}{2}$
Semicircular arc		0	$\frac{2r}{\pi}$	πr
Arc of circle		$\frac{r \sin \alpha}{\alpha}$	0	$2\alpha r$

Composite Plates and Areas



- Composite plates

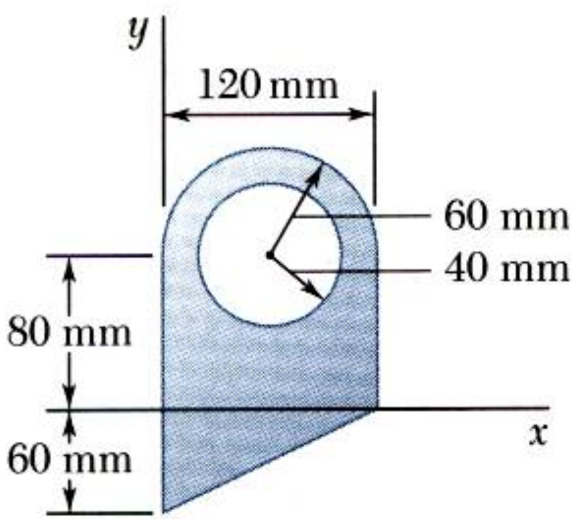
$$\bar{X}\sum W = \sum \bar{x} W$$
$$\bar{Y}\sum W = \sum \bar{y} W$$



- Composite area

$$\bar{X}\sum A = \sum \bar{x} A$$
$$\bar{Y}\sum A = \sum \bar{y} A$$

Sample Problem 5.1

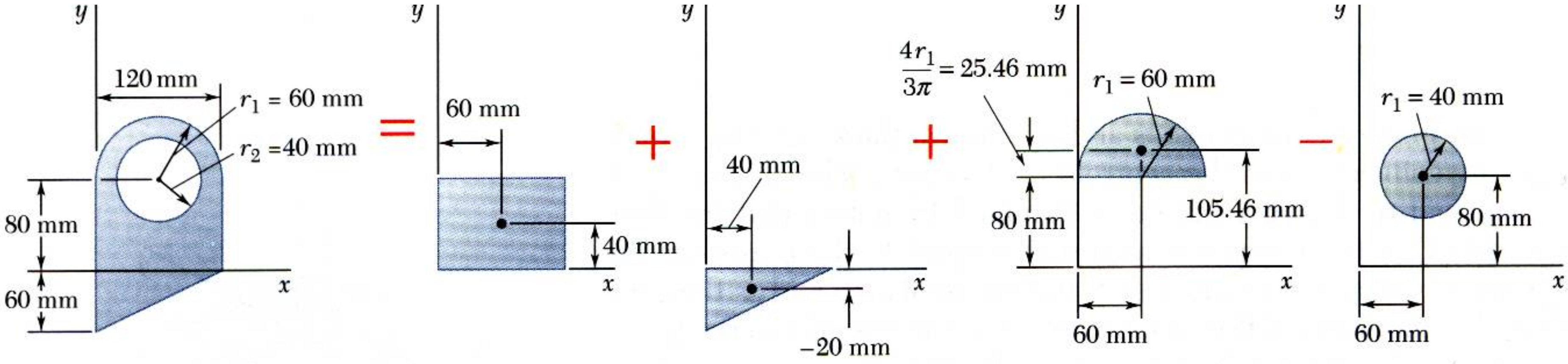


For the plane area shown, determine the first moments with respect to the *x* and *y* axes and the location of the centroid.

SOLUTION:

- Divide the area into a triangle, rectangle, and semicircle with a circular cutout.
- Calculate the first moments of each area with respect to the axes.
- Find the total area and first moments of the triangle, rectangle, and semicircle. Subtract the area and first moment of the circular cutout.
- Compute the coordinates of the area centroid by dividing the first moments by the total area.

Sample Problem 5.1



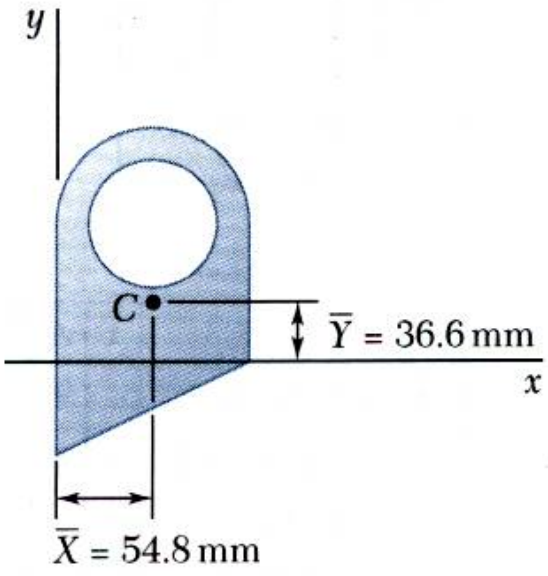
Component	A, mm²	$\bar{x}$, mm	$\bar{y}$, mm	$\bar{x}A$, mm³	$\bar{y}A$, mm³
Rectangle	$(120)(80) = 9.6 \times 10^3$	60	40	$+576 \times 10^3$	$+384 \times 10^3$
Triangle	$\frac{1}{2}(120)(60) = 3.6 \times 10^3$	40	−20	$+144 \times 10^3$	-72×10^3
Semicircle	$\frac{1}{2}\pi(60)^2 = 5.655 \times 10^3$	60	105.46	$+339.3 \times 10^3$	$+596.4 \times 10^3$
Circle	$-\pi(40)^2 = -5.027 \times 10^3$	60	80	-301.6×10^3	-402.2×10^3
	$\Sigma A = 13.828 \times 10^3$			$\Sigma\bar{x}A = +757.7 \times 10^3$	$\Sigma\bar{y}A = +506.2 \times 10^3$

- Find the total area and first moments of the triangle, rectangle, and semicircle. Subtract the area and first moment of the circular cutout.

$$\boxed{\begin{aligned} Q_x &= +506.2 \times 10^3\ \text{mm}^3 \\ Q_y &= +757.7 \times 10^3\ \text{mm}^3 \end{aligned}}$$

Sample Problem 5.1

- Compute the coordinates of the area centroid by dividing the first moments by the total area.



