The periodic tiling conjecture

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November 3, 2023

§1 Introduction

Throughout this talk, we'll work within finitely generated abelian groups (but it's possible to generalize things to other abelian groups as well).

§1.1 Translational tilings

Definition 1.1. Let G be a finitely generated abelian group, and let F be a finite subset of G. If there is a set $A \subseteq G$ such that for all $x \in G$, there is exactly one pair $(f, a) \in F \times A$ such that $x = f + a$, then we say that F tiles G, and that A is a tiling of G by F.

In other words, A is a tiling of G by F if $F \oplus A = G$. We can also consider a more analytic definition — we can restate this condition as $1_F * 1_A = 1$ (where $*$ denotes convolution, so $1_F * 1_A(x) = \sum_{f \in F} 1_A(x - f)$).

First, we'll see some examples.

Example 1.2

Let G be any finitely generated abelian group, take any element $x \in G$, and let $F = \{x\}$. Then we can take $A = G$, since if we translate $\{x\}$ by all points in G, then we'll cover every point exactly once.

Example 1.3

As a non-example, let $G = \mathbb{Z}$, and let $F = \{0, 1, 3\}$. Then it's not possible to tile $\mathbb{Z}$ with F, since we can't cover every point in $\mathbb{Z}$ without having overlapping tiles — for example, if we place down $\{0, 1, 3\}$, then we can't place down another tile that covers 2 but doesn't overlap the previous tile.

In Example 1.2 we only had one option for our tiling A, but in general we can have lots of options, as in the next example.

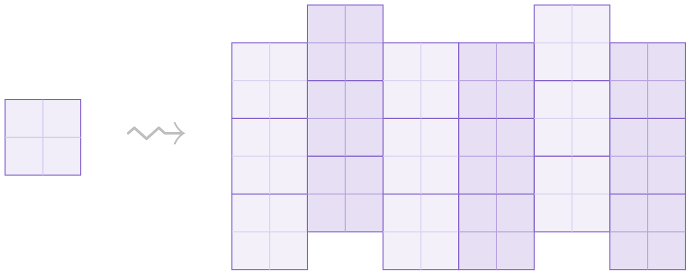
Example 1.4

Let $G = \mathbb{Z}^2$ and $F = \{0, 1\}^2$. Then we can take

$$A = \{(m, n + a(n)) \mid m, n \in 2\mathbb{Z}\}$$

for any function $a\colon \mathbb{Z} \to \{0, 1\}$.

This means we're shifting the columns of our tiling independently (where the shifts are given by a).



(We'll draw the elements of $\mathbb{Z}^2$ as square boxes.)

We'll see later that in the case $G = \mathbb{Z}$, all the possible tilings are very structured; but once we go to $\mathbb{Z}^2$, the tilings can get very messy. So the jump from $\mathbb{Z}$ to $\mathbb{Z}^2$ is already quite significant; and you can imagine that when we go to $\mathbb{Z}^3$, things get even more complicated (in general, the higher the dimension is, the worse the structure of the tilings can be).

For instance, here's an example of a messier tiling in $\mathbb{Z}^2$, where we get to shift both columns *and* rows.

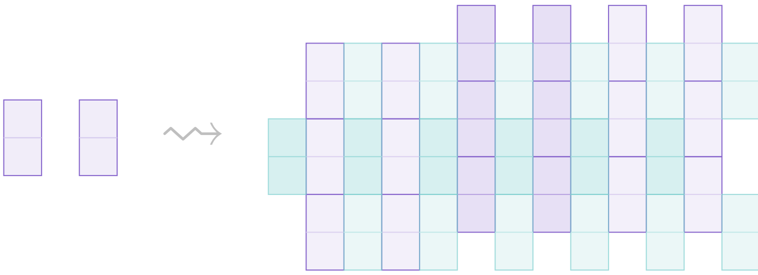
Example 1.5

Let $G = \mathbb{Z}^2$, and let $F = \{0, 2\} \times \{0, 1\}$. Then we can take

$$A = \{(4n, 2m + a(n)) \mid n, m \in \mathbb{Z}\} \cup \{(4n + 1 + 2b(m), 2m) \mid n, m \in \mathbb{Z}\}$$

for all choices of functions $a, b \in \{0, 1\}$.

Here the first set covers the even columns, while the second set covers the odd columns; and we're shifting the columns independently in the first set and the *rows* independently in the second set.



§1.2 Periodicity

Definition 1.6. If A is a tiling of G, we say A is $\langle h \rangle$-periodic (for some $h \in G$) if A is invariant under translation by h — i.e., if $A = h + A$.

Definition 1.7. We say a tiling A is periodic if there exists a lattice Λ (i.e., a subgroup of G with finite index) such that $A = \lambda + A$ for all $\lambda \in \Lambda$.

