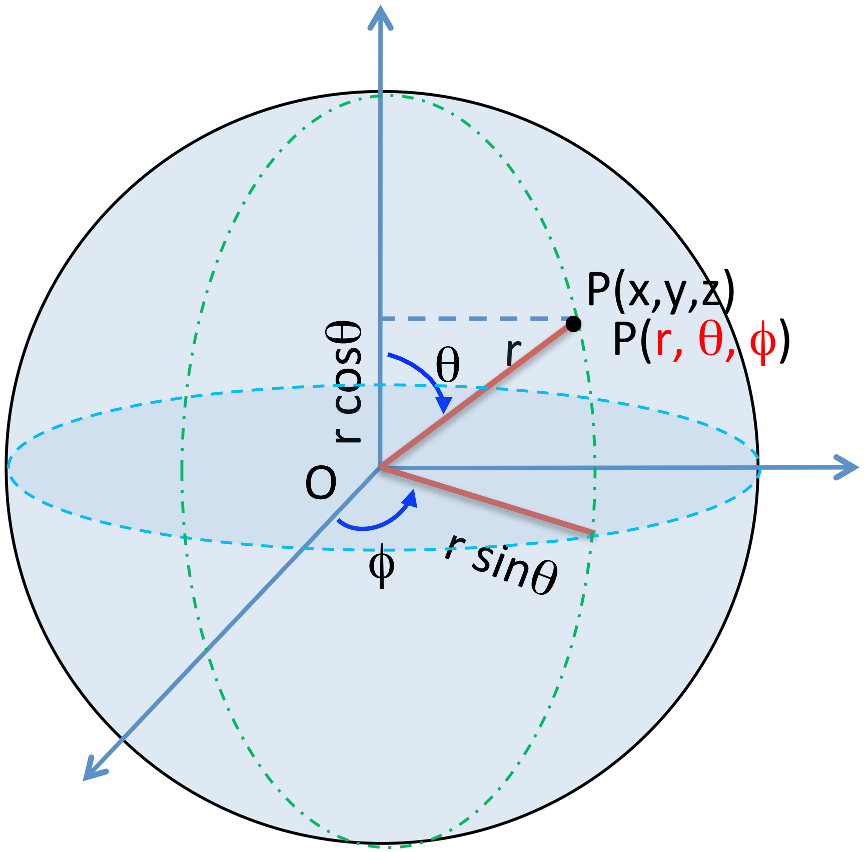
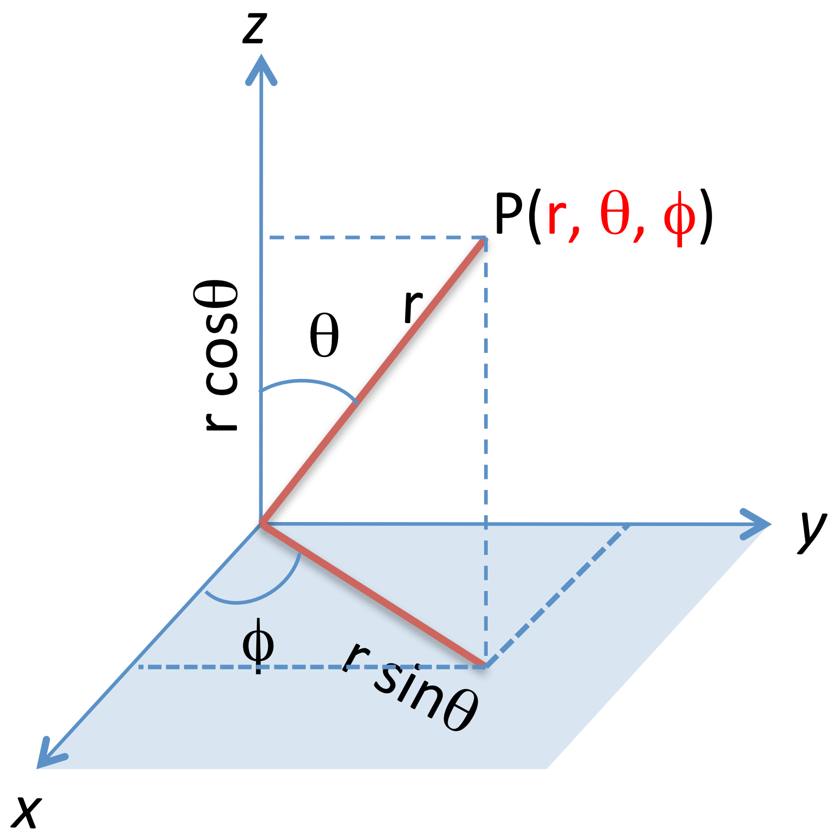
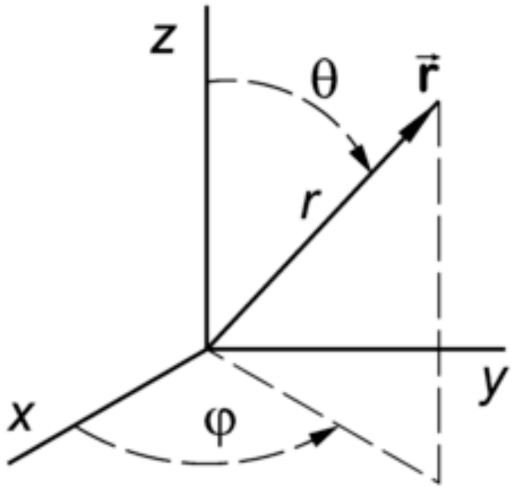
Spherical Polar Coordinates





SPHERICAL POLAR COORDINATES

$$
\begin{aligned}
x &= r\sin\theta\cos\phi \\
y &= r\sin\theta\sin\phi \\
z &= r\cos\theta
\end{aligned}
$$



$r\in[0,\infty]$
$\theta\in[0,\pi]$
$\phi\in[0,2\pi]$

$$
\begin{pmatrix} \delta x \\ \delta y \\ \delta z \end{pmatrix} = \begin{pmatrix} \sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ \cos\theta & -r\sin\theta & 0 \end{pmatrix} \begin{pmatrix} \delta r \\ \delta\theta \\ \delta\phi \end{pmatrix}
$$

$$
\begin{pmatrix} \hat{r} \\ \hat{\theta} \\ \hat{\phi} \end{pmatrix} = \begin{pmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \end{pmatrix} \begin{pmatrix} \hat{x} \\ \hat{y} \\ \hat{z} \end{pmatrix}
$$

$$
\hat{r}.\hat{\theta} = \hat{r}.\hat{\phi} = \hat{\theta}.\hat{\phi} = 0
$$

SPHERICAL POLAR COORDINATES

$$dl_r = \delta r \qquad dl_\theta = r\,\delta\theta \qquad dl_\phi = r\sin\theta\,\delta\phi$$

Infinitesimal displacement $d\vec{l} = \delta r\,\hat{r} + r\,\delta\theta\,\hat{\theta} + r\sin\theta\,\delta\phi\,\hat{\phi}$

Arclength $ds^2 = d\vec{l}\,.\,d\vec{l} = \delta r^2 + r^2\,\delta\theta^2 + r^2\sin^2\theta\,\delta\phi^2$

$$dV = dl_r\,dl_\theta\,dl_\phi = r^2\sin\theta\,\delta r\,\delta\theta\,\delta\phi$$

Area element:

$If \quad r = constant \quad (surface \ of \ a \ sphere) \quad \delta r = 0$

$$dA = dl_\theta\,dl_\phi = r^2\sin\theta\,d\theta\,d\phi$$

$If \quad \theta = constant, \quad \delta\theta = 0$

$$dA = dl_r\,dl_\phi = r\sin\theta\,dr\,d\phi$$

$If \quad \phi = constant, \quad \delta\phi = 0$

$$dA = dl_r\,dl_\theta = r\,dr\,d\theta$$

SPHERICAL POLAR COORDINATES

Gradient of a scalar field

If we have a function $T(r,\theta,\phi)$ then we want

$$dT = \frac{\partial T}{\partial r}\delta r + \frac{\partial T}{\partial \theta}\delta\theta + \frac{\partial T}{\partial \phi}\delta\phi$$
$$= \nabla T . \delta\vec{l}$$

Now, $\delta\vec{l} = \hat{r}\delta r + \hat{\theta} r\delta\theta + \hat{\phi} r\sin\theta\delta\phi$

The gradient in the spherical polar system can be identified as

$$\nabla T = \hat{r}\frac{\partial T}{\partial r} + \hat{\theta}\frac{1}{r}\frac{\partial T}{\partial \theta} + \hat{\phi}\frac{1}{r\sin\theta}\frac{\partial T}{\partial \phi}$$