$$\bar{X} = \frac{\sum \bar{x} A}{\sum A} = \frac{+757.7 \times 10^3\,\mathrm{mm}^3}{13.828 \times 10^3\,\mathrm{mm}^2}$$

$$\boxed{\bar{X} = 54.8\,\mathrm{mm}}$$

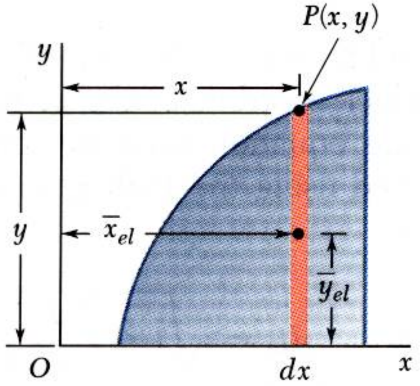
$$\bar{Y} = \frac{\sum \bar{y} A}{\sum A} = \frac{+506.2 \times 10^3\,\mathrm{mm}^3}{13.828 \times 10^3\,\mathrm{mm}^2}$$

$$\boxed{\bar{Y} = 36.6\,\mathrm{mm}}$$

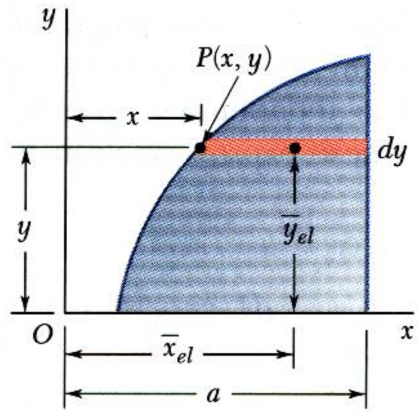
Determination of Centroids by Integration

$$\bar{x}A = \int x\,dA = \iint x\,dxdy = \int \bar{x}_{el}\,dA$$
$$\bar{y}A = \int y\,dA = \iint y\,dxdy = \int \bar{y}_{el}\,dA$$

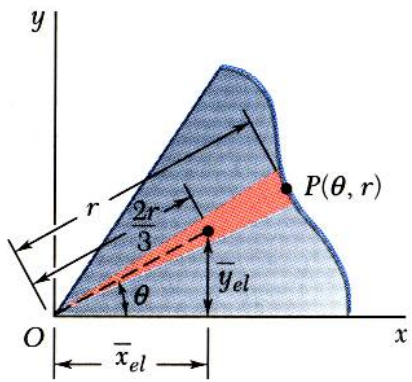
- Double integration to find the first moment may be avoided by defining *dA* as a thin rectangle or strip.



$$\bar{x}A = \int \bar{x}_{el}\,dA$$
$$= \int x\,(y\,dx)$$
$$\bar{y}A = \int \bar{y}_{el}\,dA$$
$$= \int \frac{y}{2}(y\,dx)$$

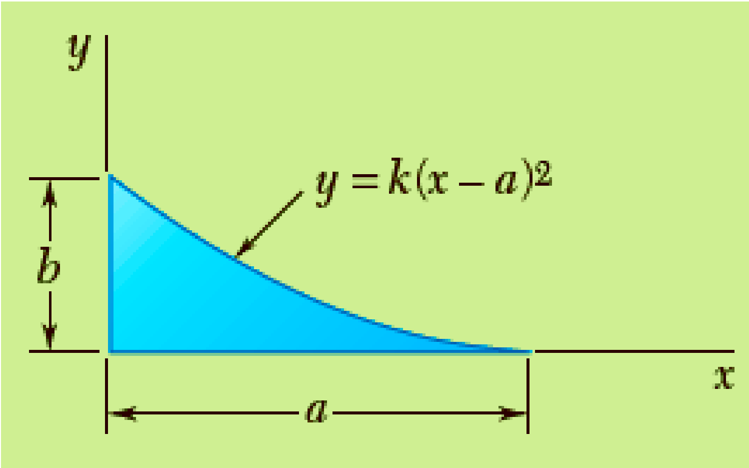


$$\bar{x}A = \int \bar{x}_{el}\,dA$$
$$= \int \frac{a+x}{2}\left[(a-x)dx\right]$$
$$\bar{y}A = \int \bar{y}_{el}\,dA$$
$$= \int y\left[(a-x)dx\right]$$



$$\bar{x}A = \int \bar{x}_{el}\,dA$$
$$= \int \frac{2r}{3}\cos \quad \left(\frac{1}{2}r^2 d \quad\right)$$
$$\bar{y}A = \int \bar{y}_{el}\,dA$$
$$= \int \frac{2r}{3}\sin \quad \left(\frac{1}{2}r^2 d \quad\right)$$

Sample Problem 5.4

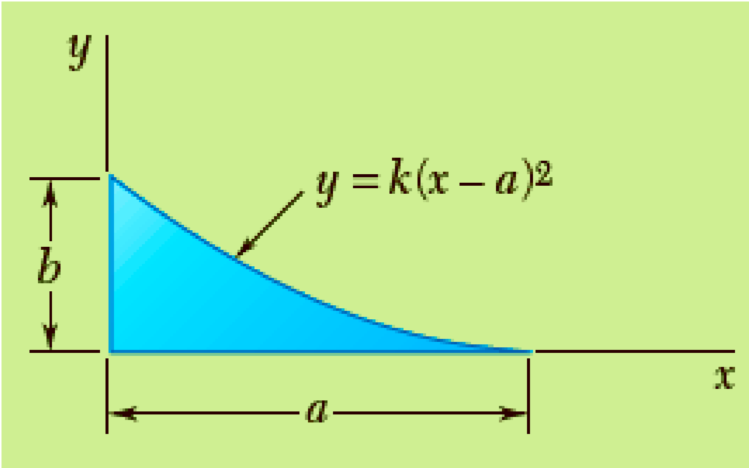


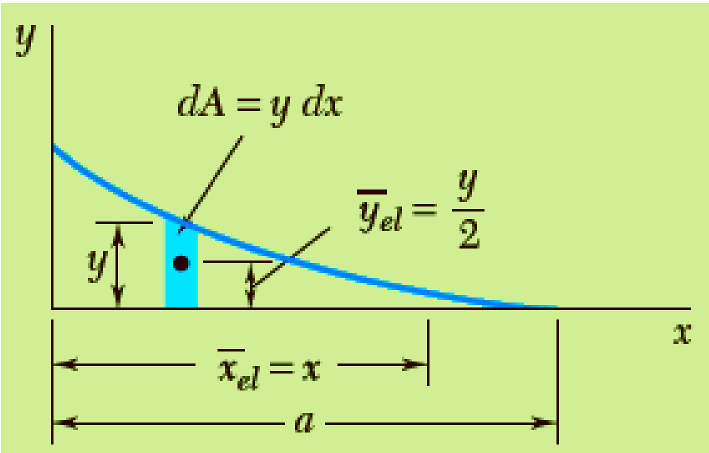
Determine by direct integration the location of the centroid of a parabolic spandrel.

SOLUTION:

- Determine the constant k.
- Evaluate the total area.
- Using either vertical or horizontal strips, perform a single integration to find the first moments.
- Evaluate the centroid coordinates.

