Pigeonhole Principle

$A \wedge B \rightarrow C$

$i \in \{1,2 \ldots n+1\}$ $j \in \{1,2 \ldots n\}$

A: Each pigeon is in at least one hole

$[P(1,1) \vee P(1,2) \ldots \vee P(1,n)] \wedge$
$[P(2,1) \vee P(2,2) \ldots \vee P(2,n)] \wedge$

$[P(n+1,1) \vee P(n+1,2) \ldots \vee P(n+1,n)]$

B: Each pigeon is in at most one hole

$[\neg(P(1,1) \wedge P(1,2)) \wedge \neg(P(1,1) \wedge P(1,3)) \ldots$

$[\neg(P(n+1,1) \wedge P(n+1,2)) \wedge \neg(P(n+1,1) \wedge P(n+1,3)) \ldots$

C: At least one hole has more than one pigeon

$[(P(1,1) \wedge P(2,1)) \vee (P(1,1) \wedge P(3,1)) \ldots$
$[(P(1,2) \wedge P(2,2)) \vee (P(1,2) \wedge P(3,2)) \ldots$

$[(P(1,n) \wedge P(2,n)) \vee (P(1,n) \wedge P(3,n)) \ldots$

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1	$(p \to q) \to q$	Assumption
2	$q \to p$	Assumption
3	$\neg p$	Assumption
4	p	Assumption
5	$\bot$	$\bot$ i 3, 4
6	q	$\bot$ e 5
7	$p \to q$	$\to$ i 4-6
8	q	MP 1, 7
9	p	MP 2, 8
10	$\bot$	$\bot$ i 3, 9
11	$\neg\neg p$	$\neg$ i 3-10
12	p	$\neg\neg$ e 11
13	$(q \to p) \to p$	$\to$ i 2-12

14 $[((p \to q) \to q] \to [(q \to p) \to p]$ $\to$ i 1-13

6 ✝

1	$A \to B$	premise	
2	$C \vee A$	premise	
3	C	assumption	
4	$B \vee C$	$\vee$ intro 3	
5	A	assumption	
6	B	MP 1, 5	
7	$B \vee C$	$\vee$ i 6	
8	$B \vee C$	$\vee$ e 2, 3-4, 5-7	

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1	$A \to C$	premise
2	$B \to C$	premise
3	$A \vee B$	assumption
4	A	assumption
5	C	MP 1, 4
6	B	assumption
7	C	MP 2, 6
8	C	$\vee$elim 3, 4-5, 6-7
9	$A \vee B \to C$	$\to$ intro