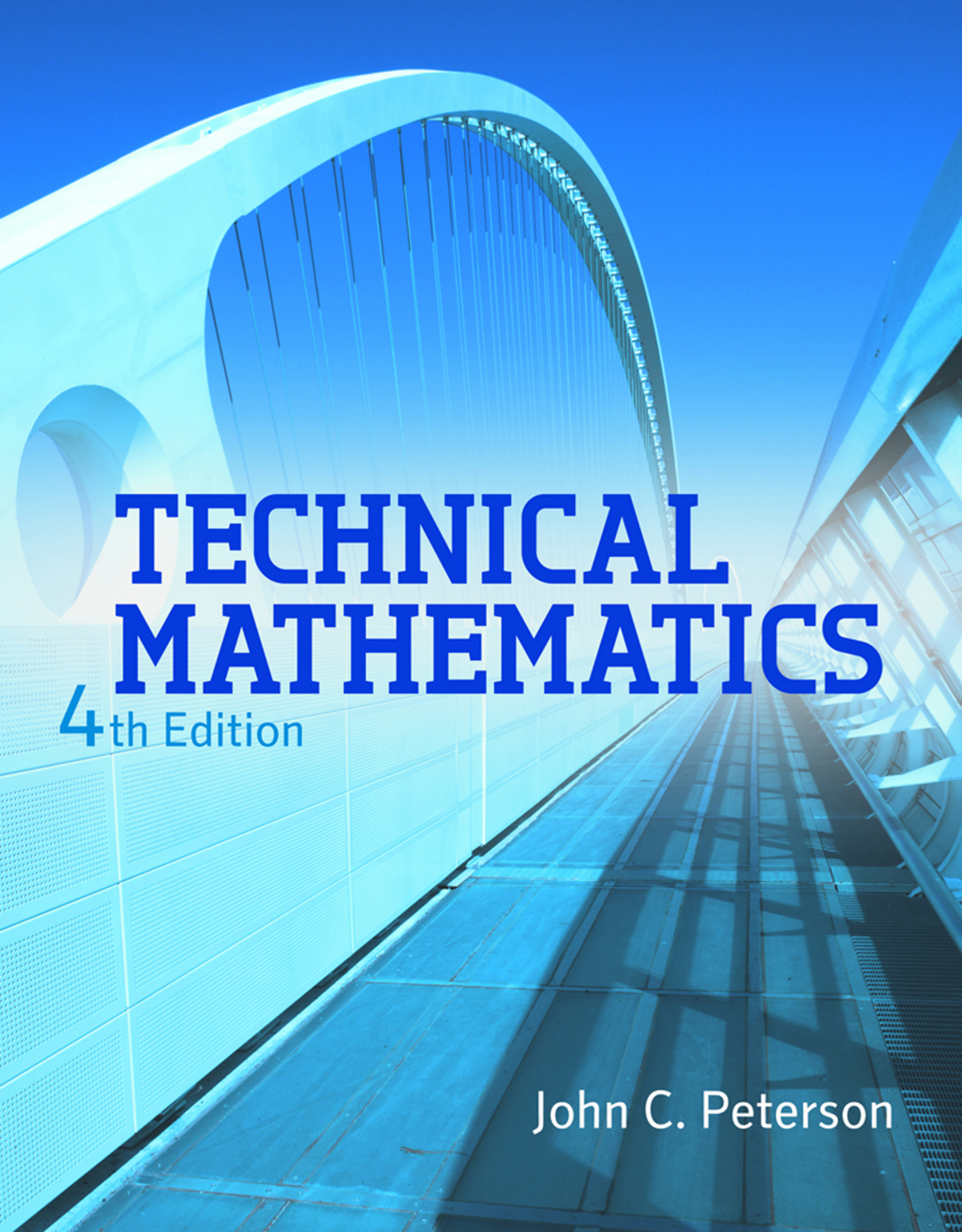


TECHNICAL MATHEMATICS

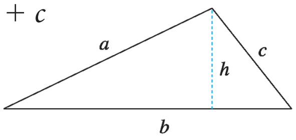
4th Edition

John C. Peterson

Triangle

$P = a + b + c$

$A = \frac{1}{2}bh$



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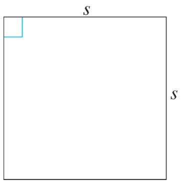
Square

$P = 4s$

$A = s^2$



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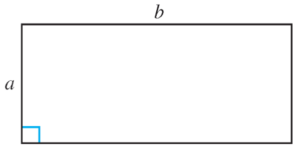
Rectangle

$P = 2(a + b)$

$A = ab$



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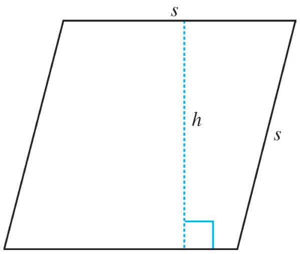
Rhombus

$P = 4s$

$A = sh$



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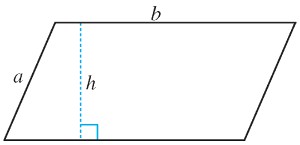
Parallelogram

$P = 2(a + b)$

$A = bh$



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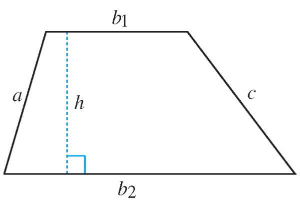
Trapezoid

$P = a + b_1 + c + b_2$

$A = \frac{1}{2}(b_1 + b_2)h$



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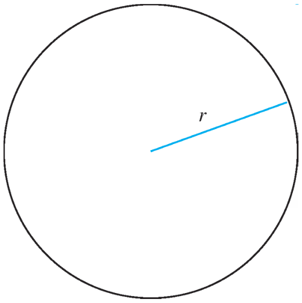
Circle

$C = 2\pi r$

$A = \pi r^2$



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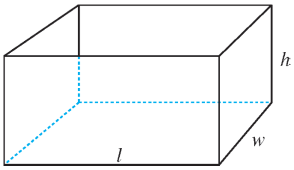
Rectangular Prism

$L = 2h(l + w)$

$T = 2(lw + lh + hw)$

$V = lwh$



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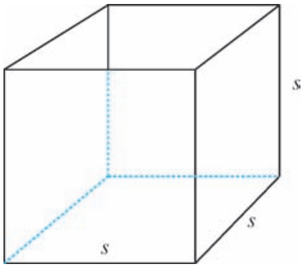
Cube

$L = 4s^2$

$T = 6s^2$

$V = s^3$



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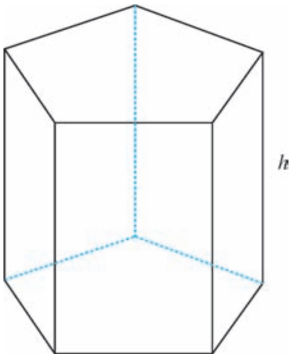
Prism

$L = ph$

$T = ph + 2B$

$V = Bh$

(B = area of base)



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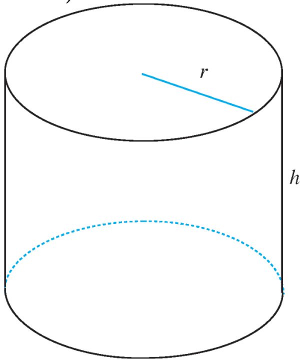
Circular Cylinder

$L = 2\pi rh$

$T = 2\pi r(r + h)$

$V = \pi r^2 h$



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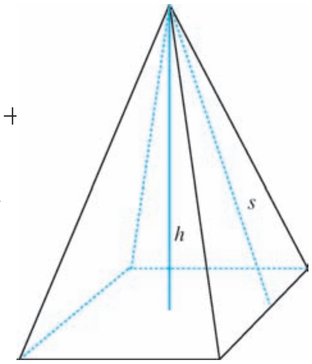
Pyramid

$L = \frac{1}{2}ps$

$T = \frac{1}{2}ps +$

$V = \frac{1}{3}Bh$



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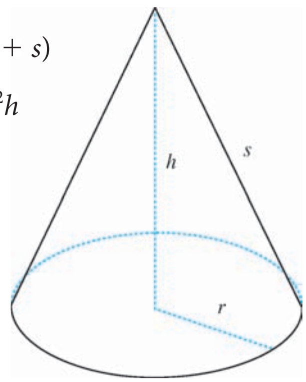
Circular Cone

$L = \pi rs$

$T = \pi r(r + s)$

$V = \frac{1}{3}\pi r^2 h$



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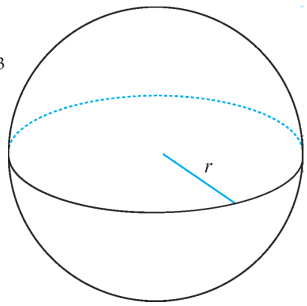
Sphere

$T = 4\pi r^2$

$V = \frac{4}{3}\pi r^3$



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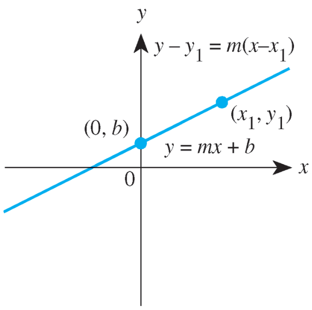
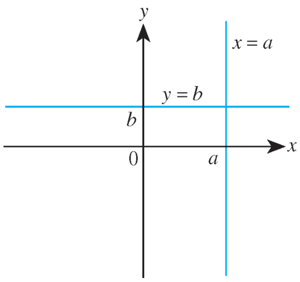
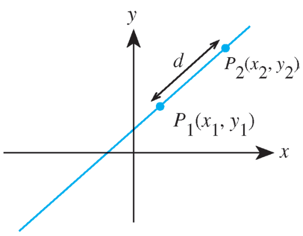
Nonvertical line, slope m



Horizontal line, vertical line



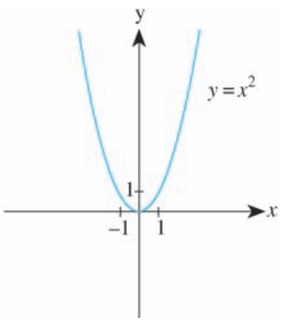
Distance formula, slope m of a nonvertical line



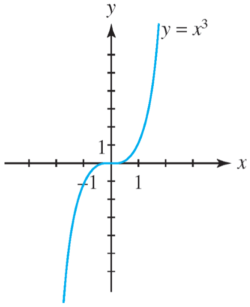
$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

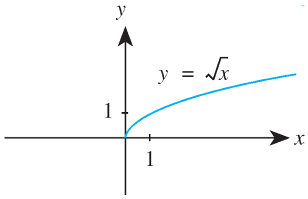
$y = x^2$



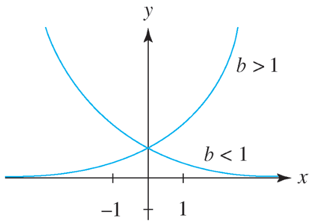
$y = x^3$



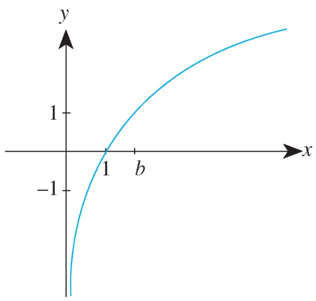
$y = \sqrt{x}$



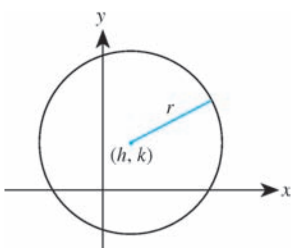
$y = b^x, b > 0, b \neq 1$



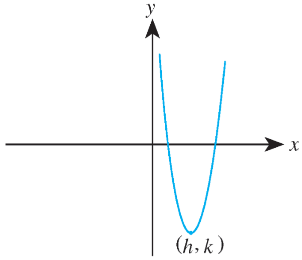
$y = \log_b x, b > 1$



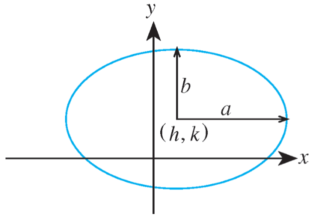
$(x - h)^2 + (y - k)^2 = r^2$



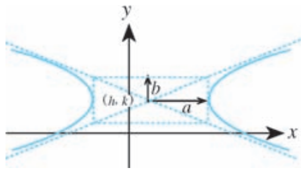
$4p(y - k) = (x - h)^2, p > 0$



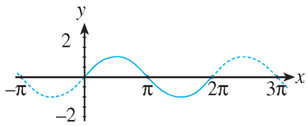
$\dfrac{(x - h)^2}{a^2} + \dfrac{(y - k)^2}{b^2} = 1, a > b$



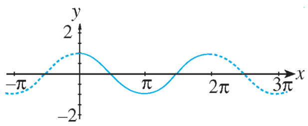
$\dfrac{(x - h)^2}{a^2} + \dfrac{(y - k)^2}{b^2} = 1$



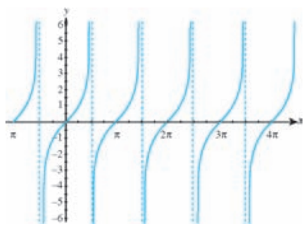
$y = \sin x$

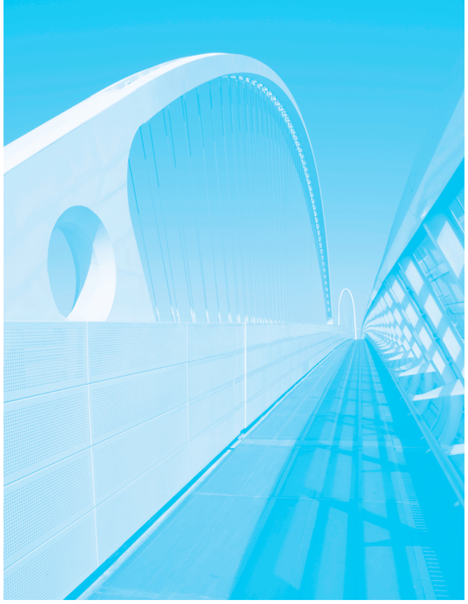


$y = \cos x$



$y = \tan x$





TECHNICAL MATHEMATICS

4th Edition

TECHNICAL MATHEMATICS

4th Edition

JOHN C. PETERSON

Chattanooga State Technical Community College

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John C. Peterson

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PREFACE

Technical Mathematics, fourth edition, is a student-oriented textbook designed to be easily read and understood by students who are preparing for technical or scientific careers. Mathematics can be a difficult subject to read or study, especially when the math terms become tangled with the technical terms used in the applications. This text offers a clear, readable development of the math concepts. But learning mathematics takes more than reading—it requires extensive practice. Therefore, *Technical Mathematics* provides many opportunities to practice problem solving. Together, these two essential elements have formed our constant focus in the development of this text: thorough, uncomplicated discussions followed by examples and problem sets that draw on real-world applications of mathematics.

ORGANIZATION AND APPROACH

The first three chapters of this text give a thorough introduction to the real number system, algebra, and geometry. Much of the material in these chapters might be familiar to your students and can be covered quickly or skipped entirely. However, Section 3.4 on the area of irregular shapes is probably new. Students will not only find these ideas useful, but they will also provide an informal introduction to calculus. As mentioned later in this Preface, Section 2.5 provides thorough coverage of dimensional analysis. Chapters 4–21 focus on the precalculus areas of mathematics through trigonometry, analytic geometry, and introductory statistics. Chapter 22 is an informal introduction to calculus. Chapters 23 through 34 cover calculus topics and are now available in eBook format or as printable PDFs through Cengage Learning's new on-line supplement, CourseMate. (We'll discuss CourseMate in greater detail and provide access information in the sections that follow.)

This text offers an integrated presentation of algebra, geometry, trigonometry, and analytic geometry. Many intuitive introductions to concepts in calculus are in the first 21 chapters. Practical applications of mathematics in the technical and scientific areas are provided throughout the text. Students also receive ample opportunities to solve problems using scientific calculators, graphing calculators, spreadsheets, and personal computers.

The author strived to make the reading and comprehension of its contents easier for students. The uncomplicated writing style makes difficult discussions easier to grasp. Each important concept is supported by numerous examples, applications, and practice problems.

MAJOR CHANGES TO THE FOURTH EDITION

There are seven major differences between the fourth edition and the third edition: (1) new section on dimensional analysis, (2) new chapter on computer number systems, (3) integration of spreadsheets as a means of graphing and solving problems, (4) calculator examples were changed from the TI-86 to the TI-84, (5) inclusion of chapter objectives in the text, (6) new chapter openers, and (7) new examples, new exercises, and updating of many of the examples and exercises. Each of these changes is discussed next.

NEW SECTION ON DIMENSIONAL ANALYSIS

Dimensional analysis has been a part of previous editions, but many users believed that it need more prominence. As a result, the material on dimensional analysis has been expanded and presented as a separate section in Chapter 2, Section 2.5.

NEW CHAPTER ON COMPUTER NUMBER SYSTEMS

Binary, octal, and hexadecimal number systems are now part of the text as Chapter 16 and should be of particular use to anyone interested in computer technology. This new chapter shows students how to convert between each of these numeration systems and the decimal systems, and how to perform basic arithmetic in each system. Some applications of each system have been included in the chapter.

INTEGRATION OF SPREADSHEETS FOR GRAPHING AND SOLVING PROBLEMS

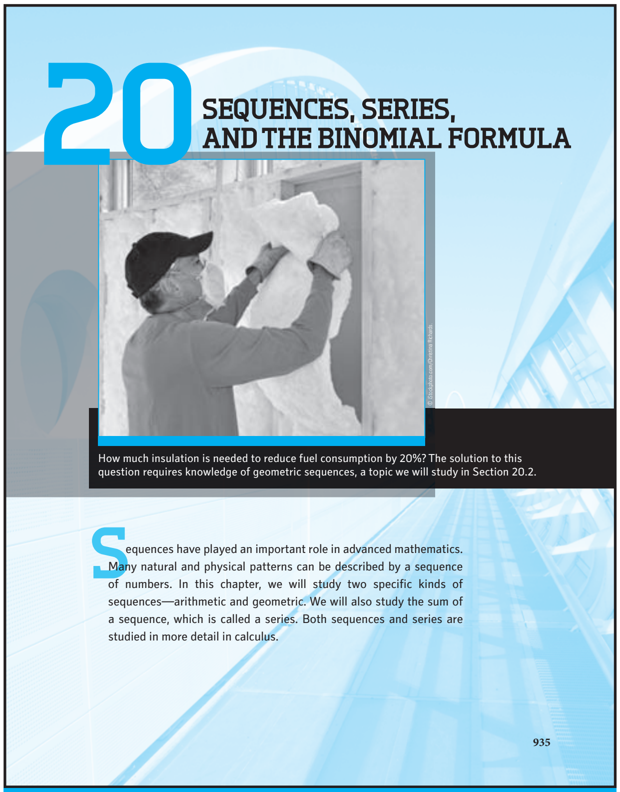
Many employers expect employees to be able to present their ideas using computers and spreadsheets. Thus, while calculators are very handy for many calculations and for graphing, we have incorporated instructions for using spreadsheets to perform many of the same duties as calculators.

CALCULATOR EXAMPLES CHANGED FROM TI-86 TO TI-84

Graphing calculators have been a part of *Technical Mathematics* since the first edition. In the third edition the TI-86 was used for most examples. However, many users prefer the TI-84, and so the examples have been rewritten with instructions for the TI-83/84. For instructors who would prefer to retain the TI-86 instructions, they can be downloaded from the Companion Website.

CHAPTER OBJECTIVES INCLUDED IN TEXT

In the past, the chapter objectives were part of the Instructor's Guide. It was believed that students would benefit from seeing these objectives, so they have been moved from the Instructor's Guide to the beginning of each chapter in the text.

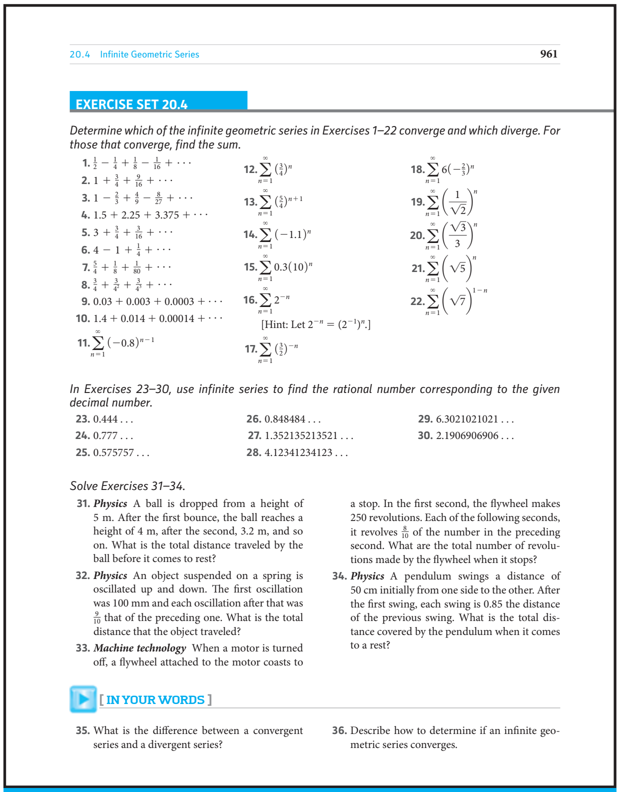

20
SEQUENCES, SERIES, AND THE BINOMIAL FORMULA

How much insulation is needed to reduce fuel consumption by 20%? The solution to this question requires knowledge of geometric sequences, a topic we will study in Section 20.2.

Sequences have played an important role in advanced mathematics. Many natural and physical patterns can be described by a sequence of numbers. In this chapter, we will study two specific kinds of sequences—arithmetic and geometric. We will also study the sum of a sequence, which is called a series. Both sequences and series are studied in more detail in calculus.

935

New Chapter Opener


20.4 Infinite Geometric Series 961

EXERCISE SET 20.4

Determine which of the infinite geometric series in Exercises 1–22 converge and which diverge. For those that converge, find the sum.

1. $\frac{1}{2} - \frac{1}{4} + \frac{1}{8} - \frac{1}{16} + \cdots$
2. $1 + \frac{3}{4} + \frac{9}{16} + \cdots$
3. $1 - \frac{2}{3} + \frac{4}{9} - \frac{8}{27} + \cdots$
4. $1.5 + 2.25 + 3.375 + \cdots$
5. $3 + \frac{3}{4} + \frac{3}{16} + \cdots$
6. $4 - 1 + \frac{1}{4} + \cdots$
7. $\frac{5}{4} + \frac{1}{8} + \frac{1}{80} + \cdots$
8. $\frac{3}{4} + \frac{3}{4^2} + \frac{3}{4^3} + \cdots$
9. $0.03 + 0.003 + 0.0003 + \cdots$
10. $1.4 + 0.014 + 0.00014 + \cdots$
11. $\sum_{n=1}^{\infty} (-0.8)^{n-1}$
12. $\sum_{n=1}^{\infty} (\frac{3}{4})^n$
13. $\sum_{n=1}^{\infty} (\frac{5}{4})^{n+1}$
14. $\sum_{n=1}^{\infty} (-1.1)^n$
15. $\sum_{n=1}^{\infty} 0.3(10)^n$
16. $\sum_{n=1}^{\infty} 2^{-n}$
[Hint: Let $2^{-n} = (2^{-1})^n$.]
17. $\sum_{n=1}^{\infty} (\frac{3}{2})^{-n}$
18. $\sum_{n=1}^{\infty} 6(-\frac{2}{3})^n$
19. $\sum_{n=1}^{\infty} \left(\frac{1}{\sqrt{2}}\right)^n$
20. $\sum_{n=1}^{\infty} \left(\frac{\sqrt{3}}{3}\right)^n$
21. $\sum_{n=1}^{\infty} \left(\sqrt{5}\right)^n$
22. $\sum_{n=1}^{\infty} \left(\sqrt{7}\right)^{1-n}$

In Exercises 23–30, use infinite series to find the rational number corresponding to the given decimal number.

23. 0.444 . . .
24. 0.777 . . .
25. 0.575757 . . .
26. 0.848484 . . .
27. 1.352135213521 . . .
28. 4.12341234123 . . .
29. 6.3021021021 . . .
30. 2.1906906906 . . .

Solve Exercises 31–34.

31. ***Physics*** A ball is dropped from a height of 5 m. After the first bounce, the ball reaches a height of 4 m, after the second, 3.2 m, and so on. What is the total distance traveled by the ball before it comes to rest?
32. ***Physics*** An object suspended on a spring is oscillated up and down. The first oscillation was 100 mm and each oscillation after that was $\frac{9}{10}$ that of the preceding one. What is the total distance that the object traveled?
33. ***Machine technology*** When a motor is turned off, a flywheel attached to the motor coasts to a stop. In the first second, the flywheel makes 250 revolutions. Each of the following seconds, it revolves $\frac{8}{10}$ of the number in the preceding second. What are the total number of revolutions made by the flywheel when it stops?
34. ***Physics*** A pendulum swings a distance of 50 cm initially from one side to the other. After the first swing, each swing is 0.85 the distance of the previous swing. What is the total distance covered by the pendulum when it comes to a rest?

[IN YOUR WORDS]

35. What is the difference between a convergent series and a divergent series?
36. Describe how to determine if an infinite geometric series converges.

Additional Exercises

NEW CHAPTER OPENERS

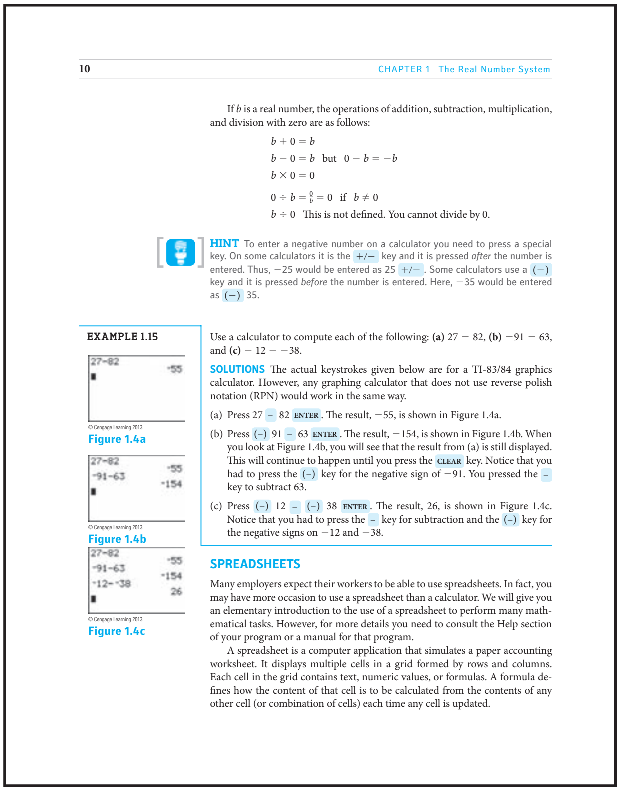
Several chapter openers have been revised with updated artwork and introductions. Other chapter openers from the first three editions remain in the text and continue to provide a set of rich examples.

NEW EXAMPLES AND EXERCISES

New exercises have been added. These not only consist of some new "routine" (drill and practice) exercises, but also applications and writing opportunities. Many of the applications provide data in tabular format. Students are asked to examine the data and determine an appropriate function for it; the application then asks some questions that require the use of that function.

CALCULUS CHAPTERS AVAILABLE ON LINE

Twelve chapters covering integral and derivative calculus topics are now available through Cengage Learning's new on line study tool and homework solution, CourseMate. Students and instructors who purchase CourseMate as a supplement can access these chapters as an interactive eBook or as printable PDFs. To learn more about CourseMate, or to access the resources for this text, please visit *www.cengagebrain.com*. At the homepage, search on the ISBN from the back cover of this book. This search will take you to the product page where resources for this title can be found.



10 CHAPTER 1 The Real Number System

If b is a real number, the operations of addition, subtraction, multiplication, and division with zero are as follows:

$$b + 0 = b$$
$$b - 0 = b \quad \text{but} \quad 0 - b = -b$$
$$b \times 0 = 0$$
$$0 \div b = \tfrac{0}{b} = 0 \quad \text{if} \quad b \neq 0$$
$$b \div 0 \quad \text{This is not defined. You cannot divide by 0.}$$

HINT To enter a negative number on a calculator you need to press a special key. On some calculators it is the +/− key and it is pressed *after* the number is entered. Thus, −25 would be entered as 25 +/−. Some calculators use a (−) key and it is pressed *before* the number is entered. Here, −35 would be entered as (−) 35.

EXAMPLE 1.15

Use a calculator to compute each of the following: **(a)** 27 − 82, **(b)** −91 − 63, and **(c)** − 12 − −38.

SOLUTIONS The actual keystrokes given below are for a TI-83/84 graphics calculator. However, any graphing calculator that does not use reverse polish notation (RPN) would work in the same way.

(a) Press 27 − 82 ENTER. The result, −55, is shown in Figure 1.4a.

(b) Press (−) 91 − 63 ENTER. The result, −154, is shown in Figure 1.4b. When you look at Figure 1.4b, you will see that the result from (a) is still displayed. This will continue to happen until you press the CLEAR key. Notice that you had to press the (−) key for the negative sign of −91. You pressed the − key to subtract 63.

(c) Press (−) 12 − (−) 38 ENTER. The result, 26, is shown in Figure 1.4c. Notice that you had to press the − key for subtraction and the (−) key for the negative signs on −12 and −38.

27-82 -55

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Figure 1.4a

27-82 -55
-91-63 -154

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Figure 1.4b

27-82 -55
-91-63 -154
-12--38 26

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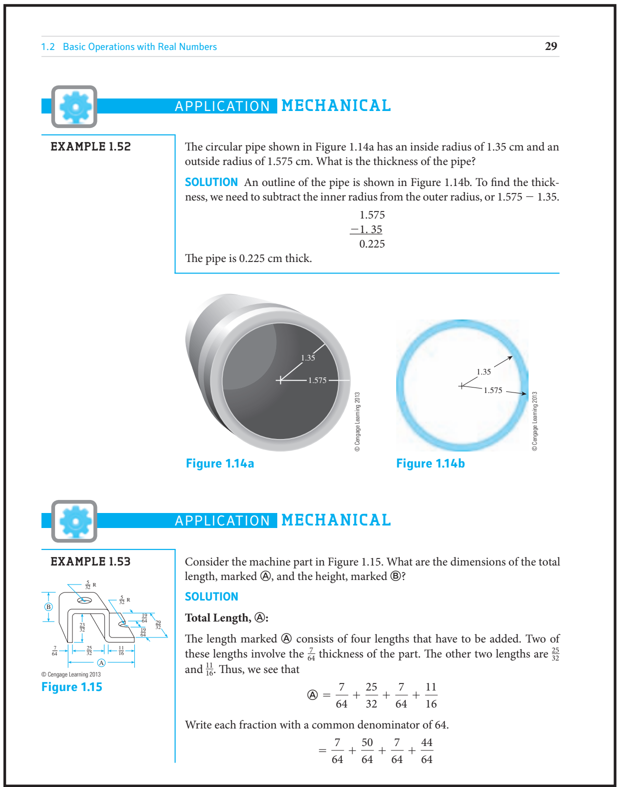
Figure 1.4c

SPREADSHEETS

Many employers expect their workers to be able to use spreadsheets. In fact, you may have more occasion to use a spreadsheet than a calculator. We will give you an elementary introduction to the use of a spreadsheet to perform many mathematical tasks. However, for more details you need to consult the Help section of your program or a manual for that program.

A spreadsheet is a computer application that simulates a paper accounting worksheet. It displays multiple cells in a grid formed by rows and columns. Each cell in the grid contains text, numeric values, or formulas. A formula defines how the content of that cell is to be calculated from the contents of any other cell (or combination of cells) each time any cell is updated.

Revision of Calculator Material



1.2 Basic Operations with Real Numbers 29

APPLICATION MECHANICAL

EXAMPLE 1.52

The circular pipe shown in Figure 1.14a has an inside radius of 1.35 cm and an outside radius of 1.575 cm. What is the thickness of the pipe?

SOLUTION An outline of the pipe is shown in Figure 1.14b. To find the thickness, we need to subtract the inner radius from the outer radius, or 1.575 − 1.35.

$$\begin{array}{r} 1.575 \\ \underline{-1.35} \\ 0.225 \end{array}$$

The pipe is 0.225 cm thick.

1.35 1.575 © Cengage Learning 2013

Figure 1.14a

1.35 1.575 © Cengage Learning 2013

Figure 1.14b

APPLICATION MECHANICAL

EXAMPLE 1.53

Consider the machine part in Figure 1.15. What are the dimensions of the total length, marked Ⓐ, and the height, marked Ⓑ?

SOLUTION

Total Length, Ⓐ:

The length marked Ⓐ consists of four lengths that have to be added. Two of these lengths involve the $\frac{7}{64}$ thickness of the part. The other two lengths are $\frac{25}{32}$ and $\frac{11}{16}$. Thus, we see that

$$Ⓐ = \frac{7}{64} + \frac{25}{32} + \frac{7}{64} + \frac{11}{16}$$

Write each fraction with a common denominator of 64.

$$= \frac{7}{64} + \frac{50}{64} + \frac{7}{64} + \frac{44}{64}$$

$\frac{5}{32}$ R $\frac{5}{32}$ R Ⓑ $\frac{23}{32}$ $\frac{19}{64}$ $\frac{19}{64}$ $\frac{29}{32}$ $\frac{7}{64}$ $\frac{25}{32}$ $\frac{11}{16}$ Ⓐ

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Figure 1.15

Problem Solving

FOLLOWING STANDARDS

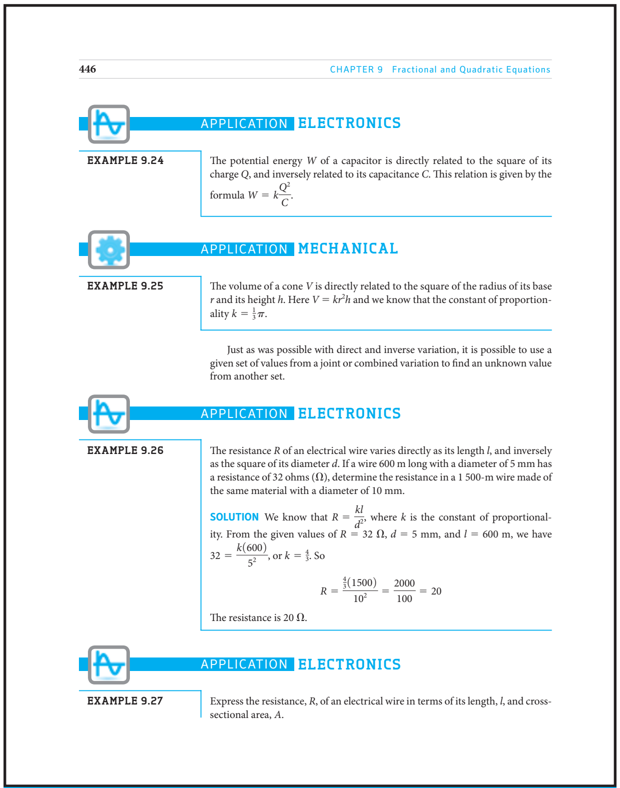
As in the previous editions, the text has tried to follow both the NCTM and AMATYC standards. This edition has also adopted many of the suggestions in AMATYC's *A Vision: Mathematics for the Emerging Technologies*. This is most prominent in the new projects, approach to problem solving, use of technology, and writing assignments.

HELP FOR TEACHING AND LEARNING

Well-written, carefully illustrated discussions form the foundation of any textbook; but your students will form their math skills with practice. We have developed a package of learning features that will inspire students to practice hard, and this will make their efforts, and yours, effective.

PROBLEM SOLVING

Students often have trouble assessing a real situation and sorting out the pertinent information. Our approach to problem solving encourages students to visualize problems, organize information, and develop their intuitive abilities. The applications, examples, and problems are intended to teach students how to interpret real-world situations. Many problems are accompanied by illustrations and photos intended to guide students in analyzing information. Problem



446 CHAPTER 9 Fractional and Quadratic Equations

APPLICATION ELECTRONICS

EXAMPLE 9.24

The potential energy W of a capacitor is directly related to the square of its charge Q, and inversely related to its capacitance C. This relation is given by the formula $W = k\frac{Q^2}{C}$.

APPLICATION MECHANICAL

EXAMPLE 9.25

The volume of a cone V is directly related to the square of the radius of its base r and its height h. Here $V = kr^2h$ and we know that the constant of proportionality $k = \frac{1}{3}\pi$.

Just as was possible with direct and inverse variation, it is possible to use a given set of values from a joint or combined variation to find an unknown value from another set.

APPLICATION ELECTRONICS

EXAMPLE 9.26

The resistance R of an electrical wire varies directly as its length l, and inversely as the square of its diameter d. If a wire 600 m long with a diameter of 5 mm has a resistance of 32 ohms (Ω), determine the resistance in a 1 500-m wire made of the same material with a diameter of 10 mm.

SOLUTION We know that $R = \frac{kl}{d^2}$, where k is the constant of proportionality. From the given values of $R = 32\ \Omega$, $d = 5$ mm, and $l = 600$ m, we have $32 = \frac{k(600)}{5^2}$, or $k = \frac{4}{3}$. So

$$R = \frac{\frac{4}{3}(1500)}{10^2} = \frac{2000}{100} = 20$$

The resistance is 20 Ω.

APPLICATION ELECTRONICS

EXAMPLE 9.27

Express the resistance, R, of an electrical wire in terms of its length, l, and cross-sectional area, A.

Visualizing Mathematics

solving is formally introduced in Sections 2.4 and 2.6, and is developed throughout the book. Section 4.6, Introduction to Modeling, presents another excellent opportunity to provide good problem-solving experiences. Hints, Notes, and Cautions are offered to hone problem-solving techniques, and many boxed features include guidelines for solving certain types of problems.

VISUALIZING MATHEMATICS

For students to develop good problem-solving skills, they must learn to visualize mathematical problems. This text offers a unique feature that will help students to see the important information in a problem. Each chapter opens with an illustration that demonstrates mathematics at work in the real world. Later in the chapter, this illustration is analyzed in an example and used to solve a problem based on the picture. When a photograph is used, a sketch placed next to the photo extracts the important mathematical elements. Where possible, this sketch is imposed over the photograph to show students how to cull the important elements from a real situation, and how to create a sketch that will help to solve a problem.

USING TECHNOLOGY IN PROBLEM SOLVING

This text emphasizes the wise use of graphing calculators because they can take the drudgery out of repetitive computations and can help students learn faster. Reducing the drudgery can help students focus on the important aspects of the operations they are performing. However, students must be aware that not all solutions are aided by technology. They must also learn how to determine whether a calculator or computer has given a correct answer, a reasonable answer, or a wrong answer.

Rather than indicate that only certain problems should be solved with a calculator, spreadsheet, or other computer software, the author has assumed that any problem can be solved with the help of one of these tools. Calculator keystroke sequences appear frequently throughout the text to help students learn to use these important tools efficiently. Most of these keystrokes are for a TI-84. This same material for different calculators can be downloaded from the text's Companion Website.

EXAMPLES

The book contains frequent, realistic examples that show how to apply mathematical concepts. Examples cover both routine mathematical manipulations and applications. Approximately 20% of these examples have been selected from trade, vocational, technical, and industrial areas to demonstrate how mathematics is applied to all fields of technology.

EXERCISES

The number of problems has increased in the fourth edition. Many of these problems show the great variety of technical and scientific applications for math. Methods for solving most of the exercises can be found by reviewing the worked examples. Answers to the odd-numbered questions appear in the back of the book.

WRITING TO LEARN MATH

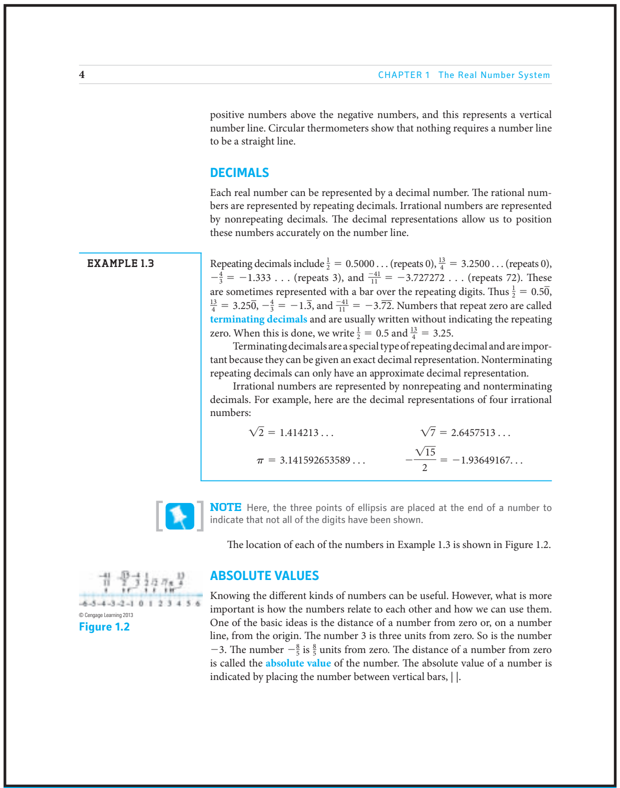
The fourth edition helps instructors incorporate NCTM and AMATYC standards and AMATYC's *A Vision: Mathematics for the Emerging Technologies* by offering writing problems at the end of each exercise section. These exercises appear under the heading "In Your Words," and will encourage students to work through their problem-solving processes verbally as well as numerically. The writing exercises can also be used to promote collaborative learning.

APPLICATIONS

Most technical math courses are designed to prepare students to solve applied problems in their chosen technical fields. For this reason, we have taken care to present many exercises and examples that show how to apply math in a variety of technical fields. These applications are highlighted in the text with special headings that indicate the applied field. The text also offers an index of the applications by field.

PEDAGOGICAL FEATURES

Throughout the book, you will find rules, formulas, guidelines, and hints that are boxed for easy identification. The boxes make the material easy to locate, and also make the book a valuable reference after the course is finished.



4 CHAPTER 1 The Real Number System

positive numbers above the negative numbers, and this represents a vertical number line. Circular thermometers show that nothing requires a number line to be a straight line.

DECIMALS

Each real number can be represented by a decimal number. The rational numbers are represented by repeating decimals. Irrational numbers are represented by nonrepeating decimals. The decimal representations allow us to position these numbers accurately on the number line.

EXAMPLE 1.3

Repeating decimals include $\frac{1}{2} = 0.5000\ldots$ (repeats 0), $\frac{13}{4} = 3.2500\ldots$ (repeats 0), $-\frac{4}{3} = -1.333\ldots$ (repeats 3), and $\frac{-41}{11} = -3.727272\ldots$ (repeats 72). These are sometimes represented with a bar over the repeating digits. Thus $\frac{1}{2} = 0.5\overline{0}$, $\frac{13}{4} = 3.25\overline{0}$, $-\frac{4}{3} = -1.\overline{3}$, and $\frac{-41}{11} = -3.\overline{72}$. Numbers that repeat zero are called **terminating decimals** and are usually written without indicating the repeating zero. When this is done, we write $\frac{1}{2} = 0.5$ and $\frac{13}{4} = 3.25$.

Terminating decimals are a special type of repeating decimal and are important because they can be given an exact decimal representation. Nonterminating repeating decimals can only have an approximate decimal representation.

Irrational numbers are represented by nonrepeating and nonterminating decimals. For example, here are the decimal representations of four irrational numbers:

$$\sqrt{2} = 1.414213\ldots \qquad \sqrt{7} = 2.6457513\ldots$$

$$\pi = 3.141592653589\ldots \qquad -\frac{\sqrt{15}}{2} = -1.93649167\ldots$$

NOTE Here, the three points of ellipsis are placed at the end of a number to indicate that not all of the digits have been shown.

The location of each of the numbers in Example 1.3 is shown in Figure 1.2.

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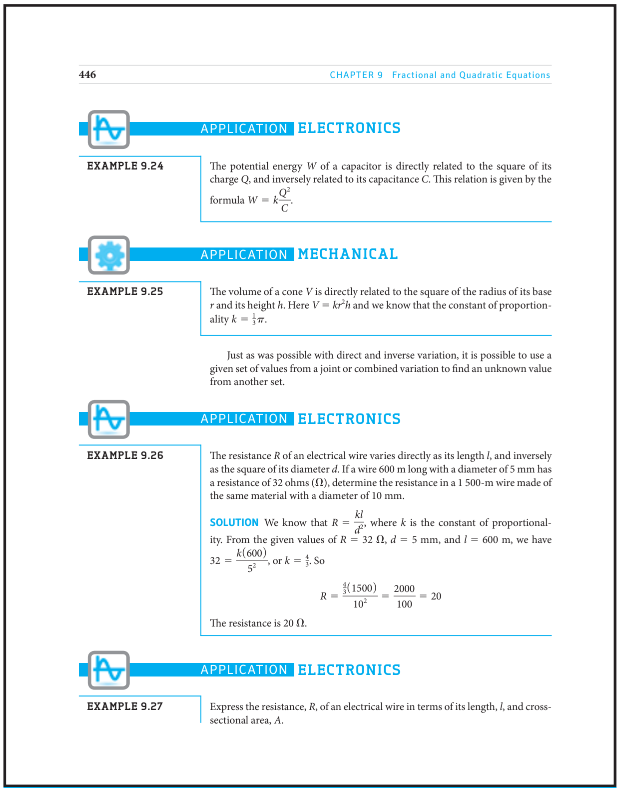
Figure 1.2

ABSOLUTE VALUES

Knowing the different kinds of numbers can be useful. However, what is more important is how the numbers relate to each other and how we can use them. One of the basic ideas is the distance of a number from zero or, on a number line, from the origin. The number 3 is three units from zero. So is the number -3. The number $-\frac{8}{5}$ is $\frac{8}{5}$ units from zero. The distance of a number from zero is called the **absolute value** of the number. The absolute value of a number is indicated by placing the number between vertical bars, $|\;|$.

Examples



446 CHAPTER 9 Fractional and Quadratic Equations

APPLICATION ELECTRONICS

EXAMPLE 9.24

The potential energy W of a capacitor is directly related to the square of its charge Q, and inversely related to its capacitance C. This relation is given by the formula $W = k\frac{Q^2}{C}$.

APPLICATION MECHANICAL

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The volume of a cone V is directly related to the square of the radius of its base r and its height h. Here $V = kr^2h$ and we know that the constant of proportionality $k = \frac{1}{3}\pi$.

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SOLUTION We know that $R = \frac{kl}{d^2}$, where k is the constant of proportionality. From the given values of $R = 32\ \Omega$, $d = 5$ mm, and $l = 600$ m, we have $32 = \frac{k(600)}{5^2}$, or $k = \frac{4}{3}$. So

$$R = \frac{\frac{4}{3}(1500)}{10^2} = \frac{2000}{100} = 20$$

The resistance is $20\ \Omega$.

APPLICATION ELECTRONICS

EXAMPLE 9.27

Express the resistance, R, of an electrical wire in terms of its length, l, and cross-sectional area, A.

Applications

Four types of icons appear frequently in the text:

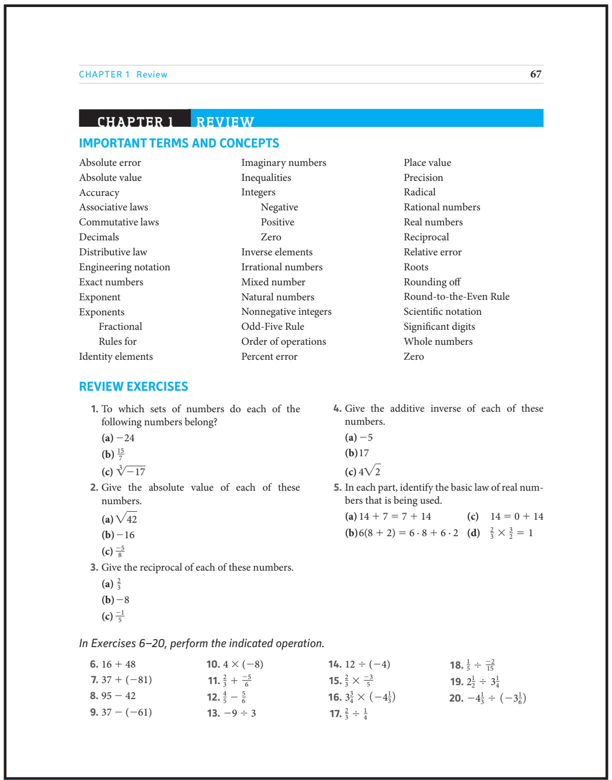
Many years of teaching experience have shown the author where students make common errors. These places are indicated by the "CAUTION" icon. This icon not only points out the potential errors, but how to avoid them.

The "HINT" icon points out a valuable technique or learning hint that students can use to solve problems.

The "NOTE" icon refreshes students' memories about a concept, or points out interesting or unusual ideas. These notes highlight ideas that might easily be overlooked, alternative ways to solve a problem, or a possible shortcut.

A set of 10 icons has been developed for the Application examples. There is an icon for each of the following areas:

- Architecture
- Automotive
- Business
- Environmental science
- Health care
- Civil engineering
- Construction
- Electronics
- Mechanical
- General technology


CHAPTER 1 Review 67

CHAPTER 1 REVIEW

IMPORTANT TERMS AND CONCEPTS

Absolute error	Imaginary numbers	Place value
Absolute value	Inequalities	Precision
Accuracy	Integers	Radical
Associative laws	Negative	Rational numbers
Commutative laws	Positive	Real numbers
Decimals	Zero	Reciprocal
Distributive law	Inverse elements	Relative error
Engineering notation	Irrational numbers	Roots
Exact numbers	Mixed number	Rounding off
Exponent	Natural numbers	Round-to-the-Even Rule
Exponents	Nonnegative integers	Scientific notation
Fractional	Odd-Five Rule	Significant digits
Rules for	Order of operations	Whole numbers
Identity elements	Percent error	Zero

REVIEW EXERCISES

1. To which sets of numbers do each of the following numbers belong?
   (a) -24
   (b) $\frac{15}{7}$
   (c) $\sqrt[3]{-17}$
2. Give the absolute value of each of these numbers.
   (a) $\sqrt{42}$
   (b) -16
   (c) $\frac{-5}{8}$
3. Give the reciprocal of each of these numbers.
   (a) $\frac{2}{3}$
   (b) -8
   (c) $\frac{-1}{5}$
4. Give the additive inverse of each of these numbers.
   (a) -5
   (b) 17
   (c) $4\sqrt{2}$
5. In each part, identify the basic law of real numbers that is being used.
   (a) $14 + 7 = 7 + 14$
   (b) $6(8 + 2) = 6 \cdot 8 + 6 \cdot 2$
   (c) $14 = 0 + 14$
   (d) $\frac{2}{3} \times \frac{3}{2} = 1$

In Exercises 6–20, perform the indicated operation.

6. $16 + 48$
7. $37 + (-81)$
8. $95 - 42$
9. $37 - (-61)$
10. $4 \times (-8)$
11. $\frac{2}{3} + \frac{-5}{6}$
12. $\frac{4}{5} - \frac{5}{6}$
13. $-9 \div 3$
14. $12 \div (-4)$
15. $\frac{2}{3} \times \frac{-3}{5}$
16. $3\frac{3}{4} \times (-4\frac{1}{3})$
17. $\frac{2}{3} \div \frac{1}{4}$
18. $\frac{1}{5} \div \frac{-2}{15}$
19. $2\frac{1}{2} \div 3\frac{1}{4}$
20. $-4\frac{1}{3} \div (-3\frac{1}{6})$

Chapter Review

CHAPTER REVIEW AND TEST

Each chapter concludes with a list of the important terms covered in the chapter, a generous set of review exercises, and a chapter test. The review exercises include both routine computations and applications. Many of these exercises draw on more than one section of the chapter, requiring students to think harder and synthesize their learning. The chapter test gives students an idea of the types of questions they might expect for a 50-min exam.

SUPPLEMENTS

The supplements package has been thoroughly revised and expanded for the fourth edition, offering instructors an array of products that allow them to tailor a course to their own student profile.

THE INSTRUCTOR COMPANION WEBSITE

The Instructor Companion Website contains the following five major items:

Solutions Manual containing fully worked solutions to all the exercises in the text. It is in pdf format so it can be downloaded to your computer or printed.

Computerized Test Bank in ExamView® software The test bank has approximately 100 questions per chapter to choose from. The test bank can be searched by section, style, or level of difficulty.

PowerPoint™ presentations of major concepts in the text. These can be downloaded and used in the classroom.

Image Library containing most of the figures in the text. Instructors can select the ones they want and either print them for student use or project them.

APPLIED MATH COURSEMATE

Each Delmar mathematics text includes access to Applied Math CourseMate, Cengage Learning's online solution for building strong math skills. Students and instructors alike will benefit from the following CourseMate resources:

- an interactive eBook including 12 calculus chapters, with highlighting, note-taking, and search capabilities
- interactive learning tools, including:
  - ✓ quizzes
  - ✓ flashcards
  - ✓ PowerPoint slides
  - ✓ skill-building games
- twelve calculus chapters available in PDF format for convenient printing

Instructors will be able to use Applied Math CourseMate to access the Instructor Resources and other classroom management tools. To access these resources, please visit *www.cengagebrain.com*. At the CengageBrain.com homepage, search for the ISBN of your title (from the back cover of your book) using the search box at the top of the page. This will take you to the product page where your resources can be found.

STUDENT COMPANION WEBSITE

In addition to the study tools and resources available in Applied Math CourseMate, students can access several free resources at The Student Companion Website. The website includes 12 additional chapters covering calculus topics, as well as sections on calculator keystroke sequences and instructions for using the TI-86 graphing calculator.

ACKNOWLEDGMENTS

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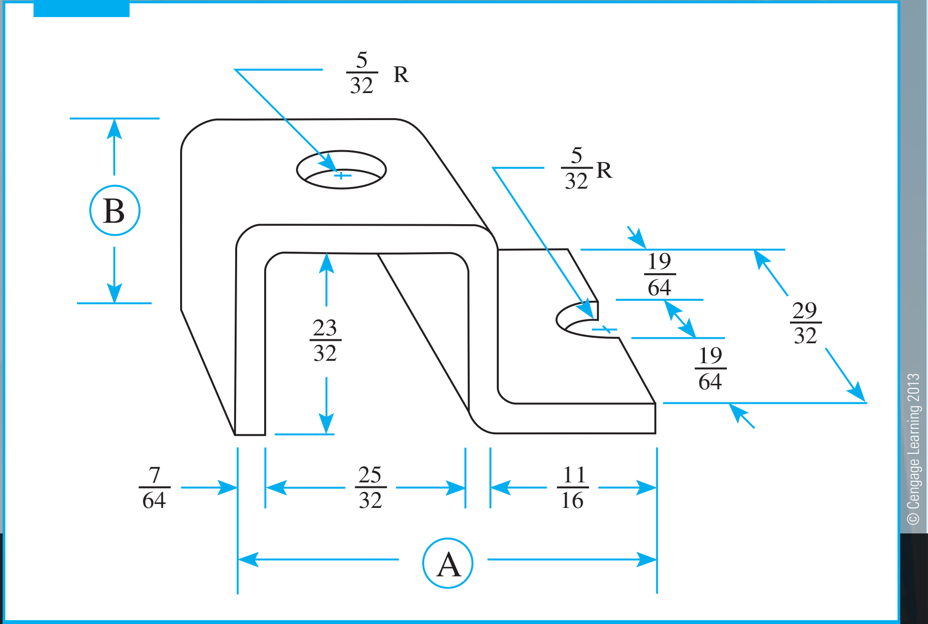
Linda Willey
Clifton Park, New York

No one could complete an effort such as this without the support, encouragement, and tolerance of an understanding family. I want to thank my wife, Dr. Marla Peterson, for giving me the time and understanding to complete this work.

John C. Peterson

1 THE REAL NUMBER SYSTEM



Technicians need to use mathematics to determine measurements such as the ones indicated by letters in this figure. In Section 1.2 we will see how to add and subtract real numbers to add these measures.

Every technician must be able to use mathematics. The science of mathematics is a universal language that helps technicians communicate and do their work. Advances in technology require that technicians know more and more mathematics. Technicians must be able to solve mathematical problems quickly and accurately.

The mathematics that are developed in this text provide the foundation for work in almost any field of technology. Perhaps as important as the actual mathematics is the ability to use calculators and computers to help work mathematics problems. As you use this book you should learn how to use calculators and computers to help you with mathematics.

OBJECTIVES

AFTER COMPLETING THIS CHAPTER, YOU WILL BE ABLE TO:

- Evaluate expressions containing fractions, decimals, exponents, roots, and absolute value using the correct order of operations without a calculator.
- Use a calculator to evaluate expressions containing the basic operations, exponents, roots, and absolute value.
- Round off decimals appropriately.
- Use significant digits properly.
- Perform operations involving scientific notation with and without a calculator.
- Evaluate expressions containing roots and radicals.
- Solve applications that involve evaluating expressions.
- Use dimensional analysis to convert between different units of measurement.

1.1 SOME SETS AND BASIC LAWS OF REAL NUMBERS

Fundamental to our work in mathematics is an understanding of the different types of numbers. One of the first sets of numbers that you learned was the **natural numbers**. These numbers are 1, 2, 3, 4, and so forth. These were the first numbers invented by people. Later, people discovered a need for the number zero (0). When 0 was added to the natural numbers, a new set of numbers, called the **whole numbers**, was formed.

INTEGERS

People continued to use whole numbers until they discovered that they were not able to solve some problems with the set of whole numbers. Problems such as $3 - 5$ required a new set of numbers the **integers**. The integers included the natural numbers, zero, and a negative value for each natural number. The set of integers consists of the numbers

$$\ldots, -4, -3, -2, -1, 0, 1, 2, 3, \ldots.$$

The negative numbers are called the **negative integers** and the natural numbers are called the **positive integers**. The number 0 is neither positive nor negative. Positive numbers can be written with a plus $(+)$ sign in front as $+1, +2, +3, \ldots$ but this is not necessary. The numbers $0, 1, 2, 3, \ldots$ are called the **nonnegative integers**.

NOTE Three points of ellipsis (. . .) in a series of numbers means that some numbers have been omitted. They are placed wherever the missing numbers would be.

EXAMPLE 1.1

The numbers 1, 14, 973, and 8,436 are all positive integers. They are also natural numbers.

The numbers -2, -18, and $-4{,}392$ are all negative integers, but not natural numbers.

RATIONAL NUMBERS

The rational numbers were created to indicate when a part of something was needed. **Rational numbers** are used to represent division of one integer by a positive or negative integer. There are both positive and negative rational numbers. Rational numbers can be expressed in more than one way. The numbers $\frac{4}{2}$, $\frac{6}{3}$, $\frac{2}{1}$, and $\frac{142}{71}$ are all different ways of writing 2.

EXAMPLE 1.2

Some positive rational numbers are 14, $\frac{7}{3}$, $\frac{2}{5}$, $\frac{271}{18}$, $\frac{4}{2}$, and 973.

Negative rational numbers include $-\frac{2}{3}$, -5, $-\frac{9}{7}$, $-\frac{5}{23}$, and $\frac{-14}{2}$.

Zero (0) is a rational number that is neither positive nor negative.

IRRATIONAL NUMBERS

At one time people thought that everything could be expressed as a rational number and that every problem would have a rational number as an answer. In fact, there is a story that the first person who proved that all numbers are not rational was executed because of this proof. Perhaps because it is not rational to execute a person just because he discovered a new group of numbers, these new numbers were called the **irrational numbers**. Some of the irrational numbers are $\sqrt{2}$, $\sqrt{5}$, π, and $-\sqrt{17}$. Although we will not prove it here, it is not possible to represent any of these numbers by the division of two integers.

REAL NUMBERS

When the rational numbers are combined with the irrational numbers, a new group of numbers is formed—the **real numbers**. Most of our work in this text uses the real numbers. Later we will introduce a new group of numbers, called the imaginary numbers, and combine the real and the imaginary numbers into another group, the complex numbers.

At times it is convenient to represent the real numbers on a line called the **number line**. The usual number line is a horizontal line that has been marked in equally spaced intervals. One of these marks is called the **origin** and is indicated by the number zero (0). The marks to the right are labeled using the positive integers. The negative integers are used to designate the marks to the left of the origin. A typical number line is shown in Figure 1.1.

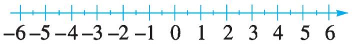


Figure 1.1

The other real numbers are located between the integers.

The fact that a number line does not have to be horizontal can be demonstrated by a thermometer. Most wall thermometers hang vertically with the positive numbers above the negative numbers, and this represents a vertical number line. Circular thermometers show that nothing requires a number line to be a straight line.

DECIMALS

Each real number can be represented by a decimal number. The rational numbers are represented by repeating decimals. Irrational numbers are represented by nonrepeating decimals. The decimal representations allow us to position these numbers accurately on the number line.

EXAMPLE 1.3

Repeating decimals include $\frac{1}{2} = 0.5000\ldots$ (repeats 0), $\frac{13}{4} = 3.2500\ldots$ (repeats 0), $-\frac{4}{3} = -1.333\ldots$ (repeats 3), and $\frac{-41}{11} = -3.727272\ldots$ (repeats 72). These are sometimes represented with a bar over the repeating digits. Thus $\frac{1}{2} = 0.5\overline{0}$, $\frac{13}{4} = 3.25\overline{0}$, $-\frac{4}{3} = -1.\overline{3}$, and $\frac{-41}{11} = -3.\overline{72}$. Numbers that repeat zero are called **terminating decimals** and are usually written without indicating the repeating zero. When this is done, we write $\frac{1}{2} = 0.5$ and $\frac{13}{4} = 3.25$.

Terminating decimals are a special type of repeating decimal and are important because they can be given an exact decimal representation. Nonterminating repeating decimals can only have an approximate decimal representation.

Irrational numbers are represented by nonrepeating and nonterminating decimals. For example, here are the decimal representations of four irrational numbers:

$$\sqrt{2} = 1.414213\ldots \qquad \sqrt{7} = 2.6457513\ldots$$

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NOTE Here, the three points of ellipsis are placed at the end of a number to indicate that not all of the digits have been shown.

The location of each of the numbers in Example 1.3 is shown in Figure 1.2.

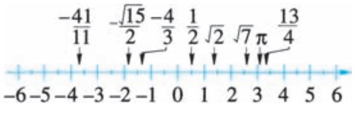

Figure 1.2

ABSOLUTE VALUES

Knowing the different kinds of numbers can be useful. However, what is more important is how the numbers relate to each other and how we can use them. One of the basic ideas is the distance of a number from zero or, on a number line, from the origin. The number 3 is three units from zero. So is the number -3. The number $-\frac{8}{5}$ is $\frac{8}{5}$ units from zero. The distance of a number from zero is called the **absolute value** of the number. The absolute value of a number is indicated by placing the number between vertical bars, $|\ |$.

EXAMPLE 1.4

The absolute value of 4 is 4, the absolute value of -9 is 9. These, and other absolute values, are written as

$$\begin{aligned} |4| &= 4 \\ |-9| &= 9 \\ \left|\frac{4}{3}\right| &= \frac{4}{3} \\ \left|-\frac{18}{7}\right| &= \frac{18}{7} \\ |-\sqrt{7}| &= \sqrt{7} \\ |0| &= 0 \end{aligned}$$

PLACE VALUE

Each position a digit can occupy has a **place value** equal to the base of the number system raised to a power. The place values and the place names most often used with the decimal numbers are shown in Figure 1.3. You might notice that as we move from right to left each place value is 10 times the value of the place to its right.

Place name		Ten thousands	Thousands	Hundreds	Tens	Units or ones		Tenths	Hundredths	Thousandths	Ten thousandths	
Place value		10^4	10^3	10^2	10^1	10^0	.	10^{-1}	10^{-2}	10^{-3}	10^{-4}	. . .
	. . .	10000	1000	100	10	1	.	0.1	0.01	0.001	0.0001	. . .

Decimal point

Figure 1.3

ROUNDING OFF NUMBERS TO A PLACE VALUE

To round off a number to a certain place value, examine the digit in the next place to the right and follow the rules in the following box.

ROUNDING NUMBERS

To round a number to a certain place value, examine the digit in the next place to the right. We will call this number the "first discarded digit" or the "test digit."

1. If the first discarded digit is less than 5 (0, 1, 2, 3, or 4), drop that digit and all the following digits. Replace any whole number places dropped with zeros.
2. If the first discarded digit is 6 or more (6, 7, 8, or 9) or a 5 followed by a nonzero digit, then add 1 to the digit in the place to which you are rounding. Drop all other following digits. Replace any whole number places dropped with zeros.

(Continues)

(Continued)

3. If the first discarded digit is a 5 followed by all zeros, then add 1 to the round-off digit if it is odd (1, 3, 5, 7, or 9) or retain the original round-off digit if it is even (0, 2, 4, 6, or 8).
4. If a number is negative, round the absolute value and then include the negative sign when you write the answer.

EXAMPLE 1.5

Round off 69,472 to the nearest hundred.

SOLUTION The digit 4 is in the hundreds place. The first discarded digit is the next digit to the right, a 7. This is more than 5, so the 4 is increased by 1 (to 5) and the 7 and any digits to its right are replaced with zeros. Thus, 69,472 rounded to the nearest hundred is 69,500.

EXAMPLE 1.6

Round off 278,491 to the nearest thousand.

SOLUTION The digit 8 is in the thousands place. The first discarded digit is 4. This is less than 5, so the 8 is not changed. The 4 and any digits to its right are replaced with zeros. Thus, 278,491 rounded to the nearest thousand is 278,000.

The third rule is often referred to as the **Round-to-the-Even** or the **Odd-Five Rule**. Rounding to the even is used to help reduce bias when several numbers are added. Examples 1.7 and 1.8 show how to use the "Round to the Even" rule.

EXAMPLE 1.7

Round off 4,392,500 to the nearest thousand.

SOLUTION The digit 2 is in the thousands place and is the round-off digit. The first discarded digit is 5 and all the numbers to its right are zeros. Since the round-off digit is even we keep it and replace the 5 with a zero. Thus, 4,392,500 rounded to the nearest thousand is 4,392,000.

EXAMPLE 1.8

Round off 34,750,000 to the nearest hundred thousand.

SOLUTION The digit 7 is in the hundred thousands place and is the round-off digit. The first discarded digit is 5 and all the numbers to its right are zeros. Since the round-off digit is odd we add 1 and replace the 7 with an 8. Thus, 34,750,000 rounded to the nearest hundred thousand is 34,800,000.

EXAMPLE 1.9

Round off −14,367 to the nearest hundred.

SOLUTION We use the rules for rounding numbers on the number 14,367, the absolute value of −14,367. To the nearest hundred this is 14,400. So, −14,367 to the nearest hundred is −14,400.

EXAMPLE 1.10

Round off 83.427 to the nearest hundredth.

SOLUTION The digit 2 is in the hundredths place. The next digit to the right is a 7. This is more than 5, so the 2 is increased by 1 (to 3) and the 7 is dropped. Thus, 83.427 rounded to the nearest hundredth is 83.43.

EXAMPLE 1.11

Round 5.2348 to the nearest hundredth.

SOLUTION Here the digit in the hundredths place is a 3. The next digit to the right is a 4. Since 4 is less than 5, the 4 and all other digits to its right are dropped. So, 5.2348 rounded off to the nearest hundredth is 5.23.

PROPERTIES OF REAL NUMBERS

Knowing the different kinds of numbers is important, but it is even more important that you can use these numbers. In order to use them, there are some basic properties of the real numbers that you should know.

1. $a + b = b + a$	Commutative property for addition
2. $ab = ba$	Commutative property for multiplication
3. $(a + b) + c = a + (b + c)$	Associative property for addition
4. $(ab)c = a(bc)$	Associative property for multiplication
5. $a(b + c) = ab + ac$	Distributive property for multiplication over addition
6. $b + 0 = b$	Identity element for addition
7. $b \cdot 1 = b$	Identity element for multiplication
8. $-b$ since $b + (-b) = 0$	Inverse element for addition (additive inverse)
9. $\frac{1}{b}$ if $b \neq 0$ since $b \cdot \frac{1}{b} = 1$	Inverse element for multiplication

CAUTION Subtraction and division are **not** commutative. Thus,

$$7 - 4 \neq 4 - 7 \quad \text{and} \quad 12 \div 3 \neq 3 \div 12$$

CAUTION Subtraction and division are **not** associative. Thus,

$$15 - (7 - 3) \neq (15 - 7) - 3 \quad \text{and} \quad 18 \div (6 \div 3) \neq (18 \div 6) \div 3$$

ORDER OF OPERATIONS

Now, let's consider the order of operations. The order of operations is very important in mathematics. Mathematicians have made some basic agreements that some operations are to be performed before others. By these agreements, operations such as addition, subtraction, multiplication, division, and raising to powers are to be performed in the following order. (Remember, not all operations are in every problem.)

ORDER OF OPERATIONS

Perform the operations in a problem in the following order.

1. Operations inside parentheses or above or below a fraction bar. Always start with the innermost parentheses and work outward.
2. Raising to a power.
3. Multiplications and divisions in the order in which they appear from left to right.
4. Additions and subtractions in the order in which they appear from left to right.

HINT Some people use the acronym "**P**lease **E**xcuse **M**y **D**ear **A**unt **S**ally" to help remember the order of operations. Here the **P** in "Please" stands for parentheses, the **E** for exponents (raise to a power), **M** and **D** for multiplication and division, and the **A** in "Aunt" and **S** in "Sally" for addition and subtraction.

Computers and most scientific calculators are programmed to perform the operations according to the previous rules. The logic used by these calculators and computers is called **algebraic logic** or the **algebraic operating system**.

CHECKING: DOES YOUR CALCULATOR USE ALGEBRAIC LOGIC?

Work this problem on your calculator: $3 + 5 \times 7$. If your calculator gives the answer 38, then it uses algebraic logic.

If your calculator gives the answer 56, then it does not use algebraic logic.

If your calculator gives an answer other than 38 or 56, then you probably made a mistake. Try again.

EXAMPLE 1.12

We will consider the problem $6 + 10 \div 2$ in two different ways. Following the rules, we should do the division first:

$$6 + 10 \div 2 = 6 + 5 = 11$$

Notice that $6 + 10 \div 2$ is performed as if it were $6 + (10 \div 2)$. If you wanted to indicate that the addition should be performed first, then parentheses would have to be added:

$$(6 + 10) \div 2 = 16 \div 2 = 8$$

As you can see, the answers 8 and 11 are not the same. Since 11 is the correct answer, it shows the importance of performing the operations in the correct order.

NOTE $(6 + 10) \div 2$ is sometimes written as $\frac{6+10}{2}$. Here we perform the operation above the fraction bar (order of operation #1) before we perform the division. Thus, $\frac{6+10}{2} = \frac{16}{2} = 8$.

EXAMPLE 1.13

Solve $6 + 12 \div 4 \times 5 - 4 \times 3$

SOLUTION Orders of operations #1 and #2 do not apply because there are no parentheses, fractions, or exponents. According to order of operation #3, we should perform the multiplication and division in order from left to right. We begin by performing the multiplication and the division operations in order from left to right. First we perform $12 \div 4$ and then multiply this answer by 5 as shown.

$$\begin{aligned} 6 + 12 \div 4 \times 5 - 4 \times 3 &= 6 + 3 \times 5 - 4 \times 3 \\ &= 6 + 15 - 4 \times 3 \\ &= 6 + 15 - 12 \\ &= 21 - 12 \\ &= 9 \end{aligned}$$

EXAMPLE 1.14

Solve $28 - (26 - (3 - (4 - 3)))$

SOLUTION We begin with the innermost set of parentheses and perform the operations inside of them. So, we begin by working $4 - 3$.

$$\begin{aligned} 28 - (26 - (3 - (4 - 3))) &= 28 - (26 - (3 - 1)) \\ &= 28 - (26 - 2) \\ &= 28 - 24 \\ &= 4 \end{aligned}$$

Notice that at each step we performed the operations in the innermost parentheses.

Sometimes different grouping symbols are used to help a person see symbols that go together. Example 1.12 might have been written as follows:

$$28 - \{26 - [3 - (4 - 3)]\}$$

You would first work the problem inside the parentheses ( ), then inside the brackets [ ], and finally, inside the braces { }.

OPERATIONS WITH ZERO

Because operations with zero can cause problems, we will review them next.

If b is a real number, the operations of addition, subtraction, multiplication, and division with zero are as follows:

$$b + 0 = b$$
$$b - 0 = b \quad \text{but} \quad 0 - b = -b$$
$$b \times 0 = 0$$
$$0 \div b = \tfrac{0}{b} = 0 \quad \text{if} \quad b \neq 0$$
$$b \div 0 \quad \text{This is not defined. You cannot divide by 0.}$$

HINT To enter a negative number on a calculator you need to press a special key. On some calculators it is the +/− key and it is pressed *after* the number is entered. Thus, −25 would be entered as 25 +/− . Some calculators use a (−) key and it is pressed *before* the number is entered. Here, −35 would be entered as (−) 35.

EXAMPLE 1.15

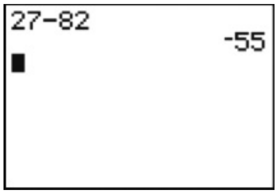

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Figure 1.4a

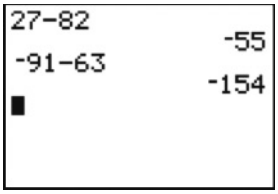

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Figure 1.4b

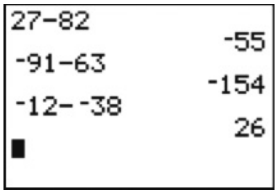

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Figure 1.4c

Use a calculator to compute each of the following: **(a)** 27 − 82, **(b)** −91 − 63, and **(c)** − 12 − −38.

SOLUTIONS The actual keystrokes given below are for a TI-83/84 graphics calculator. However, any graphing calculator that does not use reverse polish notation (RPN) would work in the same way.

(a) Press 27 − 82 ENTER. The result, −55, is shown in Figure 1.4a.

(b) Press (−) 91 − 63 ENTER. The result, −154, is shown in Figure 1.4b. When you look at Figure 1.4b, you will see that the result from (a) is still displayed. This will continue to happen until you press the CLEAR key. Notice that you had to press the (−) key for the negative sign of −91. You pressed the − key to subtract 63.

(c) Press (−) 12 − (−) 38 ENTER. The result, 26, is shown in Figure 1.4c. Notice that you had to press the − key for subtraction and the (−) key for the negative signs on −12 and −38.

SPREADSHEETS

Many employers expect their workers to be able to use spreadsheets. In fact, you may have more occasion to use a spreadsheet than a calculator. We will give you an elementary introduction to the use of a spreadsheet to perform many mathematical tasks. However, for more details you need to consult the Help section of your program or a manual for that program.

A spreadsheet is a computer application that simulates a paper accounting worksheet. It displays multiple cells in a grid formed by rows and columns. Each cell in the grid contains text, numeric values, or formulas. A formula defines how the content of that cell is to be calculated from the contents of any other cell (or combination of cells) each time any cell is updated.

The most popular spreadsheet is Microsoft's Excel, which works on both Macintosh and PC computers. Numbers is Apple Inc.'s spreadsheet software, and is part of iWork. OpenOffice.org Calc is a freely available, open-source program modeled after Microsoft Excel. Web-based online spreadsheets, such as Google Spreadsheets, are also available. The illustrations in this text were all made using Excel.

Before doing Example 1.16, we need to go over some basic terms for spreadsheets. Consider Figure 1.5. The spreadsheet is made up of columns (A, B, C, . . .) and rows (1, 2, 3, . . .). The intersection of a column and row is a cell. Cell A1 is identified in the figure. A row above the column headings includes the "Name Box" that identifies the cell that is highlighted, in this case A1, and the "Formula Bar" that displays the contents of the active cell. The row contained in the Formula Bar can be moved by placing the cursor below the dot at the far left of that row and dragging it to the desired location. Future illustrations will show the Formula Bar just above the row with the column headings.

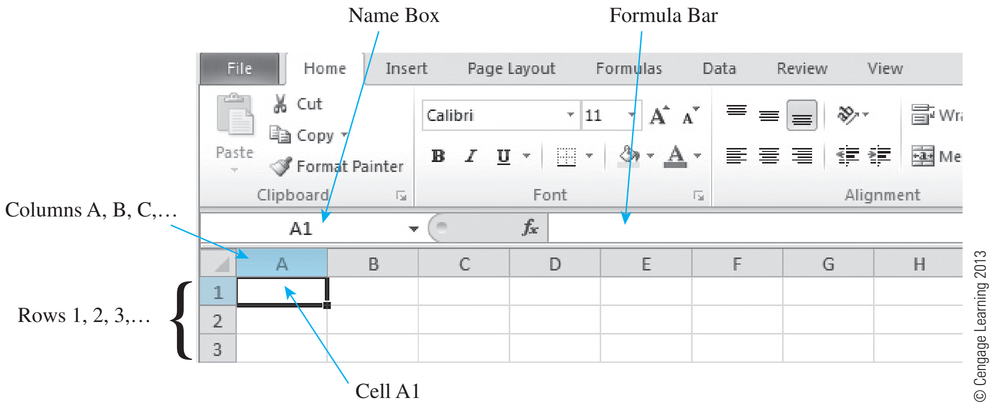

Figure 1.5

EXAMPLE 1.16

Use a spreadsheet to compute each of the following: **(a)** 27 − 82, **(b)** −91 − 63, and **(c)** −12 − −38.

SOLUTIONS An *equal sign* is used to distinguish between text and expressions in a spreadsheet cell. To perform a calculation, start with an equal sign and enter the expression.

(a) Figure 1.6a shows how to calculate 27 − 82. Enter "=27-82" in Cell A1. When you have finished typing the expression, hit ENTER or RETURN and the result will be displayed as shown in Row 1 in Figure 1.6b.

(b) Look in Cell A2 or the Formula Bar of Figure 1.6b to see how to calculate 91 − 63. Don't forget to begin with an = sign. The result is shown in Row 2 of Figure 1.6c.

(c) Row 3 in Figure 1.6c shows how to calculate −12 − −38. Notice that you had to put parentheses around (−38). The result of 26 is shown in Row 3 of Figure 1.6d.

EXAMPLE 1.16 (Cont.)

Figure 1.6e shows an alternative procedure, one you probably would not use for a simple calculation. However, we show it here to begin to understand how spreadsheet cells work.

Enter the first number in each subtraction problem in Column A and the second number in Column B. The formula "= A1 − B1" is entered in cell C1, "= A2 − B2" in C2, and "= A3 − B3" in C3. The results are the same. Cell names in spreadsheets are used as variables.

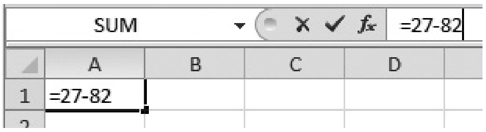

Figure 1.6a

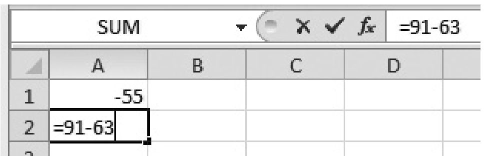

Figure 1.6b

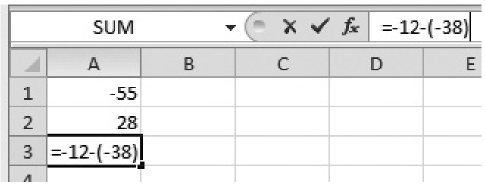

Figure 1.6c

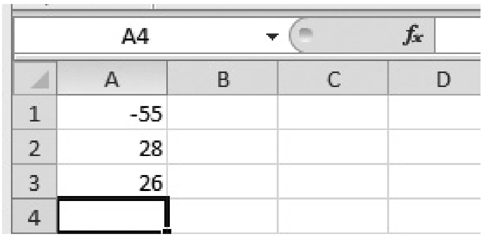

Figure 1.6d

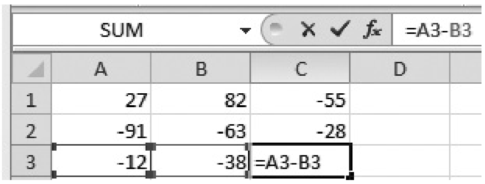

Figure 1.6e

ABSOLUTE VALUE SIGNS

Absolute value signs also act as grouping symbols and should be treated in the same manner as parentheses in the order of operations.

EXAMPLE 1.17

Use your calculator to evaluate $|3 + 4(2.6 - 12.9)|$.

SOLUTION We will show how to do this with three different calculators: The T1-83 or TI-84, TI-86, and TI-89.

Using a TI-83 or TI-84 Calculator

The absolute value function is found by pressing MATH ▶ ENTER. The screen should look something like the one in Figure 1.7a. Now press 3 + 4 (2.6 − 12.9) ENTER. The result is 38.2 as shown in Figure 1.7b.

Figure 1.7a

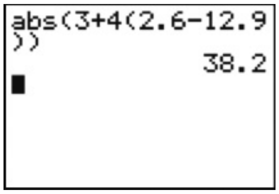

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Figure 1.7b

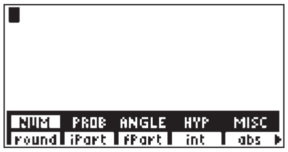

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Figure 1.7c

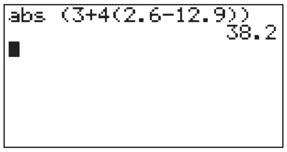

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Figure 1.7d

Using a TI-86 Calculator

To access the absolute value function, press 2nd MATH F1 [NUM]. The screen should look like the one in Figure 1.7c. The last entry in the bottom line of the screen is abs. Press F5 and abs will be displayed on the screen. Next, press (3 + 4 (2.6 − 12.9)) ENTER. The result, 38.2, is shown in Figure 1.7d.

Using a TI-89 Calculator

There are two ways you can access the absolute value function.

1. One method is to type 2nd ALPHA A B S (. The result should look like the screen in Figure 1.7e.
2. The second method is to press CATALOG. If the screen looks like the one in Figure 1.7f, then press ENTER. If, after you press CATALOG, the screen does not look like the one in Figure 1.7f, press the = [A] key and you should get a screen like the one in Figure 1.7f. Now, press ENTER. The screen should now look like the one in Figure 1.7e.

   Now, press 3 + 4 (2.6 − 12.9)) ENTER. The result, 38.2, is shown in Figure 1.7g.

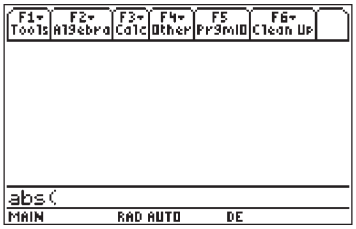

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Figure 1.7e

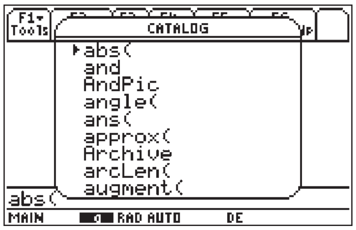

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Figure 1.7f

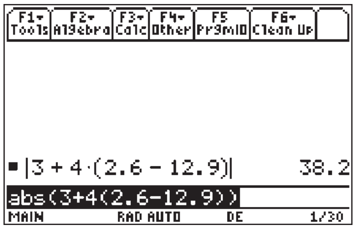

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Figure 1.7g

EXAMPLE 1.18

Use your spreadsheet to evaluate $|3 + 4(2.6 - 12.9)|$.

SOLUTION When entering an expression in a spreadsheet, enter all the operations. You have to be careful to require that the order of operations is followed. An asterisk (*) is used for multiplication. Some procedures, like the absolute value, are built-in functions and require a special set of keystrokes. In some versions of Excel, built-in functions are found by clicking on the "Paste Function" icon (see Figure 1.8a). In other versions use the "Formula Builder" as shown in Figure 1.8b. A third method is shown in Figure 1.8c. Here you first click on "Functions," and then "Math & Trig."

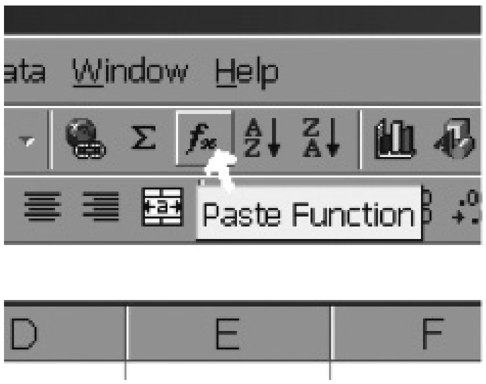

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Figure 1.8a

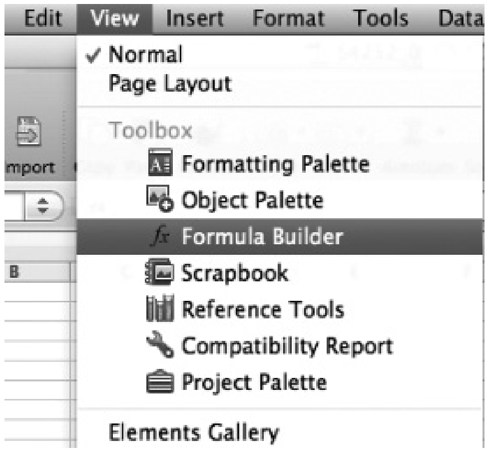

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Figure 1.8b

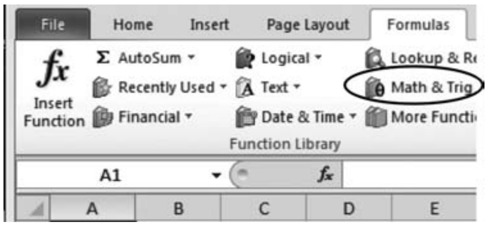

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Figure 1.8c

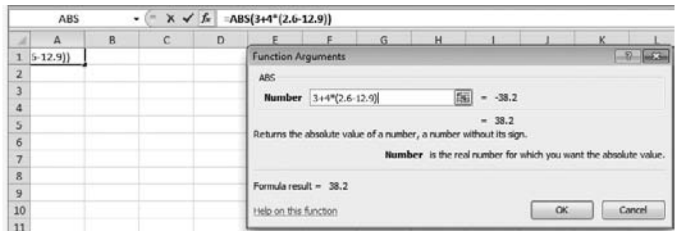

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Figure 1.8d

Either scroll down to Math & Trig where the absolute value function (ABS) is the first one on the list or type "abs" where it says "Search for a function." Double click on ABS and you should see "=ABS()" appear in the Formula Bar. There will also be a white dialogue box (rectangle) in the Formula Builder. Enter the rest of the expression in the dialogue box (the white rectangle) as shown in Figure 1.8d. Notice that what you key also appears in the Formula Bar. Be careful to insert a multiplication sign (*) for the multiplication that is understood between the 4 and the parentheses.

When you become more familiar with the built-in functions, you can simply enter them rather than going through the dialogue box. Simply enter "ABS $(3 + 4 * (2.6 - 12.9))$" in the cell. (Actually, you should enter $= \text{ABS}(3 + 4 * (2.6 - 12.9))$. We omitted the $=$ sign from the beginning of the entry. We will continue to leave it out of the instructions, but you should not leave it out when you key in the entry.) When you press ENTER or RETURN you should obtain the result 38.2.

Each number (except zero) has an inverse for multiplication. The **multiplicative inverse** of a number is the number that will give a product of 1 when multiplied by the original number. So, the multiplicative inverse of 5 is $\frac{1}{5}$ because $5 \times \frac{1}{5} = 1$. (This means that the multiplicative inverse of $\frac{1}{5}$ is 5.) The multiplicative inverse of $\frac{-3}{8}$ is $\frac{8}{-3}$ because $\frac{-3}{8} \times \frac{8}{-3} = 1$ and the multiplicative inverse of $-\frac{\sqrt{2}}{3}$ is $-\frac{3}{\sqrt{2}}$ because $-\frac{\sqrt{2}}{3} \times \left(-\frac{3}{\sqrt{2}}\right) = 1$.

You might have already noticed that the multiplicative inverse is the reciprocal of a number. This is because division is the inverse operation of multiplication.

INEQUALITIES

Some other important ideas include how numbers relate to each other. The relationship most often used is equality, or when two different numerals represent the same amount. Since $\frac{6}{3}$ and 2 both represent the same amount, we say they are equal, use the equal sign ($=$), and indicate this by writing $\frac{6}{3} = 2$.

Other relationships are concerned with numbers that are **not equal to** each other. To indicate that two numbers do not represent the same amount we use the "is not equal to" symbol, $\neq$. Thus, $3 \neq 4$ and $-5 \neq 17$ are two examples showing how to use the "is not equal to" symbol.

The $\neq$ sign is one example of a **sign of inequality**. Other signs of inequality are often used to indicate which of two numbers is the larger. A number is larger than another if it is located farther to the right on the number line. In this case we say that the number that is farther to the right on the number line **is greater than** the other number. The symbol $>$ is used to indicate that one number is greater than another. Another definition says that when the second number is subtracted from the first and the answer is positive, then the first number is greater than the second.

EXAMPLE 1.19

$7 > 5$, because $7 - 5 = 2$, a positive number
$4 > -2$, because $4 - (-2) = 6$, a positive number
$-3 > -10$, because $-3 - (-10) = 7$, a positive number

If the first number is to the left of the second number on the number line, or the answer is negative when the second number is subtracted from the first, then the first number **is less than** the second. We use the symbol $<$.

EXAMPLE 1.20

$5 < 7$, because $5 - 7 = -2$, a negative number
$-2 < 4$, because $-2 - 4 = -6$, a negative number
$-10 < -3$, because $-10 - (-3) = -7$, a negative number

There are two other inequality symbols. The symbol $\leq$ means "is less than or equal to" and the symbol $\geq$ indicates "is greater than or equal to."

EXAMPLE 1.21

$5 \geq 3$, because $5 > 3$; that is, $5 - 3 = 2$, a positive number
$3 \geq 3$, because $3 = 3$

EXAMPLE 1.22

Use $<$, $>$, and $=$ on each of the following pairs of numbers:

(a) 4 and 9.7
(b) -7 and -8.6
(c) 0 and -0.01
(d) -4.3 and -4.5
(e) $-2\frac{1}{4}$ and -2.25
(f) $\frac{4}{3}$ and $1.\overline{3}$

EXAMPLE 1.22 (Cont.)

SOLUTION

(a) $4 < 9.7$ because $4 - 9.7 = -5.7$, a negative number.

(b) $-7 > -8.6$ because $-7-(-8.6) = 1.6$, a positive number.

(c) $0 > -0.01$ because $0 - (-0.01) = 0.01$, a positive number.

(d) $-4.3 > -4.5$ because $-4.3 - (-4.5) = 0.2$, a positive number.

(e) $-2\frac{1}{4} = -2.25$ because $-2\frac{1}{4} - (-2.25) = 0$.

(f) $\frac{4}{3} = 1.\overline{3}$ because $\frac{4}{3} - 1.\overline{3} = 0$.

The symbol $\approx$ means "is approximately equal to" and is used to indicate that two numbers are near each other on the number line.

Irrational numbers and nonterminating rational numbers are often represented by decimal approximations.

EXAMPLE 1.23

Give decimal approximations for the rational numbers $\frac{29}{9}, \frac{1}{3}, -\frac{41}{11}$, and $-4.3\overline{25}$ and the irrational numbers $\sqrt{2}$, π, and e.

SOLUTION

Rational numbers	Irrational numbers
$\frac{29}{9} \approx 3.22$	$\sqrt{2} \approx 1.414$
$\frac{1}{3} \approx 0.3$	$\pi \approx 3.1416$
$-\frac{41}{11} \approx -3.73$	$e \approx 2.71828$
$-4.3\overline{25} \approx -4.325$	

RECIPROCALS

Every number, except zero, has a **reciprocal**. The reciprocal of a number is equal to 1 divided by that number. If the number is a fraction, then the reciprocal is 1 divided by the fraction or $\frac{1}{\text{fraction}}$. Thus, the reciprocal of $\frac{4}{3}$ is $\frac{1}{\frac{4}{3}} = \frac{3}{4}$.

EXAMPLE 1.24

The reciprocal of 9 is $\frac{1}{9}$.

The reciprocal of $\frac{2}{3}$ is $\frac{1}{\frac{2}{3}} = \frac{3}{2}$.

The reciprocal of $\sqrt{7}$ is $\frac{1}{\sqrt{7}}$.

The reciprocal of $\frac{-5}{2}$ is $\frac{1}{\frac{-5}{2}} = \frac{-2}{5}$.

NOTE The product of a number and its reciprocal is 1.

In this section we have introduced some of the different sets of numbers, their properties, and ways to indicate the relationship between two numbers. In the next section we will examine basic operations with real numbers.

EXERCISE SET 1.1

In Exercises 1–4, indicate all the sets of numbers to which each number belongs. Remember, a number can belong to more than one set of numbers.

1. 15
2. $\frac{-2}{3}$
3. $\frac{\sqrt{-7}}{8}$
4. 0

Round off each number in Exercises 5–18 to the indicated place value.

5. 1752 (hundreds)
6. 2461 (tens)
7. -3792 (hundreds)
8. 54,500 (thousands)
9. 327,500 (thousands)
10. 432,750 (hundreds)
11. 432,750 (thousands)
12. 432,750 (ten thousands)
13. 24.368 (hundredths)
14. 0.054 (tenths)
15. 13.7436 (hundredths)
16. -4.250 (tenths)
17. $23\frac{3}{8}''$ (nearest inch)
18. $29\frac{7}{16}$ (units)

Evaluate each of the expressions in Exercises 19–22.

19. $|15|$
20. $\left|\frac{-\sqrt{7}}{8}\right|$
21. $|0|$
22. $|4-7|$

Locate each of the numbers in Exercises 23–26 on a number line.

23. $\frac{4}{7}$
24. 2.5
25. $-\frac{8}{3}$
26. $\frac{-\pi}{3}$

In Exercises 27–38, insert the correct sign of inequality ($<$ or $>$) between the given pairs of numbers.

27. 2 and 3
28. 5 and 3
29. -4 and 7
30. 9 and -7
31. -3 and -8
32. -15 and -7
33. $\frac{-2}{3}$ and $-\frac{1}{2}$
34. $-\frac{7}{2}$ and $-\frac{8}{3}$
35. 0.7 and 0.5
36. -2.1 and -2.0
37. $|2.3|$ and $|-4.1|$
38. $|-5.1|$ and $|3.7|$

In Exercises 39–50, determine which of the basic laws of real numbers is being used.

39. $4+3=3+4$
40. $8(4\cdot 6)=(8\cdot 4)6$
41. $(9\cdot 7)=(7\cdot 9)$
42. $4(3+5)=(4\cdot 3)+(4\cdot 5)$
43. $(9+3)+6=9+(3+6)$
44. $(\pi+\sqrt{2})\sqrt{5}=\pi\sqrt{5}+\sqrt{2}\sqrt{5}$
45. $1\times 81=81$
46. $\frac{3}{16}\times\frac{16}{3}=1$
47. $-5+5=0$
48. $19-5=-5+19$
49. $(\sqrt{3}\cdot\sqrt{4})\sqrt{5}=\sqrt{3}(\sqrt{4}\cdot\sqrt{5})$
50. $16=0+16$

Name the additive inverse of each of the numbers in Exercises 51–54.

51. 91 **52.** -8 **53.** $\sqrt{2}$ **54.** $\frac{1}{3}$

Name the multiplicative inverse of each of the numbers in Exercises 55–58.

55. $\frac{1}{2}$ **56.** -5 **57.** $\dfrac{\sqrt{2}}{2}$ **58.** $-\dfrac{3}{7}$

Find the reciprocals of the numbers in Exercises 59–62.

59. -5 **60.** $\dfrac{1}{2}$ **61.** $\dfrac{17}{3}$ **62.** $\dfrac{-2}{\pi}$

Use the correct order of operations to solve Exercises 63–72.

63. $16 - 8 \div 4$

64. $16 \div 8 + 2$

65. $24 + 3 - 10 \div 5 \times 8 + 2$

66. $13 \times 7 - 26 \div 5 + 5$

67. $(7 - 2) - (3 + 8 - 7)$

68. $7 - 2 - 3 + 8 - 7$

69. $7 \times 3 + 5 \times 2$

70. $6 \times 4 - 3 \times 5$

71. $\{-[5 - (8 - 4) + (3 - 7)] - (4 - 2)\}$

72. $(14 + 3(8 - 6) + 4(9 - 5))$

Use your calculator to work Exercises 73–84. Be sure to include all parentheses.

73. $-39 - 72$

74. $-56 - -344$

75. **(a)** $243 + (-691 + 89)$
(b) $(243 + -691) + 89$

76. **(a)** $-41(63 + -177)$
(b) $-41 \times 63 + -41 \times -177$

77. **(a)** $(-342 + -18)91$
(b) $-342 \times 91 + -18 \times 91$

78. **(a)** $\frac{3}{7} \times \frac{7}{3}$ **(b)** $-\frac{4}{11} + \frac{4}{11}$

79. $|17.3 - 42.6|$

80. $|42.6 - 17.3|$

81. $-4.3\,|5.7 - 9.3|$

82. $-41\,|6.3 - 17.7|$

83. $||249 - 641| - |6.4 - 12.2||$

84. $|4| - 35.2 - 12.9| - 2.1|16.3 - 68.1||$

Solve Exercises 85–104.

85. List the following numbers in numerical order from smallest to largest: $-5, -\frac{2}{3}, |-8|, \frac{16}{3}, -|4|, \frac{-1}{3}$

86. List the following numbers in numerical order from smallest to largest: $\pi, \frac{22}{7}, -\sqrt{2}, -\sqrt{7}, \sqrt{7} - \sqrt{2}, \frac{7}{5}, -|\frac{7}{4}|$

87. ***Automotive technology*** Front-end shims come in $\frac{1}{32}$-, $\frac{1}{16}$-, and $\frac{1}{8}$-in. thicknesses.
(a) Which is the thinnest shim?
(b) Which is the thickest shim?
(c) What is the decimal size of the $\frac{1}{16}$-in. shim?

88. ***Automotive technology*** The service manager records how long each technician has worked on a job in hours written as a decimal. If you spent $\frac{3}{4}$ h to change a front engine mount, $1\frac{1}{2}$ h to replace the engine oil pan, and $2\frac{1}{4}$ h to replace the clutch, how would the service manager record these times?

89. ***Electronics*** The voltage across an element with respect to the ground is -19.4 V initially and then it changes to -16.8 V. Determine the absolute value of the change in the voltage.

90. ***Electronics*** The voltage across an element with respect to the ground is -6.5 V initially, and then it changes to 14.7 V. Determine the absolute value of the change in the voltage.

91. ***Automotive technology*** Five technicians were given the same task. By noon, Sheila had completed $\frac{5}{7}$ of her task, José had completed $\frac{3}{5}$ of his, Lamar had completed $\frac{4}{9}$ of his, Hazel had completed $\frac{2}{3}$ of hers, and Robert had completed $\frac{9}{16}$ of his task. Rank the technicians in order according to who completed most of the assigned task, with the person who had finished the most listed first.

92. ***Electronics*** The voltage across an element with respect to the ground is 23.7 V initially and then it becomes -5.2 V. Determine the absolute value of the change in the voltage.

93. ***Automotive technology*** Helen drives past mile marker 157 on the interstate highway. In the next hour, she drives 64 mi. At what mile marker is she now? (Hint: There are two correct answers.)

94. ***Electronics*** As resistors warm, the value of their resistance increases. One resistor's value changed from 14 Ω to 19 Ω. Determine the absolute value of the change in the resistance.

95. ***Construction*** A bedroom measures $12'6\frac{5}{8}'' \times 14'3\frac{5}{16}''$. What are the dimensions of the room to the nearest foot?

96. ***Construction*** A living room measures 4.848 m × 5.362 m. What are the dimensions of the room to the nearest centimeter (a centimeter is a hundredth m)?

97. ***Business*** Eight identical boxes are each packed with 12 boxes of CD-RW media and 3 CD-RW drives.

(a) Name the property that demonstrates how you can determine the total number of items in the boxes.

(b) Write a mathematical expression that shows how to apply this property to this problem.

98. ***Business*** A technical support person for an appliance company averages 3.7 min when answering questions about washers and 2.1 min for questions about dryers. During one work period 21 questions were answered about washers and 37 were answered about dryers.

(a) Name the property that demonstrates that the same amount of time was spent answering questions about washers and answering questions about dryers.

(b) Write a mathematical expression that shows how to apply this property to this problem.

99. ***Transportation*** A trucker drives 257 mi before using a rest stop and another 92 mi after the stop. On the return trip the driver stops at the same rest stop and does not stop again until she gets back to her starting point.

(a) Name the property that demonstrates that the outbound distance is the same as the return distance.

(b) Write a mathematical expression that shows how to apply this property to this problem.

100. ***Transportation*** A trucker drives 135 mi before reaching his first stop and unloading some of the cargo. He then drives 73 mi to the next stop where some more cargo is unloaded. Next, he drives 49 mi to the last stop where the truck is unloaded and some new cargo is placed on the truck. On the return trip the driver stops at each of the prior stops and loads some more cargo. He does not make any other stops until he gets back to his starting point.

(a) Name the properties that demonstrate that the outbound distance is the same as the return distance.

(b) Write a mathematical expression that shows how to apply this property to this problem.

101. Five stamping machines in a manufacturing plant produce the same product. Each machine has a counter that records the number of parts produced. Table 1.1 shows the counter reading for the beginning and end of one week's production.

TABLE 1.1 Production for One Week

	Machine 1	Machine 2	Machine 3	Machine 4	Machine 5
Counter reading beginning of week	17,855	13,947	8,132	46,937	773
Counter reading end of week	56,782	42,538	34,641	87,462	32,561

(a) How many parts were produced during the week by each machine?

(b) What was the total production for the week?

102. ***Business*** An electrical contractor has 5000 m of NM wire in stock at the beginning of a job. At different times during this job, electricians remove the following lengths from stock: 325 m, 580 m, 260 m, and 98 m. When the job is completed, 127 m are left over and returned to stock. How many meters of wire are now in stock?

103. ***Business*** A gasoline dealer estimated that during the month of July (25 business days) an average of 6,500 gal of gasoline would be sold each day. During July, a total of 178,250 gal were actually sold. How many more gallons were sold than were estimated for the month?

104. ***Electricity*** Juan is an electrician. During the first 3 months of the year he used the following amounts of cable: January: 9,430 ft; February: 7,450 ft; and March: 4,873 ft.

(a) Round the amount for each month to the nearest hundred feet.

(b) Add your answers in (a) to find the approximate amount of cable used during these 3 months.

(c) Find the actual total amount of cable used during this 3-month period.

[IN YOUR WORDS]

105. What is the meaning of *absolute value*?

106. Explain why you might want the absolute value of an answer rather than the actual value.

107. On a sheet of paper write an exercise that requires the use of one or more rational numbers to solve the exercise. On the back of the paper write your solution. Share your exercise with a friend. Rewrite your exercise or solution to clarify places where your classmate had difficulty.

108. On a sheet of paper write an exercise that requires the use of absolute values to solve the exercise. Share your exercise with a friend. Rewrite your exercise to clarify places where your classmate had difficulty.

109. Explain how you know when to use the (–) key and when to use the – key on a calculator.

110. **(a)** What is the inverse element for addition of -76?

(b) What is the inverse element for multiplication of -37?

(c) Describe how you can tell if your answers in (a) and (b) are correct.

111. Lend your calculator to someone who does not know how to use it to determine an absolute value. Ask that person to push the buttons as you explain how to use your calculator to work Exercise 79. If he or she does not get 25.3 for an answer, determine what was wrong with your directions or with which buttons the person pressed.

112. Explain to someone how to use a spreadsheet to evaluate an absolute value, such as $3 - |4 + (2 - 9)|$. Watch as he or she follows your instructions on a computer and revise your directions. Write an explanation telling how you would change your instructions the next time you need to tell someone how to use a spreadsheet.

1.2 BASIC OPERATIONS WITH REAL NUMBERS

The ability to work with numbers is very important in mathematics. As you use this book you will be using calculators and/or computers to help solve problems. In many ways, these solving technologies make it more important that you have good arithmetic skills. Both the calculator and the computer will only give the correct answer if you correctly tell the machine what to do. This section provides a brief review of arithmetic with real numbers. The first group of rules will use integers in the examples. Later in this section, we will examine the arithmetic of rational numbers.

ADDITION OF INTEGERS

RULE 1

To add two real numbers with the same sign, add the numbers and give to the sum the sign of the original numbers.

EXAMPLE 1.25

$+8 + (+7)$

SOLUTION $8 + 7 = 15$

8 and 7 both have a $+$ sign.

So, $+8 + (+7) = +15$.

EXAMPLE 1.26

$-9 + (-24)$

SOLUTION $9 + 24 = 33$

9 and 24 both have a $-$ sign.

So, $-9 + (-24) = -33$.

RULE 2

To add two real numbers with different signs, take the absolute value of both numbers, subtract the smaller absolute value from the larger, and give the answer the sign of the number with the larger absolute value.

EXAMPLE 1.27

$-8 + (+5)$

SOLUTION $|-8| = 8$

$|+5| = 5$

$8 > 5$, so the answer will be negative.
$8 - 5 = 3$ and so, $-8 + (+5) = -3$.

EXAMPLE 1.28

$+14 + (-9)$

SOLUTION $|+14| = 14$

$|-9| = 9$

$14 > 9$, so the answer will be positive.
$14 - 9 = 5$ and so $+14 + (-9) = +5$.

NOTE Remember that positive numbers are often written without a + sign.

EXAMPLE 1.29

$-19 + (-18) + 23$

SOLUTION $-19 + (-18) + 23 = -37 + 23$, because $-19 + (-18) = -37$

$= -14$

APPLICATION GENERAL TECHNOLOGY

EXAMPLE 1.30

When you started your car one morning, the temperature was $-16°C$. Later you hear someone say that the temperature has gone up $24°C$. What is the latest temperature?

SOLUTION To find the new temperature, we add the increase to the morning temperature:

$$-16 + 24 = 8$$

The new temperature is $8°C$.

APPLICATION ELECTRONICS

EXAMPLE 1.31

Figure 1.9 shows a closed circuit with two voltage sources and three resistors. The sum of voltages in such a loop must be 0. Check the voltage drops given in the figure to determine if the sum is 0.

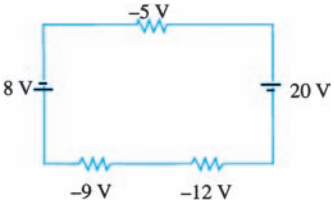

Figure 1.9

SOLUTION The voltage drops in Figure 1.9 are 8 V, -5 V, 20 V, -12 V, and -9 V. The sum of the voltage drops is

$$\begin{aligned} 8 + (-5) + 20 + (-12) + (-9) &= 3 + 20 + (-12) + (-9) \\ &= 23 + (-12) + (-9) \\ &= 11 + (-9) \\ &= 2 \end{aligned}$$

A sum of 2 V indicates an error, probably in one of the measurements.

SUBTRACTION OF INTEGERS

RULE 3

To subtract one real number from another, change the sign of the number being subtracted and then add according to Rule 2.

EXAMPLE 1.32

$(-6) - (-9)$

SOLUTION $(-6) - (-9) = (-6) + (+9)$ Change -9 to $+9$ and add.
$= 3$

EXAMPLE 1.33

$(-13) - (+7)$

SOLUTION $(-13) - (+7) = (-13) + (-7)$ Change $+7$ to -7 and add.
$= -20$

CAUTION Do not use the [–] (subtraction) key to enter a negative number into your calculator. The [–] key is used only for subtraction. A negative number is entered by using the [+/–] or [(–)] key.

EXAMPLE 1.34

$14 - 29$

SOLUTION $14 - 29 = 14 + (-29)$ Change 29 to -29 and add.
$= -15$

APPLICATION AUTOMOTIVE

EXAMPLE 1.35

When performing a front-end alignment, a technician must work with the included angle, the steering axis inclination, and the camber. To find the steering axis angle, you subtract the camber angle from the included angle. If the included angle is 5° and the camber angle is $-2°$, what is the steering axis?

SOLUTION Subtracting the camber angle from the included angle, we have

$$\begin{aligned} 5° - (-2°) &= 5° + 2° \\ &= 7° \end{aligned}$$

The steering axis angle is 7°.

MULTIPLICATION AND DIVISION OF INTEGERS

RULE 4

The product (or quotient) of two real numbers with the same sign is the product (or quotient) of their absolute values.

EXAMPLE 1.36

$(-8)(-9)$

SOLUTION
$$\begin{aligned} (-8)(-9) &= |-8| \times |-9| \\ &= 8 \times 9 = 72 \end{aligned}$$

EXAMPLE 1.37

$(-27) \div (-3)$

SOLUTION
$$\begin{aligned} (-27) \div (-3) &= |-27| \div |-3| \\ &= 27 \div 3 \\ &= 9 \end{aligned}$$

RULE 5

The product (or quotient) of two real numbers with different signs is the additive inverse of the product (or quotient) of their absolute values.

EXAMPLE 1.38

$(-8)(9)$

SOLUTION $|-8| = 8 \qquad |9| = 9 \qquad 8 \times 9 = 72$

The additive inverse of 72 is -72.

So, $(-8)(9) = -72$.

EXAMPLE 1.39

$81 \div -3$

SOLUTION $|81| = 81 \qquad |-3| = 3 \qquad 81 \div 3 = 27$

The additive inverse of 27 is -27 and so $81 \div (-3) = -27$.

APPLICATION BUSINESS

EXAMPLE 1.40

The total cost of a car, including finance charges, is \$9,216. This is to be repaid in 48 equal payments. How much is each payment?

SOLUTION To find the size of each payment, we need to divide the total cost by the number of payments.

$$9{,}216 \div 48 = 192$$

Each payment is \$192.

APPLICATION MECHANICAL

EXAMPLE 1.41

A metal plate contracts (shrinks) 0.2 mm for each degree below 60°F. If the plate is 42 mm wide at 60°F, what is its width at 35°F?

SOLUTION The temperature change from 60°F to 35°F is $60^\circ - 35^\circ = 25^\circ$F. A decrease, or shrinkage, of 0.2 mm can be written as -0.2. So, for a 25°F change in temperature, the total shrinkage is

$$-0.2 \times 25 = -5.0$$

So, the plate will shrink 5 mm and, at 35°F, its width is

$$42 - 5 \text{ mm} = 37 \text{ mm}$$

ARITHMETIC OF RATIONAL NUMBERS

We will now look at the arithmetic of rational numbers. Remember, the numerator of a rational number is the top number and the denominator is the bottom number. In the rational number $\frac{-5}{8}$, -5 is the numerator and 8 is the denominator.

Sometimes a rational number is written as a **mixed number** that combines an integer and a rational number. Examples of mixed numbers are $2\frac{1}{2}$ and $-4\frac{2}{3}$. Although it is not shown, the two parts of a mixed number are being added. Thus, $2\frac{1}{2}$ means $2 + \frac{1}{2}$ and $-4\frac{2}{3}$ means $-(4 + \frac{2}{3})$.

Every rational number can be written with no more than one negative sign. If a rational number is written with more than one negative sign, you can rewrite it by reducing the number of negative signs by two.

EXAMPLE 1.42

The following rational number has three negative signs: $-\frac{-3}{-8}$. You can delete any two of these and get $\frac{-3}{8}$ or $-\frac{3}{8}$ or $\frac{3}{-8}$ depending on which two negative signs you delete. All three of these are equivalent to the original number.

EXAMPLE 1.43

The following rational number has two negative signs: $-\frac{9}{-5}$. If you delete both you get $\frac{9}{5}$, which is equivalent to the original number.

RULE 6

To add (or subtract) two rational numbers, change both denominators to the same positive integer (the **common denominator**). Add (or subtract) the numerators, and place the result over the common denominator. In symbols, this is written as

$$\text{Addition:} \quad \frac{a}{b} + \frac{c}{d} = \frac{ad}{bd} + \frac{bc}{bd} = \frac{ad + bc}{bd}$$

$$\text{Subtraction:} \quad \frac{a}{b} - \frac{c}{d} = \frac{ad}{bd} - \frac{bc}{bd} = \frac{ad - bc}{bd}$$

EXAMPLE 1.44

$$\frac{2}{3} + \frac{-5}{6}$$

SOLUTION The denominators are 3 and 6. A common denominator of 3 and 6 is 6, so $\frac{2}{3} = \frac{4}{6}$. The problem then becomes $\frac{2}{3} + \frac{-5}{6} = \frac{4}{6} + \frac{-5}{6} = \frac{4+(-5)}{6} = \frac{-1}{6}$. So, $\frac{2}{3} + \frac{-5}{6} = \frac{-1}{6}$.

EXAMPLE 1.45

$$\frac{7}{-8} + \frac{-5}{6}$$

SOLUTION The denominators are -8 and 6. A common denominator of -8 and 6 is 24. So, $\frac{7}{-8} = \frac{-21}{24}$ and $\frac{-5}{6} = \frac{-20}{24}$. Thus,

$$\begin{aligned} \frac{7}{-8} + \frac{-5}{6} &= \frac{-21}{24} + \frac{-20}{24} \\ &= \frac{-21 + (-20)}{24} \\ &= \frac{-41}{24} \end{aligned}$$

NOTE Many common denominators of two numbers are possible. For instance, in the above example, $\frac{7}{-8} + \frac{-5}{6}$, some possible common denominators are -24, 48, -48, 72, and -72. We normally select the smallest positive common denominator because it makes computations easier.

EXAMPLE 1.46

$$\frac{3}{8} + \left(-1\frac{5}{16}\right)$$

SOLUTION A common denominator is 16. Thus, $\frac{3}{8} = \frac{6}{16}$ and $-1\frac{5}{16} = \frac{-21}{16}$. The example then becomes $\frac{6}{16} + \frac{-21}{16} = \frac{-15}{16}$.

EXAMPLE 1.47

$$\frac{-5}{7} - \frac{-8}{3}$$

SOLUTION This is a subtraction problem. We first change it to an addition problem using Rule 3: $\frac{-5}{7} - \frac{-8}{3} = \frac{-5}{7} + \frac{8}{3}$. A common denominator of 7 and 3 is 21: $\frac{-5}{7} = \frac{-15}{21}$ and $\frac{8}{3} = \frac{56}{21}$. Thus,

$$\begin{aligned}\frac{-5}{7} - \frac{-8}{3} &= \frac{-15}{21} + \frac{56}{21}\\ &= \frac{-15 + 56}{21}\\ &= \frac{41}{21}\end{aligned}$$

Technology can be used when operating with fractions. The results can be expressed in either fraction or decimal form.

EXAMPLE 1.48

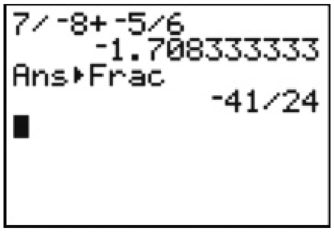

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Figure 1.10

Use your calculator to evaluate $\frac{7}{-8} + \frac{-5}{6}$.

SOLUTION On a TI-83/84 the answer will be given as a decimal and will have to be converted to a fraction. To do the computation press the following keys:

7 ÷ (–) 8 + (–) 5 ÷ 6

If you press ENTER you obtain the decimal value -1.708333333. To convert this decimal number to a fraction press MATH and either 1 or ENTER and then press ENTER. You should obtain the result shown in Figure 1.10.

You could have put all of this on one line rather than the two lines shown in Figure 1.9. However, this would not have given you the decimal approximation.

EXAMPLE 1.49

Use a spreadsheet to evaluate $\frac{7}{-8} + \frac{-5}{6}$.

SOLUTION The spreadsheet recognizes the order of operations so this statement can be entered in one cell.

Enter $= 7/-8 + -5/6$ in Cell A1 and hit ENTER or RETURN. The result, shown in Figure 1.11, is a decimal approximation to the actual result. In the next spreadsheet example, we will show how to get a fractional answer.

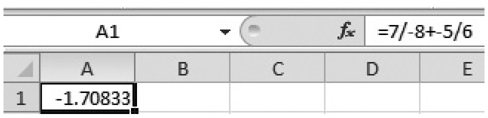

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Figure 1.11

EXAMPLE 1.50

Use your calculator to evaluate $\frac{3}{8} + \left(-1\frac{5}{16}\right)$.

SOLUTION Here $-1\frac{5}{16}$ is a mixed number. We need to remember that $-1\frac{5}{16}$ means $-(1 + \frac{5}{16})$ and use the parentheses when we enter the mixed fraction in the calculator. Figure 1.12 shows the results when using a TI-83 or TI-84.

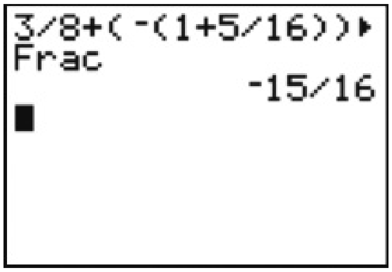

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Figure 1.12

EXAMPLE 1.51

Use a spreadsheet to evaluate $\frac{3}{8} + \left(-1\frac{5}{16}\right)$.

SOLUTION Here $-1\frac{5}{16}$ is a mixed fraction. We need to remember that $-1\frac{5}{16}$ means $-(1 + \frac{5}{16})$ and use the parentheses when we enter the mixed fraction in the cell. Figure 1.13a shows the result. (Notice the Formula Bar.) If the result is required to be written in the form of a fraction, then we must reformat the cell. With Cell A1 is highlighted, click on "General" and then scroll down to "More Number Formats" (see Figure 1.13b).

In the popup menu, click on "Fraction" and then choose the required number of digits—watching the "Sample" until it appears to be the correct display. (See Figure 1.13b.) Click the "OK" button in the lower right-hand corner of the Format Cells panel and the result should look like Figure 1.13c. From this, you can see that the the result is $-\frac{15}{16}$.

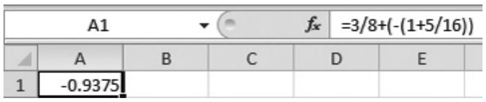

Figure 1.13a

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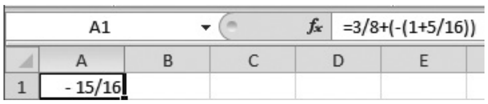

Figure 1.13c

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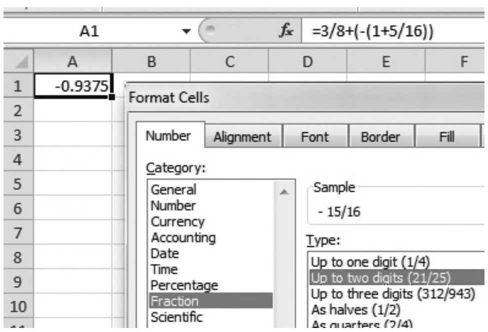

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Figure 1.13b

APPLICATION MECHANICAL

EXAMPLE 1.52

The circular pipe shown in Figure 1.14a has an inside radius of 1.35 cm and an outside radius of 1.575 cm. What is the thickness of the pipe?

SOLUTION An outline of the pipe is shown in Figure 1.14b. To find the thickness, we need to subtract the inner radius from the outer radius, or $1.575 - 1.35$.

$$\begin{array}{r} 1.575 \\ \underline{-1.35} \\ 0.225 \end{array}$$

The pipe is 0.225 cm thick.

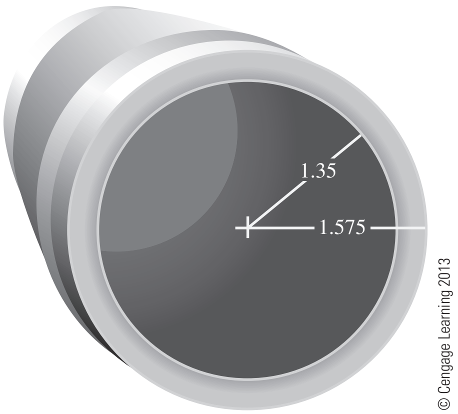

Figure 1.14a

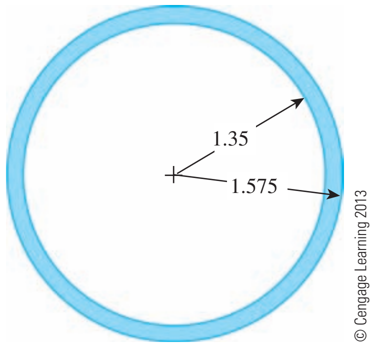

Figure 1.14b

APPLICATION MECHANICAL

EXAMPLE 1.53

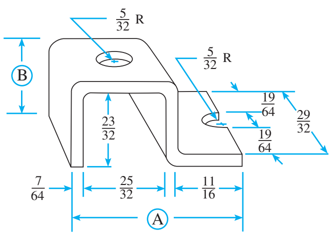

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Figure 1.15

Consider the machine part in Figure 1.15. What are the dimensions of the total length, marked Ⓐ, and the height, marked Ⓑ?

SOLUTION

Total Length, Ⓐ:

The length marked Ⓐ consists of four lengths that have to be added. Two of these lengths involve the $\frac{7}{64}$ thickness of the part. The other two lengths are $\frac{25}{32}$ and $\frac{11}{16}$. Thus, we see that

$$Ⓐ = \frac{7}{64} + \frac{25}{32} + \frac{7}{64} + \frac{11}{16}$$

Write each fraction with a common denominator of 64.

$$= \frac{7}{64} + \frac{50}{64} + \frac{7}{64} + \frac{44}{64}$$

EXAMPLE 1.53 (Cont.)

$$= \frac{7 + 50 + 7 + 44}{64}$$

$$= \frac{108}{64} = \frac{27}{16}$$

This can be written as the mixed number $1\frac{11}{16}$, so the total length of this part is either $\frac{27}{16}$ or $1\frac{11}{16}$.

Height, Ⓑ:

This length requires adding only two numbers: $\frac{7}{64}$ and $\frac{23}{32}$.

$$Ⓑ = \frac{7}{64} + \frac{23}{32}$$

$$= \frac{7}{64} + \frac{46}{64}$$

$$= \frac{7 + 46}{64} = \frac{53}{64}$$

The height of this part is $\frac{53}{64}$.

RULE 7

To multiply two rational numbers, multiply the numerators and multiply the denominators. In symbols, this is written as

$$\frac{a}{b} \times \frac{c}{d} = \frac{a \times c}{b \times d} = \frac{ac}{bd}$$

EXAMPLE 1.54

$$\frac{3}{4} \times \frac{-5}{8}$$

SOLUTION $\frac{3}{4} \times \frac{-5}{8} = \frac{3 \times (-5)}{4 \times 8} = \frac{-15}{32}$

EXAMPLE 1.55

$$\left(\frac{-7}{8}\right)(-9)$$

SOLUTION $\left(\frac{-7}{8}\right)(-9) = \frac{-7}{8} \times \frac{-9}{1} = \frac{(-7)(-9)}{8 \times 1} = \frac{63}{8}$

HINT Sometimes it is possible to "cancel" factors that appear in the numerators and denominators *before* you multiply the numbers. For example, in $\frac{5}{12} \times \frac{8}{25} = \frac{5 \times 8}{12 \times 25}$, the numerator and denominator have a factor of 5×4 in common. Thus, we can write

$$\frac{5}{12} \times \frac{8}{25} = \frac{5 \times 8}{12 \times 25} = \frac{\cancel{5} \times \cancel{4} \times 2}{3 \times \cancel{4} \times \cancel{5} \times 5} = \frac{2}{3 \times 5} = \frac{2}{15}$$

APPLICATION BUSINESS

EXAMPLE 1.56

An auto shop charges \$56 for each hour technicians work on your car. How much will the shop charge if they work on it for $3\frac{1}{4}$ hours?

SOLUTION We need to multiply $56 \times 3\frac{1}{4}$. We begin by changing 56 and $3\frac{1}{4}$ to fractions. We write 56 as $\frac{56}{1}$ and $3\frac{1}{4}$ as $\frac{13}{4}$.

$$\begin{aligned} 56 \times 3\frac{1}{4} &= \frac{56}{1} \times \frac{13}{4} \\ &= \frac{56 \times 13}{1 \times 4} \qquad \text{or} \qquad \frac{\cancel{4} \times 14 \times 13}{1 \times \cancel{4}} = \frac{14 \times 13}{1} \\ &= \frac{728}{4} = 182 \qquad\qquad\qquad\qquad = 182 \end{aligned}$$

The shop will charge \$182.

CAUTION Be careful! $56 \times 3\frac{1}{4}$ does not mean $56 \times 3 \times \frac{1}{4} = 168 \times \frac{1}{4} = \frac{168}{4}$. Remember, $3\frac{1}{4}$ is a short way of writing $3 + \frac{1}{4}$ and $56 \times 3\frac{1}{4}$ means $56 \times (3 + \frac{1}{4})$. Look back at what we did with $-1\frac{5}{16}$ in Example 1.50.

APPLICATION ENVIRONMENTAL SCIENCE

EXAMPLE 1.57

To change 14°F to its equivalent temperature in degrees Celsius, you need to compute $\frac{5}{9}(14° - 32°)$. What is this temperature?

SOLUTION

$$\begin{aligned} \frac{5}{9}(14° - 32°) &= \frac{5}{9}(-18°) \\ &= \frac{5}{9} \times \frac{-18}{1} \\ &= \frac{5 \times (-18)}{9 \times 1} \qquad \text{or} \qquad \frac{5 \times (-2) \times \cancel{9}}{\cancel{9} \times 1} = \frac{-10}{1} \\ &= \frac{-90}{9} = -10 \qquad\qquad\qquad\qquad = -10 \end{aligned}$$

So, −10°C is the same as 14°F.

RULE 8

To divide one rational number by another, multiply the first number by the reciprocal of the second. In symbols, this is written as

$$\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \times \frac{d}{c} = \frac{ad}{bc}$$

EXAMPLE 1.58

$$\frac{-5}{8} \div \frac{2}{3}$$

SOLUTION The reciprocal of $\frac{2}{3}$ is $\frac{3}{2}$ so

$$\begin{aligned}\frac{-5}{8} \div \frac{2}{3} &= \frac{-5}{8} \times \frac{3}{2} \\ &= \frac{(-5)(3)}{8 \times 2} \\ &= \frac{-15}{16}\end{aligned}$$

EXAMPLE 1.59

$$\frac{-7}{8} \div \frac{-9}{5}$$

SOLUTION The reciprocal of $\frac{-9}{5}$ is $\frac{5}{-9}$ (or $\frac{-5}{9}$) so

$$\begin{aligned}\frac{-7}{8} \div \frac{-9}{5} &= \frac{-7}{8} \times \frac{-5}{9} \\ &= \frac{(-7)(-5)}{8 \times 9} \\ &= \frac{35}{72}\end{aligned}$$

CAUTION When using technology you have to remember the order of operations and often need to insert parentheses so that the calculator or computer will get the correct result.

EXAMPLE 1.60

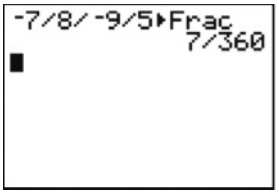

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Figure 1.16a

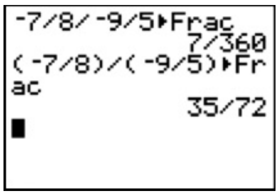

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Figure 1.16b

Use a calculator to work $\frac{-7}{8} \div \frac{-9}{5}$.

SOLUTION This is the same as Example 1.59 so we know that the answer should be $\frac{35}{72}$. In Figure 1.16a we typed the expression just as it appears in the statement of the example. When we pressed the ENTER key we saw that the calculator got the wrong answer of $\frac{7}{360}$. This is because the calculator first computed $\frac{-7}{8} \div -9$ and the answer was then divided by 5. This is the wrong procedure!

To avoid errors such as this you need to place parentheses around the divisor and the dividend. It is often a good idea to use your pencil to add parentheses so that the problem looks like $\left(\frac{-7}{8}\right) \div \left(\frac{-9}{5}\right)$. When parentheses are used we obtain the correct answer of $\frac{35}{72}$ as shown in Figure 1.16b.

EXAMPLE 1.61

Use a spreadsheet to work $\frac{-7}{8} \div \frac{-9}{5}$.

SOLUTION This is the same as Examples 1.59 and 1.60 so we know that the answer should be $\frac{35}{72}$.

The order of operations is crucial here. The problem is to divide the result of a division by the result of another division. Parentheses will be used to ensure that happens. Enter "$= (-7/9)/(-9/5)$" in Cell A1. Change the result to a fraction and it should match what is shown in Figure 1.17.

What would have happened if you had not used parentheses? Using the order of operations, the calculations are -7 divided by 8, that result is then divided by -9, and then that result is divided by 5. The result would be $\frac{7}{360}$.

This wrong answer helps to show that it is important to estimate what you expect the answer to be—even when using a computer or calculator. In this case we are dividing a number close to negative one by a number close to negative two. The result should be about one-half not 7/360, which is around 0.02.

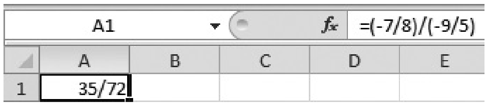

Figure 1.17

APPLICATION AUTOMOTIVE

EXAMPLE 1.62

A tank of fuel weighs $185\frac{1}{6}$ pounds. If a gallon of gasoline weighs $7\frac{1}{3}$ pounds, how many gallons are in the tank?

SOLUTION We need to divide the weight of the tank by the number of pounds for each gallon.

$$\begin{aligned} 185\frac{1}{6} \div 7\frac{1}{3} &= \frac{1{,}111}{6} \div \frac{22}{3} \\ &= \frac{1{,}111}{6} \times \frac{3}{22} \\ &= \frac{1{,}111 \times 3}{6 \times 22} \\ &= \frac{3{,}333}{132} \\ &= 25\frac{33}{132} = 25\frac{1}{4} \end{aligned}$$

The tank contains $25\frac{1}{4}$ gal of gasoline.

EXERCISE SET 1.2

Perform the indicated operation in Exercises 1–40.

1. $+27 + (+23)$
2. $8 + (-19)$
3. $27 + (-13)$
4. $-9 + (-8)$
5. $7 - 16$
6. $29 - (-8)$
7. $-8 - 16$
8. $-25 - (-13)$
9. $-37 - (-49)$
10. $(-2)(6)$
11. $(-3)(-5)$
12. $(7)(-8)$
13. $-38 \div 4$
14. $-45 \div (-9)$
15. $\frac{3}{4} + \frac{-5}{8}$
16. $-1\frac{3}{4} + \frac{-2}{3}$
17. $\frac{-9}{5} + \frac{7}{3}$
18. $\frac{-2}{3} + \frac{5}{6}$
19. $\frac{2}{5} + \frac{-1}{4}$
20. $\frac{-4}{5} + \frac{-5}{6}$
21. $\frac{3}{8} - \frac{-1}{4}$
22. $1\frac{1}{3} - \frac{-5}{6}$
23. $\frac{-9}{10} - \frac{2}{3}$
24. $-\frac{5}{16} - \frac{-3}{8}$
25. $\frac{5}{32} - \frac{1}{8}$
26. $-\frac{7}{3} - \frac{-6}{7}$
27. $\frac{-2}{3} \times \frac{4}{5}$
28. $\frac{3}{4} \times \frac{-5}{8}$
29. $\frac{-1}{8} \times \frac{-3}{4}$
30. $\frac{9}{16} \times \frac{1}{2}$
31. $-\frac{4}{3} \times \frac{5}{2}$
32. $\frac{-9}{5} \times \frac{-3}{8}$
33. $-\frac{3}{4} \div \frac{-5}{8}$
34. $1\frac{3}{4} \div \frac{-2}{3}$
35. $-\frac{3}{5} \div 4$
36. $-\frac{3}{8} \div \frac{1}{4}$
37. $-\frac{7}{5} \div \left(-\frac{5}{7}\right)$
38. $\frac{2}{3} \div \left(-\frac{7}{3}\right)$
39. $\left(-2 + \frac{2}{3}\right) \times \frac{-1}{2} - \left[\frac{3}{2} \div (-3)\right] + \frac{7}{3}$
40. $\frac{4}{3} \times \frac{-7}{8} \times \frac{3}{-5} \div \frac{4}{5} + \frac{-5}{8} - \frac{7}{8}$

Solve Exercises 41–70.

41. ***Recreation*** A ski resort received $17\frac{1}{2}$ inches of snow in December, $15\frac{7}{8}$ inches in January, $29\frac{3}{4}$ inches in February, and $15\frac{3}{8}$ inches in March. How much snow did the resort get during these four months?

42. ***Automotive technology*** The toe-in reading is $\frac{5}{32}$ on one front wheel and $\frac{3}{16}$ on the other. What is the total toe-in?

43. ***Electronics*** When two or more voltages are connected in series, the total voltage is the sum of the separate voltages. When the voltages are connected in the same direction, they are all considered to be positive. When the voltages are connected in the opposite direction, one direction is considered positive and the other negative. Find the total voltage of the batteries in Figure 1.18.

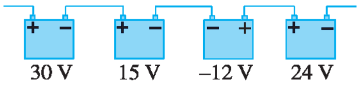

Figure 1.18

44. ***Construction*** A planer removed $\frac{1}{5}$ in. from a $1\frac{5}{8}$-in. board. Find the thickness of the finished board.

45. ***Recreation*** A $14\frac{1}{2}$-mi race has 3 checkpoints. The first checkpoint is $4\frac{5}{8}$ mi from the starting point. The second checkpoint is $3\frac{1}{5}$ mi from the first checkpoint. The third checkpoint is $3\frac{3}{8}$ mi from the second checkpoint.
    - **(a)** How many miles is it from the starting point to the second checkpoint?
    - **(b)** How many miles is it from the starting point to the third checkpoint?
    - **(c)** How many miles is it from the third checkpoint to the finish line?

46. ***Automotive technology*** An automobile traveled $427\frac{1}{5}$ mi on $13\frac{3}{4}$ gal of gasoline. How many miles did it get per gallon?

47. ***Electrical technician*** A certain job requires 37 pieces of electrical wire that are each 2 ft $3\frac{1}{2}$ in. long. What is the total length of wire that is needed?