Sample Problem 5.4





SOLUTION:

- Determine the constant k.

$$y = k(x-a)^2$$

$$b = k\,a^2 \Rightarrow k = \frac{b}{a^2}$$

$$y = \frac{b}{a^2}(x-a)^2 \quad or \quad x = \left[1 + \left(\frac{y}{b}\right)^{1/2}\right]$$

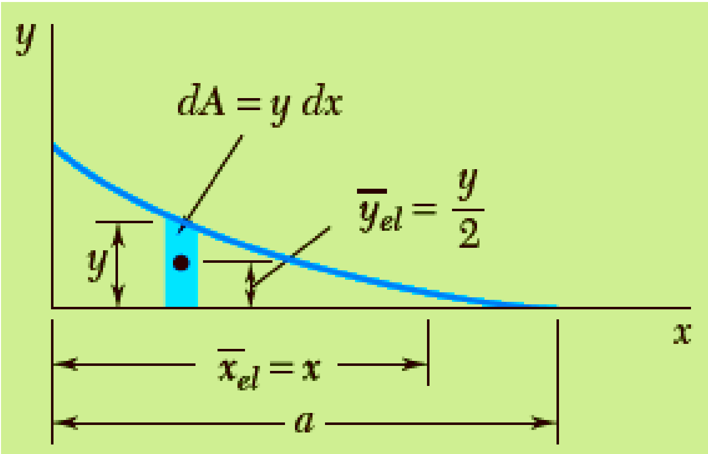
- Evaluate the total area.

$$A = \int dA$$

$$= \int y\,dx = \int_0^a \frac{b}{a^2}(x-a)^2\,dx = \frac{b}{a^2}\left[\frac{x^3}{3} - ax^2 + a^2x\right]_0^a$$

$$= \frac{ab}{3}$$

Sample Problem 5.4



- Using vertical strips, perform a single integration to find the first moments.

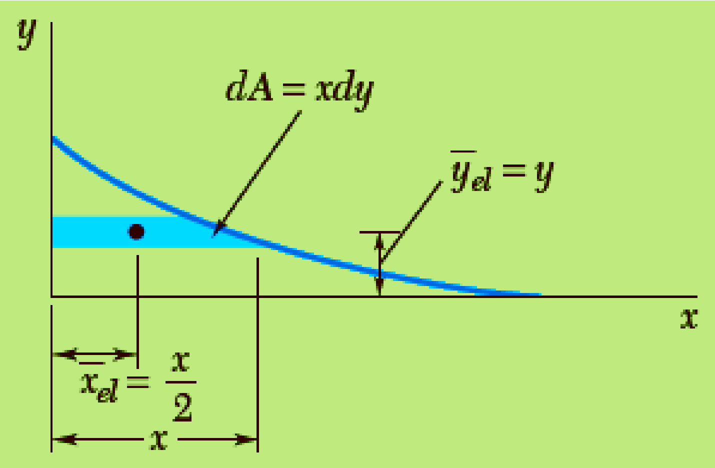
$$Q_y = \int \overline{x}_{el}\,dA = \int xy\,dx = \int_0^a x\frac{b}{a^2}\left(x - a^2\right)dx$$

$$= \frac{b}{a^2}\Big|_0^a = \frac{a^2b}{12}$$

$$Q_x = \int \overline{y}_{el}\,dA = \int \frac{y}{2}\,y\,dx = \int_0^a \frac{1}{2}\left[\frac{b}{a^2}(x-a)^2\right]^2 dx$$

$$= \frac{1}{2}\frac{b^2}{a^4}\int_0^a (x-a)^4\,dx = \frac{ab^2}{10}$$

Sample Problem 5.4



- Or, using horizontal strips, perform a single integration to find the first moments.

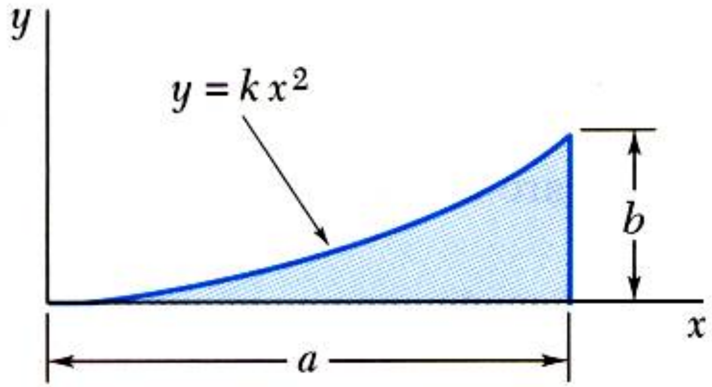
$$Q_y = \int \bar{x}_{el} dA = \int \frac{x}{2}(x)dy = \int_0^b \frac{x^2 dy}{2}$$

$$= \frac{1}{2}\int_0^b \left(a - \frac{a^2}{b^{1/2}} y^{1/2} \right) dy = \frac{a^2 b}{12}$$

$$Q_x = \int \bar{y}_{el} dA = \int yxdy = \int ya\left[1 - \left(\frac{y}{b}\right)^{1/2}\right]dy$$

$$= \int_0^b \left(ay - \frac{a}{b^{1/2}} y^{3/2} \right) dy = \frac{ab^2}{10}$$

Sample Problem 5.4



- Evaluate the centroid coordinates.

$$\bar{x}A = Q_y$$

$$\bar{x}\frac{ab}{3} = \frac{a^2 b}{12}$$

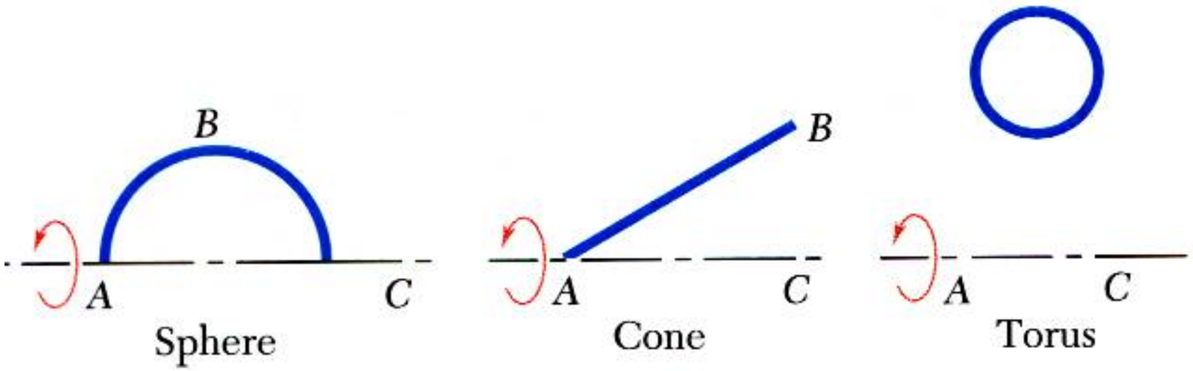
$$\boxed{\bar{x} = \frac{a}{4}}$$

$$\bar{y}A = Q_x$$

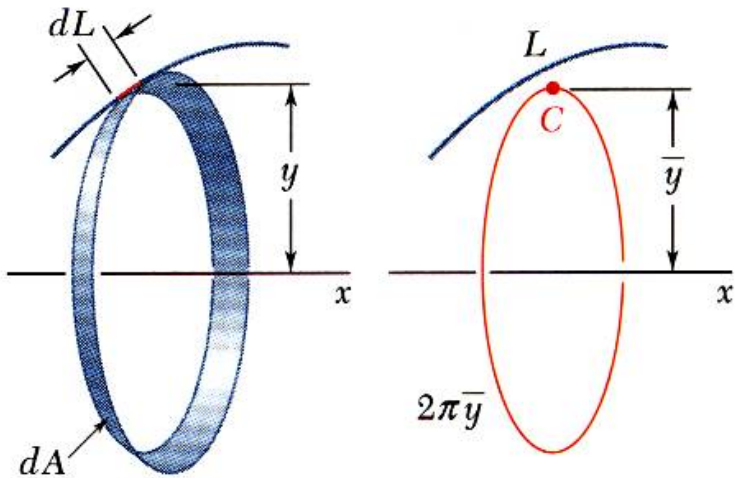
$$\bar{y}\frac{ab}{3} = \frac{ab^2}{10}$$

