Cutsets and percolation

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§1 Introduction

We'll work with a graph G which is infinite and connected; we'll always implicitly assume that all vertices have finite degree and the number of vertices is countable. For example, you could imagine $G = \mathbb{Z}^2$ (though this holds for much more general graphs G).

§1.1 Percolation

We consider Bernoulli bond percolation on G, which we denote by $\mathbb{P}_p$ — this means we take G and retain each edge with probability p. (We refer to edges that are kept as *open*, and edges that are not as *closed*.)

When you study this model on an infinite connected graph, what's interesting is that as you vary p, there's a phase transition where you go from only having finite clusters to having at least one infinite cluster. The point where this transition occurs is called the *critical point*, and is denoted by p_c.

The basic question you can ask about this is the following:

Question 1.1. Is p_c strictly between 0 and 1?

In words, p_c being strictly between 0 and 1 means that you have both a subcritical phase (where there are only finite clusters) and a supercritical phase (where there is an infinite cluster).

It turns out that usually proving $p_c > 0$ is quite straightforward — for example, this is true if the vertex degrees are bounded. The difficulty is in showing that $p_c < 1$.

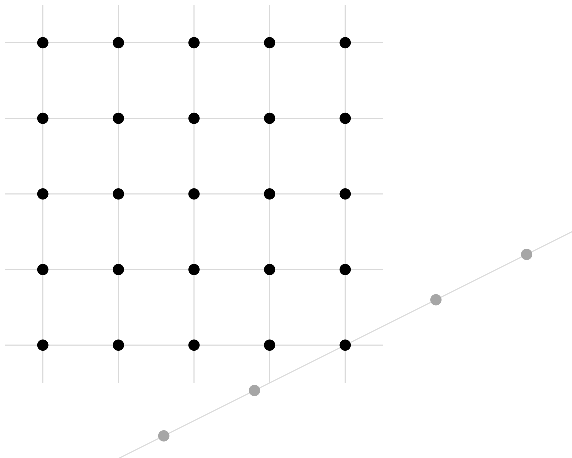
Here are a few examples:

Example 1.2

If $G = \mathbb{Z}$, it's not hard to show that $p_c = 1$ — there will be infinitely many missing edges, and these missing edges split the graph into infinitely many pieces which are all finite.

Example 1.3
If $G = \mathbb{Z}^2$, we have $p_c = \frac{1}{2}$.

Example 1.4
If G consists of $\mathbb{Z}^2$ together with an infinite 1-dimensional line, we still have $p_c = \frac{1}{2}$.



(The extra line basically has no effect; what matters is just whether the $\mathbb{Z}^2$ has an infinite cluster.)

What we'd like to answer is the following question.

Question 1.5. Is there a geometric criterion for $p_c < 1$?

In other words, if I give you a graph G, is there some geometric test that you can perform to determine exactly whether or not $p_c < 1$ (i.e., whether or not G has a phase transition)?

For example, looking at these graphs, we might want to say that $p_c < 1$ for $\mathbb{Z}^2$ but not for $\mathbb{Z}$ because $\mathbb{Z}^2$ has better isoperimetric properties than $\mathbb{Z}$. But the example made of $\mathbb{Z}^2$ with an appended line is a bit annoying from this perspective — it does have $p_c < 1$, but if you look at the infinite 1-dimensional line, it doesn't have good isoperimetric qualities.

But the thing about this example is that when the infinite cluster forms, it'll stay in $\mathbb{Z}^2$, and not go far up into the line. So we'll work with a slightly different definition of p_c.

Definition 1.6. We define p_c as the threshold for when *every* vertex has reasonable probability of being in an infinite cluster, i.e., $p_c = \inf\{p \mid \inf_{v \in V} \mathbb{P}_p[v \leftrightarrow \infty] > 0\}$.

(We write $\mathbb{P}_p[v \leftrightarrow \infty]$ for the probability that v is connected to ∞, i.e., in an infinite cluster.) So in words, we don't just want there to be an infinite component; we want *every* vertex to have reasonable probability of being in that component.