Remark 1.8. Sometimes this condition (that we have a full lattice of periods) is called being *strongly* periodic; but in the context of the periodic tiling conjecture, it's commonly just referred to as periodic.

Let's go back to our examples and check whether they satisfy periodicity.

Example 1.9

The tiling in Example 1.2 (where $A = G$) is certainly periodic.

Example 1.10

The tiling in Example 1.4 is $\langle (0,2) \rangle$-periodic. But for most choices of the function $a\colon \mathbb{Z} \to \{0,1\}$, it won't be (strongly) periodic — we won't get a full lattice of periods (we'll only have periods in the vertical direction).

Example 1.11

The tiling in Example 1.5 doesn't have a period at all (for most choices of a and b) — A isn't periodic in any direction. However, the first set in the union is $\langle (0,2) \rangle$-periodic, and the second is $\langle (4,0) \rangle$-periodic; so we *can* write A as a union of two sets, each of which is periodic in one direction. We'll call this type of structure *weakly periodic*, and we'll come back to it later.

§1.3 The periodic tiling conjecture

Conjecture 1.12 (Periodic tiling conjecture) — If F tiles G (by translation), then it must admit some tiling A which is (strongly) periodic.

In 1974, Stein suggested this conjecture in the context of finitely generated abelian groups. In 1987, Grünbaum and Shephard wrote an important book about tilings in which they implicitly suggest this conjecture. The conjecture was also formulated in a paper by Lagarias and Wang in 1996 in the context of Euclidean space.

We can think of the periodic tiling conjecture as saying that any tiling can be 'repaired' to be made periodic. For example, the tiling in Example 1.4 is not periodic, but we can shift its columns to make it periodic (while preserving the fact that it's a tiling).

Remark 1.13. Is the version of the periodic tiling conjecture for general finitely generated abelian groups equivalent to the one for $\mathbb{Z}^d$? The answer is yes, though it's not easy to show (it was proven very recently) — it's been shown that if we have a counterexample to the periodic tiling conjecture in a *quotient* of $\mathbb{Z}^d$, then we can lift it in a rigid way to get a counterexample in $\mathbb{Z}^d$. (In general, when we lift to $\mathbb{Z}^d$ we get much more room to repair the tiling, so we have to somehow do this in a rigid way; that makes this statement nontrivial.)

§2 The periodic tiling conjecture in $\mathbb{Z}$

We'll first consider the one-dimensional case, where we're trying to tile $\mathbb{Z}$. In this case, we do know the periodic tiling conjecture is true; in fact, even a much stronger statement is true.

Theorem 2.1 (Newman 1977)

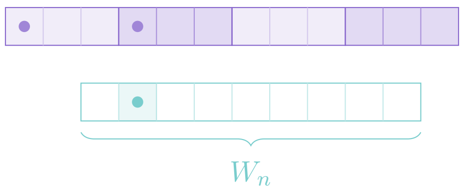
Any tiling of $\mathbb{Z}$ is periodic.

Proof. Let F tile $\mathbb{Z}$ (for convenience, we'll assume that the leftmost element of F is 0), and let A be a tiling of $\mathbb{Z}$ by F. Define

$$W = \{1, \dots, \operatorname{diam}(F)\}$$

(we use the letter W for 'window'). Then for each $n \in \mathbb{Z}$, we define the *nth window* of our tiling as

$$W_n = (A - n) \cap W.$$



Then for each n, the window W_n determines the entire tiling — i.e., given W_n, we can figure out the entire set A. To see this, W_n tells us all the points in A in a block of length $\operatorname{diam}(F)$. Then to figure out what's happening at the next point (i.e., whether the point immediately after this block should be placed in A), we simply check whether this point is already covered by one of the tiles from our block. If the answer is yes, then we can't put this point in A (as this would cause our tiles to overlap). Meanwhile, if the answer is no, then we *must* put this point in A, or else the point wouldn't get covered (since it can't get covered by a tile starting outside our length-$\operatorname{diam}(F)$ block).

But there's only finitely many possibilities for these windows, so there must be two values of n with the same window — more quantitatively, there's at most $2^{\operatorname{diam}(F)}$ possible windows, so there must exist distinct $n_1, n_2 \in \{0, \dots, 2^{\operatorname{diam}(F)}\}$ with $W_{n_1} = W_{n_2}$. And since each of these windows completely determines the entire tiling, we get that A is periodic with period $n_2 - n_1$, i.e., $A = (n_2 - n_1)\mathbb{Z} + A$. (Explicitly, this is because if we try completing A from the n_2th window, we'll get the same result as if we tried completing A from the n_1th window, but shifted over by $n_2 - n_1$.) □

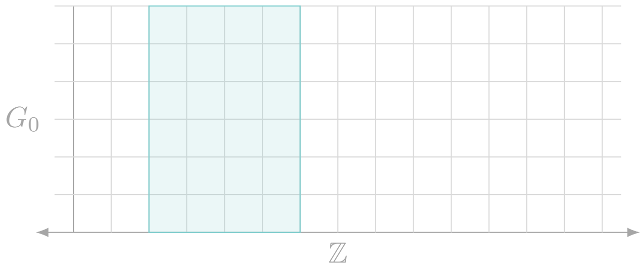
We'll see this type of pigeonhole argument many times — whenever we're able to 'push' our tiling to a one-dimensional structure, we can apply pigeonhole to get a period.