$$\boxed{\bar{y} = \frac{3}{10}b}$$

Theorems of Pappus-Guldinus



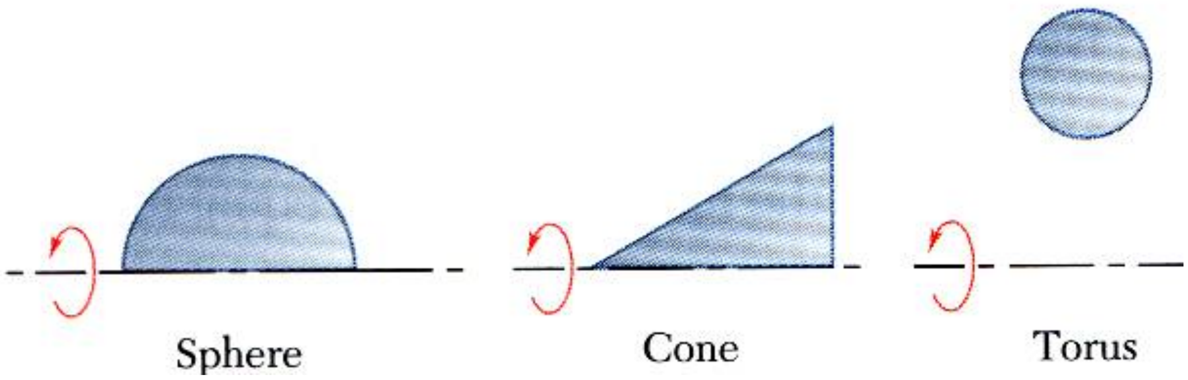
- Surface of revolution is generated by rotating a plane curve about a fixed axis.



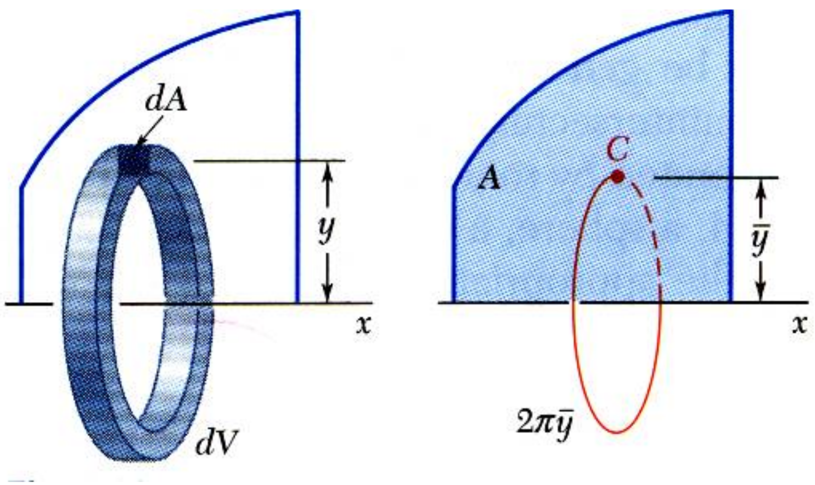
- Area of a surface of revolution is equal to the length of the generating curve times the distance traveled by the centroid through the rotation.

$$A = 2\ \bar{y}L$$

Theorems of Pappus-Guldinus



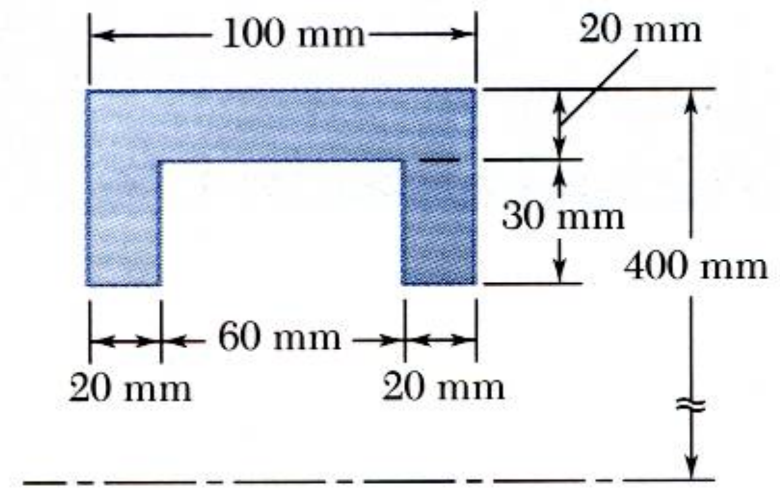
- Body of revolution is generated by rotating a plane area about a fixed axis.



- Volume of a body of revolution is equal to the generating area times the distance traveled by the centroid through the rotation.

$$V = 2\ \bar{y}A$$

Sample Problem 5.7



The outside diameter of a pulley is 0.8 m, and the cross section of its rim is as shown. Knowing that the pulley is made of steel and that the density of steel is $= 7.85\times10^3 \text{ kg/m}^3$ determine the mass and weight of the rim.

SOLUTION:

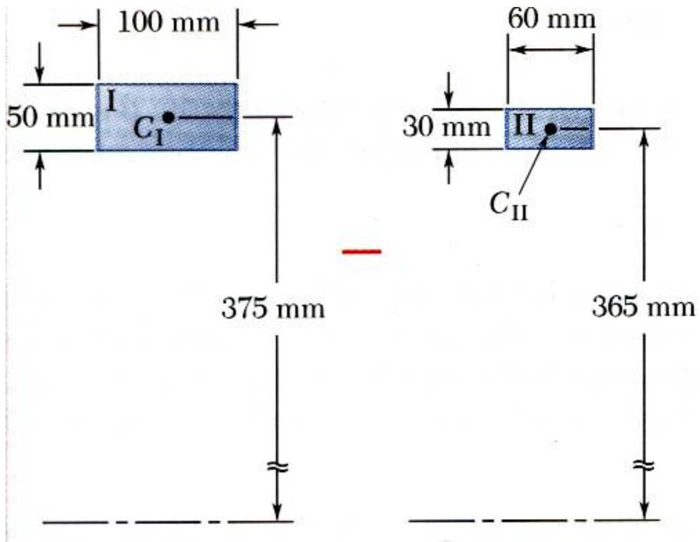
- Apply the theorem of Pappus-Guldinus to evaluate the volumes or revolution for the rectangular rim section and the inner cutout section.
- Multiply by density and acceleration to get the mass and acceleration.

