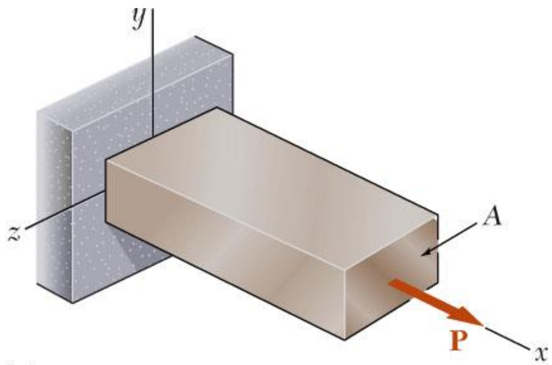
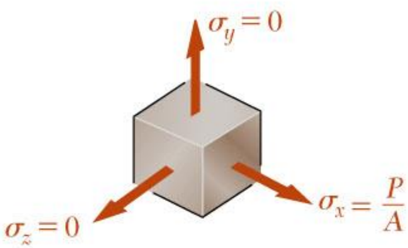
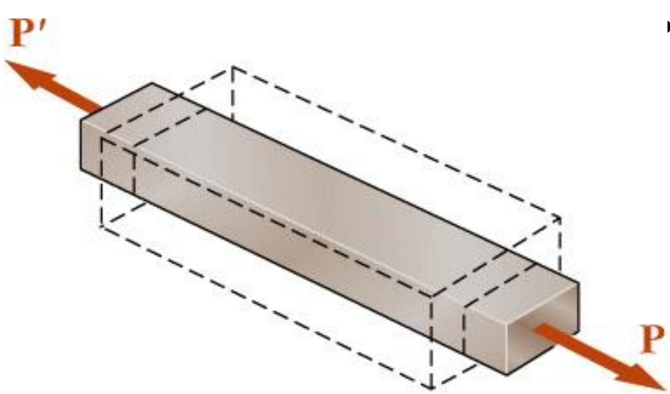
Poisson's Ratio



(a)



(b)



- For a slender bar subjected to axial loading:

$$\varepsilon_x = \frac{\sigma_x}{E} \qquad \sigma_y = \sigma_z = 0$$

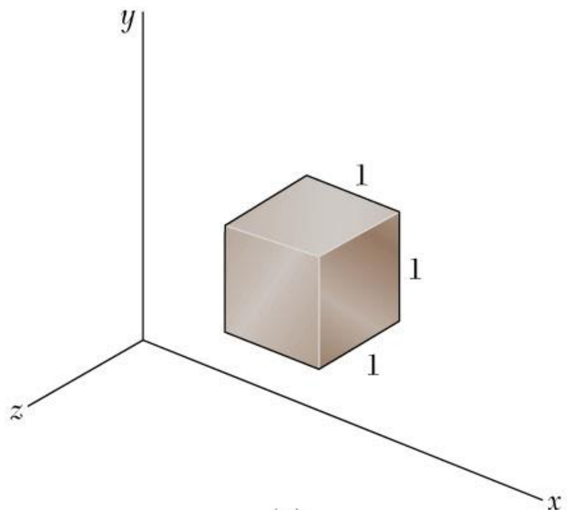
- The elongation in the x-direction is accompanied by a contraction in the other directions. Assuming that the material is isotropic (no directional dependence),

$$\varepsilon_y = \varepsilon_z \neq 0$$

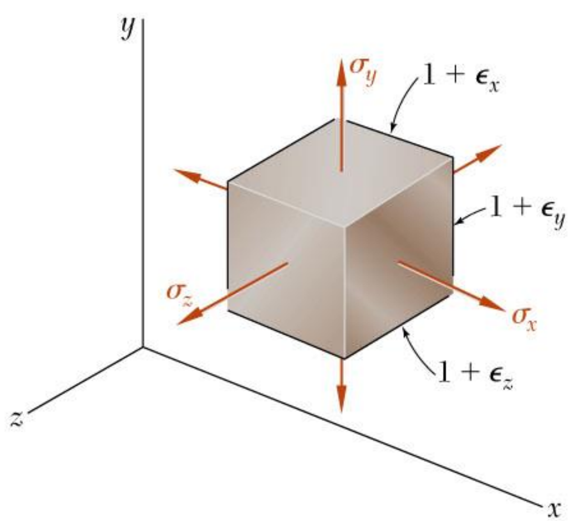
- Poisson's ratio is defined as

$$\nu = \left|\frac{\text{lateral strain}}{\text{axial strain}}\right| = -\frac{\varepsilon_y}{\varepsilon_x} = -\frac{\varepsilon_z}{\varepsilon_x}$$

Generalized Hooke's Law



(a)



(b)

- For an element subjected to multi-axial loading, the normal strain components resulting from the stress components may be determined from the *principle of superposition*. This requires:

  1) strain is linearly related to stress
  2) deformations are small

- With these restrictions:

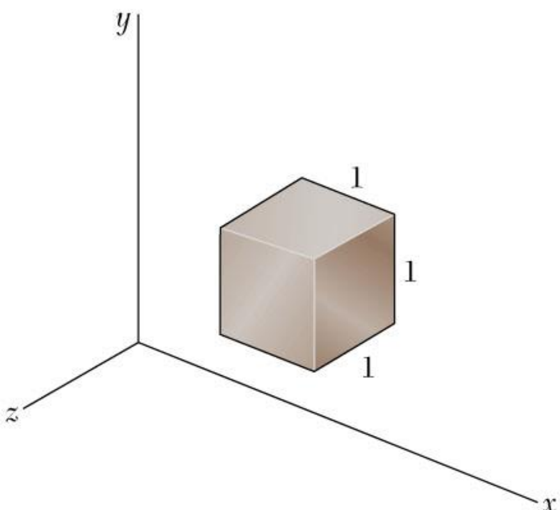
$$\varepsilon_x = +\frac{\sigma_x}{E} - \frac{\nu\sigma_y}{E} - \frac{\nu\sigma_z}{E}$$

$$\varepsilon_y = -\frac{\nu\sigma_x}{E} + \frac{\sigma_y}{E} - \frac{\nu\sigma_z}{E}$$

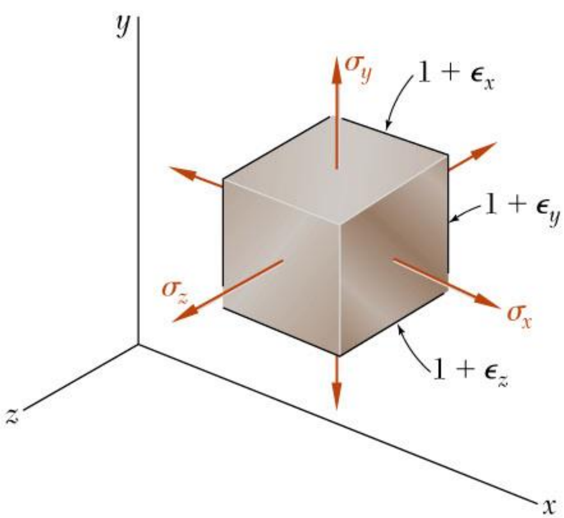
$$\varepsilon_z = -\frac{\nu\sigma_x}{E} - \frac{\nu\sigma_y}{E} + \frac{\sigma_z}{E}$$

End

Dilatation: Bulk Modulus



(a)



(b)