SPHERICAL POLAR COORDINATES

Velocity and acceleration

$$\begin{pmatrix} \hat{r} \\ \hat{\theta} \\ \hat{\phi} \end{pmatrix} = \boldsymbol{M} \begin{pmatrix} \hat{x} \\ \hat{y} \\ \hat{z} \end{pmatrix} \qquad \boldsymbol{M} = \begin{pmatrix} \sin\theta\,\cos\phi & \sin\theta\,\sin\phi & \cos\theta \\ \cos\theta\,\cos\phi & \cos\theta\,\sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \end{pmatrix}$$

$$\begin{pmatrix} \hat{x} \\ \hat{y} \\ \hat{z} \end{pmatrix} = \boldsymbol{M}^T \begin{pmatrix} \hat{r} \\ \hat{\theta} \\ \hat{\phi} \end{pmatrix} \qquad \begin{pmatrix} \dot{\hat{r}} \\ \dot{\hat{\theta}} \\ \dot{\hat{\phi}} \end{pmatrix} = \dot{\boldsymbol{M}} \begin{pmatrix} \hat{x} \\ \hat{y} \\ \hat{z} \end{pmatrix} \qquad \boldsymbol{M}^T = \begin{pmatrix} \sin\theta\,\cos\phi & \cos\theta\,\cos\phi & -\sin\phi \\ \sin\theta\,\sin\phi & \cos\theta\,\sin\phi & \cos\phi \\ \cos\theta & -\sin\theta & 0 \end{pmatrix}$$

$$\dot{\boldsymbol{M}} = \begin{pmatrix} \cos\theta\,\cos\phi\,\dot{\theta} - \sin\theta\,\sin\phi\,\dot{\phi} & \cos\theta\,\sin\phi\,\dot{\theta} + \sin\theta\,\cos\phi\,\dot{\phi} & -\sin\theta\,\dot{\theta} \\ -\sin\theta\,\cos\phi\,\dot{\theta} - \cos\theta\,\sin\phi\,\dot{\phi} & -\sin\theta\,\sin\phi\,\dot{\theta} + \cos\theta\,\cos\phi\,\dot{\phi} & -\cos\theta\,\dot{\theta} \\ -\cos\phi\,\dot{\phi} & -\sin\phi\,\dot{\phi} & 0 \end{pmatrix}$$

SPHERICAL POLAR COORDINATES

Velocity and acceleration

$$\begin{pmatrix} \dot{\hat{r}} \\ \dot{\hat{\theta}} \\ \dot{\hat{\phi}} \end{pmatrix} = \dot{\boldsymbol{M}}\,\boldsymbol{M}^T \begin{pmatrix} \hat{r} \\ \hat{\theta} \\ \hat{\phi} \end{pmatrix}$$

$$= \begin{pmatrix} 0 & \dot{\theta} & \sin\theta\,\dot{\phi} \\ -\dot{\theta} & 0 & \cos\theta\,\dot{\phi} \\ -\sin\theta\,\dot{\phi} & -\cos\theta\,\dot{\phi} & 0 \end{pmatrix} \begin{pmatrix} \hat{r} \\ \hat{\theta} \\ \hat{\phi} \end{pmatrix}$$

Why are the diagonal terms zero? Can you see the physical implication?

Notice that the matrix connecting the two vectors is anti-symmetric.

This was also the case in the plane polar co-ordinates. But we didn't mention it there.

The problem for velocity and acceleration components can now be completed...

SPHERICAL POLAR COORDINATES

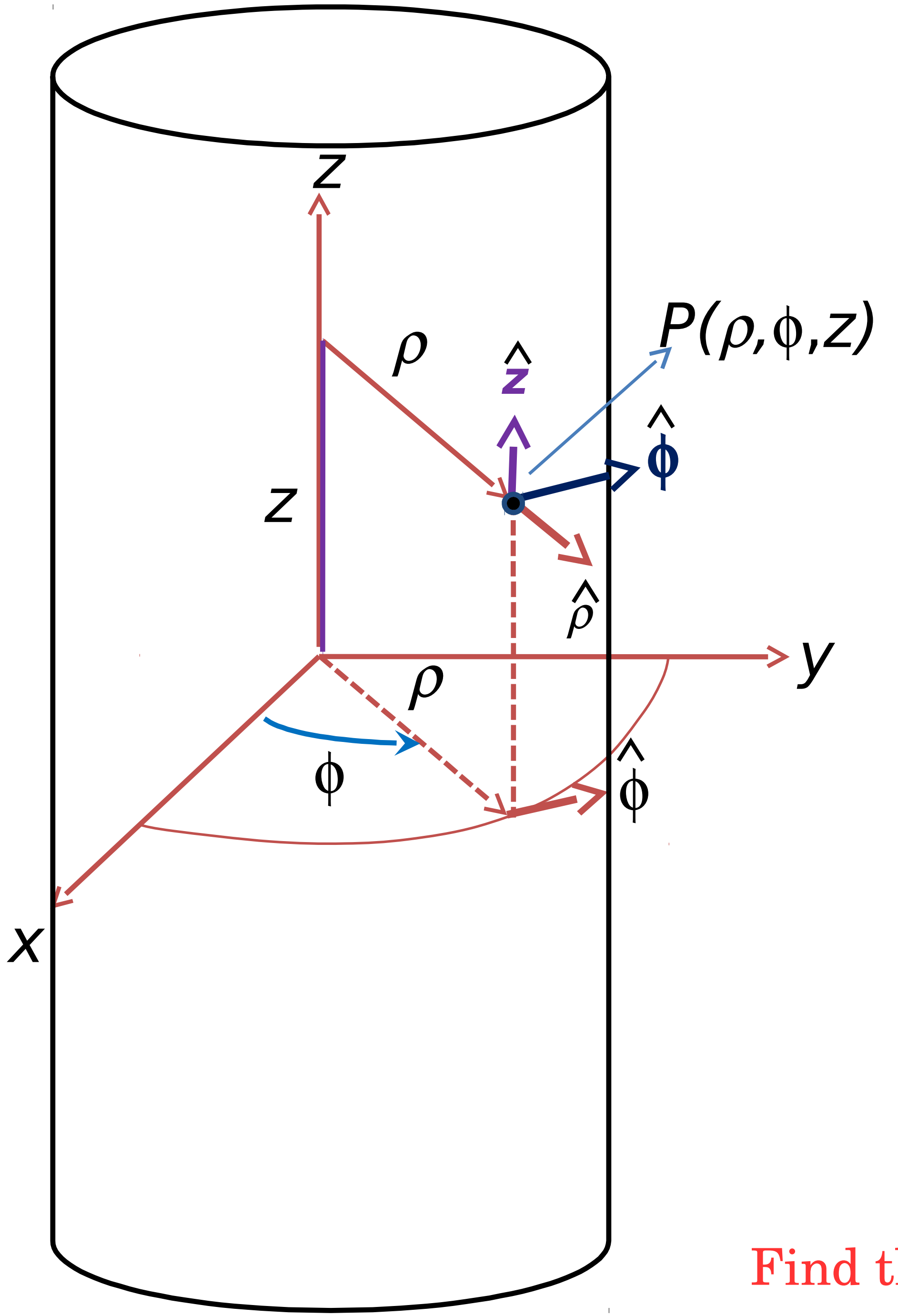
$$\vec{r} = r\,\hat{r}$$

$$\vec{v} = \dot{r}\,\hat{r} + r\,\dot{\hat{r}} = \dot{r}\,\hat{r} + r\,\dot{\theta}\,\hat{\theta} + r\sin\theta\,\dot{\phi}\,\hat{\phi}$$

$$\vec{a} = \hat{r}\left(\ddot{r} - r\,\dot{\theta}^2 - r\,\dot{\phi}^2\sin^2\theta\right) + \hat{\theta}\left(r\,\ddot{\theta} + 2\,\dot{r}\,\dot{\theta} - r\,\dot{\phi}^2\sin\theta\cos\theta\right) +$$
$$\hat{\phi}\left(r\,\ddot{\phi}\sin\theta + 2\,\dot{r}\,\dot{\phi}\sin\theta + 2r\,\dot{\theta}\,\dot{\phi}\cos\theta\right)$$

We now have all ingredients to solve dynamical coordinates in this system.

CYLINDRICAL POLAR COORDINATES



$$x = \rho\cos\phi \qquad 0 \leqslant \rho \leqslant \infty$$
$$y = \rho\sin\phi \qquad 0 \leqslant \phi \leqslant 2\pi$$
$$z = z \qquad 0 \leqslant z \leqslant \infty$$

$$\begin{pmatrix} \hat{r} \\ \hat{\phi} \\ \hat{z} \end{pmatrix} = \begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \hat{x} \\ \hat{y} \\ \hat{z} \end{pmatrix}$$

