1) Canadian federal income tax works on a graduated system, meaning the amt that you pay depends on your income. Within each tax bracket, you only pay the income tax rate on the amt of your income that falls within the range. Eg: if you earn $50,000 then you pay 15% on the first $48,535 and 20.5% on the remaining $1465.

The 2020 Canadian federal income tax rates are:

$x \leq 48,535$	15%
$48,535 < x \leq 97,069$	20.5%
$97,069 < x \leq 150,473$	26%
$150,473 < x \leq 214,368$	29%
$x > 214,368$	33%

Let $T(x)$ represent the amt of income tax (in dollars) that a Canadian person paid in 2020 based on their income, x (in dollars).

Ans:

a)

$$T(x)=\begin{cases} 0.15x & \text{if } x \le 48{,}535 \\ 0.205x(x-48535)+7280.25 & \text{if } 48{,}535 < x \le 97{,}069 \\ 0.26(x-97069)+17229.72 & \text{if } 97{,}069 < x \le 150{,}473 \\ 0.29(x-150473)+31114.76 & \text{if } 150{,}473 < x \le 214{,}368 \\ 0.33(x-214368)+49644.31 & \text{if } 214{,}368 < x \end{cases}$$

To complete the function provided, the key insight that we needed was that as soon as somebody progresses to the next tax bracket, they have to pay the maximum tax for the previous tax bracket. Thus, the constant added is the highest tax paid for the previous bracket.

b) since income = \$ 162,000,

$$T(x) = 0.29(162000-150473) + 31114.76$$

$$= 0.29(11527) + 31114.76$$

$$= 3342.83 + 31114.76$$

$$= \$\,34457.59 \quad \text{(Ans)}$$

c) Yes, $T(x)$ is continuous for all $x \in (0, \infty)$

We know a limit is continuous when

$$\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) = f(a).$$

In ~~the~~ function $T(x)$,

① $$\lim_{x \to 48535^-} T(x) = 0.15(48535) = 7280.25$$

$$\lim_{x \to 48535^+} T(x) = 0.205(48535 - 48535) + 7280.25$$

$$= 7280.25.$$

Thus,

$$\lim_{x \to 48535^-} T(x) = \lim_{x \to 48535^+} T(x)$$

② $$\lim_{x \to 97069^-} T(x) = 0.205(97069 - 48535) + 7280.25$$

$$= 9949.47 + 7280.25 = 17229.72$$

$$\lim_{x \to 97069^+} T(x) = 0.26(97069 - 97069) + 17229.72$$

$$= 17229.72$$

Thus,

$$\lim_{x \to 97069^-} T(x) = \lim_{x \to 97069^+} T(x)$$

(3)

$$\lim_{n \to 150473^-} T(n) = 0.26(150473 - 97069) + 17229.72$$
$$= 13885.04 + 17229.72 = 31114.76$$

$$\lim_{n \to 150473^+} T(n) = 0.29(150473 - 150473) + 31114.76$$
$$= 31114.76$$

Thus,

$$\lim_{n \to 150473^-} T(n) = \lim_{n \to 150473^+} T(n)$$

(4)

$$\lim_{n \to 214368^-} T(n) = 0.29(214368 - 150473) + 31114.76$$
$$= 18529.55 + 31114.76 = 49644.31$$

$$\lim_{n \to 214368^+} T(n) = 0.33(214368 - 214368) + 49644.31$$
$$= 49644.31$$

Thus,

$$\lim_{n \to 214368^-} T(n) = \lim_{n \to 214368^+} T(n)$$

$\therefore$ Thus, $T(n)$ is continuous.

d)

yes, there was a large change in ~~the~~ tax paid.

when the student earned \$ 48,000, he

paid tax = $0.15 \times 48{,}000$

$= \$7200$

when he earned \$49,000, he paid

tax = $0.205(465) + 7280.25$

$= 95.325 + 7280.25$

$= \$7375.575$

$\therefore$ Additional tax paid $= 7375.575 - 7200$

$= \$175.575$

$\therefore$ The student earned more money even after paying additional tax (\$175.575) due to the additional pay (\$1000).

(Additional salary > additional tax paid)

Thus,

Additional earnings

$= \$1000 - 175.575$

$= \$824.425$ (Ans)

2) Explain whether ~~the~~ following statement is true or false:

The IVT tells us that for all positive integers n, ~~the~~ function $f(x) = \cos x - x^n$ has at least one root.

2) The statement is True.

$f(x) = \cos x - x^n$)

To prove that $f(x)$ has atleast one root, we will first prove that $f(x)$ is continuous.

In order to prove that $f(x)$ is continuous, we will prove the continuity of $\cos x$ and x^n.

to show that $\cos x$ is continuous,

let $x = a$, where $a \in \mathbb{R}$.

