Chapter 3: Vector Spaces

1. A nonempty set V of objects (called elements or vectors) is called a vector space over the scalars $\mathbb{F}$ ($\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$) if the following axioms are satisfied.

2. Closure axioms:

   i. (closure under vector addition) For every pair of elements $x, y \in V$ there is a unique element $x + y \in V$ called the sum of x and y.

   ii. (closure under scalar multiplication of vectors by elements of $\mathbb{F}$) For every $x \in V$ and every scalar $\alpha \in \mathbb{F}$ there is a unique element $\alpha x \in V$ called the product of α and x.

3. Axioms for vector addition:

   iii. (commutative law) $x + y = y + x$ for all $x, y \in V$.

   iv. (associative law) $x + (y + z) = (x + y) + z$ for all $x, y, z \in V$.

   v. (existence of zero element) There exists an element 0 in V such that $x + 0 = 0 + x = x$ for all $x \in V$.

   vi. (existence of inverse or negatives) For $x \in V$ there exists an element written as $-x \in V$ such that $x + (-x) = 0$.

Vector Spaces: Definition

1. Axioms for scalar multiplication:

   vii. (associativity) For all $\alpha, \beta \in \mathbb{F},\ x \in V$,
   $$\alpha(\beta x) = (\alpha\beta)x.$$

   viii. (distributive law for addition in V) For all $x, y \in V$ and $\alpha \in \mathbb{F}$,
   $$\alpha(x + y) = \alpha x + \alpha y.$$

   ix. (distributive law for addition in $\mathbb{F}$) For all $\alpha,\ \beta \in \mathbb{F}$ and $x \in V$,
   $$(\alpha + \beta)x = \alpha x + \beta x.$$

   x. (existence of identity for multiplication) For all $x \in V$,
   $$1x = x.$$

2. When $\mathbb{F} = \mathbb{R}$ we say that V is a real vector space.
3. If we replace real numbers in the above definition by complex numbers then we get the definition of a complex vector space.

Vector Spaces: Examples

1. In the examples below we leave the verification of the vector addition and scalar multiplication axioms as exercises.
2. $V = \mathbb{R}$, $\mathbb{F} = \mathbb{R}$ with ordinary addition and multiplication as vector addition and scalar multiplication. This gives a real vector space.
3. $V = \mathbb{C}$, $\mathbb{F} = \mathbb{C}$ with ordinary addition and multiplication as vector addition and scalar multiplication. This gives a complex vector space.
4. $V = \mathbb{C}$, $\mathbb{F} = \mathbb{R}$ with ordinary addition and multiplication as vector addition and scalar multiplication. This gives a real vector space.
5. $V = \mathbb{R}^n = \{[a_1, a_2, \ldots, a_n] | a_1, \ldots, a_n \in \mathbb{R}\}$, $\mathbb{F} = \mathbb{R}$ with addition of row vectors as vector addition and multiplication of a row vector by a real number as scalar multiplication. This gives a real vector space.
6. We can similarly define a real vector space of column vectors with n real components.
7. Depending on the context $\mathbb{R}^n$ could refer to either row vectors or column vectors with n real components.

Vector Spaces: Examples

1. $V = \mathbb{C}^n = \{[a_1, a_2, \ldots, a_n] | a_1, \ldots, a_n \in \mathbb{C}\}$, $\mathbb{F} = \mathbb{C}$ with addition of row vectors as vector addition and multiplication of a row vector by a complex number as scalar multiplication. This gives a complex vector space.
2. We can similarly define a complex vector space of column vectors with n complex components.
3. Depending on the context $\mathbb{C}^n$ could refer to either row vectors or column vectors with n complex components.
4. Let $a < b$ be real numbers and set $V = \{f : [a, b] \longrightarrow \mathbb{R}\}$, $\mathbb{F} = \mathbb{R}$. If $f, g \in V$ then we set $(f + g)(x) = f(x) + g(x)$ for all $x \in [a, b]$. If $a \in \mathbb{R}$ and $f \in V$ then $(af)(x) = af(x)$ for all $x \in [a, b]$. This gives a real vector space. Here V is also denoted by $\mathbb{R}^{[a,b]}$.
5. Let t be an indeterminate. The set $\mathcal{P}_n(\mathbb{R}) = \{a_0 + a_1 t + \ldots + a_n t^n | a_0, a_1, \ldots, a_n \in \mathbb{R}\}$ is a real vector space under usual addition of polynomials and multiplication of polynomials with real numbers.