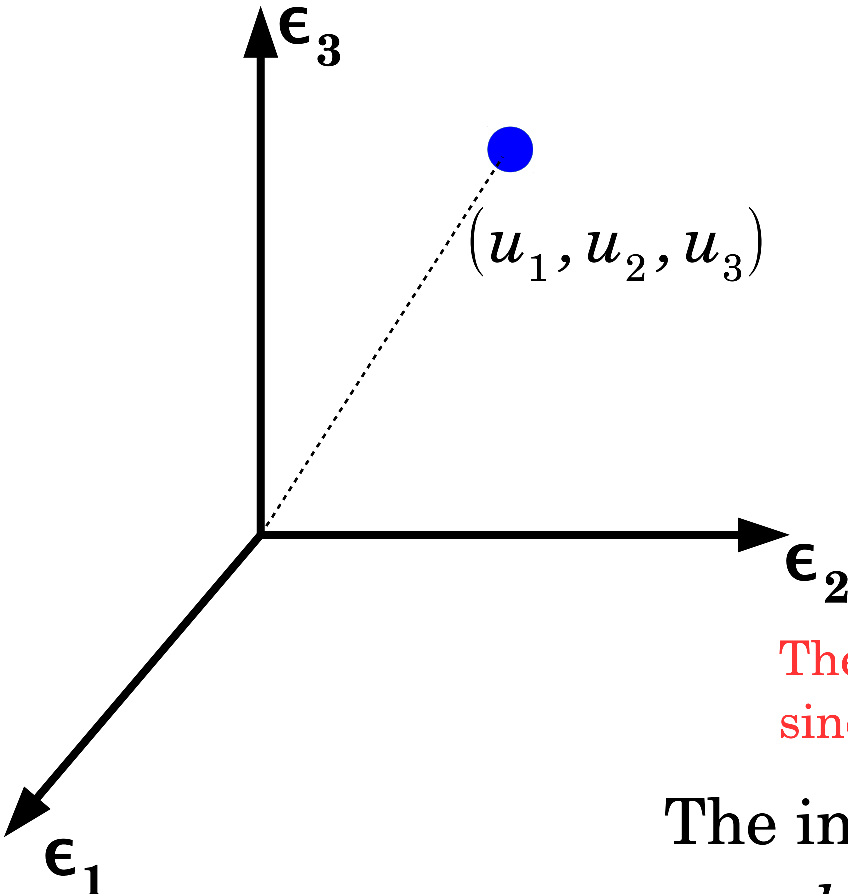
$$\delta\vec{l} = \delta\rho\,\hat{\rho} + \rho\,\delta\phi\,\hat{\phi} + \delta z\,\hat{z}$$
$$ds^2 = d\rho^2 + \rho^2 d\phi^2 + dz^2$$

$$dV = \rho\, d\rho\, d\phi\, dz$$

$$\nabla T = \hat{\rho}\frac{\partial T}{\partial \rho} + \hat{\phi}\frac{1}{\rho}\frac{\partial T}{\partial \phi} + \hat{z}\frac{\partial T}{\partial z}$$

Find the velocity and acceleration components

GENERAL ORTHOGONAL CURVILINEAR COORDINATES



Identify a point in space by its three coordinates (u_1, u_2, u_3)

If the system is orthogonal,

$$\boldsymbol{\epsilon}_i \cdot \boldsymbol{\epsilon}_j = 0 \quad \forall \quad i \neq j$$

$$\boldsymbol{\epsilon}_i \cdot \boldsymbol{\epsilon}_j = \delta_{ij}$$

The unit vectors in general are functions of position, since their direction can vary from point to point.

The infinitesimal displacement vector is

$$d\boldsymbol{l} = \boldsymbol{\epsilon}_1 h_1 du_1 + \boldsymbol{\epsilon}_2 h_2 du_2 + \boldsymbol{\epsilon}_3 h_3 du_3$$

$$h_i = \sqrt{\left(\frac{\partial x}{\partial u_i}\right)^2 + \left(\frac{\partial y}{\partial u_i}\right)^2 + \left(\frac{\partial z}{\partial u_i}\right)^2}$$

$$= \sum_i \boldsymbol{\epsilon}_i h_i du_i = \boldsymbol{\epsilon}_i h_i du_i$$

REPEATED INDEX IMPLIES SUMMATION

$$ds^2 = h_i^2 du_i^2$$

$$dV = \prod_i h_i du_i$$

GENERAL ORTHOGONAL CURVILINEAR COORDINATES

System	u_1	u_2	u_3	h_1	h_2	h_3
Cartesian	x	y	z	1	1	1
Spherical	r	θ	ϕ	1	r	$r\sin\theta$
Cylindrical	ρ	ϕ	z	1	r	1

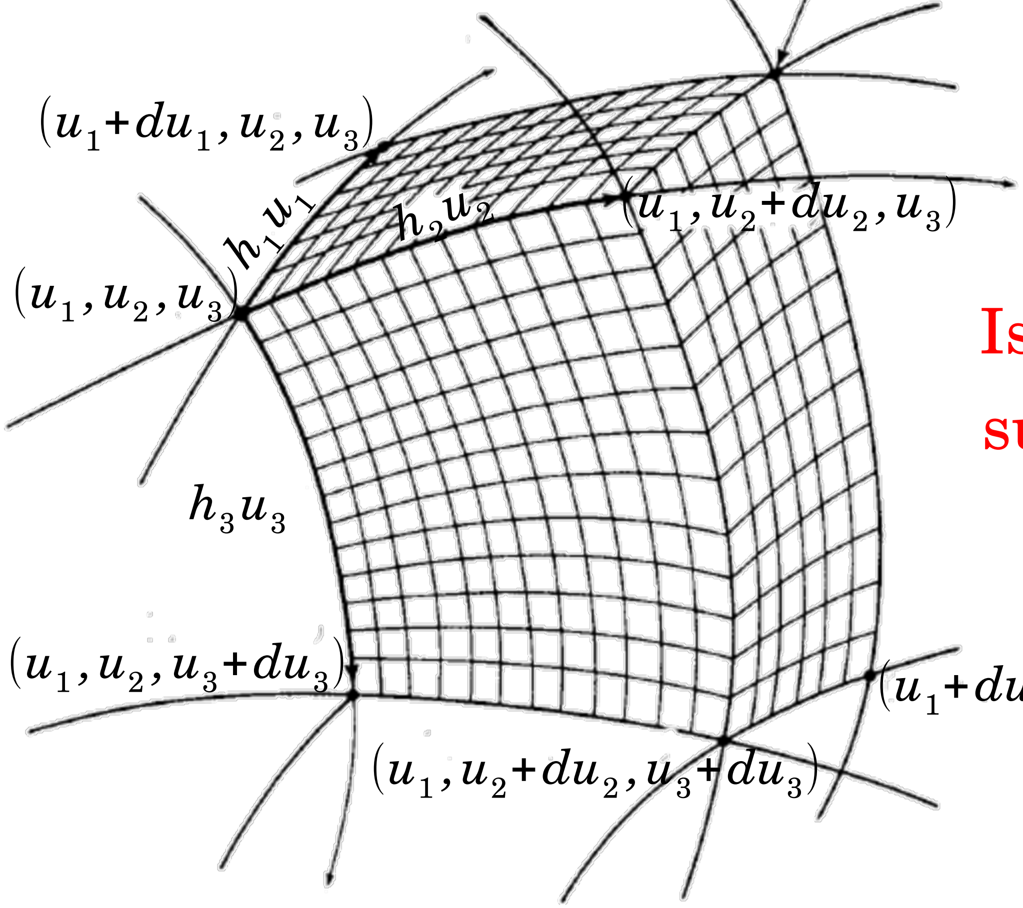
Gradient of a scalar field

$$dT = \frac{\partial T}{\partial u_i}\delta u_i = \nabla T . d\boldsymbol{l} = [(\nabla T)_i \boldsymbol{\epsilon_i}](h_i \delta u_i \boldsymbol{\epsilon_i})$$

$$\nabla T = \frac{1}{h_1}\frac{\partial T}{\partial u_1}\boldsymbol{\epsilon_1} + \frac{1}{h_2}\frac{\partial T}{\partial u_2}\boldsymbol{\epsilon_2} + \frac{1}{h_3}\frac{\partial T}{\partial u_3}\boldsymbol{\epsilon_3} = \frac{1}{h_i}\frac{\partial T}{\partial u_i}\boldsymbol{\epsilon_i}$$

Does it match with the gradient for the three coordinate systems derived previously?

DIVERGENCE IN CURVILINEAR COORDINATES



Consider a vector $\vec{\boldsymbol{F}}$.

Is it possible to have a function $\boldsymbol{X}(\boldsymbol{F})$ such that $\iiint X(\boldsymbol{F})dV = \oiint \boldsymbol{F}.d\boldsymbol{S}$?

Consider the inifinitesimal volume generated by starting at the point (u_1, u_2, u_3) and increasing each of the coordinates in succession by an inifinitesimal amount.

