Probing Neutron Star Matter Using Multi-Messenger Timing Results

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Capstone Thesis Presentation

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- Collapse of massive stars: 8–25 $M_\odot$
- Nuclear Fusion stops → Gravitational Collapse
- Outer Envelope Expelled in Supernova Explosion
- Collapse Halted by Nuclear forces
- Core Density Approaches ~10^{15} g/cm³

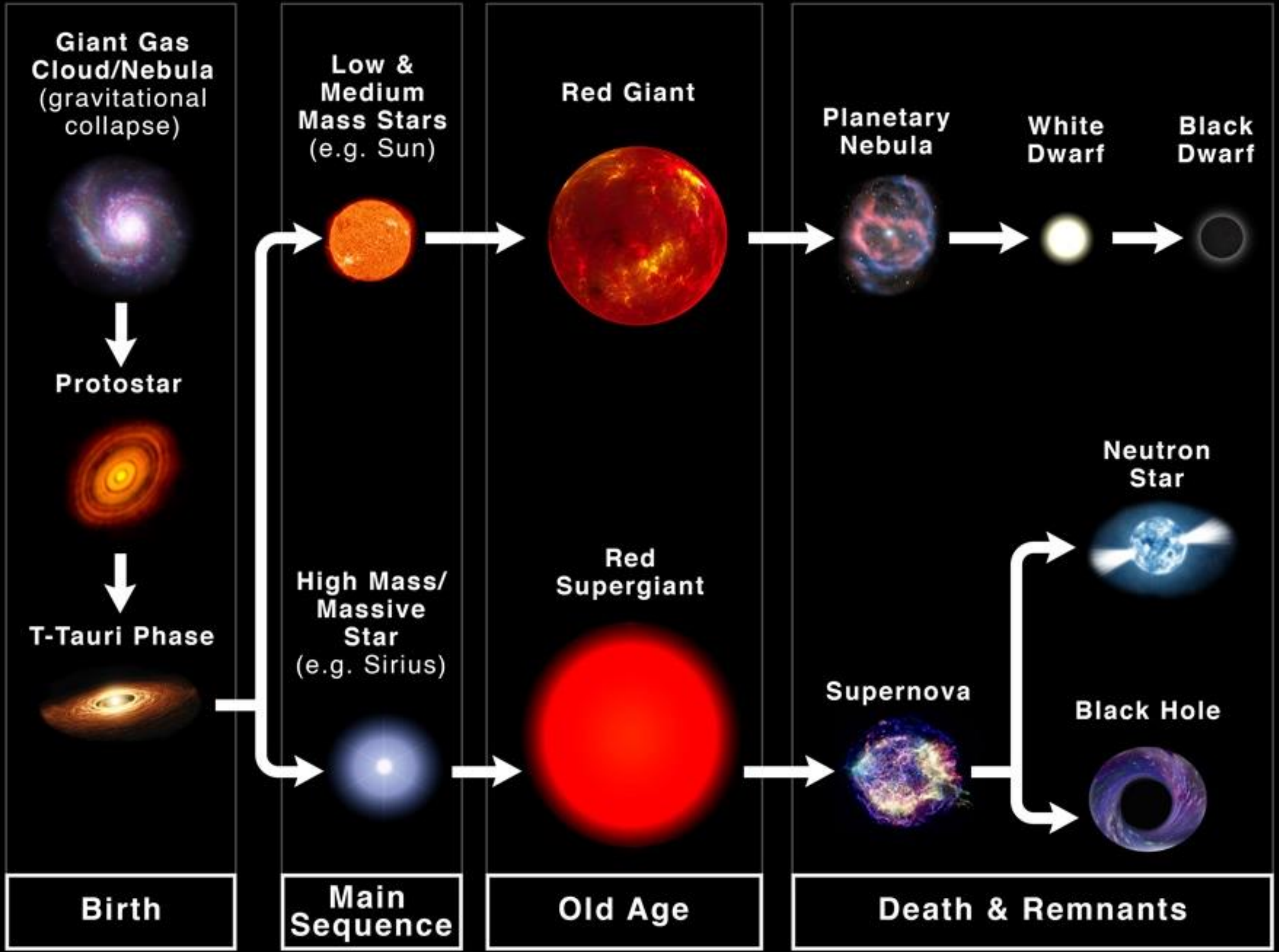

Image Source: sciencefacts.net

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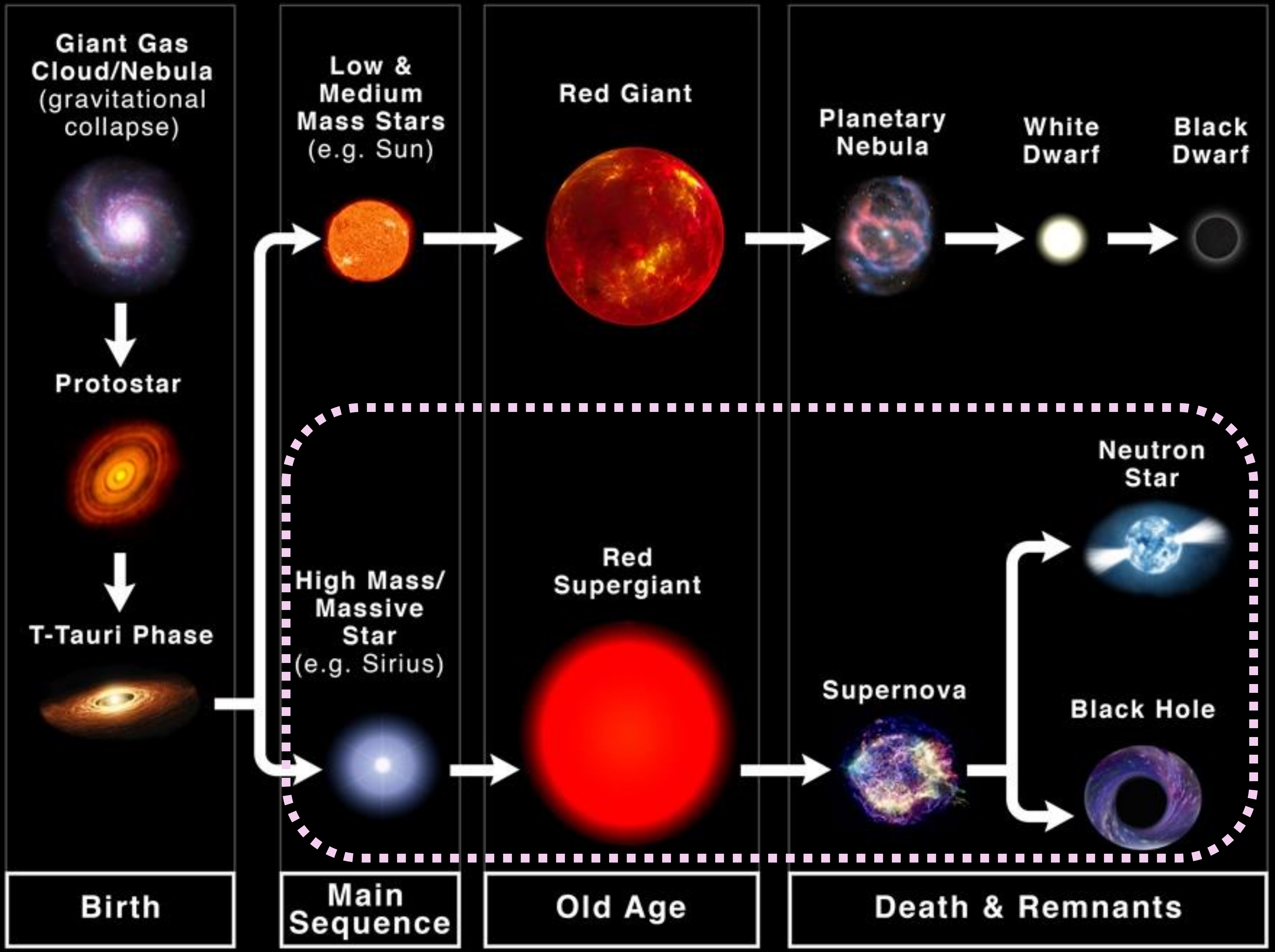

