Artin's Theorem on Induced Characters

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1 Introduction

In this paper, we will prove Artin's theorem, following [1, Chapter 9]. The statement of the theorem is as follows.

Theorem 1.1 (Artin)**.** *Let G be a finite group, and let $\mathcal{X}$ be a family of subgroups of G. Then the following two statements are equivalent:*

(i) *For every conjugacy class C in G, there is some subgroup in $\mathcal{X}$ containing an element of C. (In other words, G is the union of all the conjugates of the subgroups in $\mathcal{X}$.)*

(ii) *Every character of G can be written as a $\mathbb{Q}$-linear combination (i.e., a linear combination with rational coefficients) of characters induced from the subgroups in $\mathcal{X}$.*

We will provide two different proofs of Theorem 1.1 in Sections 2 and 3. For both proofs, it will be useful to first establish some definitions and background.

1.1 The Ring $R(G)$

Definition 1.2. For a finite group G, let $\mathcal{F}(G)$ denote the ring of $\mathbb{C}$-valued class functions on G. Let $R(G) \subseteq \mathcal{F}(G)$ denote the subring of $\mathcal{F}(G)$ generated as an abelian group by the characters of G.

In other words, if the irreducible characters of G are $\chi_1, \ldots, \chi_r$, then

$$R(G) = \mathbb{Z}\chi_1 \oplus \cdots \oplus \mathbb{Z}\chi_r \subseteq \mathcal{F}(G).$$

From this definition, it is only obvious that $R(G)$ is a group (under addition). However, $R(G)$ is in fact a ring — the product of two characters of G is again a character of G (as the product of the characters of two representations ρ and ρ' is the character of the representation $\rho \otimes \rho'$), and $R(G)$ contains the multiplicative identity $\mathbf{1}$, the function sending $x \mapsto 1$ for all $x \in G$ (as $\mathbf{1}$ is the character of the unit representation of G).

Remark 1.3. Note that $\mathcal{F}(G) = \mathbb{C} \otimes R(G)$ (which can be viewed as the space of $\mathbb{C}$-linear combinations of characters of G), as the irreducible characters $\chi_1, \ldots, \chi_r$ form a basis for $\mathcal{F}(G)$ as a $\mathbb{C}$-vector space.

1.2 Induction and Restriction

Definition 1.4. For a finite group G and subgroup $H \subseteq G$, the linear map $\operatorname{Ind}_H^G \colon \mathcal{F}(H) \to \mathcal{F}(G)$ is the map sending a class function $\varphi \colon H \to \mathbb{C}$ to the class function $\operatorname{Ind}_H^G(\varphi) \colon G \to \mathbb{C}$ defined as

$$\operatorname{Ind}_H^G(\varphi) \colon x \mapsto \frac{1}{|H|} \sum_{\substack{g \in G \\ gxg^{-1} \in H}} \varphi(gxg^{-1}).$$

It is well-known (see [1, Chapter 7.2] for a proof) that if φ is the character of a representation ρ of H, then $\mathrm{Ind}_H^G(\varphi)$ is the character of the representation of G induced from ρ. In particular, this implies Ind_H^G also provides a $\mathbb{Z}$-linear map (or group homomorphism) $R(H) \to R(G)$.

Definition 1.5. Given a finite group G and a collection of subgroups $\mathcal{X}$, define $\mathrm{Ind}_{\mathcal{X}} \colon \bigoplus_{H\in\mathcal{X}} \mathcal{F}(H) \to \mathcal{F}(G)$ as the linear map $(\varphi_H)_{H\in\mathcal{X}} \mapsto \sum_{H\in\mathcal{X}} \mathrm{Ind}_H^G(\varphi_H)$.

As above, $\mathrm{Ind}_{\mathcal{X}}$ can also be viewed as a $\mathbb{Z}$-linear map $\bigoplus_{H\in\mathcal{X}} R(H) \to R(G)$. Then the condition (ii) of Artin's theorem can be restated in terms of $\mathrm{Ind}_{\mathcal{X}}$ — it states that for every character χ of G, there is an integer d such that $d\chi$ is in the image of $\mathrm{Ind}_{\mathcal{X}} \colon \bigoplus_{H\in\mathcal{X}} R(H) \to R(G)$.

