Combinatorial Interpretations

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§1 Introduction

We can consider two types of problems:

- *Direct problems* — given a set A, compute $|A|$.
- *Inverse problems* — given a number N, find a set A such that $|A| = N$.

The second statement may sound a bit silly — after all, there certainly exists a set of any size. But even the first is underspecified (what does it mean to compute $|A|$, e.g., do we want an asymptotic answer or an exact one?) — these are both meant to be general frameworks, and we'll see a more precise definition later.

Example 1.1

- Given a polynomial, computing its Galois group is a direct problem; finding a polynomial with a given Galois group is an inverse problem (which we do not know how to do).
- Given a polytope, computing the normal vectors and volumes of its facets is a direct problem; determining whether there exists a polytope with given normal vectors and facet volumes is an inverse problem.
- A convex polytope can be projected to produce a graph. Given a graph, determining whether there exists a polytope with this graph as a projection is an inverse problem.

(As we can see in these examples, the concept of direct and inverse problems can apply to cases where we have objects other than integers in place of N.)

The set A with $|A| = N$ can be thought of as a *combinatorial interpretation* of N. Usually we'll have a family of numbers (rather than just one number N), and we would like a way to construct sets corresponding to these numbers.

Example 1.2

The *Catalan numbers* can be defined according to the formula

$$C_n = \frac{1}{n+1}\binom{2n}{n} = \binom{2n}{n} - \binom{2n}{n-1}$$

(where the first expression shows $C_n > 0$, and the second shows $C_n \in \mathbb{Z}$). One could ask for a combinatorial interpretation of the Catalan numbers, i.e., sets which are counted by these numbers. There are many such combinatorial interpretations — for example, C_n is the number of triangulations of a (convex) $(n+2)$-gon.

Example 1.3

The binomial coefficients are *unimodal* — we have $\binom{n}{0} \leq \binom{n}{1} \leq \cdots \leq \binom{n}{\lfloor n/2 \rfloor} \geq \cdots \geq \binom{n}{n}$. This is easy to prove by direct calculation, but does there exist a combinatorial interpretation of the difference

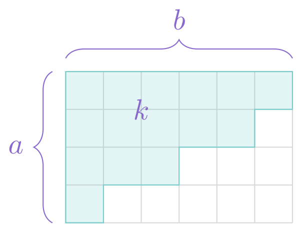
$$\binom{n}{k} - \binom{n}{k-1}$$

for $k \leq \frac{n}{2}$? The answer is yes — this counts the number of grid walks from $(0, 0)$ to $(n-k, k)$ which remain (weakly) below the line $y = x$, by the reflection principle.

§1.1 Sylvester's Theorem

We'll now see a harder question in this framework.

Definition 1.4. The number of partitions of k which fit into an $a \times b$ rectangle is denoted by $p_k(a, b)$.



Theorem 1.5 (Sylvester 1878)

For fixed $a, b \geq 1$, the numbers $p_k(a, b)$ for $k = 0, \ldots, ab$ are unimodal.

(This result was a conjecture for about 25 years, and Sylvester was very happy when he proved it; he used what we'd now call the representation theory of SL_2.)

Question 1.6. Does $p_k(a, b) - p_{k-1}(a, b)$ (for $0 \leq k \leq \frac{ab}{2}$) have a combinatorial interpretation?

The original proof doesn't give a combinatorial interpretation. It's natural to try to go from partitions of size $k-1$ to partitions of size k by adding a square, but we don't know of a way to do this.

However, there's a description of this quantity as a *Kronecker coefficient*.

Definition 1.7. The *Kronecker coefficient* $g(\lambda, \mu, \nu)$ is defined as $\langle \chi^\lambda \chi^\mu, \chi^\nu \rangle$.

Here χ^λ denotes the character of the irreducible representation of S_n corresponding to λ. We can imagine taking the characters χ^λ and χ^μ and multiplying them to produce a new character, and then writing this product as a sum of irreducible characters; then $g(\lambda, \mu, \nu)$ is the coefficient of χ^ν in this expansion.

