# CS 228 : Logic in Computer Science

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# Rules for Natural Deduction

**The rule of double negation elimination $\neg\neg e$**

$$\frac{\neg\neg\varphi}{\varphi}$$

**The rule of double negation introduction $\neg\neg i$**

$$\frac{\varphi}{\neg\neg\varphi}$$

# Rules for Natural Deduction

**The implies elimination rule or Modus Ponens MP**

$$\frac{\varphi \qquad \varphi \to \psi}{\psi}$$

# A Second Proof

- Show that $p, p \rightarrow q, p \rightarrow (q \rightarrow \neg\neg r) \vdash r$

| | | |
|---|---|---|
| 1. | $p \rightarrow (q \rightarrow \neg\neg r)$ | premise |
| 2. | | |

# A Second Proof

- Show that $p, p \to q, p \to (q \to \neg\neg r) \vdash r$

| | | |
|---|---|---|
| 1. | $p \to (q \to \neg\neg r)$ | premise |
| 2. | $p \to q$ | premise |
| 3. | | |

# A Second Proof

- Show that $p, p \rightarrow q, p \rightarrow (q \rightarrow \neg\neg r) \vdash r$

| | | |
|---|---|---|
| 1. | $p \rightarrow (q \rightarrow \neg\neg r)$ | premise |
| 2. | $p \rightarrow q$ | premise |
| 3. | $p$ | premise |
| 4. | | |

# A Second Proof

- Show that $p, p \to q, p \to (q \to \neg\neg r) \vdash r$

| | | |
|---|---|---|
| 1. | $p \to (q \to \neg\neg r)$ | premise |
| 2. | $p \to q$ | premise |
| 3. | $p$ | premise |
| 4. | $q \to \neg\neg r$ | MP 1,3 |
| 5. | | |

# A Second Proof

- Show that $p, p \to q, p \to (q \to \neg\neg r) \vdash r$

| | | |
|---|---|---|
| 1. | $p \to (q \to \neg\neg r)$ | premise |
| 2. | $p \to q$ | premise |
| 3. | $p$ | premise |
| 4. | $q \to \neg\neg r$ | MP 1,3 |
| 5. | $q$ | MP 2,3 |
| 6. | | |

# A Second Proof

- Show that $p, p \to q, p \to (q \to \neg\neg r) \vdash r$

| | | |
|---|---|---|
| 1. | $p \to (q \to \neg\neg r)$ | premise |
| 2. | $p \to q$ | premise |
| 3. | $p$ | premise |
| 4. | $q \to \neg\neg r$ | MP 1,3 |
| 5. | $q$ | MP 2,3 |
| 6. | $\neg\neg r$ | MP 4,5 |
| 7. | | |

# A Second Proof

- Show that $p, p \to q, p \to (q \to \neg\neg r) \vdash r$

| | | |
|---|---|---|
| 1. | $p \to (q \to \neg\neg r)$ | premise |
| 2. | $p \to q$ | premise |
| 3. | $p$ | premise |
| 4. | $q \to \neg\neg r$ | MP 1,3 |
| 5. | $q$ | MP 2,3 |
| 6. | $\neg\neg r$ | MP 4,5 |
| 7. | $r$ | $\neg\neg e$ 6 |

# Rules for Natural Deduction

**Another implies elimination rule or Modus Tollens MT**

$$\frac{\varphi \to \psi \qquad \neg\psi}{\neg\varphi}$$

# A Proof

- Show that $p \to \neg q, q \vdash \neg p$

| | | |
|---|---|---|
| 1. | $p \to \neg q$ | premise |
| 2. | | |

# A Proof

- Show that $p \to \neg q, q \vdash \neg p$

| | | |
|---|---|---|
| 1. | $p \to \neg q$ | premise |
| 2. | $q$ | premise |
| 3. | | |

# A Proof

- Show that $p \to \neg q, q \vdash \neg p$

| | | |
|---|---|---|
| 1. | $p \to \neg q$ | premise |
| 2. | $q$ | premise |
| 3. | $\neg\neg q$ | $\neg\neg i$ 2 |
| 4. | | |

# A Proof

- Show that $p \to \neg q, q \vdash \neg p$

| | | |
|---|---|---|
| 1. | $p \to \neg q$ | premise |
| 2. | $q$ | premise |
| 3. | $\neg\neg q$ | $\neg\neg i$ 2 |
| 4. | $\neg p$ | MT 1,3 |

# More Rules

- Can we prove $p \to q \vdash \neg q \to \neg p$?