Vector Spaces: Examples

1. $C[a,b] := \{f : [a,b] \longrightarrow \mathbb{R} | f$ is continuous on $[a,b]\}$ is a real vector space under addition and scalar multiplication defined in item 4 of the last slide.
2. $V = \{f : [a,b] \longrightarrow \mathbb{R} | f$ is differentiable at $x \in [a,b], x$ fixed$\}$ is a real vector space under the operations described in item 4 of the last slide.
3. The set of all solutions to the differential equation $y^{''} + ay^{'} + by = 0$ where $a, b \in \mathbb{R}$ form a real vector space. More generally, in this example we can take $a = a(x),\ b = b(x)$ suitable functions of x.
4. Let $V = M_{m\times n}(\mathbb{R})$ denote the set of all $m \times n$ matrices with real entries. Then V is a real vector space under usual matrix addition and multiplication of a matrix by a real number.
5. The above examples indicate that the notion of a vector space is quite general.
6. A result proved for vector spaces will simultaneously apply to all the above different examples.

Subspace of a Vector Space

1. **Exercise**: Using only the vector space axioms show that $0_{\mathbb{F}} \cdot u = 0_V$ and $\alpha \cdot 0_V = 0_V$ for all $u \in V$ and $\alpha \in \mathbb{F}$. Note that $0_{\mathbb{F}}$ is the zero element of the set $\mathbb{F}$ ($\mathbb{R}$ or $\mathbb{C}$) of the scalars and 0_V is the zero vector of the vector space V.
2. Let V be a vector space over $\mathbb{F}$.
3. A nonempty subset W of V is called a subspace of V if
   i. $0 \in W$
   ii. $u, v \in W$ implies $u + v \in W$
   iii. $u \in W, \alpha \in \mathbb{F}$ implies $\alpha u \in W$.
4. Before giving examples we discuss an important notion.
5. Linear span:
6. Let V be a vector space over $\mathbb{F}$. Let $x_1, \ldots, x_n$ be vectors in V and let $c_1, \ldots, c_n \in \mathbb{F}$.
7. The vector $\sum_{i=1}^{n} c_i x_i \in V$ is called a linear combination of x_i's and c_i is called the coefficient of x_i in this linear combination.
8. Let S be a subset of a vector space V over $\mathbb{F}$.

Subspace of a Vector Space: Linear Span

1. The **linear span** of S is the subset of all vectors in V expressible as linear combinations of finitely many elements in S, i.e.,

$$L(S) = \left\{ \sum_{i=1}^{n} c_i x_i | n \geq 1,\ x_1, x_2, \ldots, x_n \in S \text{ and } c_1, c_2, \ldots, c_n \in \mathbb{F} \right\}.$$

2. By convention the empty sum of vectors is the zero vector. Thus $L(\varnothing) = \{0\}$.
3. We say that $L(S)$ is spanned by S.
4. The linear span $L(S)$ is actually a subspace of V (why?).
5. Now, if $S \subset W \subset V$ and W is a subspace of V then $L(S) \subset W$. It follows that $L(S)$ is the smallest subspace of V containing S.
6. Let A be an $m \times n$ matrix over $\mathbb{F}$, with rows $R_1, \ldots, R_m$ and columns $C_1, \ldots, C_n$.
7. The row space of A, denoted $\mathcal{R}(A)$, is the subspace of $\mathbb{F}^n$ spanned by the rows of A.
8. The column space of A, denoted $\mathcal{C}(A)$, is the subspace of $\mathbb{F}^m$ spanned by the columns of A.

Linear Span

1. The null space of A, denoted $\mathcal{N}(A)$, is defined by $\mathcal{N}(A) = \{x \in \mathbb{F}^n : Ax = 0\}$.
2. Notice that the null space of A is the set of all solutions of the homogeneous system (of linear equations) $Ax = 0$.
3. Check that (in fact, we have already done this!) $\mathcal{N}(A)$ is a subspace of $\mathbb{F}^n$.
4. Different sets may span the same subspace.
5. For example, $L(\{e_1, e_2\}) = L(\{e_1, e_1 + e_2\}) = \mathbb{R}^2$.
6. The vector space $\mathcal{P}_n(\mathbb{R})$ is spanned by $\{1, t, t^2, \ldots, t^n\}$ and also by $\{1, (1+t), \ldots, (1+t)^n\}$ (why?).
7. We have introduced the notion of linear span of a subset S of a vector space. This raises some natural questions:
   i. Which vector spaces can be spanned by finite number of elements?
   ii. If a vector space $V = L(S)$ for a finite subset S of V then what is the size of smallest such S?
8. To answer these questions we introduce the notions of linear dependence and independence, basis and dimension of a vector space.