DIVERGENCE IN CURVILINEAR COORDINATES

Volume element $dV = (h_1 h_2 h_3) du_1 du_2 du_3$

For the front surface $d\boldsymbol{S} = -(h_2 h_3) du_2 du_3 \boldsymbol{\epsilon_1}$

Flux through the front surface $[\boldsymbol{F} . d\boldsymbol{S}]_F = -(h_2 h_3 F_1) du_2 du_3$

Flux through the back surface $[\boldsymbol{F} . d\boldsymbol{S}]_B = (h_2 h_3 F_1)\Big|_{u_1 + du_1} du_2 du_3$

$$= (h_2 h_3 F_1) du_2 du_3 + \frac{\partial}{\partial u_1}(h_2 h_3 F_1) du_1 du_2 du_3$$

Front and back together give a contribution

$$[\boldsymbol{F} . d\boldsymbol{S}]_F + [\boldsymbol{F} . d\boldsymbol{S}]_B = \frac{\partial}{\partial u_1}(h_2 h_3 F_1) du_1 du_2 du_3 = \frac{1}{h_1 h_2 h_3} \frac{\partial}{\partial u_1}(h_2 h_3 F_1) dV$$

Similarly, the left and right sides together give a contribution

$$[\boldsymbol{F} . d\boldsymbol{S}]_L + [\boldsymbol{F} . d\boldsymbol{S}]_R = \frac{1}{h_1 h_2 h_3} \frac{\partial}{\partial u_2}(h_1 h_3 F_2) dV$$

and $[\boldsymbol{F} . d\boldsymbol{S}]_T + [\boldsymbol{F} . d\boldsymbol{S}]_B = \dfrac{1}{h_1 h_2 h_3} \dfrac{\partial}{\partial u_3}(h_1 h_2 F_3) dV$

DIVERGENCE IN CURVILINEAR COORDINATES

Adding the flux from the individual sides, the total flux is

$$\boldsymbol{F}.d\boldsymbol{S}=\frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial u_1}(h_2 h_3 F_1)+\frac{\partial}{\partial u_2}(h_1 h_3 F_2)+\frac{\partial}{\partial u_3}(h_1 h_2 F_3)\right]dV$$

The coefficient of dV defines the $\nabla.\boldsymbol{F}$ in curvilinear coordinates

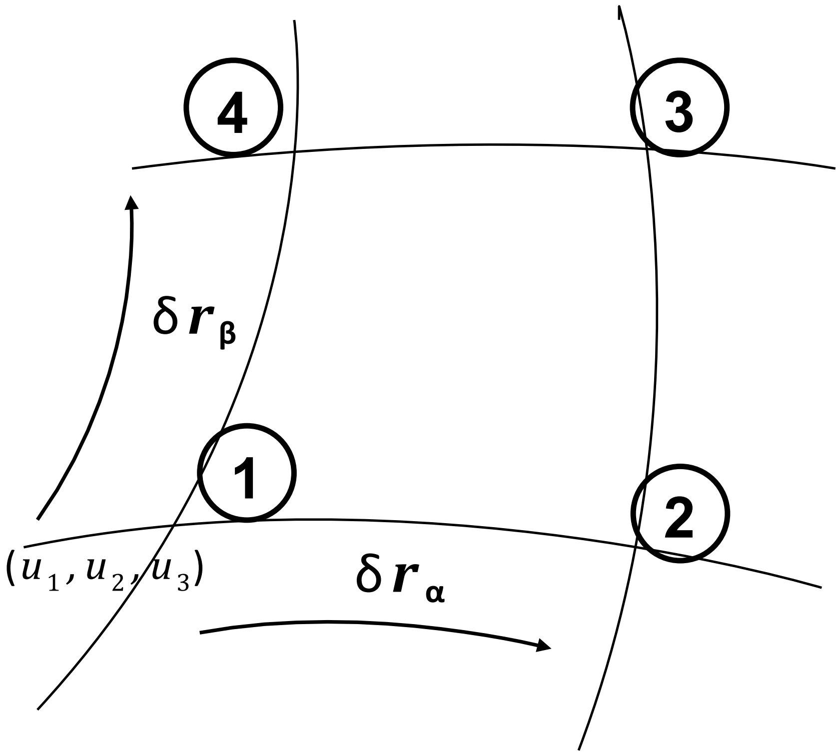
$$\nabla.\boldsymbol{F}=\frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial u_1}(h_2 h_3 F_1)+\frac{\partial}{\partial u_2}(h_1 h_3 F_2)+\frac{\partial}{\partial u_3}(h_1 h_2 F_3)\right]$$

Cartesian: $\nabla.\boldsymbol{F}=\frac{\partial F_x}{\partial x}+\frac{\partial F_y}{\partial y}+\frac{\partial F_z}{\partial z}$

Spherical: $\nabla.\boldsymbol{F}=\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 F_r)+\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta F_\theta)+\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}(F_\phi)$

Cylindrical: $\nabla.\boldsymbol{F}=\frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho F_\rho)+\frac{1}{\rho}\frac{\partial F_\theta}{\partial\theta}+\frac{\partial F_z}{\partial\phi}$

CURL OF VECTOR FIELDS



Consider a vector field $\boldsymbol{F}$.
Given two arbitrary displacements $\delta \boldsymbol{r}_\alpha$ and $\delta \boldsymbol{r}_\beta$ around the point (u_1, u_2, u_3),

Can we define a function $\boldsymbol{X}(F)$ such that

$$\iint \boldsymbol{X}(F).d\boldsymbol{S} = \oint \boldsymbol{F}.d\boldsymbol{l}$$

$$\delta \boldsymbol{r}^\alpha = \boldsymbol{\epsilon}_1 h_1 \delta u_1^\alpha + \boldsymbol{\epsilon}_2 h_2 \delta u_2^\alpha + \boldsymbol{\epsilon}_3 h_3 \delta u_3^\alpha$$

$$\delta \boldsymbol{r}^\beta = \boldsymbol{\epsilon}_1 h_1 \delta u_1^\beta + \boldsymbol{\epsilon}_2 h_2 \delta u_2^\beta + \boldsymbol{\epsilon}_3 h_3 \delta u_3^\beta$$

CURL OF VECTOR FIELDS

What is the area element?

$$d\boldsymbol{S} = \delta \boldsymbol{r}^{\alpha} \times \delta \boldsymbol{r}^{\beta} = \begin{vmatrix} \boldsymbol{\epsilon_1} & \boldsymbol{\epsilon_2} & \boldsymbol{\epsilon_3} \\ h_1 \delta u_1^{\alpha} & h_2 \delta u_2^{\alpha} & h_3 \delta u_3^{\alpha} \\ h_1 \delta u_1^{\beta} & h_2 \delta u_2^{\beta} & h_3 \delta u_3^{\beta} \end{vmatrix}$$

What is the flux of the vector ***X(F)*** through this area?

$$\begin{aligned} X(\boldsymbol{F}).d\boldsymbol{S} = \; & X_1 h_2 h_3 [\delta u_2^{\alpha} \delta u_3^{\beta} - \delta u_3^{\alpha} \delta u_2^{\beta}] \\ & - X_2 h_1 h_3 [\delta u_1^{\alpha} \delta u_3^{\beta} - \delta u_3^{\alpha} \delta u_1^{\beta}] \\ & + X_3 h_1 h_2 [\delta u_1^{\alpha} \delta u_2^{\beta} - \delta u_2^{\alpha} \delta u_1^{\beta}] \end{aligned}$$

CURL OF VECTOR FIELDS

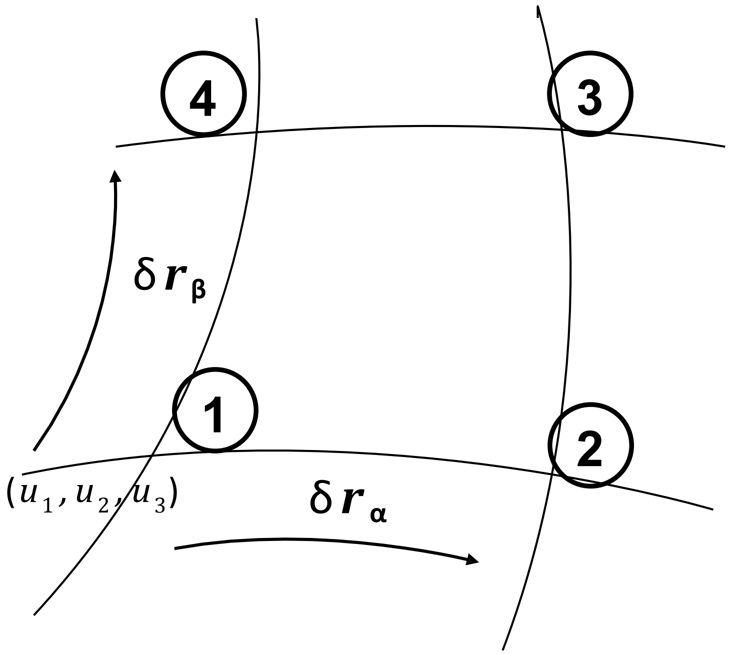
Consider the pair of paths $(1 \to 2)$ and $(3 \to 4)$

$$\boldsymbol{F}.\delta \boldsymbol{l}|_{1\to 2} = F_1 h_1 \delta u_1^\alpha + F_2 h_2 \delta u_2^\alpha + F_3 h_3 \delta u_3^\alpha$$

$$\boldsymbol{F}.\delta \boldsymbol{l}|_{3\to 4} = \left[F_i h_i + (\nabla F_i h_i).\delta \boldsymbol{r}^\beta\right](-\delta u_i^\alpha) \qquad (i=1,2,3)$$

Write contributions from $\boldsymbol{F}.\delta \boldsymbol{l}|_{2\to 3}$ & $\boldsymbol{F}.\delta \boldsymbol{l}|_{4\to 1}$ similarly

Full path gives:



$$\left(\nabla \boldsymbol{F}.\delta r^\beta\right).\delta r^\alpha - \left(\nabla \boldsymbol{F}.\delta r^\alpha\right).\delta r^\beta$$

$$= \sum_{k,i}\left[\frac{1}{h_k}\frac{\partial F_i h_i}{\partial u_k}\delta u_i^\beta\right] h_k \delta u_k^\alpha$$

$$-\sum_{k,i}\left[\frac{1}{h_k}\frac{\partial F_i h_i}{\partial u_k}\delta u_i^\alpha\right] h_k \delta u_k^\beta$$

$$= \sum_{k,i}\left[\frac{\partial F_i h_i}{\partial u_k} - \frac{\partial F_k h_k}{\partial u_i}\right]\delta u_i^\beta \delta u_k^\alpha$$