Relative to the unstressed state, the change in volume is

$$
\begin{aligned}
e &= \left[(1+\varepsilon_x)(1+\varepsilon_y)(1+\varepsilon_z)\right] - 1 = \left[1+\varepsilon_x+\varepsilon_y+\varepsilon_z\right] - 1 \\
&= \varepsilon_x+\varepsilon_y+\varepsilon_z \\
&= \frac{1-2\nu}{E}\left(\sigma_x+\sigma_y+\sigma_z\right) \\
&= \text{dilatation (change in volume per unit volume)}
\end{aligned}
$$

For element subjected to uniform hydrostatic pressure,

$$e = -p\frac{3(1-2\nu)}{E} = -\frac{p}{k}$$

$$k = \frac{E}{3(1-2\nu)} = \text{bulk modulus}$$

Subjected to uniform pressure, dilatation must be negative, therefore

$$0 < \nu < \tfrac{1}{2}$$

End

Shearing Strain

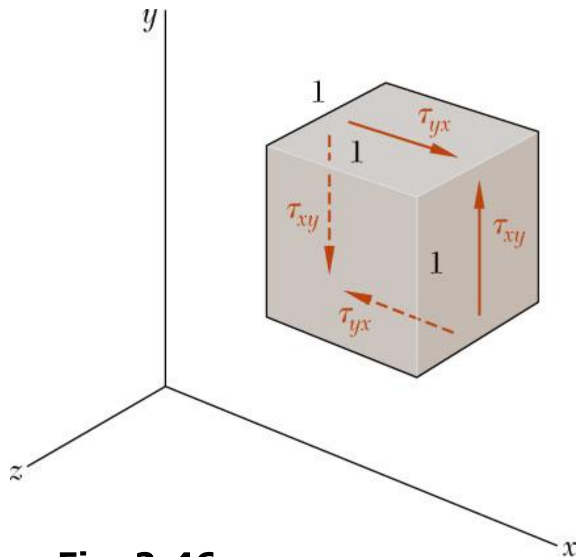


Fig. 2-46

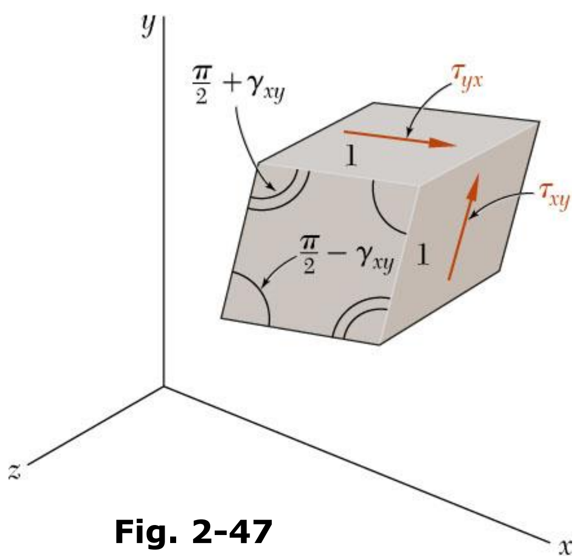


Fig. 2-47

- A cubic element subjected to a shear stress will deform into a rhomboid. The corresponding *shear* strain is quantified in terms of the change in angle between the sides,

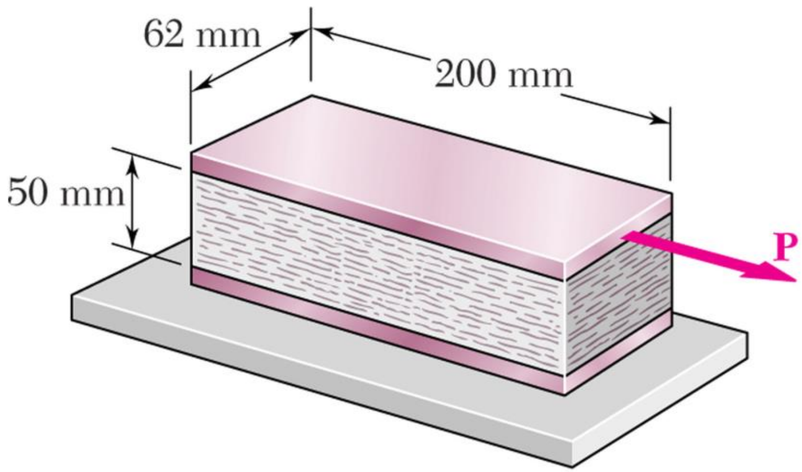
$$\tau_{xy} = f(\gamma_{xy})$$

- A plot of shear stress vs. shear strain is similar to the previous plots of normal stress vs. normal strain except that the strength values are approximately half. For small strains,

$$\tau_{xy} = G\gamma_{xy} \quad \tau_{yz} = G\gamma_{yz} \quad \tau_{zx} = G\gamma_{zx}$$

where *G* is the modulus of rigidity or shear modulus.

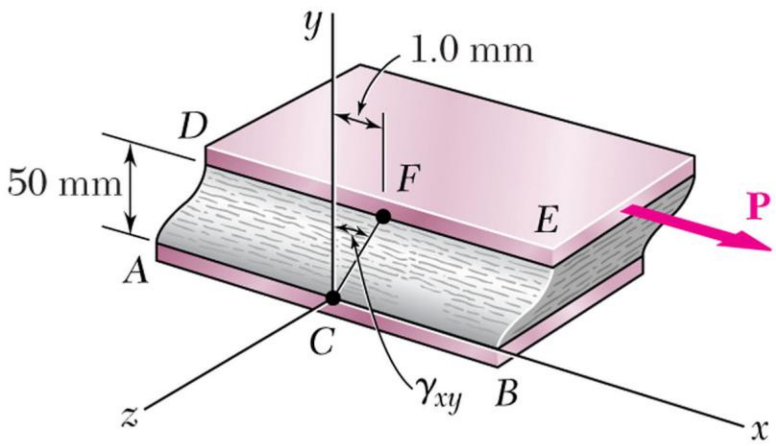
Example 2.10



A rectangular block of material with modulus of rigidity $G = 630$ MPa is bonded to two rigid horizontal plates. The lower plate is fixed, while the upper plate is subjected to a horizontal force P. Knowing that the upper plate moves through 1.0 mm. under the action of the force, determine a) the average shearing strain in the material, and b) the force P exerted on the plate.

SOLUTION:

- Determine the average angular deformation or shearing strain of the block.
- Apply Hooke's law for shearing stress and strain to find the corresponding shearing stress.
- Use the definition of shearing stress to find the force P.



- Determine the average angular deformation or shearing strain of the block.

$$\gamma_{xy} \approx \tan\gamma_{xy} = \frac{1\,\text{mm}}{50\,\text{mm}} \qquad \gamma_{xy} = 0.020\,\text{rad}$$

- Apply Hooke's law for shearing stress and strain to find the corresponding shearing stress.

$$\tau_{xy} = G\gamma_{xy} = (630\,\text{MPa})(0.020\,\text{rad}) = 12.6\,\text{MPa}$$