Sample Problem 5.7

SOLUTION:

- Apply the theorem of Pappus-Guldinus to evaluate the volumes or revolution for the rectangular rim section and the inner cutout section.

- Multiply by density and acceleration to get the mass and acceleration.

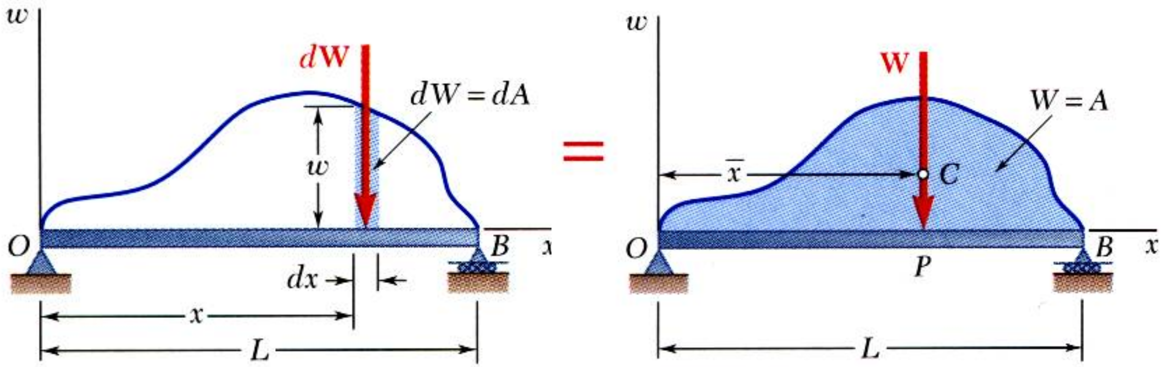


	Area, mm^2	$\bar{y}$, mm	Distance Traveled by C, mm	Volume, mm^3
I	+5000	375	$2\pi(375) = 2356$	$(5000)(2356) = 11.78 \times 10^6$
II	−1800	365	$2\pi(365) = 2293$	$(-1800)(2293) = -4.13 \times 10^6$
				Volume of rim $= 7.65 \times 10^6$

$$m = \quad V = \left(7.85\times10^3\,\mathrm{kg/m^3}\right)\left(7.65\times10^6\,\mathrm{mm^3}\right)\left(10^{-9}\,\mathrm{m^3/mm^3}\right) \qquad \boxed{m = 60.0\,\mathrm{kg}}$$

$$W = mg = (60.0\,\mathrm{kg})\left(9.81\,\mathrm{m/s^2}\right) \qquad \boxed{W = 589\,\mathrm{N}}$$

Distributed Loads on Beams



$$W = \int_0^L w\,dx = \int dA = A$$

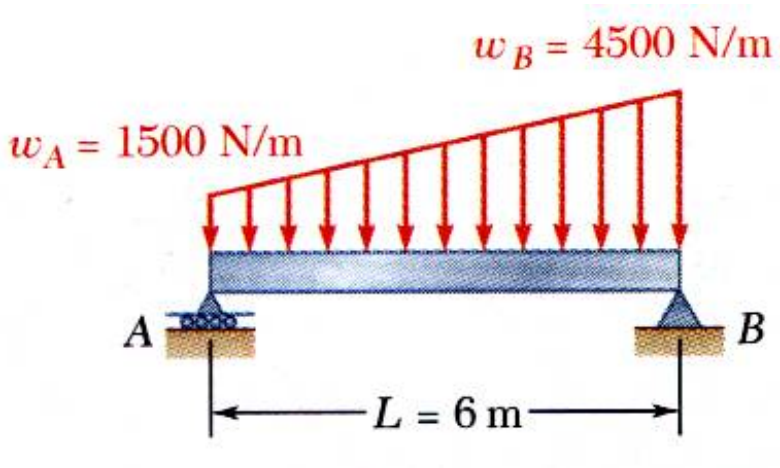
- A distributed load is represented by plotting the load per unit length, w (N/m) . The total load is equal to the area under the load curve.

$$(OP)W = \int x\,dW$$

$$(OP)A = \int_0^L x\,dA = \bar{x}A$$

- A distributed load can be replace by a concentrated load with a magnitude equal to the area under the load curve and a line of action passing through the area centroid.

Sample Problem 5.9

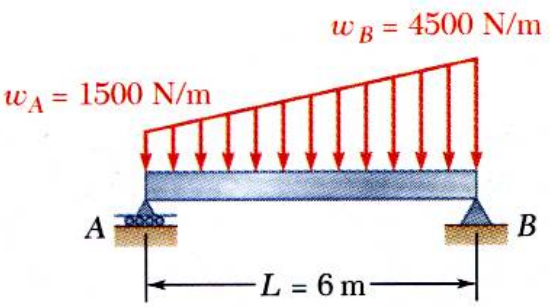


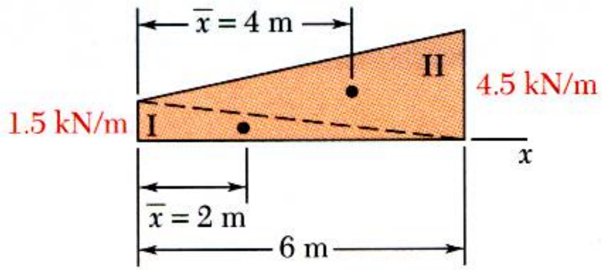
A beam supports a distributed load as shown. Determine the equivalent concentrated load and the reactions at the supports.

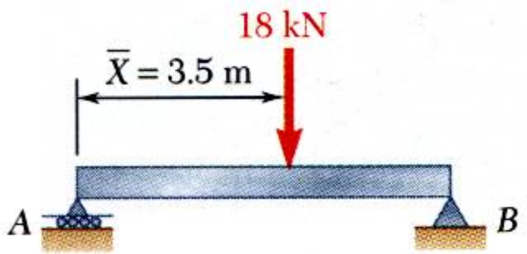
<u>SOLUTION</u>:

- The magnitude of the concentrated load is equal to the total load or the area under the curve.
- The line of action of the concentrated load passes through the centroid of the area under the curve.
- Determine the support reactions by summing moments about the beam ends.

Sample Problem 5.9







SOLUTION:

- The magnitude of the concentrated load is equal to the total load or the area under the curve.