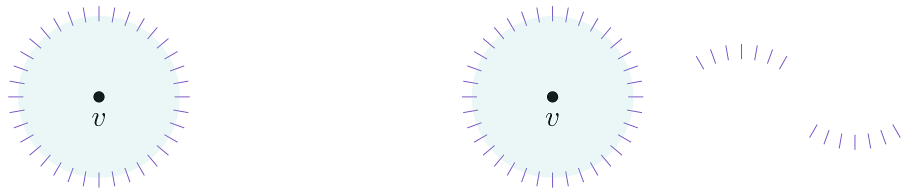
When the graph is transitive (meaning that all vertices look the same), this is the same as the usual definition of p_c. But now $\mathbb{Z}^2$ with a line appended has $p_c = 1$.

§1.2 Cutsets

There is a very old geometric criterion that's at least *sufficient* for determining that $p_c < 1$, which has to do with *cutsets*.

Definition 1.7. Given a vertex v, a cutset is a set of edges whose deletion would disconnect v from ∞ (i.e., leave v in a finite component).

This definition allows the cutset to have some extraneous edges. What we're really interested in is *minimal* cutsets — ones which don't have a proper subset that is still a cutset. (By 'minimal' we mean minimal with respect to inclusion; we don't mean that the cutset has the smallest possible number of edges.)



minimal cutset non-minimal cutset

These objects sort of pose an obstacle to having $p_c < 1$ — any time you see a cutset, if all the edges in that cutset happen to be closed (i.e., deleted), then your vertex can't be in an infinite component.

We can kind of measure the extent of this obstacle by counting cutsets — we'll look at

$$\#[\text{minimal cutsets from } v \text{ to } \infty \text{ of size } n]$$

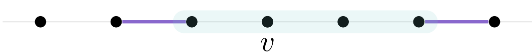
(a cutset being from v to ∞ means that it leaves v in a finite component; the *size* of a cutset is its number of edges). We're really interested in the *exponential growth* of this number — does it grow exponentially in n, or not? (Intuitively, this is because the probability a cutset gets closed is $(1-p)^n$, which is exponential.) To quantify this exponential growth, we have the following definition:

Definition 1.8. We define $\kappa = \sup_{v \in V} \sup_{n \geq 1} \#[\text{minimal cutsets from } v \text{ to } \infty \text{ of size } n]^{1/n}$.

So κ essentially measures the exponential growth rate of the number of cutsets of given size (supremized over the entire space of vertices). We're typically not interested in the exact value of κ; we're only interested in whether it's finite or infinite.

Example 1.9

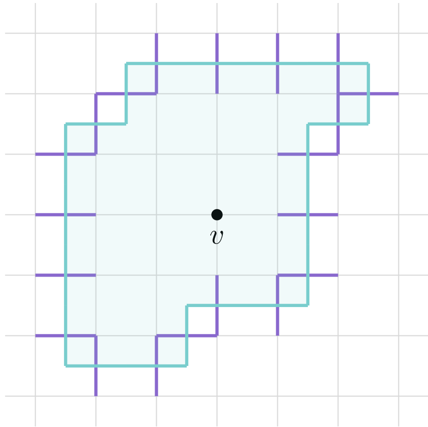
In $\mathbb{Z}$, we have $\kappa = \infty$ — if we fix a vertex v, then taking one edge on each side of v gives you a cutset of size 2. So the number of cutsets of size 2 is already ∞.



Example 1.10

In $\mathbb{Z}^2$, we have $\kappa < \infty$.

One way to prove this, which is very specific to $\mathbb{Z}^2$, is that there's a bijection between minimal cutsets and *circuits* in the dual graph — if we have a minimal cutset, then we can draw the circuit in the dual graph that goes through all its edges. So then you want to count how many circuits of length n surround a particular vertex; and it's easy to show that this grows exponentially in n, because you can construct the circuit iteratively as you go along (with a constant number of choices per step).



One result you'll see in a first course in percolation is an argument called the Peierls argument:

Theorem 1.11 (Peierls 1936)
If $\kappa < \infty$, then $p_c < 1$.

For example, this shows that $p_c < 1$ on $\mathbb{Z}^2$.

Remark 1.12. This result is actually from before the time percolation was invented, and was introduced in the context of the Ising model; but the same is true for percolation (and in fact, whether the Ising model on a graph has a nontrivial phase transition is the same as whether $p_c < 1$).