48. ***Construction*** A 4-ft by 8-ft sheet of plywood is cut into strips, each $1\frac{5}{16}$ in. by 4 ft. The saw cut is $\frac{1}{8}$-in. wide. How many strips can be cut from the sheet?

49. ***Construction*** Find the length indicated by the question mark in Figure 1.19.

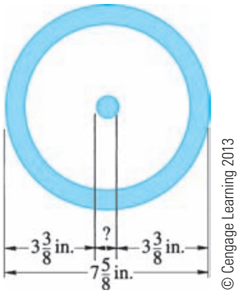

Figure 1.19

50. ***Electronics*** Determine the sum of the voltages in the closed circuit in Figure 1.20.

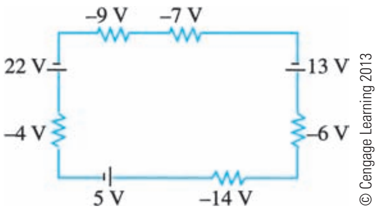

Figure 1.20

51. ***Electronics*** The voltage across an element with respect to the ground is −6.5 V initially, and then it changes to 14.7 V.
    (a) Determine the change in the voltage.
    (b) Explain the meaning of your answer in (a).
52. ***Electronics*** The voltage across an element with respect to the ground is 23.7 V initially, and then it changes to −5.2 V.
    (a) Determine the change in the voltage.
    (b) Explain the meaning of your answer in (a).
53. ***Construction*** A load of concrete is made by mixing 339 lb of aggregate, 159 lb of sand, 97 lb of cement, and 31 lb of water. What is the total weight of the mixture?
54. ***Construction*** Six pieces of wood were glued together. If each piece had a thickness of 1.625 in., what was the total thickness of the final piece of wood?
55. ***Electronics*** In a certain electrostatic field, the voltage at one point is 1,675 V, and the voltage at another point is −285 V. The potential difference between the two points is the absolute value of the difference between these two values. Determine the potential difference between the points.
56. ***Business*** A company showed a loss of $345,000 for the first half of the year and a profit of $268,700 for the second half of the year. What was the overall profit for the year?
57. ***Business*** A different company showed a profit of $2,973,400 for the first half of the year. If the overall profit for the year was $1,837,500, what was the profit for the second half of the year?
58. ***Business*** A company purchases 12 reams of paper at $11.95 per ream and 3 toner cartridges at $79.99 per cartridge. What was the total cost of the purchase before any sales tax?
59. ***Semiconductor technology*** An area in the Fab (Fabrication Department) has 8 tools, which process 11 different layers for the factory. Each time a wafer passes through the area it counts as one wafer. The area can produce 65,000 wafers a week at 100% tool availability and 100% tool utilization. How many wafers will the area have to run on each tool in order to maintain 65,000 wafers a week?
60. ***Semiconductor technology*** Suppose that the Fab (Fabrication Department) acquires another tool. If each tool can process 8,125 wafer a week, how many wafers does the area produce at 100% tool availability and 100% tool utilization?
61. ***Construction*** Use the drawing in Figure 1.21 to determine the rise and run of each step.

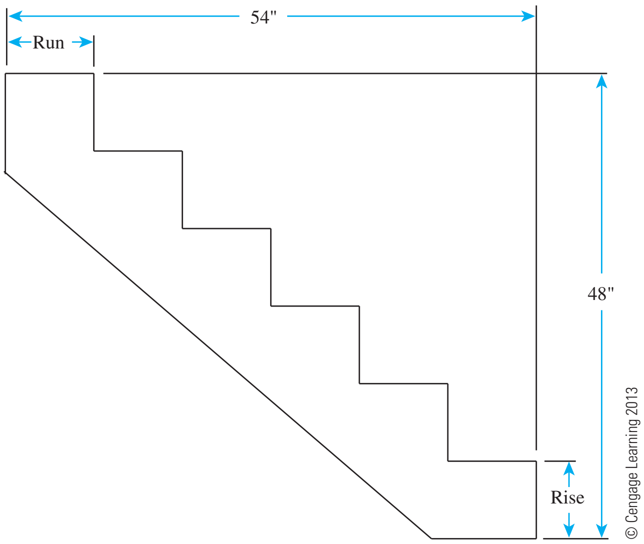

Figure 1.21

62. ***Manufacturing*** Determine the inside diameter of the bushing in Figure 1.22.

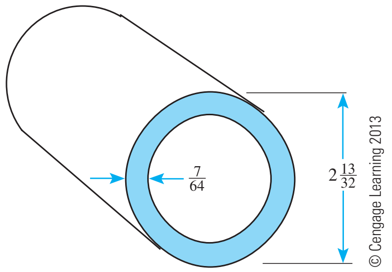

Figure 1.22

63. ***Manufacturing*** Determine the overall dimension, marked Ⓐ, of the tool in Figure 1.23.

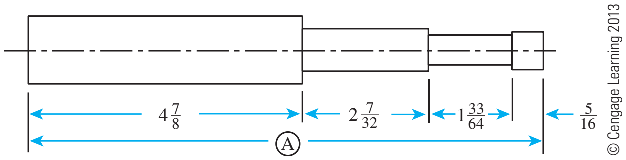

Figure 1.23

64. ***Manufacturing*** Determine each of the missing measurements marked Ⓐ and Ⓑ in Figure 1.24.

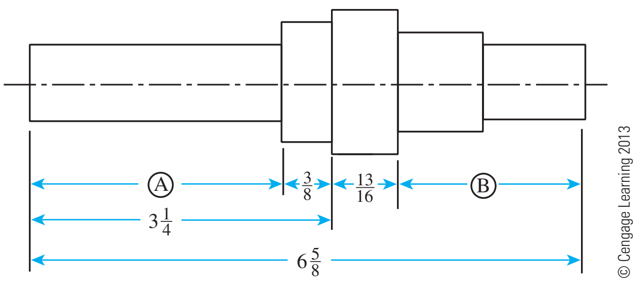

Figure 1.24

65. ***Manufacturing*** The manufacturing of a certain size denim jacket requires $3\frac{3}{8}$ yards of material including the selvage. How much material will be needed to make 90 jackets? Give (a) the total length of fabric needed and (b) the amount of fabric rounded to the nearest whole yard.

66. ***Manufacturing*** Determine the inside diameter of the bushing in Figure 1.25.

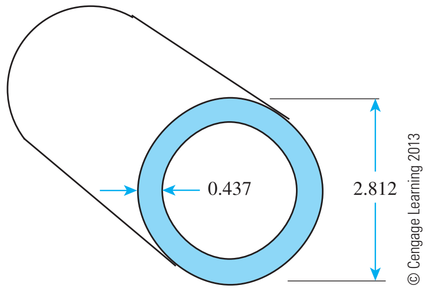

Figure 1.25

67. ***Manufacturing*** Determine each of the missing measurements marked Ⓐ, Ⓑ, and Ⓒ in Figure 1.26.

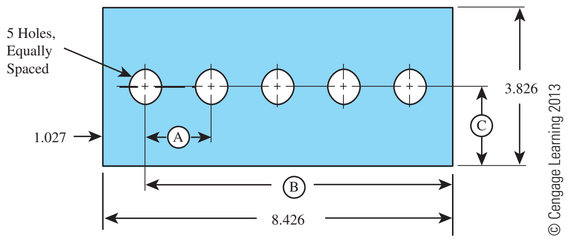

Figure 1.26

68. ***Manufacturing*** Determine each of the missing measurements marked Ⓐ, Ⓑ, Ⓒ, and Ⓓ in Figure 1.27.

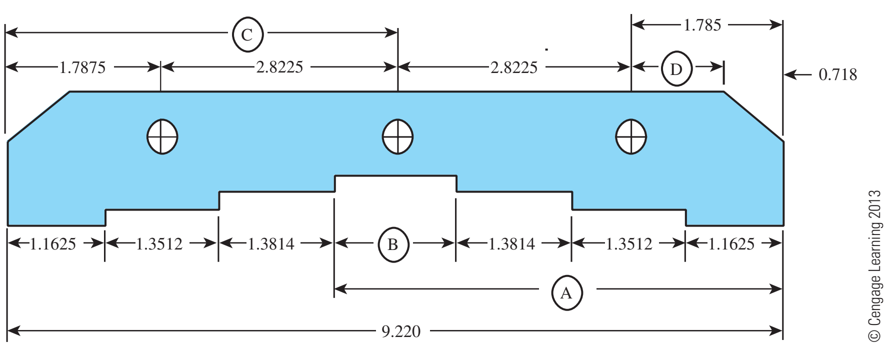

Figure 1.27

69. ***Construction*** Six wall receptacles are to be equally spaced along a wall. The first and last receptacles are to be 4 ft from each end of the wall. If the wall is $81'1\frac{1}{2}''$ long, determine the center-to-center distance between each receptacle.

70. ***Medical technology*** A bottle of a certain brand of children's medicine contains 24 teaspoons of liquid. If each dose for a 2-year-old child is $\frac{1}{2}$ teaspoon, how many doses are in this bottle?

71. ***Construction*** In finishing the interior trim of a building, a carpenter measures and saws a length of molding in three pieces as shown in Figure 1.28. Find, in inches, the length of the third piece.

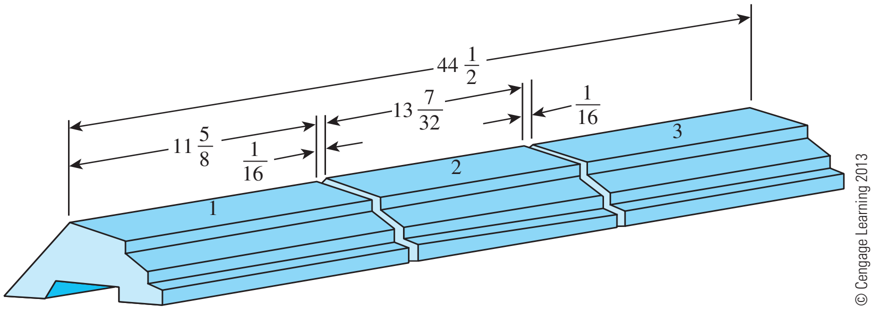

Figure 1.28

72. ***Sheet metal technology*** A sheet metal technician shears 5 pieces from a 48-in. length of sheet steel. The lengths of the sheared pieces are $5\frac{3}{8}$ in., $12\frac{1}{4}$ in., $8\frac{7}{32}$ in., $11\frac{9}{16}$ in., and $3\frac{5}{8}$ in. What is the length of sheet steel left after all pieces have been sheared?

73. ***Construction*** Dale was ordering trim to be installed around some replacement windows. Five of the windows measured $33\frac{3}{8}$ in. wide and $76\frac{1}{2}$ in. high, five measured $33\frac{3}{8}$ in. wide and $64\frac{1}{2}$ in. high, and one measured $66\frac{3}{4}$ in. wide and $76\frac{1}{2}$ in. high. The trim will only be placed along the sides and across the top of each window. Each cut removes $\frac{1}{8}$ in. of trim. The trim comes in $12\frac{1}{2}$-ft lengths. How many pieces of trim should be ordered?

74. ***Construction*** A brickmason estimates that the labor costs for a bricklaying job will take a total of 56 h. The job takes longer to complete than estimated. The hours worked each day were $7\frac{3}{4}$, $7\frac{1}{6}$, 8, $8\frac{3}{5}$, $9\frac{3}{8}$, $10\frac{1}{2}$, and $11\frac{2}{3}$. By how many hours was the job underestimated?

75. ***Sheet metal technology*** A sheet metal technician needs to cut twenty-five $3\frac{7}{16}$-in. lengths of band iron with $\frac{3}{32}$-in. of waste with each cut. The pieces are to be cut from a strip of band iron that is $117\frac{3}{4}$ in. long. How much stock is left over after all the pieces are cut?

76. ***Medical technology*** A nurse practitioner gives a patient $\frac{1}{2}$ tablet of morphine for pain. If one morphine tablet contains $\frac{1}{4}$ grain (gr) of morphine, how much morphine does the patient receive?

77. ***Medical technology*** A nurse practitioner gives a patient $2\frac{1}{2}$ tablets of ibuprofen. Each tablet contains 250 mg. How many mg of ibuprofen did the patient receive?

[IN YOUR WORDS]

78. Explain the significance of positive and negative signs when referring to "amount of change." How does this differ from the meaning of the absolute value of the amount of change?

79. Describe how to add two rational numbers. Show your description to a classmate and see if he or she understands what you wrote. Rewrite your description to clarify areas where your classmate had difficulty.

80. Describe how to multiply two rational numbers. Show your description to a classmate and see if he or she understands what you wrote. Rewrite your description to clarify any places where your classmate had difficulty.

1.3 EXPONENTS AND ROOTS

We have introduced the different types of numbers, the basic laws of arithmetic, and the basic operations with real numbers. In this section, we will learn some shorthand notation used in mathematics that will help us with the work in future sections.

EXPONENTS

It is often necessary to multiply a number, b, by itself several times. The notation b^n is used to indicate that a total of n bs are being multiplied. The number b is called the **base**, and n is called the **exponent**. We say that b^n is the "nth power of b" or "b to the nth power." When the exponent is 1, the 1 is often not written. Thus, $19^1 = 19$.

EXAMPLE 1.63

$$3^4 = 3 \times 3 \times 3 \times 3 = 81$$

The base is 3 and the exponent is 4. This is 3 to the fourth power or the fourth power of 3.

$$7^5 = 7 \times 7 \times 7 \times 7 \times 7 = 16{,}807$$

The base is 7 and the exponent is 5. This is 7 to the fifth power.

Notice how much easier it is to write 7^5 than 16,807.

$$9^2 = 9 \times 9 = 81$$

The base is 9, the exponent 2. This is the second power of 9 or, as it is more frequently called, 9 squared.

$$\left(\tfrac{5}{4}\right)^3 = \tfrac{5}{4} \times \tfrac{5}{4} \times \tfrac{5}{4} = \tfrac{125}{64}$$

The base is $\frac{5}{4}$, the exponent is 3. This is the third power of $\frac{5}{4}$ or $\frac{5}{4}$ cubed.

$$(-3)^4 = (-3)(-3)(-3)(-3) = 81$$

The base is -3, the exponent 4.

At the present time the exponent must be an integer. Later we will learn how to use and understand expressions where the exponent is any real number.

RULES OF EXPONENTS

As with all of the operations we have studied so far, there are some basic rules of exponents.

RULES OF EXPONENTS

1. $b^m b^n = b^{m+n}$
2. $(b^m)^n = b^{mn}$
3. $(ab)^n = a^n b^n$

4a. $\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}$, if $b \neq 0$

4b. $\left(\frac{1}{b}\right)^m = \frac{1}{b^m}$, if $b \neq 0$ — This is a version of 4a.

5. $\frac{b^m}{b^n} = b^{m-n}$, if $b \neq 0$
6. $b^0 = 1$, if $b \neq 0$
7. $b^{-n} = \frac{1}{b^n}$, if $b \neq 0$

EXAMPLE 1.64

Use the definition of exponents to show that, according to Rule 1, $3^5 \cdot 3^2 = 3^{5+2} = 3^7$.

SOLUTION

$$\begin{aligned} 3^5 \cdot 3^2 &= (3 \times 3 \times 3 \times 3 \times 3)(3 \times 3) \\ &= 3 \times 3 \times 3 \times 3 \times 3 \times 3 \times 3 \\ &= 3^7 \end{aligned}$$

EXAMPLE 1.65

Use Rule 1 to multiply $x^3 \cdot x^4 \cdot x$.

SOLUTION We begin by rewriting x as x^1 and then apply Rule 1.

$$x^3 \cdot x^4 \cdot x = x^3 \cdot x^4 \cdot x^1 = x^{3+4+1} = x^8$$

EXAMPLE 1.66

Use Rule 2 to expand $(x^4)^5$.

SOLUTION Rule 2 says that in order to take a power of a power you multiply the exponents. Using Rule 2, we obtain

$$(x^4)^5 = x^{4 \cdot 5} = x^{20}$$

EXAMPLE 1.67

Use Rule 3 to expand $(2 \cdot 5)^3$.

SOLUTION According to Rule 3, $(2 \cdot 5)^3 = 2^3 \cdot 5^3$.

EXAMPLE 1.68

Use Rule 3 to expand $(xy^2)^5$.

SOLUTION

$$\begin{aligned}(xy^2)^5 &= x^5(y^2)^5\\ &= x^5y^{(2\cdot 5)}\\ &= x^5y^{10}\end{aligned}$$

We see that $(xy^2)^5 = x^5y^{10}$.

Rule 4a says that to raise a quotient to a power, both the numerator and the denominator are raised to that power.

EXAMPLE 1.69

Use Rule 4a to expand $\left(\frac{2}{5}\right)^4$.

SOLUTION

$$\begin{aligned}\left(\frac{2}{5}\right)^4 &= \left(\frac{2}{5}\right)\left(\frac{2}{5}\right)\left(\frac{2}{5}\right)\left(\frac{2}{5}\right)\\ &= \frac{2\times 2\times 2\times 2}{5\times 5\times 5\times 5}\\ &= \frac{2^4}{5^4}\end{aligned}$$

EXAMPLE 1.70

According to Rule 4b, $\left(\frac{1}{3}\right)^4 = \frac{1^4}{3^4} = \frac{1}{3^4}$

Rule 5 states that to divide two powers with the same base, you subtract the exponent of the denominator from the exponent of the numerator.

There are really three cases to this rule depending on which is larger, m or n, or if they are the same. Let's look first at when $m > n$.

EXAMPLE 1.71

Apply Rule 5 to simplify $\frac{x^9}{x^4}$.

SOLUTION According to Rule 5, $\frac{x^9}{x^4} = x^{9-4} = x^5$.

When $m = n$ we have $\frac{b^n}{b^n} = b^{n-n} = b^0$. But, $\frac{b^n}{b^n} = 1$ and so $b^0 = 1$. This is Rule 6. In summary, Rule 6 states that any nonzero number raised to the zero power has a value of one.

Finally, let's look at when $m < n$.

EXAMPLE 1.72

Use Rule 5 and division of fractions on $\frac{6^3}{6^7}$ to show how Rule 7 works.

SOLUTION According to Rule 5, $\frac{6^3}{6^7} = 6^{3-7} = 6^{-4}$. But, division of fractions means

$$\begin{aligned}\frac{6^3}{6^7} &= \frac{6 \times 6 \times 6}{6 \times 6 \times 6 \times 6 \times 6 \times 6 \times 6} \\ &= \frac{1}{6 \times 6 \times 6 \times 6} = \frac{1}{6^4}\end{aligned}$$

Thus, we see that $\frac{6^3}{6^7} = 6^{-4} = \frac{1}{6^4}$.

EXAMPLE 1.73

Show that $\frac{x^5}{x^7} = x^{-2} = \frac{1}{x^2}$.

SOLUTION According to Rule 5, $\frac{x^5}{x^7} = x^{5-7} = x^{-2}$. However, when we use the rules for division of fractions we get

$$\begin{aligned}\frac{x^5}{x^7} &= \frac{x \cdot x \cdot x \cdot x \cdot x}{x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x} \\ &= \frac{1}{x \cdot x} = \frac{1}{x^2}\end{aligned}$$

We have shown that $\frac{x^5}{x^7} = x^{-2} = \frac{1}{x^2}$.

Using Examples 1.72 and 1.73, we can summarize Rule 7 by stating that any number raised to a negative exponent is the same as the reciprocal of the number raised to that positive power.

NOTE Since $n = -(-n)$, it sometimes helps to think of Rule 7 as

$$b^n = b^{-(-n)} = \frac{1}{b^{-n}}$$

EXAMPLE 1.74

Use the rules of exponents to expand $(x^2y^3zw^{-5})^{-4}$.

SOLUTION

$$(x^2y^3zw^{-5})^{-4} = \frac{1}{(x^2y^3zw^{-5})^4} = \frac{1}{x^8y^{12}z^4w^{-20}} = \frac{w^{20}}{x^8y^{12}z^4}$$

EXAMPLE 1.75

Use the note that $b^n = \dfrac{1}{b^{-n}}$ to simplify $\left(\dfrac{a^2b}{x^3w^2}\right)^{-5}$.

SOLUTION

$$\left(\frac{a^2b}{x^3w^2}\right)^{-5} = \frac{(a^2b)^{-5}}{(x^3w^2)^{-5}} = \frac{(x^3w^2)^5}{(a^2b)^5}$$
$$= \frac{x^{15}w^{10}}{a^{10}b^5}$$

EXAMPLE 1.76

Use the rules of exponents to simplify $\dfrac{(x^2y^7)^3}{(x^3y^5)^4}$.

SOLUTION

$$\frac{(x^2y^7)^3}{(x^3y^5)^4} = \frac{x^6y^{21}}{x^{12}y^{20}} = x^{6-12}y^{21-20} = x^{-6}y$$

You might want to write this answer as $\dfrac{y}{x^6}$.

CAUTION Although $b^{-n} = \dfrac{1}{b}$ and $\left(\dfrac{1}{b}\right)^{-n} = b^n$, you need to remember the order of operations if a negative exponent is placed a round some parentheses. Thus, $(a + b)^{-1} = \dfrac{1}{a + b}$, but to simplify $\left(\dfrac{1}{a} + \dfrac{1}{b}\right)^{-1}$ you must first add the fractions inside the parentheses.

$$\left(\frac{1}{a} + \frac{1}{b}\right)^{-1} = \left(\frac{b}{ab} + \frac{a}{ab}\right)^{-1}$$
$$= \left(\frac{b + a}{ab}\right)^{-1}$$
$$= \frac{ab}{b + a}$$

APPLICATION ELECTRONICS

EXAMPLE 1.77

The total resistance, R, of a series-parallel circuit is given by the formula

$$R = \left(\frac{1}{R_1} + \frac{1}{R_2}\right)^{-1} + R_3$$

Find R when $R_1 = 0.75\ \Omega$, $R_2 = 0.50\ \Omega$, and $R_3 = 0.60\ \Omega$.

SOLUTION We begin by using the formula in the Caution to add the terms in the parentheses.

$$
\begin{aligned}
R &= \left(\frac{1}{R_1} + \frac{1}{R_2}\right)^{-1} + R_3 \\
&= \left(\frac{1}{0.75} + \frac{1}{0.50}\right)^{-1} + 0.60 \\
&= \frac{(0.75)(0.50)}{0.50 + 0.75} + 0.60 \\
&= \frac{0.375}{1.25} + 0.60 \\
&= 0.30 + 0.60 = 0.90
\end{aligned}
$$

The total resistance is 0.90 Ω.

NOTE Remember when we discussed the order of operations in Section 1.1 that order #2 was raising to a power. Thus, unless grouping symbols indicate otherwise, perform all operations of raising to a power before multiplication, division, addition, or subtraction.

ROOTS

Now let's look at the reverse process of taking exponents. Remember that 3^4 means $3 \times 3 \times 3 \times 3 = 81$. Now, suppose you were asked to find the number (or numbers) that you could raise to the fourth power and get 81. This number (or numbers) would be called the fourth root (or roots) of 81. We use the symbol $\sqrt[4]{81}$ to indicate the fourth root of 81.

In general, the symbol $\sqrt[n]{b}$ is used to indicate the nth root of a number b. When $n = 2$ it is not necessary to put the 2 in the symbol. Thus, $\sqrt{9}$ means $\sqrt[2]{9}$. The $\sqrt{}$ sign is called the **root** or **radical sign**.

EXAMPLE 1.78

$\sqrt{16}$ This is the second root of 16 or, as it is more commonly called, the square root of 16.

$\sqrt[3]{8}$ This is the third root of 8 or the cube root of 8.

$\sqrt[6]{17}$ This is the sixth root of 17.

The solution to $\sqrt{16}$, or to the question "What number squared is 16?," can readily be seen to be either 4 or -4. This presents a problem. Do you write both as the solution to $\sqrt{16}$? Again, mathematicians have decided on a way to interpret $\sqrt{16}$. They have decided that $\sqrt{16} = 4$ and not -4. The symbol $-\sqrt{16}$ is used when they want the value -4.

The value of the $\sqrt[n]{b}$ is called the **principal *n*th root of *b***.

PRINCIPAL *n*th ROOT OF *b*

The principal nth root of b is positive if b is positive.
The principal nth root of b is negative if b is negative and n is odd.

EXAMPLE 1.79

$$\sqrt{16} = 4 \quad \text{and not } -4$$
$$\sqrt[3]{8} = 2 \quad \text{since } 2 \times 2 \times 2 = 8$$
$$\sqrt[3]{-8} = -2 \text{ since } -2 \times -2 \times -2 = -8$$

NOTE If you want the negative number that is the square root of b, then you must write $-\sqrt{b}$. Thus, $-\sqrt{16} = -4$.

There are n different numbers that will satisfy the definition of $\sqrt[n]{b}$. In most cases we will limit ourselves to just one of them—the principal nth root.

RULES OF ROOTS

The rules of roots are very similar to the rules of exponents.

RULES OF ROOTS

1. $\sqrt[n]{ab} = \sqrt[n]{a} \cdot \sqrt[n]{b}$
2. $\sqrt[n]{\dfrac{a}{b}} = \dfrac{\sqrt[n]{a}}{\sqrt[n]{b}}$
3. $(\sqrt[n]{b})^n = b$
4. $\sqrt[n]{b} = b^{1/n}$

EXAMPLE 1.80

According to Rule 1

$$\sqrt{75} = \sqrt{25} \cdot \sqrt{3} = 5\sqrt{3}$$
$$\sqrt[3]{40} = \sqrt[3]{8 \cdot 5} = \sqrt[3]{8}\sqrt[3]{5} = 2\sqrt[3]{5}$$

EXAMPLE 1.81

According to Rule 2

$$\sqrt[3]{\frac{5}{8}} = \frac{\sqrt[3]{5}}{\sqrt[3]{8}} = \frac{\sqrt[3]{5}}{2}.$$

EXAMPLE 1.82

According to Rule 3

$$\left(\sqrt[11]{\frac{7}{3}}\right)^{11} = \frac{7}{3}.$$

NOTE Rule 4 means that every radical or root can be written as a fractional exponent. This means that all the rules of exponents will hold for roots.

EXAMPLE 1.83

Rule 4 means that

$$27^{1/3} = \sqrt[3]{27} = 3$$
$$16^{1/4} = \sqrt[4]{16} = 2$$

APPLICATION GENERAL TECHNOLOGY

EXAMPLE 1.84

According to *Blasius's formula*, the friction factor f of flow in a smooth pipe is

$$f = \frac{0.316}{\sqrt[4]{\text{Re}}}$$

where Re is *Reynold's number*, a constant that depends on the average velocity of the fluid flow, the diameter of the pipe, and the kinematic viscosity of the fluid. Calculate f when Re = 7,340.

SOLUTION We will show how to use a calculator to solve this problem. Remember, we are calculating

$$\frac{0.316}{\sqrt[4]{7{,}340}}$$

and also remember that $\sqrt[4]{7{,}340} = 7{,}340^{1/4}$.

Now, we use our calculator:

PRESS	DISPLAY
.316 ÷ 7340 ^ (1 ÷ 4) ENTER	.0341399647

So, the friction factor is about 0.0341.

APPLICATION ENVIRONMENTAL SCIENCE

EXAMPLE 1.85

For a large rectangular orifice of width b and depth $h_2 - h_1$, the discharge, Q, is

$$Q = \frac{2}{3}C_Q b\left(h_2^{2/3} - h_1^{2/3}\right)\sqrt{2g}$$

where C_Q is the coefficient of discharge. Determine the discharge if $C_Q = 0.96$, $b = 0.200$ m, $h_2 = 0.512$ m, $h_1 = 0.385$ m, and $g = 9.81$ m/s^2.

EXAMPLE 1.85 (Cont.)

SOLUTION Again, we use a calculator to solve the problem. Remember, after the values above are substituted, we are calculating $\frac{2}{3}(0.96)(0.200)(0.512^{2/3} - 0.385^{2/3})\sqrt{2(9.81)}$

PRESS	DISPLAY
2 ÷ 3 × .96 × .2 × (.512 ^ (2 ÷ 3) − .385 ^ (2 ÷ 3)) × 2nd √ (2 × 9.81) ENTER	.0628059206

So, the discharge is about 0.063 m^3/s.

You will remember that in defining the principal *n*th root of a number, we said that if *b* was negative then *n* had to be an odd number. This means that, as yet, we have no definition for numbers such as $\sqrt{-1}$, $\sqrt{-9}$, or $\sqrt[4]{-16}$. In general, these are new types of numbers called **imaginary numbers**. They do not belong to the set of real numbers.

The special symbol *j* is used to represent $\sqrt{-1}$. (Some people use *i* for $\sqrt{-1}$.) This allows us to represent numbers such as $\sqrt{-9}$ in terms of *j*, since $\sqrt{-9} = \sqrt{(9)(-1)} = \sqrt{9}\sqrt{-1} = 3j$. We will study imaginary numbers in a later chapter. For the present time we need to remember that an even root of a negative number is not a real number.

NOTE Your calculator may give you an odd-looking display if you try to take an even root of a negative number. For example, if you try to use your calculator to determine $\sqrt{-9}$, a TI-86 will display (0, 3). This is the format the TI-86 uses for $3j$. The display on a TI-84 or TI-89 will depend on the "mode" in which the calculator is set. If it is set in real mode you will get the display in Figure 1.29a on a TI-84 and like Figure 1.29b on a TI-89. If it is set in complex rectangular mode you will see $3i$. The notations $3i$ and $3j$ mean the same thing. We will see more about complex numbers later in Chapter 14.

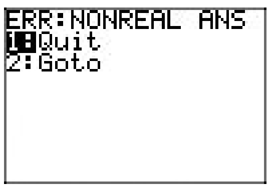

Figure 1.29a

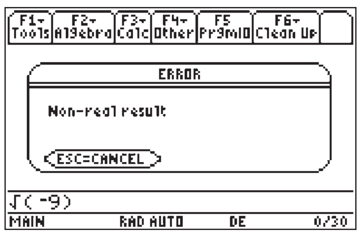

Figure 1.29b

EXERCISE SET 1.3

Evaluate Exercises 1–8.

1. 5^3
2. 3.8^0
3. $\left(\frac{2}{3}\right)^{-1}$
4. $\left(\frac{3}{5}\right)^{-2}$
5. $(-4)^2$
6. $(-5)^4$
7. $\dfrac{7}{7^3}$
8. -3^2

In Exercises 9–42, perform the indicated operations. Leave all answers in terms of positive exponents.

9. $3^2 \cdot 3^4$

10. d^8d^5

11. $2^4 \cdot 2^3 \cdot 2^5$

12. $f^3f^4f^1$

13. $\frac{2^5}{2^3}$

14. $\frac{5^{14}}{5^3}$

15. $(2^3)^2$

16. $(5^7)^3$

17. $(x^4)^5$

18. $(xy^3)^4$

19. $(a^{-2}b)^{-3}$

20. $\left(\frac{2}{3}\right)^4$

21. $\left(\frac{x}{4}\right)^3$

22. $\left(\frac{a}{b^3}\right)^5$

23. $\left(\frac{a^2b}{c^3}\right)^4$

24. 4^{-3}

25. x^{-7}

26. $\frac{1}{p^{-5}}$

27. $\left(\frac{1}{5}\right)^3$

28. $\frac{x^4}{x^2}$

29. $\frac{7^3}{7^8}$

30. $\frac{5^2}{5^{10}}$

31. $\frac{a^2y^3}{a^5y^7}$

32. $\frac{x^4yb^2}{xy^3b^5}$

33. $\frac{a^2p^5y^3}{a^6p^5y}$

34. $\frac{p^3q^4r^2}{p^4r^2}$

35. $(pr^2)^{-1}$

36. $\left(\frac{2x^2}{y}\right)^{-1}$

37. $\left(\frac{4y^3}{5^2}\right)^{-1}$

38. $\left(\frac{4x^3}{y^2}\right)^{-2}$

39. $\left(\frac{2b^2}{y^5}\right)^{-3}$

40. $(-8pr^2)^{-3}$

41. $(-b^4)^6$

42. $ap^2(-a^2p^3)^2$

Find the principal nth root of each real number in Exercises 43–58.

43. $\sqrt{25}$

44. $\sqrt{36}$

45. $\sqrt{144}$

46. $\sqrt{121}$

47. $\sqrt[3]{8}$

48. $\sqrt[3]{-64}$

49. $\sqrt[3]{-27}$

50. $\sqrt[4]{81}$

51. $\sqrt[4]{16}$

52. $\sqrt[5]{-243}$

53. $\sqrt[4]{\frac{16}{81}}$

54. $\sqrt[3]{\frac{-27}{1{,}000}}$

55. $\sqrt{0.04}$

56. $\sqrt{0.25}$

57. $\sqrt[3]{-0.001}$

58. $\sqrt[3]{0.125}$

Simplify each of the real numbers in Exercises 59–90. Use the rules for roots and the rules for exponents.

59. $\sqrt{3^2}$

60. $\sqrt[3]{5^3}$

61. $\sqrt[4]{8.32^4}$

62. $\sqrt[6]{7.91^6}$

63. $\sqrt[3]{5}\,\sqrt[3]{25}$

64. $\sqrt[5]{-3}\,\sqrt[5]{81}$

65. $\sqrt[4]{8}\sqrt[4]{9}$

66. $\sqrt[6]{12}\sqrt[6]{48}$

67. $\frac{\sqrt{75}}{\sqrt{3}}$

68. $\frac{\sqrt{112}}{\sqrt{7}}$

69. $\frac{\sqrt[3]{5}}{\sqrt[3]{40}}$

70. $\frac{\sqrt[3]{11}}{\sqrt[3]{297}}$

71. $\sqrt[3]{2^3} + \sqrt[4]{5^4}$

72. $\sqrt{3^2} - \sqrt[3]{2^3}$

73. $\sqrt[3]{\left(\frac{2}{3}\right)^3} - \sqrt[4]{\left(\frac{1}{3}\right)^4}$

74. $\sqrt[5]{\left(\frac{3}{4}\right)^5} + \sqrt[3]{\left(\frac{5}{4}\right)^3}$

75. $\sqrt{5^{2/3}}$

76. $\sqrt{7^{2/5}}$

77. $\sqrt[3]{16^{3/4}}$

78. $\sqrt[4]{27^{4/3}}$

79. $16^{3/4}$

80. $(-27)^{2/3}$

81. $(-8)^{2/3}$

82. $25^{3/2}$

83. $8^{-2/3}$

84. $9^{-3/2}$

85. $(-27)^{4/3}$

86. $(-32)^{3/5}$

87. $\sqrt{\frac{81}{(8)(0.01)}}$

88. $\sqrt[3]{\frac{(27)(0.008)^2}{0.027}}$

89. $\sqrt[3]{\frac{(0.125)^3\sqrt{144}}{\frac{3}{2}}}$

90. $\sqrt{\frac{64}{(0.25)(0.16)}}$

Solve Exercises 91–106.

91. ***Electronics*** The impedance in an RC circuit is given by the expression

$$Z_{RC} = \sqrt{R^2 + ((2\pi f C)^{-1})^2}$$

Determine the impedance if $R = 40\ \Omega$, $f = 60$ Hz, and $C = 8 \times 10^{-5}$ F.

92. ***Electronics*** The total resistance, R, of a certain series-parallel circuit is given by

$$R = \left(\frac{1}{R_1} + \frac{1}{R_2}\right)^{-1} + R_3$$

Find R when $R_1 = 0.8\ \Omega$, $R_2 = 0.45\ \Omega$, and $R_3 = 0.76\ \Omega$.

93. ***Electronics*** The total resistance, R, of a certain series-parallel circuit is given by

$$R = R_1 + \left(\frac{1}{R_2} + \frac{1}{R_3} + \frac{1}{R_4}\right)^{-1}$$

Find R when $R_1 = 6.5\ \Omega$, $R_2 = 5\ \Omega$, $R_3 = 6\ \Omega$, and $R_4 = 7.5\ \Omega$.

94. ***Business*** If P dollars are invested at an annual interest rate of r compounded n times a year, then the total amount, A, accumulated after t years is given by the formula

$$A = P\left(1 + \frac{r}{n}\right)^{nt}$$

Approximate the total amount of money accumulated after 5 years by investing \$1,200 at 3.8% ($r = 0.038$) interest compounded monthly ($n = 12$).

95. ***Electronics*** A wave consists of many crests and troughs. The distance between any two crests is called the wavelength. The frequency of a wave is the number of waves that pass a given point in a unit of time. The relationship between the wavelength λ of a wave, the frequency f in hertz (Hz), and the speed c at which the wave is propagated is given by the formula $\lambda = \frac{c}{f}$. λ and c must be in the same units of length; that is, if λ is given in meters, then c must be in meters per second (m/s) or if λ is given in feet, then c must be in feet per second (ft/s). What is the wavelength in meters of a radio wave with a frequency of 143.5 megahertz (MHz) (143.5 MHz = 143 500 000 Hz) if $c =$ 300 000 000 m/s?

96. ***Wastewater technology*** The minimum retention time (in days) of a certain waste-handling system is given by the expression below. Evaluate the given expression

$$\frac{1}{0.35\left[1 - \sqrt{\dfrac{8{,}100}{8{,}100 + 121{,}000}}\right] - 0.045}$$

97. ***Acoustics*** The speed of sound of a longitudinal sound wave in a liquid, v_L, is given by the formula

$$v_L = \sqrt{\frac{K}{\rho}}$$

where K is the bulk modulus and ρ is the mass density of the liquid. Determine the speed of sound in water if $K = 2.1 \times 10^9$ N/m^2 and $\rho =$ 1000 kg/m^3.

98. ***Wastewater technology*** For a large rectangular orifice of depth $h_2 - h_1$, the average discharge velocity is

$$\bar{v} = \frac{2C_v(h_2^{2/3} - h_1^{2/3})\sqrt{2g}}{3(h_2 - h_1)}$$

Find $\bar{v}$ (in meters per second) when C_v, the coefficient of velocity, is 0.96, $h_2 = 0.512$ m, $h_1 =$ 0.385 m, and $g = 9.81$ m/s^2.

99. ***Automotive technology*** The diameter in centimeters of a cylinder, c, is given by the formula

$$c = \sqrt{\frac{2.5\,hp}{N}}$$

where hp is the horsepower and N is the number of cylinders of the engine. If an engine has 8 cylinders and is rated at 212 hp, what is the diameter of each cylinder?

100. ***Electronics*** The amperage, A, through an electrical circuit is given by $A = \sqrt{\frac{W}{R}}$ where W is the wattage in watts and R is the resistance in ohms. If an electric heating device has a resistance of 764 Ω and uses 45 W of power, determine the amperage.

101. ***Electronics*** Determine the amperage flowing through an electrical circuit when the wattage is 220 W and the resistance is 98 Ω.

102. ***Automotive technology*** The design diameter D of the crankshaft of an automobile is determined by the formula $D^3 = T \div 0.2\,S$. Determine the diameter of a crankshaft where $S = 38\,500$ and $T = 136\,000$.

IN YOUR WORDS

103. **(a)** Use your calculator to evaluate -7^2 and $(-7)^2$.

(b) Are the answers in (a) different or the same?

(c) Explain why the answers are different or why they are the same.

104. **(a)** Use your calculator to evaluate $(2^3)^2$ and 2^{3^2}.

(b) Are the answers in (a) different or the same?

(c) Explain why the answers are different or why they are the same.

105. **(a)** Evaluate $\sqrt{81}$ and $-\sqrt{81}$.

(b) Which of your answers in (a) is the principal square root of 81?

(c) Evaluate $\sqrt[3]{-27}$ and $-\sqrt[3]{27}$.

(d) Which of your answers in (c) is the principal cube root of -27?

(e) Explain what is meant by the principal nth root of a number.

106. Explain what is meant by $b^{m/n}$.

107. **(a)** Use a calculator to evaluate 64^(1 ÷ 3).

(b) Use a calculator to evaluate 64^3 x^{-1}.

(c) Are these two answers the same?

(d) What do they represent?

1.4 SIGNIFICANT DIGITS AND ROUNDING OFF

A great deal of technical work deals with measurements. Problems in this book are in both the metric system and the customary (or English) system of measurement. If you are not familiar with the metric system of measurement, you should read Appendix A.

No measurement deals with exact numbers. For example, suppose an automotive technician says that the outside diameter of a valve stem is 7.1 mm. Although the diameter may be exactly 7.1 mm, it is more than likely to be a little more or less than 7.1 mm.

The amount of precision in such a measurement depends on the measuring instrument and the person doing the measuring. In mathematical terms, we have the following definitions of precision, significant digits, and accuracy.

PRECISION, SIGNIFICANT DIGITS, AND ACCURACY

The **precision** of a measurement is indicated by the position of the last significant digit relative to the decimal point.

The **significant digits** are those that are determined by measurement.

Accuracy refers to the number of significant digits.

A number with five significant digits, such as 7.1043, is more accurate than a number with four significant digits, such as 7.104.

GUIDELINES FOR DETERMINING WHICH DIGITS ARE SIGNIFICANT

1. All nonzero digits are significant.
2. Zero digits that lie between nonzero digits are significant. For example, 307 has 3 significant digits.
3. Zero digits that lie to the right of both the decimal point and the last nonzero digit are significant. For example, .860 has 3 significant digits and 860.00 has 5 significant digits.
4. Zeros at the beginning of a decimal fraction are not significant. For example, both .045 and 0.045 have 2 significant digits since the zeros serve only to locate the decimal point.
5. Zeros written at the end of a whole number are significant only if there is a "tilde" (~) or a "bar" (¯) over the last significant digit. For example, $980,\tilde{0}00$ and $980,\bar{0}00$ both have 4 significant digits.

Perhaps a few words about notation are needed here. A number such as 15,340,000 is written in the metric system by using spaces instead of commas, as 15 340 000. The spaces are also used for numbers smaller than one, for example, 0.00002471 would be written in the metric system as 0.000 024 71. A four-digit number in the metric system does not need to be written with the space, just as we do not always use a comma in such a number. Thus, 2400 and 2400 are both correct in the metric system and 2,400 and 2400 are both correct in the English system. As a technician, you may encounter both metric and English notations and should recognize a number in either system.

CONCEPT OF ERROR

Let us now return to the earlier measurement of the outside diameter of a valve stem. If the person who measured this diameter had used an instrument that measured in thousandths of millimeters and the diameter had measured

7.1 mm to the nearest thousandth, then the measurement should have been written 7.100 mm.

Since we have spent so much time describing approximate numbers and the method with which to indicate how carefully they were measured, the question might arise as to what is an exact number. **Exact numbers** result from a definition or from counting. For example, the number of spark plugs in an 8-cylinder vehicle is an exact number because the cylinders are counted; the length of a board is an approximate number.

Related to the ideas of accuracy and precision are the ideas of absolute, relative, and percent error. The **absolute error** is the true value subtracted from the approximate value of a number. The absolute error can be either positive or negative depending on whether the true value is smaller or larger than the approximate value.

The **relative error** is the ratio of the absolute error to the true value. Relative error is usually expressed as a percent. When relative error is written as a percent it is often called **percent error**.

THREE TYPES OF ERRORS

$$\text{Absolute error} = \text{approximate value} - \text{true value}$$

$$\text{Relative error} = \frac{\text{absolute error}}{\text{true value}}$$

$$\text{Percent error} = \text{Relative error} \times 100\%$$

EXAMPLE 1.86

The outside diameter of a valve stem was measured as 7.127 mm when the actual diameter was 7.134 6 mm. What are the absolute, relative, and percent errors?

SOLUTION

$$\begin{aligned}
\text{Absolute error} &= \text{approximate value} - \text{true value} \\
&= 7.127 - 7.134\,6 \text{ mm} \\
&= -0.007\,6 \text{ mm} \\
\text{Relative error} &= \frac{\text{absolute error}}{\text{true value}} \\
&= \frac{-0.007\,6 \text{ mm}}{7.134\,6 \text{ mm}} \\
&= -0.001\,065 \\
\text{Percent error} &= \text{Relative error} \times 100\% \\
&= -0.001\,065 \times 100\% \\
&= -0.106\,5\%
\end{aligned}$$

NOTE When computing with approximate numbers, the answer will depend on the accuracy or precision of those numbers.

WORKING WITH APPROXIMATE NUMBERS

When you add or subtract approximate numbers, the result is only as precise as the least precise number.

EXAMPLE 1.87

Add 182.7, 43.69, 2,470.765, and 0.32, and express the answer to the correct precision.

SOLUTION First, we will add the numbers:

$$\begin{array}{r} 182.7 \\ 43.69 \\ 2{,}470.765 \\ +\quad 0.32 \\ \hline 2{,}697.475 \end{array}$$

These 4 numbers have 182.7 as their least precise number, so the sum can only be significant to the tenths place. The answer would be rounded off to 2,697.5.

APPLICATION MECHANICAL

EXAMPLE 1.88

The masses of 5 pieces of metal plate are 16.63 kg, 738.6 kg, 4.314 kg, 21.645 kg, and 0.8752 kg. Find the total mass of the four pieces correct to four significant digits.

SOLUTION These 5 numbers have 738.6 as their least precise number, so the sum can only be significant to the tenths place. First, we will add the numbers:

$$\begin{array}{r} 16.63 \\ 738.6 \\ 4.314 \\ 21.645 \\ +\quad 0.8752 \\ \hline 782.0642 \end{array}$$

The answer would be rounded off to 782.1 kg.

When multiplying or dividing approximate numbers, the errors are enlarged. For this reason, the result is only as accurate as the least accurate number.

EXAMPLE 1.89

If a rectangle measures 42.37 mm long and 5.81 mm wide, it has an area of

$$\begin{array}{r} 42.37 \text{ mm} \\ \underline{\times\ 5.81 \text{ mm}} \\ 246.1697 \text{ mm}^2 \end{array}$$

Since 42.37 has 4 significant digits and 5.81 has only 3 significant digits, the product is rounded off to 3 significant digits and the area is 246 mm^2.

ROUNDING OFF NUMBERS

In general, to round off a number to a certain number of significant digits, examine the digit in the next place to the right and follow the rules for rounding in Section 1.1.

EXAMPLE 1.90

Round 7,030.6 to **(a)** 2 significant digits, **(b)** 3 significant digits, and **(c)** 4 significant digits.

SOLUTIONS

(a) The second significant digit is a 0. The digit to the right is a 3. Since 3 is less than 5, we accept the second significant digit, 0, and a tilde is placed over it to show that it is significant. Each digit between the second significant digit and the decimal point is changed to 0. So, 7,030.6 rounded to 2 significant digits is $7{,}\tilde{0}00$.

(b) The third significant digit is a 3. The digit to the right is a 0. Since 0 is less than 5, we accept the third significant digit, 3. So, 7,030.6 rounded to 3 significant digits is 7,030.

(c) The fourth significant digit is a 0. The digit to the right is a 6. Since 6 is greater than 5, we raise the fourth significant digit by 1 to 1. So, 7,030.6 rounded to 4 significant digits is 7,031.

In many scientific and engineering applications the round-off rule discussed earlier is used when the test digit is 5. This rule is known as either the **Odd-Five Rule** or the **Round-to-the-Even Rule**.

The Odd-Five Rule states that if the test digit is a 5 and it is the last nonzero digit in a number, then add 1 to the round-off digit if it is odd (1, 3, 5, 7, or 9) or retain the original round-off digit if it is even (0, 2, 4, 6, or 8).

EXAMPLE 1.91

Round off 43.725 to the nearest hundredth by the Odd-Five Rule.

SOLUTION We first note that the test digit is 5 and that it is the last nonzero digit. Next, because the round-off digit 2 is even, it is retained and the digit 5 is dropped. Thus, 43.725 would round off to the nearest hundredth to 43.72, by the Odd-Five Rule.

EXAMPLE 1.92

Round off 153.835 to the nearest hundredth by using the Odd-Five Rule.

SOLUTION Again the test digit is 5 and it is the last nonzero digit, but here the round-off digit is 3, an odd number. Using the Odd-Five Rule, we add 1 to the 3 and drop the 5, and the number is rounded off to 153.84.

NOTE Unless it is specified, we will **not** use the Odd-Five Rule in this book.

CAUTION A precaution about significant digits and electronic calculators: Most calculators display 8 or 10 digits and store 2 or 3 more for rounding-off purposes. However, this is not always the case. Some calculators employ a method called **truncation** in which any digits not displayed are discarded. Thus, 489.781 truncated to tenths is 489.7. Rounded off to tenths it would be 489.8. This can result in different answers when two different calculators are used.

ESTIMATION

As we mentioned in the introduction to Section 1.2, you will most likely be using calculators and computers in your work as a technician. You must be able to recognize when an answer provided by these tools is reasonable. The ability to estimate answers is very important in your ability to recognize when answers are reasonable. Your skill at rounding off numbers will help you estimate answers to problems.

In addition and subtraction, round off each number to one or two significant digits and add (or subtract) these rounded-off figures.

EXAMPLE 1.93

Estimate the sum of $4.7 + 8.6 + 9.2 + 4.1$.

SOLUTION The numbers in this sum, $4.7 + 8.6 + 9.2 + 4.1$, would be rounded off to $5 + 9 + 9 + 4 = 27$. The actual sum of the original four numbers is 26.6, which rounds off to 27.

EXAMPLE 1.94

Estimate the sum of $93.74 + 182.7 + 14{,}325 + 83.43$.

SOLUTION Rounding off each number to the nearest hundred, you get the estimate $100 + 200 + 14{,}300 + 100 = 14{,}700$. The actual total is 14,684.87.

We will learn how to estimate products and quotients in Section 1.5, which involves scientific notation.

ROUNDING OFF WITH A CALCULATOR

Most calculators display from 8 to 10 digits. It is often helpful to set your calculator so that it will only display a certain number of digits to the right of the decimal point. For this, you use the Fix or MODE key of the calculator. So, if you want your answer to show 3 decimal places to the right of the decimal point, set your calculator mode to MODE 3. The next example shows how to get the answer of a sum to the correct amount of precision.

NOTE You cannot use a calculator to round to a specific number of places to the left of the decimal point. For example, if you want a number rounded to the nearest hundred you will have to use the rules for rounding numbers and not rely on your calculator.

EXAMPLE 1.95

Use your calculator to add 182.7, 43.69, 2,470.765, and 0.32. Express the answer to the correct number of decimal places.

SOLUTION We worked this example earlier in Example 1.82, and found the answer to be 2,697.5. Now, let's use a calculator. The least precise number is 182.7. Set the calculator to MODE ▼ ▶ ▶ ENTER. This will cause the calculator to display the answer with only one digit to the right of the decimal point. Now we enter each number that we want to add.

PRESS	DISPLAY
182.7 +	182.7
43.69 +	226.4
2470.765 +	2697.2
0.32 =	2697.5

Once again, we get the answer of 2,697.5.

EXERCISE SET 1.4

In Exercises 1–6, determine whether the given numbers are exact or approximate.

1. There are 27 students in class.

2. The car traveled at 88 km/h.

3. A calculator is 148 mm by 79 mm by 35 mm.

4. She bought 18 bolts for $2.79.

5. Of all the people working in the United States, 7.9% are technicians.

6. A sheet of plywood is 1200 mm by 2400 mm.

Give the number of significant digits for the numbers in Exercises 7–14.

7. 6.05

8. 4,030

9. 12.0

10. 0.432

11. 4,000

12. 0.00290

13. 70.06

14. 160.070

Which numbers in Exercises 15–23 are (a) more accurate and (b) more precise? (It is possible for both numbers to be accurate or for both to be precise.)

15. 6.05; 2.8

16. 0.027; 6.324

17. 0.027; 5.01

18. 19,020; 29,000

19. $27{,}0\bar{0}0$; 37,800

20. 6,000; 0.003

21. 0.2; 86

22. 3.05; 305.00

23. 140.070; 140,070

Round off each number in Exercises 24–31 to (a) one, (b) two, and (c) three significant digits.

24. 4.362
25. 14.37
26. 4.065
27. 7.035
28. 0.006155
29. 403.2
30. 0.03725
31. 305.4

In Exercises 32–39, round off each of the numbers to (a) one, (b) two, and (c) three significant digits using the Odd-Five Rule. (These are the same numbers you rounded off in Exercises 24-31.)

32. 4.362
33. 14.37
34. 4.065
35. 7.035
36. 0.006155
37. 403.2
38. 0.03725
39. 305.4

Round off each number in Exercises 40–47 to the nearest (a) ten, (b) tenth, and (c) thousandth.

40. 25.3345
41. 89.8992
42. 125.3755
43. 237.3017
44. 96.99854
45. 437.9975
46. 12.3405
47. 78.6705

Solve Exercises 48 and 49.

48. ***Construction*** The actual length of an I-beam is 12.445 m. An engineer measures the beam as 12.45 m. What are the absolute, relative, and percent errors in this measurement?

49. ***Computer science*** A microcomputer chip is supposed to measure 24 mm long, 8 mm wide, and 3 mm thick. One chip, when measured with a micrometer, is 23.72 mm long, 8.35 mm wide, and 2.98 mm thick. What are the absolute, relative, and percent errors of each of these measurements?

Estimate the answers to Exercises 50–53, then work the exercises and round off the answers using the rules for approximate numbers.

50. $4.31 + 2.015 + 18.35$
51. $97.83 - 4.378 + 5.92$
52. $243.7 + 85.37 - 62.105 + 143.8$
53. $1{,}342.8 + 85.32 + 173.54 + 16$

Work Exercises 54–57 and round off the answers using the rules for approximate numbers.

54. 14.3×5.7
55. 20.4×50.1
56. 0.0034×2.50
57. 3.4×5.00

Solve Exercises 58–63.

58. ***Automotive technology*** The 2 front parking lamps of a car each draw a current of 0.417 A (amperes). The 2 tail lamps each draw a current of 0.457 A. The license plate lamp draws 0.736 A. Find the total current drawn by the 5 lamps. Give your answer correct to 3 significant digits.

59. ***Electronics*** An inductor has 37 layers of wire. Each layer contains 132 turns. The average length of a turn is 0.072 m. Find the length of the wire in the coil. Give your answer correct to 5 significant digits.

60. ***Construction*** A pile of lumber has 379 pieces with an average length of 11 ft $10\frac{1}{8}$ in. Find the total length of the lumber, in inches, correct to (**a**) three decimal places and (**b**) three significant digits.

61. ***Metalworking*** A milling machine cutter with 18 teeth removes 0.086 mm of steel per tooth. The cutter makes 597.3 revolutions. What length of material is removed? Round off your answer to 4 significant digits.

62. ***Electricity*** Juan is an electrician. During the second 3 months of the year he used the following amounts of Romex cable: April, 12,789 ft; May, 13,952 ft; and June, 9,374 ft.

(**a**) Round the amount for each month to the nearest hundred feet.

(b) Add your answers in (a) to find the approximate amount of cable used during these 3 months.

(c) Find the actual total amount of cable used during this 3-month period.

63. ***Electricity*** A wiring job requires 14,765.493 m of cable. Round this amount of cable to the nearest

(a) ten meters

(b) meter

(c) tenth meter

[IN YOUR WORDS]

64. On a sheet of paper, explain precision, significant digits, and accuracy. Do not look at the definitions in the book.

65. On a sheet of paper, describe how to determine absolute error and relative error. Do not look at the definitions in the book.

1.5 SCIENTIFIC AND ENGINEERING NOTATION

In scientific and technical work it is often necessary to work with very large or very small numbers. For example, the star Sirius is approximately 8 220 000 000 000 000 km from earth. On the other hand, the radius of an electron is about 0.000 000 000 000 002 82 m.

WRITING NUMBERS IN SCIENTIFIC NOTATION

Writing numbers with all these zeros is very time consuming and increases the chance of making an error by either omitting a zero or by using too many zeros. In order to save time and reduce the chance of making a mistake, scientists adopted a method for abbreviating numerals. This method is called **scientific notation**.

EXPRESSING A NUMBER IN SCIENTIFIC NOTATION

To express any number in scientific notation, write the number as the product of the significant digits written with exactly one nonzero digit to the left of the decimal point and a power of 10.

EXAMPLE 1.96

$$2{,}400 = 2.4 \times 1{,}000 = 2.4 \times 10^3$$

$$38{,}900{,}000 = 3.89 \times 10{,}000{,}000 = 3.89 \times 10^7$$

$$4{,}07\tilde{0}{,}000{,}000{,}000 = 4.070 \times 1{,}000{,}000{,}000{,}000 = 4.070 \times 10^{12}$$

$$100{,}000 = 1 \times 10^5 \text{ or } 10^5$$

$$0.036 = 3.6 \times \frac{1}{100} = 3.6 \times \frac{1}{10^2} = 3.6 \times 10^{-2}$$

$$0.000\,000\,403 = 4.03 \times \frac{1}{10\,000\,000} = 4.03 \times \frac{1}{10^7} = 4.03 \times 10^{-7}$$

$$8.2 = 8.2 \times 10^0 \text{ or } 8.2$$

Perhaps the easiest way to remember how to express a number in scientific notation is to follow these two guidelines:

GUIDELINES FOR EXPRESSING A NUMBER IN SCIENTIFIC NOTATION

1. Move the decimal point to the immediate right of the first nonzero digit.
2. Next multiply the number from Step 1 by 10^n if you shifted the decimal point n places to the left or by 10^{-n} if you shifted the decimal point n places to the right.

EXAMPLE 1.97

Express 20 500 in scientific notation.

SOLUTION $20\,500 = 2.05 \times 10^4$
Move decimal point
four places to the left.

EXAMPLE 1.98

Express 0.000 037 in scientific notation.

SOLUTION $0.000\,037 = 3.7 \times 10^{-5}$
Move decimal point
five places to the right.

Any zeros that are significant should be included when the number is written in scientific notation. For example, $0.002\,30 = 2.30 \times 10^{-3}$ and $847\tilde{0}00 = 8.470 \times 10^5$.

To change a number from scientific notation to ordinary notation you just reverse the process. To change 8.37×10^6 to ordinary notation you would move the decimal point six places to the right. So, $8.37 \times 10^6 = 8\,370\,000$. A number like 4.61×10^{-4} is changed by moving the decimal point four places to the left. Then $4.61 \times 10^{-4} = 0.000\,461$.

Obviously scientific notation relies heavily on exponents. This will give us the first chance to use the rules for exponents that we learned in Section 1.3. Also, many calculators use scientific notation to express some numbers. We will now compute products and quotients using scientific notation.

SCIENTIFIC NOTATION ON A CALCULATOR

A calculator or a computer will automatically put a number in scientific notation if the number contains too many digits to show on the screen. For example, if you multiply 250,000 and 19,700,000, the calculator displays the answer as

4.925E12

or 4.925 12

which should be interpreted as 4.925×10^{12}.

CAUTION Do not interpret the calculator display 4.925E12 or 4.925 12 as 4.925^{12}. It means 4.925×10^{12}.

Similarly, the answer to 0.000 025 ÷ 800 000 would be displayed as

3.125E−11

or 3.125 −11

which is interpreted as 3.125×10^{-11}.

NOTE Some calculators place a space between the number and the exponent whereas other calculators use the letter "E."

To enter the number in scientific notation on a calculator, you need to use the Enter Exponent key, labeled as EE, EEX, or EEP.

EXAMPLE 1.99

Enter 4.37×10^{12} on a TI-83 or TI-84 calculator.

SOLUTION

PRESS	DISPLAY
4.37 2nd EE 12 ENTER	4.37E 12

EXAMPLE 1.100

Enter 4.37×10^{12} in a spreadsheet.

SOLUTION To enter a number written in scientific notation, use the letter "E" to denote the power of ten. Enter "4.37E + 12" in Cell A1 as shown in Figure 1.30a.

As you can see, the number in the cell is still displayed in scientific notation while the value is shown in the Formula Bar. What you see in the cell may depend on the number setting for that cell. In Figure 1.30b the cell was set to General number.

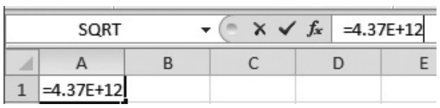

Figure 1.30a

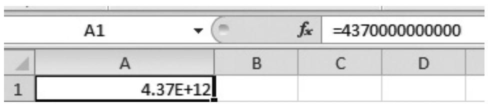

Figure 1.30b

EXAMPLE 1.101

Enter 9.87×10^{-15} on a calculator.

SOLUTION The procedure depends on the type of calculator. With some models, you use this method:

PRESS	DISPLAY
9.87 EE	9.87 00
15 +/−	9.87 −15

EXAMPLE 1.101 (Cont.)

With other calculators, such as the TI-8*x* graphics calculator, you follow these steps:

PRESS	DISPLAY
9.87 EE	9.87E
(–) 15	9.87E–15

HINT You might find it easier to use the 10^x key than the EE key.

EXAMPLE 1.102

Enter 4.37×10^{12} in a spreadsheet.

SOLUTION Enter "9.87E–15" in Cell A1 as shown in Figure 1.31.

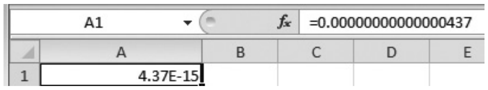

Figure 1.31

Special care has to be taken when entering a power of 10. The next example shows a correct procedure.

EXAMPLE 1.103

Enter 10^{25} into a calculator.

SOLUTION The procedure depends on the calculator. Many graphing calculators have a 10^x key. Thus, to enter 10^{25} into such a calculator, press 10^x 25 ENTER. On a TI-8*x* calculator, the result is shown as 1E25.

You may also use the EE key

PRESS	DISPLAY
1 EE	1 00 or 1E
25	1 25 or 1E25

CAUTION Remember that powers of 10, such as 10^{25}, really represent 1×10^{25}. Thus, you enter 1 EE 25 and not 10 EE 25. However, on a TI-8*x*, EE 25 will work.

NOTE Many calculators allow you to put the calculator in scientific notation mode. If this is done, all answers will be displayed in scientific notation.

PRODUCTS AND QUOTIENTS

The product of $87\,000\,000 \times 470\,000\,000\,000$ can be simplified with scientific notation.

EXAMPLE 1.104

Use scientific notation to find the product of 87 000 000 × 470 000 000 000.

SOLUTION Change each number to scientific notation and then multiply.

$$\begin{aligned} 87\,000\,000 \times 470\,000\,000\,000 &= (8.7 \times 10^7) \times (4.7 \times 10^{11}) \\ &= (8.7 \times 4.7) \times (10^7 \times 10^{11}) \\ &= 40.89 \times 10^{7+11} \\ &= 40.89 \times 10^{18} \\ &= 4.089 \times 10^{19} \end{aligned}$$

Notice that we had to change our first answer (40.89×10^{18}) to scientific notation because 40.89 is not between 1 and 10.

Similarly, a large and small number can be multiplied more easily by using scientific notation, as shown in the next example.

EXAMPLE 1.105

Solve 4 100 000 000 × 0.000 002 4.

SOLUTION

$$\begin{aligned} 4\,100\,000\,000 \times 0.000\,002\,4 &= (4.1 \times 10^9) \times (2.4 \times 10^{-6}) \\ &= (4.1 \times 2.4) \times (10^9 \times 10^{-6}) \\ &= 9.84 \times 10^3 \\ &= 9\,840 \end{aligned}$$

Division can also be simplified using scientific notation.

EXAMPLE 1.106

Solve 0.000 000 036 ÷ 0.000 012.

$$\begin{aligned} 0.000\,000\,036 \div 0.000\,012 &= (3.6 \times 10^{-8}) \div (1.2 \times 10^{-5}) \\ &= \frac{3.6 \times 10^{-8}}{1.2 \times 10^{-5}} = \frac{3.6}{1.2} \times \frac{10^{-8}}{10^{-5}} \\ &= \frac{3.6}{1.2} \times 10^{-8+5} \\ &= 3 \times 10^{-8+5} \\ &= 3 \times 10^{-3} = 0.003 \end{aligned}$$

EXAMPLE 1.107

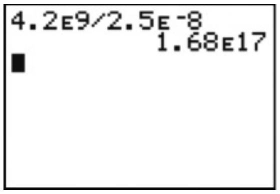

Figure 1.32

Use a calculator to solve 4 200 000 000 ÷ 0.000 000 025.

SOLUTION First, we convert each of these numbers to scientific notation.

$$4\,200\,000\,000 = 4.2 \times 10^9$$
$$0.000\,000\,025 = 2.5 \times 10^{-8}$$

Now, we use our calculator as shown in Figure 1.32.

So, the quotient is 1.68×10^{17}, which rounds to 1.7×10^{17}.

EXAMPLE 1.108

Use a spreadsheet to solve 4 200 000 000 ÷ 0.000 000 025.

SOLUTION First, convert each number to scientific notation.

$$4\ 200\ 000\ 000 = 4.2 \times 10^9$$
$$0.000\ 000\ 025 = 2.5 \times 10^{-8}$$

Now enter as shown in Figure 1.33a and the result should appear similar to Figure 1.33b.

So, the quotient is 1.68×10^{17}, which rounds to 1.68×10^{17}, which rounds off to 1.7×10^{17}

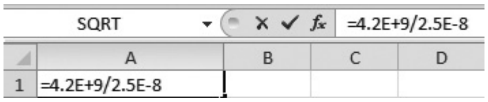

Figure 1.33a

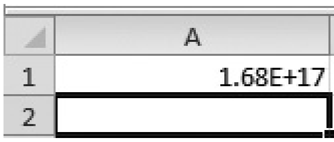

Figure 1.33b

EXAMPLE 1.109

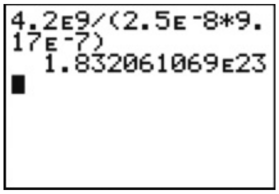

Figure 1.34

Use scientific notation to evaluate $\dfrac{4\,200\,000\,000}{0.000\,000\,025 \times 0.000\,000\,917}$.

SOLUTION First convert each number to scientific notation.

$$4\ 200\ 000\ 000 = 4.2 \times 10^9$$
$$0.000\ 000\ 025 = 2.5 \times 10^{-8}(39)$$
$$0.000\ 000\ 917 = 9.17 \times 10^{-7}$$

Now you can use your calculator as shown in Figure 1.34. Notice that parentheses were placed around the denominator so that the calculator would divide 4.2×10^9 by the product $2.5 \times 10^{-8} \times 9.17 \times 10^{-7}$.

From Figure 1.34, we see that the solution is about 1.83×10^{23}.

EXAMPLE 1.110

Use scientific notation to evaluate $\dfrac{4\,200\,000\,000}{0.000\,000\,025 \times 0.000\,000\,917}$.

SOLUTION First convert each number to scientific notation.

$$4\ 200\ 000\ 000 = 4.2 \times 10^9$$
$$0.000\ 000\ 025 = 2.5 \times 10^{-8}$$
$$0.000\ 000\ 917 = 9.17 \times 10^{-7}$$

Now enter the calculation in Cell A1 as shown in Figure 1.35a. Be careful to use parentheses to separate the numerator from the product of the two numbers in the denominator $2.5 \times 10^{-8} \times 9.17 \times 10^{-7}$.

From Figure 1.35b, we see that the solution is 1.83×10^{23}.

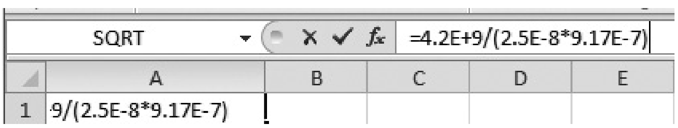

Figure 1.35a

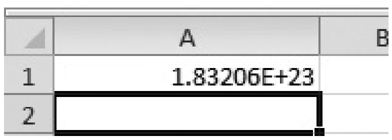

Figure 1.35b

APPLICATION ELECTRONICS

EXAMPLE 1.111

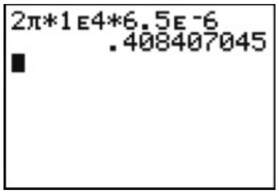

Figure 1.36

The alternating current reactance of a circuit, X_L, is given in ohms (Ω) by the formula

$$X_L = 2\pi fL$$

where f is the frequency of the alternating current in hertz (Hz), and L is the inductance of the circuit or inductor in henrys (H). Compute the inductive reactance when $f = 10\ 000$ Hz and $L = 0.000\ 006\ 5$ H.

SOLUTION We will use a TI-83 or TI-84 calculator to compute this product. First, we convert 10 000 Hz and 0.000 006 5 H to scientific notation.

$$10\ 000 = 1 \times 10^4$$
$$0.000\ 006\ 5 = 6.5 \times 10^{-6}$$

Now, we use our calculator and obtain the result shown in Figure 1.36. So, the reactance is about 0.41 Ω.

USING SCIENTIFIC NOTATION FOR ESTIMATES

Scientific notation can be used to help estimate products and quotients.

EXAMPLE 1.112

Estimate the product of $362 \times 2{,}165 \times 82$.

SOLUTION First, round off each number so that it has only one nonzero digit. Thus, 362 is rounded off to 400, 2,165 to 2,000, and 82 to 80. Express each of these in scientific notation and multiply.

$$\begin{aligned} 400 \times 2{,}000 \times 80 &= 4 \times 10^2 \times 2 \times 10^3 \times 8 \times 10^1 \\ &= (4 \times 2 \times 8) \times (10^2 \times 10^3 \times 10^1) \\ &= 64 \times 10^6 \\ &= 64{,}000{,}000 \end{aligned}$$

The actual product of $362 \times 2{,}165 \times 82$ is 64,265,860.

Much of the time the estimated product (or quotient) will not be this close to the actual product (or quotient). This method of estimation is best used to make sure that you are placing the decimal point in the correct place.

For example, $2\,965 \times 650 \times 24 = 46\ 254\ 000$. Using the estimation procedure previously described you would get

$$\begin{aligned} 3000 \times 700 \times 20 &= 3 \times 10^3 \times 7 \times 10^2 \times 2 \times 10 \\ &= (3 \times 7 \times 2) \times (10^3 \times 10^2 \times 10) \\ &= 42 \times 10^6 = 42\,000\,000 \end{aligned}$$

As you can see, 46 254 000 is quite a bit larger than 42 000 000. Yet, the estimation procedure indicated that the correct answer would have eight digits to the left of the decimal point.

ENGINEERING NOTATION

Engineering notation is very similar to scientific notation. In engineering notation, the exponents of the 10 are always multiples of three. The main advantage of engineering notation is when SI (metric) units are used. In the metric system, the most widely used prefixes are for every third power of 10.

EXPRESSING A NUMBER IN ENGINEERING NOTATION

To express any number in engineering notation, write the number as the product of a number with one, two, or three digits to the left of the decimal point and a power of 10 where the power-of-10 exponent is a multiple of three.

EXAMPLE 1.113

Express each of the following numbers in engineering notation: **(a)** 25,300, **(b)** 120,000, **(c)** $407,\tilde{0}00,000,000$, and **(d)** 0.000 000 025.

SOLUTIONS

(a) $25,300 = 25.3 \times 10^3$

(b) $120,000 = 120 \times 10^3$

(c) $407,\tilde{0}00,000,000 = 407.0 \times 10^9$

(d) $0.000\,000\,025 = 25 \times 10^{-9}$

NOTE Many calculators let you put the calculator in engineering notation mode. Most calculators abbreviate "engineering mode" with "Eng." If this is done, all answers will be displayed in engineering notation.

EXAMPLE 1.114

Use a TI-86 calculator to express the number 0.9988428928 in **(a)** "normal" mode, **(b)** scientific mode, and **(c)** engineering mode.

SOLUTION

(a) Key 0.9988428928 into the calculator and press ENTER. The result in "normal" mode is shown in the viewing window as .9988428928.

(b) Press MODE and then press the ▶ key until Sci for scientific notation is highlighted. Now press ENTER to put the calculator in scientific mode and 2nd QUIT to return to the home screen. Key 0.9988428928 into the calculator and press ENTER. The viewing window should now display 9.988428928E-1.

(c) MODE and then press the ▶ key until Eng for engineering format is highlighted. Press ENTER to put the calculator in engineering mode and return to the home screen. Now press ENTER again. The viewing window should now display 998.8428928E-3.

APPLICATION GENERAL TECHNOLOGY

EXAMPLE 1.115

The rest energy, E, of an electron with rest mass, m, is given by Einstein's equation

$$E = mc^2$$

where c is the speed of light. Find E if $m \approx 9.110 \times 10^{-31}$ kg and $c \approx 2.998 \times 10^8$ m/s. Express the answer in **(a)** engineering notation and **(b)** scientific notation.

SOLUTION Substituting the values of m and c into the given equation, we find

$$\begin{aligned} E &\approx (9.110 \times 10^{-31})\,(2.998 \times 10^8)^2 \\ &= (9.110)(2.998)^2\,(10^{-31})(10^8)^2 \\ &= (9.110)(2.998)^2\,(10^{-31})\,(10^8)\,(10^8) \\ &= (81.8807)\,(10^{-31+8+8}) \\ &= 81.8807 \times 10^{-15} \end{aligned}$$

The rest energy of an electron is about 81.8807×10^{-15} kg·m^2/s^2.

(a) Since the exponent of 10 in the number 81.8807×10^{-15} is a multiple of three, the number 81.8807×10^{-15} is already in engineering notation.

(b) In scientific notation, we have $81.8807 \times 10^{-15} = (8.18807 \times 10) \times 10^{-15} = 8.18807 \times 10^{-14}$. Thus, in scientific notation, the rest energy of an electron is about 8.18807×10^{-14} kg·m^2/s^2.

EXERCISE SET 1.5

In Exercises 1–10, change each of the numbers to scientific notation.

1. 42 000
2. 370 000 000
3. 0.000 38
4. 0.000 007 5
5. 9 807 000 000
6. 87 000 000
7. 0.000 097 0
8. 0.400
9. 4.3
10. 2.07

In Exercises 11–20, change each of the numbers to engineering notation.

11. 74 000
12. 910 000 000
13. 0.000 47
14. 0.000 007 5
15. 9 807 000 000
16. 57 000 000
17. 0.000 053 10
18. 0.700
19. 5.6
20. 3.08

In Exercises 21–28, change each number from scientific notation to ordinary notation.

21. 4.5×10^3
22. 3.7×10^5
23. 4.05×10^7
24. 3.05×10^8
25. 6.3×10^{-5}
26. 1.87×10^{-8}
27. 7.2×10
28. 9.6×10^{-1}

In Exercises 29–36, change each number from engineering notation to ordinary notation.

29. 75×10^3
30. 19×10^6
31. 47.5×10^9
32. 317.0×10^9
33. 39.20×10^{-6}
34. 1.75×10^{-3}
35. 83.15×10^0
36. 391.25×10^{-9}

In Exercises 37–58, perform the indicated calculations by using scientific notation and/or engineering notation.

37. 760 000 × 20 400 000 000

38. (43 200 000)(850 000 000)

39. 0.000 035 × 0.000 000 76

40. (0.000 42)(0.000 075)

41. (840 000 000)(0.000 35)

42. (0.000 0042)(23 000)

43. (70 400)(0.000 003 2)

44. (0.000 302)(4 370 000 000)

45. (28 800 000 000) ÷ (240 000)

46. 55 500 000 000 ÷ 370 000

47. 375 000 ÷ 150 000 000

48. 79 800 ÷ 840 000 000

49. 0.003 2 ÷ 0.000 000 16

50. 0.000 48 ÷ 0.000 000 3

51. 0.000 000 36 ÷ 0.000 2

52. 0.000 000 009 8 ÷ 0.000 014

53. (8 760 000)(245 000 000)(6 400 000 000)

54. (4 360 000)(625 000 000)(38 700 000 000)

55. $\dfrac{(250\,200)(630\,000\,000)}{0.000\,000\,006\,3}$

56. $\dfrac{(25\,200)(8\,000\,000)}{3\,970\,000\,000}$

57. $\dfrac{0.000\,005\,2 \times 480\,000\,000}{0.000\,000\,000\,006\,4}$

58. $\dfrac{96\,000\,000 \times 810\,000}{243\,000\,000\,000\,000}$

Solve Exercises 59–68.

59. Estimate the computation of $\dfrac{325 \times 86.5}{43.1}$.

60. Estimate the computation of $\dfrac{453 \times 672.4}{3.81 \times 42.3}$.

61. ***Physics*** An electron moves at the rate of 300 000 000 000 mm/s (millimeters per second). How many millimeters will it move in 0.000 003 7 s?

62. ***Astronomy*** The speed of light is approximately 300 000 km/s. How many minutes will it take for a ray of light to reach the earth from the sun, if the sun is 150 000 000 km from earth?

63. ***Physics*** One atomic mass unit (amu) is $1.660\,6 \times 10^{-27}$ kg. If 1 carbon atom has 12 atomic mass units, what is the mass of 14 000 000 carbon atoms?

64. ***Physics*** One iron atom contains 55 amu. What is the mass in kilograms of 230 000 000 iron atoms?

65. ***Physics*** The rest mass of one electron is 9.1095×10^{-31} kg and the rest mass of one neutron is 1.6750×10^{-27} kg. Which has the larger mass—one electron or one neutron? How many times heavier is it?

66. ***Electronics*** Compute the inductive reactance when $f = 10{,}000{,}000$ Hz and $L = 0.015$ H. (See Example 1.111.)

67. ***Electronics*** Compute the inductive reactance when $f = 18{,}000{,}000$ Hz and $L = 0.0025$ H. Work the exercise, and express your answer in engineering notation.

68. ***Physics*** The mass of the earth is about 5.975×10^{24} kg, and its volume is about 1.083×10^{21} m^3. Use engineering notation to compute the density of the earth. (Note: the density of an object is defined as its mass divided by its volume.)

[IN YOUR WORDS]

69. Explain why you might use scientific notation. Explain the advantages of scientific notation over ordinary notation.

70. Explain when you might use engineering notation rather than scientific notation.

CHAPTER 1 REVIEW

IMPORTANT TERMS AND CONCEPTS

Absolute error
Absolute value
Accuracy
Associative laws
Commutative laws
Decimals
Distributive law
Engineering notation
Exact numbers
Exponent
Exponents
  Fractional
  Rules for
Identity elements
Imaginary numbers
Inequalities
Integers
  Negative
  Positive
  Zero
Inverse elements
Irrational numbers
Mixed number
Natural numbers
Nonnegative integers
Odd-Five Rule
Order of operations
Percent error
Place value
Precision
Radical
Rational numbers
Real numbers
Reciprocal
Relative error
Roots
Rounding off
Round-to-the-Even Rule
Scientific notation
Significant digits
Whole numbers
Zero

REVIEW EXERCISES

1. To which sets of numbers do each of the following numbers belong?
(a) -24
(b) $\frac{15}{7}$
(c) $\sqrt[3]{-17}$

2. Give the absolute value of each of these numbers.
(a) $\sqrt{42}$
(b) -16
(c) $\frac{-5}{8}$

3. Give the reciprocal of each of these numbers.
(a) $\frac{2}{3}$
(b) -8
(c) $\frac{-1}{5}$

4. Give the additive inverse of each of these numbers.
(a) -5
(b) 17
(c) $4\sqrt{2}$

5. In each part, identify the basic law of real numbers that is being used.
(a) $14 + 7 = 7 + 14$
(b) $6(8 + 2) = 6 \cdot 8 + 6 \cdot 2$
(c) $14 = 0 + 14$
(d) $\frac{2}{3} \times \frac{3}{2} = 1$

In Exercises 6–20, perform the indicated operation.

6. $16 + 48$
7. $37 + (-81)$
8. $95 - 42$
9. $37 - (-61)$
10. $4 \times (-8)$
11. $\frac{2}{3} + \frac{-5}{6}$
12. $\frac{4}{5} - \frac{5}{6}$
13. $-9 \div 3$
14. $12 \div (-4)$
15. $\frac{2}{3} \times \frac{-3}{5}$
16. $3\frac{3}{4} \times (-4\frac{1}{3})$
17. $\frac{2}{3} \div \frac{1}{4}$
18. $\frac{1}{5} \div \frac{-2}{15}$
19. $2\frac{1}{2} \div 3\frac{1}{4}$
20. $-4\frac{1}{3} \div (-3\frac{1}{6})$

Evaluate each number in Exercises 21–26.

21. 2^5

22. $(-3)^4$

23. $(-4)^3$

24. $8^{1/3}$

25. $4^{1/2}$

26. $(-64)^{1/3}$

In Exercises 27–38, perform the indicated operations. Leave all answers in terms of positive exponents.

27. $2^5 \cdot 2^3$

28. $3^5 \cdot 3^4$

29. $2^{-3} \cdot 2^5$

30. $16^{-4} \cdot 16^{-3}$

31. $(4^3)^5$

32. $(2^{-3})^4$

33. $(4^{1/3})^3$

34. $(5^{1/4})^{2/3}$

35. $\dfrac{a^2b^3}{ab^4}$

36. $\dfrac{x^2y^3z}{x^3yz}$

37. $(ax^2)^{-2}$

38. $\dfrac{(by^{-2})^2}{(cy^{-4})^{1/4}}$

Which of the pairs of numbers in Exercises 39–42 is (a) more accurate and which is (b) more precise?

39. 2.37; 42,000

40. 0.002; 2.02

41. 7; 0.7

42. 1200; 0.0021

In Exercises 43–46, round off each number to (a) two significant digits, (b) three significant digits, (c) the nearest tenth, and (d) the nearest hundredth.

43. 7.351

44. 18.2874

45. 2.0528

46. 4.028476

Change each of the numbers in Exercises 47–54 to scientific notation.

47. 371 000 000 000

48. 2 540 000 000 000 000

49. 0.000 000 000 024

50. 0.000 000 000 000 000 000 049 1

51. 29 080 000 000 000 000

52. 753 000 000 000 000 000 000 000

53. 0.000 000 000 000 000 000 000 075

54. 0.000 000 000 000 000 193

Find the principal root of the real numbers in Exercises 55–58.

55. $\sqrt{144}$

56. $\sqrt[3]{-64}$

57. $\sqrt[4]{625}$

58. $\sqrt{\frac{36}{121}}$

Solve Exercises 59 and 60.

59. ***Business*** A wholesale parts distributor discounts the unit cost of an item \$0.14 for each time a full dozen is ordered. A manufacturer orders 418 items that normally cost \$13.79 each.

(a) What was the amount of the discount?

(b) What is the total cost of the manufacturer's order?

60. ***Nuclear physics*** The average kinetic energy of atomic particles in a certain location, E, can be determined from the absolute temperature, T, at that location by using the equipartition principle

$$E = \tfrac{3}{2}kT$$

where k is Boltzmann's constant $8.617\,33 \times 10^{-5}$ eV/K.

(a) What is the average kinetic energy in electronvolts of particles at room temperature ($T = 295$ K)?

(b) What is the average kinetic energy in electronvolts of particles at the surface of the sun ($T = 5.714 \times 10^3$ K)?

CHAPTER 1 TEST

1. Give the absolute value of each of the following:
   (a) $\frac{4}{3}$
   (b) $\frac{1}{2} - \frac{5}{8}$
   (c) -6
2. What is the reciprocal of $-\frac{3}{7}$?
3. What is the additive inverse of $-\frac{5}{8}$?

In Exercises 4–22, perform the indicated operation.

4. $35 + 76$
5. $-47 - 65$
6. -5×14
7. $\frac{5}{3} + 5\frac{2}{3}$
8. $\frac{7}{4} - \left(-\frac{3}{5}\right)$
9. $-4\frac{1}{2} \times 2\frac{1}{3}$
10. $\frac{-5}{7} \div \frac{15}{28}$
11. $-2\frac{1}{3} \div -4\frac{5}{6}$
12. Evaluate $(-5)^3$.
13. $4^{3/2}$
14. $2^6 \cdot 2^{-4}$
15. $(5^{1/4})^8$
16. $3^{5/2} \div 3^{-3/2}$
17. $\dfrac{a^3b^5}{a^4b^2}$
18. Which is more accurate: 4.516 or 37 000?
19. Which is more precise: 0.000 51 or 51.02?
20. Write 47,500,000,000,000 in scientific notation.
21. What is the principal root of $\sqrt{\frac{49}{25}}$?
22. Write 92 500 000 000 000 in engineering notation.

2 ALGEBRAIC CONCEPTS AND OPERATIONS

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In order to operate correctly, a computer relies on instructions given to it by a programmer. The computer programmer must understand algebra in order to tell the computer what to do.

Until now we have spent most of our time working with real numbers. We have learned the basic operations with real numbers, how to take powers, roots, and absolute values, and how to write real numbers in scientific notation. Now it is time to begin our work in algebra.

In this chapter, we will begin by looking at the operations of addition, subtraction, multiplication, and division of algebraic expressions. Once these four operations have been learned, we will use them to learn how to solve equations. Finally, we will use our equation-solving skills to solve many problems like those encountered in technical areas.

OBJECTIVES

AFTER COMPLETING THIS CHAPTER, YOU WILL BE ABLE TO:

- Simplify algebraic expressions by performing addition, subtraction, multiplication, and/or division on polynomials.
- Solve equations containing one variable.
- Use appropriate problem-solving techniques to solve word problems from various applications.
- Translate problems stated in words to appropriate algebraic statements.
- Use formulas to solve applications.
- Use percentages and proportions as tools to solve problems.
- Set up a ratio between two quantities.
- Convert units in a ratio.
- Find a rate given two measurements.
- Set up and solve a proportion for a missing quantity.
- Determine if two ratios are equal.
- Solve problems using correct units.

2.1 ADDITION AND SUBTRACTION

One of the first things that everyone notices about algebra is that it uses letters and other symbols to represent numbers. Algebra also uses letters and other symbols to represent unknown amounts in equations and inequalities. In the first chapter we used letters to help present some of the rules. For example, we said that $b^{-n} = \frac{1}{b^n}$, if $b \neq 0$.

VARIABLES

When a letter or other symbol is used to indicate something that can be assigned any value from a given or implied set of numbers, it is called a **variable**. We have already used letters as variables in this book. For example, we used a, b, and c as variables when we said that $a(b + c) = ab + ac$. We used b and n as variables when we said that $b^{-n} = \frac{1}{b^n}$, if $b \neq 0$. Another place where variables are used is on a calculator. For example, the x^2, $\sqrt{x}$, $1/x$, and y^x keys on a calculator all use the letter x as a variable and one uses the letter y.

CONSTANTS

Letters or other symbols may also be used to designate fixed, but unspecified, numbers called **constants**. One constant that you are already familiar with is π, which represents the number 3.14159265. . . . (The . . . at the end of this number means that there are more digits after the 5.) Another constant that we will use a lot is $e = 2.7182818284. \ldots$ Usually letters near the beginning of the alphabet, like a, b, c, d, are used to represent constants. Letters near the end of the alphabet, such as w, x, y, and z, are used to indicate variables. Thus, in the expression $ax+b$, a and b represent constants and x is a variable.

ALGEBRAIC EXPRESSIONS

The term **algebraic expression** is used for any combination of variables and constants that is formed using a finite number of operations. Examples of algebraic expressions include ax^2, $ax^2 + bx + c$, $\dfrac{x^3 + \sqrt{y}}{ax^2 - by}$, $\dfrac{4x^{-3} + 7x^2}{2x - 3w}$, and $1.8C + 32$.

ALGEBRAIC TERMS

If an algebraic expression consists of parts connected by plus or minus signs, it is called an **algebraic sum.** Each of the parts of an algebraic sum, together with the sign preceding it, is called an **algebraic term**.

EXAMPLE 2.1

What are the algebraic terms of the algebraic sum $5x^7 + 2x^3y - \dfrac{4x}{y^2}$?

SOLUTION This algebraic sum has three algebraic terms: $5x^7$, $2x^3y$, and $-\dfrac{4x}{y^2}$.

An algebraic term may consist of several **factors**. The term $5xy$ has the individual factors of 5, x, and y. Other factors are products of two or more of the individual factors. Thus, $5x$, $5y$, xy, and $5xy$ are also factors of $5xy$.

EXAMPLE 2.2

What are the factors of $\dfrac{6x^2}{y}$?

SOLUTION The individual factors are 2, 3, x, and $\dfrac{1}{y}$. Other factors are $6, 2x, 3x, 6x, x^2, 2x^2, 3x^2, 6x^2, \dfrac{2}{y}, \dfrac{3}{y}, \dfrac{6}{y}, \dfrac{x}{y}, \dfrac{2x}{y}, \dfrac{3x}{y}, \dfrac{6x}{y}, \dfrac{x^2}{y}, \dfrac{2x^2}{y}, \dfrac{3x^2}{y}$, and $\dfrac{6x^2}{y}$.

EXAMPLE 2.3

What are the factors of $\dfrac{-10(x + y)}{z}$?

SOLUTION The individual factors are -1, 2, 5, $x + y$, and $\dfrac{1}{z}$. The other factors are products of these factors. Notice that the negative sign was treated by using -1 as a factor.

Each term has two parts. One part is the coefficient and the other part contains the variables. The **coefficient** is the product of all of the constants. Normally the coefficient is written at the front of the term. A variable with no visible coefficient, such as x or y, is understood to have a coefficient of 1. The coefficient of the term $4x^2$ is 4; of $18y^3z$ is 18; of $\frac{w^2}{3}$ is $\frac{1}{3}$; and of yx^2 is 1. The coefficient of $4\pi bx$ is $4\pi b$ if we assume that b is a constant.

Terms that involve exactly the same variables raised to exactly the same power are called **like terms**. For example, $7x^2y$ and $4x^2y$ are like terms because both contain the same variables (x and y), the variable x is raised to the second power in each term and the y is raised to the first power. $4xyz^3$ and $5xyz^2$ are not like terms even though they both contain the variables x, y, and z, because one term has the variable z raised to the third power and in the other term it is raised to the second power.

MONOMIALS AND MULTINOMIALS

An algebraic expression with only one term is a **monomial**. An algebraic sum with exactly two terms is a **binomial**; an algebraic sum with three terms is a **trinomial**; and an algebraic sum with two or more terms is a **multinomial**. As you can see, binomials and trinomials are special kinds of multinomials.

EXAMPLE 2.4

$4x^2$, $\frac{7}{5}xy$, and $-8xz^3\sqrt{w}$ are all monomials.
$ax^2 + by$, $\sqrt{3}x^3z - 2w$, and $4x^2 + 2$ are binomials.
$3x^3 - 2x + 1$ and $2xy + 3yz - 4\sqrt{2y}$ are trinomials.
$16x^3 - 7\sqrt{2}y^2 + \frac{2}{3}xy - 7^5x^3y$ is a multinomial with four terms. In this last expression the coefficient of the first term is 16, the second term has a coefficient of $-7\sqrt{2}$, the third $\frac{2}{3}$, and the fourth -7^5 (or $-16{,}807$).

ADDING AND SUBTRACTING MULTINOMIALS

In order to add or subtract monomials, we add or subtract the coefficients of like terms. So, $3x^2 + 5x^2 = 8x^2$. We can justify this with the distributive property, since $3x^2 + 5x^2 = (3 + 5)x^2 = 8x^2$.

NOTE Remember, you can only add or subtract like terms.

So,

$$3x^2 + 2xy - y^3 - 8xy = 3x^2 - 6xy - y^3$$

because the only like terms are $2xy$ and $-8xy$. When these are added, we get $-6xy$. None of the other terms can be combined because they are not like terms.

EXAMPLE 2.5

$$\begin{aligned}(3x^2 + 2yx - 5y) + (-6xy + 6x^2 - 7y)\\ &= (3x^2 + 6x^2) + (2yx - 6xy) + (-5y - 7y)\\ &= 9x^2 - 4yx - 12y\end{aligned}$$

Notice that $2yx$ and $-6xy$ are like terms even though the variables are written in a different order. This is an application of the commutative law introduced in Section 1.1.

EXAMPLE 2.6

$$\begin{aligned}(4y - 3z + 4xy) - (7y - 3z - 6xy)\\ &= 4y - 3z + 4xy - 7y + 3z + 6xy\\ &= (4y - 7y) + (-3z + 3z) + (4xy + 6xy)\\ &= -3y + 10xy\end{aligned}$$

Notice that when you have a negative sign in front of a parenthesis it means to multiply *each* term inside the parentheses by -1. So,

$$-(7y - 3z - 6xy) = -7y + 3z + 6xy$$

As mentioned in Section 1.2, there are several different symbols for grouping. These symbols are parentheses ( ), brackets [ ], and braces { }. When it is necessary to simplify an expression, you should start removing grouping symbols from the inside. For example, the expression

$$5\{2x - [3x + (4x - 5) + 2] - 3x\}$$

would be simplified by first removing the parentheses and combining any like terms that are within the brackets.

$$\begin{aligned}5\{2x - [3x + (4x - 5) + 2] - 3x\} &= 5\{2x - [3x + 4x - 5 + 2] - 3x\}\\ &= 5\{2x - [7x - 3] - 3x\}\end{aligned}$$

Next, remove the brackets. Do not forget the negative sign in front of the left bracket [. This indicates that each term inside the brackets is to be multiplied by -1 when the brackets are removed.

$$\begin{aligned}&= 5\{2x - [7x - 3] - 3x\}\\ &= 5\{2x - 7x + 3 - 3x\}\\ &= 5\{-8x + 3\}\end{aligned}$$

Now, remove the braces. The 5 in front of the left brace { means that each term within the braces should be multiplied by 5.

$$\begin{aligned}&= 5\{-8x + 3\}\\ &= -40x + 15\end{aligned}$$

APPLICATION ELECTRONICS

EXAMPLE 2.7 The total resistance, R, in a parallel circuit with two resistances, R_1 and R_2, is given by the formula $\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}$. Simplify the right-hand side of this equation.

SOLUTION In order to simplify these two fractions, we need to have a common denominator. For this fraction, the common denominator is R_1R_2.

$$\frac{1}{R} = \frac{R_2}{R_1R_2} + \frac{R_1}{R_1R_2} = \frac{R_2 + R_1}{R_1R_2}$$

EXERCISE SET 2.1

In Exercises 1–4, identify the variables and constants in each expression.

1. $4xy$
2. $\frac{5x^2}{3tw}$
3. $8\pi r^2$
4. $\sqrt{7}ax + by$

In Exercises 5–8, identify the algebraic terms in each expression.

5. $3x^3 + 4x$
6. $47x^4 - 9y^5$
7. $(2x^3)(5y) + \sqrt{3}ab - \frac{7a}{b}$
8. $-3x^5 - (4a)\left(\frac{b}{5}\right) + 9\sqrt{y}$

Solve Exercises 9 and 10.

9. What are the individual factors of $\frac{-5x^3}{y^2}$?
10. What are the individual factors of $\frac{15(x - y)}{x^2 + 3y}$?

In Exercises 11–14, identify the coefficient of each expression.

11. $47xy^5$
12. $\frac{2\sqrt{3}y}{x^5}$
13. $\frac{\pi ax}{4y}$
14. $\frac{\sqrt{17}byz}{\sqrt{5}at}$

In Exercises 15–20, identify the like terms in each exercise.

15. $3x^2y$, $17x^2y$, and $12xy^2$
16. $\sqrt{5}xy$, $\frac{2}{3}ax^5y$, and $\frac{2}{3}bxy$
17. $(x + y)^2$, $2(x + y)$, $4(x + y)^3$, and $5(x + y)^2$
18. $3x^2y$, $-9xy^2$, $5(xy^2)$, and $6(x^2 + y)$
19. $5(a + b + c)$, $a + b - c$, and $-2(a + b - c)$
20. ax^2, a^2x, $5x^2$, and $5a^5x$

In Exercises 21–70, simplify each algebraic expression.

21. $4x + 7x$
22. $5y - 2y$
23. $3z - z$
24. $7w + 4w - w$
25. $8x + 9x^2 - 2x$
26. $11y - 7y + 6y^2$
27. $10w + w^2 - 8w^2$
28. $y^2 - 6y^2 + 4y$
29. $ax^2 + a^2x + ax^2$

30. $by - by^2 + by$

31. $7xy^2 - 5x^2y + 4xy^2$

32. $12wz - 8w^2z + 6w^2z$

33. $(a + 6b) - (a - 6b)$

34. $(x - 7y) - (7y - x)$

35. $(2a^2 + 3b) + (2b + 4a)$

36. $(7c^2 - 8d) + (6d - 8c)$

37. $(4x^2 + 3x) - (2x^2 - 3x)$

38. $(3y^2 - 4x) - (4y + 2x)$

39. $2(6y^2 + 7x)$

40. $5(3a + 4b)$

41. $-3(4b - 2c)$

42. $-2(-6b + 3a)$

43. $4(a + b) + 3(b + a)$

44. $2(c + d) + 8(d + c)$

45. $3(x + y) - 2(x + y)$

46. $3(x^2 - y) - 2(y + x^2)$

47. $2(a + b + c) + 3(a + b - c)$

48. $4(x - y + z) + 2(x + y - z)$

49. $3[2(x + y)]$

50. $4[3(x - y)]$

51. $3(a + b) + 4(a + b) - 2(a + b)$

52. $2(x + y) - 3(x + y) - 4(x + y)$

53. $2(a + b + c) + 3(a - b + c) + (a - b - c)$

54. $3(x - y + z) - 2(x + y - z) + 4(-x - y + z)$

55. $2(x + 3y) - 3(x - 2y) + 5(2x - y)$

56. $3(y - 2a) - (a + 3y) + 4(a - 3y)$

57. $3(x + y - z) - 2(3x + 2y - z) - 3(x - y + 4z)$

58. $5(x - y + z) - (y - x + z) + 2(x - 2y + z)$

59. $(x + y) - 3(x - z) + 4(y + 4z) - 2(x + y - 3z)$

60. $(a + b) - 2(b - c) + 4(c + 2d) - 5(d + 2c - 3b - a)$

61. $x + [3x + 2(x + y)]$

62. $x + [5y + 3(y - x)]$

63. $y - [2z - 3(y + z) + y]$

64. $2w - [4z - 5(z + w) + 2w]$

65. $[2x + 3(x + y) - 2(x - y) + y] - 2x$

66. $[4a - 8(a + b) + 2(a - b) + 3a] - 3b$

67. $-\{-[2a - (3b + a)]\}$

68. $-\{-3[4x - (5x - 4y)]\}$

69. $5a - 2\{4[a + 2(4a + b) - b] + a\} - a$

70. $7x - 3\{-[x + 2(x - y) - y] + 2x\} - 3y$

Solve the word problems in Exercises 71–76.

71. ***Civil engineering*** In order to determine the cost of widening a highway, several cost comparisons were used. These led to the following expression for determining the total cost: $p + \frac{1}{2}p + \frac{2}{3}p$. Simplify by combining like terms.

72. ***Construction*** A concrete mix has the following ingredients by volume: 1 part cement, 2 parts water, 3 parts aggregate, and 3 parts sand. These are used to produce the expression $x + 2x + 3x + 3x$. This expression is then used to determine the amount of each ingredient for a specific amount of concrete. Simplify by combining like terms.

73. ***Electronics*** The total capacitance, C_T, in a series circuit with two capacitors, C_1 and C_2, is given by the formula

$$\frac{1}{C_T} = \frac{1}{C_1} + \frac{1}{C_2}$$

Simplify the right-hand side of this equation.

74. ***Electronics*** The total inductance, L_T, in a parallel circuit with three inductors, L_1, L_2, and L_3, is given by

$$\frac{1}{L_T} = \frac{1}{L_1} + \frac{1}{L_2} + \frac{1}{L_3}$$

Simplify the right-hand side of this equation.

75. ***Business*** A employee earns a gross wage W. The employee's payroll deductions are shown in Table 2.1. The employee's net wage N is the difference between the gross wage and the total deductions.

(a) Write an equation that shows how to determine the net wage from the information in the table.

(b) Simplify the equation.

76. ***Manufacturing*** The part in Figure 2.1 is used in several different-sized assemblies. What is the total length, L, of the part?

TABLE 2.1

Type of deduction	Federal income tax	Social Security (FICA)	FICA medical	Company retirement	Health insurance	Life insurance	401(k)
Deduction amount	$0.134W$	$0.046W$	$0.011W$	$0.075W$	$0.010W$	$0.002W$	$0.085W$

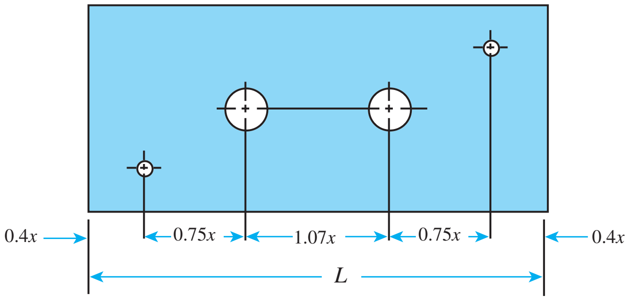

Figure 2.1

77. ***Electronics*** The total power in two circuits is given by $14(3I_a^2 + 5I_b^2)$. Use the distributive property to rewrite this expression.

78. ***Electronics*** The current in a circuit is given by $5t^2 + 7t + 3t^2 - 6t + 2$, where t is time. Simplify this expression.

[IN YOUR WORDS]

79. Explain how an algebraic term and an algebraic expression are alike and how they are different.

80. Describe what is meant by "like terms."

2.2 MULTIPLICATION

In Section 1.3 we learned some rules for exponents. The very first rule stated that $b^m b^n = b^{m+n}$. We will use this rule as a foundation for learning how to multiply multinomials. We will also use the rule for multiplying real numbers from Section 1.2.

MULTIPLYING MONOMIALS

First, we will find the product of two or more monomials. When multiplying monomials, first multiply the numerical coefficients to get the numerical coefficient of the product. Then, multiply the remaining factors using the rules of exponents.

EXAMPLE 2.8

$$(3x^3)(-7bx^4) = -21bx^{3+4} = -21bx^7$$
$$(4ax^2)(7bx^3) = 28abx^{2+3} = 28abx^5$$
$$(-5xy^2z^3)(3x^2y^2) = -15x^3y^4z^3$$
$$(\sqrt{2}x^3y)(\sqrt{3}xy^4) = \sqrt{6}x^4y^5$$

To multiply a monomial and a multinomial, use the distributive property. Distribute the monomial over the multinomial. Thus, by the distributive property $a(b + c) = ab + ac$. Each term of the multinomial will be multiplied by the monomial.

EXAMPLE 2.9

$$\begin{aligned} -7ab^2(2ax^3 - 5abx + 3b^3) &= (-7ab^2)(2ax^3) + (-7ab^2)(-5abx) \\ &\quad + (-7ab^2)(3b^3) \\ &= -14a^2b^2x^3 + 35a^2b^3x - 21ab^5 \end{aligned}$$

As you gain experience, you will find that you may not always need to write all the steps. The more you practice, the better you will become, until you may be able to write the correct answer without writing the middle expression.

MULTIPLYING MULTINOMIALS

To multiply one multinomial by another, multiply each term in one multinomial by each term in the other multinomial. Again, use the distributive property to help in the multiplication.

EXAMPLE 2.10

$$\begin{aligned} (x^2 + b)(x + c) &= x^2(x + c) + b(x + c) \\ &= x^3 + cx^2 + bx + bc \end{aligned}$$

EXAMPLE 2.11

$$\begin{aligned} (y^2 + 2f)(y^2 - 3f) &= y^2(y^2 - 3f) + 2f(y^2 - 3f) \\ &= y^4 - 3y^2f + 2y^2f - 6f^2 \\ &= y^4 - y^2f - 6f^2 \end{aligned}$$

The preceding two examples show multiplication of two binomials. The same procedure is used for multiplying any two multinomials. The next example shows how to multiply two trinomials. This example uses an extended version of the distributive property, where $a(b + c + d) = ab + ac + ad$.

EXAMPLE 2.12

$$\begin{aligned} (2x + 3y + z)(4x - 3y - z) &= 2x(4x - 3y - z) + 3y(4x - 3y - z) \\ &\quad + z(4x - 3y - z) \\ &= (8x^2 - 6xy - 2xz) + (12xy - 9y^2 - 3yz) \\ &\quad + (4xz - 3yz - z^2) \\ &= 8x^2 + 6xy + 2xz - 9y^2 - 6yz - z^2 \end{aligned}$$

Another method is to multiply in the same manner that you used with real numbers. This is shown in Example 2.13. When using this method, like terms are put in the same column.

EXAMPLE 2.13 $(x^3 + 2x^2 - 3x + 7)(5x^2 - 4x + 2)$

SOLUTION

$$\begin{array}{rrrrrr}
 & & x^3 & + \ 2x^2 & - \ 3x & + \ 7 \\
\times & & & 5x^2 & - \ 4x & + \ 2 \\
\hline
 & & 2x^3 & + \ 4x^2 & - \ 6x & + \ 14 \\
 & -\ 4x^4 & -\ 8x^3 & + \ 12x^2 & - \ 28x & \\
5x^5 & + \ 10x^4 & - \ 15x^3 & + \ 35x^2 & & \\
\hline
5x^5 & + \ 6x^4 & - \ 21x^3 & + \ 51x^2 & - \ 34x & + \ 14
\end{array}$$

THE FOIL METHOD FOR MULTIPLYING BINOMIALS

There will be many times when you will need to multiply two binomials. Because this happens so often, there are a few patterns that we should notice. The first pattern, called the **FOIL method**, is one that many people use to remember how to multiply two binomials. The letters for FOIL come from the first letters for the words First, Outside, Inside, and Last.

HINT Use the FOIL method to help you remember how to multiply binomials. FOIL indicates the order in which terms of two binomials could be multiplied.

F suggests the product of the **F**irst terms.
O suggests the product of the **O**utside terms.
I suggests the product of the **I**nside terms.
L suggests the product of the **L**ast terms.

EXAMPLE 2.14 Use the FOIL method to multiply $(x + 4)(x + 3)$.

SOLUTION

$\underbrace{(x + 4)(x}_{\text{F}} + 3)$ $\quad x^2 \quad$ the product of the **F**irst terms

$\underbrace{(x + 4)(x + 3)}_{\text{O}}$ $\quad 3x \quad$ the **O**utside terms

$(x + \underbrace{4)(x}_{\text{I}} + 3)$ $\quad 4x \quad$ the **I**nside terms

$(x + \underbrace{4)(x + 3}_{\text{L}})$ $\quad 12 \quad$ 81 the **L**ast terms

Adding and combining the terms in the right-hand column

$$x^2 + 3x + 4x + 12 = x^2 + 7x + 12$$

gives the correct answer.

This example looks much longer than it really is. Try it! You will find that this will help "foil" errors caused by not multiplying all the terms.

THE SPECIAL PRODUCT $(a + b)(a - b)$

One special product is shown by this example:

$$(x + 4)(x - 4) = x^2 - 4x + 4x - 16 = x^2 - 16$$

Notice that the two middle terms are additive inverses of each other and their sum is zero.

Let's look at the general case of this type of problem.

$$\begin{aligned}(a + b)(a - b) &= a^2 - ab + ab - b^2 \\ &= a^2 - b^2\end{aligned}$$

What is so special about this problem? Look at the two factors in the original problem. One factor is $a + b$ and the other is $a - b$. Notice that each factor has the same first term, a, and that the second terms are additive inverses of each other, b and $-b$. Whenever you have a product of two binomials in the form $a + b$ and $a - b$, the answer is $a^2 - b^2$.

DIFFERENCE OF SQUARES

$$(a + b)(a - b) = a^2 - b^2$$

EXAMPLE 2.15

$$(x + 7)(x - 7) = x^2 - 49 \qquad \text{Note that } 49 = 7^2.$$

$$(3a + b)(3a - b) = 9a^2 - b^2 \qquad \text{Note that } (3a)^2 = 9a^2.$$

$$\left(\frac{4}{3}x - 2y\right)\left(\frac{4}{3}x + 2y\right) = \frac{16}{9}x^2 - 4y^2$$

THE SPECIAL PRODUCT $(a + b)^2$

The final special product of binomials is a perfect square: $(a + b)^2$. Remember that $(a + b)^2 = (a + b)(a + b)$. If we multiply these we get

$$\begin{aligned}(a + b)^2 = (a + b)(a + b) &= a^2 + ab + ab + b^2 \\ &= a^2 + 2ab + b^2\end{aligned}$$

The answer contains the squares of the first and last terms. The middle term in the answer is twice the product of the two terms, a and b. So, when you have a square of a binomial you get $(a + b)^2 = a^2 + 2ab + b^2$ or $(a - b)^2 = a^2 - 2ab + b^2$.

SQUARE OF A BINOMIAL

$$(a + b)^2 = a^2 + 2ab + b^2$$

$$(a - b)^2 = a^2 - 2ab + b^2$$

EXAMPLE 2.16

$$(x + 3y)^2 = x^2 + 6xy + 9y^2$$
$$(t + 7)^2 = t^2 + 14t + 49$$
$$(4a - b)^2 = 16a^2 - 8ab + b^2$$
$$\left(\frac{ax}{2} - \frac{3c}{4}\right)^2 = \frac{a^2x^2}{4} - \frac{3}{4}acx + \frac{9c^2}{16} \qquad \text{Note that } 2\left(\frac{ax}{2}\right)\left(\frac{3c}{4}\right) = \frac{3}{4}acx.$$

CAUTION Remember: $(a + b)^2 \neq a^2 + b^2$ and $(a - b)^2 \neq a^2 - b^2$.

APPLICATION GENERAL TECHNOLOGY

EXAMPLE 2.17

A physics equation that describes an elastic collision is $m_1(v_a - v_b)(v_a + v_b) = m_2(v_c - v_d)(v_c + v_d)$. Perform the indicated multiplications.

SOLUTION We will begin by multiplying the expressions in parentheses.

$$m_1(v_a - v_b)(v_a + v_b) = m_2(v_c - v_d)(v_c + v_d)$$
$$m_1(v_a^2 - v_b^2) = m_2(v_c^2 - v_d^2)$$

Next, we shall multiply through the left-hand side by m_1 and the right-hand side by m_2, producing

$$m_1{v_a}^2 - m_1{v_b}^2 = m_2{v_c}^2 - m_2{v_d}^2$$

We cannot simplify this equation any more, because none of the remaining terms are like terms.

EXERCISE SET 2.2

In Exercises 1–62, perform the indicated multiplication.

1. $(a^2x)(ax^2)$
2. $(by^2)(b^2y)$
3. $(3ax)(2ax^2)$
4. $(5by)(3b^2y)$
5. $(2xw^2z)(-3x^2w)$
6. $(-4ya^2b)(6y^2b)$
7. $(3x)(4ax)(-2x^2b)$
8. $(4y)(3y^2b)(-5by^2)$
9. $2(5y - 6)$
10. $4(3x - 5)$
11. $-5(4w - 7)$
12. $-3(8 + 5p)$
13. $3x(7y + 4)$
14. $6x(8y - 7)$
15. $-5t(-3 + t)$
16. $-3n(2n - 5)$
17. $\frac{1}{2}a(4a - 2)$
18. $\frac{1}{3}x(-21x - 15)$
19. $2x(3x^2 - x + 4)$
20. $3y(4y^2 - 5y - 7)$
21. $4y^2(-5y^2 + 2y - 5 + 3y^{-1} - 6y^{-2})$
22. $5p^2(-4p^3 - 3p + 2 + p^{-1} - 7p^{-2})$
23. $(a + b)(a + c)$
24. $(s + t)(s + 2t)$
25. $(x + 5)(x^2 + 6)$
26. $(y + 3)(y^2 + 7)$
27. $(2x + y)(3x - y)$
28. $(4a + b)(8a - b)$
29. $(2a - b)(3a - 2b)$
30. $(4p + q)(3p - 2q)$
31. $(b - 1)(2b + 5)$
32. $(4x - 1)(3x - 2)$

33. $(7a^2b + 3c)(8a^2b - 3c)$

34. $(6p^2r + 2t)(5p^2r + 4t)$

35. $(x + 4)(x - 4)$

36. $(a + 8)(a - 8)$

37. $(p - 6)(p + 6)$

38. $(b - 10)(b + 10)$

39. $(ax + 2)(ax - 2)$

40. $(xy - 3)(xy + 3)$

41. $(2r^2 + 3x)(2r^2 - 3x)$

42. $(4p^3 - 7d)(4p^3 + 7d)$

43. $(5a^2x^3 - 4d)(5a^2x^3 + 4d)$

44. $(3p^2st - \frac{11}{3}w^3)(3p^2st + \frac{11}{3}w^3)$

45. $\left(\frac{2}{3}pa^2f + \frac{3}{4}tb^3\right) \times \left(-\frac{2}{3}pa^2f + \frac{3}{4}tb^3\right)$

46. $\left(\frac{\sqrt{3}}{2} + \frac{7}{5}t^2u\right) \times \left(-\frac{\sqrt{3}}{2} + \frac{7}{5}t^2u\right)$

47. $(x + y)^2$

48. $(p + r)^2$

49. $(x - 5)^2$

50. $(b - 7)^2$

51. $(a + 3)^2$

52. $(w + 5)^2$

53. $(2a + b)^2$

54. $(3c + d)^2$

55. $(3x - 2y)^2$

56. $(5a - 6f)^2$

57. $4x(x + 4)(3x - 2)$

58. $5y(y - 6)(2y + 3)$

59. $(x + y - z)(x - y + z)$

60. $(a + b + c)(a - b - c)$

61. $(a + 2)^3$

62. $(m - n)^3$

Solve Exercises 63–68.

63. ***Computer science*** The amount of time required to test a computer "chip" with n cells is given by the expression $2[n(n + 2) + n]$. Simplify this expression. (In this problem, asking you to simplify means that you are to perform the indicated multiplications and add the like terms.)

64. ***Aeronautical engineering*** An aircraft uses its radar to measure the direct echo range R to another object. If x represents the distance to the ground echo point, then the following expression results:

$$(2R - x)^2 - x^2 - R^2$$

Simplify this expression.

65. ***Construction*** In computing the center of mass of a plate of uniform density, the expression $\frac{1}{2}(y_2 - y_1)(y_2 + y_1)$ is used. Simplify this expression.

66. ***Business*** After 3 years, an investment of \$500 at an interest rate r compounded annually (once a year) is worth $500(1 + r)^3$. Perform the indicated multiplications.

67. ***Automotive technology*** Under certain conditions, position x of a motorcycle at time t is given by

$$x = 3(t + 4)(t + 1)$$

Simplify the right-hand side of this equation by multiplying the terms.

68. ***Electronics*** Electric power P in watts, W, is the product of the voltage E in volts, V, and the current I in amperes, A. If the current is increased to $4I - 5$, what is the electrical power in watts?

69. ***Landscaping*** A planter is in the shape of a trapezoid with bases $b_1 = 6'$ and $b_2 = 8'$ and height $h = 4'$. The area of a trapezoid is $A = \frac{1}{2}h(b_1 + b_2)$ so this planter has an area of $A = \frac{1}{2}(4)(6 + 8) = 28\,\text{ft}^2$. If the height and both bases are each lengthened by x ft, what is the new area?

70. ***Thermal science*** The quantity of heat, Q, transferred by radiation between two parallel equal areas, A, in time t is

$$Q = \bar{e}A\left\{\left(\frac{T_1}{100} - \frac{T_2}{100}\right)\left(\frac{T_1}{100} + \frac{T_2}{100}\right) \times \left[\left(\frac{T_1}{100}\right)^2 + \left(\frac{T_2}{100}\right)^2\right]\right\}10^8 t$$

where $\bar{e}$ is a radiation constant and T_1 and T_2 are absolute temperatures of the areas. Simplify this equation by multiplying the terms in the braces.

[IN YOUR WORDS]

71. Explain how to use the FOIL method for multiplying two binomials. (Do not look at the explanation in the book.)

72. Describe how you would multiply a binomial and a trinomial.

2.3 DIVISION

To find the quotient when one multinomial is divided by another, we will again need to use the rules of exponents. Rule 5 in Section 1.4 states that $\frac{b^m}{b^n} = b^{m-n}$, if $b \neq 0$. This is a very helpful rule in the division of algebraic expressions.

Remember that division can be indicated by $x \div y$, $\frac{x}{y}$, or x/y. You must be very careful, when reading algebra problems and writing your answers, to watch how terms are grouped. For example, in $a^2 + b/c + 3$ the only division is b divided by c. Thus, $a^2 + b/c + 3$ is equivalent to $a^2 + \frac{b}{c} + 3$. Now, consider $(a^2 + b)/c + 3$. The parentheses indicate that the entire expression $a^2 + b$ is being divided by c. So $(a^2 + b)/c + 3 = \left(\frac{a^2 + b}{c}\right) + 3$. This could be written without parentheses as $\frac{a^2 + b}{c} + 3$. Finally, $(a^2 + b)/(c + 3)$ means that the expression $a^2 + b$ is being divided by the expression $c + 3$. So, $(a^2 + b)/(c + 3) = \frac{a^2 + b}{c + 3}$.

DIVIDING A MONOMIAL BY A MONOMIAL

We begin by dividing a monomial by a monomial. Consider $16x^5 \div 8x^2$. Using the laws of exponents, we have $x^5 \div x^2 = x^{5-2} = x^3$. If there are numerical coefficients, they are divided separately. So, $16x^5 \div 8x^2 = \frac{16}{8}\frac{x^5}{x^2} = 2x^{5-2} = 2x^3$.

Each variable should be treated separately. For example, $24x^3y^2 \div 8xy^2 = \frac{24}{8}\frac{x^3}{x}\frac{y^2}{y^2} = 3x^{3-1}y^{2-2} = 3x^2y^0 = 3x^2$.

EXAMPLE 2.18

Divide $6w^2xy^4z$ by $9wx^3y^2z$.

SOLUTION

$$\begin{aligned} 6w^2xy^4z \div 9wx^3y^2z &= \frac{6w^2xy^4z}{9wx^3y^2z} \\ &= \frac{6}{9}w^{2-1}x^{1-3}y^{4-2}z^{1-1} \\ &= \frac{2}{3}wx^{-2}y^2z^0 \\ &= \frac{2wy^2}{3x^2} \end{aligned}$$

EXAMPLE 2.19

Divide $35a^2b^3c^{-2}$ by $-7a^{-3}bc^2$.

SOLUTION

$$\begin{aligned}35a^2b^3c^{-2} \div -7a^{-3}bc^2 &= \frac{35a^2b^3c^{-2}}{-7a^{-3}bc^2}\\ &= \frac{35}{-7}a^{2-(-3)}b^{3-1}c^{-2-2}\\ &= -5a^5b^2c^{-4}\\ &= \frac{-5a^5b^2}{c^4}\end{aligned}$$

CAUTION Be careful! A typical error here is cancelling noncommon factors. Do not make either of the following mistakes:

$$\frac{\cancel{x}+y}{\cancel{x}} = 1 + y$$

$$\text{or} \quad \frac{x+\cancel{y}}{\cancel{y}} = x + 1$$

These are *not correct*, as you can see by substituting some numerical values for *x* or *y*.

DIVIDING A MULTINOMIAL BY A MONOMIAL

In order to divide a multinomial by a monomial, you should divide each term of the multinomial by the monomial. You may remember that $\dfrac{a+b}{c} = \dfrac{a}{c} + \dfrac{b}{c}$. (This is part of Rule 6 in Section 1.2.)

EXAMPLE 2.20

Divide $(4r^2st^3 - 8rs^3)$ by $-2rst^2$.

SOLUTION

$$\begin{aligned}(4r^2st^3 - 8rs^3) \div -2rst^2 &= \frac{4r^2st^3 - 8rs^3}{-2rst^2}\\ &= \frac{4r^2st^3}{-2rst^2} + \frac{-8rs^3}{-2rst^2}\\ &= -2rt + 4\frac{s^2}{t^2}\end{aligned}$$

Remember to divide each term in the multinomial by the monomial.

EXAMPLE 2.21

Divide $(6xy - 4xy^2 + 9x^2y - 12x^2y^2)$ by $3xy$.

SOLUTION

$$\begin{aligned}\frac{6xy - 4xy^2 + 9x^2y - 12x^2y^2}{3xy} &= \frac{6xy}{3xy} - \frac{4xy^2}{3xy} + \frac{9x^2y}{3xy} - \frac{12x^2y^2}{3xy}\\ &= 2 - \frac{4}{3}y + 3x - 4xy\end{aligned}$$

DIVIDING A MULTINOMIAL BY A MULTINOMIAL

The next step in the division process is to divide a multinomial by a multinomial. This is not as easy as dividing by a monomial, so we will learn it in two stages. In the first stage we will learn to divide a polynomial by a polynomial. In stage two we will apply what we learned in stage one.

Before we can learn to divide a polynomial by a polynomial we need to have a definition of a polynomial. A polynomial is a special type of multinomial. An algebraic sum is a **polynomial** in x if each term is of the form ax^n, where n is a non negative integer. One example of a polynomial in x is $7x^3 - \sqrt{3}x^2 + \frac{1}{2}x - 2$. (Remember that -2 can be written as $-2x^0$.) Another polynomial in x is $7x^{15} + 2x^3 - 5x$.

Not every polynomial is a polynomial in x. For example, $4y^3 - y^2$ is a polynomial in y and $\frac{-\sqrt{3}}{2}$ is a polynomial that is a constant. The multinomial $4x^3 + 5x^2 - 3 + x^{-3}$ is *not* a polynomial because the exponent on the last term is -3, which is not a non negative integer. The multinomial $6x + \sqrt{7x} = 6x + \sqrt{7}x^{1/2}$ is not a polynomial in x because the exponent on the variable in the last term is $\frac{1}{2}$, which is not an integer.

We will now examine how to divide one polynomial by another. Study Example 2.22. Make sure you read the comments in Example 2.22, because they explain the method we use to divide polynomials.

EXAMPLE 2.22

Divide $6x^2 - 4 - 4x + 8x^4$ by $2x + 1$.

SOLUTION

1. **Write both the dividend and the divisor in decreasing order of powers**.

The dividend is $6x^2 - 4 - 4x + 8x^4$. The largest power is 4, the next largest is 2, and so the dividend should be written as $8x^4 + 6x^2 - 4x - 4$. The divisor, $2x + 1$, is already in descending order of the powers.

2. **Are there any missing terms in the dividend? If so, write them with a coefficient of 0**.

Here the dividend does not have an x^3 term. Rewrite the dividend as $8x^4 + 0x^3 + 6x^2 - 4x - 4$.

3. **Set up the problem just as you would any long division problem**.

$$2x + 1\overline{)8x^4 + 0x^3 + 6x^2 - 4x - 4}$$

4. **Divide the first term in the dividend by the first term in the divisor**.

In this example the first term in the dividend is $8x^4$ and the first term in the divisor is $2x$. The result of this division, $4x^3$, is written above the dividend directly over the term with the same power.

$$\begin{array}{r} 4x^3 \\ 2x + 1\overline{)8x^4 + 0x^3 + 6x^2 - 4x - 4} \end{array}$$

EXAMPLE 2.22 (Cont.)

5. **Multiply the divisor by this first term of the quotient. Write the product below the dividend with like terms under the like terms in the dividend. Subtract this product from the dividend**. We will call this difference the "new dividend."

$$\begin{array}{r} 4x^3 \phantom{{}+6x^2-4x-4} \\ 2x+1\overline{)8x^4+0x^3+6x^2-4x-4} \\ \underline{8x^4+4x^3\phantom{{}+6x^2-4x-4}} \\ -4x^3+6x^2-4x-4 \end{array} \quad \text{new dividend}$$

CAUTION Be careful doing the subtraction. A common error would be to get $+4x^3$ for the x^3 term instead of $-4x^3$.

6. **Repeat the last two steps until the power of the new dividend is less than the power of the divisor**. Each time the last two steps are repeated, you should divide the first term of the new dividend by the first term of the divisor as shown here.

A.

$$\begin{array}{r} 4x^3-2x^2\phantom{{}-4x-4} \\ 2x+1\overline{)8x^4+0x^3+6x^2-4x-4} \\ \underline{8x^4+4x^3\phantom{{}+6x^2-4x-4}} \\ -4x^3+6x^2-4x-4 \\ \underline{-4x^3-2x^2\phantom{{}-4x-4}} \\ 8x^2-4x-4 \end{array} \quad \text{new dividend}$$

B.

$$\begin{array}{r} 4x^3-2x^2+4x\phantom{{}-4} \\ 2x+1\overline{)8x^4+0x^3+6x^2-4x-4} \\ \underline{8x^4+4x^3\phantom{{}+6x^2-4x-4}} \\ -4x^3+6x^2-4x-4 \\ \underline{-4x^3-2x^2\phantom{{}-4x-4}} \\ 8x^2-4x-4 \\ \underline{8x^2+4x\phantom{{}-4}} \\ -8x-4 \end{array} \quad \text{new dividend}$$

C.

$$\begin{array}{r} 4x^3-2x^2+4x-4 \\ 2x+1\overline{)8x^4+0x^3+6x^2-4x-4} \\ \underline{8x^4+4x^3\phantom{{}+6x^2-4x-4}} \\ -4x^3+6x^2-4x-4 \\ \underline{-4x^3-2x^2\phantom{{}-4x-4}} \\ 8x^2-4x-4 \\ \underline{8x^2+4x\phantom{{}-4}} \\ -8x-4 \\ \underline{-8x-4} \\ 0 \end{array}$$

Stop! This is really $0x^0$ and its power, 0, is less than the power of the first term in the divisor, 1. So,

$$(8x^4+6x^2-4x-4) \div (2x+1) = 4x^3-2x^2+4x-4$$

When you divide one polynomial by another, your work should look like the final step, C. The other steps were given to help you see what we were doing. In the next example we will show it all in one step.

EXAMPLE 2.23

Divide $4x^5 - 2x^3 + 6x^2 - 8$ by $4x^2 + 2$.

SOLUTION The dividend and divisor are already in descending order. The dividend, $4x^5 - 2x^3 + 6x^2 - 8$, does not have an x^4 or an x^1 term. We should include those terms with coefficients of zero. The dividend now becomes $4x^5 + 0x^4 - 2x^3 + 6x^2 + 0x - 8$. We will now set the problem up in the standard long division format and divide.

$$
\begin{array}{r|rrrrrr}
 & & & x^3 & + 0x^2 & - x & + \frac{3}{2} \\
\cline{2-7}
4x^2 + 2 & 4x^5 & + 0x^4 & - 2x^3 & + 6x^2 & + 0x & - 8 \\
 & 4x^5 & & + 2x^3 & & & \\
\cline{2-7}
 & & 0x^4 & - 4x^3 & + 6x^2 & + 0x & - 8 \\
 & & 0x^4 & & + 0x^2 & & \\
\cline{3-7}
 & & & - 4x^3 & + 6x^2 & + 0x & - 8 \\
 & & & - 4x^3 & & - 2x & \\
\cline{4-7}
 & & & & 6x^2 & + 2x & - 8 \\
 & & & & 6x^2 & & + 3 \\
\cline{5-7}
 & & & & & 2x & - 11
\end{array}
$$

The power of the $2x$ term, 1, is less than the power of the x^2 term of the divisor, 2. This expression, $2x - 11$, is the remainder.

The remainder is written as $\dfrac{2x - 11}{4x^2 + 2}$. So,

$$
\begin{aligned}
(4x^5 - 2x^3 + 6x^2 - 8) \div (4x^2 + 2) &= x^3 + 0x^2 - x + \frac{3}{2} + \frac{2x - 11}{4x^2 + 2} \\
&= x^3 - x + \frac{3}{2} + \frac{2x - 11}{4x^2 + 2}
\end{aligned}
$$

There is a faster method of division that can be used with some polynomials. This method is known as synthetic division. We will learn about synthetic division in Chapter 17.

In the next example, we will look at a division problem in which both the dividend and the divisor have more than one variable. The method used will be very similar to the method for dividing one polynomial by another.

EXAMPLE 2.24

Divide $27x^6 - 8y^6$ by $3x^2 - 2y^2$.

SOLUTION

$$\begin{array}{r}
9x^4 + 6x^2y^2 + 4y^4 \\
3x^2 + 2y^2 \overline{)\,27x^6 \qquad\qquad\qquad - 8y^6} \\
\underline{27x^6 - 18x^4y^2 \qquad\qquad} \\
18x^4y^2 \qquad - 8y^6 \\
\underline{18x^4y^2 - 12x^2y^4 \qquad} \\
12x^2y^4 - 8y^6 \\
\underline{12x^2y^4 - 8y^6} \\
0
\end{array}$$

So, $(27x^6 - 8y^6) \div (3x^2 - 2y^2) = 9x^4 + 6x^2y^2 + 4y^4$.

EXERCISE SET 2.3

In Exercises 1–86, perform the indicated division.

1. x^7 by x^3
2. y^8 by y^6
3. $2x^6$ by x^4
4. $3w^4$ by w^2
5. $12y^5$ by $4y^3$
6. $15a^7$ by $3a^4$
7. $-45ab^2$ by $15ab$
8. $-55xy^3$ by $-11xy$
9. $33xy^2z$ by $3xyz$
10. $65x^2yz$ by $5xyz$
11. $96a^2xy^3$ by $-16axy^2$
12. $105b^3yw^2$ by $-15b^2yw$
13. $144c^3d^2f$ by $8cf$
14. $162x^2yz^3$ by $9x^2z^2$
15. $9np^3$ by $-15n^3p^2$
16. $15rs^2t$ by $-27r^2st^3$
17. $8abcdx^2y$ by $14adxy^2$
18. $9efg^2hr$ by $24e^2fh^3r$
19. $2a^3 + a^2$ by a
20. $4x^4 - x^3$ by x^2
21. $36b^4 - 18b^2$ by $9b$
22. $49y^5 + 35y^3$ by $7y^2$
23. $42x^2 + 28x$ by 7
24. $56z^6 - 48z^3$ by 8
25. $34x^5 - 51x^2$ by $17x^2$
26. $105w^6 + 63w^4$ by $21w^2$
27. $24x^6 - 8x^4$ by $-4x^3$
28. $42y^7 - 24y^5$ by $6y^4$
29. $5x^2y + 5xy^2$ by xy
30. $7a^2b - 7ab^2$ by ab
31. $10x^2y + 15xy^2$ by $5xy$
32. $25p^2q - 15pq^2$ by $5pq$
33. $ap^2q - 2pq$ by pq
34. $bx^2w + 3xw$ by xw
35. $a^2bc + abc$ by abc
36. $x^3yz - xyz$ by xyz
37. $9x^2y^2z - 3xyz^2$ by $-3xyz$
38. $12a^2b^2c + 4abc^2$ by $-4abc$
39. $b^3x^2 + b^3$ by $-b$
40. $c^5y^3 - cy^2$ by $-c$
41. $x^2y + xy - xy^2$ by xy
42. $ab^2 - ab + a^2b$ by ab
43. $18x^3y^2z - 24x^2y^3z$ by $-12x^2yz$
44. $36a^4b^2c - 27a^2b^4c^2$ by $-27a^2bc$
45. $x^2 + 7x + 12$ by $x + 3$
46. $x^2 + x - 12$ by $x + 4$
47. $x^2 - 3x + 2$ by $x - 2$
48. $x^2 - 2x - 15$ by $x - 5$
49. $x^2 + x - 2$ by $x + 2$
50. $x^2 + x - 6$ by $x + 3$
51. $6a^2 + 17a + 7$ by $3a + 7$
52. $4b^2 + 10b - 6$ by $4b - 2$
53. $8y^2 - 8y - 6$ by $2y - 3$
54. $12t^2 + t - 6$ by $4t + 3$
55. $x^3 - 5x + 2$ by $x^2 + 2x - 1$
56. $d^3 + d^2 - 3d + 2$ by $d^2 + d + 2$
57. $6a^3 - 7a^2 + 10a - 4$ by $3a^2 - 2a + 4$
58. $9y^3 - 16y + 8$ by $3y^2 + 3y - 4$
59. $4x^3 - 3x + 4$ by $2x - 1$
60. $7p^3 + 2p - 5$ by $3p + 2$
61. $r^3 - r^2 - 6r + 5$ by $r + 2$
62. $2c^3 - 3c^2 + c - 4$ by $c - 2$
63. $x^4 - 81$ by $x - 3$
64. $y^4 - 81$ by $y + 3$
65. $23x^2 - 5x^4 + 12x^5 - 12 - 14x^3 + 8x$ by $3x^2 - 2 + x$
66. $10a + 21a^3 - 35 + 6a^5 - 25a^4 + 21a^7 - 14a^6$ by $2a + 7a^3 - 7$

67. $x^2 - y^2$ by $x - y$
68. $a^2 - b^2$ by $a + b$
69. $w^3 - z^3$ by $w - z$
70. $x^3 + y^3$ by $x + y$
71. $x^3 + xy^2 + x^2y + y^3$ by $x + y$
72. $a^3 - a^2b + ab^2 - b^3$ by $a - b$
73. $c^3d^3 - 8$ by $cd - 2$
74. $e^3f^3 + 27$ by $ef + 3$
75. $x^2 - 2xy + y^2$ by $x - y$
76. $a^2 + 6ab + 9b^2$ by $a + 3b$
77. $5p^3r - 10p^2 + 15pr^2 - p^2r^2 + 2pr - 3r^3$ by $5p - r$
78. $8x^3 - 12x^2y + 6xy^2 - y^3$ by $2x - y$
79. $a^2 - 2ad - 3d^2 + 3a - 13d - 8$ by $a - 3d - 1$
80. $2x^2 - xy - 6y^2 + x + 19y - 15$ by $2x + 3y - 5$
81. $a^2f - af^2$ by $a - f$
82. $d^3 - 2dm^2 + 2m^3 - d^2m$ by $d - m$
83. $a^2 - b^2 + 2bc - c^2$ by $a - b + c$
84. $e^2 + 2eh - f^2 + h^2$ by $e + f + h$
85. $a^4 + 2a^2 - a + 2$ by $a^2 + a + 2$
86. $x^6 - x^4 + 2x^2 - 1$ by $x^3 - x + 1$

Solve Exercises 87–92.

87. ***Electronics*** The expression for the total resistance of three resistances in a parallel electrical circuit is

$$\frac{R_1R_2R_3}{R_2R_3 + R_1R_3 + R_1R_2}$$

Determine the reciprocal of this expression, and then perform the indicated division.

88. ***Thermal science*** The radiation constant, $\bar{e}$ (see Section 2.2, Exercise 70, page 82), is defined as

$$\bar{e} = \frac{\sigma}{\frac{e_1 + e_2 - e_1e_2}{e_1e_2}}$$

where e_1 and e_2 are radiation emissives of the respective areas and σ is the Stefan–Boltzmann constant. Find an equivalent formula by writing the denominator as three separate terms and simplifying those terms.

