Homework 3 solutions

S320/520

Due at the beginning of class, Thursday 17th September

Please write "S320" or "S520" at the top of your homework.

1. Trosset exercise 4.5.10

   Each of the $n = 12$ attendees represents a Bernoulli trial. The possible outcomes are attendance and nonattendance. If we designate attendance as success and nonattendance as failure, then the probability of success is $p = 0.5 \cdot 0.8 = 0.4$. Let Y denote the observed number of successes, so that $Y \sim \text{Binomial}(12, 0.4)$. Then

$$\begin{aligned} P(Y > 7) &= 1 - P(Y \leq 7) \\ &= \texttt{1 - pbinom(7, 12, 0.4)} \\ &= 0.057 = 5.7\%. \end{aligned}$$

2. Trosset exercise 4.5.13

   (a) Each challenge is a Bernoulli trial. Let $X_i = 1$ if challenge i is overruled and let $X_i = 0$ if it's not. We're assuming that $p \equiv P(X_i = 1) = 0.2$. If we also assume the X_i are independent, then

$$Y = \sum_{i=1}^{38} X_i \sim \text{Binomial}(n = 38, p = 0.2)$$

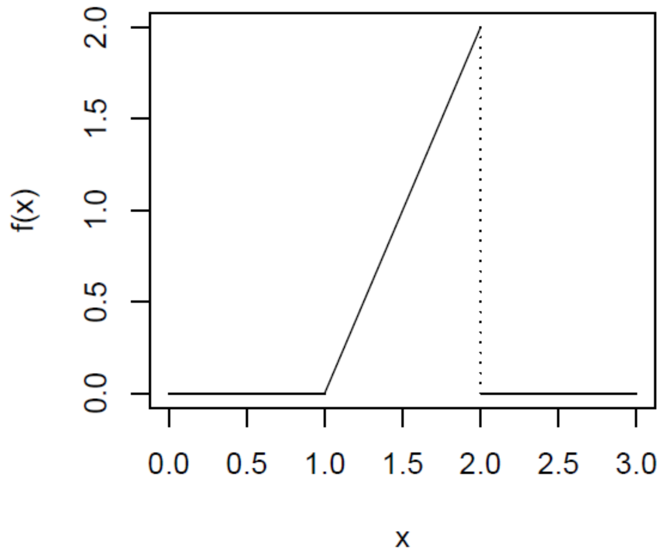
   and

$$\begin{aligned} P(Y \geq 12) &= 1 - P(P \leq 11) \\ &= \texttt{1 - pbinom(11, 28, 0.2)} \\ &= 0.0623 = 6.2\%. \end{aligned}$$

   (b) Let E denote the event in question, in which case E^c is the event that 25 consecutive matches have less than 12 overrules. By (a), the probability that one match has less than 12 overrules is 0.0623. Hence, assuming the matches are independent,

$$\begin{aligned} P(E) &= 1 - P(E^c) \\ &= 1 - (1 - 0.0623)^{25} \\ &= 0.800 = 80\%. \end{aligned}$$

3. Trosset exercise 5.6.2



(a)

Figure 1: pdf for Exercise 5.6.2.

(b) $f(x) \geq 0$ for all x and the area of the triangle is 1, so f is a pdf.

(c) This is the difference between the areas of two triangles, one with base 0.5 and height 1, and one with base 0.75 and height 1.5.

$$\begin{aligned} P(1.5 < X < 1.75) &= P(X < 1.75) - P(X < 1.5) \\ &= (0.5 \times 0.75 \times 1.5) - (0.5 \times 0.5 \times 1) \\ &= \frac{5}{16} = 0.3125. \end{aligned}$$

4. Trosset exercise 5.6.3

(a) The pdf is plotted in Figure 2. c must be nonnegative for f to be a pdf. The total area under f is the sum of two triangles:

$$(0.5 \times 1.5 \times 1.5c) + (0.5 \times 1.5 \times 1.5c) = \frac{9}{4}c.$$

This has to equal 1 for a pdf, so

$$\begin{aligned} \frac{9}{4}c &= 1 \\ c &= \frac{4}{9}. \end{aligned}$$

(b) Looking at Figure 2, it's evident the f is symmetric about $x = 1.5$. So the expected value of X must be 1.5.

(c) $P(X > 2)$ is the area under the pdf between 2 and 3, which is the area of a triangle. The base of the triangle is $3 - 2 = 1$ and the height of the triangle is $f(2) = c = 4/9$. The area is $1/2 \times 1 \times 4/9 = 2/9$.