Both proofs of Artin's theorem will involve the operation of *restriction* as well, though in different ways.

Definition 1.6. For a finite group G and subgroup $H \subseteq G$, the linear map $\mathrm{Res}_H^G \colon \mathcal{F}(G) \to \mathcal{F}(H)$ is the map sending a class function $\varphi\colon G \to \mathbb{C}$ to its restriction $\varphi|_H$.

Clearly if φ is the character of a representation ρ of G, then $\mathrm{Res}_H^G(\varphi)$ is the character of the restriction $\rho|_H$. So Res_H^G also provides a $\mathbb{Z}$-linear map $R(G) \to R(H)$.

Definition 1.7. Given a finite group G and a collection of subgroups $\mathcal{X}$, define $\mathrm{Res}_{\mathcal{X}} \colon \mathcal{F}(G) \to \bigoplus_{H\in\mathcal{X}} \mathcal{F}(H)$ as the linear map $\varphi \mapsto (\mathrm{Res}_H^G(\varphi))_{H\in\mathcal{X}}$.

As above, $\mathrm{Res}_{\mathcal{X}}$ can also be viewed as a $\mathbb{Z}$-linear map $R(G) \to \bigoplus_{H\in\mathcal{X}} R(H)$. The interplay between $\mathrm{Ind}_{\mathcal{X}}$ and $\mathrm{Res}_{\mathcal{X}}$ will be crucial to the first proof of Artin's theorem, and will be useful in the second proof as well.

2 First Proof of Artin's Theorem

In this section, we will give a proof of Artin's theorem using linear algebra. The main idea of the proof is to first reduce the condition (ii) to a statement about the surjectivity of $\mathrm{Ind}_{\mathcal{X}}$ as a map between the $\mathbb{C}$-vector spaces $\bigoplus_{H\in\mathcal{X}} \mathcal{F}(H)$ and $\mathcal{F}(G)$, and then use the fact that $\mathrm{Ind}_{\mathcal{X}}$ and $\mathrm{Res}_{\mathcal{X}}$ are adjoint operators to reduce this to a statement about the *injectivity* of $\mathrm{Res}_{\mathcal{X}}$, which is more easily seen to be equivalent to (i).

Lemma 2.1. *The following two conditions are equivalent:*

(ii) *Every character of G can be written as a $\mathbb{Q}$-linear combination of characters induced from the subgroups in $\mathcal{X}$.*

(iii) *Every class function on G can be written as a $\mathbb{C}$-linear combination of class functions induced from the subgroups in $\mathcal{X}$ — i.e., $\mathrm{Ind}_{\mathcal{X}} \colon \bigoplus_{H\in\mathcal{X}} \mathcal{F}(H) \to \mathcal{F}(G)$ is surjective.*

Proof. First it is clear that (ii) implies (iii), since the irreducible characters of G form a $\mathbb{C}$-basis of $\mathcal{F}(G)$ (so if every irreducible character of G can be written as a $\mathbb{Q}$-linear combination of characters induced from $\mathcal{X}$, then since every class function can be written as a $\mathbb{C}$-linear combination of irreducible characters of G, we obtain that every class function is a $\mathbb{C}$-linear combination of characters induced from $\mathcal{X}$).

To see that (iii) implies (ii), first note that for each $H \subseteq \mathcal{X}$, the irreducible characters of H form a $\mathbb{C}$-basis for $\mathcal{F}(H)$, so (iii) implies that the characters $\varphi_1, \ldots, \varphi_m$ induced from irreducible characters of subgroups in $\mathcal{X}$ (of which there are finitely many) span $\mathcal{F}(G)$ as a $\mathbb{C}$-vector space. Now assume for contradiction that (ii) does *not* hold, so $\varphi_1, \ldots, \varphi_m$ do not span $\mathbb{Q}\otimes R(G) \subseteq \mathcal{F}(G)$ as a $\mathbb{Q}$-vector space. Then we perform the following process — we begin with the list of induced characters $\varphi_1, \ldots, \varphi_m$, and for as long as our list of characters is not linearly independent over $\mathbb{Q}$, we remove one which is in the $\mathbb{Q}$-span of the others. This does not affect either the $\mathbb{Q}$-span or the $\mathbb{C}$-span of the characters in our list (since if one vector is in the $\mathbb{Q}$-span of the others, it is also in their $\mathbb{C}$-span). In the end, when the characters remaining are linearly independent over $\mathbb{Q}$, we must have strictly fewer than $\dim_{\mathbb{Q}}(\mathbb{Q}\otimes R(G))$ of them; but since $\dim_{\mathbb{Q}}(\mathbb{Q}\otimes R(G)) = \dim_{\mathbb{C}} \mathcal{F}(G)$, these characters cannot span $\mathcal{F}(G)$ as a $\mathbb{C}$-vector space, contradicting (iii). ☐