89. ***Thermal science*** One reference book gives the formula for the volume change of a gas under a constant pressure due to the changes in temperature from T_1 to T_2 as

$$V_2 = V_1\left(1 + \frac{T_2 - T_1}{T_1}\right)$$

where V_1 is the volume at T_1 and V_2 is the volume at T_2. Simplify the right-hand side of this equation.

90. The area of a certain rectangle is given by $x^3 + 6x^2 - 7x$, and the length of one side of the rectangle is $x + 7$.
    (a) If the area of a rectangle is the product of the length and the width, what algebraic expression describes the width of this rectangle?
    (b) If $x = 4$ ft, what are the length, width, and area of this rectangle?

91. ***Electronics*** Resistance R in ohms, Ω, is the quotient of the voltage V divided by the current I. That is, $R = \frac{V}{I}$. If the voltage is $24r^2 - 15r$ and the current is $3r$, what is the resistance?

92. ***Mechanics*** The tension T on a wire rope basket is given by the formula $T = \frac{WL}{NV}$, where W is the weight being lifted, L is the length of the chocker, N is the number of chockers, and V is the vertical distance from the load to the hook. Determine the tension if $W = 4x^2 + 2x - 2$, $L = 15x + 75$, $N = 4x + 20$, and $V = 2x^2 + x - 1$.

[IN YOUR WORDS]

93. Describe how to divide a multinomial by a monomial.

94. Describe how to divide a multinomial by a multinomial.

2.4 SOLVING EQUATIONS

Until now we have worked with real numbers and learned how to add, subtract, multiply, and divide algebraic expressions. In these next two sections we are going to begin learning how to use these skills to solve problems.

The ability to solve a problem often depends on the ability to write an equation for a problem and then solve that equation. We will use this section to learn how to solve an equation. In the next section we will start to learn how to write an equation for a problem.

EQUATIONS

An **equation** is an algebraic statement. It asserts that two algebraic expressions are equal. The two algebraic expressions are called the left-hand side (or left side) of the equation and the right-hand side (or right side) of the equation. As the name indicates, the left-hand side is to the left of the equal sign and the right-hand side is to the right of the equal sign.

$$\underbrace{4x^3 + 2x - \sqrt{3}}_{\text{left-hand side}} = \underbrace{7x^2 - 5x}_{\text{right-hand side}}$$

Some equations are true for all values of their variables. These equations are called **identities**. For example, $(x + 2)(x - 2) = x^2 - 4$ is an identity. An equation that is true for only some values of the variables and not true for the other values is a **conditional equation**. Examples of conditional equations are $4x = -24$ and $y^2 = 25$. The first is true only when $x = -6$. The second is true only when $y = 5$ or $y = -5$.

Any value that can be substituted for the variable and that makes an equation true is called a **solution** or **root** of the equation. So, 5 is a solution of $2x^2 - 5x - 5 = 20$, since $2(5^2) - 5(5) - 5 = 20$. The root of $x + 7 = 10$ is 3, and this is often indicated by the phrase, "3 satisfies the equation $x + 7 = 10$."
To solve an equation means to find all of its solutions or roots.

EQUIVALENT EQUATIONS

The purpose of this section is to help you learn how to find the solutions or roots of an equation. The techniques you learn here will be used whenever you need to solve an equation. Two equations are **equivalent** if they have exactly the same roots. There is a series of five operations that will allow you to change an equation into an equivalent equation.

To change an equation into an equivalent equation you can use any of the following five operations.

OPERATIONS FOR CHANGING EQUATIONS INTO EQUIVALENT EQUATIONS

1. Add or subtract the same algebraic expression or amount to/from both sides of the equation.
2. Multiply or divide both sides of the equation by the same algebraic expression, provided the expression does not equal zero.
3. Combine like terms on either side of the equation.
4. Replace one side (or both sides) with an identity.
5. Interchange the two sides of the equation.

In order to solve an equation you will generally have to use a combination of these five operations. You may have to use the same operation more than once. The following examples show how these operations are used. The first four examples use one operation and the other examples use a combination of operations.

EXAMPLE 2.25

Solve $x + 7 = 15$.

SOLUTION

$$x + 7 = 15$$

$$(x + 7) - 7 = 15 - 7 \qquad \text{Subtract 7 from both sides (Operation \#1).}$$

$$x = 8$$

Substituting this value in the original equation verifies that 8 is the solution, since $8 + 7 = 15$.

EXAMPLE 2.26

Solve $3x = -15$.

SOLUTION

$$3x = -15$$

$$\frac{1}{3}(3x) = \frac{1}{3}(-15) \qquad \text{Multiply both sides by } \tfrac{1}{3}\text{, the reciprocal of 3 (Operation \#2).}$$

$$x = -5$$

Checking the solution in the original equation, we see that -5 satisfies the equation, since $3(-5) = -15$.

EXAMPLE 2.27

Solve $5x - 3x - x = 20 - 12$.

SOLUTION

$$5x - 3x - x = 20 - 12$$

$$x = 8 \qquad \text{Combine like terms (Operation \#3).}$$

Checking the root in the original, we see that $5 \times 8 - 3 \times 8 - 8 = 40 - 24 - 8 = 8$. This satisfies the original equation.

EXAMPLE 2.28

Solve $x^2 + 6x + 9 = 0$.

SOLUTION

$x^2 + 6x + 9 = 0$

$(x + 3)^2 = 0$ Replace $x^2 + 6x + 9$ with $(x + 3)^2$ (Operation #4).

We will stop this example here. To continue solving this problem takes some more mathematics, which we will learn later. The important thing to notice is that we used an identity to rewrite one side of the equation.

The next four examples use combinations of the operations.

EXAMPLE 2.29

Solve $3x + 27 = 6x$.

SOLUTION

$$\begin{aligned} 3x + 27 &= 6x && \\ (3x + 27) - 3x &= 6x - 3x && \text{Subtract } 3x \text{ (Operation \#1).} \\ 27 &= 3x && \text{Combine like terms (Operation \#3).} \\ 3x &= 27 && \text{Switch sides (Operation \#5).} \\ \frac{1}{3}(3x) &= \frac{1}{3}(27) && \text{Multiply by } \tfrac{1}{3} \text{ (Operation \#2).} \\ x &= 9 && \end{aligned}$$

To check, replace the x in the original equation with the value we found, 9. The left-hand side is $3(9) + 27 = 27 + 27 = 54$ and the right-hand side is $6(9) = 54$. Since both sides give the same value, 9 is a solution.

EXAMPLE 2.30

Solve $7y + 6 = 216 - 3y$.

SOLUTION

$$\begin{aligned} 7y + 6 &= 216 - 3y && \\ 7y + 6 + 3y &= 216 - 3y + 3y && \text{Add } 3y \text{ (Operation \#1).} \\ 10y + 6 &= 216 && \text{Combine terms (Operation \#3).} \\ 10y + 6 - 6 &= 216 - 6 && \text{Subtract 6 (Operation \#1).} \\ 10y &= 210 && \text{Combine terms.} \\ \frac{10y}{10} &= \frac{210}{10} && \text{Divide by 10 (Operation \#2).} \\ y &= 21 && \end{aligned}$$

Check this answer in the original equation. When you replace y with 21 do you get the same value on both the left-hand and right-hand sides of the equation? You should.

Notice that in the next-to-last step we divided both sides by 10. We could have multiplied both sides by $\frac{1}{10}$ and arrived at the same result. Do you understand why? Both approaches are correct because division by 10 is the same as multiplication by $\frac{1}{10}$.

EXAMPLE 2.31

Solve $3(4z-8) + 2(3z + 5) = 4(z + 7)$.

SOLUTION

$3(4z - 8) + 2(3z + 5) = 4(z + 7)$	
$12z - 24 + 6z + 10 = 4z + 28$	Remove parentheses by distribution.
$18z - 14 = 4z + 28$	Combine terms.
$14z - 14 = 28$	Subtract 4z.
$14z = 42$	Add 14.
$z = 3$	Divide by 14 (or multiply by $\frac{1}{14}$).

Check this in the original equation. You should get 40 on each side of the equation.

EXAMPLE 2.32

Solve $\frac{n}{2} + 5 = \frac{2n}{3}$.

SOLUTION

$\frac{n}{2} + 5 = \frac{2n}{3}$	
$6\left(\frac{n}{2} + 5\right) = 6\left(\frac{2n}{3}\right)$	Multiply by 6. (6 is a common denominator of $\frac{n}{2}$ and $\frac{2n}{3}$.)
$3n + 30 = 4n$	
$30 = n$	Subtract $3n$.

Because most people like to give the variable first and then the value of that variable, you could now use Operation #5 and rewrite this as $n = 30$.

Check by substituting 30 for n in the original equation. The left side is $\frac{30}{2} + 5 = 15 + 5 = 20$. The right side is $\frac{2(30)}{3} = \frac{60}{3} = 20$. So, $n = 30$ satisfies the original equation.

Ratios and proportions are useful methods for solving equations.

RATIO

A **ratio** is a comparison of two quantities. If these quantities are represented by the values a and b, then the ratio comparing a to b is written $\frac{a}{b}$ or $a : b$. A **pure ratio** compares two quantities that have the same units. A pure ratio is written without units.

EXAMPLE 2.33

What is the ratio of (a) 4 m to 3 m, (b) 5 ft to 4 yards, and (c) 2 m to 35 cm?

SOLUTION We will write each of these as a pure ratio.

(a) $\frac{4\text{ m}}{3\text{ m}} = \frac{4}{3}$ or 4 : 3

(b) $\frac{5\text{ ft}}{4\text{ yards}} = \frac{5\text{ ft}}{12\text{ ft}} = \frac{5}{12}$ or 5 : 12

(c) $\frac{2\text{ m}}{35\text{ cm}} = \frac{200\text{ cm}}{35\text{ cm}} = \frac{200}{35} = \frac{40}{7}$ or 40 : 7

NOTE The rational numbers got their name because they indicate a ratio of two numbers.

Some ratios are always expressed as a quantity compared to 1, as shown in the next example.

APPLICATION MECHANICAL

EXAMPLE 2.34

A driven gear has 26 teeth and the driving gear has 8 teeth. What is the gear ratio of the driven gear to the driving gear?

SOLUTION

$$\frac{\text{driven gear}}{\text{driving gear}} = \frac{26\text{ teeth}}{8\text{ teeth}} = \frac{26}{8} = \frac{13}{4} = \frac{3.25}{1} \text{ or } 3.25 : 1$$

RATE

A **rate** is a ratio that compares different kinds of units. Common rates include miles and gallons, revolutions and minutes, and miles and hours.

APPLICATION AUTOMOTIVE

EXAMPLE 2.35

An automobile travels 243 km in 3 h. What is its rate of speed?

SOLUTION $\frac{243\text{ km}}{3\text{ h}} = \frac{81\text{ km}}{1\text{ h}} = 81\text{ km/h}$

The car is traveling at the rate of 81 km/h.

PROPORTION

A statement that says two ratios are equal is called a **proportion**. If the ratio $a : b$ is equal to the ratio $c : d$, then we have the proportion $a : b = c : d$ or

$$\frac{a}{b} = \frac{c}{d}$$

which is read, "a is to b as c is to d."

Two ratios are equal, and hence form a proportion, when the **cross-products** are equal. This is stated algebraically as shown in the box.

EQUAL RATIOS

Two ratios $a : b$ and $c : d$ are equal, if $ad = bc$.

Stated using the "fractional notation" for ratios, this says that

$$\frac{a}{b} = \frac{c}{d}, \text{ if and only if } ad = bc$$

The following diagram may help you to remember which terms to multiply and also to explain why this is called a cross-product.

$$\frac{a}{b} \quad \frac{c}{d}$$

EXAMPLE 2.36

What is the missing number in the proportion $5 : 7 = x : 42$?

SOLUTION We are given $\frac{5}{7} = \frac{x}{42}$. Cross multiplying, we get

$$\begin{aligned} 7x &= 5 \cdot 42 \\ &= 210 \\ x &= 30 \end{aligned}$$

We have determined that $5 : 7 = 30 : 42$.

APPLICATION AUTOMOTIVE

EXAMPLE 2.37

If 74.5 L of fuel are used to drive 760 km, how many liters are needed to drive 3 754 km?

SOLUTION We have the rates of 74.5 L to 760 km and x L to 3 754 km. We will form a proportion where the left-hand side equals the ratio $\frac{\text{L}}{\text{km}}$. We must then have $\frac{\text{L}}{\text{km}}$ on the right-hand side. If we set these two rates equal, we have

$$\frac{74.5 \text{ L}}{760 \text{ km}} = \frac{x \text{ L}}{3754 \text{ km}}$$

EXAMPLE 2.37 (Cont.)

The cross-product is

$$\begin{aligned} 760x &= (74.5)(3\,754) \\ &= 279\,673 \\ x &= 367.99079 \end{aligned}$$

So, it would require about 368 L of fuel to travel 3 754 km.

HINTS FOR SOLVING EQUATIONS

Several of the examples we have worked used a combination of the five operations used to make equivalent equations. Each of the examples showed how to use one or more of the hints for solving problems. As you continue through this book, you will get some more hints to help you become a better problem-solver.

HINTS FOR SOLVING EQUATIONS

1. Eliminate fractions. Multiply both sides of the equation by a common denominator.
2. Remove grouping symbols. Perform the indicated multiplications and then combine terms to remove parentheses, brackets, and braces.
3. Combine like terms whenever possible.
4. Get all terms containing the variable on one side of the equation. All other terms should be placed on the other side of the equation.
5. Check your answer in the original equation.

Here are two more examples. See if you can work each problem before studying the solution given.

EXAMPLE 2.38

Solve $\dfrac{3p + 7}{4} - \dfrac{2(p - 5)}{3} = \dfrac{p - 3}{2} + 6$.

SOLUTION

$$\frac{3p + 7}{4} - \frac{2(p - 5)}{3} = \frac{p - 3}{2} + 6$$

$$12\left[\frac{3p + 7}{4} - \frac{2(p - 5)}{3}\right] = 12\left[\frac{p - 3}{2} + 6\right]$$ Multiply both sides by 12, a common denominator of 2, 3, and 4.

$$12\left[\frac{3p + 7}{4}\right] - 12\left[\frac{2(p - 5)}{3}\right] = 12\left[\frac{p - 3}{2}\right] + 12(6)$$ Distribute the 12.

$$\begin{aligned} 3[3p + 7] - 4[2(p - 5)] &= 6[p - 3] + 12(6) \\ 9p + 21 - 4[2p - 10] &= 6p - 18 + 72 \end{aligned}$$ Remove grouping symbols.

$$\begin{aligned} 9p + 21 - 8p + 40 &= 6p - 18 + 72 && \\ p + 61 &= 6p + 54 && \text{Combine terms.} \\ 7 &= 5p && \text{Use hint \#4.} \\ \frac{7}{5} &= p && \text{Divide by 5.} \end{aligned}$$

The solution is $\frac{7}{5}$, or 1.4. Check. You should get 5.2 on both sides of the equation when you evaluate each side with $p = 1.4$. This would be a good time to practice using your calculator to see if you get 5.2 on each side of the equation.

EXAMPLE 2.39

Solve $\dfrac{3w + a}{2b} - \dfrac{4(w + a)}{b} = \dfrac{2w - 6}{3b} + \dfrac{2 + 6a}{b}$ for w.

SOLUTION There are several letters in this equation, a, b, and w, but you are asked to solve for w. This means that w is the variable and that a and b should be treated as constants.

$$\frac{3w + a}{2b} - \frac{4(w + a)}{b} = \frac{2w - 6}{3b} + \frac{2 + 6a}{b}$$

$$6b\left[\frac{3w + a}{2b} - \frac{4(w + a)}{b}\right] = 6b\left[\frac{2w - 6}{3b} + \frac{2 + 6a}{b}\right] \quad \text{Multiply both sides by a common denominator } 6b.$$

$$\begin{aligned} 6b\left[\frac{3w + a}{2b}\right] - 6b\left[\frac{4(w + a)}{b}\right] &= 6b\left[\frac{2w - 6}{3b}\right] + 6b\left[\frac{2 + 6a}{b}\right] && \\ 3[3w + a] - 6[4(w + a)] &= 2[2w - 6] + 6[2 + 6a] && \\ 9w + 3a - 24w - 24a &= 4w - 12 + 12 + 36a && \text{Remove grouping.} \\ -15w - 21a &= 4w + 36a && \text{Combine like terms.} \\ -15w - 4w &= 36a + 21a && \text{Hint \#4.} \\ -19w &= 57a && \text{Combine like terms.} \\ w &= \frac{57a}{-19} && \text{Divide both sides by } -19. \\ w &= -3a && \end{aligned}$$

Check. Substitute $-3a$ for w in the original equation. When you simplify each side, you get $\frac{4a}{b}$. Although it is easier to substitute for w at a later step, you might have made a mistake getting to that step. Sometimes you will even make an error when you copy the problem onto your paper. Always go back to the original problem to check your work.

EXERCISE SET 2.4

Solve the equations in Exercises 1–72.

1. $x - 7 = 32$
2. $y - 8 = 41$
3. $a + 13 = 25$
4. $b + 21 = 34$
5. $25 + c = 10$
6. $28 + d = 12$
7. $4x = 18$
8. $5y = 12$
9. $-3w = 24$
10. $6z = -42$
11. $21c = -14$
12. $24d = 16$
13. $\frac{p}{3} = 5$
14. $\frac{r}{5} = 4$
15. $\frac{t}{4} = -6$
16. $\frac{s}{-3} = -5$
17. $4a + 3 = 11$
18. $3b + 4 = 16$
19. $7 - 8d = 39$
20. $9 - 7c = 44$
21. $2.3w + 4.1 = 13.3$
22. $3.5z + 5.2 = 22.7$
23. $2x + 5x = 28$
24. $3y + 8y = 121$
25. $3a + 2(a + 5) = 45$
26. $4b + 3(7 + b) = 56$
27. $4(6 + c) - 5 = 21$
28. $5(7 + d) + 4 = 31$
29. $2(p - 4) + 3p = 16$
30. $7(n - 5) + 4n = 16$
31. $3x = 2x + 5$
32. $4y = 3y + 7$
33. $4w = 6w + 12$
34. $7z = 10z + 42$
35. $9a = 54 + 3a$
36. $8b = 55 + 3b$
37. $\frac{5x}{2} = \frac{4x}{3} - 7$
38. $\frac{3y}{7} = \frac{2y}{3} + 4$
39. $\frac{6p}{5} = \frac{3p}{2} + 4$
40. $\frac{5z}{3} = \frac{4z}{5} - 3$
41. $8n - 4 = 5n + 14$
42. $9p - 5 = 6p + 37$
43. $7r + 3 = 11r - 21$
44. $8s + 7 = 15s - 56$
45. $\frac{6x - 3}{2} = \frac{7x + 2}{3}$
46. $\frac{4r - 3}{3} = \frac{5r + 2}{2}$
47. $\frac{3t + 4}{4} = \frac{2t - 5}{2}$
48. $\frac{6a - 5}{3} = \frac{7a + 5}{6}$
49. $3(x + 5) = 2x - 3$
50. $2(y - 3) = 4 + 3y$
51. $\frac{x}{2} + \frac{x}{3} - \frac{x}{4} = 2$
52. $\frac{p}{2} - \frac{p}{3} - \frac{p}{4} = 3$
53. $\frac{4(a - 3)}{5} = \frac{3(a + 2)}{4}$
54. $\frac{5(b + 4)}{3} = \frac{4(b - 5)}{5}$
55. Solve $ax + b = 3ax$ for x
56. Solve $2by = 6 + 4by$ for y
57. Solve $ax - 3a + x = 5a$ for a
58. Solve $2(by - c) = 3\left(\frac{y}{2} - c\right)$ for y
59. $\frac{3}{x} + \frac{4}{x} = 3$
60. $\frac{5}{y} - \frac{3}{y} = 6$
61. $\frac{3}{4p} + \frac{1}{p} = \frac{7}{4}$
62. $\frac{6}{5q} - \frac{2}{q} = \frac{6}{5}$
63. $\frac{1}{x + 1} - \frac{2}{x - 1} = 0$
64. $\frac{3}{x + 2} - \frac{4}{x - 2} = 0$
65. $\frac{3}{2x} = \frac{1}{x + 5}$
66. $\frac{4}{3x} = \frac{2}{x + 1}$
67. $\frac{2x + 1}{2x - 1} = \frac{x - 1}{x - 3}$
68. $\frac{2x + 3}{2x + 5} = \frac{5x + 4}{5x + 2}$
69. Solve $\frac{6b - 5a}{3} + \frac{5a + b}{2} = \frac{5(a + 2b)}{4} + 5$ for a
70. Solve $\frac{3r + t}{5t} - \frac{r + 5t}{3t} = \frac{2(r - 4)}{15t} + 2$ for r
71. Solve $\frac{2z + a}{3x} - \frac{9z + a}{6x} = \frac{z - a}{2x} + \frac{4a}{3x}$ for a
72. Solve $\frac{3p + 2x}{2y} - \frac{5p - 3x}{3y} = \frac{x - 2p}{y} + \frac{3x + p}{6y}$ for x

Express each of the rates in Exercises 73–76 as a ratio.

73. \$1.38 for 16 bolts **74.** 725 rpm **75.** 86 L/km **76.** 236 mi in 4 h

Write each of the ratios in Exercises 77–80 as a ratio compared to 1.

77. 9 : 2 **78.** $\frac{7}{5}$ **79.** $\frac{23}{7}$ **80.** 37 : 4

Solve each of the proportions in Exercises 81–88.

81. $\frac{7}{8} = \frac{c}{32}$

82. $\frac{3}{b} = \frac{9}{24}$

83. $\frac{124}{62} = \frac{158}{d}$

84. $\frac{a}{3.5} = \frac{8}{20}$

85. $\frac{a}{4} = \frac{0.16}{0.15}$

86. $\frac{7.5}{10.5} = \frac{x}{6.3}$

87. $\frac{20}{b} = \frac{8}{5.6}$

88. $\frac{2.4}{10.8} = \frac{1.6}{d}$

Solve Exercises 89–90.

89. ***Meteorology*** The formula for converting Celsius temperatures to Fahrenheit temperatures is $F = \frac{9}{5}C + 32$, where F represents the Fahrenheit temperature and C, the Celsius temperature. Find the formula for converting Fahrenheit temperatures to Celsius by solving this equation for C.

90. ***Physics*** The velocity v of a falling object after t seconds is given by the formula $v = v_0 + at$, where v_0 represents the initial velocity and a is the acceleration due to gravity.

(a) Solve this equation for t in order to determine the length of time that it takes the object to reach some velocity v.

(b) Use your answer to part (a) to determine how long it takes an object to strike the ground, if its initial velocity is 12 m/s, its final velocity is 97 m/s, and the acceleration is 9.8 m/s^2.

[IN YOUR WORDS]

91. Without looking in the text, write a summary of the operations you can use for solving equations.

92. Without looking in the text, write a summary of the hints for solving equations.

93. Explain the difference between a rate and a ratio.

2.5 DIMENSIONAL ANALYSIS

Whenever you solve a problem that involves measurements or units you may need to use dimensional analysis. **Dimensional analysis**, also called **unit analysis** or **unit conversion**, multiplies unit fractions and simplifies the resulting fraction. A **unit fraction** is a fraction that has a value of 1. Since 60 sec = 1 min, the fractions

$$\frac{1 \text{ min}}{60 \text{ sec}} \text{ and } \frac{60 \text{ sec}}{1 \text{ min}}$$

are examples of unit fractions.

The identity element for multiplication is 1 which means that multiplying a quantity by 1 does not change the value of the quantity. Symbolically this was written as $a \times 1 = 1 \times a = 1$. The next example will show how this works with unit fractions.

EXAMPLE 2.40

Change 15 min to seconds.

SOLUTION The given quantity here is 15 min. Write it as a fraction with 15 min in the numerator and 1 in the denominator, $\frac{15\,\text{min}}{1}$. You will now multiply this by a unit fraction that relates minutes and seconds. Since, minutes is in the numerator of the given fraction, select the unit fraction with minute in the denominator, $\frac{60\text{ sec}}{1\text{ min}}$. This allows the minutes to "cancel."

$$\begin{aligned}\frac{15\,\cancel{\text{min}}}{1} \times \frac{60\text{ sec}}{1\,\cancel{\text{min}}} &= 15 \times 60\text{ sec} \\ &= 900\text{ sec}\end{aligned}$$

Thus, we see that 15 min is the same as 900 sec.

Sometimes it is necessary to use more than one unit fraction. For example, if you want to convert 7 h to seconds you would first convert hours to minutes and then change minutes to seconds. We show this with the map

$$\text{hours} \longrightarrow \text{minutes} \longrightarrow \text{seconds}$$

The map indicates that we will need to use the two unit fractions $\frac{60\,\text{min}}{1\,\text{h}}$ and $\frac{60\,\text{sec}}{1\,\text{min}}$ as shown in the next example.

EXAMPLE 2.41

Change 7 h to seconds.

SOLUTION The given quantity here is 7 h. Write it as the fraction $\frac{7\,\text{h}}{1}$ with 7 h in the numerator and 1 in the denominator. Multiply this by two unit fractions, one that relates hours and minutes and the other that relates minutes and seconds.

Since we are converting *from* hours we need a unit fraction relating hours and minutes that has hours in the denominator. Thus, we select $\frac{60\,\text{min}}{1\,\text{h}}$. The next unit fraction will involve seconds and minutes. Since the unit fraction $\frac{60\,\text{min}}{1\,\text{h}}$ has minutes in the numerator the next unit fraction will have minutes in the denominator, $\frac{60\,\text{sec}}{1\,\text{min}}$. This allows both the hours and the minutes to "cancel."

$$\frac{7\,\cancel{h}}{1} \times \frac{60\,\cancel{\text{min}}}{1\,\cancel{h}} \times \frac{60\,\text{sec}}{1\,\cancel{\text{min}}} = 7 \times 60 \times 60\ \text{sec}$$
$$= 25{,}200\ \text{sec}$$

This means that 7 h is the same as 25,200 sec.

In order to do unit or dimensional analysis you must decide which form of a unit fraction to use. This will allow you to use dimensional analysis with units of measure that are not familiar to you.

DIMENSIONAL ANALYSIS

- Make a plan or map showing how you will get from the current unit of measure to the desired unit.
- Write the original quantity as a fraction with a denominator of 1.
- Write unit fractions so that the units you want to convert *from* are in the denominator and the units you want to convert *to* are in the numerator.
- A partial list of values that can be used to make unit fractions is in Table A.5 (see Appendix A).

EXAMPLE 2.42

Change 53 pounds to grams.

SOLUTION We begin by looking at Table A.5. There is no conversion from pounds to grams but there is a conversion factor from pounds to kilograms. We create the following map showing that we plan to change pounds to kilograms and then convert kilograms to grams.

pounds ⟶ **kilograms** ⟶ **grams**

The given quantity here is 53 pounds. Write it as a fraction with 53 pounds in the numerator and 1 in the denominator, $\frac{53\,\text{pounds}}{1}$. Since we want to change from pounds to kilograms we use the unit fraction $\frac{0.4535\,\text{kg}}{1\,\text{pound}}$. Next, we change from kilograms to grams with the unit fraction $\frac{1000\,\text{g}}{1\,\text{kg}}$. The result is the following product.

$$\frac{53\,\cancel{\text{pounds}}}{1} \times \frac{0.4535\,\cancel{\text{kg}}}{1\,\cancel{\text{pound}}} \times \frac{1000\,\text{g}}{1\,\cancel{\text{kg}}} = 53 \times 0.4535 \times 1000\ \text{g}$$
$$= 24\,035.5\ \text{g}$$

Thus, we see that 53 pounds is about 24 035.5 g.

For the final example we will change two units of measure.

EXAMPLE 2.43

Change 112 km/h to ft/s.

SOLUTION This involves two conversions. We need to change kilometers to feet and also to change hours to seconds. The map for changing kilometers to feet is

kilometers ⟶ miles ⟶ feet

and the map for changing hours to seconds is

hours ⟶ minutes ⟶ seconds

To change 112 km to feet we first write it as the fraction $\frac{112\,\text{km}}{1}$. Since we want to change from kilometers to miles we use the unit fraction $\frac{0.62137\,\text{mi}}{1\,\text{km}}$. Next, we change from miles to feet with the unit fraction $\frac{5280\,\text{ft}}{1\,\text{mi}}$.

But, we were given 112 km/h, so we need to think of the original amount as the fraction $\frac{112\,\text{km}}{1\,\text{h}}$. To change hours to minutes we multiply by the unit fraction $\frac{1\,\text{h}}{60\,\text{min}}$. Notice that the hour is in the numerator here because the hour in 112 km/h was in the denominator. Finally, we multiply by the unit fraction $\frac{1\,\text{min}}{60\,\text{sec}}$. Combine all of these results in the following product.

$$\begin{aligned}\frac{112\,\cancel{\text{km}}}{1\,\cancel{\text{h}}} \times \frac{0.62137\,\cancel{\text{mi}}}{1\,\cancel{\text{km}}} \times \frac{5280\,\text{ft}}{1\,\cancel{\text{mi}}} \times \frac{1\,\cancel{\text{h}}}{60\,\cancel{\text{min}}} \times \frac{1\,\cancel{\text{min}}}{60\,\text{sec}} &\approx \frac{112 \times 0.62137 \times 5280\text{ ft}}{60 \times 60\,\text{sec}} \\ &\approx 102.07\text{ ft/s}\end{aligned}$$

Thus, we see that 112 km/h is about 102 ft/s.

EXERCISE SET 2.5

In Exercises 1–8, use unit fractions to complete the indicated dimension analysis.

1. Change 3 h to seconds.
2. Change 64 in. to feet.
3. Change 7656 ft to miles.
4. Change 3.2 years to hours.
5. Change 15.25 in. to centimeters.
6. Change 7656 m to miles.
7. Change 88 ft/s to miles per hour.
8. Change 65 mi per hour to kilometers per hour.

Solve Exercises 9–28.

9. ***Medical technology*** The human esophagus is about 250 mm long. How long is that in inches?
10. ***Interior decorating*** A bedroom has 225.5 ft^2. What is that in square meters?

11. ***Construction*** A broken bolt was originally 2.14 in. long and must be replaced with a bolt of the same length. How many cm long must the replacement bolt be?

12. ***Medical technology*** A child weighs 65 lb and is to receive 0.05 mg of a drug for each kg of body weight. How many milligrams of the drug should the child receive?

13. ***Mechanics*** A driven gear has 54 teeth and a driving gear has 13 teeth. What is the gear ratio of the driven gear to the driving gear?

14. A room is 12 ft wide and 15 ft long. What is the ratio of length to width?

15. ***Automotive technology*** The *steering ratio* of an automobile can be determined by dividing the total number of degrees that the steering wheel turns by the number of degrees that the wheels turn. What is the steering ratio if the steering wheel makes $4\frac{2}{3}$ complete turns while the front wheels turn 60°?

16. If the density of gold is 19 g/cm^3, what is its specific gravity?

17. ***Electricity*** The *capacitance* of a capacitor is the ratio of the charge on either plate of the capacitor to the potential difference between the plates. The unit of capacitance is the farad (F), where 1 F = 1 coulomb/volt, or 1 C/V. Because the farad is such a large unit the microfarad (μF) is used where 1 μF $= 10^{-6}$ F. If a capacitor has a charge of 5×10^{-4} C when the potential difference across the plates is 300 V, what is the capacitance of this capacitor?

18. ***Optics*** The **linear magnification**, m, of an optical system is the ratio between the size of the image and the size of the object. What is the linear magnification of a system if an object that is 80 mm long has an image that is 20 mm long?

19. ***Automotive technology*** The *compression ratio* compares the volume of a cylinder at Bottom Dead Center (BDC) to the volume at Top Dead Center (TDC). If the compression ratio is 8.7 : 1 and the volume at BDC is 96 cm^3, what is the volume at TDC?

20. ***Automotive technology*** A driver added 90 mL of fuel conditioner to 20 L of fuel. How much is needed for 54 L of fuel?

21. ***Physics*** It requires 80 calories of heat to change 1 g of ice to water without raising the temperature. How many calories are needed to change 785 kg of ice to water without changing the temperature?

22. ***Electricity*** In a transformer, an electric current in the primary wire induces a current in the secondary wire. The ratio of turns in the windings determines the ratio of voltages across them, and is expressed as the proportion $\frac{V_1}{V_2} = \frac{N_1}{N_2}$, where V represents voltage and N represents the number of turns in each coil. A transformer has 100 turns in its primary winding and 750 in its secondary winding. If the primary voltage is 120 V, what is the secondary voltage?

23. The ratio of mm to inches is 25.4 to 1. How many inches are there in 88.9 mm?

24. ***Physics*** The acceleration due to gravity is 9.780 39 m/s^2 or 32.087 8 ft/s^2 at the equator. At the north pole, the acceleration due to gravity is 9.832 17 m/s^2. What is the value in ft/s^2 at the north pole?

25. ***Automotive technology*** The ratio of gallons to liters is 0.2642 : 1. If a car's gasoline tank holds 14.2 gal, what is its capacity in liters?

26. ***Nutrition*** For a certain breakfast cereal, a serving of 1.25 oz contains 140 cal and 3 g of dietary fiber. A box of this same cereal contains 14.1 oz. How many calories and how many grams of dietary fiber are in a box of this cereal?

27. ***Semiconductor technology*** When you measure oxide on a wafer before etch, it is 5000 Angstroms thick. After 2 min of etching, it is 2000 A thick. If the *etch rate* is defined as the rate by which material is removed from a wafer. What is the etch rate for this wafer?

28. ***Semiconductor technology*** The etched patterns on a wafer are called *vias*. As an insulator or a metal is deposited on a wafer, all patterns that have been etched into the surface are filled (as illustrated in Figure 2.2a). After all of the vias in the wafer are filled with a layer, the surface must be *planarized* (or, since it is smoothed,

it is also called *polished*), as illustrated in Figure 2.2b. To create such a uniform surface, a precise incoming thickness and outgoing target thickness is required. The tool generally calculates the polish rate automatically. If the planarization rate is 1000 Angstroms/min, how much time is required on the tool if the incoming thickness is 5000 Angstroms and the desired outgoing target is 3500 Angstroms?

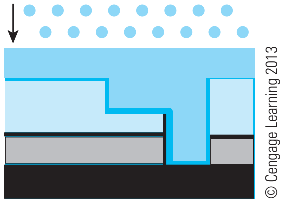

Figure 2.2a

Figure 2.2b

29. ***Construction*** The blueprint for a solar panel has a scale of 20 mm = 1.25 m. The length of the actual panel is 2.75 m and its width is 0.48 m.

(a) What is the distance on the blueprint that represents the length of the solar panel?

Round you answer to the nearest millimeter.

(b) What is the distance on the blueprint that represents the width of the solar panel?

Round you answer to the nearest millimeter.

30. ***Construction*** In excavating the foundation of a building to a 6-ft depth, 2,700 cubic yards of soil are removed. How many cubic yards are removed when excavation to a 8 ft–6 in. depth?

2.6 APPLICATIONS OF EQUATIONS

In Section 2.4 we learned how to solve some equations. But, the ability to solve equations is helpful only if you are able to take a problem and write it in the form of one or more equations. Then, once you have the equations, you can solve them to find the answer to your original problem.

Most problems you have to solve at work will be verbal problems. Someone will tell you about a problem they want you to solve or they will write part, or all, of the problem. You will have to first take this verbal problem and organize it so that it is easier to understand. Then, you will look at the problem and decide what information is important and what is not. You should then take the important information and express it in one or more equations. Once you have written the equations, the most difficult part is over. All that is left is to solve the equations and check your answers. In this section, we will focus on taking written information and writing it as an equation.

SEVEN SUGGESTIONS TO HELP SOLVE WORD PROBLEMS

Verbal, or word problems, give several numerical relationships and then ask some questions about them. You must be able to translate the word problem into equations. The fewer equations you need, the easier it will be to solve the problem. Here are some suggestions and examples to help you.

SUGGESTIONS FOR SOLVING WORD PROBLEMS

1. Read the problem carefully. Make sure you understand what the problem is asking. You may need to read the problem several times to fully understand it.

2. Clearly identify the unknown quantities. Identify each unknown quantity with a letter (or variable). Write down what each letter stands for. Use a letter for an unknown quantity that makes sense to you. For example, you might use d for distance or t for time.
3. If possible, represent all the unknowns in terms of just one variable.
4. Make a sketch (if possible) if it makes the problem clearer.
5. Analyze the problem carefully. Try to write one equation that shows how all the unknowns and knowns are related. If it is not possible to write one equation, use more. (Later we will learn how to solve several equations that show the relationship between the unknowns.)
6. Solve the equation and indicate appropriate units.
7. Check your answer in the original problem.

Once you have found an answer and checked to see that it is correct, write your solution clearly and legibly indicating the correct units. You should try to write your answer in a complete sentence using proper grammar.

EXAMPLE 2.44

In order to get an A in a word processing class, one teacher requires that you type an average of 85 words per minute for 5 different timings. Bill had speeds of 77, 78, 87, and 91 words per minute on his first 4 timings. How fast must he type on the next test in order to get an A?

SOLUTION The unknown quantity is the typing speed on the next timed typing. We will let s represent the unknown typing speed. The average for the 5 timings is the sum of these 5 timings divided by 5 or $\frac{77 + 78 + 87 + 91 + s}{5}$. This average must be 85, and so we have the equation

$$\begin{aligned}
\frac{77 + 78 + 87 + 91 + s}{5} &= 85 \\
\frac{333 + s}{5} &= 85 && \text{Combine terms.} \\
5\left(\frac{333 + s}{5}\right) &= 5 \times 85 && \text{Multiply by 5.} \\
333 + s &= 425 \\
333 + s - 333 &= 425 - 333 && \text{Subtract 333.} \\
s &= 92
\end{aligned}$$

Bill must type 92 words per minute to have an average of 85 words per minute for the 5 timings, in order to get an A in the class.

APPLICATION BUSINESS

EXAMPLE 2.45

At the end of a model year, a dealer advertises that the list prices of last year's truck models are 15% off. What was the original price of a truck that is on sale for $14,416?

SOLUTION Suppose the original price of the truck was p dollars. The discounted amount is 15% of the original price, or $0.15p$. The sale price of $14,416 is the original price p less the discounted amount, $0.15\,p$. This can be written as

$$\begin{aligned}\text{original price} - \text{discounted amount} &= \text{sale price}\\ p - 0.15p &= 14{,}416\\ 0.85p &= 14{,}416 && \text{Combine terms.}\\ p &= \frac{14{,}416}{0.85} && \text{Divide by 0.85.}\\ p &= 16{,}960\end{aligned}$$

The original price of the truck was $16,960.

You can check your answer by taking 15% of $16,960 ($0.15 \times 16{,}960 = 2{,}544$) and subtracting this from the original price ($16{,}960 - 2{,}544 = 14{,}416$). Is this the sale price?

We will now look at some types of problems that have applications to technology. The problems selected provide a variety of examples.

UNIFORM MOTION PROBLEMS

The distance an object travels is governed by the rate it is traveling and the time that it travels. This is often expressed with the formula

$$d = rt$$

where d is the distance traveled, r is the uniform or average rate of travel, and t is the time spent traveling.

APPLICATION GENERAL TECHNOLOGY

EXAMPLE 2.46

If an airplane flies 785 km/h for 4.5 h, how far does it travel?

SOLUTION We use the formula $d = rt$. We are given the rate r as 785 km/h and the time t as 4.5 h. So,

$$\begin{aligned}d &= rt\\ &= 785(4.5)\\ &= 3532.5 \text{ km}\end{aligned}$$

The plane will travel 3 532.5 km in 4.5 h.

It often helps to include the units as part of your work. In Example 2.46, we could have written

$$\begin{aligned} d &= rt \\ &= 785\frac{\text{km}}{\text{h}} \times 4.5\ \text{h} \\ &= 3\,532.5\frac{\text{km} \cdot \cancel{\text{h}}}{\cancel{\text{h}}} \\ &= 3\,532.5\ \text{km} \end{aligned}$$

This same formula, $d = rt$, can be used to find any one of the three values, distance, rate, or time, provided that we know the other two.

APPLICATION GENERAL TECHNOLOGY

EXAMPLE 2.47

If a plane travels 1,150 mi in 2.5 h, what is its average rate of speed?

SOLUTION Again, we use the formula $d = rt$. This time we have the distance d as 1,150 mi and the time t as 2.5 h.

We will show two ways you can solve this problem. One does not show the units and the second method does.

Without Units	With Units
$d = rt$	$d = rt$
$1{,}150 = r(2.5)$	$1{,}150\ \text{mi} = r(2.5\,\text{h})$
$\frac{1{,}150}{2.5} = r$	$\frac{1{,}150\ \text{mi}}{2.5\ \text{h}} = r$
$460 = r$	$460\ \text{mph} = r$

The plane averaged 460 mph.

Sometimes it is helpful to use a table to organize the information. This often makes is easier to determine the necessary equations. This is demonstrated in the next example.

APPLICATION GENERAL TECHNOLOGY

EXAMPLE 2.48

The cruising air-speed in still air of a small airplane is 295 km/h. If the wind is blowing from the east at 75 km/h, how far to the east can a pilot fly and return in 4 h?

SOLUTION This problem again uses the formula $d = rt$. We begin by determining the speed of the plane as it flies east (into the wind) and then as it returns flying west (with the wind).

EXAMPLE 2.48 (Cont.)

$$\begin{aligned}
295\text{ km/h} &= \text{speed of plane in still air}\\
75\text{ km/h} &= \text{speed of wind}\\
295\text{ km/h} - 75\text{ km/h} = 220\text{ km/h} &= \text{speed of plane flying east}\\
295\text{ km/h} + 75\text{ km/h} = 370\text{ km/h} &= \text{speed of plane flying west}
\end{aligned}$$

Next, we place the information in a table.

	Distance (d)	Rate (r)	Time (t)
Flying east	$220t$	220	t
Flying west	$370(4 - t)$	370	$4 - t$

Since the distance the pilot flies to the east is the same distance as the return flight, we know that the two quantities in the first column of the table must be the same. Thus, we have

$$\begin{aligned}
220t &= 370(4 - t)\\
&= 1480 - 370t\\
590t &= 1480\\
t &= \frac{1480}{590} \approx 2.508
\end{aligned}$$

Thus, the plane should fly about 2.50 h to the east. Notice that this was rounded *down* to 2.50 h even though we would normally have rounded 2.508 *up* to 2.51 h. The reason is because using 2.51 h would not leave the plane enough time to return.

But, the question asked for the distance the plane should fly before turning around. Since it is traveling at 220 km/h when it flies to the east, it should travel $220 \times 2.50 \approx 550$ km.

Another motion problem is demonstrated in Example 2.49.

APPLICATION GENERAL TECHNOLOGY

EXAMPLE 2.49

A car traveling on an interstate highway leaves a rest stop at 3 p.m., traveling at 75 km/h. Another car leaves the same rest stop 15 min later, headed in the same direction. If the second car travels at 100 km/h, how long will it be before it overtakes the first car?

SOLUTION Notice that the problem asks how long it will be before the cars meet, but does not say if this should be how much time or how many km. Let's find both.

The distance formula says that $d = rt$. If t is the amount of time (in hours) since the first car left, then for the first car the distance it has traveled, d, is $d = 75t$.

The distance traveled by the second car will be the same. The rate 100 km/h and the time will be different from those of the first car. The second car left

15 min or $\frac{1}{4}$h later. (Notice that we had to give this time in hours.) So, the time the second car traveled is $t - \frac{1}{4}$h. Thus, we have $d = 100(t - \frac{1}{4})$.

Both cars traveled the same distance, which means that

$$\begin{aligned} 75t &= 100\left(t - \frac{1}{4}\right) \\ 75t &= 100t - 25 \\ 75t - 100t &= 100t - 25 - 100t && \text{Subtract } 100t. \\ -25t &= -25 && \text{Combine terms.} \\ t &= 1 && \text{Divide by } -25. \end{aligned}$$

The second car will catch the first 1 h after the first car left. This will be at 4 p.m., or 45 min after the second car leaves. The distance traveled by each car is $75(1) = 75$ km.

WORK PROBLEMS

Work problems provide a different type of challenge. In these problems, we want to determine the amount of work each person or machine can do in a given amount of time.

APPLICATION BUSINESS

EXAMPLE 2.50

One printer can complete a certain job in 3 h. Another printer can do the same job in 2 h. How long would it take if both printers work on the job?

SOLUTION We will let h stand for the hours it takes both machines to complete the job. In 1 h the two machines can complete $\frac{1}{h}$ of the job. The first printer does $\frac{1}{3}$ of the job in 1 h when it works alone. The second printer does $\frac{1}{2}$ of the job in 1 h when it works alone, but together they complete $\frac{1}{h}$ of the job in 1 h. So, $\frac{1}{h} = \frac{1}{3} + \frac{1}{2}$. A common denominator is $6h$. Multiplying by $6h$ we get

$$\begin{aligned} 6 &= 2h + 3h \\ 6 &= 5h \\ \frac{6}{5} &= h \end{aligned}$$

They can complete the job together in $\frac{6}{5}$ h or in 1 h 12 min.

A slightly different work problem is shown in Example 2.51.

APPLICATION GENERAL TECHNOLOGY

EXAMPLE 2.51

An oil tanker can be emptied by the main pump in 5 h. An auxiliary pump can empty the tank in 11 h. In the main pump is started at 8 a.m. and the auxiliary pump is started 1 h later, at what time will the tanker be empty?

EXAMPLE 2.51 (Cont.)

SOLUTION We will let t represent the time in hours it takes for the main pump to complete the job. Since the auxiliary pump is turned on 1 h after the main pump it completes the job in $t - 1$ hours. From the information we know that the main pump does $\frac{1}{5}$ of the job in 1 h and the auxiliary pump does $\frac{1}{11}$ of the job in the same amount of time. Thus, we see that the main pump completes $\frac{1}{5}t$ of the job during the time it works and the auxiliary pump finishes $\frac{1}{11}(t - 1)$ of the job while it is operating. Writing this as an equation we get

$$\begin{array}{ccccc}
\text{Part of job done} & & \text{Part of job done} & & \text{Total} \\
\text{by main pump} & + & \text{by auxiliary pump} & = & \text{job} \\
\frac{1}{5}t & + & \frac{1}{11}(t-1) & = & 1 \\
11t & + & 5(t-1) & = & 55 \\
 & & 16t - 5 & = & 55 \\
 & & 16t & = & 60 \\
 & & t & = & \frac{60}{16} \\
 & & t & = & 3.75
\end{array}$$

It takes 3.75 h (or 3 h 45 min) to empty the tanker, so it is empty at 11:45 a.m.

MIXTURE PROBLEMS

In a mixture problem, two quantities are combined, or mixed, to produce a third quantity. Each quantity has a different percentage of some ingredient. The problems often ask for the percent of the ingredient in the third quantity.

APPLICATION GENERAL TECHNOLOGY

EXAMPLE 2.52

If 100 L of gasohol containing 12% alcohol is mixed with 300 L that contain 6% alcohol, how much alcohol is in the final mixture? What is the percent of alcohol in the final mixture?

SOLUTION Figure 2.3a shows the two given quantities on the left and the final mixture on the right. The first quantity has $0.12 \times 100 = 12$ L of alcohol. The second quantity has $0.06 \times 300 = 18$ L of alcohol.

The total amount of gasohol is $100 + 300 = 400$ L. The total amount of alcohol is $12 + 18 = 30$ L. The alcohol is $\frac{30}{400} = 7.5\%$ of the gasohol. (See Figure 2.3b.)

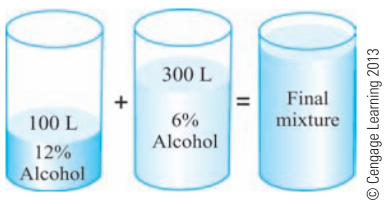

Figure 2.3a

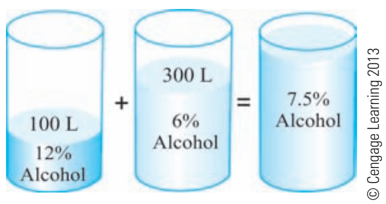

Figure 2.3b

APPLICATION AUTOMOTIVE

EXAMPLE 2.53

A 15-L cooling system contains a solution of 20% antifreeze and 80% water. In order to get the most protection, the solution should contain 60% antifreeze and 40% water. How much of the 20% solution must be replaced with pure antifreeze to get the best protection?

SOLUTION Let n = the number of liters of the original solution that will be replaced with antifreeze. The amount of original solution that is not replaced is $(15 - n)$ L. You know that you will end up with 60% of 15 L, or 9 L of antifreeze. Thus, you can write

$$\text{antifreeze left} + \text{antifreeze added} = 9\text{ L}$$

The antifreeze that is left is 20% of $15 - n$, so we can rewrite this equation as

$$\begin{aligned} 20\%(15 - n) + n &= 9 \\ 0.2(15 - n) + n &= 5 \\ 3 - 0.2n + n &= 9 \\ 3 + 0.8n &= 9 \\ 0.8n &= 6 \\ n &= \frac{6}{0.8} \\ n &= 7.5 \end{aligned}$$

So, 7.5 L of the 20% solution must be replaced with pure antifreeze.

STATICS PROBLEMS

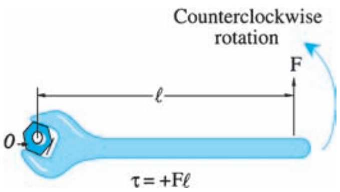

Figure 2.4a

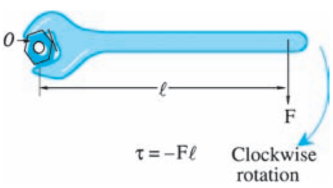

Figure 2.4b

The **moment of force**, or **torque**, about a point O is the product of the force F and the perpendicular distance ℓ from the force to the point. The distance ℓ is also called the **moment arm** of the force about a point. The Greek letter tau, τ, is used to represent torque. Thus, we have the equation

$$\tau = F\ell$$

A torque that produces a counterclockwise rotation is considered positive, as shown in Figure 2.4a. A torque that produces a clockwise rotation is considered negative. (See Figure 2.4b.)

An object is in rotational equilibrium when the sum of all the torques acting on the object about any point is zero.

The **center of gravity** of an object is the point where the object's entire weight is regarded as being concentrated. If an object is hung from its center of gravity, it will not rotate. An object can also be balanced on its center of gravity without rotating. But, what is perhaps most important in studying the equilibrium of an object, is that its weight is considered to be a downward force from its center of gravity.

APPLICATION GENERAL TECHNOLOGY

EXAMPLE 2.54

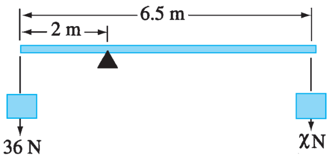

Figure 2.5

A 6.5-m rigid rod of negligible weight has a 36-N (newton) weight hung from one end. An unknown weight is hung from the other end. If the rod is balanced at a point 2 m from the end with the 36-N weight, how much is the unknown weight? (See Figure 2.5.)

SOLUTION Since the rod and weights are balanced, the torques of the two weights must be equal. On the left side the moment arm is 2 m and the force is 36 N. The torque on the left-hand side, τ_1, is $\tau_1 = 36 \text{ N} \times 2 \text{ m} = 72 \text{ N·m}$. The moment arm on the right is $6.5 \text{ m} - 2 \text{ m} = 4.5 \text{ m}$. We are to find the force, x. So, the torque on the right-hand side is $\tau_2 = (x \text{ N})(4.5 \text{ m}) = 4.5x \text{ N·m}$. Since $\tau_1 = \tau_2$ then $72 \text{ N·m} = 4.5x \text{ N·m}$ or $72 = 4.5x$. Solving this, we get $x = \dfrac{72}{4.5} = 16\text{N}$. The unknown weight is 16 N.

APPLICATION AUTOMOTIVE

EXAMPLE 2.55

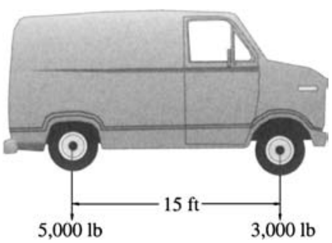

Figure 2.6

The front wheels of a truck together support 3,000 lb. Its rear wheels together support 5,000 lb. If the axles are 15 ft apart, where is the center of gravity? (See Figure 2.6.)

SOLUTION Let d represent the distance from the front axle to the center of gravity. Then $15 - d$ is the distance from the center of gravity to the rear axle. The torque on the right is $\tau_1 = 3{,}000d$ ft·lb. The torque on the left is $\tau_2 = -5{,}000(15 - d)$ ft·lb. (Notice that τ_2 is a counterclockwise rotation around the center of gravity and so it is positive. τ_1 is a clockwise rotation around the center of gravity and so it is negative.) Since the truck is in rotational equilibrium,

$$\begin{aligned}
\tau_1 + \tau_2 &= 0 \\
3{,}000d - 5{,}000(15 - d) &= 0 \\
3{,}000d - 75{,}000 + 5{,}000d &= 0 \\
-75{,}000 + 8{,}000d &= 0 \\
8{,}000d &= 75{,}000 \\
d &= \frac{75{,}000}{8{,}000} \\
d &= 9.375
\end{aligned}$$

The center of gravity is 9.375 ft or 9 ft 4.5 in. behind the front axle.

EXERCISE SET 2.6

Solve Exercises 1–35.

1. On your first 3 mathematics exams you received scores of 79, 85, and 74. What do you need to get on the next exam in order to have an 80 average?
2. There are three parts to a state certification exam. You must pass all 3 parts with an average of 75 and you cannot get below 60 on any one part. Judy's first 2 scores were 65 and 72. What must she get on the last part in order to pass?
3. If Raphael got scores of 85 and 82 on the first 2 parts of the exam described in Exercise 2, what, must he get on the third part in order to pass the exam?
4. ***Business*** A resort promised that the temperature would average 72°F during your 4-day vacation or you would get your money back. The first 3 days the average temperatures were 69°, 73°, and 68°. How warm does it have to get today for the resort to be able to keep your money?
5. ***Business*** A discount store sells personal computers for \$920. This price is 80% of the price at a wholesale store. What is the wholesale price? How much will you save at the discount store?
6. ***Business*** The personal computer discount store makes a profit of 15% based on its cost for the computer in Exercise 5. What does the computer cost the discount store?
7. An insurance company gives you \$1,839 to replace a stolen car. At the time the company tells you that the car was only worth 30% of its original cost. What was the original cost?
8. ***Automotive*** Because of inflation, the price of automotive parts increased by 3% in January and by another 4% in May. What was last year's price of parts that cost \$227.63 after the May increase?
9. ***Finance*** Sally invested \$4,500. Part of it was invested at 3.5% and part of it at 5%. After 1 year her interest was \$201. How much was invested at each rate?
10. ***Finance*** To help pay for his education, Jośe worked and invested his money. Altogether he was able to invest \$8,200. Some money he invested at 4.2% and the rest at 3.25%. He was able to earn \$320.65 in interest during the year. How much did he invest at each rate?
11. ***Business*** A factory pays time and a half for all hours over 40 h per week. Juannita makes \$8.50 an hour and 1 wk brought home \$429.25. How many hours of overtime did she work?
12. ***Business*** Mladen is paid \$8.20 an hour. He also gets paid time-and-a-half overtime when he works more than 40 h a week if it is during the week. He gets double time if the extra hours are on the weekend. One week Mladen brought home \$524.80 for 54 h of work. How many hours did he work on the weekend?
13. ***Transportation*** A freight train leaves Chicago traveling at an average rate of 38 mph. How far will it travel in 7 h?
14. ***Transportation*** How long does it take the train in Exercise 13 to travel 475 mi?
15. ***Environmental science*** An oil tanker has hit a reef and a hole has been knocked in the side of the tanker. Oil is leaking out of the tanker and forming an oil slick. The oil slick is moving toward a beach 380 km away at a rate of 12 km/d (kilometers per day). The day after the oil spill, cleanup ships leave a dock at the beach and head directly toward the oil slick at a rate of 80 km/d. How far will the ships be from the beach when they reach the oil slick?
16. A school group is traveling together on a school bus. The bus leaves a rest stop on an interstate highway at 1:00 p.m. and travels at a rate of 60 mph. One of the students did not get back on the bus before it left. A highway patrol car leaves the rest stop with the student at 1:30 p.m. and tries to catch the bus. If the patrol car averages 80 mph, at what time will it catch the bus? How far from the rest stop will the bus have traveled?

17. ***Machine technology*** The worker on machine A can complete a certain job in 6 h. The worker on machine B can do the same job in 4 h. In a rush situation, how long would it take both machines to do the job?
18. ***Construction*** Manuel can build a house in 45 days. With Errol's help they can build the house in 30 days. How long would it take Errol to build the same house by himself?
19. A tank can be filled by Pipe A in 4 h. Pipe B fills the same tank in 2 h. How long will it take to fill the tank if both pipes are used at the same time?
20. ***Energy*** A solar collector can generate 70 kJ (kilojoules) in 12 min. A second solar collector can generate the same amount of kJ in 4 min. How long will it take the two of them working together to generate 70 kJ?
21. ***Chemistry*** To generate hydrogen in a chemistry laboratory, a 40% solution of sulfuric acid is needed. You have 50 mL of an 86% solution of sulfuric acid. How many mL of water should be added to dilute the solution to the required level of sulfuric acid?
22. ***Petroleum engineering*** A petroleum distributor has two gasohol storage tanks. One tank contains 8% alcohol and the other 14% alcohol. An order is received for 1000 000 L of gasohol containing 9% alcohol. How many liters from each tank should be used to fill this order?
23. ***Chemistry*** A copper alloy that is 35% copper is to be combined with an alloy that is 75% copper. The result will be 750 kg of an alloy that is 60% copper. How many kg of each alloy should be used?
24. ***Chemistry*** How many pounds of an alloy of 20% silver must be melted with 50 lb of an alloy with 30% silver in order to get an alloy of 27% silver?
25. ***Civil engineering*** A horizontal beam of negligible weight is 20 ft long and supported by columns at each end. A weight of 850 lb is placed at one spot on the beam. The force on one end is 500 lb. What is the force on the other end? Where is the weight located?
26. ***Transportation engineering*** A loaded truck weighs 140 000 N. The front wheels of the truck support 60 000 N and the rear wheels support the rest. Where is the center of gravity if the axles are 4.2 m apart?
27. ***Physics*** Locate the center of gravity of the beam in Figure 2.7 if the beam is 12 ft long.

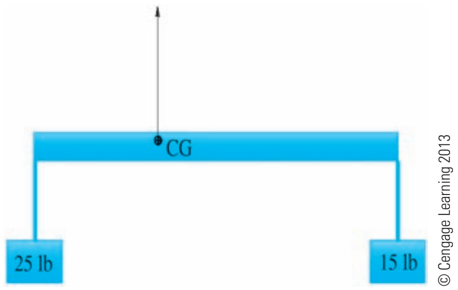

Figure 2.7

28. ***Machine technology*** Locate the center of gravity of the machine part in Figure 2.8 if it is all constructed from the same material.
29. ***Machine technology*** Locate the center of gravity of the machine part in Figure 2.9 if it is all made of the same metal. (Hint: The volume of a cylinder is $\pi r^2 h$ or $\frac{\pi}{4}d^2h$, where r is the

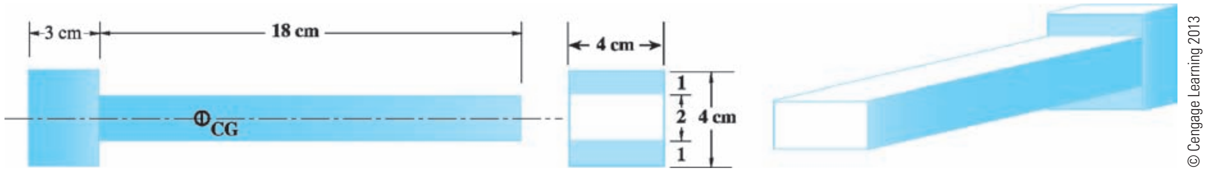

Figure 2.8

radius, d is the diameter of the circular bottom, and h is the height. This machine part is made of two circular cylinders.)

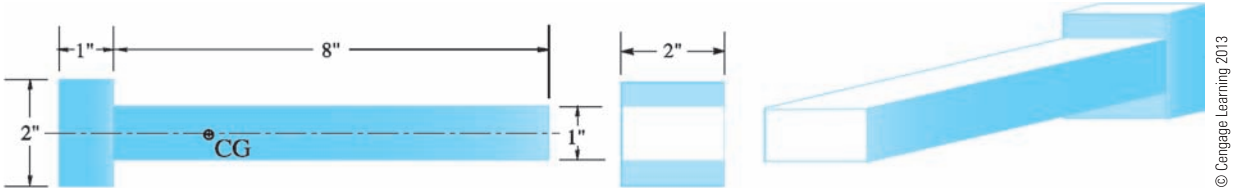

Figure 2.9

30. ***Physics*** The center of gravity of the object in Figure 2.10 is 10 cm from the left side. What is the thickness of the right side?

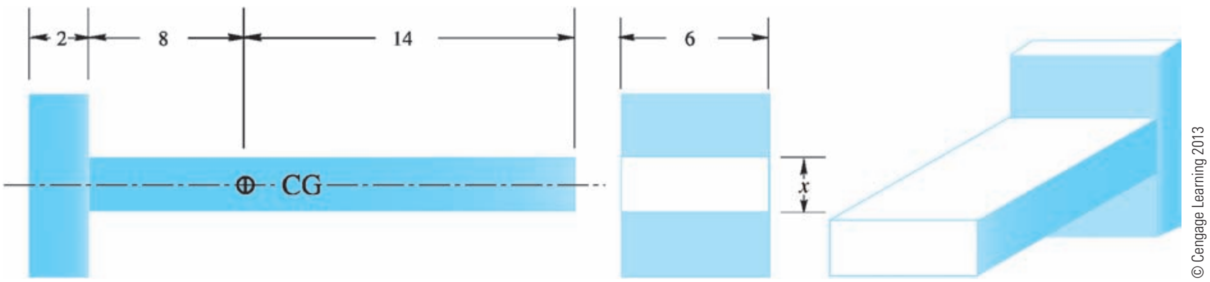

Figure 2.10

31. ***Medical technology*** How much pure alcohol must a nurse add to 10 cc (cm^3) of a 60% alcohol solution to strengthen it to a 90% solution?

32. ***Chemistry*** A chemist has 300 g of a 20% hydrochloric acid solution. She wishes to drain some off and replace it with an 80% solution in order to obtain 300 g of a 25% solution. How many grams of the 20% solution must she drain and replace with the 80% solution?

33. ***Electronics*** The voltage of V, in volts, in a circuit is the square root of the product of the power P, in watts, and the resistance R, in ohms.

(a) Write the statement as an equation.

(b) If the voltage is 240 V and the power is 2500 W, solve this equation for the resistance, R, in ohms. Round your answer to the nearest ohm.

34. ***Electronics*** The power P, in watts, of a certain circuit is the quotient of the square of the voltage V, in volts, and the resistance R, in ohms.

(a) Write this statement as an equation.

(b) If the power is 4.25 W and the resistance is 2850 Ω, solve this equation for the voltage, V. Round your answer to the nearest volt.

35. The impedance of Z is a certain circuit is the square root of the sum of the square of the resistance of R and the square of the reactance X in ohms.

(a) Write this statement as an equation.

(b) Solve this equation for the reactance X in ohms if $Z = 10\ \Omega$ and $R = 4{:}5\ \Omega$. Round your answer to the nearest tenth ohm.

[IN YOUR WORDS]

36. On a sheet of paper write a word problem. On the back of the sheet of paper, write your name and explain how to solve the problem. Give the problem you wrote to a friend and let him or her try to solve it. If your friend has difficulty understanding the problem or solving the problem, or if your friend disagrees with your solution, make any necessary changes in the problem or solution. When you have finished, give the revised problem and solution to another friend and see if he or she can solve it.

37. On a sheet of paper write a word problem that is either a uniform motion, mixture, or statics problem. (If your problem in Exercise 36 was from one of these three areas, then select a different area for this problem.) On the back of the sheet of paper, write your name and explain how to solve the problem. Give the problem you wrote to a friend and let him or her try to solve it, and follow the suggestions given in Exercise 36.

38. Consider the following problem:
How many liters of a 25% ethanol solution should be mixed with 30 L of a 37% ethanol solution in order to obtain a solution of 62% ethanol?

(a) Without working the problem explain why this does not have a solution.

(b) Use algebra to solve the problem. Explain how you can tell from the algebraic solution that this problem does not have a solution.

CHAPTER 2 REVIEW

IMPORTANT TERMS AND CONCEPTS

Algebraic expression
Algebraic sum
Algebraic term
Binomial
Center of gravity
Coefficient
Conditional equation
Constant
Cross-product
Dimensional analysis
Equation
Equivalent
Equivalent equations
Factors
FOIL method for multiplying
FOIL method for multiplying binomials
Identities
Like terms
Mixture problems
Moment arm
Moment of force
Monomial
Multinomial
Polynomial
Proportion
Rate
Ratio
Root
Solution
Solving word problems
Statics problems
Terms
Torque
Trinomial
Uniform motion problems
Unit analysis
Unit conversion
Unit fraction
Variable
Work problems

REVIEW EXERCISES

Simplify the algebraic expressions in Exercises 1–48.

1. $8y - 5y$

2. $4z + 15z$

3. $7x - 4x + 2x - 8$

4. $-9a + 4a - 3a + 2$

5. $(2x^2 + 3x + 4) + (5x^2 - 3x + 7)$

6. $(3y^2 - 4y - 3) + (5y - 3y^2 + 6)$

7. $2(8x + 4)$

8. $-3(4a - 2)$

9. $-(3x - 1)$

10. $-(2z + 5)$

11. $(4x^2 + 3x) - (2x - 5x^2 + 2)$

12. $(7y^2 + 6y) - (6y - 7y^2) + 2y - 5$

13. $2(a + b) - 3(a - b) + 4(a + b)$

14. $6(c - d) - 4(d - c) + 2(c + d)$

15. $(ax^2)(a^2x)$

16. $(cy^3)(dy)$

17. $(9ax^2)(3x)$

18. $(6cy^2z)(2cz^3)$
19. $4(5x - 6)$
20. $3(12y - 5)$
21. $2x(4x - 5)$
22. $3a(6a + a^2)$
23. $(a + 4)(a - 4)$
24. $(x - 9)(x + 9)$
25. $(2a - b)(3a - b)$
26. $(4x + 1)(3x - 7)$
27. $(3x^2 + 2)(2x^2 - 3)$
28. $(4a^3 + 2)(6a - 3)$
29. $(x + 2)^2$
30. $(3 - y)^2$
31. $5x(3x - 4)(2x + 1)$
32. $6a(4a + 3)(a - 2)$
33. $a^5 \div a^2$
34. $x^7 \div x^3$
35. $8a^2 \div 2a$
36. $27b^3 \div 3b$
37. $45a^2x^3 \div (-5ax)$
38. $52b^4c^2 \div (-4b^2)$
39. $(36x^2 - 16x) \div 2x$
40. $(39b^3 + 52b^5) \div 13b^2$
41. $(x^2 - x - 12) \div (x + 3)$
42. $(x^2 - x - 30) \div (x - 6)$
43. $(x^3 - 27) \div (x - 3)$
44. $(8a^3 + 64) \div (2a + 4)$
45. $(x^2 - y^2) \div (x + y)$
46. $(y^2 - a^2) \div (y + a)$
47. $(x^3 - 2x^2y + 2y^3 - xy^2) \div (x^2 - y^2)$
48. $(a^3 - b^3 + 2ba + 3a^2b - 3ab^2) \div (a - b)$

Solve each of the equations in Exercises 49–74.

49. $x + 9 = 47$
50. $y - 19 = -32$
51. $2x = 15$
52. $-3y = 14$
53. $\frac{x}{4} = 9$
54. $\frac{y}{3} = -7$
55. $4x - 3 = 17$
56. $7 + 8y = 23$
57. $3.4a - 7.1 = 8.2$
58. $6.2b + 19.1 = 59.4$
59. $4x + 3 = 2x$
60. $7a - 2 = 2a$
61. $4b + 2 = 3b - 5$
62. $7c + 9 = 12c - 4$
63. $3 : x = 4 : 6$
64. $x : 5 = 3 : 15$
65. $\frac{7}{9} = \frac{21}{d}$
66. $\frac{14}{6} = \frac{c}{27}$
67. $4 : 8 = 19 : x$
68. $x : 12 = 15 : 32$
69. $\frac{4(x - 3)}{3} = \frac{5(x + 4)}{2}$
70. $\frac{3(y - 7)}{5} = \frac{5(y + 4)}{2}$
71. $\frac{2}{a} - \frac{3}{a} = 5$
72. $\frac{4}{x} + \frac{5}{x} = \frac{1}{8}$
73. $\frac{3}{2a} = \frac{3}{a + 2}$
74. $\frac{9}{4b} = \frac{12}{b + 4}$

In Exercises 75–78, use unit fractions to complete the indicated conversions.

75. Change 36.8 oz to newtons.
76. Change 76.0 yards to meters.
77. Change 55.0 mph to km/min.
78. Change 8.44 pounds per square inch to newtons per square meter.

Solve Exercises 79–89.

79. ***Electricity*** The **turn ratio** in a transformer is the number of turns in the primary winding to the number of turns in the secondary winding. If the turn ratio for a transformer is 25, and there are 4,000 turns in the primary winding, how many turns are in the secondary winding?

80. ***Economics*** A worker's salary increase is proportional to the cost-of-living index. If the worker received a raise of \$12.50/wk when the index was 6.5, how much should the raise be when the index is 9.6?

81. A student received grades of 68, 70, and 74 on the first 3 exams. What does the student need to achieve on the next exam to have an average of 72?

82. ***Business*** The price of a new television was \$342.93. This price included 6.5% sales tax. How much was the television before the tax was added?

83. ***Transportation*** An airplane leaves New York City and flies at a rate of 755 km/h. How long does it take to fly 2 718 km?

84. ***Space technology*** A damaged space satellite passes over Houston at midnight and is traveling at a rate of 330 km/h. A space shuttle is attempting to overtake the satellite so it can be brought on board for repairs. The shuttle passes over Houston at 1:45 a.m., traveling in the same direction and orbit as the satellite and moving at 430 km/h. At what time does the shuttle overtake the satellite?

85. ***Chemistry*** From 50 kg of solder that is half lead and half zinc, 10 kg of solder is removed and 15 kg of lead is added. How much lead is in the final mixture? What percent of the final mixture is lead?

86. ***Physics*** A horizontal bar of negligible weight is supported by 2 columns that are 8 m apart. A load of 460 N is applied to the bar at a point x m from one end. The force on that end is 320 N. What is the force on the other end? Where was the load applied? (See Figure 2.11.)

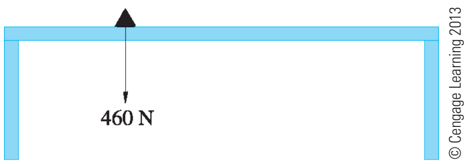

Figure 2.11

87. ***Physics*** A 490 N beam is suspended from two cables 10 m apart. A 2156-N mass is placed 4 m from one end of the beam. How much tension is on each cable? (See Figure 2.12.)

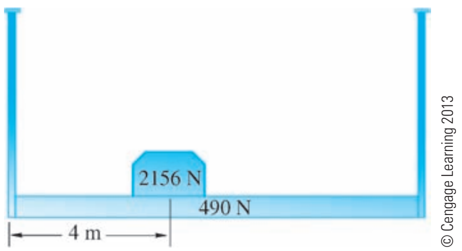

Figure 2.12

88. ***Medical technology*** A doctor ordered 20 g of a 52% solution of a certain medicine. The pharmacist has only bottles of 40% and 70% solutions. How much of each solution must be used to obtain the 20 g of the 52% solution?

89. ***Metalworking*** A solution has 1.6 lb of etchant compound dissolved in 2 gal of water. What is the concentration in g/L?

CHAPTER 2 TEST

1. Simplify $2x + 5x^2 - 5x$

2. Simplify $(4a^3 - 2b) - (3b + a^3)$

3. Solve $\frac{a}{28} = \frac{12}{20}$

4. Solve $\frac{24}{42} = \frac{78}{d}$

In Exercises 5–11, perform the indicated operation and simplify your answer.

5. $4x + [3(x + y - 2) - 5(x - y)]$

6. $(4xy^3z)\left(\frac{1}{2}xy^{-2}z^2\right)$

7. $(2b - 3)(2b + 3)$

8. $(x^3 + 3x) \div x$

9. $(6x^5 + 4x^3 - 1) \div 2x^2$

10. $(3x^3 - 2x^2 + x - 3) \div (x + 2)$

11. $\frac{y}{3y + 2} - \frac{4}{y - 1}$

Solve Exercises 12–17.

12. Solve for x: $5x - 8 = 3x$

13. Solve for x: $\frac{7x + 3}{2} - \frac{9x - 12}{4} = 8$

14. The price of a certain graphing calculator is \$74.85. This price includes a 7% sales tax. How much was the calculator before the tax was added?

15. Your automobile's cooling system, including the heater and coolant reserve system, has a capacity of 9 qt of coolant. You know that the system currently contains 50% antifreeze and 50% water. You want to remove some of the solution and replace it with antifreeze, so that the final mixture is 60% antifreeze and 40% water. How much solution should you remove?

16. The perimeter of a rectangular solar panel is 540 cm. The ratio of the length to the width is 3 : 2. What are the length and width?

17. A machinist wants to replace a damaged shaft that is 2 ft $6\frac{3}{8}$ in. long and has a diameter of 1.25 in. The catalog for the manufacturer lists the dimensions of the shaft in centimeters. What size shaft should be ordered?

3 GEOMETRY


Courtesy of U.S.Xpress

Above the entrance to the U.S. Xpress Enterprises, Inc. corporate headquarters is a large circular window. A circular arc shields the front entrance. We will examine this window in Example 3.16 and in the exercises for Section 3.3.

You have been working with geometry and geometric ideas most of your life. Geometry deals with the properties and measurements of lines, angles, plane figures, and solid figures. Geometry is a very important tool in technical mathematics. In this chapter we explore the basic geometric properties and measurements needed in technology.

OBJECTIVES

AFTER COMPLETING THIS CHAPTER, YOU WILL BE ABLE TO:

- Convert angle measurements between degrees and radians with and without a calculator.
- Find the measures of angles formed by intersecting lines.
- Use proportions to find lengths of segments formed by parallel lines.
- Find the measures of angles in a triangle.
- Find the area and perimeter of polygons.
- Find the area and circumference of circles.
- Find the approximate area of irregular shapes.
- Find the volume and surface area of solids.
- Find the approximate volume of irregular solids.
- Solve applications that involve geometric shapes.
- Know terminology related to intersecting lines, triangles, circles, and polygons.

In Chapter 4, we will begin to combine geometry and algebra. In this chapter, we will focus on the geometric ideas and skills that you will need in the following chapters and in many technical jobs.

3.1 LINES, ANGLES, AND TRIANGLES

Let's begin by discussing lines, segments, and rays. They are the building blocks for the remainder of this section. We will then look at two ways in which angles are measured. Finally, we will give a few areas in which angles and their measures are used.

LINES, SEGMENTS, AND RAYS

The basic parts of geometry are lines, angles, planes, surfaces, and the figures that they form. In this section, we will focus on the first two: lines and angles. A **line segment**, or *segment*, is a portion of a straight line between two endpoints. It is named by its endpoints, and a bar is placed over the endpoints names. For example, the segment joining the points A and B, written $\overline{AB}$, is shown in Figure 3.1a. This segment could also be named $\overline{BA}$. A *ray* or *half-line* is the portion of a line that lies on one side of a point and includes the point. For example, in Figure 3.1b, ray $\overrightarrow{CD}$ begins at point C, the *endpoint*, and passes through D. The second point used to name the ray can be any point on the ray other than the endpoint. Thus, in Figure 3.1b, $\overrightarrow{CD} = \overrightarrow{CE}$.

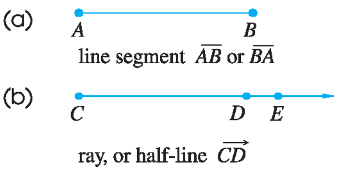

Figure 3.1

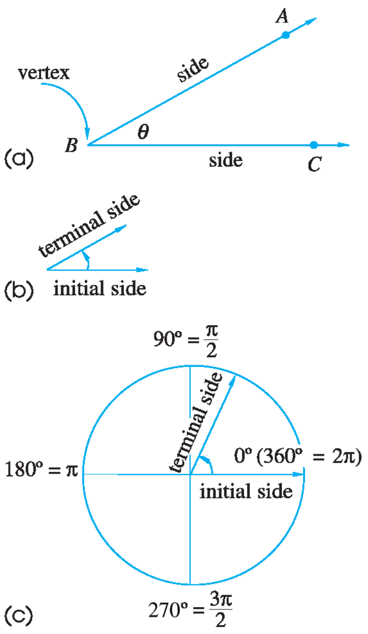

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Figure 3.2

ANGLES

An **angle** is formed by two rays that have the same endpoint. This common endpoint is called the **vertex**, and the two rays are called the *sides* of the angle. The angle in Figure 3.2a has its vertex at B. This angle has several possible names, including $\angle B$, $\angle ABC$, $\angle CBA$, and $\angle\theta$, where the symbol $\angle$ means angle.

Another way to think of an angle is to think of it as generated by moving a ray from an initial position to a terminal position as in Figure 3.2b. If this generated angle was placed inside a circle with its vertex at the center of the circle, then you would have a figure something like the one shown in Figure 3.2c.

DEGREES AND RADIANS

Angles are measured using several different systems. The two most common units are **degrees** and **radians**. We will use both in this book. Look again at Figure 3.2c. The end of the initial side is marked $0°$ ($360° = 2\pi$). This indicates that if the terminal side were to rotate completely around the circle and stop at the initial side, the size of the angle would be 360 degrees ($360°$) or 2π radians (2π rad). Later we will see a relationship between the radian and the distance around a circle.

Other important angle measures are given in relation to the distance around the circle. One-fourth of the way around the circle is $90°$ or $\frac{\pi}{2}$ rad. Halfway around the circle is $180°$ or π rad and three-fourths of the way is $270°$ or $\frac{3\pi}{2}$ rad.

NOTE If there is no degree symbol (°) after an angle measure, then assume that it is a radian measure.

Each degree is divided into 60 minutes and each minute into 60 seconds. There are 3,600 seconds in a degree. The symbol $'$ is used for minutes and $''$ for seconds. So, $60' = 1°$ and $60'' = 1'$. The use of calculators has caused decimal values for degrees to become more popular. Thus, in decimal degrees, $15' = \left(\frac{15}{60}\right)^\circ = 0.25°$ and $27'' = \left(\frac{27}{3600}\right)^\circ = 0.0075°$.

Some widely used angles besides those shown in Figure 3.2c are the $30° = \frac{\pi}{6}$, $45° = \frac{\pi}{4}$, and $60° = \frac{\pi}{3}$ angles.

There are times when you will need to convert from degrees to radians or from radians to degrees. Since $180° = \pi$ rad, then

$$1\,\text{rad} = \frac{180°}{\pi} \approx \frac{180°}{3.1416} \approx 57.296°$$

(The symbol $\approx$ means "is approximately equal to.")

and you also have

$$1° = \frac{\pi}{180°} \approx \frac{3.1416}{180°} \approx 0.01745\,\text{rad}$$

Therefore, we have the following conversion formulas.

CONVERSION BETWEEN DEGREES AND RADIANS

To change from radians to degrees multiply the number of radians by $\frac{180°}{\pi}$.

To change from degrees to radians multiply by $\frac{\pi}{180°}$.

EXAMPLE 3.1

Change **(a)** 1.89 and **(b)** $\frac{5\pi}{6}$ to degrees.

SOLUTIONS Since there is no degree symbol, these must be in radians.

(a) $1.89\,\text{rad} \approx 1.89\,\text{rad} \times \frac{180°}{\pi\,\text{rad}} \approx 108.29°$

(b) $\frac{5\pi}{6} = \frac{5\pi}{6} \times \frac{180°}{\pi} = 150°$

EXAMPLE 3.2

Change **(a)** 120° and **(b)** 82.5° to radians.

SOLUTIONS

(a) $120° = 120° \times \frac{\pi}{180°}\text{rad} = \frac{2\pi}{3}\text{rad}$

Notice here that the answer is left in terms of π and we have $120° = \frac{2\pi}{3}$, the exact answer. You may leave the answer in terms of π or compute a decimal approximation and see that $120° \approx 2.094$ rad.

(b) $82.5° = 82.5° \times \frac{\pi}{180°} = \frac{11\pi}{24} \approx 1.440\,\text{rad}$

HINT Some people find that it is easier to convert between degrees and radians by using the proportion

$$\frac{D}{180°} = \frac{R}{\pi}$$

where D represents a known or unknown number of degrees and R represents a known or unknown number of radians.

EXAMPLE 3.3

Use the proportion $\frac{D}{180°} = \frac{R}{\pi}$ to change **(a)** 120° to radians and **(b)** $\frac{5\pi}{6}$ to degrees.

SOLUTIONS

(a) Since we are asked to convert 120° to radians, we substitute 120° for D in the proportion and solve it for R as follows:

$$\frac{120°}{180°} = \frac{R}{\pi}$$

EXAMPLE 3.3 (Cont.)

$$R = \frac{120^\circ}{180^\circ}\pi$$
$$= \frac{2}{3}\pi$$

Thus, $120^\circ = \frac{2}{3}\pi = \frac{2\pi}{3}$ radians.

(b) To convert $\frac{5\pi}{6}$ to degrees substitute $\frac{5\pi}{6}$ for R in the proportion.

$$\frac{D}{180^\circ} = \frac{R}{\pi}$$
$$\frac{D}{180^\circ} = \frac{5\pi/6}{\pi}$$
$$D = \frac{5\pi}{6}\cdot\frac{180^\circ}{\pi}$$
$$= 150^\circ$$

As in Example 3.1(b), we obtain $\frac{5\pi}{6} = 150^\circ$.

TYPES OF ANGLES

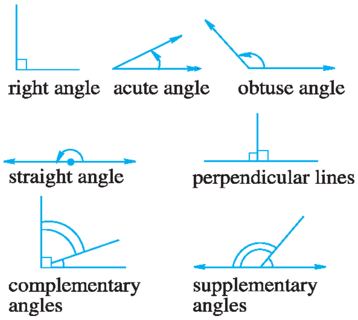

Figure 3.3

Figure 3.3 shows four basic types of angles. A **right angle** has a measure of 90° or $\frac{\pi}{2}$ rad. It is usually indicated by placing a small square at the vertex of the angle. An **acute angle** measures between 0° and 90° $\left(0 \text{ and } \frac{\pi}{2}\text{rad}\right)$ and an **obtuse angle** measures between 90° and 180° $\left(\frac{\pi}{2} \text{ and } \pi \text{ rad}\right)$. A **straight angle** has a measure of 180° or π rad. If two lines meet and form a right angle, then the lines are **perpendicular**. Two angles are **supplementary** if their sum is a straight angle (180° or π rad) and are **complementary** if their sum is a right angle $\left(90^\circ \text{ or } \frac{\pi}{2} \text{ rad}\right)$.

In geometry, when two objects are the same size we say that they are **congruent**. Two 37° angles are said to be congruent. Two segments that are 3 cm long are congruent.

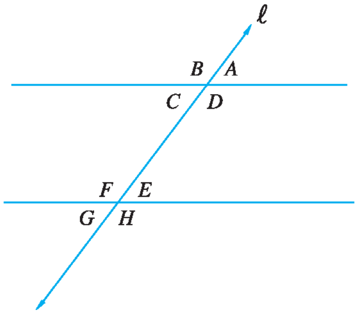

Figure 3.4

Two angles that have the same vertex and a common side are called **adjacent angles**. In Figure 3.4, $\angle A$ and $\angle B$ are adjacent angles. They are also supplementary angles. A **transversal** is a line that intersects, or crosses, two or more lines. In Figure 3.4, line ℓ is a transversal of the other two lines.

If parallel lines are intersected by two transversals, then the corresponding segments are proportional. *Corresponding segments* are the parts of the transversals between the same parallel lines. In Figure 3.5, lines P_1, P_2, and P_3 are parallel lines intersected by transversals ℓ_1 and ℓ_2. The points of intersection have been marked with capital letters. Segments $\overline{AB}$ and $\overline{DE}$ are corresponding segments, as are $\overline{BC}$ and $\overline{EF}$. Likewise, $\overline{AC}$ and $\overline{DF}$ are corresponding segments. Therefore, we have the proportion

$$\frac{AB}{DE} = \frac{BC}{EF} = \frac{AC}{DF}$$

APPLICATION CIVIL ENGINEERING

EXAMPLE 3.4 Find the distance across the lake in Figure 3.6.

SOLUTION You carefully use stakes to mark off the segments $\overline{AF}$, $\overline{BE}$, and $\overline{CD}$ so that the segments are parallel. We also make sure that A, B, C are in a line, as are D, E, and F. We know that corresponding segments are proportional, so $\frac{AB}{EF} = \frac{BC}{DE}$. Using the lengths in Figure 3.6, $\frac{100}{75} = \frac{BC}{168}$. Solving this, we get $BC = \frac{100 \times 168}{75} = 224$. It is 224 m across the lake.

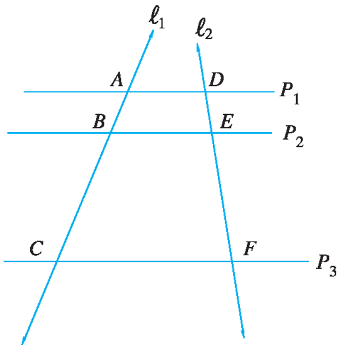

Figure 3.5

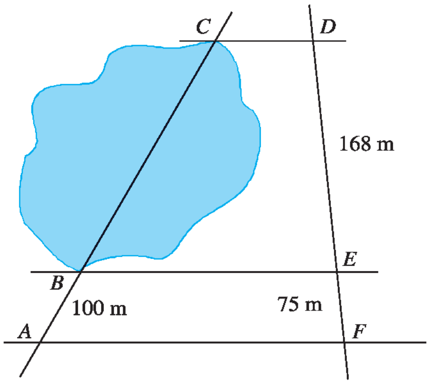

Figure 3.6

POLYGONS

A **polygon** is a figure in a plane that is formed by three or more line segments, called *sides*, joined at their endpoints. The endpoints are called *vertices* and a single endpoint is called a **vertex**.

TRIANGLES

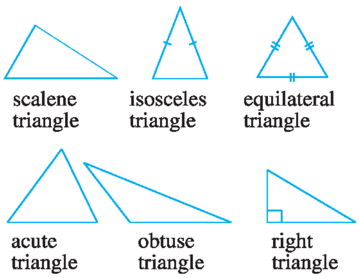

Figure 3.7

A **triangle** is a polygon that has exactly three sides. (See Figure 3.7.) Triangles are named according to a property of their sides or their angles. When classified according to its sides, a triangle is **scalene** if none of the sides are the same length, **isosceles** if two sides are the same length, and **equilateral** if all three sides are the same length. When sides are the same length, single, double, or triple marks are used to show which sides are congruent.

An **acute triangle** has three acute angles, an **obtuse triangle** has one obtuse angle, and a **right triangle** has one right angle. (See Figure 3.7.) In an equilateral triangle, all three angles are congruent, and some people called it an equiangular triangle. In an isosceles triangle the angles opposite the congruent sides are congruent.

The sum of the three angles of a triangle is 180°, or π rad. Since all three angles in an equilateral triangle are congruent, the size of each angle must be $\frac{180^\circ}{3} = 60^\circ$ or $\frac{\pi}{3}$ rad. Since a right angle is 90°, a right triangle must have two acute angles, because the other two angles, when added together, can have only a total of 90°. This means that each of the angles must be less than 90°. In a similar manner, an obtuse triangle must have two acute angles.

EXAMPLE 3.5

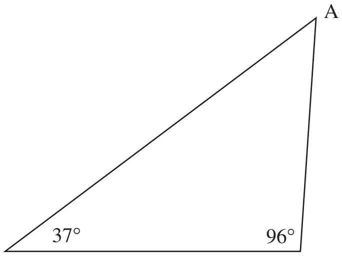

Figure 3.8

Find the size of the third angle of the triangle in Figure 3.8 if the other two angles measure 37° and 96°.

SOLUTION If we call the unknown angle $\angle A$, then

$$\angle A = 180° - 37° - 96° = 47°$$

The third angle of this triangle is 47°.

The **perimeter** of a triangle is the distance around the triangle. To find the perimeter, you add the lengths of the three sides. If the lengths of the sides are a, b, and c, then the perimeter P is $P = a + b + c$.

APPLICATION ARCHITECTURE

EXAMPLE 3.6

A triangular piece of land measures 42 m, 36.2 m, and 58.7 m on the three sides. What is the perimeter of this plot of land?

SOLUTION

$$\begin{aligned} P &= a + b + c \\ &= 42 + 36.2 + 58.7 \\ &= 136.9 \end{aligned}$$

The perimeter is 136.9 m.

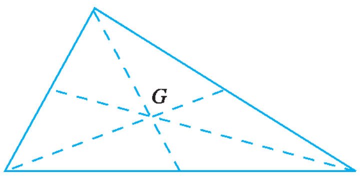

Figure 3.9

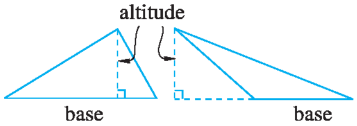

Figure 3.10

A segment drawn from a vertex to the middle (or midpoint) of the opposite side is called a *median*. If you draw all three medians of a triangle, they meet in a common point, G, called the *centroid*, as shown in Figure 3.9. The centroid is the *center of gravity* of a triangle.

A segment drawn from a vertex and perpendicular to the opposite side is an **altitude** of a triangle. The opposite side is called the *base*. (See Figure 3.10.) When solving some problems, it is sometimes necessary to extend the base so that it will intersect the altitude. The altitudes of an equilateral triangle bisect the sides of the triangle. The altitude of an isosceles triangle from the vertex of the equal sides bisects the third side of the triangle.

The area of a triangle is found by the product of $\frac{1}{2}$ and the lengths of the base and the altitude. If b is the length of the base and h the length of the altitude, then the area A is given by the formula

$$A = \frac{1}{2}bh$$

If the base and altitude are in the same units, then the area is in square units.

EXAMPLE 3.7

Find the area of a triangle with a base of 1.2 m and an altitude of 4.5 m.

SOLUTION

$$A = \frac{1}{2}bh$$
$$= \frac{1}{2}(1.2)(4.5)$$
$$= 2.7$$

The area is 2.7 m^2.

NOTE You may have noticed that m^2 was used to indicate square meters. Similar notation is used for other area units. For example, cm^2 is used for square centimeters, and ft^2 for square feet.

Sometimes the length of the altitude is not known. It is possible to find the area using **Hero's formula**, also referred to as *Heron's formula*. Instead of the altitude, you will need the lengths of the three sides and the semiperimeter. If we let a, b, and c represent the lengths of the sides, then the *semiperimeter*, s, is found using the formula $s = \frac{a + b + c}{2}$. We can find the area from Hero's formula:

$$A = \sqrt{s(s - a)(s - b)(s - c)}$$

APPLICATION CIVIL ENGINEERING

EXAMPLE 3.8

A triangular piece of land measures 51′9″, 47′3″, and 82′6″ on the three sides. What is the area of the lot of land?

SOLUTION We will let $a = 51'9'' = 51.75$ ft, $b = 47'3'' = 47.25$ ft, and $c = 82'6'' = 82.5$ ft.

$$s = \frac{a + b + c}{2} = \frac{51.75 + 47.25 + 82.5}{2} = \frac{181.5}{2} = 90.75$$
$$A = \sqrt{s(s - a)(s - b)(s - c)}$$
$$= \sqrt{90.75(90.75 - 51.75)(90.75 - 47.25)(90.75 - 82.5)}$$
$$= \sqrt{90.75(39)(43.5)(8.25)}$$
$$= \sqrt{1{,}270{,}148.3}$$
$$\approx 1{,}127.0086$$

The area is about 1,127 ft^2.

Two triangles are congruent if the corresponding angles and sides of each triangle are congruent. For example, in Figure 3.11, sides $\overline{AB}$ and $\overline{EF}$ are both

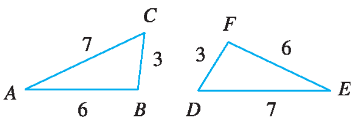

Figure 3.11

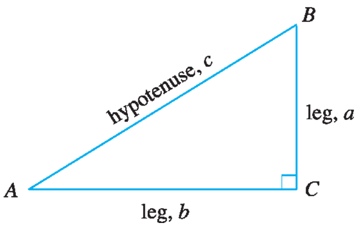

Figure 3.12

6 units long, $\overline{BC}$ and $\overline{DF}$ are both 3 units long, and $\overline{AC}$ and $\overline{DE}$ are 7 units long. If you measure the angles you will find that $\angle A \cong \angle E$, $\angle B \cong \angle F$, and $\angle C \cong \angle D$. As you can see, *congruent triangles* have the same size and shape.

We indicate that the triangles in Figure 3.11 are congruent by writing $\triangle ABC \cong \triangle EFD$. When we show that two triangles are congruent, we write it so the corresponding vertices are in the same order. Thus, writing $\triangle ABC \cong \triangle EFD$ indicates that $\angle A \cong \angle E$, $\angle B \cong \angle F$, and $\angle C \cong \angle D$.

One of the most valuable theorems in geometry involves the right triangle. In a right triangle, the side opposite the right angle is called the **hypotenuse**, as shown in Figure 3.12. The other two sides are the *legs*. The hypotenuse is the longest side of a right triangle. If the lengths of the two legs are a and b and the length of the hypotenuse is c, then the **Pythagorean theorem** is stated as follows.

PYTHAGOREAN THEOREM

ΔABC is a right triangle with a hypotenuse of length c and legs of lengths a and b, if and only if

$$a^2 + b^2 = c^2$$

EXAMPLE 3.9

A right triangle has legs of length 6 in. and 8 in. What is the length of the hypotenuse?

SOLUTION If we let $a = 6$ and $b = 8$, then, by using the Pythagorean theorem, we have

$$\begin{aligned} c^2 &= a^2 + b^2 \\ &= 6^2 + 8^2 \\ &= 36 + 64 \\ &= 100 \\ c &= \sqrt{100} = 10 \end{aligned}$$

The hypotenuse is 10 in. long.

EXAMPLE 3.10

The hypotenuse of a right triangle is 7.3 cm long. One leg is 4.8 cm long. What is the length of the other leg?

SOLUTION We are given $c = 7.3$ and $a = 4.8$, and so, by using the Pythagorean theorem, we obtain

$$\begin{aligned} c^2 &= a^2 + b^2 \\ (7.3)^2 &= (4.8)^2 + b^2 \\ (7.3)^2 - (4.8)^2 &= b^2 \end{aligned}$$

or

$$\begin{aligned} b^2 &= (7.3)^2 - (4.8)^2 \\ &= 53.29 - 23.04 \\ &= 30.25 \\ b &= \sqrt{30.25} = 5.5 \end{aligned}$$

The length of the remaining side is 5.5 cm.

APPLICATION CONSTRUCTION

EXAMPLE 3.11

A television tower is 208 ft tall. If a cable could be strung in a straight line from the top of the tower to a point on the ground $13\bar{0}$ m from the base of the tower, how long would the cable have to be? (See Figure 3.13.)

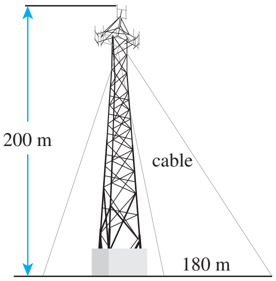

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Figure 3.13

SOLUTION The cable, tower, and ground form a right triangle. We know that the height of the tower is 200 m. This is one leg of the triangle. The other leg is 130 m. The cable will form the hypotenuse. So,

$$\begin{aligned} c^2 &= a^2 + b^2 \\ c^2 &= 130^2 + 200^2 \\ &= 16\,900 + 40\,000 \\ &= 56\,900 \\ c &= \sqrt{56\,900} \approx 238.54 \end{aligned}$$

Since the given data, $20\bar{0}$ m and $13\bar{0}$ m, have three significant digits, the cable would be about 239 m long.

The last example might not have been exactly realistic. It failed to account for any sag in the cable or any additional cable that would be needed to fasten it at each end. Later, we will find some methods to get more accurate answers to this problem.

Remember that the Pythagorean theorem can be used only with right triangles. The following exercises provide some additional examples that use the Pythagorean theorem.

EXERCISE SET 3.1

Convert each of the angle measures in Exercises 1–14 from degrees to radians or from radians to degrees without using a calculator. (You may leave your answers in terms of π.)

1. $15°$
2. $75°$
3. $210°$
4. $10°45'$
5. $85.4°$
6. $48.6°$
7. $163.5°$
8. $242°\,35'$
9. $\frac{4\pi}{3}$
10. $\frac{\pi}{6}$
11. 1.3π
12. 2.15
13. 0.25
14. 1.1

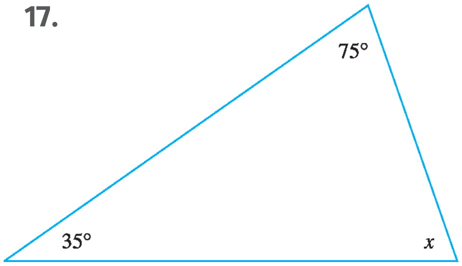
Solve Exercises 15 and 16.

15. Find the supplement of a 35° angle in **(a)** degrees and **(b)** radians.

16. Find the complement of a 65° angle in **(a)** degrees and **(b)** radians.

Find the indicated variables in Exercises 17–24. Identical marks on segments or angles indicate that they are congruent.

17.



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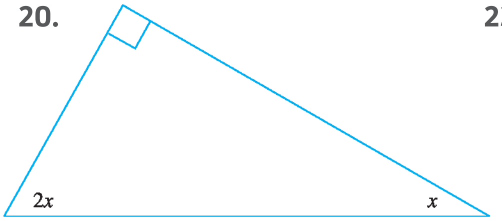

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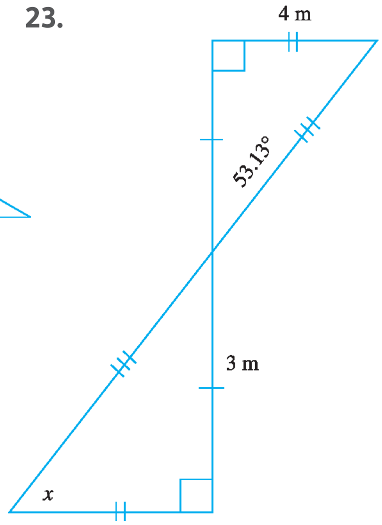

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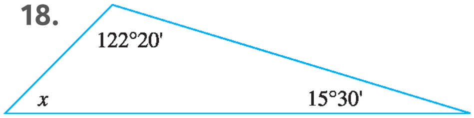

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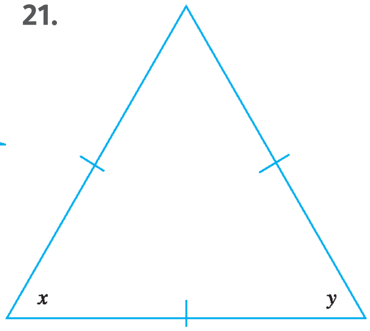

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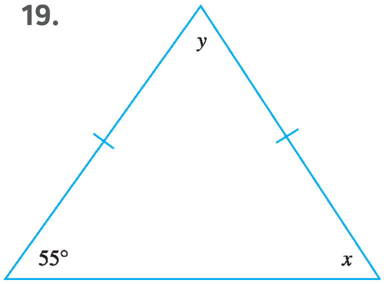

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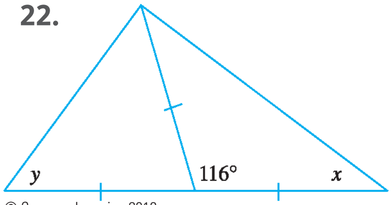

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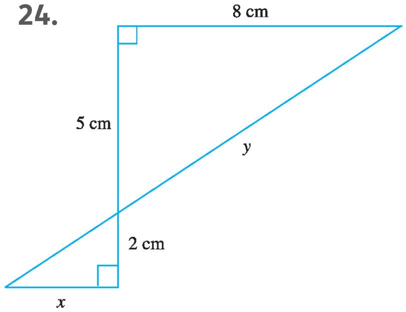
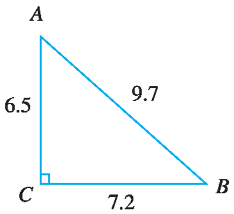


Find the area and perimeter of ABC in Exercises 25–28.

25.



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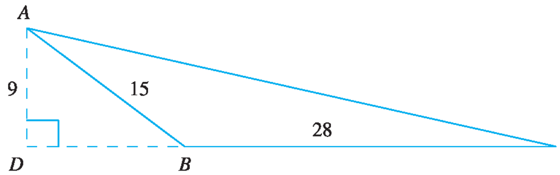

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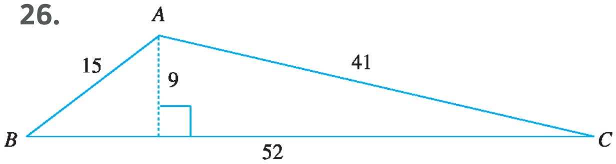

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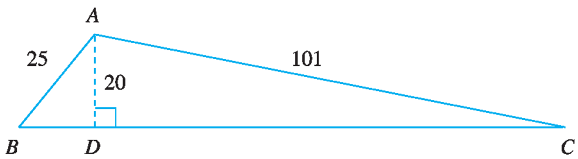

Solve Exercises 29–54.

29. What is the distance from A to B in Figure 3.14 if lines ℓ_1, ℓ_2, and ℓ_3 are parallel?

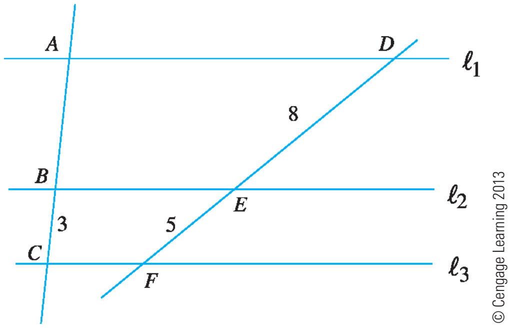

Figure 3.14

30. In Figure 3.15, if the lines ℓ_1, ℓ_2, and ℓ_3 are parallel, what is the distance from A to B?

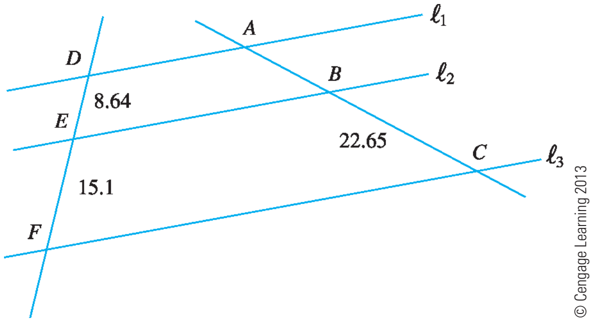

Figure 3.15

31. In Figure 3.16, angles A and B are the same size. Find the sizes of angles A and B.

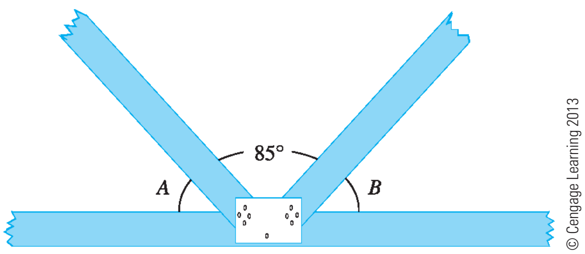

Figure 3.16

32. ***Civil engineering*** Figure 3.17 shows part of a bridge that is made of suspended cables hung from a girder to the deck of the bridge. What is the length from A to B?

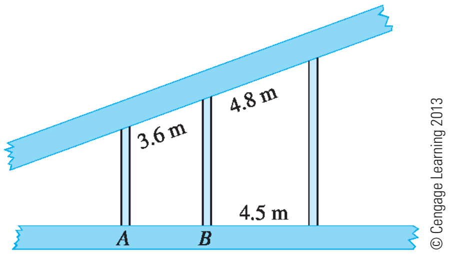

Figure 3.17

33. ***Electronics*** The angular frequency ω of an ac current is $\omega = 2\pi f$ rad/s, where f is the frequency. If $f = 60$ Hz for ordinary house current, what is the angular frequency?

34. ***Mechanical technology*** An angular velocity ω of a rigid object, such as a drive shaft, pulley, or wheel, is related to the angle θ through which the body rotates in a period of time t by the formula $\theta = \omega t$ or $\omega = \dfrac{\theta}{t}$.

(a) Find the angular velocity of a gear wheel that rotates 285° in 0.6 s.

(b) Find the angular velocity of a wheel that rotates $\frac{11\pi}{16}$ rad in 0.9 s.

35. ***Metalworking*** A machinist needs to weld a piece of pipe to two existing parallel pipes, as shown in Figure 3.18. What is the size of angle x?

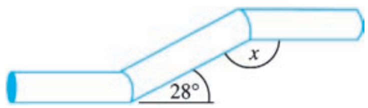

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Figure 3.18

36. ***Programming*** Use the conversion factors at the beginning of this section to write a program for, your graphing calculator that will **(a)** convert degrees to radians and **(b)** convert radians to degrees.

37. ***Landscape architecture*** A triangular piece of land measures 23.2 m, 47.6 m, and 62.5 m. **(a)** How much fencing is needed to enclose this land? **(b)** How many square meters of sod would be needed to sod the entire piece of land?

38. The shadow of a building is 123′6″. At the same time, the shadow of a yard stick is 2′3″. What is the height of the building?

39. A ladder 15 m long reaches the top of a building when its foot is 5 m from the building. How high is the building?

40. ***Mechanical engineering*** A triangular metal plate was made by cutting a rectangular plate along a diagonal. If the rectangular piece was 23 cm long and 16 cm high, what is the area of the plate?

41. ***Transportation engineering*** A traffic light support is to be suspended parallel to the ground. It reaches diagonally across the intersection of two perpendicular streets. One street is 45 ft wide and the other is 62 ft wide. Determine the length of the support.

42. ***Construction*** A house is 8 m wide and the ridge is 3 m higher than the side walls. If the rafters, r, extend 0.5 m beyond the sides of the house, how long are the rafters? (See Figure 3.19.)

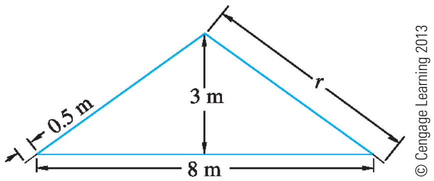

Figure 3.19

43. ***Construction*** Several cross beams are placed across an A-frame house. The highest beam is placed 2 600 mm from the ridge and the top of this beam is 2 400 mm long. The top of the next beam is placed 2 800 mm below the bottom of the top beam and the top of the third is 2 800 mm below the bottom of the second. The bottom of the third beam is 900 mm above the ground. Each beam is 300 mm thick. **(a)** How long are the tops of the second and third beams? **(b)** How far apart is the base of the A-frame? (See Figure 3.20.)

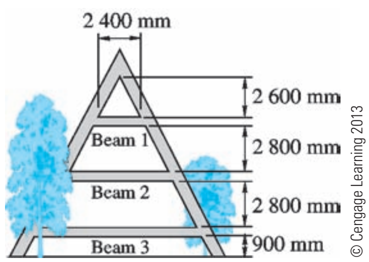

Figure 3.20

44. ***Construction*** An antenna 175 m high is supported by cables positioned around the antenna. One cable is 50 m from the base of the antenna, one is 75 m, and the third is 100 m. How long is each cable?

45. ***Electrical engineering*** A 30 m length of conduit is bent as shown in Figure 3.21. What is the length of the offset, x?

46. ***Electrical engineering*** If it were possible to run a straight conduit from A to B in Figure 3.21, how much conduit would be saved?

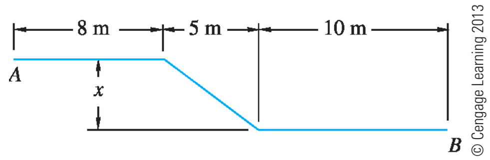

Figure 3.21

47. ***Construction*** A gas pipeline was constructed across a ravine by going down one side, across the bottom, and up the other side, as shown in Figure 3.22. **(a)** How much pipe was used to get from A to B? **(b)** How much would have been needed if it would have been possible to go directly from A to B?

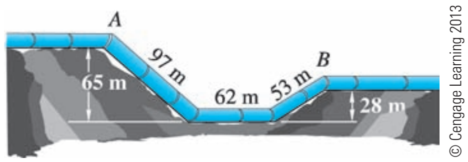

Figure 3.22

48. ***Physics*** The effect of a moving load on the stress beam is shown in the influence diagram of Figure 3.23. If the distance from A to B is 24 ft, find the lengths of $\overline{BC}$ and $\overline{CD}$. (Assume that $\overline{AF}$ is parallel to $\overline{EG}$.)

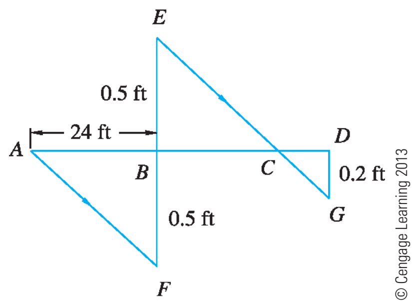

Figure 3.23

49. ***Civil engineering*** A building, located on level ground, casts a shadow of 128.45 m. At the same time, a meter stick casts a shadow of 1.75 m. What is the height of the building?

50. ***Construction*** Find the length of the brace in Figure 3.24.

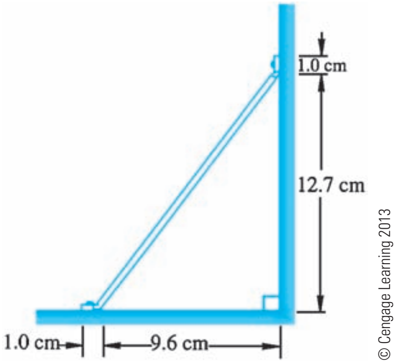

Figure 3.24

51. ***Construction*** Maria has a can of paint. Each can will cover about 425 ft^2. She needs to paint four walls. Two of the walls are 8 ft high and 10 ft long and the other two walls are 8 ft high and 14 ft long.

(a) Does Maria have enough paint?

(b) If not, how much more does she need? If she has enough, about how many more square feet could she paint?

52. ***Electronics*** Olé and Hilkka are going to install solar roof tiles on their house. Each tile measures $59'' \times 17''$ and weighs 5 lb/ft^2. Their roof is rectangular and measures 39 ft-4 in long and 18 ft-5 wide.

(a) What is the area of their roof to the nearest tenth of a square foot?

(b) How many solar roof tiles will it take to completely cover the roof?

(c) What is the total weight of these solar roof tiles?

53. ***Construction*** A ramp at a loading dock has the dimensions shown in Figure 3.25. To the nearest tenth of a foot, what is the horizontal distance x that the ramp covers?

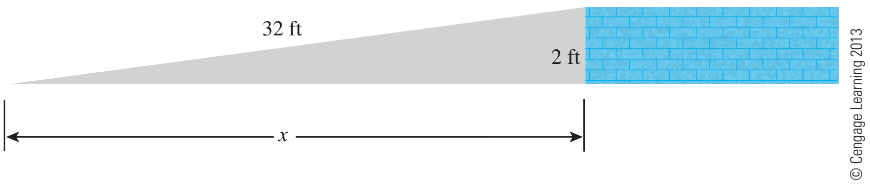

Figure 3.25

54. ***Metalworking*** Three holes are drilled in a metal plate as shown in Figure 3.26. Determine the dimensions of A and B to one decimal point.

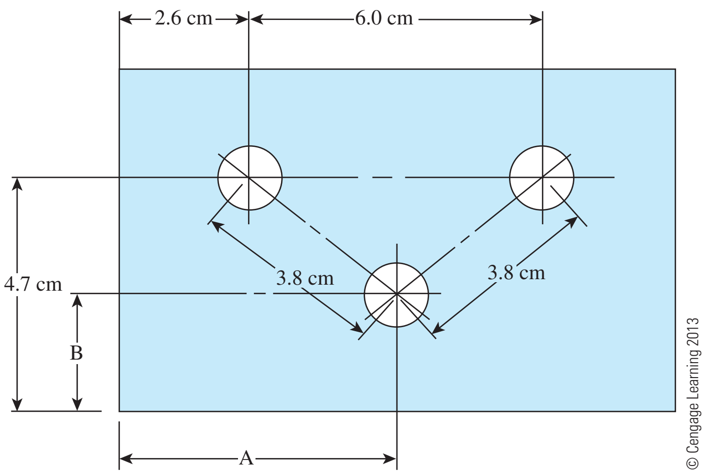

Figure 3.26

[IN YOUR WORDS]

55. **(a)** Without looking in the text, describe the relationship between degrees and radians.

(b) Explain how you can use the relationship between degrees and radians to convert between the units.

56. **(a)** Draw a diagram that shows parallel lines intersected by two transversals.

(b) Explain how you can use parallel lines and transversals to find some distances that are difficult to measure directly.

57. Describe an acute, obtuse, and right triangle. How are they alike? How are they different?

3.2 OTHER POLYGONS

In Section 3.1 we looked at the simplest type of polygon, the triangle. In this section, we will look at polygons that have more than three sides.

After the triangle, the most commonly used polygons are those with four sides. A polygon with exactly four sides is a **quadrilateral**. Other polygons that we will use are the **pentagon**, which has exactly five sides, the **hexagon** with six sides, and the **octagon** with eight sides. Any polygon that has all congruent sides and congruent angles is a *regular polygon*. For example, equilateral is another name for a regular triangle.

QUADRILATERALS

All quadrilaterals have four sides. Like triangles, different names are given to the quadrilaterals with special properties. The properties deal with the lengths of the sides, if the sides are parallel, and the sizes of the angles. Among the special types of quadrilaterals are the ones shown in Figure 3.27.

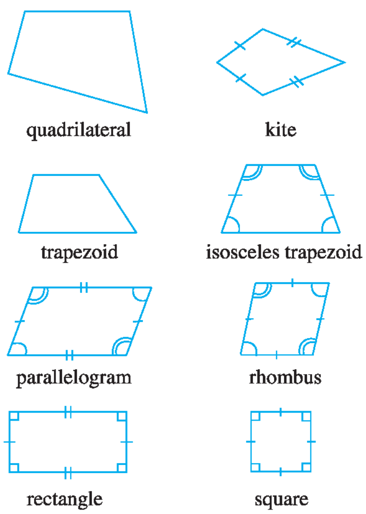

Figure 3.27

The **kite** has two pairs of adjacent sides congruent. A **trapezoid** has at least one pair of opposite sides parallel. If a trapezoid also has a pair of congruent sides that are not parallel, it is an *isosceles trapezoid* and has the congruent angles indicated in Figure 3.27. A **parallelogram** is a quadrilateral with both pairs of opposite sides parallel. As a result, the opposite sides are also congruent.

A **rhombus** is a parallelogram that has all four sides congruent. A **rectangle** is a parallelogram that has four right angles. A **square** is a rectangle that has four congruent sides. As you can see in Figure 3.27, a square is a special kind of rhombus.

To find the perimeter of a quadrilateral, you must add the lengths of the four sides. In the case of a rhombus or square, this is made easier by the fact that all four sides are the same length. So, if s is the length of one side of a rhombus or square, the perimeter is $4s$. A parallelogram, rectangle, and kite are not much more difficult, since each has two pairs of congruent sides. If the lengths of the sides that are not congruent are a and b, then the perimeter is $2(a + b)$.

In most cases, the area of a quadrilateral depends on the length of one or more sides and the distance between this side and its opposite side. If the height h is the distance between the sides and the length of the base is b, then the area of a rectangle, square, rhombus, or parallelogram can all be expressed as $A = bh$. In a trapezoid you must use the average length of the two parallel sides, $\frac{1}{2}(b_1 + b_2)$, and the height, h. So, for a trapezoid, $A = \frac{1}{2}h(b_1 + b_2)$. A kite is an exception, since its area can be found by multiplying $\frac{1}{2}$ by the lengths of the diagonals, d_1 and d_2. So, the area of a kite is $\frac{1}{2}d_1 d_2$. All of this is summarized in Figure 3.28, where the parts of each figure are labeled and the formulas for the perimeter and area of each polygon are given.

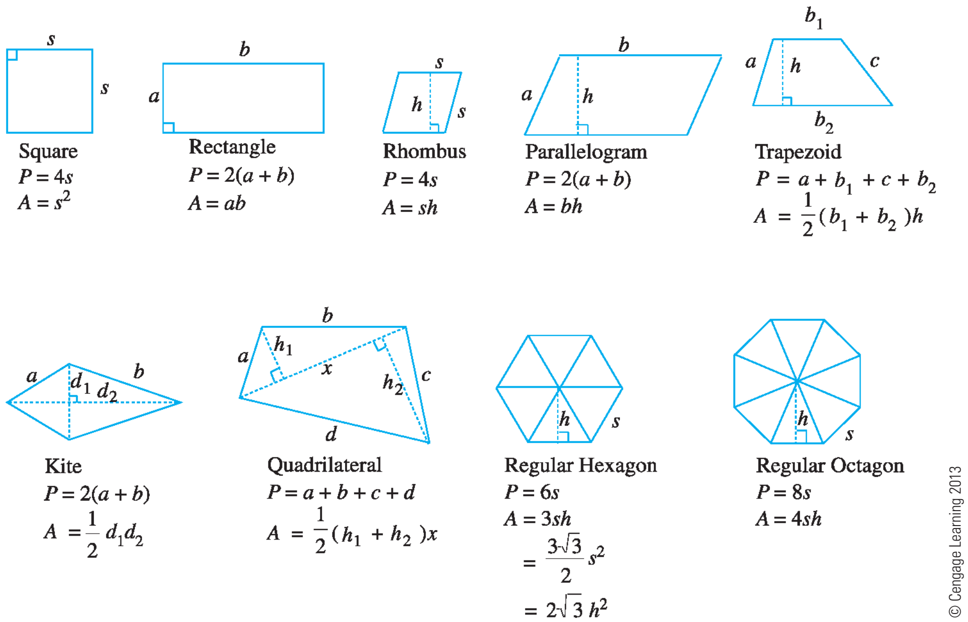

Figure 3.28

AREA AND PERIMETER OF OTHER POLYGONS

Figure 3.28 shows the formulas for the perimeter and area of a general quadrilateral. Also included are the area and perimeter for a regular hexagon and a regular octagon. In general, with a regular polygon of n sides, you can divide it into n congruent triangles, each with height h and base with length s. The perimeter of the polygon is ns and the area is $\frac{1}{2}nsh$. In Figure 3.28, two additional formulas for the area of a regular hexagon are given. These come from the fact that it is divided into six equilateral triangles and by using the Pythagorean theorem.

EXAMPLE 3.12

Find the perimeter and area of a rectangle with a length of 16 m and a width of 9 m.

SOLUTION

$$\begin{aligned} P &= 2(a + b) \\ &= 2(16 + 9) \\ &= 2(25) = 50 \end{aligned} \qquad \begin{aligned} A &= ab \\ &= 16 \cdot 9 \\ &= 144 \end{aligned}$$

The perimeter is 50 meters and the area is 144 square meters. These answers are usually written as 50 m and 144 m^2.

EXAMPLE 3.13

A trapezoid has bases of 14 and 18 in. and a height of 7 in. What is its area?

SOLUTION

$$
\begin{aligned}
A &= \frac{1}{2}h(b_1 + b_2) \\
&= \frac{1}{2}(7)(14 + 18) \\
&= \frac{1}{2}(7)(32) \\
&= 112
\end{aligned}
$$

The area is 112 square inches (often written as 112 in.^{2}).

APPLICATION MECHANICAL

EXAMPLE 3.14

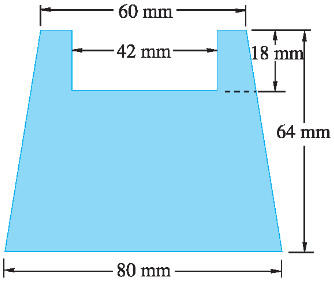

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Figure 3.29

A metal plate is made by cutting a rectangle out of an isosceles trapezoid, using the dimensions shown in Figure 3.29. (a) Find the area and (b) the perimeter of this metal plate.

SOLUTIONS

(a) To find the area of this plate, we first find the area of the trapezoid and then of the rectangle. The area of the plate is the difference between the two.

$$
\begin{aligned}
A_{\text{trapezoid}} &= \frac{1}{2}h(b_1 + b_2) \\
&= \frac{1}{2}(64)(60 + 80) \\
&= 4480 \\
A_{\text{rectangle}} &= bh \\
&= (42)(18) \\
&= 756 \\
A_{\text{plate}} &= A_{\text{trapezoid}} - A_{\text{rectangle}} \\
&= 4480 - 756 \\
&= 3724
\end{aligned}
$$

The area of this plate is 3724 mm^2.

(b) We have all the measurements we need to find the perimeter except for the lengths of the two slanted sides of the trapezoid. For this we will use the Pythagorean theorem. Look at Figure 3.30. The plate has been drawn inside a rectangle. Because the plate is an isosceles trapezoid, we know that the two lengths marked x are the same length. We also know that 2x 1 60 5 80, so x 5 10 mm. The slanted sides of the trapezoid are the hypotenuse c of the two shaded triangles in Figure 3.30. The legs of these triangles are 10 and 64 so

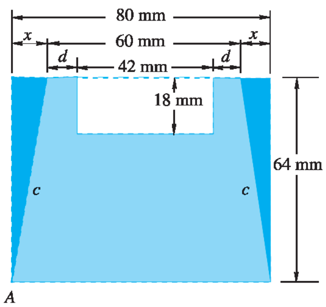

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Figure 3.30

$$c^2 = 10^2 + 64^2$$
$$= 100 + 4\,096$$
$$= 4\,196$$
$$c = \sqrt{4\,196} \approx 64.78$$

If we assume that the rectangular cutout is centered, then

$$d = \left(\frac{60 - 42}{2}\right) = 9\,\text{mm}$$

The perimeter can be found by starting at point A and adding the lengths of the sides in a clockwise direction:

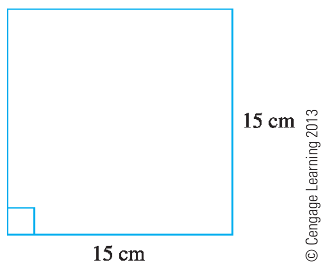
$$P = 64.78 + 9 + 18 + 42 + 18 + 9 + 64.78 + 80$$
$$= 305.56$$

The perimeter of this plate is approximately 305.56 mm.

EXERCISE SET 3.2

Find the perimeter and area of each polygon in Exercises 1–10. Use the formulas in Figure 3.28.

1.



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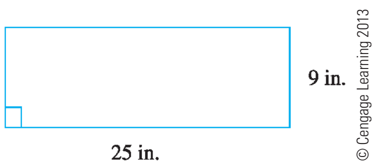

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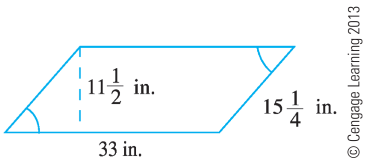

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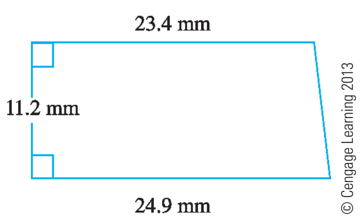

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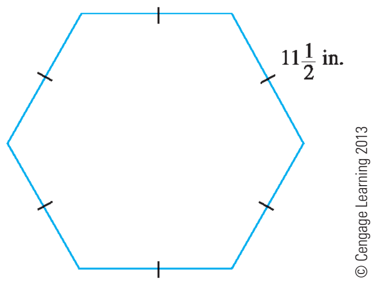

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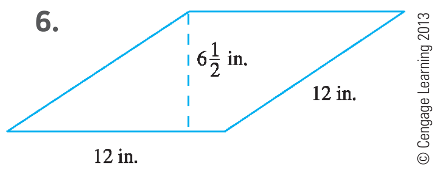

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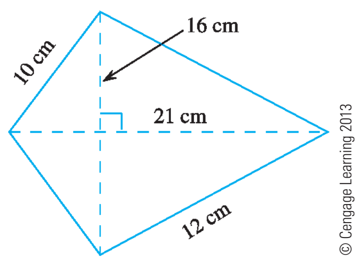

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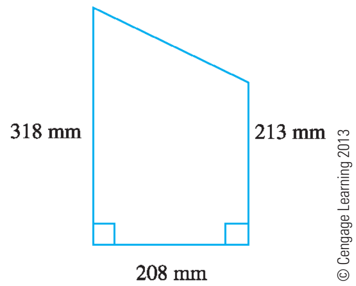

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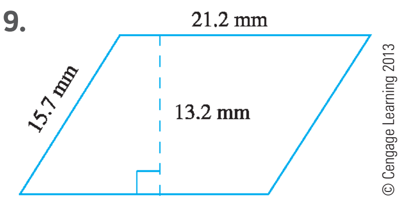

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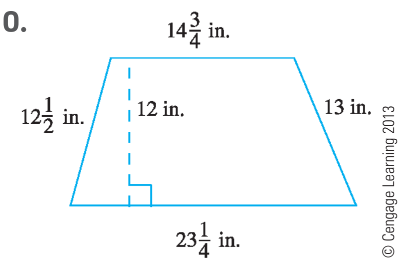

Solve Exercises 11–22.

11. ***Construction*** Find the area of the L-shaped patio in Figure 3.31.

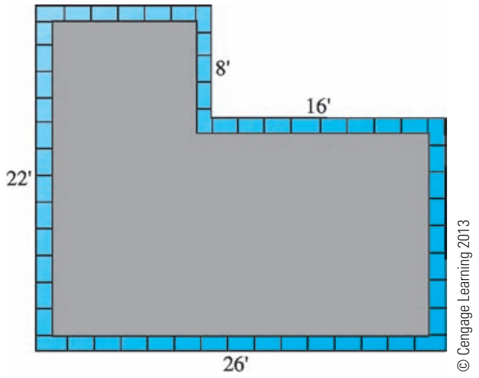

Figure 3.31

12. ***Construction*** You plan to put a brick border around the patio in Figure 3.31. Each brick is 8 in. long and 4 in. wide. If the bricks are placed end to end, how many bricks are needed?

13. ***Mechanical engineering*** What is the area of the cross-section of the I-beam in Figure 3.32a? (Hint: Think of it as a rectangle and subtract the areas of the two trapezoids. See Figure 3.32b.)

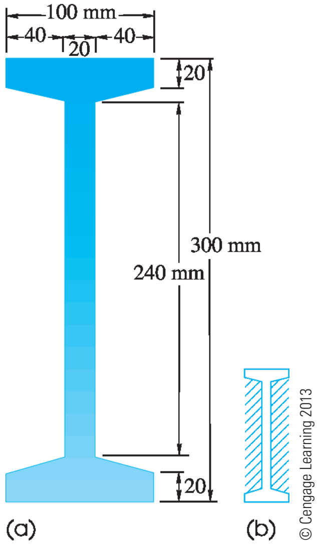

Figure 3.32

14. ***Civil engineering*** Find the area of the concrete highway support shown in Figure 3.33.

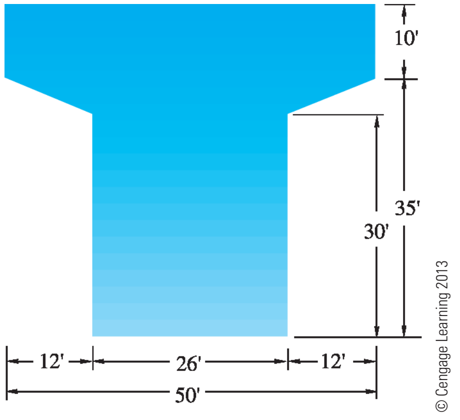

Figure 3.33

15. ***Machine technology*** A hexagonal bolt measures $\frac{7}{8}''$ across the short distance. (See Figure 3.34.) What are the perimeter and area of this bolt?

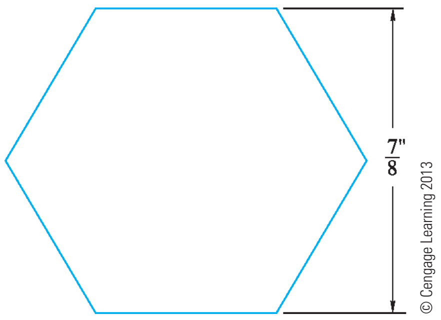

Figure 3.34

16. ***Machine technology*** A hexagonal bolt measures 15 mm across the short distance. What are the perimeter and area of this bolt?

17. ***Civil engineering*** A river bed is going to be constructed in the shape of a trapezoid. The height of the trapezoid is designed to contain flood waters between the dykes, which form the walls. The trapezoid has the dimensions given in Figure 3.35. **(a)** What is the cross-sectional area? **(b)** How many linear feet of

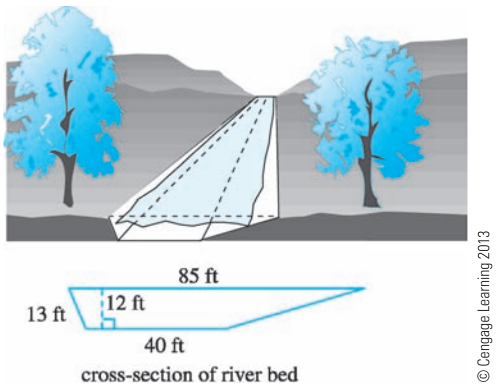

Figure 3.35

concrete are needed to surface the walls and the bottom?

18. ***Interior design*** What will it cost to carpet the floor of a rectangular room that is $20'6''$ by $15'3''$ at \$ 6.75 a square yard? (There are 9 ft^2 in 1 yd^2.)

19. ***Civil engineering*** A structural supporting member is made in the shape of an angle as shown in Figure 3.36. What is the cross-sectional area?

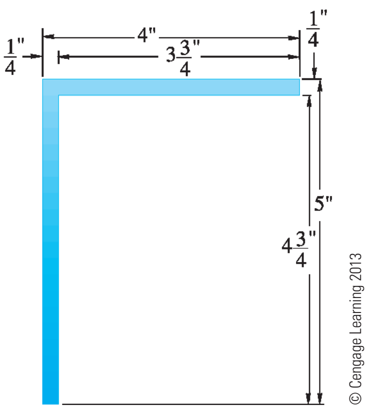

Figure 3.36

20. ***Architecture*** How many $9''$ square tiles will cover a floor $12'$ by $17'3''$?

21. ***Civil engineering*** Find the area of the cross-section of the structural tee in Figure 3.37.

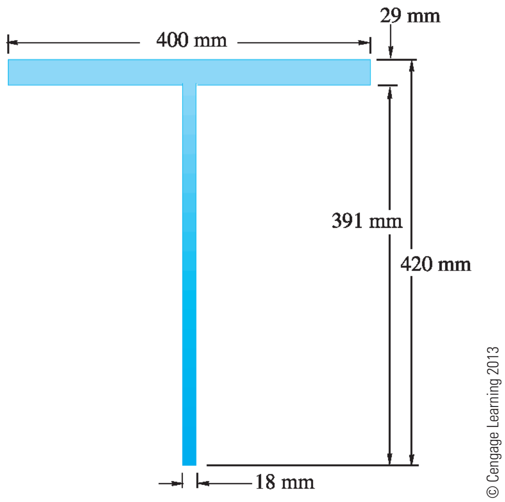

Figure 3.37

22. ***Architecture*** Find the area of the side of the house in Figure 3.38.

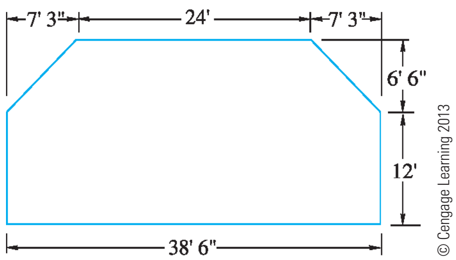

Figure 3.38

23. ***Construction*** The estimation of the quantity of brick to complete a wall with a thickness of a single brick is derived from multiplying the square feet of wall area by the number of bricks required per square foot. The number of bricks per square foot depends on the size of the brick. Wall openings such as doors and windows must be factored in the calculations or material estimates can be too high. For the house in Figure 3.39, the entire wall will be bricked except for the windows and the door.

(a) Determine the total area that will be bricked. Round your answer to the nearest square foot.

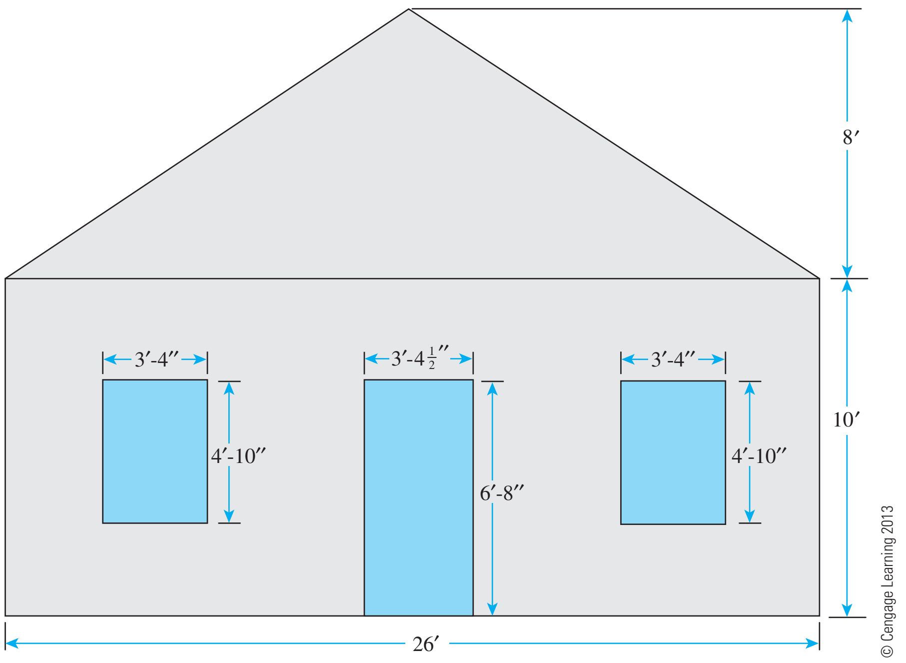

Figure 3.39

(b) How many standard-size bricks will be needed for this wall if there are 6.75 bricks per square foot?