Proof. Fix a vertex v; if it's not in an infinite component, then there must be some minimal cutset around v which is closed. The number of minimal cutsets of size n is at most κ^n, and the probability such a cutset is closed is $(1-p)^n$, so this means

$$\mathbb{P}[v \nleftrightarrow \infty] \leq \sum_n \kappa^n (1-p)^n.$$

And by making p sufficiently close to 1, you can make $\kappa(1-p)$ less than 1; then you have a positive chance of having an infinite component. $\square$

The authors have two theorems with very similar proofs; the first is the converse of this statement.

Theorem 1.13 (Easo–Severo–Tassion)
If $p_c < 1$, then $\kappa < \infty$.

So this gives a geometric criterion for determining whether $p_c < 1$ — if I give you a graph and you want to see whether $p_c < 1$, it suffices to count the number of minimal cutsets. (If you want, you can also identify minimal cutsets with connected regions without holes — minimal cutsets are the boundaries of such things.)

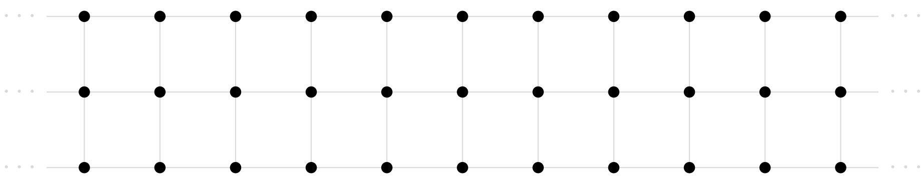
§1.3 Percolation in transitive graphs

If you don't work with percolation on general graphs, then the next result may look a bit out of the blue; so we'll give some context of what other results are known and why this is interesting.

Percolation was originally studied on $\mathbb{Z}^2$, and more generally on $\mathbb{Z}^d$. But in 1996, Benjamini and Schramm started a project on percolation on more general graphs — in particular, on arbitrary *transitive* graphs (graphs where all vertices look the same — for example, a square lattice, a regular tree, or any Cayley graph of a finitely generated group).

Question 1.14. For which transitive graphs is $p_c < 1$?

We saw that if our graph is 1-dimensional (e.g., $\mathbb{Z}$), then $p_c = 1$. In general, if the graph sort of looks like $\mathbb{Z}$ (for example, if it's a cylinder), then it's still true that $p_c = 1$.



In general, the way you capture what it means to 'look like' $\mathbb{Z}$ is to be *quasi-isometric* to $\mathbb{Z}$ (written $G \simeq \mathbb{Z}$). Their conjecture was that these examples that look like $\mathbb{Z}$ are in fact the *only* transitive graphs with $p_c = 1$.

Conjecture 1.15 (Benjamini–Schramm) — For transitive G, we have $G \not\simeq \mathbb{Z}$ if and only if $p_c < 1$.

Three years later, there was a stronger conjecture:

Conjecture 1.16 (Babson–Benjamini) — For all transitive $G \not\simeq \mathbb{Z}$, we have $\kappa < \infty$.

So this conjecture is that for such graphs, not only is $p_c < 1$, but also the number of cutsets should grow exponentially. This second conjecture implies the first (by the Peierls argument). But it's also a purely *deterministic* statement — it's just about the number of cutsets in fixed transitive graphs. (The conjecture was originally stated for Cayley graphs, and then generalized to all transitive graphs.)

In this business (when working with transitive graphs), there's a simplifying assumption you can make. Given any infinite graph, you can associate to it something called its *isoperimetric dimension* (denoted by $\operatorname{isodim}(G)$). For example, in $\mathbb{Z}^2$, any set with $\Theta(n^2)$ vertices has a boundary of size $\Omega(n)$; so we say that $\mathbb{Z}^2$ satisfies a 2-dimensional isoperimetric inequality. And 2 is the supremal constant that you can put here, so we say $\mathbb{Z}^2$ has isoperimetric dimension 2. Similarly, we say $\mathbb{Z}^d$ has isoperimetric dimension d.

And for the purposes of proving statements like Conjectures 1.15 and 1.16 (for transitive graphs), you can assume that $\operatorname{isodim}(G) = \infty$. This is because by work of previous authors, if G satisfies some isoperimetric inequality, then it's essentially like the Cayley graph of something called a *nilpotent group*. And these things are always finitely presented, so they behave a bit like $\mathbb{Z}^2$ (in particular, in $\mathbb{Z}^2$ minimal cutsets were kind of 'connected' in that you could relate them to circuits, and the same will be true here).

