18.217 Lecture Notes

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Lecture notes for the MIT class **18.217** (Combinatorial Theory), taught by Professor Alexander Postnikov. All errors are my responsibility.

Contents

§1 Introduction

This class will be about Young tableaux. There are several recommended (but not required) textbooks:

- *Enumerative Combinatorics, Volume 2* by Richard Stanley. (Young tableaux appear specifically in Chapter 7.)
- *Young Tableaux* by William Fulton.
- *The Symmetric Group* by Bruce Sagan.

To start with, we'll define what Young tableaux are.

Definition 1.1

A **partition** $\lambda \vdash n$ is a sequence of positive integers $(\lambda_1, \lambda_2, \ldots, \lambda_k)$ such that $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k$ and $\lambda_1 + \lambda_2 + \cdots + \lambda_k = n$.

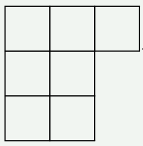
We can think of partitions as ways of writing n as the sum of positive integers, where order does not matter.

Definition 1.2

The **Young diagram** associated with a partition λ is the diagram which has λ_1 boxes in its first row, λ_2 in its second, and so on.

Example 1.3

The partition $(3, 2, 2) \vdash 7$ corresponds to the Young diagram



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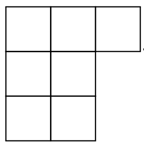
In an abuse of notation, we'll identify the Young diagram with the partition — we may write

$$\lambda = (3, 2, 2) =$$



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Then a Young tableaux is a way of writing numbers inside a Young diagram, following certain rules.

Definition 1.4

A **standard Young tableau** of shape λ is a way of writing the numbers 1, 2, $\ldots$, n in the boxes of the Young diagram associated to λ (without repetition) such that the numbers increase in each row and each column.

We often abbreviate "standard Young tableau" as SYT.

Example 1.5

One standard Young tableau of shape $(3, 2, 2)$ is

$$\begin{array}{|c|c|c|} \hline 1 & 2 & 4 \\ \hline 3 & 6 \\ \cline{1-2} 5 & 7 \\ \cline{1-2} \end{array}.$$

We'll later discuss *semistandard* Young tableaux as well (where repetition is allowed).

Young tableaux are interesting because they appear in several different areas of math:

- The representation theory of GL_n, SL_n, and S_n.
- Symmetric functions (polynomials which remain invariant when we permute their variables).
- Geometry, in particular Schubert calculus (for example, this involves Grassmanians and flag manifolds, and relates to Schubert polynomials).

This is part of why we have three textbooks — Stanley's textbook focuses mostly on symmetric functions, Fulton's on geometry, and Sagan's on representations of S_n. These perspectives are closely related — for example, we can go from representation theory to symmetric functions by taking the *characters* of the representations.

These perspectives all have extensions. In particular, Young tableaux can be identified with Gelfand–Tsetlin patterns, which provides a polytopal interpretation of Young tableaux. There are various bijections, such as the Robinson–Schensted–Knuth correspondence, which can be understood from the point of view of polytopes — for example, we can cut up one polytope and rearrange its pieces to produce another (this is sometimes called *piecewise linear combinatorics*). This then is related to tropical geometry, and then to birational combinatorics, cluster algebras, and other topics. We probably won't have time to see *all* these connections, but there are many topics in different areas of math that Young tableaux relate to.

§2 Counting Standard Young Tableaux

§2.1 The Catalan Numbers

Question 2.1. Consider the partition

$$\lambda = (n, n) = \begin{array}{|c|c|c|c|c|c|}\hline & & & \cdots & & \\ \hline & & & \cdots & & \\ \hline \end{array}$$

of $2n$. How many standard Young tableaux of shape λ are there?

This has a very nice answer — in fact, the answer is one of the most famous sequences in combinatorics.

Theorem 2.2

The number of standard Young tableaux of shape (n, n) is the nth Catalan number $C_n = \frac{1}{n+1}\binom{2n}{n}$.

The Catalan numbers have *many* combinatorial interpretations (Richard Stanley has a list of about 200 interpretations). They're most commonly defined as the number of *Dyck paths* of length $2n$:

Definition 2.3

A **Dyck path** of length $2n$ is a path from $(0, 0)$ to $(2n, 0)$ consisting of n up-steps $(1, 1)$ and n down-steps $(1, -1)$, such that the path always remains weakly above the x-axis.

Definition 2.4

The nth **Catalan number** C_n is the number of Dyck paths of length $2n$.

So Theorem 2.2 has two parts — we want to show that the number of standard Young tableaux of shape (n, n) is equal to the number of Dyck paths of length $2n$, and that this number of Dyck paths is $\frac{1}{n+1}\binom{2n}{n}$.

Proposition 2.5

There exists a bijection between standard Young tableaux of shape (n, n) and Dyck paths of length $2n$.

Proof. We'll first illustrate the bijection by example:

Example 2.6

When $n = 5$, the standard Young tableau

1	2	4	7	9
3	5	6	8	10

corresponds to the Dyck path $++-+--+-+-$:

In general, we read numbers one at a time. For each i, if i appears in the first row then we take an up-step, and if i appears in the second row then we take a down-step. This gives a path from $(0,0)$ to $(2n,0)$. After every step k, the number of $i \le k$ in the first row is at least the number of $i \le k$ in the second, so the path must stay weakly above the x-axis. $\square$

So then the number of standard Young tableaux of shape (n,n) is C_n, and it remains to prove the formula for C_n. There are many different proofs of this formula (by generating functions, or by the reflection method), but Professor Postnikov's favorite is the following proof by cyclic shifts:

Proposition 2.7

For all n, we have $C_n = \frac{1}{n+1}\binom{2n}{n}$.

Proof. We can rewrite our formula as

$$\frac{1}{n+1}\binom{2n}{n} = \frac{1}{2n+1}\binom{2n+1}{n}.$$

This suggests a nice combinatorial interpretation — $\binom{2n+1}{n}$ is the number of *all* lattice paths from $(0,0)$ to $(2n+1,-1)$, meaning paths with n up-steps and $(n+1)$ down-steps.

Example 2.8

One such path for $n = 5$ is $+---++-++--$:

Call such a path an *almost Dyck path* if only the last step goes below the x-axis — so an almost Dyck path is a Dyck path with one extra down-step at the end. Then we want to show that the probability that a randomly chosen path to $(2n+1,-1)$ is an almost Dyck path is exactly $\frac{1}{2n+1}$.