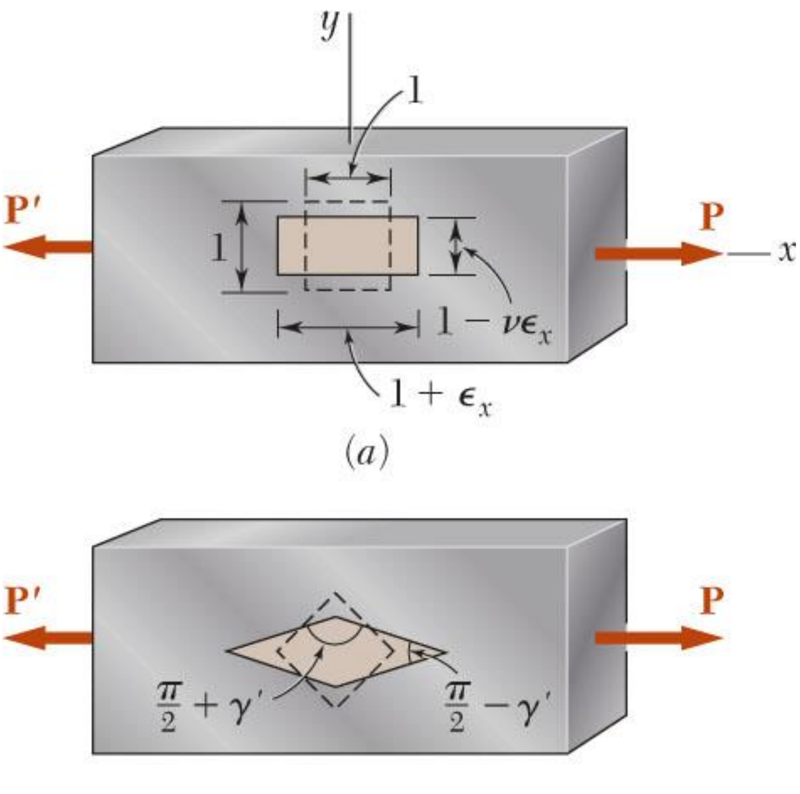
- Use the definition of shearing stress to find the force *P*.

$$P = \tau_{xy}A = \left(12.6\times 10^{6}\,\text{Pa}\right)(0.2\,\text{m})(0.062\,\text{m}) = 156.2\times 10^{3}\,\text{N}$$

$$\boxed{P = 156.2\,\text{kN}}$$

End

Relation Among E, ν, and G



(a)

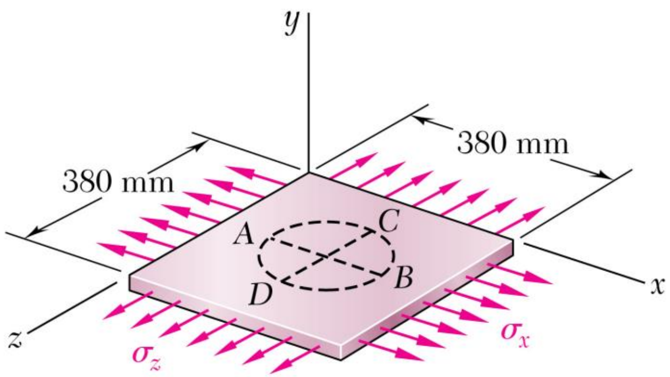
(b)

- An axially loaded slender bar will elongate in the axial direction and contract in the transverse directions.
- An initially cubic element oriented as in top figure will deform into a rectangular parallelepiped. The axial load produces a normal strain.
- If the cubic element is oriented as in the bottom figure, it will deform into a rhombus. Axial load also results in a shear strain.
- Components of normal and shear strain are related,

$$\frac{E}{2G} = (1+n)$$

End

Sample Problem 2.5



A circle of diameter *d* = 225 mm is scribed on an unstressed aluminum plate of thickness *t* = 18 mm. Forces acting in the plane of the plate later cause normal stresses σ_x = 84 MPa and σ_z = 140 MPa.

For *E* = 70 GPa and ν = 1/3, determine the change in:

a) the length of diameter *AB*,

b) the length of diameter *CD*,

c) the thickness of the plate, and

d) the volume of the plate.

SOLUTION:

- Apply the generalized Hooke's Law to find the three components of normal strain.

$$\varepsilon_x = +\frac{\sigma_x}{E} - \frac{\nu\sigma_y}{E} - \frac{\nu\sigma_z}{E}$$

$$= \frac{1}{70\times 10^3\,\mathrm{MPa}}\left[(84\,\mathrm{MPa}) - 0 - \frac{1}{3}(140\,\mathrm{MPa})\right]$$

$$= +0.533\times 10^{-3}\,\mathrm{mm/mm}$$

$$\varepsilon_y = -\frac{\nu\sigma_x}{E} + \frac{\sigma_y}{E} - \frac{\nu\sigma_z}{E}$$

$$= -1.067\times 10^{-3}\,\mathrm{mm/mm}$$

$$\varepsilon_z = -\frac{\nu\sigma_x}{E} - \frac{\nu\sigma_y}{E} + \frac{\sigma_z}{E}$$

$$= +1.600\times 10^{-3}\,\mathrm{mm/mm}$$

- Evaluate the deformation components.

$$\delta_{B/A} = \varepsilon_x d = (+0.533\times 10^{-3}\,\mathrm{mm/mm})(225\,\mathrm{mm})$$

$$\boxed{\delta_{B/A} = +0.12\,\mathrm{mm}}$$

$$\delta_{C/D} = \varepsilon_z d = (+1.600\times 10^{-3}\,\mathrm{mm/mm})(225\,\mathrm{mm})$$

$$\boxed{\delta_{C/D} = +0.36\,\mathrm{mm}}$$

$$\delta_t = \varepsilon_y t = (-1.067\times 10^{-3}\,\mathrm{mm/mm})(18\,\mathrm{mm})$$

$$\boxed{\delta_t = -0.0192\,\mathrm{mm}}$$

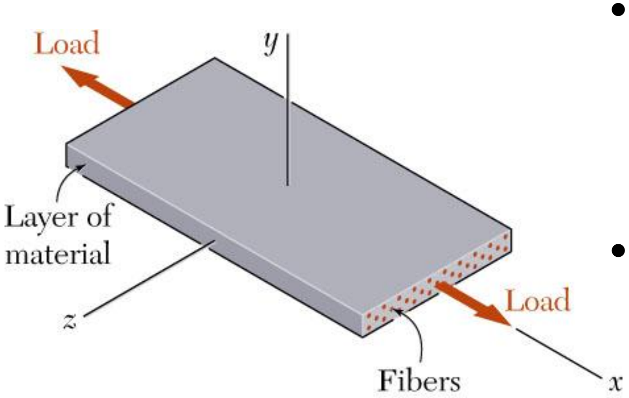
- Find the change in volume

$$e = \varepsilon_x + \varepsilon_y + \varepsilon_z = 1.067\times 10^{-3}\,\mathrm{mm^3/mm^3}$$

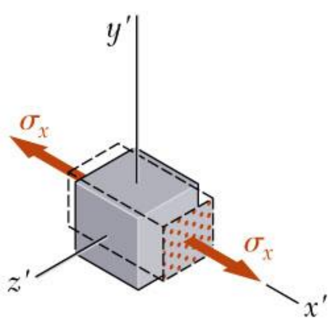
$$\Delta V = eV = 1.067\times 10^{-3}(380\times 380\times 18)\,\mathrm{mm^3}$$

$$\boxed{\Delta V = +2733\,\mathrm{mm^3}}$$

Composite Materials



(a)



(b)

- *Fiber-reinforced composite materials* are formed from *lamina* of fibers of graphite, glass, or polymers embedded in a resin matrix.
- Normal stresses and strains are related by Hooke's Law but with directionally dependent moduli of elasticity,