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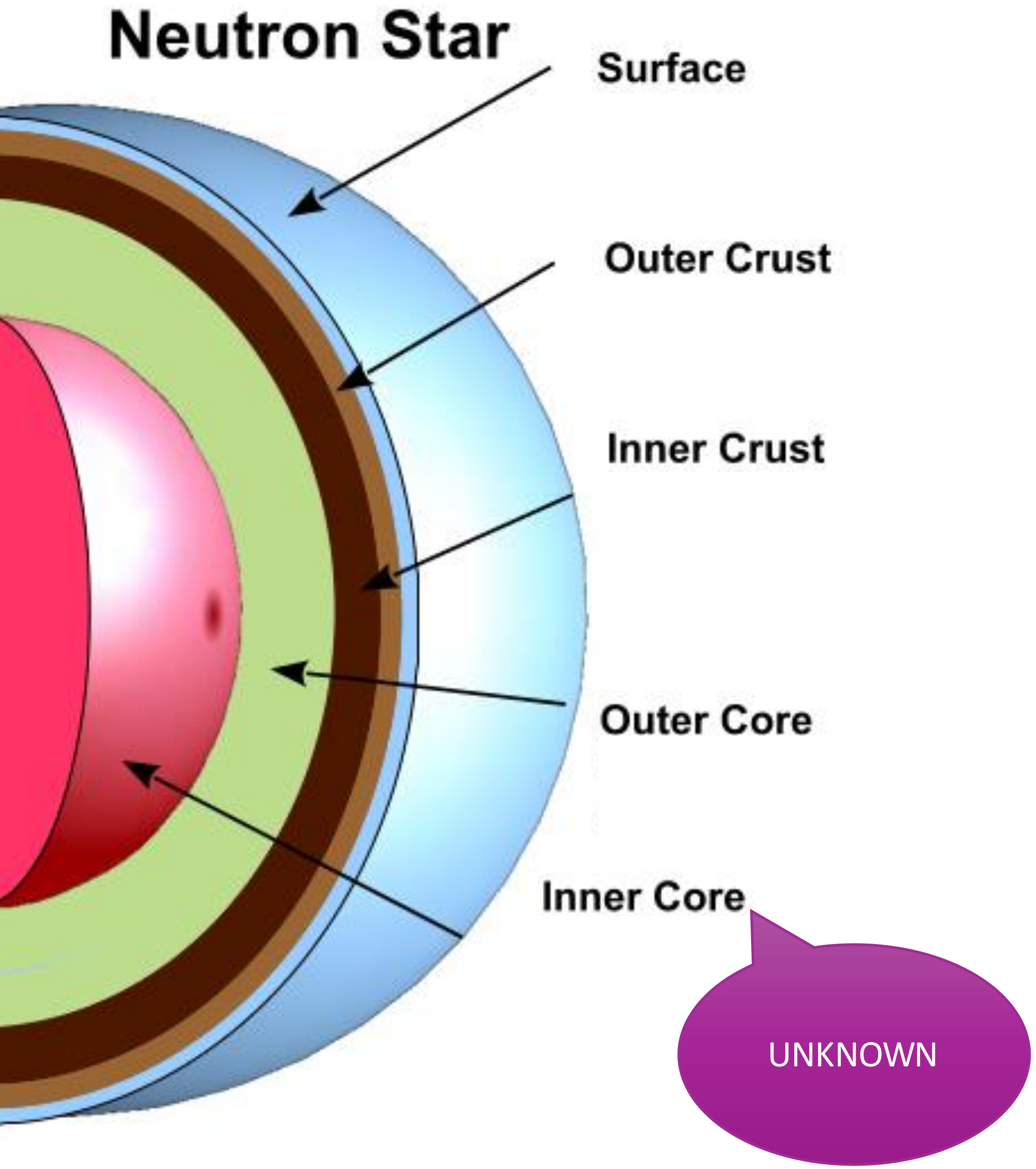


Image Source: astrobites.org

Equation of State

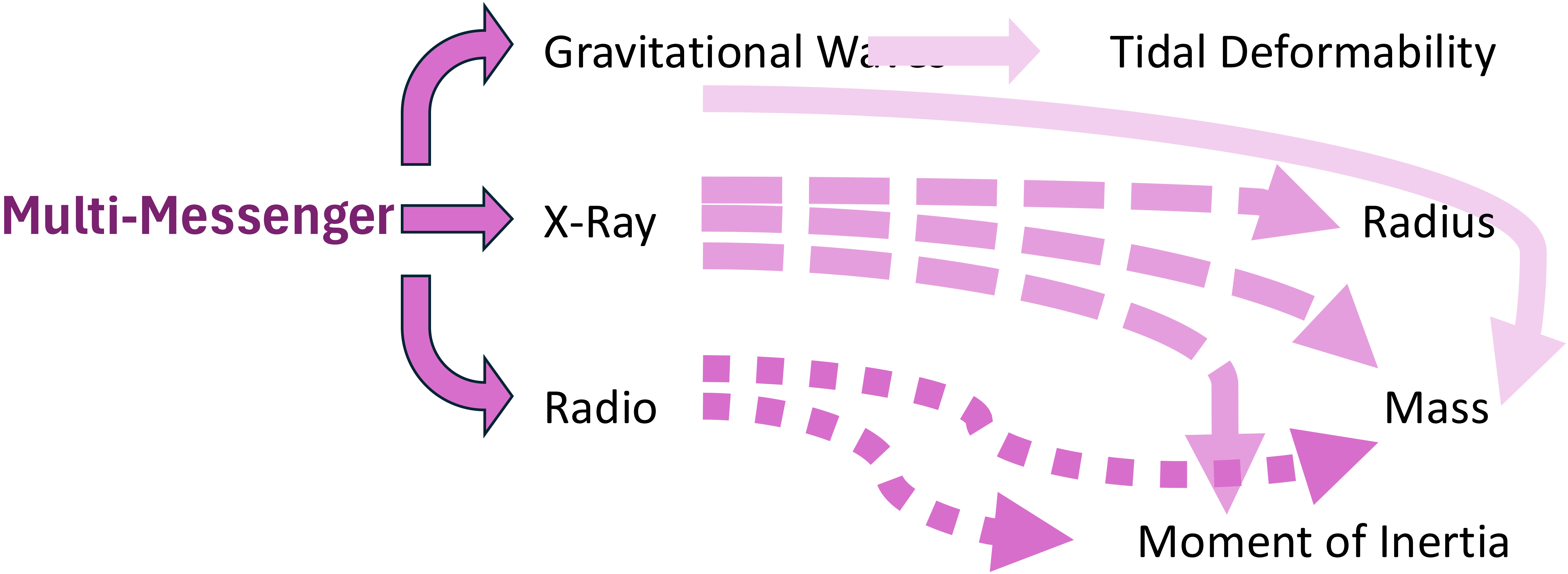
- Pressure density density relation containing information about composition

- **EoS is not known** from first principles at high density — multiple models exist –
    - Vary in assumed **constituents**: nucleons, hyperons, quark matter, boson condensates
    - Differ in **interactions and stiffness**, predicting different mass-radius relations.

Helps us understand fundamental Physics!

Probing Neutron Star Matter Using Multi-Messenger Timing Results

- Constraining the EoS is difficult due to uncertain nuclear interactions at high densities — demands multiple observational constraints.



Modelling
Neutron Star Structure –
Mass Radius Calculation

A star's structure is governed by the hydrostatic equilibrium equation

→ Balances gravitational force and pressure gradient

$$\frac{dP}{dr} = -\frac{GM(r)\rho(r)}{r^2}$$

and the mass profile equation:

$$\frac{dM(r)}{dr} = 4\pi r^2 \rho(r)$$

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In a Neutron Star, general relativistic effects become essential

→ Gravity is extremely strong; pressure itself contributes to gravity

→ The Tolman–Oppenheimer–Volkoff (TOV) equation is used (non-rotating approximation)

$$\frac{dP}{dr} = -\frac{G\left[M(r) + \frac{4\pi r^3 P(r)}{c^2}\right]\left[\rho(r) + \frac{P(r)}{c^2}\right]}{r^2\left[1 - \frac{2GM(r)}{rc^2}\right]}$$

TOV equations are coupled, nonlinear differential equations requiring numerical methods for solutions.

No closed-form analytical expression exists for realistic equations of state.

Hence, tabulated EoS are used, with interpolated values for solving.

Integration using RK45 Algorithm

Start with initial conditions

Choose central density ρ_c

Use EoS to compute corresponding central pressure P_c

Cubic spline interpolation in log-space to obtain pressure and density values from the tabulated EoS

Use the EoS (interpolated values) to compute density $\rho(r)$ from the pressure $P(r)$ at each radius r in the TOV equations

Repeat the process outward, adjusting $P(r)$ and $\rho(r)$ at each step until $P(r)$ falls to ≈ 0

Modelling
Neutron Star Structure –
Tidal Deformability Calculation

Tidal deformability measures how much an object, like a neutron star, deforms in response to the gravitational field of a companion – calculated for no-rotation case

$$\Lambda = \lambda \left(\frac{GM}{c^5}\right)^{-5}$$

The above expression is for dimensionless tidal deformability which can be obtained from GW observations