These conjectures were open for a while, until a breakthrough in 2018.

Theorem 1.17 (Duminil-Copin–Goswami–Raoufi–Severo–Yadin 2018)
If $\operatorname{isodim}(G) > 4$, then $p_c < 1$.

This theorem holds for *all* graphs, not just transitive ones. Together with the above assumption (that for transitive graphs we can assume $\operatorname{isodim}(G) = \infty$), this resolves Conjecture 1.15.

Now we'll state the second theorem from the paper (which has essentially the same proof as the first):

Theorem 1.18 (Easso–Severo–Tassion)
If $\operatorname{isodim}(G) > 2$, then $\kappa < \infty$.

Again, this holds for all (not necessarily transitive) graphs; and together with the above assumption, this resolves Conjecture 1.16. This is also a purely deterministic statement — it's about cutsets in a graph satisfying some isoperimetric inequality.

This also gives a new proof of Theorem 1.17, because of the Peierls argument. Surprisingly, the proof of Theorem 1.18 is extremely short and elementary. The original proof of DGRSY was very influential, but quite complex: The strategy involved comparing percolation to another model on the *Gaussian free field*, and doing some interpolation where you increase the amount of percolation and dial down the amount of Gaussian free field, or vice versa. This argument relied on very special properties of the Gaussian free field. (This went on to be very influential because it led to other papers about interpolation schemes.) But the proof of Theorem 1.18 is purely elementary and one page, so this gives you a one-page elementary proof of Theorem 1.17 as well.

In the rest of the talk, we'll explain what the key trick is, and sketch the full proof of Theorem 1.13 and briefly explain the tweaks needed for Theorem 1.18.

One thing the authors couldn't do, which is still open:

> **Question 1.19.** Can we prove Theorem 1.18 under the hypothesis $\operatorname{isodim}(G) > 1$ (rather than 2)?

(For *transitive* graphs, this would be equivalent to not being quasi-isomorphic to G.) This would be interesting because Theorem 1.18 is a purely geometric statement about connected sets without holes in a graph satisfying some isoperimetric inequality, so it seems like something we should be able to understand; but we don't understand its exact relation to isoperimetry.

§2 Proof of Theorem 1.13

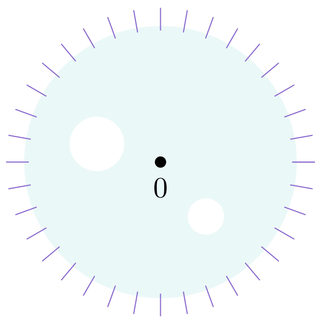
For Theorem 1.13, we get to assume that our graph has $p_c < 1$. This means there is some $p < 1$ which is supercritical, which means there is a uniform lower bound on the probability that a vertex is connected to ∞ — i.e., we have

$$\vartheta = \inf_v \mathbb{P}_p[v \leftrightarrow \infty] > 0.$$

And our goal is to prove that the number of minimal cutsets grows at most exponentially.

§2.1 Generating a random cutset

Fix some vertex 0. Imagine that we run percolation with parameter p and look at the cluster that 0 belongs to; we'll call this cluster C. We write $\partial_\infty C$ to denote the *external boundary* of C — this means we're taking the boundary, but we don't include edges that go into a hole on the inside, and only include edges that could lead you out to ∞ without reentering the cluster.



This set $\partial_\infty C$ is always a minimal cutset; so this procedure is a way of sampling a *random* minimal cutset.

Remark 2.1. We're in the regime $p > p_c$ where percolation occurs, so it's possible that our vertex 0 is in an infinite cluster; then $\partial_\infty C$ won't be defined. In this case, we can just say that the sampling procedure returns nothing.

How does generating a random minimal cutset help you *count* the number of minimal cutsets? The idea is that we'd like to prove that for any particular minimal cutset Π, the probability it gets outputted is at least exponential in its size.