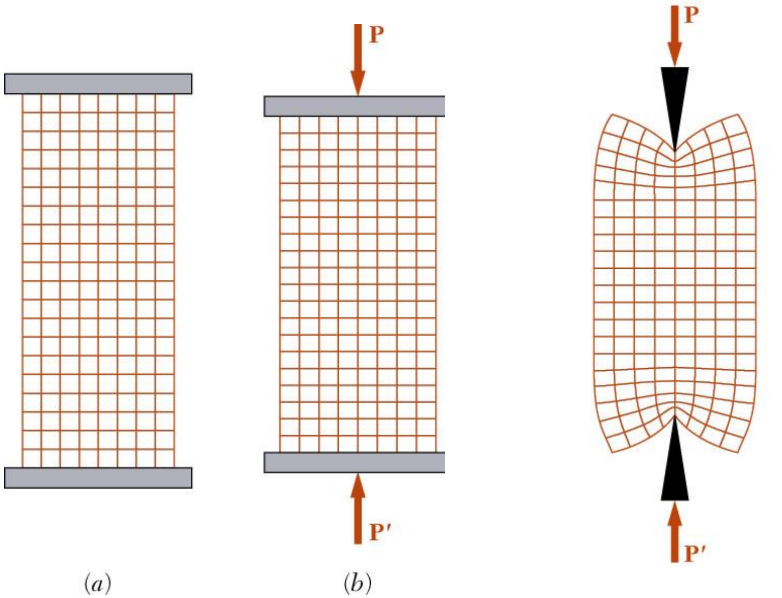
$$E_x = \frac{\sigma_x}{\varepsilon_x} \quad E_y = \frac{\sigma_y}{\varepsilon_y} \quad E_z = \frac{\sigma_z}{\varepsilon_z}$$

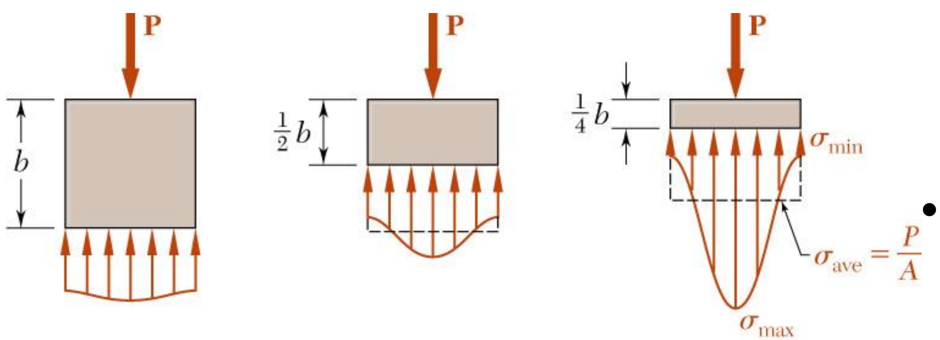
- Transverse contractions are related by directionally dependent values of Poisson's ratio, e.g.,

$$\nu_{xy} = -\frac{\varepsilon_y}{\varepsilon_x} \quad \nu_{xz} = -\frac{\varepsilon_z}{\varepsilon_x}$$

- Materials with directionally dependent mechanical properties are *anisotropic*.

Saint-Venant's Principle





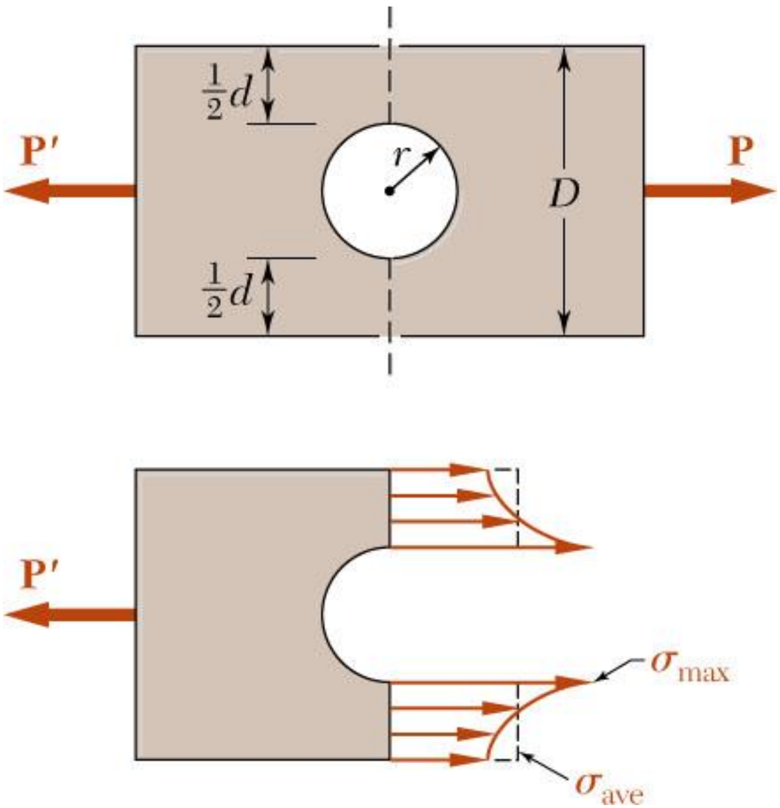
$\sigma_{min} = 0.973\sigma_{ave}$ $\sigma_{max} = 1.027\sigma_{ave}$

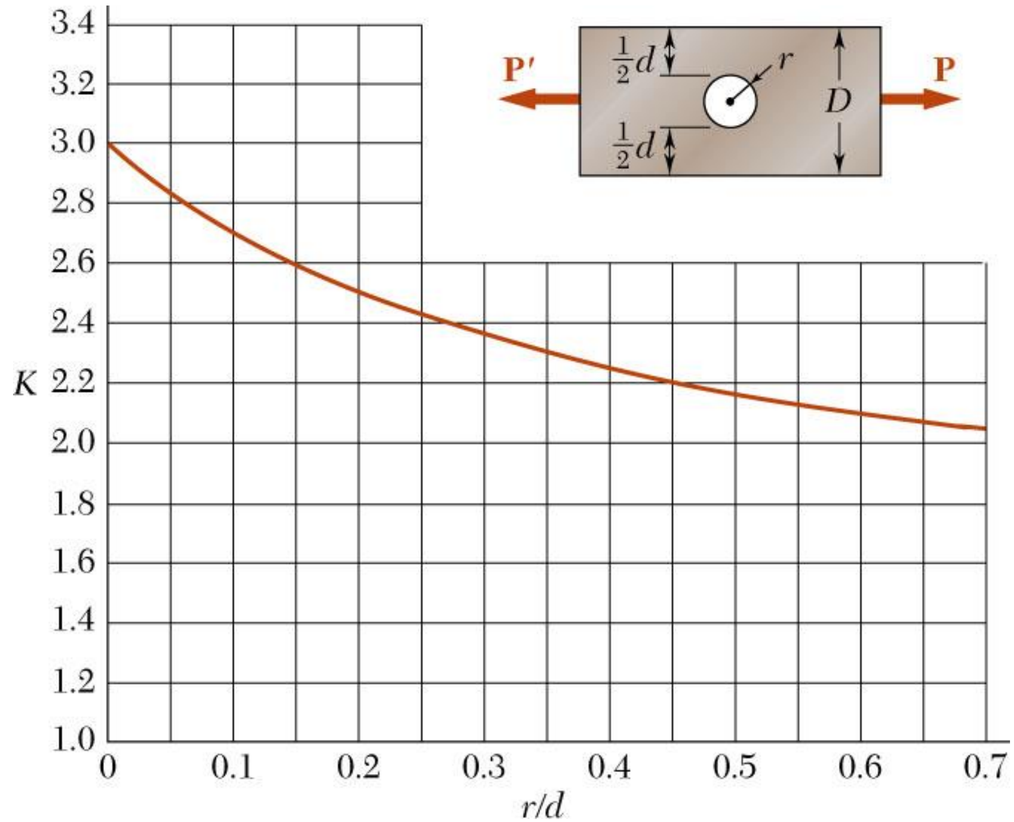
$\sigma_{min} = 0.668\sigma_{ave}$ $\sigma_{max} = 1.387\sigma_{ave}$

$\sigma_{min} = 0.198\sigma_{ave}$ $\sigma_{max} = 2.575\sigma_{ave}$

- Loads transmitted through rigid plates result in uniform distribution of stress and strain.
- Concentrated loads result in large stresses in the vicinity of the load application point.
- Stress and strain distributions become uniform at a relatively short distance from the load application points.
- **Saint-Venant's Principle:** Stress distribution may be assumed independent of the mode of load application except in the immediate vicinity of load application points.