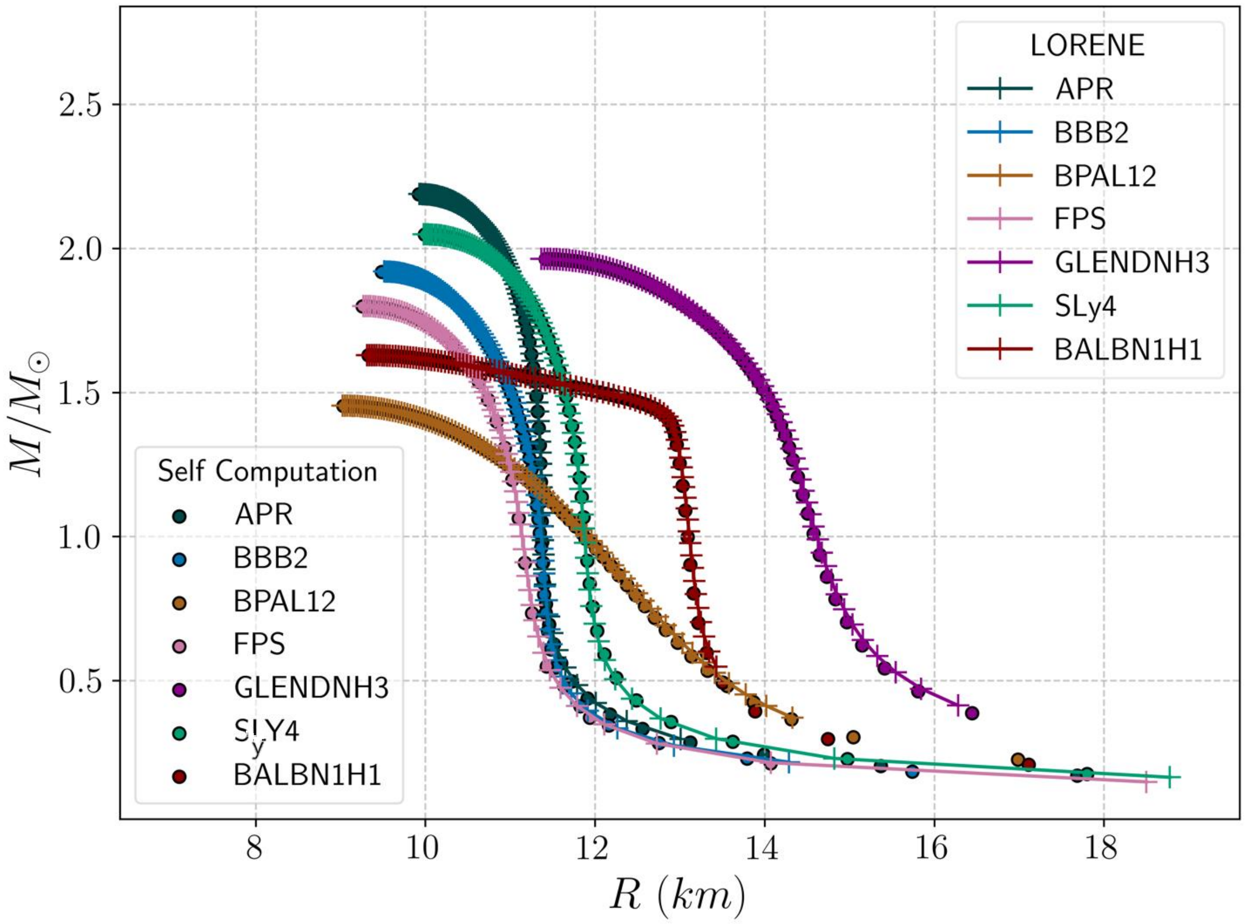
$$\lambda = \frac{2}{3} R^5 k_2$$

k_2 is the Tidal Love Number which measures the rigidity of a star in response to an external tidal potential.

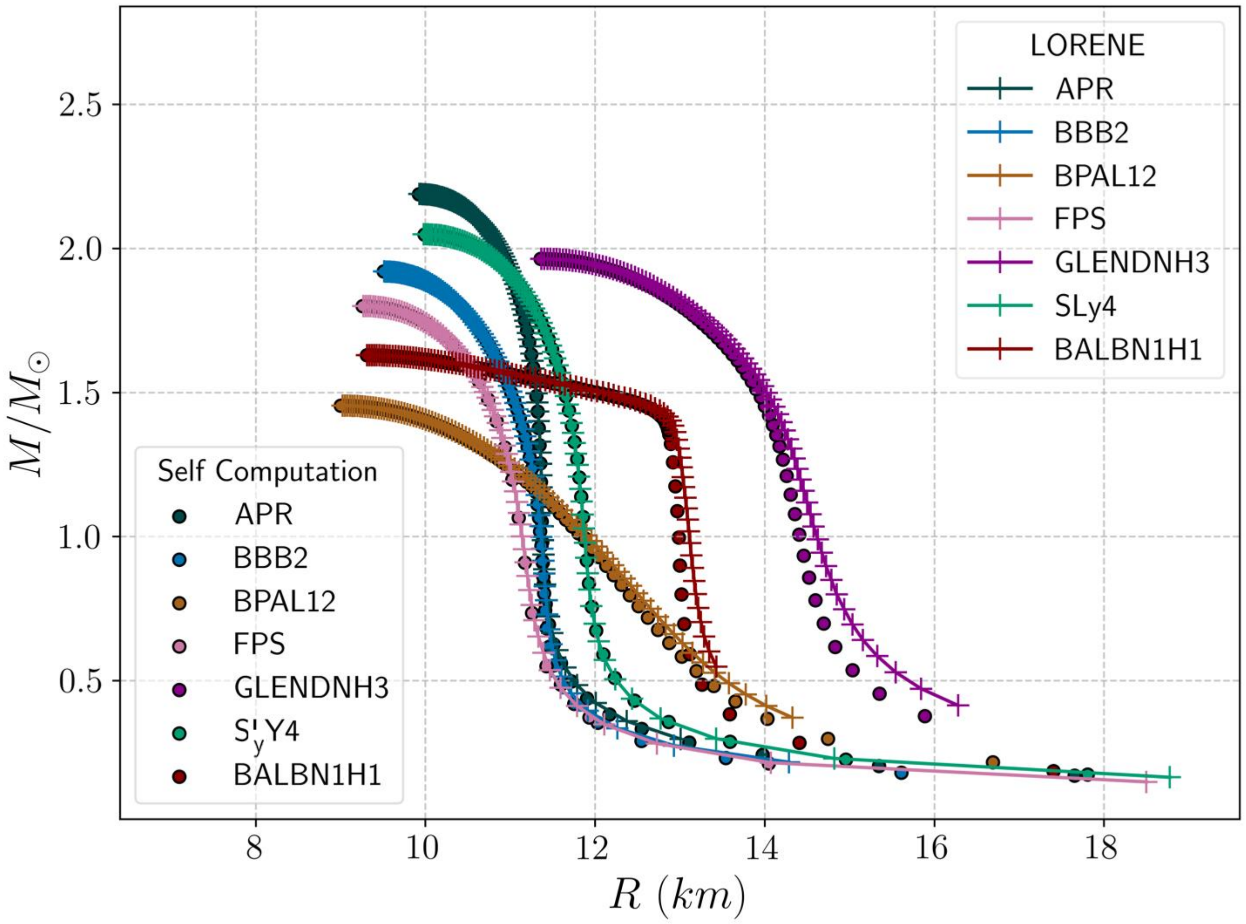
Moment of Inertia Calculations - LORENE

- Solves Einstein’s field equations directly
- Accounts for rotation in neutron stars (unlike Schwarzschild metric, which assumes static, non-rotating stars)
- Has pre-built equations of state
- Linear interpolation of EoS — errors only at very low masses (not relevant for observed neutron stars)

