Topological Methods in Combinatorics

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We'll discuss combinatorial problems that can be solved by topological methods — especially ones where the topological methods are a bit unexpected.

§1 Colorful Radon

We'll start with the Colorful Radon theorem (by Lovász 1992).

Theorem 1.1

Imagine that we are given $d+1$ pairs of points in $\mathbb{R}^d$; we think of each pair as being of a different color. Then there is a colorful partition such that their convex hulls intersect.

So if $d = 2$, we have three points in the plane; we want to split them into two triangles (each consisting of one point of each color) such that the triangles intersect.

Since we have a finite number of points in $\mathbb{R}^d$, this can be solved using basic linear algebra. But let's see the proof that Lovász had.

Proof. Start with out set of points in $\mathbb{R}^d$. The first thing we do is represent them in $\mathbb{R}^{d+1}$. We have $d+1$ axes — one for each pair. For the blue points, we take two opposite points on the corresponding axis of $\mathbb{R}^{d+1}$ which are symmetric about the origin; we do the same for each pair (and we remember which blue point corresponds to which original blue point, and so on).

If we take a colorful triangle in $\mathbb{R}^d$, we know which colorful triangle upstairs it corresponds to. So now let's look at all possible colorful triangles downstairs. Then upstairs we end up with the boundary of an octahedron $\mathbb{O}^d$, which if you squint a bit is basically a d-dimensional sphere.

This gives us a continuous function $f\colon \mathbb{S}^d \to \mathbb{R}^d$ (by extending to all points). Then we can use the Borsat–Ulam theorem:

Theorem 1.2

If $f\colon \mathbb{S}^d \to \mathbb{R}^d$ is continuous, then there exists $x \in \mathbb{S}^d$ with $f(x) = f(-x)$.

But if we look at two antipodal points here, the opposite point will correspond to the complementary simplex. So we get exactly the result that we want. $\square$

§2 Necklace Splitting Theorem

This is due to Golberg and West (1986).

> **Question 2.1.** Suppose we're given an open necklace (a segment), and we have some pearls on the necklace; some are blue, yellow, and green, such that there is an even number of each type.
>
> Suppose we have two thieves who steal an open necklace with m kinds of pearls, and they want to distribute it — each wants exactly half of each kind of pearl. One way to do this is to cut the necklace between all pairs of pearls and just do the division; but they care about the minimum number of cuts.
>
> So what is the minimum number of cuts needed to evenly divide the necklace?

One 'bad' necklace is if we have all the blue pearls grouped together, all the green pearls grouped, and all the yellow pearls grouped; this means we always need at least m cuts.

> **Theorem 2.2**
>
> It is always possible to find a fair partition with at most m cuts.

This is a completely combinatorial problem; it only depends on the pearls and their order.

We'll actually prove a slightly different result — instead of pearls, imagine we have m absolutely continuous measures in $[0, 1]$. Our necklace is the interval $[0, 1]$; instead of green pearls we'll have integrals.

This statement came out way before, due to Hobby and Rice 1965; it's actually equivalent (by the usual approximation algorithms — approximating a measure by finite sets of points, or approximating finite sets of points by measures concentrated near points).

So we'll prove this problem.

Proof. Say that we're aiming for a partition with m cuts.

In the previous problem, we looked at the space of all possible simplices, and parametrized it as a nice space (an octahedron, which was basically a sphere). Here we want to parametrize cutting $[0, 1]$ into m pieces. So let's look at the lengths of the pieces — let's say they have lengths $x_1, \ldots, x_{m+1}$, which are nonzero and sum to 1. Then if we just look at these segment lengths, the set of vectors $(x_1, \ldots, x_{m+1})$ satisfying $\sum x_i = 1$ is Δ^m, the m-dimensional simplex. But we want to do a bit more — we also want to know who gets each piece. So we give these pieces signs — we write $+$ if A gets the piece, and $-$ if B gets the piece. Then we have a bunch of numbers $y_1, \ldots, y_{m+1}$ (including the signs in the x_i); then we want to consider

$$\{(y_1, \ldots, y_{m+1}) \mid \sum |y_i| = 1\}.$$

This set is again an octahedron $\mathbb{O}^m \cong S^m$.