Linearly Dependent and Independent subsets of V.S.

1. **Linear independence:**
2. Let V be a vector space.
3. A subset $S \subset V$ is called linearly dependent (L.D.) if there exist distinct elements $v_1, v_2, \ldots, v_n \in S$ (for some $n \geq 1$) and scalars $\alpha_1, \alpha_2, \ldots, \alpha_n$ not all zero such that
$$\alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_n v_n = 0.$$
4. A set S is called linearly independent (L.I.) if it is not linearly dependent, i.e., for all $n \geq 1$ and for all distinct $v_1, v_2, \ldots, v_n \in S$ and scalars $\alpha_1, \alpha_2, \ldots, \alpha_n$
$$\alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_n v_n = 0 \Longrightarrow \alpha_i = 0, \text{ for all } i.$$
5. Elements of a linearly independent set are called linearly independent.
6. Note that the empty set is linearly independent.
7. Linearly independent sets are important because each one of them gives us data that we cannot obtain from any linear combination of the others.

L.D. and L.I. subsets of V.S.: Remarks and Examples

1. **Proposition:** The following statements are true.
   i. Any subset of V containing a linearly dependent set is linearly dependent.
   ii. Any subset of a linearly independent set in V is linearly independent.
   iii. It can be seen that S is linearly dependent $\iff$ either $0 \in S$ or a vector in S is a linear combination of other vectors in S.

2. **Proof:** Exercise.

3. Examples:
   i. Consider the vector space $\mathbb{R}^n$ and let $S = \{e_1, e_2, \ldots, e_n\}$. Then S is linearly independent. Indeed, if $\alpha_1 e_1 + \alpha_2 e_2 + \ldots + \alpha_n e_n = 0$ for some scalars $\alpha_1, \alpha_2, \ldots, \alpha_n$ then $(\alpha_1, \alpha_2, \ldots, \alpha_n) = 0$.
   ii. Let $S := \left\{ \begin{bmatrix} 1 & 2 \end{bmatrix}^t, \begin{bmatrix} 2 & 1 \end{bmatrix}^t, \begin{bmatrix} 1 & -1 \end{bmatrix}^t \right\} \subset \mathbb{R}^{2\times 1}$.
   Then the set S is linearly dependent since

$$\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \end{bmatrix}. \text{ Clearly, } \begin{bmatrix} 1 \\ 2 \end{bmatrix} - \begin{bmatrix} 2 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

L.D. and L.I. subsets of V.S.: Examples

iii. Let S denote the subset of $\mathbb{R}^{1\times 4}$ consisting of the vectors $\begin{bmatrix}1 & 0 & 0 & 0\end{bmatrix}, \begin{bmatrix}1 & 1 & 0 & 0\end{bmatrix}, \begin{bmatrix}1 & 1 & 1 & 0\end{bmatrix}$ and $\begin{bmatrix}1 & 1 & 1 & 1\end{bmatrix}$. Then S is linearly independent.

  To see this, let $\alpha_1, \alpha_2, \alpha_3, \alpha_4 \in \mathbb{R}$ be such that $\alpha_1 \begin{bmatrix}1 & 0 & 0 & 0\end{bmatrix} + \alpha_2 \begin{bmatrix}1 & 1 & 0 & 0\end{bmatrix} + \alpha_3 \begin{bmatrix}1 & 1 & 1 & 0\end{bmatrix} + \alpha_4 \begin{bmatrix}1 & 1 & 1 & 1\end{bmatrix} = \begin{bmatrix}0 & 0 & 0 & 0\end{bmatrix}$.