CURL OF VECTOR FIELDS

Comparing coefficients,

$$X_1 h_2 h_3 = \left[\frac{\partial F_3 h_3}{\partial u_2} - \frac{\partial F_2 h_2}{\partial u_3} \right]$$

The function X(F) is then,

So $\quad X(\boldsymbol{F}) = \dfrac{1}{h_1 h_2 h_3} \begin{vmatrix} h_1 \boldsymbol{\epsilon_1} & h_2 \boldsymbol{\epsilon_2} & h_3 \boldsymbol{\epsilon_3} \\ \dfrac{\partial}{\partial u_1} & \dfrac{\partial}{\partial u_2} & \dfrac{\partial}{\partial u_3} \\ h_1 F_1 & h_2 F_2 & h_3 F_3 \end{vmatrix} \equiv \begin{cases} \boldsymbol{\nabla} \times \boldsymbol{F} \\ curl\, \boldsymbol{F} \\ rot\, \boldsymbol{F} \end{cases}$

We have $\iint \boldsymbol{\nabla} \times \boldsymbol{F} . d\boldsymbol{S} = \oint \boldsymbol{F} . d\boldsymbol{l} \qquad (Stoke's\ theorem)$

GRADIENT, DIVERGENCE, CURL – GENERALIZED COORDINATES

$$\nabla T = \frac{1}{h_1}\frac{\partial T}{\partial u_1}\boldsymbol{\epsilon_1} + \frac{1}{h_2}\frac{\partial T}{\partial u_2}\boldsymbol{\epsilon_2} + \frac{1}{h_3}\frac{\partial T}{\partial u_3}\boldsymbol{\epsilon_3} \;=\; \frac{1}{h_i}\frac{\partial T}{\partial u_i}\boldsymbol{\epsilon_i}$$

$$\nabla . \boldsymbol{F} = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial u_1}(h_2 h_3 F_1) + \frac{\partial}{\partial u_2}(h_1 h_3 F_2) + \frac{\partial}{\partial u_3}(h_1 h_2 F_3)\right]$$

$$\begin{aligned}\nabla \times \boldsymbol{F} = \frac{1}{h_1 h_2 h_3}\Bigg[& h_1 \boldsymbol{\epsilon_1}\left\{\frac{\partial}{\partial u_2}(h_3 F_3) - \frac{\partial}{\partial u_3}(h_2 F_2)\right\} \\ & + h_2 \boldsymbol{\epsilon_2}\left\{\frac{\partial}{\partial u_3}(h_1 F_1) - \frac{\partial}{\partial u_1}(h_3 F_3)\right\} \\ & + h_3 \boldsymbol{\epsilon_3}\left\{\frac{\partial}{\partial u_1}(h_2 F_2) - \frac{\partial}{\partial u_2}(h_1 F_1)\right\}\Bigg]\end{aligned}$$

IS EVERYTHING CONSISTENT? - GRADIENT

$$f = \rho^3 \sin\phi$$

$$\nabla f = 3\rho^2 \sin\phi \, \hat{\rho} + \rho^2 \cos\phi \, \hat{\phi}$$

$$f = x^2 y + y^3$$

$$\nabla f = 2xy\,\hat{x} + (3y^2 + x^2)\,\hat{y}$$

$$\begin{aligned}\nabla f &= 3\rho^2 \sin\phi(\cos\phi\,\hat{x} + \sin\phi\,\hat{y}) + \rho^2\cos\phi(-\sin\phi\,\hat{x} + \cos\phi\,\hat{y}) \\ &= [2\rho^2 \sin\phi\cos\phi]\,\hat{x} + [3\rho^2\sin^2\phi + \rho^2\cos^2\phi]\,\hat{y} \\ &= 2xy\,\hat{x} + (3y^2 + x^2)\,\hat{y}\end{aligned}$$

$$f = \frac{1}{r}$$

$$\nabla f = -\frac{1}{r^2}\hat{r}$$

$$f = \frac{1}{\sqrt{x^2 + y^2 + z^2}}$$

$$\nabla f = -\frac{1}{r^2}(\sin\theta\cos\phi\,\hat{x} + \sin\theta\sin\phi\,\hat{y} + \cos\theta\,\hat{z}) = -\frac{1}{r^2}\left(\frac{x}{r}\hat{x} + \frac{y}{r}\hat{y} + \frac{z}{r}\hat{z}\right)$$

$$= -\left(\frac{x}{(x^2+y^2+z^2)^{3/2}}\hat{x} + \frac{y}{(x^2+y^2+z^2)^{3/2}}\hat{y} + \frac{z}{(x^2+y^2+z^2)^{3/2}}\hat{z}\right)$$

IS EVERYTHING CONSISTENT? - DIVERGENCE

$$\vec{F} = \rho^3 \hat{\rho} + \rho z \hat{\phi} + \rho z \sin\phi \hat{z}$$

$$\nabla . \vec{F} = \frac{1}{\rho}\left\{\frac{\partial}{\partial \rho}(\rho F_\rho) + \frac{\partial}{\partial \phi}(F_\phi) + \frac{\partial}{\partial z}(\rho F_z)\right\} = 4\rho^2 + \rho \sin\phi$$

$$\vec{F} = (x^3 + x y^2 - y z)\hat{x} + (x^2 y + y^3 + xz)\hat{y} + y z \hat{z}$$

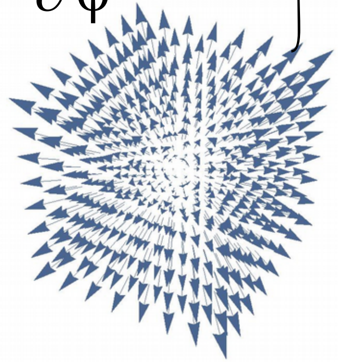
$$\nabla . \vec{F} = 4(x^2 + y^2) + y$$

$$\vec{F} = r \hat{\boldsymbol{r}}$$

$$\nabla . \vec{F} = \frac{1}{r^2 \sin\theta}\left\{\frac{\partial}{\partial r}(r^2 \sin\theta F_r) + \frac{\partial}{\partial \theta}(r \sin\theta F_\theta) + \frac{\partial}{\partial \phi}(r F_\phi)\right\} = 3$$

$$\vec{F} = x \hat{x} + y \hat{y} + z \hat{z}$$

$$\nabla . \vec{F} = 3$$



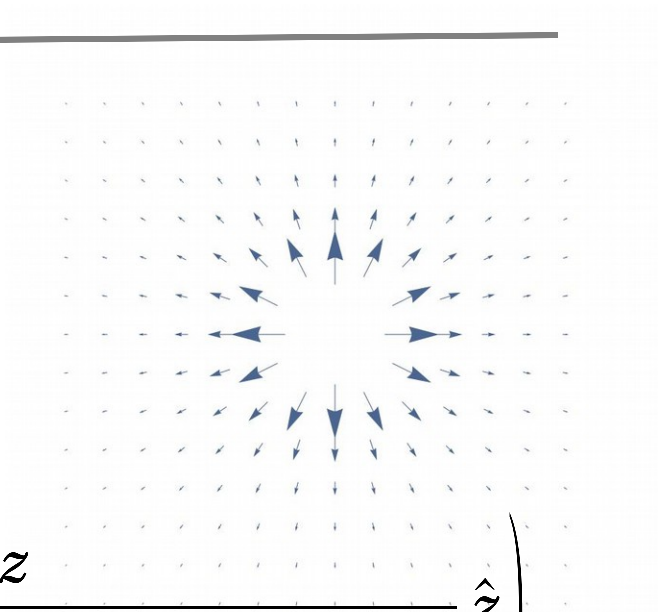
IS EVERYTHING CONSISTENT? - CURL

$$\boldsymbol{F}=\rho^3\hat{\rho}+\rho z\hat{\phi}+\rho z\sin\phi\,\hat{z}$$

$$\nabla\times\boldsymbol{F} = \frac{1}{\rho}\begin{vmatrix} \hat{\boldsymbol{\rho}} & \rho\hat{\boldsymbol{\phi}} & \hat{\boldsymbol{z}} \\ \dfrac{\partial}{\partial\rho} & \dfrac{\partial}{\partial\phi} & \dfrac{\partial}{\partial z} \\ \rho^3 & \rho^2 z & \rho z\sin\phi \end{vmatrix} = (z\cos\phi-\rho)\hat{\rho}+z\sin\phi\,\hat{\phi}+2z\hat{z}$$