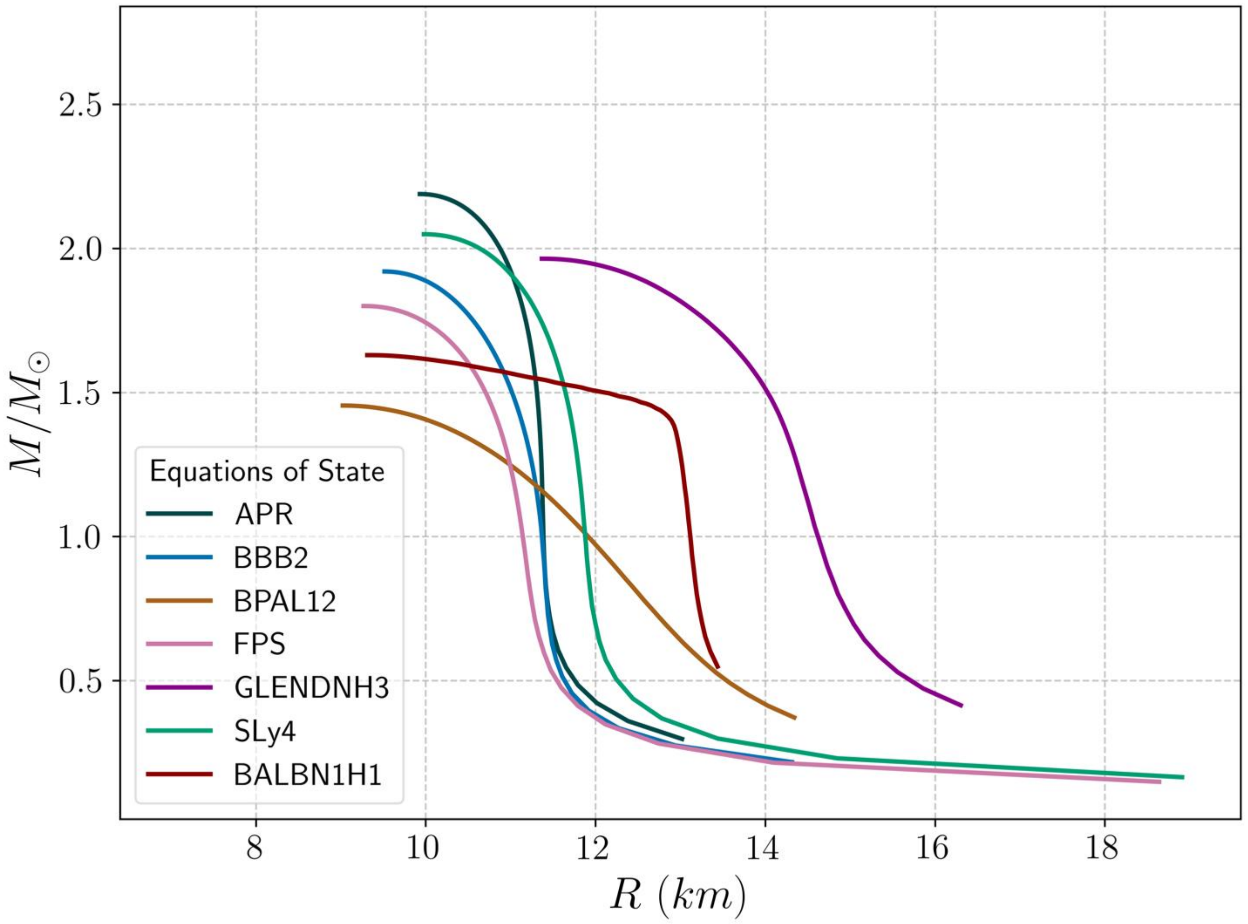
Results



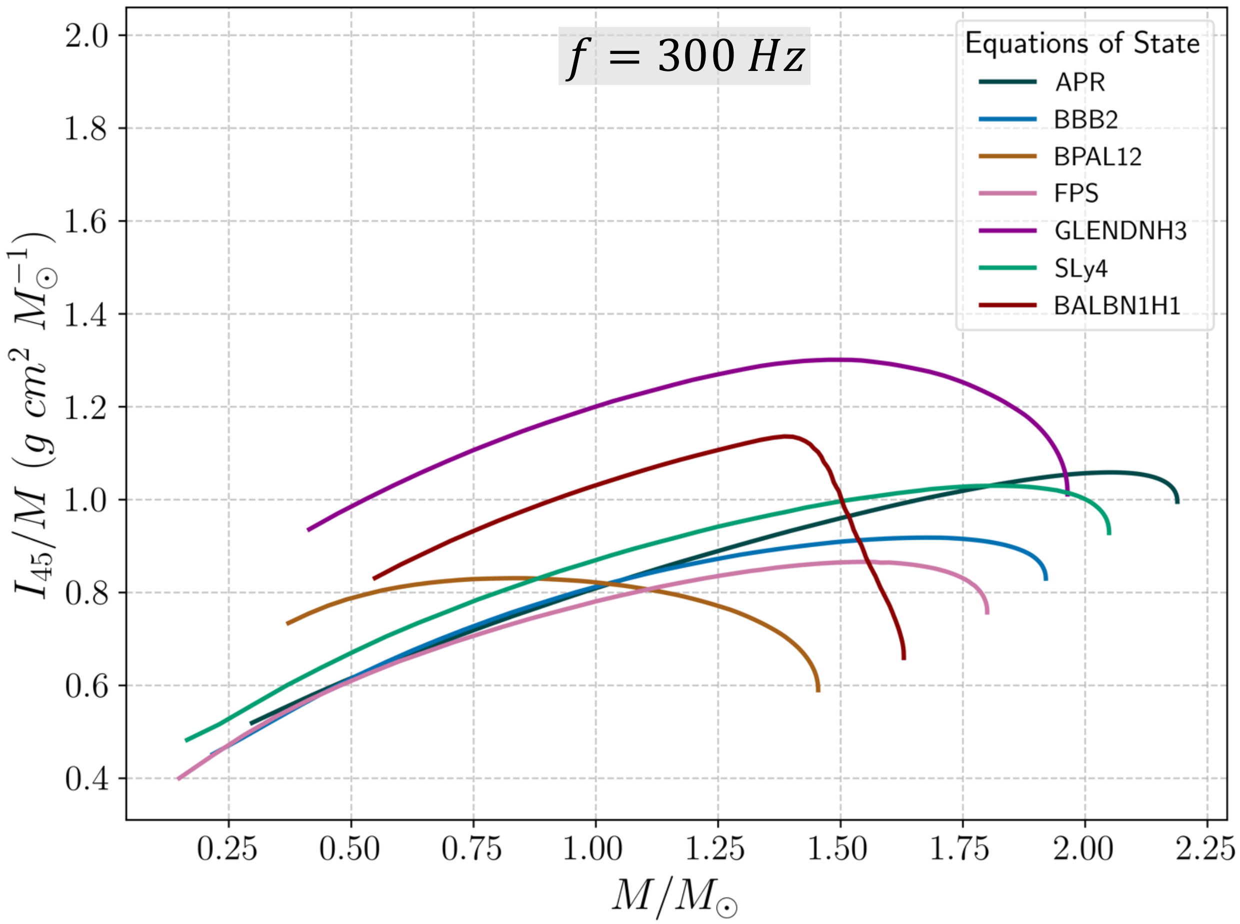
Mass-radius relation obtained from TOV equations using linear interpolation, compared with results generated by the LORENE software for the non-rotating case



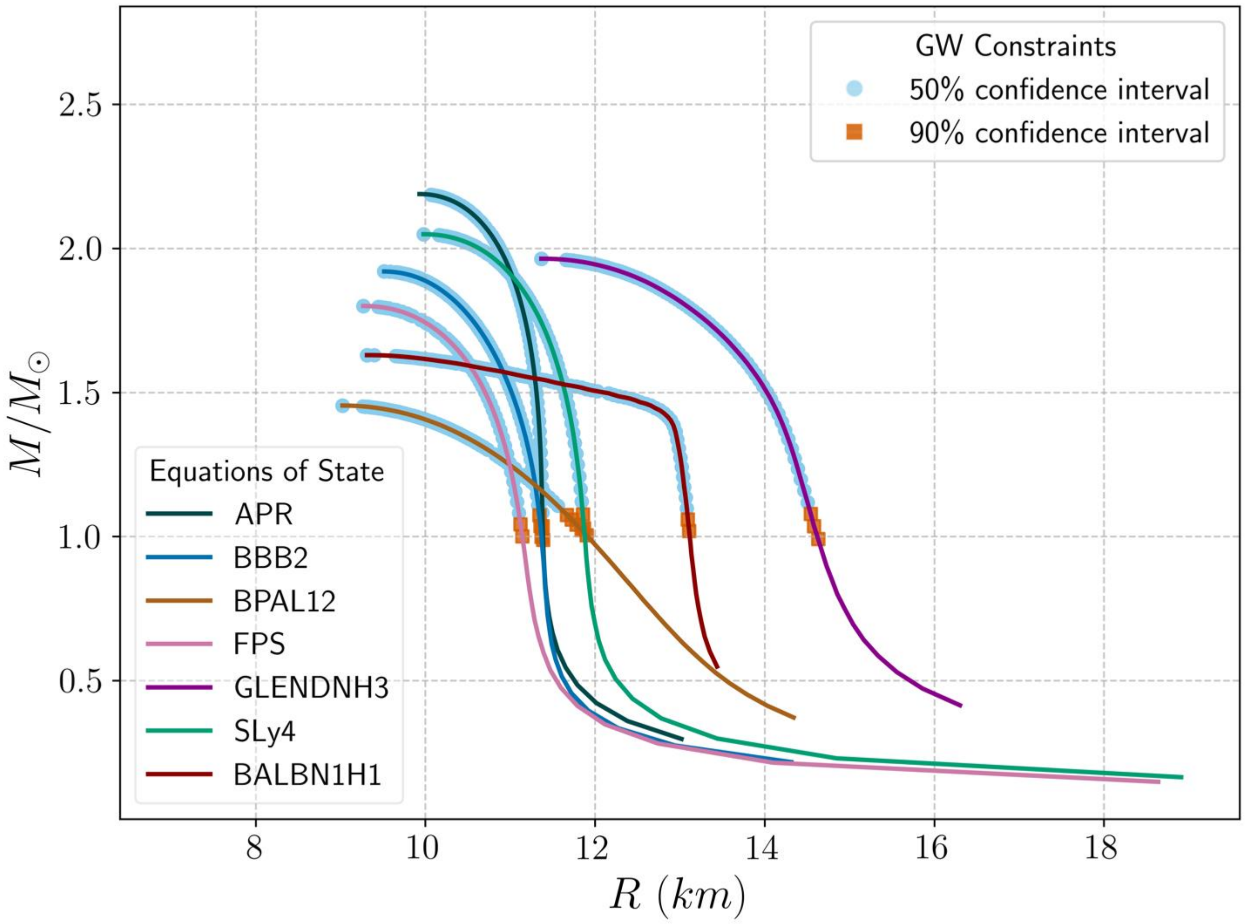
Mass-radius relation obtained from TOV equations using cubic spline interpolation, compared with results generated by the LORENE software for the non-rotating case