We have a sphere, and we have the earlier result of spheres, so we just need to make a function. We have m measures $\mu_1, \ldots, \mu_m$, so we can define

$$f(y) = (\mu_1(A), \ldots, \mu_{m+1}(A))$$

as the amount that Thief A gets (the person with positive signs). Then we can apply BU, which gives us a point with $f(y) = f(-y)$. We have $f(-y) = (\mu_1(B), \ldots, \mu_{m+1}(B))$, since flipping the signs just corresponds to changing who gets each part. So this says that they'll get the same value for measure 1, the same value for measure 2, and so on. $\square$

§3 Baskets and Fruits

Question 3.1. We are given n baskets, and k types of fruit; each basket has some amount of fruit. We're going to choose some baskets, and e have two goals: first, we want to have at least half of each kind of fruit. Second, we want to have as few baskets to carry as possible.

So each basket might have some amount of bananas, some amount of blueberries, and so on.

We'll assume $n \not\equiv k \pmod 2$ (so one is even and one is odd) — when they have the same parity, our solution might be off by one basket.

First, what would be a bad set of baskets and fruits? We could have $k-1$ fruits each in one basket, and the remaining fruit distributed uniformly among the other baskets. Then we'd need to pick these $k-1$ baskets and half of the remaining ones; this requires us to have

$$(k-1) + \left\lceil \frac{n-k+1}{2} \right\rceil$$

baskets. So we'll try to get something close to this.

Proof. Take n points in $\mathbb{R}^k$ in general position (no hyperplane contains more than k of the points). For each point we will assign one of the baskets; that gives us k measures in $\mathbb{R}^k$ (each fruit represents a measure concentrated at the corresponding point). (Each fruit has some fraction of the fruit in the corresponding basket, and we can assign that value to the corresponding point; that's a measure in $\mathbb{R}^k$, because the value of a set is the sum of the amounts of fruit.)

For example, if we have one apple in the first basket and two in the second, then for the apple measure we put a weight of 1 in the point corresponding to the first basket, and a weight of 2 in the point corresponding to the second.

Theorem 3.2 (Ham Sandwich)

Given k finite measures in $\mathbb{R}^k$, there exists a hyperplane H such that $\mu(H^+) \geq \mu(\mathbb{R}^k)/2$ and $\mu(H^-) \geq \mu(\mathbb{R}^k)/2$, where H^+ and H^- denotes the closed half-spaces corresponding to H.

We usually see this written for measures which are absolutely continuous with respect to the Lebesgue measure, in which case you can get exactly 1/2.

(By *finite* we mean that $\mu(\mathbb{R}^k) < \infty$.)

Now taking our k measures and applying the ham sandwich theorem — we chose our points in general position, so H contains at most k baskets. Then one of the open half-spaces contains at most $\left\lfloor \frac{n-k}{2} \right\rfloor$ points; if we take that closed half-space, it has at most

$$k + \left\lfloor \frac{n-k}{2} \right\rfloor$$

of the points (and by Ham Sandwich it has at least half of every fruit). This number is equal to the original one when $n \not\equiv k \pmod 2$ (they're off by 1 otherwise). $\square$

Remark 3.3. The problem appears in math olympiads; there are also non-topological solutions (which are nice when $k = 2$).

§4 The ham sandwich theorem

We'll now discuss the ham sandwich theorem (from 1938). Back then, the author imagined splitting a leg of ham between two people, such that they had the same amount of meat, bone, and so on (the name has evolved into the ham sandwich theorem from then).