Stress Concentration: Hole



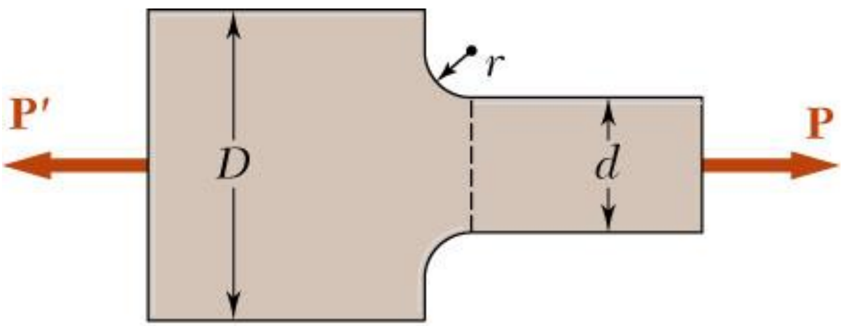


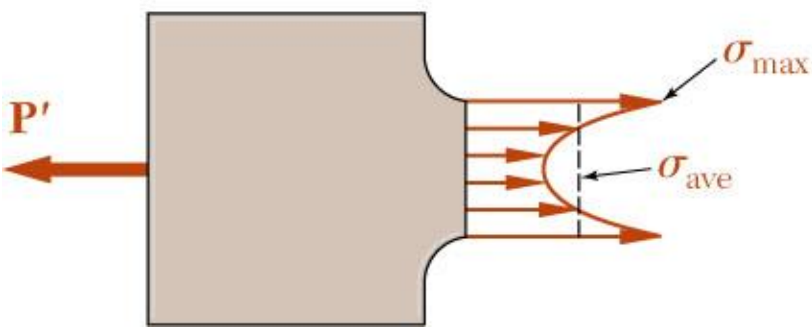
***(a)* Flat bars with holes**

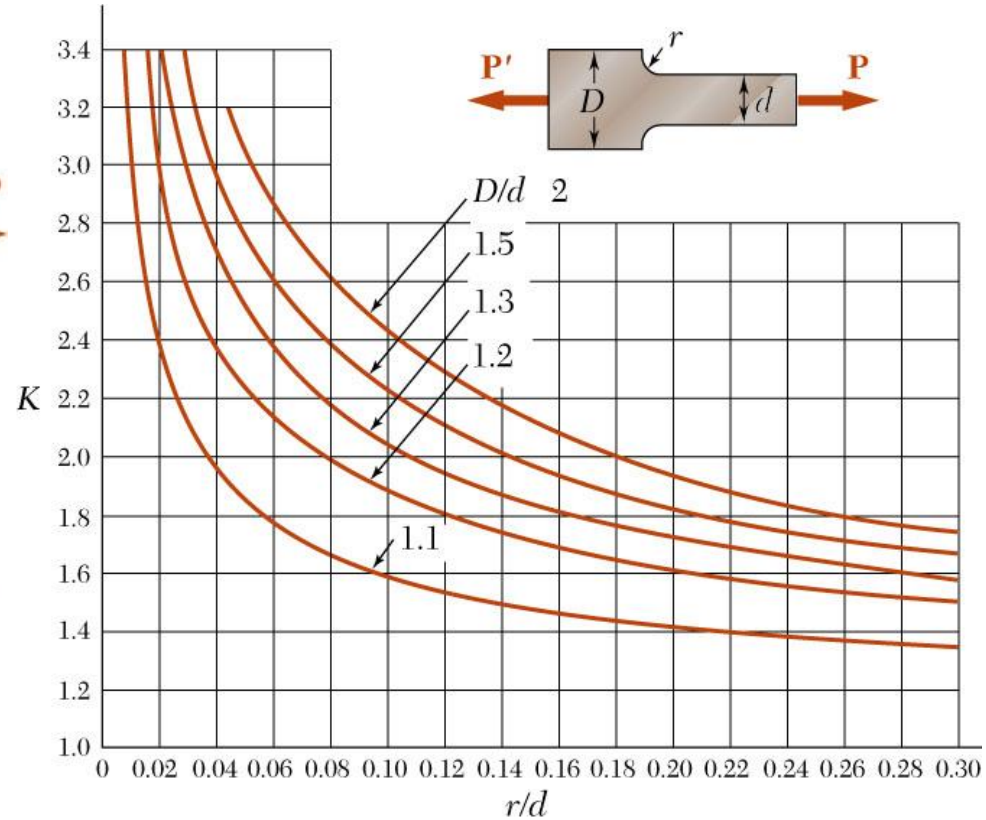
Discontinuities of cross section may result in high localized or *concentrated* stresses.