Goal 2.2. Show there exists a constant $c > 0$ such that for every minimal cutset Π, we have

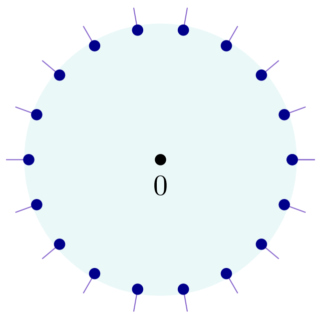
$$\mathbb{P}[\partial_\infty C = \Pi] \geq c^{|\Pi|}.$$

If we can do this, then this gives an upper bound on the number of minimal cutsets, because the probabilities of all the outcomes of a random variable have to sum to 1 — we have $\sum_\Pi \mathbb{P}[\partial_\infty C = \Pi] \leq 1$, so if every minimal cutset Π of size n has probability at least c^n of getting outputted, then the *number* of cutsets has to be at most c^{-n}. So we'd get $\kappa \leq 1/c$.

This was the trick that was missing — to try to count the number of minimal cutsets by trying to construct a random one. (The authors were inspired by Karger's algorithm from CS.) We'll prove Goal 2.2 for the rest of the talk; but the most important part is just coming up with this as the goal.

§2.2 Setup

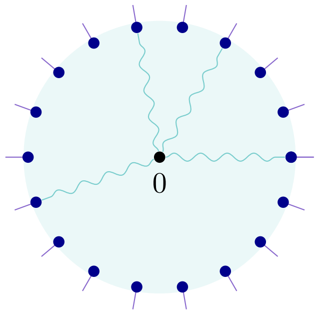
To fix some notation, we'll call our vertex 0, and we'll fix a target minimum cutset Π. We'll write B to denote the inner vertex boundary of Π (i.e., the set of vertices on the inside endpoints of the edges in Π), and I to denote the interior of the component enclosed by Π.



(Here the purple edges are Π, the light blue region is I, and the dark blue vertices are B.)

Our goal is to show that Π gets outputted with exponentially large probability. So let's consider, what are sufficient conditions on the percolation such that this target cutset gets outputted (i.e., $\partial_\infty C = \Pi$)?

- First, all the edges in Π need to be closed.
- Second, every vertex on the boundary should be connected to 0 through I. (We *don't* necessarily need all the vertices on the interior to be connected to 0, because we ignore holes; but we do need the boundary vertices to be connected to 0.)



We'll write $\mathbb{P}$ to denote the percolation $\mathbb{P}_p$ restricted to only the edges in the interior of Π (since only this interior matters for the second event).

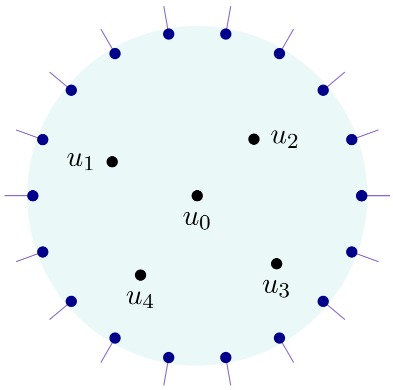
§2.3 Well-spaced sequences

The key definition is the notion of a *well-spaced sequence*. Suppose we pick a maximal sequence of vertices $u_0, u_1, \ldots, u_k \in I$ with $u_0 = 0$ such that for all i, we have

$$\frac{p\vartheta}{2} \leq \mathbb{P}[u_{i+1} \leftrightarrow \{u_0, \ldots, u_i\}] \leq \frac{\vartheta}{2}.$$

So we start with 0 and keep picking points u_1, u_2, and so on, such that each point has a reasonably good but not too high probability of connecting to the set of all previous points (the exact bounds are not super important). By *maximal*, we mean that we keep doing this until we can't append any more vertices; and we call the last vertex u_k.

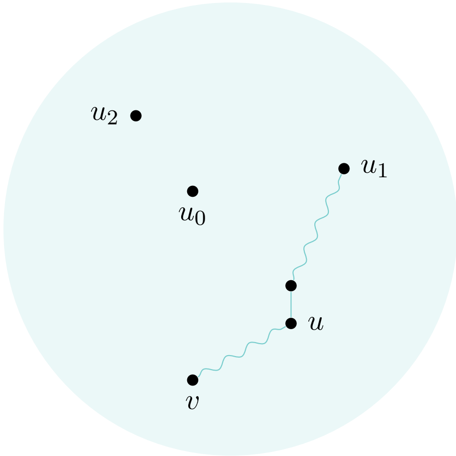
Intuitively, these points sort of form a 'net' — you can think of them as sort of filling up space without being too close to each other.



