Unit - II
2.12 DC Transient Analysis – RLC

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Syllabus

UNIT – II **14 Periods**

DC Circuit Analysis: Voltage source and current sources, ideal and practical, Kirchhoff's laws and applications to network solutions using mesh analysis, - Simplifications of networks using series- parallel, Star/Delta transformation, DC circuits-Current-voltage relations of electric network by mathematical equations to analyse the network (Superposition theorem, Thevenin's theorem, Maximum Power Transfer theorem), Transient analysis of R-L, R-C and R-L-C Circuits.

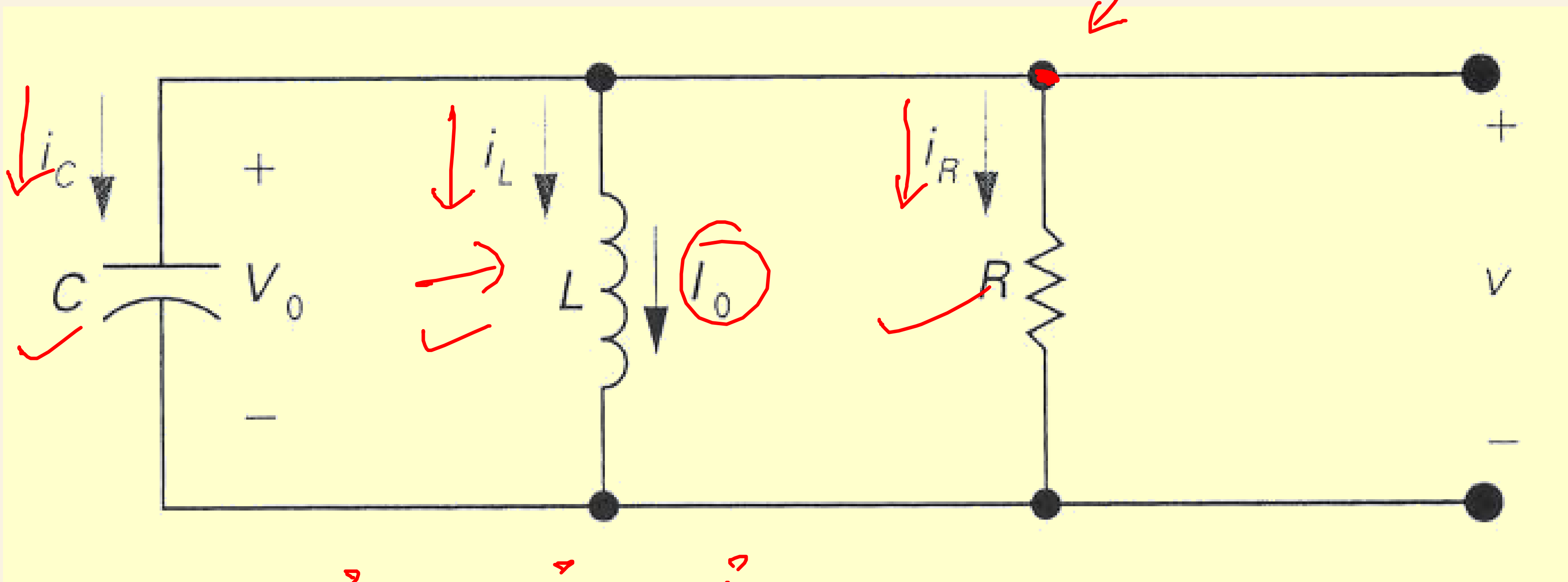
AC Steady-state Analysis: AC waveform definitions - Form factor - Peak factor - study of R-L - R-C -RLC series circuit - R-L-C parallel circuit - phasor representation in polar and rectangular form - concept of impedance - admittance - active - reactive - apparent and complex power - power factor, Resonance in R-L-C circuits - 3 phase balanced AC Circuits

Second order response for RLC

- RLC circuit: consists of resistor, inductor and capacitor
- Second order response : response from RLC circuit
- Type of RLC circuit:
  1. Series RLC
  2. Parallel RLC

Natural response of parallel RLC

KCL



$$i_R + i_C + i_L = 0$$

$$\frac{v}{R} + C \cdot \frac{dv}{dt} + \frac{1}{L}\int v\,dt = 0$$

- Summing all the currents away from node,

$$\frac{V}{R} + \frac{1}{L}\int_0^t v d\tau + I_0 + C\frac{dv}{dt} = 0$$

Differentiate 't'

$$\frac{1}{R} \cdot \frac{dv}{dt} + \frac{1}{L} v + C \cdot \frac{d^2 v}{dt^2} = 0$$

- Differentiating once with respect to t,

$v = Ae^{st}$

$$\frac{1}{R}\frac{dv}{dt}+\frac{v}{L}+C\frac{d^2v}{dt^2}=0$$

$\div C$

$$\frac{d^2v}{dt^2}+\frac{1}{RC}\frac{dv}{dt}+\frac{v}{LC}=0$$

- Assume that $v = Ae^{st}$

$$As^2 e^{st} + \frac{As}{RC}e^{st} + \frac{A}{LC}e^{st} = 0$$

$$Ae^{st}\underbrace{\left(s^2 + \frac{s}{RC} + \frac{1}{LC}\right)}_{\text{characteristic equation}} = 0$$