$$\boldsymbol{F}=(x^3+xy^2-yz)\hat{x}+(x^2y+y^3+xz)\hat{y}+yz\hat{z}$$

$$\boldsymbol{F}=r^k\hat{\boldsymbol{r}}$$

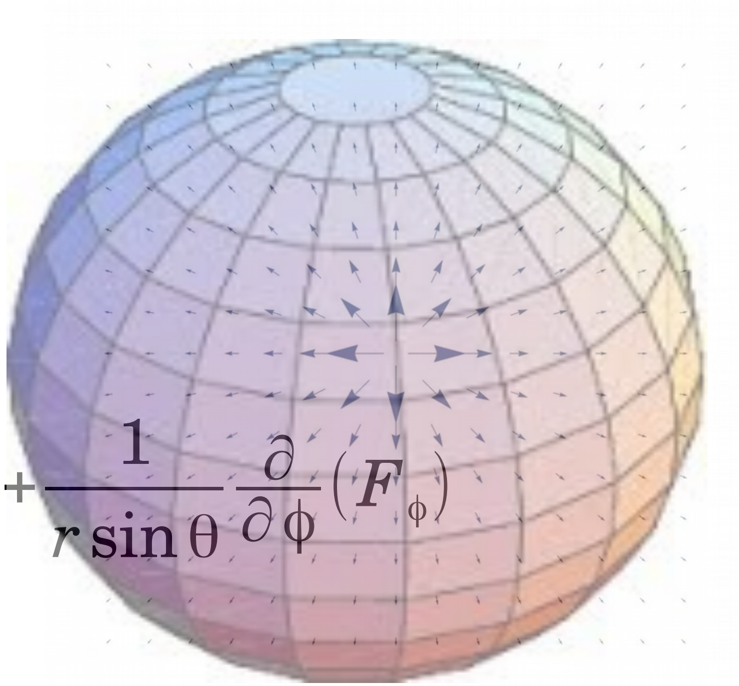


$$\nabla\times\boldsymbol{F} = \frac{1}{r^2\sin\theta}\begin{vmatrix} \hat{\boldsymbol{r}} & r\hat{\boldsymbol{\theta}} & r\sin\theta\hat{\boldsymbol{\phi}} \\ \dfrac{\partial}{\partial r} & \dfrac{\partial}{\partial\theta} & \dfrac{\partial}{\partial\phi} \\ r^k & r.0 & r\sin\theta.0 \end{vmatrix} = 0$$

$$\boldsymbol{F}=\left(\frac{x}{(x^2+y^2+z^2)^{(k+1)/2}}\hat{x}+\frac{y}{(x^2+y^2+z^2)^{(k+1)/2}}\hat{y}+\frac{z}{(x^2+y^2+z^2)^{(k+1)/2}}\hat{z}\right)$$

DIVERGENCE OF 1/R²

Consider the vector function $\boldsymbol{v} = \frac{\hat{\boldsymbol{r}}}{r^2}$



What is $\nabla . \boldsymbol{v}$?

$$\nabla . \boldsymbol{F} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 F_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta F_\theta) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}(F_\phi)$$

$$\nabla . \boldsymbol{v} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{1}{r^2}\right) = 0$$

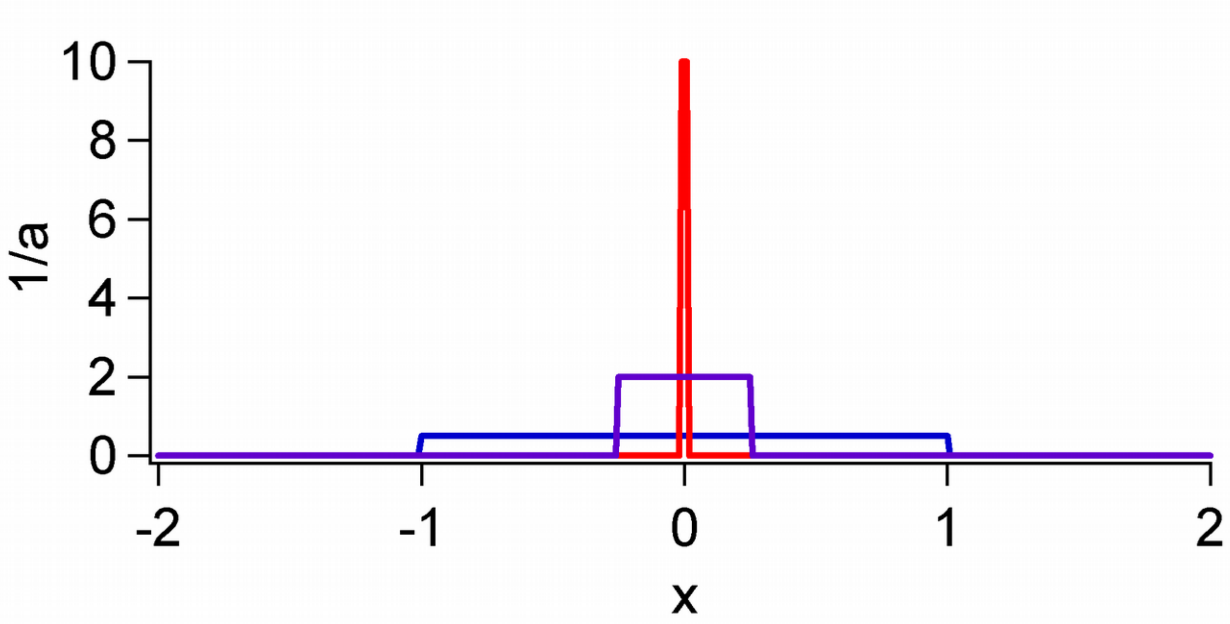
$$\iiint \nabla . \boldsymbol{v}\, dV = 0$$

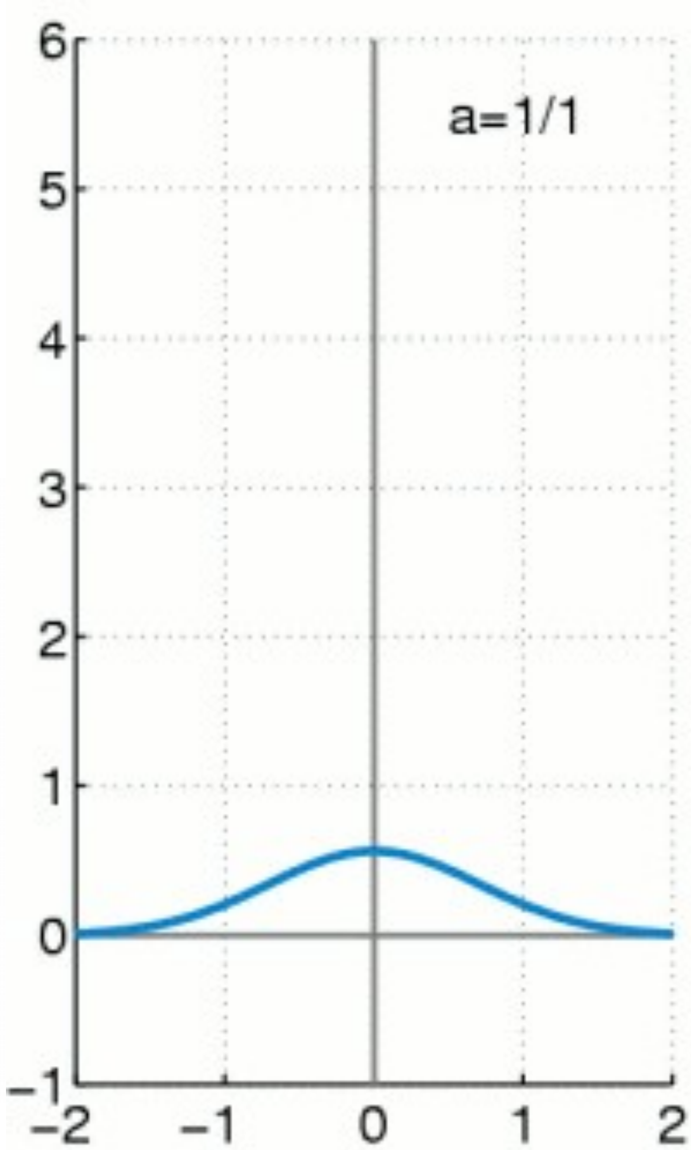
Something is wrong!

$$\oint \boldsymbol{v} . d\boldsymbol{a} = \iint \left(\frac{1}{R^2}\hat{\boldsymbol{r}}\right) . (R^2 \sin\theta\, d\theta\, d\phi\, \hat{\boldsymbol{r}}) = \left(\int_0^\pi \sin\theta\, d\theta\right)\left(\int_0^{2\pi} d\phi\right) = 4\pi$$

THE DIRAC DELTA FUNCTION

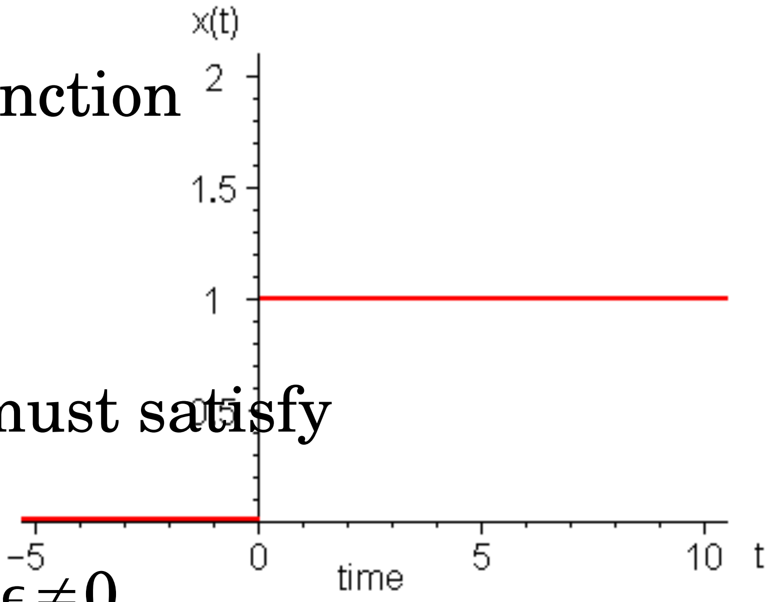
$$\int_{-\infty}^{\infty} \delta(x - x_0) f(x)\, dx = f(x_0)$$