One thing that's intuitively clear from this picture is that there shouldn't be a large 'hole' in the graph where you don't see any of these vertices — if there is a hole, we should be able to add another vertex. To make this precise:

> **Claim 2.3** — For all $v \in I$, we have $\mathbb{P}[v \leftrightarrow \{u_0, \ldots, u_k\}] \geq \vartheta/2$.

Proof. Assume we have a vertex v which *doesn't* have good probability of connecting to the points we've chosen. Now consider a deterministic path from v to the set of chosen points. As we walk along this path, at some point the probability (of the current vertex being connected to this set) has to flip from being small (i.e., less than $\vartheta/2$) to big (i.e., greater than $\vartheta/2$).



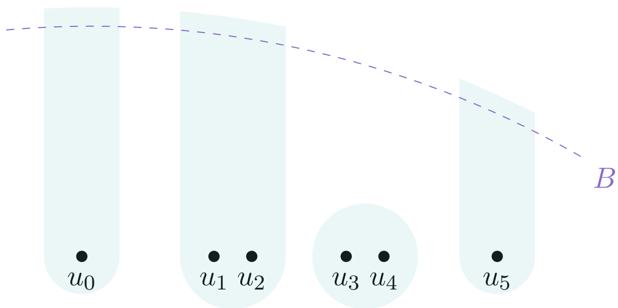
Then if we consider the vertex u at which the probability flips, the probability at u is less than $\vartheta/2$, but it's also not too small (the connection probabilities change by at most p at each step along the path). So we could append u to our sequence, contradicting its maximality. □

Something that's less obvious is that the number of vertices in this sequence (i.e., k) is actually at most linear in the size of the boundary (which is at most n).

Claim 2.4 — We have $k \leq (2/\vartheta)n$.

(The specific constant $2/\vartheta$ is not important.) This is maybe surprising because from our picture, we might expect the number of points to be bounded by the 'volume' of the region (or the size of its interior), but it's actually bounded by the *boundary*.

Proof. For each i, let n_i be the number of clusters which intersect both $\{u_0, \ldots, u_i\}$ and the boundary B. So n_0 is the number of clusters that touch both u_0 and the boundary, which is 0 if u_0 isn't connected to the boundary and 1 if it is. Then for n_1, we also look at u_1; if it's in the same cluster as u_0 then nothing happens, but if it's in a different cluster also touching the boundary then we get a new cluster. And so on.



(For example, here the sequence would be 1, 2, 2, 2, 2, 3.)

We can imagine tracking this number of clusters as we increment i one step at a time, and looking at the expected increment size $\mathbb{E}[n_{i+1} - n_i]$. The definition of n_{i+1} is the same as that of n_i, except that now we also get to count clusters that contain u_{i+1}. So if u_{i+1} is in a *new* cluster that connects to the boundary — i.e., it connects to the boundary but not $\{u_0, \ldots, u_i\}$ — then we get an increment of 1. This means

$$\mathbb{E}[n_{i+1} - n_i] \geq \mathbb{P}[u_{i+1} \leftrightarrow B] - \mathbb{P}[u_{i+1} \leftrightarrow \{u_0, \ldots, u_i\}].$$

But the probability u_{i+1} connects to the boundary is at least ϑ, because every vertex has probability at least ϑ of reaching ∞ (since we're in the percolating regime), and if you reach ∞ then you certainly reach the boundary. Meanwhile, by construction, the probability u_{i+1} is connected to the previous vertices is at most $\vartheta/2$. So we get

$$\mathbb{E}[n_{i+1} - n_i] \geq \vartheta - \frac{\vartheta}{2} = \frac{\vartheta}{2}.$$