To do so, we'll form groups of $2n+1$ paths, by grouping together all *cyclic shifts* of one path:

Example 2.9

Suppose our initial path is $+--++--$. Then we have a group consisting of its seven cyclic shifts:

$$\begin{gathered}
+--++-- \\
--++--+ \\
-++--+- \\
++--+-- \\
+--+--+ \\
--+--++ \\
-+--++-
\end{gathered}$$

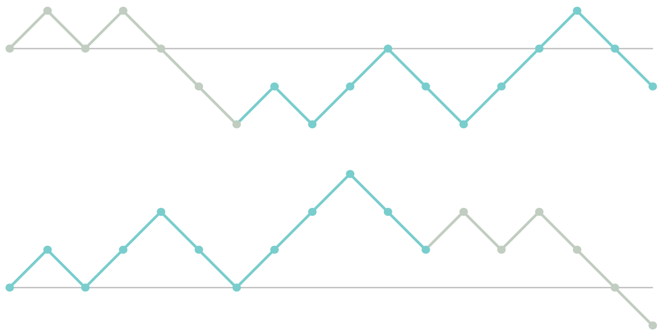
Claim — Each group has exactly $2n+1$ elements — in other words, for any path, all its $2n+1$ cyclic shifts are distinct.

Proof. If two cyclic shifts were the same, then our sequence would be nontrivially periodic — it must consist of the same sequence repeated k times, for some $k > 1$. But there are n up-steps and n down-steps, so k would have to divide n and $n+1$; since $\gcd(n, n+1) = 1$, this is a contradiction. ■

Claim — Exactly one of these cyclic shifts is an almost Dyck path.

In the above example, the cyclic shift $++--+--$ is the only almost Dyck path.

Proof. To produce a cyclic shift of a given path p, we split p into two parts as $p' \circ p''$, and reverse the order of these two parts to get $p'' \circ p'$.



Then the one cyclic shift which produces an almost Dyck path is cutting at the *first minimal point* (as shown above) — it's fairly easy to show that this cyclic shift works, and no others do. ■

So then we can partition the $\binom{2n+1}{n}$ paths into groups of $2n+1$ where exactly one path in each group is an almost Dyck path. So the number of almost Dyck paths, and therefore the number of Dyck paths, is $\frac{1}{2n+1}\binom{2n+1}{n}$. □

§2.1.1 Queue-Sortable Permutations

We've previously mentioned connections from Young tableaux to many different areas of math. But they also appear naturally in computer science — in the third volume of Donald Knuth's famous *The Art of Computer Programming* (which is about sorting), a large part of the book discusses standard and semistandard Young tableau and in particular the Robinson–Schensted–Knuth correspondence (which will be a central topic in this class).

One connection comes from attempting to sort permutations with *queues*. A **queue** is a first-in first-out data structure — we can insert entries at the back of the queue and remove them from the front.

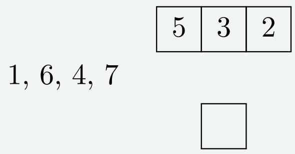
Question 2.10. Which permutations can we sort using some queues?

If we have only one queue, then we can't sort any permutation (except the identity) — all elements come out in the same order they came in, so we can't change the permutation at all.

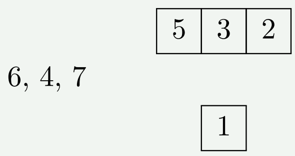
A more interesting case is when we have *two* queues — so we read our permutation one by one, and place numbers into *either* the first or second queue (and we can remove numbers from either queue at any time as well). We say a permutation is **queue-sortable** if it can be sorted using two queues.

Example 2.11

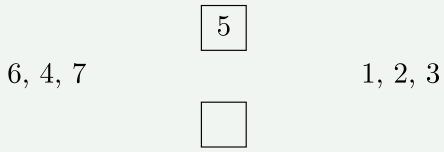
The permutation 2, 3, 5, 1, 6, 4, 7 is queue-sortable. To sort it, we can first insert each of 2, 3, and 5 into the first queue:



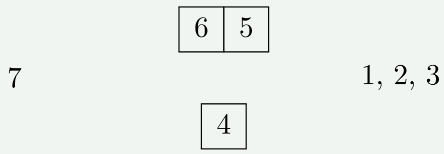
Then we insert 1 into the second queue:



Now we remove 1 from the second queue, then 2 and 3 from the first queue:



Then we add 6 to the first queue, and add 4 to the second:



Finally we remove 4 from the second queue, remove 5 and 6 from the first queue, add 7 to the first queue, and remove it. This produces the sorted permutation 1, 2, 3, 4, 5, 6, 7.

Theorem 2.12

The number of queue-sortable permutations of $[n]$ is C_n.

We can also ask how many permutations we can sort with k queues. In general, there isn't a nice exact answer; but there is an *asymptotic* answer, which can be found using techniques we'll discuss.

§2.2 The Hook Length Formula

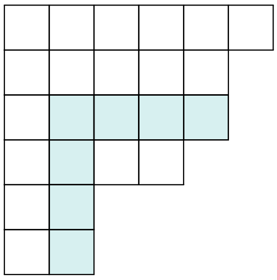
Notation 2.13. For a partition λ, f^λ is the number of standard Young tableaux of shape λ.

In this language, we've shown that $f^{(n,n)} = C_n$. So in some sense, we can think of standard Young tableaux as a generalization of Dyck paths, and f^λ as a generalization of the Catalan numbers. It turns out that there is also a nice formula for f^λ in general.

Definition 2.14

Given a partition $\lambda \vdash n$, for each box $(i, j) \in \lambda$, its **hook length** h_{ij} is the number of boxes below it and to its right, including itself.

For example, the following is a hook, and the corresponding square has hook length 7:



Theorem 2.15 (Frame–Robinson–Thrall)

For any partition λ, we have

$$f^\lambda = \frac{n!}{\prod_{(i,j)\in\lambda} h_{ij}}.$$

Example 2.16

Consider the partition $\lambda = (3, 2)$. The hook lengths are

4	3	1
2	1	

so the hook length formula gives

$$f^{(3,2)} = \frac{5!}{1 \cdot 1 \cdot 2 \cdot 3 \cdot 4} = 5.$$

The original proof of the hook length formula is quite complicated, but people have found nicer proofs. For example, there is a probabilistic proof based on a random walk on the Young diagram (presented in **18.212** last semester). The proof we will see today is the one Professor Postnikov thinks is the best proof — it's the easiest to understand and most conceptual. This proof is especially nice because understanding the essence of the proof also gives a lot of other constructions — the idea relates to RSK, the octahedral recurrence, cluster algebras, and other topics.

This proof is based on *polytopes*.

§2.2.1 Construction of Polytopes

Fix a partition $\lambda \vdash n$. We'll start by defining two polytopes in $\mathbb{R}^n$:

- The first polytope Δ_λ is the set of (x_{ij}) for $(i, j) \in \lambda$ such that $x_{ij} \geq 0$ and $\sum h_{ij} x_{ij} \leq 1$.
- The second polytope P_λ is the set of (y_{ij}) for $(i, j) \in \lambda$ such that $y_{ij} \geq 0$, entries are nondecreasing in rows and columns, and $\sum y_{ij} \leq 1$.