LHL at $x = a$:

$$\lim_{x \to a^-} g(x) = \lim_{x \to a^-} \cos x$$

$$= \lim_{h \to 0} \cos(a-h) = \cos a$$

RHL at $x = a$:

$$\lim_{x \to a^+} g(x) = \lim_{x \to a^+} \cos x$$

$$= \lim_{h \to 0} \cos(a+h) = \cos a$$

$$g(a) = \cos a$$

since, LHL = RHL , $g(x)$ is continuous.

To show that x^n is continuous,

let $x = c$ where $c \in R$.

$$\lim_{x \to c} h(x) = \lim_{x \to c} x^n = c^n$$

$$h(c) = c^n$$

$\therefore$ $h(x) = x^n$ is continuous at $x = c$.

$f(x) = g(x) + (-h(x))$, where {The negative of a continuous function is also continuous.}

both $g(x)$ and $h(x)$ are continuous.

Since ~~the~~ sum of two continuous function is continuous, $f(x)$ is continuous.

Now,

$$F(0) = \cos 0 - 0$$
$$= 1 > 0$$
$$f\left(\frac{\pi}{2}\right) = \cos\left(\frac{\pi}{2}\right) - \left(\frac{\pi}{2}\right)^n$$
$$= 0 - \left(\frac{\pi}{2}\right)^n < 0$$ {for all $n > 0$}

We see that $f(x)$ is continuous, and $f(x)$ is positive at $x = 0$ and negative at $f\left(\frac{\pi}{2}\right)$

Thus, there has to be a point between 0 and $\frac{\pi}{2}$ when $f(x) = 0$.

$\therefore f(x)$ has atleast one root.

3) Find ~~the~~ derivative of $f(x) = \frac{1}{\sqrt{x^2+1}}$ at $x = 0$ using ~~the~~ definition of ~~the~~ derivative as a limit. Be sure to justify your computations. Specifically, if at any point you evaluate a limit by inserting a value, explain why it is possible to do that.

$$f(x) = \frac{1}{\sqrt{x+2}}$$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{\frac{1}{\sqrt{x+h+2}} - \frac{1}{\sqrt{x+2}}}{h}$$

$$= \lim_{h \to 0} \frac{\sqrt{x+2} - \sqrt{x+h+2}}{(\sqrt{x+h+2})(\sqrt{x+2})\,h}$$

$$= \lim_{h \to 0} \frac{\sqrt{x+2} - \sqrt{x+h+2}}{(\sqrt{x+h+2})(\sqrt{x+2})h} \times \frac{\sqrt{x+2} + \sqrt{x+h+2}}{\sqrt{x+2} + \sqrt{x+h+2}}$$

$$= \lim_{h \to 0} \frac{\cancel{x} + 2 - \cancel{x} - h - 2}{(\sqrt{x+h+2})(\sqrt{x+2})h(\sqrt{x+2} + \sqrt{x+h+2})}$$

$$\Rightarrow \frac{-\cancel{h}}{\cancel{h}(\sqrt{x+h+2})(\sqrt{x+2})(\sqrt{x+2} + \sqrt{x+h+2})}$$

$$= \frac{-1}{(2\sqrt{x+2})(x+2)}$$

$$\Rightarrow \frac{-1}{2(x+2)^{3/2}}$$

when $x = 0$,

$$f'(x) = \frac{-1}{2 \cdot 2^{3/2}}$$

$$= \frac{-1}{2^{5/2}} \quad \text{(Ans)}$$

4) suppose $f(x) = 2ax^3 + b^3x - 4c$

where a, b, c are real constants.

suppose that f has a horizontal

tangent line at $x = 3$, and that

~~the~~ tangent line to f at $x = 0$ is

given by $y = 8x + 4$.

find a, b, c and explain your

reasoning.

$f'(x) = 6ax^2 + b^3$ (by chain rule)

slope of ~~the~~ tangent = Derivative ~~of~~

~~the~~ curve at that point.

At $x = 3$,

$6a(3)^2 + b^3 = 0$

$b^3 = -54a$

At $x = 0$,

$b^3 = 8$

$b = 2$ (Ans 1)

$8 = -54a$

$a = \frac{-\cancel{8}\,4}{\cancel{54}\,27} = -6.75$ (Ans 2)

At $x = 0$,

$f(x) = -4c$.

when $x = 0$,

eq of ~~tangent~~ $= 4$

$\Rightarrow \quad -4c = 4$

$\Rightarrow \quad c = -1$ (Ans)

5) $f(x) = e^{x^2}$

$f''(x) = ?$

$f'(x) = e^{x^2} \cdot 2x$

$f''(x) = e^{x^2} \cdot 2 + 2x\, e^{x^2} \cdot 2x$

$= 2e^{x^2} + 4x^2 e^{x^2}$

(Ans)