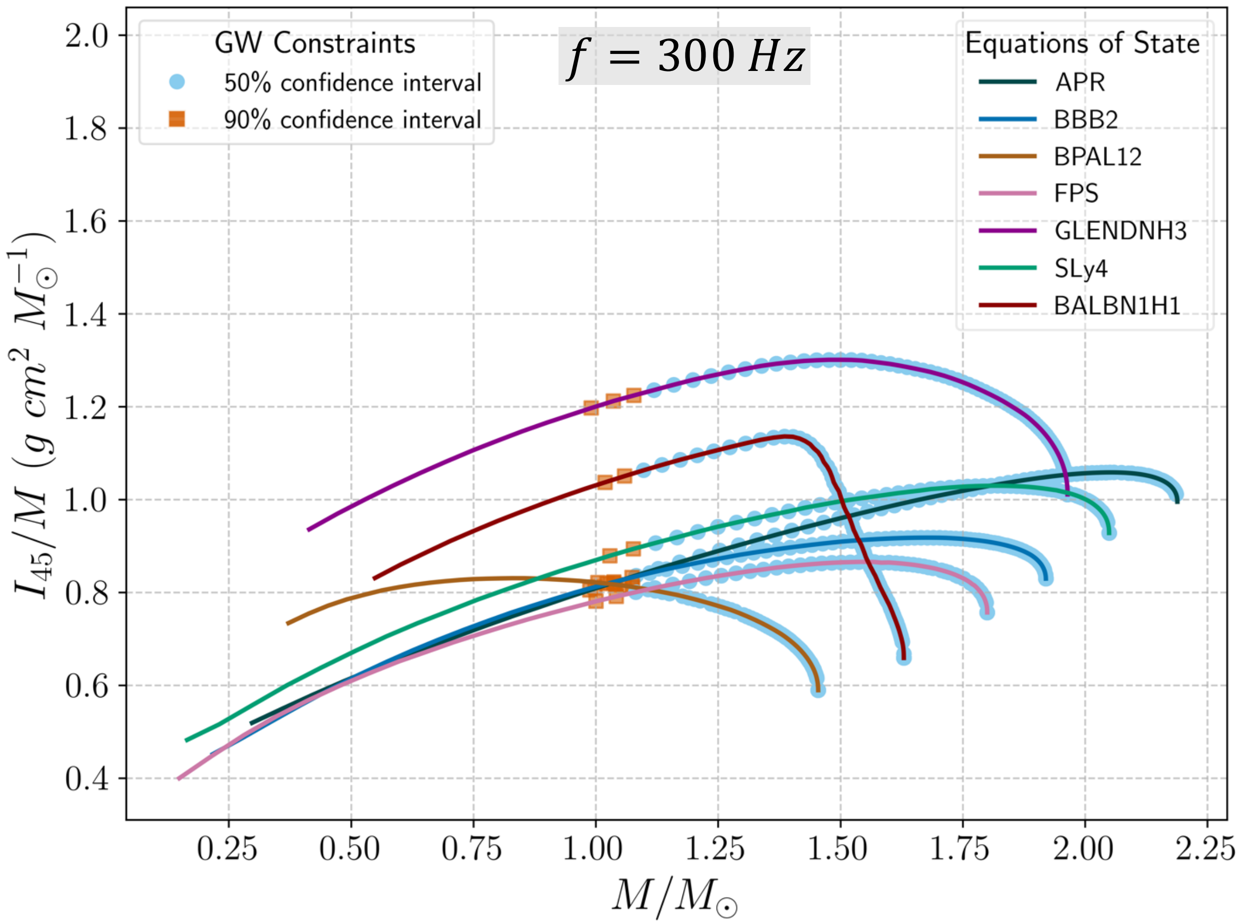
Mass vs. Radius plot obtained by solving neutron star structure



Moment of Inertia vs. Mass plot obtained from LORENE



Mass vs. Radius plot +
Tidal Deformability constraints obtained from GW170817



Moment of Inertia vs. Mass plot obtained from LORENE
+
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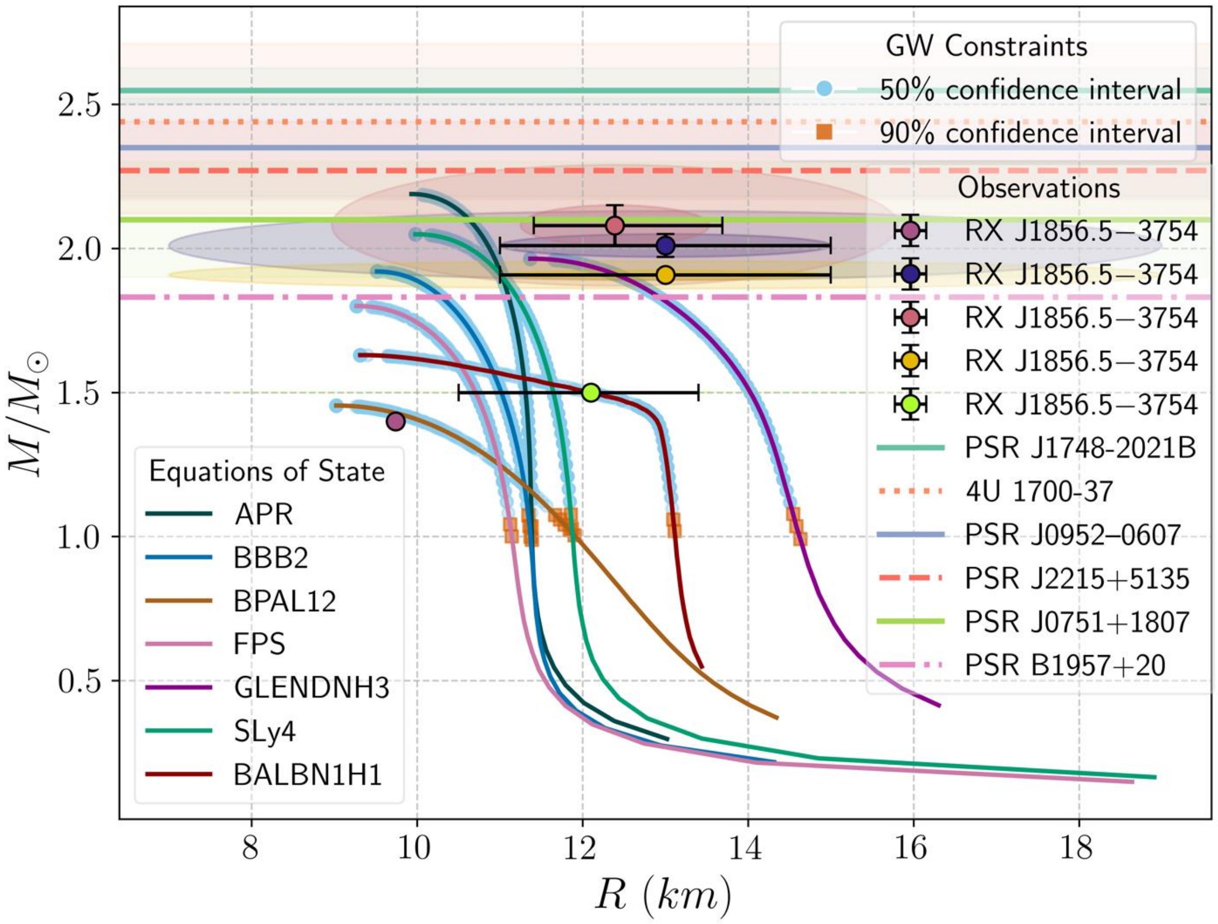
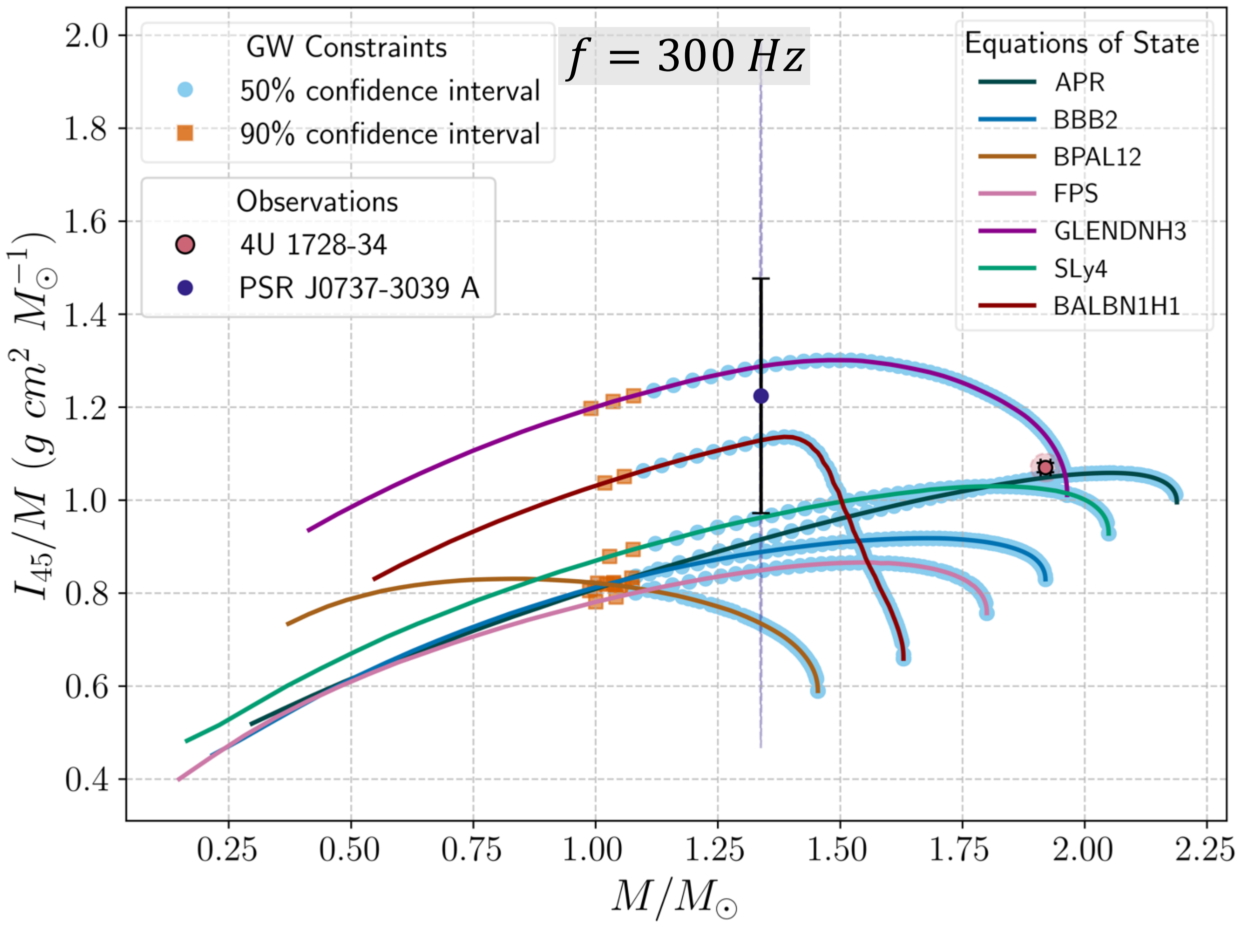