We can write our coordinates inside the Young diagram of λ:

$$\begin{array}{|c|c|c|c|c|c|}
\hline
x_{11} & x_{12} & x_{13} & \cdots & & \\
\hline
x_{21} & x_{22} & \cdots & & & \\
\hline
x_{31} & \cdots & & \multicolumn{3}{c}{} \\
\cline{1-3}
\vdots & & & \multicolumn{3}{c}{} \\
\cline{1-3}
 & & \multicolumn{4}{c}{} \\
\cline{1-2}
\end{array}$$

Example 2.17

Take the partition $\lambda = (3, 2)$. Then

$$\Delta_\lambda = \left\{ \begin{array}{|c|c|c|} \hline x_{11} & x_{12} & x_{13} \\ \hline x_{21} & x_{22} & \multicolumn{1}{c}{} \\ \cline{1-2} \end{array} \;\middle|\; x_{ij} \geq 0 \text{ and } 4x_{11} + 3x_{12} + x_{13} + 2x_{21} + x_{22} \leq 1 \right\},$$

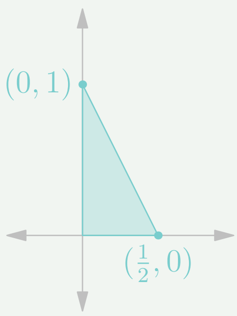
and meanwhile

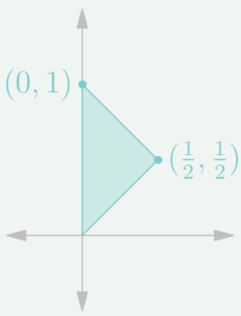
$$P_\lambda = \left\{ \begin{array}{|c|c|c|} \hline y_{11} & y_{12} & y_{13} \\ \hline y_{21} & y_{22} & \multicolumn{1}{c}{} \\ \cline{1-2} \end{array} \;\middle|\; y_{ij} \geq 0, \begin{array}{ccccc} y_{11} & \leqslant & y_{21} & \leqslant & y_{22} \\ \rotatebox{90}{$\geqslant$} & & \rotatebox{90}{$\geqslant$} & & \\ y_{21} & \leqslant & y_{22} & & \end{array}, \sum y_{ij} \leq 1 \right\}.$$

These polytopes are five-dimensional, so we can't draw them on the blackboard. Let's instead look at a more trivial example, so that we can actually draw the polytopes:

Example 2.18

Take the partition $\lambda = (2)$. Then Δ_λ is defined by the conditions $x_1, x_2 \geq 0$ and $2x_1 + x_2 \leq 1$, while P_λ is given by the conditions $0 \leq y_1 \leq y_2$ and $y_1 + y_2 \leq 1$.





In this case, both polytopes have area $\frac{1}{4}$. In fact, there's a simple map sending Δ_λ to P_λ that preserves area — we can send $(x_1, x_2) \mapsto (y_1, y_2) = (x_1, x_1 + x_2)$. It turns out that this generalizes to all λ!

First let's see the connection between these polytopes and the hook length formula, by calculating their volumes. First, Δ_λ is a simplex — it's the standard coordinate simplex with its coordinates rescaled, so

$$\operatorname{Vol}(\Delta_\lambda) = \frac{1}{\prod_{(i,j) \in \lambda} h_{ij}} \operatorname{Vol}\left\{(x_{ij}) \mid x_{ij} \geq 0 \text{ and } \sum x_{ij} \leq 1\right\} = \frac{1}{\prod_{(i,j) \in \lambda} h_{ij}} \cdot \frac{1}{n!}$$

(since $\frac{1}{h_{ij}}$ are the factors we're rescaling each coordinate by).

Meanwhile to find $\mathrm{Vol}(P_\lambda)$, note that the number of possible orderings of the y_{ij} is exactly f^λ (since each ordering corresponds to a standard Young tableau), so

$$\mathrm{Vol}(P_\lambda) = f^\lambda \cdot \mathrm{Vol}\left\{(y_1, \ldots, y_n) \mid 0 \le y_1 \le \cdots \le y_n \text{ and } y_1 + \cdots + y_n \le 1\right\}.$$

But this new figure is exactly $\frac{1}{n!}$ of the standard coordinate simplex, since the standard coordinate simplex (corresponding to points with $y_i \ge 0$ and $y_1 + \cdots + y_n \le 1$) can be split into $n!$ equal pieces depending on the order of the y_i, and this is one of those pieces. So then

$$\mathrm{Vol}(P_\lambda) = f^\lambda \cdot \frac{1}{n!} \cdot \mathrm{Vol}\left\{(y_{ij}) \mid y_i \ge 0 \text{ and } \sum y_i \le 1\right\} = \frac{f^\lambda}{(n!)^2}.$$

So then the hook length formula is equivalent to stating that the volumes of our two polytopes are equal!

§2.2.2 A Bijection Between Polytopes

So far, all we've done is reformulated the hook length formula into a polytopal form — we now want to show that $\mathrm{Vol}(\Delta_\lambda) = \mathrm{Vol}(P_\lambda)$. In order to show that two polytopes have the same volume, we can try to construct a volume-preserving map between them.

Theorem 2.19

There exists a continuous bijective piecewise-linear volume-preserving map $\varphi_\lambda \colon \Delta_\lambda \to P_\lambda$.

In Example 2.18, we found a *linear* map between Δ_λ and P_λ, but this usually isn't possible — Δ_λ is always a simplex, while P_λ may not be. But the theorem states that it is always possible to find a *piecewise* linear map — we can break Δ into polytope pieces and define a linear map on each of those pieces (where all these linear maps are volume-preserving).

Student Question. *Why is it important that φ_λ is continuous?*

Answer. It isn't necessary for proving that $\mathrm{Vol}(\Delta_\lambda) = \mathrm{Vol}(P_\lambda)$. But it's nice that we have this additional property of continuity — and the map φ_λ that we'll construct can actually be used to prove many *other* facts as well, not just the hook length formula.

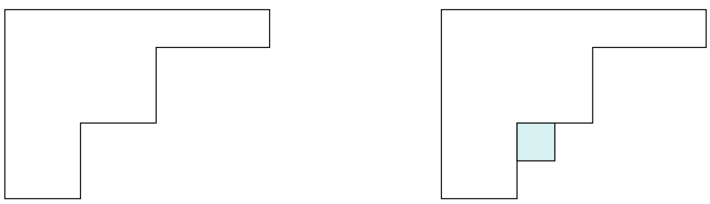
We'll actually construct a map between the two *cones*

$$\varphi_\lambda \colon \left\{ \boxed{x_{ij}} \;\middle|\; x_{ij} \ge 0 \right\} \to \left\{ \rotatebox{90}{$\leqslant$}\ \overset{\leqslant}{\boxed{y_{ij}}} \;\middle|\; y_{ij} \ge 0 \right\}$$

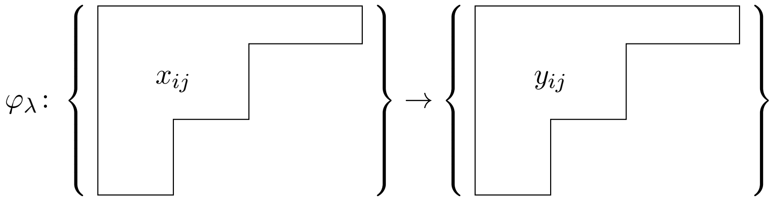
such that $\sum h_{ij} x_{ij} = \sum y_{ij}$ — then φ_λ must also map $\Delta_\lambda \to P_\lambda$, since these two finite pieces of the cones will be defined by corresponding constraints.

