Expanding polynomials with additive structure

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Background

Expanding polynomials

A polynomial $f \in \mathbb{R}[x, y]$ is **expanding** if for any sets $A, B \subseteq \mathbb{R}$ of similar size, the set

$$f(A, B) = \{f(a, b) \mid a \in A,\ b \in B\}$$

is much larger — i.e., if $|A| = |B| = n$, then $|f(A, B)| = \omega(n)$.

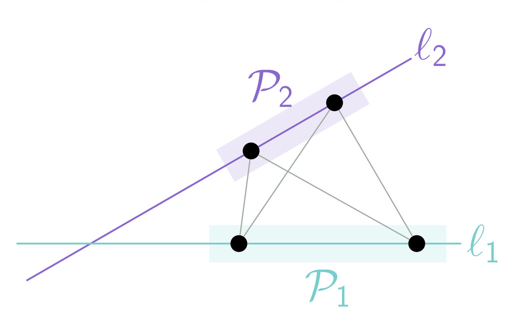
- For example, $f(x, y) = x + y$ is not expanding — if we take $A = B = \{1, 2, \dots, n\}$, then $|f(A, B)| = 2n - 1$.
- Similarly, $f(x, y) = (x + 1)y$ is not expanding — we can take $A = \{2^1 - 1, 2^2 - 1, \dots, 2^n - 1\}$ and $B = \{2^1, 2^2, \dots, 2^n\}$.
- It turns out that all polynomials which don't look like these are expanding! Elekes–Ronyai [1] showed that all polynomials are expanding except those of the forms
$$f(x, y) = g(x) + h(y) \text{ or } f(x, y) = g(x)h(y).$$
For example, $f(x, y) = x^2 + xy$ is expanding.

Once we know which polynomials are expanding, we can try to prove *quantitative* bounds on expansion.

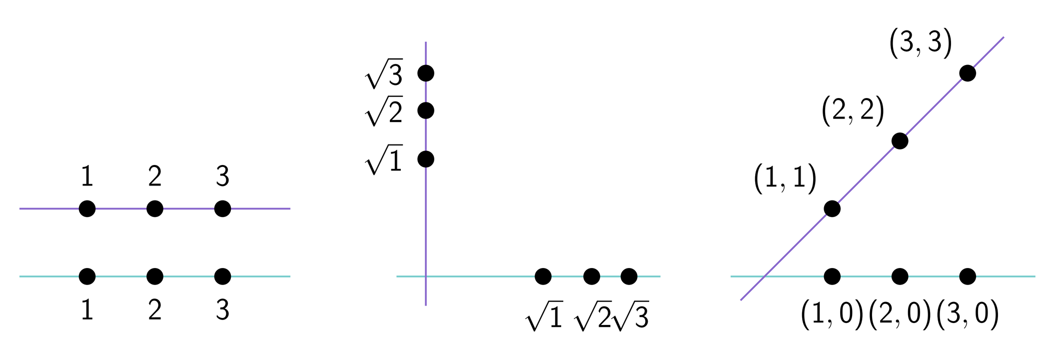
- Raz–Sharir–Solymosi [2] proved that if f isn't of one of these special forms, then $|f(A, B)| \gtrsim |A|^{2/3} |B|^{2/3}$.
- Solymosi–Zahl [3] improved this to $|f(A, B)| \gtrsim |A|^{3/4} |B|^{3/4}$.

Distinct distances between two lines

Expanding polynomials have a wide variety of applications. For example: Given two lines ℓ_1 and ℓ_2, if we place a set of n points $\mathcal{P}_1$ on ℓ_1 and $\mathcal{P}_2$ on ℓ_2, what's the minimum possible number of distinct distances between $\mathcal{P}_1$ and $\mathcal{P}_2$?

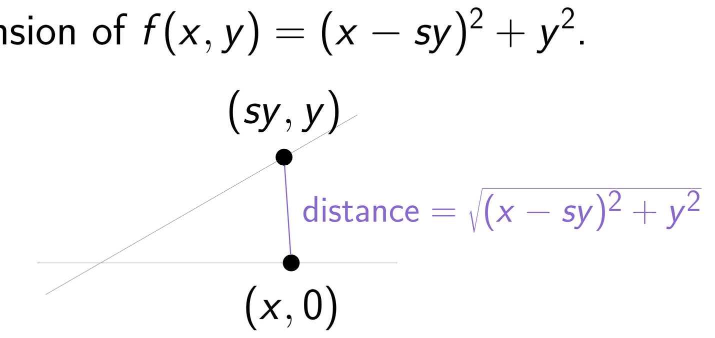


- If the lines are parallel or orthogonal, there are constructions with roughly n distances. Otherwise, the best construction we know has roughly $n^2/\sqrt{\log n}$ distances.



the parallel case — the perpendicular case — the general case

- If we call the lines $y = 0$ and $x = sy$, this problem is about the expansion of $f(x, y) = (x - sy)^2 + y^2$.



- The best lower bound we know is $n^{3/2}$; this comes from the Solymosi–Zahl expanding polynomials bound.

Our work

Question: If we know A and B have additive structure, can we prove better expansion bounds?

- We think of A as additively structured if its *sumset*
$$A + A = \{a_1 + a_2 \mid a_1, a_2 \in A\}$$
is small. For example, an arithmetic progression (which is very structured) has the smallest possible sumset, while a set of arbitrary numbers (which is very unstructured) has the largest possible sumset.
- So we want to prove better lower bounds on $|f(A, B)|$ when $A + A$ and $B + B$ are small.
- We do this for polynomials of the form $f(x, y) = g(x + p(y)) + h(y)$. (This includes polynomials $f(x, y) = g(x) + h(y)$, which are not expanding in general.)