Theorem 1.8 (Pak–Panora 2013)

The quantity $p_k(a, b) - p_{k-1}(a, b)$ is equal to the Kronecker coefficient $g(a^b, a^b, (n-k, k))$.

Here a^b denotes the $a \times b$ rectangle, and $(n-k, k)$ denotes a partition with parts $n-k$ and k (where $n = ab$).

Is this a combinatorial interpretation of $p_k(a,b) - p_{k-1}(a,b)$? The answer is no, for reasons we'll see later.

This was used as a lemma to prove the following result.

Theorem 1.9

For all $a, b \geq 8$, we have $g(a^b, a^b, (n-k, k)) \geq 1$.

§2 The Complexity Class #P

We'll now see a way to formalize what it means to be a combinatorial interpretation — we'll say a function is a combinatorial interpretation if it's a member of the complexity class #P.

Definition 2.1. The complexity class NP consists of decision problems for which a good object can be verified in polynomial time.

Example 2.2

Some examples of problems in NP:

- Deciding whether a graph G has a Hamiltonian cycle — this is because given a subset of edges, checking whether it forms a Hamiltonian cycle takes polynomial time.
- Deciding whether a graph G has a proper 3-coloring — this is because given a coloring, checking whether it is proper takes polynomial time.

Definition 2.3. The complexity class #P consists of problems where we count the number of good objects, for which deciding whether a given object is good takes polynomial time.

Example 2.4

The problems in #P corresponding to the above problems in NP:

- Counting the number of Hamiltonian cycles in a graph G.
- Counting the number of proper 3-colorings of a graph G.

In this definition, the first two examples we saw — the Catalan numbers as the number of triangulations of a $(n+2)$-gon, and $\binom{n}{k} - \binom{n}{k-1}$ as the number of grid walks below the diagonal — are both combinatorial interpretations.

There's actually an easy combinatorial interpretation of $p_k(a,b) - p_{k-1}(a,b)$ under this definition — there exists a polynomial-time algorithm to compute $p_k(a,b)$. So we can take the combinatorial interpretation to be the set of numbers $\{1, 2, \ldots, p_k(a,b) - p_{k-1}(a,b)\}$ — given any number, we can verify whether it's in this interval in polynomial time by simply computing $p_k(a,b) - p_{k-1}(a,b)$.

However, the following question is open.

Question 2.5 (Open Problem). Does $g(\lambda, \mu, \nu)$ have a combinatorial interpretation?

Unlike the above case, this Kronecker coefficient can't be computed in polynomial time. Note that it's defined as

$$g(\lambda, \mu, \nu) = \frac{1}{n!} \sum_{\sigma \in S_n} \chi^\lambda(\sigma)\chi^\mu(\sigma)\chi^\nu(\sigma),$$

and just looking at this sum, it's not obvious whether it's even an integer or nonnegative (both are true, because of representation theory); if we can't even see that, how can we prove it's counting something? (The second point is more interesting than the first — if we could prove even that $n!g(\lambda, \mu, \nu)$ had a combinatorial interpretation, this would be a big deal.)

Conjecture 2.6 — $\{g(\lambda, \mu, \nu)\} \notin \#\mathsf{P}$.

(The goal is to prove this under assumptions similar to $\mathsf{P} \neq \mathsf{NP}$.)

§3 Some Examples

§3.1 Matchings

Let $G = (V, E)$ be a (simple) graph, and let m_k be the number of *k-matchings* — k-tuples of edges in E where no two share a vertex.

Theorem 3.1 (Heilmann–Lieb 1972)
For all k, $m_k(G)^2 - m_{k+1}(G)m_{k-1}(G) \geq 0$.

In other words, the numbers $m_k(G)$ have log-concavity.