6) $k(x) = \dfrac{5x^2}{4x+3}$

$\Rightarrow \quad k'(x) = \dfrac{(4x+3)10x - 5x^2(4)}{(4x+3)^2}$

$$= \frac{40x^2 + 30x - 20x^2}{(4x+3)^2}$$

$$= \frac{20x^2 + 30x}{(4x+3)^2}$$

7) find $\frac{dy}{dx}$ if $y = (\ln x)^x$

$$\ln y = x \ln(\ln x)$$

$$\Rightarrow \frac{1}{y}\frac{dy}{dx} = \frac{x}{\ln x \times x} + \ln(\ln x)$$

$$\Rightarrow \frac{1}{y}\frac{dy}{dx} = \frac{1}{\ln x} + \ln(\ln x)$$

$$\Rightarrow \frac{dy}{dx} = (\ln x)^x \left[\frac{1}{\ln x} + \ln(\ln x)\right]$$

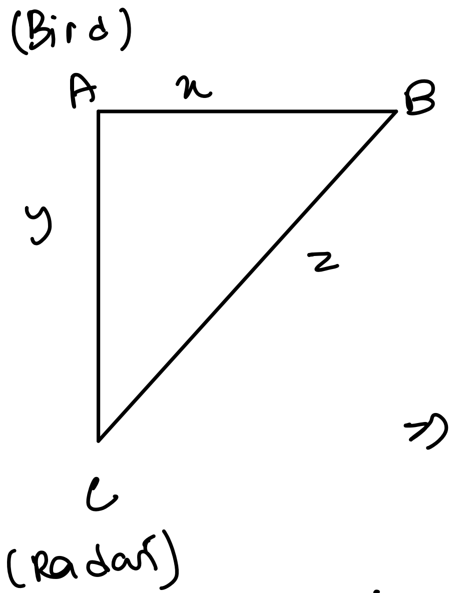
(Ans)

8) suppose a migrating bird flies at a constant altitude of 8 km, with a velocity of 36 km/h. At time $t = 0$, the bird passes directly above a radar station, where t is measured in hours.

a) How fast is the distance between the bird and the radar station changing after 10 minutes?

b) How fast is the distance between the bird and the radar station changing at time $t = 0$, when the bird is directly above the radar station?
Explain if this is reasonable.

a)



using pyth theorem,

$$x^2 + y^2 = z^2 \quad -(1)$$

$$\Rightarrow 2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 2z\frac{dz}{dt}$$

$$\Rightarrow \frac{dz}{dt} = \frac{x}{z}\frac{dx}{dt} + \frac{y}{z}\frac{dy}{dt}$$

when $t = \frac{1}{6}$,

x = 8 kms

y = 6 kms

Since, distance = speed x time

$$z = \sqrt{64 + 36}$$

$$= \sqrt{100} = 10 \text{ km}$$

$\frac{dh}{dt} = 0$ km/h (since ~~the~~ altitute of ~~the~~ bird is constant)

$\frac{dy}{dt} = 36$ km/h

$\therefore \frac{dz}{dt} = \frac{0 + 36 \times 6}{10}$

$= \frac{216}{10} = 21.6$ km/h

The rate at which ~~the~~ distance bet ~~the~~ bird and radar is changing is 21.6 km/hrs (Ans)

b) At $t = 0$, $\frac{dy}{dt} = 36$ km/hrs

This means ~~that~~ at this point ~~the~~ bird starts to move away at 36km/h. (~~the~~ bird was not moving before this)

$\therefore$ since, ~~there~~ is no relative speed to ~~the~~ previous part a, the rate of change of speed bet ~~the~~ radar & bird is 0km/hr.

a) find $\frac{dy}{dx}$:

$$8y^2 + 8\sin(2y) = \sqrt{2}y + 5x$$

$$\Rightarrow 16y\frac{dy}{dx} + 8\cos 2y \frac{dy}{dx}\cdot 2 = \sqrt{2}\frac{dy}{dx} + 5$$

$$\Rightarrow \frac{dy}{dx}\left[16y + 16\cos 2y - \sqrt{2}\right] = 5$$

$$\Rightarrow \frac{dy}{dx} = \frac{5}{16y + 16\cos 2y - \sqrt{2}} \quad \text{(ans)}$$

10) $f(3) = 2$, $f'(3) = 5$, find $f(3.2)$.

$$f(3.2) = f(3 + 0.2)$$

$$\simeq f(3) + 0.2\, f'(3)$$

$$= 2 + 0.2 \times 5$$

$$= 3 \quad \text{(ans)}$$

11) The position of a particle moving along a co-ordinate axis is given by

$s(t) = t^3 - 9t^2 + 24t + 4 \quad , t \geq 0$

a) find $v(t)$.

b) At what time is the particle at rest?

c) On what intervals is the particle moving from left to right? from right to left?