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- Can we prove $p \rightarrow q \vdash \neg q \rightarrow \neg p$?
- So far, no proof rule that can do this.
- Given $p \rightarrow q$, let us assume $\neg q$. Can we then prove $\neg p$?

# More Rules

- Can we prove $p \to q \vdash \neg q \to \neg p$?
- So far, no proof rule that can do this.
- Given $p \to q$, let us assume $\neg q$. Can we then prove $\neg p$?
- Yes, using MT.

# The implies introduction rule $\rightarrow i$

- $p \rightarrow q \vdash \neg q \rightarrow \neg p$

| | | |
|---|---|---|
| 1. | $p \rightarrow q$ | premise |
| 2. | $\neg q$ | assumption |
| 3. | $\neg p$ | MT 1,2 |
| 4. | $\neg q \rightarrow \neg p$ | $\rightarrow i$ 2-3 |

# More on $\rightarrow i$

- $\vdash (q \rightarrow r) \rightarrow [(\neg q \rightarrow \neg p) \rightarrow (p \rightarrow r)]$

| | | |
|---|---|---|
| 1. | *true* | premise |
| 2. | | |

# More on $\rightarrow i$

- $\vdash (q \rightarrow r) \rightarrow [(\neg q \rightarrow \neg p) \rightarrow (p \rightarrow r)]$

| | | |
|---|---|---|
| 1. | *true* | premise |
| 2. | $q \rightarrow r$ | assumption |
| 3. | | |

# More on $\rightarrow i$

- $\vdash (q \rightarrow r) \rightarrow [(\neg q \rightarrow \neg p) \rightarrow (p \rightarrow r)]$

| | | |
|---|---|---|
| 1. | $true$ | premise |
| 2. | $q \rightarrow r$ | assumption |
| 3. | $\neg q \rightarrow \neg p$ | assumption |
| 4. | | |

# More on $\rightarrow i$

- $\vdash (q \rightarrow r) \rightarrow [(\neg q \rightarrow \neg p) \rightarrow (p \rightarrow r)]$

| | | |
|---|---|---|
| 1. | $true$ | premise |
| 2. | $q \rightarrow r$ | assumption |
| 3. | $\neg q \rightarrow \neg p$ | assumption |
| 4. | $p$ | assumption |
| 5. | | |

# More on $\rightarrow i$

- $\vdash (q \rightarrow r) \rightarrow [(\neg q \rightarrow \neg p) \rightarrow (p \rightarrow r)]$

| | | |
|---|---|---|
| 1. | $true$ | premise |
| 2. | $q \rightarrow r$ | assumption |
| 3. | $\neg q \rightarrow \neg p$ | assumption |
| 4. | $p$ | assumption |
| 5. | $\neg\neg p$ | $\neg\neg i$ 4 |
| 6. | | |

# More on $\rightarrow i$

- $\vdash (q \rightarrow r) \rightarrow [(\neg q \rightarrow \neg p) \rightarrow (p \rightarrow r)]$

| | | |
|---|---|---|
| 1. | $true$ | premise |
| 2. | $q \rightarrow r$ | assumption |
| 3. | $\neg q \rightarrow \neg p$ | assumption |
| 4. | $p$ | assumption |
| 5. | $\neg\neg p$ | $\neg\neg i$ 4 |
| 6. | $\neg\neg q$ | MT 3,5 |
| 7. | | |

# More on $\rightarrow i$

- $\vdash (q \rightarrow r) \rightarrow [(\neg q \rightarrow \neg p) \rightarrow (p \rightarrow r)]$

| | | |
|---|---|---|
| 1. | $true$ | premise |
| 2. | $q \rightarrow r$ | assumption |
| 3. | $\neg q \rightarrow \neg p$ | assumption |
| 4. | $p$ | assumption |
| 5. | $\neg\neg p$ | $\neg\neg i$ 4 |
| 6. | $\neg\neg q$ | MT 3,5 |
| 7. | $q$ | $\neg\neg e$ 6 |
| 8. | | |