Proof. Suppose that $\mu_1, \ldots, \mu_k$ are finite and absolutely continuous measures in $\mathbb{R}^k$. (By absolutely continuous, we mean with respect to the Lebesgue measure; it's enough for all hyperplanes to have measure 0.) We'll prove there's a halving hyperplane for all the measures.

As before, we want to parametrize the space of partitions or objects we're dealing with (and it's again going to be a sphere).

Take μ_k to be a special measure, and consider $\mathbb{S}^{k-1} \subseteq \mathbb{R}^k$. Given a direction $v \in \mathbb{S}^{k-1}$, we can look at the hyperplanes perpendicular to v, and slide them until the moment it splits μ_k into two.

There might be a range for which we can do this (if the measure has nothing in a slice); if we can do this, we pick the middle of that range.

So given $v \in \mathbb{S}^{k-1}$, we define H_v to be this halving hyperplane orthogonal to v for μ_k. Let A be the side on the direction of v, and B the side on the direction of $-v$.

All hyperplanes which split μ_k by half are parametrized in this way; we now want to construct a function $f\colon \mathbb{S}^{k-1} \to \mathbb{R}^{k-1}$. We define

$$f(v) = (\mu_1(A) - \mu_1(B), \ldots, \mu_{k-1}(A) - \mu_{k-1}(B)).$$

So we have a function $\mathbb{S}^{k-1} \to \mathbb{R}^{k-1}$. This is an odd function, meaning that $f(-v) = -f(v)$ — so if $f(v) = f(-v)$ (and there must exist some such v by BU), then $f(v) = 0$. (This is the usual presentation of BU — if we have an odd function $\mathbb{S}^d \to \mathbb{R}^d$, then it must have a zero.) Here that means we have the same measure in each of these $k-1$ measures; and by construction we're also halving the special measure that we took out. □

§5 Test map scheme

This showcases the general method we follow here. You usually see 'test map scheme' about splitting measures, but general topological methods in combinatorics use similar ideas. Whenever presented with a problem like this, you can:

1. First try to find a space X parametrizing the objects in the problem.
2. Find a space Y that gives you information about X. (If you're splitting many measures, this might tell you if you're giving more to A than B, or how many you're giving to A. In colorful Radon, it told you where a point on the octahedron was mapped.) This induces a natural function $f\colon X \to Y$.
3. We then want to study the maps $f\colon X \to Y$ as a topological problem, and see if we can get something interesting. THere are some usual things you're looking for:
   - Maybe X and Y have some symmetry (half-spaces can be flipped, if we have thieves we can flip who gets each piece); if we have a group G acting on both X and Y, this usually turns out to be an equivariant map (where applying it to our point before or after applying f gives the same result). (In our examples G was $\{\pm 1\}$, but in the necklace splitting problem with more than two thieves, you'll have other groups — with n thieves you might have S_n be the group you're looking for, and the topological result corresponding to BU is more complicated.) Then good things happen.

The next step from BU is the following:

Theorem 5.1 (Bold 1985)
Suppose that X and Y are G-spaces (where G is a nontrivial group, and a G-space means that G has a free action on each one — i.e., $gx \neq x$ for all $x \in X$ and all $g \in G \setminus \{e\}$; this action should be continuous for each g) and are paracompact, and:

- Y is at most n-dimensional;
- X is n-connected.

Then there is no continuous G-equivariant map $f\colon X \to Y$.

(We won't worry about what paracompact means; all our relevant sets will satisfy it.)

The idea we should have in our minds is that when $G = \mathbb{Z}_2$, $X = \mathbb{S}^{n+1}$, and $Y = \mathbb{S}^n$, this is exactly the BU theorem. (By n-connected, what we really want is that the first n reduced homology groups cancel. There is an easier description where whenever you map to a lower-dimensional sphere, there is no homotopy.)

This is one of the nicer generalizations of BU — you just have to check these two parameters and that the parameters are free. (You only need to check free on X.)