Now summing this over all k, we get

$$\mathbb{E}[n_k] \geq \frac{k\vartheta}{2}.$$

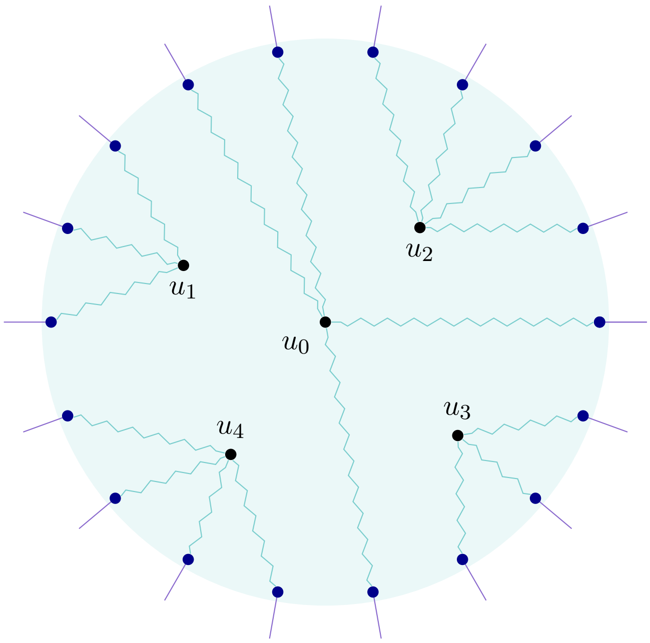
But the punchline is that n_k can't be greater than the size of the boundary — it's counting clusters that cross from one of these vertices $u_0, \ldots, u_k$ to the boundary, and the number of clusters touching the boundary is at most the size of the boundary. So we always have $n_k \leq n$. This means $n \geq k\vartheta/2$, as desired. □

§2.4 The conclusion

Now we're essentially done. Our goal was to prove an exponential lower bound on the probability that we output this particular cutset Π when we run our percolation experiment, i.e., $\mathbb{P}[\partial_\infty C = \Pi]$. So let's look at this blob and the net we've picked, and try to use this to estimate $\mathbb{P}[\partial_\infty C = \Pi]$.

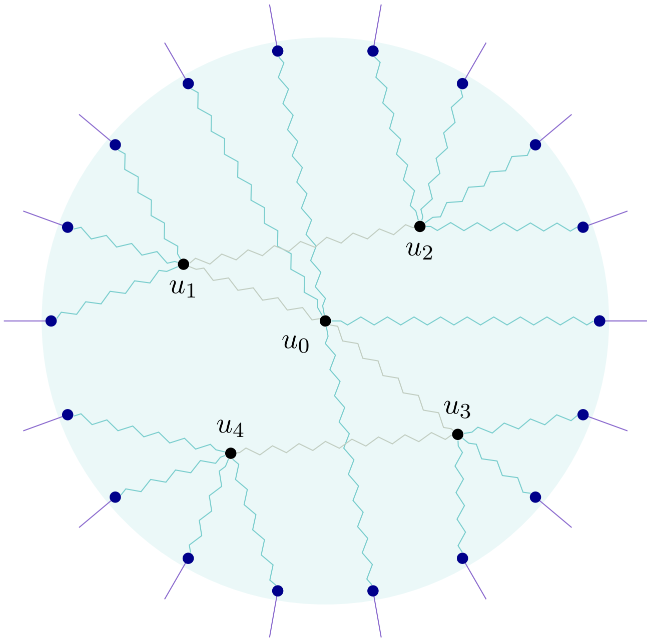
First, we have to close all the edges on the boundary; the probability of this is $(1-p)^n$. Meanwhile, to make $0 = u_0$ connected to every vertex on the boundary, it suffices for two things to happen:

- Every vertex on the boundary should be connected to at least one vertex u_i in our net.



  For each vertex on the boundary, we showed (Claim 2.3) that the probability it connects to some u_i is always at least some constant, specifically $\vartheta/2$ — we actually showed this for *all* vertices on the interior. So the probability that this occurs is at least $(\vartheta/2)^n$.

- We also want u_1 to be connected to u_0, u_2 to be connected to u_1 or u_0, u_3 to u_0, u_1, or u_2, and so on — this guarantees that all the vertices $u_0, \ldots, u_k$ are connected to each other, and together with the above event, this means u_0 is connected to every vertex on the boundary.



> And each of these steps occurs with probability at least $p\vartheta/2$, by the way we defined our sequence. So the probability this occurs is at least $(p\vartheta/2)^k$. And we showed (Claim 2.4) that k is linear in n, which means this is at least $(p\vartheta/2)^{2n/\vartheta}$.