Using a similar pigeonhole argument, we can prove the following generalization.

Theorem 2.2 (Greenfeld–Tao 2022)

The periodic tiling conjecture holds in $\mathbb{Z} \times G_0$ for any *finite* abelian group G_0.

Proof sketch. Imagine that we plot $\mathbb{Z} \times G_0$ with $\mathbb{Z}$ on the x-axis and G_0 on the y-axis. We'll then take our windows to look like rectangles.



There's still finitely many possible windows, which means we can again find two identical ones. This won't necessarily imply that the tiling is periodic, but it does mean we can *repair* the tiling to be periodic. (We won't go into too much detail now because we'll return to this when proving the periodic tiling conjecture in $\mathbb{Z}^2$ later on.) $\square$

Remark 2.3. These proofs give a bound on the period which is exponential in $\operatorname{diam}(F)$. But in reality, we don't know tilings which need such huge periods — all the tilings we know of have periods linear in $\operatorname{diam}(F)$. So if we want to get good *quantitative* bounds, this probably isn't the right way to argue.

§3 The periodic tiling conjecture for prime tiles

We can also prove the periodic tiling conjecture for any tile with *prime* cardinality, and the proof is again surprisingly simple.

Theorem 3.1 (Szegedy 1998)

The periodic tiling conjecture is true (in any G) for tiles F where $|F|$ is prime.

Proof. Let F be a tile with $|F| = p$ (where p is prime). We'll make use of the convolution condition for tilings, which states that there exists a set A such that

$$1_F * 1_A = 1.$$

Now we'll keep convolving by 1_F — we'll do this $p-1$ times on both sides to get

$$(1_F)^{*p} * 1_A = |F|^{p-1} \tag{1}$$

(since $1_F * 1 = |F|$). Next, we'll perform a trick that we'll see again later — we take this equation mod p. The right-hand side is 0, as $|F| = p$; to handle the left-hand side, we'll use the following observation.

Claim 3.2 — We have $(1_F)^{*p} = 1_{pF}$.

Proof. We can think of this as the Frobenius endomorphism. More explicitly, we can write $1_F = \sum_{f \in F} \delta_f$, where δ_f is the indicator function of the singleton $\{f\}$. We're trying to convolve $(\sum_f \delta_f)^{*p}$ and take the result mod p; and if we open up the parentheses (i.e., expand everything out), all terms cancel out mod p except the ones of the form $(\delta_f)^{*p}$, which means we get $(\sum_{f\in F} \delta_f)^{*p} = \sum_{f \in F} (\delta_f)^{*p}$. But we have $(\delta_f)^{*p} = \delta_{fp}$ (since convolving with δ_f once corresponds to shifting our function's input by f, so convolving p copies of it produces the function that corresponds to shifting by f for p times). $\square$

Plugging this into (1) gives

$$1_{pF} * 1_A \equiv 0 \pmod{p}.$$

Writing this convolution more explicitly, this means that for every $x \in G$, we have

$$\sum_{f \in F} 1_A(x - pf) \equiv 0 \pmod{p}.$$

Now we can fix $f_0 \in F$ and $a \in A$ and set $x = a + pf_0$; plugging this in gives

$$\sum_{f \in F} 1_A(a + pf_0 - pf) \equiv 0 \pmod{p}.$$

But we know that at least one of these terms is 1 — namely, the term with $f = f_0$ (for which we simply get $1_A(a) = 1$). On the other hand, there are only p terms in the sum, each of which is 0 or 1; this means as an integer, the sum is at most p. So for it to be 0 mod p, it actually has to be *exactly* p, and all the terms in the sum must be 1 — so we get that $a + pf_0 - pf \in A$ for all $f \in F$.

And this is true for *all* $a \in A$ and $f_0 \in F$, so we get that $A = h + A$ for all $h \in p(F - F)$ (where $F - F$ denotes the difference set of F). This gives many periods — if $F - F$ contains enough vectors to span a lattice, then this means the tiling must *already* be periodic. If not, then we still get a periodic tiling of a smaller subgroup, and we can independently shift its cosets to repair the entire tiling to be periodic. ☐

Remark 3.3. The original proof was a bit more complicated.