Question 3.2. Is the difference $m_k(G)^2 - m_{k+1}(G)m_{k-1}(G)$ (considered as a function ϕ from (G, k) to a nonnegative integer) in $\#\mathsf{P}$?

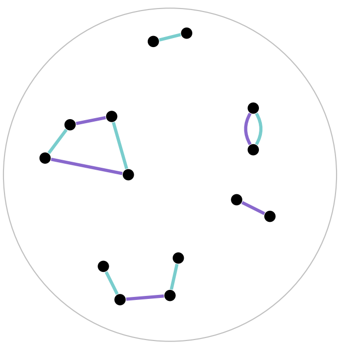
We can't perform the same trick as with p_k where we simply compute all the quantities in polynomial time — computing $m_k(G)$ is $\#\mathsf{P}$-complete.

The original proof was by linear algebra; this isn't easy to convert into a combinatorial interpretation (because dealing with eigenvalues is generally not combinatorial). But the answer is yes.

Theorem 3.3 (Krattenhaler 1991)
We have $m_k(G)^2 - m_{k+1}(G)m_{k-1}(G) \in \#\mathsf{P}$.

What this means is that Krattenhaler gave a proof of the theorem which *can* be converted into a combinatorial interpretation.

Proof. Take a $(k-1)$-matching M and a $(k+1)$-matching M', and overlay them in separate colors (here the $(k-1)$-matching is shown in purple and the $(k+1)$-matching in blue).



We want to construct an injection from such configurations to overlays of a purple and blue k-matching.

This configuration will consist of some cycles with an equal number of blue and purple edges (possibly of length 2, corresponding to an edge in both matchings); some paths with an even number of edges, which have an equal number of blue and purple edges; and some paths with an odd number of edges, which have one excess blue or purple edge; suppose we have ℓ excess-blue and $\ell - 2$ excess-purple paths. We're not going to touch the cycles or even paths; meanwhile, we want to flip the colors of some of the odd paths so that we end up with $\ell - 1$ excess-blue and $\ell - 1$ excess-purple paths instead.

The main point is that $\binom{2\ell-2}{\ell} \leq \binom{2\ell-2}{\ell-1}$ — if we imagine fixing all the edges involved in the picture and the colors of the cycles and even paths, then the configurations we start with correspond to size-ℓ subsets of the $2\ell - 2$ odd paths (representing the set that are colored to be excess-blue), and the configurations we're trying to end with correspond to size-$(\ell - 1)$ subsets of these paths.

This proves the inequality, and $\binom{2\ell-2}{\ell-1} - \binom{2\ell-2}{\ell}$ *does* have a combinatorial interpretation (for example, the one from the reflection principle). This combinatorial interpretation tells us which paths to flip in order to produce $\ell - 1$ excess-blue and $\ell - 1$ excess-purple paths (the ones which correspond to elements we flip in our injection from size-ℓ subsets to size-$(\ell - 1)$ subsets based on the reflection principle).

So this gives a combinatorial interpretation of $m_k(G)^2 - m_{k-1}(G)m_{k+1}(G)$, as desired. (Explicitly, if we use the reflection principle as our injection, then this quantity counts the number of blue-purple configurations as above which have $\ell - 1$ excess-blue and $\ell - 1$ excess-purple paths for some ℓ, and for which the subset of odd paths which are excess-blue corresponds to a walk remaining below the diagonal.) □

Remark 3.4. There do exist things we might intuitively think of as 'combinatorial interpretations' that aren't captured by this definition of #P — for example, 2^{2^n} is the number of Boolean functions on n Boolean variables. These more complicated combinatorial objects would correspond to another complexity class in place of #P (just as there are other complexity classes than P or NP in decision problems), such as #Exp.

Something like 'the number of Boolean functions on n Boolean variables' is simple to state in English, so it seems like something we'd want to consider a combinatorial interpretation; on the other hand, being able to describe something in words isn't necessarily the same as being able to compute it, and can lead to issues like the 'smallest uninteresting number.' And there exist such things in decision problems as well — for example, by having the decision problem be about all the subgraphs of a graph.