$$K = \frac{\sigma_{max}}{\sigma_{ave}}$$

Stress Concentration: Fillet

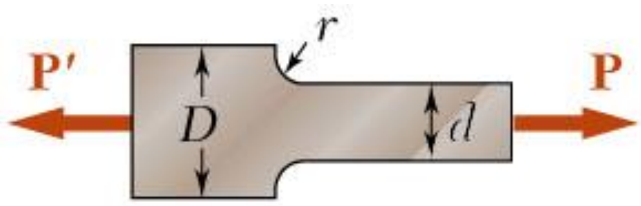






(b) **Flat bars with fillets**

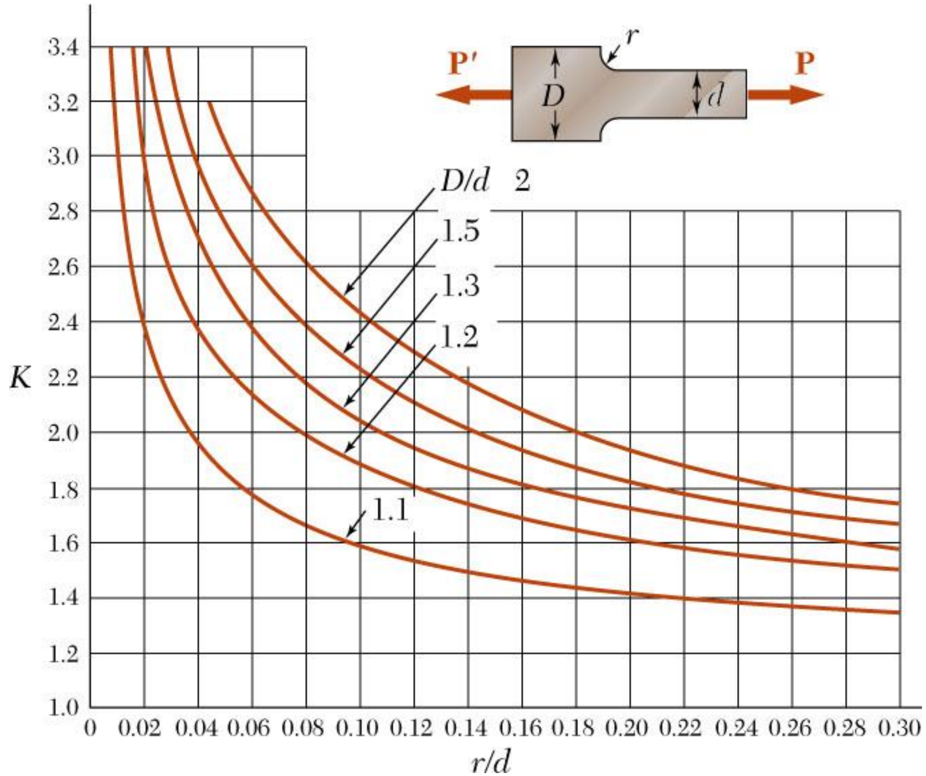
Example 2.12



Determine the largest axial load P that can be safely supported by a flat steel bar consisting of two portions, both 10 mm thick, and respectively 40 and 60 mm wide, connected by fillets of radius r = 8 mm. Assume an allowable normal stress of 165 MPa.

SOLUTION:

- Determine the geometric ratios and find the stress concentration factor from Fig. 2.64*b*.
- Find the allowable average normal stress using the material allowable normal stress and the stress concentration factor.
- Apply the definition of normal stress to find the allowable load.



***(b)* Flat bars with fillets**

- Determine the geometric ratios and find the stress concentration factor from Fig. 2.64*b*.

$$\frac{D}{d} = \frac{60\,\text{mm}}{40\,\text{mm}} = 1.50 \qquad \frac{r}{d} = \frac{8\,\text{mm}}{40\,\text{mm}} = 0.20$$

$$K = 1.82$$

- Find the allowable average normal stress using the material allowable normal stress and the stress concentration factor.

$$\sigma_{\text{ave}} = \frac{\sigma_{\max}}{K} = \frac{165\,\text{MPa}}{1.82} = 90.7\,\text{MPa}$$

- Apply the definition of normal stress to find the allowable load.

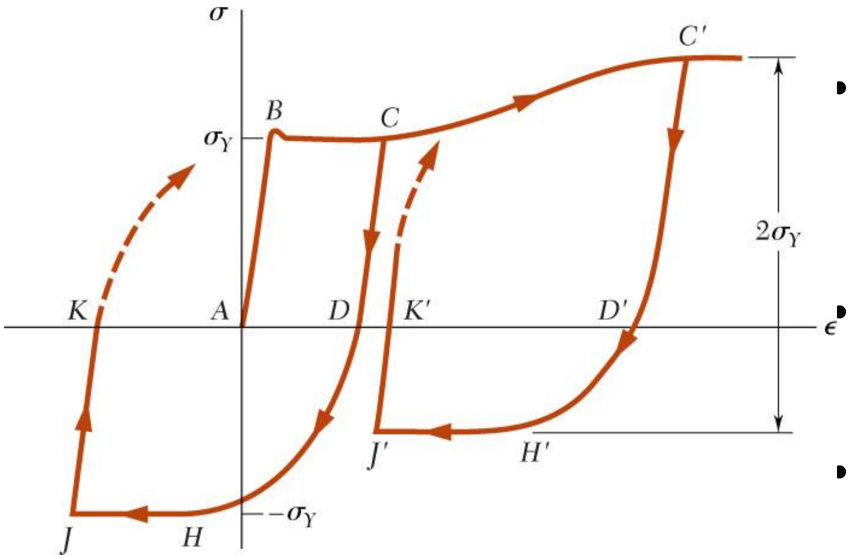
$$P = A\sigma_{ave} = (40\,\text{mm})(10\,\text{mm})(90.7\,\text{MPa})$$

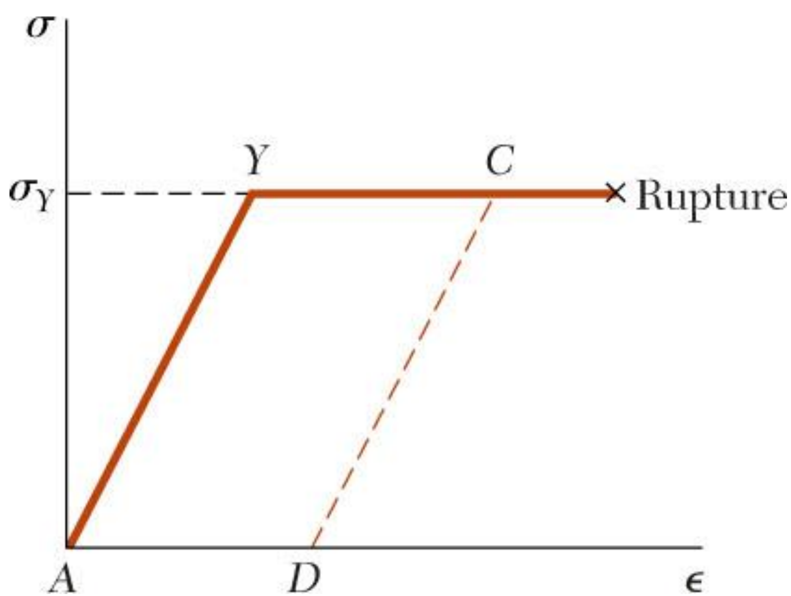
$$= 36.3 \times 10^3\,\text{N}$$

$$\boxed{P = 36.3\,\text{kN}}$$

End

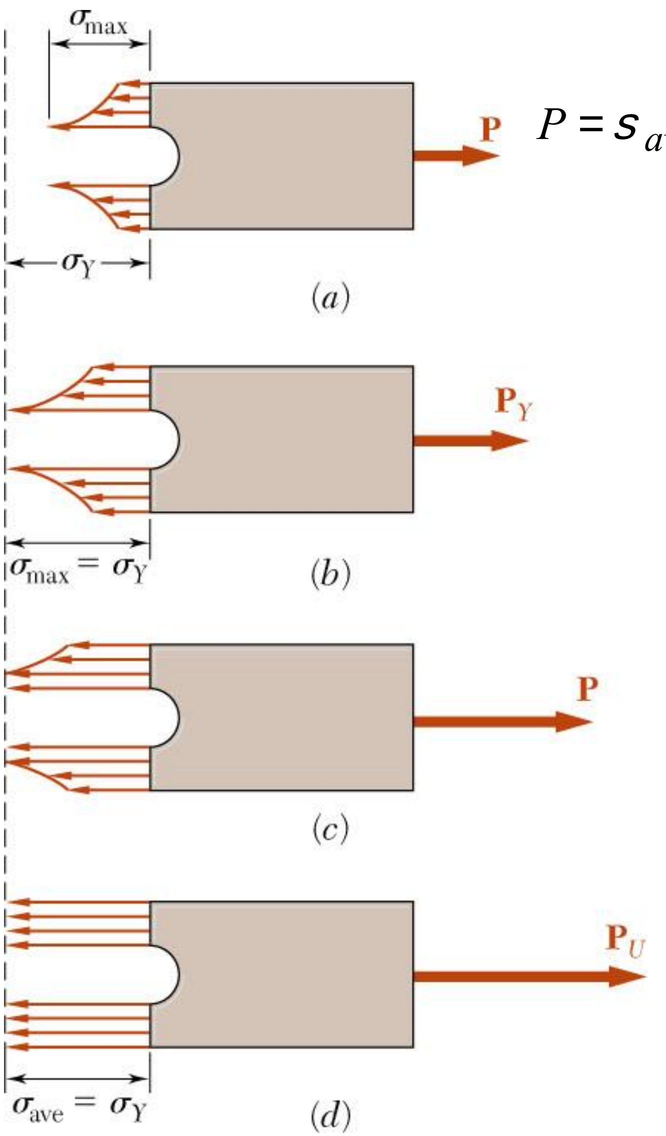
Elastoplastic Materials





- Previous analyses based on assumption of linear stress-strain relationship, i.e., stresses below the yield stress
- Assumption is good for brittle material which rupture without yielding
- If the yield stress of ductile materials is exceeded, then plastic deformations occur
- Analysis of plastic deformations is simplified by assuming an idealized *elastoplastic material*
- Deformations of an elastoplastic material are divided into elastic and plastic ranges
- Permanent deformations result from loading beyond the yield stress