We'll use this to get a solution with more thieves. First we'll see a second solution for two thieves, because it is nice:

Proof for Two Thieves. Recall that we had our m types of pearls in $[0,1]$; we're going to send $[0,1] \to \mathbb{R}^m$ by embedding this in the moment curve. THe moment curve is a curve in $\mathbb{R}^n$ parametrized as $(\lambda, \lambda^2, \lambda^3, \ldots, \lambda^m)$ for $\lambda \in [0,1]$. Now we have m kinds of pearls in $\mathbb{R}^m$, and we can apply the ham sandwich theorem.

The cool thing about this curve is that a hyperplane can only cut it in m places; those are the m cuts you make (every segment on one side of the hyperplane you give to person A, and the second to B). Here you need to do the continuous version so that you're not cuttingexactly through a pearl ;the measures you get will not be absolutely continuous (because they're on a curve), but they do satisfy that the mass of a hyperplane is 0, which is good enough. $\square$

Now we'll prove the necklace splitting problem with more thieves. Let p be a prime number, and take p thieves. (We'll later talk about what happens when the number of thieves is not prime.)

Proof. We first need to parametrize the space of partitions. Before, when we were doing this problem for two thieves, we took a cut, assigned ± 1 to each, and got a partition. Now we want to label each with $1, \ldots, p$. This leads to the *topological join* operation — given X and Y two topological wpafes, we define $X * Y$ by embedding them in very high-dimensional space to be in general position, and thhen tkaing the convex hull of X and Y.

What this is doing is
$$X * Y = X \times Y \times [0,1]/\sim$$
where $(x, y, 0) \sim (x, y', 0)$ (on the side of 0 we don't care about Y, so we just get a copy of X), and $(x, y, 1) \sim (x', y, 1)$. (There is no other overlap.)

For example, $\mathbb{Z}_2 * \mathbb{Z}_2 * \mathbb{Z}_2$ (the topological join is associative), where $\mathbb{Z}_2$ is a 2-point space, means we get a quadrilateral (all linear combinations of a point in x and a point in y); if we join another $\mathbb{Z}_2$ then we get an octahedron (putting one point ahead and one behind). In particular $(\mathbb{Z}_2)^{*n+1} \cong \mathbb{S}^n$.

Claim 5.2 — The space of partitions we want is $(\mathbb{Z}_p)^{*(c+1)}$ where c is the number of cuts.

Proof. Let's think about $\mathbb{Z}_p * \mathbb{Z}_p$ — this is telling us that we have an element $\lambda_1 \in \mathbb{Z}_p$ and an element $\lambda_2 \in \mathbb{Z}_p$, and a coefficient $\alpha \in [0, 1]$. We think of this as the partition $[\alpha \mid 1 - \alpha]$ where the first piece of length α goes to person λ_1, and the second piece to person λ_2. If $\alpha = 0$ then you don't care who receives the first piece (since it has length 0); if $\alpha = 1$ you similarly don't care who receives the second piece. □

For two thieves we said m cuts was the worst case scenario; what do we think will happen for p thieves? Again, the worst example we had was when all the pearls of one type were clumped together; then each one needs to be split into p pieces, so we need at least $p - 1$ cuts in each type of pearl. So we need at least $(p - 1)m$ cuts; the question is, is that enough?

So that's what we're aiming for, which means the space we consider is $(\mathbb{Z}_p)^{*[(p-1)m+1]}$. This is at least $(p - 1)m - 1$-connected (the connectedness grows, just like with $\mathbb{Z}_2$ where we started with a disconnected set, then got a connected set, then a 1-connected set, and so on). (This may or may not be off by 1.) For example $\mathbb{Z}_p$ is disconnected (-1-connected).

Now we have our space, and we need to make a test map — we need a space Y that tells us if our partition is good.

To do this, suppose we have our people $A_1, \ldots, A_p$; and say that given a partiion, A_1 is what the person 1 gets, A_2 is what the person 2 gets, and so on.

We also have an action of $\mathbb{Z}_p$ on $(\mathbb{Z}_p)^{*(c+1)}$ which is free — this is the only reason we needed p to be a prime number.