- Characteristic equation is zero:

$$\left(s^2 + \frac{s}{RC} + \frac{1}{LC}\right) = 0$$

- The two roots:

$$s_1 = -\frac{1}{2RC} + \sqrt{\left(\frac{1}{2RC}\right)^2 - \frac{1}{LC}}$$

$$s_2 = -\frac{1}{2RC} - \sqrt{\left(\frac{1}{2RC}\right)^2 - \frac{1}{LC}}$$

- The natural response of series RLC:

$$v = A_1 e^{s_1 t} + A_2 e^{s_2 t}$$

- The two roots:

$$s_1 = -\alpha + \sqrt{\alpha^2 - \omega_0{}^2}$$

$$s_2 = -\alpha - \sqrt{\alpha^2 - \omega_0{}^2}$$

- where:

$$\alpha = \frac{1}{2RC} \qquad \omega_0 = \frac{1}{\sqrt{LC}}$$

- Summary

Parameter	Terminology	Value in natural response
s_1, s_2	Characteristic equation	$\begin{cases} s_1 = -\alpha + \sqrt{\alpha^2 - \omega_0^2} \\ s_2 = -\alpha - \sqrt{\alpha^2 - \omega_0^2} \end{cases}$
α	Neper frequency	$\alpha = \dfrac{1}{2RC}$
ω_0	Resonant radian frequency	$\omega_0 = \dfrac{1}{\sqrt{LC}}$

- The two roots, s_1 and s_2 depend on the value of α and ω_o.
- 3 possible conditions are:

1. If $\omega_o < \alpha^2$, the voltage response is overdamped
2. If $\omega_o > \alpha^2$, the voltage response is underdamped
3. If $\omega_o = \alpha^2$, the voltage response is critically damped

Overdamped voltage response

- Overdamped voltage expression:

$$v = A_1 e^{s_1 t} + A_2 e^{s_2 t}$$

$v(0^+) = A_1 + A_2$

$\frac{dv(0^+)}{dt} = s_1 A_1 + s_2 A_2$

- The constants of A1 and A2 can be obtained from these two equations:

$$v(0^+) = A_1 + A_2 \rightarrow (1)$$

$$\frac{dv(0^+)}{dt} = s_1 A_1 + s_2 A_2 \rightarrow (2)$$

- Whereby, v_o^+ is obtained from the circuit at the instant when the switching occurred.

- The value of $v(0+) = V_0$ and the initial value of dv/dt is

$$\frac{dv(0^+)}{dt} = \frac{i_C(0^+)}{C}$$

The process for finding the overdamped response, $v(t)$:

1. Find the roots of the characteristic equation, $s1$ and $s2$, using the value of R, L and C.
2. Find $v(0+)$ and $dv(0+)/dt$ using circuit analysis.

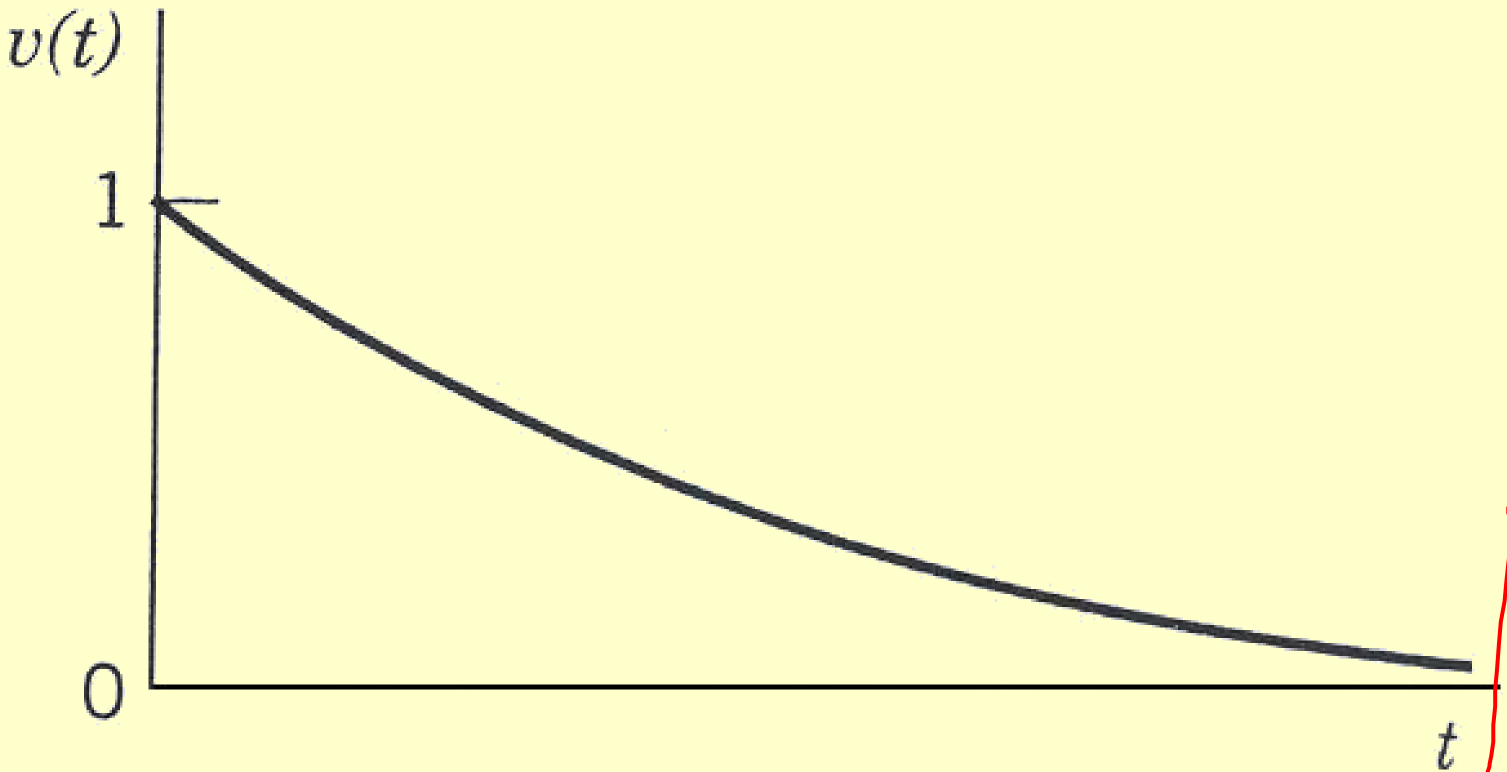
3. The values of A1 and A2 is obtained by solving the equation below:

$$v(0^+) = A_1 + A_2$$

$$\frac{dv(0^+)}{dt} = \frac{i_C(0^+)}{C} = s_1 A_1 + s_2 A_2$$

4. Substitute the value for s_1, s_2, A_1 dan A_2 to determine the expression for v(t) for t $\geq$ 0.

- The response of overdamped voltage for v(0) = 1V and i(0) = 0



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Delta PRO

https://time.com/collection/best-inventions-2021/

Underdamped voltage response

- When $\omega_o^2 > \alpha^2$, the roots of the characteristic equation are complex and the response is underdamped.

- The roots s1 and s2 as,

$$s_1 = -\alpha + \sqrt{-(\omega_0{}^2 - \alpha^2)}$$

$$= -\alpha + j\sqrt{\omega_0{}^2 - \alpha^2}$$

$$= -\alpha + j\omega_d$$

$$s_2 = -\alpha - j\omega_d$$

- ω_d : damped radian frequency

- The underdamped voltage response of a parallel RLC circuit is

$$v(t) = B_1 e^{-\alpha t} \cos \omega_d t + \cancel{B_2 e^{-\alpha t} \sin} \omega_d t$$

$v(0^+) = V_0 = B_1$

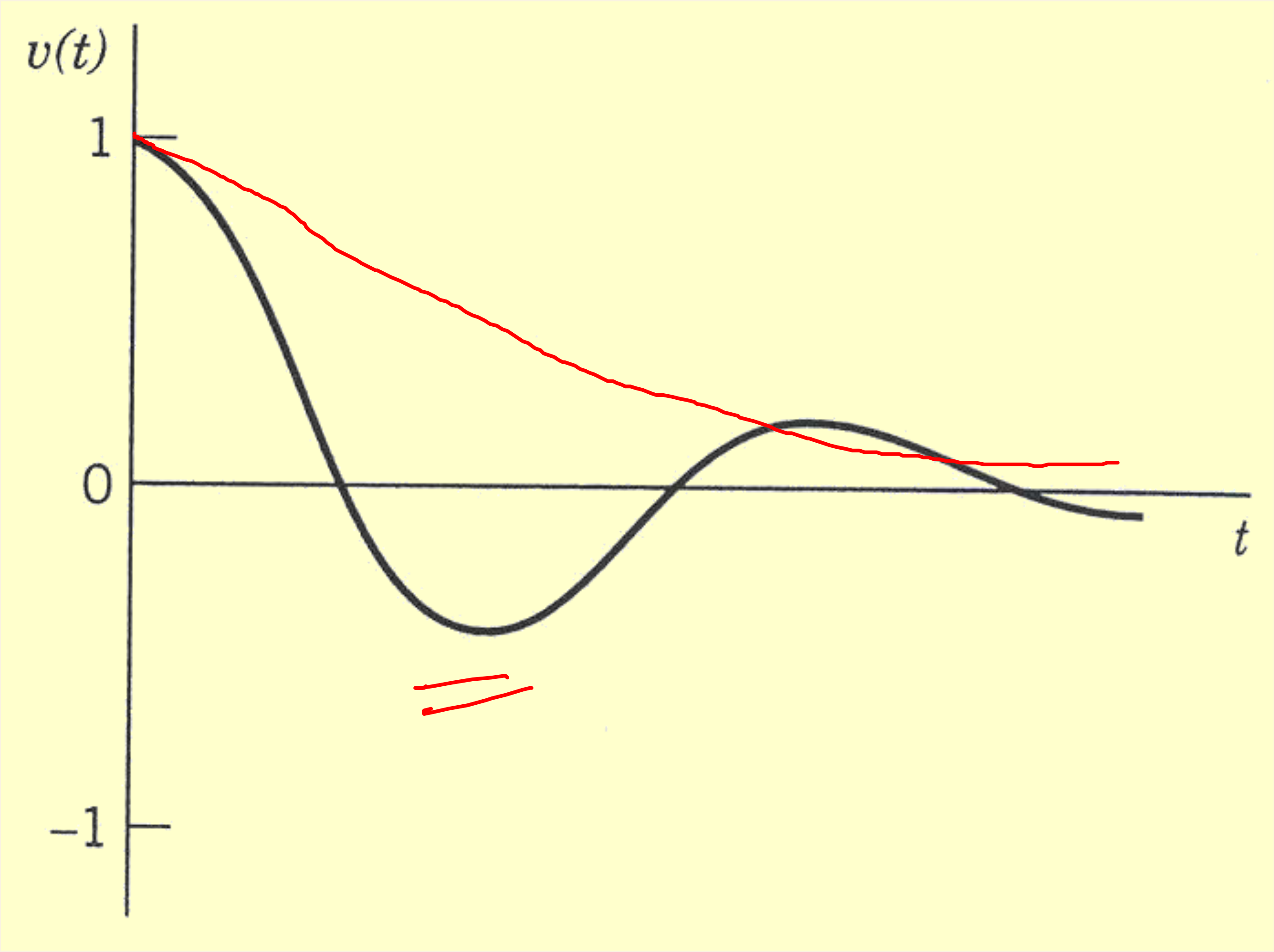
- The constants B1 and B2 are real number.