Lemma 2.1 means that in order to prove Artin's theorem, it suffices to prove that (i) and (iii) are equivalent; this can be done by considering the *adjoint* map.

Definition 2.2. Let V and W be vector spaces with inner products $\langle -, - \rangle_V$ and $\langle -, - \rangle_W$. For a linear map $T\colon V \to W$, the *adjoint* of T is the unique linear map $T^*\colon W \to V$ such that for all $v \in V$ and $w \in W$ we have $\langle v, T^* w\rangle_V = \langle Tv, w\rangle_W$.

Fact 2.3. For finite-dimensional vector spaces V and W, a linear map $T\colon V \to W$ is surjective if and only if its adjoint $T^*\colon W \to V$ is injective.

Proof. First assume T^* is *not* injective, so there exists a nonzero vector $w \in \ker T^*$. Then for all $v \in V$ we have $\langle Tv, w\rangle_W = \langle v, T^* w\rangle_V = \langle v, 0\rangle = 0$. This means w is orthogonal to $\operatorname{im}(T)$, so since w is nonzero, we cannot have $\operatorname{im}(T) = W$.

Conversely, assume T is *not* surjective, so there exists a nonzero vector w in the orthogonal complement of $\operatorname{im}(T)$. Then for all $v \in V$ we have $\langle v, T^* w\rangle_V = \langle Tv, w\rangle_W = 0$; this means we must have $T^* w = 0$, so T^* has nonzero kernel. $\square$

In our situation, the $\mathbb{C}$-vector space $\mathcal{F}(G)$ has the standard inner product defined as

$$\langle \varphi, \psi\rangle_G = \frac{1}{|G|}\sum_{x \in G} \varphi(x)\overline{\psi(x)}.$$

This can be used to obtain a standard inner product on $\bigoplus_{H\in\mathcal{X}} \mathcal{F}(H)$, defined as

$$\langle (\varphi_H)_{H\in\mathcal{X}}, (\psi_H)_{H\in\mathcal{X}}\rangle = \sum_{H\in\mathcal{X}} \langle \varphi_H, \psi_H\rangle_H.$$

Fact 2.4. The map $\operatorname{Ind}_{\mathcal{X}}\colon \bigoplus_{H\in\mathcal{X}} \mathcal{F}(H) \to \mathcal{F}(G)$ has adjoint $\operatorname{Res}_{\mathcal{X}}\colon \mathcal{F}(G) \to \bigoplus_{H\in\mathcal{X}} \mathcal{F}(H)$.

Proof. This follows directly from Frobenius reciprocity (see [1, Section 7.2, Theorem 13]) — Frobenius reciprocity states that Res_H^G is the adjoint of Ind_H^G for all subgroups $H \subseteq G$, and summing over $H \in \mathcal{X}$ gives that $\operatorname{Res}_{\mathcal{X}}$ is the adjoint of $\operatorname{Ind}_{\mathcal{X}}$. $\square$

We are now in a position to prove Artin's theorem using these results.

Proof of Theorem 1.1. By Lemma 2.1 we have that (ii) and (iii) are equivalent, and by Fact 2.3 and Fact 2.4, the condition (iii) is equivalent to the injectivity of $\operatorname{Res}_{\mathcal{X}}$; so it suffices to prove that (i) is equivalent to the injectivity of $\operatorname{Res}_{\mathcal{X}}$.