§4 The periodic tiling conjecture in $\mathbb{Z}^2$

In the rest of the talk, we'll prove the periodic tiling conjecture in $\mathbb{Z}^2$.

Theorem 4.1 (Bhattacharya 2020)
The periodic tiling conjecture is true in $\mathbb{Z}^2$.

The original proof by Bhattacharya uses spectral analysis and ergodic theory; we'll give an alternative proof that is purely combinatorial.

Remark 4.2. So far, we've only been working with discrete sets; but what if we wanted to work with non-discrete sets, such as the plane $\mathbb{R}^2$? We don't know whether the periodic tiling conjecture is true in $\mathbb{R}^2$ — the reason for this is that an important step in the proof for $\mathbb{Z}^2$ is the *dilation lemma*, which gives a lot of structural information about a tiling (in all dimensions), and we don't know how to prove it in other settings such as $\mathbb{R}^2$.

However, the periodic tiling conjecture *is* known to be true in $\mathbb{R}$. This doesn't follow from the argument we gave for $\mathbb{Z}$, but it was proven by Lagarias and Wang (in the same paper in which they suggested the conjecture for Euclidean space) — they use Fourier analysis to show that all tilings on $\mathbb{R}$ are 'rational' in some sense, which allows us to reduce to the case of $\mathbb{Z}$.

§4.1 The dilation lemma

The first ingredient in the proof is the dilation lemma, which is a very strong statement that has been proven in many different contexts.

Lemma 4.3 (Dilation lemma)

Let $F \subseteq G$ be finite, and let $A \subseteq G$. If $F \oplus A = G$, then $rF \oplus A = G$ for every r coprime to $|F|$.

In words, this states that if A is a tiling of G by F, then A is also a tiling of G by every *dilation* of F where the dilation factor r is coprime to $|F|$.

The reason this is so strong is that if we know that A is a tiling by F, then it tells us that A is also a tiling by *many* other sets. And we can take r to be bigger and bigger, which makes our tile rF sparser and sparser (while A doesn't change); heuristically, this means A should have 'long-term correlations' — by taking r large, we can get relationships between even points in A that are very far away from each other.

This can be thought of as one source of motivation for the periodic tiling conjecture — because if A is periodic, then it *does* exhibit this phenomenon of having long-term correlations. The dilation lemma alone isn't strong enough to prove the periodic tiling conjecture, but we'll later see some sorts of long-term correlations that we *can* prove using it.

Remark 4.4. Bhattacharya proved this lemma for $\mathbb{Z}^2$, but his proof can be adapted to work for any G; the proof we'll present is a simplified version.

Remark 4.5. There's lots of statements that make sense according to geometric intuition but are hard to prove. This statement is sort of the opposite — it's hard to see using geometric intuition, but it's not that hard to prove analytically.

Proof. We'll start by considering the case where r is a prime p which is coprime to $|F|$. We'll again use the convolution condition for a tiling — the statement that F tiles G by A means that $1_F * 1_A = 1$. And we'll again use the trick of applying a $(p-1)$-convolution of 1_F to both sides to get

$$(1_F)^{*p} * 1_A = |F|^{p-1},$$

and taking both sides mod p. On the right-hand side, we have $|F|^{p-1} \equiv 1 \pmod{p}$ by Fermat's little theorem (as $|F|$ is coprime to p). Meanwhile, on the left-hand side, we again have $(1_F)^{*p} = 1_{pF}$ by the Frobenius endomorphism; so this means

$$1_{pF} * 1_A \equiv 1 \pmod{p}.$$

This is *almost* the statement we wanted to prove (we wanted to show that pF tiles G by A, or equivalently $1_{pF} * 1_A = 1$), but right now we have an annoying mod p that we need to get rid of. First, we can say that

$$1_{pF} * 1_A \geq 1$$

(since $1_{pF} * 1_A$ takes on nonnegative integer values, so if these values are always 1 mod p, then they're certainly at least 1 as integers). We *want* to say that $1_{pF} * 1_A$ is always *exactly* 1, and to do so we'll use a counting argument — we can convolve both sides with 1_F to get that

$$1_{pF} * (1_F * 1_A) \geq 1 * 1_F = |F|.$$

But we know $1_F * 1_A = 1$, so the left-hand side is $1_{pF} * 1 = |pF| = |F|$. This means that equality must hold everywhere, so $1_{pF} * 1_A$ must be *exactly* 1 (on all inputs).

