CS 228 : Logic in Computer Science

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Propositional Logic

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- Parantheses as required
- Example : $[p \wedge (q \vee r)] \rightarrow [\neg r \wedge p]$
- $\neg$ binds tighter than $\vee, \wedge$, which bind tighter than $\rightarrow$. In the absence of parantheses, $p \rightarrow q \rightarrow r$ is read as $p \rightarrow (q \rightarrow r)$

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- It is raining, and Tia is outside, and is not wet. $\psi = (R \wedge \mathit{TiaOut} \wedge \neg \mathit{TiaWet})$
- So, Tia has her rain gear with her. *RG*
- Thus, $\chi = \varphi \wedge \psi \rightarrow RG$. You can deduce *RG* from $\varphi \wedge \psi$.
- Is χ valid? Is χ satisfiable?

Two Examples of Natural Deduction

Solve Sudoku

Consider the following kid's version of Sudoku.

	2	4	
1			3
4			2
	1	3	

Rules:

- Each row must contain all numbers 1-4
- Each column must contain all numbers 1-4
- Each 2×2 block must contain all numbers 1-4
- No cell contains 2 or more numbers

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    $[P(1,1,3) \vee P(1,2,3) \vee P(1,3,3) \vee P(1,4,3)] \wedge$
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  - Row 2: $[P(2,1,1) \vee \ldots$
  - Row 3: $[P(3,1,1) \vee \ldots$
  - Row 4: $[P(4,1,1) \vee \ldots$

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- Column 2: $[P(1,2,1) \vee \ldots$
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- Upper left block contains all numbers 1-4:

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$$[P(1,1,2) \vee P(1,2,2) \vee P(2,1,2) \vee P(2,2,2)] \wedge$$
$$[P(1,1,3) \vee P(1,2,3) \vee P(2,1,3) \vee P(2,2,3)] \wedge$$
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- Upper right block contains all numbers 1-4:

$$[P(1,3,1) \vee P(1,4,1) \vee P(2,3,1) \vee P(2,4,1)] \wedge \dots$$

- Lower left block contains all numbers 1-4:

$$[P(3,1,1) \vee P(3,2,1) \vee P(4,1,1) \vee P(4,2,1)] \wedge \dots$$

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No cell contains 2 or more numbers

- For cell(1,1):

$$P(1,1,1) \rightarrow [\neg P(1,1,2) \wedge \neg P(1,1,3) \wedge \neg P(1,1,4)] \wedge$$

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- Similar for other cells

Encoding as Propositional Satisfiability

Encoding Initial Configuration:

$$P(1,2,2) \wedge P(1,3,4) \wedge P(2,1,1) \wedge P(2,4,3) \wedge$$
$$P(3,1,4) \wedge P(3,4,2) \wedge P(4,2,1) \wedge P(4,3,3)$$

Solving Sodoku

To solve the puzzle, just conjunct all the above formulae and find a satisfiable truth assignment!

Gold Rush

(**Box1**)	*The gold is not here*
(**Box2**)	*The gold is not here*
(**Box3**)	*The gold is in Box 2*

Only one message is true; the other two are false. Which box has the gold?

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  - Conjunct all these, and call the formula φ.
  - Is there a unique satisfiable assignment for φ?
  - For example, is $M1 = \textit{true}$ a part of the satisfiable assignment?

A Proof Engine

- Given a formula φ in propositional logic, how to "prove" φ if φ is valid?
- Given premises $p, \neg q$, can we prove or deduce $\neg(p \wedge q)$?
- Given premises $\neg p$ and $\neg q$, can we deduce $\neg(p \vee q)$?
- Given nothing, can we deduce or prove $p \wedge q \to p$, or $p \to p \vee q$?
- What is the proof engine?

Proof Engine : Natural Deduction

- Natural Deduction as a technique to “prove” valid formulae
- Understand this proof engine
- Show that this proof engine is sound and complete
  - Completeness: Any fact that can be captured using propositional logic can be proved by the proof engine
  - Soundness: Any formula that is proved to be valid by the proof engine is indeed valid

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- $\varphi_1, \ldots, \varphi_n \vdash \psi$: This is called a sequent. $\varphi_1, \ldots, \varphi_n$ are premises, and ψ, the conclusion.
- Given $\varphi_1, \ldots, \varphi_n$, we can deduce or prove ψ. What was the sequent in Tia's case?

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- $\varphi, \psi \vdash RG$

What can we prove?

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- $\neg p \to q, q \to r, \neg r \vdash p$ is a sequent. How do you prove this?
- Proof rules to be carefully chosen, for instance you should not end up proving something like $p \land q \vdash \neg q$

The Rules of the Proof Engine

Rules for Natural Deduction

The and introduction rule denoted $\wedge i$

$$\frac{\varphi \qquad \psi}{\varphi \wedge \psi}$$

Rules for Natural Deduction

The and elimination rule denoted $\wedge e_1$

$$\frac{\varphi \wedge \psi}{\varphi}$$

The and elimination rule denoted $\wedge e_2$

$$\frac{\varphi \wedge \psi}{\psi}$$

A first proof using $\wedge i, \wedge e_1, \wedge e_2$

- Show that $p \wedge q, r \vdash q \wedge r$

1.	$p \wedge q$	premise
2.		

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2.	r	premise
3.	q	$\wedge e_2$ 1
4.		

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1.	$p \wedge q$	premise
2.	r	premise
3.	q	$\wedge e_2$ 1
4.	$q \wedge r$	$\wedge i$ 3,2