We'll construct φ_λ by induction on $|\lambda|$. The base case is when λ is the empty partition $\varnothing$ of 0 — then both polytopes are a point.

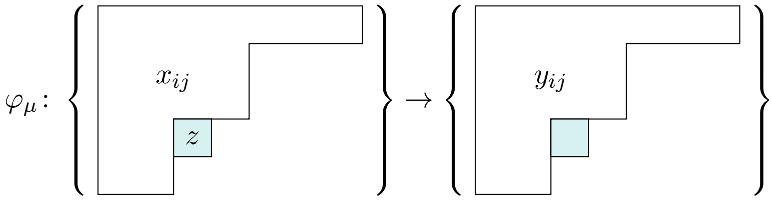
For the inductive step, suppose we've already constructed φ_λ, and we want to construct φ_μ for a partition μ which comes from adding one box to λ.



Now we want to modify our map

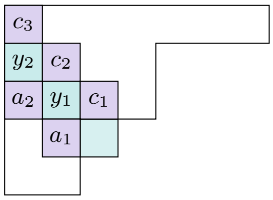


into a new map

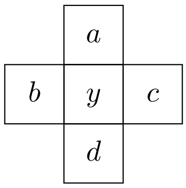


such that the conditions are all preserved.

It turns out that we can do this without modifying very many of the y_{ij} — we'll only modify the entries on the diagonal of the new square, and this modification will only depend on the diagonals above and below it. Label our diagonal entries y_1, y_2, $\ldots$, and the diagonals above and below it a_1, a_2, $\ldots$ and c_1, c_2, $\ldots$ respectively.



For each entry y in this diagonal, suppose that initially we have



in φ_λ. Then we're going to send y to some new value y^*. To motivate the choice of y^*, our constraints y are $y \geq \max(a, b)$ and $y \leq \min(c, d)$ (and since we're not modifying any of a, b, c, and d, then we have the same constraints on y^*). So we would like a bijective transformation that sends this interval $[\max(a, b), \min(c, d)]$ to itself. The simplest such map is reflecting the entire interval, so we'll take

$$y^* = \max(a, b) + \min(c, d) - y.$$

This operation is called a *toggle move*.

So then we perform a toggle move on each of the entries on our diagonal, sending each y_i to $y_i^* = \max(a_{i-1}, c_{i-1}) + \min(a_i, c_i) - y_i$. Finally, we also need to define one new entry, in the box we added. If in the (x_{ij}) this box contains z, then in this box we place $\max(a_1, c_1) + z$ — this is motivated by the fact that this entry must be at least a_1 and c_1, and should relate somehow to z.

When we try to perform this construction, it's possible that some of the values we're attempting to use don't exist (our indices may be out of bounds). In that case, we use 0 instead.

We'll soon check all the relevant properties, but first we'll see a few examples.

Example 2.20

For the partition $\lambda = (1)$, our map is simply

$$\varphi_{(1)} : \begin{array}{|c|}\hline x_{11} \\ \hline\end{array} \mapsto \begin{array}{|c|}\hline x_{11} \\ \hline\end{array}.$$

Example 2.21

For the partition $\lambda = (2)$, we can begin with (1) and add one box on the right:

$$\begin{array}{|c|c|}\hline x_{11} & x_{12} \\ \hline\end{array}.$$

This box doesn't have any other entries on its diagonal, so we don't need to modify any of the already existing y_{ij}; meanwhile we add a new entry here, to get the map

$$\varphi_{(2)} : \begin{array}{|c|c|}\hline x_{11} & x_{12} \\ \hline\end{array} \mapsto \begin{array}{|c|c|}\hline x_{11} & x_{11} + x_{12} \\ \hline\end{array}.$$

Example 2.22

For the partition $\lambda = (2, 1)$, we can begin with (2) and add one box in the second row:

$$\begin{array}{|c|c|}\hline x_{11} & x_{12} \\ \hline x_{21} \\ \cline{1-1}\end{array}.$$

Again there are no other boxes on the diagonal, so we simply add a new entry to $\varphi_{(2)}$ to get

$$\varphi_{(2,1)} : \begin{array}{|c|c|}\hline x_{11} & x_{12} \\ \hline x_{21} \\ \cline{1-1}\end{array} \mapsto \begin{array}{|c|c|}\hline x_{11} & x_{11} + x_{12} \\ \hline x_{11} + x_{21} \\ \cline{1-1}\end{array}.$$

So far, all of our maps have been linear — in fact, this is true for any λ shaped like a hook. We'll now see a more interesting example:

Example 2.23

For the partition $\lambda = (2, 2)$, we can begin with $(2, 1)$ and add a box in the corner:

$$\begin{array}{|c|c|}\hline x_{11} & x_{12} \\ \hline x_{21} & x_{22} \\ \hline\end{array}.$$

Then we need to perform a toggle move on y_{11}, replacing x_{11} with $\min(x_{11} + x_{12}, x_{11} + x_{21}) - x_{11} = \min(x_{12}, x_{21})$, so we get the map

$$\varphi_{(2,2)} : \begin{array}{|c|c|}\hline x_{11} & x_{12} \\ \hline x_{21} & x_{22} \\ \hline\end{array} \mapsto \begin{array}{|c|c|}\hline \min(x_{12}, x_{21}) & x_{11} + x_{12} \\ \hline x_{11} + x_{21} & x_{11} + \max(x_{12}, x_{21}) + x_{22} \\ \hline\end{array}.$$

In this example, since $\min(a,b)+\max(a,b) = a+b$, we have

$$\begin{aligned}\sum y_{ij} &= \min(x_{12}, x_{21}) + \max(x_{12}, x_{21}) + x_{22} + x_{11} + x_{21} + x_{11} + x_{12} \\ &= 3x_{11} + 2x_{21} + 2x_{12} + x_{22},\end{aligned}$$

which is exactly $\sum h_{ij}x_{ij}$. So φ_λ has the property we want in this example.

Student Question. *If we add boxes in a different order when inductively constructing φ_λ, do we still get the same map?*

Answer. The answer is yes — to see this, suppose it's possible to add box a and then b, or b and then a. Then a and b can't be in adjacent diagonals, so they don't influence each other — this means either order would produce the same result. Given this, we can use the diamond lemma to deduce that the map is independent of the order in which we add boxes.