# More on $\rightarrow i$

- $\vdash (q \rightarrow r) \rightarrow [(\neg q \rightarrow \neg p) \rightarrow (p \rightarrow r)]$

| | | |
|---|---|---|
| 1. | $true$ | premise |
| 2. | $q \rightarrow r$ | assumption |
| 3. | $\neg q \rightarrow \neg p$ | assumption |
| 4. | $p$ | assumption |
| 5. | $\neg\neg p$ | $\neg\neg i$ 4 |
| 6. | $\neg\neg q$ | MT 3,5 |
| 7. | $q$ | $\neg\neg e$ 6 |
| 8. | $r$ | MP 2,7 |

# More on $\rightarrow i$

- $\vdash (q \rightarrow r) \rightarrow [(\neg q \rightarrow \neg p) \rightarrow (p \rightarrow r)]$

| | | |
|---|---|---|
| 1. | $true$ | premise |
| 2. | $q \rightarrow r$ | assumption |
| 3. | $\neg q \rightarrow \neg p$ | assumption |
| 4. | $p$ | assumption |
| 5. | $\neg\neg p$ | $\neg\neg i$ 4 |
| 6. | $\neg\neg q$ | MT 3,5 |
| 7. | $q$ | $\neg\neg e$ 6 |
| 8. | $r$ | MP 2,7 |
| 9. | $p \rightarrow r$ | $\rightarrow i$ 4-8 |

# More on $\rightarrow i$

- $\vdash (q \rightarrow r) \rightarrow [(\neg q \rightarrow \neg p) \rightarrow (p \rightarrow r)]$

| | | |
|---|---|---|
| 1. | $true$ | premise |
| 2. | $q \rightarrow r$ | assumption |
| 3. | $\neg q \rightarrow \neg p$ | assumption |
| 4. | $p$ | assumption |
| 5. | $\neg\neg p$ | $\neg\neg i$ 4 |
| 6. | $\neg\neg q$ | MT 3,5 |
| 7. | $q$ | $\neg\neg e$ 6 |
| 8. | $r$ | MP 2,7 |
| 9. | $p \rightarrow r$ | $\rightarrow i$ 4-8 |
| 10. | $(\neg q \rightarrow \neg p) \rightarrow (p \rightarrow r)$ | $\rightarrow i$ 3-9 |
| 11. | | |

# More on $\rightarrow i$

- $\vdash (q \rightarrow r) \rightarrow [(\neg q \rightarrow \neg p) \rightarrow (p \rightarrow r)]$

| | | |
|---|---|---|
| 1. | *true* | premise |
| 2. | $q \rightarrow r$ | assumption |
| 3. | $\neg q \rightarrow \neg p$ | assumption |
| 4. | $p$ | assumption |
| 5. | $\neg\neg p$ | $\neg\neg i$ 4 |
| 6. | $\neg\neg q$ | MT 3,5 |
| 7. | $q$ | $\neg\neg e$ 6 |
| 8. | $r$ | MP 2,7 |
| 9. | $p \rightarrow r$ | $\rightarrow i$ 4-8 |
| 10. | $(\neg q \rightarrow \neg p) \rightarrow (p \rightarrow r)$ | $\rightarrow i$ 3-9 |
| 11. | $(q \rightarrow r) \rightarrow [(\neg q \rightarrow \neg p) \rightarrow (p \rightarrow r)]$ | $\rightarrow i$ 2-10 |

# Transforming Proofs

- Knowing the proof of $\vdash (q \to r) \to [(\neg q \to \neg p) \to (p \to r)]$, can you prove

# Transforming Proofs

- Knowing the proof of $\vdash (q \to r) \to [(\neg q \to \neg p) \to (p \to r)]$, can you prove
  - $q \to r \vdash (\neg q \to \neg p) \to (p \to r)$

# Transforming Proofs

- Knowing the proof of $\vdash (q \to r) \to [(\neg q \to \neg p) \to (p \to r)]$, can you prove
  - $q \to r \vdash (\neg q \to \neg p) \to (p \to r)$
  - $(q \to r), (\neg q \to \neg p) \vdash p \to r$