Plastic Deformations



$$P = \sigma_{ave} A = \frac{\sigma_{max} A}{K}$$

- Elastic deformation while maximum stress is less than yield stress

$$P_Y = \frac{\sigma_Y A}{K}$$

- Maximum stress is equal to the yield stress at the maximum elastic loading

- At loadings above the maximum elastic load, a region of plastic deformations develop near the hole

$$P_U = \sigma_Y A$$
$$= K P_Y$$

- As the loading increases, the plastic region expands until the section is at a uniform stress equal to the yield stress

Residual Stresses

- When a single structural element is loaded uniformly beyond its yield stress and then unloaded, it is permanently deformed but all stresses disappear. This is not the general result.

- *Residual stresses* will remain in a structure after loading and unloading if
  - only part of the structure undergoes plastic deformation
  - different parts of the structure undergo different plastic deformations

- Residual stresses also result from the uneven heating or cooling of structures or structural elements

Example 2.14, 2.15, 2.16

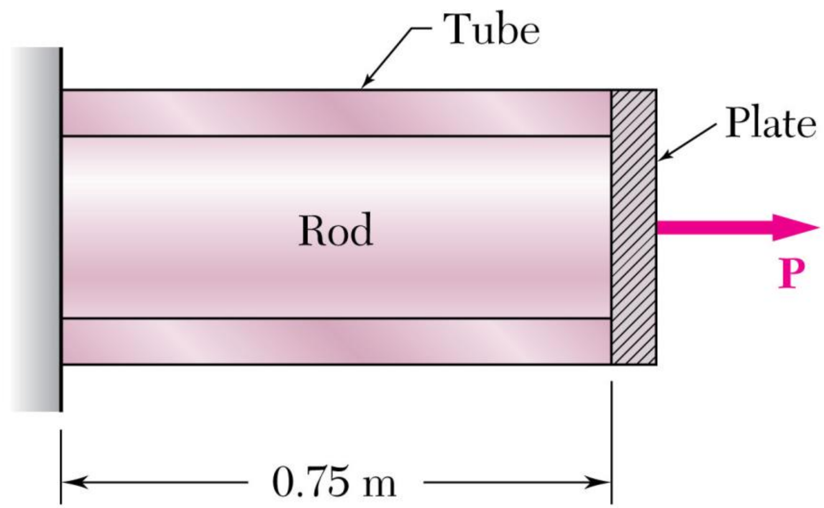
A cylindrical rod is placed inside a tube of the same length. The ends of the rod and tube are attached to a rigid support on one side and a rigid plate on the other. The load on the rod-tube assembly is increased from zero to 25 kN and decreased back to zero.

a) draw a load-deflection diagram for the rod-tube assembly

b) determine the maximum elongation

c) determine the permanent set

d) calculate the residual stresses in the rod and tube.

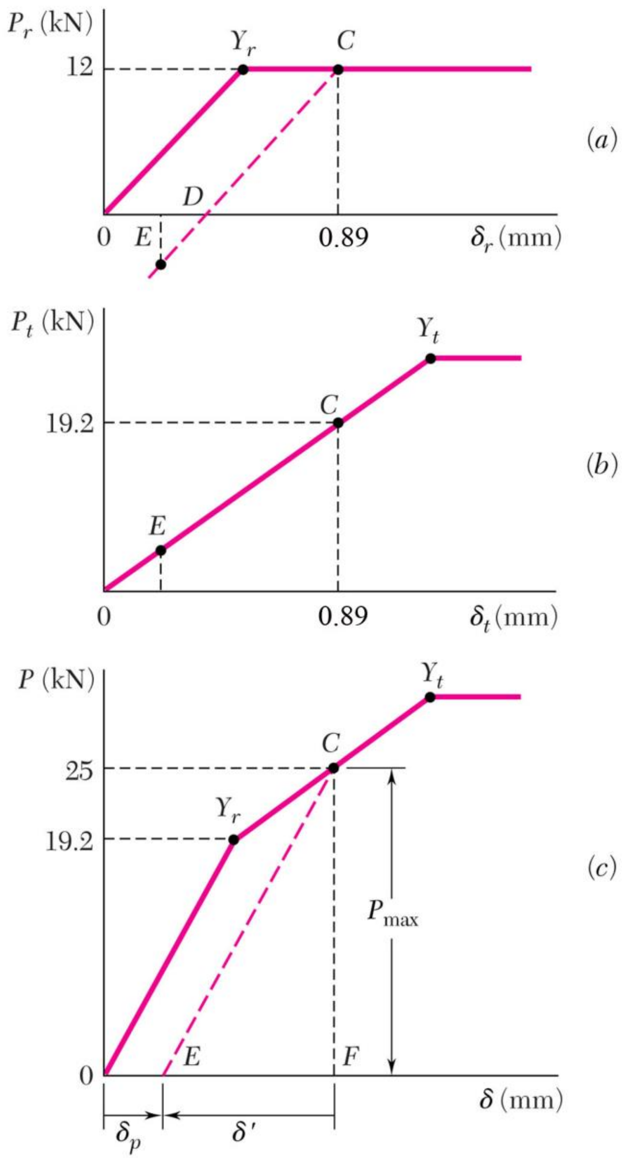


$A_r = 48\ \text{mm}^2$ $\quad A_t = 62\ \text{mm}^2$

$E_r = 210\ \text{GPa}$ $\quad E_t = 105\ \text{GPa}$

$(\sigma_r)_Y = 250\ \text{MPa}$ $\quad (\sigma_t)_Y = 310\ \text{MPa}$

Example 2.14, 2.15, 2.16



a) Draw a load-deflection diagram for the rod-tube assembly

$$\left(P_r\right)_Y = \left(\sigma_r\right)_Y A_r = \left(250\times 10^6\,\mathrm{Pa}\right)\left(48\times 10^{-6}\,\mathrm{m}^2\right) = 12\,\mathrm{kN}$$

$$\left(\delta_r\right)_Y = \left(\varepsilon_r\right)_Y L = \frac{\left(\sigma_r\right)_Y}{E_r} L = \frac{250\times 10^6\,\mathrm{Pa}}{210\times 10^9\,\mathrm{Pa}}\left(750\,\mathrm{mm}\right)$$

$$= 0.89\mathrm{mm}$$

$$\left(P_t\right)_Y = \left(\sigma_t\right)_Y A_t = \left(310\times 10^6\,\mathrm{Pa}\right)\left(62\times 10^{-6}\,\mathrm{m}^2\right) = 19.2\,\mathrm{kN}$$