- The two simultaneous equation that determine B_1 and B_2 are:

$$v(0^+) = V_0 = B_1$$

$$\frac{dv(0^+)}{dt} = \frac{i_C(0^+)}{C} = -\alpha_1 B_1 + \omega_d B_2$$

Response of underdamped voltage for v(0) = 1V and i(0) = 0



Critically Damped voltage response

- A circuit is critically damped when $\omega_o^2 = \alpha^2$ ($\omega_o = \alpha$). The two roots of the characteristic equation are real and equal:

$$s_1 = s_2 = -\alpha = -\frac{1}{2RC}$$

- The solution for the voltage is

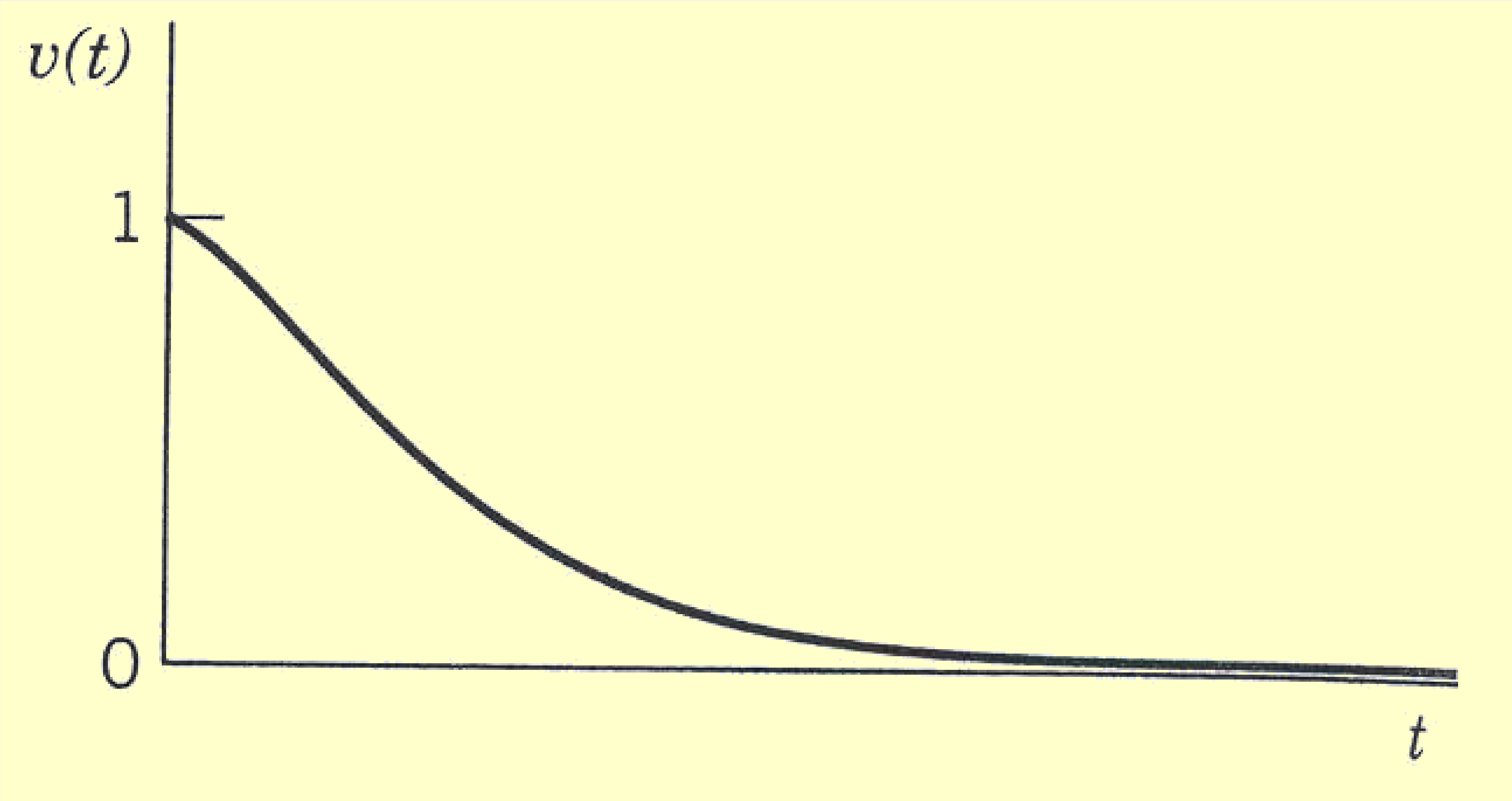
$$v(t) = D_1 t\, e^{-\alpha t} + D_2\, e^{-\alpha t}$$