The advantage of the definition in terms of #P is that it makes the problem precise in terms of counting complexity, so that we can prove things about it.

As a story, about 300 years ago Gauss proved two theorems in one paper — that a 17-gon can be constructed with a ruler and compass, and that a 7-gon cannot. The 17-gon became extremely popular, while the 7-gon didn't — because people generally wanted a construction (or combinatorial interpretation) more than to know it's not possible. But later people changed their mind — proving that a 7-gon can't be constructed is useful.

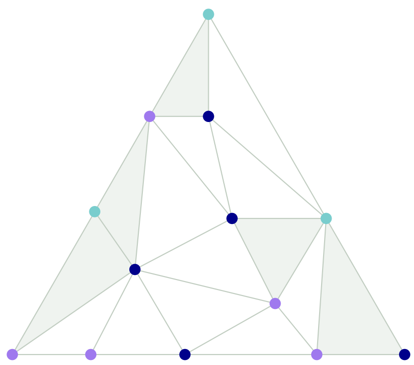
In order to prove these results, Gauss formalized what we mean by a geometric construction. If you then use an object called a *trisector* (which it is possible to build as a physical instrument), then it *is* possible to construct a 7-gon (and you can solve which polygons can or can't be constructed with a trisector).

The point of this story is that you can always enlarge the set of tools you have, which leads to more theorems about possibility and impossibility — but you have to start somewhere (with some way of formalizing the idea that lets you potentially prove impossibility).

§3.2 Sperner's Lemma

Consider a triangle which has been split into smaller triangles. We color the edges with three colors (1, 2, and 3, here shown in blue, purple, and green) such that the entire first side of the original triangle is colored 1, the second side is colored 2, and the third side is colored 3. (Note: I think either I wrote this down incorrectly and we're coloring vertices instead of edges, or there's more conditions I'm missing.)

Lemma 3.5 (Sperner's Lemma)
Suppose a triangle is split into smaller triangles, and each vertex is colored with one of the colors 1, 2, and 3, such that the three vertices of the triangle are colored 1, 2, and 3 respectively, and on each outer edge, every vertex has the same color as one of the endpoints. Then there exists a triangle with all three colors.



For a coloring χ, let $t(\chi)$ denote the number of triangles with vertices of all three colors.

Question 3.6. Is $t(\chi) - 1$ in #P?

Of course if you're given the coloring explicitly, then you can simply compute $t(\chi)$ in polynomial time. But instead suppose that the coloring is given by some kind of function — imagine that we have a giant graph where the vertices are binary sequences of length n, and we are given some circuit which computes the color of every vertex. (This means we can still check a given triangle quickly, but the number of triangles is no longer polynomial in the input size, so we can't go through and count all of them in polynomial time.)

The answer to this question is yes. To see why, we can write $t(\chi) = t_+(\chi) + t_-(\chi)$, where $t_+(\chi)$ denoes the number of triangles with 1, 2, 3 in counterclockwise order, and $t_-(\chi)$ the number of triangles with 1, 2, 3 in clockwise order. Then $t_+(\chi) - t_-(\chi) = 1$ — this follows by induction (removing a vertex). So

$$t(\chi) - 1 = 2t_-(\chi),$$

which does have a combinatorial interpretation (since for every triangle, we can check whether it has colors 1, 2, 3 in clockwise order).

§3.3 Parity

Fact 3.7 — In any graph, the number of odd-degree vertices is even.

Question 3.8. Is $\frac{1}{2}\#\{\text{odd-degree vertices}\}$ in #P?

If the graph is given explicitly to you, then again you can simply count the number of odd-degree vertices. But suppose it's instead a massive graph given by some function (where the vertices correspond to binary sequences and the edges are given by an algorithm or Turing machine which computes the neighbors of a given vertex in polynomial time — assume that the graph has polynomial degree), so that you can't.