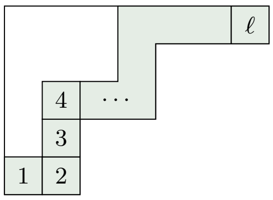
It's easy to see that φ_λ satisfies most of the conditions of Theorem 2.19 — it's bijective because all steps are reversible, it's piecewise linear because min and max are piecewise linear, and it's not hard to check that it's volume-preserving (in particular, toggle moves are volume-preserving). So the property that it remains to prove is the following:

Lemma 2.24

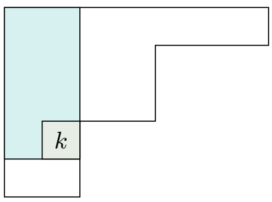
If φ_λ sends $\boxed{x_{ij}} \mapsto \boxed{y_{ij}}$, then $\sum h_{ij}x_{ij} = \sum y_{ij}$.

In fact, a stronger lemma is true. We want to prove one linear relation between the x_{ij} and y_{ij}, but there are actually *many* linear relations!

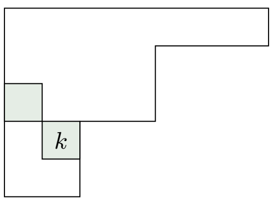
To describe these linear relations, we focus on the border ribbon of our Young diagram λ:



Number the boxes on this ribbon 1 through ℓ. For each box k, let R_k be the rectangle which starts at k and goes as far up and left as possible:



Meanwhile, let D_k be the diagonal starting at box k:



Lemma 2.25

For each k, we have $\sum_{(i,j)\in R_k} x_{ij} = \sum_{(i,j)\in D_k} y_{ij}$.

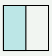
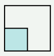
In other words, the rectangular sums of the x_{ij} equal the diagonal sums of the y_{ij}. Denote these sums by r_k and d_k respectively.

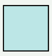
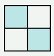
Example 2.26

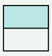
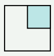
For the partition $\lambda = (2,2)$, in Example 2.23 we found

$$\varphi_{(2,2)} : \begin{array}{|c|c|}\hline x & y \\ \hline z & t \\ \hline \end{array} \mapsto \begin{array}{|c|c|}\hline \min(y,z) & x+y \\ \hline x+z & x+\max(y,z)+t \\ \hline \end{array}.$$

The rectangular sums in (x_{ij}) and diagonal sums in (y_{ij}) are:

 $r_1 = x+z$  $d_1 = x+z$

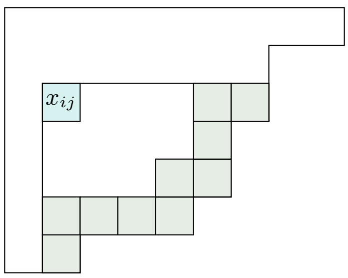
 $r_2 = x+y+z+t$  $d_2 = \min(y,z)+x+\max(y,z)+t = x+y+z+t$

 $r_3 = x+y$  $d_3 = x+y$.

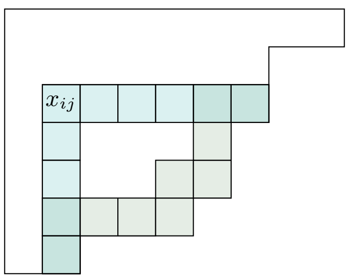
We can see that $r_k = d_k$ for all k.

First let's see why the second lemma implies the first.

Proof of Lemma 2.25 $\implies$ Lemma 2.24. First, since every y_{ij} appears in exactly one diagonal d_k, we have $\sum d_k = \sum y_{ij}$. Meanwhile, we have $\sum r_k = \sum a_{ij}x_{ij}$ for some multiplicities a_{ij}.



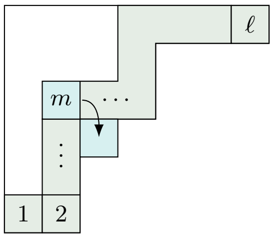
The rectangle R_k contains x_{ij} if and only if box k is between the jth column and the ith row (inclusive) — or in other words, in the rectangle starting at x_{ij} and extending as far down and right as possible. But the number of boxes in this segment of the ribbon is exactly the hook length of x_{ij}:



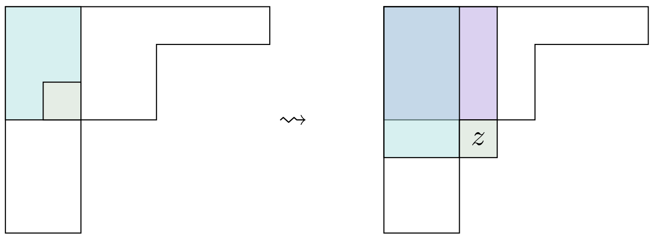
So the the multiplicity of x_{ij} in $\sum r_k$ is exactly h_{ij}, and $\sum r_k = \sum h_{ij}x_{ij}$. Then since $\sum d_k = \sum r_k$ by Lemma 2.25, we have $\sum y_{ij} = \sum h_{ij}x_{ij}$, as desired. $\square$

Now it remains to prove Lemma 2.25. The proof is actually quite simple.

Proof of Lemma 2.25. We use induction — suppose we begin with a partition λ with sums $r_1, \ldots, r_\ell$ and $d_1, \ldots, d_\ell$, and we add one extra box in the mth diagonal.



Then all the rectangular sums except r_m stay the same, since none of the other rectangles change (and none of the values in x_{ij} change, apart from the new one we added); suppose r_m becomes $\tilde{r}_m$. Similarly, all the diagonal sums except d_m stay the same, since we're only modifying the entries y_{ij} in diagonal D_m; suppose d_m becomes $\tilde{d}_m$. Then it suffices to show that r_m and d_m change "in the same way."



When we add our box, the rectangle R_m grows by one row and one column. To construct the new rectangle, we can take R_{m-1} and R_{m+1}, subtract the overlap — which is exactly the initial rectangle R_m — and add in the new box. So we have

$$\tilde{r}_m = r_{m-1} + r_{m+1} - r_m + z.$$

Meanwhile in the (y_{ij}), suppose that the $(m-1)$st diagonal for λ was $a_1, a_2, \ldots,$ the mth diagonal was $b_1, b_2, \ldots,$ and the $(m+1)$st diagonal was $c_1, c_2, \ldots,$ so that $d_{m-1} = \sum a_i$, $d_m = \sum b_i$, and $d_{m+1} = \sum c_i$. Then each toggle move replaces b_i with $\max(a_{i-1}, c_{i-1}) + \min(a_i, c_i) - b_i$, so we now have

$$\tilde{d}_m = (\max(a_1, c_1) + z) + (\max(a_2, c_2) + \min(a_1, c_1) - b_1) + (\max(a_3, c_3) + \min(a_2, c_2) - b_2) + \cdots.$$

But $\max(a_i, c_i) + \min(a_i, c_i) = a_i + c_i$, so then we can eliminate all the maxes and mins to get

$$\tilde{d}_m = (a_1 + a_2 + \cdots) + (c_1 + c_2 + \cdots) - (b_1 + b_2 + \cdots) + z,$$

so then $\tilde{d}_m = d_{m-1} + d_{m+1} - d_m + z$. But this is exactly the same as the formula for $\tilde{r}_m$! So the rectangular sums and diagonal sums change in the exact same way, which means they must remain equal. $\square$