(c) How many engineered/oversize bricks will be needed for this wall if there are 5.8 bricks per square foot?

(d) How many economy-size bricks will be needed for this wall if there are 4.5 bricks per square foot?

24. ***Construction*** Since brick must be cut at openings and waste occurs, estimations can be simplified if wall opening dimensions exclude inches and fractions, using only the foot measurements. For example, a door opening measuring $3'–4\frac{1}{2}''$ wide and $6'–8''$ high can be considered as $3' \times 6'$ for estimating purposes. Use this estimation method for the house in Figure 3.39 and

(a) Determine the total area that will be bricked. Round your answer to the nearest square foot.

(b) How many standard-size bricks will be needed for this wall if there are 6.75 bricks per square foot?

(c) How many engineered/oversize bricks will be needed for this wall if there are 5.8 bricks per square foot?

(d) How many economy-size bricks will be needed for this wall if there are 4.5 bricks per square foot?

[IN YOUR WORDS]

25. (a) Describe how a trapezoid and a parallelogram are different.

(b) Can a trapezoid ever be a parallelogram? Explain your answer.

26. Describe what is meant by area and perimeter.

3.3 CIRCLES

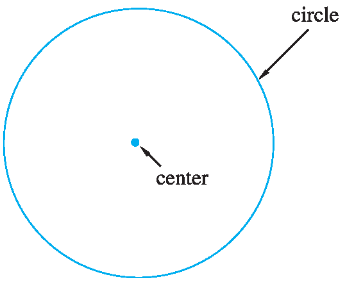

Figure 3.40

Until now, the geometric figures we have studied have all been made of segments joined at their endpoints. A circle is a different type of geometric figure. Like the polygons, a circle is part of a plane. A **circle** is made up of the set of points that are all the same distance from a point called the **center**, as shown in Figure 3.40.

PARTS OF A CIRCLE

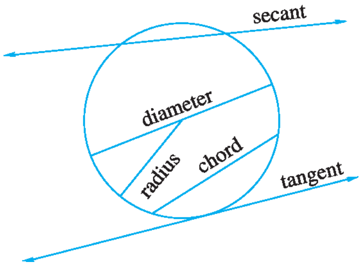

Figure 3.41

There are many parts to a circle. Some of them are shown in Figure 3.41. The **radius** is a segment with one endpoint at the center and the other on the circle. A **chord** is any segment with both endpoints on the circle. A **diameter** is a chord through the center of the circle. A **secant** is a line that intersects a circle twice, and a **tangent** intersects the circle in exactly one point. A tangent is perpendicular to the radius at its point of tangency. The **circumference** of a circle is the name for the perimeter and is equal to the distance around the circle.

Figure 3.42 shows a few more parts of the circle that will be used later. A **central angle** is an angle with a vertex at the center of the circle. An **arc** is a section of a circle and is often described in terms of the size of its central angle. Thus, we might refer to a 20° arc or an arc of $\frac{\pi}{9}$ rad. An arc with length equal to the radius is 1 rad.

A central angle divides a circle into a *minor arc* and a *major arc*. We also may refer to an arc by its endpoints. The minor arc in Figure 3.42 is identified as $\widehat{AB}$. The major arc is identified as ACB, where A and B are the endpoints and C is any other point on the major arc. The length of an arc is denoted by placing an m in front of the name of the arc. Thus, $m\,\widehat{AB}$ is the length of $\widehat{AB}$. A **sector** is the region inside the circle and is bounded by a central angle and an arc.

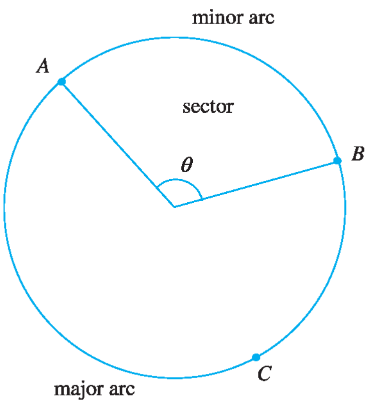

Figure 3.42

CIRCUMFERENCE AND AREA OF CIRCLES

The formulas for the circumference and area of a circle involve the use of the irrational number π. The circumference C of a circle is

$$C = \pi d \qquad \text{or} \qquad C = 2\pi r, \qquad (\text{where } d = 2r)$$

where d is the length of a diameter and r is the length of a radius.

The area of a circle is

$$A = \pi r^2$$

Some mechanics use the formula

$$A \approx 0.785d^2$$

Since $d = 2r$, you can see that $d^2 = 4r^2$ and so $r^2 = \dfrac{d^2}{4}$. The area of the circle can be written as $A = \pi\left(\dfrac{d^2}{4}\right) = \frac{\pi}{4}d^2$. Since $\frac{\pi}{4} \approx 0.785$, we have $A \approx 0.785d^2$.

CIRCUMFERENCE AND AREA OF A CIRCLE

A circle with radius r and diameter d has a circumference C and an area A where

$$C = 2\pi r = \pi d$$

$$\text{and } A = \pi r^2 = \frac{\pi}{4}d^2$$

EXAMPLE 3.15

Find the circumference and area of a circle with **(a)** diameter 7.00 in. and **(b)** radius 8.3 mm.

SOLUTIONS (a)

$$\begin{aligned} d &= 7.00\,\text{in.} \\ C &= \pi d \\ &= \pi(7) \\ &\approx 21.99\,\text{in.} \\ A &= \pi r^2 \\ &= \pi\left(\frac{7}{2}\right)^2 \\ &\approx 38.48\,\text{in.}^2 \end{aligned}$$

(b)

$$\begin{aligned} r &= 8.3\,\text{mm} \\ C &= 2\pi r \\ &= 2\pi(8.3) \\ &\approx 52.2\,\text{mm} \\ A &= \pi r^2 \\ &= \pi(8.3)^2 \\ &\approx 216.4\,\text{mm}^2 \end{aligned}$$

This example raises a question. What value should you use for π? If you use 3.14 or $3\frac{1}{7}$, your answers may differ slightly from those shown. They were obtained using a value programmed into a calculator and recalled by pressing the π key, with the resulting display of 3.1415927. As you can see, $\pi \neq 3.14$. We use 3.14 as an approximation of π.

APPLICATION ARCHITECTURE

EXAMPLE 3.16

The corporate headquarters of U.S. Xpress Enterprises, Inc. in Chattanooga, Tennessee, is shown in Figure 3.43. The building has a circular window in the

Courtesy of U.S.Xpress

Figure 3.43

front of the building that has a diameter, including the frame, of $23'11\frac{1}{2}''$. What are the **(a)** circumference and **(b)** area of the window and frame together?

SOLUTIONS

In order to solve this problem we must first convert $23'11\frac{1}{2}''$ to feet. Using dimension analysis we see that $11\frac{1}{2}'' = 11.5'' = \frac{11.5\ \cancel{\text{in.}}}{1} \times \frac{1\ \text{ft}}{12\ \cancel{\text{in.}}} \approx 0.958\overline{3}\ \text{ft}$. Thus, the diameter is $23.958\overline{3}$ ft.

(a) To find the circumference we use the formula $C = \pi d$. For this window the circumference is $C = \pi \cdot 23.958\overline{3} \approx 75.27$ ft.

(b) For the area we will use the formula $A = \frac{\pi}{4}d^2$.

$$\begin{aligned} A &= \frac{\pi}{4}d^2 \\ &= \frac{\pi}{4}(23.958\overline{3})^2 \\ &\approx 450.82 \end{aligned}$$

The area of this window and frame is about 450.82 ft^2.

ARC LENGTH

The arc length s is a direct result of the size of the angle that determines the arc. An angle of 2π rad is a complete revolution and has an arc length equal to the circumference, 2πr. An angle of π rad is half a revolution, and so its arc length is πr. Similarly, an angle of 3 rad has an arc length of $3r$. In general, a central angle of θ rad has an arc length of θr. This gives us the following formula.

ARC LENGTH

An arc formed from a circle of radius r and central angle of θ rad has an arc length s, where

$$s = \theta r$$

Similarly, the area of a sector of a circle can be derived from the formula for the area of a circle. A complete circle has an angle of 2π rad and an area of $\pi r^2 = \frac{1}{2}(2\pi)r^2$. A semicircle, or half a circle, has an angle of π rad and an area of $\frac{1}{2}\pi r^2$. In general, a central angle of θ rad forms a sector with area $\frac{1}{2}\theta r^2 = \frac{1}{2}r^2\theta$.

AREA OF A SECTOR

A sector formed by a circular arc with radius r and a central angle of θ rad has an area, A, where

$$A = \frac{1}{2}r^2\theta$$

NOTE In both of the previous formulas, the central angle θ *must* be in radians.

APPLICATION ARCHITECTURE

EXAMPLE 3.17

A landscaper is going to put some plastic edging around a pie-shaped flower bed. The flower bed is formed from a circle with radius 9.0 ft and a central angle of 105°. How much edging, in feet, will be needed?

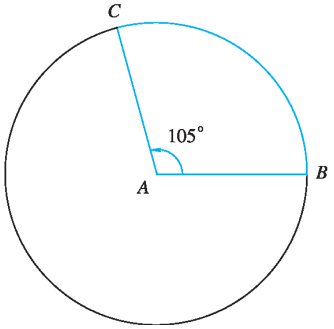

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Figure 3.44

SOLUTION A sketch of the flower bed is shown in Figure 3.44. The edging will go from C to A then to B and back to C around the arc of the circle. The length of edging needed is $CA + AB + m\widehat{BC} = 9.0 + 9.0 + m\widehat{BC} = 18.0 + m\widehat{BC}$. All we need to determine is the length of the arc $\widehat{BC}$.

To determine the length of the arc, we will use $s = \theta r$. Here, $r = 9.0$ and $\theta = 105°$. To use this formula, we must convert 105° to radians. Using the proportion $\frac{D}{180°} = \frac{R}{\pi}$, with $D = 105°$, we obtain $R = \frac{7\pi}{12}$. Thus,

$$\begin{aligned} m\,\widehat{BC} &= r\theta \\ &= 9.0\left(\frac{7\pi}{12}\right) \\ &= \frac{21.0\pi}{4} \approx 16.5\,\text{ft} \end{aligned}$$

So, the length of edging needed is $18.0 + 16.5 = 34.5$ ft.

APPLICATION MECHANICAL

EXAMPLE 3.18

A machine shop is installing two pulleys with radii 570 mm and 130 mm, respectively. The pulleys, as shown in Figure 3.45, are 1250 mm apart and the length AB is 1170 mm. If $\angle AOP = 70°$, what is the length of the driving belt?

SOLUTION To solve this problem we must add the lengths of the straight sections of the belt, AB and CD, and the two arc lengths, s_1 and s_2. We are given $AB = 1170$ mm. So, $CD = 1170$. To find s_1 we will use $s_1 = r\theta$. We are

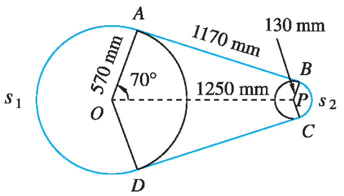


Figure 3.45

given $r = 570\,\text{mm}$, $\theta = 360° - \angle AOD = 360° - 140° = 220° \approx 3.84\,\text{rad}$. This means that

$$\begin{aligned} s_1 &= r\theta \\ &= 570(3.84) \\ &= 2188.8\,\text{mm} \end{aligned}$$

Now, $s_2 = r\theta$, where $r_2 = 130$ mm and $\theta_2 = 140° \approx 2.44$ rad. So,

$$\begin{aligned} s_2 &= r_2\theta_2 \\ &= (130)(2.44) \\ &= 317.2\,\text{mm} \end{aligned}$$

The length of the belt is $1170 + 317.2 + 1170 + 2188.8 = 4846$ mm.

APPLICATION MECHANICAL

EXAMPLE 3.19

A pie-shaped piece is going to be cut out of a circular piece of metal. The radius of the circle is 32.00 mm and the central angle of the sector is 80°. What is the area of this sector? (See Figure 3.46.)

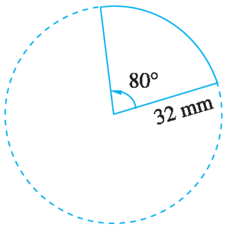


Figure 3.46

SOLUTION We will use the formula $A = \frac{1}{2}r^2\theta$. We are given $r = 32.00$ mm and $\theta = 80° \approx 1.396$ rad, so

$$A = \frac{1}{2}(32.0)^2(1.396) \approx 714.752$$

The area is about 714.8 mm^2.

APPLICATION BUSINESS

EXAMPLE 3.20

Two competing pizza companies sell pizza by the slice. At Checker's Pizza, a typical slice of pizza is a sector with a central angle of 60° formed from an 8″ radius pizza. A slice sells for \$1.75. Pizza Plus makes each slice from a 10″ radius pizza with a central angle of 45°. A slice sells for \$2.10. At which company do you get the most pizza for the price?

SOLUTION At Checker's, a sector with $\theta = 60° = \frac{\pi}{3}$ and $r = 8''$ has an area $A = \frac{1}{2}r^2\theta = \frac{1}{2}(8^2)\frac{\pi}{3} = \frac{32\pi}{3}\,\text{in.}^2$ At \$1.75 per slice, this is $\$1.75 \div \frac{32\pi}{3} \approx \$0.052/\text{in.}^2$

At Pizza Plus, a sector with $\theta = 45° = \frac{\pi}{4}$ and $r = 10''$ has an area $A = \frac{1}{2}r^2\theta = \frac{1}{2}(10^2)\frac{\pi}{4} = \frac{25\pi}{2}\,\text{in.}^2$ At \$2.10 per slice, this is $\$2.10 \div \frac{25\pi}{2} \approx \$0.053/\text{in.}^2$

Since the Checker's Pizza is $\$0.052/\text{in.}^2$ and the Pizza Plus slice cost $\$0.053/\text{in.}^2$, a slice from Checker's Pizza is the better bargain.

EXERCISE SET 3.3

Find the area and circumference of the circles in Exercises 1–8 with the given radius or diameter. (You may leave answers in terms of π.)

1. $r = 4$ cm
2. $d = 16$ in.
3. $r = 5$ in.
4. $d = 23$ mm
5. $r = 14.2$ mm
6. $r = 13\frac{1}{4}$ in.
7. $d = 24.20$ mm
8. $d = 23\frac{1}{2}$ in.

Solve Exercises 9–22.

9. ***Interior design*** **(a)** What is the area of a circular table top with a diameter of 48 in.? **(b)** How much metal edging would be needed to go around this table?

10. ***Electricity*** A coil of bell wire has 42 turns. The diameter of the coil is 0.5 m. How long is the wire on this coil?

11. ***Industrial engineering*** Two circular drums are to be riveted together, as shown in Figure 3.47, with the rivets spaced 75 mm apart. How many rivets will be needed?

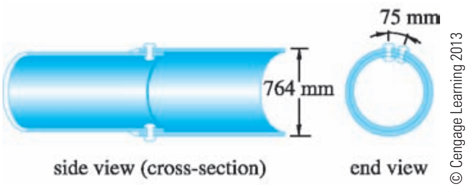


Figure 3.47

12. ***Industrial design*** Two pulleys each with a radius of $3'2''$ have their centers $11'9\frac{1}{4}''$ apart, as shown in Figure 3.48. What is the length of the belt needed for these pulleys?

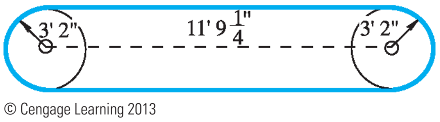


Figure 3.48

13. ***Sheet metal technology*** A sheet of copper has been cut in the shape of a sector of a circle. It is going to be rolled up to form a cone. **(a)** What is the arc length, $\widehat{AB}$, of the sector? **(b)** What is the area of this piece of copper? (See Figure 3.49.)

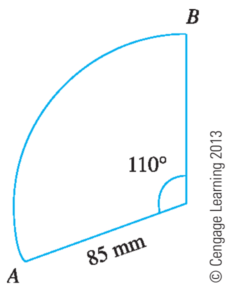


Figure 3.49

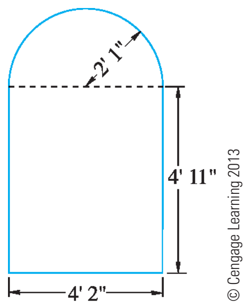


Figure 3.50

14. ***Architecture*** **(a)** Find the length of molding used around the window opening in Figure 3.50. **(b)** What is the area of the glass needed for this window?

15. ***Architecture*** The top of a stained glass window has the shape shown in Figure 3.51. The triangle is an equilateral triangle and the two arcs have their centers at the opposite vertices. That is, the arc from B to C is from a circle with center A. **(a)** What is the amount of molding needed for this window? **(b)** What is the area of the glass needed for this window?

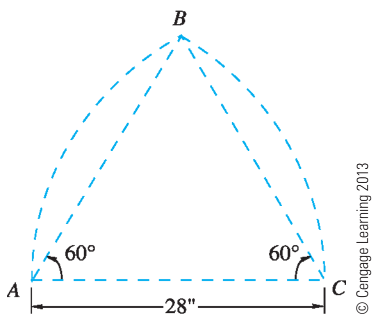


Figure 3.51

16. ***Architecture*** **(a)** Find the length of molding needed to go around the window molding in

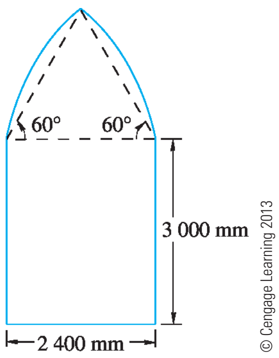


Figure 3.52

Figure 3.52. **(b)** What is the area of the glass needed for this window? (The radii for the arcs are 2 400 mm.)

17. ***Industrial design*** **(a)** What is the area of the table top in Figure 3.53? **(b)** How much metal edging would be required for this table?

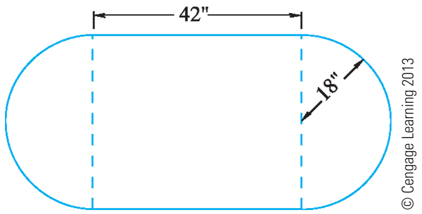


Figure 3.53

18. ***Civil engineering*** A cross-section of pipe is shown in Figure 3.54. What is the area of the cross-section?

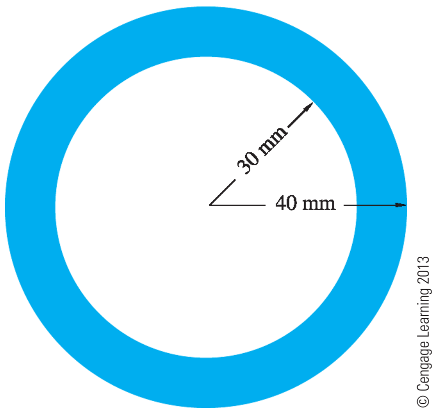


Figure 3.54

19. ***Electronics*** The resistance in a circuit can be reduced by 50% by replacing a wire with another wire that has twice the cross-sectional area. What is the diameter of a wire that will replace one that has a diameter of 8.42 mm?

20. ***Space technology*** A communications satellite is orbiting the earth at a fixed altitude above the equator. If the radius of the earth is 3,960 mi and the satellite is in direct communication with $\frac{1}{3}$ of the equator, what is the height h of the satellite? (See Figure 3.55.)

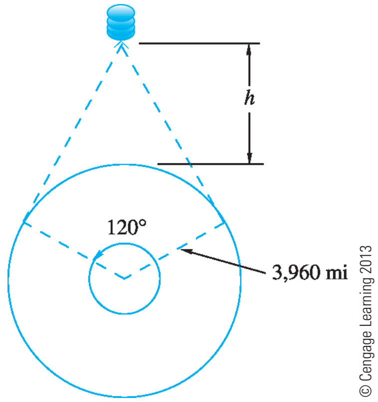


Figure 3.55

21. **(a)** What is the absolute error between $3\frac{1}{7}$ and π?

(b) What is the percent error between $3\frac{1}{7}$ and π?

22. **(a)** What is the absolute error between 3.14 and π?

(b) What is the percent error between 3.14 and π?

23. ***Electricity*** The outside diameter of an electrical conduit is $2\frac{1}{2}$ in. The conduit is 1/8 in. thick. What is the inside diameter of the conduit?

24. ***Electricity*** Juan is going to install some lights around a circular pond. If the lights are placed no more than 4 ft-3 in. apart and the pond has a diameter of 12 ft, how many lights will he need?

Figure 3.56 shows a 200-mm wafer placed on top of a 300- mm wafer. Notice that the term "200-mm wafer" refers to a wafer with a diameter of 200 mm. Use the figure to work Exercises 25–28.

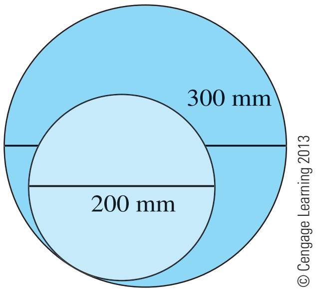


Figure 3.56

25. ***Semiconductor technology*** What is the total area of a 200-mm wafer?

26. ***Semiconductor technology*** What is the total area of a 300.0-mm wafer?

27. ***Semiconductor technology*** The entire surface of a wafer cannot be used to make chips. Each wafer has a border about 1.2 mm wide called the *exclusion area* that cannot be used. If the manufacturable area of a wafer is the total area with the exclusion area subtracted,

(a) What is the manufacturable area on a 200.0-mm wafer?

(b) What is the manufacturable area on a 300.0-mm wafer?

28. ***Semiconductor technology*** How much more manufacturability does a 300-mm wafer provide over a 200-mm wafer?

[IN YOUR WORDS]

29. Explain what is meant by an arc and by a sector of a circle.

30. Write an application in your technology area of interest that requires you to use some of the circle concepts in this section. Give your problem to a classmate and see if he or she understands and can solve your problem. Rewrite the problem as necessary to remove any difficulties encountered by your classmate.

3.4 THE AREA OF IRREGULAR SHAPES

Many shapes that have to be measured are not regular. In this section we will look at some ways to approximate the area of an irregular shape.

RECTANGULAR APPROXIMATION

Probably the easiest method would be to use the areas of several rectangles to approximate the area of an irregular shape. However, this is not very accurate. One method might be to make sure that each rectangle is entirely inside the irregular shape, as in Figure 3.57. As you can see, there is quite a bit of area that is not included. You could get a better approximations by using more (and therefore narrower) rectangles. Rather than do this, we will use another quadrilateral—the trapezoid.

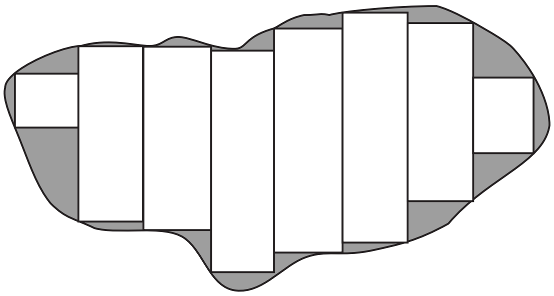


Figure 3.57

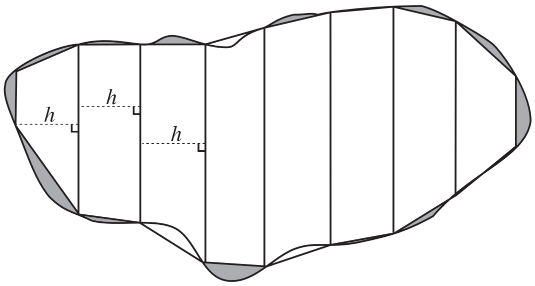


Figure 3.58

TRAPEZOIDAL APPROXIMATION

A trapezoid has one pair of opposite sides that are parallel. To use *trapezoidal approximation* we first divide the shape into sections by drawing parallel line segments that are the same distance apart. Each segment should be drawn from one side of the shape to the other. Next, we connect adjacent endpoints on one side of the figure and then connect those on the other side. When you have finished your figure should be filled with trapezoids and look something like the one in Figure 3.58. The distance between the parallel segments is marked h.

Next, measure the length of each segment. In Figure 3.59 these are labeled $y_0, y_1, y_2, \ldots, y_{n-2}, y_{n-1}$, and y_n. The area of the first trapezoid is $\frac{1}{2}h(y_0 + y_1)$, the area of the second trapezoid is $\frac{1}{2}h(y_1 + y_2)$, the area of the third trapezoid is $\frac{1}{2}h(y_2 + y_3)$, and so on. When we add each of these areas we get an approximation for the total area of the figure. This approximation method is called the trapezoidal rule.

THE TRAPEZOIDAL RULE

$$A_t \approx \frac{h}{2}(y_0 + 2y_1 + 2y_2 + 2y_3 + \cdots + 2y_{n-2} + 2y_{n-1} + y_n)$$

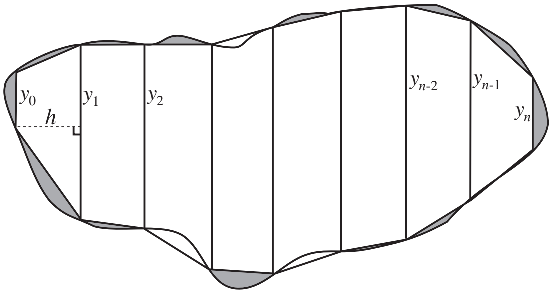


Figure 3.59

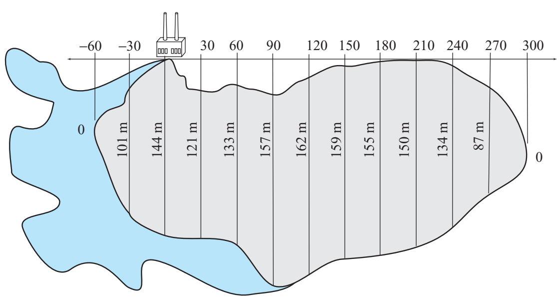


Figure 3.60

APPLICATION ENVIRONMENTAL SCIENCE

EXAMPLE 3.21

A malfunction causes a sewage treatment plant to discharge raw sewage into a lake. Because of breezes and the current only the grey portion of the lake in Figure 3.60 is polluted. If all distances are in meters, determine the area of the spill.

SOLUTION The spill has been divided into 12 sections. The distance between the parallel vertical lines is 30 m, so $h = 30$. The left-most point of the spill is y_0 and it is 0. The first segment is 101 m across the spill, so we let $y_1 = 101$. Similarly, $y_2 = 144, \ldots, y_{11} = 87$ and $y_{12} = 0$. Substituting the numbers in the trapezoidal rule we have

$$\begin{aligned} A_t &\approx \frac{h}{2}(y_0 + 2y_1 + 2y_2 + 2y_3 + \cdots + 2y_{n-2} + 2y_{n-1} + y_n) \\ &= \frac{30}{2}[0 + 2(101) + 2(144) + 2(121) + 2(133) + 2(157) + 2(162) \\ &\quad + 2(159) + 2(155) + 2(150) + 2(134) + 2(87) + 0] \\ &= (15)(3006) \\ &= 45\,090\,\text{m}^2 \end{aligned}$$

Rounding the answer to two significant digits we see that about 45 000 m^2 of the lake are polluted.

APPLICATION CONSTRUCTION

EXAMPLE 3.22

The following table gives the results of a series of drillings to determine the depth of the bedrock at a building site. These drillings were taken along a straight line down the middle of the lot where the building will be placed. In the table, x is the distance from one end of the building site and y is the corresponding depth. Both x and y are given in feet.

x	0	20	40	60	80	100	120	140	160
y	33	35	40	45	42	38	46	40	48

Approximate the area of this cross-section.

SOLUTION The total length of the interval is 160 ft. This has been divided into 8 segments, with each segment 20 ft long, so $h = 20$. This time $y_0 = 33$, $y_1 = 35, \ldots, y_8 = 48$.

Once again, using the trapezoidal rule for approximating the area, we get

$$A_t \approx \frac{h}{2}(y_0 + 2y_1 + 2y_2 + 2y_3 + \cdots + 2y_{n-2} + 2y_{n-1} + y_n)$$

$$= \frac{20}{2}[33 + 2(35) + 2(40) + 2(45) + 2(42) + 2(38)$$

$$+ 2(46) + 2(40) + 48]$$

$$= (10)(653)$$

$$= 6{,}530\,\text{ft}^2$$

This cross-section has an area of about 6,530 ft^2.

SIMPSON'S APPROXIMATION

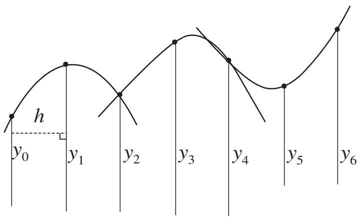


Figure 3.61

The trapezoid rule uses the nonparallel line segments of trapezoids to form the outline of a curve. An even better method would be to use curves rather than segments. A method, called **Simpson's rule**, uses curves called parabolas to try to outline the curve. Each parabola passes through three points as shown in Figure 3.61.

To use Simpson's rule we first divide the shape into sections by drawing parallel line segments that are the same distance apart. Each segment should be drawn from one side of the shape to the other. The distance between the parallel segments is marked h. This is exactly the same way you began the trapezoidal rule. However, for Simpson's rule you must have an even number of sections.

SIMPSON'S RULE

$$A_S \approx \frac{h}{3}(y_0 + 4y_1 + 2y_2 + 4y_3 + 2y_4 + \cdots + 2y_{n-2} + 4y_{n-1} + y_n)$$

where n is an even number.

APPLICATION CONSTRUCTION

EXAMPLE 3.23

Use Simpson's rule to find the area of the cross-section for Example 3.22. The table of data has been repeated below.

x	0	20	40	60	80	100	120	140	160
y	33	35	40	45	42	38	46	40	48

SOLUTION The total length of the interval is 160 ft. This has been divided into 8 segments, with each segment 20 ft long, so $h = 20$. This time $y_0 = 33$, $y_1 = 35, \ldots, y_8 = 48$.

EXAMPLE 3.23 (Cont.)

Using Simpson's rule to approximate the area, we get

$$\begin{aligned} A_S &\approx \frac{h}{3}(y_0 + 4y_1 + 2y_2 + 4y_3 + \cdots + 2y_{n-2} + 4y_{n-1} + y_n) \\ &= \frac{20}{3}[33 + 4(35) + 2(40) + 4(45) + 2(42) + 4(38) \\ &\quad + 2(46) + 4(40) + 48] \\ &= \frac{20}{3}(969) \\ &= 6{,}460\,\text{ft}^2 \end{aligned}$$

Using Simpson's rule the cross-section has an area of about 6,460 ft^2. You will recall that the trapezoidal rule gave an approximation of 6,530 ft^2.

APPLICATION CONSTRUCTION

EXAMPLE 3.24

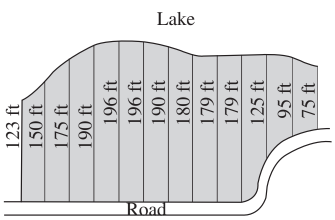


Figure 3.62

A parking lot at a state park needs to be paved. In order to know how much asphalt will be needed to pave this lot, its area will have to be determined. This area will then be multiplied by the thickness of the asphalt to obtain the total volume of the asphalt needed.

In order to find the area, the contractor drew a base line through the "middle" of the parking lot and then drew lines perpendicular to the base line every 50 feet. Each of these lines was drawn until it reached the other side of the parking lot, as shown in Figure 3.62. The following table shows the lengths labeled y (in feet) of each of these lines.

x	0	50	100	150	200	250	300	350	400	450	500	550	600
y	123	150	175	190	196	196	190	180	179	179	125	95	75

Use the data in the table to approximate the area of the parking lot using both **(a)** the trapezoidal rule and **(b)** Simpson's rule. **(c)** Contractors measure area in square yards, so convert each of these areas to square yards.

SOLUTIONS We are given that the length of each interval is 50 ft, and so $h = 50$.

(a) We first use the trapezoidal rule to approximate the area.

$$\begin{aligned} A_t &\approx \frac{h}{2}(y_0 + 2y_1 + 2y_2 + 2y_3 + \cdots + 2y_{n-2} + 2y_{n-1} + y_n) \\ &= \frac{50}{2}[123 + 2(150) + 2(175) + 2(190) + 2(196) + 2(196) + 2(190) \\ &\quad + 2(180) + 2(179) + 2(179) + 2(125) + 2(95) + 75] \\ &= (25)(3{,}908) \\ &= 97{,}700 \end{aligned}$$

Thus, we see that, according to the trapezoidal rule, the area of this parking lot is approximately 97,700 ft^2.

(b) Next, we approximate the area using Simpson's rule.

$$A_S \approx \frac{h}{3}(y_0 + 4y_1 + 2y_2 + 4y_3 + 2y_4 + \cdots + 2y_{n-2} + 4y_{n-1} + y_n)$$

$$= \frac{50}{3}[123 + 4(150) + 2(175) + 4(190) + 2(196) + 4(196) + 2(190) + 4(180) + 2(179) + 4(179) + 2(125) + 4(95) + 75]$$

$$= \frac{50}{3}(5{,}888)$$

$$= 98{,}133$$

Using Simpson's rule the area is about 98,133 ft^2.

(c) There are 9 square feet in 1 square yard. So, to convert each of these areas to yd^2 we divide by 9 with the following results:

Trapezoidal rule: $\frac{97{,}700}{9} \approx 10{,}856\,\text{yd}^2$

Simpson's rule: $\frac{98{,}133}{9} \approx 10{,}904\,\text{yd}^2$

EXERCISE SET 3.4

In Exercises 1–4, a table gives the distance across an irregular shape. Use the trapezoidal rule to determine the area of each shape. Assume that all measures are in feet.

1.

x	0	5	10	15	20	25	30	35	40	45	50
y	7.2	8.7	9.6	12.8	25.6	34.8	33.7	27.3	19.5	12.4	7.5

2.

x	0	4	8	12	16	20	24	28	32
y	25.3	32.9	39.5	28.7	35.3	46.7	52.1	46.8	39.1

3.

x	0	0.5	1.0	1.5	2.0	2.5	3.0	3.5	4.0
y	23.5	32.9	30.3	27.6	25.3	26.7	21.6	19.8	21.8

4.

x	3	5	7	9	11	13	15	17	19
y	53.4	51.7	50.4	49.3	52.5	54.3	53.2	51.7	50.9

In Exercises 5–8, a table gives the distance across an irregular shape. Use Simpson's rule to determine the area of each shape. Assume that all measures are in feet.

5.

x	0	5	10	15	20	25	30	35	40	45	50
y	7.2	8.7	9.6	12.8	25.6	34.8	33.7	27.3	19.5	12.4	7.5

6.

x	0	4	8	12	16	20	24	28	32
y	25.3	32.9	39.5	28.7	35.3	46.7	52.1	46.8	39.1

7.

x	0	0.5	1.0	1.5	2.0	2.5	3.0	3.5	4.0
y	23.5	32.9	30.3	27.6	25.3	26.7	21.6	19.8	21.8

8.

x	3	5	7	9	11	13	15	17	19
y	53.4	51.7	50.4	49.3	52.5	54.3	53.2	51.7	50.9

Solve Exercises 9–14.

9. ***Landscaping*** Determine the area of the fish pond in Figure 3.63.
10. ***Recreation*** Determine the area of the golf green in Figure 3.64.
11. ***Recreation*** Determine the area of the brake arm for a bicycle coaster brake in Figure 3.65. The vertical line segments are 0.25 in. apart. You will need to measure the length of each segment.
12. ***Aerodynamics*** Determine the area of the airfoil in Figure 3.66.
13. ***Recreation*** Determine the area of the bicycle Biopace chain ring in Figure 3.67. Note that this chain ring is not circular. You will need to decide how many vertical lines to use and then draw and measure them.
14. ***Environmental science*** Determine the area of ground covered by the Southeast Plain Coast aquifer. It is the portion shaded in color in Figure 3.68. Decide whether you want to use mi^2 or km^2 and use the scale at the bottom left of the map. You will need to decide how many vertical lines to use.

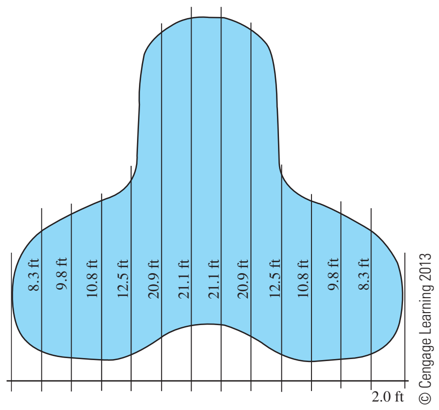


Figure 3.63

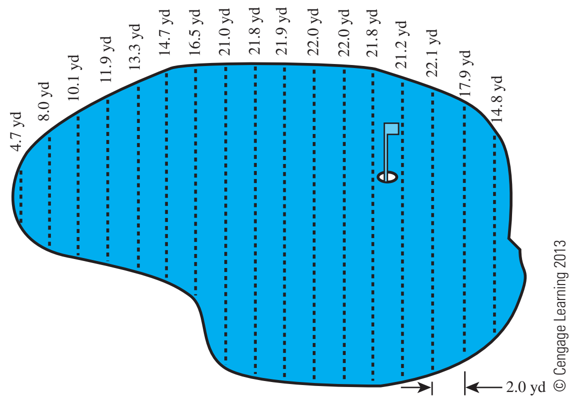


Figure 3.64

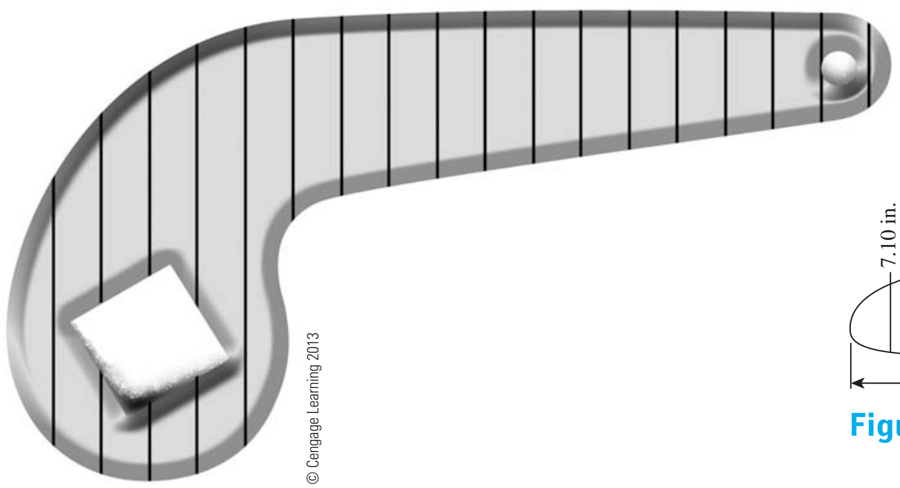


Figure 3.65

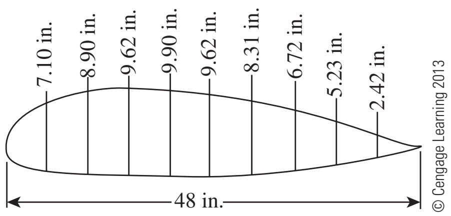


Figure 3.66

Figure 3.67

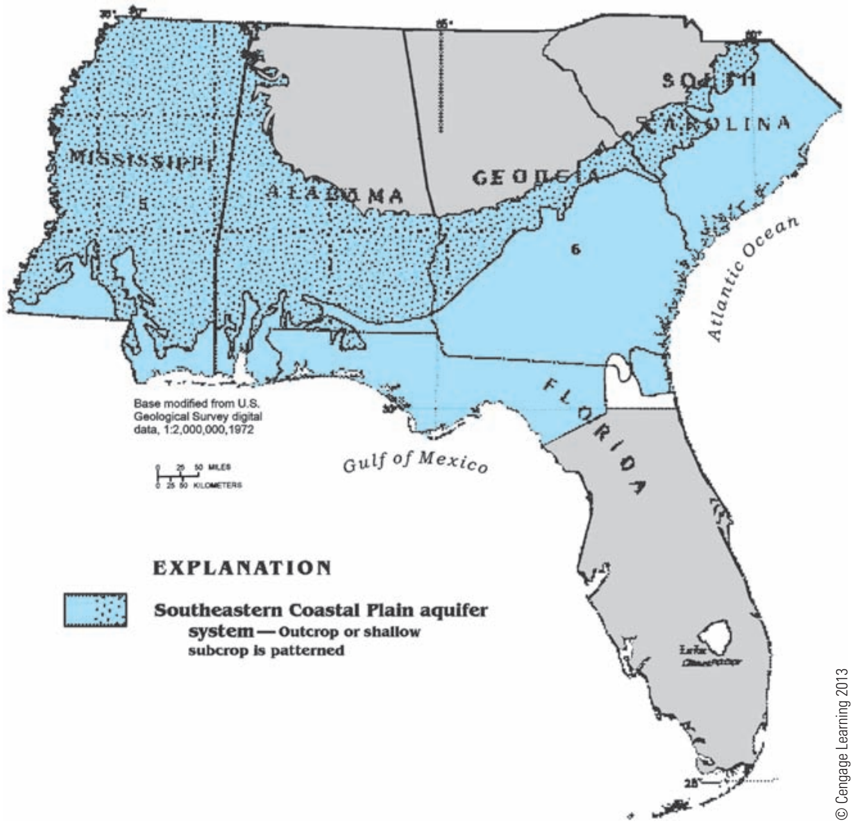


Figure 3.68

[IN YOUR WORDS]

15. Describe how to use the procedures of this chapter to find the area of an irregular-shaped region.

16. What changes would you have to make in the procedures of this section if the trapezoids were not all the same width?

3.5 GEOMETRIC SOLIDS

The geometric figures we have looked at thus far have all been plane figures, that is, figures that can be drawn in two dimensions. But, we live in a three-dimensional world. Most of the objects we work with can be characterized as three-dimensional solids. The plane geometric figures we have studied are what the objects look like on one side when they are taken apart, or "sliced" into cross-sections.

CYLINDERS

A **cylinder** is a solid whose ends, or *bases*, are parallel congruent plane figures arranged in such a manner that the segments connecting corresponding points on the bases are parallel. Several cylinders are shown in Figure 3.69. A *circular* *cylinder* is a cylinder in which both bases are circles. A *right circular cylinder* is the most common type of cylinder and is formed when the bases are perpendicular to the elements. The *height* or *altitude* of a cylinder is a segment that is perpendicular to both bases.

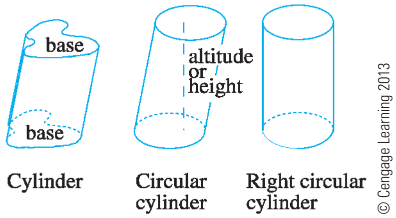


Figure 3.69

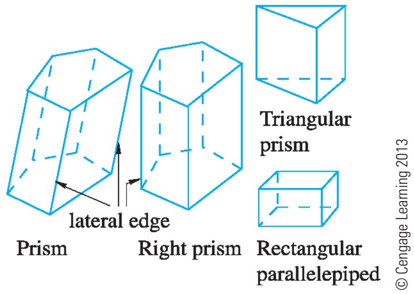


Figure 3.70

PRISMS

As shown in Figure 3.70, a **prism** is a solid with ends, or *bases*, that are parallel congruent polygons and with sides, called *faces* or *lateral faces*, that are parallelograms. The segments that form the intersections of the lateral faces are called the *lateral edges*. The height, or altitude, of a prism is the distance between the bases. A *right prism* has its bases perpendicular to its lateral edges; hence, its faces are rectangles.

Prisms get their names from their bases. If the bases are regular polygons, then the prism is a *regular prism*. A *triangular prism* has triangles for bases and a *rectangular prism* has rectangles for bases. The most common prisms are the right rectangular prisms, which are called *rectangular parallelepipeds*, and the right square prism, more commonly known as a *cube*.

There are two kinds of areas that are usually associated with any solid figure. The *lateral area* is the sum of the areas of all the sides. The **total surface area** is the lateral area plus the area of the bases. The volume of a cylinder or prism is the area of the base B times the height.

LATERAL AREA, TOTAL SURFACE AREA, AND VOLUME OF A CYLINDER OR PRISM

The lateral area, total surface area, and volume of a cylinder or prism are given by the following formulas

Solid	Lateral Area L	Total Surface Area T	Volume V
Prism	ph	$ph + 2B$	Bh
Cylinder	$2\pi rh$	$2\pi r(r + h)$	πr^2h

where p is the perimeter of a base of the prism, h is the height, r is the radius of a base of the cylinder, and B is the area of a base.

EXAMPLE 3.25

Find the **(a)** lateral area, **(b)** total surface area, and **(c)** volume of the triangular prism shown in Figure 3.71.

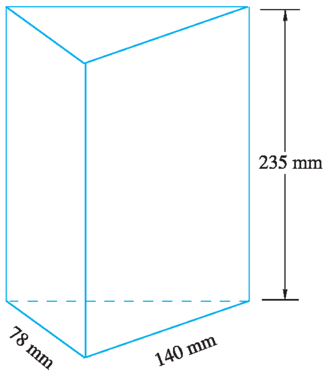


Figure 3.71

SOLUTIONS

(a) Lateral area $L = ph$, where p is the perimeter of the base and h the height of the prism. We are given $h = 235$ and add the lengths of the triangle's sides to obtain $p = 78 + 140 + 196 = 414$.

$$\begin{aligned} L &= ph \\ &= 414 \times 235 \\ &= 97\,290 \end{aligned}$$

The lateral area is 97 290 mm^2.

(b) Total surface area $T = L + 2B$, where B is the area of a base. Using Hero's formula with $s = \frac{p}{2} = \frac{414}{2} = 207$, we get

$$\begin{aligned} B &= \sqrt{s(s-a)(s-b)(s-c)} \\ &= \sqrt{207(207-78)(207-140)(207-196)} \\ &= \sqrt{207(129)(67)(11)} \\ &= \sqrt{19\,680\,111} \\ &\approx 4\,436.23 \\ T &= 2B + L \\ &= 2(4\,436.23) + 97\,290 \\ &= 8\,872.46 + 97\,290 \\ &= 106\,162.46 \end{aligned}$$

The total surface area is 106 162.46 mm^2.

(c) Volume $V = Bh$

$$\begin{aligned} &= 4\,436.23 \times 235 \\ &= 1\,042\,514.1 \end{aligned}$$

The volume is 1 042 514.1 mm^3.

CONES

A **cone** is formed by drawing segments from a plane figure, the *base*, to a point called the *vertex*. The vertex cannot be in the same plane as the base. The *altitude* is a segment from the vertex and perpendicular to the base. The most common cones are the *circular cone* and the *right circular cone*. Both have a base that is a circle. In a right circular cone, the altitude intersects the base at its center. The *slant height* of a right circular cone is a segment from the vertex to a point on the circumference. (See Figure 3.72.)

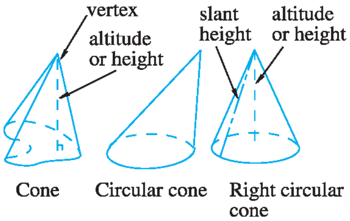


Figure 3.72

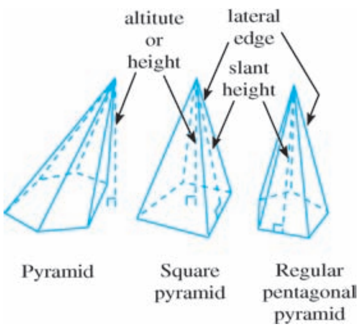


Figure 3.73

PYRAMIDS

A *pyramid* is a special type of cone, which has a base that is a polygon. A typical pyramid and some of its parts are shown in Figure 3.73. Each side of a pyramid is a triangle and is called a *lateral face*. The lateral faces meet at the *lateral edges*. As with prisms, pyramids are classified according to the shape of their base. A *regular pyramid* has a regular polygon for a base and an altitude that is perpendicular to the base at its center. The slant height of a regular pyramid is the altitude of any of the lateral faces.

LATERAL AREA, TOTAL SURFACE AREA, AND VOLUME OF CONES OR PYRAMIDS

The lateral area, total surface area, and volume of a right circular cone or regular pyramid are given by the following formulas

Solid	Lateral Area	Total Surface Area	Volume
	L	T	V
Pyramid	$\frac{1}{2}ps$	$\frac{1}{2}ps + B$	$\frac{1}{3}Bh$
Cone	πrs	$\pi r(r + s)$	$\frac{1}{3}\pi r^2 h$

where p is the perimeter of a base of the pyramid, h is the height or altitude, s is the slant height, r is the radius of a base of the cone, and B is the area of a base.

APPLICATION MECHANICAL

EXAMPLE 3.26

How many square inches of metal are needed to make a cone-shaped container like the one in Figure 3.74. What is the volume of this cone?

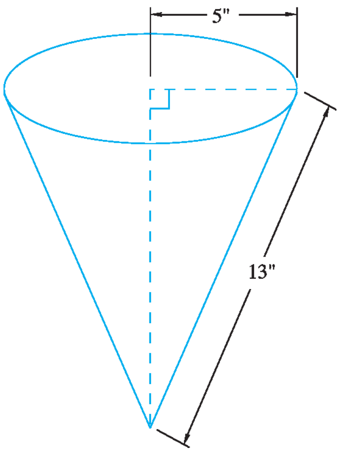


Figure 3.74

SOLUTION The amount of metal needed is the lateral area, L. We know that

$$L = \pi rs$$

where r is the radius of the base and s is the slant height of the cone.

$$\begin{aligned} L &= \pi(5)(13) \\ &= 65\pi \approx 204.2 \end{aligned}$$

It will take about 204.2 in.^{2} of metal.

$V = \frac{1}{3}\pi r^2 h$, where h is the height of the cone. The slant height, a radius, and the altitude form a right triangle with the slant height equal to the hypotenuse. So, $h^2 + 5^2 = 13^2$ or $h = 12$.

$$\begin{aligned} V &= \frac{1}{3}\pi(5)^2(12) \\ &= 100\pi \approx 314.2 \end{aligned}$$

The volume is about 314.2 in.^{3}

FRUSTUMS

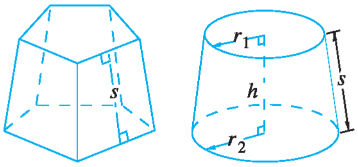


Figure 3.75

A **frustum** of a cone or a pyramid is formed by a plane parallel to the base that intersects the solid between the vertex and the base. We will limit our study to frustums of right circular cones or right regular pyramids, as shown in Figure 3.75. If the height of the frustum is h and the areas of the bases are B_1 and B_2 then the volume V is given by

$$V = \frac{h}{3}\left(B_1 + B_2 + \sqrt{B_1B_2}\right)$$

The lateral area L is given using p_1 and p_2 as the perimeter of the bases and s, the slant height:

$$L = \frac{s}{2}(p_1 + p_2)$$

For the frustum of a cone with bases of radii r_1 and r_2, p_1 and p_2 are actually the circumferences of the bases. Thus, $p_1 = 2\pi r_1$ and $p_2 = 2\pi r_2$ and the formula becomes $L = \frac{s}{2}(2\pi r_1 + 2\pi r_2)$ or

$$L = \pi s(r_1 + r_2)$$

APPLICATION CIVIL ENGINEERING

EXAMPLE 3.27

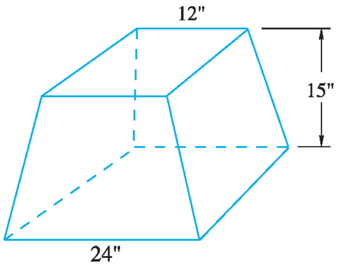


Figure 3.76

The concrete base of a light pole is constructed in the form of the frustum of a square pyramid, as shown in Figure 3.76. What is the volume of the base for the light pole?

SOLUTION The volume of a frustum of a square pyramid is given by the formula $V = \frac{h}{3}(B_1 + B_2 + \sqrt{B_1B_2})$. We must first find the area of the bases. If B_1 is the area of the top base, then $B_1 = 12^2 = 144$ in.^{2} and $B_2 = 24^2 = 576$ in.^{2} So,

$$\begin{aligned} V &= \frac{h}{3}\left(B_1 + B_2 + \sqrt{B_1B_2}\right) \\ &= \frac{15}{3}\left(144 + 576 + \sqrt{144 \cdot 576}\right) \\ &= \frac{15}{3}(144 + 576 + 288) \\ &= 5{,}040 \end{aligned}$$

The volume is 5,040 in.^{3}

APPLICATION MECHANICAL

EXAMPLE 3.28

The photo in Figure 3.77a shows an oven hood. The hood was made by combining two prisms and the frustum of a square pyramid. In Figure 3.77b, a line drawing of the hood is shown with the measurements indicated. The bottom of the pyramid is 72 in. on each of the two longer sides and 36 in. on each of the two shorter sides. The top base measures 18 in. on each side. The slant height for the longer sides is 15 in. For the shorter sides, the slant height is 29.5 in. The top prism is 36 in. high and the bottom prism is 6 in. high. **(a)** How much metal will it take to form the outside of the part of the oven hood formed by the frustum of the pyramid? **(b)** How much metal will it take to form the outside of the entire oven hood?

Figure 3.77a

SOLUTIONS

(a) The amount of metal needed can be found by determining the lateral area L of this pyramid. Notice that this is a frustum of a rectangular pyramid and not of a square pyramid. This means that we cannot use the above formula for the lateral area of the frustum of a pyramid. Since we cannot use the formula, we will think of this frustum as if it were made of four trapezoids, as shown in Figure 3.77c. These four trapezoids are two pairs of congruent trapezoids, so we need only find the areas of the left two trapezoids, marked T_1 and T_2, and double their total area to get the lateral surface area of the frustum in the oven hood.

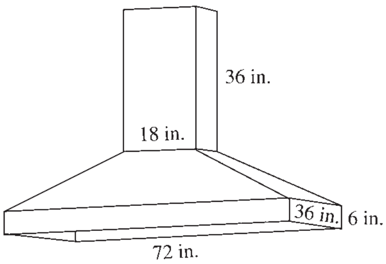


Figure 3.77b

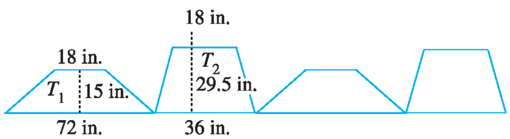


Figure 3.77c

The left trapezoid T_1, has an area L_1 of

$$\begin{aligned} L_1 &= \frac{1}{2}(b_1 + b_2)h \\ &= \frac{1}{2}(18 + 72)15 \\ &= 675 \end{aligned}$$

Trapezoid T_2 has an area of $L_2 = \frac{1}{2}(36 + 18)(29.5) = 796.5\,\text{in.}^2$ The total lateral surface area for the frustum and hence the amount of metal it will take to make this portion of the oven hood is $2(675 + 796.5) = 2{,}943\ \text{in.}^2$

(b) To get the amount of metal it will take to form the outside of the entire oven hood, we add the answer to (a) to the lateral surface areas of the two prisms. The top prism has a square base that is 18 in. on each side and the prism is 36 in. high. So, it has a perimeter of $4 \times 18 = 72$ in. and its lateral area is $L_t = 36 \times 72 = 2{,}592$ in.^{2} The bottom prism is a rectangular prism with a perimeter of $72 + 36 + 72 + 36 = 216$ in. and a height of 6 in. Hence, its lateral surface area is $L_b = 6 \times 216 = 1{,}296$ in.^{2} The total amount of metal needed to form the outside of this oven hood is then $2{,}943 + 2{,}592 + 1{,}296 = 6{,}831$ in.^{2}

SPHERES

The final geometric solid that we will consider is the sphere. A *sphere* consists of all the points in space that are a fixed distance from some fixed point called the center. For a sphere with radius r, the volume and surface area are given by the following formulas.

SURFACE AREA AND VOLUME OF A SPHERE

A sphere with radius r has a surface area S and volume V where

$$S = 4\pi r^2$$

$$V = \frac{4}{3}\pi r^3$$

APPLICATION CIVIL ENGINEERING

EXAMPLE 3.29

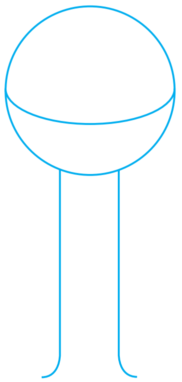


Figure 3.78

A water tower, like the one in Figure 3.78, is in the shape of a sphere on top of a cylinder. Most of the water is stored in the sphere, which has a radius of 30 ft. How much water can the tower hold?

SOLUTION We want to determine the volume of a sphere with a radius of 30 ft. The formula for the volume of a sphere is $V = \frac{4}{3}\pi r^3$. Since $r = 30$ ft, we have

$$\begin{aligned} V &= \frac{4}{3}\pi(30)^3 \\ &= \frac{4}{3}\pi(27{,}000) \\ &= 36{,}000\pi \\ &\approx 113{,}097.34 \end{aligned}$$

The water tower will hold approximately 113,097 ft^3 of water.

A summary of all the formulas for the areas and volumes of the solid figures is given in Table 3.1.

TABLE 3.1 Areas and Volumes of Solid Figures

Solid	Lateral Area (L)	Total Surface Area (T)	Volume (V)
Rectangular prism	$2h(l + w)$	$2lw + 2lh + 2hw$ $= 2(lw + lh + hw)$	lwh
Cube	$4s^2$	$6s^2$	s^3
Prism	ph	$ph + 2B$	Bh
Cylinder	$2\pi rh$	$2\pi r(r + h)$	$\pi r^2 h$
Pyramid	$\frac{1}{2}ps$	$\frac{1}{2}ps + B$	$\frac{1}{3}Bh$
Cone	πrs	$\pi r(r + s)$	$\frac{1}{3}\pi r^2 h$
Frustum of pyramid	$\frac{s}{2}(p_1 + p_2)$	$\frac{s}{2}(p_1 + p_2) + B_1 + B_2$	$\frac{h}{3}(B_1 + B_2 + \sqrt{B_1B_2})$
Frustum of cone	$\pi s(r_1 + r_2)$	$\pi[r_1(r_1 + s) + r_2(r_2 + s)]$	$\frac{\pi h}{3}(r_1^2 + r_2^2 + r_1r_2)$
Sphere	not applicable	$4\pi r^2$	$\frac{4}{3}\pi r^3$

VOLUMES OF IRREGULAR SHAPES

We can use the trapezoidal rule and Simpson's rule to approximate the volume of an irregular solid figure. Rather than find the length across the object at various places you need to find the cross-sectional area at several places. In Example 3.30 the cross-sectional areas are given, but you may need to calculate or approximate them.

APPLICATION **CONSTRUCTION**

EXAMPLE 3.30

A road is going to be constructed through a hill. In order to determine the volume of material (soil, rock, etc.) that must be removed, a cross-section was surveyed every 100 feet and the area of each cross-section is given in the table below. Use the trapezoidal rule to determine the volume of material that has to be removed.

x(ft)	0	100	200	300	400	500	600	700	800
A(ft^2)	0	1681.0	2394.4	2525.6	2369.8	2107.4	1869.6	1713.8	1648.2

x(ft)	900	1000	1100	1200	1300	1400	1500	1600	1700
A(ft^2)	1682.4	1664.6	1640.0	1541.6	1320.2	975.8	565.8	180.4	0

SOLUTION The length of each interval is 100 ft, and so $h = 100$. We use the trapezoidal rule but change the ys to As since they represent area rather than length.

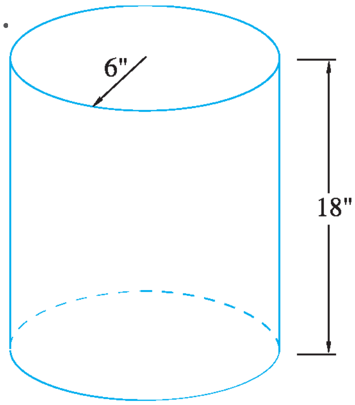
$$
\begin{aligned}
V_t &\approx \frac{h}{2}(A_0 + 2A_1 + 2A_2 + 2A_3 + \cdots + 2A_{n-2} + 2A_{n-1} + A_n) \\
&= \frac{100}{2}[0 + 2(1681.0) + 2(2394.4) + 2(2525.6) + 2(2369.8) \\
&\quad + 2(2107.4) + 2(1869.6) + 2(1713.8) + 2(1648.2) + 2(1682.4) \\
&\quad + 2(1664.6) + 2(1640.0) + 2(1541.6) + 2(1320.2) + 2(975.8) \\
&\quad + 2(565.8) + 2(180.4) + 0] \\
&= (50)(51{,}761.2) \\
&= 2{,}588{,}060
\end{aligned}
$$

According to the trapezoidal rule, about 2,588,060 ft^3 of material must be removed to build this portion of the road.

EXERCISE SET 3.5

Find the lateral area, total surface area, and volume of each of the solids in Exercises 1–8.

1.



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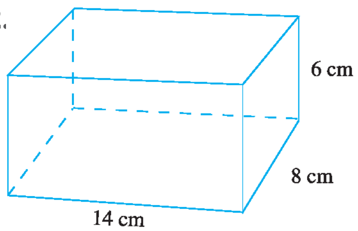
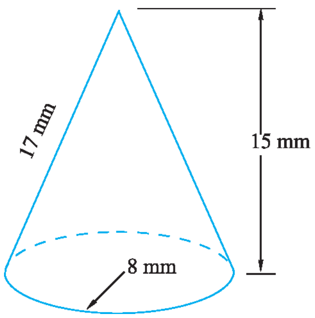


3.



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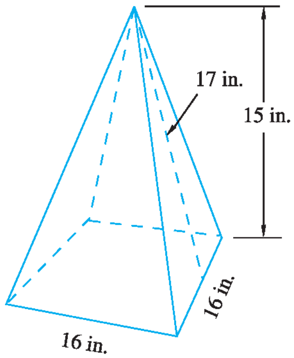

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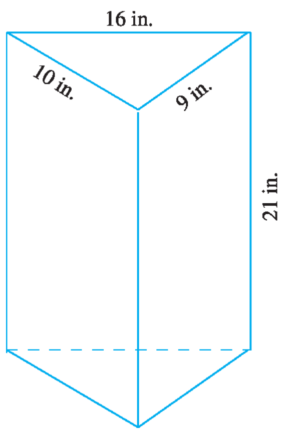
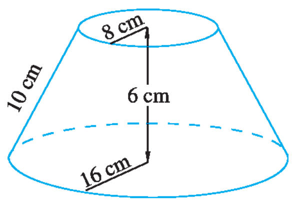


6.



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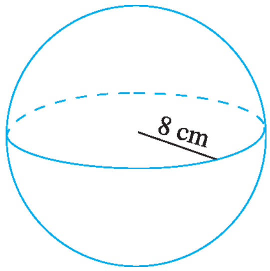
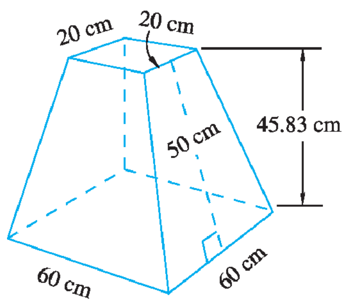


8.



Solve Exercises 9–32.

9. ***Civil engineering*** The cross-section of a road is shown in Figure 3.79. Find the number of cubic yards of concrete it will take to pave 1 mi of this road. (There are 27 ft^3 in 1 yd^3 and 5,280 ft in 1 mi.)

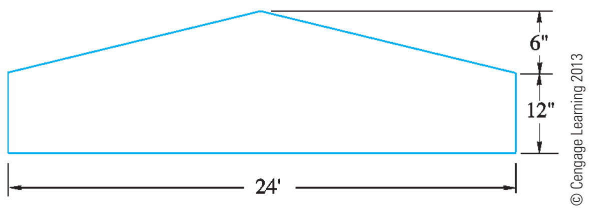


Figure 3.79

10. ***Civil engineering*** The cross-section of an I-beam is shown in Figure 3.80. **(a)** What is the volume of this beam if it is 10 m long? **(b)** How much paint, in cm^2, will be needed for this beam? **(c)** If 1 cm^3 of steel has a mass of 0.008 kg, what is the mass of this beam?

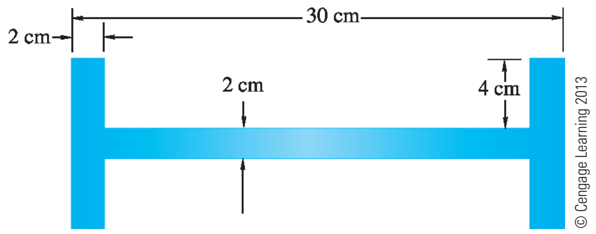


Figure 3.80

11. ***Transportation*** A railroad container car in the shape of a rectangular parallelepiped is 30 ft long, 10 ft wide, and 12 ft high. **(a)** How much can the container car hold? **(b)** How many square feet of aluminum were required to make the car?

12. ***Energy*** A cylindrical gas tank has a radius of 48 ft and a height of 140 ft. What are the volume and total surface area of the tank?

13. ***Product design*** A cylindrical soup can has a diameter of 66 mm and a height of 95 mm. **(a)** How much soup can the can hold (in mm^3)? **(b)** How many square millimeters of paper are needed for the label if the ends overlap 5 mm?

14. ***Mechanics*** A cross-section of pipe is shown in Figure 3.81. If the pipe is 2 m long, what is the volume of the material needed to make the pipe?

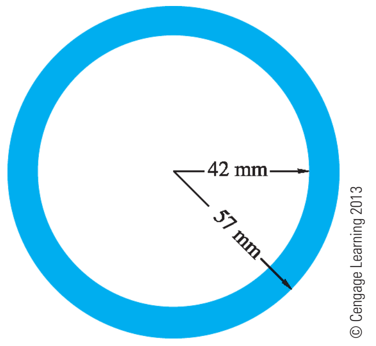


Figure 3.81

15. ***Energy*** A spherical fuel tank has a radius of 10 m. What is its volume?

16. ***Sheet metal technology*** A vent hood is made in the shape of a frustum of a square pyramid that is open at the top and bottom, as indicated in Figure 3.82. If the slant height is 730 mm, how much metal did it take to make this vent?

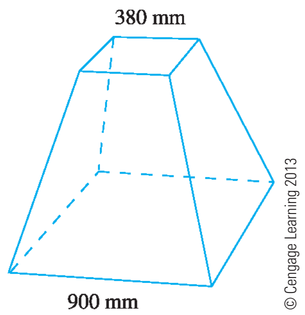


Figure 3.82

17. ***Civil engineering*** The concrete highway support in Figure 3.83 is 2 ft 6 in. thick. How many cubic yards of concrete are needed to make one support? (27 ft^3 = 1 yd^3.)

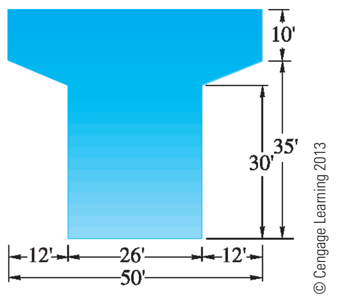


Figure 3.83

18. ***Construction*** How many cubic feet of dirt had to be excavated to dig a $\frac{1}{4}$-mi (1,320-ft) section of the river bed in Figure 3.84?

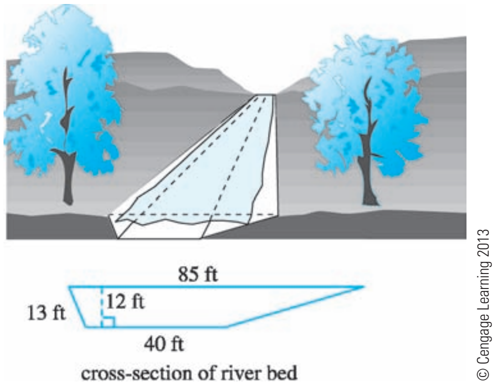


Figure 3.84

19. ***Sheet metal technology*** What is the volume of the cone that was made from the piece of copper in Figure 3.85?

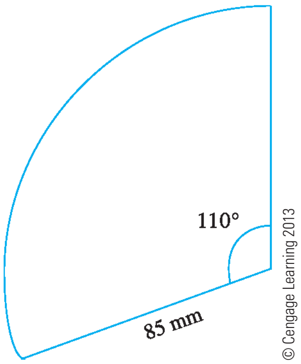


Figure 3.85

20. ***Sheet metal technology*** A manufacturer has an order for 500 tubs in the shape of a frustum of a cone that is 490 mm across at the top, 380 mm across at the bottom, and 260 mm deep. How much material will be needed for the sides of the tubs?

21. ***Sheet metal technology*** How many square inches of tin are required to make a funnel in the shape of a frustum of a cone that has a top and bottom with diameters of 3 in. and 8 in., respectively, and a slant height of 12 in.?

22. ***Sheet metal technology*** The funnel in Figure 3.86 has a diameter 3 cm at B and 1 cm at A. How much metal is needed to make this funnel?

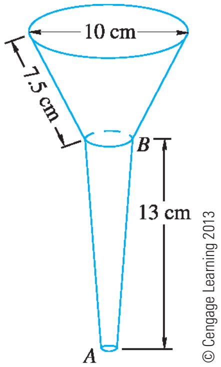


Figure 3.86

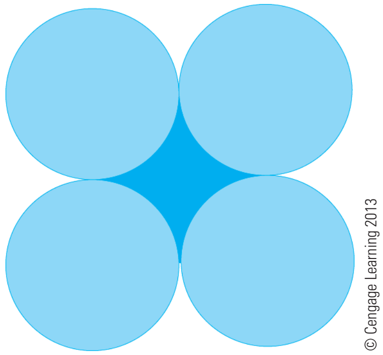


Figure 3.87

23. ***Agricultural technology*** Figure 3.87 shows the top view of four grain elevators. When the elevators are filled, the grain will overflow to fill the space in the middle. The radius of each elevator is 3 m and the height is 10 m. What is the total volume that can be held by the four elevators and the space in the middle if the grain is leveled at the top?

24. ***Automotive technology*** The piston displacement is the volume of the cylinder with a given bore (diameter) and piston stroke (height). **(a)** If the bore is 7 cm and the stroke is 8 cm, what is the displacement for this piston? **(b)** If this is a 6-cylinder engine, what is the total engine displacement?

25. ***Construction*** A pile of sand falls naturally into a cone. If a pile is 4 ft high and 12 ft in diameter, how many cubic feet of sand are there?

26. ***Construction*** A grain silo has the shape of a right circular cylinder topped by a hemisphere. The cylindrical part of the silo has a height of 40 ft and radius of 8 ft. What is the surface area of the silo?

27. ***Sheet metal manufacturing*** A tray for electronics parts is going to be made from the piece of sheet metal shown in Figure 3.88 by folding the metal on the dashed lines and welding the ends that meet.

(a) What is the name of the completed figure?

(b) What is the area of the bottom of this tray?

(c) What is the surface area of the tray? (Remember, the tray does not have a top.)

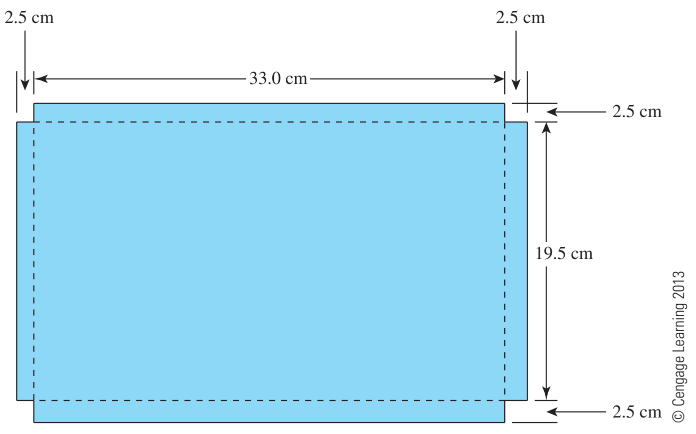


Figure 3.88

28. ***Electricity*** The core of a cylindrical electromagnet is 9.25 in. long with a radius of 15.4 in. If it is entirely covered with one layer of insulation paper, what is the area of the paper?

29. ***Construction*** A rectangular swimming pool, with a width of 6.4 m and a cross-section as shown in Figure 3.89, is to be lined with epoxy paint. Note that the entire sides of the pool including the portion above the water will need to be painted.

(a) What is the area of the surfaces that will need to be painted?

(b) If a minimum dry coat thickness of 3.0 mm is required and the paint will shrink 25% as it dries, how many liters of paint will be needed?

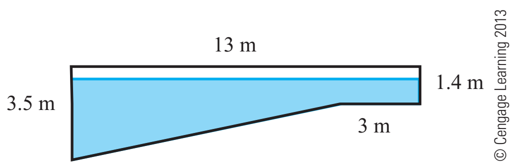


Figure 3.89

30. ***Recreation*** An automatic chlorination and purification system is being designed for the outdoor pool in Figure 3.90. The actual design of the system will depend on the volume of water in the pool. The width of the pool has been measured at 2-meter intervals. The walls of the pool are vertical.

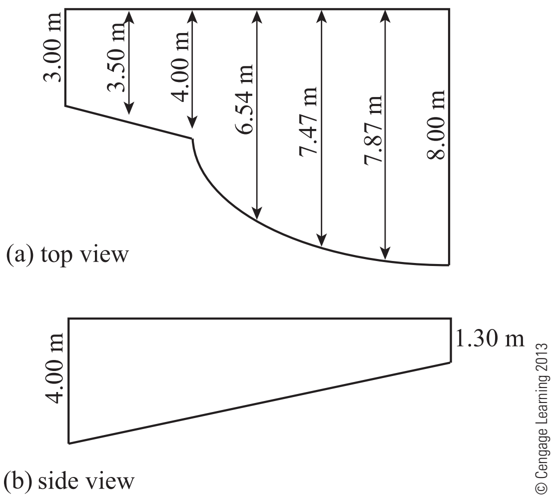


Figure 3.90

(a) Use Simpson's rule to estimate the area of the pool.

(b) Use the trapezoidal rule to estimate the volume of water the pool will hold. Assume that the water will go to the top of the pool.

31. ***Construction*** A tunnel needs to be cut through a mountain. The cross-section of the tunnel will have an area of about 46.6 m^2, and its length will be 4327 m. How much rock will need to be removed?

32. ***Construction*** In Example 3.24 we used the trapezoidal rule to find the area of a parking lot at 10,856 yd^2. The lot is to be paved with asphalt. It will require an 8″ thick crushed rock base, 2″ of asphalt binder, and $1\frac{1}{2}$″ of asphalt topping. The paving contractor measures the amount of material in "yards," where a yard is a square yard of material 2″ thick. A truck will hold around 20 tons of asphalt, which will be enough to cover about 200 yards.

(a) How many truck loads will be needed for the base?

(b) How many truck loads will be needed for the binder?

(c) How many truck loads will be needed for the topping?

33. ***Construction*** A pilaster is shown in Figure 3.91. Except for the very top and bottom portions the cross-sections of the pilaster are circles. Diameters of the post were measured every inch.

(a) Use the diameters every two inches and the trapezpoidal rule to approximate the volume of this 22″ section of the pilaster.

(b) Why can't you use these measures with Simpson's rule to get an estimate of the volume?

34. ***Construction*** A pilaster is shown in Figure 3.91. Except for the very top and bottom portions the cross-sections of the pilaster are circles. Diameters of the post were measured every inch.

(a) Use the diameters every inch and the trapezoidal rule to approximate the volume of the 22″ section of the pilaster.

(b) Use the diameters every inch and Simpson's rule to approximate the volume of the 22″ section of the pilaster.

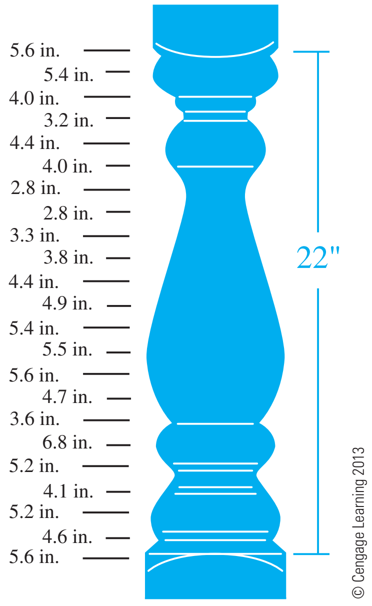


Figure 3.91

[IN YOUR WORDS]

35. Describe how cylinders and prisms are alike and how they are different.

36. Describe how cylinders and cones are alike and how they are different.

3.6 SIMILAR GEOMETRIC SHAPES

In this section, we will examine similar triangles and other geometric figures. We begin with the concept of continued proportion.

CONTINUED PROPORTION

A **continued proportion** is a proportion that involves six or more quantities. If there are six quantities, a continued proportion is of the form $a : b : c = x : y : z$. This is a shorthand notation for writing $\frac{a}{x} = \frac{b}{y} = \frac{c}{z}$. We will use continued proportions when we study similar figures and later when we study trigonometry.

EXAMPLE 3.31

Solve the proportion $2:5:7 = x:32:z$.

SOLUTION We will solve this continued proportion in stages. Remember that $2:5:7 = x:32:z$ is a short way of writing $\frac{2}{x} = \frac{5}{32} = \frac{7}{z}$. We will first find x.

Working with the proportion from the two ratios on the left, we obtain the proportion $\frac{2}{x} = \frac{5}{32}$. The product of the extremes is $2 \times 32 = 64$ and is equal to the product of the means, $5x$. If $5x = 64$, then $x = 12.8$.

Next, we work with the proportion from the two ratios on the right of the continued proportion. This proportion is $\frac{5}{32} = \frac{7}{z}$ or $5z = 7(32) = 224$, and so $z = 44.8$.

Thus, the solution to the proportion $2:5:7 = x:32:z$ is $2:5:7 = 12.8:32:44.8$.

EXAMPLE 3.32

A 520-mm wire is to be cut into three pieces so that the ratio of the lengths is to be 6 : 4 : 3. How long should each piece be?

SOLUTION The length of each piece is a multiple of some unknown length, x. If we represent the length of each piece by $6x$, $4x$, and $3x$, then we have the total length of $6x + 4x + 3x = 13x$. But the length of the wire is 520 mm. Thus,

$$13x = 520$$
$$x = 40$$

The lengths must then be $6(40) = 240$, $4(40) = 160$, and $3(40) = 120$. The proportion would be $6:4:3 = 240:160:120$.

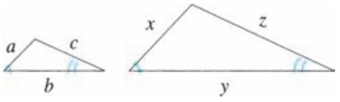


Figure 3.92

SIMILAR TRIANGLES

Two triangles that have the same shape are **similar triangles**. In similar triangles the corresponding angles are congruent and the corresponding sides are proportional. In Figure 3.92, since the triangles are similar, $\frac{a}{x} = \frac{b}{y} = \frac{c}{z}$ or $a:b:c = x:y:z$.

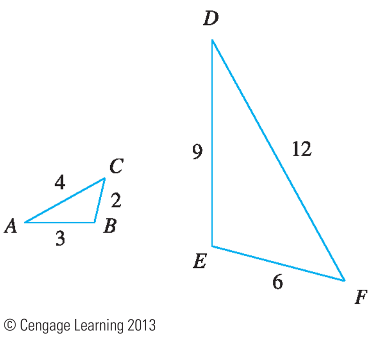


Figure 3.93

In Figure 3.93, $\angle A \cong \angle D$, $\angle B \cong \angle E$, and $\angle C \cong \angle F$. Since the corresponding angles are the same size, the triangles are similar. Congruent triangles are a special case of similar triangles in which the corresponding sides are the same length. In Figure 3.93, the sides of the larger triangle are three times as large as those of the smaller triangle. Thus, the ratios of the corresponding sides of the larger triangle to the smaller triangle are $\frac{12}{4} = \frac{9}{3} = \frac{6}{2} = \frac{3}{1}$.

APPLICATION CONSTRUCTION

EXAMPLE 3.33

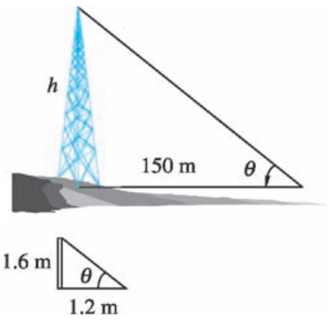


Figure 3.94

A television tower casts a shadow that is $15\bar{0}$ m long. At the same time, a vertical pole that is 1.60 m high casts a shadow 1.20 m long. How high is the television tower? (See Figure 3.94.)

SOLUTION We have two similar triangles, so the corresponding sides are proportional. If we call the unknown height h, then we have

$$\begin{aligned} \frac{h}{1.6} &= \frac{150}{1.2} \\ h &= \frac{150(1.6)}{1.2} \\ &= 200 \end{aligned}$$

The height of the tower is 200 m.

EXAMPLE 3.34

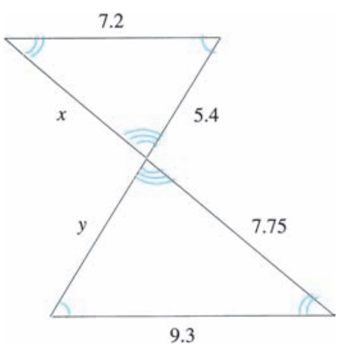


Figure 3.95

Find the lengths of the unknown sides of the similar triangles in Figure 3.95.

SOLUTION The corresponding sides of these triangles are proportional, thus $9.3 : 7.75 : y = 7.2 : x : 5.4$. Solving for x, produces

$$\begin{aligned} \frac{9.3}{7.2} &= \frac{7.75}{x} \\ 9.3x &= (7.75)(7.2) \\ &= 55.8 \\ x &= 6 \end{aligned}$$

Now, solving for y, we have the proportion

$$\begin{aligned} \frac{9.3}{7.2} &= \frac{y}{5.4} \\ 7.2y &= (9.3)(5.4) \\ &= 50.22 \\ y &= 6.975 \end{aligned}$$

OTHER SIMILAR FIGURES

Similar figures other than triangles can be more difficult to work with. If the corresponding angles of a triangle are congruent, we know that the triangles are similar. This is not true for other figures. But it is true that, for two similar figures, the distance between any two points on one figure is proportional to the distance between any two corresponding points on the other figure. This is true for any two similar figures whether they are plane figures or solid ones.

EXAMPLE 3.35

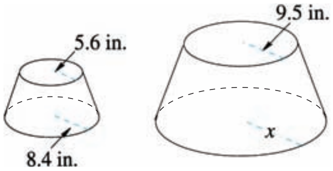


Figure 3.96

Figure 3.96 shows two similar frustums of cones. The radii of the bases of the smaller frustum are given as is the radius of the smaller base for the larger frustum. What is the radius of the bottom base of the larger frustum?

SOLUTION Since these are similar figures, we know that the ratios of the radius of the small base to the radius of the large base must be the same for each figure. Thus,

$$\frac{5.6}{8.4} = \frac{9.5}{x}$$
$$5.6x = (8.4)(9.5)$$
$$= 79.8$$
$$x = 14.25\text{in.}$$

The radius of the bottom base of the larger frustum is 14.25 in.

If we know that two figures are similar, then it is easy to find the area or volume of one if we know the area or volume of the other.

AREAS OF SIMILAR FIGURES

Areas of similar figures are related to each other as the *squares* of any two corresponding dimensions. This is true for both plane and solid figures. It is also true for plane areas, lateral surface areas, total surface areas, or cross-sectional areas. For example, suppose two circles are similar. If one circle has a radius of r_1 and the other a radius of r_2, then the ratio of their areas is

$$\frac{A_1}{A_2} = \frac{\pi r_1^2}{\pi r_2^2} = \frac{r_1^2}{r_2^2}$$

Thus, the ratio of the areas of two circles is the same as the ratio of the squares of their radii.

EXAMPLE 3.36

The lateral surface area of the smaller frustum in Figure 3.96 is 220 in.^{2} What is the lateral surface area of the larger figure?

SOLUTION We will let L represent the lateral surface area that we are to find and use the proportion

$$\frac{220}{L} = \frac{5.6^2}{9.5^2}$$
$$5.6^2L = (220)(9.5)^2$$
$$31.36\,L = 220(90.25)$$
$$= 19{,}855$$
$$L = 633.13138$$

The lateral surface area of the larger figure is about 633.13 in.^{2}

VOLUMES OF SIMILAR FIGURES

If two solid figures are similar, then their volumes are related to each other as the *cubes* of any two corresponding dimensions. Say two spheres are similar. If one sphere has a radius of r_1 and the other a radius of r_2, then the ratio of their volumes is

$$\frac{V_1}{V_2} = \frac{\frac{4}{3}\pi r_1^3}{\frac{4}{3}\pi r_2^3} = \frac{r_1^3}{r_2^3}$$

EXAMPLE 3.37

The volume of the smaller frustum in Figure 3.96 is 646.6 in.^{3} What is the volume of the larger figure?

SOLUTION We will let V represent the volume that we are to find and use the proportion

$$\begin{aligned} \frac{646.6}{V} &= \frac{5.6^3}{9.5^3} \\ 5.6^3 V &= (646.6)(9.5)^3 \\ 175.616V &= 554{,}378.67 \\ V &= 3{,}156.7663 \end{aligned}$$

The volume of the larger frustum is about 3,156.77 in.^{3}

SCALE DRAWINGS

Perhaps the most common use of similar figures applies when using scale drawings. Scale drawings are used in maps, blueprints, engineering drawings, and other figures. The ratio of distances on the drawing to corresponding distances on the actual object is called the scale of the drawing.

APPLICATION ARCHITECTURE

EXAMPLE 3.38

A rectangular building $200' \times 145'$ is drawn to a scale of $\frac{1}{4}'' = 1'0''$. What is the size of the rectangle that will represent this building on the blueprint?

SOLUTION This is the continuing proportion $\frac{\frac{1}{4}''}{1'0''} = \frac{x}{200'} = \frac{y}{145'}$. Solving each of these we get $x = 50''$ and $y = 36\frac{1}{4}''$.

The building will be represented by a rectangle that is $50'' \times 36\frac{1}{4}''$.

APPLICATION CIVIL ENGINEERING

EXAMPLE 3.39

A map has a scale of 1 : 24 000. What is the actual distance of a map distance of 32 mm?

EXAMPLE 3.39 (Cont.)

SOLUTION We will use the proportion $\frac{1}{24\ 000} = \frac{32\,\text{mm}}{x}$.

$$x = (32\,\text{mm})(24\ 000) = 768\ 000\,\text{mm}$$
$$= 768\,\text{m}$$

So, 32 mm on the map represents an actual distance of 768 m.

EXERCISE SET 3.6

The pairs of figures in each of Exercises 1–6 are similar. Find the lengths of the unknown sides.

1.

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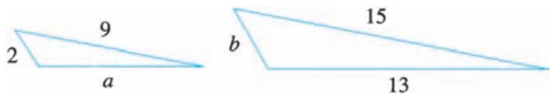
2.



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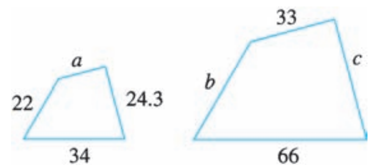

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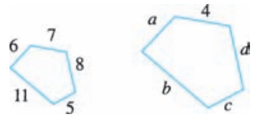

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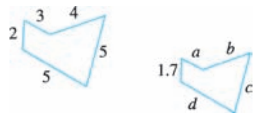

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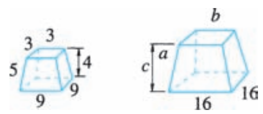

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Solve each of the proportions in Exercises 7–10.

7. $2 : 5 : 9 = x : 14 : z$

8. $a : 6 : 9 = 20 : y : 45$

9. $a : b : 7.5 = 5.25 : 8.4 : 4.5$

10. $12.4 : y : z = 15.5 : 52.75 : 84$

Solve Exercises 11–24.

11. ***Architectural technology*** The floor plan of a building has a scale of $\frac{1}{8}$ in. = 1 ft. One room of the floor plan has an area of 20 in.^{2} What is the actual room area in square inches? What is the area in square feet? (Hint: 144 in.^{2} = 1 ft^2)

12. ***Wastewater technology*** A pipe at a sewage treatment plant is 75 mm in diameter and discharges 2000 L of water in a given period of time. If a pipe is to discharge 3000 L in the same period of time, what is its diameter?

13. A square bar of steel, 38 mm on a side, has a mass of 22 kg. What is the mass of another bar of the same length that measures 19 mm on a side?

14. ***Agricultural technology*** It cost \$982 to fence in a circular field that has an area of 652 ft^2. What will it cost to enclose another circular field with three times as much area?

15. ***Construction*** It requires 700 L to paint a spherical tank that has a radius of 20 m. How much paint will be needed to paint a tank with a radius of 35 m?

16. ***Environmental technology*** A water tank that is 12 m high has a volume of 20 kiloliters (kL) or 20 000 L. What is the volume of a similar tank that is 30 m high?

17. ***Sheet metal technology*** A cylinder has a capacity of 3 930 mm^3. Its diameter is 15 mm.

What is the volume of a similar container with a diameter of 5 mm?

18. ***Sheet metal technology*** A cylindrical container has a diameter of 4″ and a height of 5″. A similar container has a diameter of 2.5″. What is the height of the second container and the volume of each?

19. ***Sheet metal technology*** A sphere with a 10-cm radius has a volume of 4 188.79 cm^3 and a surface area of 1 256.64 cm^2. Find **(a)** the surface area of a sphere with a radius of 2 cm and **(b)** the volume of a sphere with a radius of 2 cm.

20. ***Product design*** A cylindrical soup can has a diameter of 66.0 mm and a height of 95.0 mm, and holds about 325.013 5 cm^3. The soup company wants to market a can that holds twice as much soup in a cylindrical can that is similar in size to the present can. What should be the dimensions of the new can?

21. ***Broadcasting*** The viewing size of a television screen refers to the length of a diagonal of the screen. The *aspect ratio* of a television screen is the ratio of the width to the height. A wide-screen TV has an aspect ratio of 16 : 9. What are the length and width of the screen of a 57″ wide-screen TV?

22. ***Broadcasting*** A traditional TV screen has an aspect ratio of 4 : 3. What are the length and width of the screen of a 57″ traditional TV?

23. ***Nutrition*** For a certain breakfast cereal, a serving size is 30 g and contains 130 mg of sodium and 14 g of dietary fiber. A box of this same cereal contains 453 g. How much sodium and how much dietary fiber is in a box of this cereal?

24. ***Nutrition*** For a certain brand of soup, a serving size is 240 g and contains 80 calories, 210 mg or 9% daily value of sodium, and 21 g or 7% daily value of carbohydrates. Based on a daily diet of 2000 calories, how many mg of sodium and how many grams of carbohydrates should a person have in his or her diet?

[IN YOUR WORDS]

25. Describe how you would use similar figures to make a scale drawing.

26. **(a)** If the lengths of each side of a triangle are twice as long as the corresponding sides of a similar triangle, how are their areas related?

(b) If the lengths of each side of a triangle are three times as long as the corresponding sides of a similar triangle, how are their areas related?

(c) Describe how the areas between any two similar triangles are related to the lengths of their sides.

CHAPTER 3 REVIEW

IMPORTANT TERMS AND CONCEPTS

Adjacent angles
Altitude
Angle
- Acute
- Obtuse
- Right
- Straight

Arc
Arc length
Area
Central angle
Chord
Circumference
Complementary angles

Cone
Congruent angles
Congruent polygons
Congruent triangles
Continued proportion
Corresponding angles
Cylinder

Degree
Diameter
Frustum
Hero's formula
Hexagon
Hypotenuse
Isosceles trapezoid
Lateral angle
Line segment
Octagon
Parallel lines
Parallelogram
Pentagon
Perimeter
Perpendicular lines
Polygon
Prism
Pythagorean theorem
Quadrilateral
Radians
Radius
Rectangle
Regular polygon
Rhombus
Scale drawings
Secant
Similar figures
  Area
  Volume
Similar polygons
Similar triangles
Simpson's rule
Square
Supplementary angles
Tangent
Total surface area
Transversal
Trapezoid
Trapezoidal rule
Triangle
  Acute
  Equilateral
  Isosceles
  Obtuse
  Right
  Scalene
Vertex

REVIEW EXERCISES

Convert each of the angle measures in Exercises 1–4 to either radians or degrees, without using a calculator. When you have finished, check your work by using a calculator.

1. $27°$

2. $212°$

3. 1.1π

4. 0.75

Solve Exercises 5 and 6.

5. What is the supplement of a $137°$ angle?

6. What is the complement of a $\frac{\pi}{6}$ angle?

Solve each of the proportions in Exercises 7–8.

7. $7 : 24.5 : x = 8 : y : 42$

8. $\dfrac{12.5}{x} = \dfrac{y}{47} = \dfrac{8}{5}$

Solve Exercise 9.

9. What is the distance from A to B in Figure 3.97, if lines ℓ_1, ℓ_2, and ℓ_3 are parallel?

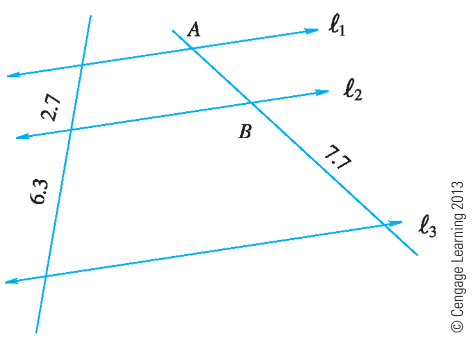
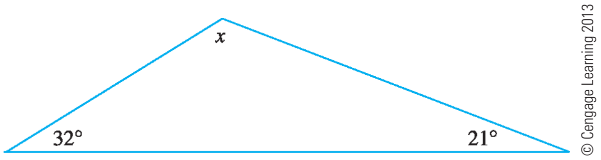


Figure 3.97

Find the variables indicated in Exercises 10 and 11.

10.



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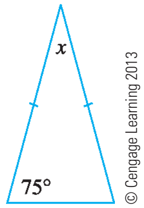

Find the length of the missing side in the right triangles in Exercises 12 and 13.

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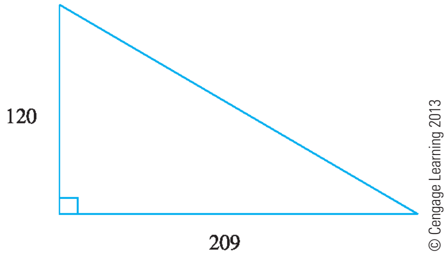

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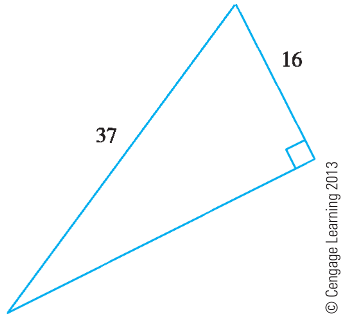
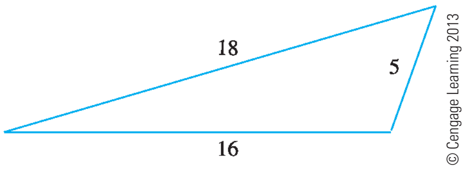


Find the area and perimeter or circumference of each of the figures in Exercises 14–21.

14.



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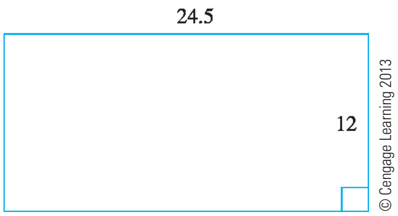

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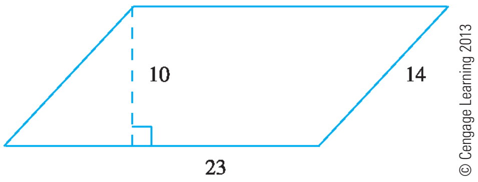

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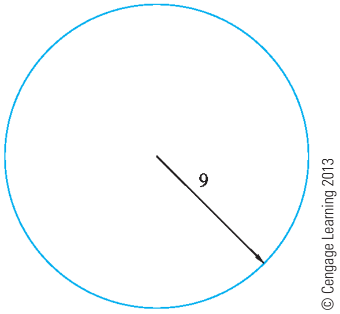

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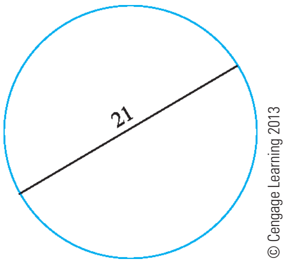

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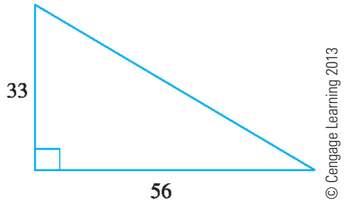

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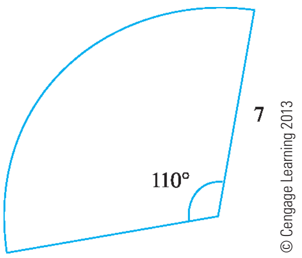

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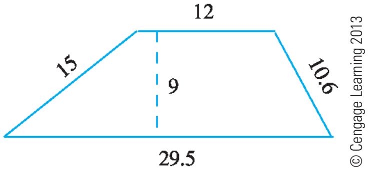
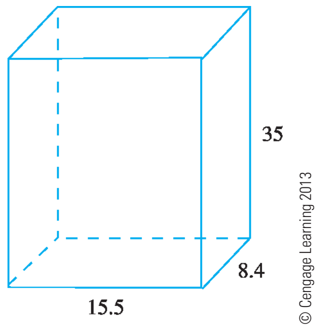


Find the lateral area, total surface area, and volume of each of the solid figures in Exercises 22–29.

22.



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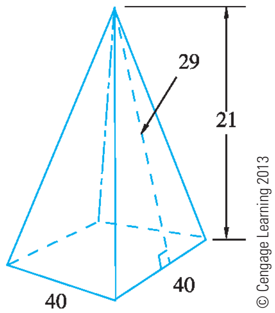

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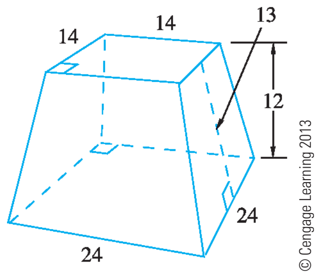

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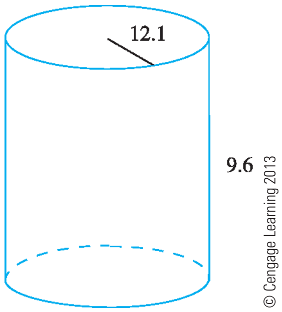

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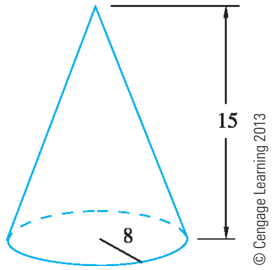

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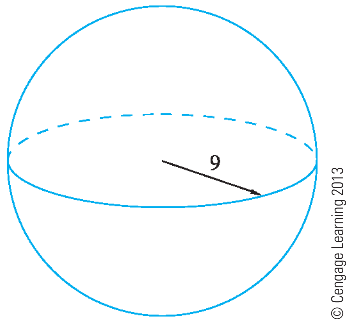

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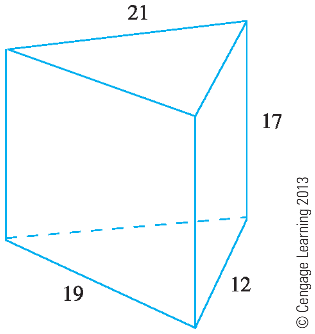

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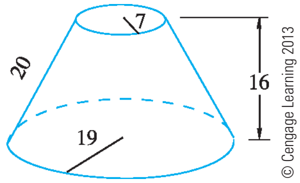


Solve Exercises 30–37.

30. ***Construction*** An antenna 175 m high is supported by cables positioned at three positions around the antenna. At each position, four cables go to various heights of the antenna. One set of cables is attached to the antenna 50 m above the ground, one set is attached 100 m above the ground, the third set is attached 150 m above the ground, and the fourth set is attached to the top of the antenna. If each of the three positions is located 75 m from the base of the antenna, what is the total length of all the cables used to support the antenna?

31. ***Civil engineering*** Find the distance across the lake in Figure 3.98.

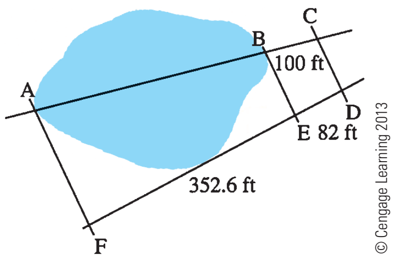


Figure 3.98

32. ***Metalworking*** A metal washer is in the shape of a circular cylinder with a (circular cylindrical) hole punched in the middle. Each washer has a diameter of 3.20 cm, the hole has a diameter of 1.05 cm, and the washer is 0.240 cm thick. The washers are made by feeding a 1-m strip of metal that is the same width and thickness as a washer into a stamping machine. For safety reasons, the last 5 cm of each strip are not fed into the machine.

(a) How many strips of metal will be needed to make (stamp) 100,000 washers?

(b) How much actual metal is required for these washers?

(c) How much scrap metal is generated in the production of these washers?

33. ***Electricity*** The *turn ratio* in a transformer is the number of turns in the primary winding to the number of turns in the secondary winding. If the turn ratio for a transformer is 25, and there are 4,000 turns in the primary winding, how many turns are in the secondary winding?

34. The longest side of triangle A is 180 mm. Triangle B has sides of 4, 5, and 8 mm. Triangles A and B are similar. What are the lengths of the other two sides of triangle A?

35. ***Physics*** The **theoretical mechanical advantage (TMA)** of an inclined plane or ramp is equal to the ratio between its length and height. A ramp 90 ft long slopes down 5 ft to the edge of a lake. What is the TMA of the ramp?

36. ***Machine technology*** The efficiency of a machine is the ratio between its actual mechanical advantage (AMA) and its TMA. The AMA of the boat ramp in Exercise 35 is 16 because of the friction in a boat trailer's wheels. What is the efficiency of the ramp?

37. ***Landscaping*** The playing field of Oriole Park at Camden Yards in Baltimore, Maryland, is shown in Figure 3.99.

(a) Approximate the area of the playing field.

(b) Approximate the area of the portion of the field covered with grass.

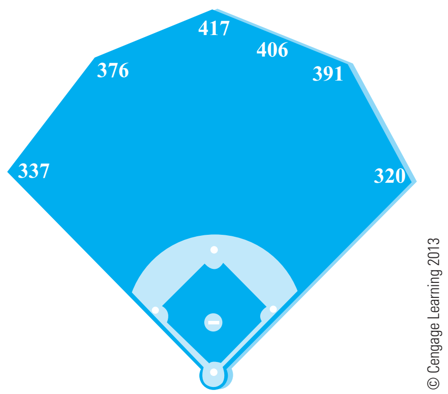


Figure 3.99

CHAPTER 3 TEST

1. Convert 35° to radians.

2. Convert $\frac{7\pi}{15}$ to degrees.

3. What is the supplement of a 76° angle?

4. Solve the proportion $6 : 8 : x = y : 14 : 24.5$.

5. Solve the proportion $\dfrac{a}{9} = \dfrac{b}{30} = \dfrac{75}{45}$.

6. In Figure 3.100, what is the length of $\overline{AB}$, if ℓ_1, ℓ_2, and ℓ_3 are parallel?

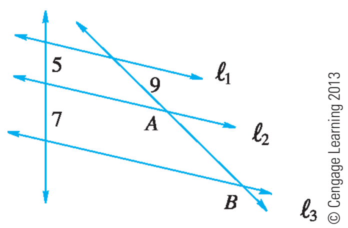


Figure 3.100

7. Determine the length of side a in Figure 3.101.

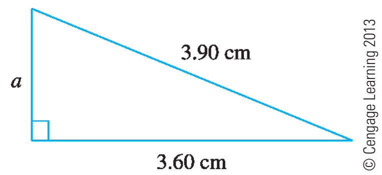
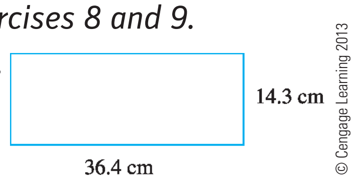


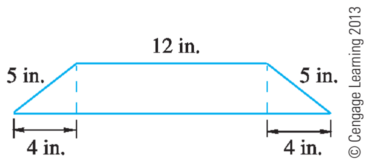
Figure 3.101

Find the perimeter of each of the figures in Exercises 8 and 9.

8.

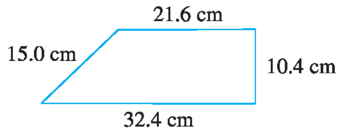


9.

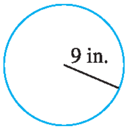


Find the area of each of the figures in Exercises 10 and 11.

10.



11.



Solve Exercises 12–18.

12. A building casts a shadow of 120 m. At the same time, an antenna that is 14 m high casts a shadow of 11.6 m. How tall is the building?

13. An automobile tire has a diameter of 62 cm. How many revolutions must the tire make when the car travels 25 m in a straight line?

14. What is the volume of the box in Figure 3.102?

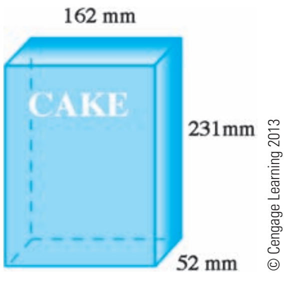


Figure 3.102

15. A spherical storage tank has a diameter of 35 ft. What is its volume?

16. The part of a cylindrical soup can that is covered by the label is 9.5 cm tall and has a diameter of 6.5 cm. What is the area of a label that covers the entire side of the can and that needs a 0.8-cm overlap to glue the ends of the label?

17. The perimeter of a rectangular solar panel is 540 cm. The ratio of the length to the width is 3 : 2. What are the length and width?

18. The area of a geometric figure varies directly as the square of any dimension. Two similar triangles have corresponding sides of length 12 m and 18 m. The smaller triangle has an area of 72 m^2. What is the area of the other triangle?

4 FUNCTIONS AND GRAPHS

Some technicians have to wear protective gear when they take samples at a pollution site. In Sections 4.4 and 4.5, we will see how to use functions and their graphs to help analyze these samples and predict the effects of pollution.

In Chapter 2, we looked at algebraic equations. We learned some ways to solve equations, and we learned how we could take some information and write an equation that showed how that information was related.

Some of the equations involved variables that stood for things related to each other. One equation used variables to show the relationship between distance, rate, and time. Torque, force, and moment arm length were other relationships in which variables were used. In each of these last two relationships, once we knew the values of two of the variables we could determine the third.

OBJECTIVES

AFTER COMPLETING THIS CHAPTER, YOU WILL BE ABLE TO:

- Determine if a relation is a function.
- Recognize functions when defined by a table of values, with symbols, or as graphs.
- Define function in your own words.
- Select an application from you field of study and construct a table of values, an equation, and a graph to illustrate a function.
- Define the independent and dependent variables in an application and list any restrictions on the independent variable.
- Understand and apply functional notation.
- Solve an equation for one variable in terms of others.
- Find the inverse function if it exists.
- Plot ordered pairs.
- Scale the x- and y-axes using suitable values to appropriately display the information provided.
- Construct a table of values, both by hand and using a spreadsheet, in order to graph a function.
- Graph a function by hand, using a spreadsheet, and using a graphing utility.
- Identify the intercepts on a graph as well as find the intercepts from the equation.
- Graph a line and find the slope of the line.
- Solve an equation using a table of values generated on a spreadsheet and by finding the intercept using a graphing utility.

4.1 RELATIONS AND FUNCTIONS

In this section, we will learn to use a particular kind of relationship called a function. The key to the mathematical analysis of a technical problem is your ability to recognize the relationship between the variables that describe the problem. Such a relationship often takes the form of a formula that expresses one variable as a function of another variable. We will be studying the basic ideas of a function and, later in the chapter, show how we can draw a picture or graph of a function. These basic ideas will form the foundation for much of the work we will do later in this book.

RELATIONS

We have used the word *relationship* very loosely. A **relation** in mathematics is used to represent a relationship between sets of numbers, variables, or objects. A relation is often written as a set of ordered pairs or by a rule that describes how the items are related. In mathematics this rule may be given as an equation, an inequality, or a system of equations or inequalities. A **function** is a special kind of relation.

EXAMPLE 4.1

$a = 5b + 1$ is a relation between a and b. Pick a value for b, say $b = 3$, then $a = 5(3) + 1 = 16$. On the other hand, we could pick a value for a, like $a = 2.7$, and determine that $b = 0.34$.

EXAMPLE 4.2

The equation $I = 2.54C$ expresses a relation between the two variables I and C. So, if you select a value for C, say $C = 5$, you would get $I = 2.54(5) = 12.7$. You could also pick a value for I, say $I = 8.89$. Then $8.89 = 2.54C$ and $C = \frac{8.89}{2.54} = 3.5$.

EXAMPLE 4.3

The equation $v = \frac{1}{2}t^2 + 9t + 1$ describes a relation between the two variables v and t. If you let $t = 4$, then you get $v = \frac{1}{2}(4^2) + 9(4) + 1 = 45$. Let $v = 21$, then $t = 2$ or $t = -20$ will both work.

EXAMPLE 4.4

$x^2 + y^2 = 64$ is a relation between x and y. Pick a value for x between -8 and 8. If you pick $x = 0$ then $0 + y^2 = 64$ or $y^2 = 64$, which means $y = \pm 8$. If $x = 1$, then $y = \pm\sqrt{63}$. If $x = -8$, then $y = 0$. In general, $y = \pm\sqrt{64 - x^2}$. Notice that you cannot select a number smaller than -8 or greater than 8 because that would require taking the square root of a negative number.

ORDERED PAIRS

In addition to defining a relation by an equation, we may also define it as a set of **ordered pairs**. For example, the solutions to $I = 2.54C$ are pairs of numbers. If we agree to write the ordered pairs as (C, I), where the first number is the C-value and the second number is the I-value, then these ordered pairs also define the relation between C and I. Some of these ordered pairs are (1, 2.54), (2, 5.08), (2.5, 6.35), (5, 12.7), (3.5, 8.89), and $(-3, -7.62)$.

In technical work you often work with ordered pairs of numbers. An example of a set of ordered pairs of numbers is given in Table 4.1. In this table pressures in kilopascals (kPa) and the corresponding pressure in pounds per square inch (psi) are given. For each pressure in psi there is a corresponding pressure in kPa.

Ordered pairs can be represented in table form or with each corresponding ordered pair written between parentheses. If we use the ordered pairs from

TABLE 4.1 Inflation Pressure Conversion Chart (kilopascals to psi)

kPa	psi	kPa	psi
140	20	215	31
145	21	220	32
155	22	230	33
160	23	235	34
165	24	240	35
170	25	250	36
180	26	275	40
185	27	310	45
190	28	345	50
200	29	380	55
205	30	415	60

Conversion: 6.9 kPa = 1 psi

1991 Buick Skyhawk Service Manual. Flint, Michigan: Service Department, Buick Division, General Motors Corporation, 1991. Figure 3 Inflation Pressure Conversion, p. 3E-2

Table 4.1 we would write a kPa pressure as the first number and the corresponding psi pressure as the second number. The first four ordered pairs from Table 4.1 would be written as (140, 20), (145, 21), (155, 22), and (160, 23).

Whenever you have a set of ordered pairs of numbers (x, y) such that for each value of x there corresponds exactly one value of y, you have a function. You will notice that at the bottom of Table 4.1, the function is given in equation form, 6.9 kPa = 1 psi. We might rewrite this as $k = 6.9p$, where p represents a pressure in psi and k stands for a pressure in kPa. In this case, the ordered pairs are of the form (k, p).

DOMAIN AND RANGE

For any relation, the set of all possible first component values is called the **domain** and the set of all second component values that can result from using values in the domain is called the **range**.

EXAMPLE 4.5

The set of ordered pairs

$$\{(2, 5), (3, 8), (4, 12), (6, -12)\}$$

expresses a relation between the set of first components, or domain, $\{2, 3, 4, 6\}$, and the set of second components, or range, $\{5, 8, 12, -12\}$.

NOTE The domain and range are important ideas that will help us better understand a relation, especially when we begin graphing, and they will help us to make better use of graphing calculators and computer graphing software.

EXAMPLE 4.6

(a) In Example 4.4 where $x^2 + y^2 = 64$, both the domain and range were the real numbers between, and including, -8 and 8.

(b) In Example 4.3, the equation $v = \frac{1}{2}t^2 + 9t + 1$ expressed how the first component v was obtained from the second component t. The domain contains all the real numbers. The range would be all real numbers v greater than or equal to -39.5, or $v \geq -39.5$.

(c) If $y = \sqrt{x^2 - 81}$, then the domain would be all real numbers such that $x^2 - 81 \geq 0$. That means that the domain is the values of x larger than or equal to 9 or less than or equal to -9, or $x \leq -9$ or $x \geq 9$. The range would be the nonnegative real numbers, or $y \geq 0$.

(d) If $y = \dfrac{5}{x-3}$, the domain would be all real numbers except 3, because if $x = 3$, then $x - 3 = 0$ and $\dfrac{5}{x-3}$ is not defined. (Remember, division by zero is not defined.) The range is all real numbers except 0, because a fraction is 0 only when the numerator is 0, but this numerator is always 5 (and so, never 0).

FUNCTIONS

The relation in Example 4.4 was $x^2 + y^2 = 64$. In this relation there are two values of y for each value of x, unless $x = 8$ or $x = -8$. If a relation between x and y gives only one value of y for each value of x, then we say that y is a *function of x*. Every function is a relation. However, as Example 4.4 demonstrated, not every relation is a function.

FUNCTION

The variable y is a function of x if a relation between x and y produces exactly one value of y for each value of x.

EXAMPLE 4.7

$y = 7x - 2$ is a function described by an equation. Pick any value for x, say -4. Then $y = 7(-4) - 2 = -28 - 2 = -30$.

For each value of x there is only one value of y.

EXAMPLE 4.8

The table below is an example of a function defined by a table.

x	2	3	4	6
y	5	8	12	-12

We saw this same function earlier in Example 4.5 as a set of ordered pairs. Tables and ordered pairs are useful for describing functions that involve only a few values.

As we have seen, a function can be described by a set of ordered pairs (Example 4.5), a table of values (Example 4.8), or an equation (Example 4.7). We will see later that a function can also be described by a graph.

EXAMPLE 4.9

The equation $x^2 + y^4 = 5$ does not describe a function. Pick any value of x, say 2. Then $y = -1$ and $y = 1$ both make this equation true. Since this particular value of x, 2, produces more than one value of y that makes the equation true, we have shown that this equation is not a function of x.

NOTE Whenever specific values (numbers) are used to show that an equation is not true, we say that we have given a *counterexample*. The numbers we used in Example 4.9 provided a counterexample to show that $x^2 + y^4 = 5$ is *not* a function.

EXAMPLE 4.10

The set of ordered pairs (0, 1), (2, 5), (2, −8) is not a function because there are two different second values, 5 and −8, for the first number, 2.

INDEPENDENT AND DEPENDENT VARIABLES

For the function $y = 7x - 2$, the variable x is called the **independent variable** and y is the **dependent variable**. These terms are used because the value of y depends upon the value selected for x. Since the value selected for x can be freely chosen, x is called the independent variable. These terms are arbitrary. After all, if we had written the function as $x = \frac{1}{3}(y + 5)$, then y would have been the independent variable and x the dependent variable.

FINDING DOMAINS AND RANGES

Since any function can be described by a set of ordered pairs, a list, a table, an equation, or a graph, each of these can be used to determine the domain and range of the function. Here we will show how to determine the domain of a function defined by a set of ordered pairs, a list, a table, and an equation. Later in the chapter we will show how to use a graph to help determine the domain.

Finding the range is not always as easy as finding the domain. Here we will show you some techniques that you can use. You will find others as you progress through the text.

USING A TABLE, LIST, OR SET OF ORDERED PAIRS TO DETERMINE DOMAIN AND RANGE

A table or list can be used to determine the domain and range when it is possible to list all the ordered pairs in the function.

EXAMPLE 4.11

Determine the domain and range of the function defined by the table in Example 4.8.

SOLUTION The table is shown below.

x	2	3	4	6
y	5	8	12	-12

Here, the first elements are in the top row of the table, the row labeled x. The numbers in this row are the domain of the function: $\{2, 3, 4, 6\}$. The second elements, in the bottom row, are the elements of the range. So the range is $\{5, 8, 12, -12\}$. Usually we like to list these in increasing order. In that case the range would be given as $\{-12, 5, 8, 12\}$.

EXAMPLE 4.12

Determine the domain and range of the function defined by the set of ordered pairs: $\{(-3, 30), (-2, 10), (-1, -2), (0, -6), (\frac{1}{2}, -5), (1, -2), (2, 10), (3, 30)\}$.

SOLUTION Because this function has just eight ordered pairs, it is easy to list all the first elements. This list of first elements is the domain. Thus, the domain is $\{-3, -2, -1, 0, \frac{1}{2}, 1, 2, 3\}$. Similarly, the range is the list of all the second elements; that is, $\{-6, -5, -2, 10, 30\}$. Notice that we did not list an element of the range the second time it was used.

To use a table for this example, you would present the data as in

x	-3	-2	-1	0	$\frac{1}{2}$	1	2	3
y	30	10	-2	-6	-5	-2	10	30

The domain is the set of numbers in the first row, and the range is found from the second row.

USING AN EQUATION TO DETERMINE DOMAIN AND RANGE

Many times a function is defined by an equation. When this is done you will need to use algebra and the equation to help determine the domain and range.

EXAMPLE 4.13

Show how to use the equation of a function to determine the domain and range of the function $y = 2x + 7$.

SOLUTION It is possible to replace x with any real number and get a value of y. Thus, we conclude that the domain of this function is all real numbers.

The equation says that each value of x will be doubled and then have 7 added to it. If you want some particular value of y, say -23, you could replace y with that number and solve for x. In the case of $y = -23$, we would get $-23 = 2x + 7$ or $-30 = 2x$, and so $x = -15$. Since we can do this for any value of y, we conclude that the range is all real numbers.

Domain: All real numbers
Range: All real numbers

EXAMPLE 4.14

Show how to use the equation of a function to determine the domain and range of the function $y = x^2 - 3$.

SOLUTION It is possible to replace x with any real number and get a value of y. Thus, we conclude that the domain of this function is all real numbers.

The equation says that each value of x will be squared and then have -3 added to it. If you square a positive or negative number, you get a positive number. If you square 0, then you get 0. Thus, if any real number is squared it is greater than or equal to 0. If -3 is added to this number, then the sum is greater than or equal to -3. Thus, we conclude that the range is all real numbers greater than or equal to -3. We often write this as $\{y : y \geq -3\}$. This is read as "the set of all y's such that y is greater than or equal to -3."

Domain: All real numbers

Range: All real numbers greater than or equal to -3 or $\{y : y \geq -3\}$

EXAMPLE 4.15

Show how to use the equation of a function to determine the domain and range of the function $y = -\sqrt{x}$.

SOLUTION Because you can only take the square root of a nonnegative real number, the domain is $\{x : x \geq 0\}$.

The principal square root of a nonnegative number is nonnegative. Thus, for any x in the domain, $\sqrt{x} \geq 0$. But, we want the range of $y = -\sqrt{x}$. Since multiplying a positive number by -1 results in a negative number, we can see that the range is $\{y : y \leq 0\}$.

EXAMPLE 4.16

Show how to use the equation of a function to determine the domain and range of the function in Example 4.6c: $y = \sqrt{x^2 - 81}$.

SOLUTION To find the domain, we need to find all real numbers where $x^2 - 81 \geq 0$. These are $x \leq -9$ or $x \geq 9$. The range is all nonnegative numbers, or $y \geq 0$.

EXAMPLE 4.17

Show how to use the equation of a function to determine the domain and range of the function $y = \dfrac{5}{x - 2}$.

SOLUTION If $x = 2$, then $\dfrac{5}{x - 2}$ is not defined, so 2 is not in the domain. In fact, the domain consists of all real numbers except 2. We can write this as $\{x : x \neq 2\}$. Since the numerator is 5 it can never be zero, $y \neq 0$. Indeed, the range is all real numbers y where $y \neq 0$. We can write this as $\{y : y \neq 0\}$.

EXAMPLE 4.18

Show how to use the equation of a function to determine the domain and range of the function $y = \dfrac{x}{x + 3}$.

SOLUTION If $x = -3$, then $\frac{x}{x+3}$ is not defined, so -3 is not in the domain. In fact, the domain consists of all real numbers except -3. We can write this as $\{x : x \neq -3\}$.

To help find the range, we will simplify $\frac{x}{x+3}$ by dividing. When we do this, we obtain $\frac{x}{x+3} = 1 + \frac{-3}{x+3}$. Since the numerator of $\frac{-3}{x+3}$ cannot be zero, then $\frac{-3}{x+3}$ cannot be zero. Hence, $\frac{x}{x+3} = 1 + \frac{-3}{x+3}$ can never be one. From this we see that the range is all real numbers y where $y \neq 1$. We can write this as $\{y : y \neq 1\}$.

CAUTION As indicated below, you must not simplify the equation before you determine the domain and range of a function.

EXAMPLE 4.19

The functions given by $y = \frac{x^2 - 4}{x - 2}$ and $y = x + 2$ agree at all values of $x \neq 2$. However, they are not equal sets of points. The reason is that they have different domains. The first function's natural domain is all real numbers $x \neq 2$, or $\{x : x \neq 2\}$. The domain of the second function is all real numbers. Thus, the functions are not equal since only the second one contains a point with x-coordinate 2, namely $(2, 4)$.

HINT Unless there are some other restrictions, the domain of a function defined by an equation includes all real numbers except:

- Any numbers for which a denominator is zero
- Any numbers for which the expression defined by the equation is not a real number

EXAMPLE 4.20

Create a function whose domain is $\{x : x \neq 3, x \neq -2\}$.

SOLUTION The notation $\{x : x \neq 3, x \neq -2\}$ means that $x = 3$ cannot be in the domain and neither can $x = -2$.

The easiest way to do this is to write the function using a fraction with a denominator that is zero when $x = 3$ and $x = -2$. One solution is $y = \frac{1}{(x-3)(x+2)}$. A second solution is $y = \frac{1}{x-3} + \frac{1}{x+2} = \frac{2x-1}{(x-3)(x+2)}$.

Two other solutions are given by the functions $y = \frac{x^2}{(x-3)(x+2)}$ and $y = \frac{x^2 - 4}{(x-3)(x+2)(x+2)}$.

NOTE Since no restrictions were placed on the range, the numerator can be anything that is defined for all real numbers.

EXAMPLE 4.21

Create a function whose domain is $\{x : x \geq -3\}$.

SOLUTION Restricting the domain to $x \geq -3$ is the same as saying we want $x + 3 \geq 0$. We know that $\sqrt{x+3}$ is defined only if $x + 3 \geq 0$ and so one function with the desired domain is $y = \sqrt{x+3}$.

EXAMPLE 4.22

Create a function whose domain is $\{x : x \leq 7, x \neq -3\}$.

SOLUTION As in Example 4.20, we can eliminate $x = -3$ by putting an $x + 3$ in the denominator.

Restricting the rest of the domain to $x \leq 7$ is the same as saying we want $x - 7 \leq 0$ (or $-(x - 7) = 7 - x \geq 0$). We know that $\sqrt{7-x}$ is defined only if $7 - x \geq 0$. Let this be the numerator.

One function with domain $\{x : x \leq 7, x \neq -3\}$ is $y = \dfrac{\sqrt{7-x}}{x+3}$.

NOTE Are the conditions in Example 4.22 still satisfied if $\sqrt{7-x}$ is in the denominator, that is, if $y = \dfrac{1}{(x+3)\sqrt{7-x}}$? No! Then 7 would not be in the domain and the domain would be $\{x : x < 7, x \neq -3\}$.

FUNCTIONAL NOTATION

A function is often identified by a letter or group of letters, such as f, g, h, j, F, G, C, tan, ln, ABS, or $\cos^{-1}$. If x is the independent variable and y the dependent variable, then the number that corresponds to y is designated as $f(x)$, $g(x)$, $h(x)$, $F(x)$, or $\tan(x)$ depending on how the function is identified.

The notation $f(x)$ is read "f of x" or "f at x." We say y is a function of x and write $y = f(x)$. Thus the function $y = 2x + 7$ in Example 4.13 could be written as $f(x) = 2x + 7$ and the function $y = x^2 - 3$ in Example 4.14 as $g(x) = x^2 - 3$.

The independent variable does not have to be x. We often use functions of other variables. Thus, $f(y) = 5y + \frac{1}{2}$ is a function of y and $h(t) = \frac{4}{3}t^2 - 2t + 1$ is a function of t.

If you want to know the value of a function at a specific point, then function notation is very useful. For example, suppose you have the function $f(x) = 2x^2 - 3$ and you want to know what value of f corresponds to $x = 4$. You would then be asking for $f(4)$ and you should substitute 4 for each x in the function. The result is

$$\begin{aligned} f(4) &= 2(4^2) - 3 \\ &= 2(16) - 3 \\ &= 29 \end{aligned}$$

The value of $f(4)$ is 29. This gives the ordered pair $(4, f(4))$ or $(4, 29)$.

CAUTION The notation $f(4)$ does not mean $f \cdot (4) = 4f$. The notation $f(4)$ asks for the value of the function f when $x = 4$.

EXAMPLE 4.23

If $f(x) = 3x^2 - 5x + 2$, find $f(-2)$.

SOLUTION Substitute -2 for each value of x, with the result

$$\begin{aligned} f(-2) &= 3(-2)^2 - 5(-2) + 2 \\ &= 3(4) - (-10) + 2 \\ &= 12 + 10 + 2 \\ &= 24 \end{aligned}$$

So, $f(-2) = 24$.

EXAMPLE 4.24

If $d(t) = 88t - 2.5$, find $d(4.5)$.

SOLUTION Substitute 4.5 for the variable t, with the result

$$\begin{aligned} d(4.5) &= 88(4.5) - 2.5 \\ &= 396 - 2.5 \\ &= 393.5 \end{aligned}$$

Thus, we have found that $d(4.5) = 393.5$.

EXAMPLE 4.25

If $g(x) = 3x^2 - 2x + 1$, then find $g(5 + h)$.

SOLUTION

Replace each x with $5 + h$.

$$\begin{aligned} g(5 + h) &= 3(5 + h)^2 - 2(5 + h) + 1 \\ &= 3(25 + 10h + h^2) - 2(5 + h) + 1 \\ &= (75 + 30h + 3h^2) - (10 + 2h) + 1 \\ &= 75 + 30h + 3h^2 - 10 - 2h + 1 \\ &= 66 + 28h + 3h^2 \end{aligned}$$

So, $g(5 + h) = 66 + 28h + 3h^2$.

APPLICATION ARCHITECTURE

EXAMPLE 4.26

If 100 m of fencing is used to enclose a rectangular yard, then the resulting area of the fenced yard is given by the function

$$A(x) = x(50 - x)$$

where x is the length of the rectangle. **(a)** What is the area when the length is 10 m? **(b)** What is the area when the length is 30 m? **(c)** What restrictions must be placed on the domain x so that the problem makes physical sense?

EXAMPLE 4.26 (Cont.)

SOLUTIONS

(a) When the length is 10 m, we have $x = 10$, and the area is

$$\begin{aligned} A(10) &= x(50 - x) \\ &= 10(50 - 10) \\ &= 10(40) = 400\,\text{m}^2 \end{aligned}$$

(b) Here $x = 30$ m, so the area is

$$\begin{aligned} A(30) &= x(50 - x) \\ &= 30(50 - 30) \\ &= 30(20) = 600\,\text{m}^2 \end{aligned}$$

(c) The domain should be $0 < x < 50$. If $x \leq 0$ or $x \geq 50$, we would obtain a length or width that is either 0 or a negative number. This would produce an area that is either 0 or a negative number, which would not make physical sense.

APPLICATION CIVIL ENGINEERING

EXAMPLE 4.27

The water pressure on the base of a dam is a function of the depth of the water. The weight density of water is 9 800 newtons per square meter (9 800 N/m^2). The water pressure P beneath the surface of the water is $P(d) = 9\,800d$, where d represents the depth of the water in meters. What is the pressure at the base of a dam that is 20.5 m below the water's surface?

SOLUTION We know $P(d) = 9\,800d$ and we are given $d = 20.5$ m. Substituting, we obtain $P(d) = 9\,800(20.5) = 200\,900$. So, the pressure is 200 900 N/m^2 or 200 900 Pa.

Very often, we do not bother to write a formula as a function. For instance, in the last example, we had the function $P(d) = 9\,800d$. Most of the time we simply write this as $P = 9\,800d$.

EXPLICIT AND IMPLICIT FUNCTIONS

When a function is expressed in the form $y = f(x)$, it is called an *explicit function*. For example, the equation $y = 5x - 7$ defines y explicitly as a function of x. If we call this function f, then we have the explicit function $f(x) = 5x - 7$.

If the relationship between x and y is not of this form, we say that x and y are related implicitly or that it is an *implicit function*. For example, if we write the above equation as $y - 5x = -7$, or $5x - y = 7$, or $5x - y - 7 = 0$, then the new equation defines y implicitly as a function of x. Solving for y gives the original explicit function we had before.

Some implicit functions can be used more easily if they are written as explicit functions. However, being able to write a function in terms of one (or more) explicit functions is *very* important. In most cases, in order to graph a function on a graphing calculator or with computer graphing software, you must be able to write it in explicit form.

EXAMPLE 4.28

(a) $4x - y = 2$ is an implicit equation. If we solve for y, we get an explicit function, $y = 4x - 2$.

(b) $x^2 + y^2 = 64$ is an implicit equation. It defines two explicit functions, $y = \sqrt{64 - x^2}$ and $y = -\sqrt{64 - x^2}$.

(c) $x + 3y^2 - y = 7$ is an implicit function that defines two explicit functions, $y = \frac{1}{6} + \sqrt{\frac{85}{36} - \frac{x}{3}}$ and $y = \frac{1}{6} - \sqrt{\frac{85}{36} - \frac{x}{3}}$.

(d) $x^3 - xy + 5y^4 = 7$ is an implicit function that does not define an explicit function.

Implicit functions can be written in functional notation. For example, $4x - y = 2$ could be written as a function of the two variables x and y, by writing $f(x, y) = 4x - y - 2$ and $x^2 + y^2 = 36$ as $g(x, y) = x^2 + y^2 - 36$. There is nothing that restricts a function to two variables. Thus, we could have the implicit function $x^2 + y^2 + z = 81$ and represent it as a function of three variables, or $h(x, y, z) = x^2 + y^2 + z - 81$.

Just as we learned that if $f(x) = 7x - 2$, then $f(-1) = 7(-1) - 2 = -9$, we can substitute values for the variables in functions of more than one variable.

EXAMPLE 4.29

Let $f(x, y, z) = 2x - \frac{1}{2}y + z^2 + 4$ and find $f(1, -2, 3)$.

SOLUTION Here we substitute $x = 1$, $y = -2$, and $z = 3$, with the result

$$\begin{aligned} f(1,-2,3) &= 2(1) - \frac{1}{2}(-2) + 3^2 + 4 \\ &= 2 + 1 + 9 + 4 \\ &= 16 \end{aligned}$$

So, we have determined that $f(1, -2, 3) = 16$.

EXERCISE SET 4.1

Which of the relations in Exercises 1–8 are also functions of x?

1. $y = 10x + 2$

2. $y = 17 - 3x$

3. $y^2 = x^2 - 5$

4. $y = x^3 - 2$

5. $y = \sqrt{x} + 5$

6. $y = \sqrt[3]{x}$

7. $y^2 = 2x - 7$

8. $y = \pm\sqrt{x^2} - 5$

Solve Exercises 9–16.

9. Is the set of ordered pairs (0, 0), (1, 1), (2, 8), (3, 27), and (4, 64) a function? Why or why not?

10. Is the set of ordered pairs (−2, 4), (−1, 1), (0, 0), (1, 1), and (2, 4) a function? Why or why not?

11. Is the set of ordered pairs (4, −2), (1, −1), (0, 0), (1, 1), and (4, 2) a function? Why or why not?

12. Does this table describe a function? Explain.

x	−40	−30	−20	−10	0	10	20	30	40
y	−40	−22	−4	14	32	50	68	86	104

13. Give the domain and range of the function defined by these ordered pairs: $(-3, -7)$, $(-2, -5)$, $(-1, -3)$, $(0, -1)$, $(1, 1)$, and $(2, 3)$.

14. Give the domain and range of the function defined by these ordered pairs: $(-3, 3)$, $(-2, -2)$, $(-1, -5)$, $(0, -6)$, $(1, -5)$, $(2, -2)$, $(3, 3)$.

15. Give the domain and range of the function defined by the values in this table.

x	-3	-2	-1	0	1	2	3
y	-25	-6	1	2	3	10	25

16. Give the domain and range of the function defined by the values in this table.

x	-2	$-\frac{3}{2}$	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	$\frac{3}{2}$	2
y	20	5	-1	$-\frac{7}{3}$	-5	-2	5	25	5

In Exercises 17–28, determine the domain of each function. In Exercises 17–22, also determine the range of each function.

17. $y = x + 2$

18. $y = x - 7$

19. $y = \dfrac{x}{x - 5}$

20. $y = \dfrac{x + 1}{x + 13}$

21. $y = \sqrt{x} - 2$

22. $y = x^2$

23. $y = \dfrac{15}{(x - 2)(x + 3)}$

24. $y = \dfrac{-9}{(x + 1)(x - 5)}$

25. $y = \dfrac{x + 1}{(x + 1)^2(x - 5)}$

26. $y = \dfrac{x - 2}{(x - 2)^2(x + 3)}$

27. $y = \dfrac{x^2 - x - 2}{(x - 2)^2(x + 4)^3}$

28. $y = \dfrac{x^2 + 3x - 10}{(x - 2)^2(x + 5)^3}$

In Exercises 29–36, create a function with the given domain.