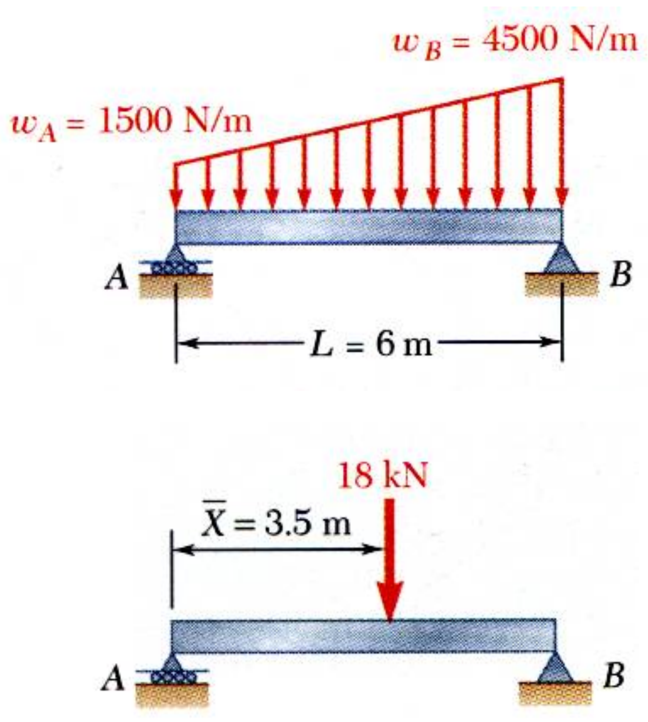
$$\boxed{F = 18.0\,\mathrm{kN}}$$

- The line of action of the concentrated load passes through the centroid of the area under the curve.

$$\bar{X} = \frac{63\,\mathrm{kN}\cdot\mathrm{m}}{18\,\mathrm{kN}} \qquad \boxed{\bar{X} = 3.5\,\mathrm{m}}$$

Component	A, kN	$\bar{x}$, m	$\bar{x}A$, kN · m
Triangle I	4.5	2	9
Triangle II	13.5	4	54
	$\Sigma A = 18.0$		$\Sigma\bar{x}A = 63$

Sample Problem 5.9



- Determine the support reactions by summing moments about the beam ends.

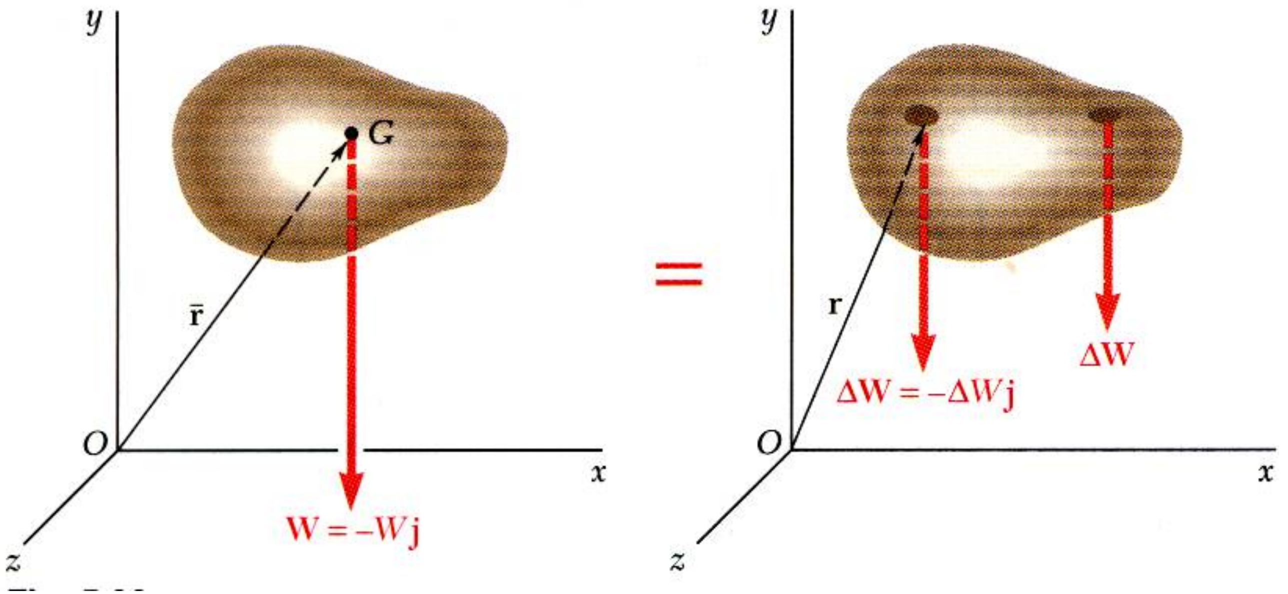
$$\sum M_A = 0: \quad B_y(6\text{ m}) - (18\text{ kN})(3.5\text{ m}) = 0$$

$$\boxed{B_y = 10.5\text{ kN}}$$

$$\sum M_B = 0: \quad -A_y(6\text{ m}) + (18\text{ kN})(6\text{ m} - 3.5\text{ m}) = 0$$

$$\boxed{A_y = 7.5\text{ kN}}$$

Center of Gravity of a 3D Body: Centroid of a Volume



- Center of gravity *G*

$$-W\vec{j} = \sum\left(-\Delta W\vec{j}\right)$$

$$\vec{r}_G \times\left(-W\vec{j}\right) = \sum\left[\vec{r}\times\left(-\Delta W\vec{j}\right)\right]$$
$$\vec{r}_G W \times\left(-\vec{j}\right) = \left(\sum\vec{r}\Delta W\right)\times\left(-\vec{j}\right)$$

$$W = \int dW \qquad \vec{r}_G W = \int \vec{r}\, dW$$

- Results are independent of body orientation,

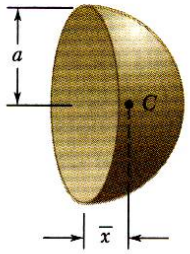
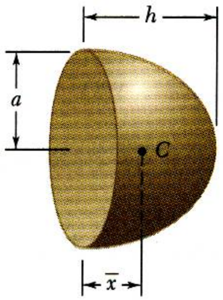
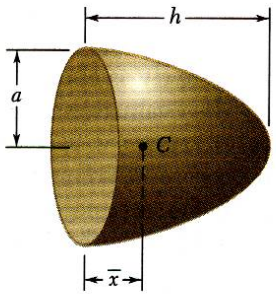
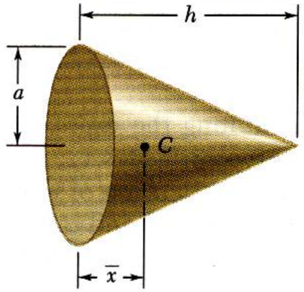
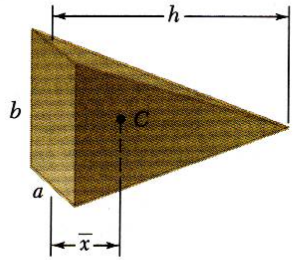
$$\bar{x}W = \int x\,dW \qquad \bar{y}W = \int y\,dW \qquad \bar{z}W = \int z\,dW$$