  Then $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = 0$, $\alpha_2 + \alpha_3 + \alpha_4 = 0$, $\alpha_3 + \alpha_4 = 0$ and $\alpha_4 = 0$, that is, $\alpha_4 = \alpha_3 = \alpha_2 = \alpha_1 = 0$.

iv. Let V be the vector space of all continuous functions from $\mathbb{R}$ to $\mathbb{R}$. Let $S = \{1, \cos^2 t, \sin^2 t\}$. Then the relation $\cos^2 t + \sin^2 t - 1 = 0$ shows that S is linearly dependent.

v. Let $\alpha_1 < \alpha_2 < \ldots < \alpha_n$ be real numbers. Let $V = \{f : \mathbb{R} \longrightarrow \mathbb{R} | f \text{ is continuous}\}$. Consider the set $S = \{e^{\alpha_1 x}, e^{\alpha_2 x}, \ldots, e^{\alpha_n x}\}$.

We show that S is linearly independent by induction on n. Let $n = 1$ and $\beta e^{\alpha_1 x} = 0$. Since $e^{\alpha_1 x} \neq 0$ for any x, we get $\beta = 0$. Now assume that the assertion is true for $n - 1$ and

$$\beta_1 e^{\alpha_1 x} + \ldots + \beta_n e^{\alpha_n x} = 0.$$

Then $\beta_1 e^{(\alpha_1 - \alpha_n)x} + \ldots + \beta_n e^{(\alpha_n - \alpha_n)x} = 0$.

Let $x \longrightarrow \infty$ to get $\beta_n = 0$ (why?). Now apply induction hypothesis to get $\beta_1 = \ldots = \beta_{n-1} = 0$.

L.D. and L.I. subsets of V.S.: Examples

vi. Let $\mathcal{P}$ denote the vector space of all polynomials $p(t)$ with real coefficients.

Then the set $S = \{1, t, t^2, \ldots\}$ is linearly independent.

For, suppose that $0 \leq n_1 < n_2 < \ldots < n_r$ and

$$\alpha_1 t^{n_1} + \alpha_2 t^{n_2} + \ldots + \alpha_r t^{n_r} = 0$$

for certain real numbers $\alpha_1, \alpha_2, \ldots, \alpha_r$.

Differentiate the left hand side polynomial of the above equation n_r times to get $\alpha_r = 0$. Continuing this way we see that all $\alpha_1, \alpha_2, \ldots, \alpha_r$ are zero.

Bases and Dimension

1. Bases and dimension are two important notions in the study of vector spaces.
2. As we have seen already a vector space may be realized as linear span of several sets of different sizes.
3. We study properties of the smallest sets whose linear span is the given vector space.

Definition

A subset S of a vector space V is called a **basis** of V if elements of S are linearly independent and $V = L(S)$. A vector space V possessing a finite basis is called **finite dimensional**. Otherwise V is called **infinite dimensional.**

4. Let $\{v_1, \ldots, v_n\}$ be a basis of a finite dimensional vector space V. Then every $v \in V$ can be **uniquely** (why?) expressed as $v = a_1 v_1 + \cdots + a_n v_n$, for scalars $a_1, \ldots, a_n$.

Bases and Dimension

1. We show that all bases of a finite dimensional vector space have same cardinality (i.e., they contain the same number of elements).
2. For this we prove the following result.

Lemma

Let $S = \{v_1, v_2, \ldots, v_k\}$ be a subset of a vector space V. Then any $k+1$ elements in $L(S)$ are linearly dependent.

3. Proof. Let $T = \{u_1, \ldots, u_{k+1}\} \subseteq L(S)$. Write

$$u_i = \sum_{j=1}^{k} a_{ij} v_j, \quad i = 1, \ldots, k+1.$$

4. Consider the $(k+1) \times k$ matrix $A = (a_{ij})$.

Bases and Dimension: Proof continues...

1. Since A has more rows than columns there exists (why?) a nonzero row vector $c = [c_1, \dots, c_{k+1}]$ such that $cA = (A^t c^t)^t = 0$, i.e., for $j = 1, \dots k$

$$\sum_{i=1}^{k+1} c_i a_{ij} = 0.$$

2. We now have

$$\sum_{i=1}^{k+1} c_i u_i \quad = \quad \sum_{i=1}^{k+1} c_i (\sum_{j=1}^{k} a_{ij} v_j) = \sum_{j=1}^{k} (\sum_{i=1}^{k+1} c_i a_{ij}) v_j = 0$$

where NOT all c_j's are 0. Hence T is linearly dependent $\square$

Bases and Dimension

Theorem

Any two bases of a finite dimensional vector space have same number of elements.

Proof.