But this is clear — a class function $\varphi \in \mathcal{F}(G)$ is in $\ker \operatorname{Res}_{\mathcal{X}}$ if and only if its restriction to every subgroup $H \in \mathcal{X}$ is identically zero. If (i) is true, then any such φ must be zero on every conjugacy class $C \subseteq G$ (as one of the subgroups in $\mathcal{X}$ contains an element of C, and the restriction of φ to that subgroup must be zero). Conversely, if (i) is not true, then we can let C be a conjugacy class such that no subgroup $H \in \mathcal{X}$ contains an element of C, and define φ to be the class function which is 1 on C and 0 everywhere else; then φ is a nonzero element of $\ker \operatorname{Res}_{\mathcal{X}}$. $\square$

3 Second Proof of Artin's Theorem

In this section, we will give a second proof of the implication (i) $\implies$ (ii) of Artin's theorem. This proof will in fact prove a slightly stronger statement — that for $\mathcal{X}$ satisfying (i), for every character χ of G we can write $n\chi$ as a $\mathbb{Z}$-linear combination of characters induced from subgroups $H \in \mathcal{X}$, where $n = |G|$ (in other words, we can obtain rational coefficients in (ii) whose denominators divide n). The idea of the proof is to first reduce to the case where $\mathcal{X}$ is the collection of all cyclic subgroups of G, in which case we can find an explicit construction.

3.1 Reduction to the Cyclic Case

In this subsection, we will prove the implication (i) $\implies$ (ii) of Artin's theorem assuming the following proposition (in other words, we will show that the general case follows from the special case where $\mathcal{X}$ is the collection of all cyclic subgroups of G), which we will prove in the next subsection.

Proposition 3.1. *Let G be a finite group with order n. Then for every character χ of G, the character $n\chi$ can be written as a $\mathbb{Z}$-linear combination of characters induced from cyclic subgroups of G.*

This reduction will follow directly from the following lemma.

Lemma 3.2. *Let G be a finite group, and let $H \subseteq G$ be a subgroup of G and $A \subseteq G$ be a subgroup of G which is conjugate to a subgroup of H. Then every character of G induced from A is also a character induced from H.*

Proof. First, if A and A' are conjugate, then they clearly induce the same characters; so we may assume A itself is a subgroup of H. Then for every character φ of A, we have $\operatorname{Ind}_A^G(\varphi) = \operatorname{Ind}_H^G(\operatorname{Ind}_A^H(\varphi))$. $\square$

Proof of (i) $\implies$ (ii) assuming Proposition 3.1. Given any collection of subgroups $\mathcal{X}$ satisfying (i), every cyclic subgroup $A \subseteq G$ must be conjugate to a subgroup of some $H \in \mathcal{X}$ (since if A is generated by x, then by (i) there must exist $H \in \mathcal{X}$ containing a conjugate of x, and therefore the cyclic subgroup generated by this conjugate of x). By Lemma 3.2 this means every character induced from a cyclic subgroup can also be induced from a subgroup $H \in \mathcal{X}$. Proposition 3.1 states that for every character χ of G we can write $n\chi$ as a linear combination of characters induced from cyclic subgroups $A \subseteq G$, so this means we can write $n\chi$ as a linear combination of characters induced from subgroups $H \in \mathcal{X}$ as well. $\square$

3.2 Proof of the Cyclic Case

In this subsection, we will prove Proposition 3.1. We will first prove that $n\mathbf{1}$ (the constant function $x \mapsto n$) can be written as a $\mathbb{Z}$-linear combination of characters induced from cyclic subgroups of G, using the following explicit class functions.

Definition 3.3. For a cyclic group A, define the class function $\theta_A \colon A \to \mathbb{C}$ as

$$\theta_A(x) = \begin{cases} |A| & \text{if } x \text{ generates } A \\ 0 & \text{otherwise.} \end{cases}$$

Lemma 3.4. *For a finite group G with order n, we have*

$$n\mathbf{1} = \sum_A \operatorname{Ind}_A^G(\theta_A),$$

where the sum is over all cyclic subgroups $A \subseteq G$.