To see a situation in which this problem naturally arises, consider the following problem.

Theorem 3.9

In a 3-regular graph, for any given edge e, the number of Hamiltonian cycles containing e is even.

Question 3.10. Is $\frac{1}{2}\#\{\text{Hamiltonian cycles containing } e\}$ in $\#\mathsf{P}$?

Here we're given the graph explicitly.

This is actually a special case of the above problem — you can construct a graph whose vertices are all the Hamiltonian cycles.

Conjecture 3.11 — The answer is no — i.e,. $\frac{1}{2}\#\{\text{Hamiltonian cycles containing } e\} \notin \#\mathsf{P}$.

It's known that the answer to the previous question — asking for $\frac{1}{2}$ of the number of odd-degree vertices — is no, conditional on an assumption similar to $\mathsf{P} \neq \mathsf{NP}$. On the other hand, the answer to this one — with Hamiltonian cycles — is not known. (We know it's a subproblem of the previous question, but not whether it's a *complete* subproblem.) There's another conjecture that states that this is PPA-complete; if it is, then it would also be possible to prove that this is not in $\#\mathsf{P}$.

§3.4 Correlation Inequalities

Theorem 3.12 (Kleitman 1966)

If $\mathcal{A}$ is the set of planar subgraphs of K_n and $\mathcal{B}$ is the set of triangle-free subgraphs of K_n, then

$$|\mathcal{A}| \cdot |\mathcal{B}| \leq 2^{\binom{n}{2}} |\mathcal{A} \cap \mathcal{B}|.$$

In other words, this states that if we choose a subgraph H of K_n uniformly at random, then

$$\mathbb{P}(H \text{ is planar} \mid H \text{ is triangle-free}) \geq \mathbb{P}(H \text{ is planar}).$$

This applies more generally to any downwards-closed properties (if you remove an edge from a planar or triangle-free graph, it remains planar or triangle-free).

Question 3.13. Is the difference $|\mathcal{A}| + |\mathcal{B}| - 2^{\binom{n}{2}} |\mathcal{A} \cap \mathcal{B}|$ in $\#\mathsf{P}$?

The answer is yes — this is not obvious, but Kleitman's original proof shows that it is.

On the other hand, it's been proved that the FKG and AD inequalities (which generalize Kleitman–Harris) are not in $\#\mathsf{P}$, under the assumption that $\mathsf{PH} \neq \Sigma_2$.

§3.5 Posets

Definition 3.14. For a finite poset $P = (X, \prec)$, a *linear extension* of P is a bijective function $f\colon X \to [n]$ (where $n = |X|$) such that whenever $x \prec y$ we have $f(x) < f(y)$. The number of linear extensions of P is denoted by $e(P)$.

Fact 3.15 — $e(P) - 1$ is in #P.

Proof. Given a poset P, we can compute the lexicographically minimal linear extension in polynomial time. So we can say $e(P) - 1$ counts the number of linear extensions which are *not* the lexicographically minimal one (since given an extension, we can check that it's not lexicographically minimal). □

Given a poset, for each element x, we define $b(x)$ as $\#\{y \in X \mid y \succeq x\}$.

Theorem 3.16
For any poset P, we have

$$e(P) \geq \frac{n!}{\prod_{x \in X} b(x)}.$$

(This is in some sense a statement about correlation — any x must be assigned the smallest number among all y with $y \succeq x$, which occurs with probability $\frac{1}{b(x)}$; so this sort of means that there's a positive correlation between this being true for x and some other element y.)

Fact 3.17 — $e(p) \cdot \prod_{x \in X} b(x) - n!$ is in #P.

This is because the original proof actually gives an injection (this isn't obvious).