# Transforming Proofs

- Knowing the proof of $\vdash (q \to r) \to [(\neg q \to \neg p) \to (p \to r)]$, can you prove
  - $q \to r \vdash (\neg q \to \neg p) \to (p \to r)$
  - $(q \to r), (\neg q \to \neg p) \vdash p \to r$
  - $(\neg q \to \neg p) \vdash [(q \to r) \to (p \to r)]$

# Transforming Proofs

- Knowing the proof of $\vdash (q \to r) \to [(\neg q \to \neg p) \to (p \to r)]$, can you prove
  - $q \to r \vdash (\neg q \to \neg p) \to (p \to r)$
  - $(q \to r), (\neg q \to \neg p) \vdash p \to r$
  - $(\neg q \to \neg p) \vdash [(q \to r) \to (p \to r)]$
  - $(q \to r), (\neg q \to \neg p), p \vdash r$
- Knowing the proof of any of the above 4 sequents, can you prove $\vdash (q \to r) \to [(\neg q \to \neg p) \to (p \to r)]$?

# Transforming Proofs

- Knowing the proof of $\vdash (q \to r) \to [(\neg q \to \neg p) \to (p \to r)]$, can you prove
  - $q \to r \vdash (\neg q \to \neg p) \to (p \to r)$
  - $(q \to r), (\neg q \to \neg p) \vdash p \to r$
  - $(\neg q \to \neg p) \vdash [(q \to r) \to (p \to r)]$
  - $(q \to r), (\neg q \to \neg p), p \vdash r$
- Knowing the proof of any of the above 4 sequents, can you prove $\vdash (q \to r) \to [(\neg q \to \neg p) \to (p \to r)]$?
- Transform any proof $\varphi_1, \ldots, \varphi_n \vdash \psi$ to $\vdash \varphi_1 \to (\varphi_2 \to \ldots (\varphi_n \to \psi) \ldots))$ by adding $n$ lines of the rule $\to i$
- See an example : transform proof of $p \to (q \to r) \vdash (p \wedge q) \to r$ into that of $\vdash p \to (q \to r) \to [(p \wedge q) \to r]$

# An Example for Proof Transformation

- $p \rightarrow (q \rightarrow r) \vdash (p \land q) \rightarrow r$

| | | |
|---|---|---|
| 1. | $p \rightarrow (q \rightarrow r)$ | premise |
| 2. | | |

# An Example for Proof Transformation

- $p \to (q \to r) \vdash (p \wedge q) \to r$

| | | |
|---|---|---|
| 1. | $p \to (q \to r)$ | premise |
| 2. | $p \wedge q$ | assumption |
| 3. | | |

# An Example for Proof Transformation

- $p \to (q \to r) \vdash (p \wedge q) \to r$

| | | |
|---|---|---|
| 1. | $p \to (q \to r)$ | premise |
| 2. | $p \wedge q$ | assumption |
| 3. | $p$ | $\wedge e_1$ 2 |
| 4. | | |

# An Example for Proof Transformation

- $p \to (q \to r) \vdash (p \wedge q) \to r$

| | | |
|---|---|---|
| 1. | $p \to (q \to r)$ | premise |
| 2. | $p \wedge q$ | assumption |
| 3. | $p$ | $\wedge e_1$ 2 |
| 4. | $q$ | $\wedge e_2$ 2 |
| 5. | | |

# An Example for Proof Transformation

- $p \to (q \to r) \vdash (p \wedge q) \to r$

| | | |
|---|---|---|
| 1. | $p \to (q \to r)$ | premise |
| 2. | $p \wedge q$ | assumption |
| 3. | $p$ | $\wedge e_1$ 2 |
| 4. | $q$ | $\wedge e_2$ 2 |
| 5. | $q \to r$ | MP 1,3 |
| 6. | | |

# An Example for Proof Transformation

- $p \to (q \to r) \vdash (p \land q) \to r$

| | | |
|---|---|---|
| 1. | $p \to (q \to r)$ | premise |
| 2. | $p \land q$ | assumption |
| 3. | $p$ | $\land e_1$ 2 |
| 4. | $q$ | $\land e_2$ 2 |
| 5. | $q \to r$ | MP 1,3 |
| 6. | $r$ | MP 4,5 |
| 7. | | |