a) velocity (v) :

$v(t) = s'(t) = 3t^2 - 18t + 24$

b) Particle is at rest when

$v(t) = 0$,

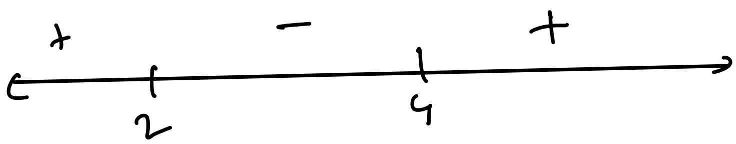
$3t^2 - 18t + 24 = 0$

$\Rightarrow 3(t-2)(t-4) = 0$

$t = 2, 4.$ (ans)

c) when $v(t) > 0$, particle is moving from left to right.

and when $v(t) < 0$, particle is moving from right to left.



<u>left to right :</u> $[0, 2) \cup (4, \infty)$

<u>right to left :</u> $(2, 4)$

12) $\frac{dy}{dx} = ?$ when $f(x) = 5x^3 \sin x$

$$\frac{dy}{dx} = 5x^3 \cos x + 15 \sin x \, x^2$$ (ans)

13) $y = \sin x$

Find $\frac{d^{74}}{dx^{74}} (\sin x)$

$$\frac{d^{74}}{dx^{74}}(\sin x) = \frac{d^{72+2}}{dx^{72+2}}(\sin x)$$

$$= \frac{d^2}{dx^2}(\sin x) = -\sin x$$

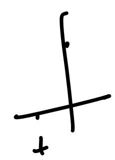
(Ans)

(ii) The position of a ball at time t is given by $s(t) = \sin 2t + \cos 3t$. What is its velocity at time $t = \frac{\pi}{3}$?

$$v(t) = s'(t)$$

$$= 2\cos 2t - 3\sin 3t$$

$$= 2\cos\frac{2\pi}{3} - 3\sin\pi$$



$$= -\cancel{2} \times \frac{1}{\cancel{2}} - 0$$

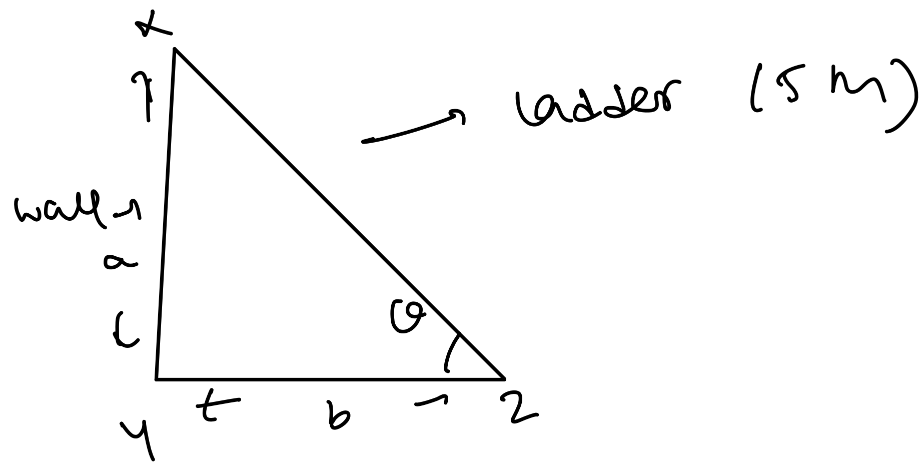
$\sin(2 \times 90^\circ + 0)$

$\sin 0$

$$= -1$$

(Ans)

(5) A 5 m long ladder is placed against a tall vertical wall to make a right-angled Δle. Find the max possible perimeter of this Δle. provide reasonable explanations for the same.



$\sin\theta = \frac{a}{5}$, $a = 5\sin\theta$

$\cos\theta = \frac{b}{5}$, $b = 5\cos\theta$

P of a Δle = a + b + hypotenuse

= a + b + 5

$$P(\theta) = 5\sin\theta + 5\cos\theta + 5$$

$$P'(\theta) = 5\cos\theta - 5\sin\theta$$

$$P'(\theta) = 0$$

$$5\cos\theta = 5\sin\theta$$

$$\Rightarrow \quad \theta = \frac{\pi}{4}$$

$$a = 5\sin\frac{\pi}{4} = \frac{5}{\sqrt{2}}$$

$$b = 5\sin\frac{\pi}{4} = \frac{5}{\sqrt{2}} \quad \text{(Ans)}$$

$$P = \frac{5\sqrt{2}}{2} + \frac{5\sqrt{2}}{2} + 5$$

$$= \frac{10\sqrt{2} + 10}{2} = 5(\sqrt{2} + 1)$$

$= 5 \times 2.41$

$= 12.05$ (Ans)

— X —