Finally, we've now proven the statement when r is *prime* (and coprime to $|F|$); and we can use the fundamental theorem of arithmetic and iterate this argument to get it for *all* r coprime to $|F|$ (we factor r as a product of primes and use this argument to dilate F by each of those primes one at a time). $\square$

§4.2 A structure theorem

Now as a corollary of the dilation lemma, we can prove the following structure theorem, which gives an instance of long-term correlation in A. (This comes from a paper where Greenfeld and Tao suggested an alternative proof for Bhattacharya's result — Bhattacharya also had a structure theorem, but it was a bit weaker, and this stronger structure theorem allowed them to significantly simplify the proof.)

Theorem 4.6 (Greenfeld–Tao 2021)
Let $F \subseteq G$ be finite and $A \subseteq G$, and suppose $0 \in F$. If $1_F * 1_A = 1$, then we can write

$$1_A = 1 - \sum_{f \in F \setminus \{0\}} \varphi_f$$

where each φ_f is a function $G \to [0,1]$ which is $\langle |F|\, f \rangle$-periodic.

(We can always assume without loss of generality that $0 \in F$.)

In words, this says that if A is a tiling of G by F, then we can always decompose 1_A in this form, which intuitively means that there are long-term correlations inside A. Note that if we have such a decomposition, then we can collect collinear terms to write

$$1_A = 1 - \sum_{j=1}^{m} \psi_j$$

where each $\psi_j \colon G \to [0,1]$ is a $\langle h_j \rangle$-periodic function for some mutually incommensurable (i.e., not collinear) elements $h_1, \ldots, h_m \in G$. (We're simply taking the decomposition from Theorem 4.6 and collecting collinear terms; the h_j's may not themselves be elements of F, but they have the same *directions* as elements of F, since collecting collinear terms preserves direction.)

Remark 4.7. It's not obvious that the φ_j's are $[0,1]$-valued (they come from collecting several $[0,1]$-valued functions, so they're certainly nonnegative, but it's not clear that they're bounded by 1). But we can see this by noting that the left-hand side 1_A is $[0,1]$-valued, while on the right-hand side we're subtracting these functions φ_j from 1; so if some φ_j exceeded 1, then the left-hand side would become negative at some point.

Remark 4.8. In the paper, the authors were very quantitative, because they wanted a result that gave good bounds on the period. Here we won't be quantitative, to avoid getting too technical.

Proof. We know that $1_F * 1_A = 1$. We'll now apply the dilation lemma with dilation factors $r_n = n\,|F| + 1$ for each $n \in \mathbb{N}$ (these numbers are all coprime to $|F|$); so we get that

$$1_{r_n F} * 1_A = 1$$

for every n. We can then expand out the left-hand side as $\sum_{f \in F} 1_A * \delta_{r_n f}$, and pulling out the term with $f = 0$, which is $1_A * \delta_0 = 1_A$, and rearranging gives that

$$1_A = 1 - \sum_{f \in F \setminus \{0\}} 1_A * \delta_{r_n f}.$$

Now for each f, we define a sequence of functions $(S_m^f)_{m \in \mathbb{N}}$ by

$$S_m^f = \frac{1}{m} \sum_{n=1}^{m} 1_A * \delta_{r_n f}.$$

Each function S^f_m takes values in $[0,1]$. Then by compactness, this sequence (S^f_m) has a convergent subsequence; let φ_f be the function that this subsequence converges to, which also takes values in $[0,1]$. And the sequence (r_n) is an arithmetic progression with difference $|F|$, so φ_f is $\langle |F|\, f\rangle$-periodic (explicitly, this is because if we expand out $S^f_m(x) - S^f_m(x + |F|\, f)$ and plug in the definition of r_n, we'll get a telescoping series; this means $|S^f_m(x) - S^f_m(x + f\,|F|)| \leq 2/m$, and taking $m \to \infty$ gives that $|\varphi_f(x) - \varphi_f(x + f\,|F|)| = 0$).

And we have $1_A = 1 - \sum_{f \in F \setminus \{0\}} 1_A * \delta_{r_n f}$ for all n, and averaging over n gives $1_A = 1 - \sum_{f \in F\setminus\{0\}} S^f_m$; finally, taking the limit (along the appropriate subsequence) gives that $1_A = 1 - \sum_{f \in F \setminus \{0\}} \varphi_f$, as desired. $\square$

Theorem 4.6 is almost an immediate consequence of the dilation lemma, but it's quite strong. In particular, in the two-dimensional case it's *super* strong — it essentially gives one-dimensional periodicity, and in two dimensions this is almost what we want.

