Homework 2 solutions

S320/520

Due at the beginning of class, Thursday 10th September

1. *A ticket is to be drawn at random from a box containing eight tickets. Each ticket should have two numbers; however, three tickets have one blank space:*

   $$(1, -), (1, 2), (1, 2), (1, 3), (3, 1), (3, 2), (3, -), (3, -)$$

   What numbers should go in the first, second, and third blank spaces to make the first number of the ticket drawn and the second number of the ticket drawn independent?

   We require the conditional probability distribution of the second number to be the same regardless of whether the first number is a 1 or 3. To do this, we need a 1 in the first blank space, and a 2 and 3 in the second and third blank spaces (either way around.)

2. Trosset exercise 3.7.8

   (a) $P(+|D^c) = 1 - P(-|D^c) = 1 - 0.82 = 0.18$

   (b) $P(-|D) = 1 - P(+|D) = 1 - 0.62 = 0.38$

   (c) See Figure 1.

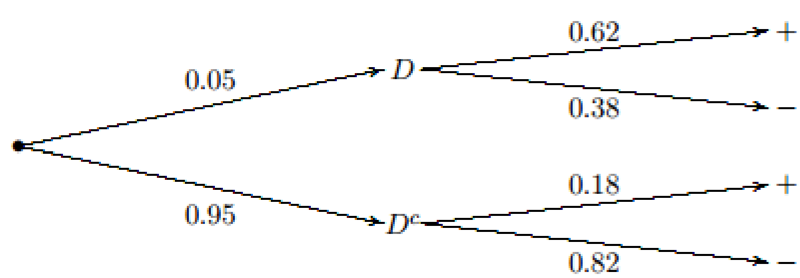

Figure 1: Tree diagram for Exercise 3.7.8(c).

   (d)

$$\begin{aligned} P(+) &= P(+ \cap D) + P(+ \cap D^c) \\ &= 0.05 \cdot 0.62 + 0.95 \cdot 0.18 \\ &= 0.202 \end{aligned}$$

(e)

$$P(D|+) = \frac{P(D \cap +)}{P(+)} = \frac{0.05 \cdot 0.62}{0.202} \approx 0.153$$

3. Trosset exercise 3.7.14

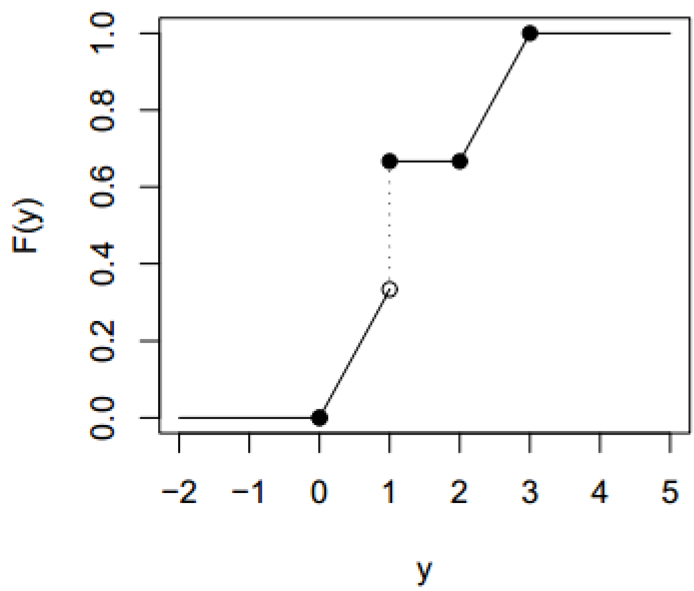

Figure 2: Graph of CDF for Exercise 3.7.14.

(a)

$$P(X > 0.5) = 1 - P(X \leq 0.5)1 - F(0.5) = 1 - \frac{1}{6} = \frac{5}{6}$$

(b)

$$\begin{aligned} P(2 < X \leq 3) &= P(X \leq 3) - P(X \leq 2) \\ &= F(3) - F(2) \\ &= 1 - \frac{2}{3} = \frac{1}{3} \end{aligned}$$

(c)

$$\begin{aligned} P(0.5 < X \leq 2.5) &= P(X \leq 2.5) - P(X \leq 0.5) \\ &= F(2.5) - F(0.5) \\ &= \frac{5}{6} - \frac{1}{6} \\ &= \frac{2}{3} \end{aligned}$$

(d) $P(X = 1)$ is the height of the jump at $x = 1$. This is $1/3$.