Theorem (Das–Pohoata–Sheffer 2025+)

Let f be a polynomial of the form $f(x, y) = g(x + p(y)) + h(y)$ where $\deg g, \deg h \geq 2$, and let A and B be sets of similar size such that $A + A$ and $B + B$ are reasonably small. Then

$$|f(A, B)| \gtrsim \frac{|A|^{256/121 - o(1)} |B|^{74/121 - o(1)}}{|A + A|^{108/121} |B + B|^{24/121}}.$$

For example, when $|A| = |B|$, $|A + A| \approx |A|$, and $|B + B| \approx |B|$, this gives

$$|f(A, B)| \gtrsim |A|^{18/11 - o(1)}.$$

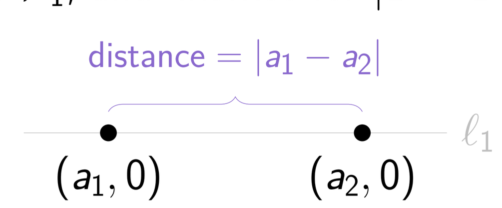
Some applications

Distinct distances between two lines

In the above theorem, we can replace $A + A$ with

$$A - A = \{a_1 - a_2 \mid a_1, a_2 \in A\}.$$

Geometrically, $|A - A|$ is twice the number of distinct distances between points in $\mathcal{P}_1$, and likewise for $|B - B|$ and $\mathcal{P}_2$.



So our bound, applied to

$$f(x, y) = (x - sy)^2 + y^2,$$

says that if there's few distances between points on the *same* line, then there must be lots of distances between points on *different* lines.

Corollary

Suppose ℓ_1 and ℓ_2 are not parallel, and $\mathcal{P}_1 \subseteq \ell_1$ and $\mathcal{P}_2 \subseteq \ell_2$ are sets of n points such that the numbers of distinct distances within $\mathcal{P}_1$ and within $\mathcal{P}_2$ are at most m. Then

$$\#(\text{distances between } \mathcal{P}_1 \text{ and } \mathcal{P}_2) \gtrsim \frac{n^{30/11 - o(1)}}{m^{12/11}}.$$

This bound is an improvement on $n^{3/2}$ when $n \leq m \leq n^{9/8 - o(1)}$. (It also applies when the lines are orthogonal.)

Sum-product-type bounds

The **sum-product phenomenon** is that a set A can't be both additively and multiplicatively structured: $A + A$ and $A \cdot A$ can't both be small.

- Erdős and Szemerédi showed that we always have
$$\max\{|A + A|, |A \cdot A|\} \gtrsim |A|^{1+c}$$
for some absolute constant $c > 0$, and conjectured that
$$\max\{|A + A|, |A \cdot A|\} \gtrsim |A|^{2 - o(1)}.$$
- The current best bound shows that we can take c slightly greater than $1/3$.

This holds more generally — for any convex function g, the sets A and $g(A)$ can't both be very additively structured. (The sum-product problem corresponds to $g(x) = -\log x$.)

- The best bound, due to Stevens and Warren [4], was
$$\max\{|A + A|, |g(A) + g(A)|\} \gtrsim |A|^{49/38 - o(1)}.$$
- We get a slight improvement when g is a polynomial, by taking $f(x, y) = g(x) + g(y)$ and $A = B$ in our theorem.

Corollary

For any polynomial g with $\deg g \geq 2$,

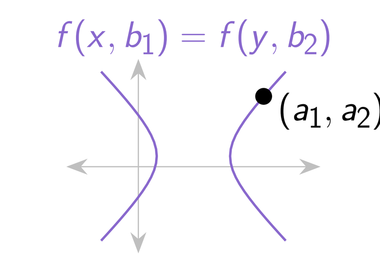
$$\max\{|A + A|, |g(A) + g(A)|\} \gtrsim |A|^{30/23 - o(1)}.$$

(We have $49/38 \approx 1.289$ and $30/23 \approx 1.304$.)

Proof ideas

Previous work on expanding polynomials

- Raz, Sharir, and Solymosi [2] looked at the **energy**
$$\#\{(a_1, b_1, a_2, b_2) \mid f(a_1, b_1) = f(a_2, b_2)\}.$$
If $f(A, B)$ is small, then this energy is large; so it's enough to prove an *upper* bound on the energy.
- They turned this into an **incidence geometry** problem by using (a_1, a_2) to define a point, and (b_1, b_2) to define a curve.



Then the energy is the number of incidences between these points and curves (point-curve pairs where the point lies on the curve); tools from incidence geometry provide an upper bound on this number.

- Solymosi and Zahl [3] improved this with **proximity**: they only considered 4-tuples with a_1 close to a_2 and b_1 to b_2.
- This shrinks the number of points and curves, so it shrinks the upper bound. It also shrinks the lower bound, but by less — this is because among solutions to $f(a_1, b_1) = f(a_2, b_2)$, the two proximity conditions are very well-correlated.

Incorporating additive structure

- Each $a \in A$ can be written as $\alpha - a'$ with $\alpha \in A + A$ and $a' \in A$ in at least $|A|$ ways. So if the energy is large and $A + A$ is small, there are lots of solutions to
$$f(\alpha_1 - a_1', b_1) = f(\alpha_2 - a_2', b_2).$$
- We also upper-bound the number of solutions using tools from incidence geometry, using (α_1, α_2) to define a point and (a_1', a_2', b_1, b_2) to define a curve.
- We incorporate proximity by restricting to 6-tuples where α_1 is close to α_2, a_1' to a_2', and b_1 to b_2.

Acknowledgements

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[2] Orit E. Raz, Micha Sharir, and József Solymosi.
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