§2.2.3 Another Application

We've completed the proof of the hook length formula, but that isn't the end of the story — we proved the identity that we needed (Lemma 2.24), but we also proved many *other* identities as well (in Lemma 2.25), and we can use these to prove other claims than the hook length formula. In fact, this construction implies *many* different results. Here is one example, which will be on the problem set:

Exercise 2.27. Suppose that λ is a $m \times n$ rectangle with $m \leq n$. Then consider standard Young tableaux of shape λ. For each such tableau T, let its entries on the main diagonal be $a_1 < a_2 < \cdots < a_m$, and define its *weight* as

$$\mathrm{wt}(T) = \frac{1}{\prod_{i=1}^{m} i^{a_{i+1}-a_i}}$$

(where a_{m+1} is defined as $a_{mn} + 1$). Prove that if we sum over all T of shape λ, then

$$\sum \mathrm{wt}(T) = 1.$$

Example 2.28

When λ is a 2×3 rectangle, the standard Young tableau are

$$\begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 & 5 & 6 \\ \hline\end{array}\,, \quad \begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 3 & 5 & 6 \\ \hline\end{array}\,, \quad \begin{array}{|c|c|c|}\hline 1 & 3 & 4 \\ \hline 2 & 5 & 6 \\ \hline\end{array}\,, \quad \begin{array}{|c|c|c|}\hline 1 & 2 & 5 \\ \hline 3 & 4 & 6 \\ \hline\end{array}\,, \quad \begin{array}{|c|c|c|}\hline 1 & 3 & 5 \\ \hline 2 & 4 & 6 \\ \hline\end{array}\,.$$

We always have $a_1 = 1$, and a_2 is 5 for the first three tableaux and 4 for the last two. So the first three tableaux have weight

$$\frac{1}{1^{5-1} \cdot 2^{7-5}} = \frac{1}{4},$$

and the last two have weight

$$\frac{1}{1^{4-1} \cdot 2^{7-4}} = \frac{1}{8}.$$

So $\sum \mathrm{wt}(T) = \frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{8} + \frac{1}{8} = 1$.

In fact, if λ is a $2 \times n$ rectangle for any n, then as we've seen earlier, standard Young tableaux are in bijection with Dyck paths. The weight we get will always be of the form $\frac{1}{2^k}$, where k depends on the number of up-steps in the first two diagonals. In this case there's a probabilistic interpretation of the weight in terms of the Dyck path.

§3 Robinson–Schensted–Knuth Correspondence

Another identity involving the numbers f^λ is the following:

Theorem 3.1

We have $\sum_{\lambda \vdash n} (f^\lambda)^2 = n!$.

This identity has many nice interpretations. In particular, it's related to the representation theory of S_n:

Fact 3.2 — The irreducible representations of S_n (up to isomorphism) can be labelled by Young diagrams with n boxes, or equivalently partitions $\lambda \vdash n$. Furthermore, if V_λ is the irreducible representation corresponding to λ, then $\dim V_\lambda = f^\lambda$.

Fact 3.3 — For *every* finite group, the sum of squares of the dimensions of irreducible representations is the order of the group.

So this identity directly follows from representations of S_n. But we'd like to see this identity combinatorially.

§3.1 Robinson–Schensted Correspondence

The Robinson–Schensted correspondence proves $\sum_{\lambda \vdash n} (f^\lambda)^2 = n!$ by constructing a bijection between sets counted by the left-hand and right-hand sides: it produces a bijection

$$S_n \to \{(P, Q) \mid P \text{ and } Q \text{ are SYT of the same shape } \lambda \vdash n\}.$$

The classical construction of the Robinson–Schensted correspondence works via the Schensted insertion algorithm: we read the entries of w in order and insert them into a tableau P, called the *insertion tableau*. Meanwhile we'll use Q as the *recording tableau*, which keeps track of the order in which boxes were added to our Young diagram. We insert entries in the following way:

Algorithm 3.4 (Schensted Insertion) — Suppose we currently have the intermediate tableau T, and we want to insert a. Then:

(1) Set $i := 1$ and $x := a$.

(2) If all entries in the ith row are at most x (or if the ith row is empty), then add a new box at the end of the ith row, fill it with x, and stop.

(3) Otherwise, find the leftmost entry y in the ith row such that $y > x$. Replace y with x, set $x := y$ and $i := i + 1$, and return to step (2).

Note that the distinction between weak and strict inequalities doesn't really matter here, since all entries of a standard Young tableau are distinct. However, the distinction *will* matter in the case of *semistandard* Young tableaux (where entries can be repeated).

Example 3.5

Suppose that we want to insert 3 into the tableau

$$\begin{array}{|c|c|c|c|}\hline 1 & 4 & 5 & 9 \\ \hline 2 & 7 \\ \cline{1-2} 6 \\ \cline{1-1} 10 \\ \cline{1-1}\end{array}.$$

We first attempt to insert 3 into the first row. The leftmost entry greater than 3 is 4, so 3 bumps out 4. Now we have

$$\begin{array}{|c|c|c|c|}\hline 1 & 3 & 5 & 9 \\ \hline 2 & 7 \\ \cline{1-2} 6 \\ \cline{1-1} 10 \\ \cline{1-1}\end{array}$$

and we want to insert 4 into the second row. The leftmost entry in the second row greater than 4 is 7, so 4 bumps out 7, and now we have

$$\begin{array}{|c|c|c|c|}\hline 1 & 3 & 5 & 9 \\ \hline 2 & 4 \\ \cline{1-2} 6 \\ \cline{1-1} 10 \\ \cline{1-1}\end{array}$$

and we want to insert 7 into the third row. All elements in the third row are at most 7, so we can simply insert 7 at the end. So

$$\begin{array}{|c|c|c|c|}\hline 1 & 3 & 5 & 9 \\ \hline 2 & 4 \\ \cline{1-2} 6 & 7 \\ \cline{1-2} 10 \\ \cline{1-1}\end{array}$$

is the final result of the insertion.

Algorithm 3.6 (Robinson–Schensted Correspondence) — Suppose we have a permutation $w = w_1 \cdots w_n$. We initially start with P and Q both empty. Then we insert w_1, w_2, ... into P in order, so that

$$P = ((((\emptyset \leftarrow w_1) \leftarrow w_2) \leftarrow \cdots) \leftarrow w_n).$$

When we insert w_i into P, we insert i into Q in the same position as the new box added to P.

So we build P in the order w_1, w_2, ... and Q in the order 1, 2, ..., while always keeping them the same shape.