1. Suppose S and T are bases of a finite dimensional vector space V, i.e., S and T both are linearly independent and $L(S) = V = L(T)$.
2. Suppose $|S| < |T|$. Since $T \subset L(S) = V$, T is linearly dependent.
3. This is a contradiction. Similarly, $|T| < |S|$ also gives a contradiction.
4. Hence $|T| = |S|$. $\square$

Definition

The number of elements in a basis of a finite-dimensional vector space V is called the **dimension** of V. It is denoted by $\dim V$.

Bases and Dimension: Examples

Examples:

i. The n "coordinate vectors" $e_1, e_2, \ldots, e_n$ in $\mathbb{R}^n$ form a basis of $\mathbb{R}^n$.

ii. Let A be an $n \times n$ matrix. Then the columns of A form a basis of $\mathbb{F}^n$ iff the linear system $Ax = 0$ has only the zero solution (why?) iff A is invertible.

iii. $\mathcal{P}_n(\mathbb{R}) = \{a_0 + a_1 t + \ldots + a_n t^n | \ a_0, a_1, \ldots, a_n \in \mathbb{R}\}$ is spanned by $S = \{1, t, t^2, \ldots, t^n\}$. Since S is linearly independent, $\dim \mathcal{P}_n(\mathbb{R}) = n + 1$.

iv. Let $M_{m\times n}(\mathbb{F})$ denote the vector space of all $m \times n$ matrices with entries in $\mathbb{F}$.

Let e_{ij} denote the $m \times n$ matrix with 1 in $(i, j)^{\text{th}}$ position and 0 elsewhere.

If $A = (a_{ij}) \in M_{m\times n}(\mathbb{F})$ then $A = \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} e_{ij}$.

It is easy to see that the mn matrices e_{ij} are linearly independent. Hence $M_{m\times n}(\mathbb{F})$ is an $mn-$dimensional vector space.

1. What is the dimension of $M_{n\times n}(\mathbb{C})$ as a real vector space?

Bases and Dimension

Proposition

Suppose V is a finite dimensional vector space. Let S be a linearly independent subset of V. Then S can be enlarged to a basis of V.

1. **Proof.** Suppose that $\dim V = n$ and S has less than n elements.
2. Let $v \in V \setminus L(S)$. Then $S \cup \{v\}$ is a linearly independent subset of V (why?). Continuing this way we can enlarge S to a basis of V.
3. What if $|S| = n$? Is it possible that $|S| > n$? $\square$
4. Gauss elimination, row space and column space:

Lemma

Let A be an $m \times n$ matrix over $\mathbb{F}$ and E a non-singular (that is, invertible) $m \times m$ matrix over $\mathbb{F}$. Then

i. $\mathcal{R}(A) = \mathcal{R}(EA)$. *Hence* $\dim \mathcal{R}(A) = \dim \mathcal{R}(EA)$.

ii. *Let $1 \leq i_1 < i_2 < \cdots < i_k \leq n$. Columns $\{i_1, \ldots, i_k\}$ of A are linearly independent if and only if columns $\{i_1, \ldots, i_k\}$ of EA are linearly independent. Hence* $\dim \mathcal{C}(A) = \dim \mathcal{C}(EA)$.

1. **Proof.**

i. Note that $R(EA) \subseteq R(A)$ (why?) since every row of EA is a linear combination of the rows of A. Similarly,

$$R(A) = R(E^{-1}(EA)) \subseteq R(EA).$$

ii. Suppose columns $\{i_1, \ldots, i_k\}$ of A are linearly independent.

Then

$$\begin{array}{rc} & \alpha_1 (EA)_{i_1} + \alpha_2 (EA)_{i_2} + \cdots + \alpha_k (EA)_{i_k} = 0 \\ \text{iff} & E(\alpha_1 A_{i_1} + \alpha_2 A_{i_2} + \cdots + \alpha_k A_{i_k}) = 0 \\ \text{iff} & E^{-1}(E(\alpha_1 A_{i_1} + \alpha_2 A_{i_2} + \cdots + \alpha_k A_{i_k})) = 0 \\ \text{iff} & \alpha_1 A_{i_1} + \alpha_2 A_{i_2} + \cdots + \alpha_k A_{i_k} = 0 \\ \text{iff} & \alpha_1 = \cdots = \alpha_k = 0. \end{array}$$

2. Thus columns $\{i_1, \ldots, i_k\}$ of EA are linearly independent. The proof of the converse is similar. □

Bases and Dimension: Row and Column spaces of a Matrix

Theorem

Let A be an $m \times n$ matrix. Then $\dim \mathcal{R}(A) = \dim \mathcal{C}(A)$.