For our construction, take a partition $Q \in (\mathbb{Z}_p)^{*((p-1)m+1)}$. Ideally, we want to assign $\mu_1(A_1)$, $\mu_1(A_2)$, ..., $\mu_1(A_p)$ — we want to see how much each person is getting — and same for the others. So we define

$$f(Q) = \begin{bmatrix} \mu_1(A_1) & \mu_1(A_2) & \cdots & \mu_1(A_p) \\ \mu_2(A_1) & \mu_2(A_2) & \cdots & \mu_2(A_p) \\ \vdots & & & \\ \mu_m(A_1) & \mu_m(A_2) & \cdots & \mu_m(A_p) \end{bmatrix}$$

We can view this as an element of $\mathbb{R}^{pm}$.

THis dimension is too big, so we need to do a dimension reduction — here we have an action of $\mathbb{Z}_p$ which exchanges the columns, and this action is equivariant (it corresponds to just relabelling who gets each piece).

In each of the rows, if we are given a row $[\beta_1, \ldots, \beta_p]$, we can let $k = \sum \beta_i$ be the sum of the row, and we can replace it by $[\beta_1 - \frac{k}{p}, \ldots, \beta_p - \frac{k}{p}]$. What advantage do we have here? Now since the sum is 0, thsi is actually an element of $\mathbb{R}^{p-1}$ embedded into $\mathbb{R}^p$ (as the set of vectors with sum of coordinates 0). So this gives us a new function $G\colon (\mathbb{Z}_p)^{*(p-1)m+1} \to \mathbb{R}^{(p-1)m}$. What we're looking for is now a zero; if we find a zero, we're happy.

We an take
$$h(Q) = \frac{g(Q)}{\|g(Q)\|},$$
which is defined for every partition since we don't have a zero. So H goes from the space we want — $h\colon (\mathbb{Z}_p)^{*(p-1)m+1} \to \mathbb{S}^{(p-1)m+1}$. Now we can apply Bol, and we are happy. This is the solution when p is prime; what happens if it is not prime? □

This is the solution when p is prime; what happens if it is not prime?

There are other things using similar methods. Usually people prove it when one of hte parameters is 2, then p, then a prime power; and the case with no conditions is much more difficult. This problem is an exception, where you don't actually need more topology than this.

Claim 5.3 — If we can solve the problem for a thieves and for b thieves, we can solve it for ab thieves (where the claim is that for k thieves, we need $(k-1)m$ cuts).

Proof. Clump our thieves into b groups of a thieves each. THen we get our interval, and we make our first set of cuts to split between each of the groups of thieves — this takes $(b-1)m$ cuts.

THen each group takes the intervals assigned to them, puts them back to back as a necklace, and makes cuts there; this means they need $(a-1)m$ cuts.

SO we'll need

$$(b-1)m + b(a-1)m$$

cuts, which comes out to exactly $(ab-1)m$ cuts. □

Interestingly, this is different from Alon's solution in 1987 — he did use topological methods, but instead of using results like this one that guarantee no equivariant maps, with other parameters you can guarantee that there *are* equivariant maps, and he used one of those ofrce the partition.

§6 Method complexity

The starting point is the BU theorem; then you have Dold's theorem, and some degree arguments (about maps of spheres). Then you start getting into the fancier things, like characteristic classes, index theories, or spectral sequences and things like that. What we've done is much closure to the beginning.

Left of the middle between BU and Dold, there is a nice book that has almost everything you need; someone advancing these methods called them 'kids' stuff' because he found the book on ebay for kids. (It's nice to be on this side because computational geometry people care about these, and want algorithms; it's easier to do that with BU proofs.)