§4.3 Deducing the periodic tiling conjecture for $\mathbb{Z}^2$

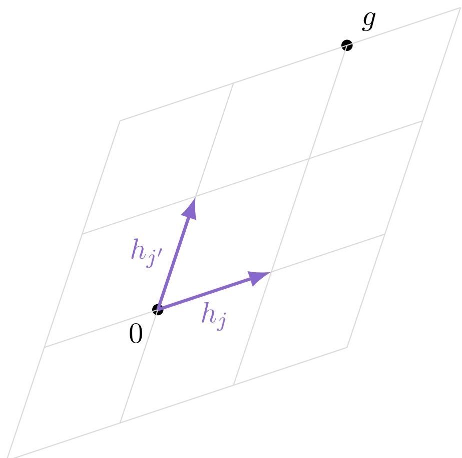
We'll now use this to prove the periodic tiling conjecture for $\mathbb{Z}^2$. Suppose that $F \oplus A = \mathbb{Z}^2$; we want to show that F admits some periodic tiling. (The original tiling A might not itself be periodic, but we want to show that we can *repair* it to be periodic.)

First, we can use Theorem 4.6 to write

$$1_A = 1 - \sum_{j=1}^{m} \psi_j$$

where the functions $\psi_j \colon \mathbb{Z}^2 \to [0,1]$ are periodic in incommensurable directions — i.e., each ψ_j is $\langle h_j \rangle$-periodic, and $h_1, \ldots, h_m$ are incommensurable.

Now we can choose a nonzero point $g \in \mathbb{Z}^2$ such that $g \in \langle h_j, h_{j'} \rangle$ for all distinct $1 \leq j, j' \leq m$, and g is incommensurable with all of $h_1, \ldots, h_m$. (We can do this because h_j and h'_j are not collinear, so their span $\langle h_j, h_{j'} \rangle$ is some lattice; and we're simply taking a point in the intersection of all these lattices — which is still a lattice — that's not collinear with any of $h_1, \ldots, h_m$.)



In particular, this means that for each $1 \leq j \leq m-1$, there exist $a_j, b_j \in \mathbb{Z}$ such that $a_j h_j = g + b_j h_m$.

Now we'll take our equation $1_A = 1 - \sum_j \psi_j$ and take discrete derivatives, which are defined as follows.

Definition 4.9. For $h \in \mathbb{Z}^2$, we define the discrete derivative of f in the direction h, denoted $\Delta_h(f)$, by

$$\Delta_h(f)(x) = f(x) - f(x - h).$$

We'll take discrete derivatives of 1_A in the directions a_jh_j for all j (iteratively) — we have

$$(\Delta_{a_1h_1}\cdots\Delta_{a_{m-1}h_{m-1}})(1_A) = \sum_j(\Delta_{a_1h_1}\cdots\Delta_{a_{m-1}h_{m-1}})(\psi_j),$$

and since ψ_j is $\langle h_j\rangle$-periodic and taking a discrete derivative in the direction h kills $\langle h\rangle$-periodic functions, all the terms in this sum are killed except the one with $j = m$. And we have $a_jh_j = g + b_jh_m$, and since ψ_m is $\langle h_m\rangle$-periodic, the b_jh_m terms don't matter and taking a discrete derivative in the direction a_jh_j is the same as taking one in the direction g; so we end up with

$$(\Delta_{a_1h_1}\cdots\Delta_{a_{m-1}h_{m-1}})(1_A) = (\Delta_g)^{m-1}(\psi_m).$$

Now the point is that the left-hand side takes integer values, so we have

$$(\Delta_g)^{m-1}(\psi_m) \equiv 0 \pmod 1,$$

which means that ψ_m must be a polynomial mod 1 (of degree at most $m-2$) — more precisely, for every $x \in \mathbb{Z}^2$, the function $p_{x,m}\colon \mathbb{Z} \to [0,1]$ defined by $p_{x,m}(n) = \psi_m(x + ng)$ is a polynomial mod 1.

And by symmetry, we can do the same for each ψ_j (in place of ψ_m) and get the same conclusion — so for each $x \in \mathbb{Z}^2$ and $1 \leq j \leq m$, the function $p_{x,j}\colon \mathbb{Z} \to [0,1]$ defined as

$$p_{x,j}(n) = \psi_j(x + ng)$$

is a polynomial mod 1.

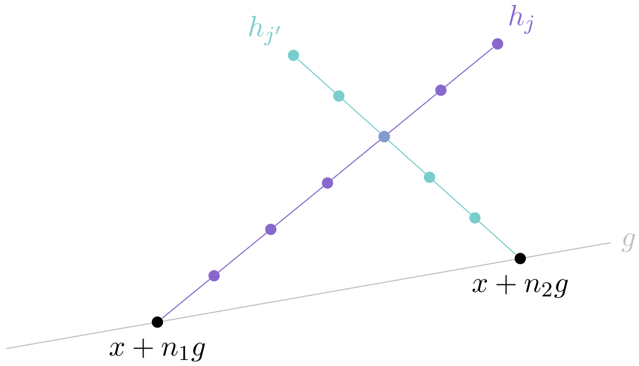
Now that we have polynomials, we get a dichotomy — a theorem of Weyl states that a polynomial taking inputs in $\mathbb{Z}$ (and values in the torus, i.e., mod 1) is either periodic or equidistributed — so either its output is super structured or totally unstructured.