1. **Proof**. Apply row operations to reduce A to the RCF U. That is, $U = EA$ where E is an invertible matrix which is product of elementary matrices.
2. Suppose U has r nonzero rows. Thus U has r pivotal columns.
3. Then (why?) the r nonzero rows of U form a basis of $\mathcal{R}(A)$ (thanks to the last lemma). Let $k_1, \ldots, k_r$ be the pivotal columns of U.
4. Then (why?) columns $k_1, \ldots, k_r$ of A form a basis of $\mathcal{C}(A)$ (thanks to the last lemma again). Thus $\dim \mathcal{R}(A) = \dim \mathcal{C}(A)$. $\square$
5. Example: Let A be a 4×6 matrix whose RCF is
$$U = \begin{bmatrix} 1 & 2 & 3 & 0 & 5 & 0 \\ 0 & 0 & 0 & 1 & 7 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$
6. Columns A_1, A_4, A_6 of A form a basis of $\mathcal{C}(A)$ and the first 3 rows of U form a basis of $\mathcal{R}(A)$.

Bases and Dimension: Rank-Nullity Theorem for a Matrix

Definition

The **rank** of an $m \times n$ matrix A, denoted by $r(A)$ or rank A, is $\dim \mathcal{R}(A) = \dim \mathcal{C}(A)$. The **nullity** of A is the dimension of the nullspace $\mathcal{N}(A)$.

1. **The Rank-Nullity Theorem**:

Theorem

Let A be an $m \times n$ matrix. Then

$$\textit{rank } A + \textit{ nullity } A = n.$$

2. **Proof.** Let $r = r(A)$. Reduce A to its RCF (or even REF) U using elementary row operations. Then U has r nonzero rows, r pivotal columns $k_1, k_2, \ldots, k_r$ and $n - r$ non-pivotal columns $l_1, l_2, \ldots, l_{n-r}$.

3. We need to show that $\dim \mathcal{N}(A) = \dim \mathcal{N}(U) = n - r$.

Rank in terms of determinants

1. For this, we just need to check that the set $S = \{s_{l_1}, s_{l_2}, \ldots, s_{l_{n-r}}\}$ of the $n-r$ basic solution vectors of the linear system $Ax = 0$ is linearly independent (why?) (since $L(S) = \mathcal{N}(A)$).
2. Now recall how the basic solution vectors s_{l_j} for $1 \leq j \leq n-r$ are defined.
3. The linearly independence of the set $\{s_{l_1}, s_{l_2}, \ldots, s_{l_{n-r}}\}$ directly follows from their definitions (why?). Hence $\dim \mathcal{N}(A) = n - r$. $\square$
4. **Remark**: Now, you got the answer to the question "why are the $n-r$ solution vectors $s_{l_1}, s_{l_2}, \ldots, s_{l_{n-r}}$ called basic solution vectors for the linear system $Ax = 0$?".
5. We now characterize rank A in terms of minors of A. Recall that a minor of order r of A is a submatrix of A consisting of r rows and r columns of A.

Theorem

An $m \times n$ matrix A has rank $r \geq 1$ iff $\det M \neq 0$ for some order r minor M of A and $\det N = 0$ for all order $r+1$ minors N of A.

Rank in terms of determinants: Proof

1. **Proof.** Let the rank of A be $r \geq 1$.
2. Then some r columns of A are linearly independent.
3. Let B be the $m \times r$ matrix consisting of these r columns of A.
4. Then rank $(B) = r$ and thus some r rows of B will be linearly independent.
5. Let C be the $r \times r$ matrix consisting of these r rows of B.
6. Then $\det(C) \neq 0$ (why?), since C is invertible as nullity $(C) = r - r = 0$ and hence $Cx = 0$ has only the zero solution.
7. Let N be a $(r+1) \times (r+1)$ minor of A.
8. Without loss of generality we may take N to consist of the first $r+1$ rows and columns of A (why?), since the interchanges of rows or interchanges of columns does not change the rank of the matrix.
9. Suppose $\det(N) \neq 0$. Then the $r+1$ rows of N, and hence the first $r+1$ rows of A, are linearly independent, a contradiction.
10. The converse is left as an exercise. $\square$