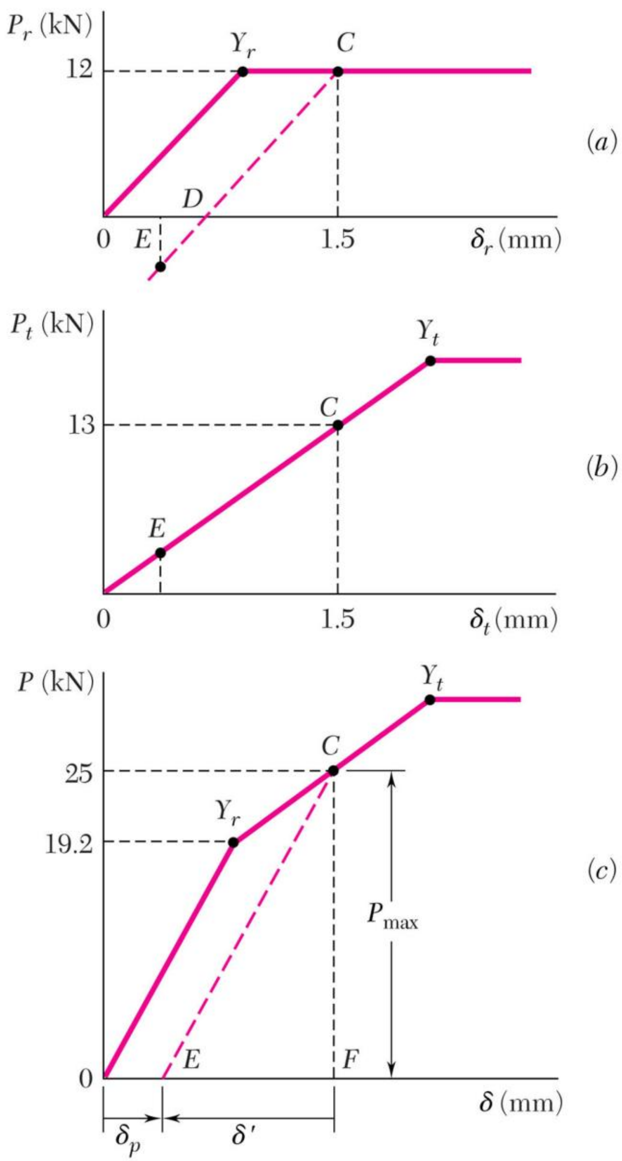
$$\left(\delta_t\right)_Y = \left(\varepsilon_t\right)_Y L = \frac{\left(\sigma_t\right)_Y}{E_t} L = \frac{310\times 10^6\,\mathrm{Pa}}{105\times 10^9\,\mathrm{Pa}}\left(750\,\mathrm{mm}\right)$$

$$= 2.21\mathrm{mm}$$

$$P = P_r + P_t$$

$$\delta = \delta_r = \delta_t$$

Example 2.14, 2.15, 2.16



At a load of P = 25 kN, the rod has reached the plastic range while the tube is still in the elastic range

$$P_r = (P_r)_Y = 12\,\text{kN}$$

$$P_t = P - P_r = (25 - 12)\,\text{kN} = 13\,\text{kN}$$

$$\sigma_t = \frac{P_t}{A_t} = \frac{13\,\text{kN}}{62 \times 10^{-6}\,\text{m}^2} = 210\,\text{MPa}$$

$$\delta_t = \varepsilon_t L = \frac{\sigma_t}{E_t} L = \frac{210 \times 10^6\,\text{Pa}}{105 \times 10^9\,\text{Pa}} 750\,\text{mm}$$

$$\boxed{\delta_{max} = \delta_t = 1.5\,\text{mm}}$$

The rod-tube assembly unloads along a line parallel to $0Y_r$

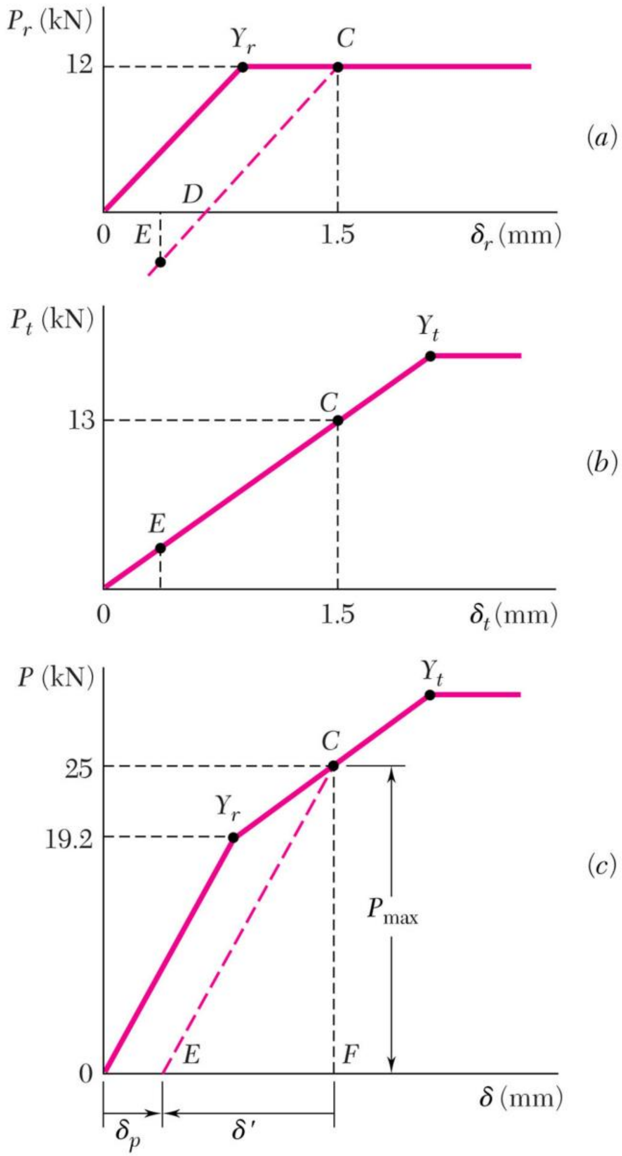
$$m = \frac{19.2\,\text{kN}}{0.89\,\text{mm}} = 21.6\,\text{kN/mm} = \text{slope}$$

$$\delta' = -\frac{P_{max}}{m} = -\frac{25\,\text{kN}}{21.6\,\text{kN/mm}} = -1.16\,\text{mm}$$

$$\delta_p = \delta_{max} + \delta' = (1.5 - 1.16)\,\text{mm}$$

$$\boxed{\delta_p = 0.34\,\text{mm}}$$

Example 2.14, 2.15, 2.16



Calculate the residual stresses in the rod and tube.

Calculate the reverse stresses in the rod and tube caused by unloading and add them to the maximum stresses.

$$\varepsilon' = \frac{\delta'}{L} = \frac{-1.16\,\text{mm}}{750\,\text{mm}} = -1.55 \times 10^{-3}\ \text{mm/mm}$$

$$\sigma'_r = \varepsilon' E_r = \left(-1.55 \times 10^{-3}\right)\left(210\,\text{GPa}\right) = -325.5\,\text{MPa}$$

$$\sigma'_t = \varepsilon' E_t = \left(-1.55 \times 10^{-3}\right)\left(105\,\text{Gpa}\right) = -162.75\,\text{MPa}$$

$$\sigma_{residual,r} = \sigma_r + \sigma'_r = \left(250 - 325.5\right)\text{MPa} = -75.5\,\text{MPa}$$

$$\sigma_{residual,t} = \sigma_t + \sigma'_t = \left(210 - 162.75\right)\text{MPa} = 47.25\,\text{MPa}$$