Next, we'll show that equidistribution *can't* occur in this setting, which will mean that all these polynomials $p_{x,j}$ have to be periodic. This follows from the following geometric observation.

Claim 4.10 — For all $x \in \mathbb{Z}^2$, all $j \neq j'$, and all $n_1, n_2 \in \mathbb{Z}$, we have

$$p_{x,j}(n_1) + p_{x,j'}(n_2) \leq 1.$$

Proof. Recall that we defined $p_{x,j}(n) = \varphi_j(x + ng)$. Consider the coset $x + \langle g\rangle$ (we can think of this as a line in the direction of g) and the two points $x + n_1g$ and $x + n_2g$ on this line. We also know ψ_j is periodic in the direction of h_j, and $\psi_{j'}$ in the direction of $h_{j'}$; so we can imagine drawing the lines through these two points in the directions h_j and $h_{j'}$, and ψ_j and $\psi_{j'}$ (respectively) are constant along these lines.



And since $g \in \langle h_j, h_{j'} \rangle$, these two lines intersect at some point $y \in \mathbb{Z}^2$. Then φ_j and $\varphi_{j'}$ take the same values at $x + n_1 g$ and $x + n_2 g$ (respectively) as at y. And the sums of their values at y must be at most 1 (because $1_A(y) = 1 - \sum_{j=1}^m \varphi_j(y)$, so the sum of *any* subset of the φ_j's at a given point is bounded by 1), so the same must be true of our original points — i.e., we have

$$\varphi_j(x + n_1 g) + \varphi_{j'}(x + n_2 g) = \varphi_j(y) + \varphi_{j'}(y) \le 1. \qquad \square$$

This doesn't allow any $p_{x,j}$ to be equidistributed — assume for contradiction that even one function $p_{x,j}$ is equidistributed. Then we have

$$1_A(x + ng) = 1 - \sum_j \varphi_j(x + ng).$$

The left-hand side is $\{0, 1\}$-valued, so if one of the terms on the right-hand side is equidistributed (as a function of n), then another term must be as well (in order to cancel it out so that the right-hand side ends up being $\{0, 1\}$-valued as well). But on the other hand, if $p_{x,j}$ is equidistributed, then it must take on values at least $1 - \varepsilon$ at some point (for any ε); and then Claim 4.10 means that the other polynomials $p_{x,j'}$ must all be bounded by ε *everywhere*, which means they can't be equidistributed.

Remark 4.11. This argument relies crucially on the fact that every point is covered *once* by our tiling. If instead of considering a tiling of level 1, we considered a tiling of level 2 (where each point is covered exactly *twice*), then it's not so easy to show that there's no equidistribution scenarios (we don't know whether this is true or not). In fact, if we consider tilings of level 4, then there *are* examples with equidistribution scenarios (and these can be extended to any level that's a multiple of 4).

We *have* proved the periodic tiling conjecture for tilings of all levels, but the proof for higher levels uses probabilistic techniques.

So we've shown that no $p_{x,j}$ can be equidistributed mod 1, and it follows (from Weyl's theorem) that every $p_{x,j}$ is periodic mod 1.

We'd like to eliminate the mod 1 (i.e., to get a statement about $p_{x,j}$ that isn't mod 1). Consider the coset $\langle g \rangle + x$. There's two cases — first, if all the functions $p_{x,j}$ have supremum strictly less than 1 (meaning that they're always in $[0, 1 - \varepsilon]$ for some ε), then we can ignore the mod 1 and get that each $p_{x,j}$ is periodic, which means $\sum_j p_{x,j}(n)$ is as well.

Otherwise, if there's some p_{x,j_0} with supremum 1, then Claim 4.10 implies that the other functions $p_{x,j}$ must be identically 0, which means $\sum p_{x,j} = p_{x,j_0}$; and since $\sum p_{x,j} = 1 - 1_A$ is $\{0, 1\}$-valued, this means p_{x,j_0} is as well, so it's actually the indicator function of some set. And φ_{j_0} (the function used to define p_{x,j_0}) is $\langle h_{j_0} \rangle$-periodic. We can use this to eventually find a lattice Λ such that the intersection of A with each coset $x + \Lambda$ is 1-periodic.

Using this, we can split A up as a disjoint union

$$A = A_1 \sqcup \cdots \sqcup A_t$$

where each set A_i is periodic in one direction. Recall that in Example 1.5, we saw a tiling that could be split up into two pieces, each of which was periodic in one direction (which we abbreviate as 1-periodic); what we've shown here is that this behavior is true for *every* tiling in $\mathbb{Z}^2$ (i.e., that it can be decomposed into 1-periodic elements).