Example 3.7

Suppose our permutation is

$$w = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 4 & 1 & 6 & 5 & 2 \end{bmatrix}.$$

We start with both P and Q empty. Then we insert 3, which produces

$$P = \begin{ytableau} 3 \end{ytableau} \text{ and } Q = \begin{ytableau} 1 \end{ytableau}.$$

Then we insert 4 into the first row, to get

$$P = \begin{ytableau} 3 & 4 \end{ytableau} \text{ and } Q = \begin{ytableau} 1 & 2 \end{ytableau}.$$

Then we insert 1 into P. This bumps 3 (the smallest entry strictly greater than 1) from the first row, so now we have

$$P = \begin{ytableau} 1 & 4 \\ 3 \end{ytableau} \text{ and } Q = \begin{ytableau} 1 & 2 \\ 3 \end{ytableau}.$$

Now we csn insert 6 into the first row to get

$$P = \begin{ytableau} 1 & 4 & 6 \\ 3 \end{ytableau} \text{ and } Q = \begin{ytableau} 1 & 2 & 4 \\ 3 \end{ytableau}.$$

Then we insert 5 into the first row. This bumps 6, giving

$$P = \begin{ytableau} 1 & 4 & 5 \\ 3 & 6 \end{ytableau} \text{ and } Q = \begin{ytableau} 1 & 2 & 4 \\ 3 & 5 \end{ytableau}.$$

Finally we insert 2, which bumps 4, and inserting 4 into the second row bumps 6, so

$$P = \begin{ytableau} 1 & 2 & 5 \\ 3 & 4 \\ 6 \end{ytableau} \text{ and } Q = \begin{ytableau} 1 & 2 & 4 \\ 3 & 5 \\ 6 \end{ytableau}$$

is the end result.

To see that this correspondence is a bijection, it's not hard to check that all steps are reversible.

This proves the identity in Theorem 3.1, but it's actually much more powerful — the correspondence has many additional properties as well. For example, the length of the first row of λ is the size of the longest increasing sequence of w, and the length of the first column is the size of the longest *decreasing* sequence. ore generally, the sum of the first k rows of λ is the maximal number of entries that can be covered with k increasing subsequences, and the same is true for columns and decreasing subsequences.

Example 3.8

In the above example, the longest increasing subsequence is 3, 4, 6 or 3, 4, 5, which both have 3 entries, corresponding to the first row having 3 boxes. Similarly, the longest decreasing subsequence is 6, 5, 2, corresponding to the first column having 3 boxes.

Another feature of this bijection is that if $w \mapsto (P, Q)$, then $w^{-1} \mapsto (Q, P)$.

Both these properties look mysterious — in this construction, P and Q play very different roles, so it's not clear why we have some sort of symmetry between them. So it would be nice to find a construction of this correspondence that makes this symmetry more clear. We'll see such a construction soon, but first we'll generalize to *semistandard* Young tableaux.

§3.2 Semistandard Young Tableaux

Definition 3.9

A **semistandard Young tableau** is a labelling of the boxes of a Young diagram with positive integers, such that entries are *weakly increasing* in rows and *strictly increasing* in columns.

So a semistandard Young tableau is similar to a standard Young tableau, but numbers can be repeated or missing.

Example 3.10

One example of a semistandard Young tableau is

$$\begin{array}{|c|c|c|c|c|c|} \hline 1 & 1 & 1 & 2 & 2 & 4 \\ \hline 2 & 2 & 4 & 5 & 7 \\ \cline{1-5} 4 & 5 & 5 & 7 \\ \cline{1-4} \end{array}.$$

Note that the entries can be arbitrarily large — we could have replaced 7 by 70.

Definition 3.11

The **shape** of a semistandard Young tableau is the Young diagram λ. Its **weight** is the sequence $\beta = (\beta_1, \beta_2, \ldots)$ where β_i is the number of i's in the tableau.

Example 3.12

In Example 3.10 we have $\lambda = (6, 5, 4)$ and $\beta = (3, 4, 0, 3, 3, 0, 2)$.

Remark 3.13. Some people use the word *content* instead of *weight*. Professor Postnikov doesn't like this because the word *content* is reserved for something else — the content of a box in a Young diagram is $i - j$ (where i and j refer to its row and column), which will be important later.

§3.3 Robinson–Schensted–Knuth Correspondence

The Robinson–Schensted–Knuth correspondence (abbreviated RSK) generalizes the Robinson–Schensted correspondence to semistandard Young tableaux. First, fix a number n, which represents the maximal possible entry that can appear in the tableau. (For the tableau in Example 3.10, we could set n to be any integer at least 7.)

RSK then provides a bijection between $n \times n$ matrices with entries in $\mathbb{Z}_{\geq 0}$, and pairs (P, Q) of semistandard Young tableau of the same shape. We also fix two vectors $\alpha = (\alpha_1, \ldots, \alpha_n)$ and $\beta = (\beta_1, \ldots, \beta_n)$ with $\sum \alpha_i = \sum \beta_i$. Then more specifically, RSK provides a bijection between $n \times n$ matrices with column sums $\alpha_1, \ldots, \alpha_n$ and row sums $\beta_1, \ldots, \beta_n$, and semistandard Young tableau such that the weight of P is α and the weight of Q is β.

Remark 3.14. Sometimes the correspondence is stated for $m \times n$ matrices instead. But we can always transform general matrices into square matrices by inserting some number of 0's in the end, which has no effect.

The first step of RSK is to convert the matrix into a 'generalized permutation' in the following way: for each matrix entry a in (i, j), we write down a copies of $(i, j)^t$, arranged in the lexicographical ordering.

Example 3.15

For $n = 3$, we would replace the matrix

$$\begin{bmatrix} 0 & 1 & 0 \\ 2 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 2 & 2 & 2 & 3 & 3 & 3 & 3 \\ 2 & 1 & 1 & 2 & 1 & 1 & 1 & 3 \end{bmatrix}.$$

Now if our $n \times n$ matrix is sent to

$$A \rightsquigarrow \begin{bmatrix} u_1 & u_2 & \cdots & u_N \\ w_1 & w_2 & \cdots & w_N, \end{bmatrix}$$

then P is again the result of the sequence of insertions

$$(((\emptyset \leftarrow w_1) \leftarrow w_2) \leftarrow \cdots) \leftarrow w_N$$

using the Schensted insertion algorithm (now we have to be careful about whether inequalities are weak or strict), and when we insert w_i, in Q we add the new box created and fill it with u_i.

Example 3.16

Consider the matrix in Example 3.15.

We insert 2, 1, 1, 2, 1, 1, 1, 3 into P (and record the numbers above them in Q). We start out with both P and Q empty, and after the first step we get

$$P = \begin{array}{|c|}\hline 2 \\ \hline\end{array} \text{ and } Q = \begin{array}{|c|}\hline 1 \\ \hline\end{array}.$$

We then insert 1, which bumps 2, so now we have

$$P = \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline\end{array} \text{ and } Q = \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline\end{array}.$$

Now we insert 1 again (and record 2) to get

$$P = \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 \\ \cline{1-1}\end{array} \text{ and } Q = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 \\ \cline{1-1}\end{array}.$$

Then we insert 2 (and record 2) to get

$$P = \begin{array}{|c|c|c|}\hline 1 & 1 & 2 \\ \hline 2 \\ \cline{1-1}\end{array} \text{ and } Q = \begin{array}{|c|c|c|}\hline 1 & 2 & 2 \\ \hline 2 \\ \cline{1-1}\end{array}.$$

Now we insert 1 (and record 3). This bumps the first number in the first row *strictly* greater than it, which is 2; so we then insert 2 into the second row, giving

$$P = \begin{array}{|c|c|c|}\hline 1 & 1 & 1 \\ \hline 2 & 2 \\ \cline{1-2}\end{array} \text{ and } Q = \begin{array}{|c|c|c|}\hline 1 & 2 & 2 \\ \hline 2 & 3 \\ \cline{1-2}\end{array}.$$

The remaining two 1's and 3 can be inserted into the top row, so

$$P = \begin{array}{|c|c|c|c|c|c|} \hline 1 & 1 & 1 & 1 & 1 & 3 \\ \hline 2 & 2 \\ \cline{1-2} \end{array} \text{ and } Q = \begin{array}{|c|c|c|c|c|c|} \hline 1 & 2 & 2 & 3 & 3 & 3 \\ \hline 2 & 3 \\ \cline{1-2} \end{array}$$

is the final result.

