Homework 4 solutions

S320/520

Due at the beginning of class, Thursday 24th September

Please write "S320" or "S520" at the top of your homework. All students should answer all questions. Trosset question numbers refer to the hardcover textbook. Show all working and give R code where appropriate.

1. (a) *Suppose that buses go past my stop exactly 30 minutes apart. I arrive at the stop at a completely random time during the day. What is the expected length of time I will have to wait for a bus?*

   The length of time has a Uniform$[0, 30)$ distribution, so the expected value is 15 minutes.

   (b) *Suppose that buses go past my father's stop at exactly ten minutes past the hour and thirty minutes past the hour (e.g. 9:10, 9:30) every hour. My father arrives at his stop at a completely random time during the day. What is the expected length of time he will have to wait for a bus?*

   Thinking about the PDF, there's a 2/3 chance the waiting time will be between 0 and 20 minutes, and a 1/3 chance the time will be between 20 and 40 minutes. So the expected waiting time is $(2/3 \times 10) + (1/3 \times 30) = 16\ 2/3$ minutes. (See question 5 for a more rigorous explanation.)

2. Trosset exercise 5.6.6

   No, Y is not normal. For instance, a normal random variable can take all real values, whereas Y can't take negative values.

3. Trosset exercise 5.6.7 (use R)

   (a) `pnorm(0, mean=-5, sd=10)` = 69%

   (b) `1 - pnorm(5, mean=-5, sd=10)` = 16%

   (c) `pnorm(7, mean=-5, sd=10) - pnorm(-3, mean=-5, sd=10)` = 31%

   (d) `pnorm(5, mean=-5, sd=10) - pnorm(-15, mean=-5, sd=10)` = 68%

   (e) `pnorm(1, mean=-5, sd=10) + 1 - pnorm(5, mean=-5, sd=10)` = 88%

4. Trosset exercise 5.6.8

   (a) Expected value 4, variance 25

   (b) Expected value -3, variance 16

   (c) Expected value -2, variance 25

(d) Expected value 2, variance 36

(e) Expected value -4, variance 100

5. *Let X be a random variable with PDF*

$$f(x) = \begin{cases} \frac{1}{30} & 0 \leq x < 20 \\ \frac{1}{60} & 20 \leq x < 40 \\ 0 & \text{otherwise.} \end{cases}$$

(a) *Find the CDF of X, $F(y)$, for all y.*

$$F(y) = \begin{cases} 0 & y < 0 \\ \frac{y}{30} & 0 \leq y < 20 \\ \frac{y+20}{60} & 20 \leq y < 40 \\ 1 & y \geq 40. \end{cases}$$

(b) *Find y such that $F(y) = 0.5$. Is this larger than, smaller than, or the same as EX?*
To meet this condition, $y/30 = 0.5$, so $y = 15$. We argued in question 1(b) that $EX = 50/3$. To show this formally:

$$\begin{aligned} EX &= \int_0^{40} x \cdot f(x)\, dx \\ &= \int_0^{20} \frac{1}{30} x\, dx + \int_{20}^{40} \frac{1}{60} x\, dx \\ &= \left[x^2/60\right]_0^{20} + \left[x^2/120\right]_{20}^{40} \\ &= (20/3 - 0) + (40/3 - 10/3) \\ &= 50/3. \end{aligned}$$

In any case, y is smaller.

6. (Compulsory for S520, optional for half-credit for S320.) Trosset exercise 5.6.4

(a) X measures the distance that a point in a disc of radius 1 lies from its center, so $0 \leq X \leq 1$.

(b) Let A_1 denote the concentric disc of radius 0.5. Then

$$P(X \leq 0.5) = P(A_1) = \frac{\text{area}(A_1)}{\pi} = \frac{\pi 0.5^2}{\pi} = 0.25.$$

(c) Let A_1 denote the concentric disc of radius 0.7. Then

$$P(X \leq 0.7) = P(A_2) = \frac{\text{area}(A_2)}{\pi} = \frac{\pi 0.7^2}{\pi} = 0.49$$

and

$$P(0.5 < X \leq 0.7) = P(A_2) - P(A_1) = 0.49 = 0.25 = 0.24.$$

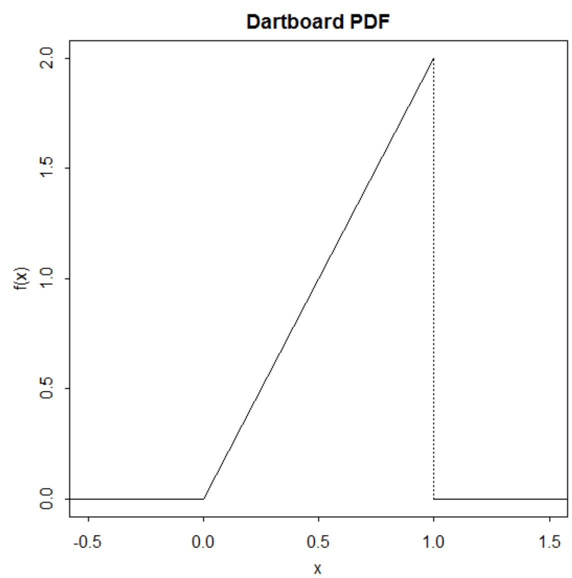

Figure 1: PDF for Trosset exercise 5.6.4.

(d)

$$F(y) = \begin{cases} 0 & y < 0 \\ y^2 & 0 \leq y < 1 \\ 1 & y \geq 1 \end{cases}$$

(e) The pdf is the derivative of the cdf.

$$f(x) = \begin{cases} 2x & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$