29. $\{x : x \neq 5\}$

30. $\{x : x \neq -7\}$

31. $\{x : x \neq -1, x \neq 2\}$

32. $\{x : x \neq -3, x \neq 1, x \neq 4\}$

33. $\{x : x \geq 5\}$

34. $\{x : x \leq 7\}$

35. $\{x : x \geq -1, x \neq 4\}$

36. $\{x : x > -5, x \neq 3\}$

In Exercises 37–44, given the function $f(x) = 3x - 2$, determine the following.

37. $f(0)$

38. $f(2)$

39. $f(-3)$

40. $f(\frac{4}{3})$

41. $f(0) + f(-3)$

42. $f(b)$

43. $f(b + 3)$

44. $f(x + h) - f(x)$

In Exercises 45–54, given the function $g(x) = x^2 - 5x$, determine the following.

45. $g(0)$

46. $g(-3)$

47. $g(2)$

48. $g(0.4)$

49. $g(\frac{2}{5})$

50. $g(-x)$

51. $g(x - 5)$

52. $g(3 - t)$

53. $g(4m^2)$

54. $g(x + h) - g(x)$

In Exercises 55–60, given the function $F(x) = \dfrac{2 - x}{x^2 + 2}$, determine the following.

55. $F(0)$

56. $F(2)$

57. $F(1.5)$

58. $F(\frac{1}{3})$

59. $F(2x)$

60. $\dfrac{F(2x)}{F(x)}$

Given the function $C(f) = \frac{5}{9}(f - 32)$, determine the following in Exercises 61–64.

61. $C(32)$ **62.** $C(98.6)$ **63.** $C(-40)$ **64.** $C(72)$

Which of the equations in Exercises 65–68 are in explicit form and which are in implicit form? If possible, rewrite any implicit equations in explicit form.

65. $y = x^2 + 5$ **66.** $y + 2x = 7$ **67.** $y = x^2y + x$ **68.** $y = x^3 - 5x + 3$

In Exercises 69–72, if $f(x, y) = 3x - 2y + 4$, determine the indicated value.

69. $f(1, 0)$ **70.** $f(0, 1)$ **71.** $f(-1, 2)$ **72.** $f(2, -3)$

In Exercises 73–76, if $g(x, y) = x^2 - y^2 + 2xy$, determine the indicated value.

73. $g(-2, 0)$ **74.** $g(0, -2)$ **75.** $g(5, 4)$ **76.** $g(-3, 5)$

In Exercises 77–80, if $h(x, y, z) = 3x - 4y - 2z + xyz$, determine the indicated value.

77. $h(-1, 2, 3)$ **78.** $h(2, -1, -3)$ **79.** $h(3, 2, -1)$ **80.** $h(a, a^2, a^3)$

Solve Exercises 81–86.

81. ***Petroleum engineering*** The volume V of a cylindrical storage tank 40 ft high is a function of its radius and is described by $V = 40\pi r^2$. The total surface area S of the steel needed to construct this same tank is given by $S = 2\pi r(r + 40)$. Two tanks are designed. One has a radius of 20 ft and the other a radius of 30 ft. Compute the volume and total surface area of each tank.

82. ***Dynamics*** The distance s that a free-falling object travels is a function of the time t since it started falling, and is described by the function $s(t) = 4.91t^2$, where s is measured in meters and t in seconds. **(a)** How far will an object fall in 1 s? **(b)** How far will it fall in 2 s? **(c)** How far will it fall in 5 s?

83. ***Business*** The cost of renting a chain saw is \$20 for the first 4 hours and \$5 for each hour after 4. If $R(h)$ represents the rental charge for h hours, then $R(h) = 20 + 5(h - 4)$. Determine the cost of renting this chain saw for 7 hours.

84. ***Landscape architecture*** A rectangular yard is to be enclosed by 450 ft. of fencing.

(a) Write an equation for the area of the yard as a function of the yard's width.

(b) What is the area if the width is 50 ft?

(c) What is the area if the width is 150 ft?

(d) What is the area if the width is 250 ft?

(e) What restrictions must be placed on the domain x so that the problem makes physical sense?

85. ***Recreation*** The water pressure at the base of a dam is a function of the depth of the water. The weight density of water is 62.4 lb/ft^3. Thus, the water pressure, P, beneath the surface of the water is $P(d) = 62.4d$, where d represents the depth of the water in feet. What is the pressure on a scuba diver who is 80 ft below the water's surface?

86. ***Medical technology*** At various times in a child's life, the average number of millions of red blood cells in each mm^3 of the child's blood is given in the following table.

Age	Birth	2 days	14 days	3 mo	6 mo	1 yr	2 yr	4 yr	8 − 21 yr
Red cells/mm^3 (in millions)	5.1	5.3	5.0	4.3	4.6	4.7	4.8	4.8	5.1

(a) What is the domain?

(b) What is the range?

[IN YOUR WORDS]

87. Explain how to find the domain of a function when it is described by

(a) a table

(b) a set of elements

(c) an equation

88. **(a)** Can a relation ever be a function? Explain your answer.

(b) Can a function ever be a relation? Explain your answer.

4.2 OPERATIONS ON FUNCTIONS; COMPOSITE FUNCTIONS

This section introduces inverse functions and some operations on functions. Most of the operations involve the usual operations of addition, subtraction, multiplication, and division; but a new operation, called composition, will be introduced.

INVERSE FUNCTIONS

We often use a pair of functions that are natural opposites of each other. By a natural opposite, we mean that one of the functions "undoes" the result of the other function, much the same as subtraction "undoes" the result of addition and division "undoes" multiplication. Two functions with this relationship are called **inverse functions**.

To determine an inverse function of $y = f(x)$, you first solve for x. This gives an inverse relation for y. If this relation is also a function, say $g(y)$, then $g(y)$ is the inverse function of $f(x)$. We use the notation $f^{-1}(x)$ to designate the inverse function of the function $f(x)$.

Figure 4.1 shows that the domain of f is the range of f^{-1} and the range of f is the domain of f^{-1}.

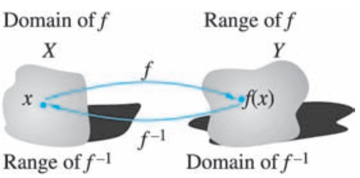


Figure 4.1

CAUTION Do not confuse the −1 in f^{-1} with a negative exponent. The symbol f^{-1} does not represent $\frac{1}{f}$.

GUIDELINES FOR FINDING f^{-1}

1. Solve the function $y = f(x)$ for x. Let $x = f^{-1}(y)$.
2. Exchange x and y to get $y = f^{-1}(x)$.
3. Check the domains and ranges: The domain of f and the range of f^{-1} should be the same, as should the domain of f^{-1} and the range of f.

EXAMPLE 4.30

Find the inverse function for each of the functions: **(a)** $f(x) = 4x - 5$, **(b)** $g(x) = x^3 + 5$, and **(c)** $h(x) = x^2$.

SOLUTIONS

(a) Let $y = f(x) = 4x - 5$. Solving for x, we get $x = \dfrac{y + 5}{4}$ and we write $f^{-1}(y) = \dfrac{y + 5}{4}$. Since, by tradition, we usually let x represent the independent variable, we normally rewrite this as $f^{-1}(x) = \dfrac{x + 5}{4}$.

(b) Let $y = g(x) = x^3 + 5$. Then $x = \sqrt[3]{y - 5}$ and $g^{-1}(y) = \sqrt[3]{y - 5}$, or, more traditionally, $g^{-1}(x) = \sqrt[3]{x - 5}$.

(c) Let $y = h(x) = x^2$. Then $x = \pm\sqrt{y}$, but this is not the inverse function of $h(x)$. Why? Because the domain of h is all real numbers and the range of $x = \sqrt{y}$ is the nonnegative real numbers.

The range of an inverse function is the domain of the original function. Another way of saying this is that an inverse function "undoes" what the function did. In symbols, we would write $f^{-1}(f(x)) = x$. One way to see if you have the inverse of a function is to test some values in this equation.

To see what this means, let's look at the first function in Example 4.30: $f(x) = 4x - 5$. We said that $f^{-1}(x) = \dfrac{x + 5}{4}$. If we try the value $x = 2$ in $f(x) = 4x - 5$, we get $f(2) = 4(2) - 5 = 3$. Now evaluate

$$\begin{aligned} f^{-1}(3) &= \frac{3 + 5}{4} \\ &= \frac{8}{4} \\ &= 2 \end{aligned}$$

Good! We started and ended with 2, or $f^{-1}(f(2)) = f^{-1}(3) = 2$.

Try another number, say -10.

$$f(-10) = 4(-10) - 5$$
$$= -45$$
$$\text{and } f^{-1}(-45) = \frac{-45 + 5}{4}$$
$$= \frac{-40}{4}$$
$$= -10$$

Again it checks! $f^{-1}(f(-10)) = f^{-1}(-45) = -10$.

Look at the third function in Example 4.30: $y = h(x) = x^2$. We solved for x and obtained $x = \sqrt{y}$. Now check some values. $h(2) = 2^2 = 4$ and $\sqrt{4} = 2$, so this checks. Try a negative number, like -10. $h(-10) = (-10)^2 = 100$ and $\sqrt{100} = 10$. It does not work. What we hoped would be the inverse returned a value of 10 instead of -10.

How can we tell if a function has an inverse? We will explore that idea later in this chapter.

OPERATIONS ON FUNCTIONS

Two functions, f and g, can be added, subtracted, multiplied, and divided according to the following definitions.

BASIC OPERATIONS ON FUNCTIONS

If f and g are two functions, then their sum $(f + g)$, their difference $(f - g)$, their product $(f \cdot g)$, and their quotient $\left(f/g \text{ or } \dfrac{f}{g}\right)$ are defined by

$$(f + g)(x) = f(x) + g(x)$$
$$(f - g)(x) = f(x) - g(x)$$
$$(f \cdot g)(x) = f(x) \cdot g(x)$$
$$(f/g)(x) = \left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$$

The domain of $f + g$, $f - g$, and $f \cdot g$ is the set of numbers that belong to both the domain of f and the domain of g. For the quotient function f/g, the domain also excludes all numbers where $g(x) = 0$.

EXAMPLE 4.31

Find each function, $(f+g)(x)$, $(f-g)(x)$, $(f \cdot g)(x)$, and $(f/g)(x)$ and their domains, if $f(x) = x^2 - 16$ and $g(x) = x - 4$.

SOLUTION

$$\begin{aligned}(f+g)(x) = f(x) + g(x) &= (x^2 - 16) + (x - 4) \\ &= x^2 - 16 + x - 4 \\ &= x^2 + x - 20 \\ (f-g)(x) = f(x) - g(x) &= (x^2 - 16) - (x - 4) \\ &= x^2 - 16 - x + 4 \\ &= x^2 - x - 12 \\ (f \cdot g)(x) = f(x) \cdot g(x) &= (x^2 - 16)(x - 4) \\ &= x^3 - 4x^2 - 16x + 64 \\ (f/g)(x) = \frac{f(x)}{g(x)} &= \frac{x^2 - 16}{x - 4} \\ &= \frac{(x+4)(x-4)}{x-4} = x + 4\end{aligned}$$

The domain of all the new functions, except for f/g, is the set of real numbers. Since $g(x) = 0$ when $x = 4$, the domain of f/g is all real numbers except $x = 4$.

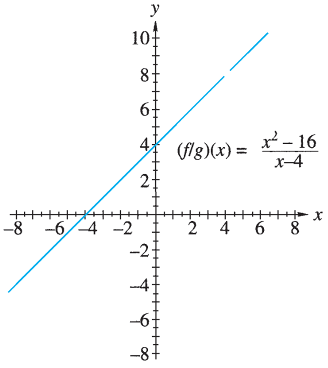


Figure 4.2a

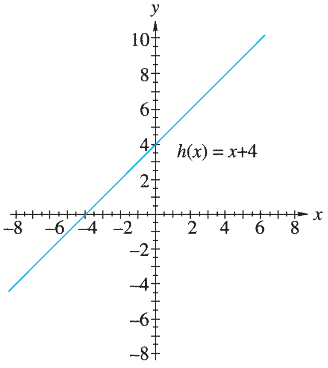


Figure 4.2b

Notice in Example 4.31 that the function $(f/g)(x) = x + 4$, if $x \neq 4$, and the function $h(x) = x + 4$ are identical except at $x = 4$. Because $(f/g)(x)$ is not defined when $x = 4$, its graph, as shown in Figure 4.2a, looks like the line $y = x + 4$ with a "hole" when $x = 4$. The graph of $h(x) = x + 4$ is shown in Figure 4.2b.

EXAMPLE 4.32

Find each function $(f+g)(x)$, $(f-g)(x)$, $(f \cdot g)(x)$, and $(f/g)(x)$ and their domains, if $f(x) = \sqrt{x+7}$ and $g(x) = \sqrt{x-5}$.

SOLUTIONS

$$\begin{aligned}(f+g)(x) &= \sqrt{x+7} + \sqrt{x-5} \\ (f-g)(x) &= \sqrt{x+7} - \sqrt{x-5} \\ (f \cdot g)(x) &= (\sqrt{x+7})(\sqrt{x-5}) = \sqrt{(x+7)(x-5)} \\ &\qquad = \sqrt{x^2 + 2x - 35} \\ (f/g)(x) &= \frac{\sqrt{x+7}}{\sqrt{x-5}} = \sqrt{\frac{x+7}{x-5}}\end{aligned}$$

The domain of f is all real numbers $x \geq -7$. The domain of g is all real numbers $x \geq 5$. So, the domain of each new function is all real numbers $x \geq 5$ except for f/g; since $g(5) = 0$, the domain of f/g is all real numbers $x > 5$.

COMPOSITE FUNCTIONS

There is another important way in which two functions f and g can be combined to form a new function. This new function is called the composite function. An example of a composite function is $y = (x - 7)^3$. If we let $u = g(x) = x - 7$, then $y = f(u) = u^3$. You could have also written this as $y = f(u) = f(g(x)) = (x - 7)^3$.

COMPOSITE FUNCTION

If f and g are two functions, the **composite function**, $f \circ g$, is defined by

$$(f \circ g)(x) = f(g(x))$$

where the domain of $f \circ g$ is all numbers x in the domain of g where $g(x)$ is in the domain of f.

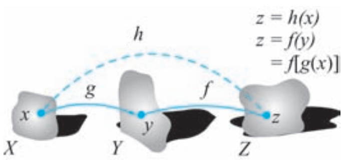


Figure 4.3

The sketch in Figure 4.3 shows a function g, which assigns to each element x of set X, some element y of set Y. The figure also shows a function f that takes each element of set Y and assigns to it a value z of set Z. Using both f and g, an element x in X is assigned to an element z in Z. This process is a new function h, which takes an element x in X and assigns to it an element z in Z.

CAUTION We do not perform composition in the same left-to-right direction as we do in reading. In the composition $f \circ g$, function g is performed first.

The definition of the domain is rather cumbersome, but it will make sense after a few examples.

EXAMPLE 4.33

Let $f(x) = x - 2$ and let $g(x)$ be defined by the following table.

x	-2	-1	0	1	2	3	4
$g(x)$	-7	3	0	-2	5	1	-7

Describe **(a)** $(f \circ g)(x)$ and **(b)** $(g \circ f)(x)$.

SOLUTIONS

(a) To find $(f \circ g)(x)$, replace each x in the expression for $f(x)$ with $g(x)$. Thus,

$$(f \circ g)(x) = f(g(x)) = g(x) - 2.$$

At this point, we can give no further description for $f \circ g$ in the form of an equation. We can, however, use a table to describe the composition.

x	-2	-1	0	1	2	3	4
$g(x)$	-7	3	0	-2	5	1	-7
$(f \circ g)(x)$	-9	1	-2	-4	3	-1	-9

This table lists all possible values for $(f \circ g)(x)$. Thus, we see that the domain of $f \circ g$ is $\{-2, -1, 0, 1, 2, 3, 4\}$, and the range of $f \circ g$ is $\{-9, -4, -2, -1, 1, 3\}$.

(b) To find $(g \circ f)(x) = g(f(x)) = g(x - 2)$, we use the following table.

x	-2	-1	0	1	2	3	4	5	6
$f(x) = x - 2$	-4	-3	-2	-1	0	1	2	3	4
$(g \circ f)(x) = g(x-2)$			-7	3	0	-2	5	1	-7

This table lists all possible values for $(g \circ f)(x)$. Notice that two of the cells in the table are empty. Why? Consider $x = -2$. It is in the domain of f and $f(-2) = -4$. But -4 is not in the domain of g. The same argument can be made if $x = -1$. Since the domain of f is not specified, we can assume it is all real numbers. In this case, the domain of f actually is all real numbers. From the table, we see that the domain of $g \circ f$ is $\{0, 1, 2, 3, 4, 5, 6\}$, and the range of $g \circ f$ is $\{-7, -2, 0, 1, 3, 5\}$.

EXAMPLE 4.34

If $f(x) = x^2$ and $g(x) = x - 4$, find **(a)** $(f \circ g)(x)$, **(b)** $(g \circ f)(x)$, **(c)** $(f \circ g)(5)$, and **(d)** $(g \circ f)(5)$

SOLUTIONS

(a) $(f \circ g)(x) = f(g(x)) = f(x - 4)$
$$= (x - 4)^2$$

(b) $(g \circ f)(x) = g(f(x)) = g(x^2) = x^2 - 4$

(c) $(f \circ g)(5) = (5 - 4)^2 = (1)^2 = 1$

(d) $(g \circ f)(5) = 5^2 - 4 = 25 - 4 = 21$

Because the domains of both of these functions are the entire set of real numbers, the domains of both $f \circ g$ and $g \circ f$ are all real numbers.

NOTE Examples 4.34(c) and (d) show that in most cases $(f \circ g)(x) \neq (g \circ f)(x)$.

EXAMPLE 4.35

If $f(x) = x^2 - 9$ and $g(x) = \sqrt{x + 4}$, then find **(a)** $(f \circ g)(x)$, **(b)** $(g \circ f)(x)$, **(c)** $(f \circ g)(2)$, **(d)** $(g \circ f)(2)$, and **(e)** the domains of $f \circ g$ and $g \circ f$.

SOLUTIONS

(a) $(f \circ g)(x) = f(g(x)) = f(\sqrt{x + 4}) = (\sqrt{x + 4})^2 - 9$
$$= (x + 4) - 9$$
$$= x - 5$$

(b) $(g \circ f)(x) = g(f(x)) = g(x^2 - 9) = \sqrt{(x^2 - 9) + 4}$
$$= \sqrt{x^2 - 5}$$

(c) $(f \circ g)(2) = 2 - 5 = -3$

EXAMPLE 4.35 (Cont.)

(d) $(g \circ f)(2) = \sqrt{2^2 - 5} = \sqrt{-1}$, which is not a real number. Therefore, 2 is not in the domain of $g \circ f$.

(e) The domain of f is all real numbers and the domain of g is all real numbers $x \geq -4$. The domain of $f \circ g$ will be all numbers $x \geq -4$. The domain of $g \circ f$ will be the real numbers x, where $f(x) \geq -4$ or where $x^2 - 9 \geq -4$. Simplifying this, we see that $x^2 \geq 5$, which further simplifies to $x \geq \sqrt{5}$ or $x \leq -\sqrt{5}$.

APPLICATION ENVIRONMENTAL SCIENCE

EXAMPLE 4.36

An oil well in the Gulf of Mexico is leaking. There is no wind or current, so the oil is spreading on the water as a circle. At any time t, in minutes, after the leak began, the radius in meters of the circular oil slick is $r(t) = 12t$. If we know that the area of a circle of radius r is $A = \pi r^2$, **(a)** find the function $A(t)$, the area of the oil slick as a function of t; **(b)** determine the area of the slick, 15 min after the leak began; and **(c)** determine the area of the slick, 1 h after it began.

SOLUTIONS

(a) We have $r(t) = 12t$ and $A(r) = \pi r^2$. Then $A(t) = (A \circ r)(t) = A(r(t)) = A(12t) = \pi(12t)^2 = 144\pi t^2 \text{m}^2$.

(b) When $t = 15$ min, $A(15) = 144\pi(15^2) = 32\,400\pi$ m^2.

(c) When $t = 1$ h, we must use 1 h $= 60$ min, because $A(t)$ is for t in minutes. $A(1 \text{ h}) = A(60) = 144\pi(60^2) = 518\,400\pi$ m^2.

EXERCISE SET 4.2

Find the inverse of each of the functions or relations in Exercises 1–8.

1. $y = x - 5$
2. $y = x + 7$
3. $y = 2x - 8$
4. $y = 5x + 12$
5. $6y = 2x - 4$
6. $9y = 3x + 5$
7. $y^2 = x - 5$
8. $y = x^3 + 7$

In Exercises 9–20, let $f(x) = x + 3$ and let $g(x)$ be defined by the following table.

x	-3	-2	-1	0	1	2	3	4
$g(x)$	-10	5	0	-1	5	1	-10	7

Determine the following.

9. $(f + g)(x)$
10. $(f - g)(x)$
11. $(g - f)(x)$
12. $(f \circ g)(x)$
13. $(f / g)(x)$
14. $(g / f)(x)$
15. $(f \circ g)(x)$
16. $(g \circ f)(x)$
17. The domain and range of f and g
18. The domain and range of **(a)** $f + g$, **(b)** $f - g$, and **(c)** $f \cdot g$.
19. The domain and range of **(a)** f / g and **(b)** g / f.
20. The domain and range of **(a)** $f \circ g$ and **(b)** $g \circ f$

In Exercises 21–38, let $f(x) = x^2 - 1$ and $g(x) = 3x + 5$. Determine the following.

21. $(f + g)(x)$
22. $(f + g)(4)$
23. $(f - g)(x)$
24. $(f - g)(4)$
25. $(g - f)(x)$
26. $(g - f)(5)$
27. $(f \cdot g)(x)$
28. $(f \cdot g)(5)$
29. $(f / g)(x)$
30. $(f / g)(-2)$
31. $(g / f)(x)$
32. $(g / f)(-2)$
33. The domains of f and g
34. The domains of f / g and g / f
35. $(f \circ g)(x)$
36. $(f \circ g)(-3)$
37. $(g \circ f)(x)$
38. $(g \circ f)(-3)$

In Exercises 39–56, let $f(x) = 3x - 1$ and $g(x) = 3x^2 + x$. Determine the following.

39. $(f + g)(x)$
40. $(f + g)(4)$
41. $(f - g)(x)$
42. $(f - g)(-2)$
43. $(g - f)(x)$
44. $(g - f)(-2)$
45. $(f \cdot g)(x)$
46. $(f \cdot g)(-1)$
47. $(f / g)(x)$
48. $(f / g)(-1)$
49. $(g / f)(x)$
50. $(g / f)(-1)$
51. $(f \circ g)(x)$
52. $(f \circ g)(5)$
53. $(g \circ f)(x)$
54. $(g \circ f)(5)$
55. The domains of f and g
56. The domains of f / g and g / f

In Exercises 57 and 58, use the given tabular values of $g(x)$ and $(f \circ g)(x)$ to determine values for $f(x)$.

57. The first two tables below represent the functions g and $f \circ g$, respectively. Use the information in these tables to complete the table for f.

x	1	2	3	4	5	6	7
$g(x)$	3	5	8	5	7	2	6

x	1	2	3	4	5	6	7
$(f \circ g)(x)$	2	7	9	7	5	5	4

x	1	2	3	4	5	6	7
$f(x)$							

58. The first two tables below represent the functions f and $f \circ g$, respectively. Use the information in these tables to complete the table for g.

x	1	2	3	4	5	6	7
$f(x)$	4	5	8	5	7	2	6

x	1	2	3	4	5	6	7
$(f \circ g)(x)$	2	7	9	7	5	5	4

x	1	2	3	4	5	6	7
$g(x)$							

Solve Exercises 59–70.

59. ***Business*** The cost of manufacturing an item is the sum of the fixed costs and the variable costs. Fixed costs are the costs the business must meet just to stay in business, such as rent and insurance. Variable costs are the costs, such as material, labor, and lighting, of producing just one item. Suppose the fixed cost, in dollars, of manufacturing computer chips is $F(n) =$ \$7,500/week and the variable cost is \$15 per chip. **(a)** Find the cost function $C(n)$, in dollars, of manufacturing n computer chips per week. **(b)** How much will it cost to manufacture 100 chips? **(c)** How much will it cost to manufacture 1,000 chips?

60. ***Business*** In manufacturing a certain product, a company must pay a set-up cost of $S(n) = \dfrac{2{,}750{,}000}{n}$ dollars, where n is the number of units it produces in a production run, and with $1 \le n \le 25{,}000$. The fixed cost is $F(n) =$ \$12,500/week and the variable cost is \$8/item. **(a)** Determine the cost function, $C(n)$, of setting up and manufacturing n items in a week. **(b)** What is the cost of manufacturing 100 items? **(c)** of manufacturing 1,000 items?

61. ***Business*** A company's profit is equal to the revenue it generates minus the cost of producing and selling the product. The revenue function for a certain product is $R(n) = 90n$ and its cost function is $C(n) = 30n + 275$, where n is the number of units made and sold. **(a)** If R and C are in dollars, determine the profit function, $P(n)$, for this product. **(b)** What is the profit if you make and sell 50 units?

62. ***Business*** The demand function $n = f(p)$ relates the number of units n that can be sold to the price p charged for each unit. The revenue is equal to the price per unit multiplied by the quantity sold. A manufacturer is currently selling its calculators for p dollars each. Based on past experience, the company believes that the weekly demand for the calculator is related to its price by the equation $f(p) = 750 - 4.2p$. **(a)** What is the revenue function? **(b)** How many calculators can the company expect to sell if it charges \$100 for each calculator? **(c)** What is the revenue from the \$100 calculators? **(d)** How many calculators can it expect to sell at \$80 each? **(e)** What is the revenue for the \$80 calculator?

63. ***Environmental science*** The population P of a certain species of large fish depends on the number n of a smaller fish on which it feeds, with

$$P(n) = 300n^2 - 50n$$

The number of smaller fish depends on the amount a of its food supply, a type of plankton, with

$$n(a) = 7a + 4$$

Find the relationship between the population P of the larger fish and the amount a of plankton available.

64. ***Meteorology*** A spherical weather balloon is inflated so the radius r is changed at a constant rate of 3 cm/s.

(a) Find an algebraic expression for $V(t)$, the volume of the balloon as a function of time t in seconds.

(b) Determine the volume of the balloon after 10 s.

65. ***Business*** The number n of cars produced at a certain factory each day is given by $n(t) = 85t - 5t^2$, where t represents the hours since midnight and $0 \le t < 24$. If the cost C in dollars of producing n cars is given by $C(n) = 16{,}250 + 9{,}500n$, write C as a function of time since midnight.

66. ***Finance*** Currency traders often move investments from one country to another in order to make a profit. Table 4.2 gives the exchange rate for U.S. dollars, the European Union's euro, and the Mexican peso. In January 2011, for example, 1 U.S. dollar could purchase 0.7593 euros or 12.1974 pesos. Similarly, 1 euro purchases 16.065 pesos or 1.3171 dollars. Suppose you have the following functions:

TABLE 4.2 Exchange Rate for U.S. Dollars, European Euro, and Mexican Peso, January 10, 2011

	Amount Purchased		
Amount Invested	**Dollars**	**Euro**	**Peso**
1 dollar	1.00000	0.7593	12.1974
1 euro	1.3171	1.00000	16.065
1 peso	0.0820	0.0622	1.0000

$de(x)$ = Number of euros that can be bought with x dollars

$dp(x)$ = Number of pesos that can be bought with x dollars

$pe(x)$ = Number of euros that can be bought with x pesos

(a) Find formulas for de, pe, and dp.

(b) Evaluate $dp(pe(1000))$.

(c) Describe the meaning of the answer to **(b)** in writing.

(d) Compare $dp(pe(1000))$ and $de(1000)$. Are they the same? If not, why do you think they might not be the same?

67. ***Environmental science*** The $A(V)$ surface area in square feet covered by an oil spill on still water is given by the formula $A(V) = 10^5V^{3/4}C$, where $C = 0.1643$ is a constant and V is the volume of the spill in gallons. The volume of a particular spill is given by the function $V(t) = 225\sqrt{t}$, where t is the number of hours since the spill began. Determine the surface area of this spill as a function of t.

68. ***Business*** The total number of pieces of mail delivered each year by the U.S. Postal Service can be modeled as $T(t) = -0.0300t^3+8.7048t^2-836.5644t+26{,}826.4856$ billion pieces of mail, where t is the number of years since 1990. The number of pieces of periodicals delivered by the U.S. Postal Service can be modeled as $P(t) = -0.0004t^3 + 0.1148t^2 - 10.7320t + 346.3238$ billion periodicals, where t is the number of years since 1990. Write an equation for the number of pieces of mail other than periodicals delivered by the U.S. Postal Service since 1990.

69. ***Environmental science*** The Chézy-Manning equation

$$Q = \frac{c_1}{n}AR^{2/3}\sqrt{s}$$

is used to estimate the fluid flow in a pipe, where Q is the flow rate in cubic units per second, $c_1 = 1$ in the metric system and $c_1 = 1.486$ in the English system, n is a friction factor based on the roughness of the pipe, A is the cross-sectional area, R is the hydraulic radius, $R = \dfrac{A}{p}$ where p is the wetted perimeter, and s is the slope of the pipe.

Consider a half-full sewer pipe with a slope of 1% (that is, $s = 0.01$), $A = 1.2m^2$ is the cross-sectional area of the pipe, $p = 4.65$ m, and $n = 0.015 + 0.0014t$ where t is the age of the pipe in t years.

(a) Find the flow rate of this pipe when it was new.

(b) Determine the flow rate of this pipe when it was 10 years old.

(c) Rewrite the Chézy-Manning equation in terms of t rather than n.

70. ***Environmental science*** When a rectangular channel is used, the Chézy-Manning equation becomes

$$Q = \frac{whc_1}{n}\left(\frac{wh}{w + 2h}\right)^{2/3}\sqrt{s}$$

where Q is the flow rate in cubic units per second, $c_1 = 1$ in the metric system and $c_1 = 1.486$ in the English system, n is a friction factor based on the roughness of the channel, w is the width of the channel, h is the depth of the water in the channel, and s is the slope of the pipe.

Consider a channel with a slope of 0.5% (that is, $s = 0.005$), $w = 3.2$ ft, and $n = 0.017 + 0.0012t$, where t is the age of the channel in t years.

(a) Find the flow rate of this pipe when it was new and $h = 1.2$ ft.

(b) Determine the flow rate of this pipe when it was new and $h = 2.8$ ft.

(c) Rewrite this Chézy-Manning equation in terms of t rather than n.

[IN YOUR WORDS]

71. Explain the difference between the composite functions $f \circ g$ and $g \circ f$.

72. Explain how to determine the domain of a composite function $f \circ g$.

4.3 RECTANGULAR COORDINATES

In Chapter 1 we graphed some points on a number line. In this section we will expand our idea of graphing to a plane.

Why is a graph so important? A graph gives us a picture of an equation. By looking at a graph, we can often get a better idea of what we can expect an equation to do. The graphs will also help us find solutions to some of our problems.

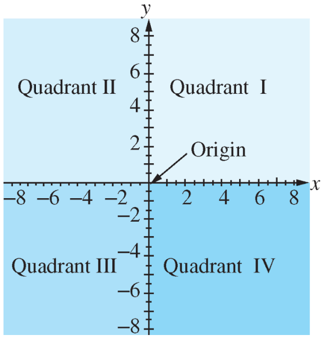


Figure 4.4

In Section 1.1 we learned that we could represent the numbers by points on a line. To get a graph in a plane, we need two number lines. The two number lines are usually drawn perpendicular to each other and intersect at the number zero as shown in Figure 4.4. The point of intersection is called the *origin*. If one of the lines is horizontal, then the other is vertical. The horizontal number line is called the ***x*-axis** and the vertical number line the ***y*-axis**. This is called the **rectangular coordinate system** or the **Cartesian coordinate system** in honor of the man who invented it, René Descartes.

On the x-axis, positive numbers are to the right of the origin and negative numbers to the left. On the y-axis, positive numbers are above the origin and negative values are below it. The two axes divide the plane into four regions called **quadrants**, with the first quadrant in the upper right section of the plane and the others numbered in a counterclockwise rotation around the origin. (See Figure 4.4.)

Suppose that P is a point in the plane. The coordinates of P can be determined by drawing a perpendicular line segment from P to the x-axis. If this perpendicular segment meets the x-axis at the value a, then the x-coordinate of the point P is a. Now draw a perpendicular segment from P to the y-axis. It meets the y-axis at b, so the y-coordinate of P is b. The coordinates of P are the ordered pair (a, b).

NOTE The x-coordinate is always listed first. The order in which the coordinates are written is very important. In most cases, the point (a, b) is different than the point (b, a). (See Figure 4.5.)

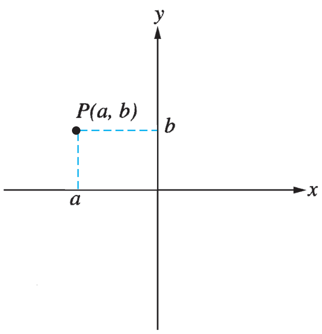


Figure 4.5

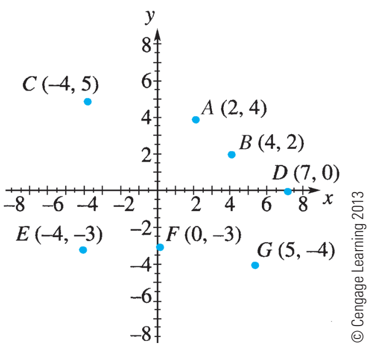


Figure 4.6

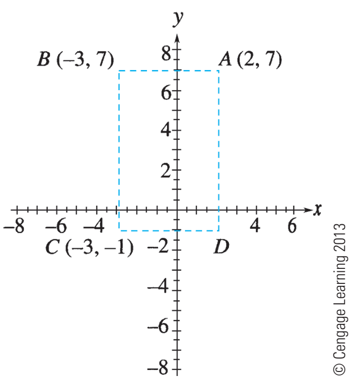


Figure 4.7

EXAMPLE 4.37

The positions of $A(2, 4)$, $B(4, 2)$, $C(-4, 5)$, $D(7, 0)$, $E(-4, -3)$, $F(0, -3)$, and $G(5, -4)$ are shown in Figure 4.6. Note that $A(2, 4)$ and $B(4, 2)$ are different points as are $C(-4, 5)$ and $G(5, -4)$.

EXAMPLE 4.38

$A(2, 7)$, $B(-3, 7)$, and $C(-3, -1)$ are three vertices of a rectangle as shown in Figure 4.7. What are the coordinates of the fourth vertex, D?

SOLUTION If we plot points A, B, and C, we can see that the missing vertex D will have the same x-coordinate as A, 2, and the same y-coordinate as C, -1. So, D has the coordinates $(2, -1)$.

EXAMPLE 4.39

Some ordered pairs for the equation $y = 2x + 7$ are $(-2,3), (-1,5), (0,7)$, $(\frac{1}{2},8), (1, 9)$, and $(2, 11)$. Plot these ordered pairs on a rectangular coordinate system.

SOLUTION The ordered pairs are plotted on the graph in Figure 4.8.

APPLICATION CONSTRUCTION

EXAMPLE 4.40

The following table shows the results of a series of drillings used to determine the depth of the bedrock at a building site. These drillings were taken along a straight line down the middle of the lot, from the front to the back, where the building will be placed. In the table, x is the distance from the front of the parking lot and y is the corresponding depth. Both x and y are given in feet.

x	0	20	40	60	80	100	120	140	160
y	33	35	40	45	42	38	46	40	48

Plot these ordered pairs on a rectangular coordinate system. Connect the points in order to get an estimate of the profile of the bedrock.

SOLUTION The ordered pairs are plotted on the graph in Figure 4.9. The points have been connected in order. Note that this graph is "upside down." That is, the top of the dirt is along the x-axis, while the bedrock, which is below ground, is above the x-axis.

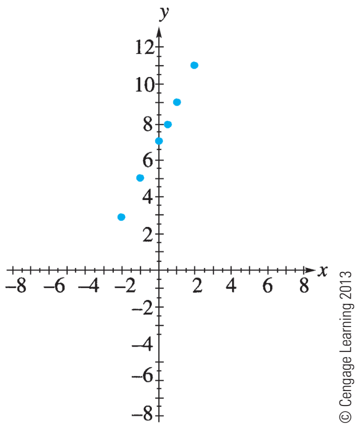


Figure 4.8

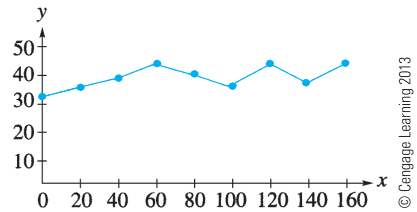


Figure 4.9

EXERCISE SET 4.3

In Exercises 1–10, plot the points on a rectangular coordinate system.

1. $(4, 5)$
2. $(1, 7)$
3. $(7, 1)$
4. $(-2, 4)$
5. $(-3, -5)$
6. $(6, -1)$
7. $(0, 0)$
8. $(-\frac{5}{2}, 3)$
9. $(2, -\frac{3}{2})$
10. $(4.5, -1.5)$

Solve Exercises 11–28.

11. If $A(2, 5)$, $B(-1, 5)$, and $C(2, -4)$ are three vertices of a rectangle, what are the coordinates of the fourth vertex?

12. The points $(-3, 6)$, $(-2, 1)$, $(-1, -2)$, $(0, -3)$, $(1, -2)$, $(2, 1)$, and $(3, 6)$ are some ordered pairs of the equation $y = x^2 - 3$. Plot these ordered pairs on the same rectangular coordinate system.

13. Plot the ordered pairs $(-5, 3)$, $(-3, 3)$, $(0, 3)$, $(1, 3)$, $(4, 3)$, and $(6, 3)$. What do all of these points have in common?

14. Plot the ordered pairs $(-2, 6)$, $(-2, 4)$, $(-2, 1)$, $(-2, -1)$, $(-2, -3)$, and $(-2, -4)$. What do all of these points have in common?

15. ***Automotive technology*** Graph these ordered pairs from the conversion formula 6.9 kPa = 1 psi: (138, 20), (172.5, 25), (207, 30), (241.5, 35), (276, 40), (345, 50), (414, 60). Use a ruler and draw the segment connecting (138, 20) and (414, 60). What seems to happen?

16. Where are all the points whose x-coordinates are 0?

17. Where are all the points whose y-coordinates are -2?

18. Where are all the points whose y-coordinates are 7?

19. Where are all the points whose x-coordinates are -5?

20. Where are all the points whose y-coordinates are 0?

21. Where are all the points (x, y) for which $x > -3$?

22. Where are all the points (x, y) for which $x > 0$ and $y < 0$?

23. Where are all the points (x, y) for which $x > 1$ and $y < -2$?

24. ***Automotive technology*** If the antifreeze content of the coolant is increased, the boiling point of the coolant is also increased, as shown in the table below.

Percent antifreeze in coolant	0	10	20	30	40	50	60	70	80	90	100
Boiling temperature °F	210	212	214	218	222	228	236	246	258	271	330

(a) Plot these ordered pairs on a rectangular coordinate system.

(b) What will be the boiling point of the coolant if the system is filled with a recommended solution of 50% water and 50% antifreeze?

(c) What will be the boiling point of the coolant if the system is filled with a recommended solution of 40% water and 60% antifreeze?

25. ***Construction*** Refrigerant 407C is often used in residential and commercial air-conditioners and heat pumps the temperature-pressure relationship of refrigerant R-407C is very important to maintain proper operation and for diagnosis. The table below indicates the pressure of R-407C at various temperatures.

Temperature °F	– 20	– 10	0	10	20	30	40	50	60	70	80	90
Vapor pressure (psi)	6.6	12.5	19.6	28.0	38.0	49.6	63.1	78.7	96.8	117	140	166

(a) Plot these ordered pairs on a rectangular coordinate system. Connect the ordered pairs in order to help you answer **(b)** and **(c)**.

(b) One day the early morning temperature was 45°F. What was the vapor pressure, in psi, when the temperature was 45°F?

(c) If a technician connected a pressure gauge to an air-conditioning system filled with R-407C on a 90°F summer day, what pressure, in psi, would the gauge indicate?

26. ***Automotive technology*** Recent automotive air-conditioning systems use R-134a rather than R-12. This action is based on evidence indicating that R-12 is causing depletion of the earth's protective ozone layer. As with R-12, the temperature-pressure relationship of refrigerant R-134a is very important to maintain proper operation and diagnosis. The table below indicates the pressure of R-134a at various temperatures.

Temperature (°C)	−20	−10	0	10	20	30
R-134a pressure (kPa)	31.4	99.4	191.4	312.9	469.5	666.7

(a) Plot these ordered pairs on a rectangular coordinate system. Connect the ordered pairs in order to help you answer parts (b), (c), and (d).

(b) One day the early morning temperature was 5°C. What was the pressure, in kPa, when the temperature was 5°C?

(c) If a technician connected a pressure gauge to an air-conditioning system filled with R-134a on a 35°C summer day, what pressure, in kPa, would the gauge indicate?

(d) Suppose a technician connected a pressure gauge to an air-conditioning system filled with R-134a and got a reading of 500 kPa. What is the temperature in °C?

27. ***Machine technology*** The following table gives actual recording times and counter values as obtained on one particular cassette tape deck.

Time (minutes)	1	2	3	4	5	10	15	20	25	30
Counter reading	30	60	88	115	141	262	369	466	556	640

(a) Plot these ordered pairs on a rectangular coordinate system. Connect the ordered pairs in order to help you answer parts (b) and (c).

(b) What would you expect the counter to read after 12.5 min?

(c) How much time has passed if the counter has a value of 500?

28. ***Forestry*** The following table gives the girth of a pine tree measured in feet at shoulder height and the amount of lumber in board feet that was finally obtained.

Girth (ft)	15	18	20	22	25	28	33	35	40	43
Lumber (bd ft)	9	27	42	60	90	126	198	231	324	387

(a) Plot these ordered pairs on a rectangular coordinate system. Connect the ordered pairs in order to help you answer parts (b) and (c).

(b) How many board feet of lumber would you expect to obtain from a tree with a girth of 30 ft?

(c) What was the girth of a tree that produced 275 board feet of lumber?

[IN YOUR WORDS]

29. Look again at Example 4.40. Describe how you would have changed the graph, or the way the data was recorded, so that the graph would appear rightside up instead of upside down.

30. Gather some data from your field of interest. Write a problem that will require that data to be graphed. Then write some questions that are answered by reading the graph. Put your answers on the back of the paper. Give your problem to a classmate and ask him or her to solve the problem. Rewrite your exercise or solution to clarify places where your classmate had difficulty.

4.4 GRAPHS

In the last section, we learned how to graph a point on a plane. Earlier, we said that a function could be represented by a graph. In this section, we will see how we can use graphs to represent functions.

GRAPH OF AN EQUATION

The graph of an equation in two variables x and y is formed by all the points $P(x, y)$ whose coordinates (x, y) satisfy the given equation.

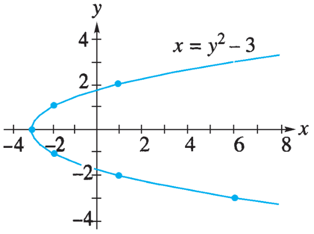


Figure 4.10

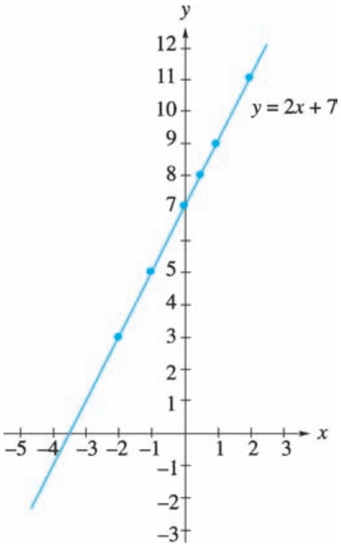


Figure 4.11

To graph an equation such as $x = y^2 - 3$, you can set up a table of values, plot the points in the table, and then connect the points smoothly. To fill the table, we will select values for y and use the equation to solve for the corresponding values for x. Because the ordered pairs have x listed first, we will list x as the top value in the table.

x	6	1	-2	-3	-2	1	6
y	-3	-2	-1	0	1	2	3

These points are plotted in Figure 4.10 and a smooth curve has been used to connect the points. Remember, this equation has an infinite number of possible ordered pairs. We have selected only seven of them. We must plot enough points to be confident that the points give an outline of the curve that is complete enough for us to tell what the actual curve looks like. Later, you will learn some ways to use mathematics to help determine when you have a sufficient number of points to sketch a curve.

In Figure 4.8 we plotted six ordered pairs for the equation $y = 2x + 7$: $(-2, 3)$, $(-1, 5)$, $(0, 7)$, $\left(\frac{1}{2}, 8\right)$, $(1, 9)$, and $(2, 11)$. If we connect these points, we see that we get the straight line in Figure 4.11. In fact, any equation of the form $Ax + By + C = 0$ where A, B, and C are constants and both A and B are not 0 is a **linear equation** whose graph is a straight line. The equation $y = 2x + 7$ is also a function and can be written as $f(x) = 2x + 7$.

INTERCEPTS

The graph of the equation $y = 2x + 7$ crosses both the x-axis and the y-axis. These points are called the **intercepts**. The graph crosses the y-axis at $(0, 7)$, so we say that the **y-intercept** is 7. It crosses the x-axis at $\left(-\frac{7}{2}, 0\right)$, so the **$x$-intercept** is $-\frac{7}{2}$.

HINT Any time you have a linear equation, you will only need to plot two points in order to sketch the graph of that equation. The intercepts are often the easiest two points to determine and use when graphing a linear equation.

EXAMPLE 4.41

Plot the graph of $2x + 3y = 12$.

SOLUTION This is a linear equation, so we will use two points to determine the line. If we find the intercepts, we get the points (6, 0) and (0, 4). These points and the line they determine are shown in Figure 4.12.

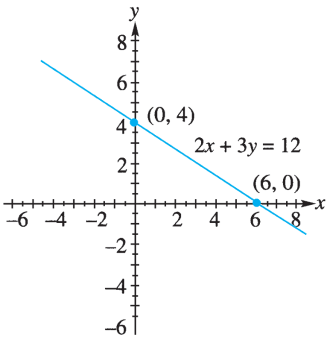


Figure 4.12

SLOPE

One important property of the graph of a straight line is its slope. The slope refers to the steepness or inclination of the line. The idea of the slope is so important that we will keep coming back to it. In fact, it is a major topic in calculus.

SLOPE

The **slope**, m, of a straight line is a measure of its steepness with respect to the x-axis. If (x_1, y_1) and (x_2, y_2) are two different points on a line, then the slope of the line is defined as

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

provided $x_2 - x_1 \neq 0$.

HINT Some people like to remember the slope as the "rise over the run" or $\frac{\text{rise}}{\text{run}}$. Here "rise" means the vertical change and "run" the horizontal change.

EXAMPLE 4.42

What is the slope of the line $2x + 3y = 12$ in Example 4.41?

SOLUTION We know that (6, 0) and (0, 4) are two points on this line. If we let $x_1 = 6$, $y_1 = 0$, $x_2 = 0$, and $y_2 = 4$, then

$$\begin{aligned} m &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{4 - 0}{0 - 6} = \frac{4}{-6} = -\frac{2}{3} \end{aligned}$$

The slope of this line is $m = -\frac{2}{3}$.

EXAMPLE 4.43

Graph $4y - 6x = 12$ and find the intercepts and the slope.

SOLUTION This is a straight line. The intercepts are (0, 3) and (−2, 0). Plotting these two points and the line through them gives the graph in Figure 4.13. Letting $x_1 = 0$, $y_1 = 3$, $x_2 = -2$, and $y_2 = 0$, we determine that the slope is

$$m = \frac{0 - 3}{-2 - 0} = \frac{-3}{-2} = \frac{3}{2}$$

EXAMPLE 4.44

Graph $8x = 2 + y$, and find the intercepts and the slope.

SOLUTION Again, this is a straight line. If we let $x = 0$, then we get $0 = 2 + y$ or $y = -2$, and so one intercept is $(0, -2)$. If we let $y = 0$, then $8x = 2$ and $x = \frac{1}{4}$. Thus, $(\frac{1}{4}, 0)$ is the other intercept. The slope is

$$m = \frac{-2 - 0}{0 - \frac{1}{4}} = \frac{-2}{-\frac{1}{4}} = 8$$

Plotting the two intercepts and drawing the line through them produces the line in Figure 4.14.

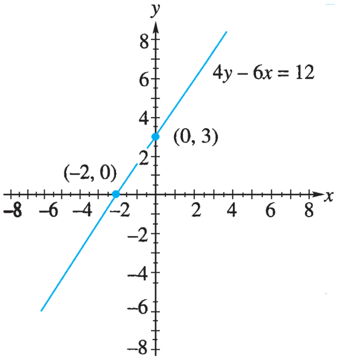


Figure 4.13

Look again at the last three examples. In Example 4.42, the slope of the line was $-\frac{2}{3}$. Notice that this line falls as the x-values get larger. In Example 4.43, the slope of the line was $\frac{3}{2}$. In Example 4.44, the slope of the line was 8. Both of these lines rose as the values of x got larger. Notice that the line in Example 4.44 was steeper than the line in Example 4.43.

In general, as the values of x increase, a straight line rises if it has a positive slope and falls if it has a negative slope. This is demonstrated in Figures 4.15 and 4.16.

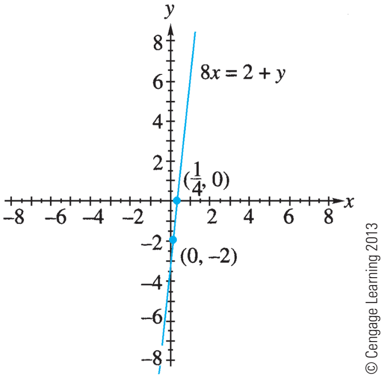


Figure 4.14

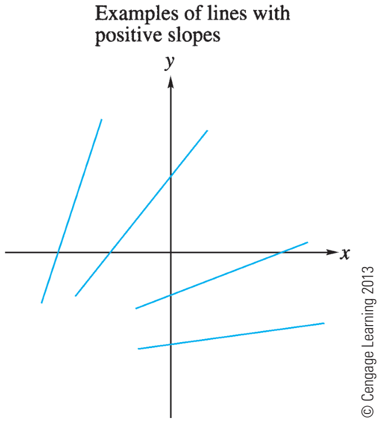


Figure 4.15

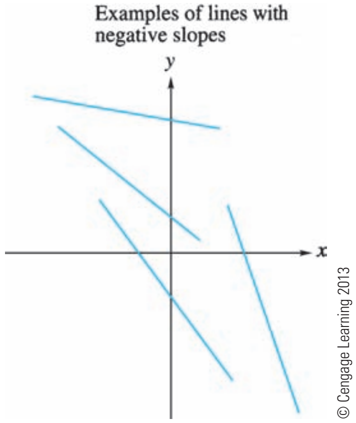


Figure 4.16

GRAPHS THAT ARE NOT LINES

Not every graph is a straight line. The next four examples show some graphs of functions that are not lines. This is followed by a test that tells how to use a graph to determine if it is a graph of a function.

EXAMPLE 4.45

Graph the function from Example 4.14, $y = x^2 - 3$.

SOLUTION Some ordered pairs that satisfy $y = x^2 - 3$ are shown in the following table.

x	-3	-2	-1	0	1	2	3
y	6	1	-2	-3	-2	1	6

If we plot these seven points and connect them with a smooth curve, we get the graph in Figure 4.17. Notice that this graph has a y-intercept at $y = -3$ and that it has two x-intercepts, when $x = \sqrt{3}$ and when $x = -\sqrt{3}$.

EXAMPLE 4.46

Graph the function $f(x) = x^2 - 2x$.

SOLUTION Here is a table of some of the values for this function.

x	-3	-2	-1	0	1	2	3	4	5
$f(x)$	15	8	3	0	-1	0	3	8	15

These points and the curve connecting them are shown in Figure 4.18. The y-intercept is $y = 0$ and the x-intercepts are $x = 0$ and $x = 2$. This graph passes through the origin, so the y-intercept is also an x-intercept.

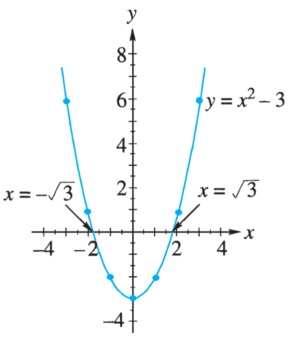


Figure 4.17

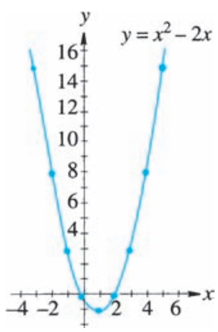


Figure 4.18

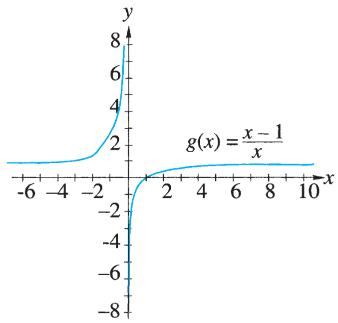


Figure 4.19

EXAMPLE 4.47

Graph the function $g(x) = \dfrac{x - 1}{x}$.

SOLUTION Note that this function is not defined when $x = 0$. A partial table of values for this function follows.

x	-5	-4	-3	-2	-1	-0.4	-0.1
$g(x)$	1.2	1.25	$1.3\overline{3}$	1.5	2	3.5	11

x	0	0.1	0.4	1	2	3	4	5
$g(x)$	DNE*	-9	-1.5	0	0.5	$1.6\overline{6}$	0.75	0.80

*DNE means Does Not Exist.

The graph is shown in Figure 4.19. Since this graph is not defined when $x = 0$, it does not have a y-intercept. It has one x-intercept, $x = 1$.

Another interesting graph is shown by Example 4.48.

EXAMPLE 4.48

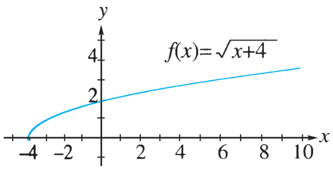


Figure 4.20

Graph the function $f(x) = \sqrt{x+4}$.

SOLUTION You can only take the square root of a nonnegative number. Thus, the function is only defined for $x \geq -4$, and the domain of the function is $\{x : x \geq -4\}$. A partial table of values for this function follows. Its graph is shown in Figure 4.20.

x	-4	-3	-2	-1	0	1	2	3	4	5
y	0	1	$\sqrt{2} \approx$ 1.414	$\sqrt{3} \approx$ 1.732	2	$\sqrt{5} \approx$ 2.236	$\sqrt{6} \approx$ 2.449	$\sqrt{7} \approx$ 2.646	$\sqrt{8} \approx$ 2.828	3

The y-intercept is $y = 2$ and the only x-intercept is $x = -4$.

APPLICATION BUSINESS

EXAMPLE 4.49

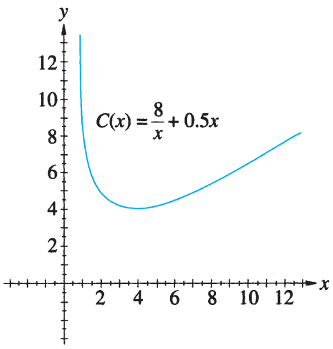


Figure 4.21

The total cost to manufacture x items of a certain product is given by $C(x) = \dfrac{8}{x} + 0.5x$, where $x > 0$. Set up a partial table of values and sketch the graph of this function.

SOLUTION This function is only defined for $x > 0$. A partial table of values is shown. Its graph is shown in Figure 4.21.

x	1	2	3	4	5	6	7	8	9	10
y	8.5	5	$4\frac{1}{6}$	4	4.1	$4\frac{1}{3}$	$4\frac{9}{14}$	5	$5\frac{7}{18}$	5.8

VERTICAL LINE TEST

A graph can provide us with an easy test to see if the equation that has been graphed is a function. This test is called the **vertical line test**.

VERTICAL LINE TEST

The vertical line test states that a graph is the graph of a function if no vertical line intersects the graph more than once.

To use the vertical line test, you look at a graph and determine if there are any places where a vertical line would intersect the graph more than once. If you cannot find any such places, then this is the graph of a function. If you can, then this is not the graph of a function. Figures 4.22a and 4.22b give some examples. The graph in Figure 4.22a is not the graph of a function, because the vertical line intersects the graph three times. On the other hand, the graph in Figure 4.22b is the graph of a function because no vertical line can intersect the graph more than once.

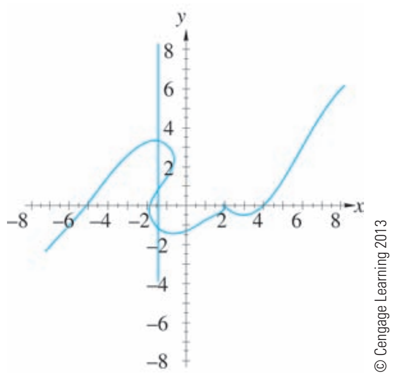


Figure 4.22a

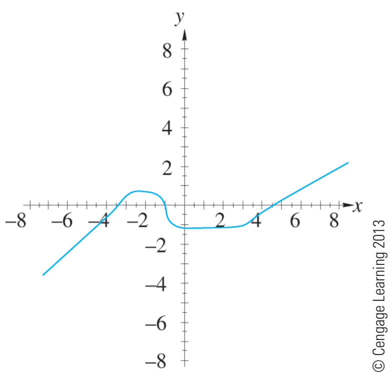


Figure 4.22b

APPLICATION BUSINESS

EXAMPLE 4.50

The concentration in parts per million (ppm) of a certain pollutant m mi from a certain factory is given by

$$C(m) = \frac{50}{m^2 + 4}, \; m \geq 0$$

Set up a partial table of values and sketch the graph of this function.

SOLUTION This function is defined for values of $m \geq 0$. A partial table of values follows. The graph of the function is shown in Figure 4.23. Notice that this graph passes the vertical line test.

m	0	1	2	3	4	5	6
$C(m)$	12.5	10	6.25	3.85	2.5	1.72	1.25

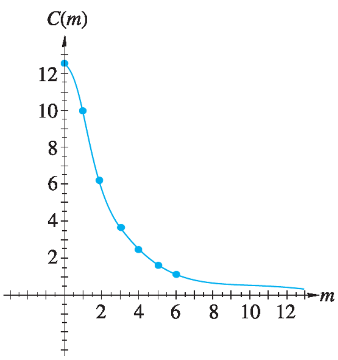


Figure 4.23

USING A GRAPH TO DETERMINE THE DOMAIN AND RANGE OF A FUNCTION

We just used the vertical line test to determine if a graph represents a function. We can use a variation of that test to help us determine the domain of the function. To determine the domain of a function, think of vertical lines drawn from each point on the x-axis. If a vertical line intersects the graph of a function, then the x-coordinate determined by that line is in the domain. If the vertical line does not intersect the graph, then the x-coordinate is not in the domain.

We can also use a graph to help us determine the range of a function. To determine the range of a function, think of horizontal lines drawn from each point on the y-axis. If a horizontal line intersects the graph, then the y-coordinate determined by that line is in the range. If it does not intersect the graph, then the y-coordinate is not in the range.

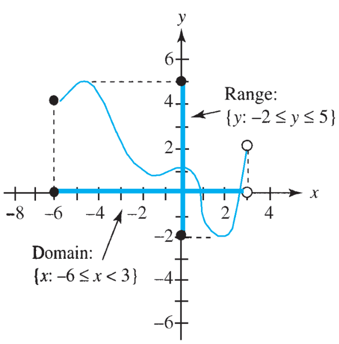


Figure 4.24

EXAMPLE 4.51

Determine the domain and range of the function shown by the graph in Figure 4.24.

SOLUTION Because the graph extends from $x = -6$ to, but not including, $x = 3$, the domain is $\{x : -6 \leq x < 3\}$. Since the graph extends from $y = 5$ to $y = -2$, the range is $\{y : -2 \leq y \leq 5\}$. Notice that the range was not determined by the y-values when $x = -6$ and when $x = 3$.

EXAMPLE 4.52

Show how to use the graph of a function to determine the domain and range of the functions in Example 4.19, $y = x + 2$ and $y = \dfrac{x^2 - 4}{x - 2}$.

SOLUTION The graph of $y = x + 2$ is shown in Figure 4.25a and the graph of $y = \dfrac{x^2 - 4}{x - 2}$ is shown in Figure 4.25b.

The graph of the equation $y = x + 2$ is the line shown in Figure 4.25a. Although we cannot see the entire graph, we know that this is a line and are led to believe that any vertical line will intersect this graph exactly once. Similarly, it appears as if any horizontal line will intersect the graph. Hence, the domain and range are both the set of real numbers.

Look at the graph of $y = \dfrac{x^2 - 4}{x - 2}$ in Figure 4.25b. There appears to be a "hole" in the graph at (2, 4). Because the denominator is 0 when $x = 2$, the domain of this function is $\{x : x \neq 2\}$. If you draw a vertical line through the "hole," it will hit the x-axis at $x = 2$. This graphically shows that $x = 2$ is not in the domain of this function. If you draw a horizontal line through the "hole" at (2, 4), it hits the y-axis at $y = 4$. This leads us to conclude that the range is $\{y : y \neq 4\}$.

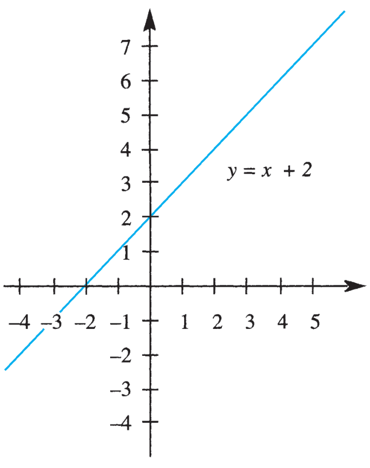


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Figure 4.25a

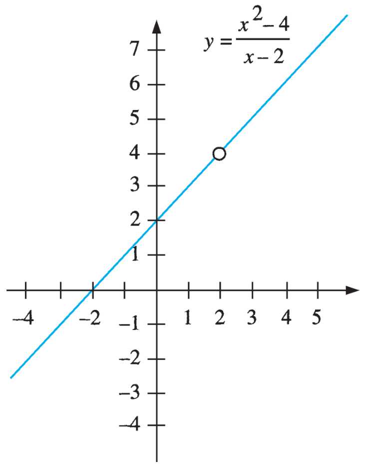

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Figure 4.25b

CAUTION You must be careful when you use the graph of a function to determine the domain and the range. Most graphs that you will draw or see show only part of the graph. You must use your algebra skills, your experience, *and* the graph to fully determine the range.

The next section will further explore how you can determine the domain and range with the aid of a graphing calculator.

EXERCISE SET 4.4

For each of Exercises 1–10, (a) determine the intercepts, (b) calculate the slope, and (c) draw the graphs of the linear equations.

1. $y = x$

2. $y = 2x$

3. $y = -3x$

4. $y = \frac{1}{2}x - 3$

5. $y = -4x + 2$

6. $x + y = 2$

7. $x - y = 2$

8. $2x + y = 1$

9. $3x - 6y = 9$

10. $3x - 6y = -6$

In Exercises 11–20, (a) set up a table of values, (b) graph the given functions, and (c) determine the domain and range of each function.

11. $y = x^2$

12. $y = x^2 + 3$

13. $y = x^2 - 2$

14. $y = x^2 + 2x + 1$

15. $y = x^2 - 6x + 9$

16. $y = -x^2 + 2$

17. $y = \dfrac{1}{x}$

18. $y = \dfrac{1}{x + 3}$

19. $y = x^3$

20. $y = 3\sqrt{x}$

In Exercises 21–26, set up a table of values and graph each of the equations.

21. $y^2 + x^2 = 25$

22. $y = \sqrt{25 - x^2}$

23. $y = |x + 2|$

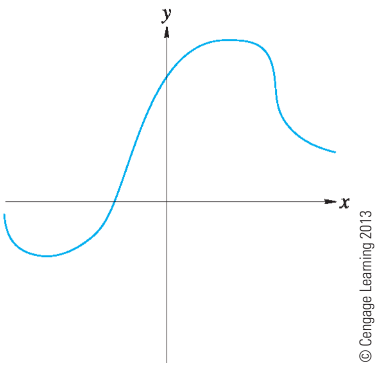
24. $\dfrac{x^2}{4} + \dfrac{y^2}{9} = 1$

25. $\dfrac{x^2}{4} - \dfrac{y^2}{9} = 1$

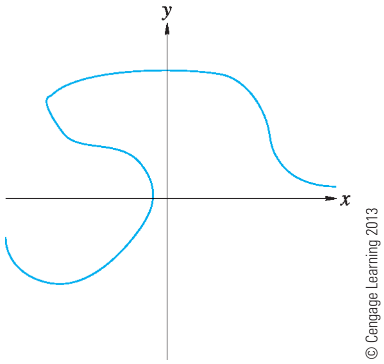
26. $y = x^3 - x^2$

Which of the graphs in Exercises 27–30 are functions?

27.



28.



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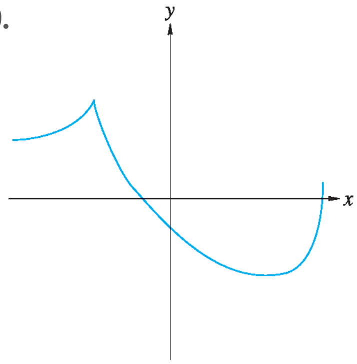

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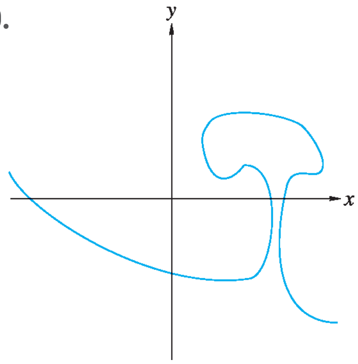
30.



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Solve Exercises 31–34.

31. ***Ecology*** An ecologist is investigating the effects of air pollution from an industrial city in the plants surrounding the city. She estimates that the percentage of diseased plants is given by

$$p(k) = \frac{25}{2k + 1}$$

(a) Set up a table of values and graph the equation for $0 \leq k \leq 9$.

(b) Does your graph pass the vertical line test?

32. ***Ecology*** A second ecologist thinks that the function in the previous problem does not give the correct percentages. She thinks that the percentage of diseased plants k km from the city is given by

$$p(k) = \frac{25}{\sqrt{k + 1}}$$

Make a table of values and graph the function for $0 \leq k \leq 9$.

33. ***Business*** Based on past experience, a company decides that the weekly demand (in thousands) for a new microwaveable food product is $d(p) = -p^2 + 2.5p + 10$, where p is the price (in dollars) of the product.

(a) What is the domain of this function?

(b) Make a table of values for each \$0.25 change in price for $0 < p \leq \$2.00$.

(c) Sketch a graph of this function.

(d) What appears to be the price that will result in the most sales?

34. ***Automotive technology*** In a chrome-electroplating process, the mass m in grams of the chrome plating increases according to the formula $m = 1 + 2^t/2$, where t is the time in minutes.

(a) Set up a table of values and graph the equation for $1 \leq t \leq 15$.

(b) How long does it take to form 100 g of plating?

[IN YOUR WORDS]

35. Describe how to use the graph of a function to help determine the range of the function.

36. Explain what the slope of a line tells you.

4.5 CALCULATOR GRAPHS AND SOLVING EQUATIONS GRAPHICALLY

A graphing calculator or computer can remove much of the drudgery of plotting equations by hand. In this section we will use a graphing calculator to demonstrate some mathematical concepts and to show how it can relieve you of some work. Moreover, we will show you how to interpret the information displayed on the machine.

Examples in this section were run on a Texas Instruments TI-84 graphing calculator. However, you could have used a different graphing calculator or a computer program. Rather than teach you how to write a graphing program for a computer, we will assume you have access to a program that can be used to graph functions. Some, such as Excel, are computer "spreadsheets" that have graphing capabilities. Other more specialized programs include Mathematica, Derive, and Maple, all of which are classified as computer algebra systems.

A purpose of this book is not to explain how to use the graphing capabilities of these programs, but to help you use these computer programs to graph curves and to interpret their graphs.

USING A GRAPHING CALCULATOR

We begin with a discussion on using graphing calculators. The examples and the instructions in this book are meant to supplement the user's guide for your calculator, not to replace it. You should always consult the user's guide to get answers to "how to" questions.

All examples described in this section used a Texas Instruments TI-84 graphing calculator. Directions for using some other calculators can be found on the web site for the text.

The first equation to graph will be of a straight line, $y = 2x + 7$. Before we begin, we must determine the domain of this function and what portion we want to see. The viewing range of this function is partially determined by the allowable values of y that can appear on the calculator's screen.

EXAMPLE 4.53

Use a calculator to sketch the graph of $f(x) = 2x + 7$.

SOLUTION We begin by entering the function. First, press **Y =** and then type 2x + 7. For the x key you can press the key labeled **X, T, θ, n**. What you see should look like Figure 4.26a.

The domain of this function is all real numbers. We can only view a portion of the domain in the viewing window. We select $-5 \le x \le 4$.

To set the size of the viewing window press the **WINDOW** key. The part of the x-coordinate system that will be shown on the screen is indicated by Xmin (the smallest value of x displayed on the screen), Xmax (the corresponding largest x-value), and Xscl (the x-axis scale or, more precisely, the distance between tick marks on the x-axis). The y-coordinate is similarly restricted by Ymin, Ymax, and Yscl. For this function we use the values in Figure 4.26b.

To display the graph press **GRAPH**. The result is shown in Figure 4.26c.

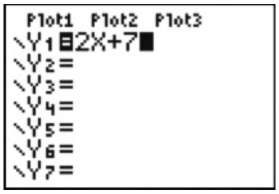


Figure 4.26a

Figure 4.26b

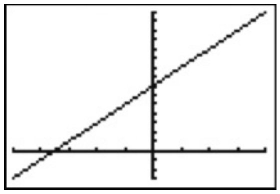


Figure 4.26c

EXAMPLE 4.54

Use a graphing calculator to sketch the graph of $g(x) = 0.2\,x^2 + 15$

SOLUTION Again, press **Y =**. If you want to erase the previous function, press **CLEAR**. If you want to graph this new function and the previous function, press **ENTER** and then type $0.2x^2 + 15$. To get x^2 you can type either **X, T, θ, n** **x^2** or **X, T, θ, n** **^** **2**. When you press graph you get the result shown in Figure 4.27a.

This is not very satisfactory because we don't see any of the graph. This is because we used the same window settings as in the previous example. There are several ways to correct this. One way is to decide what x-values you want to see in the viewing windows. Suppose for this example, you wanted to see the graph when $-6 \le x \le 5$. Press **WINDOW** and set xMin $= -6$ and xMax $= 5$. If you then use ZoomFit by pressing **ZOOM** and holding down the **▼** key until you see 0 : ZoomFit, as shown in Figure 4.27b. Pressing **ENTER** graphs the function and gives Figure 4.27c.

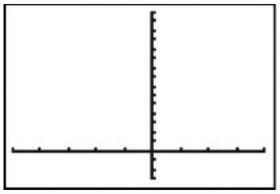


Figure 4.27a

Figure 4.27b

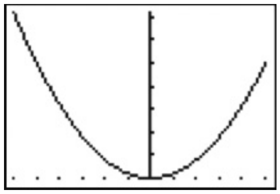


Figure 4.27c

Whenever we are not using the default window settings on a calculator, we will adopt the notation [Xmin, Xmax, Xscl] × [Ymin, Ymax, Yscl]. Thus, the viewing window used in Figure 4.27c is $[-6, 5, 1] \times [15, 22.2, 1]$. If both Xscl and Yscl are 1 we may shorten this to $[-6, 5] \times [15, 22.2]$. The default window settings, written ZStandard for Zoom Standard, are $[-10, 10, 1] \times [-10, 10, 1]$.

EXAMPLE 4.55

Use a graphing calculator to sketch the graph of $f(x) = \dfrac{x^2 + 1}{x + 2}$.

SOLUTION In graphing this function you must be careful to insert enough parentheses and to insert them in the right spots. Remember that everything over the fraction bar is one group and everything under the fraction bar is another group. Parentheses are used to group terms. So, we end up writing the equation as

$$y = (x^2 + 1)/(x + 2)$$

When this is graphed on a TI-83 or TI-84 in the standard viewing window. you get the graph in Figure 4.28a. (You get the standard viewing window by pressing ZOOM 6 [6 : ZStandard].)

Sometimes, as in Figure 4.28a, the calculator draws in some points (usually shown as vertical lines) that are really not part of the graph. This may also happen with some computer graphing software. It may be possible to adjust the viewing window, as in Figure 4.28b, to eliminate these extraneous points. Another possibility is to set the graph format to draw individual dots (Dot on a TI-83 or TI-84) rather than connecting the dots. When this is done you get a graph like the one in Figure 4.28c. On a TI-83 or TI-84, you get the calculator to draw individual dots by pressing MODE ▼ ▼ ▼ ▼ ► ENTER .

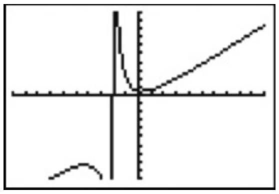
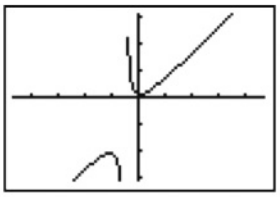
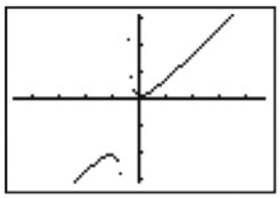


Figure 4.28a



[−18.8, 18.8, 4] x [−12.4, 12.4, 4]

Figure 4.28b



[−18.8, 18.8, 4] x [−12.4, 12.4, 4]

Figure 4.28c

SOLVING EQUATIONS GRAPHICALLY

To solve an equation graphically means to locate the x-intercepts of the graph. For a function these points are called the **zeros** of the function because these are the values of x where $f(x) = 0$. For other places these are called the *solutions* or *roots* of the equation.

STEPS TO SOLVING AN EQUATION GRAPHICALLY

1. Write the equation in the form [expression in x] $= 0$ or $f(x) =$ [expression in x].
2. Graph $y =$ [expression in x].
3. Find the x-intercepts of the graph. These are the solutions of the equation.

EXAMPLE 4.56

Graphically approximate roots of $x^2 - 3x - 1 = 0$.

SOLUTION First, set $f(x) = x^2 - 3x - 1$. Then set up a partial table of values for the function f.

x	-2	-1	0	1	2	3	4	5
$y = f(x)$	9	3	-1	-3	-3	-1	3	9

Graphing the curve determined by these points, we get the curve in Figure 4.29. From the graph we can see that there appear to be roots at $x = -0.25$ and $x = 3.25$. If we evaluate the function at these two values, we get $f(-0.25) = -0.1875$ and $f(3.25) = -0.1875$. This shows two things. First, it shows that -0.25 and 3.25 are *not* roots of this function, since $f(x) \neq 0$ at either of these two points. Second, it does show that the roots of this function are "close" to -0.25 and 3.25.

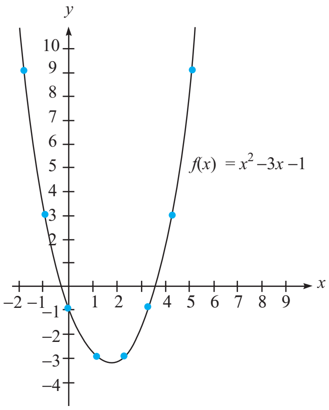


Figure 4.29

If we wanted more accurate approximations of the roots of the function in Example 4.56, we could substitute different values for x until we got an approximation that was as accurate as we wanted. In a later chapter, we will find out how to determine the exact roots to this function. In the next example, we will show how to use a graphing calculator to approximate this function's roots.

EXAMPLE 4.57

Use a graphing calculator to find the approximate roots of $f(x) = x^2 - 3x - 1$.

SOLUTION This is the same function we graphed in Example 4.56. We already know that the roots are near $x = -0.25$ and $x = 3.25$. When you graph the function using the calculator's default window settings, you obtain the graph in Figure 4.30a on a TI-83 or TI-84.

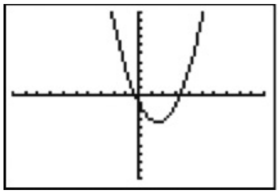


Figure 4.30a

As you can see from this figure, the graph crosses the x-axis at two points; these are the x-intercepts or roots. On a TI-83 or TI-84 press **2nd** **TRACE** [CALC] **2** [2:zero]. [Note: Some calculators, here the TI-83 or TI-84, use the term "zero" rather than root.] This function has two roots, and we will find the one on the left first. At the bottom left of the calculator screen you should see the question `Left Bound?` as in Figure 4.30b. Use the ◀ or ▶ to move the cursor to the *left* of the root you are trying to find. When the cursor is left of this root press **ENTER** and you see something like Figure 4.30c.

Now, you should see the question `Right Bound?` at the bottom left of the screen. Use the ▶ key to move the cursor to the *right* of the root you are trying to find as in Figure 4.30c. Again, press **ENTER**.

You should now see the question `Guess?`. Move the cursor near the x-intercept and press **ENTER** a third time. Shortly, you should see Figure 4.30d on your calculator screen. This indicates that the root is $x \approx -0.3024456$. (A TI-83 gives an answer of $x \approx -0.3027756377$.) The screen also displays $y = 0$.

Now, approximate the other root of this function. Press **2nd** **TRACE** [CALC] **2** [2:zero] and repeat the process as shown in Figure 4.30e–Figure 4.30g. The second root is $x \approx 3.3027756$. (Notice in the text that the TI-83 found that the root is $x \approx 3.3027756377$.)

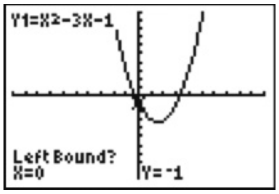


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Figure 4.30b

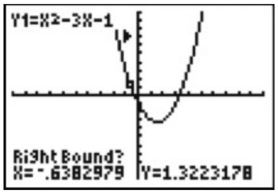

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Figure 4.30c

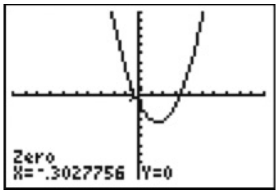

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Figure 4.30d

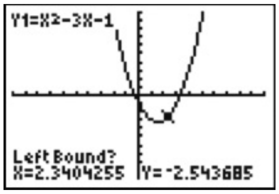

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Figure 4.30e

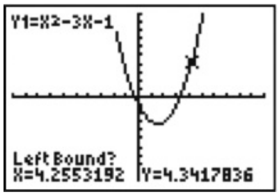

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Figure 4.30f

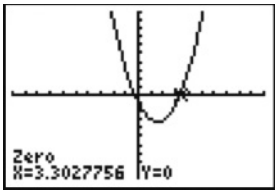

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Figure 4.30g

EXAMPLE 4.58

Use a graphing calculator or computer graphing software with "zoom" capability to determine any x-intercepts of $y = x^2 + 1.5$.

SOLUTION The graph of this function is shown in Figure 4.31. As you can see, the graph does not cross the x-axis. Thus, this function does not have any real roots.

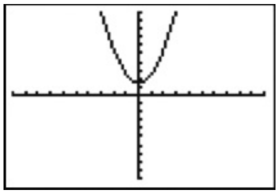
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Figure 4.31

The graph of an equation can tell the approximate roots or, as in Example 4.58, if the equation has no real roots. There are many times when we will need more accurate values of roots and a quicker procedure than that of graphing the equation. In later chapters, we will learn some alternative methods that can be used to find the roots of an equation.

In Section 4.2, we discussed the inverse of a function. We said that two functions, f and g, are inverses of each other if $f[g(x)] = x$ for every value of x in the domain of g and $g[f(x)] = x$ for every value of x in the domain of f. We also said that if g was the inverse of f, we would write $g(x) = f^{-1}(x)$.

The graph of a function can be used to determine if the function has an inverse and to sketch the graph of the inverse function. You may remember that the vertical line test indicates that if no vertical line intersected a graph more than once, then the graph represents a function. A similar test can be used to determine if a function has an inverse.

HORIZONTAL LINE TEST

The **horizontal line test** states that a function has an inverse function if no horizontal line intersects its graph at more than one point.

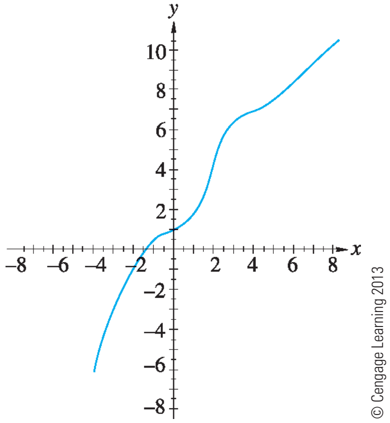

Figure 4.32a

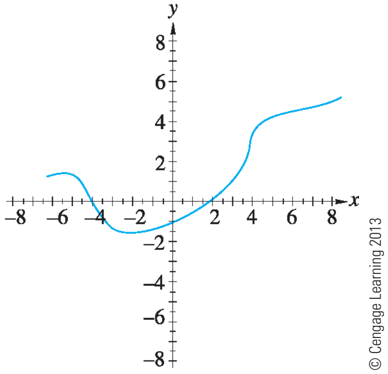

Figure 4.32b

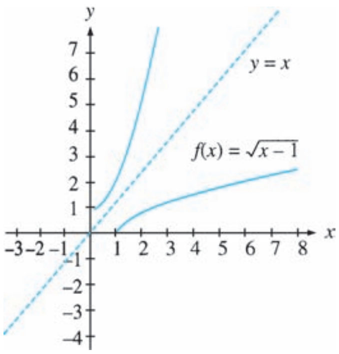

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Figure 4.33

For example, Figures 4.32a and 4.32b show the graphs of two functions. The function in Figure 4.32a has an inverse function, because it is not possible to draw a horizontal line that will intersect the graph at more than one point. The function in Figure 4.32b does not have an inverse because it is possible to draw a horizontal line that will intersect the graph twice. In fact, the x-axis intersects the graph in Figure 4.32b in at least two places.

Now let's use our graphing ability to graph the inverse of a function. The graph of $f(x) = \sqrt{x-1}$ is shown as the lower curve in Figure 4.33. Using the horizontal line test, we can see that this function has an inverse. Now, draw the line $y = x$. (In Figure 4.33 this is the dashed line.) Suppose you placed a mirror on the line $y = x$. The image you would see is indicated by the upper curve in Figure 4.33.

What this essentially does is take any point (a, b) on the function and convert it to its mirror image (b, a). Remember that if (a, b) is a point on a function f, then $f(a) = b$; and if (b, a) is a point on some function g, then $g(b) = a$. But, since $f^{-1}[f(a)] = f^{-1}(b) = a$, then $f^{-1}(b) = g(b)$, and we can see that g must be f^{-1}.

EXAMPLE 4.59

Graph the inverse function of $y = x^3$.

SOLUTION The graph of $y = x^3$ is given in Figure 4.34a and the line $y = x$ is shown by a dashed line. The reflection of $y = x^3$ over the line $y = x$ is shown in Figure 4.34b and is the graph of $y = \sqrt[3]{x}$. If $f(x) = x^3$, then $f^{-1}(x) = \sqrt[3]{x}$.

APPLICATION ENVIRONMENTAL SCIENCE

EXAMPLE 4.60

In Example 4.50, we made the following partial table for the concentration in parts per million (ppm) of a certain pollutant m mi from a certain factory as given by the function

$$C(m) = \frac{50}{m^2 + 4}, \quad m \geq 0$$

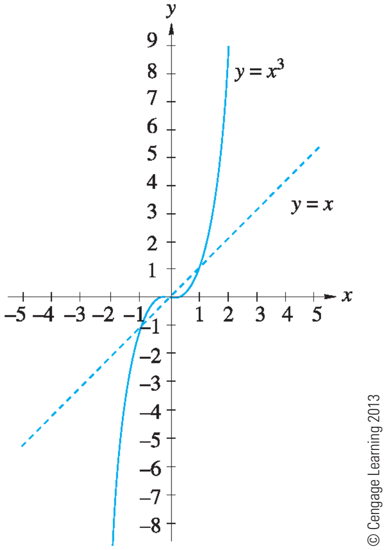


Figure 4.34a

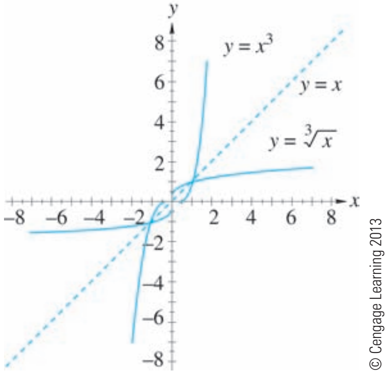


Figure 4.34b

m	0	1	2	3	4	5	6
$C(m)$	12.5	10	6.25	3.85	2.5	1.72	1.25

Next, we used our table to sketch the graph of this function.

(a) Use the table to sketch the graph of the inverse of this function.

(b) Use the table or the graph to estimate how far you are from the factory, if the concentration of the pollutant is 3.08 ppm.

(c) Determine the equation that describes the inverse function of C.

(d) Use your equation for C^{-1} to determine $C^{-1}(3.08)$.

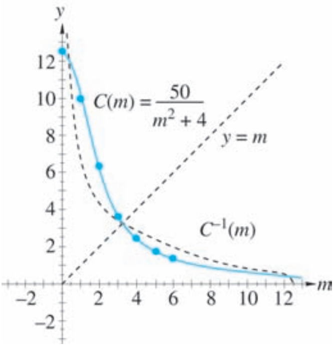


Figure 4.35a

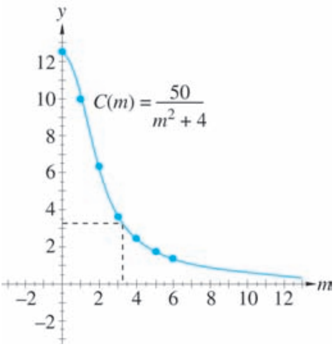


Figure 4.35b

SOLUTIONS

(a) The graph of the function $y = C(m)$ is shown in color in Figure 4.35a. The reflection of the graph over the line $y = m$ produces the inverse function shown by the dotted curve in the same figure.

(b) We will use the graph of the original function to determine $C^{-1}(3.08)$. Move up the vertical axis for the graph of $C(m)$ until you reach the point that is approximately at 3.08 on the vertical axis. Draw a horizontal line from this point until it reaches the graph of C. Draw a vertical line until it crosses the horizontal axis. This should be near 3.5 as shown in Figure 4.35b. Thus, $C^{-1}(3.08) \approx 3.5$ and we conclude that the pollutant was collected about 3.5 mi from the factory.

(c) As described in Section 4.4, we will let $y = \dfrac{50}{m^2 + 4}$. Solving for m, we obtain $m = \sqrt{\dfrac{50}{y} - 4}$, for $0 < y < 12.5$. Thus, $C^{-1}(m) = \sqrt{\dfrac{50}{m} - 4}$, for $0 < m \le 12.5$.

EXAMPLE 4.60 (Cont.)

(d) Substituting 3.08 in the formula for C^{-1}, we obtain

$$C^{-1}(3.08) = \sqrt{\frac{50}{3.08} - 4} \approx 3.50$$

Thus, the formula and the graph give us approximately the same answers.

APPLICATION ENVIRONMENTAL SCIENCE

EXAMPLE 4.61

Use a graphing calculator to rework parts of Example 4.60.

(a) Sketch the graph of $C(m) = \frac{50}{m^2 + 4}$, $m \geq 0$.

(b) Use the graph and your calculator to estimate how far you are from the factory if the concentration of the pollutant is 3.08 ppm.

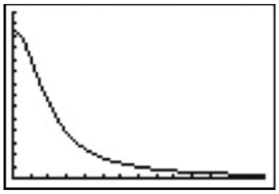


Figure 4.36

SOLUTIONS

(a) You must make two changes before you begin graphing. First, you must replace $C(m)$ with y. Second, your calculator will only allow you to use certain variables when you graph a function. At present the only variable you can use is x. So, rewrite the given function as $y = \frac{50}{x^2 + 4}$, $x \geq 0$. Don't forget the correct use of parentheses. You will want to graph this as $y = \frac{50}{(x^2 + 4)}$. Also, to get the graph to look like the one in Figure 4.35b, change the viewing window to $[0, 14, 1] \times [0, 14, 1]$. The result is the graph shown in Figure 4.36.

(b) We want to know when the concentration of the pollutant is 3.08 ppm. This means we want to know the value of x when $y = 3.08$. Press TRACE and move the cursor until you have a y-value near 3.08. On this particular calculator, the y-values change from around 2.98 to 3.17. Zooming in around one of these points and using the trace function, we see that when $y \approx 3.08$, then $x \approx 3.4976$. Once again, we conclude that the pollutant was collected about 3.5 mi from the factory.

EXERCISE SET 4.5

In Exercises 1–26, graph each of these functions with a calculator or computer. Compare the machine's graph with the graph you made of the same functions in Section 4.4.

1. $y = x$
2. $y = 2x$
3. $y = -3x$
4. $y = \frac{1}{2}x - 3$
5. $y = -4x + 2$
6. $x + y = 2$
7. $x - y = 2$
8. $2x + y = 1$
9. $3x - 6y = 9$
10. $3x - 6y = -6$
11. $y = x^2$
12. $y = x^2 + 3$
13. $y = x^2 - 2$
14. $y = x^2 + 2x + 1$
15. $y = x^2 - 6x + 9$
16. $y = -x^2 + 2$
17. $y = \frac{1}{x}$
18. $y = \frac{1}{x + 3}$
19. $y = x^3$

20. $y = \sqrt[3]{x}$

21. $y^2 + x^2 = 25$ (HINT: Solve for y, then graph two equations.)

22. $y = \sqrt{25 - x^2}$

23. $y = |x + 2|$ (HINT: Use the **Abs** key. Read you manual to see how to access this on your computer.)

24. $\dfrac{x^2}{4} + \dfrac{y^2}{9} = 1$ (HINT: Solve for y, then graph two equations.)

25. $\dfrac{x^2}{4} - \dfrac{y^2}{9} = 1$ (HINT: Solve for y, then graph two equations.)

26. $y = x^3 - x^2$

The equations in Exercises 27–44 all have at least one root between −10 and 10. Write each equation in the form y = f (x). Graph each equation to find the approximate value of the roots. If possible, use a graphing calculator or a computer graphing program to help you.

27. $2x + 5 = 0$

28. $5x - 9 = 0$

29. $x^2 - 9 = 0$

30. $4x^2 - 10 = 0$

31. $x^2 = 5x$

32. $x^2 = 4x - 3$

33. $x^2 + 5x - 3 = 0$

34. $3x^2 + 5x - 3 = 0$

35. $20x^2 + 21x = 54$

36. $x^4 - 4x^3 - 25x^2 + 5x + 6 = 0$

37. $\sqrt{x + 4} = 0$

38. $\sqrt{x - 2.5} = 0$

39. $\sqrt{x + 1} = 3$

40. $\sqrt{x - 3} = 2$

41. $\dfrac{x}{x + 1} = -3$

42. $\dfrac{x}{x - 2} = 5$

43. $\dfrac{x}{x - 2} = 2 - x^2$

44. $\dfrac{x^2 + 5}{x - 3} = x^2 - 2$

In Exercises 45–60, graph each of the functions with a calculator or on a computer. Use the graph to help determine the domain and range of each function.

45. $y = x^2 - 4x$

46. $y = x^2 + 6x$

47. $y = x^2 - 10x + 37$

48. $y = 0.1x^2 - 20$

49. $y = -x^2 - 4x$

50. $y = -x^2 + 6x$

51. $y = \sqrt{x + 5}$

52. $y = \sqrt{x - 5}$

53. $y = \sqrt{9 - x}$

54. $y = -\sqrt{5 - x}$

55. $y = \dfrac{x + 2}{x}$

56. $y = \dfrac{x}{x + 2}$

57. $y = \dfrac{x^2 - 1}{x + 2}$

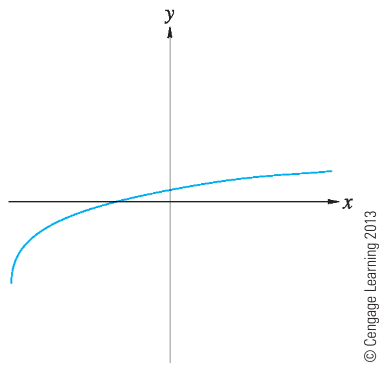
58. $y = \dfrac{x + 2}{x^2 + 3}$

59. $y = \dfrac{x^2 - 4}{x + 2}$

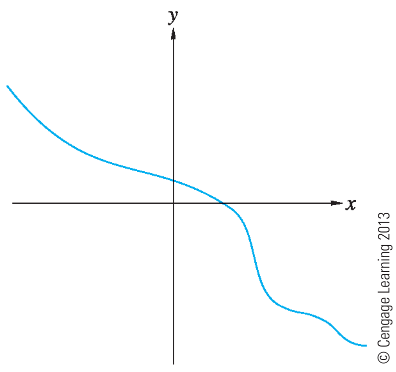
60. $y = \dfrac{x + 2}{x^2 - 4}$

Use the horizontal line test to determine whether or not the functions graphed in Exercises 61–64 have inverse functions.

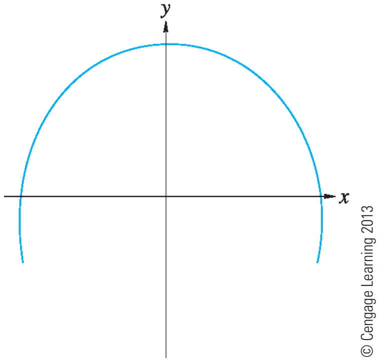
61.



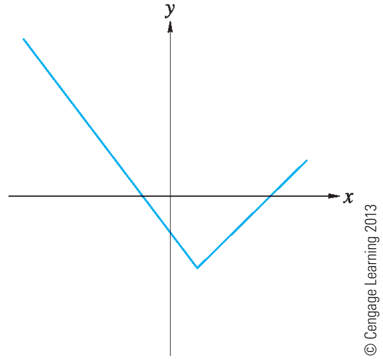
62.



63.

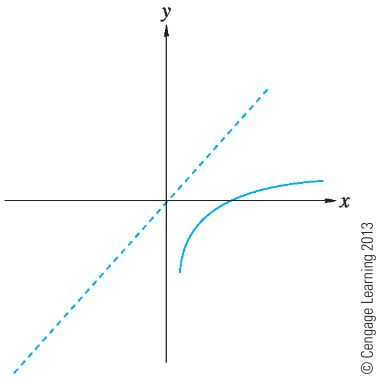


64.

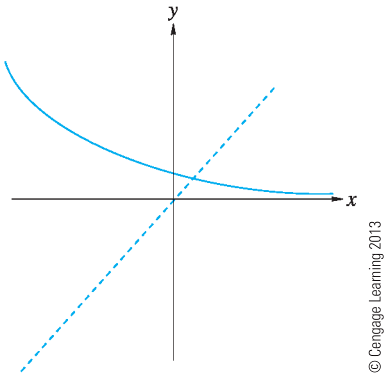


Each of the functions in Exercises 65–68 has an inverse. Sketch the graph of the inverse of each function.

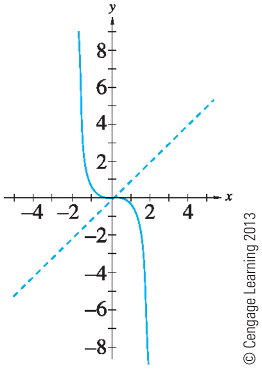
65.



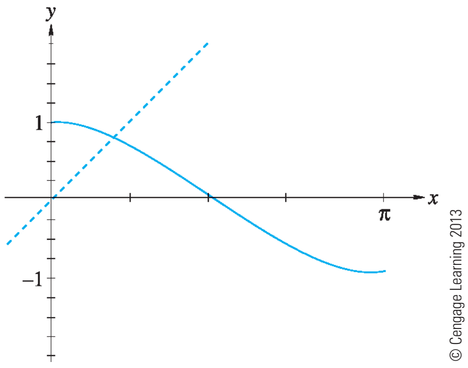
66.



67.



68.



Solve Exercises 69–72.

69. ***Automotive technology*** Study the following table. It shows that if the antifreeze content of a coolant is increased, the boiling point of the coolant is also increased.

Percent antifreeze in coolant	0	10	20	30	40	50	60	70	80	90	100
Boiling temperature °F	210	212	214	218	222	228	236	246	258	271	330

(a) What percent of antifreeze will be in the coolant if the boiling point of the system is 218°F?

(b) What percent of antifreeze will be in the coolant if the boiling point of the system is 236°F?

(c) What percent of antifreeze will be in the coolant if the boiling point of the system is 250°F?

70. ***Environmental science*** The population P of a certain species of animal depends on the number n of a smaller animal on which it feeds, with

$$P(n) = 5\sqrt{n} - 10$$

(a) Determine the inverse function for P.

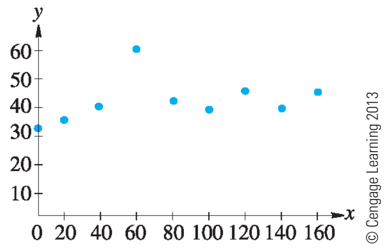
(b) If the population of P is 5, how many of the small animals are there?

(c) Graph the inverse function of P.

71. ***Construction*** This table contains the results of a series of drillings used to determine the depth of the bedrock at a building site. Drillings were taken along a straight line down the middle of the lot, where the building will be placed. In the table, x is the distance from the front of the building site and y is the corresponding depth. Both x and y are given in feet.

x	0	20	40	60	80	100	120	140	160
y	33	35	40	45	42	38	46	40	48

These ordered pairs have been plotted on the following graph. Does this graph have an inverse function?



72. ***Environmental science*** Suppose a cost-benefit model is given by

$$C(x) = \frac{6.4x}{100 - x}$$

where $C(x)$ is the cost in millions of dollars for removing x percent of a given pollutant.

(a) What is the cost of removing 30% of the pollutant?

(b) What is the cost of removing 60% of the pollutant?

(c) What is the inverse function of C?

(d) If a community can only spend \$12,000,000, what percent of the pollutant can be removed?

(e) Graph the given function and its inverse function.

[IN YOUR WORDS]

73. Explain the notation $[-15, 15, 2] \times [-10, 20, 4]$.

74. Write a description that tells how to zoom in and zoom out on your calulator.

75. Explain how to use your calculator to solve an equation.

76. On a sheet of paper, explain the horizontal line test. Do not look at the definition in the book.

4.6 USING A SPREADSHEET TO GRAPH AND SOLVE EQUATIONS GRAPHICALLY

A computer spreadsheet can remove much of the drudgery of plotting equations by hand. In this section we will use a spreadsheet to demonstrate some mathematical concepts and to show how it can relieve you of some work. Moreover, we will show you how to interpret the information displayed on the machine.

Using a spreadsheet to graph an equation of a function is much different than using a graphing calculator. In order to graph a function we must first create a set of ordered pairs for the function—just as you would by hand except the calculations are done by the computer. In Example 4.5 we showed how this is done.

The first equation to graph will be of a straight line, $y = 2x + 7$. Before we begin to graph this function, or any function, we must determine the domain of this function and what portion we want to see. The viewing range of this function is partially determined by the allowable values of y that can appear on the calculator's screen.

EXAMPLE 4.62

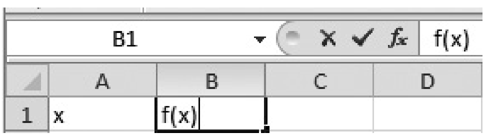
Use a spreadsheet to sketch the graph of $f(x) = 2x + 7$.

SOLUTION We will need two columns. We will use Column A for the x-values and Column B for the values of the function. Enter x in Cell A1 and enter f(x) in Cell B1. (See Figure 4.37a.) Notice the row above the column headings. In the left cell in says B1, because you have clicked on Cell B1. To the right of that cell is an equal sign and a cell with f(x) displayed. This cell, which we will call the "Formula Bar," shows what is entered in Cell B1, in this case, f(x).

Next, enter appropriate values in Column A beginning in Cell A2. If you wish to examine the behavior of the function in the interval between -5 and 4, then enter values between -5 and 4. The more values you enter, the more accurate your graph will be. (for right now, enter each number one by one in each cell. Later, we will find shortcuts for this tedious process.) (See Figure 4.37b.)

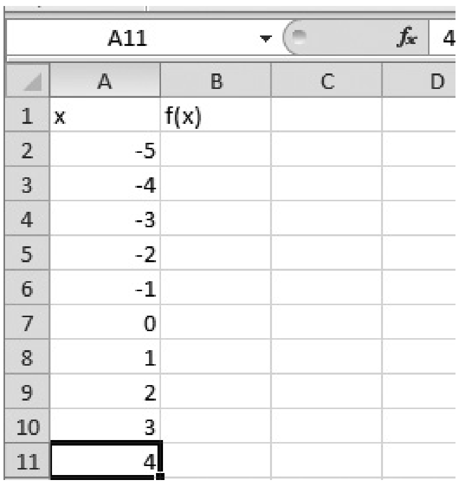
After entering the x-values, we now will calculate the values of the function. Move to Cell B2. (Place the mouse on Cell B2 and left click, or use the arrows to move to the highlighted cell.) To enter a formula in Cell B2, start with an equal sign and then type the remainder of the function, the part on the right-hand side of the equal sign. However, we want the value of the function displayed in Cell B2 to use the value of x in the cell immediately to the left, A2. So instead of using x in the formula we will use A2 as the variable. Enter =2*A2+7 in Cell B2. (See Figure 4.37c.)

It is important to understand that the value in Cell B2 is calculated using the value of x in Cell A2. Another way to explain this is that the value of Cell B2 is calculated using the value of x in the cell just to the left of it. This is called relative position—an important concept in spreadsheets. The formula that you want to have in Cell B3 is =2*A3+7, in B4 we want =2*A4+7, and so on. However we



	A	B	C	D
1	x	f(x)		

Figure 4.37a



	A	B	C	D
1	x	f(x)		
2	-5			
3	-4			
4	-3			
5	-2			
6	-1			
7	0			
8	1			
9	2			
10	3			
11	4			

Figure 4.37b

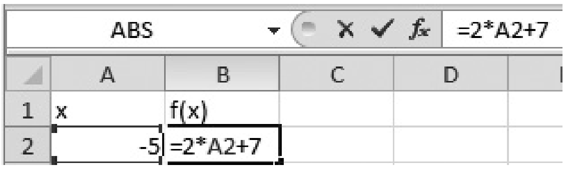


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Figure 4.37c

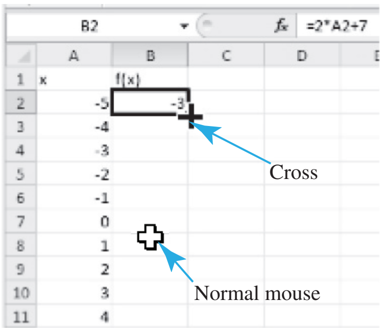

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Figure 4.37d

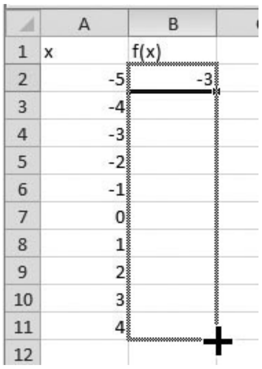

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Figure 4.37e

do not want to type all of these functions. There is an easier way. It can be done very easily by *copying* instead of going to each cell and typing the appropriate formula.

Place the cursor on the bottom right corner of Cell B2. (See Figure 4.37d.) As you move the cursor to that position, it changes shape. When you see the cursor change shape to a cross, press and hold the left mouse button down and drag the cursor down Column B through Row 11 (the extent of the x-values). When you release the mouse the formula in Cell B2 will have been copied down the page.

Figure 4.37e shows the first step after you click and drag the mouse down Column B. When you release the mouse, your spreadsheet should look like what appears in Figure 4.37f.

Examine the formulas in Column B. As the formulas were copied, the formulas changed so that the value used in the formula was also in the cell immediately to the left of the formula. In every case the calculation is 2 times the value in the cell immediately to the left plus 7. You can see this by clicking on one of the cells in column B. For example, if you click on cell B4 you will see =2*A4+7 in the Formula Bar.

We now have a limited table of values for the function $f(x) = 2x + 7$.

To graph the function, we need to highlight the table we just constructed. Left click on Cell A1 and hold the left mouse key down as you move the cursor to Cell B11. As you move the cursor, the cells should become highlighted. (See Figure 4.37g.)

Release the mouse key (leaving the cells highlighted) and move the cursor up to "Chart Wizard." Locating Chart Wizard will depend on the version of Excel that you are using. On some versions it will appear as a small icon on the top standard menu. In the version used for this text, it appears on one of the toolbars on the worksheet, as shown in Figure 4.37h. If this does not work, then look on the Insert menu and click on "Chart. . . ".

Click on Chart Wizard to begin the process of constructing a graph of the function. The first step (see Figure 4.37i), is to select the type of graph. Choose `XY (Scatter)` at the far right, as shown in Figure 4.37i, and choose the sub-type that places a smooth line between the data points (see Figure 4.37j).

Please note that we did not choose the chart type "Line" because this chart type will not always space the points on the x-axis correctly.

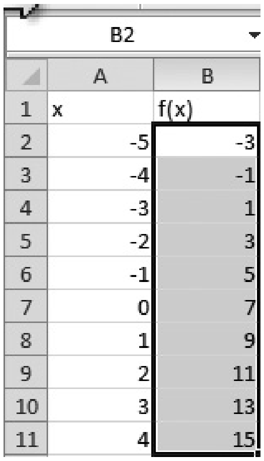

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Figure 4.37f

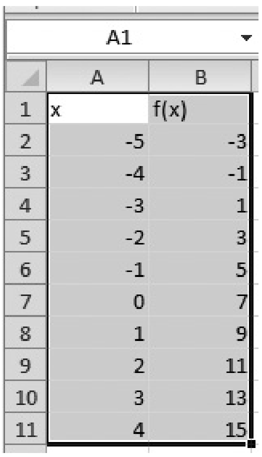

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Figure 4.37g

EXAMPLE 4.62 (Cont.)

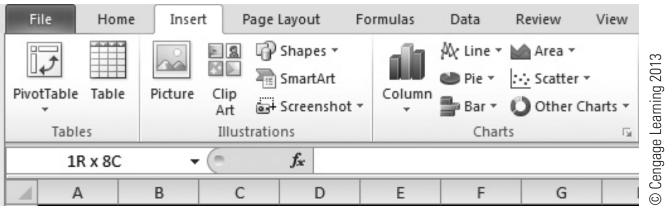

Figure 4.37h

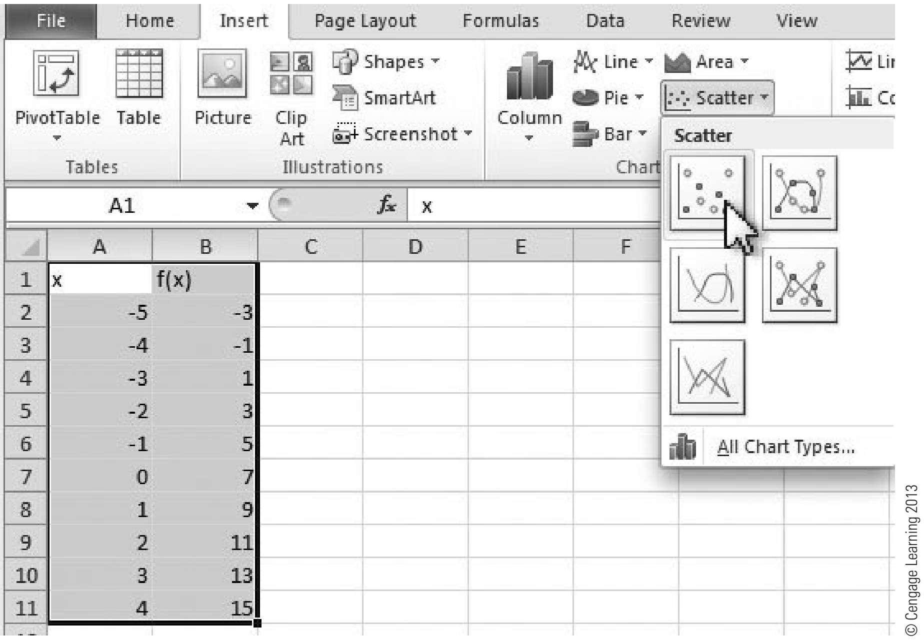

Figure 4.37i

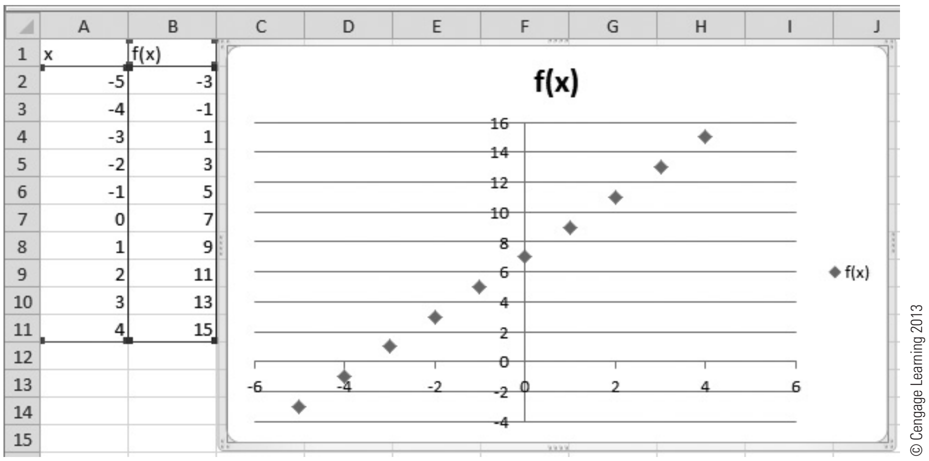

Figure 4.37j

Work your way through the steps to insert the graph into the current worksheet. The result is shown in Figure 4.37k. The graph in Figure 4.37j has been dragged so that it is next to the table of values in the worksheet.

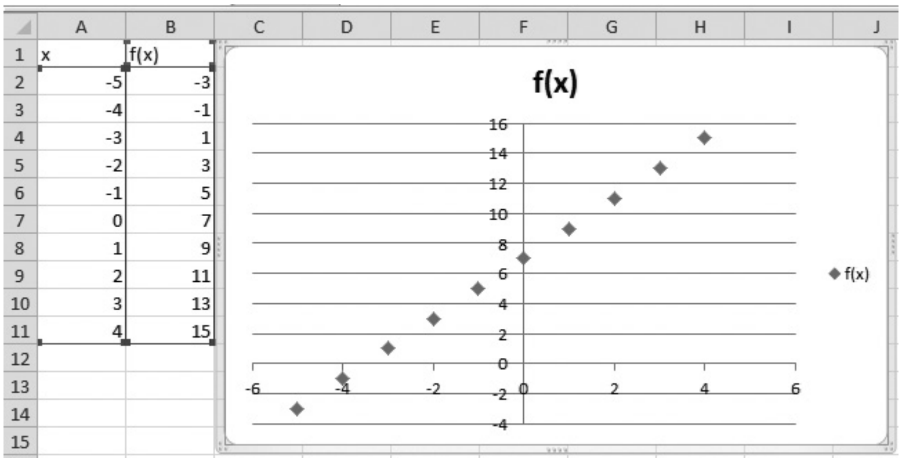

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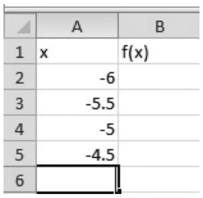
Figure 4.37k

EXAMPLE 4.63



	A	B
1	x	f(x)
2	-6	
3	-5.5	
4	-5	
5	-4.5	
6		

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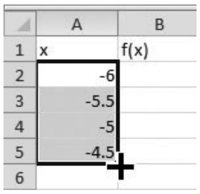
Figure 4.38a



	A	B
1	x	f(x)
2	-6	
3	-5.5	
4	-5	
5	-4.5	
6		

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Figure 4.38b

Use a spreadsheet to sketch the graph of $g(x) = 0.2x^2 + 15$.

SOLUTION The first step to graph this function is to make a table of values. But, we are going to need more values since this function is more involved than in the previous example. Instead of typing in each value of x we wish to use, we are going to let the spreadsheet "fill" in the values using the pattern we establish.

First enter x and f(x) in Cells A1 and B1, respectively. Then enter -6 in Cell A2, -5.5 in Cell A3, -5 in Cell A4, and -4.5 in Cell A5. (See Figure 4.38a.)

Since the pattern is obvious we are going to ask the spreadsheet to extend that pattern until we get to an x-value of 5. You do that by highlighting the first four cells of values (it doesn't have to be four) and then dragging down Column A. The first step is to highlight the Cells A2, A3, A4, and A5 and then place the cursor on the bottom right corner of that rectangle. (See Figure 4.38b.)

Drag the cursor down Column A until the value of 5 appears. When you release the mouse key, the column is filled with the values from -6 to 5 in increments of 0.5, as shown in Figure 4.38c.

Next, enter the function in Cell B2. The function has an exponent that must be entered using the caret. Enter =.2*A2^2+15, as shown in the Function Bar in Figure 4.38d.

After typing in the function, copy that cell down Column B, as far as there are values in Column A. After the formula is being copied down Column B, you should see a Column B filled with values as in Figure 4.38e.

When the table of values has been completed, the graph can be constructed. Follow the same steps as before: highlight the table, click on Chart Wizard, select `XY (Scatter)`, and follow the steps to produce the graph shown in Figure 4.38f.

You can change the appearance of the graph by selecting different parts of the graph. For example, to see some of the options for changing the x-axis, first left click on the values for the x-axis and then right click[1] while the x-axis is highlighted.[2]

[1] Macintosh computers do not come with a mouse that has a right button. Instead of right clicking, hold down the CONTROL key and click the mouse to achieve the same function. If the computer has a trackpad instead of a mouse, press the trackpad with two fingers.
[2] On some computers you need only double click on the numbers below the x-axis.

EXAMPLE 4.63 (Cont.)

(You can tell the x-axis is highlighted by the two dots on either end.) A right click will bring up a pop up menu. Select Format Axis (or Axis Options).

Selecting Format Axis will cause another pop up menu to be displayed. The options for changing the x-axis are shown on the tabs. The best way to learn what these options do is to try them.

You can do the same with the y-axis: left click on it to select it, right click to bring up the Format Axis (or Axis Options) menu, and so on. You can also change the format of the graph by right clicking on other parts, such as the Plot Area or the Chart Area.

One more important point while we are on this example. If you are not satisfied with the graph, you can change one or more of the x-values and the graph will change along with the values.

In Figure 4.38g the last x-value in the table, 5, was changed to 10. Press the TAB key and you will see the corresponding value for $f(x)$ change. Then click in the graph area and the graph will be extended to include these new coordinates. This is one way to change the graph once you've completed the process. In the next example, we will discuss another way to make the graph more flexible to meet your needs.

A	B
x	f(x)
-6	
-5.5	
-5	
-4.5	
-4	
-3.5	
-3	
-2.5	
-2	
-1.5	
-1	
-0.5	
0	
0.5	
1	
1.5	
2	
2.5	
3	
3.5	
4	
4.5	
5	

Figure 4.38c

SQRT =.2*A2^2+15

	A	B	C	D	E
1	x	f(x)			
2	-6	=.2*A2^2+15			

Figure 4.38d

	A	B
1	x	f(x)
2	-6	22.2
3	-5.5	21.05
4	-5	20
5	-4.5	19.05
6	-4	18.2
7	-3.5	17.45
8	-3	16.8
9	-2.5	16.25
10	-2	15.8
11	-1.5	15.45
12	-1	15.2
13	-0.5	15.05
14	0	15
15	0.5	15.05
16	1	15.2
17	1.5	15.45
18	2	15.8
19	2.5	16.25
20	3	16.8
21	3.5	17.45
22	4	18.2
23	4.5	19.05
24	5	20

Figure 4.38e

	A	B
1	x	f(x)
2	-6	22.2
3	-5.5	21.05
4	-5	20
5	-4.5	19.05
6	-4	18.2
7	-3.5	17.45
8	-3	16.8
9	-2.5	16.25
10	-2	15.8
11	-1.5	15.45
12	-1	15.2
13	-0.5	15.05
14	0	15
15	0.5	15.05
16	1	15.2
17	1.5	15.45
18	2	15.8
19	2.5	16.25
20	3	16.8
21	3.5	17.45
22	4	18.2
23	4.5	19.05
24	5	20

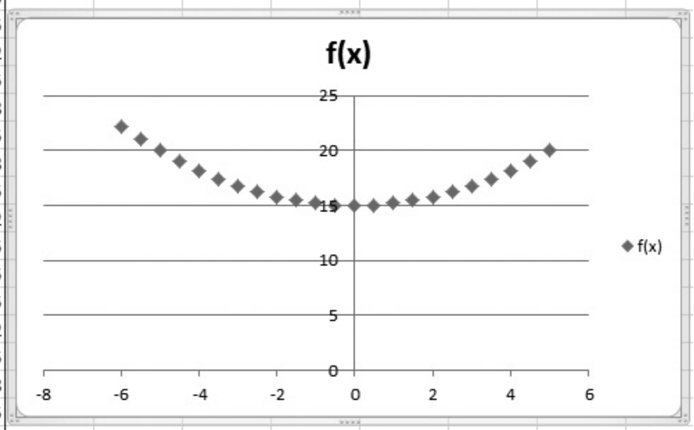


Figure 4.38f

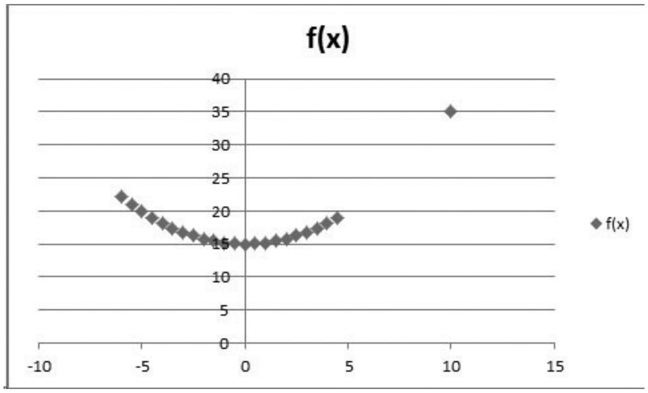


Figure 4.38g

EXAMPLE 4.64

Use a spreadsheet to sketch the graph of $f(x) = \dfrac{x^2 + 1}{x + 2}$.

SOLUTION This function is a little more complex than the previous two examples.

Choosing the appropriate values of x may be difficult.

We will start the process of graphing this function by formatting the worksheet. Enter `Initial Value of x` in Cell D1 and `Increment for x` in D2. Change the width of Column D to 1.4 inches and the width of Columns A and B to 0.4 inches. (See Figure 4.39a.)

Now we are ready to construct a table of values. For starters, enter -5 in Cell E1 and 0.5 in Cell E2. Obviously, that means we are going to construct a table of values starting at -5 with an increment of 0.5. To do that, we have to put a formula in Cell A1 that refers to Cell E1. Place =E1 in Cell A1. (See Figure 4.39b.)

We want the value in Cell A2 to be the value in Cell A1 *plus* the increment. So, enter =A1+\$E\$2 in cell A2. The dollars signs *anchor* the cell reference so that as we copy the formula down the page, the spreadsheet always adds the value in E2 to the cell immediately above it.

Now, copy the formula in Cell A2 down Column A until you get to row 110. Once you copy the formula, release the mouse key and the cells are filled with values from -5 to 44.5 in increments of 0.5.

Now type the function in Cell B1. Parentheses will be needed to ensure that the order of operations will be followed. Remember that everything over the fraction bar is one group and everything under the fraction bar is another group. The function is entered as = (A1^2+1) / (A1+2).

	A	B	C	D
1				Initial value of x
2				Increment for x

Figure 4.39a

	A	B	C	D	E
1	=E1			Initial value of x	-10
2				Increment for x	0.5

Figure 4.39b

EXAMPLE 4.64 (Cont.)

Copy the formula in Cell B1 down Column B through row 110 to match the x-values in Column A through row 110. One of the first things you notice is the Cell B17. As you begin to investigate, you realize that the value -2 will produce division by zero. The #### or #DIV/0! denotes an undefined number, one that cannot be displayed. We will address this problem later.

Construct the graph using the steps discussed in the previous two examples. You should get a result similar to the one shown in Figure 4.39c.

The area that we are concerned about occurs around $x = -2$. The many points beyond $x = 10$ don't help with the analysis of the graph. This is where the starting point and the increment come back into play. We can change the viewing window by changing the increment. Enter 0.2 in Cell E2.

What you get is not pretty! (See Figure 4.39d.) The values of the function around $x = -2$ cause the spreadsheet to expand the limits of the y-axis to very large values. So we must adjust the scale on both axes to eliminate that problem.

Click on the y-axis and choose Format Axis (or Axis Options) and then choose Scale. Enter the following values: Minimum: -12, Maximum: 12, Major unit: 4, and Minor unit: 4. Leave the entry Horizontal (Category) axis crosses at: 0.0 alone Use a similar process to change the scale on the x-axis to -10 to 10. At the same time, change the initial value of x to -10 by entering -10 in Cell E1.

The resulting graph will still have a line that attempts to connect the points at the value $x = -2$. But as you can see from the table of values, that really doesn't occur. So, we will override the connect the dots feature and get a more accurate representation of the graph. Left click on the graph to highlight the graph. When the graph is highlighted, right click and you will see the pop up menu "Format Data Series." Select the tab labeled "Patterns" and in the "Line" column select "None." If you wish, you can change the shape or size of the dots in the "Marker" column. When you close the "Format Data Series" menu, you should obtain a graph like the one shown in Figure 4.39e.

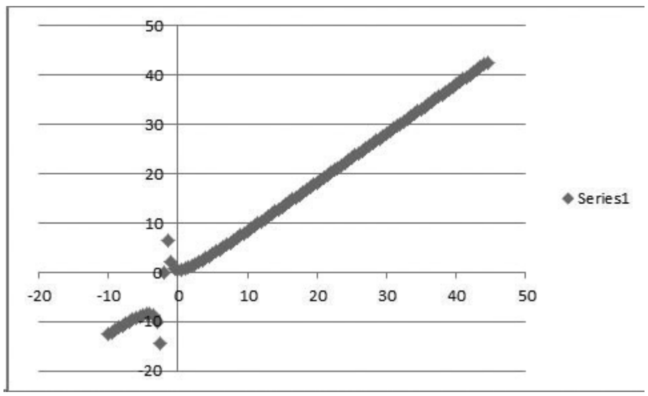


Figure 4.39c

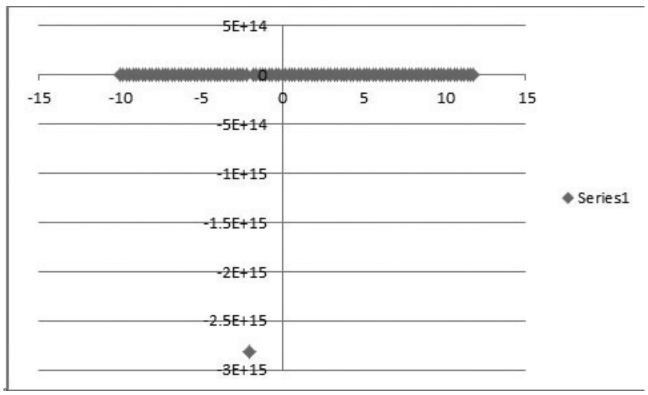


Figure 4.39d

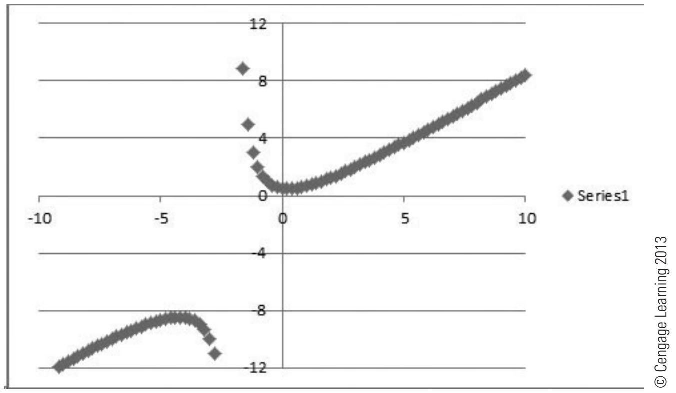


Figure 4.39e

SOLVING EQUATIONS GRAPHICALLY

To solve an equation graphically means to locate the x-intercepts of the graph. For a function these points are called the *zeros* of the function because these are the values of x where $f(x) = 0$. Other places these are called the *solutions* or *roots* of the equation.

STEPS TO SOLVING AN EQUATION GRAPHICALLY

1. Write the equation in the form [expression in x] $= 0$ or $f(x) =$ [expression in x].
2. Graph $y =$ [expression in x].
3. Find the x-intercepts of the graph. These are the solutions of the equation.

EXAMPLE 4.65

Graphically approximate the roots of $x^2 - 3x - 1 = 0$.

SOLUTION First, set $f(x) = x^2 - 3x - 1$. Then set up a partial table of values for the function. (See Figure 4.40a.)

From the graph in Figure 4.40b, we can see that the graph crosses the x-axis between $x = -1$ and $x = 0$ so there must be a solution between $x = -1$ and $x = 0$. For the same reason, there must be a solution between $x = 3$ and $x = 4$. From the graph it appears that the roots are at $x = -0.25$ and $x = 3.25$.

If we evaluate the function at these two values, we get $f(-0.25) = -0.1875$ and $f(3.25) = -0.1875$. This shows two things. First, it shows that -0.25 and 3.25 are not roots of this function, since $f(x) \neq 0$ at either of these two points. Second, it does show that the roots of this function are "close" to -0.25 and 3.25.

	A	B
1	x	y-f(x)
2	-2	9
3	-1	3
4	0	-1
5	1	-3
6	2	-3
7	3	-1
8	4	3
9	5	9

Figure 4.40a

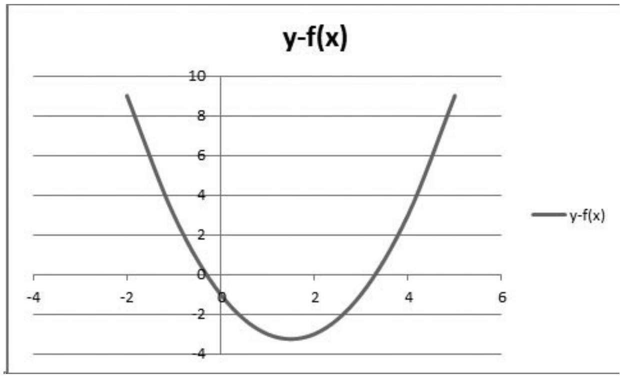


Figure 4.40b

EXAMPLE 4.66

Use a spreadsheet to find the approximate roots of $f(x) = x^2 - 3x - 1$.

SOLUTION This is the same function we graphed in Example 4.65. We already know that the roots are near $x = -0.254$ and $x = 3.25$.

To find the solutions using a spreadsheet, we use a table like the one we constructed in Example 4.64. (See Figure 4.41a.)

Using the table in Figure 4.41a, you can see that a root exists between $x = -0.4$ and $x = -0.3$. We can "zoom" in closer to that root by changing the initial value of x to -0.4 and the increment to 0.01. Then use the results of that table to zoom in even further—getting a good approximation for the root. The tables in Figure 4.41b show that the approximate value of a solution is $x = -0.303$.

Using a similar process, we find another approximate solution at $x = 3.303$. (See Figure 4.41c.)

If the roots needed to be more accurate, you would simply repeat the process until the desired accuracy is obtained.

	A	B
1	Initial value of x	-1
2	Increment for x	0.1
3	x	y=f(x)
4	-1	3
5	-0.9	2.51
6	-0.8	2.04
7	-0.7	1.59
8	-0.6	1.16
9	-0.5	0.75
10	-0.4	0.36
11	-0.3	-0.01
12	-0.2	-0.36
13	-0.1	-0.69
14	-1.38778E-16	-1
15		

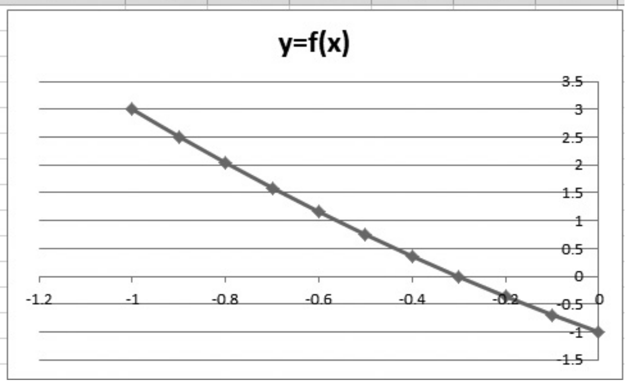


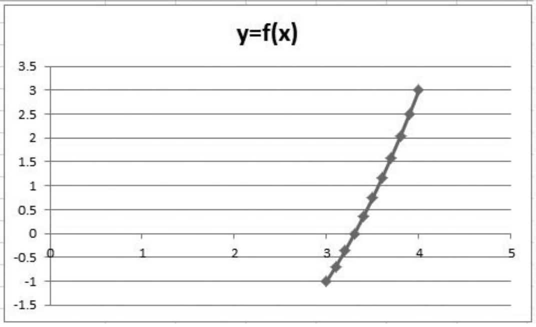
Figure 4.41a

	A	B
1	Initial value of x	-0.4
2	Increment for x	0.01
3	x	y=f(x)
4	-0.4	0.36
5	-0.39	0.3221
6	-0.38	0.2844
7	-0.37	0.2469
8	-0.36	0.2096
9	-0.35	0.1725
10	-0.34	0.1356
11	-0.33	0.0989
12	-0.32	0.0624
13	-0.31	0.0261
14	-0.3	-0.01

	A	B
1	Initial value of x	-0.31
2	Increment for x	0.001
3	x	y=f(x)
4	-0.31	0.0261
5	-0.309	0.022481
6	-0.308	0.018864
7	-0.307	0.015249
8	-0.306	0.011636
9	-0.305	0.008025
10	-0.304	0.004416
11	-0.303	0.000809
12	-0.302	-0.0028
13	-0.301	-0.0064
14	-0.3	-0.01

Figure 4.41b

	A	B
1	Initial value of x	3
2	Increment for x	0.1
3	x	y=f(x)
4	3	-1
5	3.1	-0.69
6	3.2	-0.36
7	3.3	-0.01
8	3.4	0.36
9	3.5	0.75
10	3.6	1.16
11	3.7	1.59
12	3.8	2.04
13	3.9	2.51
14	4	3
15		



	A	B
1	Initial value of x	3.3
2	Increment for x	0.01
3	x	y=f(x)
4	3.3	-0.01
5	3.31	0.0261
6	3.32	0.0624
7	3.33	0.0989
8	3.34	0.1356
9	3.35	0.1725
10	3.36	0.2096
11	3.37	0.2469
12	3.38	0.2844
13	3.39	0.3221
14	3.4	0.36

	A	B
1	Initial value of x	3.3
2	Increment for x	0.001
3	x	y=f(x)
4	3.3	-0.01
5	3.301	-0.0064
6	3.302	-0.0028
7	3.303	0.000809
8	3.304	0.004416
9	3.305	0.008025
10	3.306	0.011636
11	3.307	0.015249
12	3.308	0.018864
13	3.309	0.022481
14	3.31	0.0261

Figure 4.41c

EXAMPLE 4.67

Use a spreadsheet to determine any x-intercepts of $y = x^2 + 1.5$.

SOLUTION A table of values for $f(x) = x^2 + 1.5$ is shown in Figure 4.42a. From that table you can see that the y-values will never equal zero, another way of saying that the graph will not have an x-intercept. That conjecture is supported by the graph shown in Figure 4.42b. Thus, the function does not have any real roots.

	A	B
1	Initial value of x	-5
2	Increment for x	1
3	x	y=f(x)
4	-5	26.5
5	-4	17.5
6	-3	10.5
7	-2	5.5
8	-1	2.5
9	0	1.5
10	1	2.5
11	2	5.5
12	3	10.5
13	4	17.5
14	5	26.5

Figure 4.42a

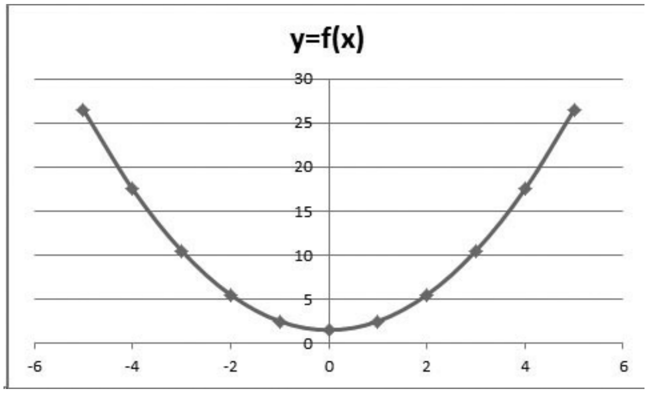


Figure 4.42b

EXAMPLE 4.68

Use a spreadsheet to rework the following parts of Example 4.60.

(a) Sketch the graph of $C(m) = \dfrac{50}{m^2 + 4}$, $m \geq 0$.

(b) Use the graph and your calculator to estimate how far you are from the factory if the concentration of the pollutant is 3.08 ppm.

SOLUTION

(a) To sketch the graph, we replace $C(m)$ with y and construct a table of values as shown in Figure 4.43a.