Putting this all together, we get that

$$\mathbb{P}[\partial_\infty C = \Pi] \geq (1-p)^n \left(\frac{\vartheta}{2}\right)^n \left(\frac{p\vartheta}{2}\right)^{2n/\vartheta}.$$

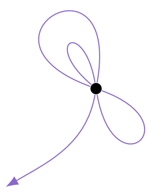
This is an exponential lower bound, as we wanted.

> **Remark 2.5.** To be precise, we're using FKG here. The boundary being closed is independent from everything else (which only depends on the interior), but the other events aren't independent of each other. But those events are all increasing — opening more edges only helps — and increasing events are positively correlated in percolation (by FKG).

§3 Proof idea for Theorem 1.18

Now we'll talk about how to prove Theorem 1.18. For Theorem 1.13, the trick was to count the number of minimal cutsets by running percolation and taking the boundary of a percolation cluster, which gave a way of randomly sampling a minimal cutset (and then we could bound the probabilities of each output).

But here's an alterative way to sample a cutset. Imagine that we run a simple random walk on our graph, starting with some vertex; this walk might leave, then return to that vertex, leave again, return, and so on.



As long as the graph is transient, eventually the walk leaves forever. So we look at this last return time and delete everything that comes after. Then we get some connected finite set, and we can take its external boundary (meaning that we ignore a loop within a loop, for example).

So this procedure also generates a random cutset. And you can again prove (using a similar argument) that every cutset gets outputted with exponentially good probability, assuming we have some uniform bound on transience. The condition $\operatorname{isodim}(G) > 2$ guarantees that the graph is transient, so this works and we get $\kappa < \infty$ (and so $p_c < 1$).

§4 Concluding remarks

It's surprising that the construction of a random minimal cutset used to prove Theorem 1.18 and the one used in Theorem 1.13 look nothing alike — somehow the exponential bound is so coarse that lots of things work. There are other ways of constructing random minimal cutsets (for Theorem 1.18) that work as well.

- You could take your vertex and draw a large ball, and pick a uniform random spanning tree; there's a way to build a cutset out of this, and it turns out this also works.
- You could take a Gaussian free field on the graph (this is some kind of random field; the definition is not super important for now). This induces a percolation configuration (by taking all the edges where the field stays high), and you can take the cluster containing a particular point, as in the proof of Theorem 1.13. Here knowing that the graph is transient actually lets you deduce that the Gaussian free field sort of percolates, and then you could run a similar argument to the one for Theorem 1.13.

So lots of things work, which is surprising.

As one other comment, knowing that the number of minimal cutsets is bounded exponentially actually gives you much more than just $p_c < 1$. The Peierls argument is one application of such an exponential bound, and it tells you that if p is close enough to 1, then we get an infinite cluster. But through this, we can prove much stronger facts. For example, not only is there an infinite cluster, but the finite clusters are really tiny (as tiny as you could ask for — the authors get sharp asymptotics). And if you take the union of the infinite clusters, the geometry of this union closely resembles the geometry of the base graph in many ways (for example, the behavior of a simple random walk will be similar). This was previously unknown even for general Cayley graphs.

Here's another fun corollary coming from this. If you care about random walks on general graphs, it's a basic result that as soon as $\operatorname{isodim}(G) > 2$, the simple random walk is transient. For example, a random walk in $\mathbb{Z}^3$ is transient; a random walk in $\mathbb{Z}^2$ is not transient, so this bound is sharp.

But what if you take a graph G with $\operatorname{isodim}(G) > 2$ and perturb it? More specifically, you go to every edge and assign independent and identically distributed random weights, according to any law you like (we don't require this law to have finite moments or anything like that). For example, maybe one edge gets assigned 3, another gets 2.7, and so on. And then you can do a random walk on this perturbed graph, where a lower weight means the edge is harder to cross.

> **Question 4.1.** Is it true that even after this noising, the random walk is still transient?

The answer is yes. But the only way we know how to prove it is through the results of this paper (regarding cutsets) — we know G has an exponential number of cutsets, which tells you about how percolation behaves close to 1, and from that you can read off this fact about random walks.