# An Example for Proof Transformation

- $p \to (q \to r) \vdash (p \wedge q) \to r$

| | | |
|---|---|---|
| 1. | $p \to (q \to r)$ | premise |
| 2. | $p \wedge q$ | assumption |
| 3. | $p$ | $\wedge e_1$ 2 |
| 4. | $q$ | $\wedge e_2$ 2 |
| 5. | $q \to r$ | MP 1,3 |
| 6. | $r$ | MP 4,5 |
| 7. | $p \wedge q \to r$ | $\to i$ 2-6 |

# Transform proof

- $\vdash [p \to (q \to r)] \to [(p \land q) \to r]$

| | | |
|---|---|---|
| 1. | *true* | premise |
| 2. | $p \to (q \to r)$ | assumption |
| 3. | | |

# Transform proof

- $\vdash [p \to (q \to r)] \to [(p \wedge q) \to r]$

| | | |
|---|---|---|
| 1. | *true* | premise |
| 2. | $p \to (q \to r)$ | assumption |
| 3. | $p \wedge q$ | assumption |
| 4. | | |

# Transform proof

- $\vdash [p \to (q \to r)] \to [(p \wedge q) \to r]$

| | | |
|---|---|---|
| 1. | $true$ | premise |
| 2. | $p \to (q \to r)$ | assumption |
| 3. | $p \wedge q$ | assumption |
| 4. | $p$ | $\wedge e_1$ 3 |
| 5. | | |

# Transform proof

- $\vdash [p \to (q \to r)] \to [(p \land q) \to r]$

| | | |
|---|---|---|
| 1. | $true$ | premise |
| 2. | $p \to (q \to r)$ | assumption |
| 3. | $p \land q$ | assumption |
| 4. | $p$ | $\land e_1$ 3 |
| 5. | $q$ | $\land e_2$ 3 |
| 6. | | |

# Transform proof

- $\vdash [p \to (q \to r)] \to [(p \wedge q) \to r]$

| | | |
|---|---|---|
| 1. | $true$ | premise |
| 2. | $p \to (q \to r)$ | assumption |
| 3. | $p \wedge q$ | assumption |
| 4. | $p$ | $\wedge e_1$ 3 |
| 5. | $q$ | $\wedge e_2$ 3 |
| 6. | $q \to r$ | MP 2,4 |
| 7. | | |

# Transform proof

- $\vdash [p \to (q \to r)] \to [(p \wedge q) \to r]$

| | | |
|---|---|---|
| 1. | *true* | premise |
| 2. | $p \to (q \to r)$ | assumption |
| 3. | $p \wedge q$ | assumption |
| 4. | $p$ | $\wedge e_1$ 3 |
| 5. | $q$ | $\wedge e_2$ 3 |
| 6. | $q \to r$ | MP 2,4 |
| 7. | $r$ | MP 5,6 |
| 8. | | |

# Transform proof

- $\vdash [p \to (q \to r)] \to [(p \land q) \to r]$

| | | |
|---|---|---|
| 1. | $true$ | premise |
| 2. | $p \to (q \to r)$ | assumption |
| 3. | $p \land q$ | assumption |
| 4. | $p$ | $\land e_1$ 3 |
| 5. | $q$ | $\land e_2$ 3 |
| 6. | $q \to r$ | MP 2,4 |
| 7. | $r$ | MP 5,6 |
| 8. | $p \land q \to r$ | $\to i$ 3-7 |
| 9. | $[p \to (q \to r)] \to [(p \land q) \to r]$ | $\to i$ 2-8 |

# More Rules

## The or introduction rule $\vee i_1$

$$\frac{\varphi}{\varphi \vee \psi}$$

## The or introduction rule $\vee i_2$

$$\frac{\psi}{\varphi \vee \psi}$$

# More Rules

**The or elimination rule $\vee e$**

$$\frac{\varphi \vee \psi \qquad \varphi \vdash \chi \qquad \psi \vdash \chi}{\chi}$$