Tidal Deformability constrained Mass vs. Radius + constraints from X-Ray and Radio timing observations



Tidal Deformability constrained Moment of Inertia vs. Mass plot + constraints from X-Ray and Radio timing observations

Results so far

Stiffer equations of state like APR, SLy, and GLENDNH3 fit the mass-radius and MOI observational constraints well.

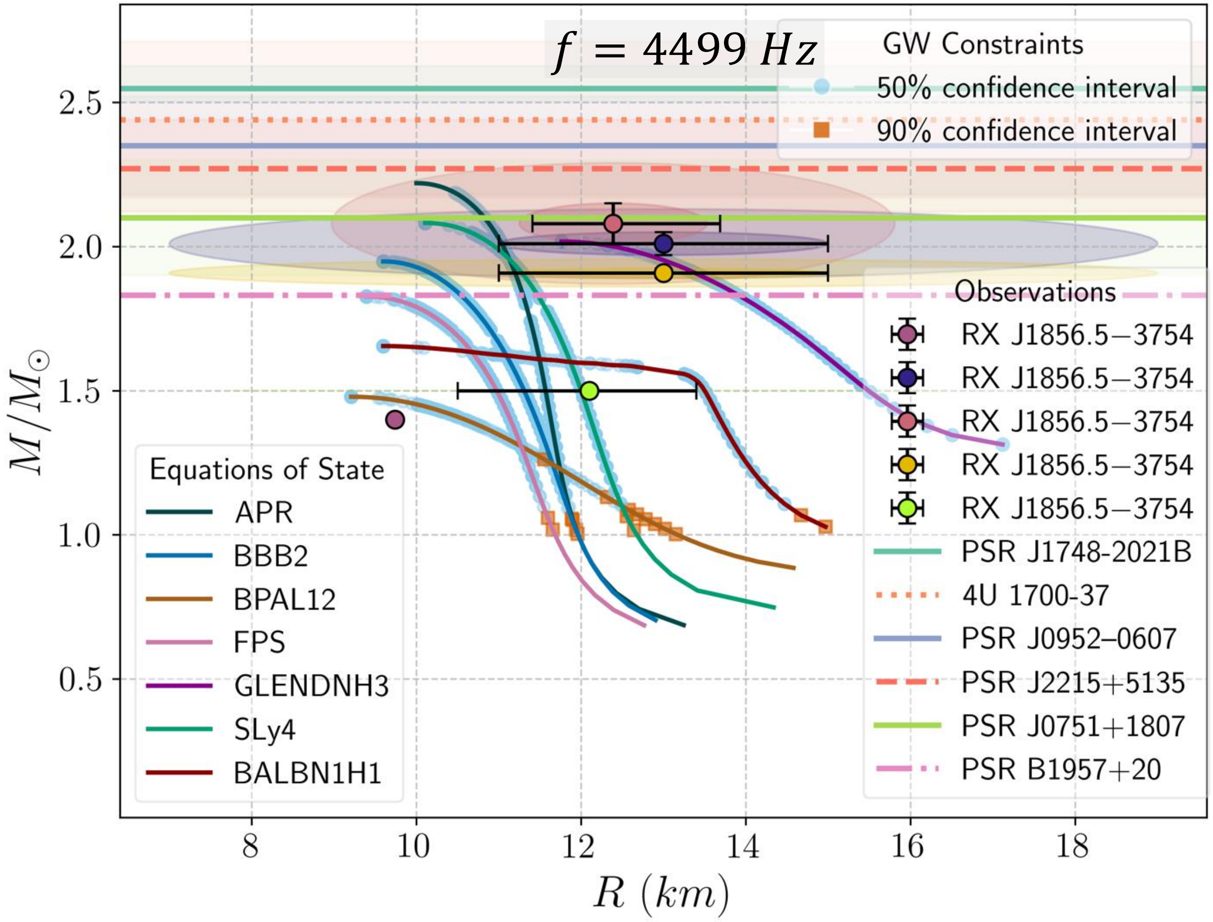
Exotic matter EoS (those with hyperons, mesons, etc.) like BPAL12, BLBN1H1 tend to predict lower maximum masses, and are increasingly disfavoured by current observations.

Gravitational wave constraints and observational data suggest neutron stars are unlikely to exist below ~ 1 $M_{\odot}$

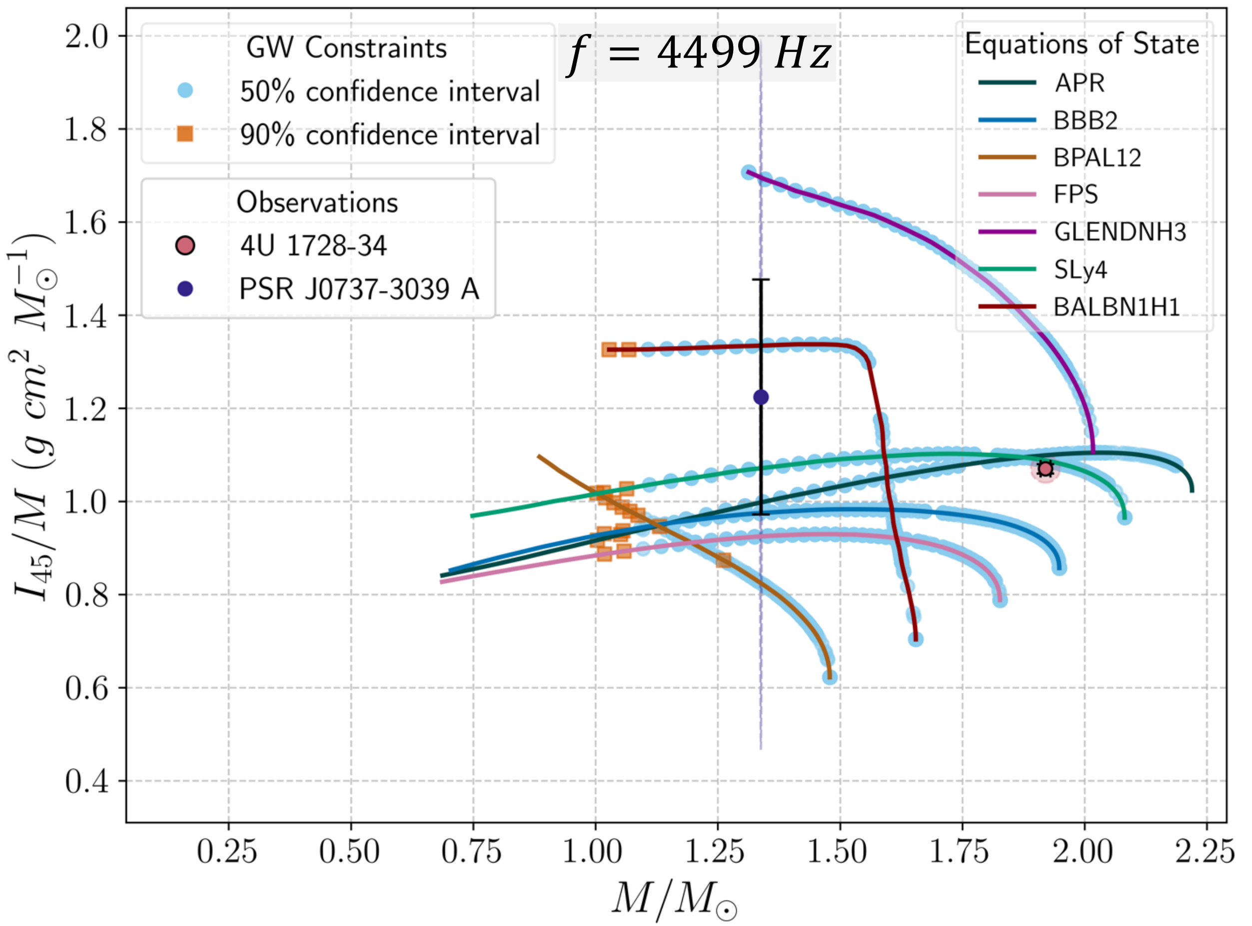
Future Directions

- Study more sophisticated EoS, including temperature-dependent models.
- Apply Bayesian inference to statistically constrain EoS using observational data.
- Extend observational inputs by analyzing X-ray binary QPO data to estimate moment of inertia.
- Explore high-rotation cases and assess how constraints shift with rapid rotation.