4. Trosset exercise 4.5.1

   (a)

$$f(x) = \begin{cases} 0.1 & x = 1 \\ 0.4 & x = 3 \\ 0.4 & x = 4 \\ 0.1 & x = 6 \\ 0 & \text{otherwise} \end{cases}$$

   (b)

$$F(x) = \begin{cases} 0 & x < 1 \\ 0.1 & 1 \le x < 3 \\ 0.5 & 3 \le x < 4 \\ 0.9 & 4 \le x < 6 \\ 1 & x \ge 6 \end{cases}$$

   (c) $EX = 1 \cdot 0.1 + 3 \cdot 0.4 + 4 \cdot 0.4 + 6 \cdot 0.1 = 3.5$

   (d)

$$\begin{aligned} E\left[X^2\right] &= 1^2 \cdot 0.1 + 3^2 \cdot 0.4 + 4^2 \cdot 0.4 + 6^2 \cdot 0.1 \\ &= 13.7 \\ \operatorname{Var} X &= E\left[X^2\right] - (EX)^2 \\ &= 13.7 - 3.5^2 = 1.45 \end{aligned}$$

   (e) $\sigma = \sqrt{\operatorname{Var} X} \approx 1.2$

5. Trosset exercise 4.5.2

   (a)

$$f(x) = \begin{cases} 0.3 & x = 1 \\ 0.25 & x = 2 \\ 0.2 & x = 3 \\ 0.15 & x = 4 \\ 0.1 & x = 5 \\ 0 & \text{otherwise} \end{cases}$$

   (b)

$$F(x) = \begin{cases} 0 & x < 1 \\ 0.3 & 1 \le x < 2 \\ 0.55 & 2 \le x < 3 \\ 0.75 & 3 \le x < 4 \\ 0.9 & 4 \le x < 5 \\ 1 & x \ge 5 \end{cases}$$

(c) $EX = 1 \cdot 0.3 + 2 \cdot 0.25 + 3 \cdot 0.2 + 4 \cdot 0.15 + 5 \cdot 0.1 = 2.5$

(d)

$$\begin{aligned}
E\left[X^2\right] &= 1^2 \cdot 0.3 + 2^2 \cdot 0.25 + 3^2 \cdot 0.2 + 4^2 \cdot 0.15 + 5^2 \cdot 0.1 \\
&= 8 \\
\operatorname{Var} X &= E\left[X^2\right] - (EX)^2 \\
&= 8 - 2.5^2 = 1.75
\end{aligned}$$

(e) $\sigma = \sqrt{\operatorname{Var} X} \approx 1.32$

6. (Compulsory for S520, optional for S320.) Trosset exercise 4.5.4

(a)

$$P(2 \text{ students}) = \frac{\text{number of juries with 2 students}}{\text{number of juries}}$$

There are $C(25, 12)$ ways of choosing the jury. If the two students are chosen, then there are $C(23, 10)$ ways of choosing the other members of the jury. So:

$$\begin{aligned}
P(2 \text{ students}) &= \frac{23!}{10!13!} \Big/ \frac{25!}{12!13!} \\
&= \frac{11}{50} = 0.22.
\end{aligned}$$

(b) There are three kinds of ways of having exactly twice as many retired persons as employed persons.

- 8 retired, 4 employed, 0 others

$$\text{Number of ways} = C(12, 8) \cdot C(6, 4) \cdot C(7, 0) = 7425$$

- 6 retired, 3 employed, 3 others

$$\text{Number of ways} = C(12, 6) \cdot C(6, 3) \cdot C(7, 3) = 646800$$

- 4 retired, 2 employed, 6 others

$$\text{Number of ways} = C(12, 4) \cdot C(6, 2) \cdot C(7, 6) = 51975$$

So the probability of exactly twice as many retired persons as employed persons is

$$\frac{7425 + 646800 + 51975}{C(25, 12)} \approx 13.6\%.$$