As another problem about posets, fix $x \in X$, and for $k \in [n]$ define

$$N(k) = \#\{\text{linear extensions of } P \text{ with } f(x) \geq k\}.$$

Theorem 3.18 (Stanley's inequality)
For all k, $N(k)^2 \geq N(k-1)N(k+1)$.

Question 3.19. Is $N(k)^2 - N(k-1)N(k+1)$ in #P?

This is an open problem; Prof. Pak conjectures that the answer is no.

Stanley's proof uses standard inequalities applied to polytopes, but these standard inequalities aren't combinatorial (so the original proof doesn't give a combinatorial interpretation).

§3.6 Spanning Forests

For a graph $G = (V, E)$ and positive integer k, let $f_k(G)$ denote the number of spanning forests in G with k edges.

Theorem 3.20 (AHK 2018)
For all k, $f_k(G)^2 \geq f_{k-1}(G)f_{k+1}(G)$.

The proof of this is difficult (it was a conjecture for 40 or 50 years).

Conjecture 3.21 — The difference $f_k(G)^2 - f_{k-1}(G)f_{k+1}(G)$ is not in #P.

§4 An Impossibility Result

Now we'll see something that we *can* prove.

Theorem 4.1 (IPP 2023)
We have $(\chi^\lambda(\mu))^2 \notin \#\mathsf{P}$ (under the assumption $\mathsf{PH} \neq \Sigma_2$).

The Murnaghan–Nakayama rule says that $\chi^\lambda(\mu)$ is the *difference* of two counts (both of tableaux with certain properties). It may be positive or negative, so we might want to have a combinatorial interpretation of its square (which is nonnegative); however, they proved one doesn't exist.

The idea of their proof is to first write $\chi^\lambda(\mu)$ as a difference $f - g$, where f is the number of 3–SAT solutions to some formula φ, and g is the number of 3–SAT solutions to another formula ψ.

Now suppose that $\chi^\lambda(\mu)^2 = (f-g)^2$ is in $\#\mathsf{P}$. Then $f \neq g$ has a weakness — there is an object which we can check in polynomial time which would tell us that $f \neq g$. This is weird because it would mean that looking at φ and ψ, there's a polynomial-time weakness that would tell us they have different numbers of 3–SAT solutions, and this shouldn't be possible. (This is how the $\mathrm{PH} \neq \Sigma_2$ assumption comes up.)

Remark 4.2. Regarding a different open problem (about posets), they found a proof which makes a strong inductive assumption on properties of some explicit matrices you can construct from the poset; then the proof is conceptually easy. (This is how many results are proved — the Kleitman–Harris inequality, and AD and FKG, are proven by making the result stronger and then proving it by induction.)

They did this in order to prove that the quantity considered is in $\#\mathsf{P}$. This *almost* worked — nearly everything in the proof is combinatorial, but for one technical point where you have to take a limit. (This is because there's an issue where an eigenvalue can be 0, and you need a strict inequality for the induction to work; so you prove the statement for a slightly deformed statement and then take the limit.) This seems like a trivial step, but it prevents the proof from showing that it's in $\#\mathsf{P}$.

Remark 4.3. There are two related numbers

$$a(\lambda) = \sum_{\mu \vdash n} \chi^\lambda(\mu) \text{ and } b(\lambda) = \sum_{\mu \vdash n} \chi^\mu(\lambda).$$

It is known that $b(\lambda) \in \#\mathsf{P}$, because it counts something — the number of $\sigma \in S_n$ such that the cycle type of σ^2 is λ. On the other hand, it's an open problem to determine whether $a(\lambda) \in \#\mathsf{P}$. (Stanley formulated the question, in 2000, as whether these numbers have a combinatorial interpretation.)

(This is a natural question — one expression is a sum over rows of the character table, and the other is a sum over columns. Both quantities are nonnegative integers, as they're multiplicities of the irreducible characters in a conjugate representation.)

Prof. Pak wrote a 60-page survey of this topic called *What is a combinatorial interpretation?*