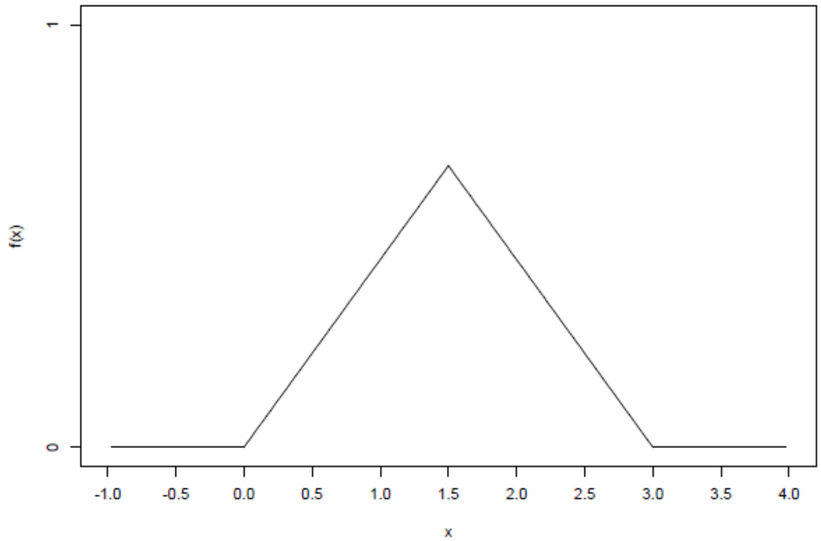


Figure 2: pdf for Exercise 5.6.3.

(d) Figure 3 plots the two pdfs on top of each other. Both have the same expected value: $EX = EY = 1.5$. However, the values of X tend to cluster a bit nearer 1.5 than do the values of X. So Y has the larger variance.

(e) Firstly, if $y < 0$, then $F(y) = 0$, and if $y > 3$, then $F(y) = 1$.
If $0 \leq y \leq 1.5$, then $F(y)$ is the area of a triangle:

$$\begin{aligned} F(y) &= P(X \leq y) \\ &= \frac{1}{2} \cdot y \cdot cy \\ &= \frac{2y^2}{9}. \end{aligned}$$

If $1.5 \leq y \leq 3$, then $F(y)$ is one minus the area of a triangle. The base of the triangle is $3 - y$ and the height is $c(3 - y)$.

$$\begin{aligned} F(y) &= 1 - P(X > y) \\ &= 1 - \frac{1}{2} \cdot (3 - y) \cdot c(3 - y) \\ &= 1 - \frac{c}{2}(3 - y)^2 \\ &= 1 - \frac{2}{9}(3 - y)^2 \end{aligned}$$

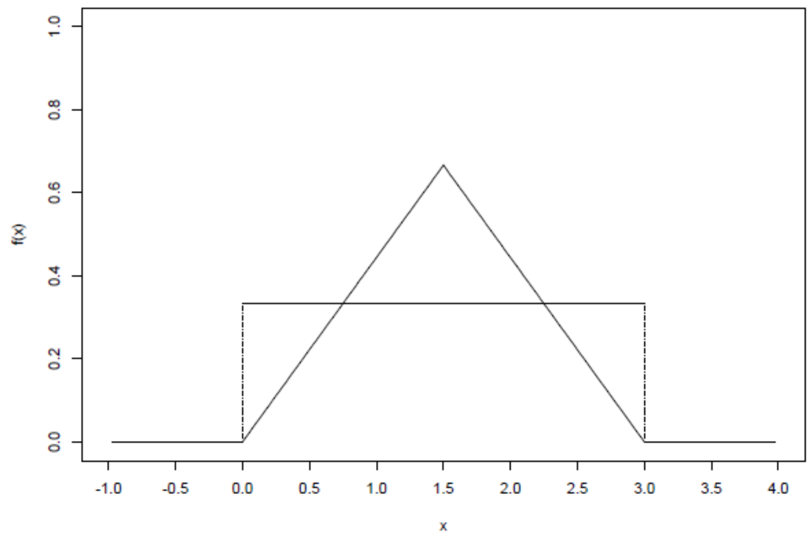


Figure 3: pdfs for Exercise 5.6.3(d).

One way of writing all of this down formally is:

$$F(y) = \begin{cases} 0 & y < 0 \\ \frac{2y^2}{9} & 0 \leq y < 1.5 \\ 1 - \frac{2}{9}(3-y)^2 & 1.5 \leq y3 \\ 1 & y \geq 3 \end{cases}.$$

5. Let X be a continuous random variable with probability density function (PDF)

$$f(x) = \begin{cases} 2k & 0 \leq x < 3 \\ 3k & 3 \leq x < 5 \\ 0 & \text{otherwise.} \end{cases}$$

where k is a constant.

(a) Find k.

The pdf consists of two rectangles, one with area $6k$ and the other also with area $6k$. The total area is 1, so $12k = 1$, so $k = 1/12$.

(b) Find $F(4)$, the cumulative distribution function at $x = 2$.

We wish to find $P(X \leq 4)$. Split this up into two parts: $P(0 < X < 3)$ and $P(3 < X < 4)$. $P(0 < X < 3)$ is the area of a rectangle with width 3 and height $2k$, giving area $6k$. $P(3 < X < 4)$ is the area of a rectangle with width 1 and height $3k$, giving area $3k$. So the total probability is $9k$, which is $3/4$.

(c) Find the expected value of X.

Half the time we have a variable uniform on $[0, 3)$, and the other half the time we have a variable uniform on $[3, 5)$. A variable uniform on $[0, 3)$ has an expected value of 1.5, while a variable uniform on $[3, 5)$ has an expected value of 4. The overall expected value is halfway in between: $(1.5 + 4)/2 = 2.75$.