(b) We want to know when the concentration of the pollutant is 3.08 ppm. This means we need to know the value of x when $y = 3.08$. One way to do that is to change the initial value and increment until a y-value of 3.08 appears. The tables in Figure 4.43b show those steps. Once again, we conclude that the pollutant was collected about 3.5 miles from the factory.

A second way to find the value of x that will make $y = 3.08$ is to use `Goal Seek`. (The directions for using Goal Seek are in the Excel Help menu.) Figures 4.43c and 4.43d show two of the steps that use Goal Seek to find the solution. Using Goal Seek we see that $m \approx 3.497576$ gives a value of $C(m) \approx 3.08$.

	A	B
1	Initial value of m	0
2	Increment for x	1
3	m	y=C(m)
4	0	12.5
5	1	10
6	2	6.25
7	3	3.846154
8	4	2.5
9	5	1.724138
10	6	1.25
11	7	0.943396
12	8	0.735294
13	9	0.588235
14	10	0.480769

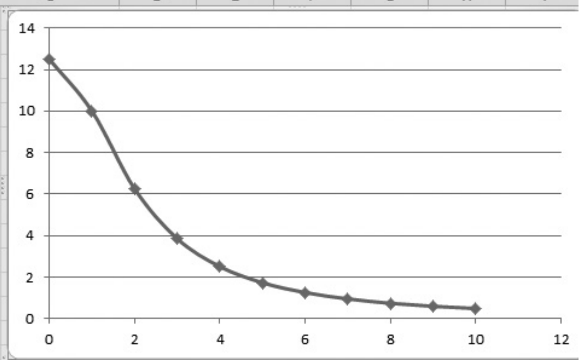


Figure 4.43a

	A	B
1	Initial value of m	3.2
2	Increment for x	0.1
3	m	y=C(m)
4	3.2	3.511236
5	3.3	3.357958
6	3.4	3.213368
7	3.5	3.076923
8	3.6	2.948113

	A	B
1	Initial value of m	3.495
2	Increment for x	0.001
3	m	y=C(m)
4	3.495	3.08356
5	3.496	3.082231
6	3.497	3.080903
7	3.498	3.079575
8	3.499	3.078249

Figure 4.43b

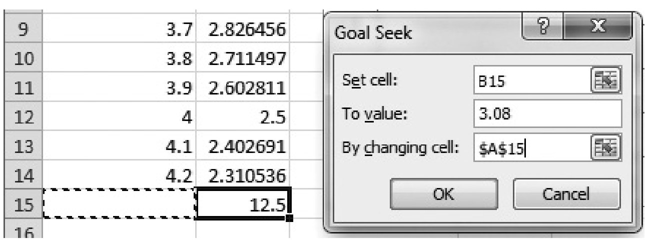


Figure 4.43c

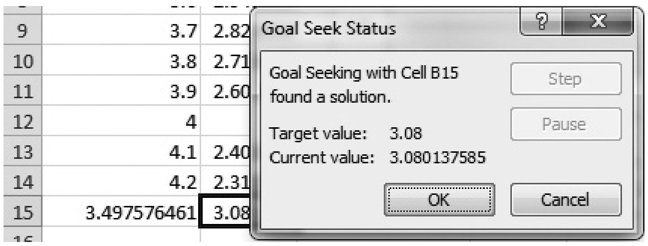


Figure 4.43d

EXERCISE SET 4.6

In Exercises 1–26 graph each of these functions with a computer spreadsheet. Compare the spreadsheet graph with the graph you made of the same functions in Section 4.4.

1. $y = x$
2. $y = 2x$
3. $y = -3x$
4. $y = \frac{1}{2}x - 3$
5. $y = -4x + 2$
6. $x + y = 2$
7. $x - y = 2$
8. $2x + y = 1$
9. $3x - 6y = 9$
10. $3x - 6y = -6$
11. $y = x^2$
12. $y = x^2 + 3$
13. $y = x^2 - 2$
14. $y = x^2 + 2x + 1$
15. $y = x^2 - 6x + 9$
16. $y = -x^2 + 2$
17. $y = \dfrac{1}{x}$
18. $y = \dfrac{1}{x + 3}$
19. $y = x^3$
20. $y = \sqrt[3]{x}$
21. $y^2 + x^2 = 25$ (HINT: Solve for y, then graph two equations.)
22. $y = \sqrt{25 - x^2}$
23. $y = |x + 2|$
24. $\dfrac{x^2}{4} + \dfrac{y^2}{9} = 1$ (HINT: Solve for y, then graph two equations.)
25. $\dfrac{x^2}{4} - \dfrac{y^2}{9} = 1$ (HINT: Solve for y, then graph two equations.)
26. $y = x^3 - x^2$

The equations in Exercises 27–44 all have at least one root between −10 and 10. Write each equation in the form y = f (x). Use a spreadsheet to graph each equation to find the approximate value of the roots.

27. $2x + 7 = 0$
28. $4x - 9 = 0$
29. $x^2 - 16 = 0$
30. $20 - 4x^2 = 0$
31. $x^2 = 7x$
32. $x^2 = 3x - 2$
33. $x^2 = 5x + 6$
34. $4x^2 + 3x - 4 = 0$
35. $10x^2 + 18x = 9$
36. $x^4 + 5x^3 + 4x^2 - 5x - 6 = 0$
37. $\sqrt{x + 4} = 0$
38. $\sqrt{x - 1.4} = 0$
39. $\sqrt{x + 1} = 2$
40. $\sqrt{5 - x} = 3$
41. $\dfrac{x}{x - 1} = 3$
42. $\dfrac{x}{x + 3} = -5$
43. $\dfrac{x}{x + 3} = 5 - x^2$
44. $\dfrac{x^2 + 3}{x - 2} = x^2 - 3$

4.7 INTRODUCTION TO MODELING

We are not always given a function or an equation and asked to use it. Instead, we collect some data and have to interpret it. The first step is to organize the data in a list or table. Next, graph the points and use a calculator, computer program, or spreadsheet to draw a curve through the points. The points seldom lie on a "nice" curve. You can connect the points using a series of line segments in what is called a *broken line graph*.

A broken line graph makes the graph easier to see but is not very useful. If at all possible we want to develop a **model** for the data. A model is a function that comes close to going through the data points. It may not go through them all—in fact it may not go through any of them—but it comes the closest of any given type of function. Determining these models uses a technique known as **regression**, and until it was built into calculators it was very difficult to do.

A LOOK TO THE FUTURE

The next example will show a plot of some data, a broken line graph for that data, and one model that seems to fit the data. We will use that model to predict an outcome that is not in the table. This example is intended as a demonstration of what we will be doing later in this section and later in the book.

APPLICATION **BUSINESS**

EXAMPLE 4.69

Table 4.3 gives the annual domestic fuel consumption, in billion gallons, of all motor vehicles in the United States for selected years from 1970 through 1999.

TABLE 4.3 Annual Motor Vehicle Fuel Consumption in the United States, 1970–2005

Year	1970	1975	1980	1985	1990	1995	2000	2005
Fuel consumption (billion gallons)	92.3	109.0	115.0	121.3	130.8	143.8	162.5	174.8

(a) Plot the points.

(b) Connect the points to form a broken line graph.

(c) Determine a regression curve that will fit the data.

(d) Use the regression curve to determine the fuel consumption in 1998 and 2000.

SOLUTIONS:

Using a calculator:

(a) We have to first enter the data into the calculator as two lists. Press STAT 4 2nd L1 , 2nd L1 ENTER to clear lists L1 and L2 in the calculator. Then press 1 [EDIT]. Place the independent variable, in this case the years, in the

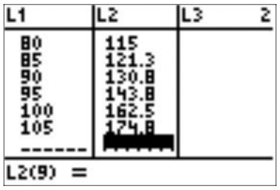


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Figure 4.44a

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Figure 4.44b

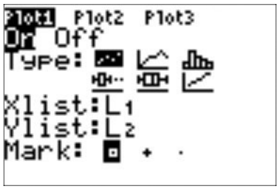

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Figure 4.44c

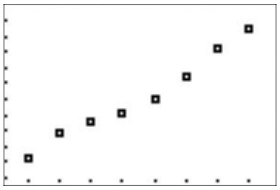
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Figure 4.44d

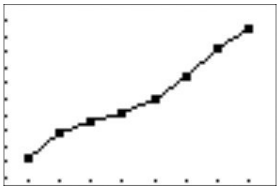
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Figure 4.44e

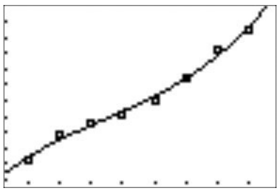
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Figure 4.44f

L1 column and place the dependent variable, the fuel consumption, in the L2 column. When you are finished the result should look like Figure 4.44a. The first two rows in each column cannot be seen. We used 70 for 1970, 75 for 1975, and so on with 100 for the year 2000. Note that the entry for 1970 is 92.3. The fuel data is given in billion gallons so 92.3 represents 92,300,000,000 gallons. If you do these transformations of the data you must be sure to indicate that fact when you write the linear equation and also when you use the equation to make predictions.

To graph these data we use the statistical plotting feature of the calculator. First press **Y =** and delete any functions that are listed. Next, we press **2nd** **Y =** [STAT PLOT]. The result, shown in Figure 4.44b, shows three possible stat plot screens. Press **ENTER** or **1** to select the first plot screen. The result is shown in Figure 4.44c.

There are several plot types. The first is the scatter plot. When it is selected, as shown in Figure 4.44c, and the data in Table 4.3 is plotted using the zoom setting 9 : ZoomStat you obtain the result in Figure 4.44d.

(b) Change the plot type to the second plot type. This one connects the dots. When this is plotted you obtain the broken line graph in Figure 4.44e.

(c) A cubic regression that fits this data is $C(t) \approx 0.0016t^3 - 0.3989t^2 + 34.5959t - 926.65$ billion gallons t years after 1900. When the actual regression equation produced by the calculator is graphed on the scatter plot we obtain the graph in Figure 4.44f.

(d) We can use the regression formula stored in the calculator to approximate the fuel consumption in 1997 and 2002. According to the calculator (see Figure 4.44g), $C(97) = 149.3$. Note here that in the regression formula t represented years after 1900 and so 1997 would use the value $t = 97$. Similarly, fuel consumption in 2003 would be $C(103) \approx 168.8$ billion gallons.

The actual fuel consumption for 1997 was 150.4 billion gallons, very close to the predicted value. The actual value for 2003 was 170.0 billion gallons.

Using a spreadsheet: Note that 1970 is entered as 70 and the entry for 1970 is 92.3. The fuel data is given in billion gallons, so 92.3 represents 92,300,000,000 gallons.

(a) Enter the data in Table 4.3 in the first two columns of the spreadsheet. The result should look something like Figure 4.44h.

(b) Graphing the data in these the first two columns of the spreadsheet produces the graph in Figure 4.44i. The graph is difficult to read, so the scale of the horizontal axis is changed so the minimum value is 65 and the vertical axis scale is changed so the minimum is 80. Finally, click on the "◇ Fuel consumption" expression to the right of the graph and press the **RETURN** key. This removes the "◇ Fuel consumption" expression from the graph, with a result looking something like Figure 4.44j.

(c) We will use cubic regression. Right click on one of the data points and select Add Trendline. Then select Polynomial and Order 3, as shown in

EXAMPLE 4.69 (Cont.)

Figure 4.44k, which means the model will be a cubic. Under Options, choose Display Equation on Chart. The result is shown in Figure 4.44l.

Looking at Figure 4.44l, we can see that a cubic regression that fits this data is $C(t) \approx 0.0016t^3 - 0.3989t^2 + 34.596t - 926.65$ billion gallons t years after 1900.

(d) We now need to use this cubic regression equation to determine the billions of gallons of fuel consumed in the years 1997 and 2003. Copy the equation for the model into Cell D1, using C1 as the variable, as shown in Figure 4.44m. Enter 97 (for 1997) in Cell C1 and the result is approximately 136.2 billions gallons of fuel were consumed in 1997. Change the value in Cell C1 to 103 and the regression equation says that about 153.2 billions gallons of fuel were consumed in 2003.

The actual fuel consumption for 1997 was 150.4 billion gallons, very close to the predicted value. The actual value for 2003 was 170.0 billion gallons.

The approximations given by the calculator and the spreadsheet differ for both 1997 and 2003. This is because the two regression equations use different numbers of decimal places.

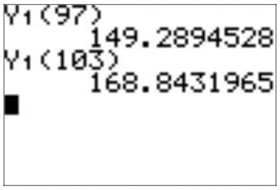

Figure 4.44g

	A	B
1	Year	Fuel Consumption
2	70	92.3
3	75	109
4	80	115
5	85	121.3
6	90	130.8
7	95	143.8
8	100	162.5
9	105	174.8

Figure 4.44h

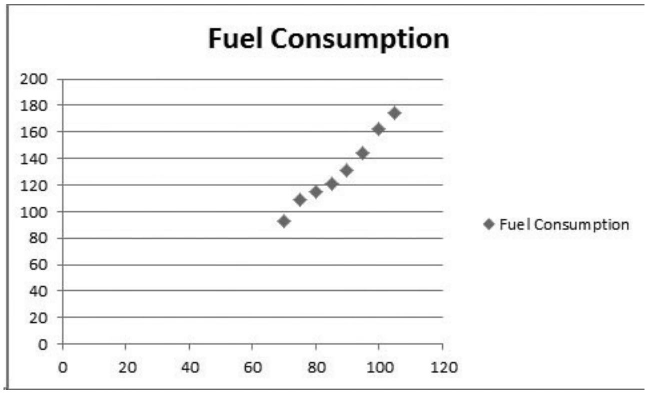

Figure 4.44i

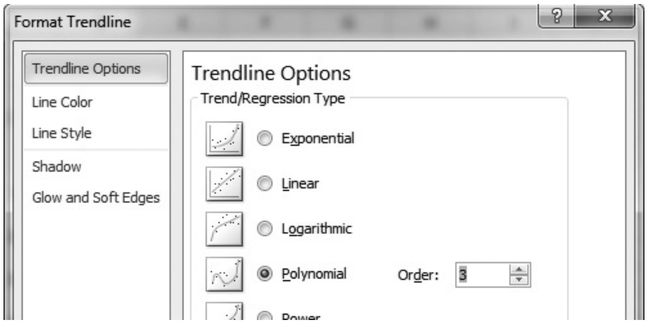

Figure 4.44k

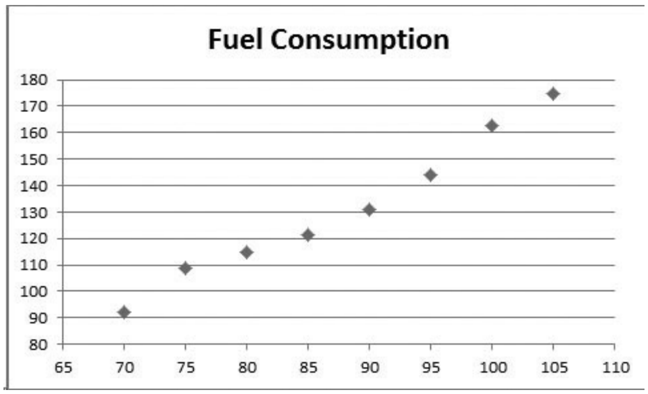

Figure 4.44j

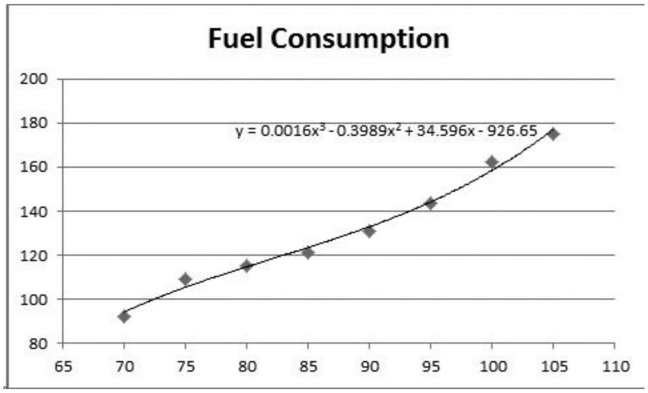

Figure 4.44l

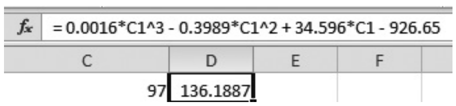

Figure 4.44m

LINEAR CURVE FITTING

In Example 4.60 we looked at the graph of some data, had the calculator determine the formula for a curve that would come close to passing through the data, and made some predictions based on the formula. Now let's look at this more closely. We will begin with some data that seems to fit a line. This process is called *linear curve fitting* or **linear regression**.

Graphing calculators and many computer software programs with graphing ability contain built-in programs that find the "best fit" equation for a collection of points in a scatter diagram. Different software has different names for this ability. For example, in Excel it is referred to as "Add Trendline" and a TI-84 calls it LinR or LinReg.

APPLICATION BUSINESS

EXAMPLE 4.70

The total vehicle highway miles of travel in the United States from 1980–1999, in billions of miles, is shown in Table 4.4.

(a) Plot the points.

(b) Determine a linear regression curve that will fit the data.

TABLE 4.4 Motor Vehicle Miles of Travel

Year	1980	1985	1990	1995	2000	2003	2005	2007
Billion Miles	1527	1775	2144	2423	2747	2890	2989	3030

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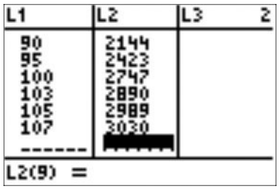

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Figure 4.45a

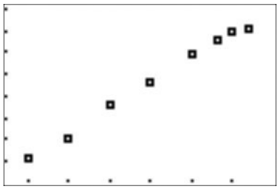

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Figure 4.45b

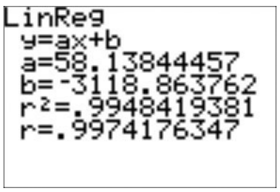

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Figure 4.45c

SOLUTIONS:

Using a calculator:

(a) As in the previous example, we have to enter the data as lists. Begin by pressing the STAT key and clearing lists L1 and L2. Next, enter the data in Table 4.4 in L1 and L2. Place the independent variable, in this case the years, in the L1 list and place the dependent variable, the number of miles, in the L2 list. In Figure 4.45a you will see that only the last two digits of the year were entered in L1.

A scatter plot of these data is shown in Figure 4.45b.

To get the regression equation, press STAT ▶ 4 [LinReg (ax+b)] ENTER with the result in Figure 4.45c. The second, third, and fourth lines of Figure 4.45c indicate the linear regression line is $y = ax + b$ and that $a = 58.138444567$ and $b = -3118.8637817209$. We would probably write our answer as the function $M(t) = 58.138444567t - 3118.8637817209$ billion miles t years after 1900.

To graph the regression line with the original data, first press Y = CLEAR and then press VARS 5 [5 : Statistics] ▶ ▶ 1 [1 : RegEq]. This pastes the regression equation on the Y1 line of the Y = screen. Notice that the regression equation shows more digits for a and b than were given in Figure 4.45c.

Now press GRAPH to get the result in Figure 4.45d. As we will see later when we study linear equations, 58.14 is the slope of this line and -3118.9 is the y-intercept.

EXAMPLE 4.70 (Cont.)

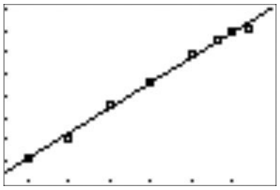

Figure 4.45d

	A	B
1	Year	Miles Traveled
2	80	1527
3	85	1775
4	90	2144
5	95	2423
6	100	2747
7	103	2890
8	105	2989
9	107	3030

Figure 4.45e

Using a spreadsheet:

(a) Enter the data in Table 4.4 in the first two columns of the spreadsheet. The result should look something like Figure 4.45e.

(b) Graphing the data in these the first two columns of the spreadsheet produces the graph in Figure 4.45f. The graph is difficult to read. Change the scale of the horizontal axis so the minimum value is 80 and change the scale on the vertical axis so the minimum is 1500. Finally, click on the "◇ Miles traveled" expression to the right of the graph and press the **RETURN** key. The result should now look something like Figure 4.45g.

(c) We will use linear regression. Right click on one of the data points and select Add Trendline. Click on Type and then select Linear. Under Options, choose Display Equation on Chart. The result is shown in Figure 4.45h.

Looking at Figure 4.44h, we can see that a linear regression that fits this data is $M(t) \approx 58.138t - 3118.9$ billion miles t years after 1900. As we will see later when we study linear equations, 58.138 is the slope of this line and -3118.9 is the y-intercept.

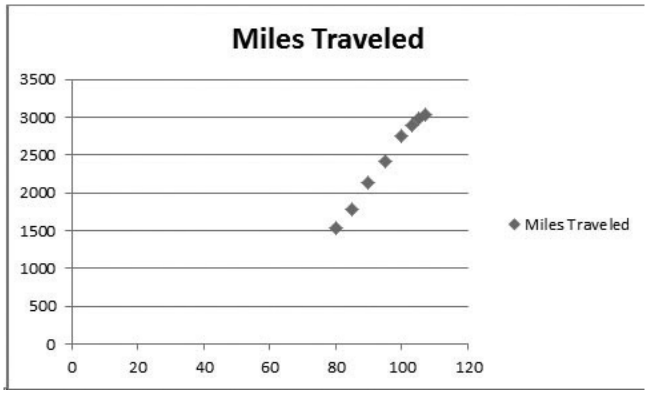

Figure 4.45f

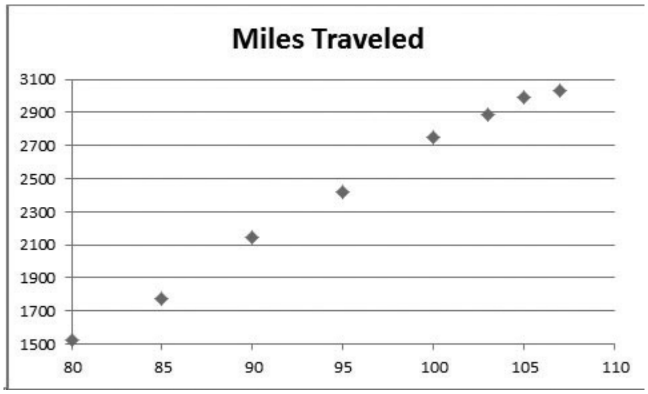

Figure 4.45g

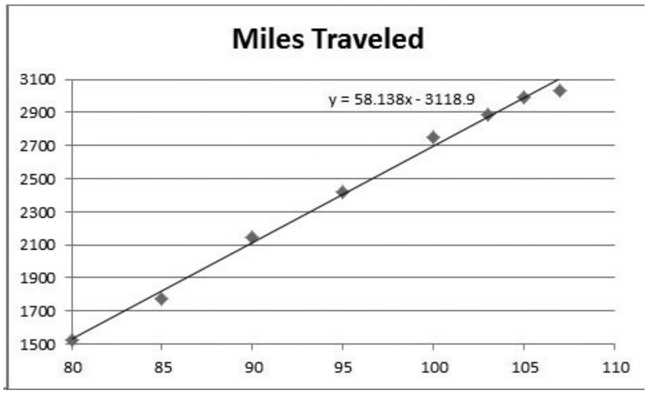

Figure 4.45h

CORRELATION COEFFICIENTS

Look again at Figure 4.45c. The next to the last line of output is corr $= .998684099$. This number, called the **correlation coefficient**, $\boldsymbol{r}$, where $-1 \leq r \leq 1$, is a measure of the strength of the linear relation that exists between the given data for the two variables. The closer $|r|$ is to 1, the more perfect the linear relationship between the variables. If r is negative then the line has negative slope. If $-0.75 < r < 0.75$, the correlation is considered to be poor. Graphs of points with different values of r are shown in Figures 4.46a–d.

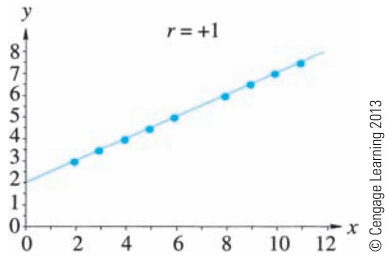

Figure 4.46a

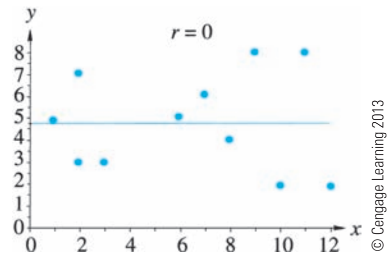

Figure 4.46b

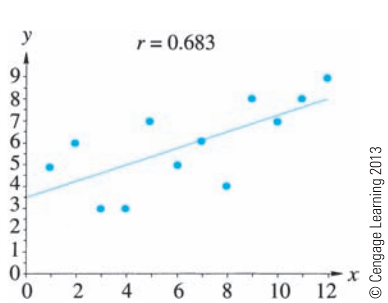

Figure 4.46c

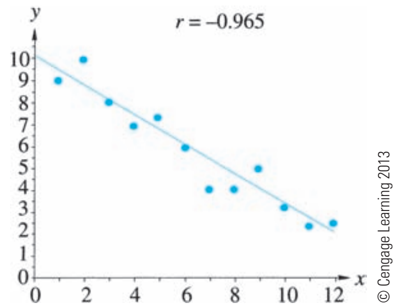

Figure 4.46d

INTERPOLATION AND EXTRAPOLATION

When you predict values of the dependent variable for independent values that are within the interval of the given data in your table or scatter plot, you are using a process called **interpolation**. On the other hand, when you are predicting values of the dependent variable for values of the independent variable that are outside the interval of the given data, you are using a process call **extrapolation**. Because you do not know what happens outside the given data, estimates obtained through extrapolation should be viewed with more caution than estimates found with interpolation.

APPLICATION **BUSINESS**

EXAMPLE 4.71

Use the regression curve from Example 4.70 to determine the total vehicle highway miles of travel in the United States for 1998 and 2010.

EXAMPLE 4.71 (Cont.)

SOLUTION In Example 4.70, we entered the regression equation from Example 4.70 as Y1 in the Y=. We can use this to evaluate $M(1998)$ and $M(2010)$.

1. Use the function feature of the calculator Key. VARS ► 1 [1 : Function] ENTER (98) ENTER and VARS ► 1 [1 : Function] ENTER (110) ENTER on the home screen of the calculator. You should obtain something like Figure 4.47a.
2. Use the table feature of the calculator. Since you want the data for 1998 and 2010 press 2nd WINDOW [TBLSET] and set TblStart = 98. If you set ΔTbl = 2 and press 2nd GRAPH [TABLE] you should obtain Figure 4.47b.

On a spreadsheet enter the regression formula in Cell D1 and copy it to Cell D2. Enter the two years, 98 and 110, in Cells C1 and C2. The result are shown in Figure 4.47c.

All three procedures yield the same results. According to our regression formula and interpolation, all the motor vehicles in the United States traveled around 2579 billion miles in 1998. (The actual value for 1998 was 2,632 billion miles.) In 2010, extrapolation predicts that the vehicles traveled about 3276 billion miles.

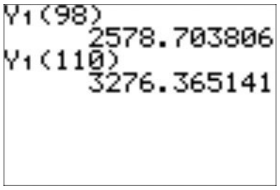

Figure 4.47a

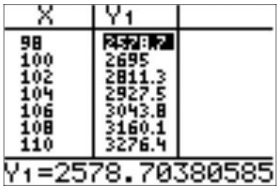

Figure 4.47b

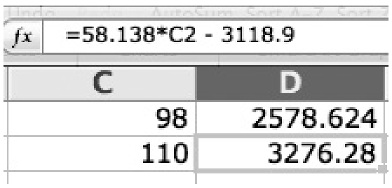

Figure 4.47c

APPLICATION HEALTHCARE

EXAMPLE 4.72

The following table gives the height, in inches, and the weight, in pounds, for 12 students.

Student	A	B	C	D	E	F	G	H	I	J	K	L
Height	66	67	68	69	70	71	72	70	72	71	72	73
Weight	130	140	180	160	185	190	200	195	195	210	210	180

(a) Plot the points.

(b) Determine a linear regression curve that will fit the data.

(c) Determine the correlation coefficient for this curve.

(d) Use the regression curve to predict the weight of a student who is 6′6″ tall.

SOLUTIONS

(a) A graph of these 12 points is shown in Figure 4.48a. A quick glance at this graph tells you that these points do not all lie on the same straight line. But, in general, the taller a person is, the more he or she weighs. Thus, we think that it is possible that there might be some straight line that would come close to these points and from which none of the points differ by very much.

(b) The calculator gives a regression curve of $y \approx -492.54331 + 9.61417323x$. We will write this as the function $w(h) \approx -492.54331 + 9.61417323h$, where h is the height in inches and w is the weight in pounds. A graph of the original 12 points and the regression line is in Figure 4.48b.

(c) The correlation coefficient is about $r = 0.82$.

(d) A student who is 6′6″ tall is 78 inches tall. Evaluating, we obtain $w(78) \approx 257$. So, a student who is 6′6″ tall would weigh about 257 pounds.

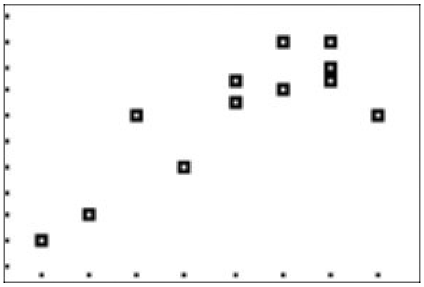

Figure 4.48a

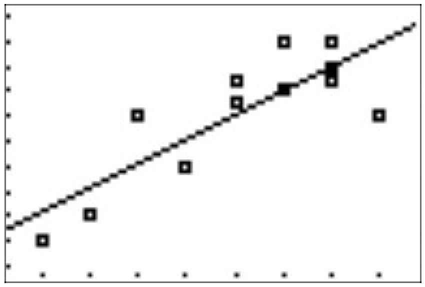

Figure 4.48b

EXERCISE SET 4.7

For each table in Exercises 1–14, (a) plot the points, (b) determine a linear regression curve that will fit the data, (c) determine the correlation coefficient for the given data, and (d) plot the linear regression curve with the original data.

1.

x	5	6	8	9	12	14
y	2	3	7	7	8	10

2.

x	1	3	5	7	9
y	16	45	86	104	132

3.

x	2	4	6	8	10
y	30	55	93	122	132

4.

x	68.5	67.2	67.7	63.8	69.8
y	33.6	35.0	30.2	30.0	33.2

x	64.7	66.4	69.1	65.3	64.8
y	30.8	30.2	33.3	32.9	37.3

5. ***Machine technology*** A tensile ring is to be calibrated by measuring the deflection in thousandths of an inch at various loads in thousands of pounds. The following results were obtained by applying increasingly larger load forces from 1,000 to 12,000 lb.

load force (lb × 1,000)	1	2	3	4	5	6
deflection (in × 0.001)	16	34	44	76	84	98

load force	7	8	9	10	11	12
deflection	108	126	137	158	165	185

6. ***Business*** A vending machine company studied the relationship between maintenance cost and dollar sales for its machines. Here are the results from nine machines:

maintenance cost	95	105	85	130	145
sales	1,100	1,250	600	890	1,450

maintenance cost	125	90	110	90
sales	1,500	800	1,300	870

7. ***Energy technology*** An engineering company studied the relationship between the air velocity and the evaporation of burning fuel droplets in an impulse engine. The following results were obtained:

air velocity (mm/sec)	200	600	1000
evaporation coefficient (mm^2/s)	0.18	0.37	0.39

air vel.	1400	1800	2200	2600	3000
evap. coeff.	0.75	0.82	0.93	1.15	1.42

8. ***Automotive engineering*** An automobile company studied the relationship between highway mileage and automobile weight.

highway mpg	42	37	45	40	36	35	45	43	30	25
weight (lb × 100)	21	23	24	22	25	24	19	20	25	26

9. ***Police science*** The following information was collected on the age, in years, and the BAC (blood alcohol count) when convicted DUI jail inmates were first arrested for driving under the influence of alcohol:

Age	16.8	53.4	35.6	45.3	18.9	20.0	37.4	56.6	22.6	26.8
BAC	0.19	0.21	0.22	0.16	0.22	0.26	0.14	0.16	0.19	0.25

10. ***Insurance*** An insurance company noted the following age and vehicle speed for motorists stopped for speeding in a 55-mph speed zone:

Age	42	63	37	74	32	27	18	21	32	16
Speed	62	57	62	59	67	64	65	72	61	82

Age	16	20	42	63	37	28	24	32	27	19
Speed	75	64	65	59	63	72	62	62	67	74

11. ***Energy*** The annual number of billion kWh of electricity used by households in the United States is shown in the following table.

Year	1990	1995	2000	2007	2003	2004	2005	2006	2007	2008
Annual electricity used (billion kWh)	924.0	1042.5	1192.4	1275.8	1292.0	1359.2	1351.5	1392.2	1379.3	

12. ***Business*** The average annual amount of U.S. exports to Mexico in billions of dollars is shown in the table below:

Year	2004	2005	2006	2007	2008
U.S. exports to Mexico (billion dollars)	111	120	134	136	151

13. ***Automotive technology*** Several different makes and models of automobiles were selected. The displacement in liters of each engine and the average miles per gallon (mpg) of the that automobile are given in the table below:

Engine displacement	1.5	1.7	1.9	2.2	2.3	2.8	3.5	3.8	4.3	4.5	4.9
Average mpg	43.1	41.9	38.7	36.1	34.7	33.0	32.1	28.6	26.5	27.1	26.1

14. ***Health technology*** The following data lists the life expectancy, in years, of people born in various years. (Hint: Enter the years as 20, 30, etc. rather than 1920, 1930, etc.)

Year	1920	1930	1940	1950	1965	1970
Life expectancy	54.1	59.7	62.9	68.2	70.2	70.8

Year	1975	1980	1985	1990	1995	2000	2005
Life expectancy	72.6	73.7	74.7	75.4	75.8	76.8	77.4

For each table in Exercises 15–22, (a) plot the points, (b) determine a linear regression curve that will fit the data, (c) determine the slope of the regression line, (d) determine the y-intercept of the regression line, (e) use the regression curve to complete the table, and (f) if possible, determine the actual results for the missing data.

15. ***Physics*** The data in this table show the relationship between degrees Celsius and degrees Fahrenheit.

Celsius	−40	−20	0	20	25	30	35	40	50	100
Fahrenheit	−40		32	68				104		212

16. ***Physics*** A 10,000-psia digiquartz sensor has drifted, and its responses at known pressures are given in the following table.

Actual pressure (psia)	0	2,000	4,000	6,000	8,000	10,000
Indicated pressure	−12.5	1988.4	3984.7	5982.7		9985.6

17. ***Recreation*** The data in this table contain the Olympic results in men's 50-meter freestyle swimming. (This event was not held from 1908–1984.)

Year	1904	1988	1992	1996	2000	2004	2008	2012
Time (sec)	28	22.14	21.91	22.13		21.93	21.30	

18. ***Recreation*** These data are for the Olympic results in the women's 200-meter dash.

Year	1948	1952	1956	1960	1964	1968	1972
Time (sec)	24.4	23.7	23.4		23	22.5	22.4

Year	1976	1980	1984	1988	1992	1996	2000
Time (sec)	22.37		21.81	21.34	21.81	22.12	

19. ***Environmental science*** Municipal solid waste generation in millions of tons (121.9 represents 121,900,000) from 1970 to 2007 is given in the following table.

Year	1970	1980	1985	1990	1995	2000	2004	2005	2006	2007
Waste	121.9	151.6	164.4	205.2	211.4	239.1	249.8	250.4	254.2	254.1

20. ***Environmental science*** The concentration of carbon dioxide (CO_2), in ppm (parts per million), at the Mauna Loa, Hawaii, observatory are given in the following table.

Year	1975	1980	1985	1990	1995	2000	2002	2005	2008	2012
CO_2	331	339		354	361	369	373	380	385	

21. ***Environmental science*** World natural gas production in quadrillion Btu (British thermal unit) is given in the following table.

Year	1980	1990	1995	2000	2001	2002	2003	2004	2005	2006
Production	54.7	76.1	80.4	91.0	93.3	96.3		101.5	104.8	107.2

22. ***Health technology*** The number of CAT scans, in thousands, conducted on males in the United States is given in the following table.

Year	1990	1993	1995	1997	1998	1999	2000	2006
CAT scans	736	565	473		462	408	345	366

[IN YOUR WORDS]

23. Describe how to use your calculator to enter and graph data from a table.

24. Describe how to use your calculator to produce the equation for a linear regression curve.

25. Explain the correlation coefficient. How do you use your calculator to find it? What does it tell you? What are its possible values?

CHAPTER 4 REVIEW

IMPORTANT TERMS AND CONCEPTS

Cartesian coordinate system	Intercepts	Rectangular coordinate system
Composite function	Interpolation	Regression
Correlation coefficient	Inverse function	Relation
Dependent variable	Linear equation	Slope
Domain	Linear regression	Vertical line test
Extrapolation	Model	x-axis
Function	Ordered pairs	x-intercept
Horizontal line test	Quadrants	y-axis
Independent variable	Range	y-intercept

REVIEW EXERCISES

For Exercises 1–6, (a) graph each of the relations; (b) determine the domain, range, x-intercept, and y-intercept of each relation; (c) use the vertical line test to determine if each relation is a function; (d) use the horizontal line test to determine if each function has an inverse function; and (e) graph each inverse function that exists.

1. $y = 8x - 7$

2. $y = 2x^2 - 4$

3. $y = \sqrt{x} - 3$

4. $y = \frac{1}{2}x^3 + 2$

5. $y = x^2 - 2x$

6. $y = x^2 + 4$

In Exercises 7–12, given the function $f(x) = 4x - 12$, determine the following.

7. $f(0)$

8. $f(-2)$

9. $f(3)$

10. $f(a)$

11. $f(a - 2)$

12. $f(x + h)$

In Exercises 13–19, given the function $g(x) = \dfrac{x^2 - 9}{x^2 + 9}$, determine the following.

13. $g(0)$

14. $g(3)$

15. $g(-3)$

16. $g(-2)$

17. $g(4)$

18. $g(-5)$

19. Use your values from Exercises 13–18 to graph $g(x)$. From your graph, determine the zeros of g.

In Exercises 20–33, let $f(x) = 4x - 12$ and $g(x) = \dfrac{x^2 - 9}{x^2 + 9}$. Determine the following.

20. $(f + g)(x)$

21. $(f + g)(3)$

22. $(f - g)(x)$

23. $(f - g)(-2)$

24. $(f \cdot g)(x)$

25. $(f \cdot g)(0)$

26. $(f / g)(x)$

27. $(f / g)(5)$

28. $(g / f)(x)$

29. $(g / f)(2)$

30. $(f \circ g)(x)$

31. $(f \circ g)(4)$

32. $(g \circ f)(x)$

33. $(g \circ f)(3)$

In Exercises 34–40, graph each of these functions or relations with a calculator or computer.

34. $y = \dfrac{5}{x - 4}$

35. $y = -\dfrac{7}{x + 3}$

36. $y = \sqrt{2x - 3}$

37. $y = \sqrt{5 - 4x}$

38. $y = \dfrac{x}{\sqrt{x + 2}}$

39. $x^2 + y^2 = 9$

40. $\frac{1}{2}x^2 + y^2 = 16$

Each of the equations in Exercises 41–44 has a root between -10 and 10. Write each equation in the form $y = f(x)$. Graph each equation to find the approximate values of the roots.

41. $4x + 7y = 0$

42. $x^2 - 20 = 0$

43. $2x^2 + 10x + 4 = 0$

44. $8x^3 - 20x^2 - 34x + 21 = 0$

Solve Exercises 45–49.

45. ***Business*** The manager of a videotape store has found that n videotapes can be sold if the price is $P(n) = 35 - \dfrac{n}{20}$ dollars.

(a) What price should be charged in order to sell 101 videotapes? 350 videotapes? 400 videotapes?

(b) Find an expression for the revenue from the sale of n vidoetapes, where revenue = demand × price.

(c) How much revenue can be expected if 101 videotapes are sold? if 350 are sold? if 400 are sold?

46. ***Business*** A videotape store has learned that the function $R(n) = 30n - \frac{n^2}{20}$ is a good predictor of its revenue, in dollars, from the sale of n tapes. The cost of operating the store is given by $C(n) = 550 + 10n$.

(a) If the profit P is given by $P(n) = R(n) - C(n)$, what is the profit function?

(b) How much profit will the store make if it sells 30 videotapes? 100 videotapes? 150 videotapes? 300 videotapes? 400 videotapes?

47. ***Medical technology*** A measure of cardiac output can be determined by injecting a dye into a vein near the heart and measuring the concentration of the dye. In a normal heart, the concentration of the dye is given by the function

$$h(t) = -0.02t^4 + 0.2t^3 - 0.3t^2 + 3.2t$$

where t is the number of seconds since the dye was injected. Set up a partial table of values for $0 \le t \le 10$, and sketch the graph of this function.

48. ***Automotive technology*** The distance s, in feet, needed to stop a car traveling v mph is given by

$$s(v) = 0.04v^2 + v$$

(a) Set up a partial table of values for $s(v)$ with $0 \le v \le 70$.

(b) What was the velocity of a car that took 265 ft. to stop?

(c) Sketch the graph of s and s^{-1} on the same set of axes.

49. ***Business*** A microcomputer leasing company conducted periodic maintenance visits to the places where it rented its microcomputers. The company collected the following data on the number of microcomputers at each location and the total time, in minutes, needed to complete the maintenance.

No. of microcomputers	4	6	2	5
Time for maintenance (min)	205	282	93	237

No. of microcomputers	7	7	6	3
Time for maintenance (min)	324	336	277	153

No. of microcomputers	8	5	3	1
Time for maintenance (min)	368	242	140	75

(a) Graph these data.

(b) Find the linear regression curve for the given data.

(c) Graph the linear regression curve with the original data.

(d) What is the correlation coefficient?

CHAPTER 4 TEST

Use the function $f(x) = 7x - 5$ in Exercises 1 and 2, and determine the indicated value.

1. $f(-2)$

2. $f(3 - a)$

In Exercises 3 and 4, let $g(x) = \dfrac{x^2 - 2x - 15}{x + 3}$ and determine the indicated value.

3. $g(0)$

4. $g(5)$

Solve Exercise 5.

5. (a) Graph $h(x) = \frac{1}{2}\sqrt{x + 4} - 3$.

(b) What is the domain of h?

(c) What is the range of h?

(d) What is the x-intercept of h?

(e) What is the y-intercept of h?

In Exercises 6–11, let $f(x) = 3x - 15$ and $g(x) = \dfrac{x-5}{x+5}$, and determine the indicated value.

6. $(f+g)(x)$
7. $(f-g)(x)$
8. $(f \cdot g)(x)$
9. $(f/g)(x)$
10. $(f \circ g)(x)$
11. $(g \circ f)(x)$

In Exercises 12–16, sketch the graph of each of the following functions or relations.

12. $y = 3x - 4$
13. $f(x) = x^2 - 2x + 1$
14. $g(x) = \dfrac{x+5}{x-1}$
15. $\dfrac{x^2}{x+1} = 2x - 1 + y$
16. $\sqrt{x^2 - 1} = 3 - x + y$

Solve Exercises 17–19.

17. A researcher in physiology has decided that the function $r(s) = -s^2 + 12s - 20$ is a good mathematical model for the number of impulses fired after a nerve has been stimulated. Here, r is the number of responses per millisecond (ms) and s is the number of milliseconds since the nerve was stimulated.

(a) Graph this function.

(b) How many responses can be expected after 3 ms?

(c) If there are 16 responses, how many ms have elapsed since the nerve was stimulated?

(d) If there are 12 responses, how many ms have elapsed since the nerve was stimulated?

18. The cost of removing a certain pollutant is given by

$$C(x) = \frac{5x}{100 - x}$$

where $C(x)$ is the cost in thousands of dollars of removing x percent of the pollutant.

(a) Graph this function.

(b) How much will it cost to remove 50% of the pollutant?

(c) How much will it cost to remove 90% of the pollutant?

(d) What is the inverse function of C?

(e) If you had only \$15,000, how much of the pollutant could you remove?

19. ***Business*** An automobile rental agency wanted to see if there was a relationship between the minimum cost to rent a car and the number of cars rented. The following data were collected on the minimum cost and the number of cars rented each week for a 12-week period.

Minimumcost (\$)	19.95	20.95	18.95	19.95
Number of cars rented	265	232	293	237

Minimumcost (\$)	19.95	21.95	22.95	18.95
Number of cars rented	224	186	167	346

Minimumcost (\$)	19.95	20.95	22.95	19.95
Number of cars rented	318	242	140	275

(a) Graph these data.

(b) Find the equation of the linear regression curve for the given data.

(c) Graph the linear regression curve with the original data.

(d) What is the correlation coefficient?

5 AN INTRODUCTION TO TRIGONOMETRY AND VARIATION

The angle an access ramp makes with the ground must be 4.8° or less. Trigonometry allows you to determine how much room is needed for such a ramp.

Until now, we have studied only linear functions. But much of the mathematics used by people working in technical areas involves triangles.

In this chapter, we will learn about a new type of function—the trigonometric function. Trigonometric functions were originally developed to describe the relationship between the sides and angles of triangles. But, as so often happens in mathematics, other uses were discovered for these functions. Technical and scientific areas rely a great deal on trigonometric functions.

We will begin our study of trigonometry with the study of angles and angular measurements. We will then define the six trigonometric functions and their inverse functions, and study how we can use trigonometry to solve problems involving right triangles. (Chapter 8 will expand the applications to all types of triangles.) This chapter also provides some good opportunities to use our calculators and computers in new ways.

OBJECTIVES

AFTER COMPLETING THIS CHAPTER, YOU WILL BE ABLE TO:

- Convert angles between decimal degrees, radians, and degrees, minutes, and seconds.
- Draw (sketch by hand) an angle in standard position using either the measure of the angle in degrees or in radians or using the coordinates of a point on the terminal side of the angle.
- Find the trigonometric functions of an angle using a point on the terminal side of the angle and using a calculator or a spreadsheet.
- Find the acute angle (the reference angle) that produces a given value of a trigonometric function and use that angle to find the other angle (within one revolution) that has the same value of that trigonometric function (evaluate inverse trigonometric functions).
- Find the missing sides and angles of a right triangle.
- Find the algebraic sign of a given trigonometric function for an angle in any quadrant.
- Find the value of a trigonometric function, given any angle, using a calculator or spreadsheet.
- Find the value of inverse trigonometric functions.
- Convert between angular velocity measured in radians per second or in degrees per second and revolutions per second.
- Apply the trigonometric functions to applications of circular motion and to mechanics.

5.1 ANGLES, ANGLE MEASURE, AND TRIGONOMETRIC FUNCTIONS

At the beginning of this section, we will review our knowledge of angles and how they are measured. From this introduction we will quickly move into a study of the trigonometric functions.

POSITIVE AND NEGATIVE ANGLES

In Chapter 3, we gave two definitions of an angle. One of these was to think of generating a ray from an initial position to a terminal position. One revolution is the amount a ray would turn to return to its original position. As can be seen in Figure 5.1, if the rotation of the terminal side from the initial side is counterclockwise, the angle is a positive angle, but if the rotation is clockwise, the angle is a negative angle.

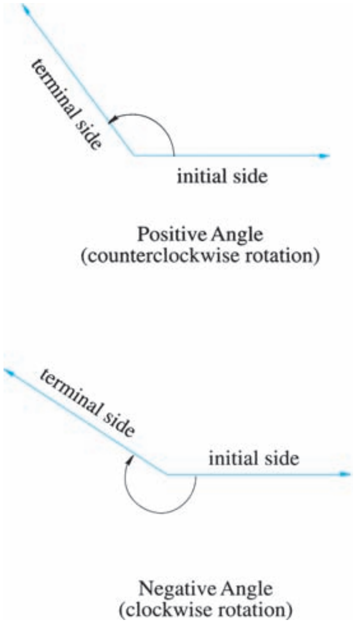

Figure 5.1

DEGREES AND RADIANS

Angles are measured using several different systems. The two most common are degrees and radians. In Chapter 3, we discussed how to change from degrees to radians and from radians to degrees. This is an important skill. We will briefly review this technique, but for more details you should refer to Section 3.1.

A degree is $\frac{1}{360}$ of a circle and the symbol ° is used to indicate degree(s). A radian is $\dfrac{1}{2\pi}$ of a circle. An entire circle contains 360° or 2π rad.

DEGREE-RADIAN CONVERSIONS

To convert from degrees to radians, multiply the number of degrees by $\dfrac{\pi}{180^\circ}$.

To convert from radians to degrees, multiply the number of radians by $\dfrac{180^\circ}{\pi}$.

Some scientific calculators have a key that will allow you to convert between degrees and radians. (Consult your owner's manual to see if your calculator can do this.)

EXAMPLE 5.1

Convert 72° to radians.

SOLUTION

$$\begin{aligned} 72^\circ &= 72^\circ\left(\frac{\pi}{180^\circ}\right) \\ &= \frac{72\pi}{180} \\ &= \frac{2\pi}{5}\text{ rad} \approx 1.25664\text{ rad} \end{aligned}$$

EXAMPLE 5.2

Convert 0.62 rad to degrees.

SOLUTION

$$\begin{aligned} 0.62\text{ rad} &= 0.62 \times \frac{180^\circ}{\pi} \\ &\approx 35.52352^\circ \end{aligned}$$

EXAMPLE 5.3

Convert $\frac{3\pi}{4}$ rad to degrees.

SOLUTION We multiply by $\dfrac{180^\circ}{\pi}$.

$$\frac{3\pi}{4} = \frac{3\pi}{4} \times \frac{180^\circ}{\pi}$$

$$= \frac{3}{4} \times 180°$$
$$= 135°$$

If you cannot remember these conversion values and if your calculator is not handy, use your knowledge of proportions. Since $180° = \pi$ rad, we can use the proportion given in the box.

CONVERTING BETWEEN DEGREES AND RADIANS

To convert d degrees to r radians (or vice versa), use the proportion

$$\frac{d}{180°} = \frac{r}{\pi}$$

When you use this ratio, you will know the value of either d or r and want to find the other.

EXAMPLE 5.4

Use this ratio to convert 72° to radians.

SOLUTION Here $d = 72°$, so

$$\frac{72°}{180°} = \frac{r}{\pi}$$

and $\frac{72°}{180°}\pi = \frac{2}{5}\pi \approx 1.25664$ gives you the same answer we got in Example 5.1. Notice that 72° is exactly $\frac{2}{5}\pi$, whereas 1.25664 is an approximation.

EXAMPLE 5.5

Use the ratio method to convert $\frac{3\pi}{4}$ to degrees.

SOLUTION

This is the same problem we worked in Example 5.3. Here $r = \frac{3\pi}{4}$, so

$$\frac{d}{180°} = \frac{3\pi/4}{\pi}$$

and

$$d = \frac{3\pi/4}{\pi} \cdot 180°$$
$$= \frac{3\pi}{4} \cdot \frac{1}{\pi} \cdot 180°$$
$$= \frac{3\cancel{\pi}}{4} \cdot \frac{1}{\cancel{\pi}} \cdot 180°$$

EXAMPLE 5.5 (Cont.)

$$= \frac{3}{4} \cdot 180°$$
$$= 135°.$$

Thus, we see that $\frac{3\pi}{4} = 135°$.

COTERMINAL ANGLES

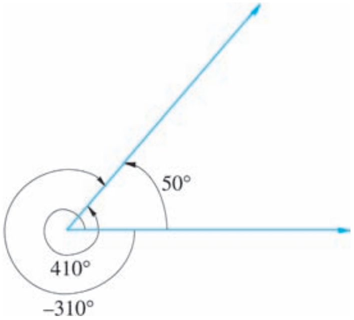

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Figure 5.2

If two angles have the same initial side and the same terminal side, they are **coterminal angles**. An example of coterminal angles is given in Figure 5.2. One way to find a coterminal angle of a given angle is to add 360° to the original angle. In Figure 5.2, the original angle is 50° and one coterminal angle is 50° + 360° = 410°. In fact, you could add any integer multiple of 360° to the original angle to find a coterminal angle. So, 50° + 4(360°) = 50° + 1,440° = 1,490° is another coterminal angle of a 50° angle.

In the same way, you could subtract an integer multiple of 360° from the original angle to get a coterminal angle. Thus, 50° − 360° = −310° is a coterminal angle of a 50° angle, and so is 50° − 2(360°) = 50° − 720° = −670°.

An angle is in **standard position** if its vertex is at the origin of a rectangular coordinate system and its initial side coincides with the positive *x*-axis. The angle is determined by the position of the terminal side. The angle is said to be in a certain quadrant if its terminal side lies in that quadrant. If the terminal side coincides with one of the coordinate axes, the angle is a **quadrantal angle**.

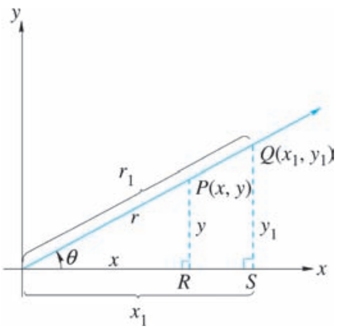

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Figure 5.3

Consider an angle θ in a standard position and let $P(x, y)$ be a fixed point on the terminal side of θ, as in Figure 5.3. We will call r the distance from O to P. From the Pythagorean theorem we know $r = \sqrt{x^2 + y^2}$. The length r is also called the *radius vector*. Suppose $Q(x_1, y_1)$ is any other point on the terminal side θ. If PR and QS are both perpendicular to the x-axis, then ΔPOR and ΔQOS are similar. As you remember from our discussion on proportions (Section 2.4), the corresponding sides of similar triangles are proportional. So, the ratios of the corresponding sides are equal. For example, $\frac{y}{r} = \frac{y_1}{r_1}$ and $\frac{x}{y} = \frac{x_1}{y_1}$. As long as θ does not change, these ratios will not change.

THE TRIGONOMETRIC FUNCTIONS

There are six possible ratios of two sides of a triangle. Each of these ratios has been given a name. These names (and their abbreviations) are sine (sin), cosine (cos), tangent (tan), cosecant (csc), secant (sec), and cotangent (cot). Because these ratios depend on the size of angle θ, they are written sin θ, cos θ, and so on. So, what you have are ratios that are functions of θ—the trigonometric functions. The six trigonometric, or trig, functions are defined using the triangle in Figure 5.4.

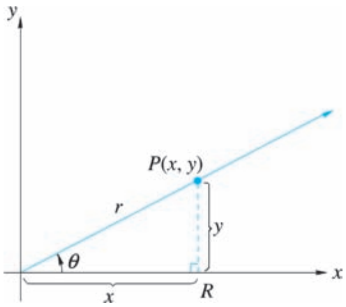

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Figure 5.4

TRIGONOMETRIC FUNCTIONS

$$\sin\theta = \frac{y}{r} \qquad \csc\theta = \frac{r}{y}$$

$$\cos\theta = \frac{x}{r} \qquad \sec\theta = \frac{r}{x}$$

$$\tan\theta = \frac{y}{x} \qquad \cot\theta = \frac{x}{y}$$

EXAMPLE 5.6

Given the point (5, 12) on the terminal side of an angle θ, find the six trigonometric functions of θ.

SOLUTION A sketch of the angle is in Figure 5.5. Since $x = 5$ and $y = 12$, we can find the radius vector r by using the Pythagorean theorem.

$$r = \sqrt{x^2 + y^2} = \sqrt{5^2 + 12^2} = \sqrt{25 + 144} = \sqrt{169} = 13$$

$$\sin\theta = \frac{y}{r} = \frac{12}{13} \qquad \csc\theta = \frac{r}{y} = \frac{13}{12}$$

$$\cos\theta = \frac{x}{r} = \frac{5}{13} \qquad \sec\theta = \frac{r}{x} = \frac{13}{5}$$

$$\tan\theta = \frac{y}{x} = \frac{12}{5} \qquad \cot\theta = \frac{x}{y} = \frac{5}{12}$$

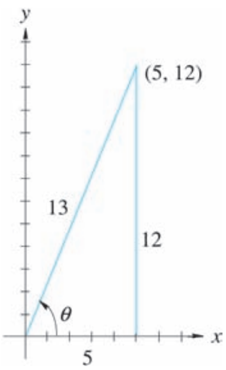

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Figure 5.5

NOTE The values of x and y may be positive, negative, or zero; but r is always positive or zero. Remember, r is never negative because $r = \sqrt{x^2 + y^2}$ and the expression $\sqrt{x^2 + y^2}$ is never negative.

Of course, there is nothing to restrict the terminal side to the first quadrant. In the next two examples, the terminal side is in Quadrants II and IV, respectively.

EXAMPLE 5.7

Given the point $(-8, 15)$ on the terminal side of an angle θ, find the six trigonometric functions of θ.

SOLUTION Here we have $x = -8$ and $y = 15$, so the radius vector r is given by $\sqrt{(-8)^2 + (15)^2} = \sqrt{64 + 225} = \sqrt{289} = 17$ as shown in Figure 5.6. The six trigonometric functions are

$$\sin\theta = \frac{15}{17} \qquad \csc\theta = \frac{17}{15}$$

$$\cos\theta = \frac{-8}{17} \qquad \sec\theta = \frac{-17}{8}$$

$$\tan\theta = -\frac{15}{8} \qquad \cot\theta = -\frac{8}{15}$$

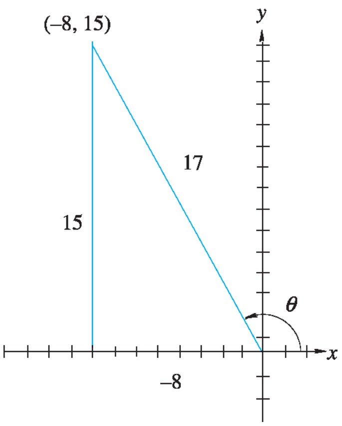

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Figure 5.6

As a final example, we will consider a case where the radius vector is not an integer. Note that none of the sides of the triangle have to be integers.

EXAMPLE 5.8

Given the point $(3, -5)$ on the terminal side of an angle θ, find the six trigonometric functions of θ.

SOLUTION Since $x = 3$ and $y = -5$ the radius vector, r, is $\sqrt{9 + 25} = \sqrt{34}$, as shown in Figure 5.7. The trigonometric functions of θ are

$$\sin\theta = \frac{-5}{\sqrt{34}} \approx -0.8575 \qquad \csc\theta = -\frac{\sqrt{34}}{5} \approx -1.1662$$

$$\cos\theta = \frac{3}{\sqrt{34}} \approx 0.5145 \qquad \sec\theta = \frac{\sqrt{34}}{3} \approx 1.9437$$

$$\tan\theta = -\frac{5}{3} \approx -1.6667 \qquad \cot\theta = -\frac{3}{5} = -0.6$$

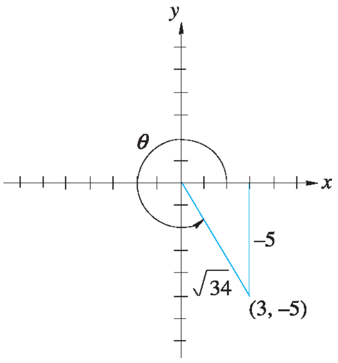

Figure 5.7

EXERCISE SET 5.1

Convert each of the angle measures in Exercises 1–8 from degrees to radians.

1. $90°$ **2.** $45°$ **3.** $80°$ **4.** $-15°$ **5.** $155°$ **6.** $-235°$ **7.** $215°$ **8.** $180°$

Convert each of the angle measures in Exercises 9–16 from radians to degrees.

9. 2 **10.** 3 **11.** 1.5 **12.** π **13.** $\frac{\pi}{3}$ **14.** $\frac{5\pi}{6}$ **15.** $-\frac{\pi}{4}$ **16.** -1.3

In Exercises 17–20, (a) draw each of the angles in standard position; (b) draw an arrow to indicate the rotation; (c) for each angle, find two other angles, one positive and one negative, that are coterminal with the given angle. (Note: There are many possible correct answers.)

17. $150°$ **18.** $315°$ **19.** $-135°$ **20.** $-30°$

Each point in Exercises 21–30 is on the terminal side of angle θ in standard position. Find the six trigonometric functions of the angle θ associated with each of these points.

21. $(4, 3)$ **22.** $(-6, -8)$ **23.** $(8, -15)$ **24.** $(-20, 21)$ **25.** $(1, 2)$ **26.** $(-2, 4)$ **27.** $(10, -8)$ **28.** $(-5, -2)$ **29.** $(\sqrt{11}, 5)$ **30.** $(-6, \sqrt{13})$

Find the trigonometric functions that exist for each of the quadrantral angles θ when drawn in standard position for each of the points in Exercises 31–34.

31. $(3, 0)$ **32.** $(0, -4)$ **33.** $(-5, 0)$ **34.** $(0, 6)$

For each of Exercises 35–40, there is a point P on the terminal side of an angle θ in standard position. From the information given, determine the six trigonometric functions of θ.

35. $x = 6, r = 10, y > 0$ **36.** $y = -9, r = 15, x > 0$ **37.** $x = -20, r = 29, y > 0$ **38.** $y = 5, r = 13, x < 0$ **39.** $x = -7, r = 8, y < 0$ **40.** $y = 5, r = 30, x < 0$

[IN YOUR WORDS]

41. **(a)** Write an explanation of how to convert from degrees to radians.

(b) Ask a classmate to use your written explanation to convert 50° to radians.

(c) Did your classmate get $50^\circ \approx 0.87266$? If not, either your classmate did not follow your directions or your explanation needs to be rewritten. Decide where the error was made and make the necessary corrections.

42. **(a)** Draw a right triangle on the xy-coordinate system with an acute angle at the origin, one leg on the positive x-axis, and the other leg vertical. Label the angle at the origin θ, the hypotenuse r, the horizontal leg x, and the vertical leg y.

(b) Define each of the trigonometric functions of θ in terms of x, y, and r.

(c) Compare your drawing in (a) with Figure 5.4. Compare your definitions in (b) with those in the "Trigonometric Functions" box near Figure 5.4.

5.2 VALUES OF THE TRIGONOMETRIC FUNCTIONS

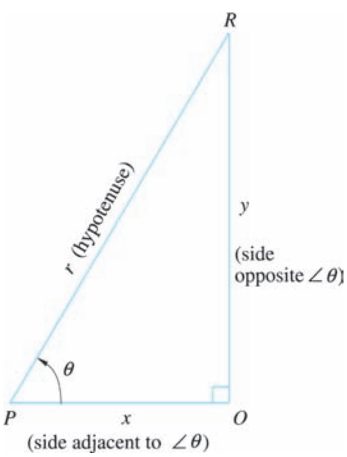

Figure 5.8

We have learned how to calculate the values of the trigonometric functions for an angle θ in standard position when we are given a point on the terminal side of θ. This is not always the most convenient way to find the values of the trigonometric functions for θ.

If we have an angle θ in the standard position and draw the triangle as we did in Figure 5.4, we get a picture that helps us to determine the values of the trigonometric functions for θ. If we look at just $\triangle POR$, we get a figure much like the one in Figure 5.8. The length of the hypotenuse is r, the length y is for the side opposite angle θ, and the other side x is the side adjacent to angle θ. We can use these descriptions of the sides to rephrase our definitions for the trigonometric function.

TRIGONOMETRIC FUNCTIONS

$$\sin\theta = \frac{y}{r} = \frac{\text{side opposite } \theta}{\text{hypotenuse}} \qquad \csc\theta = \frac{r}{y} = \frac{\text{hypotenuse}}{\text{side opposite } \theta}$$

$$\cos\theta = \frac{x}{r} = \frac{\text{side adjacent to } \theta}{\text{hypotenuse}} \qquad \sec\theta = \frac{r}{x} = \frac{\text{hypotenuse}}{\text{side adjacent to } \theta}$$

$$\tan\theta = \frac{y}{x} = \frac{\text{side opposite } \theta}{\text{side adjacent to } \theta} \qquad \cot\theta = \frac{x}{y} = \frac{\text{side adjacent to } \theta}{\text{side opposite } \theta}$$

With these relationships, we can find the trigonometric functions of any angle of a right triangle.

EXAMPLE 5.9

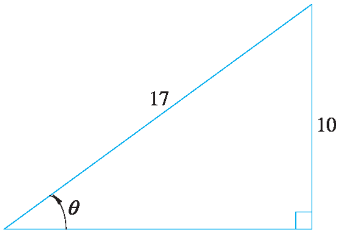

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Figure 5.9

Determine the values of the trigonometric functions for an angle of a triangle with a hypotenuse of 17 and the opposite side of length 10, as shown in Figure 5.9.

SOLUTION The length of the adjacent side x is missing. Use the Pythagorean theorem, $x = \sqrt{r^2 - y^2}$. Since $r = 17$ and $y = 10$, $x = \sqrt{17^2 - 10^2} = \sqrt{189} = 3\sqrt{21}$. The trigonometric functions are

$$\sin\theta = \frac{\text{side opposite}}{\text{hypotenuse}} = \frac{10}{17} \qquad \csc\theta = \frac{\text{hypotenuse}}{\text{side opposite}} = \frac{17}{10}$$

$$\cos\theta = \frac{\text{side adjacent}}{\text{hypotenuse}} = \frac{3\sqrt{21}}{17} \qquad \sec\theta = \frac{\text{hypotenuse}}{\text{side adjacent}} = \frac{17}{3\sqrt{21}}$$

$$\tan\theta = \frac{\text{side opposite}}{\text{side adjacent}} = \frac{10}{3\sqrt{21}} \qquad \cot\theta = \frac{\text{side adjacent}}{\text{side opposite}} = \frac{3\sqrt{21}}{10}$$

We could have worked the problem had we been given the value of one of the trigonometric functions rather than the lengths of two of the sides.

EXAMPLE 5.10

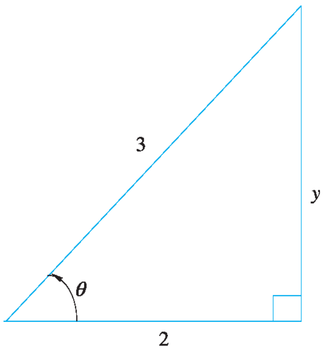

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Figure 5.10

In Figure 5.10, if θ is an angle of a triangle and $\cos\theta = \frac{2}{3}$, what are the values of the other trigonometric functions?

SOLUTION The length y of the side opposite θ is $\sqrt{3^2 - 2^2} = \sqrt{5}$. Thus, we have the values for the other functions

$$\sin\theta = \frac{\sqrt{5}}{3} \qquad \csc\theta = \frac{3}{\sqrt{5}} \qquad \sec\theta = \frac{3}{2}$$

$$\tan\theta = \frac{\sqrt{5}}{2} \qquad \cot\theta = \frac{2}{\sqrt{5}}$$

Look at Figure 5.11. This is the same triangle that was in Figure 5.8. There is another acute angle in that triangle, labeled ϕ. Angles θ and ϕ are complementary. Remember, complementary angles are two angles that measure 90° when added. So, $\theta + \phi = 90°$. What are the trigonometric functions for ϕ? The side opposite ϕ is x and the side adjacent to ϕ is y, so

$$\sin\phi = \frac{x}{r} \qquad \csc\phi = \frac{r}{x}$$

$$\cos\phi = \frac{y}{r} \qquad \sec\phi = \frac{r}{y}$$

$$\tan\phi = \frac{x}{y} \qquad \cot\phi = \frac{y}{x}$$

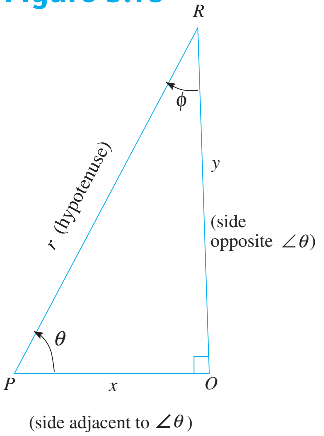

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Figure 5.11

From this, we get the principle of cofunctions of complementary angles.

TRIGONOMETRIC FUNCTIONS OF COMPLEMENTARY ANGLES

If θ and ϕ are complementary angles, then

$$\sin\theta = \cos\phi \qquad \tan\theta = \cot\phi \qquad \csc\theta = \sec\phi$$

You may have begun to notice a pattern with the trigonometric functions. The values of some of the trig functions are the reciprocals of other functions. These are known as the **reciprocal identities**. There are three reciprocal identities. (An identity is an equation that is true for every value in the domain of its variables.)

RECIPROCAL IDENTITIES

$$\csc\theta = \frac{1}{\sin\theta} \qquad \sec\theta = \frac{1}{\cos\theta} \qquad \cot\theta = \frac{1}{\tan\theta}$$

There is also a relationship between the sin θ, cos θ, and the tan θ. This relationship, and its reciprocal, are known as the **quotient identities**.

QUOTIENT IDENTITIES

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \qquad \cot\theta = \frac{\cos\theta}{\sin\theta}$$

None of these identities is true when the denominator is zero.

EXAMPLE 5.11

If $\sin\theta = \frac{1}{2}$ and $\cos\theta = \frac{\sqrt{3}}{2}$, find the values of the other four trigonometric functions.

SOLUTION Using the reciprocal and quotient identities we have

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{\frac{1}{2}}{\frac{\sqrt{3}}{2}} = \frac{1}{\sqrt{3}}$$

EXAMPLE 5.11 (Cont.)

$$\cot\theta = \frac{1}{\tan\theta} = \frac{1}{\frac{1}{\sqrt{3}}} = \frac{\sqrt{3}}{1} = \sqrt{3}$$

$$\csc\theta = \frac{1}{\sin\theta} = \frac{1}{\frac{1}{2}} = \frac{2}{1} = 2$$

$$\sec\theta = \frac{1}{\cos\theta} = \frac{1}{\frac{\sqrt{3}}{2}} = \frac{2}{\sqrt{3}}$$

TRIGONOMETRIC VALUES

Until now, we have been finding the values of trigonometric functions without knowing the size of the angle θ. But, many times we are given the value of θ and are asked to determine the values of the trigonometric functions for that angle.

There are two basic methods used to find values of trigonometric functions. By far, the easiest way is with a calculator or a computer. The other basic method is with a table of trigonometric values. There are some trigonometric angles that seem to be used quite frequently. Some people learn the values of these basic angles so they can readily use these values when they are needed. We will concentrate on calculators and computers for determining the value of a trigonometric function.

Before you use a calculator or a computer, you need to decide if the angle is measured in degrees or radians. Most calculators can compute the trigonometric functions of an angle whether it is in degrees or radians, as long as the calculator is set to work in that mode.

GRAPHING CALCULATORS

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Figure 5.12

Different models of calculators have different symbols or different locations for the symbols that let you know which mode the calculator is in. For example, on a graphing calculator, such as the Texas Instruments TI-83 or TI-84, press MODE. The TI-83 or TI-84 will display a screen like that in Figure 5.12. (The calculator displayed in Figure 5.12 is in degree mode.)

Use the cursor keys ▲, ▼, ◄, and ► to darken the appropriate degree or radian mode. Press ENTER to set the calculator in that mode. If your calculator is not a TI-83 or TI-84, make sure that you consult the owner's manual for your calculator.

TRIGONOMETRIC VALUES ON A CALCULATOR

Some calculators have three trigonometric function keys— SIN, COS, and TAN. If you want one of the other trigonometric functions, you will have to use one of these keys and the $1/x$ or x^{-1} key. Here you need to know the reciprocal identities discussed earlier in this section. You must press the $1/x$ or x^{-1} key *after* you have pressed the trigonometric function key.

CAUTION Make sure that your calculator is set in the correct degree or radian mode before you begin to work the problem.

EXAMPLE 5.12

Determine the trigonometric functions of 48.9°.

SOLUTION Put the calculator in degree mode, and then proceed as follows.

Function	ENTER	DISPLAY
sin 48.9°	SIN 48.9 ENTER	0.753563
cos 48.9°	COS 48.9 ENTER	0.6573752
tan 48.9°	TAN 48.9 ENTER	1.1463215
csc 48.9°	(SIN 48.9) x^{-1} ENTER	1.3270284
sec 48.9°	(COS 48.9) x^{-1} ENTER	1.5212012
cot 48.9°	(TAN 48.9) x^{-1} ENTER	0.8723556

HINT You do not need to change modes to find the trigonometric function of an angle measured in degrees. There is a degree symbol on the ANGLE menu of a TI-83 or TI-84 graphics calculator. You can use the calculator's degree symbol to get the correct answer from either degree or radian mode. To get sin 48.9° on a TI-83/84, press SIN 48.9 2ND APPS [ANGLE] 1) ENTER. The calculator should display the result .7535633923.

To find the values of the trigonometric functions for an angle in radians, put the calculator in the radian mode. Then proceed as in Example 5.13.

EXAMPLE 5.13

Use a calculator to determine the trigonometric functions of 0.65 rad.

SOLUTION Put the calculator in radian mode and then proceed as follows.

Function	ENTER	DISPLAY
sin 0.65	SIN 0.65 ENTER	0.6051864
cos 0.65	COS 0.65 ENTER	0.7960838
tan 0.65	TAN 0.65 ENTER	0.7602044
csc 0.65	(SIN 0.65) x^{-1} ENTER	1.6523834
sec 0.65	(COS 0.65) x^{-1} ENTER	1.2561492
cot 0.65	(TAN 0.65) x^{-1} ENTER	1.3154357

HINT Just as you do not need to change modes to find the trigonometric function of an angle measured in degrees, you also do not need to change modes to find the trigonometric function of an angle measured in radians. The radian symbol on the TI-83 or TI-84 calculator is an *r*. You can use the calculator's radian symbol to get the correct answer from either degree or radian mode. To get sin 0.65 on a

TI-83/84, press SIN .65 2ND APPS [ANGLE] 3) ENTER. The calculator should display `sin .65'` on one line, with the result, .6051864057, on the next line.

SPREADSHEETS AND THE TRIGONOMETRIC FUNCTIONS

When performing calculations with trigonometric functions, it is important to know that the spreadsheet is always in radian "mode," using the terminology of the graphing calculator. There is no way to switch modes but it is fairly easy to convert degrees to radians.

To find sin $\left(\frac{\pi}{3}\right)$ enter the expression almost as it is written. Remember to enter π as either PI( ) or pi( ) (See Figure 5.13a.) You can enter the expression in upper case letters as shown in Figure 5.13a or in lower case letters: sin (pi ( ) /3).

A1 | f_x =SIN(PI()/3)

	A	B	C	D	E
1	0.866025				

Figure 5.13a

To find the sin (60°), you must convert 60° to radians by entering RADIANS (60) either within the expression (see Figure 5.13b) or separately (see Figure 5.13c).

A1 | f_x =SIN(RADIANS(60))

	A	B	C	D	E
1	0.866025				

Figure 5.13b

A3 | f_x =SIN(A2)

	A	B	C	D
1		The expression in Cell A2		
2	1.047198	= RADIANS(60)		
3	0.866025			

Figure 5.13c

We will rework Examples 5.12 and 5.13 using a spreadsheet rather than a calculator.

EXAMPLE 5.14

Use a spreadsheet to determine the trigonometric functions of 48.9°.

SOLUTION Remember that a spreadsheet is automatically in radian mode, so you must tell the program that you want answers for degrees. The input and the results are shown in Figure 5.14.

	A	B	C
1	Function	Enter	Result
2	sin 48.9°	=SIN(RADIANS(48.9))	0.753563
3	cos 48.9°	=COS(RADIANS(48.9))	0.657375
4	TAN 48.9°	=TAN(RADIANS(48.9))	1.146322
5	CSC 48.9°	=1/SIN(RADIANS(48.9))	1.327028
6	SEC 48.9°	=1/COS(RADIANS(48.9))	1.521201
7	COT 48.9°	=1/TAN(RADIANS(48.9))	0.872356

Figure 5.14

EXAMPLE 5.15

Use a spreadsheet to determine the trigonometric functions of 0.65 rad.

SOLUTION Remember that a spreadsheet is automatically in radian mode. The input and the results are shown in Figure 5.15.

	A	B	C
1	Function	Enter	Result
2	sin 0.65	=SIN(0.65)	0.605186
3	cos 0.65	=COS(0.65)	0.796084
4	TAN 0.65	=TAN(0.65)	0.760204
5	CSC 0.65	=1/SIN(0.65)	1.652383
6	SEC 0.65	=1/COS(0.65)	1.256149
7	COT 0.65	=1/TAN(0.65)	1.315436

Figure 5.15

EXERCISE SET 5.2

In Exercises 1–6, find the six trigonometric functions for angle θ for the indicated sides of $\triangle ABC$. (See Figure 5.16)

1. $a = 5, c = 13$

2. $a = 7, b = 8$

3. $a = 1.2, c = 2$

4. $a = 2.1, b = 2.8$

5. $a = 1.4, b = 2.3$

6. $c = 3.5, b = 2.1$

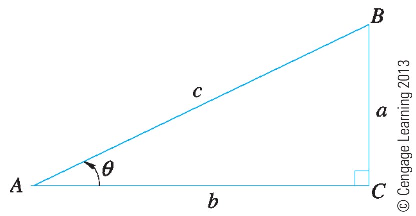

Figure 5.16

In Exercises 7–12, let θ be an angle of a right triangle with the given trigonometric function. Find the values of the other trigonometric functions.

7. $\cos\theta = \frac{8}{17}$

8. $\tan\theta = \frac{21}{20}$

9. $\sin\theta = \frac{3}{5}$

10. $\sec\theta = \frac{10}{6}$

11. $\csc\theta = \frac{29}{20}$

12. $\csc\theta = \frac{15}{12}$

In Exercises 13–16, the values of two of the trigonometric functions of angle θ are given. Use this information to determine the values of the other four trigonometric functions for θ.

13. $\sin\theta = 0.866, \cos\theta = 0.5$

14. $\sin\theta = 0.975, \cos\theta = 0.222$

15. $\sin\theta = 0.085, \sec\theta = 1.004$

16. $\csc\theta = 4.872, \cos\theta = 0.979$

Use a calculator or a spreadsheet to determine the values of each of the indicated trigonometric functions in Exercises 17–32.

17. $\sin 18.6°$

18. $\cos 38.4°$

19. $\tan 18.3°$

20. $\sin 20°15'$

21. $\tan 76°32'$

22. $\sec 24°\ 14'$

23. $\cot 82.6°$

24. $\csc 19°50'$

25. sin 0.25 rad

26. cos 0.4 rad

27. tan 0.63 rad

28. sec 1.35 rad

29. cot 1.43 rad

30. sin 1.21 rad

31. csc 0.21 rad

32. tan 1.555 rad

Solve Exercises 33–36.

33. ***Dynamics*** If there is no air resistance, a projectile fired at an angle θ above the horizontal with an initial velocity of v_0 has a range R of

$$R = \frac{v_0^2}{g}\sin 2\theta$$

A football is thrown with initial velocity of 15 m/s at an angle of 40° above the horizontal. If $g = 9.8$ m/s^2, how far will the football travel?

34. ***Dynamics*** If the football in Exercise 33 had been thrown at an angle of 45° above the horizontal, how much further, or shorter, would the ball have traveled? If the angle had been 48°, how would the results have differed?

35. ***Electricity*** In an ac circuit that contains resistance, inductance, and capacitance in series, the angle of the applied voltage v and the voltage drop across the resistance V_R is the *phase angle* ϕ, and $V_R = V\cos\phi$. If the phase angle is 32° and the applied voltage is 5.8 V, what is the effective voltage across the resistor V_R?

36. ***Electricity*** Consider an ac circuit that contains resistance, inductance, and capacitance in series, as in Exercise 35. If the phase angle is 71° and the applied voltage is 11.25 V, what is the effective voltage across the resistor, V_R?

[IN YOUR WORDS]

37. **(a)** Draw a right triangle with acute angle θ. Label the hypotenuse, adjacent side, and opposite side.

(b) Define each of the trigonometric functions of θ in terms of the hypotenuse, the adjacent side, and the opposite side.

(c) Compare your drawing in (a) with Figure 5.8. Check your definitions in (b) with those in the box near Figure 5.8.

38. **(a)** What are the reciprocal identities?

(b) What are the quotient identities?

5.3 THE RIGHT TRIANGLE

In Section 5.2, we learned how to determine trigonometric functions given the lengths of two sides of a right triangle. We are now ready to begin finding the unknown parts of a triangle if we know the length of one side and the size of one of the acute angles. Once we can do that, we will begin to apply our knowledge to solving some problems that involve triangles.

Look at the right triangle in Figure 5.17. The right angle is labeled C and the other two vertices A and B. The side opposite each of these vertices is labeled with the lowercase version of the same letter. Thus, side a is opposite $\angle A$, side b is opposite $\angle B$, and side c is opposite $\angle C$. Since this is a right triangle, angles A and B are complementary.

We now have the tools to solve right triangles. To solve a triangle means to find the sizes of all unknown sides and angles.

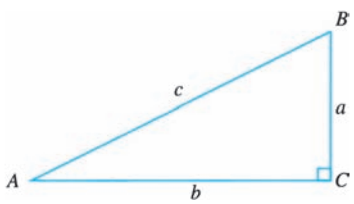

Figure 5.17

EXAMPLE 5.16

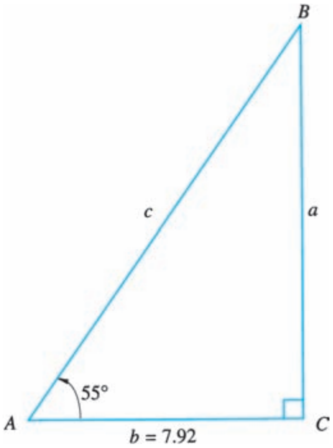

Figure 5.18

Given $A = 55°$ and $b = 7.92$, solve the right triangle ABC.

SOLUTION This data chart contains the information that we know and shows what we have left to find. Notice that, since this is a right triangle, we wrote that $C = 90°$.

sides	angles
$a =$ ____	$A = 55°$
$b = 7.92$	$B =$ ____
$c =$ ____	$C = 90°$

Since $A = 55°$, then $B = 90° - 55° = 35°$. If you look at Figure 5.18 you see that $\tan A = \frac{a}{b}$ or $a = b \tan A$. We know $A = 55°$ and $b = 7.92$, so $a = 7.92 \tan 55° \approx 11.310932$ or about 11.31. You could use $\sec A = \frac{c}{b}$, to determine c, since $c = b \sec A$. But, your calculator does not have a sec key so we will find c using $\cos A$.

$$\cos A = \frac{b}{c}$$

$$\text{so } c = \frac{b}{\cos A}$$

$$= \frac{7.92}{\cos 55°}$$

$$\approx 13.81$$

Look at the data chart again. This time all of the parts of the chart have been filled in, so we know that we have solved for all the missing parts of this triangle. From the completed chart, we can easily see the measures of any part of the triangle.

sides	angles
$a \approx 11.31$	$A = 55°$
$b = 7.92$	$B = 35°$
$c \approx 13.81$	$C = 90°$

We could have used our derived value of a and the Pythagorean theorem to find c, or we could have used a and one of the other trigonometric functions.

HINT It is usually best not to use the derived values to calculate other values because of errors that can be introduced by rounding.

By the way, if you use the Pythagorean theorem to check the results in Example 5.16, you will see that $(7.92)^2 + (11.31)^2 \neq (13.81)^2$. The Pythagorean theorem says that these should be equal, but because we are using rounded-off values, they are not. However, they are equal to three significant digits.

USING THE 2ND OR INV KEY ON A CALCULATOR

Until now, we have calculated the values of a trigonometric function from the size of the angle. But, what if we know the value of a trigonometric function and want to know the size of the angle? Here you press the INV or 2nd key on the calculator before you enter the function key.

EXAMPLE 5.17

Use a graphing calculator to determine the angles that have each of the following values of trigonometric functions: $\sin A = 0.785$, $\cos B = 0.437$, $\tan C = 4.213$, and $\csc D = 8.915$.

SOLUTION

Function	ENTER	DISPLAY
$\sin A = 0.785$	2nd SIN 0.785	51.720678
$\cos B = 0.437$	2nd COS 0.437	64.087375
$\tan C = 4.213$	2nd TAN 4.213	75.547345
$\csc D = 8.915$	2nd SIN 8.915 x^{-1}	6.4404506
or	2nd SIN (1 ÷ 8.915)	6.4404506

So, $A \approx 51.72°$, $B \approx 64.09°$, $C \approx 76.65°$, and $D \approx 6.44°$. Notice that to find angle D, we had to use the 2nd SIN keys with the reciprocal of 8.915. This is because $\csc D = 8.915$ is the same as $\frac{1}{\sin D} = 8.915$ or $\sin D = \frac{1}{8.915}$. Hence, $D = \sin^{-1}\frac{1}{8.915}$, which may be entered in your calculator as either

2nd SIN 8.915 x^{-1}

or 2nd SIN (1 ÷ 8.915)

EXAMPLE 5.18

Solve right triangle ABC ($\triangle ABC$), if $a = 23.5$ and $c = 42.7$, as shown in Figure 5.19.

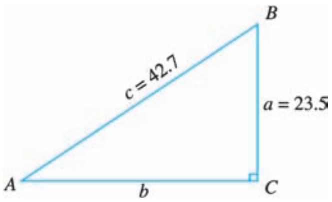

Figure 5.19

SOLUTION The following data chart shows the information we are starting with and what we have to find.

sides	angles
$a = 23.5$	$A =$ ____
$b =$ ____	$B =$ ____
$c = 42.7$	$C = 90°$

From the Pythagorean theorem we know that $b^2 = c^2 - a^2 = (42.7)^2 - (23.5)^2 = 1271.04$, so $b = 35.7$. Now, $\sin A = \frac{23.5}{42.7} \approx 0.5503513$. Pressing 2nd SIN 0.5503513, we get 33.391116. Thus, $A \approx 33.4°$ and $B \approx 90° - 33.4° = 56.6°$. The completed data chart follows.

sides	angles
$a = 23.5$	$A \approx 33.4°$
$b \approx 35.7$	$B \approx 56.6°$
$c = 42.7$	$C = 90°$

HINT Two possible shortcuts could have been taken in Example 5.18 when you determined the size of *A*.

1. After you calculated $\sin A = \frac{23.5}{42.7} \approx 0.5503513$, you could have used the ability of a graphing calculator to keep the last answer in memory. Thus, rather than press 2nd SIN 0.5503513, you could have pressed

   2nd SIN 2nd ANS ENTER

2. An even faster method would have performed the calculation in one step:

   2nd SIN (23.5 ÷ 42.7) ENTER

EXAMPLE 5.19

Solve for the right triangle *ABC* if $B = 0.62$ rad and $a = 23.85$, as shown in Figure 5.20.

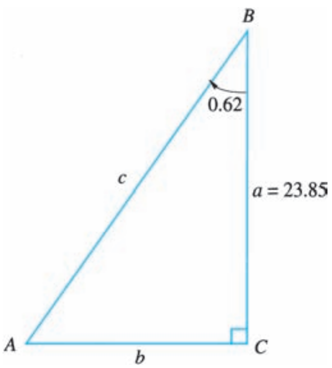

Figure 5.20

SOLUTION The data chart for this triangle is

sides	angles
$a = 23.85$	$A =$ ____
$b =$ ____	$B = 0.62$
$c =$ ____	$C = \frac{\pi}{2} \approx 1.57$

In this example, we can use angle *B* as our reference angle. So, $\tan B = \frac{b}{a}$ and $b = a \tan B = (23.85) \tan 0.62 = 17.02673$, or $b \approx 17.03$. (Did you remember to put your calculator in radian mode?)

Also, $\cos B = \frac{a}{c}$, so

$$\begin{aligned} c &= \frac{a}{\cos B} \\ &= \frac{23.85}{\cos 0.62} \\ &\approx 29.30413 \\ \text{or} \quad &\approx 29.30 \end{aligned}$$

Finally, we have $\tan A = \frac{23.85}{17.03} = 1.4004698$, and pressing 2nd TAN 1.4004698, we get 0.9507055, or $A \approx 0.95$ rad.

EXAMPLE 5.19 (Cont.)

Remember from Exercise 1 in Exercise Set 5.1 that $90° \approx 1.57$ rad. If you used your calculator to convert 90° to radians, you could then have found A as $1.57 - 0.62 = 0.95$.

The completed data chart for this example is

sides	angles
$a = 23.85$	$A \approx 0.95$
$b \approx 17.03$	$B = 0.62$
$c \approx 29.30$	$C = \frac{\pi}{2} \approx 1.57$

APPLICATION CONSTRUCTION

EXAMPLE 5.20

A concrete access ramp to a building is being built. The ramp will be 48″ wide and make a 4.8° angle with the ground. If it ends at a platform that is $23\frac{1}{4}''$ above the ground, how much concrete will be needed?

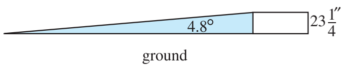

Figure 5.21

SOLUTION A side view of the ramp is in Figure 5.21. We are interested only in the concrete needed for the shaded part of the figure. The ramp is in the shape of a triangular prism. We want to first find the area of the triangle that forms the base of the prism. We know the height of the triangle is $23\frac{1}{4}''$. We need to find the length of the side that lies on the ground. Here we have

$$\tan 4.8° = \frac{23\frac{1}{4}''}{\text{bottom}}$$

$$\text{bottom} = \frac{23\frac{1}{4}''}{\tan 4.8°}$$

$$\approx 276.876 \text{ inches}$$

The area of the base of the prism is $A = \frac{1}{2}\left(23\frac{1}{4}''\right)(276.876'')$, and the volume of the prism is

$$V = 48''\ A = 48''\left[\frac{1}{2}\left(23\frac{1''}{4}\right)(276.876'')\right]$$

$$\approx 154{,}496.8 \text{ in.}^3$$

Concrete is sold in cubic yards, so we need to convert this answer to cubic yards. There are 36″ in one yard, which gives the following conversion.

$$V = 154{,}496.8 \text{ in.}^3$$

$$= \frac{154{,}496.8 \text{ in.}^3}{1} \quad \frac{1 \text{ yd}^3}{(36 \text{ in.})^3}$$

$$\approx 3.31 \text{ yd}^3$$

It will take about 3.31 yd³ of concrete to build this ramp.

ANGLES OF ELEVATION OR DEPRESSION

Frequently when solving problems involving trigonometry, we have to use the **angle of elevation** (see Figure 5.22), which is the angle, measured from the horizontal, through which an observer would have to elevate his or her line of sight in order to see an object. Similarly, the **angle of depression** is the angle, measured from the horizontal, through which an observer has to lower his or her line of sight in order to see an object. (See Figure 5.23.)

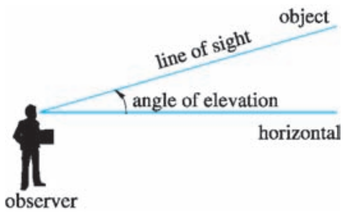

Figure 5.22

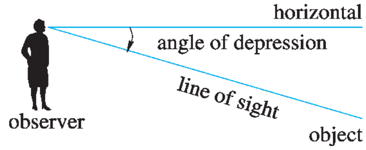

Figure 5.23

APPLICATION CIVIL ENGINEERING

EXAMPLE 5.21

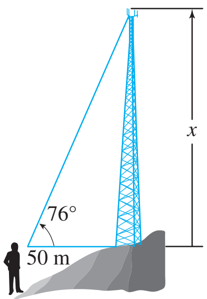

Figure 5.24

A person is standing 50 m from the base of a tower at eye level with the base of the tower. (See Figure 5.24.) The angle of elevation to the top of the tower is 76°. How high is the tower?

SOLUTION We have a right triangle as sketched in Figure 5.24. The height of the tower is labeled x, so we have $\tan 76° = \frac{x}{50}$, or

$$x = 50\tan 76°$$
$$= 200.53905$$

The tower is about 200.5 m high.

APPLICATION CIVIL ENGINEERING

EXAMPLE 5.22

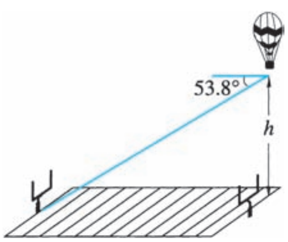

Figure 5.25

Two people are in a hot air balloon. One of them is able to get a sighting from the gondola of the balloon as it passes over one end of a football field, as shown in Figure 5.25. The angle of depression to the other end of the football field is 53.8°. This person knows that the length of the football field, including the end zones, is 120 yd. How high was the balloon when it went over the football field?

SOLUTION We want to determine the height of the right triangle. The height has been labeled h in Figure 5.25. So, we have $\tan 53.8° = \frac{h}{120}$, or

$$h = 120\tan 53.8°$$
$$= 163.9592079$$

The balloon was about 164 yd high.

APPLICATION CONSTRUCTION

EXAMPLE 5.23 A surveyor marks off a right-angle corner of a rectangular house foundation. In sighting on the diagonally opposite corner of the foundation, the line of sight has to move through an angle of 35° as shown in Figure 5.26a. If the length of the short side of the foundation is 46.3 ft, what is the length of the long side?

SOLUTION We have the situation sketched in Figure 5.26b. (Notice that in Figure 5.26b, the right angle does not look like a right angle.) But, this is confusing; so in Figure 5.26c the triangle has been rotated to the more "typical" position with the right angle along the bottom. Since $\tan 35° = \frac{46.3}{x}$, we have

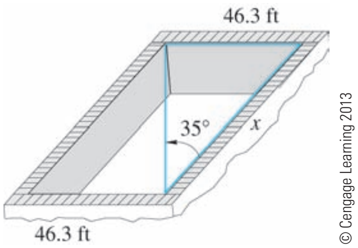

Figure 5.26a

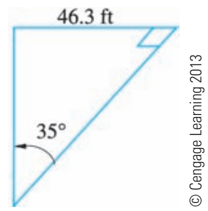

Figure 5.26b

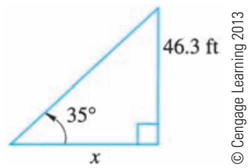

Figure 5.26c

$$x = \frac{46.3}{\tan 35°}$$
$$= 66.123253$$

So the long side is about 66.1 ft.

EXERCISE SET 5.3

Find the acute angles for the trigonometric functions given in Exercises 1–12. (Give answers to the nearest 0.01.)

1. $\sin A = 0.732$
2. $\cos B = 0.285$
3. $\tan C = 4.671$
4. $\sin D = 0.049$
5. $\cos E = 0.839$
6. $\tan F = 0.539$
7. $\sec G = 3.421$
8. $\csc H = 1.924$
9. $\cot I = 0.539$
10. $\sec J = 1.734$
11. $\csc K = 4.761$
12. $\cot L = 4.021$

In Exercises 13–30, sketch each right triangle and solve it. Use your knowledge of significant figures to round off appropriately.

13. $A = 16.5°$, $a = 7.3$
14. $B = 53°$, $b = 9.1$
15. $A = 72.6°$, $c = 20$
16. $B = 12.7°$, $a = 19.4$
17. $A = 43°$, $b = 34.6$
18. $B = 67°$, $c = 32.4$
19. $A = 0.92$ rad, $a = 6.5$
20. $B = 1.13$ rad, $b = 24$
21. $A = 0.15$ rad, $c = 18$
22. $B = 0.23$ rad, $a = 9.7$
23. $A = 1.41$ rad, $b = 40$
24. $A = 1.15$ rad, $c = 18$

25. $a = 9, b = 15$

26. $a = 19.3, c = 24.4$

27. $b = 9.3, c = 18$

28. $a = 14, b = 9.3$

29. $a = 20, c = 30$

30. $b = 15, c = 25$

Solve Exercises 31–44.

31. ***Electricity*** In an ac circuit that has inductance x_L and resistance R, the phase angle can be determined from the equation $\tan\phi = \dfrac{x_L}{R}$. If $x_L = 12.3\ \Omega$ and $R = 19.7\ \Omega$, what is the phase angle?

32. ***Navigation*** From the top of a lighthouse, the angle of depression to the waterline of a boat is 23.2°. If the lighthouse is 222 ft high, how far away is the ship from the bottom of the lighthouse?

33. ***Navigation*** An airplane is flying at an altitude of 700 m when the copilot spots a ship in distress at an angle of depression of 37.6°. How far is it from the plane to the ship?

34. ***Construction*** A bridge is to be constructed across a river. As shown in Figure 5.27, a piling is to be placed at point A and another at C. To find the distance between A and C a surveyor locates point B exactly 95 m from C so that $\angle C$ is a right angle. If $\angle B$ is 57.62°, how far apart are the pilings?

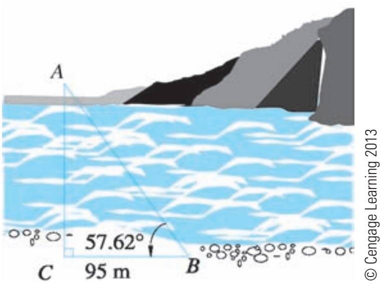

Figure 5.27

35. A vector is usually resolved into component vectors that are perpendicular to each other. Thus, you get a rectangle with one side F_x, the horizontal component, and the other F_y, the vertical component, as shown in Figure 5.28. The original vector F is the diagonal of the rectangle. If $F = 10$ N and $\theta = 30°$, what are the horizontal and vertical components of this force?

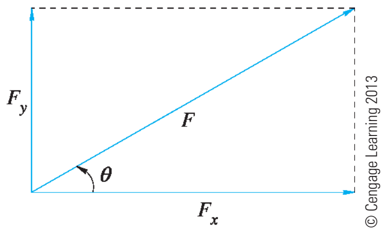

Figure 5.28

36. ***Physics*** Two forces, one of 20 lb and the other of 12 lb, act on a body in directions perpendicular to each other. The magnitude of the resultant is the length of the diagonal of the rectangle formed from the two perpendicular forces. What is the magnitude of the resultant of these forces? What is the size of the angle the resultant of these forces makes with the smaller force?

37. ***Transportation engineering*** Highway curves are usually banked at an angle ϕ. The proper banking angle for a car making a turn of radius r at velocity v is $\tan\phi = \dfrac{v}{gr}$, where $g = 32\,\text{ft/s}^2$ or $9.8\,\text{m/s}^2$. Find the proper banking angle for a car moving at 55 mph to go around a curve 1,200 ft in radius. (First, convert 55 mph to ft/s.)

38. ***Transportation engineering*** Find the proper banking angle for a car moving 88 km/h to go around a curve 500 m in radius. (First, convert 88 km/h to m/s.)

39. ***Transportation engineering*** Two straight highways A and B intersect at an angle 67°. A service station is located on highway A 300 m from the intersection. What is the location of the point on highway B that is closest to the service station? How near is it?

40. ***Air traffic control*** One way of measuring the ceiling, or height of the bottom of the clouds, at an airport is to focus a light beam straight up at the clouds at a known distance from an observation point. The angle of elevation from the observation point to the light beam on the clouds is θ. Find the ceiling height if the observation point is 950 ft from the source of the light beam and $\theta = 58.6°$.

41. ***Forestry*** Using a clinometer and tape measure, a forest ranger locates the top of a tree at a 67.3° angle of elevation from a point 25.75 ft from the base of the tree and on the same level as the base. How tall is the tree?

42. ***Fire science*** A 32.0-m ladder on a fire truck can be extended at an angle of 65.0°. The base of the ladder is located at a point that is 2.7 m above the ground.
   (a) How close to the base of a building must the base of the ladder be located for the end of the ladder to reach the building?
   (b) A person needing to be rescued is located 35.85 m above the ground. Can a 1.92-m tall firefighter reach the person if the firefighter (foolishly) stands on the very top rung of the ladder?

43. ***Construction*** The roof in Figure 5.29 is 8 ft high at its apex. If the roof is 36 ft wide, what is the angle of incline of the roof?

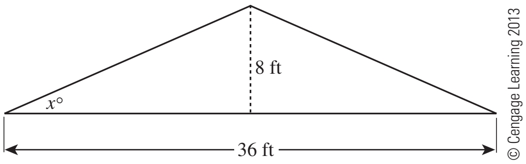

Figure 5.29

44. ***Electricity*** A furnace 4′0″ wide and 3′8″ high is placed in an attic with the front side 2 ft back from a line dropped from the apex. The gabled roof has an incline of 22° and covers a width span of 33 ft (Figure 5.30). What is the clearance between the top back edge of the furnace and the roof?

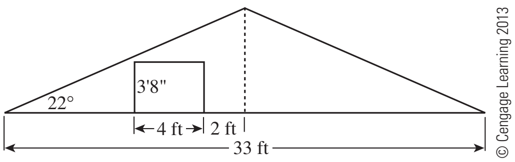

Figure 5.30

[IN YOUR WORDS]

45. Describe what it means to solve a triangle.

46. What is the difference between an angle of elevation and an angle of depression? How can you tell one from another?

5.4 TRIGONOMETRIC FUNCTIONS OF ANY ANGLE

In Sections 5.2 and 5.3, we have concentrated on the trigonometric functions for angles between 0° and 90° or 0 and $\frac{\pi}{2}$ rad. We will now examine other angles.

If you remember, we originally defined the trigonometric functions in terms of a point on the terminal side of an angle in standard position. We want to return to that definition as we look at angles that are not acute.

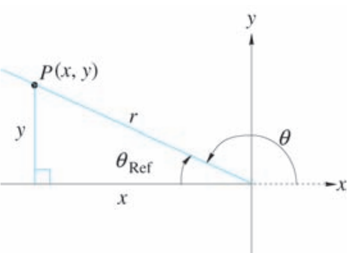

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Figure 5.31

REFERENCE ANGLES

Consider angle θ in Figure 5.31. Point P is on the terminal side and is in the second quadrant. If a perpendicular line is dropped from P to the x-axis, a triangle is formed with an acute angle θ_{Ref}. If we use the definition of the trigonometric functions from Section 5.1, we seem to get the same values we got for angle θ of the triangle.

REFERENCE ANGLE

The acute angle, θ_{Ref}, between the terminal side of θ and the x-axis is called the **reference angle** for θ. Thus, we can see that the trigonometric functions of θ are numerically the same as those of its reference angle, θ_{Ref}. The reference angle θ_{Ref} for any angle θ in each of four quadrants is shown in Figures 5.32a–d.

EXAMPLE 5.24

Find the reference angle θ_{Ref} for each angle θ: (**a**) $\theta = 75°$, (**b**) $\theta = 218°$, (**c**) $\theta = 320°$, (**d**) $\theta = \frac{3\pi}{4}$, (**e**) $\theta = \frac{7\pi}{6}$.

SOLUTIONS The solutions will use the guidelines demonstrated in Figure 5.32.

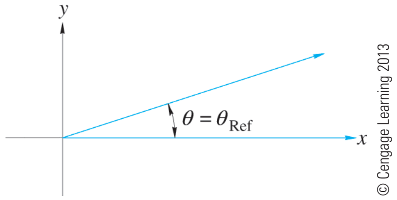

Figure 5.32a

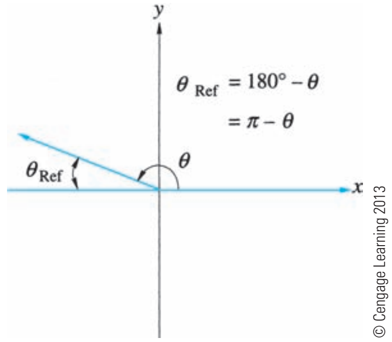

Figure 5.32b

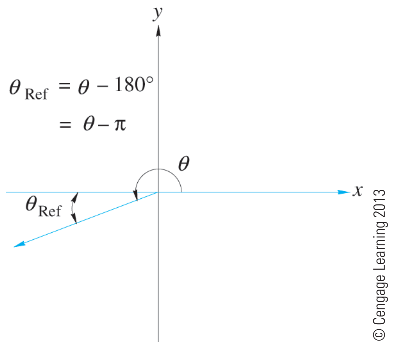

Figure 5.32c

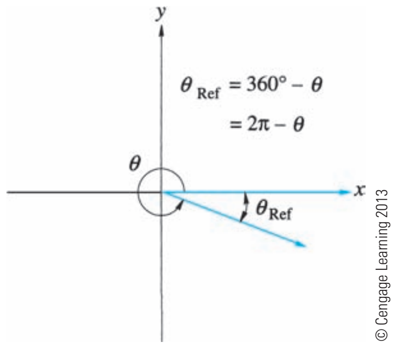

Figure 5.32d

(a) 75° is in Quadrant I, so $\theta_{Ref} = 75°$.

(b) 218° is Quadrant III, so $\theta_{Ref} = 218° - 180° = 38°$.

(c) 320° is in Quadrant IV, so $\theta_{Ref} = 360° - 320° = 40°$.

(d) $\frac{3\pi}{4}$ is in Quadrant II, so $\theta_{Ref} = \pi - \frac{3\pi}{4} = \frac{4\pi}{4} - \frac{3\pi}{4} = \frac{\pi}{4}$.

(e) $\frac{7\pi}{6}$ is in Quadrant III, so $\theta_{Ref} = \frac{7\pi}{6} - \pi = \frac{7\pi}{6} - \frac{6\pi}{6} = \frac{\pi}{6}$.

HINT The reference angle is always measured from the *x*-axis and never from the *y*-axis. The "bowtie" in Figure 5.33 may help you remember how to determine the reference angle.

EXAMPLE 5.25

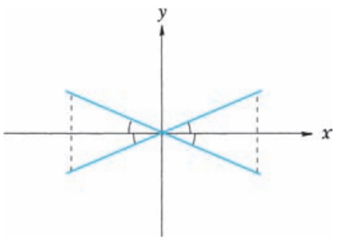

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Figure 5.33

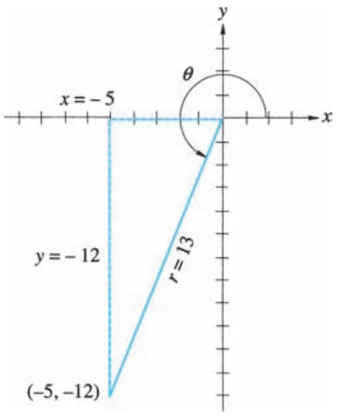

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Figure 5.34

A point on the terminal side of an angle θ has the coordinates $(-5, -12)$. Write the six trigonometric functions of θ to three decimal places.

SOLUTION A sketch of the angle is in Figure 5.34. This angle is in the third quadrant. From the Pythagorean theorem we get

$$r = \sqrt{(-5)^2 + (-12)^2} = \sqrt{25 + 144} = \sqrt{169} = 13$$

So, the trigonometric functions of θ are

$$\sin\theta = \frac{y}{r} = \frac{-12}{13} \approx -0.923 \qquad \csc\theta = \frac{r}{y} = \frac{13}{-12} \approx -1.083$$

$$\cos\theta = \frac{x}{r} = \frac{-5}{13} \approx -0.385 \qquad \sec\theta = \frac{r}{x} = \frac{13}{-5} = -2.6$$

$$\tan\theta = \frac{y}{x} = \frac{-12}{-5} = 2.4 \qquad \cot\theta = \frac{x}{y} = \frac{-5}{-12} \approx 0.417$$

As you can see, the sine and cosine and their reciprocals are negative. The tangent and its reciprocal are positive.

In our previous work with right triangles, we found that the values of the trigonometric functions were always positive. You can see here and from our work in Section 5.1 that they can sometimes be negative. Whenever the angle is in the second, third, or fourth quadrant, some of the trigonometric functions will be negative.

EXAMPLE 5.26

Use a calculator to find the values of the following trigonometric functions: sin 215°, cos 110°, tan 332°, csc 163°, sec 493°, and cot (−87°).

SOLUTION We find these the same way we used the calculator to find the values of angles between 0° and 90°:

Function	ENTER	DISPLAY
sin 215°	SIN 215 ENTER	−0.5735764
cos 110°	COS 110 ENTER	−0.3420201
tan 332°	TAN 332 ENTER	−0.5317094
csc 163°	(SIN 163) x^{-1} ENTER	3.4203036
sec 493°	(COS 493) x^{-1} ENTER	−1.4662792
cot (−87°)	(TAN (−) 87) x^{-1} ENTER	−0.0524078

EXAMPLE 5.27

Use a spreadsheet to find the values of the following trigonometric functions: sin 215°, cos 110°, tan 332°, csc 163°, sec 493°, cot(−87°).

SOLUTION We find these the same way we used a spreadsheet to find the values of angles between 0° and 90°. These are the same values and functions that we evaluated with a calculator in the previous example. The input and the results are shown in Figure 5.35.

	A	B	C
1	Function	Enter	Result
2	sin 215°	=SIN(RADIANS(215))	-0.57358
3	cos 110°	=COS(RADIANS(110))	-0.34202
4	tan 332°	=TAN(RADIANS(332))	-0.53171
5	csc 163°	=1/SIN(RADIANS(163))	3.42030
6	sec 493°	=1/COS(RADIANS(493))	-1.46628
7	cot (-87°)	=1/TAN(RADIANS(-87))	-0.05241

Figure 5.35

There are times when we need to know the sign of an angle and we may not have a calculator or computer spreadsheet handy. One case might be if you are using the table of trigonometric functions. But you will need it more often when you have to find the size of an angle and you know the value of a trigonometric function. You will see in Section 5.5 that the calculator or spreadsheet will give you an answer, but it may not be the correct answer to the problem. You will have to determine the correct answer from the calculator's answer and from your knowledge of the signs of the trigonometric functions.

To find the sign of a trigonometric function, make a sketch of the angle. You do not need a very accurate sketch, but make sure you get the angle in the correct quadrant. Draw the triangle for the reference angle and note the signs of x and y. Remember that the radius vector r is always positive. Then set up the ratio using the two values from x, y, and r that are appropriate for this angle.

EXAMPLE 5.28

What is the sign of sec 215°?

SOLUTION Examine the sketch in Figure 5.36. We have drawn a right triangle for the reference triangle and labeled the sides with the sign in parentheses after the letter x, y, or r. Since $\sec 215° = \frac{r}{x}$, where r is positive and x is negative, the sign of the sec 215° is negative.

Figure 5.37 summarizes the signs of the trigonometric functions in each quadrant. If a function is not mentioned in a quadrant, then the function is negative. Thus in Quadrant II, since they are not mentioned, the cos, tan, cot, and sec are all negative.

We will finish this section with a discussion of how to find the values of trigonometric functions from a table. If an angle is not an acute angle, you need to use the reference angle and the coterminal angle.

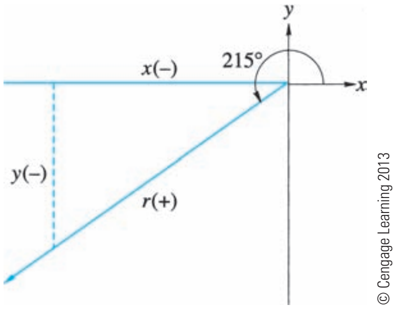

Figure 5.36

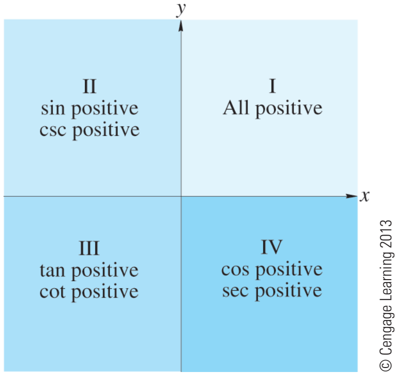

Figure 5.37

If an angle is negative, or if it is a positive angle with measure more than 360° or 2π rad, replace it with a coterminal angle that is between 0° and 360° or 0 and 2π rad. To do this, you will add or subtract multiples of 360° if the angle is in degrees, or 2π if it is in radians, until you have the desired coterminal angle.

EXAMPLE 5.29

For each angle find a coterminal angle between 0° and 360° or 0 and 2π rad: **(a)** 1,292°, **(b)** −683°, **(c)** $\frac{17}{4}\pi$, **(d)** $-\frac{5\pi}{3}$, **(e)** −8.7 rad.

SOLUTIONS

(a) $1{,}292° - 3(360°) = 1{,}292° - 1{,}080° = 212°$

(b) $-683° + 2(360°) = -683° + 720° = 37°$

(c) $\frac{17}{4}\pi - 2(2\pi) = \frac{17}{4}\pi - 2\left(\frac{8\pi}{4}\right) = \frac{17\pi}{4} - \frac{16\pi}{4} = \frac{\pi}{4}$

(d) $-\frac{5\pi}{3} + 1(2\pi) = \frac{-5\pi}{3} + \frac{6\pi}{3} = \frac{\pi}{3}$

(e) $-8.7 + 2(2\pi) = -8.7 + 12.566371 = 3.8663706$ rad

The value of any trigonometric function of an angle θ is the same as the value of the function for the reference angle θ_{Ref}, except for an occasional change in the algebraic sign. If the angle is not between 0° and 360° or 0 and 2π rad, first find the coterminal angle in that interval and then find the reference angle of the coterminal angle.

INVERSE TRIGONOMETRIC FUNCTIONS

In our earlier work with inverse trigonometric functions, we were concerned with right triangles. Thus, all the angles we had to find were acute angles. As we just saw, simply knowing the value of a trigonometric function for an angle did not allow us to determine the size of the angle. For example, if $\sin\theta = 0.2437$, we know that θ must be in Quadrant I or II. These are the only quadrants where sine is positive. This means that $\theta = 14.105021°$ or

$\theta = 180° - 14.105021° = 165.894979°$. It is possible that θ could be any angle coterminal with either of these two angles.

In a similar manner, if $\tan\theta = -1.5$, then using a calculator you obtain $\theta = -56.309932°$. This is coterminal with $\theta = 303.690068°$. This angle is in Quadrant IV. The tangent is also negative for values in Quadrant II. In Quadrant II, the angle would be 123.69007°.

USING CALCULATORS FOR INVERSE TRIGONOMETRIC FUNCTIONS

Using a calulator as we did in previous examples is certainly fast and accurate. But, it may not always give us the angle that we want. For example, it will never give an angle in Quadrant III. Why is this? It goes back to the basic idea of a function—the idea that for each different value of x there is exactly one value of y.

Because people wanted inverse trigonometric *functions* they had to restrict the answers to exactly one value of y for each value of x. That is why, when you use your calculator to determine inverse sin 0.5, you obtain 30° by pressing **2nd** **SIN** 0.5.

This is nothing new. At one time we studied a function $f(x) = x^2$ and its inverse $f^{-1}(x) = \sqrt{x}$. We knew that $f(-5) = f(5) = 5^2 = 25$, but that $f^{-1}(25) = \sqrt{25} = 5$. The inverse function of x^2, $\sqrt{x}$, only gave the principal value of the square root. In the same way, the inverse trigonometric functions have only the *principal value* of the angle defined for each value of x. Thus, we have the following inverse trigonometric functions.

INVERSE TRIGONOMETRIC FUNCTIONS

If $\sin\theta = x$, then $\arcsin x = \theta$, where	$-90° \le \theta \le 90°$
or	$-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$
If $\cos\theta = x$, then $\arccos x = \theta$, where	$0° \le \theta \le 180°$
or	$0 \le \theta \le \pi$
If $\tan\theta = x$, then $\arctan x = \theta$, where	$-90° < \theta < 90°$
or	$\frac{-\pi}{2} < \theta < \frac{\pi}{2}$
If $\csc\theta = x$, then $\operatorname{arccsc} x = \theta$, where	$-90° \le \theta \le 90°, \theta \ne 0°$
or	$-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}, \theta \ne 0$
If $\sec\theta = x$, then $\operatorname{arcsec} x = \theta$, where	$0° \le \theta \le 180°, \theta \ne 90°$
or	$0 \le \theta \le \pi, \theta \ne \frac{\pi}{2}$
If $\cot\theta = x$, then $\operatorname{arccot} x = \theta$, where	$0° < \theta < 180°$
or	$0 < \theta < \pi$

As you can see, for positive values you always get an angle in the first quadrant. If you want the inverse value of a negative number, then you will get an angle in the fourth quadrant for INV SIN and INV TAN and an angle in the second quadrant for INV COS. Working with a calculator will also show you that the angles in the fourth quadrant are given as negative angles. Thus on a calculator, arcsin $(-0.5) = -30°$ and not 330°.

In the previous list, we used the symbol arcsin x to represent the inverse of the sin θ. Arcsin is an accepted symbol for the inverse of the sine function. Another symbol that is often used is $\sin^{-1}x$. Just as we used $f^{-1}(x) = \sqrt{x}$, here we use $f^{-1}(x) = \sin^{-1}x$. Similarly, $\arccos x = \cos^{-1}x$, $\arctan x = \tan^{-1} x$, and so on. In fact, above the SIN, COS, and TAN keys on your calculator you should see $\sin^{-1}$, $\cos^{-1}$, and $\tan^{-1}$, respectively.

USING SPREADSHEETS FOR INVERSE TRIGONOMETRIC FUNCTIONS

The spreadsheet function for $\sin^{-1}$ (or arcsin) is ASIN. Similarly, ACOS and ATAN are used for $\cos^{-1}$ and $\tan^{-1}$, respectively. The returned angle is given in radians. If you want your answer in degrees, then multiply the result by 180/PI( ) or use the DEGREES function.

EXAMPLE 5.30

Use a spreadsheet to find $\sin^{-1} 0.5$ in **(a)** radians and **(b)** degrees.

SOLUTIONS

(a) Since we want this answer in radians all we need to do is enter ASIN(0.5) as shown in Figure 5.38a. From the figure we see that $\sin^{-1} 0.5 \approx 0.52360$ radians.

	A	B
1	Enter	Result
2	=ASIN(0.5)	0.52360

Figure 5.38a

(b) This answer is to be in degrees. As noted, we have two options: multiply the result by 180/PI( ) or use the DEGREES. Each of these is shown in Figure 5.38b, where we see that $\sin^{-1} 0.5 = 30°$.

	A	B
1	Enter	Result
2	=ASIN(0.5)*180/PI()	30.0
3	=DEGREES(ASIN(0.5))	30.0

Figure 5.38b

FINDING ALL ANGLES FOR INVERSE TRIGONOMETRIC FUNCTIONS

Since inverse trigonometric functions only give one answer to problems like $\sin\theta = \frac{1}{2}$ where $\theta = \sin^{-1}\frac{1}{2}$, we must develop a method for finding other angles that satisfy this equation. We will use our knowledge of reference angles and the sign of the functions in the four quadrants to do this.

To find the reference angle of an equation of the form $\sin\theta = n$, where n is a number in the range of sine, we will use the absolute value of n, $|n|$. Then the desired answers can be found by adding or subtracting the reference angle from 180° (or π) or subtracting from 360° (or 2π).

EXAMPLE 5.31

Find all angles θ in the interval $(0^\circ, 360^\circ)$ where $\cos\theta = -0.42$.

SOLUTION Here, we have $n = -0.42$, so $|n| = |-0.42| = 0.42$ and $\theta_{\text{Ref}} = \cos^{-1} 0.42 = 65.2^\circ$. Since n was negative, we have answers in Quadrants II and III, the quadrants where cosine is negative.

$$\theta_{\text{II}} = 180^\circ - \theta_{\text{Ref}} = 180^\circ - 65.2^\circ = 114.8^\circ$$
$$\theta_{\text{III}} = 180^\circ + \theta_{\text{Ref}} = 180^\circ + 65.2^\circ = 245.2^\circ$$

The answers are 114.8° and 245.2°.

EXAMPLE 5.32

Find all angles θ in the interval $(0, 2\pi)$ where $\cot\theta = -1.73$.

SOLUTION Here, we have $n = -1.73$, so $|n| = |-1.73| = 1.73$ and $\theta_{\text{Ref}} = \cot^{-1} 1.73 = \tan^{-1}\frac{1}{1.73} = 0.5241$. Since n was negative, we have answers in Quadrants II and IV, the quadrants where cotangent is negative.

$$\theta_{\text{II}} = \pi - \theta_{\text{Ref}} = \pi - 0.5241 \approx 2.6175$$
$$\theta_{\text{IV}} = 2\pi - \theta_{\text{Ref}} = 2\pi - 0.5241 \approx 5.7591$$

The answers are 2.6175 rad and 5.7591 rad.

In the previous two examples, we only found two answers. There are, however, an infinite number of angles coterminal with these answers. If we want to represent all these answers, we proceed as follows.

For Example 5.31:

$$114.8^\circ + 360^\circ k \text{ and } 245.2^\circ + 360^\circ k, \text{ where } k \text{ is an integer}$$

For Example 5.32:

$$2.6175 + 2\pi k \text{ rad and } 5.7591 + 2\pi k \text{ rad, where } k \text{ is an integer}$$

EXERCISE SET 5.4

Give the reference angle for each of the angles in Exercises 1–8.

1. 87°
2. 200°
3. 137°
4. 298°
5. $\frac{9\pi}{8}$ rad
6. 2.1 rad
7. 4.5 rad
8. 5.85 rad

Give the coterminal angle for each of the angles in Exercises 9–16.

9. $518°$
10. $1{,}673°$
11. $-871°$
12. $-137°$
13. 7.3 rad
14. $\frac{16\pi}{3}$ rad
15. -2.17 rad
16. -8.43 rad

State which quadrant or quadrants the terminal side of θ is in for each of the angles, expressions, or trigonometric functions in Exercises 17–38.

17. $165°$
18. $285°$
19. $-47°$
20. $312°$
21. $250°$
22. $197°$
23. $98°$
24. $-177°$
25. $\sin\theta = \frac{1}{2}$
26. $\tan\theta = \frac{3}{4}$
27. $\cot\theta = -2$
28. $\cos\theta = -0.25$
29. $\sec\theta = 4.3$
30. $\cos\theta = 0.8$
31. $\csc\theta = -6.1$
32. $\sin\theta = -\dfrac{\sqrt{3}}{2}$
33. $\sin\theta$ is positive.
34. $\tan\theta$ is negative.
35. $\cos\theta$ is positive.
36. $\sec\theta$ is negative.
37. $\csc\theta$ is negative and $\cos\theta$ is positive.
38. $\tan\theta$ and $\sec\theta$ are negative.

State whether each of the expressions in Exercises 39–46 is positive or negative.

39. $\sin 105°$
40. $\sec 237°$
41. $\tan 372°$
42. $\cos(-53°)$
43. $\cos 1.93$ rad
44. $\cot 4.63$ rad
45. $\sin 215°$
46. $\csc 5.42$ rad

Find the values of each of the trigonometric functions in Exercises 47–68.

47. $\sin 137°$
48. $\cos 263°$
49. $\tan 293°$
50. $\sin 312°$
51. $\tan 164.2°$
52. $\csc 197.3°$
53. $\sin 2.4$ rad
54. $\cos 1.93$ rad
55. $\tan 6.1$ rad
56. $\sec 4.32$ rad
57. $\tan 1.37$ rad
58. $\sin 3.2$ rad
59. $\sin 415.5°$
60. $\tan 512.1°$
61. $\cot -87.4°$
62. $\cos 372.1°$
63. $\cos 357.3°$
64. $\sin 6.5$ rad
65. $\tan 8.35$ rad
66. $\sin 9.42$ rad
67. $\cos - 0.43$ rad
68. $\sec 9.34$ rad

In Exercises 69–76, find, to the nearest tenth of a degree, all angles θ, where $0° \le \theta < 360°$, with the given trigonometric function.

69. $\sin\theta = \frac{1}{2}$
70. $\tan\theta = \frac{3}{4}$
71. $\cot\theta = -2$
72. $\cos\theta = -0.25$
73. $\sec\theta = 4.3$
74. $\cos\theta = 0.8$
75. $\csc\theta = -6.1$
76. $\sin\theta = -\dfrac{\sqrt{3}}{2}$

In Exercises 77–84, find, to the nearest hundredth of a radian, all angles θ, where $0 \le \theta < 2\pi$, with the given trigonometric function.

77. $\sin\theta = 0.75$
78. $\tan\theta = 1.6$
79. $\cot\theta = -0.4$
80. $\cos\theta = 0.25$
81. $\csc\theta = 4.3$
82. $\sin\theta = -0.08$
83. $\sec\theta = 2.7$
84. $\cos\theta = -0.95$

Evaluate each of the functions in Exercises 85–92. Write each answer to the nearest tenth of a degree.

85. $\arcsin 0.84$
86. $\arccos(-0.21)$
87. $\arctan 4.21$
88. $\text{arccot}(-0.25)$
89. $\sin^{-1} 0.32$
90. $\cos^{-1} 0.47$
91. $\tan^{-1}(-0.64)$
92. $\csc^{-1}(-3.61)$

Evaluate each of the functions in Exercises 93–100. Write each answer to the nearest hundredth of a radian.

93. $\arccos(-0.33)$
94. $\arctan 1.55$
95. $\arccos 0.29$
96. $\text{arcsec}(-3.15)$
97. $\sin^{-1} 0.95$
98. $\cos^{-1}(-0.67)$
99. $\tan^{-1} 0.25$
100. $\cot^{-1}(-0.75)$

Solve Exercises 101–110.

101. ***Electronics*** The intensity, I, of a sinusoidal current in an ac circuit is given by $I = I_{max} \sin \theta$, where I_{max} is the maximum intensity of the current. Find I when $I_{max} = 32.65$ mA and $\theta = 132.0°$.

102. ***Electronics*** The potential voltage, V, of a sinusoidal current in an ac circuit is given by $V = V_{max} \sin \theta$, where V_{max} is the maximum potential. Find V when $V_{max} = 115.2$ V and $\theta = 113.4°$.

103. ***Metalworking*** Find the length of the piece of metal ABC in Figure 5.39.

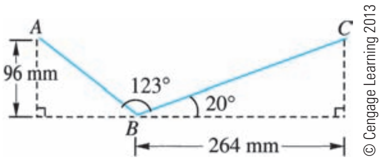

Figure 5.39

104. ***Metalworking*** Figure 5.40 shows the outline of an isosceles trapezoid that is to be cut from a metal sheet. Find the size of $\angle\theta$.

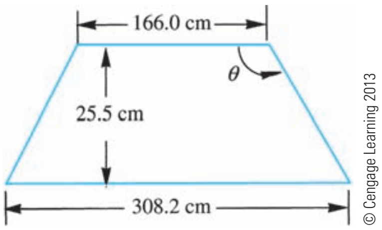

Figure 5.40

105. ***Landscape architecture*** Figure 5.41 shows the outline of some ground that needs to be sodded. How many square feet of sod are needed to completely cover the ground?

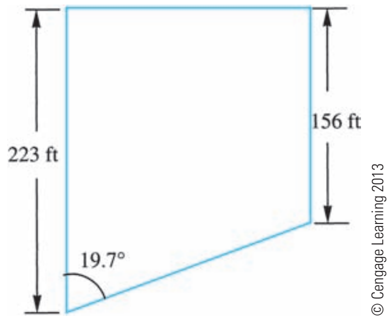

Figure 5.41

106. ***Metalworking*** Find the length of side AB on the metal sheet shown in Figure 5.42.

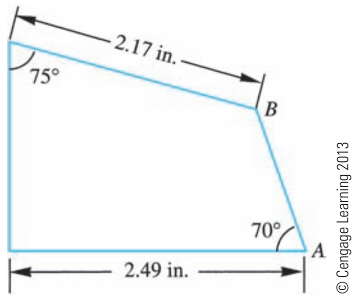

Figure 5.42

107. ***Electronics*** The phase angle, θ, in an RC circuit between the capacitive reactance, X_C, and the resistance, R, when resistance is in series with capacitive reactance, can be found by using $\theta = \tan^{-1}\left(-\frac{X_C}{R}\right)$. Find θ when $X_C = 33\ \Omega$ and $R = 40\ \Omega$.

108. ***Electronics*** The phase angle, θ, in an inductive circuit between the impedance, Z, and the resistance, R, can be found by using $\theta = \cos^{-1}\left(\frac{Z}{R}\right)$. Find θ when $Z = 30\ \Omega$ and $R = 50\ \Omega$.

109. ***Automotive technology*** Find the angle, θ, of the diesel engine valve face in Figure 5.43.

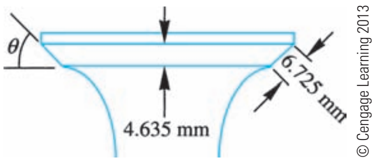

Figure 5.43

110. ***Dynamics*** The maximum height, h, of a projectile with initial velocity, v_0, at angle θ is given by

$$h = \frac{v_0^2 \sin^2\theta}{2g}$$

where g is the acceleration due to gravity.

(a) Solve for θ.

(b) If $v_0 = 172.5$ m/s and $g = 9.8$ m/s^2, at what angle was the projectile shot if its maximum height was 431.1 m?

[IN YOUR WORDS]

111. Explain what is meant by a reference angle?

112. Describe how you can tell the signs of the six trigonometric functions of a particular angle if you know the quadrant in which the angle lies.

113. Describe the difference between $\sin^{-1}\theta$ and $(\sin\theta)^{-1}$.

114. Describe the difference between $\cos^{-1}\theta$ and $(\sec\theta)^{-1}$.

5.5 APPLICATIONS OF TRIGONOMETRY

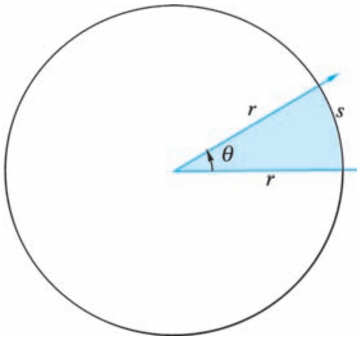

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Figure 5.44

In earlier sections, we looked at some applications of trigonometry, particularly applications dealing with right triangles. In this section, we will look at some additional applications of trigonometry.

We defined a radian as $\frac{1}{2\pi}$ of a circle. Another way to look at a radian is to examine the circle in Figure 5.44. If r is the radius of the circle and s is the length of the arc opposite θ, then $\theta = \dfrac{s}{r}$ or $\theta r = s$, where θ is in radians.

We can use this relationship to find the area of the sector of a circle. For the shaded sector in Figure 5.44, the area $A = \dfrac{rs}{2} = \dfrac{r^2\theta}{2} = \dfrac{1}{2}r^2\theta$. Both of these formulas work if the angles are in radians.

EXAMPLE 5.33

Find the arc length and area of a sector of a circle, with radius 2.7 cm and central angle 1.3 rad.

SOLUTION To find the arc length we use $s = \theta r$ with $\theta = 1.3$ rad and $r = 2.7$ cm.

$$\begin{aligned} s &= \theta r \\ &= (1.3)(2.7) \\ &= 3.51\,\text{cm} \end{aligned}$$

The area is found using $A = \frac{1}{2}r^2\theta$.

$$A = \frac{1}{2}r^2\theta$$
$$= \frac{1}{2}(2.7)^2(1.3)$$
$$= 4.7385 \text{ or } 4.74\,\text{cm}^2$$

Radian measure was developed with the aid of a circle and is used to measure angles. One application of radian measure involves rotational motion. When we work with motion in a straight line, we use the formula $d = vt$ or distance = velocity (or rate)$\times$ time. We assume that the rate is constant.

Suppose instead that we had an object moving around a circle at a constant rate of speed. If we let s represent the distance around the circle and v the velocity (or rate), then $s = vt$ or $v = \frac{s}{t}$. Since $s = \theta r$, where r is the radius of the circle, we have $v = \frac{\theta r}{t}$. The centripetal acceleration is $a_c = \frac{v^2}{r} = \frac{\theta^2 r}{t^2}$.

APPLICATION GENERAL TECHNOLOGY

EXAMPLE 5.34

A ball is spun in a horizontal circle 60 cm in radius at the rate of one revolution every 3 s. What is the ball's velocity and centripetal acceleration?

SOLUTION The velocity $v = \frac{s}{t}$. We know that $t = 3$ s, but we have to find s. In one revolution, the ball covers one complete circle, so $\theta = 2\pi$, and since $r =$ 60 cm, we have $s = \theta r = 120\pi$. Thus,

$$v = \frac{s}{t}$$
$$= \frac{120\pi\,\text{cm}}{3\,\text{s}}$$
$$= 40\pi\,\text{cm/s} = 125.66\,\text{cm/s}$$

The centripetal acceleration is

$$a_c = \frac{v^2}{r} = \frac{(40\pi\,\text{cm/s})^2}{60\,\text{cm}}$$
$$= 263.19\,\text{cm/s}^2$$

The angular distance D (measured at the earth's center) between two points P_1 and P_2 on the earth's surface is determined by

$$\cos D = \sin L_1 \sin L_2 + \cos L_1 \cos L_2 \cos(M_1 - M_2)$$

where L_1, L_2 and M_1, M_2 are the respective latitudes and longitudes of two points. The latitude is the angle measured at the earth's surface between a point on the earth and the equator on the same meridian. Meridians are imaginary lines on the surface of the earth, which pass through both the north and south poles. (See Figure 5.45.)

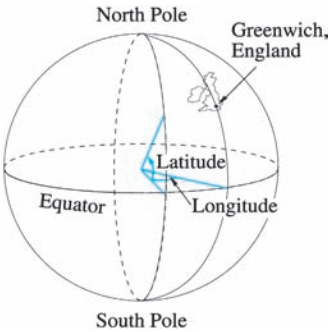

Figure 5.45

The longitude is the angle between the meridian passing through a point on the earth and the 0° meridian passing through Greenwich, England.

APPLICATION CIVIL ENGINEERING

EXAMPLE 5.35

If Knoxville, Tennessee, is at latitude 36° N and longitude 83°55′ W, and Denver, Colorado, is at latitude 39°44′ N and longitude 104°59′ W, and the radius of the earth is 3,960 mi, how far is it from Knoxville to Denver?

SOLUTION We have $L_1 = 36°$, $L_2 = 39°44'$, $M_1 = 83°55'$, and $M_2 = 104°59'$, so $M_1 - M_2 = -21°04'$.

$$\begin{aligned}
\cos D &= (\sin 36°)[\sin(39°44')] + (\cos 36°)(\cos 39°44')[\cos(-21°04')] \\
&= (0.5878)(0.6392) + (0.8090)(0.7690)(0.9332) \\
&= 0.9563 \\
D &= \cos^{-1}(0.9563) \\
&= 17.00 \\
&= 0.2967\,\text{rad}
\end{aligned}$$

Notice that we had to convert our answer for D from degrees to radians.

Since $r = 3{,}960$ mi, the distance from Knoxville to Denver is

$$s = \theta r = (0.2967)(3{,}960) = 1{,}174.93 \text{ mi}$$

This would, of course, be air miles and not the distance you would travel by automobile.