THE DIRAC DELTA FUNCTION

Consider a simpler example: the step function



$$x(t) = \begin{cases} 0 & t<0 \\ 1 & t>0 \end{cases}$$

$$\frac{dx}{dt} = ?$$ looks like zero everywhere but must satisfy

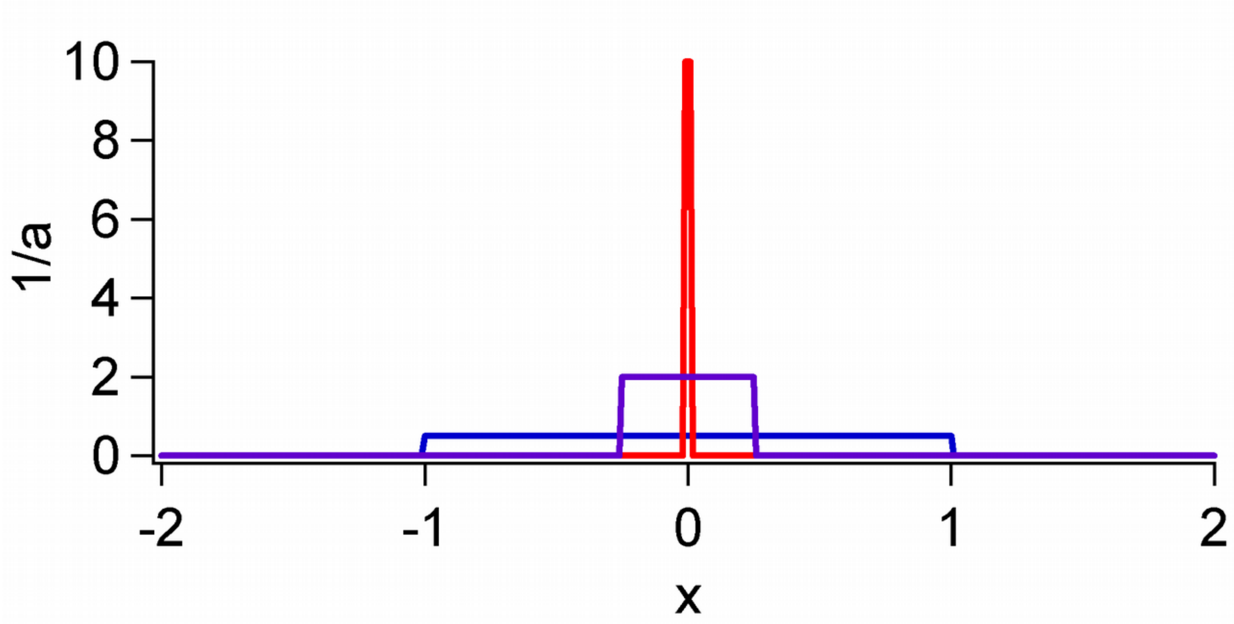
$$\int_{0-|\epsilon|}^{0+|\epsilon|} \left(\frac{dx}{dt}\right) dt = x(|\epsilon|) - x(-|\epsilon|) = 1 \quad \forall \quad \epsilon \neq 0$$

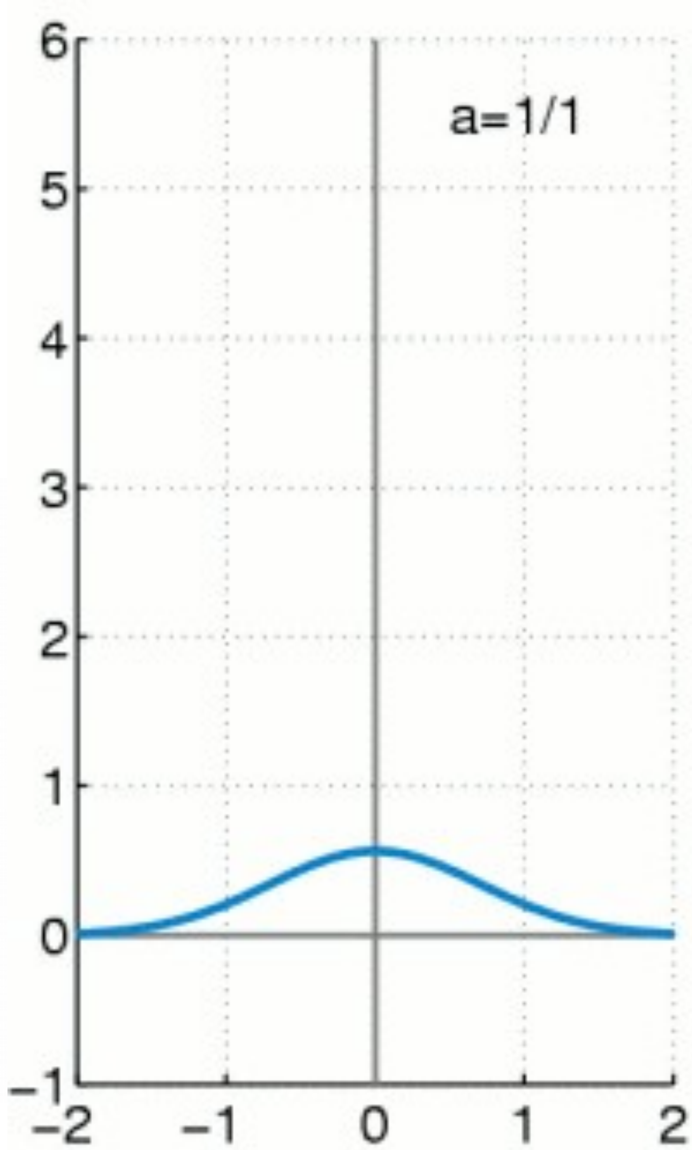
Such integrable singularities are treated by ***defining*** the δ function

$$\int_a^b \delta(x-x_0) f(x)\, dx = \begin{cases} f(x_0), & if\ x_0\ is\ within\ the\ limits \\ 0 & otherwise \end{cases}$$

The Delta function is not a proper function, but rather the limit of a sequence of functions. It’s called a generalized function or distribution, and is properly defined only within an integral.

$$\int_{-\infty}^{\infty} \delta(x - x_0) f(x)\, dx = f(x_0)$$





THE 3D DIRAC DELTA FUNCTION

$$\delta(\boldsymbol{r}) = \delta(x)\,\delta(y)\,\delta(z)$$

$$\int_{\text{all space}} \delta(\boldsymbol{r})\, dV = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \delta(x)\,\delta(y)\,\delta(z)\, dx\, dy\, dz = 1$$

$$\int_{\text{all space}} f(\boldsymbol{r})\,\delta(\boldsymbol{r}-\boldsymbol{a})\, dV = f(\boldsymbol{a})$$

$$\iiint \nabla \cdot \frac{\hat{\boldsymbol{r}}}{\boldsymbol{r}^2}\, dV = 4\pi$$

$$\nabla \cdot \frac{\hat{\boldsymbol{r}}}{\boldsymbol{r}^2} = 4\pi\,\delta^3(\boldsymbol{r})$$

Prove:

$$\delta(-x) = \delta(x)$$

$$\delta(\alpha x) = \frac{\delta(x)}{|\alpha|}$$

$$x \frac{d}{dx}(\delta(x)) = -\delta(x)$$

UNIQUE SPECIFICATION OF A VECTOR FIELD

If we specify the curl of a function $\nabla \times \boldsymbol{F} = \boldsymbol{C}$,
can we determine the function $\boldsymbol{F}$?

NO. $\boldsymbol{F}' = \boldsymbol{F} + \nabla \phi$ will give same result!

If we specify the divergence of a function $\nabla . \boldsymbol{F} = D$,
can we determine the function $\boldsymbol{F}$?

NO. $\boldsymbol{F}' = \boldsymbol{F} + \nabla \times \boldsymbol{A}$ will give same result!

If we specify the divergence **AND** the curl of a function

$$\nabla \times \boldsymbol{F} = \boldsymbol{C} \quad \textbf{AND} \quad \nabla . \boldsymbol{F} = D,$$

can we determine the function $\boldsymbol{F}$?