- For homogeneous bodies,

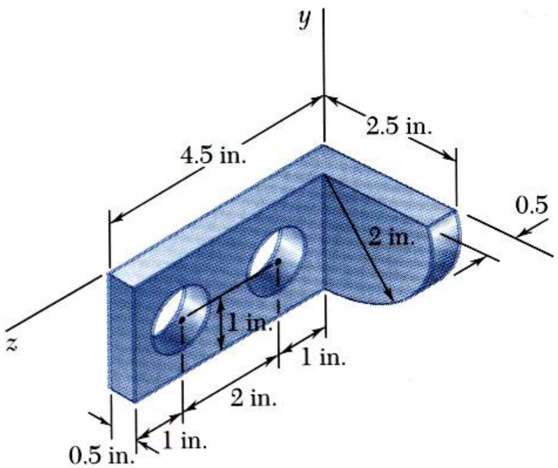
$$W = \quad V \text{ and } dW = \quad dV$$

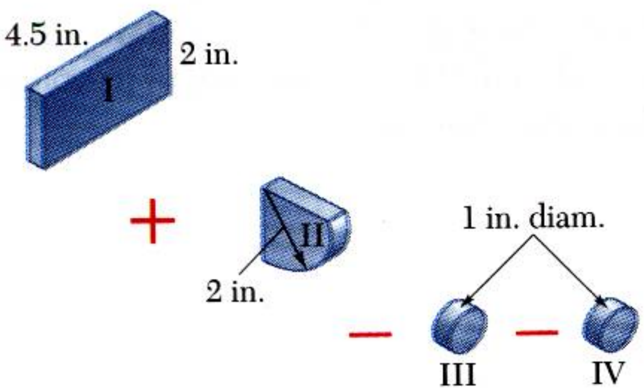
$$\bar{x}V = \int x\,dV \qquad \bar{y}V = \int y\,dV \qquad \bar{z}V = \int z\,dV$$

Centroids of Common 3D Shapes

Shape		$\bar{x}$	Volume
Hemisphere		$\frac{3a}{8}$	$\frac{2}{3}\pi a^3$
Semiellipsoid of revolution		$\frac{3h}{8}$	$\frac{2}{3}\pi a^2 h$
Paraboloid of revolution		$\frac{h}{3}$	$\frac{1}{2}\pi a^2 h$
Cone		$\frac{h}{4}$	$\frac{1}{3}\pi a^2 h$
Pyramid		$\frac{h}{4}$	$\frac{1}{3}abh$

Composite 3D Bodies





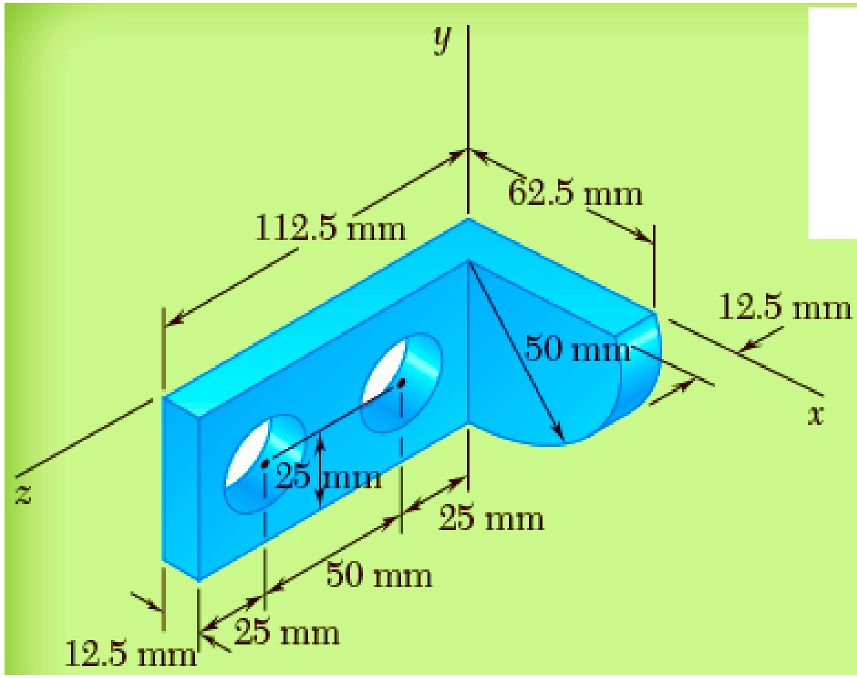
- Moment of the total weight concentrated at the center of gravity G is equal to the sum of the moments of the weights of the component parts.

$$\bar{X}\sum W = \sum \bar{x} W \quad \bar{Y}\sum W = \sum \bar{y} W \quad \bar{Z}\sum W = \sum \bar{z} W$$

- For homogeneous bodies,

$$\bar{X}\sum V = \sum \bar{x} V \quad \bar{Y}\sum V = \sum \bar{y} V \quad \bar{Z}\sum V = \sum \bar{z} V$$

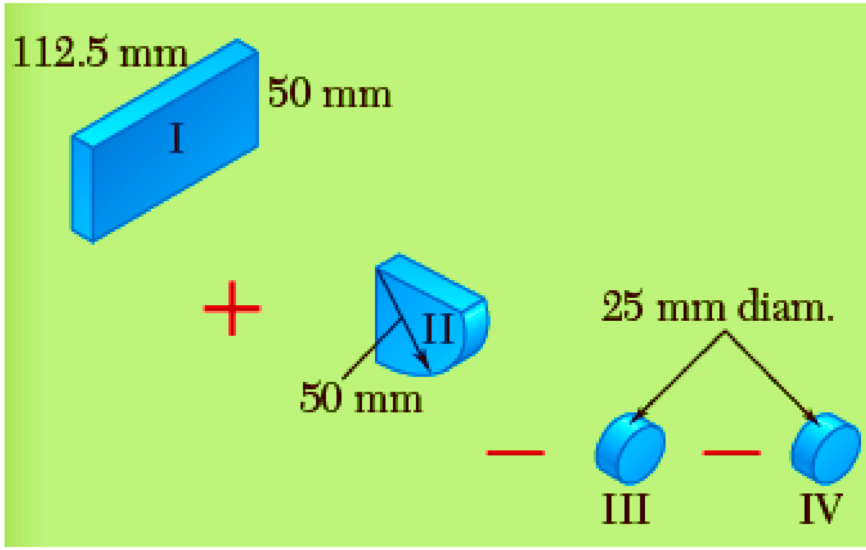
Sample Problem 5.12



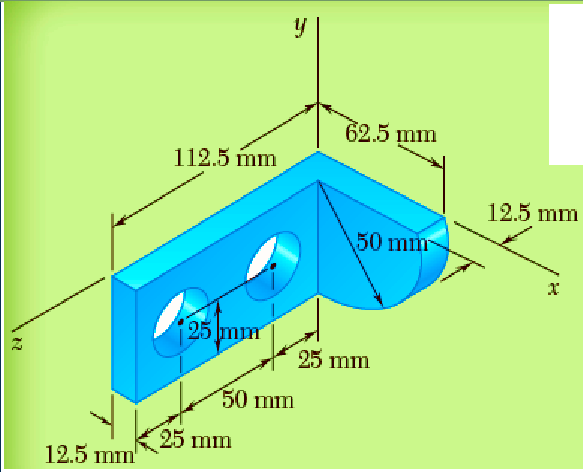
Locate the center of gravity of the steel machine element. The diameter of each hole is 25 mm.