- The two simultaneous equation that determine D1 and D2 are,

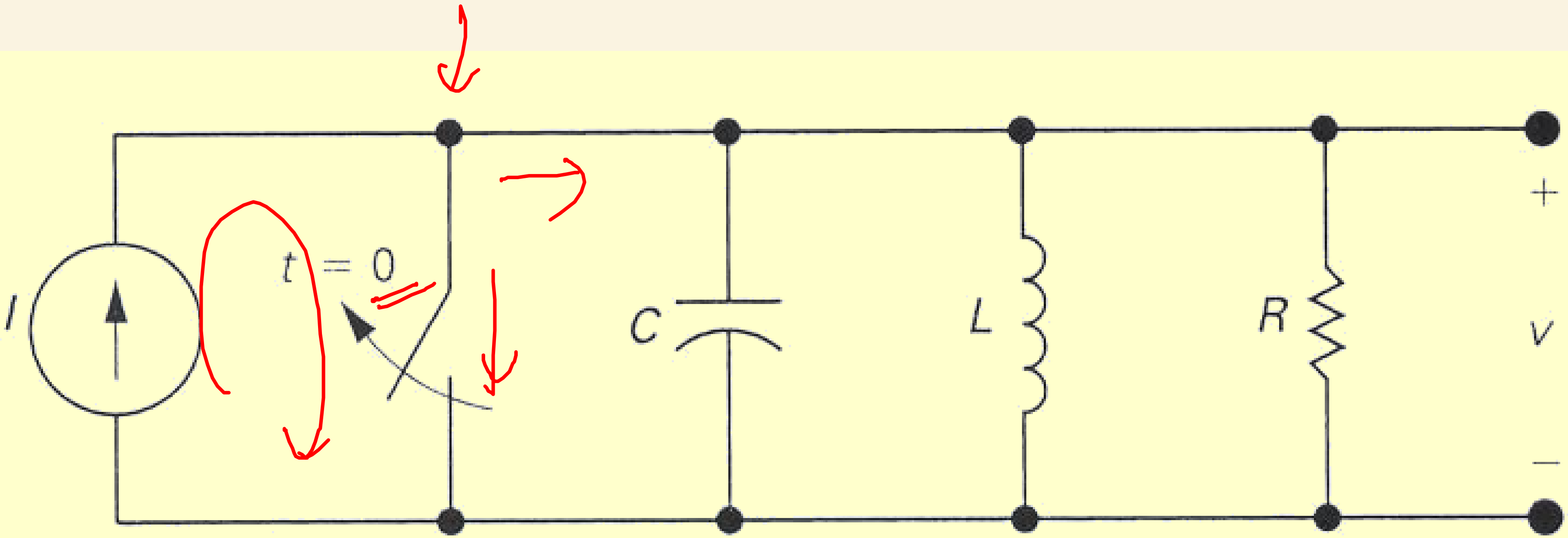
$$v(0^+) = V_0 = D_2$$

$$\frac{dv(0^+)}{dt} = \frac{i_C(0^+)}{C} = D_1 - \alpha D_2$$

Response of the critically damped voltage for v(0) = 1V and i(0) = 0



The step response of a parallel RLC circuit



$$I = i_L + i_C + i_R$$

- From the KCL,

$$i_L + i_R + i_C = I$$

$$i_L + \frac{v}{R} + C\frac{dv}{dt} = I$$

- Because $$v = L\frac{di}{dt}$$

- We get $$\frac{dv}{dt} = L\frac{d^2 i_L}{dt^2}$$

- Thus,

$$i_L + \frac{L}{R}\frac{di_L}{dt} + LC\frac{d^2 i_L}{dt^2} = I$$

$$\frac{d^2 i_L}{dt^2} + \frac{1}{RC}\frac{di_L}{dt} + \frac{i_L}{LC} = \frac{I}{LC}$$

- There are two approaches to solve the equation:
  - direct approach
  - indirect approach.

Indirect Approach

- From the KCL:

$$\frac{1}{L}\int_0^t v\,d\tau + \frac{v}{R} + C\frac{dv}{dt} = I$$

- Differentiate once with respect to t:

$$\frac{v}{L} + \frac{1}{R}\frac{dv}{dt} + C\frac{d^2v}{dt^2} = 0$$

$$\frac{d^2v}{dt^2} + \frac{1}{RC}\frac{dv}{dt} + \frac{v}{LC} = 0$$

- The solution for v depends on the roots of the characteristic equation:

$$v = A_1 e^{s_1 t} + A_2 e^{s_2 t}$$

$$\mathrm{v} = \mathrm{B}_1 \mathrm{e}^{-\alpha \mathrm{t}} \cos \omega_\mathrm{d} \mathrm{t} + \mathrm{B}_2 \mathrm{e}^{-\alpha \mathrm{t}} \sin \omega_\mathrm{d} \mathrm{t}$$

$$v = D_1 t\, e^{-\alpha t} + D_2\, e^{-\alpha t}$$

- Substitute into KCL equation :

$$i_L = I + A_1' e^{s_1 t} + A_2' e^{s_2 t}$$

$$i_L = I + B_1' e^{-\alpha t} \cos \omega_d t + B_2' e^{-\alpha t} \sin \omega_d t$$

$$i_L = I + D_1' t e^{-\alpha t} + D_2' e^{-\alpha t}$$

Direct Approach

- It is much easier to find the primed constants directly in terms of the initial values of the response function.

$$A_1', A_2', B_1', B_2', D_1', D_2'$$

- The primed constants could be determined from:

$$i_L(0) \quad \text{and} \quad \frac{di_L(0)}{dt}$$

- The solution for a second-order differential equation consists of the forced response and the natural response.

- If I_f and V_f is the final value of the response function, the solution for the step function can be written as:

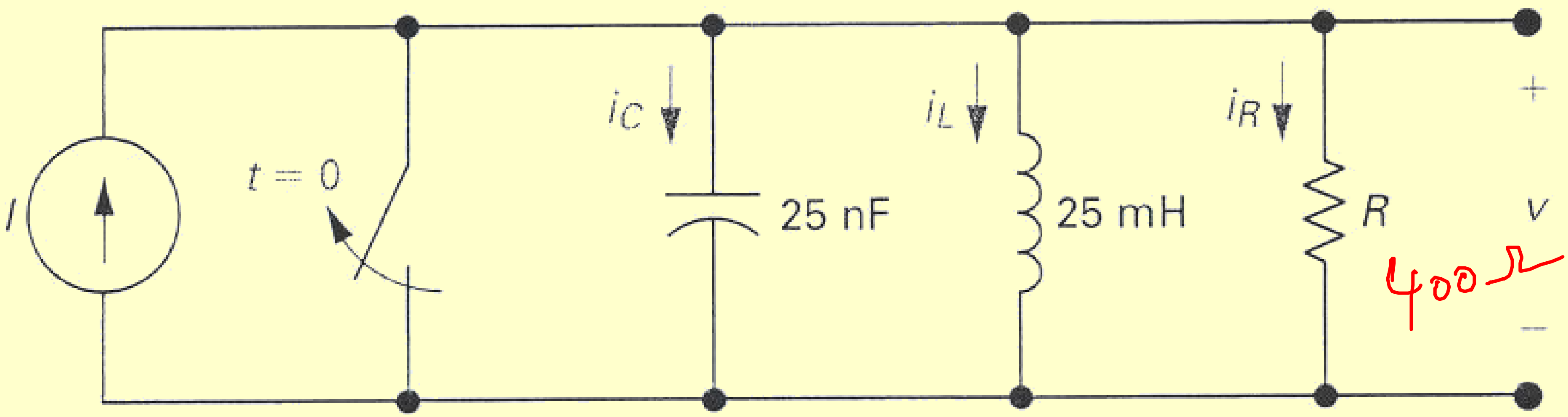
$$i = I_f + \left\{ \begin{matrix} \text{Function of the same form} \\ \text{as the natural response} \end{matrix} \right\}$$