One use of radian measure involves rotational motion. Remember that for motion in a straight line, distance = velocity (or rate) × time or $d = vt$, where we assumed that the rate remains constant. Now, if the motion is in a circular direction instead of a straight line, we would have the formula $s = vt$. Since $s = \theta r$ and $v = \frac{s}{t}$, then $v = \frac{\theta r}{t} = r\left(\frac{\theta}{t}\right)$. This ratio, $\frac{\theta}{t}$, is called the *angular velocity* and is usually denoted by the Greek letter ω (omega). Thus it is known that angular velocity ω of an object that turns through the angle θ in time t is given by $\omega = \frac{\theta}{t}$. The linear velocity v of a point that moves in a circle of radius r with uniform angular velocity ω is $v = \omega r$. A rotating body with an angular velocity that changes from ω_0 to ω_f in the time interval t has an angular acceleration of $\alpha = \frac{\omega_f - \omega_0}{t}$.

APPLICATION MECHANICAL

EXAMPLE 5.36

A steel cylinder 8 cm in diameter is to be machined in a lathe. If the desired linear velocity of the cylinder's surface is to be 80.5 cm/s, at how many rpm should it rotate?

SOLUTION We have $v = \omega r$. We want to find ω and $\omega = \frac{v}{r}$, where $v =$ 80.5 cm/s and $r = 4$ cm.

$$\begin{aligned}\omega &= \frac{80.5\,\text{cm/s}}{4\,\text{cm}}\\ &= 20.125\,\text{rad/s}\end{aligned}$$

Now, 1 rpm is 2π rad/60 s, so using dimensional analysis we have

$$\begin{aligned}\omega &= \frac{20.125\,\text{rad}}{1\,\text{sec}} \cdot \frac{60\,\text{sec}}{1\,\text{min}} \cdot \frac{1\,\text{rev}}{2\pi\,\text{rad}}\\ &= \frac{20.125 \times 60\,\text{rev}}{2\pi\,\text{min}}\\ &\approx 192.22\,\text{rpm}\end{aligned}$$

APPLICATION MECHANICAL

EXAMPLE 5.37

An engine requires 6 s to go from its idling speed of 600 rpm to 1500 rpm. What is its angular acceleration?

SOLUTION The initial velocity of the engine is

$$\begin{aligned}\omega_0 &= 600\,\text{rpm}\\ &= \frac{600\,\text{rev}}{1\,\text{min}} \cdot \frac{2\pi\,\text{rad}}{1\,\text{rev}} \cdot \frac{1\,\text{min}}{60\,\text{sec}}\\ &\approx 62.83\,\text{rad/s}\end{aligned}$$

and the final velocity is

$$\begin{aligned}\omega_f &= 1500\,\text{rpm}\\ &= \frac{1500\,\text{rev}}{1\,\text{min}} \cdot \frac{2\pi\,\text{rad}}{1\,\text{rev}} \cdot \frac{1\,\text{min}}{60\,\text{sec}}\\ &\approx 157.058\,\text{rad/s}\end{aligned}$$

The angular acceleration is

$$\begin{aligned}\alpha &= \frac{\omega_f - \omega_0}{t}\\ &= \frac{157.05 - 62.82}{6}\\ &= \frac{94.23\,\text{rad/s}}{6\,\text{s}}\\ &= 15.705\,\text{rad/s}^2\end{aligned}$$

EXERCISE SET 5.5

Solve Exercises 1–30.

1. ***Transportation engineering*** A circular highway curve has a radius of 1050.250 m and a central angle of 47° measured to the center line of the road. What is the length of the curve?
2. ***Physics*** An object is moving around a circle of radius 8 in. with an angular velocity of 5 rad/s. What is the linear velocity?
3. ***Physics*** A flywheel rotates with an angular velocity of 30 rpm. If its radius is 12 in., what is the linear velocity of the rim?
4. ***Industrial technology*** A pulley belt 4 m long takes 2 s to complete one revolution. If the radius of the pulley is 180 mm, what is the angular velocity of a point on the rim of the pulley?
5. In a circle of radius 15 cm, what is the length of the arc intercepted by a central angle of $\frac{5\pi}{8}$?
6. In a circle of radius 235 mm, what is the length of the arc intercepted by a central angle of 85°?
7. What is the area of the sector in Exercise 5?
8. What is the area of the sector in Exercise 6?
9. ***Physics*** A clock pendulum 1.2 m long oscillates through an angle of 0.07 rad on each side of the vertical. What is the distance the end of the pendulum travels from one end to the other?
10. ***Space technology*** A communication satellite is in orbit at 22,300 mi above the surface of the earth. The satellite is in permanent orbit above a certain point on the equator. Thus, the satellite makes one revolution every 24 h. What is the angular velocity of the satellite? What is its linear velocity? (The radius of the earth at the equator is 3,963 mi.)
11. ***Acoustical engineering*** A phonograph record is 175.26 mm in diameter and rotates at 45 rpm. **(a)** What is the linear velocity of a point on the rim? **(b)** How far does this point travel in 1 min?
12. ***Acoustical engineering*** A phonograph turntable rotating at 4.2 rad/s makes 4 complete turns before it stops after 11.92 s. What is its angular acceleration?
13. ***Navigation*** City A is at 35°30′ N, 78°40′ W and City B is at 40°40′ N, 88°50′ W. What is the angular distance between the two cities? How far is it between the cities?
14. ***Mechanical engineering*** A 1,200-rpm motor is directly connected to a 12-in.-diameter circular saw blade. What is the linear velocity of the saw's teeth in ft/min?
15. ***Computer science*** The outer track on a $5\frac{1}{4}''$ diameter diskette for a microcomputer is $2\frac{1}{2}''$ from the center of the diskette. If 8 bytes of data can be stored on $\frac{1}{4}''$ of track, how many bytes can be stored in the length of the track subtended by $\frac{\pi}{3}$ rad?
16. ***Computer science*** Some diskettes are divided into 16 sectors per track. What is the length of one sector of the track in Exercise 15?
17. ***Mechanical engineering*** A flywheel makes 850 rev in 1 min. How many degrees does it rotate in 1 s?
18. ***Mechanical engineering*** If the flywheel in Exercise 17 has a 15-in. diameter, what is the linear velocity of a point on its outer edge?
19. ***Energy technology*** A wind generator has blades 4.2 m long. What is the speed of a blade tip when the blades are rotating at 20 rpm?
20. ***Physics*** A 2,500-newton weight is resting on an inclined plane that makes an angle of 23° with the horizontal. Find the component F_x and F_y of the weight parallel to and perpendicular to the surface of the plane, as shown in Figure 5.46.

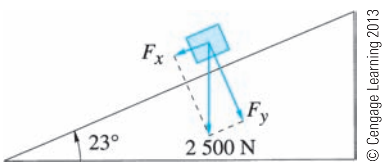

Figure 5.46

21. ***Navigation*** The arctic circle is approximately at latitude 66.5° N. If the radius of the earth is 6,370 km, how far is the arctic circle from the equator?

22. ***Optics*** When unpolarized light strikes an interface between two materials, *Brewster's law* states that the reflected light is completely polarized perpendicular to the plane of incidence if the angle of incidence, θ, is given by

$$\tan\theta = \frac{n_b}{n_a}$$

If sunlight is reflected off commercial plate glass, at what angle of reflection is the light completely polarized? Here, for air $n_a = 1.00$ and for commercial plate glass, $n_b = 1.516$.

23. ***Optics*** The intensity, I, of a light beam transmitted by a pair of polarizing filters is given by the equation

$$I = I_m \cos^2\theta$$

where I_m is the maximum intensity and θ is the angle between the filters. At what angle does the intensity drop to three-fourths of the maximum intensity?

24. ***Construction*** A roof that slopes at 23° to the horizontal is 14.5 m long and has a slant height of 9.2 m. How large an area does the roof actually cover? (Hint: Find the area of rectangle $ABCD$ in Figure 5.47)

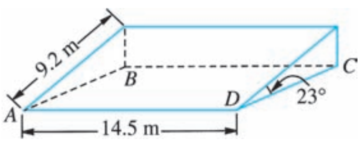

Figure 5.47

25. ***Space technology*** The global positioning system (GPS) consists of 24 satellites, each orbiting the earth in a circular orbit at an altitude of 11,000 mi. Each satellite orbits the earth once every 12 hr. If the earth's radius is 3,960 mi, determine:

(a) The angular speed of each GPS satellite in rad/s

(b) The linear speed of each GPS satellite in mi/s.

26. ***Navigation*** A nautical mile may be defined as the arc length intercepted on the surface of the earth by a central angle with measure $1'$. Suppose that the radius of the earth is 3,960 (statute) miles and that one statute mile is 5,280 ft.

(a) Determine the number of feet in a nautical mile.

(b) How much longer (in ft) is a nautical mile than a statute mile?

27. ***Construction*** **(a)** Find the width L of the taper shown in Figure 5.48. **(b)** Determine the length of the slanted sides of the taper.

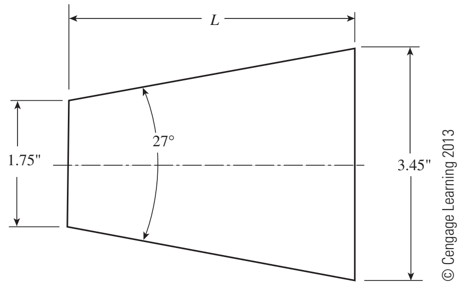

Figure 5.48

28. ***Electricity*** An electrician must bend a pipe to make a 1.8-ft rise in a 6.3-ft horizontal distance, as shown in Figure 5.49. **(a)** What is the angle at A? **(b)** What is the angle at B?

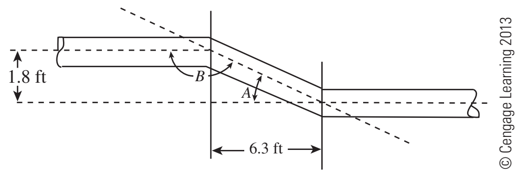

Figure 5.49

29. ***Graphic design*** A graphic artist is designing a Web page that will include the animation of a piston, crank, and wheel. The crank is attached to the wheel and piston as shown in Figures 5.50a and 5.50b. As the wheel rotates, the crank moves the piston. $\angle A$ will vary from 0 to 2π radians. To make the motion appear realistic, the artist needs to find an expression for $\angle B$. Express $\angle B$ in terms of the length of the crank, c, the radius of the wheel, r, and $\angle A$.

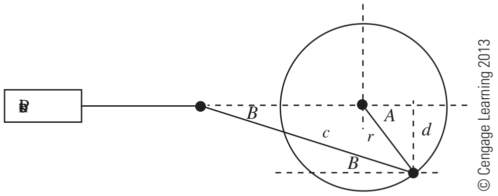

Figure 5.50a

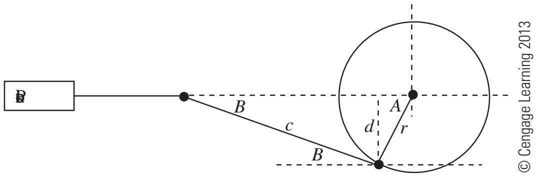

Figure 5.50b

30. ***Drafting*** A drafter is designing a gauge like the one in Figure 5.51 to accurately measure the distance between pins. Find the lengths A and B to the nearest 0.001 inch.

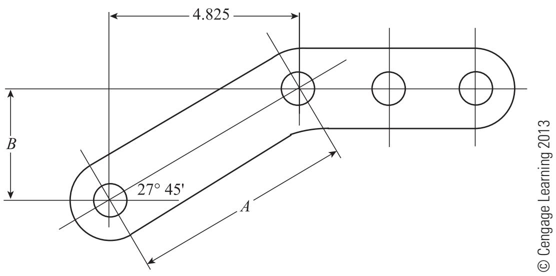

Figure 5.51

31. A security camera is is positioned to get a view of people using an ATM machine. As shown in Figure 5.52, the camera is located 5 ft-3 in. above and 12 ft-8 in. to the right of a normal person's head.

(a) How far, to the nearest inch, is the camera from the person's head?

(b) What is the angle θ to the nearest tenth degree that the camera is depressed below the horizontal?

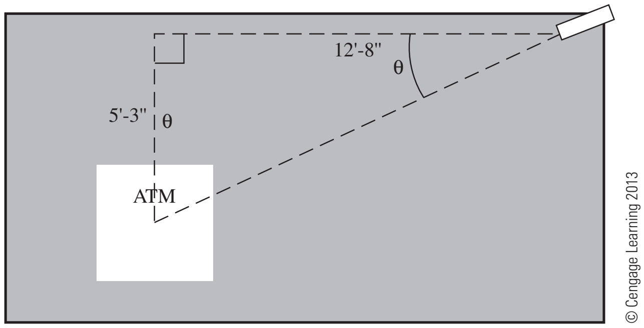

Figure 5.52

32. Two utility poles are on opposite sides of an interstate highway, much like those in Figure 5.53. In order to find the distance between the poles, Ali paced off 90 ft along the highway and perpendicular to a line joining the two poles. The angle between the poles is measured at 68°. Determine the distance between the two poles to the nearest tenth foot.

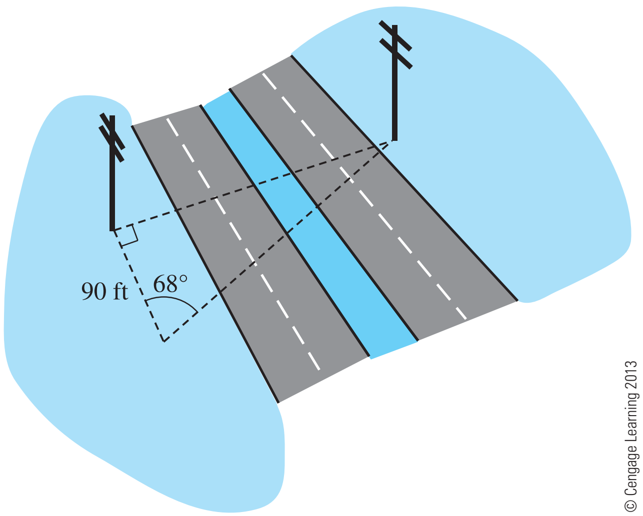

Figure 5.53

[IN YOUR WORDS]

33. Write a word problem in your technology area of interest that requires you to use trigonometry. On the back of the sheet of paper, write your name and explain how to solve the problem by using trigonometry. Give the problem you wrote to a friend and let him or her try to solve it. If your friend has difficulty understanding the problem or solving the problem, or disagrees with your solution, make any necessary changes in the problem or solution. When you have finished, give the revised problem and solution to another friend and see if he or she can solve it.

34. As in Exercise 33, write a word problem in your technology area that requires you to use trigonometry. This problem should use a different trigonometric function from the one in Exercise 33.

CHAPTER 5 REVIEW

IMPORTANT TERMS AND CONCEPTS

Angle(s)
- Complementary
- Coterminal
- Negative
- Of depression
- Of elevation
- Positive
- Quadrantal
- Reference
- Standard position

Degrees

Inverse trigonometric functions
- Arccos or $\cos^{-1}$
- Arccot or $\cot^{-1}$
- Arccsc or $\csc^{-1}$
- Arcsec or $\sec^{-1}$
- Arcsin or $\sin^{-1}$
- Arctan or $\tan^{-1}$

Quotient identities

Radians

Reciprocal identities

Trigonometric functions
- Cosecant (csc)
- Cosine (cos)
- Cotangent (cot)
- Secant (sec)
- Sine (sin)
- Tangent (tan)

REVIEW EXERCISES

Convert each of the angle measures in Exercises 1–6 from degrees to radians.

1. 60°
2. 198°
3. 325°
4. 180°
5. −115°
6. 435°

Convert each of the angle measures in Exercises 7–12 from radians to degrees.

7. $\frac{3\pi}{4}$ rad
8. 1.10 rad
9. 2.15 rad
10. $\frac{7\pi}{3}$ rad
11. −4.31 rad
12. 5.92 rad

For Exercises 13–24, (a) tell which quadrant the angle is in, (b) give the reference angle for the given angle, and (c) give two coterminal angles, one positive and one negative.

13. 60°
14. 198°
15. 325°
16. 180°
17. −115°
18. 435°
19. $\frac{3\pi}{4}$ rad
20. 1.10 rad
21. 2.15 rad
22. $\frac{7\pi}{3}$ rad
23. −4.31 rad
24. 5.92 rad

Each point in Exercises 25–30 is on the terminal side of an angle θ in standard position. Find the six trigonometric functions of the angle θ associated with each of these points.

25. $(3, -4)$
26. $(5, 12)$
27. $(-20, 21)$
28. $(-4, -7)$
29. $(7, 1)$
30. $(-12, 8)$

In Exercises 31–36, find the trigonometric functions of angle θ at vertex A of triangle ABC for the indicated sides.

31. $a = 8, c = 17$
32. $a = 5, b = 12$
33. $b = 8, c = \frac{40}{3}$
34. $a = 6, c = 7.5$
35. $b = 7, c = 18.2$
36. $a = 42, b = 44.1$

In Exercises 37–39, let θ be an angle of a triangle with the given trigonometric function. Find the values of the other trigonometric functions.

37. $\sin\theta = \dfrac{12.8}{27.2}$
38. $\tan\theta = \dfrac{16}{16.8}$
39. $\sec\theta = \dfrac{4}{2.5}$

In Exercises 40–42, the values of two trigonometric functions of an angle θ are given. Use this information to determine the values of the other four trigonometric functions for θ.

40. $\sin\theta = 0.532, \tan\theta = 0.628$
41. $\sin\theta = 0.5, \cos\theta = 0.866$
42. $\cos\theta = 0.680, \csc\theta = 1.364$

Use a calculator or spreadsheet to determine the values of each of the indicated trigonometric functions in Exercises 43–50.

43. $\sin 45°$
44. $\cos 82.5°$
45. $\tan 213.5°$
46. $\sec(-81°)$
47. $\cos 2.3$ rad
48. $\sin 4.75$ rad
49. $\tan(-3.2)$ rad
50. $\csc 0.21$ rad

In Exercises 51–54, find, to the nearest tenth of a degree, all angles θ, where $0° \leq \theta < 360°$, with the given trigonometric function.

51. $\sin\theta = 0.5$
52. $\tan\theta = 2.5$
53. $\cos\theta = -0.75$
54. $\csc\theta = 3.0$

In Exercises 55–58, find, to the nearest hundredth of a radian, all angles θ, where $0 \leq \theta < 2\pi$, with the given trigonometric function.

55. $\cos\theta = -0.5$
56. $\sin\theta = 0.717$
57. $\tan\theta = -0.95$
58. $\sec\theta = 2.25$

Evaluate each of the functions in Exercises 59–64. Write each answer to the nearest tenth of a degree.

59. arcsin 0.866

60. arccos 0.5

61. arctan (-1)

62. $\cos^{-1}(-0.707)$

63. $\sin^{-1} 0.385$

64. $\cot^{-1}(3.5)$

Solve Exercises 65–80.

65. ***Optics*** The index of refraction n of a medium is the ratio of the speed of light c in air to the speed of light in the medium c_m. According to Snell's law, the angles of incidence i and refraction r (see Figure 5.54) of a light ray are related by the formula

$$n = \frac{\sin i}{\sin r}$$

The index of refraction for water is 1.33. If a light beam enters a lake at an angle of incidence of 30°, what is the angle of refraction?

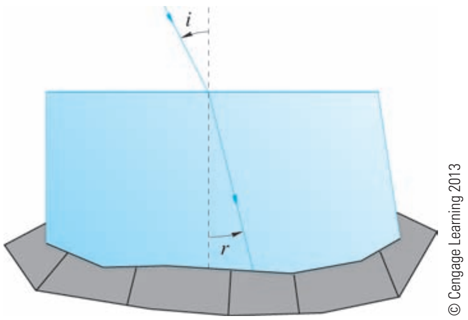

Figure 5.54

66. ***Optics*** A beam of light enters a pane of glass at an angle of incidence of 0.90 rad. If the angle of refraction is 0.55 rad, what is the index of refraction of the glass?

67. ***Electronics*** The current in an ac circuit varies with time t and the maximum value of the current I_m according to the formula

$$I = I_m \sin \omega t$$

where ω is the angular frequency of the alternating current. Find I, if $I_m = 12.6$ A and ωt is $\frac{\pi}{5}$ rad.

68. ***Machine technology*** An engine valve is shown in Figure 5.55. What is the angle θ of the valve face, in degrees?

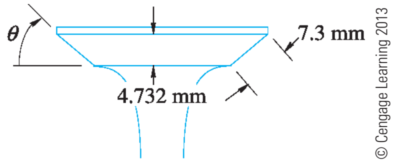

Figure 5.55

69. ***Electronics*** If resistor and capacitor are connected in series to an ac power source, the phase angle θ is given by the formula

$$\tan \theta = \frac{V_L}{V_R}$$

where θ is the phase angle between the total voltage V and the resistive voltage V_R. If V_L, the voltage across the inductor, is 44 V and $V_R =$ 50 V, what is the phase angle θ? What is the total voltage V? (See Figure 5.56.)

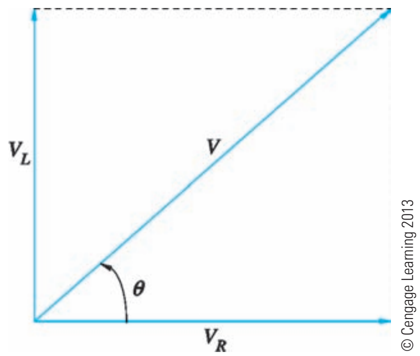

Figure 5.56

70. ***Machine technology*** A tapered shaft in the shape of an equilateral trapezoid is shown in Figure 5.57. What is the height of the shaft?

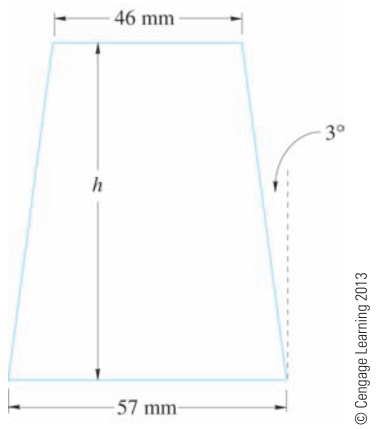

Figure 5.57

71. ***Transportation engineering*** A guard rail is going to be constructed around the outer edge of a highway curve. If the curve has a radius of 900 ft and an angle of 37°, how many feet of railing are needed?

72. ***Computer technology*** The drive speed of a disk drive for a microcomputer is 300 rpm. How many radians does it rotate in 1 s? What is the linear velocity of a point on the rim of a $5\frac{1}{4}''$ diskette?

73. ***Space technology*** Earth is approximately 93,000,000 mi from the sun and revolves around the sun in an almost circular orbit every 365 days. What is the approximate linear speed in miles per hour of Earth in its orbit?

74. ***Construction*** A guy wire for an electric pole is anchored to the ground at a point 15 ft from the base of the pole. The wire makes an angle of 68° with the level ground. How high up the pole is the wire attached?

75. Find the size of angle x in Figure 5.58 to the nearest tenth of a degree.

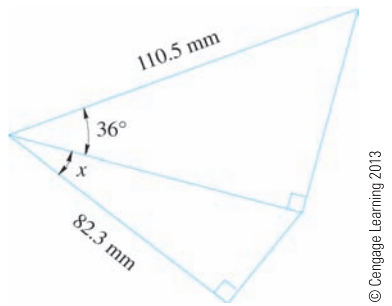

Figure 5.58

76. ***Forestry*** A tree 56 ft tall casts a shadow 43 ft long. What is the angle of the elevation of the sun at that moment?

77. ***Construction*** The light on the top of a tower is at an angle of elevation of 53.7°, when a person is 150 ft from the base of the tower. How high is the light?

78. Determine the size of angle θ in Figure 5.59 to the nearest tenth of a degree.

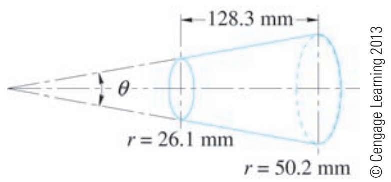

Figure 5.59

79. ***Automotive technology*** A 3,500-lb automobile rests on a ramp that makes an angle of 34.3° with the horizontal. Determine the components F_x and F_y, as shown in Figure 5.60.

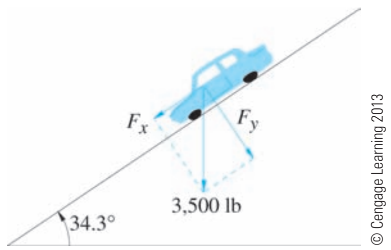

Figure 5.60

80. ***Construction*** Two guy wires from the top of a tower are anchored in the ground 35 m apart and in direct line with the tower. If the wires make angles of 36.4° and 23.7° with the top of the tower as shown in Figure 5.61, what is the height of the tower?

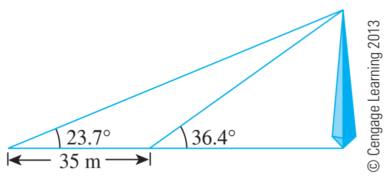

Figure 5.61

CHAPTER 5 TEST

1. Convert $50°$ to radians.
2. Convert $\frac{4\pi}{3}$ to degrees.
3. **(a)** In what quadrant is a $237°$ angle?
   (b) Give the reference angle for a $237°$ angle.
4. The point $(-5, 12)$ is on the terminal side of an angle θ in standard position. Give the six trigonometric functions of this angle.
5. If angle θ is at vertex A of right $\triangle ABC$ with sides $a = 9$ and $c = 23.4$, then determine
   (a) $\sin\theta$
   (b) $\tan\theta$
6. If $\cot\theta = \frac{6.75}{30}$, then determine
   (a) $\tan\theta$
   (b) $\sin\theta$
7. Determine each of the indicated values:
   (a) $\sin 53°$
   (b) $\tan(-112°)$
   (c) $\sec 127°$
   (d) $\cos 4.2$
8. Find, to the nearest tenth of a degree, all angles θ, where $0° \leq \theta < 360°$, with the given function.
   (a) $\tan\theta = 1.2$
   (b) $\sin\theta = -0.72$
9. A guy wire for a tower is anchored to the ground 135 ft from the base of the tower. The wire makes an angle of $53°$ with the level ground. How high up the tower is the wire attached?
10. The drive speed of a computer hard disk drive is 3,600 rpm. How many radians does it rotate in 1 s?

6 SYSTEMS OF LINEAR EQUATIONS AND DETERMINANTS

Cell phones have become a necessary part of our everyday and working lives. How can you determine which cell phone plan is best for you? In this chapter we will look at ways systems of linear equations can help you make a better decision.

Many technical problems require us to consider the effects of several conditions and variables simultaneously. We often need to use more than one equation to show how these variables are related. When this happens, we need to find the solutions that satisfy all of these equations.

For example, in order to determine how many computers can be made using several parts, we have to consider how many of each part is available and the number needed for each computer.

In Chapter 2, we introduced the idea of solving equations. In Chapter 4 we showed how we could use graphs to help find an equation's roots. In this chapter, we will use our algebraic skills to solve linear equations and to solve systems of two or more linear equations.

OBJECTIVES

AFTER COMPLETING THIS CHAPTER, YOU WILL BE ABLE TO:

- Find the slope of a line from the equation or from two points on the line.
- Find the intercepts and graph a line given its equation.
- Identify the numerical value as well as the units for the slope of a linear relationship.
- Find the equation of the line of best fit from a data set using an intuitive procedure.
- Find an approximate graphical solution to a system of two equations.
- Solve a system of two equations using algebraic procedures.
- Use technology to solve a system of equations.
- Write a system of equations to describe a given application and solve those equations.

6.1 LINEAR EQUATIONS

You may remember that a linear equation is the equation of a straight line. In general, an equation is a **linear equation** if each term contains only one variable, to the first power, or the term is a constant.

EXAMPLE 6.1

The equation $4x + 5 = 25$ is a linear equation in one variable, x. We learned how to solve this equation in Chapter 2. The solution is $x = 5$.

EXAMPLE 6.2

The equation $4x - 5y = 3$ is a linear equation in two variables, x and y. We learned how to solve this equation in Chapter 4. The solution to this equation is all the points that are on the line $y = \frac{4}{5}x - \frac{3}{5}$.

EXAMPLE 6.3

The equation $9x + 2y - 7z = 4$ is a linear equation in three variables x, y, and z. We have not learned how to solve an equation of this type.

A linear equation can have any number of variables. The previous examples show linear equations in 1, 2, and 3 variables. But, a linear equation could just as easily have 4 or 14 variables.

In Chapter 4, we found that the slope of a line that went through two points (x_1, y_1) and (x_2, y_2) was given by the formula $m = \frac{y_2 - y_1}{x_2 - x_1}$. The slope tells us how steep the line is and whether the line is rising or falling.

POINT-SLOPE EQUATION OF A LINE

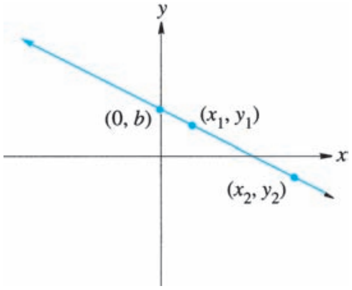

Figure 6.1

If we know the slope of a line and we know one of the points on that line, we can then determine an equation for the line. Any point on the line is of the form (x, y). Suppose we know that a specific point (x_1, y_1) is on the line and that the slope of the line is m. (See Figure 6.1.) From the equation for the slope we know that

$$m = \frac{y - y_1}{x - x_1}$$

$$\text{or } m(x - x_1) = y - y_1$$

Rewriting this equation gives us the point-slope form of a linear equation.

THE POINT-SLOPE FORM OF A LINEAR EQUATION

If (x_1, y_1) is a point on a line and the slope of the line is m, then

$$y - y_1 = m(x - x_1)$$

is known as the **point-slope equation** of a straight line.

EXAMPLE 6.4

Find the equation of the line through the point (4, 5) with a slope of $\frac{2}{3}$.

SOLUTION We are told that the slope is $\frac{2}{3}$ and that a point on the line is (4, 5). So we have $m = \frac{2}{3}, x_1 = 4$, and $y_1 = 5$. Putting these values in the point-slope form of a linear equation, we get $y - 5 = \frac{2}{3}(x - 4)$.

EXAMPLE 6.5

Find the equation of the line through the points (2, 3) and (5, 9).

SOLUTION We are not given the slope, but since we have two points we can find it. The slope is

$$m = \frac{9 - 3}{5 - 2} = \frac{6}{3} = 2$$

The point (2, 3) is on the line, so we can let $x_1 = 2$, $y_1 = 3$, and using the point-slope equation of a line, we get

$$y - 3 = 2(x - 2).$$

We do not have to use the point (2, 3). We can use any known point on the line. If we had used the point (5, 9), then $x_1 = 5$, $y_1 = 9$, and we would get the equation

$$y - 9 = 2(x - 5)$$

We can show that this is equivalent to the previous equation.

SLOPE-INTERCEPT EQUATION OF A LINE

We learned in Chapter 4 that the y-intercept is the point where the graph crosses the y-axis. The x-coordinate of this point is 0, and if the y-intercept is at b, then the point $(0, b)$ is on the line, as shown in Figure 6.1. If the slope of the line is m, then $y - b = m(x - 0)$. This simplifies to the following equation.

THE SLOPE-INTERCEPT FORM OF A LINEAR EQUATION

The **slope-intercept form** of the equation for a line is

$$y = mx + b$$

where m is the slope and b is the y-intercept.

Notice that you can easily tell the slope and the y-intercept by looking at a linear equation written in the slope-intercept form.

EXAMPLE 6.6

Find the equation of the straight line with a slope of -7 and a y-intercept of 4.

SOLUTION Here $m = -7$ and $b = 4$. Using the slope-intercept form, we get $y = -7x + 4$.

EXAMPLE 6.7

What are the slope and y-intercept of the line $2x - 5y - 10 = 0$?

SOLUTION If we solve this equation for y, it will then be in the slope-intercept form of the line. We can then determine the answers by looking at the equation for the line.

$$\begin{aligned} 2x - 5y - 10 &= 0 \\ -5y &= -2x + 10 \\ y &= \frac{2}{5}x - 2 \end{aligned}$$

This is now in the slope-intercept form for the line, $y = mx + b$. The coefficient of x is the slope and the constant is the y-intercept. So the slope, m, is $\frac{2}{5}$ and the y-intercept, b, is -2.

If we wanted to graph the line in Example 6.7, we would need a second point. Select a value for x and solve for y. For example, if $x = 10$, then $y = \frac{2}{5}(10) - 2 = 4 - 2 = 2$. So, the point (10, 2) is on this line.

Sometimes the easiest point to use is the y-intercept. Use the idea of slope as $\frac{\text{rise}}{\text{run}}$, and, from the y-intercept $(0, -2)$, "run" 5 units to the right to $x = 5$ and then "rise" 2 units upward to $y = -2 + 2 = 0$. This gives the point (5, 0), which is also on the line.

HINT If you use the idea of slope as $\frac{\text{rise}}{\text{run}}$ you should always "run" to the right. Then, if the slope is positive, "rise" upward, or, if the slope is negative, "rise" downward.

We could also let $y = 0$ and find the x-intercept. From the original equation, $2x - 5y - 10 = 0$, we have $2x - 5(0) - 10 = 0$ or $2x - 10 = 0$ or $x = 5$. This means that the point (5, 0) is also on this line. Plotting these points and drawing the line through them produces the graph in Figure 6.2.

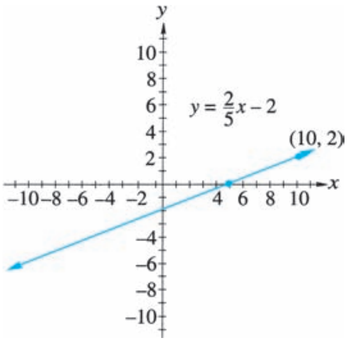

Figure 6.2

USING THE ANGLE OF THE LINE

We now use our knowledge of trigonometry to measure the angle of a line in order to complete the methods for finding the equation of a line.

Look at the two lines pictured in Figure 6.3. Line A intersects the x-axis at an angle of α, and line B intersects the x-axis at an angle of β. Together, a point on the line and the angle completely describe a line. Imagine one of the lines pivoting around the y-intercept. As the angle changes, the slope changes. There must be a relationship between the angle and the slope. What is it?

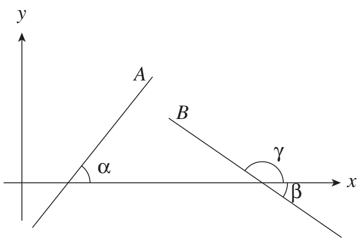

Figure 6.3

Trigonometry has the answer. Look at the right triangle shown in Figure 6.4. The vertices of the triangle are named A, B, and C, with a 90° angle at C. The legs of the triangle (the two shorter sides) are $\overline{AC}$ and $\overline{BC}$, and the hypotenuse (the longest side and the one opposite the 90° angle) is $\overline{AB}$. The angle at A is labeled α and its tangent is defined as the ratio $\frac{BC}{AC}$. From Chapter 5 we know that $\tan A = \frac{BC}{AC}$.

Look at the tangent ratio and the ratio $\frac{BC}{AC}$. They should look familiar. The tangent ratio is the slope of the line that contains the hypotenuse $\overline{AB}$.

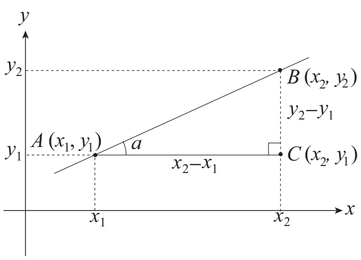

Figure 6.4

The one difficulty comes in using the correct angle, since any line makes four angles when it crosses another line. Look at Figure 6.3. The angles marked α and β are the best ones to use: the ones that make an acute angle formed by the line and the *positive* x-axis. We measure these angles according to the direction the positive x-axis is rotated to reach the other side of the angle. Tradition says that a counter-clockwise rotation is in a positive direction and a clockwise turn is a negative direction. Thus, in Figure 6.3, α is a positive angle and β is a negative angle.

There are times when it is more convenient to use the supplement of an angle such as β. In this case, you would use the angle marked γ. We will discuss this more in a later chapter.

In summary, the slopes of the lines in Figure 6.3 can be written as $m = \tan \alpha$ for line A and $m = \tan \beta$ or $m = \tan \gamma$ for line B. We will now use this idea relating slope and the tangent function to develop a new equation for a line.

Suppose that we have a line that passes through a point (x_1, y_1) and it makes an angle of α with the positive x-axis. Since the slope of this line is $m = \tan \alpha$, then we can substitute $\tan \alpha$ for m in the point-slope equation for a line. This produces

$$\begin{aligned} y - y_1 &= m(x - x_1) \\ y &= y_1 + (x - x_1)m \\ &= y_1 + (x - x_1)\tan\alpha \\ &= y_1 + x\tan\alpha - x_1\tan\alpha \\ &= x\tan\alpha + y_1 - x_1\tan\alpha \\ &= x\tan\alpha + b \end{aligned}$$

We were able to perform the last step because $y_1 - x_1 \tan \alpha$ is just a number, which we called b. We summarize these new formulas in the following box.

THE ANGLE-POINT FORM OF THE LINEAR EQUATION

If a line passes through the point (x_1, y_1) and makes an angle of α with the positive x-axis, then the equation of the line is

$$\begin{aligned} y &= y_1 + (x - x_1)\tan\alpha \\ \text{or} \quad y &= x\tan\alpha + b \end{aligned}$$

where b is the y-intercept.

EXAMPLE 6.8

A line passes through the point $(2, -4)$ and makes an angle of 65° with the positive x-axis. What is the equation of the line?

SOLUTION A graph of the given information is shown in Figure 6.5. The slope of the line is $\tan 65°$. Check the calculator to verify that it is in degree mode. If it is not, change it.

Since we know the size of the angle the line makes with the positive x-axis, it remains only to apply the angle-point form of the linear equation. The equation of this line is

$$\begin{aligned} y &= y_1 + (x - x_1)\tan\alpha \\ &= -4 + (x - 2)\tan 65° \\ &\approx -4 + (x - 2)2.1445 \end{aligned}$$

When this is written in slope-intercept form the equation is $y \approx 2.1445x - 8.289$.

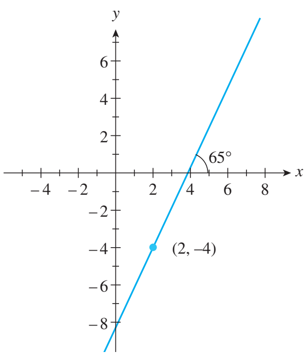

Figure 6.5

EXAMPLE 6.9

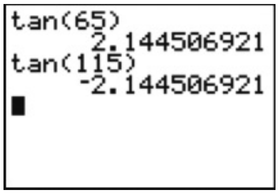

Figure 6.6a

	A	B
1	=TAN(Radians(115))	-2.144507
2	=TAN(Radians(65))	2.144507

Figure 6.6b

A line passes through the point (2, -4) and makes an angle of 115° with the positive x-axis. What is the equation of the line?

SOLUTION The slope of the line is given as tan 115°.

Using a calculator:

To approximate tan 115°, make sure that your calculator is in degree mode and press TAN 115 ENTER. The result is shown in Figure 6.6a.

Using a spreadsheet:

To approximate tan 115° on spreadsheet, you must first convert 115° to radians. This is done in one step using the TAN and RADIANS functions as shown in Cell A2 of Figure 6.6b.

Notice that the values of tan 65° and tan 115° differ only in their sign. You'll learn the reason for that when you study some more trigonometry in Chapter 10.

Now that we have the slope, we can apply the point-slope form of the linear equation to obtain

$$\begin{aligned} y &= -4 + (x - 2)\tan 115^\circ \\ &\approx -4 + (x - 2)(-2.1445) \end{aligned}$$

This is $y \approx -2.1445x + 0.2890$ when it is written in slope-intercept form.

The initial graphs of $y \approx 2.1445x - 8.289$ and $y \approx -2.1445x + 0.2890$ using as TI-83/84 graphing calculator are in Figure 6.7a and using a spreadsheet in Figure 6.7c. Notice that the ones drawn using a calculator (Figure 6.7a) appear to be almost perpendicular. This is because we used the calculator's standard viewing window. If you select the ZSQUARE [Zoom Square] window you obtain the graphs in Figure 6.7b.

The graphs drawn using a spreadsheet, shown in Figure 6.7c, are not any better. Although the lines do not appear to be perpendicular, they also do not appear to meet the x-axis at 65° and 115° angles. If the y-axis is adjusted so its minimum and maximum values are the same as those on the x-axis (-10 and 10), you get the result in Figure 6.7d. You can also see them drawn on a coordinate system in Figure 6.7e.

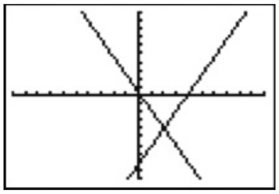

Figure 6.7a

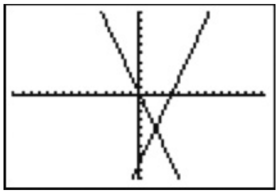

Figure 6.7b

	A	B	C
1	x	y1	y2
2	-9	19.58958	-27.5896
3	-7	15.30056	-23.3006
4	-3	6.722535	-14.7225
5	-2	4.578028	-12.578
6	-1	2.433521	-10.4335
7	0	0.289014	-8.28901
8	1	-1.85549	-6.14451
9	2	-4	-4
10	3	-6.14451	-1.85549
11	7	-14.7225	6.722535
12	9	-19.0115	11.01155
13			
14			
15			

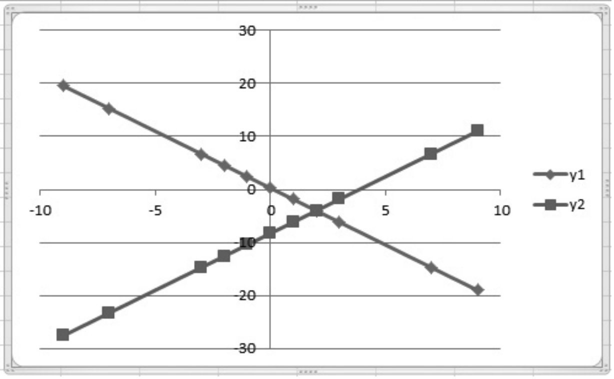

Figure 6.7c

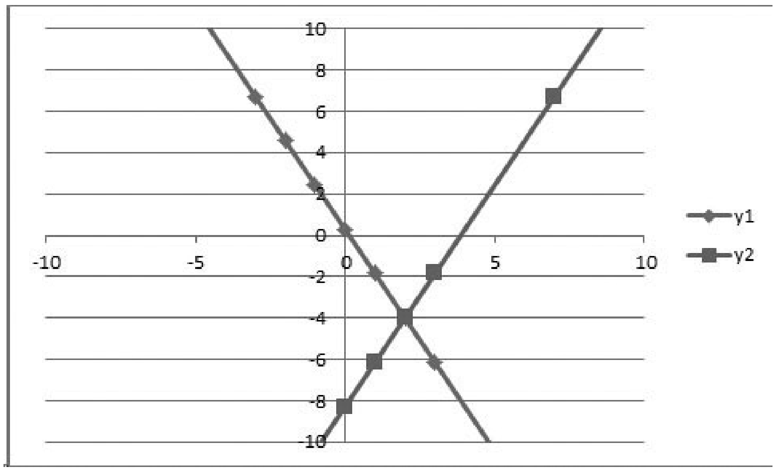

Figure 6.7d

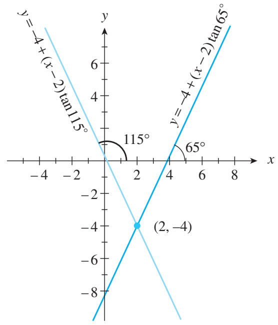

Figure 6.7e

EXAMPLE 6.10

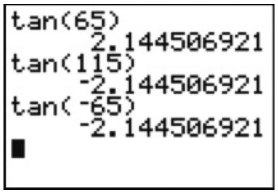

Figure 6.8

A line passes through the point $(2, -4)$ and makes an angle of $-65°$ with the positive x-axis. What is the equation of the line?

SOLUTION The slope of the line is $\tan(-65°)$ and the approximate value of $\tan(-65°)$ is shown in Figure 6.8. Notice that $\tan(-65°) = \tan 115°$. This is because the difference in the measures of their angles is $115° - (-65°) = 180°$.

Again, if we apply the angle-point form of the linear equation we see that the equation of this line is $y \approx -4 + (x - 2)(-2.1445)$ or, when written in slope-intercept form, is $y \approx -2.1445x + 0.2890$.

APPLICATION CONSTRUCTION

EXAMPLE 6.11

An access ramp to a building makes a 4.8° angle with the ground. What is the slope of this ramp?

SOLUTION The slope is $\tan 4.8°$. Using a calculator, you should obtain $\tan 4.8° \approx 0.08397$. Thus, the slope of this ramp is about 0.0840.

APPLICATION CONSTRUCTION

EXAMPLE 6.12

If the access ramp in Example 6.11 runs 25 ft along the ground, how high is it when it reaches the building?

SOLUTION Figure 6.9a is a drawing of the problem as it was described. In Figure 6.9b, we have put this on a rectangular coordinate system with the bottom of the ramp along the positive x-axis starting at the origin. The top of the ramp is at the point marked B. We know that the x-coordinate of B is 25. We want to find the y-coordinate of B.

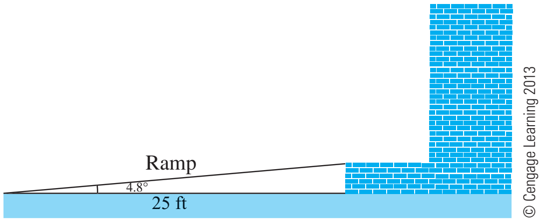

Figure 6.9a

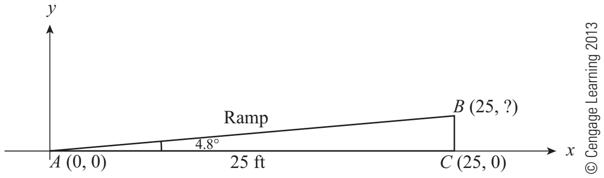

Figure 6.9b

EXAMPLE 6.12 (Cont.)

The ramp starts at the origin, and so we know that the equation of the line that forms the top of the ramp is $y = mx$ or $y = (\tan 4.8°)x$. Since $x = 25$, we have

$$\begin{aligned} y &= (\tan 4.8°)(25) \\ &\approx 2.099 \end{aligned}$$

Thus, the top of the ramp is about 2.099 ft or 25.2 in. above the ground.

EXAMPLE 6.13

A sketch of the line $y = 2.13x + 1.7$ is in Figure 6.10. Determine the angle that the line makes with the positive x-axis.

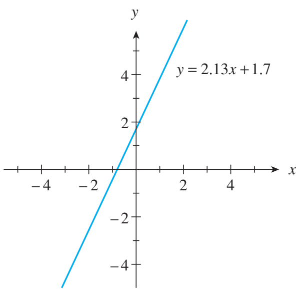

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Figure 6.10

SOLUTION This line has a slope of 2.13. We know that the slope of a line is the tangent of the angle α that the line makes with the x-axis, so $m = \tan \alpha$. Here we have $m = 2.13$. Thus, we have the equation $2.13 = \tan \alpha$.

We want the angle whose tangent is 2.13. To determine this angle, use $\tan^{-1}$ and you will see that the angle is approximately 64.85°.

APPLICATION CIVIL ENGINEERING

EXAMPLE 6.14

A small boat leaves a dock and heads due east at 9.2 kn (knots—1 kn ≈ 1.15 mi/h). A current of 3.7 kn (north) is at right angles to the heading of the boat, as shown in Figure 6.11. At what angle α does the current change the heading of the boat?

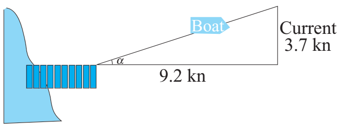

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Figure 6.11

SOLUTION The angle α that indicates the amount that this boat is deflected from its heading is the slope of the line that forms its actual direction. Thus, $\tan \alpha = \frac{3.7}{9.2}$, and so $\alpha = \tan^{-1}\left(\frac{3.7}{9.2}\right) \approx 21.91°$. The boat's heading is changed by about 21.91° to the north.

A **horizontal line** is parallel to the x-axis and has a slope of 0. This means that any horizontal line can be written as $y = 0 \cdot x + b = b$ or simply $y = b$. Notice that all points on a horizontal line have the same y-coordinate, b.

A **vertical line** has an undefined slope. This means that a vertical line cannot be written in slope-intercept form. But, all points on a vertical line have the

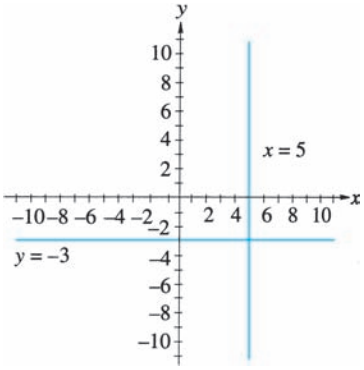

Figure 6.12

same x-coordinate. If we call this x-coordinate a, we can then write the equation of this vertical line as $x = a$.

HORIZONTAL AND VERTICAL LINES

A horizontal line is parallel to the x-axis, has a slope of 0, and can be written as $y = b$.
A vertical line is perpendicular to the x-axis, has an undefined slope, and can be written as $x = a$.

EXAMPLE 6.15

Graph the lines $x = 5$ and $y = -3$.

SOLUTION The line $x = 5$ is a vertical line with x-intercept 5. The line $y = -3$ is a horizontal line with y-intercept -3. These two lines are graphed in Figure 6.12.

EXERCISE SET 6.1

In Exercises 1–10, determine the slope of the line through the given pair of points.

1. $(2, 5), (3, 8)$
2. $(4, 7), (-2, 1)$
3. $(1, 8), (5, 3)$
4. $(0, 4), (5, 0)$
5. $(9, 3), (2, -7)$
6. $(-6, 1), (2, -5)$
7. $(0, 5), (4, 5)$
8. $(-3, 5), (-3, 7)$
9. $(2, 17), (2, \frac{3}{4})$
10. $(2, -7), (-7, -7)$

In Exercises 11–28, determine an equation in point-slope form for the line that satisfies the given data.

11. $m = 4$, point: $(5, 3)$
12. $m = -3$, point: $(-6, 1)$
13. $m = \frac{2}{3}$, point: $(1, -5)$
14. $m = \frac{3}{2}$, point: $(0, 5)$
15. $m = 0$, point: $(2, -5)$
16. m undefined, point: $(-3, 6)$
17. $m = -\frac{5}{3}$, point: $(2, 0)$
18. $m = -\frac{3}{4}$, point: $(-3, -1)$
19. m undefined, point: $(7, -4)$
20. $m = 0$, point: $(-16, 5)$
21. points: $(1, 5)$ and $(-3, 2)$
22. points: $(-5, 6)$ and $(1, -6)$
23. $(-5, 3), (7, 3)$
24. $(\frac{2}{3}, \frac{3}{4}), (0, \frac{25}{8})$
25. point: $(-4, 7)$, angle of 27° with the positive x-axis
26. point: $(5.3, -1.7)$, angle of $-75.8°$ with the positive x-axis
27. point: $(-2.1, -3.7)$, angle of 63.1° with the positive x-axis
28. point: $(9.1, 1.25)$, angle of 15.6° with the positive x-axis

In Exercises 29–32, determine an equation in slope-intercept form for the line that satisfies the given information.

29. $m = 2, b = 4$
30. $m = -3, b = 5$
31. $m = 5, b = -3$
32. $m = -4, b = -2$

In Exercises 33–40, rewrite each equation in slope-intercept form, find the slope m and y-intercept b, and sketch the graph.

33. $y - 3x = 6$
34. $2x - y = 5$
35. $2y - 5x = 8$
36. $2x - 3y = 9$
37. $5x - 2y - 10 = 0$
38. $4y - 3x - 4 = 0$
39. $x = -3y + 7$
40. $3x = 5y - 6$

Determine the angle that each of the following lines makes with the positive x-axis.

41. $y = -0.35x + 2.95$
42. $y = 1.9x - 6.31$
43. $2x - 3.2y = 12$
44. $4x + 1.7y - 9.2 = 0$

For each equation in Exercises 45–48, (a) give the slope and (b) graph the line determined by the equation.

45. $y = 4$
46. $x = -2$
47. $x = \frac{9}{2}$
48. $y = -7$

Solve Exercises 49–56.

49. ***Machine technology*** A grinding machine operates at 1780 rev/min. The surface speed s in cm/s is given by the formula $s = \frac{1780\pi d}{60}$, where d is the diameter in cm. What is the slope of this equation? (Assume that d is on the horizontal axis.)

50. ***Meteorology*** The relationship between the Fahrenheit and Celsius temperatures is linear. If the Fahrenheit temperatures are put on the horizontal or x-axis and the Celsius temperatures are put on the vertical or y-axis, then two points are $(-40, -40)$ and $(32, 0)$.

(a) What is the slope of this line?
(b) What is the y-intercept?
(c) Write the equation in point-slope form and in slope-intercept form.
(d) Graph the line.

51. ***Physics*** A spring coil has an unstretched or natural length of L_0, and requires a force F of kx to stretch it x units beyond its natural length. The letter k represents a constant known as the *spring constant*. The distance the spring is stretched, x, is equal to $L - L_0$, where L is the length of the spring when it is stretched.

(a) Write an equation in slope-intercept form for the force F in terms of k, L, and L_0.
(b) If $k = 4.5$ and $L_0 = 6$ cm, write an equation in slope-intercept form for the force F in terms of L.
(c) Graph the equation in (b).

52. ***Physics*** The pressure P at a depth h in a liquid depends on the density of the liquid, ρ. In a certain liquid at 4 ft, the pressure is 250 lb/ft^2. At 9 ft the pressure is 562.5 lb/ft^2. Write an equation in point-slope form for the pressure in terms of the depth.

53. ***Navigation*** An airplane is heading due east, according to the compass, at 327 mph. A crosswind of 57 mph is blowing directly from the north. At what angle is the wind pushing the plane off its compass heading?

54. ***Construction*** The rafters in a roof are placed at an angle of 18.7° with the horizontal. If the rafter has to cover a horizontal distance, the run, of 16′2″, what is the length of the rafter?

55. ***Construction*** The rafters in the roof in Figure 6.13 have a run of 14′6″ and a rise of 4′2″. What angle does the rafter make with the horizontal?

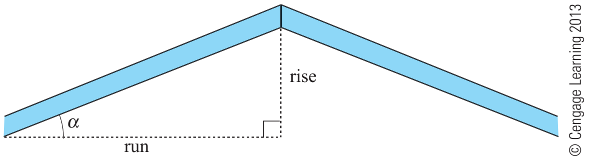

Figure 6.13

56. ***Health technology*** The following data lists the life expectancy, in years, of people born in various years. (Hint: Enter the years as 20, 30, etc. rather than 1920, 1930, etc.)

Year	1920	1930	1940	1950	1965	1970	1980	1990	1995	2000	2005
Life expectancy	54.1	59.7	62.9	68.2	70.2	70.8	73.7	75.4	75.8	76.8	77.4

(a) Determine a linear regression curve that will fit the data.

(b) Determine the angle that the line makes with the positive horizontal axis.

57. ***Energy technology*** The following data lists the amount of renewable energy produced in the United States in quadrillion British thermal units (Btu) for various years.

Year	1960	1970	1975	1980	1985	1990	1995	2000	2005	2008
Production	2.93	4.08	4.72	5.49	6.19	6.21	6.70	6.26	6.41	7.32

(a) Determine a linear regression curve that will fit the data.

(b) Determine the angle that the line makes with the positive horizontal axis.

58. ***Environmental technology*** The following data lists the amount of sulfur dioxide emissions in the United States in million tons for various years.

Year	1970	1980	1990	1995	2000	2003	2004	2005	2006	2007
SO_2	31.2	25.9	23.1	18.6	16.3	14.8	14.7	14.7	13.5	12.9

(a) Determine a linear regression curve that will fit the data.

(b) Determine the angle that the line makes with the positive horizontal axis.

[IN YOUR WORDS]

59. (a) Explain what it means for a line to have a slope of 0.

(b) Explain what it means for a line to have an undefined slope.

60. Explain how you can determine the equation for a line if you know the coordinates of two points on that line.

6.2 GRAPHICAL AND ALGEBRAIC METHODS FOR SOLVING TWO LINEAR EQUATIONS IN TWO VARIABLES

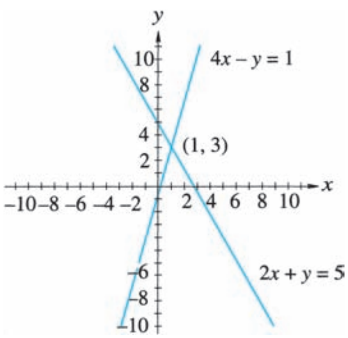

Figure 6.14

In this section, we will begin to look at methods for solving a system of simultaneous linear equations. Simultaneous linear equations are equations containing the same variables, such as

$$2x + y = 5$$
$$4x - y = 1$$

Our task in this section is to determine all the points, or ordered pairs, that these two equations have in common. We will begin by looking at a way to use graphing to help find these common points. Since we are looking for a common point of these two lines, this point will be where the two lines intersect if we graph each line. A graph of these two lines is shown in Figure 6.14.

As you can see, the lines appear to meet at point (1, 3). A quick check of both equations will show that point (1, 3) is on both lines. To check, substitute $x = 1$ and $y = 3$ in the first equation. We obtain

$$2(1) + 3 = 2 + 3 = 5$$

Substituting $x = 1$ and $y = 3$ in the second equation produces

$$4(1) - 3 = 4 - 3 = 1$$

But, as we saw in Chapter 4, graphical methods are not always accurate ways to determine the roots to an equation. Graphical methods are also not very accurate ways to determine the common solutions of simultaneous equations. What we need are some algebraic methods for solving a system of equations. We will learn two methods in this section. Both methods involve solving for one of the variables by eliminating the other variable. These are called **elimination methods**.

SUBSTITUTION METHOD

The first elimination method involves elimination by substitution. It is generally called the **substitution method**.

SUBSTITUTION METHOD FOR SOLVING A SYSTEM OF LINEAR EQUATIONS

To use the substitution method to solve a system of linear equations:

1. Solve one equation for one of the variables.
2. Substitute the solution from Step 1 into the other equation and solve for the remaining variable.
3. Substitute the value from Step 2 into the equation from Step 1 and solve for the other variable.

In the substitution method, we change two equations in two variables into one equation in one variable. The next two examples show how to use the substitution method to solve systems of two equations in two variables.

EXAMPLE 6.16

Use the substitution method to solve the following system of equations.

$$\begin{cases} 2x + y = 5 & (1) \\ 4x - y = 1 & (2) \end{cases}$$

SOLUTION We will solve the first equation for y.

$$y = 5 - 2x \qquad (3)$$

Substitute this solution for y in equation (2). Equation (2) becomes

$$4x - (5 - 2x) = 1$$

Solve this equation for x.

$$4x - 5 + 2x = 1$$
$$6x = 6$$
$$x = 1$$

We then substitute this solution for x in equation (3) and find

$$y = 5 - 2(1)$$
$$y = 3$$

The solution is (1, 3), which was the same answer we got by graphing.

EXAMPLE 6.17

Use the substitution method to solve the system of equations.

$$\begin{cases} -2x+2y=5 & (1) \\ x+6y=1 & (2) \end{cases}$$

SOLUTION This time we will solve equation (2) for x and get

$$x = 1 - 6y \qquad (3)$$

Substituting this value, $1 - 6y$, for x in equation (1) we get

$$-2(1 - 6y) + 2y = 5$$
$$-2 + 12y + 2y = 5$$
$$-2 + 14y = 5$$
$$14y = 7$$
$$y = \tfrac{7}{14} = \tfrac{1}{2}$$

Replacing the y in equation (3) with $\frac{1}{2}$ we get

$$x = 1 - 6(\tfrac{1}{2})$$
$$x = 1 - 3$$
$$x = -2$$

The solution appears to be $(-2, \frac{1}{2})$.

To be certain that we have the correct solution, we should substitute $(-2, \frac{1}{2})$ into the original equations—equations (1) and (2)—and see if this solution satisfies both of these equations. It is very important that you always check your work by using the *original* equations. If you made any errors, you might not detect them unless you check your work in the original problem.

ADDITION METHOD

The second algebraic method for solving a system of linear equations is normally called the **addition method**. Technically, its name is the *elimination method by addition and subtraction*.

ADDITION METHOD FOR SOLVING A SYSTEM OF LINEAR EQUATIONS

To use the addition method to solve a system of linear equations:

Add or subtract the two equations in order to eliminate one of the variables.

It is sometimes necessary to multiply the original equations by a constant before it is possible to eliminate one of the variables by adding or subtracting the equations.

The next two examples will show how the addition method is used. These are the same examples that we worked with when using the substitution method.

EXAMPLE 6.18

Use the addition method to solve the following system.

$$\begin{cases} 2x + y = 5 & (1) \\ 4x - y = 1 & (2) \end{cases}$$

SOLUTION Equation (1) has a $(+y)$ term and equation (2) has a $(-y)$ term. If we add equations (1) and (2), the new equation will not have a y term.

$$\begin{array}{rrl} & 2x + y = 5 & \\ & 4x - y = 1 & \\ \hline \text{adding} & 6x \quad\; = 6 & (1) + (2) \\ \text{or} & x = 1 & (3) \end{array}$$

Substituting this value of x into equation (1) we get

$$2(1) + y = 5$$
$$y = 3$$

So, the solution is $x = 1$ and $y = 3$ or the ordered pair $(1, 3)$.

EXAMPLE 6.19

Use the addition method to solve the following system.

$$\begin{cases} -2x + 2y = 5 & (1) \\ x + 6y = 1 & (2) \end{cases}$$

SOLUTION Equations (1) and (2) do not have any terms that are equal so we will have to multiply at least one equation by a constant. If we multiply equation (2) by 2, the x-term will become $2x$, which is the additive inverse of the x term in equation (1). After multiplication, equation (2) becomes

$$2x + 12y = 2 \qquad (3)$$

and adding equations (1) and (3) we get

$$\begin{array}{lrl} & -2x + 2y = 5 & (1) \\ & 2x + 12y = 2 & (3) \\ \hline \text{adding} & 14y = 7 & (1) + (3) \\ \text{or} & y = \dfrac{1}{2} & (4) \end{array}$$

Substituting this value for y into equation (2) we get

$$x + 6\left(\tfrac{1}{2}\right) = 1$$
$$x + 3 = 1$$
$$x = -2$$

Thus, we have found $x = -2, y = \frac{1}{2}$, and the solution is $\left(-2, \frac{1}{2}\right)$, the same answer we got in Example 6.17.

You may get very strange-looking results when you attempt to solve a system of equations. For example, sometimes all the variables vanish. Consider the next example.

EXAMPLE 6.20

Use the addition method to solve the following system.

$$\begin{cases} 2x + 3y = 6 & (1) \\ 4x + 6y = 30 & (2) \end{cases}$$

SOLUTION If we multiply equation (1) by -2, the terms containing the variable x will be additive inverses of each other. This multiplication makes equation (1) into

$$\begin{array}{lrl} & -4x - 6y = -12 & (3) \\ \text{and adding} & 4x + 6y = 30 & (2) \\ \hline & 0 = 18 & (3) + (2) \end{array}$$

Now we know that $0 \neq 18$, so something must be wrong. If we check our work, there do not appear to be any errors. Let's graph these equations. The graph in Figure 6.15 indicates the problem. The lines are parallel. They will never intersect, so there is no solution to this system of equations.

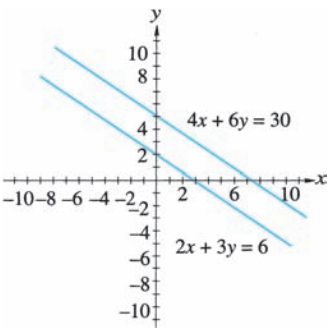

Figure 6.15

When two lines are parallel they will not intersect. Since there is no solution, the equations in the system are said to be **inconsistent**. When you try to solve a system of linear equations that is inconsistent, you will get an untrue equation. In Example 6.20, this equation was $0 = 18$.

In Example 6.21, we examine another kind of system of linear equations in which all the variables vanish.

EXAMPLE 6.21

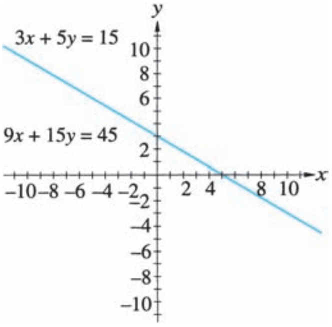

Figure 6.16

Solve the following system of linear equations.

$$\begin{cases} 3x + 5y = 15 & (1) \\ 9x + 15y = 45 & (2) \end{cases}$$

SOLUTION Multiplying equation (1) by -3, we get

$$-9x - 15y = -45 \quad (3)$$

and adding this to equation (2) we have the following:

$$\begin{aligned} -9x - 15y &= -45 && (3) \\ 9x + 15y &= 45 && (2) \\ \hline 0 &= 0 && (3) + (2) \end{aligned}$$

It is certainly true that $0 = 0$. But, how will that help us solve this system of equations? Once again, we will turn to graphing to help us solve this system. (See Figure 6.16.) The graphs of these two equations are exactly the same. Thus, there are an unlimited number of solutions to both equations. In fact, any ordered pair that satisfies one of the equations will satisfy the other.

If the graphs of two equations coincide, we say that the equations are **dependent**. In the case of dependent equations, every solution to one equation will be a solution to the other equation. When you try to solve a system of dependent equations, you will get an equality of two constants. In Example 6.21, this was $0 = 0$.

If we solve equation (1) in Example 6.21 for y, we get $y = -\frac{3}{5}x + 3$. (You get the same result if you solve equation (2) for y.) Hence, every ordered pair (a, b) of the form $\left(a, -\frac{3}{5}a + 3\right)$ is a solution of the given system.

Most of the systems of equations we will work with are consistent. A system of linear equations is **consistent** if it has exactly one point as the solution.

In this section, we looked at three methods for solving systems of linear equations—the graphical method, the substitution method, and the elimination method. In the next section, we will learn about a new way of working with numbers that will help us find an easier method for solving systems of linear equations.

EXERCISE SET 6.2

In Exercises 1–8, graphically solve each system of equations. Estimate each answer to the nearest tenth if necessary. Check your answers, but remember that your estimate may not check exactly.

1. $\begin{cases} x + y = 6 \\ x - y = 2 \end{cases}$

2. $\begin{cases} x + y = 8 \\ 2x - y = 1 \end{cases}$

3. $\begin{cases} 2x + 3y = 15 \\ 4x - 4y = 10 \end{cases}$

4. $\begin{cases} 3x + 2y = 2 \\ 2x - 6y = -39 \end{cases}$

5. $\begin{cases} 3x + 5y = 32 \\ 10x - 5y = -32 \end{cases}$

6. $\begin{cases} -5x + 11y = 11 \\ 5x + 2y = -11 \end{cases}$

7. $\begin{cases} 5x + 10y = 0 \\ 3x - 4y = 11 \end{cases}$

8. $\begin{cases} 4x + y = -8 \\ 3x + 2y = 0 \end{cases}$

In Exercises 9–20, use the substitution method to solve each system of equations.

9. $\begin{cases} y = 3x - 4 \\ x + y = 8 \end{cases}$

10. $\begin{cases} x = -2y + 12 \\ x + y = 5 \end{cases}$

11. $\begin{cases} y = -2x - 2 \\ 3x + 2y = 0 \end{cases}$

12. $\begin{cases} x = 7 + 2y \\ 3x + 4y = 1 \end{cases}$

13. $\begin{cases} 2x + 5y = 6 \\ x - y = 10 \end{cases}$

14. $\begin{cases} 3x - 2y = 5 \\ -7x + 4y = -7 \end{cases}$

15. $\begin{cases} 2x + 3y = 3 \\ 6x + 4y = 15 \end{cases}$

16. $\begin{cases} 2x + 2y = -3 \\ 4x + 9y = 5 \end{cases}$

17. $\begin{cases} 4.8x - 1.3y = 16.9 \\ -7.2x - 2.8y = -9.2 \end{cases}$

18. $\begin{cases} 2.3x + 1.7y = 8.5 \\ -6.7x + 3.7y = 38.4 \end{cases}$

19. $\begin{cases} 4.2x + 3.7y = 10.79 \\ 6.5x - 0.3y = -15.24 \end{cases}$

20. $\begin{cases} 0.75x + 1.5y = -0.225 \\ 3.13x + 1.74y = 13.073 \end{cases}$

In Exercises 21–30, use the addition method to solve each system of equations.

21. $\begin{cases} x + y = 9 \\ x - y = 5 \end{cases}$

22. $\begin{cases} 2x + 3y = 5 \\ -2x + 5y = 3 \end{cases}$

23. $\begin{cases} -x + 3y = 5 \\ 2x + 7y = 3 \end{cases}$

24. $\begin{cases} 3x - 2y = 8 \\ 5x + y = 9 \end{cases}$

25. $\begin{cases} 3x - 2y = -15 \\ 5x + 6y = 3 \end{cases}$

26. $\begin{cases} 2x - 3y = 11 \\ 6x - 5y = 13 \end{cases}$

27. $\begin{cases} 3x - 5y = 37 \\ 5x - 3y = 27 \end{cases}$

28. $\begin{cases} x + \frac{1}{2}y = 7 \\ 4x - 2y = 5 \end{cases}$

29. $\begin{cases} 6.4x - 1.7y = 66.7 \\ -4.2x + 5.1y = -62.1 \end{cases}$

30. $\begin{cases} 1.6x - 2.9y = -2.645 \\ 2.4x + 1.4y = 11.27 \end{cases}$

In Exercises 31–40, solve each system of equations by either the substitution method or the addition method. Graph each system of equations.

31. $\begin{cases} 2x + 3y = 5 \\ x - 2y = 6 \end{cases}$

32. $\begin{cases} 2x - 3y = -14 \\ 3x + 2y = 44 \end{cases}$

33. $\begin{cases} 8x + 3y = 13 \\ 3x + 2y = 11 \end{cases}$

34. $\begin{cases} 6x + 12y = 7 \\ 8x - 15y = -1 \end{cases}$

35. $\begin{cases} 10x - 9y = 18 \\ 6x + 2y = 1 \end{cases}$

36. $\begin{cases} 4x - 5y = 7 \\ -8x + 10y = -30 \end{cases}$

37. $\begin{cases} x - 9y = 0 \\ \frac{x}{3} = 2y + \frac{1}{3} \end{cases}$

38. $\begin{cases} 5x + 3y = 7 \\ \frac{3}{2}x - \frac{3}{4}y = 9\frac{1}{4} \end{cases}$

39. $\begin{cases} 4.9x + 1.7y = 10.6 \\ 3.6x - 1.2y = 14.4 \end{cases}$

40. $\begin{cases} 3.14x + 4.57y = -7.9 \\ 2.48x + 11.84y = 15.16 \end{cases}$

Solve Exercises 41–48.

41. ***Land management*** The perimeter of a rectangular field is 36 km. The length of the field is 8 km longer than the width. What are the length and width of the field? (Hint: To find the length and width of this rectangular field, you need to solve this system of linear equations where L represents the length of the field and w the width.)

$$\begin{cases} 2L + 2w = 36 \\ L = w + 8 \end{cases}$$

42. ***Land management*** The perimeter of a rectangular field is 72 mi. The length of the field is 9 mi longer that the width. What are the length and width of the field?

43. ***Land management*** The perimeter of a rectangular field is 45 km. The length is 3 times the width. What are the length and width of the field?

44. ***Land management*** The perimeter of a field in the shape of an isosceles triangle is 96 yd.

The length of each of the two equal sides is $1\frac{1}{2}$ times the length of the third side. What are the lengths of the three sides of this field?

45. ***Petroleum technology*** Two different gasohol mixtures are available. One mixture contains 5% alcohol and the other, 13% alcohol. In order to determine how much of each mixture should be used to get 10 000 L of gasohol containing 8% alcohol, you would solve the following equations:

$$\begin{cases} x + y = 10\,000 \\ 0.05x + 0.13y = (0.08)(10\,000) \end{cases}$$

where x is the number of liters of 5% gasohol mixture and y is the number of liters of 13% gasohol mixture. Determine the number of liters of each mixture that is needed.

46. ***Petroleum technology*** Two different gasohol mixtures are available. One mixture contains 4% alcohol and the other, 12% alcohol. How much of each mixture should be used to get 20 000 L of gasohol containing 9% alcohol?

47. ***Physics*** Two forces are applied to the ends of a beam. The force on one end is 8 kg and the force at the other end is not known. The unknown force is 5 m from the centroid; we do not know the distance of the 8-kg force from the centroid. If an additional force of 4 kg is applied to the 8-kg force, the unknown force must be increased by 3.2 kg for equilibrium to be maintained. Find the unknown mass and distance. (See Figure 6.17.)

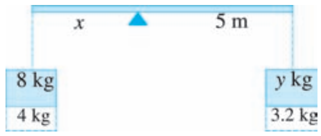

Figure 6.17

48. ***Automotive technology*** A 12-L cooling system is filled with 25% antifreeze. How many liters must be replaced with 100% antifreeze to raise the strength to 45% antifreeze?

In Exercises 49–51, use the following information.

The current that flows in each branch of a complex circuit can be found by applying **Kirchhoff's rules** to the circuit. The first rule applies to the junction of three or more wires as in Figure 6.18. The second applies to loops (circuits) or closed paths in the circuit.

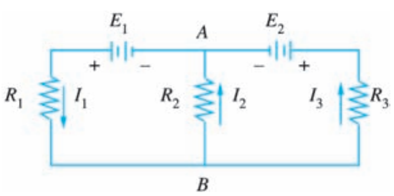

Figure 6.18

Rule 1: (Junction rule) The sum of the currents that flow into a junction is equal to the sum of the currents that flow out of the junction. In Figure 6.18, this means that $I_1 = I_2 + I_3$.

Rule 2: (Circuit rule) The sum of the voltages around any closed loop equals zero. In Figure 6.18, for the left loop this means that $E_1 = I_1R_1 + I_2R_2$ and for the right loop that $E_2 = R_2I_2 - R_3I_3$.

49. ***Electronics*** Find the currents in the three resistors of the circuit shown in Figure 6.18, given that $E_1 = 8$ V, $E_2 = 5$ V, $R_1 = 3\ \Omega$, $R_2 = 5\ \Omega$, and $R_3 = 6\ \Omega$. [Use $E_1 = I_1R_1 + I_2R_2$ and $E_2 = R_2I_2 - R_3(I_1 - I_2)$.]

50. ***Electronics*** In Figure 6.18, if $E_1 = 10$ V, $E_2 = 15$ V, $R_1 = 2\ \Omega$, $R_2 = 4\ \Omega$, and $R_3 = 8\ \Omega$, find I_1, I_2, and I_3.

51. ***Electronics*** In Figure 6.19, we have $I_2 = I_1 + I_3$, $E_1 = R_1I_1 + R_2I_2$, and $E_2 = R_3I_3 + R_2I_2$. If $E_1 = 6$ V, $E_2 = 10$ V, $R_1 = 8\ \Omega$, $R_2 = 4\ \Omega$, and $R_3 = 7\ \Omega$, find I_1, I_2, and I_3.

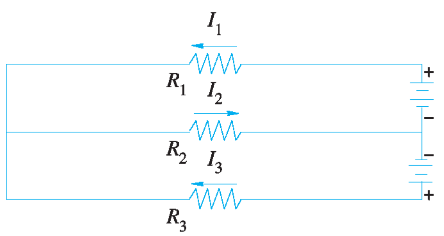

Figure 6.19

[IN YOUR WORDS]

52. Describe the advantages and disadvantages of the substitution method compared to the addition method.

53. Describe what it means for a system of equations to be consistent or inconsistent.

54. Describe how to use the substitution method for solving a system of linear equations.

6.3 ALGEBRAIC METHODS FOR SOLVING THREE LINEAR EQUATIONS IN THREE VARIABLES

In Section 6.2 we learned to solve a system of two linear equations in two variables by a graphical method and by two algebraic methods of elimination. In this section, we will expand these algebraic techniques to allow us to solve a system with three linear variables. These methods will be used later to allow us to solve n equations with n variables.

The graph of a linear equation in three variables is a plane. Graphing a system of three equations in three variables requires the ability to graph three planes and their intersections. We cannot graph three planes so that we can tell where they intersect. As a result, we will not consider graphical solutions to a system of three equations in three variables.

SUBSTITUTION METHOD

We will solve the same system of equations using both elimination methods. The first example uses the substitution method.

EXAMPLE 6.22

Solve this system of equations using the substitution method.

$$\begin{cases} x + 2y - 2z = 3 & (1) \\ 2x - y + 3z = -5 & (2) \\ 4x - 3y + z = 7 & (3) \end{cases}$$

SOLUTION If we solve equation (3) for z we get

$$z = 3y - 4x + 7 \qquad (4)$$

Substituting this value of z in equation (2) changes it to

$$2x - y + 3(3y - 4x + 7) = 2x - y + 9y - 12x + 21 = -5$$

Combining terms results in

$$-10x + 8y = -26 \qquad (5)$$

Solving equation (5) for y produces

$$8y = 10x - 26$$

$$\text{or} \quad y = \tfrac{5}{4}x - \tfrac{13}{4} \qquad (6)$$

EXAMPLE 6.22 (Cont.)

Substituting the value for z from equation (4) and the value for y from equation (6) in equation (1) we get

$$\begin{aligned} x + 2\left(\tfrac{5}{4}x - \tfrac{13}{4}\right) - 2(3y - 4x + 7) &= 3 \\ x + \tfrac{5}{2}x - \tfrac{13}{2} - 6y + 8x - 14 &= 3 \\ \tfrac{23}{2}x - 6y &= \tfrac{47}{2} \qquad (7) \end{aligned}$$

Substituting the value of y from equation (6) into equation (7) results in

$$\begin{aligned} \tfrac{23}{2}x - 6\left(\tfrac{5}{4}x - \tfrac{13}{4}\right) &= \tfrac{47}{2} \\ \tfrac{23}{2}x - \tfrac{15}{2}x + \tfrac{39}{2} &= \tfrac{47}{2} \\ \tfrac{8}{2}x &= \tfrac{8}{2} \\ \text{or} \qquad x &= 1 \end{aligned}$$

Using this value of $x = 1$ in equation (6) produces

$$y = \tfrac{5}{4} - \tfrac{13}{4} = -\tfrac{8}{4} = -2$$

Finally, using $x = 1$ and $y = -2$ in equation (4) we get

$$z = 3(-2) - 4(1) + 7 = -3$$

So, $x = 1$, $y = -2$, and $z = -3$.

ADDITION METHOD

There was no particular reason to begin by solving equation (3) for z in the previous example. We could just as easily have started by solving equation (1) for x or equation (2) for y or even equation (1) for y or z. Let's solve the same system of equations using the addition method.

EXAMPLE 6.23

Solve this system of equations using the addition method.

$$\begin{cases} x + 2y - 2z = 3 & (1) \\ 2x - y + 3z = -5 & (2) \\ 4x - 3y + z = 7 & (3) \end{cases}$$

SOLUTION We will begin by eliminating one of the variables. When this is done, we will have two equations with two variables. Let's start by eliminating the variable z. To do this, we multiply equation (3) by 2 and add this to equation (1).

$$\begin{aligned} 8x - 6y + 2z &= 14 && \text{(3) multiplied by 2} \\ x + 2y - 2z &= 3 && \text{(1)} \\ \hline 9x - 4y \qquad &= 17 && \text{Adding to get equation (4)} \end{aligned}$$

Next, we multiply equation (3) by 3 and subtract equation (2) from this new equation.

$$\begin{aligned} 12x - 9y + 3z &= 21 && \text{(3) multiplied by 3} \\ 2x - y + 3z &= -5 && \text{(2)} \\ \hline 10x - 8y \qquad &= 26 && \text{Subtracting to get equation (5)} \end{aligned}$$

We now have two equations, (4) and (5), with two variables, x and y.

$$9x - 4y = 17 \qquad (4)$$
$$10x - 8y = 26 \qquad (5)$$

If we multiply equation (4) by -2 and add equation (5) to that equation, we will eliminate the variable y.

$$\begin{array}{rcrl} -18x + 8y &=& -34 & \quad \text{(4) multiplied by } -2 \\ 10x - 8y &=& 26 & \quad (5) \\ \hline -8x &=& -8 & \quad \text{Adding to get (6)} \end{array}$$

Solving equation (6) for x, we get $x = 1$. Substituting this value in (4), we get $9 - 4y = 17$ or $-4y = 8$, which simplifies to $y = -2$. Then, if we substitute $x = 1$ and $y = -2$ in equation (3), we get $4 + 6 + z = 7$, or $z = -3$. Again, we get the solution $x = 1$, $y = -2$, and $z = -3$.

As you can see, the elimination method by addition and subtraction is often an easier method to use. We will use this method again in the next example.

APPLICATION BUSINESS

EXAMPLE 6.24

By volume, one alloy is 70% copper, 20% zinc, and 10% nickel. A second alloy is 60% copper and 40% nickel. A third alloy is 30% copper, 30% nickel, and 40% zinc. How much of each must be mixed in order to get 1000 mm^3 of a final alloy that is 50% copper, 18% zinc, and 32% nickel?

SOLUTION We must first determine the equations that are needed to solve this problem. If we let a represent the volume of the first alloy in the final alloy, b the volume of the second, and c the volume of the third, then we know that the total volume of the final alloy, 1000 mm^3, is $a + b + c$.

We know that the final alloy contains 50% or 500 mm^3 of copper and that this is $0.7a + 0.6b + 0.3c$. Also, 18% or 180 mm^3 of the final solution is zinc, so $0.2a + 0.4c = 180$. This gives you a system of three linear equations in three variables.

$$\begin{cases} a + b + c = 1000 & (1) \\ 0.7a + 0.6b + 0.3c = 500 & (2) \\ 0.2a + 0.4c = 180 & (3) \end{cases}$$

We can also establish a fourth equation for the amount of nickel in the final solution, 32% or 320 mm^3. This is

$$0.1a + 0.4b + 0.3c = 320 \qquad (4)$$

We do not need equation (4) to solve the problem, but we can use it to check our answers.

Since equation (3) does not contain variable b, we will combine equations (1) and (2) to eliminate this variable.

EXAMPLE 6.24 (Cont.)

$$\begin{array}{rcrl}
0.7a + 0.6b + 0.3c &=& 500 & \text{(2)} \\
\underline{0.6a + 0.6b + 0.6c} &=& \underline{600} & \text{(1) multiplied by 0.6} \\
0.1a \qquad\quad - 0.3c &=& -100 & \text{Subtract to get (5).}
\end{array}$$

If we now multiply equation (5) by 2 and subtract this from equation (3), we will eliminate variable *a*.

$$\begin{array}{rcrl}
0.2a + 0.4c &=& 180 & \text{(3)} \\
\underline{0.2a - 0.6c} &=& \underline{-200} & \text{(5) multiplied by 0.2} \\
c &=& 380 & \text{Subtract to get } c.
\end{array}$$

So, alloy c is 380 mm^3. Substituting this in (5) we get

$$\begin{aligned}
0.1a - 0.3(380) &= -100 \\
0.1a - 114 &= -100 \\
0.1a &= 14 \\
a &= 140
\end{aligned}$$

Then, substituting these values for a and c in equation (1) we get $140 + b + 380 = 1\,000$ or $b = 480$. The answer: we need 140 mm^3 of alloy a, 480 mm^3 of alloy b, and 380 mm^3 of alloy c. If you put these values in equation (4) you will see that they check.

APPLICATION BUSINESS

EXAMPLE 6.25

A trucking company has three sizes of trucks, large (L), medium (M), and small (S). The trucks are needed to move some packages, which come in three different shapes. We will call these three different shaped packages A, B, and C. From experience, the company knows that these trucks can hold the combination of packages as shown in this chart.

	Size of Truck		
	Large	Medium	Small
Package A	12	8	0
Package B	10	5	4
Package C	8	7	6

The company has to deliver a total of 64 A packages, 77 B packages, and 99 C packages. How many trucks of each size are needed, if each truck is fully loaded?

SOLUTION From the table we can see that the 64 A packages must be arranged with 12 on each large truck, 8 on each medium truck, and 0 on each small truck. We can write this as

$$12L + 8M = 64$$

Similarly, the B packages satisfy $10L + 5M + 4S = 77$ and the C packages satisfy $8L + 7M + 6S = 99$. Thus, we have the system

$$\begin{cases} 12L + 8M = 64 & (1) \\ 10L + 5M + 4S = 77 & (2) \\ 8L + 7M + 6S = 99 & (3) \end{cases}$$

Since equation (1) does not contain variable S, we will combine equations (2) and (3) to eliminate it.

$$\begin{aligned} 30L + 15M + 12S &= 231 && \text{(2) multiplied by 3} \\ 16L + 14M + 12S &= 198 && \text{(3) multiplied by 2} \\ 14L + M &= 33 && \text{Subtracting to get (4)} \end{aligned}$$

Now multiply equation (4) by 8 and subtract equation (1), and variable M is eliminated.

$$\begin{aligned} 112L + 8M &= 264 && \text{(4) multiplied by 8} \\ 12L + 8M &= 64 && \text{(1)} \\ \hline 100L &= 200 && \text{Subtract.} \end{aligned}$$

So, $L = \frac{200}{100} = 2$. Substituting this in (1), we get

$$\begin{aligned} 24 + 8M &= 64 \\ 8M &= 64 - 24 \\ 8M &= 40 \\ M &= 5 \end{aligned}$$

And finally, substituting these values for L and M in (2), we obtain

$$\begin{aligned} 10(2) + 5(5) + 4S &= 77 \\ 20 + 25 + 4S &= 77 \\ 4S &= 77 - 45 \\ &= 32 \\ S &= 8 \end{aligned}$$

So, a total of 2 large, 5 medium, and 8 small trucks is needed.

As with a system of two equations in two variables, it is possible to have a system of three equations with three variables that is either inconsistent or dependent. If an elimination method results in an equation of the type $0x + 0y + 0z = c$, or $0 = c$, where $c \neq 0$, then the system is **inconsistent** and has no solutions. Graphically, this would mean that the plane of one equation was parallel to the plane of another equation or the planes intersect in three pairs of parallel lines.

If the elimination method results in an equation of the type $0x + 0y + 0z = 0$, or $0 = 0$, then two or more of the equations graph the same plane and the system is **dependent**.

EXAMPLE 6.26

Solve the system.

$$\begin{cases} 3x - y - z = 5 & (1) \\ x - 5y + z = 3 & (2) \\ x + 2y - z = 1 & (3) \end{cases}$$

EXAMPLE 6.26 (Cont.)

SOLUTION Adding equations (1) and (2) produces

$$4x - 6y = 8 \tag{4}$$

and adding equations (2) and (3) gives

$$2x - 3y = 4 \tag{5}$$

If we multiply equation (5) by 2 and subtract that result from equation (4), we obtain

$$0 = 0$$

Thus, we see that the given system of equations is dependent; and every ordered triple (a, b, c) of the form $\left(a, \frac{2}{3}a - \frac{4}{3}, \frac{7}{3}a - \frac{11}{3}\right)$ is a solution of the given equation.

EXERCISE SET 6.3

In Exercises 1–4, use the substitution method to solve each system of equations.

1. $\begin{cases} 2x + y + z = 7 \\ x - y + 2z = 11 \\ 5x + y - 2z = 1 \end{cases}$ **2.** $\begin{cases} x + y + 2z = 0 \\ 2x - y + z = 6 \\ 4x + 2y + 2z = 0 \end{cases}$ **3.** $\begin{cases} 2x - y - z = -8 \\ x + y - z = -9 \\ x - y + 2z = 7 \end{cases}$ **4.** $\begin{cases} x + y + 5z = -10 \\ x - y - 5z = 11 \\ -x + y - 5z = 13 \end{cases}$

In Exercises 5–14, use the addition method to solve each system of equations. (Exercises 5–8 are the same as Exercises 1–4.)

5. $\begin{cases} 2x + y + z = 7 \\ x - y + 2z = 11 \\ 5x + y - 2z = 1 \end{cases}$ **8.** $\begin{cases} x + y + 5z = -10 \\ x - y - 5z = 11 \\ -x + y - 5z = 13 \end{cases}$ **11.** $\begin{cases} 3x - y - 2z = 11 \\ -x + 3y + 2z = -1 \\ 2x - 2y - 4z = 17 \end{cases}$ **13.** $\begin{cases} 2x + 3y + 3z = 9 \\ 5x - 2y + 8z = 6 \\ 4x - y + 5z = -1 \end{cases}$

6. $\begin{cases} x + y + 2z = 0 \\ 2x - y + z = 6 \\ 4x + 2y + 2z = 0 \end{cases}$ **9.** $\begin{cases} x + y + z = 2 \\ 8x - 2y + 4z = -3 \\ 6x - 4y - 3z = 3 \end{cases}$ **12.** $\begin{cases} x - 2y + z = -4 \\ 2x + y + 3z = 5 \\ 6x + 3y + 12z = 6 \end{cases}$ **14.** $\begin{cases} x + 2y + 3z = 4 \\ 2x - 3y - 4z = -1 \\ 3x - 4y + 5z = 6 \end{cases}$

7. $\begin{cases} 2x - y - z = -8 \\ x + y - z = -9 \\ x - y + 2z = 7 \end{cases}$ **10.** $\begin{cases} x + y - z = 7 \\ 8x + 4y + 2z = 21 \\ 4x + 3y + 6z = 2 \end{cases}$

Solve Exercises 15–22.

15. ***Electronics*** Kirchhoff's law for current states that the sum of the currents into and out of any point equals zero. Applying this to junction A in Figure 6.20 produces the equation $I_1 - I_2 + I_3 = 0$. Kirchhoff's voltage law states that the sum of the voltages around any closed loop equals zero. Applying this first to the left loop and then the right loop in Figure 6.20 results in the equations $6I_1 + 6I_2 = 18$ and $6I_2 + I_3 = 14$. What are the values of the currents I_1, I_2, and I_3? (Note that electromotive force E equals current I times resistance, or $E = IR$.)

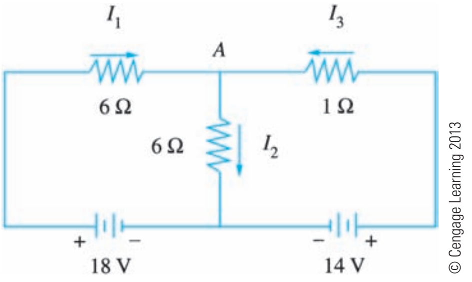

Figure 6.20

16. ***Electronics*** Applying Kirchhoff's laws to Figure 6.21 produces the following equations.

$$\begin{cases} I_1 + I_2 - I_3 = 0 \\ 3I_1 - 5I_2 - 10 = 0 \\ 5I_2 + 6I_3 - 5 = 0 \end{cases}$$

Find the currents associated with I_1, I_2, and I_3.

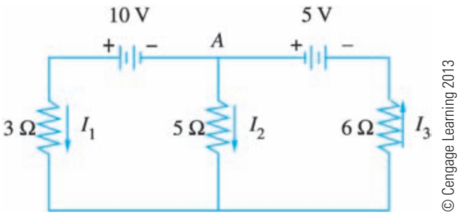

Figure 6.21

17. The standard equation for a circle is $x^2 + y^2 + ax + by + c = 0$. A circle passes through the points $P(5, 1)$, $Q(-2, -6)$, and $R(-1, -7)$. When the x- and y-coordinates for point P are put in the standard equation, it becomes $5^2 + 1^2 + a \cdot 5 + b \cdot 1 + c = 0$, or $5a + b + c + 26 = 0$. Use the coordinates of Q and R to obtain two more versions of the standard equation for this circle and then solve your system of equations for a, b, and c.

18. Another circle passes through the points $S(4, 16)$, $T(-6, -8)$, and $U(11, -1)$. Find the values of a, b, and c.

19. ***Transportation*** A trucking company has three sizes of trucks, large (L), medium (M), and small (S). Experience has shown that the large truck can carry 7 of container A, 6 of container B, and 4 of container C. The medium truck can carry 6 of A, 3 of B, and 2 of C, and the small truck can carry 8 of A, 1 of B, and 2 of C. How many trucks of the three sizes are needed to deliver 64 of container A, 33 of B, and 26 of C?

20. ***Petroleum technology*** Three crude oils are to be mixed and loaded aboard a supertanker that can carry 450 000 tonnes (metric tons, t). The crudes contain the following percentages of light-, medium-, and heavy-weight oils:

	Light	Medium	Heavy
Crude oil A	10%	20%	70%
Crude oil B	30%	40%	30%
Crude oil C	43%	44%	13%

How many tonnes of each crude should be mixed so that the new mixtures contain 24% light-, 32% medium-, and 44% heavy-weight oils?

21. ***Landscape architecture*** By weight, a 10-10-10 fertilizer contains 10% nitrogen, 10% phosphorous, and 10% potash. A 12-0-6 fertilizer contains 12% nitrogen, no phosphorous, and 6% potash. A landscaper has three types of fertilizer in stock: one is 10-12-15, a second is 10-0-5, and a third is 30-6-15. How much of each must be mixed in order to get 400 lb of fertilizer that is 16-3-9?

22. ***Business*** A company makes three types of patio furniture: chairs, tables, and recliners. Each requires the number of units of wood, plastic, and aluminum shown below:

	Wood	Plastic	Aluminum
Chair	1 unit	1 unit	2 units
Table	2 units	4 units	5 units
Recliner	1 unit	2 units	3 units

The company has in stock 500 units of wood, 900 units of plastic, and 1,300 units of aluminum. For its end-of-season production, the company wants to use all its stock. To do this, how many chairs, tables, and recliners should it make?

[IN YOUR WORDS]

23. **(a)** Which do you find easier to use: the substitution method or the addition method?

(b) Write an explanation defending your position.

24. Under what conditions is the substitution method easier than the addition method?

25. Explain what it means for a system of equations to be dependent.

6.4 DETERMINANTS AND CRAMER'S RULE

We've learned to solve two equations in two variables and to solve three equations in three variables. Many problems result in having to solve four equations in four variables or five equations in five variables.

Whenever you have to solve more than two equations in two variables, the substitution and the addition methods become very long and difficult. There are other methods that are easier, especially when used with a calculator or a computer. This section gives the background material for one of the easier methods. We begin by giving the definition of a determinant.

If a, b, c, and d are any four real numbers, then the symbol

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

is called a 2×2 **determinant** or a *determinant of the second order*. The numbers a, b, c, and d are called the **elements** or **entries** of the determinant. The value of a determinant is the number $ad - cb$, so we have the following statement.

EVALUATING A 2 × 2 DETERMINANT

If a, b, c, and d are any four real numbers, then

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - cb$$

As a memory aid, you might want to draw the diagonals of the determinant.

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - cb$$

EXAMPLE 6.27

Evaluate the determinant $\begin{vmatrix} 7 & 5 \\ 2 & 6 \end{vmatrix}$.

SOLUTION $\begin{vmatrix} 7 & 5 \\ 2 & 6 \end{vmatrix} = 7(6) - 2(5) = 42 - 10 = 32$

EXAMPLE 6.28

Evaluate the determinant $\begin{vmatrix} -6 & -2 \\ 3 & 4 \end{vmatrix}$.

SOLUTION $\begin{vmatrix} -6 & -2 \\ 3 & 4 \end{vmatrix} = (-6)(4) - 3(-2) = -24 - (-6) = -18$

Determinants have many useful properties and can be used to solve simultaneous equations, as we will see later in this section.

The determinants that we have used have all been 2×2 (read 2-by-2), or second-order determinants. A 3×3 (3-by-3) determinant or a *determinant of the third order* is represented by the symbol

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

where a_1, a_2, a_3, b_1, b_2, b_3, c_1, c_2, and c_3 are any real numbers. We will learn how to evaluate a 3×3 determinant later in this section.

As we mentioned, the numbers $a_1, a_2, \ldots, c_3$, which form the determinant, are called the elements or entries. The **rows** are numbered from top to bottom.

a	b	Row 1	a_1	b_1	c_1
c	d	Row 2	a_2	b_2	c_2
		Row 3	a_3	b_3	c_3

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The *columns* are numbered from left to right.

a	b	
c	d	
Column 1	Column 2	Column 3
a_1	b_1	c_1
a_2	b_2	c_2
a_3	b_3	c_3

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A determinant has the same number of rows and columns.

MINORS

Each element in a determinant has a minor associated with it. The **minor** of a given element is the determinant that is formed by deleting the row and column in which the element lies.

EXAMPLE 6.29

Consider the determinant $\begin{vmatrix} 2 & 4 & -4 \\ 5 & 8 & 1 \\ -6 & -3 & 7 \end{vmatrix}$.

(a) The minor of the first element, 2, is found by crossing out the first row and first column.

$$\begin{vmatrix} \cancel{2} & \cancel{4} & \cancel{-4} \\ \cancel{5} & 8 & 1 \\ \cancel{-6} & -3 & 7 \end{vmatrix}$$

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The minor of the element 2 is $\begin{vmatrix} 8 & 1 \\ -3 & 7 \end{vmatrix} = 56 + 3 = 59$.

EXAMPLE 6.29 (Cont.)

(b) The minor of the element 8 is found by crossing out the second row and the second column.

$$\begin{vmatrix} 2 & 4 & -4 \\ 5 & 8 & 1 \\ -6 & -3 & 7 \end{vmatrix}$$

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The minor of 8 is $\begin{vmatrix} 2 & -4 \\ -6 & 7 \end{vmatrix} = 14 - 24 = -10.$

(c) The minor of 1 is found by crossing out the second row and third column.

$$\begin{vmatrix} 2 & 4 & -4 \\ 5 & 8 & 1 \\ -6 & -3 & 7 \end{vmatrix}$$

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The minor of 1 is $\begin{vmatrix} 2 & 4 \\ -6 & -3 \end{vmatrix} = -6 + 24 = 18.$

COFACTORS

Each element also has a **cofactor**. The value of the cofactor is determined by first adding the number of the row and the number of the column where the element is located. If this sum is even, the value of the cofactor is equal to the value of the minor for that element. If the sum is odd, the value of the cofactor is then -1 times the value of the minor for that element.

EXAMPLE 6.30

We will consider the same determinant we used in Example 6.29.

(a) The cofactor of 2 is 59. The element 2 is in row 1 and column 1. Since $1 + 1$ is an even number and the minor for 2 had a value of 59 [see Example 6.29(a)] its cofactor has a value of 59.

(b) The element 8 is in row 2, column 2. Since $2 + 2 = 4$, an even number, the cofactor of 8 is -10, the same as its minor.

(c) The element 1 is in row 2, column 3 and $2 + 3 = 5$, an odd number. The value of the minor of 1 is 18, so the cofactor of 1 is $(-1)18 = -18$.

The following box describes how to evaluate any determinant by using a cofactor.

EVALUATING A DETERMINANT

To evaluate any determinant

(1) select any row or column of the determinant,

(2) multiply each element of that row or column by its cofactor, and

(3) add the results.

Notice that this procedure allows you to choose which row or column you want to use to evaluate the determinant.

EXAMPLE 6.31

Evaluate $\begin{vmatrix} 2 & 4 & -4 \\ 5 & 8 & 1 \\ -6 & -3 & 7 \end{vmatrix}$.

SOLUTION We will evaluate this determinant by using, or expanding, on the second row. (We picked this row because we found the cofactor of the last two elements in Example 6.30.) The cofactor of 5 is $(-1)\begin{vmatrix} 4 & -4 \\ -3 & 7 \end{vmatrix} = (-1)$ $(28 - 12) = -16$. The cofactor of 8 is -10 and the cofactor of 1 is -18. So, the value of this determinant is

$$\begin{aligned} \begin{vmatrix} 2 & 4 & -4 \\ 5 & 8 & 1 \\ -6 & -3 & 7 \end{vmatrix} &= 5(-16) + 8(-10) + 1(-18) \\ &= -80 + -80 + -18 \\ &= -178 \end{aligned}$$

EXAMPLE 6.32

Evaluate $\begin{vmatrix} 2 & 1 & -5 & -2 \\ 1 & 0 & 4 & 5 \\ 7 & 2 & 1 & 0 \\ 5 & 0 & -3 & 2 \end{vmatrix}$.

SOLUTION Since we can select any row or column, let's select the one that has the most zeros. The second column of this determinant has two zeros, so we shall use column 2 to evaluate it.

$$\begin{aligned} \begin{vmatrix} 2 & 1 & -5 & -2 \\ 1 & 0 & 4 & 5 \\ 7 & 2 & 1 & 0 \\ 5 & 0 & -3 & 2 \end{vmatrix} &= -1\begin{vmatrix} 1 & 4 & 5 \\ 7 & 1 & 0 \\ 5 & -3 & 2 \end{vmatrix} + 0\begin{vmatrix} 2 & -5 & -2 \\ 7 & 1 & 0 \\ 5 & -3 & 2 \end{vmatrix} \\ &\quad -2\begin{vmatrix} 2 & -5 & -2 \\ 1 & 4 & 5 \\ 5 & -3 & 2 \end{vmatrix} + 0\begin{vmatrix} 2 & -5 & -2 \\ 1 & 4 & 5 \\ 7 & 1 & 0 \end{vmatrix} \\ &= -1\begin{vmatrix} 1 & 4 & 5 \\ 7 & 1 & 0 \\ 5 & -3 & 2 \end{vmatrix} - 2\begin{vmatrix} 2 & -5 & -2 \\ 1 & 4 & 5 \\ 5 & -3 & 2 \end{vmatrix} \end{aligned}$$

Now we need to evaluate each of these 3×3 determinants. We will evaluate the first determinant along column three.

EXAMPLE 6.32 (Cont.)

$$\begin{vmatrix} 1 & 4 & 5 \\ 7 & 1 & 0 \\ 5 & -3 & 2 \end{vmatrix} = 5\begin{vmatrix} 7 & 1 \\ 5 & -3 \end{vmatrix} - 0\begin{vmatrix} 1 & 4 \\ 5 & -3 \end{vmatrix} + 2\begin{vmatrix} 1 & 4 \\ 7 & 1 \end{vmatrix}$$

$$\begin{aligned} &= 5[7(-3) - 5(1)] - 0 + 2[1(1) - 7(4)] \\ &= 5[-21 - 5] - 0 + 2[1 - 28] \\ &= 5(-26) + 2(-27) \\ &= -130 - 54 = -184 \end{aligned}$$

The other 3 × 3 determinant is evaluated along row one:

$$\begin{vmatrix} 2 & -5 & -2 \\ 1 & 4 & 5 \\ 5 & -3 & 2 \end{vmatrix} = 2\begin{vmatrix} 4 & 5 \\ -3 & 2 \end{vmatrix} - (-5)\begin{vmatrix} 1 & 5 \\ 5 & 2 \end{vmatrix} + (-2)\begin{vmatrix} 1 & 4 \\ 5 & -3 \end{vmatrix}$$

$$\begin{aligned} &= 2[4(2) - (-3)5] + 5[1(2) - 5(5)] - 2[1(-3) - 5(4)] \\ &= 2(8 + 15) + 5(2 - 25) - 2(-3 - 20) \\ &= 2(23) + 5(-23) - 2(-23) \\ &= 46 - 115 + 46 \\ &= -23 \end{aligned}$$

So, the value of the original determinant is

$$-1\begin{vmatrix} 1 & 4 & 5 \\ 7 & 1 & 0 \\ 5 & -3 & 2 \end{vmatrix} - 2\begin{vmatrix} 2 & -5 & -2 \\ 1 & 4 & 5 \\ 5 & -3 & 2 \end{vmatrix} = -1(-184) - 2(-23)$$

$$\begin{aligned} &= 184 + 46 \\ &= 230 \end{aligned}$$

Needless to say, the last example was quite long. We will next show how to evaluate this same determinant using a calculator and a spreadsheet.

EVALUATING DETERMINANTS ON A CALCULATOR

Many of today's scientific calculators allow you to evaluate a determinant quickly (and accurately). The following procedure describes how this is done on a Texas Instruments TI-83 or 84 graphics calculator.

To evaluate a determinant you will need to use the matrix features of this calculator. We will learn more about a matrix in Chapter 18.

EXAMPLE 6.33

Use a graphing calculator to evaluate the determinant in Example 6.32:

$$\begin{vmatrix} 2 & 1 & -5 & -2 \\ 1 & 0 & 4 & 5 \\ 7 & 2 & 1 & 0 \\ 5 & 0 & -3 & 2 \end{vmatrix}$$

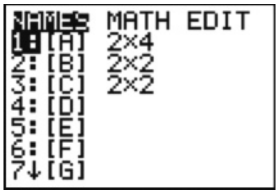

Figure 6.22a

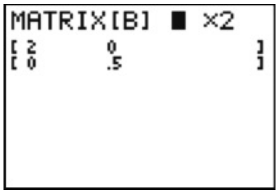

Figure 6.22b

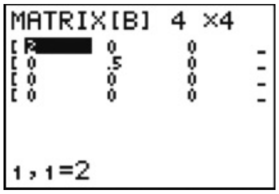

Figure 6.22c

Figure 6.22d

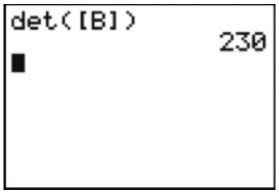

Figure 6.22e

SOLUTION The matrix operations on the TI-83 or TI-84 are accessed by pressing the MATRIX key over the x^{-1} key. So, begin by pressing 2nd x^{-1} [MATRIX]. You should see a display like the one shown in Figure 6.22a. Across the top of the screen are the titles of three menus. The NAMES menu is highlighted. This menu shows the names and the sizes of the ten matrices that can be stored in a TI-83/84 calculator. For example, the first matrix is named "matrix [A]." Its size is 2 × 4, which means that it has 2 rows and 4 columns. (What you actually see depends on whether your calculator has previously been used to enter a matrix.)

To the right of the NAMES menu is the MATH menu. It deals with several operations on matrices. We will use this menu later. The other menu, EDIT, allows us to define or modify a matrix. To evaluate a determinant, we enter it in the calculator as a matrix. To do this, you first select the EDIT menu by pressing the ► key twice or by pressing the ◄ key once.

We are now ready to define the determinant we want to evaluate. The TI-83/84 calculator can store ten matrices, which it names [A] through [J]. Indicate which matrix you want to define. Press the number shown at the left of each of these matrix names. We will name our matrix [B]. Pressing a 2 results in the screen display shown in Figure 6.22b.

The blinking cursor, shown by a black rectangle in Figure 6.22b, is on the row dimension of the determinant. We must either accept or change the dimension on the top line. To accept the number, press ENTER. To change the number, enter the numbers of rows in your determinant, and then press ENTER. We want to evaluate a 4 × 4 determinant, so press 4 ENTER 4 ENTER. You should now see the display in Figure 6.22c.

We are now ready to enter the elements of our determinant. Start with the element in the upper left-hand corner, 2, and press 2 ENTER. Next enter the second element in the top row, 1, by pressing 1 ENTER; the third element in the top row, −5, by pressing (–) 5 ENTER; and so on until all 16 elements have been entered. When you complete a row, the calculator will go to the left-most element in the next row.

When you have finished entering the elements, return to the HOME screen by pressing 2nd QUIT.

Now, you are ready to evaluate this determinant. To evaluate a determinant, you want the MATH menu for matrices. To get this, press MATRX ►. The result is shown in Figure 6.22d. There are many matrix operations listed. The one we want, determinant (or "det"), is listed first, so press 1.

If you named your matrix [B], now press MATRX 2) ENTER. The result, shown in Figure 6.22e, shows that the value of the determinant of [B] is 230.

EVALUATING DETERMINANTS WITH A SPREADSHEET

Evaluating a determinant with a spreadsheet uses a built-in function of the spreadsheet program. Example 6.34 explains the procedure.

EXAMPLE 6.34

Use a spreadsheet to evaluate the determinant is Example 6.32:

$$\begin{vmatrix} 2 & 1 & -5 & -2 \\ 1 & 0 & 4 & 5 \\ 7 & 2 & 1 & 0 \\ 5 & 0 & -3 & 2 \end{vmatrix}$$

	A	B	C	D
1	2	1	-5	-2
2	1	0	4	5
3	7	2	1	0
4	5	0	-3	2

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Figure 6.23a

SOLUTION First, enter the determinant in rows and columns as shown in Figure 6.23a.

Next, move to another cell and select the Function Builder option in the Tool Box. The symbol for the function builder is a box labeled f_x as shown in Figure 6.23b.

In the box below "Search for a function:" type Matrix and click **GO** and you should get a list of functions dealing with a matirx (see Figure 6.23c.). MDETERM is the Excel function for the matrix determinant of an array.

Click MDETERM and in the white box to the right of "Array:" type in A1:D4, the array to be evaluated, as shown in Figure 6.23d. (If you don't want to type the cell names you can drag your cursor over the 16 cells of the matrix. The value of the determinant is shown at the bottom left where it says "Function Result =". If you want it in the spreadsheet, press **OK** on the computer).

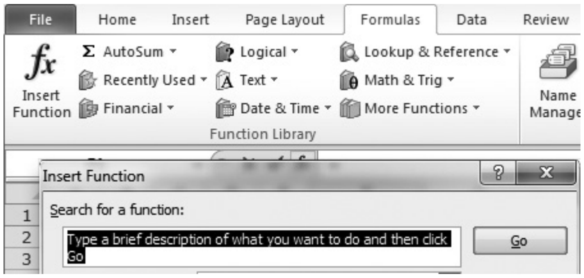

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Figure 6.23b

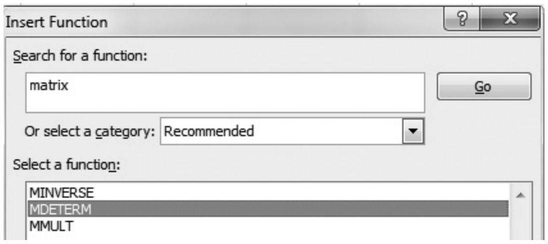

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Figure 6.23c

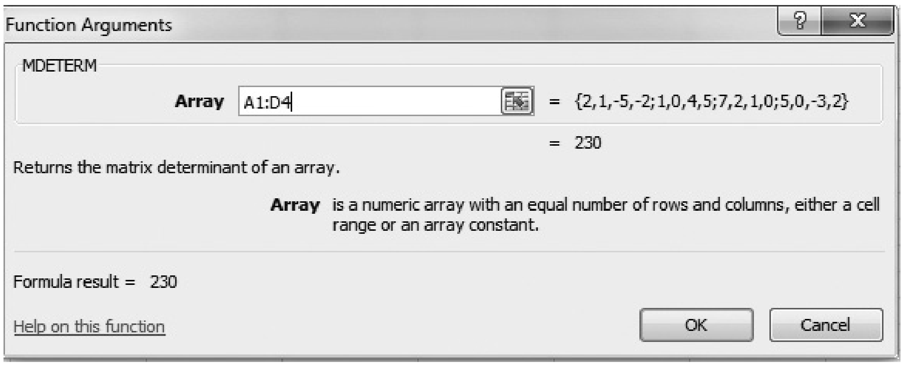

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Figure 6.23d

USING CRAMER'S RULE TO SOLVE SYSTEMS OF LINEAR EQUATIONS

Now we will learn how to use determinants to solve systems of linear equations. We will first work with second-order determinants.

Consider the system of linear equations

$$\begin{cases} ax + by = h \\ cx + dy = k \end{cases}$$

If we multiply the first equation by d and the second equation by b, we get

$$\begin{cases} adx + bdy = hd \\ bcx + bdy = bk \end{cases}$$

Subtracting the second equation from the first eliminates the y terms and produces the equation

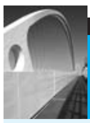
$$adx - bcx = hd - bk$$
$$\text{or} \quad (ad - bc)x = hd - bk$$

Now, this last equation could have been written using determinants as $\begin{vmatrix} a & b \\ c & d \end{vmatrix} x = \begin{vmatrix} h & b \\ k & d \end{vmatrix}$, and so

$$x = \frac{\begin{vmatrix} h & b \\ k & d \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}} \text{ provided that } \begin{vmatrix} a & b \\ c & d \end{vmatrix} \neq 0$$

Similarly, we can show that $y = \dfrac{\begin{vmatrix} a & h \\ c & k \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}}$.

These last two equations are cumbersome, so we'll abbreviate the determinants by letting $D = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$, $D_x = \begin{vmatrix} h & b \\ k & d \end{vmatrix}$, and $D_y = \begin{vmatrix} a & h \\ c & k \end{vmatrix}$. Then we have $x = \dfrac{D_x}{D}$ and $y = \dfrac{D_y}{D}$. This is **Cramer's rule** for solving two linear equations in two variables.



CRAMER'S RULE FOR SOLVING A 2 × 2 LINEAR SYSTEM

The unique solution to the linear system

$$\begin{cases} a_1x + b_1y = k_1 \\ a_2x + b_2y = k_2 \end{cases}$$

(Continues)

(Continued)

is

$$x = \frac{D_x}{D} = \frac{\begin{vmatrix} k_1 & b_1 \\ k_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} \text{ and } y = \frac{D_y}{D} = \frac{\begin{vmatrix} a_1 & k_1 \\ a_2 & k_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}$$

provided that

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \neq 0$$

The determinant D is called the **coefficient determinant**, because its entries are the coefficients of the variables in the system of linear equations:

$$\begin{cases} a_1x + b_1y = k_1 \\ a_2x + b_2y = k_1 \end{cases} \qquad D = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$$

The determinant D_x is obtained by replacing the first column of D (the coefficients of x) with the constants on the right in the system of equations. Similarly, the determinant D_y is obtained by replacing the second column of D (the coefficients of y) with the constants on the right in the system.

Shading the constants in the system, we see that D_x and D_y are obtained as follows:

$$\begin{cases} a_1x + b_1y = k_1 \\ a_2x + b_2y = k_2 \end{cases} \qquad D_x = \begin{vmatrix} k_1 & b_1 \\ k_2 & b_2 \end{vmatrix} \qquad D_y = \begin{vmatrix} a_1 & k_1 \\ a_2 & k_2 \end{vmatrix}$$

NOTE You cannot use Cramer's rule when $D = 0$. When $D = 0$, the equations in the system are either inconsistent or dependent.

EXAMPLE 6.35

Use Cramer's rule to solve $\begin{cases} 2x - y = 9 \\ x + 5y = 21 \end{cases}$.

SOLUTION Here we have $D = \begin{vmatrix} 2 & -1 \\ 1 & 5 \end{vmatrix} = 11$. This leads to the following two determinants. (The constants have been shaded to help you see the substitutions.)

$$D_x = \begin{vmatrix} 9 & -1 \\ 21 & 5 \end{vmatrix} = 66 \text{ and } D_y = \begin{vmatrix} 2 & 9 \\ 1 & 21 \end{vmatrix} = 33.$$

Using Cramer's rule, we determine that $x = \frac{D_x}{D} = \frac{66}{11} = 6$ and $y = \frac{D_y}{D} = \frac{33}{11} = 3$.

EXAMPLE 6.36

Use Cramer's rule to solve $\begin{cases} 3x+y=10 \\ 6x+2y=15 \end{cases}$.

SOLUTION In this system, $D = \begin{vmatrix} 3 & 1 \\ 6 & 2 \end{vmatrix} = 0$, so Cramer's rule does not apply. Graphing these two lines would show that they are parallel, and so the system is inconsistent.

APPLICATION CONSTRUCTION

EXAMPLE 6.37

A contractor mixes some aggregate with cement to make concrete. Two mixtures are on hand. One of the mixtures, which we will call Mixture A, is 40% sand and 60% aggregate. The other mixture, Mixture B, is 70% sand and 30% aggregate. How much of each should be used to get a 500-lb mixture that is 46.2% sand and 53.8% aggregate?

SOLUTION Let a represent the amount of Mixture A and b the amount of Mixture B used in the final mixture. The final 500-lb mixture will contain 46.2%, or 231 lb, of sand and 53.8%, or 269 lb, of aggregate. Thus, we get two equations. The first

$$0.4a + 0.7b = 231 \tag{1}$$

represents the amounts of sand from each of Mixtures A and B that are needed to make the 231 lb in the final mixture.

The second equation

$$0.6a + 0.3b = 269 \tag{2}$$

represents the amount of aggregate from Mixtures A and B needed to make the aggregate in the final mixture.

Thus, we have the system

$$\begin{cases} 0.4a+0.7b=231 \\ 0.6a+0.3b=269 \end{cases}$$

Here,

$D = \begin{vmatrix} 0.4 & 0.7 \\ 0.6 & 0.3 \end{vmatrix} = -0.3, D_a = \begin{vmatrix} 231 & 0.7 \\ 269 & 0.3 \end{vmatrix} = -119$, and $D_b = \begin{vmatrix} 0.4 & 231 \\ 0.6 & 269 \end{vmatrix}$ $= -31$. As a result, we have $a = \dfrac{D_a}{D} = \dfrac{-119}{-0.3} \approx 396.7$ and $b = \dfrac{D_b}{D} = \dfrac{-31}{-0.3} \approx$ 103.3. The final mixture should contain about 396.7 lb of Mixture A and 103.3 lb of Mixture B.

USING CRAMER'S RULE ON A SYSTEM OF THREE LINEAR EQUATIONS

To extend Cramer's rule to a system of three linear equations in three variables, you use a similar procedure, as shown in the following box.

CRAMER'S RULE FOR SOLVING A 3 × 3 LINEAR SYSTEM

The unique solution to the linear system

$$\begin{cases} a_1x + b_1y + c_1z = k_1 \\ a_2x + b_2y + c_2z = k_2 \\ a_3x + b_3y + c_3z = k_3 \end{cases}$$

is

$$x = \frac{D_x}{D} \quad y = \frac{D_y}{D} \quad \text{and} \quad z = \frac{D_z}{D}$$

where

$$D_x = \begin{vmatrix} k_1 & b_1 & c_1 \\ k_2 & b_2 & c_2 \\ k_3 & b_3 & c_3 \end{vmatrix}, \ D_y = \begin{vmatrix} a_1 & k_1 & c_1 \\ a_2 & k_2 & c_2 \\ a_3 & k_3 & c_3 \end{vmatrix}, \ D_z = \begin{vmatrix} a_1 & b_1 & k_1 \\ a_2 & b_2 & k_2 \\ a_3 & b_3 & k_3 \end{vmatrix}$$

and provided that

$$D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \neq 0$$

If $D = 0$, the equations are either inconsistent or dependent.

NOTE

- If the determinant of the denominator, D, is not zero, there is a unique solution to the system of equations.
- If all determinants are zero, the system is dependent, that is, there is an infinite number of solutions.
- If the determinant of the denominator, D, is zero, and any of the determinants in the numerator is not zero, the system is inconsistent; that is, there is no solution.

EXAMPLE 6.38

Use Cramer's rule to solve the system of linear equations:

$$\begin{cases} 3x - 2y - z = 2 \\ -x - 3y + z = 10 \\ 2x + 3y - 2z = -14 \end{cases}$$

SOLUTION For this example, we have

$$D = \begin{vmatrix} 3 & -2 & -1 \\ -1 & -3 & 1 \\ 2 & 3 & -2 \end{vmatrix} = 6$$

Since $D \neq 0$, the system is consistent, and we can solve it by Cramer's rule. Solving for D_x, D_y, and D_z, we obtain

$$D_x = \begin{vmatrix} 2 & -2 & -1 \\ 10 & -3 & 1 \\ -14 & 3 & -2 \end{vmatrix} = 6$$

$$D_y = \begin{vmatrix} 3 & 2 & -1 \\ -1 & 10 & 1 \\ 2 & -14 & 2 \end{vmatrix} = -12$$

$$\text{and } D_z = \begin{vmatrix} 3 & -2 & 2 \\ -1 & -3 & 10 \\ 2 & 3 & -14 \end{vmatrix} = 30$$

So, $x = \dfrac{D_x}{D} = \dfrac{6}{6} = 1$, $y = \dfrac{D_y}{D} = \dfrac{-12}{6} = -2$, and $z = \dfrac{D_z}{D} = \dfrac{30}{6} = 5$.

APPLICATION ELECTRONICS

EXAMPLE 6.39

An applied electromotive force (emf) of 200 V produces a current of 40 mA in the simple series circuit shown in Figure 6.24. The voltage drop across resistor R_1 is 60 V more than the combined voltage drop across R_2 and R_3. The voltage drop across R_3 is one-fourth the drop across R_2. What are the values of the three resistors in ohms?

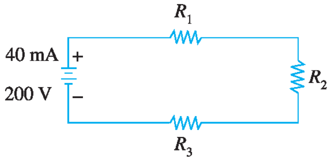


Figure 6.24

SOLUTION Since the voltage drop across resistor R_1 is 60 V more than the combined voltage drop across R_2 and R_3, we have

$$V_1 = 60 + V_2 + V_3 \tag{1}$$

We also are given

$$V_3 = \frac{1}{4}V_2 \tag{2}$$

The current of 40 mA $= 40 \times 10^{-3}$ A $= 0.040$ A. We can use Ohm's law to find the total external resistance, R_T, in the circuit.

$$\begin{aligned} E &= IR_T \\ 200 &= 0.040R_T \\ R_T &= \frac{200}{0.040} \\ &= 5\,000 \end{aligned}$$

Thus, from Kirchhoff's laws,

$$R_1 + R_2 + R_3 = 5\,000 \tag{3}$$

EXAMPLE 6.39 (Cont.)

If we use Ohm's law, we can use the following equations to express the voltage drop across each resistor:

$$V_1 = 0.040R_1$$
$$V_2 = 0.040R_2$$
$$V_3 = 0.040R_3$$

Substituting these values of V_1, V_2, and V_3 in equations (1) and (2), we obtain

$$0.04R_1 = 60 + 0.04R_2 + 0.04R_3 \tag{4}$$

$$0.04R_3 = \frac{1}{4}0.04R_2$$

$$\text{or} \quad 4R_3 = R_2 \tag{5}$$

We are now ready to use Cramer's rule to solve the system of equations:

$$\begin{cases} R_1 + R_2 + R_3 = 5{,}000 \\ 0.04\,R_1 = 60 + 0.04\,R_2 + 0.04\,R_3 \\ 4\,R_3 = R_2 \end{cases}$$

$$\text{or} \quad \begin{cases} R_1 + R_2 + R_3 = 5{,}000 \\ 0.04\,R_1 - 0.04\,R_2 - 0.04\,R_3 = 60 \\ R_2 - 4\,R_3 = 0 \end{cases}$$

Thus, we have

$$D = \begin{vmatrix} 1 & 1 & 1 \\ 0.04 & -0.04 & -0.04 \\ 0 & 1 & -4 \end{vmatrix} = 0.4$$

$$D_{R_1} = \begin{vmatrix} 5000 & 1 & 1 \\ 60 & -0.04 & -0.04 \\ 0 & 1 & -4 \end{vmatrix} = 1\,300$$

$$D_{R_2} = \begin{vmatrix} 1 & 5\,000 & 1 \\ 0.04 & 60 & -0.04 \\ 0 & 0 & -4 \end{vmatrix} = 560$$

$$\text{and} \quad D_{R_3} = \begin{vmatrix} 1 & 1 & 5\,000 \\ 0.04 & -0.04 & 60 \\ 0 & 1 & 0 \end{vmatrix} = 140$$

As a result, we obtain

$$R_1 = \frac{D_{R_1}}{D} = \frac{1\,300}{0.4} = 3\,250$$

$$R_2 = \frac{D_{R_2}}{D} = \frac{560}{0.4} = 1\,400$$

$$\text{and} \quad R_3 = \frac{D_{R_3}}{D} = \frac{140}{0.4} = 350$$

Thus, the three resistors have values of 3 250 Ω, 1 400 Ω, and 350 Ω.

EXERCISE SET 6.4

In Exercises 1–8, evaluate each determinant.

1. $\begin{vmatrix} 2 & 3 \\ 4 & -1 \end{vmatrix}$

2. $\begin{vmatrix} 4 & -5 \\ 6 & 2 \end{vmatrix}$

3. $\begin{vmatrix} 5 & -1 \\ 8 & 1 \end{vmatrix}$

4. $\begin{vmatrix} 9 & -1 \\ -2 & 3 \end{vmatrix}$

5. $\begin{vmatrix} 4 & 7 \\ -3 & 1 \end{vmatrix}$

6. $\begin{vmatrix} -1 & -4 \\ 0 & 1 \end{vmatrix}$

7. $\begin{vmatrix} -9 & \frac{1}{2} \\ 2 & 1 \end{vmatrix}$

8. $\begin{vmatrix} 6 & -3 \\ \frac{1}{3} & -\frac{2}{3} \end{vmatrix}$

In each of Exercises 9–14, use the determinant $\begin{vmatrix} 4 & 2 & -1 \\ 3 & 7 & -4 \\ -2 & 1 & 1 \end{vmatrix}$ *and for the indicated position find (a) the element in that position, (b) the minor of that element, and (c) the cofactor of that element.*

9. Row 1, Column 2

10. Row 2, Column 1

11. Row 3, Column 2

12. Row 2, Column 3

13. Row 3, Column 1

14. Row 1, Column 3

Evaluate each of the determinants in Exercises 15–32. You will probably want to use a calculator or a spreadsheet.

15. $\begin{vmatrix} 2 & -5 & 8 \\ 16 & 4 & 3 \\ 2 & -5 & 8 \end{vmatrix}$

16. $\begin{vmatrix} 1 & -2 & 3 \\ 2 & -4 & 6 \\ 0 & 3 & 5 \end{vmatrix}$

17. $\begin{vmatrix} 1 & 2 & 5 \\ -1 & -2 & 3 \\ 3 & 6 & 15 \end{vmatrix}$

18. $\begin{vmatrix} 2 & 4 & 3 \\ 0 & 1 & 19 \\ 0 & 0 & -3 \end{vmatrix}$

19. $\begin{vmatrix} 5 & 2 & 3 \\ 4 & 5 & -6 \\ 2 & 5 & -9 \end{vmatrix}$

20. $\begin{vmatrix} 9 & 3 & 3 \\ -2 & 0 & 6 \\ 6 & 3 & 3 \end{vmatrix}$

21. $\begin{vmatrix} 4 & 3 & 9 \\ -4 & -6 & 16 \\ 2 & 3 & 2 \end{vmatrix}$

22. $\begin{vmatrix} 9 & 18 & 1 \\ 0 & -7 & 5 \\ 0 & 0 & 2 \end{vmatrix}$

23. $\begin{vmatrix} 2.41 & -3.5 & -5.3 \\ 6.02 & -7.01 & -4.26 \\ 9.1 & -3.2 & -4.5 \end{vmatrix}$

24. $\begin{vmatrix} \sqrt{3} & \frac{1}{2} & -5 \\ \frac{1}{4} & \sqrt{2} & 4 \\ \frac{2}{5} & 7 & -1 \end{vmatrix}$

25. $\begin{vmatrix} 0.071 & -0.069 & -1.095 \\ 0.202 & 0.420 & -0.100 \\ 0.066 & -0.303 & -2.093 \end{vmatrix}$

26. $\begin{vmatrix} \sqrt{7} & -3 & \sqrt{5} \\ -1.4 & \sqrt{6} & 2 \\ \sqrt{8} & 3 & -6 \end{vmatrix}$

27. $\begin{vmatrix} 1 & 2 & 3 & 4 \\ 5 & 0 & 7 & 0 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{vmatrix}$

28. $\begin{vmatrix} 2 & 0 & 3 & 1 \\ 4 & 0 & 2 & 5 \\ 9 & -5 & 0 & -2 \\ 0 & 3 & 1 & -6 \end{vmatrix}$

29. $\begin{vmatrix} 3 & 2 & 4 & \sqrt{6} \\ 9 & 8 & \frac{1}{3} & -1 \\ 0 & 2.4 & 0 & -3.1 \\ 6 & -7 & \pi & 9 \end{vmatrix}$

30. $\begin{vmatrix} \sqrt{3} & \sqrt{2} & \frac{1}{4} & 1 \\ 2 & 0 & -7 & -\sqrt{3} \\ \pi & 12 & -\sqrt{2} & 0 \\ 1 & 11 & 3 & 9 \end{vmatrix}$

31. $\begin{vmatrix} 2.1 & 5.7 & 3.4 & 6 \\ 0.1 & 0.3 & 7.1 & 3.5 \\ 1.05 & 2.85 & 1.7 & 3 \\ 18 & \frac{1}{3} & 6.73 & 9 \end{vmatrix}$

32. $\begin{vmatrix} -0.7 & 0.1 & 0.1 & 0.3 \\ 1.2 & 5.6 & \frac{1}{3} & \frac{3}{17} \\ 0.01 & 0.17 & 7.4 & 3.1 \\ 9.4 & 35 & 0 & 0.9 \end{vmatrix}$

In Exercises 33–50, use Cramer's rule, when applicable, to solve each system of linear equations.

33. $\begin{cases} 2x + y = 7 \\ 3x - 2y = -7 \end{cases}$

34. $\begin{cases} 3x + y = 3 \\ 2x - 3y = 13 \end{cases}$

35. $\begin{cases} 4x + 3y = 4 \\ 3x - 2y = -14 \end{cases}$

36. $\begin{cases} \frac{1}{2}x - \frac{2}{3}y = \frac{3}{4} \\ \frac{1}{3}x + 2y = \frac{5}{6} \end{cases}$

37. $\begin{cases} 1.2x + 3.7y = 9.1 \\ 4.3x - 5.2y = 8.3 \end{cases}$

38. $\begin{cases} 0.02x - 1.22y = 3.74 \\ -0.14x + 0.32y = -1.32 \end{cases}$

39. $\begin{cases} 2.5x + 3.8y = 9.3 \\ 0.5x + 0.76y = -2.44 \end{cases}$

40. $\begin{cases} 4.3x - 2.7y = -4 \\ 3.5x + 4.2y = -1 \end{cases}$

41. $\begin{cases} -5.3x + 2.1y = 4.6 \\ 6.2x - 3.1y = 6 \end{cases}$

42. $\begin{cases} 1.3x - 0.8y = 2.9 \\ 1.7x - 0.7y = 0.4 \end{cases}$

43. $\begin{cases} 6x + 2.5y = 8.2 \\ 13.8x + 5.75y = 18.86 \end{cases}$

44. $\begin{cases} 3.5x - 6.5y = 22.45 \\ 5.5x + 3.3y = -1.21 \end{cases}$

45. $\begin{cases} 4x + 3y = 2 \\ 3x - 2y = -24 \end{cases}$

46. $\begin{cases} x + 3y - z = 7 \\ 5x - 7y + z = 3 \\ 2x - y - 2z = 0 \end{cases}$

47. $\begin{cases} x - 0.5y + 1.5z = 2 \\ x + 0.5y - 3.75z = 0.25 \\ -3x - 2.5y + 4z = -0.75 \end{cases}$

48. $\begin{cases} x + y - z = -4.7 \\ 0.5x - 3.5y + 2.4z = 17.4 \\ 1.5x + 4.5y - 3.6z = -21.6 \end{cases}$

49. $\begin{cases} 2x + y + z + w = 1.2 \\ 7x + 3y + 5z + 3w = -7.2 \\ -31x - 7y + 9z - 7w = -42.3 \\ 0.4x + 2y + 4z + 5w = 30 \end{cases}$

50. $\begin{cases} 2a + 3b - c + d = -5 \\ 4a + 5b + 2c - d = 4 \\ -2a - b - c - d = 1 \\ 6a + 7b + c - 4d = 2 \end{cases}$

Solve Exercises 51–57.

51. ***Electronics*** Applying Kirchhoff's laws to a certain circuit results in the following system of equations.

$$\begin{cases} I_1 + I_2 + I_3 = 0 \\ 8I_1 - 10I_3 = 8 \\ 6I_1 - 3I_2 = 12 \end{cases}$$

Solve for I_1, I_2, and I_3.

52. ***Physics*** The displacement s of an object moving in a straight line under constant acceleration from an initial position s_0 is given by the formula

$$s = s_0 + v_0 t + \frac{1}{2}at^2$$

where t is elapsed time in seconds, v_0 is the initial velocity, and a the acceleration. The results of three measurements are shown in the table. Find the constants s_0, v_0, and a.

Time, t (s)	2	5	7
Displacement, s (m)	212	156.5	70.5

53. ***Metallurgy*** An alloy is composed of three metals A, B, and C. The percentage of each metal is given by the following system of equations.

$$\begin{cases} A + B + C = 100 \\ A - 2B = 0 \\ -4A + C = 0 \end{cases}$$

Determine the percentage of each metal in the alloy.

54. ***Automotive technology*** A petroleum engineer was testing three different gasoline mixtures A, B, and C in the same car and under the same driving conditions. She noticed that the car traveled 90 km further when it used mixture B than when it used mixture A. Using the fuel C, the car traveled 130 km more than when it used fuel B. The total distance traveled was 1900 km. Find the distance traveled on the three fuels.

55. ***Construction technology*** If three cables are joined at a point and three forces are applied so the system is in equilibrium, the following system of equations results.

$$\begin{cases} \frac{6}{7}F_B - \frac{2}{3}F_C = 2{,}000 \\ -F_A - \frac{3}{7}F_B + \frac{1}{3}F_C = 0 \\ \frac{2}{7}F_B - \frac{2}{3}F_C = 1{,}200 \end{cases}$$

Determine the three forces, F_A, F_B, and F_C, measured in newtons (N).

56. ***Environmental science*** To control ice and protect the environment, a certain city determines that the best mixture to be spread on roads consists of 5 units of salt, 6 units of sand, and 4 units of a chemical inhibiting agent. Three companies, A, B, and C, sell mixtures of these elements according to the following table:

	Salt	Sand	Inhibiting Agent
Company A	2	1	1
Company B	2	2	2
Company C	1	5	1

(a) In what proportion should the city purchase from each company in order to spread the best mixture? (Assume that the city must buy complete truckloads.)

(b) If the city expects to need 3,297,283 units for the winter, how many units should be bought from each company?

