



CS 228 : Logic in Computer Science

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First-Order Logic : Syntax

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- ▶ Atleast 5 students scored more than 90 marks in a class of 50
- ▶ All words starting with the letter a , ending with the letter b , have even length

Signatures

- ▶ A **vocabulary** or **signature** τ is a set consisting of
 - ▶ constants c_1, c_2, \dots
 - ▶ Relation symbols R_1, R_2, \dots , each with some arity k , denoted R_i^k
- ▶ We look at finite signatures
- ▶ $\tau = (E^2, F^3)$ is a finite signature with two relations, E with arity 2 and F with arity 3

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- ▶ The symbols (and) called **paranthesis**

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- ▶ If φ and ψ are wff, then $\varphi \rightarrow \psi$ is a wff
- ▶ If φ is a wff and x is a variable, then $(\forall x)\varphi$ is a wff

Logical Abbreviations : Boolean Connectives

- ▶ $\neg\varphi = \varphi \rightarrow \perp$
- ▶ $\top = \neg\perp$
- ▶ $\varphi \vee \psi = \neg\varphi \rightarrow \psi$
- ▶ $\varphi \wedge \psi = \neg(\neg\varphi \vee \neg\psi)$
- ▶ $\exists x.\varphi = \neg(\forall x.\neg\varphi)$
- ▶ Precedence of operators : $\neg > \wedge > \vee > \rightarrow > \forall$

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- ▶ $\forall x \forall y \forall z (R(x, y) \rightarrow (R(y, z) \rightarrow R(x, z)))$ Transitivity

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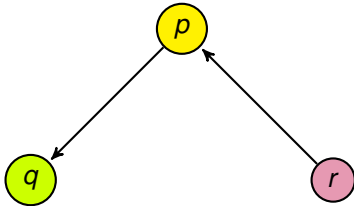
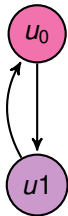
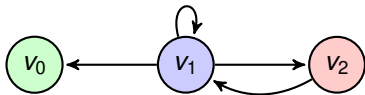
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 - ▶ The structure \mathcal{A} is finite if A (or $u(\mathcal{A})$) is finite

Examples of Structures

A Graph

- ▶ A set V of vertices
- ▶ A set $E \subseteq V \times V$ of edges



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 - ▶ $\mathcal{G} = (V = \{1, 2, 3, 4\}, E^{\mathcal{G}} = \{(1, 2), (2, 3), (3, 4), (1, 1)\})$. We could just as well draw the graph for convenience.
 - ▶ $\forall x \exists y (E(x, y))$

A totally ordered set

- ▶ A set S with an order relation
- ▶ Relates any two elements of S
- ▶ Examples : $(\mathbb{N}, <)$, $(\mathbb{R}, <)$

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 - ▶ $\mathcal{O} = (O = \{1, 2, 4\}, <^{\mathcal{O}} = \{(1, 2), (1, 4), (2, 4)\}, S^{\mathcal{O}} = \{(1, 2)\})$
 - ▶ $\exists x \neg \exists y (S(x, y))$
- ▶ Can you write a **Partial Order** as a structure, where the universe consists of all subsets of a given finite set?

Words

- ▶ A word is a sequence of symbols over a (finite) alphabet
- ▶ Alphabet $\Sigma = \{a, b, c\}$
- ▶ Some words over Σ : *b, aaa, abababa, cacbccc*
- ▶ The length of a word is the number of symbols in it

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 - ▶ $Q_a^{\mathcal{W}}$ is the set of positions labeled a in W
 - ▶ $Q_b^{\mathcal{W}}$ is the set of positions labeled b in W
 - ▶ The structure with $u(\mathcal{W}) = \{0, 1, 2, \dots, 8\}$,
 $Q_a^{\mathcal{W}} = \{0, 1, 4, 6, 8\}$, $Q_b^{\mathcal{W}} = \{2, 3, 5, 7\}$,
 - ▶ $<^{\mathcal{W}} = \{(0, 1), (0, 2), \dots, (7, 8)\}$, $S^{\mathcal{W}} = \{(0, 1), (1, 2), \dots, (7, 8)\}$ uniquely defines the word $W = aabbababa$.
 - ▶ $\forall x(Q_b(x) \rightarrow \exists y(x < y \wedge Q_a(y)))$