Analytical problems.

1. Solve the recurrance relation.

a) $x(n) = x(n-1)+5$ for $n>1$ with $x(1) = 0$

① Write down the first two terms to identify the Pattern.

$x(1) = 0$
$x(2) = x(1)+5 = 5$
$x(3) = x(2)+3 = 10$
$x(4) = x(3)+5 = 15$.

② Identify the pattern (or) the general term

→ The first term $x(1) = 0$

The common difference $d = 5$.

The general formula for the n^{th} term of A.P is

* $x(n) = x(1) + (n-1)d$

Substituting the given Values

$x(n) = 0 + (n-1) \cdot 5 = 5(n-1)$

The solution is

$x(n) = 5(n-1)$.

b) $x(n) = 3x(n-1)$ for $n>1$ with $x(1) = 4$

① write down the first two terms to identify the pattern

$x(1) = 4$
$x(2) = 3x(1) = 3 \cdot 4 = 12$
$x(3) = 3x(2) = 36$
$x(4) = 108$.

② Identify the general term
$x(1) = 4$

common ration $= 3$.

General formula for n^{th} term → $x(n) = x(1) \cdot r^{n-1}$.

Substituting the given values

$$x(n) = 4 \cdot 3^{n-1}$$

The solution is

$$x(n) = 4 \cdot 3^{n-1}.$$

c) $x(n) = x(n/2) + n$ for $n > 1$ with $x(1) = 1$ (solve for $n = 2^k$)

For $n = 2^k$, we can write recurrence in terms of k.

1) substitute $n = 2^k$ in the recurrence

$$x(2^k) = x(2^{k-1}) + 2^k$$

2) write down the first few terms to identify the pattern

$$x(1) = 1$$
$$x(2) = x(2^1) = 3$$
$$x(4) = (x(2^2) = x(2) + 4 = 3 + 4 = 7$$
$$x(8) = x(2^3) = x(4) + 8 = 15.$$

3) Identify the general term by finding the pattern

we observe that :-

$$x(2^k) = x(2^{k-1}) + 2^k.$$

we sum the series.

$$x(2^k) = x(2^{k-1}) + 2^k$$

we sum these.

$$x(2^k) = 2^k + 2^{k-1} + 2^{k-2} \ldots$$

The geometric series with the term $a = 2$ and last term 2^k except for the additional $+1$ terms.

The sum of a geometric series with ratio $r = 2$ is given by

$$S = a \frac{r^n - 1}{r - 1}$$

$a = 2$, $r = 2$ $n = k$

$$S = 2\frac{2^{k}-1}{2-1} = 2(2^k - 1)$$
$$= 2^{k+1}.$$

Adding the +1 term

$$x(2^k) = 2^{k+1} - 2 + 1 = 2^{k+1} - 1$$

solution is

$$x(2^k) = 2^{k+1} - 1$$

d) $x(n) = x(n/3) + 1$ for $n > 1$ with $x(1) = 1$ (solve for $n = 3^4$).

For $n = 3^k$, we can write the recurrance in terms of k.

1) Substitute $n = 3^k$ in the recurrence

$$x(3^k) = x(3^{k-1}) + 1$$

2) write down the first few terms to identify the pattern

$$x(1) = 1$$
$$x(3) = x(3^1) = x(1) + 1 = 1 + 1 = 2$$
$$x(9) = x(3^2) = x(3) + 1 = 2 + 1 = 3$$
$$x(27) = x(3^3) = x(9) + 1 = 3 + 1 = 4.$$

3) Identify the general term:

we observe that:

$$x(3^k) = x(3^{k-1}) + 1$$

Summing up the series

$$x(3^k) = 1 + 1 + 1 + \cdots + 1$$
$$x(3^k) = k + 1$$

The soln is

$$x(3^k) = k + 1$$

2\. Evaluate the following recurrances complexity.

1) $T(n) = T(n/2) + 1$, where $n = 2k$ for all $k \geq 0$.

The recurrance relation can be solved using iteration method.

1) Substitute $n = 2^k$ in the recurrance.

$$\text{for } k=0 \quad T(2^0) = T(1) = T(1)$$

$$k' = 1 = T(2^1) = T(1) + 1$$

$$k = 2 = T(2^3) = T(8) = T(n) + 1 = (T(1)+1)$$

$$= T(1) + 2$$

$$k = 3 = T(2^3) = T(8) = T(n) + 1 = T(1) + 2 = T(1) + 3$$

2) generalise the pattern.

$$T(2^k) = T(1) + k$$

Since $n = 2^k$, $k = \log_2 n$

4) Assume $T(1)$ is a constant c.

$$T(n) = c + \log_2 n$$

The solution is

$$T(n) = O(\log n)$$

(ii) $T(n) = T(n/3) + T(2n/3)$ where c is constant and n is input size.

The recurrence can be solved using the master's theorem for divide & conquer recurrence of the form.

$$T(n) = aT(n/b) + f(n)$$

where $a = 2$, $b = 3$ & $f(n) = cn$

Lets determine the value of $\log_b a$:

$$\log_b a = \log_3 2$$

Using the properties of algorithms.

$$\log_3 2 = \frac{\log 2}{\log 3}$$

Now we compare $f(n) = cn$ with $n^{\log_3 2}$:

$$f(n) = O(n)$$
$$n = n^1$$

Since $\log_3 2$ we are in the third case of masters theorem. $f(n) = O(n^c)$ with $c > \log_b a$.

The solution is :

$$T(n) = O(f(n)) = O(cn) = O(n)).$$

3) Consider the following recursive algorithm.

min[A[0... n-1]]
if n=1 return A[0]
Else temp = min(A[0.. n-2])
if temp <= A(n-1) return temp
else
return A[n-1]

a) what does this algorithm compute ?

The given algorithm, min[A (0,...n-1]) computes the minimum value in the array 'A' from index '0' for 'n-1' it does this by recursively finding the minimum value in the sub array A[0..n-2] and then comparing it with the last element A[n-1] to determine the overall maximum value.

b) Setup a recurrance relation for the algorithm basic operation count & solve it.

The solution is

$T(n) = n$

This means the algorithm performs n basic operations for an input array of size n.

4. Analyze the order of growth.

1) $F(n) = 2n^2+5$ & $g(n) = 7n$ the $\Omega(g(n))$ notation.

To analyze the order of growth and use the Ω notation, we need to compare the given function $F(n)$ & $g(n)$

given functions:

$F(n) = 2n^2+5$

$g(n) = 7n$.

Order of growth using $\Omega(g(n))$ notation:

The notation $\Omega g(n)$ describes a lower bound on the growth rate that for sufficiently large n, $f(n)$, grows at least as far as $g(n)$.

$F(n) = c \cdot g(n)$.

Lets analyze $F(n) = 2n^2+5$ with respect to $g(n) = 7n$.

1) Identify Dominant terms:

→ The dominant terms in $F(n)$ is $2n^2$ since it grows faster then the constant terms as n increases.

→ The dominant term in $g(n)$.