$$v = V_f + \left\{ \begin{matrix} \text{function of the same form} \\ \text{as the natural response} \end{matrix} \right\}$$

Example 1
(Step response of parallel RLC)

The initial energy stored in the circuit is zero. At $t = 0$, a DC current source of 24mA is applied to the circuit. The value of the resistor is 400Ω.

1. What is the initial value of i_L?
2. What is the initial value of di_L/dt?
3. What is the roots of the characteristic equation?
4. What is the numerical expression for $i_L(t)$ when $t \geq 0$?



I
t = 0
iC
25 nF
iL
25 mH
iR
R
400Ω
+
v
−

Solution

1. No energy is stored in the circuit prior to the application of the DC source, so the initial current in the inductor is zero. The inductor prohibits an instantaneous change in inductor current, therefore $i_L(0)=0$ immediately after the switch has been opened.

2. The initial voltage on the capacitor is zero before the switch has been opened, therefore it will be zero immediately after. Because

$$v = L\frac{di_L}{dt}$$

thus

$$\frac{di_L(0^+)}{dt} = 0$$

3. From the circuit elements,

$$\omega_0^{\ 2} = \frac{1}{LC} = \frac{10^{12}}{(25)(25)} = 16 \times 10^8$$

$$\alpha = \frac{1}{2RC} = \frac{10^9}{(2)(400)(25)}$$

$$= 5 \times 10^4 \, rad/s$$

$$\alpha^2 = 25 \times 10^8$$

- Thus the roots of the characteristic equation are real,

$$s_1 = -5\times10^4 + 3\times10^4$$
$$= -20\,000\ rad/s$$
$$s_2 = -5\times10^4 - 3\times10^4$$
$$= -80\,000\ rad/s$$

4. The inductor current response will be overdamped.

$$i_L = I_f + A_1' e^{s_1 t} + A_2' e^{s_2 t}$$

- Two simultaneous equation:

$$i_L(0) = I_f + A_1' + A_2' = 0$$

$$\frac{di_L(0)}{dt} = s_1 A_1' + s_2 A_2' = 0$$

$$A_1' = -32mA \qquad A_2' = 8mA$$

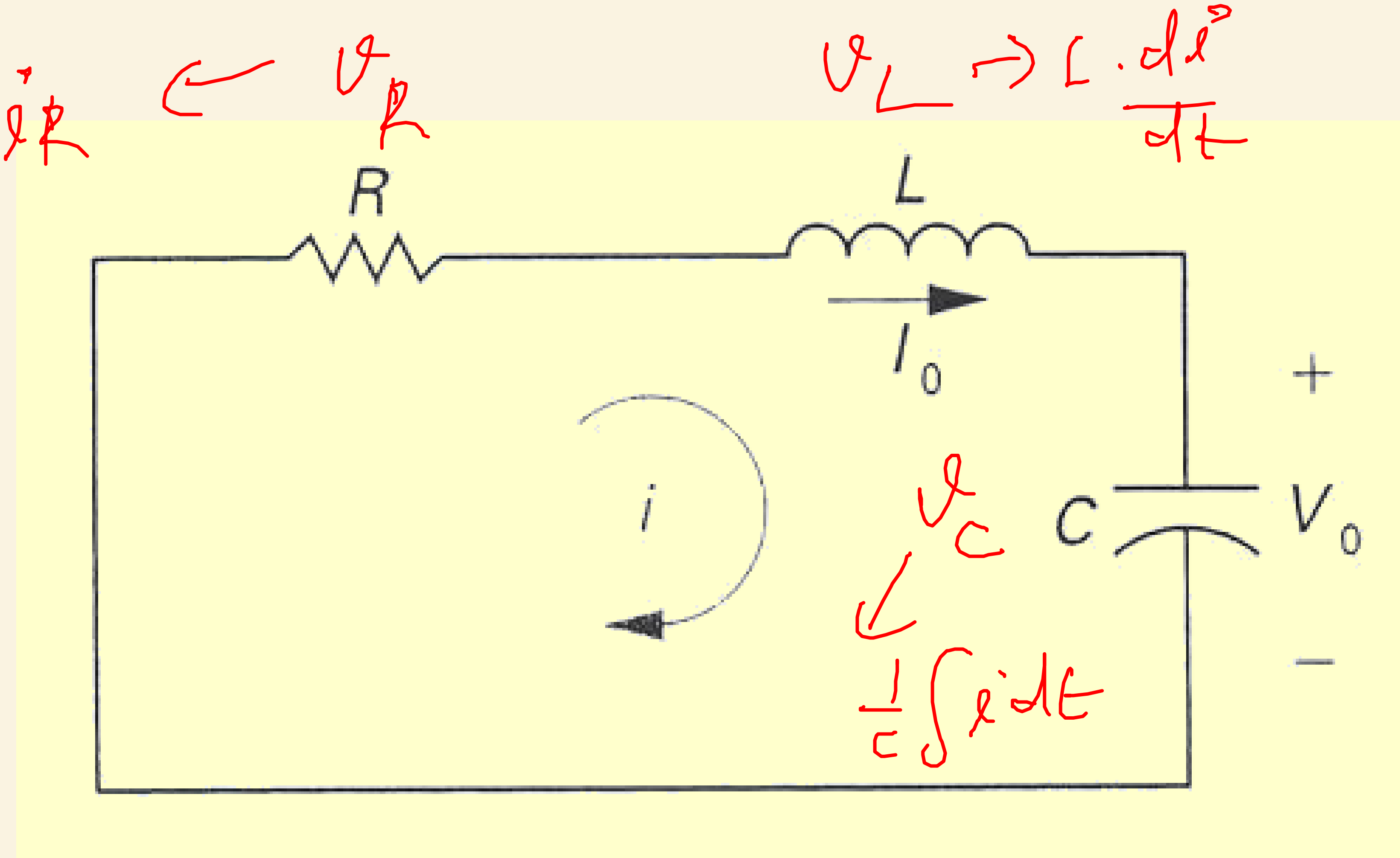
- Numerical solution:

$$i_L(t) = \begin{pmatrix} 24 - 32e^{-20000t} \\ +8e^{-80000t} \end{pmatrix} mA$$

Natural response of a series RLC

- The procedures for finding the natural response of a series RLC circuit is similar as the natural response of a parallel RLC circuit as both circuits are described by the similar form of differential equations.

Series RLC circuit



- Summing the voltage around the loop,

$$Ri + L\frac{di}{dt} + \frac{1}{C}\int_0^t i\,d\tau + V_0 = 0$$

- Differentiate once with respect to t,

$$R\frac{di}{dt}+L\frac{d^2i}{dt^2}+\frac{i}{C}=0$$

$$\frac{d^2i}{dt^2}+\frac{R}{L}\frac{di}{dt}+\frac{i}{LC}=0$$

- The characteristic equation for the series RLC circuit is,

$$s^2 + \frac{R}{L}s + \frac{1}{LC} = 0$$

- The roots of the characteristic equation are,

$$s_{1,2} = -\frac{R}{2L} \pm \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}}$$

@

$$s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_0{}^2}$$

- Neper frequency (α) for series RLC,

$$\alpha = \frac{R}{2L} rad/s$$

And the resonant radian frequency,

$$\omega_0 = \frac{1}{\sqrt{LC}} rad/s$$

The current response will be overdamped, underdamped or critically damped according to,

$$\omega_0{}^2 < \alpha^2$$

$$\omega_0{}^2 > \alpha^2$$

$$\omega_0{}^2 = \alpha^2$$

- Thus the three possible solutions for the currents are,

$$i(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}$$

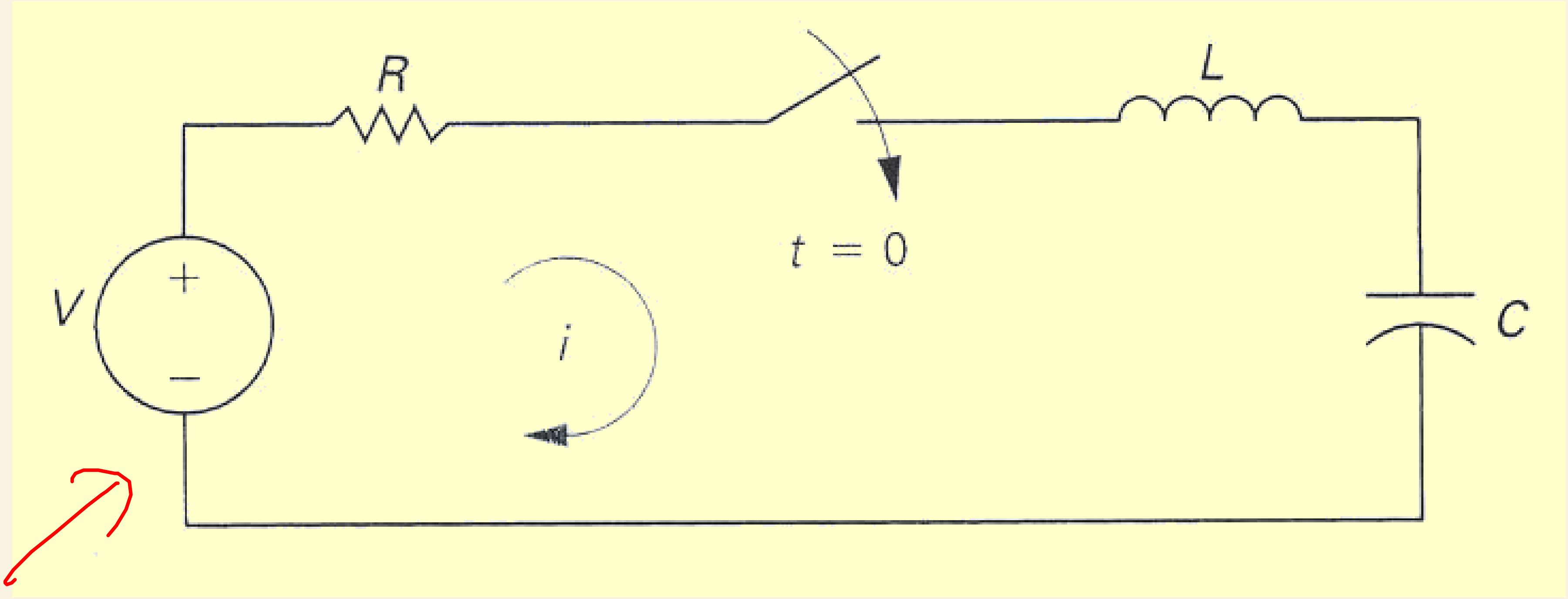
$$i(t) = B_1 e^{-\alpha t} \cos \omega_d t + B_2 e^{-\alpha t} \sin \omega_d t$$

$$i(t) = D_1 t\, e^{-\alpha t} + D_2\, e^{-\alpha t}$$

Step response of series RLC

- The procedures are the same as the parallel circuit.