Finally, we'll use a pigeonhole argument to repair our tiling to be periodic.

Lemma 4.12

If $F \oplus A = \mathbb{Z}^2$ is weakly periodic (i.e., we can write A as a disjoint union of 1-periodic sets), then there exists a (strongly) periodic set A' such that $F \oplus A' = \mathbb{Z}^2$.

Roughly speaking, the reason why we can use pigeonhole here is that we have a decomposition into 1-periodic pieces, so we already have information in one dimension; then there's only one direction we need to figure out, so we get to apply a one-dimensional argument to each of our pieces.

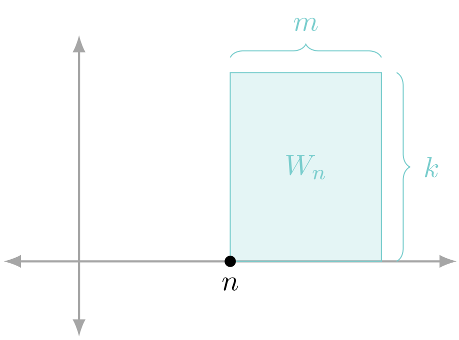
To prove Lemma 4.12, we'll need a claim whose proof we'll skip.

Claim 4.13 — If we have a decomposition $A = A_1 \sqcup \cdots \sqcup A_t$ (where each A_i is 1-periodic and A is a tiling), then for all $1 \leq j \leq t$, the set E_j for which $1_F * 1_{A_i} = 1_{E_j}$ is (strongly) periodic.

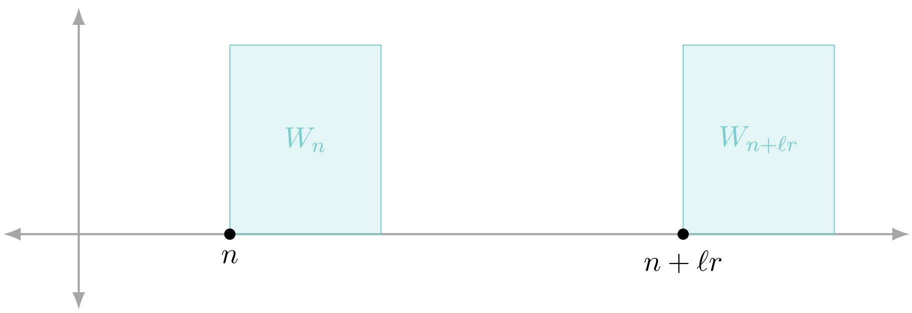
Intuitively, if we think about the Fourier side of things, when we apply a Fourier transform to a convolution of two functions, we get the product of their individual transforms. Here the left-hand side is definitely 1-periodic in physical space, so on the Fourier side it'll be supported on finitely many parallel lines. And we also have $\sum 1_{E_j} = 1_F * 1_A = 1$, so the Fourier transform of this sum will be the Dirac δ-function at 0. But each 1_{E_j} is supported on some set of parallel lines in different directions, and then we can use an inclusion-exclusion argument to show that they're actually supported only on the intersection points of these lines. (It's also not hard to prove the claim in physical space.)

Now we'll repair each of our pieces A_j to be periodic. First, by taking an affine transformation, we can assume that $h_j = (0, k)$. Then since E_j is doubly periodic, we can find some ℓ for which it's $\ell\mathbb{Z}^2$-periodic. Now take m such that $F \subseteq \{0, \ldots, m-1\} \times \mathbb{Z}$, and for each n, consider the window

$$W_n = (\{1, \ldots, m\} \times \{1, \ldots, k\}) \cap (A_j - (n, 0)).$$



There are finitely many possibilities for such a window (specifically, there's at most 2^{km} possible windows), so we can find two far-away windows that look the same — more specifically, we can find some n and $r > m$ such that $W_n = W_{n+\ell r}$.



Now we first define A'_j to match A_j on $\{n+1, \ldots, n+\ell r\} \times \mathbb{Z}$, and we complete it to be ℓr-periodic in the horizontal direction. Then because of our identical windows, we get that

$$1_{A_j} * 1_F(x) = 1_{A_{j'}} * 1_F(x)$$

for all $x \in \{n+1+m, \ldots, n+\ell r+m\} \times \mathbb{Z}$ (using the windows). And the right-hand side is $\ell\mathbb{Z}^2$ periodic; meanwhile, the left-hand side is ℓr-periodic, so we get that they're equal everywhere. (This means $A_{j'}$ really does repair our tiling A_j in the sense that they cover the same points, and by definition $A_{j'}$ is periodic in both the vertical and horizontal directions.)