Proof. Applying the formula for Ind_A^G given in Definition 1.4, for all $x \in G$ we have

$$\sum_A \operatorname{Ind}_A^G(\theta_A)(x) = \sum_A \frac{1}{|A|} \sum_{\substack{g \in G \\ gxg^{-1} \in A}} \theta_A(gxg^{-1}) = \sum_{g \in G} \sum_{A \ni gxg^{-1}} \frac{\theta_A(gxg^{-1})}{|A|}$$

(swapping the order of summation to first sum over all g, and then over all cyclic subgroups A containing gxg^{-1}). But for each $g \in G$, there is a unique cyclic subgroup A generated by gxg^{-1}; this subgroup will contribute 1 to the summation, and all others will contribute 0. So each $g \in G$ has a total contribution of 1, which means $\sum_A \operatorname{Ind}_A^G(\theta_A)(x) = \sum_{g \in G} 1 = n$. $\square$

Lemma 3.4 almost provides a way to write $n\mathbf{1}$ as a $\mathbb{Z}$-linear combination of characters induced from cyclic subgroups of G, but we do not yet know that the functions θ_A are $\mathbb{Z}$-linear combinations of characters of A. However, Lemma 3.4 provides a way to inductively prove this as well.

Lemma 3.5. *For any cyclic group A, the class function θ_A belongs to $R(A)$.*

Proof. We use induction on $|A|$. The base case $|A| = 1$ is clear, as θ_A is the constant function $\mathbf{1}$. For the inductive step, assume that we have proven $\theta_B \in R(B)$ for all cyclic groups B with $|B| < |A|$. Then by Lemma 3.4 we have

$$|A|\,\mathbf{1} = \sum_{B \subseteq A} \operatorname{Ind}_B^A(\theta_B) = \theta_A + \sum_{B \subsetneq A} \operatorname{Ind}_B^A(\theta_B),$$

where the first sum is over all cyclic subgroups $B \subseteq A$ and the second is over all *strict* subgroups $B \subsetneq A$. But $\mathbf{1} \in R(A)$, and since $\theta_B \in R(B)$ for all strict subgroups $B \subsetneq A$ by the inductive hypothesis, we have $\operatorname{Ind}_B^A(\theta_B) \in R(A)$ for all such B as well. This implies $\theta_A \in R(A)$, as desired. $\square$

Together, Lemmas 3.4 and 3.5 prove Proposition 3.1 in the case $\chi = \mathbf{1}$. To prove Proposition 3.1 in general, we need the following well-known fact.

Fact 3.6. For a finite group G and subgroup $H \subseteq G$, for any class functions φ on H and ψ on G, we have

$$\operatorname{Ind}_H^G(\varphi \cdot \operatorname{Res}_H^G \psi) = \operatorname{Ind}_H^G(\varphi) \cdot \psi.$$

Proof. Applying the formula in Definition 1.4, for all $x \in G$ we have

$$\operatorname{Ind}_H^G(\varphi \cdot \operatorname{Res}_H^G \psi)(x) = \frac{1}{|H|} \sum_{\substack{g \in G \\ gxg^{-1} \in H}} \varphi(gxg^{-1})\psi(gxg^{-1}),$$

$$\operatorname{Ind}_H^G(\varphi) \cdot \psi(x) = \frac{1}{|H|} \sum_{\substack{g \in G \\ gxg^{-1} \in H}} \varphi(gxg^{-1})\psi(x).$$

But $\psi(gxg^{-1}) = \psi(x)$ for all $g, x \in G$ because ψ is a class function on G, so the two expressions are equal. $\square$

Proof of Proposition 3.1. First, Fact 3.6 implies that for all subgroups $H \subseteq G$, the image of $\operatorname{Ind}_H^G \colon R(H) \to R(G)$ is an ideal of $R(G)$ — it is clearly closed under addition by the $\mathbb{Z}$-linearity of Ind_H^G, while Fact 3.6 implies it is closed under multiplication by all $\psi \in R(G)$. So for any collection $\mathcal{X}$ of subgroups of G, the image of $\operatorname{Ind}_{\mathcal{X}} \colon \bigoplus_{H \in \mathcal{X}} R(H) \to R(G)$ is an ideal of $R(G)$ as well.

Now taking $\mathcal{X}$ to be the collection of all cyclic subgroups, by Lemmas 3.4 and 3.5 we have that $\operatorname{im}(\operatorname{Ind}_{\mathcal{X}})$ contains $n\mathbf{1}$, so since $\operatorname{im}(\operatorname{Ind}_{\mathcal{X}})$ is an ideal, it must also contain $n\chi$ for all $\chi \in R(G)$. $\square$

References

[1] Serre, Jean-Pierre. *Linear Representations of Finite Groups*. Springer, New York, 1977.