# Or Elimination Example

- $q \rightarrow r \vdash (p \vee q) \rightarrow (p \vee r)$

| | | |
|---|---|---|
| 1. | $q \rightarrow r$ | premise |
| 2. | | |

# Or Elimination Example

- $q \rightarrow r \vdash (p \vee q) \rightarrow (p \vee r)$

| | | |
|---|---|---|
| 1. | $q \rightarrow r$ | premise |
| 2. | $p \vee q$ | assumption |
| 3. | | |

# Or Elimination Example

- $q \to r \vdash (p \lor q) \to (p \lor r)$

| | | |
|---|---|---|
| 1. | $q \to r$ | premise |
| 2. | $p \lor q$ | assumption |
| 3. | $p$ | assumption |
| 4. | | |

# Or Elimination Example

- $q \to r \vdash (p \lor q) \to (p \lor r)$

| | | |
|---|---|---|
| 1. | $q \to r$ | premise |
| 2. | $p \lor q$ | assumption |
| 3. | $p$ | assumption |
| 4. | $p \lor r$ | $\lor i_1$ 3 |
| 5. | | |

# Or Elimination Example

- $q \to r \vdash (p \vee q) \to (p \vee r)$

| | | |
|---|---|---|
| 1. | $q \to r$ | premise |
| 2. | $p \vee q$ | assumption |
| 3. | $p$ | assumption |
| 4. | $p \vee r$ | $\vee i_1$ 3 |
| 5. | $q$ | assumption |
| 6. | | |

# Or Elimination Example

- $q \to r \vdash (p \vee q) \to (p \vee r)$

| | | |
|---|---|---|
| 1. | $q \to r$ | premise |
| 2. | $p \vee q$ | assumption |
| 3. | $p$ | assumption |
| 4. | $p \vee r$ | $\vee i_1$ 3 |
| 5. | $q$ | assumption |
| 6. | $r$ | MP 1,5 |
| 7. | | |

# Or Elimination Example

- $q \to r \vdash (p \vee q) \to (p \vee r)$

| | | |
|---|---|---|
| 1. | $q \to r$ | premise |
| 2. | $p \vee q$ | assumption |
| 3. | $p$ | assumption |
| 4. | $p \vee r$ | $\vee i_1$ 3 |
| 5. | $q$ | assumption |
| 6. | $r$ | MP 1,5 |
| 7. | $p \vee r$ | $\vee i_2$ 6 |

# Or Elimination Example

- $q \to r \vdash (p \vee q) \to (p \vee r)$

| | | |
|---|---|---|
| 1. | $q \to r$ | premise |
| 2. | $p \vee q$ | assumption |
| 3. | $p$ | assumption |
| 4. | $p \vee r$ | $\vee i_1$ 3 |
| 5. | $q$ | assumption |
| 6. | $r$ | MP 1,5 |
| 7. | $p \vee r$ | $\vee i_2$ 6 |
| 8. | $p \vee r$ | $\vee e$ 2, 3-4, 5-7 |

# Or Elimination Example

- $q \to r \vdash (p \vee q) \to (p \vee r)$

| | | |
|---|---|---|
| 1. | $q \to r$ | premise |
| 2. | $p \vee q$ | assumption |
| 3. | $p$ | assumption |
| 4. | $p \vee r$ | $\vee i_1$ 3 |
| 5. | $q$ | assumption |
| 6. | $r$ | MP 1,5 |
| 7. | $p \vee r$ | $\vee i_2$ 6 |
| 8. | $p \vee r$ | $\vee e$ 2, 3-4, 5-7 |
| 9. | | |

# Or Elimination Example

- $q \to r \vdash (p \lor q) \to (p \lor r)$

| | | |
|---|---|---|
| 1. | $q \to r$ | premise |
| 2. | $p \lor q$ | assumption |
| 3. | $p$ | assumption |
| 4. | $p \lor r$ | $\lor i_1$ 3 |
| 5. | $q$ | assumption |
| 6. | $r$ | MP 1,5 |
| 7. | $p \lor r$ | $\lor i_2$ 6 |
| 8. | $p \lor r$ | $\lor e$ 2, 3-4, 5-7 |
| 9. | $(p \lor q) \to (p \lor r)$ | $\to i$ 2-8 |