Future Directions



Mass vs. Radius plot for a rapidly rotating neutron star (4499 Hz) with observational constraints



Moment of Inertia vs. Mass plot for a rapidly rotating neutron star (4499 Hz) with observational constraints

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Thank You

Interpolation

Moment of Inertia (MOI) and tidal deformability calculations are especially sensitive to smoothness, since they involve second derivatives of the metric — cubic spline helps maintain this regularity.

Linear Interpolation

Between two points (x_0, y_0) and (x_1, y_1),
the interpolated value at x is:

$$y(x) = y_0 + \frac{(y_1 - y_0)(x - x_0)}{(x_1 - x_0)}$$

•It's piecewise straight lines connecting the data points.
•First derivative (slope) is discontinuous at data points.
•No curvature is captured.

Cubic Spline Interpolation

Instead of straight lines, a cubic polynomial is fitted between each pair of points.
Between x_i and $x_{i\,+\,1}$, you use:

$$y(x) = ai(x - xi)^3 + bi(x - xi)^2 + ci(x - xi) + di$$

Coefficients a_i, bi, ci, di are chosen such that:
•The curve is continuous.
•The first and second derivatives are also continuous at every data point.
•Smooth transitions without sharp kinks.

Integration

RK4 (Fourth-Order Runge-Kutta)

•Fixed step size.

•4 function evaluations per step; accuracy $\mathcal{O}(h^4)$.

•Good for smooth, well-behaved problems.

•**Problem** : In TOV, pressure and density change slowly in the core but extremely rapidly near the surface.

•Fixed step size either:

- Wastes effort (small steps where not needed), or
- Misses sharp features (if too large).

RK45 (Fehlberg / Dormand–Prince)

•Adaptive step size method.

•Computes both 4th and 5th order estimates at each step.

•If they differ too much, step is rejected and recomputed smaller.

•**Advantage:**

Automatically shrinks step size near rapid changes (surface) and grows it in smooth regions (core).

TOV equations are coupled, nonlinear differential equations requiring numerical methods for solutions.

No closed-form analytical expression exists for realistic equations of state due to complex microphysics.

Both $P(\rho)$ and $\rho(P)$ are necessary to solve TOV equations – but realistic EoS are usually not invertible.

Hence, tabulated EoS are used, with interpolated values for solving.

Piecewise polytropes can be used to approximate results for easier and faster computation.

$$k_2 = \frac{8C^5}{5}(1-2C)^2\left[2+2C(y-1)-y\right] \times \{2C\left[6-3y+3C(5y-8)\right]$$
$$+4C^3\left[13-11y+C(3y-2)+2C^2(1+y)\right] + 3(1-2C)^2\left[2-y+2C(y-1)\right]\ln(1-2C)\}^{-1}$$

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$$y = r\frac{V(R)}{H(R)}$$

Potential calculated from the metric – comes from solving Einstein's equations

$$C = \frac{GM(R)}{Rc^2}$$

Compactness of the star – calculated from the final mass M and radius R

$$\frac{d\,V(R)}{dr} = -V\left(\frac{2}{r} + z\left(\alpha\frac{m}{r^2} + 2\pi r\alpha\left(\frac{P}{c^2} - \rho\right)\right)\right)$$

$$-H\left(\frac{-6z}{r^2} + 2\pi\alpha z\left(5\rho + \frac{9P}{c^2} + \frac{\rho + \frac{P}{c^2}}{d\rho/dP}\right) - z^2\left(-\alpha\left(4\pi r\rho - \frac{m}{r^2}\right)\right)^2\right)$$

$$z = \left(1 - \alpha\frac{m}{r}\right)^{-1}$$

$$\alpha = \frac{2G}{c^2}$$

$$\frac{d\,H(R)}{dr} = V$$

These coupled differential equations are also solved along with the TOV equations to calculate tidal deformability

Spectral Methods - LORENE

Numerical techniques solving differential equations by expanding solutions in terms of basis functions (e.g., Fourier series, Chebyshev polynomials).

Advantages:

High Accuracy: Exponential convergence for smooth solutions.
Efficient: Fewer degrees of freedom needed for high precision.
Global Convergence: Captures global behaviour, not just local, Exponential convergence for smooth problems: error $\sim e^{-N}$, where N is the number of terms.

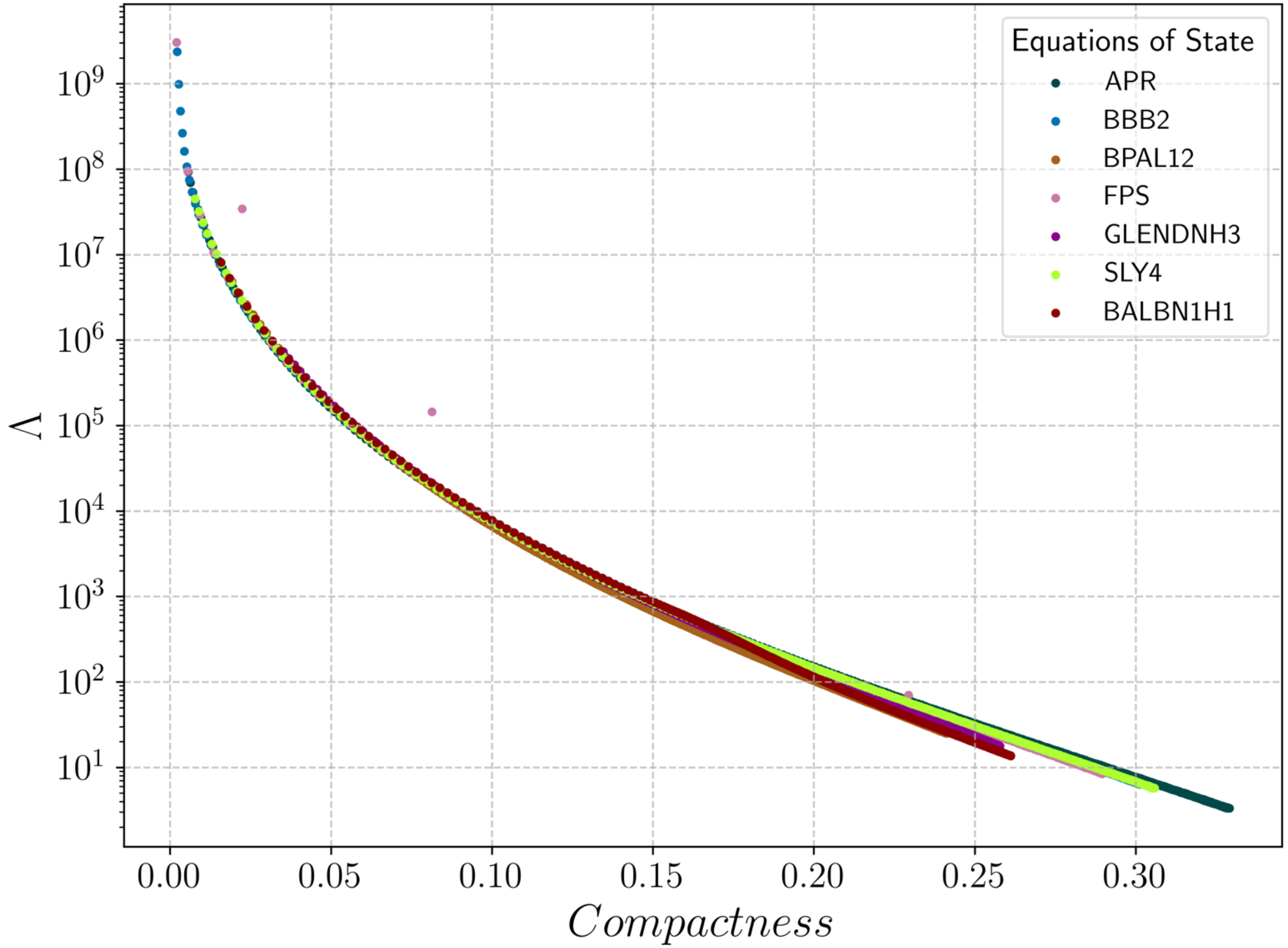
$$u(x) \approx \sum_{n=0}^{N} a_n \phi_n(x)$$

$u(x)$ is the solution.
$\phi_n(x)$ are the basis functions (e.g., sinusoids, Chebyshev polynomials).
a_n are the expansion coefficients.