RSK is also a bijection, but there's a difficulty that wasn't present in the case of standard Young tableaux: when we're trying to reverse the last step, there are *several* maximal entries in Q, so we need to be able to tell which one was added last. But it's not hard to check that the boxes with a given entry in Q must have been added from left to right, so the box that was added last is the *rightmost* box with the maximal entry. This lets us invert the correspondence, as before.

This correspondence has similar properties to the Robinson–Schensted correspondence for standard Young tableaux — the length of the first row of λ is the size of the longest *weakly* increasing subsequence (of $w_1, \ldots, w_n$), and the length of the first column is the size of the longest *strictly* decreasing subsequence. Also, if $A \mapsto (P, Q)$, then $A^\intercal \mapsto (Q, P)$. In the specific case where A is a *permutation* matrix (so all rows and columns sum to 1), P and Q are standard Young tableaux, and the correspondence is the same as the Robinson–Schensted correspondence.

§3.4 Gelfand–Tsetlin Patterns

We'll soon see an alternative way to look at RSK, and to explain this construction, we'll define another combinatorial object.

Definition 3.17

A **Gelfand–Tsetlin pattern** is a triangular array of numbers where the first row is a partition $(\lambda_1, \ldots, \lambda_n)$ (here 'partition' means that entries are weakly decreasing, but are allowed to be 0), the second row is a partition $(\mu_1, \ldots, \mu_{n-1})$, and so on (where the last row has one number), and consecutive rows are *interlaced* — we have

$$\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq \cdots \geq \mu_{n-1} \geq \lambda_n,$$

and the same is true for the other rows.

In other words, a Gelfand–Tsetlin pattern (abbreviated as GT pattern) is a triangle of numbers which weakly decrease going both southeast and northeast:

$$\begin{array}{ccccccccc}
\lambda_1 & & \lambda_2 & & \lambda_3 & & \cdots & & \lambda_n \\
& \mu_1 & & \mu_2 & & \cdots & & \mu_{n-1} & \\
& & \nu_1 & & \cdots & & \nu_{n-2} & & \\
& & & \ddots & & \iddots & & & \\
& & & & \omega_1 & & & &
\end{array}$$

Proposition 3.18

There is a bijection between semistandard Young tableaux with entries in $\{1, 2, \ldots, n\}$ and Gelfand–Tsetlin patterns of size n.

We'll illustrate this by example:

Example 3.19

Let $n = 7$ and take the tableau

1	1	1	2	2	4
2	2	4	5	7	
4	5	5	7		

.

We start by writing the number of 1's in the bottom row; here that number is 3. Then we look at the shape formed by the 1's and 2's:

1	1	1	2	2	4
2	2	4	5	7	
4	5	5	7		

This shape is $(5, 2)$, so we write down $(5, 2)$ in the second-last row. Next we look at the shape formed by 1, 2, and 3; this shape is also $(5, 2)$, so we write down $(5, 2, 0)$ in the third-last row. We keep doing this, and the end result is the following:

$$\begin{array}{ccccccccccccc}
6 & & 5 & & 4 & & 0 & & 0 & & 0 & & 0 \\
& 6 & & 4 & & 3 & & 0 & & 0 & & 0 & \\
& & 6 & & 4 & & 3 & & 0 & & 0 & & \\
& & & 6 & & 3 & & 1 & & 0 & & & \\
& & & & 5 & & 2 & & 0 & & & & \\
& & & & & 5 & & 2 & & & & & \\
& & & & & & 3 & & & & & &
\end{array}$$

In general, the ith row of the Gelfand–Tsetlin pattern from the bottom is the shape of the Young diagram formed by the numbers 1, 2, ..., i in the tableau — so the jth entry in the ith row is the number of entries at most i in the jth row of the tableau. This example has a lot of 0's because a generic semistandard Young tableau with numbers up to 7 can have up to 7 rows, but this one only has 3 rows.

Example 3.20

Consider the third entry in the fifth row from the bottom:

$$\begin{array}{ccccccccccccc}
6 & & 5 & & 4 & & 0 & & 0 & & 0 & & 0 \\
& 6 & & 4 & & 3 & & 0 & & 0 & & 0 & \\
& & 6 & & 4 & & 3 & & 0 & & 0 & & \\
& & & 6 & & 3 & & 1 & & 0 & & & \\
& & & & 5 & & 2 & & 0 & & & & \\
& & & & & 5 & & 2 & & & & & \\
& & & & & & 3 & & & & & &
\end{array}$$

To find this entry, we count the number of entries less than or equal to 5 in the third row:

1	1	1	2	2	4
2	2	4	5	7	
4	5	5	7		

Question 3.21. Given a Gelfand–Tsetlin pattern p, what are the shape and weight of the corresponding semistandard Young tableau?

The shape is easy to describe — it's the top row of p (since the top row by definition records the shape of the entire tableau). Meanwhile, to find the weight β, by definition β_1 is the number of 1's in the tableau, which is the entry in the bottom row. Then $\beta_1 + \beta_2$ is the number of 1's and 2's, which is the sum of entries in the second-last row. In general, $\beta_1 + \cdots + \beta_i$ is the ith row sum of p from the bottom.

We now have an alternative way of thinking about semistandard Young tableaux. Both semistandard Young tableaux and Gelfand–Tsetlin patterns are arrays of numbers satisfying some inequalities. But semistandard Young tableaux have weak inequalities in rows and strict inequalities in columns, while Gelfand–Tsetlin patterns only have weak inequalities. The nice thing about this is that we can think of the entries in a Gelfand–Tsetlin pattern as a collection of $\binom{n}{2}$ variables satisfying some inequalities, and if we let the entries be *real*, then we get a polytope. Then semistandard Young tableaux correspond to the lattice points on this polytope. (Creating a polytope directly from semistandard Young tableaux would work less well, because the strict inequalities would mean we'd have to remove some faces.)

§3.5 RSK for Gelfand–Tsetlin Patterns

As we've seen earlier, RSK gives a bijection sending $n \times n$ matrices with nonnegative integer entries to pairs (P, Q) of semistandard Young tableaux of the same shape λ.