Series RLC circuit



- Use KVL,

$$\boxed{v = Ri + L\frac{di}{dt} + v_C}$$

- The current, *i* is related to the capacitor voltage (v_C) by expression,

$$i = C\frac{dv_C}{dt}$$

- Differentiate once i with respect to t

$$\frac{di}{dt} = C\frac{d^2 v_C}{dt^2}$$

- Substitute into KVL equation,

$$\frac{d^2 v_C}{dt^2} + \frac{R}{L}\frac{dv_C}{dt} + \frac{v_C}{LC} = \frac{V}{LC}$$

- Three possible solutions for v_C are,

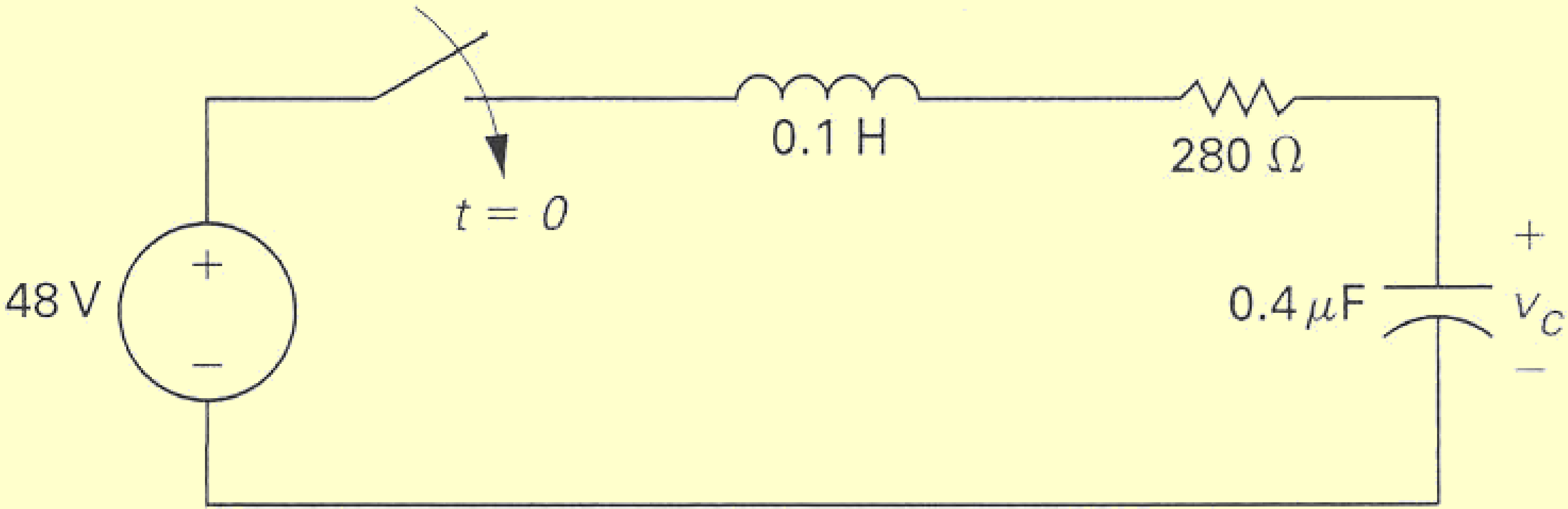
$$v_C = V_f + A_1' e^{s_1 t} + A_2' e^{s_2 t}$$

$$v_C = V_f + B_1' e^{-\alpha t} \cos\omega_d t + B_2' e^{-\alpha t} \sin\omega_d t$$

$$v_C = V_f + D_1' t\, e^{-\alpha t} + D_2' e^{-\alpha t}$$

Example 2
(step response of series RLC)

- No energy is stored in the 100mH inductor or 0.4μF capacitor when switch in the circuit is closed. Find $v_C(t)$ for $t \geq 0$.



t = 0
0.1 H
280 Ω
48 V
+
−
0.4 μF
+
v_C
−

Solution

- The roots of the characteristic equation:

$$s_1 = -\frac{280}{0.2} + \sqrt{\left(\frac{280}{0.2}\right)^2 - \frac{10^6}{(0.1)(0.4)}}$$

$$= (-1400 + j4800)\, rad/s$$

$$s_2 = (-1400 - j4800)\, rad/s$$

- The roots are complex, so the voltage response is underdamped. Thus:

$$v_C = 48 + B_1' e^{-1400t} \cos 4800t + B_2' e^{-1400t} \sin 4800t \qquad t \ge 0$$

- No energy is stored in the circuit initially, so both $v_C(0)$ and $dv_C(0^+)/dt$ are zero. Then:

$$v_C(0) = 0 = 48 + B_1'$$

$$\frac{dv_C(0^+)}{dt} = 0 = 4800 B_2' - 1400 B_1'$$

- Solving for B_1'and B_2'yields,

$$B_1^{'} = -48V$$

$$B_2^{'} = -14V$$

- Thus, the solution for $v_C(t)$,

$$v_C(t) = \begin{pmatrix} 48 - 48e^{-1400t}\cos 4800t \\ -14e^{-1400t}\sin 4800t \end{pmatrix} V$$

$$for \qquad t \geq 0$$

Summary

Transients → RLC

KCL / KVL

↓

Second order eqn

↓

char. eqn

↓

roots → over / under / critically → Solution → Initial conditions → Constant