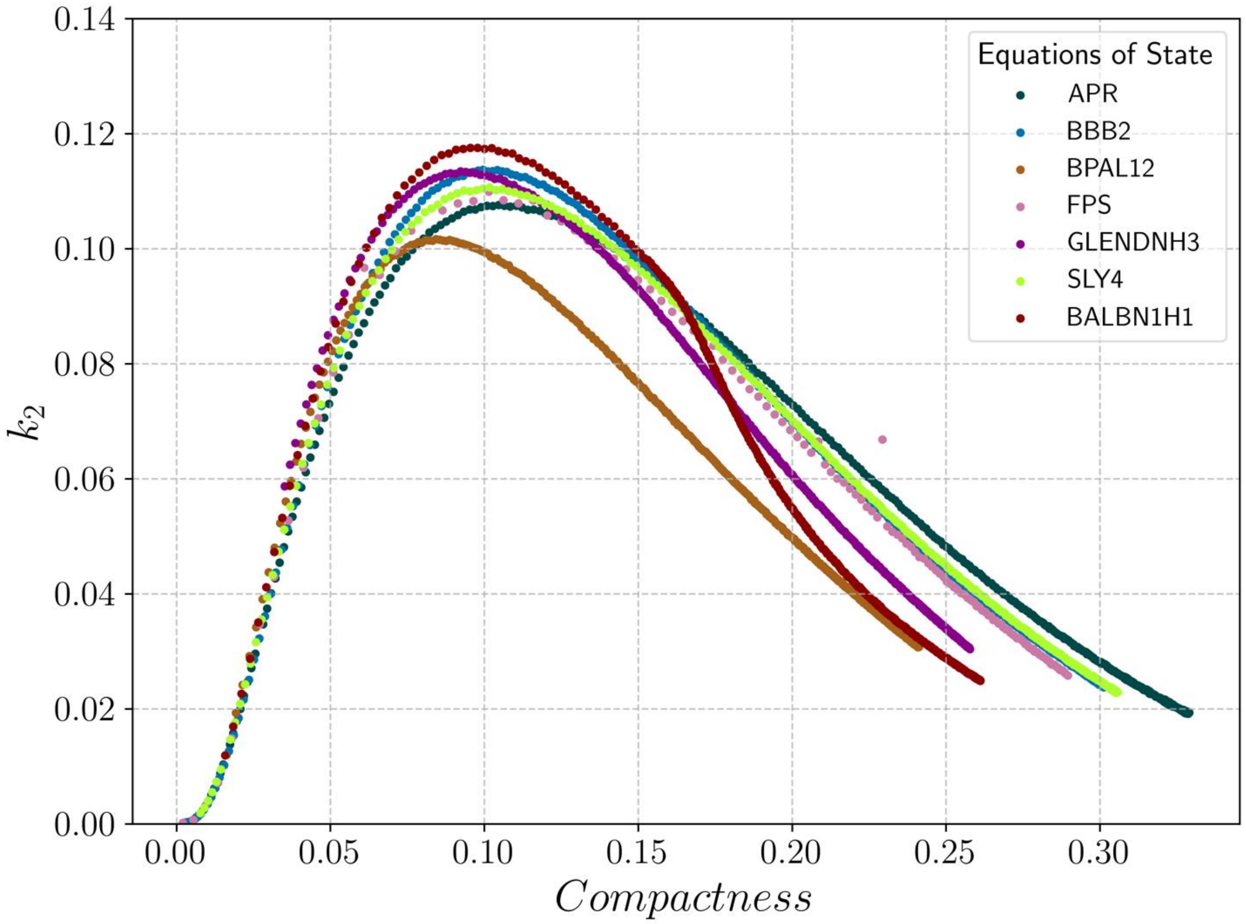
Usage in LORENE:

Precision: Essential for accurately solving complex Einstein field equations taking into account complex geometries and utilising symmetries
Efficiency: Faster convergence compared to traditional methods (e.g., finite differences).

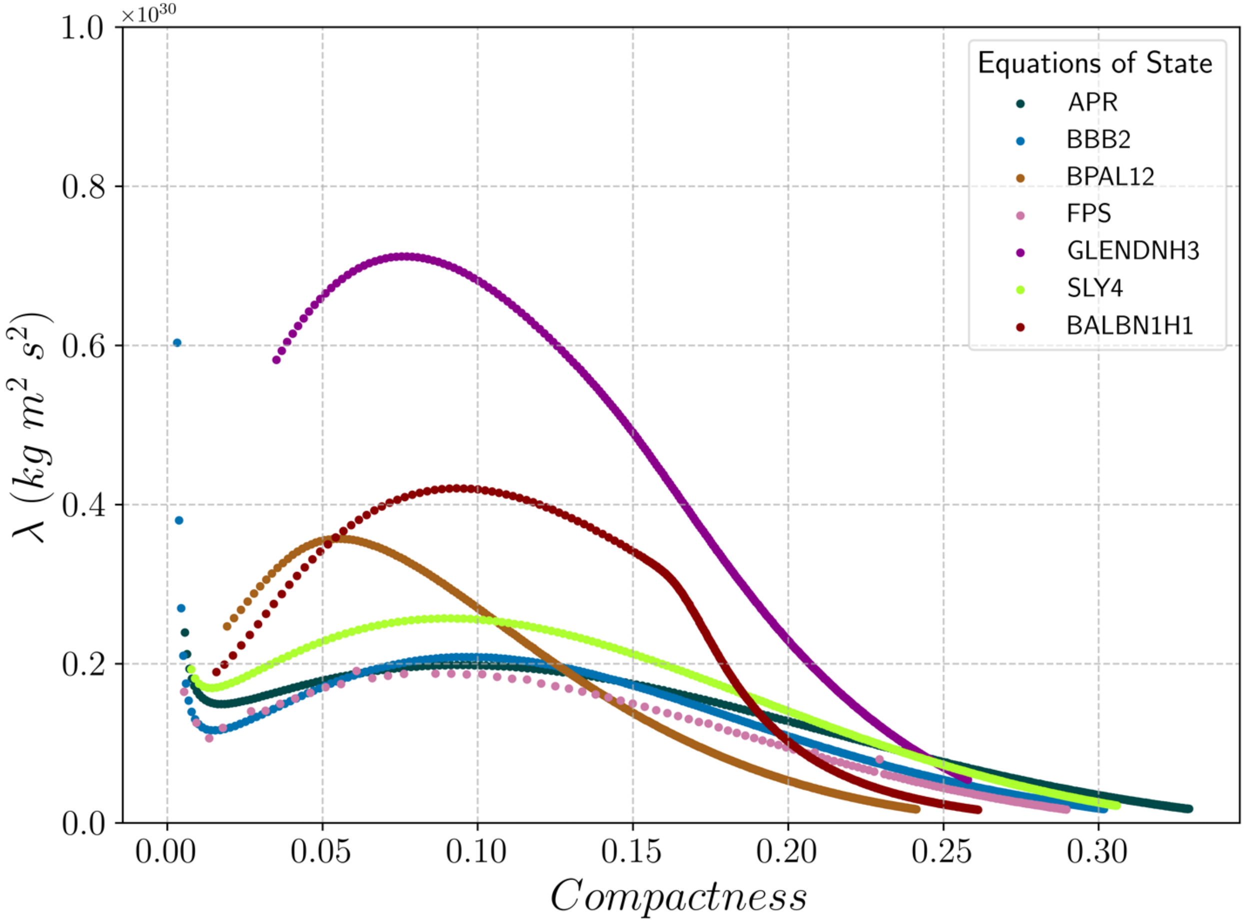
Additional Plots



Dimensionless Tidal Deformability vs Compactness plot obtained by solving neutron star structure



Tidal Love Number vs Compactness plot obtained by solving neutron star structure for the non-rotating case



Tidal Deformability vs Compactness plot obtained by solving neutron star structure for the non-rotating case

Observational Data Analysis – X-Ray

Data: X-ray flux over time and energy from isolated neutron stars or neutron stars in binaries (especially LMXBs); observed by NICER, XMM-Newton, Chandra.

(A) Thermonuclear Bursts:

Accreting neutron stars undergo nuclear bursts; X-rays fit a blackbody spectrum.

Correct for distance, gravitational redshift, and atmosphere composition to constrain mass and radius.

(B) Pulse Profile Modelling (NICER):

Rotating neutron stars with hot spots emit pulsed X-rays.

Pulse shape distorted by light bending (depends on compactness MRRM); modeling pulse profiles extracts mass and radius.

Moment of Inertia (rare):

In special binaries (e.g., PSR J0737−3039), spin-orbit coupling subtly alters orbital motion.

Long-term measurements can extract II, though usually inferred by combining mass, radius, and spin data.

Source info needed: Distance (parallax or cluster membership), atmosphere composition, rotation axis inclination, background noise levels.

Observational Data Analysis – Radio

Data: Radio pulse arrival times from pulsars, timed to sub-microsecond precision; sources are radio pulsars (especially in binaries) observed by Arecibo, Green Bank, FAST, and SKA

Analysis: Pulsar spins produce regular radio pulses; orbital motion modulates pulse timing.
From **Keplerian** (period, eccentricity) and **post-Keplerian** (relativistic) parameters, we measure neutron star and companion masses.

Shapiro Delay: Extra pulse delay through companion's gravity: $\Delta t \propto -2\frac{GMc}{c^3}\ln(1+\sin i)$

Orbital Decay: Shrinkage due to gravitational wave emission, mass-dependent.

Periastron Advance: Shift of orbital closest point, depending on mass.

Source info needed: Binary inclination, orbital parameters; distance (helpful for corrections but not critical).

Observational Data Analysis - GW

Data: Gravitational wave strain h(t)h(t) from binary neutron star mergers (e.g., GW170817), detected by LIGO, Virgo, and KAGRA.

Features: Frequency chirp — signal increases in frequency and amplitude as stars spiral inward.

Analysis: Neutron stars tidally deform each other, modifying the inspiral phase.

Compare observed waveforms with models through **Bayesian parameter estimation** to extract probability distributions for Λ.

Source info needed: Rough mass estimates and assumption that the objects are neutron stars ($\sim 1.2 - 2\, M_{\odot}$).