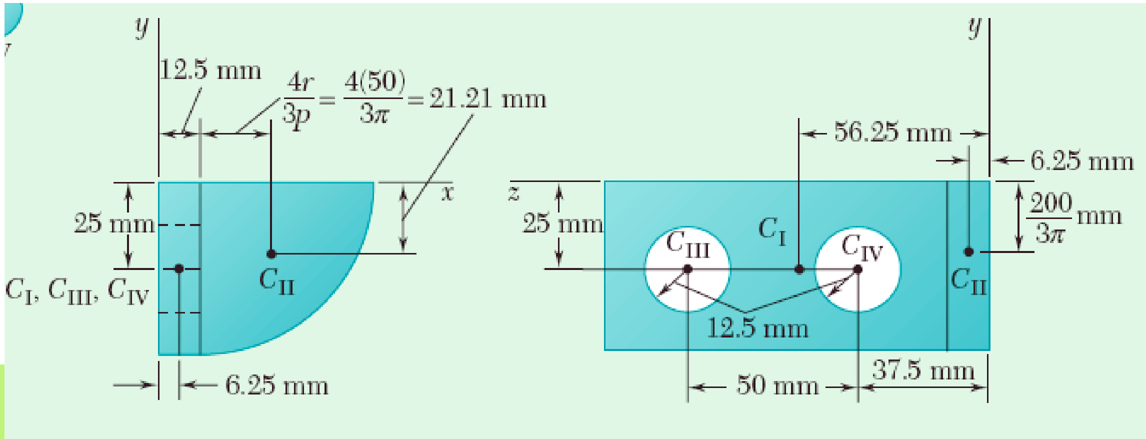
SOLUTION:

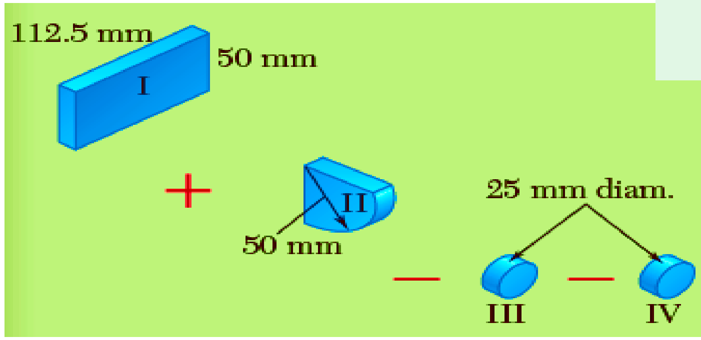
- Form the machine element from a rectangular parallelepiped and a quarter cylinder and then subtracting two 25 mm. diameter cylinders.



Sample Problem 5.12



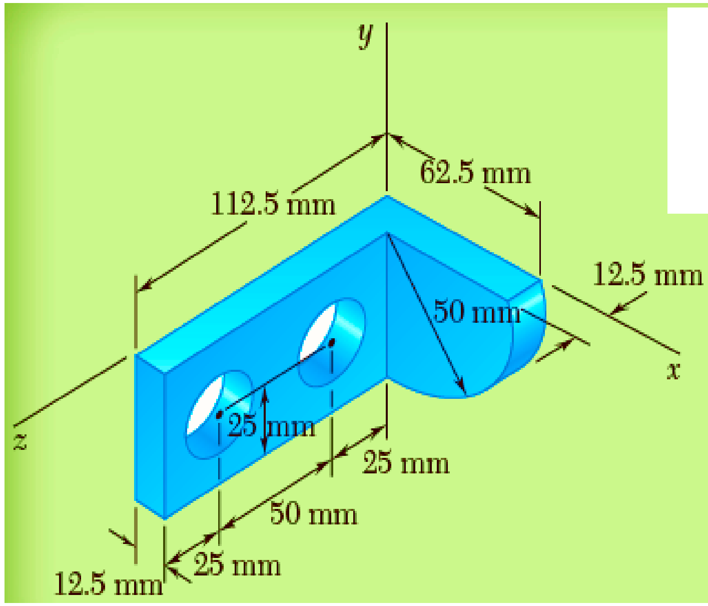




	V, mm$^3 \times 10^3$	$\bar{x}$, mm	$\bar{y}$, mm	$\bar{z}$, mm	$\bar{x}V$, mm^4	$\bar{y}V$, mm^4	$\bar{z}V$, mm^4
I	$(112.5)(50)(12.5) = 70312.5$	6.25	−25	56.25	439453	−1757813	3955078
II	$\frac{1}{4}\pi(50)^2(12.5) = 24553.6$	33.71	−21.21	6.25	827702	520782	153460
III	$-\pi(12.5)^2(12.5) = -6138.4$	6.25	−25	87.5	−38365	153460	−537110
IV	$-\pi(12.5)^2(12.5) = -6138.4$	6.25	−25	37.5	−38365	153460	−230190
	$\Sigma V = 82589.3$				$\Sigma\bar{x}V = 1200425$	$\Sigma\bar{y}V = -1971662$	$\Sigma\bar{z}V = 3341238$

Sample Problem 5.12

	V, $mm^3 \times 10^3$	$\bar{x}$, mm	$\bar{y}$, mm	$\bar{z}$, mm	$\bar{x}V$, mm^4	$\bar{y}V$, mm^4	$\bar{z}V$, mm^4
I	$(112.5)(50)(12.5) = 70312.5$	6.25	-25	56.25	439453	-1757813	3955078
II	$\frac{1}{4}\pi(50)^2(12.5) = 24553.6$	33.71	-21.21	6.25	827702	520782	153460
III	$-\pi(12.5)^2(12.5) = -6138.4$	6.25	-25	87.5	-38365	153460	-537110
IV	$-\pi(12.5)^2(12.5) = -6138.4$	6.25	-25	37.5	-38365	153460	-230190
	$\Sigma V = 82589.3$				$\Sigma\bar{x}V = 1200425$	$\Sigma\bar{y}V = -1971662$	$\Sigma\bar{z}V = 3341238$



$$\bar{X} = \sum \bar{x}V \Big/ \sum V = \left(1200425\ \text{mm}^4\right) \Big/ \left(82589.3\ \text{mm}^3\right)$$

$$\boxed{\bar{X} = 14.53\ \text{mm}}$$

$$\bar{Y} = \sum \bar{y}V \Big/ \sum V = \left(-1971662\ \text{mm}^4\right) \Big/ \left(82589.3\ \text{mm}^3\right)$$

$$\boxed{\bar{Y} = 23.9\ \text{mm}}$$

$$\bar{Z} = \sum \bar{z}V \Big/ \sum V = \left(3341238\ \text{mm}^4\right) \Big/ \left(82589.3\ \text{mm}^3\right)$$

$$\boxed{\bar{Z} = 40.5\ \text{mm}}$$