YES! (Provided the boundary conditions are specified)

HELMHOLTZ'S THEOREM

HELMHOLTZ'S THEOREM

If $\quad \nabla \times \boldsymbol{F} = \boldsymbol{C}$ **AND** $\nabla . \boldsymbol{F} = D$,

Note that, necessarily $\nabla . \boldsymbol{C} = 0$

We claim $\quad \boldsymbol{F} = -\nabla U + \nabla \times \boldsymbol{W}$

where $\quad U(\boldsymbol{r}) = \dfrac{1}{4\pi} \displaystyle\int \frac{D(\boldsymbol{r}')}{|\boldsymbol{r} - \boldsymbol{r}'|} dV'$

and $\quad \boldsymbol{W}(\boldsymbol{r}) = \dfrac{1}{4\pi} \displaystyle\int \frac{\boldsymbol{C}(\boldsymbol{r}')}{|\boldsymbol{r} - \boldsymbol{r}'|} dV'$

$$\nabla . \boldsymbol{F} = -\nabla^2 U = -\frac{1}{4\pi} \int D(\boldsymbol{r}') \nabla^2 \left(\frac{1}{|\boldsymbol{r} - \boldsymbol{r}'|} \right) dV' = \int D(\boldsymbol{r}') \delta^3(\boldsymbol{r} - \boldsymbol{r}') dV'$$

$$= D(\boldsymbol{r})$$

HELMHOLTZ'S THEOREM

$$\nabla \times \boldsymbol{F} = \nabla \times (\nabla \times \boldsymbol{W}) = -\nabla^2 \boldsymbol{W} + \nabla (\nabla . \boldsymbol{W})$$

$$-\nabla^2 \boldsymbol{W} = -\frac{1}{4\pi} \int \boldsymbol{C}(\boldsymbol{r}') \nabla^2 \left(\frac{1}{|\boldsymbol{r} - \boldsymbol{r}'|} \right) dV' = \int \boldsymbol{C}(\boldsymbol{r}') \delta^3 (\boldsymbol{r} - \boldsymbol{r}') dV' = \boldsymbol{C}(\boldsymbol{r})$$

$$4\pi \nabla . \boldsymbol{W} = \int \boldsymbol{C}(\boldsymbol{r}') . \nabla \left(\frac{1}{|\boldsymbol{r} - \boldsymbol{r}'|} \right) dV' = -\int \boldsymbol{C}(\boldsymbol{r}') . \nabla' \left(\frac{1}{|\boldsymbol{r} - \boldsymbol{r}'|} \right) dV'$$

$$= \int \frac{1}{|\boldsymbol{r} - \boldsymbol{r}'|} \nabla' . \boldsymbol{C}(\boldsymbol{r}') dV - \oint \frac{1}{|\boldsymbol{r} - \boldsymbol{r}'|} \boldsymbol{C} . d\boldsymbol{a}$$

$$= 0 \qquad (\text{if C goes to zero sufficiently rapidly})$$

$$\nabla . (f \vec{A}) = f (\nabla . \vec{A}) + \vec{A} . (\nabla f)$$

How rapidly? **C** and D must go to zero more rapidly than $1/r^2$

HELMHOLTZ'S THEOREM

Is this solution unique?

YES, as long as the vector field $\boldsymbol{F}(\boldsymbol{r})$ itself goes to zero at infinity

If the divergence D(**r**) and the curl **C(r)** of a vector function **F(r)** are specified, and if they both go to zero faster than $1/r^2$ as r goes to infinity, and if **F(r)** itself goes to zero as r goes to infinity, then **F(r)** is uniquely given by

$$\boldsymbol{F} = -\nabla U + \nabla \times \boldsymbol{W}$$

MULTIPLE VECTOR PRODUCTS – ε δ (LEVI-CIVITA)

Write the dot product as $\boldsymbol{A}.\boldsymbol{B} = \delta_{ij} A_i B_j$ $\quad$ *where* $\delta_{ij} = \begin{cases} 1 & if \quad i = j \\ 0 & if \quad i \neq j \end{cases}$

Write the cross product as

$$\boldsymbol{A} \times \boldsymbol{B} \mid_i = \epsilon_{ijk} A_j B_k \quad where \quad \epsilon_{ijk} = \begin{cases} 1 \;\; if \;\; ijk \;\; is \;\; an \;\; even \;\; permutation \;\; of \;\; 123 \\ -1 \;\; if \;\; ijk \;\; is \;\; an \;\; odd \;\; permutation \;\; of \;\; 123 \\ 0 \;\; otherwise \end{cases}$$

Convince yourself that $\epsilon_{ijk} = \boldsymbol{\epsilon_i} . \boldsymbol{\epsilon_j} \times \boldsymbol{\epsilon_k}$

This works with operators also: $with \;\; x_i \;\; for \;\; x, y, z$

$$\nabla . \boldsymbol{A} = \frac{\partial A_i}{\partial x_i}$$

$$\nabla \times \boldsymbol{A} \mid_i = \epsilon_{ijk} \frac{\partial A_k}{\partial x_j}$$

MULTIPLE VECTOR PRODUCTS – ε δ (LEVI-CIVITA)

Consider a vector triple product $\boldsymbol{A} \times \left(\boldsymbol{B} \times \boldsymbol{C}\right)$

$$
\begin{aligned}
\boldsymbol{A} \times \left(\boldsymbol{B} \times \boldsymbol{C}\right)\big|_i &= \epsilon_{ijk} A_j \left(\boldsymbol{B} \times \boldsymbol{C}\right)_k \\
&= \epsilon_{ijk} A_j \left(\epsilon_{kpq} B_p C_q\right) \\
&= \epsilon_{kij} \epsilon_{kpq} A_j B_p C_q
\end{aligned}
$$

What to do with a product like $\epsilon_{ijk} \epsilon_{lpq}$?

This is either -1 or 0 or 1

$$
\epsilon_{ijk} \epsilon_{lpq} = \begin{vmatrix} \delta_{il} & \delta_{ip} & \delta_{iq} \\ \delta_{jl} & \delta_{jp} & \delta_{jq} \\ \delta_{kl} & \delta_{kp} & \delta_{kq} \end{vmatrix}
$$

MULTIPLE VECTOR PRODUCTS – ε δ (LEVI-CIVITA)

$$\begin{aligned}
\epsilon_{kij}\epsilon_{kpq} &= \delta_{kk}\delta_{ip}\delta_{jq} + \delta_{kp}\delta_{iq}\delta_{jk} + \delta_{kq}\delta_{ik}\delta_{jp} \\
&\quad -\delta_{kk}\delta_{iq}\delta_{jp} - \delta_{kp}\delta_{ik}\delta_{jq} - \delta_{kq}\delta_{ip}\delta_{jk} \\
&= \delta_{kk}\left(\delta_{ip}\delta_{jq}-\delta_{iq}\delta_{jp}\right)+\delta_{kp}\left(\delta_{iq}\delta_{jk}-\delta_{ik}\delta_{jq}\right)+\delta_{kq}\left(\delta_{ik}\delta_{jp}-\delta_{ip}\delta_{jk}\right) \\
&= 3\left(\delta_{ip}\delta_{jq}-\delta_{iq}\delta_{jp}\right)+\left(\delta_{iq}\delta_{jp}-\delta_{ip}\delta_{jq}\right)+\left(\delta_{iq}\delta_{jp}-\delta_{ip}\delta_{jq}\right) \\
&= \delta_{ip}\delta_{jq}-\delta_{iq}\delta_{jp}
\end{aligned}$$

$$\mathit{Using\ the\ last\ result}\left(\mathit{with}\quad i=p\right)$$

$$\begin{aligned}
\epsilon_{kij}\epsilon_{kiq} &= \delta_{ii}\delta_{jq}-\delta_{iq}\delta_{ji} \\
&= 3\delta_{jq}-\delta_{jq} \\
&= 2\delta_{jq}
\end{aligned}$$

$$\mathit{Using\ the\ last\ result}\left(\mathit{with}\quad j=q\right)$$

$$\begin{aligned}
\epsilon_{kij}\epsilon_{kij} &= 2\delta_{jj} \\
&= 6
\end{aligned}$$

Successive summation over indices.

The first sum is most frequently encountered. It allows you to write a cross product in terms of dot product like terms .

PROOF OF VECTOR IDENTITIES

$$[\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\boldsymbol{A})]_i=\epsilon_{ijk}\partial_j\epsilon_{kmn}\partial_m A_n=\epsilon_{kij}\epsilon_{kmn}\partial_j\partial_m A_n$$
$$=(\delta_{im}\delta_{jn}-\delta_{in}\delta_{jm})\partial_j\partial_m A_n=\partial_j\partial_i A_j-\partial_j\partial_j A_i$$
$$=[\boldsymbol{\nabla}(\boldsymbol{\nabla}.A)-\nabla^2\boldsymbol{A}]_i$$

$$[\boldsymbol{\nabla}\times(\boldsymbol{A}\times\boldsymbol{B})]_i=\epsilon_{ijk}\partial_j\epsilon_{kmn}A_m B_n=\epsilon_{kij}\partial_j\epsilon_{kmn}\partial_j A_m B_n$$
$$=(\delta_{im}\delta_{jn}-\delta_{in}\delta_{jm})\partial_j(A_m B_n)$$
$$=\partial_j(A_i B_j)-\partial_j(A_j B_i)$$
$$=B_j\partial_j A_i+A_i\partial_j B_j-B_i\partial_j A_j-A_j\partial_j B_i$$
$$=(\vec{B}.\nabla)\vec{A}+\vec{A}(\nabla.\vec{B})-\vec{B}(\nabla.\vec{A})-(\vec{A}.\nabla)\vec{B}$$