Remark 6.1. One of the standard proofs of BU uses degree, so why did we put degree further down? Here's an example of what we mean:

Suppose we have d measures in $\mathbb{R}^d$ (as in ham sandwich). Usually with $d+1$ measures you can't find a halving hyperplane, since for e.g. $d=2$ we can take three measures each concentrated in a blob at a separate vertex of a triangle. But you can find a *disk*. The proof is the nice map you wuold expect by adding on ecoordinate — e.g. $(x,y) \mapsto (x,y,x^2+y^2)$. Now you have $d+1$ measures in $\mathbb{R}^{d+1}$, and you apply Ham Sandwich there and project back.

What if we don't wnat a sphere, but a regular hypercube (which you can move and translate and blow up)? In the plane you have three measures, and you want a square containing exactly half of each. This is true but still unpublished; but if you try to apply standard BU tricks, they don't work. The method is usually not more complicated, but the proof is more tricky.

We'll prove this for squares (it's more complicated for general hypercubes). Suppose we have 3 meausres in $\mathbb{R}^2$, and we want a square.

We parametrize the space of squares by $S^1 \times [0,1]$. If you give me a direction v and a $\alpha \in [0,1]$, I take $\frac{\alpha}{1-\alpha}$ (scaling v). Then I take the squrae centered at this point that's perpendicular to v, and I blow it up until it contains half of the last measure μ_3. (This means we're only considering squares which are 'oriented towards the origin.')

For any square c (we assume our measures are probability measures, by scaling), we define

$$f(c) = (\mu_1(c) - \frac{1}{2}, \mu_2(c) - \frac{1}{2})$$

(so we want to see, do we have half of μ_1 and half of μ_2). If at any point we have a zero, we're done. Otherwise we can do the same reduction — define $g = f/\|f\|$, so weg et $g\colon \mathbb{S}^1 \times [0,1] \to \mathbb{S}^1$ (where the second $[0,1]$ corresponds to how far along to infinity our center is).

At 0, what happens? $g(v,0) = g(-v,0)$, because if we are at 0 then we are taking the square centered at the origin, and if we flip the direction of v then we're taking the same square. But $g(v,1) = -g(-v,1)$ — as $\alpha \to 1$ the point is going further and further away, and the square keeps growing and growing; so eventually we're basically just approaching the halving plane for μ_3, and if we do it in the other direction then we're taking the same halving plane, but from the other side. So if we cal one side A and the other side B (with $\mu(A) + \mu(B) = 1$), then $\mu(A) - \frac{1}{2} = -(\mu(B) - \frac{1}{2})$. So if we flip sides, then we are changing the sign.

Now $g|_0$ is a function that satisfies $g(v,0) = g(-v,0)$; it's a function $\mathbb{S}^1 \to \mathbb{S}^1$, so it must have even degree. $g|_1$ satisfies $g(v,1) = -g(-v,1)$, so it has odd degree. So we have a homotopy between an even-degree and odd-degree function, which cannot happen.

So it's not that degree is more advanced, but if you have to use it the proofs are usually trickier.

§7 Linear algebra proof of colorful Radon

Finally, we'll prove the original problem without topology. Recall we had $d+1$ pairs of points, and we wanted to split them into two colorful simplices whose convex hulls intersect.

Call the first pair (x_1, y_1), …, (x_{d+1}, y_{d+1}). We consider the vectors $x_i - y_i$; this gives us $d+1$ vectors in $\mathbb{R}^d$, so they are linearly dependent — there exist α_1, …, α_{d+1}, not all 0, such that

$$\sum \alpha_i (x_i - y_i) = 0.$$

The trick in the proof is — what happens if we find a negative coefficient/ If $\alpha_i < 0$, then we can swap the sign of α_i and the names of x_i and y_i; we do this for every negative thing. Now all the coefficients are nonnegative, and they're not all 0; so we can assume WLOG that $\sum \alpha_i = 1$ (by scaling) and the $\alpha_i \geq 0$. Then we can move all the y_i's to the other side, and we have a point that is in the convex hull of both the x's and the y's (we have $\sum \alpha_i x_i = \sum \alpha_i y_i$).