57. ***Automotive technology*** The relationship between the velocity, v, of a car (in mph) and the distance, d (in ft), required to bring it to a complete stop is known to be of the form $d = av^2 + bv + c$, where a, b, and c are constants. Use the following data to determine the values of a, b, and c. When $v = 20$, then $d = 40$; when $v = 55$, then $d = 206.25$; and when $v = 65$, then $d = 276.25$.

[IN YOUR WORDS]

58. Explain how to use Cramer's rule.

59. **(a)** Explain what it means if the coefficient determinant $D = 0$.

(b) Explain what it means if all the determinants are 0.

(c) Explain what it means if none of the determinants is 0.

CHAPTER 6 REVIEW

IMPORTANT TERMS AND CONCEPTS

Addition method for solving systems of linear equations
Angle-point form of a linear equation
Coefficient determinant
Cofactor of determinant
Consistent equations
Cramer's rule for solving linear equations
Dependent equations
Determinant
Elements of a determinant
Elimination methods
- Addition
- Substitution

Entries of a determinant
Horizontal line
- Slope of

Kirchhoff's law
Kirchhoff's rules
Linear equation
  Point-slope form
  Slope-intercept form
Linear system of equations
  Addition method for solving
  Consistent
  Cramer's rule for solving
  Dependent
  Inconsistent
  Substitution method for solving
Minor of a determinant
Point-slope form of a linear equation
Rows of a determinant
Slope
  Of a horizontal line
  Of a vertical line
Slope-intercept form of a linear equation
Substitution method for solving systems of linear equations
Vertical line
  Slope of

REVIEW EXERCISES

Evaluate each of the determinants in Exercises 1–8.

1. $\begin{vmatrix} 2 & 5 \\ -3 & 6 \end{vmatrix}$

2. $\begin{vmatrix} 4 & -7 \\ 3 & -6 \end{vmatrix}$

3. $\begin{vmatrix} 5 & 9 \\ 8 & 1 \end{vmatrix}$

4. $\begin{vmatrix} 3 & -12 \\ 0 & 5 \end{vmatrix}$

5. $\begin{vmatrix} 6 & -2 \\ -4 & 3 \end{vmatrix}$

6. $\begin{vmatrix} 8 & 9 \\ -1 & -2 \end{vmatrix}$

7. $\begin{vmatrix} 9 & 2 & 1 \\ 3 & -4 & 6 \\ 7 & 2 & 1 \end{vmatrix}$

8. $\begin{vmatrix} 9 & -2 & 3 \\ 4 & -5 & 2 \\ 3 & 2 & 1 \end{vmatrix}$

In Exercises 9–12, (a) determine the slope of the line through the given pair of points and then (b) find the equations of the line in point-slope form.

9. $(-2, 3), (5, 1)$ **10.** $(8, -4), (3, -2)$ **11.** $(9, -1), (-2, 4)$ **12.** $(-4, -3), (2, -1)$

In Exercises 13–16, rewrite each of the equations in slope-intercept form.

13. $4x - 2y = 6$ **14.** $9x + 3y = 5$ **15.** $4x = 5y - 8$ **16.** $7x + 3y - 8 = 0$

In Exercises 17–18, write the equation in slope-intercept form of each line that passes through the given point at the given angle to the positive x-axis.

17. Point: $(-3.5, 7.2)$, angle: $9.7°$ **18.** Point: $(12.8, -5.9)$, angle: $-52.6°$

In Exercises 19–20, determine the angle the given line makes with the positive x-axis.

19. $4.2x - 7.9y = 12.5$ **20.** $19.6x + 2.8y + 9.1 = 0$

In Exercises 21–22, solve each system of equations graphically.

21. $\begin{cases} 3x + y = 4 \\ 4x + y = 10 \end{cases}$ **22.** $\begin{cases} 6x + 2y = 5 \\ 3x - 4y = 7 \end{cases}$

In Exercises 23–26, use the substitution method to solve each system of equations.

23. $\begin{cases} 3x + y = 5 \\ 4x - y = 16 \end{cases}$

24. $\begin{cases} 3x - 2y = 4 \\ x + 8y = 3 \end{cases}$

25. $\begin{cases} 4x - y = 7 \\ 3x + 2y = 5 \end{cases}$

26. $\begin{cases} 6x + y - 5 = 0 \\ 4y - 3x = -7 \end{cases}$

In Exercises 27–32, use the addition method to solve each system of equations.

27. $\begin{cases} x - y = -2 \\ x + y = 8 \end{cases}$

28. $\begin{cases} 6x + 5y = 7 \\ 3x - 7y = 13 \end{cases}$

29. $\begin{cases} x + \frac{1}{2}y = 2 \\ 3x - y = 1 \end{cases}$

30. $\begin{cases} \frac{1}{6}x + \frac{1}{4}y = \frac{1}{3} \\ \frac{1}{4}x - \frac{1}{2}y = 1 \end{cases}$

31. $\begin{cases} 3x + 2y + 3z = -7 \\ 5x - 3y + 2z = -4 \\ 7x + 4y + 5z = 2 \end{cases}$

32. $\begin{cases} x + y + z = 2.7 \\ 6x + 7y - 5z = -8.8 \\ 10x + 16y - 3z = -6.4 \end{cases}$

In Exercises 33–38, use Cramer's rule to solve each system of equations.

33. $\begin{cases} 3x - 2y = 4 \\ 5x + 2y = 12 \end{cases}$

34. $\begin{cases} 4x + 3y = 27 \\ 2x - 5y = -19 \end{cases}$

35. $\begin{cases} 4x - 3y = 9 \\ 11x + 4y = 7 \end{cases}$

36. $\begin{cases} 2x - 5y = -8 \\ 4x + 6y = 17 \end{cases}$

37. $\begin{cases} 5x + 3y - 2z = 5 \\ 3x - 4y + 3z = 13 \\ x + 6y - 4z = -8 \end{cases}$

38. $\begin{cases} 5x - 2y + 3z = 6 \\ 6x - 3y + 4z = 10 \\ -4x + 4y - 9z = 4 \end{cases}$

Solve Exercises 39–41.

39. ***Business*** A store owner makes a special blend of coffee from Colombian Supreme costing \$4.99/lb and Mocha Java costing \$5.99/lb. The mixture sells for \$5.39/lb. If this mixture is made in 50-lb batches, how many pounds of each type should be used?

40. ***Business*** A computer company makes two kinds of computers. One, a personal computer (PC), uses 4 Type A chips and 11 Type B chips. The other, a business computer (BC), uses 9 Type A chips and 6 Type B chips. The company has 670 Type A chips and 1,055 Type B chips in stock. How many PC and BC computers can the company make so that all the chips are used?

41. ***Business*** An office building has 146 rooms made into 66 offices. The smallest offices each have 1 room and rent of \$300 per month. The middle-sized offices have 2 rooms each and rent for \$520 per month. The largest offices have 3 rooms each and rent for \$730 each. If the rental income is \$37,160 per month, how many of each type of offices are there?

CHAPTER 6 TEST

1. **(a)** Determine the slope of the line through the points $(-4, 2)$ and $(5, 6)$.

(b) Write the equation of the line through the points in **(a)**.

2. **(a)** Rewrite the equation $6x - 5y = 12$ in slope-intercept form.

(b) What is the slope of this line?

(c) What is the y-intercept?

(d) Sketch the graph of this line.

3. Determine the slope-intercept form for the equation of the line that passes through the point $(5, -8)$ and makes an angle of $29.6°$ with the positive x-axis.

4. Use the addition method to solve the following system.

$$\begin{cases} 2x + 3y = 1 \\ -3x + 6y = 16 \end{cases}$$

5. Evaluate the determinant.

$$\begin{vmatrix} 4 & -2 \\ 5 & 6 \end{vmatrix}$$

6. Evaluate the determinant.

$$\begin{vmatrix} 4 & 2 & -1 \\ 3 & 0 & 4 \\ -3 & 1 & 1 \end{vmatrix}$$

7. Solve the following system graphically.

$$\begin{cases} 2x + 3y = 5 \\ x - 4y = -14 \end{cases}$$

8. Solve the following system by the substitution method.

$$\begin{cases} 4x + 3y = 9 \\ 2x + y = 2 \end{cases}$$

9. Use Cramer's rule to solve the following system.

$$\begin{cases} 2x - 4y = 2 \\ -6x + 8y = -9 \end{cases}$$

10. Solve the following system.

$$\begin{cases} -x + 3y - 2z = 7 \\ 3x + 4y - 7z = -8 \\ x + 2y + z = 2 \end{cases}$$

11. Solve the following system.

$$\begin{cases} 3x_1 + 3x_2 - x_3 + x_4 = -25 \\ 4x_1 + 5x_2 - x_3 - 2x_4 = -8 \\ x_1 + x_2 - x_3 + x_4 = 1 \\ 6x_1 + 7x_2 - x_3 - 3x_4 = 17 \end{cases}$$

12. Three machine parts cost a total of \$60. The first part costs as much as the other two together. The cost of the second part is \$3 more than twice the cost of the third part. How much does each part cost?

13. Find the currents of the system shown in Figure 6.25.

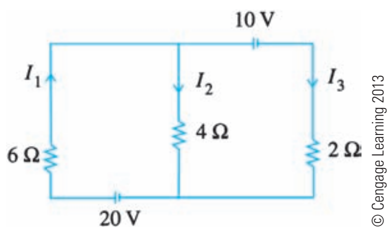


Figure 6.25

7 FACTORING AND ALGEBRAIC FRACTIONS

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Heavy rains often bring flooding. The height of the flood water depends on the volume of surface runoff. In Section 7.6, we will learn some of the basic algebra for estimating the volume of surface runoff.

This chapter uses the algebraic foundations established in Chapter 2. In Chapter 2, we learned how to multiply and divide algebraic expressions. In this chapter we will learn how to find the factors that, when multiplied together, give the original expression. Once we have the ability to factor algebraic expressions, we will use it to help solve second-degree or quadratic equations.

We will also use factoring to help add, subtract, multiply, divide, and simplify algebraic fractions. In spite of all the scientific and technical examples we have used that involve linear equations, many scientific and technical situations must be described with quadratic equations or with algebraic fractions.

OBJECTIVES

AFTER COMPLETING THIS CHAPTER, YOU WILL BE ABLE TO:

- Multiply polynomials.
- Factor polynomials using algebraic and graphical techniques.
- Simplify, multiply, divide, add, and subtract algebraic fractions.
- Simplify complex fractions.
- Use graphical techniques as a tool for checking.
- Use rational and polynomial functions in problem solving.

7.1 SPECIAL PRODUCTS

In mathematics we use certain products so often that we need to take the time to learn them. Success in factoring depends on your ability to recognize patterns in any form that they appear. All of these special products were developed in Section 2.2, but most were not stated in the general form. We will present each special product, give some examples of how it is used, and then summarize all of the special products at the end of this section.

The first special product is the distributive law of multiplication over addition, which we saw in Chapter 1.

DISTRIBUTIVE LAW

$$a(x + y) = ax + ay \qquad \#1$$

EXAMPLE 7.1

$$\begin{aligned} 8(4m + 3n) &= 8(4m) + 8(3n) \\ &= 32m + 24n \end{aligned}$$

Here $a = 8$, $x = 4m$, and $y = 3n$. As you can see, the variable x can represent a product of a constant and a variable.

EXAMPLE 7.2

$$\begin{aligned} 3x(2y + 5ax) &= (3x)(2y) + (3x)(5ax) \\ &= 6xy + 15ax^2 \end{aligned}$$

The term a in the distributive law can represent a constant, a variable, or the product of a constant and a variable.

The next special product is formed by multiplying the sum and difference of two terms. It is equal to the square of the first term minus the square of the second term.

DIFFERENCE OF TWO SQUARES

$$(x + y)(x - y) = x^2 - y^2 \qquad \#2$$

EXAMPLE 7.3

$$\begin{aligned}(x + 6)(x - 6) &= x^2 - 6^2\\ &= x^2 - 36\end{aligned}$$

Notice that each term to the right of the equal sign is a perfect square.

EXAMPLE 7.4

$$\begin{aligned}(2a + 3\sqrt{y})(2a - 3\sqrt{y}) &= (2a)^2 - (3\sqrt{y})^2\\ &= 4a^2 - 9y\end{aligned}$$

Again, once you realize that $y = (\sqrt{y})^2$, you can see that each term to the right of the equal sign is a perfect square. As you gain practice, you will be able to omit the middle step.

The third and fourth special products will be considered together.

PERFECT SQUARE TRINOMIALS

$$(x + y)^2 = x^2 + 2xy + y^2 \qquad \#3$$

$$(x - y)^2 = x^2 - 2xy + y^2 \qquad \#4$$

Note that both of these demonstrate that the square of a binomial is a **trinomial**.

CAUTION Be careful, many students make the mistake of thinking that $(x + y)^2 = x^2 + y^2$. This is not true, as you can easily verify by using the FOIL method from Chapter 2.

EXAMPLE 7.5

$$\begin{aligned}(p + 7)^2 &= p^2 + 2(7)p + 7^2\\ &= p^2 + 14p + 49\end{aligned}$$

Here $x = p$ and $y = 7$.

EXAMPLE 7.6

$$\begin{aligned}(2a - 5t^3)^2 &= (2a)^2 - 2(2a)(5t^3) + (5t^3)^2 \\ &= 4a^2 - 20at^3 + 25t^6\end{aligned}$$

Here $x = 2a$ and $y = 5t^3$.

The next special product is the product of two binomials that have the same first term. It can be verified by using the FOIL method.

TWO BINOMIALS, SAME FIRST TERM

$$(x + a)(x + b) = x^2 + (a + b)x + ab \qquad \#5$$

EXAMPLE 7.7

$$\begin{aligned}(p + 3)(p + 7) &= p^2 + (3 + 7)p + (3)(7) \\ &= p^2 + 10p + 21\end{aligned}$$

EXAMPLE 7.8

$$\begin{aligned}(r + 2)(r - 8) &= r^2 + (2 - 8)r + (2)(-8) \\ &= r^2 - 6r - 16\end{aligned}$$

In this example, $a = 2$ and $b = -8$. Remember that $r - 8$ can be written as $r + (-8)$.

The next special product is a general version of the last one. Again, this special product can be checked by using the FOIL method.

GENERAL QUADRATIC TRINOMIAL

$$(ax + b)(cx + d) = acx^2 + (ad + bc)x + bd \qquad \#6$$

EXAMPLE 7.9

Use Special Product 6 to multiply $(4x + 2)(3x + 5)$.

SOLUTION Here $a = 4$, $b = 2$, $c = 3$, and $d = 5$, so

$$\begin{aligned}(4x + 2)(3x + 5) &= 4 \cdot 3x^2 + (4 \cdot 5 + 2 \cdot 3)x + 2 \cdot 5 \\ &= 12x^2 + (20 + 6)x + 10 \\ &= 12x^2 + 26x + 10\end{aligned}$$

EXAMPLE 7.10

Multiply $(2x - 5)(2x + 9)$.

SOLUTION Here both a and c have the same value. That is, $a = c = 2$. Also, $b = -5$, and $d = 9$. The General Quadratic Trinomial formula (Special Product 6) produces

$$\begin{aligned}(2x - 5)(2x + 9) &= (2x)^2 + [(2)(9) + (-5)(2)]x + (-5)(9)\\ &= 4x^2 + [18 - 10]x + (-45)\\ &= 4x^2 + 8x - 45\end{aligned}$$

EXAMPLE 7.11

Multiply $(7x + 3y)(5x - 2y)$.

SOLUTION Here $a = 7$, $b = 3y$, $c = 5$, and $d = -2y$. Using Special Product 6 produces

$$\begin{aligned}(7x + 3y)(5x - 2y) &= (7 \cdot 5x^2) + [7(-2y) + (3y)5]x + (3y)(-2y)\\ &= 35x^2 + (-14y + 15y)x + (-6y^2)\\ &= 35x^2 + xy - 6y^2\end{aligned}$$

Of course, there is nothing to prevent the combination of two or more of these special products.

NOTE Remember the order of operations in Chapter 1. Powers are executed before multiplication or division. So, in the following example, we first square $(3x - 5)$ and then multiply that result by $4x$.

EXAMPLE 7.12

$$\begin{aligned}4x(3x - 5)^2 &= 4x(9x^2 - 30x + 25)\\ &= 36x^3 - 120x^2 + 100x\end{aligned}$$

EXAMPLE 7.13

$$\begin{aligned}(x^2 + 9)(x + 3)(x - 3) &= (x^2 + 9)(x^2 - 9)\\ &= x^4 - 81\end{aligned}$$

In this example, the left-hand side has the special product $(x + 3)(x - 3) = x^2 - 9$. When these are multiplied, we get another special product: $(x^2 + 9)(x^2 - 9)$. The time spent looking at a problem for special products can save some computation and time. It can also reduce errors.

There are four other special products that you will use less often than the ones already listed. They will be presented in pairs. The first two are the cubes of a sum or difference.

PERFECT CUBES

$$(x + y)^3 = x^3 + 3x^2y + 3xy^2 + y^3 \quad \#7$$

$$(x - y)^3 = x^3 - 3x^2y + 3xy^2 - y^3 \quad \#8$$

EXAMPLE 7.14

$$\begin{aligned}(x + 5)^3 &= x^3 + 3x^2(5) + 3x(5^2) + 5^3 \\ &= x^3 + 15x^2 + 75x + 125\end{aligned}$$

EXAMPLE 7.15

$$\begin{aligned}(2a - 4)^3 &= (2a)^3 - 3(2a)^2(4) + 3(2a)(4^2) - (4^3) \\ &= 8a^3 - 48a^2 + 96a - 64\end{aligned}$$

The last two special products also deal with cubes.

SUM AND DIFFERENCE OF TWO CUBES

$$(x + y)(x^2 - xy + y^2) = x^3 + y^3 \quad \#9$$

$$(x - y)(x^2 + xy + y^2) = x^3 - y^3 \quad \#10$$

EXAMPLE 7.16

$$\begin{aligned}(x - 3)(x^2 + 3x + 9) &= x^3 - (3)^3 \\ &= x^3 - 27\end{aligned}$$

EXAMPLE 7.17

$$\begin{aligned}(2d + 5)(4d^2 - 10d + 25) &= (2d)^3 + (5)^3 \\ &= 8d^3 + 125\end{aligned}$$

APPLICATION BUSINESS

EXAMPLE 7.18

An open box is going to be formed by cutting a square out of each corner of a rectangular piece of cardboard and folding up the sides as shown in Figure 7.1a. Suppose the piece of cardboard measures 10 cm × 12 cm. (a) Determine a

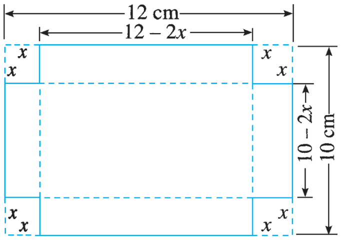

Figure 7.1a

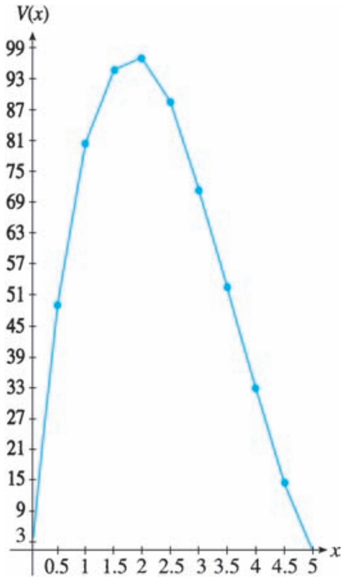

Figure 7.1b

function for the volume of the box. (b) What is the domain of this function? (c) Set up a partial table of values and sketch the graph of this function. (d) What size square seems to produce a box with the largest volume?

SOLUTIONS

(a) If the square measures x cm $\times$ x cm, then the lengths of the sides that will be folded up are $10 - 2x$ cm and $12 - 2x$ cm. So, the volume of the box is given by

$$\begin{aligned} V(x) &= x(10 - 2x)(12 - 2x) \\ &= 120x - 44x^2 + 4x^3 \end{aligned}$$

(b) The domain of the function is all real numbers. But, we cannot cut out a square that measures 0 cm or less and we cannot cut out a square that measures 5 cm or more. So, the "realistic" domain is represented by $\{x : 0 < x < 5\}$. Notice how the factored form helped us determine the domain. The expanded form will be more useful later in this book when we get to calculus.

(c) We have the following *partial* table of values.

x	0.5	1.0	1.5	2.0	2.5	3.0	3.5	4.0	4.5
$V(x)$	49.5	80.0	94.5	96.0	87.5	72.0	52.5	32.0	13.5

A graph formed from this table is shown in Figure 7.1b.

(d) From looking at the table, or the graph, it *seems* as if a 2 cm $\times$ 2 cm square will produce a box with the largest volume. Calculus can be used to show that $x \approx 1.8$ cm will produce the largest volume: 96.768 cm^3.

Thus, we have a total of 10 special products. Study them. Become familiar with them. Learn to recognize them and to use them to help simplify your work. The 10 special products are listed here together.

THE SPECIAL PRODUCTS

$$a(x + y) = ax + ay \qquad \#1$$

$$(x + y)(x - y) = x^2 - y^2 \qquad \#2$$

$$(x + y)^2 = x^2 + 2xy + y^2 \qquad \#3$$

$$(x - y)^2 = x^2 - 2xy + y^2 \qquad \#4$$

$$(x + a)(x + b) = x^2 + (a + b)x + ab \qquad \#5$$

$$(ax + b)(cx + d) = acx^2 + (ad + bc)x + bd \qquad \#6$$

$$(x + y)^3 = x^3 + 3x^2y + 3xy^2 + y^3 \qquad \#7$$

$$(x - y)^3 = x^3 - 3x^2y + 3xy^2 - y^3 \qquad \#8$$

$$(x + y)(x^2 - xy + y^2) = x^3 + y^3 \qquad \#9$$

$$(x - y)(x^2 + xy + y^2) = x^3 - y^3 \qquad \#10$$

EXERCISE SET 7.1

In Exercises 1–44, find the indicated products by direct use of one of the 10 special products. It should not be necessary to write intermediate steps.

1. $3(p + q)$
2. $7(2 + x)$
3. $3x(5 - y)$
4. $2a(4 + a^2)$
5. $(p + q)(p - q)$
6. $(3a + b)(3a - b)$
7. $(2x - 6p)(2x + 6p)$
8. $\left(\frac{y}{2} + \frac{2p}{3}\right)\left(\frac{y}{2} - \frac{2p}{3}\right)$
9. $(r + w)^2$
10. $(q - f)^2$
11. $(2x + y)^2$
12. $(\frac{1}{2}a + b)^2$
13. $(\frac{2}{3}x + 4b)^2$
14. $(a - \frac{1}{2}y)^2$
15. $(2p - \frac{3}{4}r)^2$
16. $(\frac{2}{3}r - 5t)^2$
17. $(a + 2)(a + 3)$
18. $(x + 5)(x + 7)$
19. $(x - 5)(x + 2)$
20. $(a + 9)(x - 12)$
21. $(2a + b)(3a + b)$
22. $(4x + 2)(3x + 5)$
23. $(3x + 4)(2x - 5)$
24. $(3m - n)(4m + 2n)$
25. $(a + b)^3$
26. $(r - s)^3$
27. $(x + 4)^3$
28. $(2a + b)^3$
29. $(2a - b)^3$
30. $(3x + 2y)^3$
31. $(3x - 2y)^3$
32. $(4r - s)^3$
33. $(m + n)(m^2 - mn + n^2)$
34. $(a + 2)(a^2 - 2a + 4)$
35. $(r - t)(r^2 + rt + t^2)$
36. $(h - 3)(h^2 + 3h + 9)$
37. $(2x + b)(4x^2 - 2xb + b^2)$
38. $(3a + \frac{2}{3}c)(9a^2 - 2ac + \frac{4}{9}c^2)$
39. $(3a - d)(9a^2 + 3da + d^2)$
40. $\left(\frac{2e}{5} - \frac{5r}{4}\right)\left(\frac{4e^2}{25} + \frac{er}{2} + \frac{25r^2}{16}\right)$
41. $3(a + 2)^2$
42. $5(2a - 4)^2$
43. $\frac{5r}{t}\left(t + \frac{r}{5}\right)^2$
44. $(x^2 + 4)(x + 2)(x - 2)$

In Exercises 45–60, perform the indicated operations.

45. $(x^2 - 6)(x^2 + 6)$
46. $\left(\frac{2a}{b} + \frac{b}{2}\right)^2$
47. $\left(\frac{3x}{y} - \frac{y}{x}\right)\left(\frac{3x}{y} + \frac{y}{x}\right)x^2y^2$
48. $(x + 1)(x + 2)(x + 3)$
49. $(x - y)^2 - (y - x)^2$
50. $(x + y)^2 - (y - x)^2$
51. $(x + 3)(x - 3)^2$
52. $(y - 4)(y + 4)^2$
53. $r(r - t)^2 - t(t - r)^2$
54. $5(x + 4)(x - 4)(x^2 + 16)$
55. $(5 + 3x)(25 - 15x + 9x^2)$
56. $(2 - \sqrt{x})(2 + \sqrt{x})(4 + x)$
57. $[(x + y) - (w + z)]^2$
58. $[(r + s) + (t + u)]^2$
59. $(x + y - z)(x + y + z)$
60. $(a + b + 2)(a - b + 2)$

Solve Exercises 61–68.

61. ***Electronics*** The impedance, z, of an ac circuit at frequency f is given by the formula $z^2 = R^2 + (x_L - x_C)^2$, where R is the resistance, x_L is the inductive resistance, and x_C is the capacitive resistance at the frequency f. Use a special product to expand the equation.

62. ***Physics*** The kinetic energy, KE, of an object is given by the formula $KE = \frac{1}{2}mv^2$, where m is the mass of the object and v its velocity. If the velocity of an object at any time, t, is given by the equation $v = 3t + 1$, find an equation for the kinetic energy in terms of m and t and expand your result using a special formula.

63. ***Physics*** The magnitude of the centripetal acceleration of a body in uniform circular motion is given by the formula $a_c = \frac{v^2}{r}$, where v is the velocity of the body and r is the radius of the circular path. If the velocity at any given time t is expressed as $v = 2t^2 - t$, find an equation for the centripetal acceleration in terms of r and t, and expand your result.

64. ***Energy*** The work, W, done by a steam turbine in a certain period of time is given by the formula $W = \frac{1}{2}m(v_1 - v_2)(v_1 + v_2)$, where m is the mass of the steam that passes through the turbine during that period, v_1 is the velocity of the steam when it enters, and v_2 the velocity when it leaves. Simplify this formula by multiplying the factors together.

65. ***Architectural technology*** When the maximum deflections of two similar cantilever beams bearing the load at the ends are compared, the difference in deflection is given by

$$d = \left(\frac{P}{3EI}\right)(l_1 - l_2)(l_1^2 + l_1 l_2 + l_2^2)$$

Simplify this formula by multiplying the two factors containing l_1.

66. ***Architectural technology*** When the maximum deflections of two similar cantilever beams bearing distributed loads are compared, the difference in deflection is given by

$$d = \left(\frac{w}{8EI}\right)(l_1 - l_2)(l_1 + l_2)(l_1^2 + l_2^2)$$

Simplify this formula by multiplying the three factors containing l_1.

67. ***Metallurgy*** When the temperature of a bar of length L_0 is changed by $T°$ its new length is given by the formula $L = L_0(1 + \alpha T)$, where α is the coefficient of thermal expansion. Multiply this expression.

68. ***Metallurgy*** The object in Figure 7.2 is a cube with a cube removed from the inside. The volume of the material used to construct the object is given by

$$V = (s_2 - s_1)(s_2^2 + s_1 s_2 + s_1^2)$$

Rewrite this formula by multiplying these two factors.

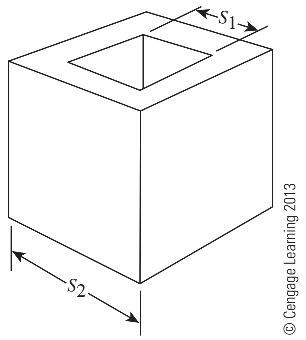

Figure 7.2

[IN YOUR WORDS]

69. Describe how you can remember the expansions of the perfect square trinomials.

70. Describe how you can remember the expansions of the perfect cubes.

7.2 FACTORING

Sometimes it is important to determine what expressions were multiplied together to form a product, instead of multiplying quantities together as we did in the special products. Recall from Section 2.1 that an algebraic term consists of several factors. For example, the term $5xy$ has factors of 1, 5, x, y, $5x$, $5y$, xy, and $5xy$. In Section 7.1, we learned to multiply several factors together to form the special products. In this section, we will begin to learn how to factor an algebraic expression. Once we know how to factor we will be able to solve more complicated problems. Determining the factors of an algebraic expression is called *factoring*.

When factoring a polynomial, we will use only those factors that are also polynomials. Further, we will continue factoring a polynomial until the only remaining factors are 1 and -1. When this has been completed, we will be able to say that the polynomial has been factored completely and all of the factors will be *prime factors*.

COMMON FACTORS

The simplest type of factoring is the reverse of the distributive law of multiplication. This type of factoring is called removing a **common factor**. If each term in an expression contains the same factor, then this is a common factor and it can be factored out using the distributive law.

COMMON FACTOR

$$ax + ay = a(x + y)$$

EXAMPLE 7.19

In the expression $6a + 3a^2$, each term contains a factor of $3a$.

$$\begin{aligned} 6a + 3a^2 &= 2(3a) + a(3a) \\ &= (2 + a)3a \\ &= 3a(2 + a) \end{aligned}$$

EXAMPLE 7.20

$8y^2 + 2y = 2y(4y + 1)$

Remember that $2y = (2y)(1)$. When you factor $2y$, you are left with a factor of 1.

EXAMPLE 7.21

$12x^5y + 8x^3y^2 = 4x^3y(3x^2 + 2y)$

You could have factored this in several other ways, such as $2xy(6x^4 + 4x^2y)$. This would not have been considered completely factored, since $6x^4 + 4x^2y$ can be factored further.

You may not see how to completely factor out all of the common factors in one step. For instance, in the last example you might have first written

$$12x^5y + 8x^3y^2 = 2xy(6x^4 + 4x^2y)$$

and then noticed that you could factor $6x^4 + 4x^2y$ as $2x^2(3x^2 + 2y)$. Then you would have had

$$\begin{aligned} 12x^5y + 8x^3y^2 &= 2xy(6x^4 + 4x^2y) \\ &= 2xy(2x^2)(3x^2 + 2y) \\ &= 4x^3y(3x^2 + 2y) \end{aligned}$$

Notice that in the last step the two monomials were combined.

EXAMPLE 7.22

Factor $6x^2y + 9xy^2z - 3xyz$.

SOLUTION We can see that each term contains a multiple of 3, as well as an x and y factor. If we factor $3xy$ out of each term, we have $(3xy)(2x) + (3xy)(3yz) + (3xy)(-z) = 3xy(2x + 3yz - z)$. So, $6x^2y + 9xy^2z - 3xyz = 3xy(2x + 3yz - z)$.

The easiest factors to locate (and sometimes the easiest to overlook) are the monomials or common factors.

HINT You should always begin factoring an algebraic expression by looking for common factors. Once you have factored out all common factors, the remaining expression is easier to factor.

USING THE SPECIAL PRODUCTS

Special product #2 gives us our second important form of factoring. Since $(x + y)(x - y) = x^2 - y^2$, we can reverse this to get

DIFFERENCE OF TWO SQUARES

$$x^2 - y^2 = (x + y)(x - y)$$

As you can see, to factor the difference between two squares you get factors that are the sum and difference of the quantities.

EXAMPLE 7.23

Factor $x^2 - 25$.

SOLUTION
$$\begin{aligned} x^2 - 25 &= x^2 - 5^2 \\ &= (x + 5)(x - 5) \end{aligned}$$

You would have had to notice that 25 was 5^2 before you could recognize that this was the difference of two squares.

EXAMPLE 7.24

Factor $9a^2 - 49b^4$.

SOLUTION
$$\begin{aligned} 9a^2 - 49b^4 &= (3a)^2 - (7b^2)^2 \\ &= (3a + 7b^2)(3a - 7b^2) \end{aligned}$$

Again, you have to recognize the perfect squares: $9a^2 = (3a)^2$ and $49b^4 = (7b^2)^2$.

EXAMPLE 7.25

Factor $27x^3 - 75xy^2$.

SOLUTION
$$\begin{aligned} 27x^3 - 75xy^2 &= 3x(9x^2 - 25y^2) \\ &= 3x(3x + 5y)(3x - 5y) \end{aligned}$$

This example demonstrates the value of first looking for common factors. There is a common factor of $3x$. When the common factor is factored out, it is easier to see that the remaining factor is the difference of two squares.

EXAMPLE 7.26

Factor $5x^4 - 80y^4$.

SOLUTION
$$\begin{aligned} 5x^4 - 80y^4 &= 5(x^4 - 16y^4) \\ &= 5[(x^2)^2 - (4y^2)^2] \\ &= 5(x^2 + 4y^2)(x^2 - 4y^2) \\ &= 5(x^2 + 4y^2)(x + 2y)(x - 2y) \end{aligned}$$

Again, you should have noticed that there was a common factor of 5 and that each of the remaining terms was a perfect square. After factoring out $x^4 - 16y^4$ you were not finished, because $x^2 - 4y^2$ is also the difference of two squares. The other factor $x^2 + 4y^2$ cannot be factored further.

NOTE Remember: $x^2 + y^2 \neq (x + y)^2$. In fact, $x^2 + y^2$ cannot be factored using real numbers.

APPLICATION ELECTRONICS

EXAMPLE 7.27

In order to shield it from stray electromagnetic radiation, an electronic device is housed in a metal canister that is shaped like a right circular cylinder. The surface area of the cylinder is given by $A = 2\pi r^2 + 2\pi rh$, where r is the radius of the base and h is the height of the cylinder. Factor the right-hand side of this formula.

SOLUTION This formula has common factors of 2π and r. Factoring, we obtain

$$\begin{aligned} A &= 2\pi r^2 + 2\pi rh \\ &= 2\pi r(r + h) \end{aligned}$$

EXERCISE SET 7.2

Completely factor each of the expressions in Exercises 1–40.

1. $6x + 6$
2. $12x + 12$
3. $12a - 6$
4. $15d - 5$
5. $4x - 2y + 8$
6. $6a + 9b - 3c$
7. $5x^2 + 10x + 15$
8. $16a^2 + 8b - 24$
9. $10x^2 - 15$
10. $14x^4 + 21$
11. $4x^2 + 6x$
12. $8a - 4a^2$
13. $7b^2y + 28b$
14. $9ax^2 + 27bx$
15. $3ax + 6ax^2 - 2ax$
16. $6by - 12b^2y + 7by^2$
17. $4ap^2 + 6a^2pq + 8apq^2$
18. $12p^2r^2 - 8p^3r + 24pr^2$
19. $a^2 - b^2$
20. $p^2 - r^2$
21. $x^2 - 4$
22. $a^2 - 16$
23. $y^2 - 81$
24. $m^2 - 49$
25. $4x^2 - 9$
26. $49y^2 - 64$
27. $9a^4 - b^2$
28. $16t^6 - a^2$
29. $25a^2 - 49b^2$
30. $121r^2 - 81t^2$
31. $144 - 25b^4$
32. $81 - 49r^4$
33. $5a^2 - 125$
34. $7x^2 - 63$
35. $28a^2 - 63b^4$
36. $81x^2 - 36t^6$
37. $a^4 - 81$
38. $b^4 - 256$
39. $16x^4 - 256y^4$
40. $25a^5 - 400ab^8$

Solve Exercises 41–50.

41. The total surface area of a cone is given by the formula $\pi r^2 + \pi r\sqrt{h^2 + r^2}$, where r is the radius of the base and h is the height of the cone. Factor this expression.

42. ***Thermodynamics*** The amount of heat that must be added to a metal object in order for it to melt is given by the formula $Q = mc\Delta t + mL_f$. Factor the right side of this equation.

43. ***Wastewater technology*** According to Bernoulli's equation, if a fluid of density d is flowing horizontally in a pipe and its pressure and velocity at one location are p_1 and v_1, respectively, and at a second location, they are p_2 and v_2, then the difference in their pressures is given by $p_1 - p_2 = \frac{1}{2}dv_2{}^2 - \frac{1}{2}dv_1{}^2$. Factor the right-hand side of this equation.

44. The cross-sectional area A of a tube can be determined from the formula $A = \pi R^2 - \pi r^2$, where R is the outside radius and r is the inside radius of the tube. Factor the right-hand side of this equation.

45. ***Acoustical engineering*** The angular acceleration α of a stereo turntable during a time period can be determined using the formula

$$\alpha = \frac{\omega_f{}^2 - \omega_0{}^2}{2\theta}$$

where ω_0 is the angular velocity at the beginning of the time interval, ω_f is the angular velocity at the end of the time interval, and θ is the number of radians the turntable rotated during the interval. Factor the right-hand side of this equation.

46. ***Thermodynamics*** A black body is a hypothetical body that absorbs, without reflection, all of the electromagnetic radiation that strikes its surface. The energy E radiated by a black body is given by $E = e\sigma T^4 - e\sigma T_0^4$, where T and T_0 are the absolute temperatures of the body and the surroundings, respectively, σ is the Stefan-Boltzmann constant, and e is the emissivity of the body. Factor the right-hand side of this equation.

47. ***Energy*** The work-energy equation for rotational motion is

$$W = \tfrac{1}{2}I\omega_2^2 - \tfrac{1}{2}I\omega_1^2$$

Factor the right-hand side of this equation.

48. ***Optics*** The *lensmaker's equation*

$$f^{-1} = nr_1^{-1} - nr_2^{-1} - r_1^{-1} + r_2^{-1}$$

gives the focal length f of a very thin lens.

(a) Rewrite the lensmaker's equation by factoring the right-hand side.

(b) Write your answer in (a) using only positive exponents.

49. ***Environmental technology*** At a certain time a circular oil spill has a radius of r meters. Some time later the radius has increased by 45 m. The change in the area of this spill is given by

$$\Delta A = \pi(r + 45)^2 - \pi r^2$$

Simplify and factor this expression.

50. ***Metallurgy*** The volume of the washer in Figure 7.3 is given by the formula

$$V = \pi r_2^2 h - \pi r_1^2 h$$

Factor this expression.

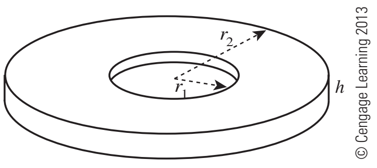

Figure 7.3

[IN YOUR WORDS]

51. Explain how you recognize when an expression is a difference of two squares and then how you would factor it.

52. Describe what you look for when removing a common factor.

7.3 FACTORING TRINOMIALS

In Section 7.2, we introduced the idea of factoring and learned how to factor two types of problems. These problems were based on the first two special products that we learned in Section 7.1.

An algebraic expression that has three terms is called a **trinomial**. Special products #3 through #6 all resulted in quadratic trinomials. In this section we will focus on factoring quadratic trinomials with the purpose of reversing one of these four special products.

Not all quadratic trinomials can be factored using real numbers. The general quadratic trinomial is of the form $ax^2 + bx + c$, where a, b, and c represent constants.

NOTE You can determine if a quadratic trinomial can be factored by examining the discriminant $b^2 - 4ac$. If the discriminant is a perfect square, then it is possible to factor the quadratic using rational numbers.

DISCRIMINANT

The **discriminant** of the trinomial $ax^2 + bx + c$ is $b^2 - 4ac$.
If $b^2 - 4ac > 0$, then $ax^2 + bx + c$ can be factored using real numbers.
If $b^2 - 4ac$ is a perfect square, then $ax^2 + bx + c$ can be factored using rational numbers.
If $b^2 - 4ac < 0$, then $ax^2 + bx + c$ cannot be factored using real numbers.

EXAMPLE 7.28

Can the trinomial $4x^2 + 3x - 7$ be factored?

SOLUTION In this quadratic trinomial $a = 4$, $b = 3$, and $c = -7$, so

$$\begin{aligned} b^2 - 4ac &= (3)^2 - 4(4)(-7) \\ &= 9 - (-112) \\ &= 121 \end{aligned}$$

Using our knowledge (or a calculator) we see that $\sqrt{121} = 11$, so this trinomial can be factored using rational numbers. Later in this section, our job will be to find its factors.

EXAMPLE 7.29

Can the quadratic trinomial $5x^2 - 3x - 7$ be factored?

SOLUTION Here $a = 5$, $b = -3$, and $c = -7$, so

$$\begin{aligned} b^2 - 4ac &= (-3)^2 - 4(5)(-7) \\ &= 9 - (-140) \\ &= 149 \end{aligned}$$

Using a calculator, we find that $\sqrt{149} \approx 12.207$ and is not a perfect square. Thus, it is possible to factor this quadratic equation with real numbers.

We will skip special products #3 and #4 and consider special product #5: $(x + a)(x + b) = x^2 + (a + b)x + ab$. If you examine special products #3 and #4 you can see that they are just special cases of #5. In the special product where $(x + a)(x + b) = x^2 + (a + b)x + ab$, the leading coefficient (the coefficient of the x^2 term) is 1. This makes the job of factoring somewhat easier, because all we need to do is determine a and b. Notice that $a + b$ is the coefficient of the x-term and ab is the constant. Thus, we have the reverse of special product #5.

$$x^2 + (a + b)x + ab = (x + a)(x + b) \qquad \#5$$

Of course, the difficulty is determining the values of a and b.

EXAMPLE 7.30

Factor $x^2 - 3x - 10$.

SOLUTION Since the value of the discriminant is $49 = 7^2$, we know that this can be factored using rational numbers. From the formula in special product #5, we know that it factors into $(x + a)(x + b)$, where $a + b = -3$ and $ab = -10$.

What possible choices are there for a and b? From the factors of -10 we have the following four possible pairs.

Possible pairs of factors that satisfy $ab = -10$			Sum $(a + b)$
-10	and	1	-9
10	and	-1	9
-5	and	2	-3
5	and	-2	3

As you can see, only one of these pairs adds to -3, the pair of -5 and 2. So, if we let $a = -5$ and $b = 2$, then we have

$$\begin{aligned} x^2 - 3x - 10 &= (x - 5)(x + 2) \\ &= (x + 2)(x - 5) \end{aligned}$$

EXAMPLE 7.31

Factor $x^2 + 10x + 16$.

SOLUTION The discriminant is 36, a perfect square, so this can be factored using rational numbers into the form $(x + a)(x + b)$. We want to find a and b so that $a + b = 10$ and $ab = 16$. Again, we will begin with the factors of the product $ab = 16$.

Possible pairs of factors that satisfy $ab = 16$			Sum $(a + b)$
16	and	1	17
-16	and	-1	-17
8	and	2	10
-8	and	-2	
4	and	4	
-4	and	-4	

We stopped once we saw that the pairs of 8 and 2 added to 10, because we had found the pair that worked. If we let $a = 8$ and $b = 2$, we have factored the trinomial as

$$\begin{aligned} x^2 + 10x + 16 &= (x + 8)(x + 2) \\ &= (x + 2)(x + 8) \end{aligned}$$

A check to see if the constant is positive or negative will give you some help in factoring. If the constant is positive, then the factors will have the same sign—both will be positive or both negative. If the constant term is negative, then the two factors will have different or unlike signs—one will be positive and the other negative.

EXAMPLE 7.32

Factor each trinomial completely.

Here the constant term is positive.

(a) $x^2 + 7x + 12 = (x + 4)(x + 3)$ Both factors have positive signs.
(b) $x^2 - 7x + 12 = (x - 4)(x - 3)$ Both factors have negative signs.

Here the constant term is negative.

(c) $x^2 + x - 12 = (x + 4)(x - 3)$
(d) $x^2 - x - 12 = (x - 4)(x + 3)$
Factors have unlike signs—one is positive and one is negative.

Factoring a quadratic trinomial with a leading coefficient that is not 1 is not as easy. Here we are looking at the reverse of special product #6, or

$$acx^2 + (ad + bc)x + bd = (ax + b)(cx + d) \qquad \#6$$

There are three techniques used to factor these quadratic trinomials. We will look at two of them in this section and examine the third in Section 7.4.

The first technique is called trial and error. In using the trial-and-error method, we make use of the fact that ac is the leading coefficient, bd is the constant term, and $ad + bc$ is the middle coefficient. We can also use our knowledge of signs to help find the signs of the factors.

EXAMPLE 7.33

Factor $3x^2 - 8x + 4$.

SOLUTION The discriminant is $16 = 4^2$, so this equation will factor using rational numbers. Since 3 is a prime number, we know that its only factors are 3 and 1. Also, since the constant is positive, both factors have the same sign. Thus, we know that the factors are $(3x + b)(x + d)$. Now, all we need to find are b and d.

Since $bd = 4$, the possible pairs of factors are $-4, -1$; $4, 1$; $2, 2$; and $-2, -2$. Next, we know that $3d + b = -8$ and the only choices from the pairs of factors that satisfy this are $b = -2$ and $d = -2$. Thus, the factors of $3x^2 - 8x + 4$ are $(3x - 2)(x - 2)$. You should multiply these factors together to check that their product is the original trinomial.

EXAMPLE 7.34

Factor $6x^2 + 7x - 20$.

SOLUTION The discriminant is $529 = 23^2$, so this equation will factor using rational numbers. The leading coefficient, 6, has factors of 6 and 1, and 2 and 3. We will try 6 and 1.

Since the constant term is negative, the factors will have unlike signs. Thus, we have $(6x + b)(x + d)$, where either b or d is negative, $bd = -20$, and $6d + b = 7$. The possible choices for the pairs b and d are 1, 20; 2, 10; and 4, 5, where one is positive and the other is negative.

If we try $b = 4$ and $d = -5$, we get $6(-5) + 4 = -26$. Since this is not 7, this is not the correct solution. All other possible combinations for b and d also fail. Thus, we must make another choice for a and c.

If $a = 2$ and $c = 3$, we have $(2x + b)(3x + d)$. If $b = 5$ and $d = -4$, we get $ad + bc = 2(-4) + (5)(3) = -8 + 15 = 7$. This is the correct coefficient for the x-term, so the factors of $6x^2 + 7x - 20$ are $(2x + 5)$ and $(3x - 4)$.

As you can see, the trial-and-error method can be very long and frustrating, but some people learn to factor a quadratic trinomial quickly by using this method.

The other method we will look at in this section is called either the "grouping" or "split-the-middle" method. It is longer than the trial-and-error method, but it is a "sure-fire" technique. We will use the grouping method on the same problem that we just worked.

EXAMPLE 7.35

Factor $6x^2 + 7x - 20$ using the grouping method.

SOLUTION

Step 1: Multiply the leading coefficient and the constant term: $(6)(-20) = -120$.

Step 2: Find two factors of this product whose sum is the coefficient of the middle term of the trinomial. For this problem we want to find factors p and q, where $pq = -120$ and $p + q = 7$. We get $p = 15$ and $q = -8$.

Step 3: Rewrite the trinomial by splitting the middle term into $px + qx$ and grouping the first two terms and the last two terms. In this problem, $7x = 15x - 8x$ and the trinomial becomes $(6x^2 + 15x) + (-8x - 20)$.

Step 4: Distribute the common factors from each grouping.

$$(6x^2 + 15x) + (-8x - 20) = 3x(2x + 5) - 4(2x + 5)$$

Step 5: Distribute the common factor $2x + 5$ from the entire expression.

$$3x(2x + 5) - 4(2x + 5) = (3x - 4)(2x + 5)$$

These are the required factors. Thus, we have factored $6x^2 + 7x - 20$ as $(3x - 4)(2x + 5)$.

EXAMPLE 7.36

Factor $21x^2 - 41x + 10$ using the grouping method.

SOLUTION

Step 1: $(21)(10) = 210$

Step 2: $210 = (-35)(-6)$ and $-35 - 6 = -41$

Step 3: $(21x^2 - 35x) + (-6x + 10)$

Step 4: $7x(3x - 5) - 2(3x - 5)$

Step 5: $(7x - 2)(3x - 5)$

NOTE Remember to look for any common factors before you start to use either the trial-and-error or the grouping method.

EXAMPLE 7.37

Factor $20x^3 + 22x^2 - 12x$.

SOLUTION There is a common factor of $2x$, so

$$\begin{aligned} 20x^3 + 22x^2 - 12x &= 2x(10x^2 + 11x - 6) \\ &= 2x(5x - 2)(2x + 3) \end{aligned}$$

Not all trinomials have just one variable. The grouping method is the best method to use when factoring trinomials that have more than one variable. For example, as shown in the next example, $2x^2 - 17xy + 36y^2$ can be factored using the grouping method.

EXAMPLE 7.38

Factor $2x^2 - 17xy + 36y^2$ completely.

SOLUTION

Step 1: $2x^2(36y^2) = 72x^2y^2$

Step 2: $72x^2y^2 = (-9xy)(-8xy)$ and $-9xy - 8xy = -17xy$

Step 3: $(2x^2 - 9xy) + (-8xy + 36y^2)$

EXAMPLE 7.38 (Cont.)

Step 4: $x(2x - 9y) - 4y(2x - 9y)$

Step 5: $(x - 4y)(2x - 9y)$

As a check, multiplying the two factors in Step 5 gives the original expression, and so we can say that $2x^2 - 17xy + 36y^2 = (x - 4y)(2x - 9y)$.

It is also possible to factor trinomials with powers greater than 2 if one exponent is twice the other.

EXAMPLE 7.39

Factor $x^8 + 5x^4 + 6$.

SOLUTION
$$\begin{aligned} x^8 + 5x^4 + 6 &= (x^4)^2 + 5x^4 + 6 \\ &= (x^4 + 3)(x^4 + 2) \end{aligned}$$

Two other useful methods are based on special products #9 and #10. These are the sum and difference of two cubes.

SUM OF TWO CUBES

$$(x + y)(x^2 - xy + y^2) = x^3 + y^3 \qquad \#9$$

DIFFERENCE OF TWO CUBES

$$(x - y)(x^2 + xy + y^2) = x^3 - y^3 \qquad \#10$$

EXAMPLE 7.40

Factor $8x^3 + 125$.

SOLUTION $8x^3 = (2x)^3$ and $125 = 5^3$, so this is the sum of two cubes, and $8x^3 + 125 = (2x)^3 + 5^3 = (2x + 5)(4x^2 - 10x + 25)$.

APPLICATION MECHANICAL

EXAMPLE 7.41

The volume V of a box of height x can be given by $V(x) = 4x^3 - 64x^2 + 252x$. Factor the right-hand side of this equation to determine the "realistic" domain for the height x.

SOLUTION First, we notice that each term on the right-hand side of $V(x) = 4x^3 - 64x^2 + 252x$ has a common factor of $4x$. So, $V(x) = 4x(x^2 - 16x + 63)$. The quadratic expression factors as $x^2 - 16x + 63 = (x - 9)(x - 7)$. So, the completely factored form is

$$V(x) = 4x(x - 9)(x - 7)$$

From this, we see that the domain is $0 < x < 7$ or $x > 9$. We will see later that $x > 9$ will not satisfy many methods used to make such a box.

The six special factors are listed here. Study them, and learn to recognize them either as factors or as one of the special factors.

THE SPECIAL FACTORS

$$ax + ay = a(x + y) \qquad \#1$$

$$x^2 - y^2 = (x + y)(x - y) \qquad \#2$$

$$x^2 + (a + b)x + ab = (x + a)(x + b) \qquad \#5$$

$$acx^2 + (ad + bc)x + bd = (ax + b)(cx + d) \qquad \#6$$

$$x^3 + y^3 = (x + y)(x^2 - xy + y^2) \qquad \#9$$

$$x^3 - y^3 = (x - y)(x^2 + xy + y^2) \qquad \#10$$

EXERCISE SET 7.3

Determine if each of the trinomials in Exercises 1–6 can be factored.

1. $x^2 + 9x - 8$
2. $x^2 + 7x - 8$
3. $3x^2 - 10x - 8$
4. $2x^2 + 16x + 14$
5. $5x^2 + 23x + 18$
6. $7x^2 - 5x + 16$

Factor each of the following trinomials completely.

7. $x^2 + 7x + 10$
8. $x^2 + 8x + 15$
9. $x^2 - 12x + 27$
10. $x^2 - 14x + 33$
11. $x^2 - 27x + 50$
12. $x^2 + 19x + 48$
13. $x^2 - x - 2$
14. $x^2 - 4x - 5$
15. $x^2 - 3x - 10$
16. $p^2 - 16p + 64$
17. $r^2 + 10r + 25$
18. $v^2 - 14v + 49$
19. $a^2 + 22a + 121$
20. $e^2 + 26e + 169$
21. $f^2 - 30f + 225$
22. $3x^2 + 4x + 1$
23. $6y^2 - 7y + 1$
24. $3p^2 + 5p + 2$
25. $7t^2 + 9t + 2$
26. $5a^2 + 14a - 3$
27. $7b^2 - 34b - 5$
28. $2y^2 + y - 6$
29. $4e^2 + 19e - 5$
30. $6m^2 - 19m + 3$
31. $3u^2 + 10u + 8$
32. $7r^2 + 13r - 2$
33. $9t^2 - 25t - 6$

34. $4x^2 + 8x + 3$
35. $6x^2 + 13x - 5$
36. $8y^2 - 8y - 6$
37. $15a^2 - 16a - 15$
38. $15d^2 + 16d - 15$
39. $15e^2 + 34e + 15$
40. $14a^2 - 39a + 10$
41. $10x^2 - 19x + 6$
42. $3x^2 + 18x + 27$
43. $3r^2 - 18r - 21$
44. $15x^2 + 50x + 35$
45. $49t^4 - 105t^3 + 14t^2$
46. $2y^4 - 9y^2 + 7$
47. $6x^2 - 11xy - 10y^2$
48. $4p^2 + 20pq + 25q^2$
49. $8a^2 - 14ab - 9b^2$
50. $6d^9 + 15d^5e^2 + 6de^4$
51. $a^3 - b^3$
52. $y^3 - 8$
53. $8x^3 - 27$
54. $64p^3 + 125t^6$

Solve Exercises 55–62.

55. ***Electronics*** The current i, in amperes, in a certain circuit varies with time, in seconds, according to the equation

$$i = 0.7t^2 - 2.1t - 2.8$$

Factor the right-hand side of this equation.

56. ***Sheet metal technology*** A box with an open top is made from a rectangular sheet of metal by cutting equal-sized squares from the corners and folding up the sides. If the length of the side of a square that is removed is x, then the volume of this box is given by $V = 180x - 58x^2 + 4x^3$. Factor this expression.

57. ***Business*** The cost C for a certain company to produce n items is given by the equation $C(n) = 0.0001n^3 - 0.2n^2 - 3n + 6{,}000$. Factor the right-hand side of this equation.

58. ***Dynamics*** A ball is thrown upward with a speed of 48 ft/s from the edge of a cliff 448 ft above the ground. The distance s of this ball above the ground at any time t is given by $s(t) = -16t^2 + 48t + 448$. Factor the right-hand side of this equation.

59. ***Ecology*** An ecology center wants to make an experimental garden. A gravel border of uniform width will be placed around the rectangular garden. The garden is 10 m long and 6 m wide. The builder has only enough gravel to cover 36 m^2 to the desired depth. In order to determine the width of the border, the equation

$$(6 + 2x)(10 + 2x) - 60 = 36$$

must be solved. **(a)** Simplify this equation and **(b)** factor your answer.

60. ***Fire science*** The flow rate, Q, in a certain hose can be found by solving the equation $2Q^2 + Q - 21 = 0$. Factor the left-hand side of this equation.

61. ***Industrial design*** A pizza box is to be made from a rectangular piece of cardboard that measures 18″ × 36″. In order for the cardboard to fold into a box with a lid, six squares are cut from the cardboard as shown in Figure 7.4. The volume of the box depends on the size of the six squares and is given by

$$V = 324x - 63.0x^2 + 3.0x^3$$

Factor the right-hand side of this equation.

62. ***Civil engineering*** The deflection, Δ, of a certain beam is given by the equation

$$\delta = 9x^2 - 30xL + 24L^2$$

Factor the right-hand side of this equation.

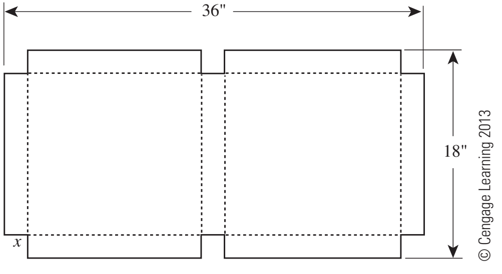

Figure 7.4

[IN YOUR WORDS]

63. Explain how to use the discriminant to find if a trinomial can be factored by **(a)** rational numbers and **(b)** real numbers.

64. Explain how you recognize when an expression is a difference of two cubes and then how you would factor it.

7.4 FRACTIONS

Working with algebraic expressions in technical situations often requires work with algebraic fractions. Working with algebraic fractions is very similar to working with fractions based on real numbers. In this section, we will work with some fundamental properties of fractions, and in Sections 7.5 and 7.6 we will use these properties on the basic operations of addition, subtraction, multiplication, and division with fractions. We begin by stating the Fundamental Principle of Fractions.

FUNDAMENTAL PRINCIPLE OF FRACTIONS

Multiplying or dividing both the numerator and denominator of a fraction by the same number, except zero, results in a fraction that is equivalent to the original fraction. Two fractions are equivalent if their cross-products are equal.

EXAMPLE 7.42

Since $\frac{3}{4} = \frac{3 \cdot 5}{4 \cdot 5} = \frac{15}{20}$ then the fractions $\frac{15}{20}$ and $\frac{3}{4}$ are equivalent by the Fundamental Principle of Fractions.

You can check their cross-products to verify that $\frac{3}{4}$ and $\frac{15}{20}$ are equivalent. Since $(3)(20) = 60$ and $(4)(15) = 60$, their cross-products are equal and the fractions are equivalent.

EXAMPLE 7.43

$$\frac{a}{x} = \frac{a(xy)}{x(xy)} = \frac{axy}{x^2y}$$

SOLUTION We can verify that $\frac{a}{x} = \frac{axy}{x^2y}$ by comparing their cross-products:

$$a(x^2y) = ax^2y \qquad \text{and} \qquad x(axy) = ax^2y$$

Since the cross-products are equal, the fractions are equivalent as well.

EXAMPLE 7.44

$$\frac{x+3}{x-2} = \frac{(x+3)(x-3)}{(x-2)(x-3)} = \frac{x^2-9}{x^2-5x+6}$$

As long as $x \neq 3$, we can say that $\frac{x+3}{x-2} = \frac{x^2-9}{x^2-5x+6}$.

Saying that two fractions are equivalent is another way of saying that they represent the same number. Remembering the rules for signed numbers in Section 1.2, we stated that the number of negative signs can be reduced (or increased) by twos without changing the value of the expression. The Fundamental Principle of Fractions says essentially the same thing:

EXAMPLE 7.45

$$\frac{-a}{b} = \frac{-a(-1)}{b(-1)} = \frac{a}{-b}$$

One of the most important applications of the Fundamental Principle of Fractions is in reducing a fraction to *lowest terms* or *simplest form*. A fraction is in lowest terms when the numerator and denominator have no factors in common other than $+1$.

EXAMPLE 7.46

Reduce $\frac{14ab^2x}{7ax^3}$ to simplest form.

SOLUTION The numerator and denominator have a common factor of $7ax$. We can then write $\frac{14ab^2x}{7ax^3} = \frac{7ax(2b^2)}{7ax(x^2)} = \frac{2b^2}{x^2}$. Notice that we used a part of the Fundamental Principle of Fractions that we had not used before. We *divided* both the numerator and the denominator by $7ax$, the common factor.

EXAMPLE 7.47

Reduce $\frac{x^2(x-3)}{x^2-9}$ to lowest terms.

SOLUTION We need to find a common factor of both the numerator and the denominator; x^2 is *not* that common factor. While x^2 is a factor of the numerator, it is not a factor of the denominator. However, the denominator will factor into $(x-3)(x+3)$, so

$$\frac{x^2(x-3)}{x^2-9} = \frac{x^2(x-3)}{(x-3)(x+3)} = \frac{x^2}{x+3}$$

Again, we used the Fundamental Principle of Fractions and divided both the numerator and denominator by $x-3$.

NOTE Some factors differ only in sign. Although this difference may not seem like much, it is often overlooked. In particular, you should note that

$$x - y = (-1)(-x + y) = -(-x + y) = -(y - x)$$

Of these four, the first and last are the ones you will use the most. Remember, $x - y$ and $y - x$ differ only in sign.

EXAMPLE 7.48

Reduce $\frac{5x - xy}{3y - 15}$ to lowest terms.

SOLUTION First, factor the numerator and the denominator:

$$\frac{5x - xy}{3y - 15} = \frac{x(5 - y)}{3(y - 5)}$$

Since $5 - y = -(y - 5)$, replace $5 - y$ with $-(y - 5)$:

$$\frac{x(5 - y)}{3(y - 5)} = \frac{-x(y - 5)}{3(y - 5)} = \frac{-x}{3}$$

Is $\frac{-x}{3}$ equal to $\frac{5x - xy}{3y - 15}$? You can always verify your answer by checking the cross-products. In this case, the cross-products are equal.

EXAMPLE 7.49

Reduce $\frac{2x^3 + 4x^2 - 30x}{15x^2 + x^3 - 2x^4}$ to simplest form.

SOLUTION First remove the common factors:

$$\frac{2x^3 + 4x^2 - 30x}{15x^2 + x^3 - 2x^4} = \frac{2x(x^2 + 2x - 15)}{x^2(15 + x - 2x^2)}$$

Factor each of the expressions in parentheses:

$$\frac{2x(x^2 + 2x - 15)}{x^2(15 + x - 2x^2)} = \frac{2x(x + 5)(x - 3)}{x^2(5 + 2x)(3 - x)}$$

Notice that $x - 3 = -(3 - x)$ and rewrite the numerator:

$$\frac{-2x(x + 5)(3 - x)}{x^2(5 + 2x)(3 - x)}$$

Divide both numerator and denominator by $x(3 - x)$, obtaining

$$\frac{-2x(x + 5)(3 - x)}{x^2(5 + 2x)(3 - x)} = \frac{-2(x + 5)}{x(5 + 2x)}$$

and we have the final result:

$$\frac{2x^3 + 4x^2 - 30x}{15x^2 + x^3 - 2x^4} = \frac{-2(x + 5)}{x(5 + 2x)}$$

EXERCISE SET 7.4

In Exercises 1–10, multiply the numerator and denominator of each fraction by the given factor. Check the cross-products to verify that your answer is equivalent to the given fraction.

1. $\frac{7}{8}$ (by 5)

2. $\frac{-5}{9}$ (by -4)

3. $\dfrac{x}{y}$ (by a)

4. $\dfrac{r}{t}$ (by z)

5. $\dfrac{x^2y}{a}$ (by $3ax$)

6. $\dfrac{a^3b}{ca}$ (by $3ab$)

7. $\dfrac{4}{x-y}$ (by $x + y$)

8. $\dfrac{a+b}{4}$ (by $a - 4$)

9. $\dfrac{a+b}{a-b}$ (by $a + b$)

10. $\dfrac{x-2}{x+3}$ (by $x - 3$)

In Exercises 11–18, divide the numerator and denominator of each of the fractions by the given factor. Check the cross-products to verify that your answer is equivalent to the given fraction.

11. $\frac{38}{24}$ (by 2)

12. $\frac{51}{119}$ (by 17)

13. $\dfrac{3x^2}{12x}$ (by $3x$)

14. $\dfrac{15a^3x^2}{3a^4x}$ (by $3a^3x$)

15. $\dfrac{4(x+2)}{(x+2)(x-3)}$ (by $x + 2$)

16. $\dfrac{7(x-3)(x+5)}{14(x+5)(x-1)}$ [by $7(x + 5)$]

17. $\dfrac{x^2-16}{x^2+8x+16}$ (by $x + 4$)

18. $\dfrac{(x-a)(x-b)(x-c)}{(x-c)(x-b)(x-d)}$ [by $(x - c)(x - b)$]

In Exercises 19–40, reduce each fraction to lowest terms.

19. $\dfrac{4x^2}{12x}$

20. $\dfrac{9y}{3y^2}$

21. $\dfrac{x^2+3x}{x^3+5x}$

22. $\dfrac{y^2-4y}{2y+y^3}$

23. $\dfrac{6m^2-3m^3}{9m+18m^3}$

24. $\dfrac{4r^2+12r^3}{8r+12r^2}$

25. $\dfrac{x^2+3x}{x^2-9}$

26. $\dfrac{a^2-9a}{a^2-81}$

27. $\dfrac{2b^2-10b}{3b^2-75}$

28. $\dfrac{4e^2-196}{14e-2e^2}$

29. $\dfrac{z^2-9}{z^2-6z+9}$

30. $\dfrac{x^2-16}{x^2+8x+16}$

31. $\dfrac{x^2+4x+3}{x^2+7x+12}$

32. $\dfrac{a^2-5a+6}{a^2+5a-14}$

33. $\dfrac{2x^2+9x+4}{x^2+9x+20}$

34. $\dfrac{15m^2-22m-5}{3m^2+4m-15}$

35. $\dfrac{12y^3+12y^2+3y}{6y^2-3y-3}$

36. $\dfrac{45x^2-60x+20}{6x^2+5x-6}$

37. $\dfrac{x^3y^6-y^3x^6}{2x^3y^4-2x^4y^3}$

38. $\dfrac{x^3-y^3}{y^2-x^2}$

39. $\dfrac{x^2-y^2}{x+y}$

40. $\dfrac{y-x}{x^2-y^2}$

Solve Exercises 41–44.

41. ***Construction*** The safe load, p (in pounds), when using a drop hammer pile driver, can be determined by the formula:

$$p = \frac{6whs + 6wh}{3s^2 + 6s + 3}$$

Simplify this expression.

42. ***Construction*** The safe load, p (in pounds), when using a steam pile driver, can be determined by the formula:

$$p = \frac{2whs + 2whk + 2amhs + 2amhk - 2bhs - 2bhk}{s^2 + 2sk + k^2}$$

Simplify this expression.

43. ***Physics*** The change in volume, ΔV, of a gas under constant pressure involves the equation:

$$\Delta V = V_1\left(1 + \frac{T_2 - T_1}{T_1}\right) - V_2\left(\frac{T_2 - T_1}{T_2} - 1\right)$$

Simplify the right-hand side of this equation.

44. ***Physics*** Suppose that two elastic bodies with masses m and m_0, respectively, collide. If each body was moving at velocity v toward the other before the collision, then the rebound velocity of the body with mass m_0 is given by

$$\left(\frac{m}{m + m_0} - \frac{m_0}{m + m_0}\right)v + \frac{2vm_0}{m + m_0}$$

Simplify this expression.

[IN YOUR WORDS]

45. Explain the Fundamental Principle of Fractions.

46. Describe how to use the Fundamental Principle of Fractions to simplify a fraction.

7.5 MULTIPLICATION AND DIVISION OF FRACTIONS

The ability to simplify fractions is a skill that will be helpful in this section and the next, as we learn to operate with fractions. After that, it is a skill that will be required throughout this book.

In Section 1.2, we learned the basic operations with real numbers. Among those were Rules 7 and 8, which dealt with multiplying and dividing rational numbers. To refresh your memory, they are repeated here.

RULE 7

To multiply two rational numbers, multiply the numerators and multiply the denominators.

EXAMPLE 7.50

$$\frac{3}{4} \times \frac{-5}{8} = \frac{3 \times (-5)}{4 \times 8} = \frac{-15}{32}$$

RULE 8

To divide one rational number by another, multiply the first by the reciprocal of the second.

EXAMPLE 7.51

Compute $\frac{-5}{8} \div \frac{2}{3}$.

SOLUTION The reciprocal of $\frac{2}{3}$ is $\frac{3}{2}$, so

$$\frac{-5}{8} \div \frac{2}{3} = \frac{-5}{8} \times \frac{3}{2} = \frac{(-5)(3)}{(8)(2)} = \frac{-15}{16}$$

NOTE You do not need common denominators when multiplying or dividing two rational numbers.

If we express these two rules symbolically, we will have the rules for multiplying or dividing any two fractions, whether they are rational numbers or algebraic fractions. The rule for multiplying two rational numbers can be restated as the following.

MULTIPLYING FRACTIONS

If $\frac{a}{b}$ and $\frac{c}{d}$ are fractions, then their product is

$$\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$$

Similarly, we can state the following rule for dividing one rational number by another.

DIVIDING FRACTIONS

If $\frac{a}{b}$ and $\frac{c}{d}$ are fractions, then their quotient is

$$\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \times \frac{d}{c} = \frac{ad}{bc}$$

EXAMPLE 7.52

Find the product of $\frac{3a^2}{x}$ and $\frac{7y}{5p^3}$.

SOLUTION $\frac{3a^2}{x} \cdot \frac{7y}{5p^3} = \frac{(3a^2)(7y)}{(x)(5p^3)} = \frac{21a^2y}{5xp^3}$

EXAMPLE 7.53

Find the product of $\frac{x-2}{x+3}$ and $\frac{x+2}{x-5}$.

SOLUTION

$$\frac{x-2}{x+3} \cdot \frac{x+2}{x-5} = \frac{(x-2)(x+2)}{(x+3)(x-5)}$$
$$= \frac{x^2-4}{x^2-2x-15}$$

EXAMPLE 7.54

Find the quotient when $\frac{7x^2}{3a}$ is divided by $\frac{5y}{4x}$.

SOLUTION

$$\frac{7x^2}{3a} \div \frac{5y}{4x} = \frac{7x^2}{3a} \cdot \frac{4x}{5y}$$
$$= \frac{(7x^2)(4x)}{(3a)(5y)}$$
$$= \frac{28x^3}{15ay}$$

EXAMPLE 7.55

Find $\frac{x-2}{x+3} \div \frac{x-4}{x-3}$.

SOLUTION

$$\frac{x-2}{x+3} \div \frac{x-4}{x-3} = \frac{x-2}{x+3} \cdot \frac{x-3}{x-4} = \frac{(x-2)(x-3)}{(x+3)(x-4)}$$
$$= \frac{x^2-5x+6}{x^2-x-12}$$

NOTE It is often beneficial to leave the answer in factored form. For example, in Example 7.55, you might have wanted to leave the answer as $\frac{(x-2)(x-3)}{(x+3)(x-4)}$. This version has two multiplication operations, one division operation, three subtraction operations, and one addition operation, for a total of seven operations. The answer $\frac{x^2-5x+6}{x^2-x-12}$ has 3 multiplication operations, one division operation, three subtractions, and one addition, for a total of eight operations. In computer programming, more operations require more computer time, which costs more money.

CAUTION When you divide, make sure that you multiply by the reciprocal of the *divisor*. Do not invert the dividend.

The following examples use all of the skills we have learned for simplifying fractions.

EXAMPLE 7.56

Multiply $\dfrac{x^2 - 9}{4x - 8}$ and $\dfrac{2x + 8}{x + 3}$.

SOLUTION If we proceed as before, we get

$$\frac{x^2 - 9}{4x - 8} \cdot \frac{2x + 8}{x + 3} = \frac{(x^2 - 9)(2x + 8)}{(4x - 8)(x + 3)} = \frac{2x^3 + 8x^2 - 18x - 72}{4x^2 + 4x - 24}$$

Although this is correct, it is not the easiest approach. It is a good idea to study a problem for a few seconds before you start to work it. If we had stopped to factor these fractions we could have saved some work:

$$\begin{aligned}
\frac{x^2 - 9}{4x - 8} \cdot \frac{2x + 8}{x + 3} &= \frac{(x + 3)(x - 3)}{4(x - 2)} \cdot \frac{2(x + 4)}{(x + 3)} \\
&= \frac{2(x + 3)(x - 3)(x + 4)}{4(x - 2)(x + 3)} \\
&= \frac{(x - 3)(x + 4)}{2(x - 2)} \\
\text{or} \qquad &= \frac{x^2 + x - 12}{2x - 4}
\end{aligned}$$

EXAMPLE 7.57

Compute $\dfrac{2x^2 + 9x - 5}{3x^2 - 3x - 60} \cdot \dfrac{3x + 12}{2x + 10}$.

SOLUTION

$$\begin{aligned}
\frac{2x^2 + 9x - 5}{3x^2 - 3x - 60} \cdot \frac{3x + 12}{2x + 10} &= \frac{(2x - 1)(x + 5)(3)(x + 4)}{3(x - 5)(x + 4)(2)(x + 5)} \\
&= \frac{2x - 1}{2(x - 5)}
\end{aligned}$$

The common factor $3(x + 5)(x + 4)$ is easily seen using this procedure.

CAUTION Be sure to factor the numerator and denominator first. Only common factors can be "canceled."

HINT When a polynomial is factored, all the $+$ and $-$ signs are inside parentheses.

EXAMPLE 7.58

Compute $\dfrac{x^2 - 9}{6x^2 - 21x} \div \dfrac{(x + 3)^2}{2x - 7}$.

SOLUTION

$$\begin{aligned}\frac{x^2 - 9}{6x^2 - 21x} \div \frac{(x + 3)^2}{2x - 7} &= \frac{x^2 - 9}{6x^2 - 21x} \cdot \frac{2x - 7}{(x + 3)^2} \\ &= \frac{(x - 3)(x + 3)(2x - 7)}{3x(2x - 7)(x + 3)(x + 3)} \\ &= \frac{x - 3}{3x(x + 3)} = \frac{x - 3}{3x^2 + 9x}\end{aligned}$$

There are times when it is just as useful to leave the final answer in the factored form rather than multiplying the factors together. Thus, we could have left the last answer in the form $\dfrac{x - 3}{3x(x + 3)}$.

EXAMPLE 7.59

Simplify $\dfrac{\dfrac{6x}{x^2 - 4}}{\dfrac{2x^2 + 10x}{x + 2}}$.

SOLUTION We have to remember that $\frac{a}{b}$ means $a \div b$. So, this is the division problem:

$$\begin{aligned}\frac{6x}{x^2 - 4} \div \frac{2x^2 + 10x}{x + 2} &= \frac{6x}{x^2 - 4} \cdot \frac{x + 2}{2x^2 + 10x} \\ &= \frac{6x}{(x - 2)(x + 2)} \cdot \frac{x + 2}{2x(x + 5)} \\ &= \frac{6x(x + 2)}{(x - 2)(x + 2)2x(x + 5)} \\ &= \frac{3}{(x - 2)(x + 5)}\end{aligned}$$

EXERCISE SET 7.5

In Exercises 1–46, perform the indicated operation and simplify.

1. $\dfrac{2}{x} \cdot \dfrac{5}{y}$
2. $\dfrac{4}{y} \cdot \dfrac{x}{3}$
3. $\dfrac{4x^2}{5} \cdot \dfrac{3}{y^3}$
4. $\dfrac{7x}{6} \cdot \dfrac{5y}{2t}$
5. $\dfrac{3}{x} \div \dfrac{7}{y}$
6. $\dfrac{a}{3} \div \dfrac{b}{4}$
7. $\dfrac{2x^2}{3} \div \dfrac{7y}{4x}$
8. $\dfrac{9x}{2y} \div \dfrac{4y}{3a}$
9. $\dfrac{2x}{3y} \cdot \dfrac{5}{4x^2}$

10. $\dfrac{3xy}{7} \cdot \dfrac{14x}{5y^2}$

11. $\dfrac{3a^2b}{5d} \cdot \dfrac{25ad^2}{6b^2}$

12. $\dfrac{x^2y^2t}{abc} \cdot \dfrac{b^2c}{y^3t}$

13. $\dfrac{3y}{5x} \div \dfrac{15x^2}{8xy}$

14. $\dfrac{4y^2}{7x} \div \dfrac{8y^3}{21x}$

15. $\dfrac{3x^2y}{7p} \div \dfrac{15x^2p}{7y^2}$

16. $\dfrac{9xyz}{7a} \div \dfrac{3ayz}{14z}$

17. $\dfrac{4y + 16}{5} \cdot \dfrac{15y}{3y + 12}$

18. $\dfrac{x^2 + 3x}{6a} \cdot \dfrac{a^2}{x^2 - 9}$

19. $\dfrac{a^2 - b^2}{a + 3b} \cdot \dfrac{5a + 15b}{a + b}$

20. $(x + y)\dfrac{x^2 + 2x}{x^2 - y^2}$

21. $\dfrac{x^2 - 100}{10} \div \dfrac{2x + 10}{15}$

22. $\dfrac{5a^2}{x^2 - 49} \div \dfrac{25ax - 25a}{x^2 + 7x}$

23. $\dfrac{4x^2 - 1}{9x - 3x^2} \div \dfrac{2x + 1}{x^2 - 9}$

24. $\dfrac{x + y}{3x - 3y} \div \dfrac{(x + y)^2}{x^2 - y^2}$

25. $\dfrac{a^2 - 8a}{a - 8} \cdot \dfrac{a + 2}{a}$

26. $\dfrac{49 - x^2}{x + y} \cdot \dfrac{x}{7 - x}$

27. $\dfrac{2a - b}{4a} \cdot \dfrac{2a - b}{4a^2 - 4ab + b^2}$

28. $\dfrac{x^4 - 81}{(x - 3)^2} \cdot \dfrac{x - 3}{4 - x^2}$

29. $\dfrac{y^2}{x^2 - 1} \div \dfrac{y^2}{x - 1}$

30. $\dfrac{m^2 - 49}{m^2 - 5m - 14} \div \dfrac{m + 7}{2m^2 - 13m - 7}$

31. $\dfrac{2y^2 - y}{4y^2 - 4y + 1} \div \dfrac{y^2}{8y - 4}$

32. $\dfrac{a - 1}{a^2 - 1} \div \dfrac{(a - 1)^2}{a^2 - 1}$

33. $\dfrac{x^2 - 3x + 2}{x^2 + 5x + 6} \cdot \dfrac{x + 3}{3x - 6}$

34. $\dfrac{2x + 2}{x^2 + 2x - 8} \cdot \dfrac{x^2 - 4}{x^2 + 4x + 4}$

35. $\dfrac{x^2 + xy - 6y^2}{x^2 + 6xy + 8y^2} \cdot \dfrac{x^2 - 9xy + 20y^2}{x^2 - 4xy - 21y^2}$

36. $\dfrac{y^2 + 14xy + 49x^2}{y^2 - 7xy - 30x^2} \cdot \dfrac{y^2 - 100x^2}{y^3 + 7xy^2}$

37. $\dfrac{9x^2 - 25}{x^2 + 6x + 9} \div \dfrac{3x + 5}{x + 3}$

38. $\dfrac{x^2 - 16}{x^2 - 6x + 8} \div \dfrac{x^3 + 4x^2}{x^2 - 9x + 14}$

39. $\dfrac{x^2 + 4xy + 4y^2}{x^2 - 4y^2} \div \dfrac{x^2 + xy - 2y^2}{x^2 - xy - 2y^2}$

40. $\dfrac{p^3 - 27q^3}{3p^2 + 9pq + 27q^2} \div \dfrac{9q^2 - p^2}{6p + 18q}$

41. $\dfrac{x + y}{4x - 4y} \div \left[\dfrac{(x + y)^2}{x^2 - y^2} \cdot \dfrac{x^3 - y^3}{x^3 + y^3}\right]$

42. $\dfrac{2.4m^2n}{0.8mn^2} \div \left[\dfrac{0.6m}{0.3n} \cdot \dfrac{3.6m}{2.4n}\right]$

43. $\dfrac{x^2 - 25}{5x^2 - 24x - 5} \cdot \dfrac{2x^2 + 12x + 2}{6x^2 - 12x} \div \dfrac{x^2 + 6x + 1}{5x^2 - 9x - 2}$

44. $\dfrac{a - b}{a^3 - b^3} \cdot \dfrac{a^3 + ab^2}{4a + 4b} \div \dfrac{a^4 - b^4}{8a^2 - 8b^2}$

45. $\left(\dfrac{2x^2 - 5x - 3}{x^2 - x - 12} \div \dfrac{2x^2 + 5x + 2}{3x + 9}\right) \div \dfrac{x^2 - 9}{x^2 - 2x - 8}$

46. $\dfrac{a^2 + 4a - 5}{a^2 - 3a - 4} \cdot \dfrac{a^2 + 3a - 28}{(a - 3)^2} \div \dfrac{a^2 + 12a + 35}{a^2 - 2a - 3}$

Solve Exercises 47–50.

47. ***Construction*** The volume strain of a beam is given by the expression:

$$\frac{(a^3 - a'^3)/a^3}{(a^2 - a'^2)/a^2}$$

Simplify this expression.

48. ***Electronics*** The charge on a capacitor is given by $Q = CV$. The energy stored in the capacitor is given by $E = \frac{1Q^2}{2V}$. Find the ratio of charge to energy.

49. ***Physics*** The quotient

$$\frac{8\pi ne^2w}{mv^2 - mvw^2} \div \frac{2mv^2 - 2\pi ne^2}{mv^2}$$

is used in the study of electromagnetic processes in space. Divide and simplify the expression.

50. ***Physics*** In a mass spectrometer the radius of curvature of a charged particle depends on its mass. If the instrument is properly calibrated, then measuring the radius allows the mass of small particles to be measured. The mass spectrometer uses the equation $R = \frac{my}{qB}$ and the frequency of oscillation is given by $f = \frac{v}{2\pi R}$. Substitute the value of R from the first equation into the second equation, and then divide and simplify the expression.

[IN YOUR WORDS]

51. **(a)** Explain how to divide two fractions: $\frac{a}{b} \div \frac{c}{d}$.

(b) What do you think is the most common mistake people make when they divide two fractions?

52. **(a)** Explain how to multiply two fractions: $\frac{a}{b} \times \frac{c}{d}$.

(b) What do you think is the most common mistake people make when they multiply two fractions?

7.6 ADDITION AND SUBTRACTION OF FRACTIONS

In Section 7.5, we learned how to multiply and divide two fractions. In this section, we will look at two other operations with fractions—addition and subtraction.

As we mentioned earlier, much of this work with algebraic fractions is patterned after our work with rational numbers from Section 1.2. Rule 6 dealt with the addition and subtraction of rational numbers. This rule is repeated here.

RULE 6

To add (or subtract) two rational numbers, change both denominators to the same positive integer (the common denominator), add (or subtract) the numerators, and place the result over the common denominator.

EXAMPLE 7.60

Perform the indicated operations and simplify (a) $\frac{2}{3} + \frac{-5}{6}$ and (b) $\frac{-5}{7} - \frac{-8}{3}$.

SOLUTIONS

(a) A common denominator of 3 and 6 is 6, so

$$\frac{2}{3} + \frac{-5}{6} = \frac{4}{6} + \frac{-5}{6} = \frac{4 + (-5)}{6} = \frac{-1}{6}$$

(b) A common denominator of 7 and 3 is 21, so

$$\frac{-5}{7} = \frac{-15}{21} \quad \text{and} \quad \frac{-8}{3} = \frac{-56}{21}$$

As a result, we obtain

$$\frac{-5}{7} - \frac{-8}{3} = \frac{-15}{21} - \frac{-56}{21} = \frac{-15 - (-56)}{21} = \frac{41}{21}$$

Before we restate Rule 6 in symbols, we will consider a special case of adding or subtracting fractions. If two fractions have the same denominator, then you need only add or subtract the numerators. Symbolically, this is represented as

$$\frac{a}{c} + \frac{b}{c} = \frac{a + b}{c} \quad \text{and} \quad \frac{a}{c} - \frac{b}{c} = \frac{a - b}{c}$$

EXAMPLE 7.61

Simplify (a) $\frac{7}{2x} + \frac{9y}{2x}$ and (b) $\frac{x}{x + 5} - \frac{2x - y}{x + 5}$.

SOLUTIONS

(a) $$\frac{7}{2x} + \frac{9y}{2x} = \frac{7 + 9y}{2x}$$

(b) $$\frac{x}{x + 5} - \frac{2x - y}{x + 5} = \frac{x - (2x - y)}{x + 5} = \frac{x - 2x + y}{x + 5} = \frac{-x + y}{x + 5}$$

If the denominators are not the same, then addition and subtraction become somewhat more complicated. As Rule 6 indicates, you need to find a common denominator and rewrite each fraction as an equivalent fraction with this common denominator. The quickest way to find a common denominator is to multiply the denominators together. This method is demonstrated next.

CAUTION When adding or subtracting fractions, remember to multiply both the numerator and denominator of a fraction by the same quantity.

ADDING AND SUBTRACTING FRACTIONS

The sum of two fractions $\frac{a}{b}$ and $\frac{c}{d}$ is

$$\frac{a}{b} + \frac{c}{d} = \frac{ad}{bd} + \frac{bc}{bd} = \frac{ad + bc}{bd}$$

The difference of two fractions $\frac{a}{b}$ and $\frac{c}{d}$ is

$$\frac{a}{b} - \frac{c}{d} = \frac{ad}{bd} - \frac{bc}{bd} = \frac{ad - bc}{bd}$$

As you will see in Examples 7.63 and 7.66, this may not produce the lowest common denominator. You must also remember that both the numerator and denominator must be multiplied by the same quantity.

EXAMPLE 7.62

Simplify $\frac{3}{x+5} + \frac{x}{x-5}$.

SOLUTION

$$\begin{aligned}
\frac{3}{x+5} + \frac{x}{x-5} &= \frac{3}{x+5} \cdot \frac{x-5}{x-5} + \frac{x}{x-5} \cdot \frac{x+5}{x+5} \\
&= \frac{3(x-5)}{(x+5)(x-5)} + \frac{x(x+5)}{(x+5)(x-5)} \\
&= \frac{3(x-5) + (x^2+5x)}{(x+5)(x-5)} \\
&= \frac{3x - 15 + x^2 + 5x}{(x+5)(x-5)} \\
&= \frac{x^2 + 8x - 15}{x^2 - 25}
\end{aligned}$$

EXAMPLE 7.63

Simplify $\frac{2x}{x+3} - \frac{x-4}{x^2-9}$.

SOLUTION A common denominator of these fractions is $(x+3)(x^2-9)$. We begin by rewriting each fraction with this common denominator and then subtract the two fractions:

$$\begin{aligned}
\frac{2x}{x+3} - \frac{x-4}{x^2-9} &= \frac{2x}{x+3} \cdot \frac{x^2-9}{x^2-9} - \frac{x-4}{x^2-9} \cdot \frac{x+3}{x+3} \\
&= \frac{2x(x^2-9)}{(x+3)(x^2-9)} - \frac{(x-4)(x+3)}{(x+3)(x^2-9)}
\end{aligned}$$

EXAMPLE 7.63 (Cont.)

$$= \frac{(2x^3 - 18x) - (x^2 - x - 12)}{(x + 3)(x^2 - 9)}$$

$$= \frac{2x^3 - x^2 - 17x + 12}{x^3 + 3x^2 - 9x - 27}$$

We will see this same problem later in Example 7.66. At that time you will not only see an easier way to work the problem but that it simplifies to $\dfrac{2x^2 - 7x + 4}{x^2 - 9}$.

LEAST COMMON DENOMINATOR

Perhaps you wondered if the last answer was in simplest form. It is not, since

$$\frac{2x^3 - x^2 - 17x + 12}{x^3 + 3x^2 - 9x - 27} = \frac{(x + 3)(2x^2 - 7x + 4)}{(x + 3)(x^2 - 9)} = \frac{2x^2 - 7x + 4}{x^2 - 9}$$

If we want our answers in the simplest form, then simply multiplying the denominators together is not the best method to use. What we need to do is determine the **least common denominator**, or LCD, of the fractions to be added or subtracted.

There are three steps to determining the LCD:

HOW TO FIND THE LEAST COMMON DENOMINATOR

(1) Factor the denominator of each of the fractions in the problem.

(2) Determine the different factors and the highest power of each factor that occurs in any denominator.

(3) Multiply the distinct factors from Step 2 after each has been raised to its highest power.

EXAMPLE 7.64

Find the LCD of the fractions $\dfrac{7x + 1}{x^4 + x^3}$, $\dfrac{14}{x^3 - 4x^2 + 4x}$, and $\dfrac{9}{2x^2 - 2x - 4}$.

SOLUTION

Step 1: Factor the denominator of each of the fractions: $\dfrac{7x + 1}{x^3(x + 1)}$, $\dfrac{14}{x(x - 2)^2}$, and $\dfrac{9}{2(x + 1)(x - 2)}$.

Step 2: List each factor and the highest exponent of each.

Factor	Highest exponent	Final factors
2	1	2^1
x	3	x^3
$x+1$	1	$(x+1)^1$
$x-2$	2	$(x-2)^2$

Step 3: The LCD is $2x^3(x+1)(x-2)^2$.

EXAMPLE 7.65

Find the least common denominator of $\dfrac{2x}{x^2+5x+6}$, $\dfrac{x-3}{x^3+2x^2}$, and $\dfrac{x^2+x}{x^3+6x^2+9x}$.

SOLUTION

Step 1: Factor each denominator: $\dfrac{2x}{(x+2)(x+3)}$, $\dfrac{x-3}{x^2(x+2)}$, and $\dfrac{x^2+x}{x(x+3)^2}$.

Step 2: List each factor and the highest exponent of each.

Factor	Highest exponent	Final factors
$x+2$	1	$x+2$
$x+3$	2	$(x+3)^2$
x	2	x^2

Step 3: The LCD is $x^2(x+2)(x+3)^2$.

Now we have the foundation for a much better way to add or subtract algebraic fractions, or any fractions. For each fraction, multiply both the numerator and denominator by a quantity that makes the denominator equal to the LCD of the fractions being added or subtracted. Then, add or subtract the numerators; place the result over the common denominator; and, if possible, simplify.

EXAMPLE 7.66

Calculate $\dfrac{2x}{x+3} - \dfrac{x-4}{x^2-9}$.

SOLUTION This is the same difference we were asked to compute in Example 7.63. First we find the LCD, which is $(x+3)(x-3)$.

We rewrite the first fraction as $\dfrac{2x}{x+3} = \dfrac{2x(x-3)}{(x+3)(x-3)}$. The second fraction, $\dfrac{x-4}{x^2-9}$, is already written with the common denominator. So,

$$\begin{aligned}\frac{2x}{x+3} - \frac{x-4}{x^2-9} &= \frac{2x(x-3)}{(x+3)(x-3)} - \frac{x-4}{x^2-9}\\ &= \frac{2x(x-3)-(x-4)}{x^2-9}\end{aligned}$$

EXAMPLE 7.66 (Cont.)

$$= \frac{(2x^2 - 6x) - (x - 4)}{x^2 - 9}$$
$$= \frac{2x^2 - 7x + 4}{x^2 - 9}$$

This was the same problem we worked in Example 7.63. Notice how much simpler this answer looks compared to the answer we found before.

EXAMPLE 7.67

Calculate $\frac{2x}{x+2} + \frac{x}{x-2} - \frac{1}{x^2-4}$.

SOLUTION The LCD of these fractions is $(x + 2)(x - 2)$. So,

$$\frac{2x}{x+2} = \frac{2x(x-2)}{(x+2)(x-2)},$$
$$\frac{x}{x-2} = \frac{x(x+2)}{(x-2)(x+2)},$$
$$\text{and } \frac{1}{x^2-4} = \frac{1}{(x-2)(x+2)}.$$

Thus, we get

$$\frac{2x}{x+2} + \frac{x}{x-2} - \frac{1}{x^2-4} = \frac{2x(x-2)}{(x+2)(x-2)} + \frac{x(x+2)}{(x-2)(x+2)} - \frac{1}{x^2-4}$$
$$= \frac{(2x^2 - 4x) + (x^2 + 2x) - 1}{x^2 - 4}$$
$$= \frac{3x^2 - 2x - 1}{x^2 - 4}$$

APPLICATION ENVIRONMENTAL SCIENCE

EXAMPLE 7.68

In order to estimate the runoff from a rainstorm, the formula

$$Q = P - I_a - s + \frac{s^2}{P - I_a + s}$$

is used, where Q is the amount of runoff, P is the rainfall, s is the potential maximum retention after runoff begins, and I_a is the initial abstraction and all values are in inches. Rewrite the right-hand side of this formula as a single fraction.

SOLUTION If we represent $P - I_a$ with x then this formula can be written as

$$Q = x - s + \frac{s^2}{x + s}$$

Writing $x - s$ as a fraction with a denominator of $x + s$ will require the use of the difference of two squares in the numerator:

$$Q = \frac{(x - s)(x + s)}{x + s} + \frac{s^2}{x + s}$$
$$= \frac{x^2 - s^2}{x + s} + \frac{s^2}{x + s}$$
$$= \frac{x^2}{x + s}$$

Replacing x with $P - I_a$ we obtain the final simplification:

$$Q = \frac{(P - I_a)^2}{P - I_a + s}$$

COMPLEX FRACTIONS

A **complex fraction** is a fraction in which the numerator, the denominator, or both contain a fraction. There are two methods that are commonly used to simplify complex fractions.

Method 1: Find the LCD of all the fractions that appear in the numerator and denominator. Multiply both the numerator and denominator by the LCD.

Method 2: Combine the terms in the numerator into a single fraction. Combine the terms in the denominator into a single fraction. Divide the numerator by the denominator.

We will work each of the next two examples using both methods. Then you will be better able to select the method you prefer.

EXAMPLE 7.69

Simplify $\dfrac{2 + \frac{1}{x}}{x - \frac{2}{x^2}}$.

SOLUTION

Method 1: The LCD of 2, $\frac{1}{x}$, x, and $\frac{2}{x^2}$ is x^2, so

$$\frac{2 + \frac{1}{x}}{x - \frac{2}{x^2}} = \frac{2 + \frac{1}{x}}{x - \frac{2}{x^2}} \cdot \frac{x^2}{x^2}$$
$$= \frac{2x^2 + x}{x^3 - 2}$$
$$= \frac{x(2x + 1)}{x^3 - 2}$$

EXAMPLE 7.69 (Cont.)

Method 2:

$$2 + \frac{1}{x} = \frac{2x}{x} + \frac{1}{x} = \frac{2x + 1}{x}$$

$$x - \frac{2}{x^2} = \frac{x^3}{x^2} - \frac{2}{x^2} = \frac{x^3 - 2}{x^2}$$

$$\frac{2 + \frac{1}{x}}{x - \frac{2}{x^2}} = \frac{\frac{2x + 1}{x}}{\frac{x^3 - 2}{x^2}} = \frac{2x + 1}{x} \div \frac{x^3 - 2}{x^2}$$

$$= \frac{2x + 1}{x} \cdot \frac{x^2}{x^3 - 2} = \frac{x(2x + 1)}{x^3 - 2}$$

EXAMPLE 7.70

Simplify $\dfrac{\frac{1}{2x} - \frac{6}{y}}{\frac{1}{x} + \frac{2}{3y}}$.

SOLUTION

Method 1: The LCD of $\frac{1}{2x}, \frac{6}{y}, \frac{1}{x}$, and $\frac{2}{3y}$ is 6xy, so

$$\frac{\frac{1}{2x} - \frac{6}{y}}{\frac{1}{x} + \frac{2}{3y}} = \frac{\left(\frac{1}{2x} - \frac{6}{y}\right)}{\left(\frac{1}{x} + \frac{2}{3y}\right)} \cdot \frac{6xy}{6xy}$$

$$= \frac{3y - 36x}{6y + 4x} = \frac{3(y - 12x)}{2(3y + 2x)}$$

Method 2:

$$\frac{1}{2x} - \frac{6}{y} = \frac{y}{2xy} - \frac{12x}{2xy} = \frac{y - 12x}{2xy}$$

$$\frac{1}{x} + \frac{2}{3y} = \frac{3y}{3xy} + \frac{2x}{3xy} = \frac{3y + 2x}{3xy}$$

$$\frac{\frac{1}{2x} - \frac{6}{y}}{\frac{1}{x} + \frac{2}{3y}} = \frac{\frac{y - 12x}{2xy}}{\frac{3y + 2x}{3xy}}$$

$$= \frac{y - 12x}{2xy} \div \frac{3y + 2x}{3xy}$$

$$= \frac{y - 12x}{2xy} \cdot \frac{3xy}{3y + 2x}$$
$$= \frac{3(y - 12x)}{2(3y + 2x)}$$

APPLICATION ELECTRONICS

EXAMPLE 7.71

If four resistances are connected in a series-parallel circuit, the total resistance R is given by the equation

$$R = \frac{1}{R_1 + R_2} + \frac{1}{R_3 + R_4}$$

where R_1, R_2, R_3, and R_4 represent four resistances. Simplify this equation by adding the right-hand side.

SOLUTION We find that the LCD of the right-hand side of the given equation is $(R_1 + R_2)(R_3 + R_4)$. So,

$$R = \frac{1}{R_1 + R_2} \cdot \frac{R_3 + R_4}{R_3 + R_4} + \frac{1}{R_3 + R_4} \cdot \frac{R_1 + R_2}{R_1 + R_2}$$
$$= \frac{R_3 + R_4}{(R_1 + R_2)(R_3 + R_4)} + \frac{R_1 + R_2}{(R_1 + R_2)(R_3 + R_4)}$$
$$= \frac{R_1 + R_2 + R_3 + R_4}{(R_1 + R_2)(R_3 + R_4)}$$

EXERCISE SET 7.6

In Exercises 1–44, perform the indicated operations and simplify.

1. $\frac{2}{7} + \frac{5}{7}$
2. $\frac{4}{5} + \frac{-11}{5}$
3. $\frac{7}{3} - \frac{5}{3}$
4. $\frac{-2}{9} - \frac{8}{9}$
5. $\frac{1}{2} + \frac{1}{3}$
6. $\frac{3}{4} + \frac{-2}{3}$
7. $\frac{4}{5} - \frac{2}{3}$
8. $-\frac{5}{7} - \frac{3}{5}$
9. $\frac{1}{x} + \frac{5}{x}$
10. $\frac{2}{y} + \frac{-5}{y}$
11. $\frac{4}{a} - \frac{3}{a}$
12. $\frac{-5}{p} - \frac{-7}{p}$
13. $\frac{2x}{y} + \frac{3x}{y}$
14. $\frac{4p}{q} - \frac{6p}{q}$
15. $\frac{3r}{2t} + \frac{-r}{2t} - \frac{5r}{2t}$
16. $\frac{3x}{2y} - \frac{5x}{2y} + \frac{x}{2y}$
17. $\frac{3}{x + 2} + \frac{x}{x + 2}$
18. $\frac{5}{y - 3} + \frac{y}{y - 3}$
19. $\frac{t}{t + 1} - \frac{2}{t + 1}$
20. $\frac{a}{b - 3} - \frac{4}{3 - b}$
21. $\frac{y - 3}{x + 2} + \frac{3 + y}{x + 2}$
22. $\frac{x + 4}{x - 2} + \frac{x - 5}{x - 2}$
23. $\frac{x + 2}{a + b} - \frac{x - 5}{a + b}$
24. $\frac{x + 4}{y - 5} - \frac{2 - x}{y - 5}$

25. $\frac{2}{x} + \frac{3}{y}$

26. $\frac{x}{y} + \frac{5}{x}$

27. $\frac{a}{b} - \frac{4}{d}$

28. $\frac{2x}{y} - \frac{3y}{x}$

29. $\frac{3}{x(x+1)} + \frac{4}{x^2-1}$

30. $\frac{5}{y(x+1)} + \frac{x}{y(x+2)}$

31. $\frac{2}{x^2-1} - \frac{4}{(x+1)^2}$

32. $\frac{6}{y-2} - \frac{3}{y+2}$

33. $\frac{x}{x^2-11x+30} + \frac{2}{x^2-36}$

34. $\frac{a}{a^2-9a+18} + \frac{a}{a^2-9}$

35. $\frac{2}{x^2-x-6} - \frac{5}{x^2-4}$

36. $\frac{b}{b^2-10b+21} - \frac{b}{b^2-9}$

37. $\frac{x-1}{3x^2-13x+4} + \frac{3x+1}{4x-x^2}$

38. $\frac{x-3}{x^2+3x+2} + \frac{2x-5}{x^2+x-2}$

39. $\frac{x-3}{x^2-1} + \frac{2x-7}{x^2+5x+4}$

40. $\frac{x+4}{x^2-9} - \frac{x-3}{x^2+6x+9}$

41. $\frac{y+3}{y^2-y-2} - \frac{2y-1}{y^2+2y-8}$

42. $\frac{1}{(a-b)(a-c)} + \frac{1}{(b-a)(b-c)} - \frac{1}{(b-c)(a-c)}$

43. $\frac{x}{(x^2+3)(x-1)} + \frac{3x^2}{(x-1)^2(x+2)} - \frac{x+2}{x^2+3}$

44. $\frac{2x-1}{x^2+5x+6} - \frac{x-2}{x^2+4x+3} + \frac{x-4}{x^2+3x+2}$

Use Method 1 to simplify each of the complex fractions in Exercises 45–50.

45. $\dfrac{1+\frac{2}{x}}{1-\frac{3}{x}}$

46. $\dfrac{x+\frac{1}{x}}{2-\frac{1}{x}}$

47. $\dfrac{x-1}{1+\frac{1}{x}}$

48. $\dfrac{x^2-25}{\frac{1}{x}-\frac{1}{5}}$

49. $\dfrac{\frac{x}{x+y}-\frac{y}{x-y}}{\frac{x}{x+y}+\frac{y}{x-y}}$

50. $\dfrac{x+3-\frac{16}{x+3}}{x-6+\frac{20}{x+6}}$

Use Method 2 to simplify each of the complex fractions in Exercises 51–56.

51. $\dfrac{1+\frac{3}{x}}{1+\frac{2}{x}}$

52. $\dfrac{y+\frac{1}{y}}{3+\frac{2}{y}}$

53. $\dfrac{t-1}{t+\frac{1}{t}}$

54. $\dfrac{x^2-36}{\frac{1}{6}-\frac{1}{x}}$

55. $\dfrac{\frac{x}{x-y}-\frac{y}{x+y}}{\frac{1}{x-y}+\frac{1}{x+y}}$

56. $\dfrac{t-5+\frac{25}{t-5}}{t+3+\frac{10}{t-3}}$

Solve Exercises 57–64.

57. ***Electronics*** If two resistors, R_1 and R_2, are connected in parallel, the equivalent resistance of the combination can be found using $\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}$. Add the right-hand side of the equation.

58. ***Optics*** The lensmaker's equation states that if p is the object distance, q the image distance, and f the focal length of a lens, then $\frac{1}{p} + \frac{1}{q} = \frac{1}{f}$. Simplify the left-hand side of this equation.

59. ***Electronics*** If three capacitors with capacitance C_1, C_2, and C_3 are connected together in series, then they can be replaced by a single capacitor of capacitance C. The value of C can be determined from the equation $\frac{1}{C} = \frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3}$. Simplify the right-hand side of the equation.

60. ***Transportation*** A car travels the first part of a trip for a distance d_1 at velocity v_1, and the second part of the trip it travels d_2 at the velocity v_2. The average speed for these two parts of the trip is given by $\dfrac{d_1 + d_2}{\frac{d_1}{v_1} + \frac{d_2}{v_2}}$. Simplify this fraction.

61. ***Civil engineering*** The Gordon-Rankine formula for intermediate steel columns is

$$P = A\left(\frac{k}{1 + \frac{L^2}{kr^2}}\right)$$

Simplify the right-hand side of this formula.

62. ***Electrical engineering*** In calculating the electric intensity of a field set up by a dipole, the following expression is used.

$$\frac{1}{\left(r - \frac{d}{2}\right)^2} - \frac{1}{\left(r + \frac{d}{2}\right)^2}$$

Simplify this expression.

63. ***Electrical engineering*** Millman's theorem provides a shortcut for finding the common voltage, V, across any number of parallel branches with different voltage sources. If there are three branches, then the common voltage is

$$V = \frac{\frac{V_1}{R_1} + \frac{V_2}{R_2} + \frac{V_3}{R_3}}{\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}}$$

Simplify the right-hand side of this equation.

64. ***Electronics*** The expression

$$\frac{1}{\left(r - \frac{d}{2}\right)^2} - \frac{1}{\left(r + \frac{d}{2}\right)^2}$$

is used to calculate the electric intensity in a field set up by a dipole. Combine the two terms and simplify the resulting expression.

[IN YOUR WORDS]

65. **(a)** What is a common denominator?

(b) Explain how to find the least common denominator.

66. Describe a complex algebraic fraction. Give examples of fractions that are complex algebraic fractions, and give examples of some that are not. How are they different?

CHAPTER 7 REVIEW

IMPORTANT TERMS AND CONCEPTS

Binomial
Common factor
Denominator
Discriminant
Factor
Fractions
Addition
Complex
Division
Equivalent
Multiplication
Reducing
Subtraction
Least common denominator (LCD)
Numerator
Trinomial

REVIEW EXERCISES

In Exercises 1–10, find the indicated products by direct use of one of the special products.

1. $5x(x - y)$
2. $(3 + x)^2$
3. $(x - 2y)^3$
4. $(x + y)(x - 6)$
5. $(2x + 3)(x - 6)$
6. $(x + 7)(x - 7)$
7. $(x^2 - 5)(x^2 + 5)$
8. $(7x - 1)(x + 5)$
9. $(2 + x)^3$
10. $(x - 7)^2$

Completely factor each of the expressions in Exercises 11–20.

11. $9 + 9y$
12. $x^2 - 4$
13. $7x^2 - 63$
14. $x^2 - 12x + 36$
15. $x^2 - 11x + 30$
16. $x^2 + 15x + 36$
17. $x^2 + 6x - 16$
18. $x^2 - 4x - 45$
19. $2x^2 - 3x - 9$
20. $8x^3 + 6x^2 - 20x$

In Exercises 21–26, reduce each fraction to lowest terms.

21. $\dfrac{2x}{6y}$
22. $\dfrac{7x^2y}{9xy^2}$
23. $\dfrac{x^2 - 9}{(x + 3)^2}$
24. $\dfrac{x^2 - 4x - 45}{x^2 - 81}$
25. $\dfrac{x^3 + y^3}{x^2 + 2xy + y^2}$
26. $\dfrac{x^3 - 16x}{x^2 + 2x - 8}$

In Exercises 27–40, perform the indicated operations and simplify.

27. $\dfrac{x^2}{y} \cdot \dfrac{3y^2}{7x}$
28. $\dfrac{x^2 - 9}{x + 4} \cdot \dfrac{x^3 - 16x}{x - 3}$
29. $\dfrac{4x}{3y} \div \dfrac{2x^2}{6y}$
30. $\dfrac{x^2 - 25}{x^2 - 4x} \div \dfrac{2x^2 + 2x - 40}{x^3 - x}$

31. $\dfrac{4x}{y} + \dfrac{3x}{y}$

32. $\dfrac{4}{x-y} + \dfrac{6}{x+y}$

33. $\dfrac{3(x-3)}{(x+2)(x-5)^2} + \dfrac{4(x-1)}{(x+2)^2(x-5)}$

34. $\dfrac{8a}{b} - \dfrac{3}{b}$

35. $\dfrac{x}{y+x} - \dfrac{x}{y-x}$

36. $\dfrac{2(x+3)}{(x+1)^2(x+2)} - \dfrac{3(x-1)}{(x+1)(x+2)^2}$

37. $\dfrac{x^2-5x-6}{x^2+8x+12} + \dfrac{x^2+7x+6}{x^2-4x-12}$

38. $\dfrac{2x-1}{4x^2-12x+5} - \dfrac{x+1}{4x^2-4x-15}$

39. $\dfrac{x^2-5x-6}{x^2+8x+12} \div \dfrac{x^2+7x+6}{x^2-4x-12}$

40. $\dfrac{x^2+x-2}{7a^2x^2-14a^2x+7a^2} \cdot \dfrac{14ax-28a}{1-2x+x^2}$

Simplify each of the complex fractions in Exercises 41–46.

41. $\dfrac{\dfrac{1}{x}-\dfrac{1}{y}}{\dfrac{1}{x}+\dfrac{1}{y}}$

42. $\dfrac{\dfrac{1}{x}+\dfrac{1}{y}}{x+y}$

43. $\dfrac{\dfrac{1}{x}-\dfrac{1}{y}}{\dfrac{x-y}{xy}}$

44. $\dfrac{1-\dfrac{1}{x}}{x-2+\dfrac{1}{x}}$

45. $\dfrac{\dfrac{x}{1+x}-\dfrac{1-x}{x}}{\dfrac{x}{1+x}+\dfrac{1-x}{x}}$

46. $\dfrac{x-\dfrac{xy}{x-y}}{\dfrac{x^2}{x^2-y^2}-1}$

CHAPTER 7 TEST

1. Multiply $(x+5)(x-3)$.
2. Multiply $(2x-3)(2x+3)$.
3. Multiply $(3x^2-4)(2-5x)$.
4. Multiply $(x-4)^3$.
5. Completely factor $2x^2-128$.
6. Completely factor $x^2-12x+32$.
7. Completely factor $10x^2+x-21$.
8. Completely factor x^3-125.
9. Reduce $\dfrac{x^2-25}{x^2+6x+5}$ to lowest terms.
10. Simplify $\dfrac{3(a+b)^3-x(a+b)}{a^2-b^2}$.
11. Calculate $\dfrac{3x}{x+2} \cdot \dfrac{x-1}{x+2}$.
12. Calculate $\dfrac{2x+6}{x-2} \div \dfrac{3x+9}{x^2-4}$.
13. Calculate $\dfrac{6}{x-5} + \dfrac{x^2-2x}{x-5}$.
14. Calculate $\dfrac{2x}{x+3} - \dfrac{x+4}{x-2}$.
15. Simplify $\dfrac{x-\dfrac{1}{x}}{x-\dfrac{2}{x+1}}$.
16. Reduce $\dfrac{1}{2x+1} - \dfrac{2}{4x^2+4x+1}$.

17. The average rate, r, for a round trip with a one-way distance d is

$$r = \frac{2d}{\dfrac{d}{r_1} + \dfrac{d}{r_2}}$$

Simplify this complex fraction.

18. The total resistance, R, in a parallel electrical circuit with three resistances is given by

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}$$

Express the sum on the right-hand side as a single fraction.

8 VECTORS AND TRIGONOMETRIC FUNCTIONS

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An accident investigator might use a special wheel tape to measure skid marks at an accident. In Section 8.1, you will use vectors to determine the vertical and horizontal forces the investigator exerts on the wheel.

Many applications of mathematics involve quantities that have both a magnitude (or size) and a direction. The quantities include force, displacement, velocity, acceleration, torque, and magnetic flux density. In this chapter, we will focus on ways to represent these quantities and how they can be used in conjunction with trigonometry to solve problems. These methods will require that you learn about vectors and how to use them with trigonometry.

OBJECTIVES

AFTER COMPLETING THIS CHAPTER, YOU WILL BE ABLE TO:

- Resolve a vector into components.
- Combine the components of a vector into a resultant.
- Solve practical problems using vectors.
- Solve oblique triangles using the law of sines.
- Solve oblique triangles using the law of cosines.
- Solve applied problems requiring oblique triangles.

8.1 INTRODUCTION TO VECTORS

Our study will begin with the introduction of two quantities: scalars and vectors. After we finish this introductory material, you will be ready to learn some of the basic operations with scalars and vectors. We will use these operations to help solve some applied problems.

SCALARS

Quantities that have size or magnitude, but no direction are called **scalars**. Time, volume, mass, speed, distance, and temperature are some examples of scalars.

VECTORS

A **vector** is a quantity that has both magnitude and direction. The **magnitude of the vector** is indicated by its length. The **direction of a vector** is often given by an angle. A vector is usually pictured as an arrow with the arrowhead pointing in the direction of the vector. Force, velocity, acceleration, torque, and electric and magnetic fields are all examples of vectors.

Vectors are usually represented with boldface letters such as **a** or **A** or by letters with arrows over them, $\vec{a}$ or $\vec{A}$. If a vector extends from a point A, called the **initial point**, to a point B, called the **terminal point**, then the vector can be represented by $\overrightarrow{AB}$ with the arrowhead over the terminal point B. In written work, it is hard to write in boldface, so you should use $\vec{a}$ or $\vec{b}$ or $\overrightarrow{AB}$.

The magnitude of a vector **A** is usually denoted by |**A**| or A. The magnitude of a vector is never negative. Two vectors, $\overrightarrow{AB}$ and $\overrightarrow{CD}$, are *equal* or *equivalent*, if they have the same magnitude and direction, and we write $\overrightarrow{AB} = \overrightarrow{CD}$. In Figure 8.1, $\overrightarrow{AB} = \overrightarrow{CD}$ but $\overrightarrow{AB} \neq \overrightarrow{EF}$ because, although they have the same magnitude,

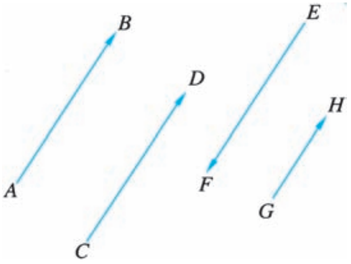

Figure 8.1

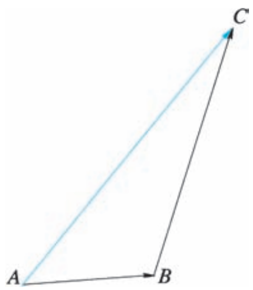

Figure 8.2a

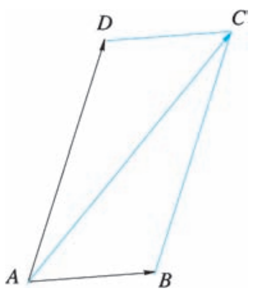

Figure 8.2b

they are not in the same direction. Also, $\overrightarrow{AB} \neq \overrightarrow{GH}$, because they do not have the same magnitude, even though they are in the same direction.

RESULTANT VECTORS

If $\overrightarrow{AB}$ is the vector from A to B and $\overrightarrow{BC}$ is the vector from B to C, then the vector $\overrightarrow{AC}$ is the **resultant vector** and represents the sum of $\overrightarrow{AB}$ and $\overrightarrow{BC}$.

$$\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC}$$

The sum of two vectors is shown geometrically in Figure 8.2a. It is also possible to add vectors such as $\overrightarrow{AB} + \overrightarrow{AD}$, as shown in Figure 8.2b. Here $\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{AD}$. As you can probably guess from the shape of Figure 8.2b, this method of adding vectors is called the **parallelogram method**.

The vectors that are being added do not need an endpoint in common. For example, to find $\overrightarrow{AB} + \overrightarrow{EF}$ in Figure 8.3a we would find the vector $\overrightarrow{BC}$, which is equivalent to $\overrightarrow{EF}$. That is, $\overrightarrow{BC} = \overrightarrow{EF}$. We are then back to the first method and $\overrightarrow{AB} + \overrightarrow{EF} = \overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$, as shown in Figure 8.3b.

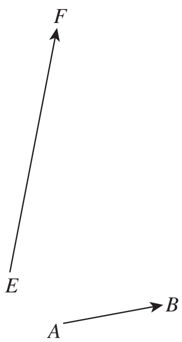

Figure 8.3a

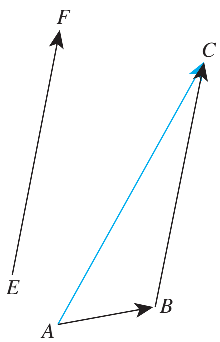

Figure 8.3b

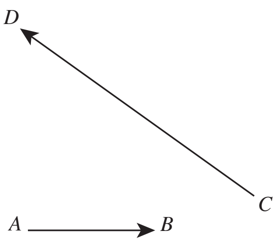

Figure 8.4a

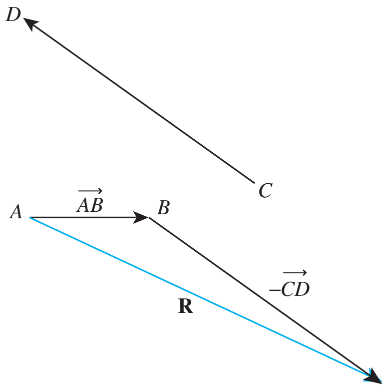

Figure 8.4b

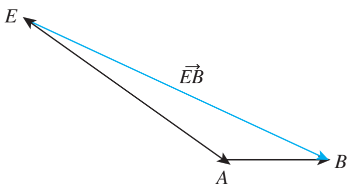

Figure 8.5

Vectors can also be subtracted. Here you need to realize that the vector **–V** has the opposite direction of the vector **V**. An example of vector subtraction, $\overrightarrow{AB} - \overrightarrow{CD} = \vec{R}$, is shown in Figures 8.4a and 8.4b.

Another way to represent subtraction of vectors is shown in Figure 8.5. If we want to subtract $\overrightarrow{AB} - \overrightarrow{CD}$ as in Figure 8.4b, we could draw $\overrightarrow{AE}$, where $\overrightarrow{AE} = \overrightarrow{CD}$. Then $\overrightarrow{EB}$ would be the vector that represents $\overrightarrow{AB} - \overrightarrow{CD} = \overrightarrow{AB} - \overrightarrow{AE}$. Notice that in Figure 8.5, $\overrightarrow{EB}$ is drawn from the terminal point of the second vector in the difference ($\overrightarrow{AE}$) to the terminal point of the first vector ($\overrightarrow{AB}$).

A vector can be multiplied by a scalar. Thus, 2**a** has twice the magnitude but the same direction as **a** and 5**a** has five times the magnitude and the same direction as **a** (see Figure 8.6).

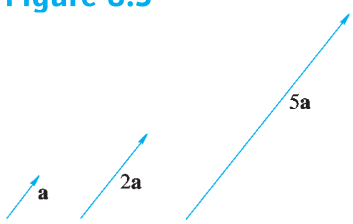

Figure 8.6

EXAMPLE 8.1

Use the vectors **A** and **B** in Figure 8.7a to determine vector 2**A** + **B**.

SOLUTION We begin by sketching **A** and placing it so that its initial point is at the terminal point of the original **A**. This new vector is 2**A** and is shown in Figure 8.7b.

Next, we move vector **B** so that its initial point is at the terminal point of vector 2**A** as shown in Figure 8.7c. The resultant vector, 2**A** + **B**, is drawn from the initial point of 2**A** to the terminal point of the moved vector **B**.

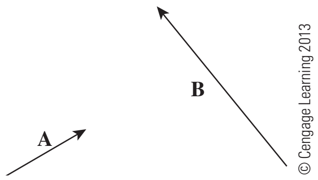

Figure 8.7a

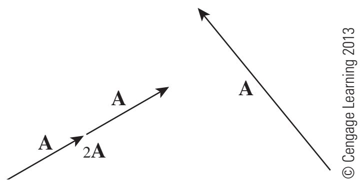

Figure 8.7b

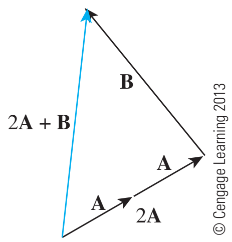

Figure 8.7c

COMPONENT VECTORS

In addition to adding two vectors together to find a resultant vector, we often need to reverse the process and think of a vector as the sum of two other vectors. Any two vectors that can be added together to give the original vector are called **component vectors**. To resolve a vector means to replace it by its component vectors. Usually a vector is resolved into component vectors that are perpendicular to each other.

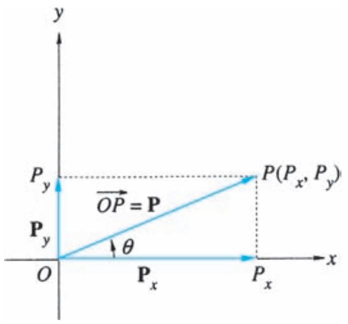

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Figure 8.8

POSITION VECTOR

If P is any point in a coordinate plane and O is the origin, then $\overrightarrow{OP}$ is called the **position vector** of P. It is relatively easy to resolve a position vector into two component vectors by using the x- and y-axis. These are called the *horizontal* (or x-) *component* and the *vertical* (or y-) *component*. In Figure 8.8, $\mathbf{P}_x$ is the horizontal component of $\overrightarrow{OP}$ and $\mathbf{P}_y$ is the vertical component.

FINDING COMPONENT VECTORS

If you study Figure 8.8, you can see that the coordinates of P are (P_x, P_y). (Remember that P_x is the magnitude of vector $\mathbf{P}_x$.) Since every position vector has O as its initial point, it is easier to refer to a position vector by its terminal point. From now on we will refer to $\overrightarrow{OP}$ as **P**.

If the angle that a position vector **P** makes with the positive x-axis is θ, then the components of **P** are found as follows.

COMPONENTS OF A VECTOR

A position vector **P** that makes an angle θ with the positive x-axis can be resolved into component vectors $\mathbf{P}_x$ and $\mathbf{P}_y$ along the x- and y-axis, respectively, with magnitudes P_x and P_y, where

$$P_x = P\cos\theta$$

$$\text{and}\quad P_y = P\sin\theta$$

EXAMPLE 8.2

Resolve a vector 12.0 units long and at an angle of 150° into its horizontal and vertical components.

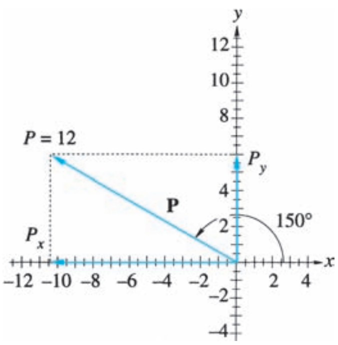

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Figure 8.9

SOLUTION Consider this to be a position vector and put the initial point at the origin. The vector will look like vector **P** in Figure 8.9. We are told that $P = 12$ and $\theta = 150°$, so

$$\begin{aligned} P_x &= 12\cos 150° \\ &= -10.392 \\ \text{and}\quad P_y &= 12\sin 150° \\ &= 6 \end{aligned}$$

We can resolve **P** into two component vectors. One component is along the negative x-axis and has an approximate magnitude of 10.4 units. The other component is along the positive y-axis and has a magnitude of 6.00 units.

EXAMPLE 8.3

Vector **P** is shown in Figure 8.10a. Resolve **P** into its horizontal and vertical components.

SOLUTION This vector is in Quadrant III, so both components will be negative. The reference angle is 67°. We want to know the angle that this vector makes with the positive x-axis. Since the vector is in Quadrant III, $\theta = 180° + 67° = 247°$. We see that $P = 130$, so we determine

$$\begin{aligned} P_x &= 130\cos 247° \\ &\approx -50.7950 \\ &\approx -50.8 \\ P_y &= 130\sin 247° \\ &\approx -119.6656 \\ &\approx -120 \end{aligned}$$

Thus, **P** has been resolved into component vectors $\mathbf{P}_x$ of magnitude 50.8 units along the negative x-axis and $\mathbf{P}_y$ of magnitude 120 units along the negative y-axis, as shown in Figure 8.10b.

EXAMPLE 8.3 (Cont.)

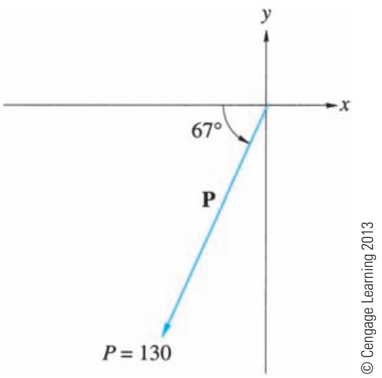


Figure 8.10a

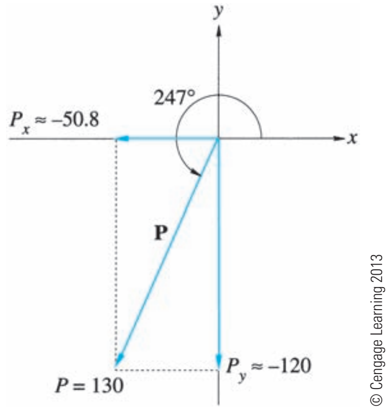


Figure 8.10b

APPLICATION GENERAL TECHNOLOGY

EXAMPLE 8.4

The police officer in Figure 8.11a is measuring skid marks at an accident scene by pushing a wheel tape with a force of 10 lb and holding the handle at an angle of 46° with the ground. Resolve this into its horizontal and vertical component vectors.

SOLUTION A vector diagram has been drawn over the illustration of the police officer operating the wheel tape. The vector diagram is shown alone in Figure 8.11b. The initial point of the vector is at the officer's hand and the terminal point is at the hub, or axle, of the wheel tape. We have placed the origin of our coordinate system at the initial point of the vector, and the horizontal or x-axis is parallel to the ground.

The vector makes an angle of $360° - 46° = 314°$ with the positive x-axis. Thus, we have $P = 10$ and $\theta = 314°$, so

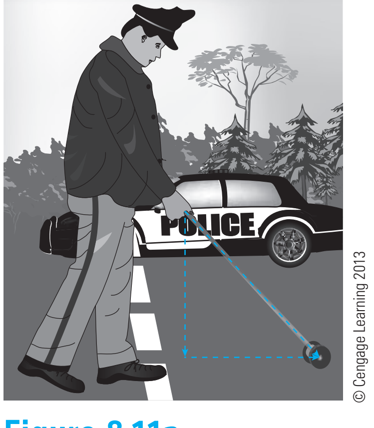


Figure 8.11a

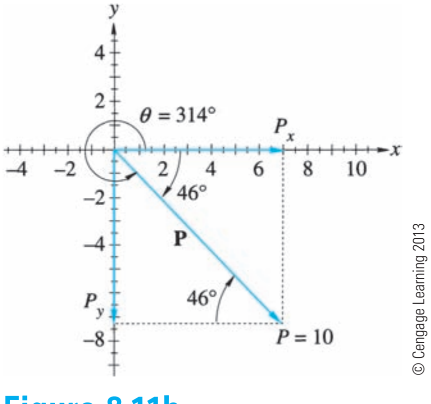


Figure 8.11b

$$P_x = 10 \cos 314°$$
$$\approx 6.9466$$
$$\text{and} \quad P_y = 10 \sin 314°$$
$$\approx -7.1934$$

The police officer exerts a horizontal force of about 6.9 lb and a vertical force of approximately 7.2 lb.

APPLICATION CONSTRUCTION

EXAMPLE 8.5

A cable supporting a television tower exerts a force of 723 N at an angle of 52.7° with the horizontal, as shown in Figure 8.12. Resolve this force into its vertical and horizontal components.

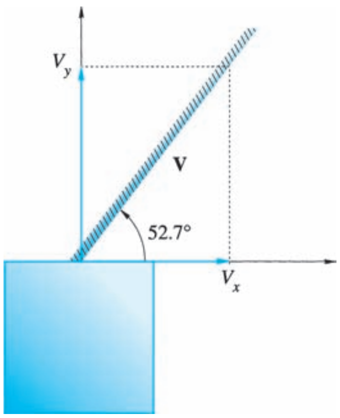


Figure 8.12

SOLUTION A vector diagram has been drawn in Figure 8.12. If the cable is represented by vector **V**, then the horizontal component is V_x and the vertical component is V_y. In this example, $\theta = 52.7°$, and so,

$$V_x = 723 \cos 52.7°$$
$$\approx 438.1296$$
$$\text{and} \quad V_y = 723 \sin 52.7°$$
$$\approx 575.1273$$

We see that this cable exerts a horizontal force of approximately 438 N and a vertical force of about 575 N.

FINDING THE MAGNITUDE AND DIRECTION OF VECTORS

If we have the horizontal and vertical components of a vector **P**, then we can use the components to determine the magnitude and direction of the resultant vector.

MAGNITUDE AND DIRECTION OF A VECTOR

If P_x is the horizontal component of vector **P** and P_y is its vertical component, then

$$|\mathrm{P}| = P = \sqrt{P_x^2 + P_y^2}$$

$$\text{and} \quad \theta_{\text{Ref}} = \tan^{-1}\left|\frac{P_y}{P_x}\right|$$

EXAMPLE 8.6

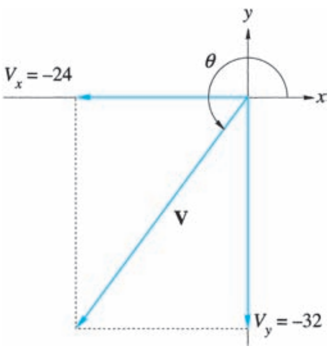


Figure 8.13

A position vector **V** has its horizontal component $V_x = -24$ and its vertical component $V_y = -32$. What are the direction and magnitude of **V**?

SOLUTION A sketch of this problem in Figure 8.13 shows that **V** is in the third quadrant. Since $\theta_{\text{Ref}} = \tan^{-1}\left|\frac{-32}{-24}\right| \approx 53.13°$, we see that $\theta \approx 180° + 53.13° = 233.13°$.

We know that

$$\begin{aligned} \mathbf{V} &= \sqrt{V_x^2 + V_y^2} \\ &= \sqrt{(-24)^2 + (-32)^2} = \sqrt{1{,}600} \\ &= 40 \end{aligned}$$

So, the magnitude of **V** is 40 and **V** is at an angle of 233.13°.

APPLICATION GENERAL TECHNOLOGY

EXAMPLE 8.7

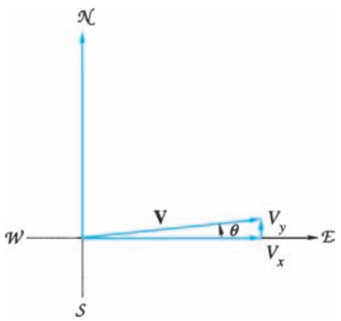


Figure 8.14

A pilot heads a jet plane due east at a ground speed of 425.0 mph. If the wind is blowing due north at 47 mph, find the true speed and direction of the jet.

SOLUTION A sketch of this situation is shown in Figure 8.14. If **V** represents the vector with components V_x and V_y, then we are given $V_x = 425.0$ and $V_y = 47$. Thus,

$$\begin{aligned} \mathbf{V} &= \sqrt{V_x^2 + V_y^2} \\ &= \sqrt{425.0^2 + 47^2} \\ &\approx 427.59 \end{aligned}$$

$$\text{and} \quad \theta = \tan^{-1}\left(\frac{47}{425}\right)$$

$$= \tan^{-1} 0.110588$$

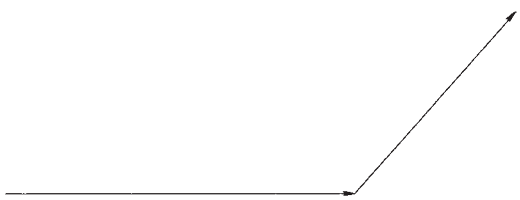
$$\text{and so,} \quad \theta = 6.31°$$

The jet is flying at a speed of approximately 427.6 mph in a direction 6.31° north of due east.

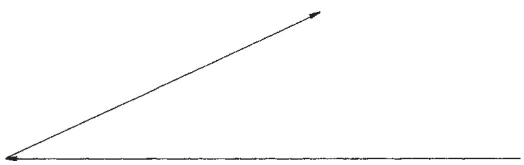
EXERCISE SET 8.1

In Exercises 1–4, add the given vectors by drawing the resultant vector.

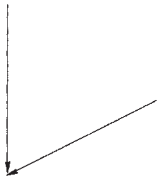
1.



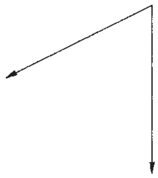
2.



3.



4.



For Exercises 5–20, trace each of the vectors A–D in Figure 8.15. Use these vectors to find each of the indicated sums or differences.

Figure 8.15

5. $\mathbf{A} + \mathbf{B}$

6. $\mathbf{B} + \mathbf{D}$

7. $\mathbf{A} + \mathbf{B} + \mathbf{C}$

8. $\mathbf{B} + \mathbf{C} + \mathbf{D}$

9. $\mathbf{A} - \mathbf{B}$

10. $\mathbf{C} - \mathbf{D}$

11. $\mathbf{B} - \mathbf{A}$

12. $\mathbf{D} - \mathbf{C}$

13. $\mathbf{A} + 3\mathbf{B}$

14. $\mathbf{C} + 2\mathbf{D}$

15. $2\mathbf{C} - \mathbf{D}$

16. $3\mathbf{B} - \mathbf{A}$

17. $\mathbf{A} + 2\mathbf{B} - \mathbf{C}$

18. $\mathbf{A} + 3\mathbf{C} - \mathbf{D}$

19. $2\mathbf{B} - \mathbf{C} + 2.5\mathbf{D}$

20. $4\mathbf{C} - 3\mathbf{B} - 2\mathbf{D}$

In Exercises 21–26, use trigonometric functions to find the horizontal and vertical components of the given vectors.

21. Magnitude 20, $\theta = 75°$

22. Magnitude 16, $\theta = 212°$

23. Magnitude 18.4, $\theta = 4.97$ rad

24. Magnitude 23.7, $\theta = 2.22$ rad

25. $V = 9.75$, $\theta = 16°$

26. $P = 24.6$, $\theta = 317°$

In Exercises 27–30, the horizontal and vertical components are given for a vector. Find the magnitude and direction of each resultant vector.

27. $A_x = -9$; $A_y = 12$

28. $B_x = 10$; $B_y = -24$

29. $C_x = 8$; $C_y = 15$

30. $D_x = -14$; $D_y = -20$

Solve Exercises 31–38.

31. ***Navigation*** A ship heads into port at 12.0 km/h. The current is perpendicular to the ship at 5 km/h. What is the resultant velocity of the ship?

32. ***Navigation*** A pilot heads a jet plane due east at a ground speed of 756.0 km/h. If the wind is blowing due north at 73 km/h, find the true speed and direction of the jet.

33. ***Construction*** A cable supporting a tower exerts a force of 976 N at an angle of 72.4° with the horizontal. Resolve this force into its vertical and horizontal components.

34. ***Construction*** A sign of mass 125.0 lb hangs from a cable. A worker is pulling the sign horizontally by a force of 26.50 lb. Find the force and the angle of the resultant force on the sign.

35. ***Medical technology*** A ramp for the physically challenged makes an angle of 4.8° with the horizontal. A woman and her wheelchair weigh 153 lb. What are the components of this weight parallel and perpendicular to the ramp?

36. ***Electronics*** If a resistor, a capacitor, and an inductor are connected in series to an ac power source, then the effective voltage of the source is given by the vector **V**, where **V** is the sum of the vector quantities V_R and $V_L - V_C$, as shown in Figure 8.16. The phase angle, ϕ, is the angle between **V** and V_R, where $\tan \phi = \frac{V_L - V_C}{V_R}$. If the effective voltages across the circuit components are $V_R = 12$ V, $V_C = 10$ V, and $V_L = 5$ V, determine the effective voltage and the phase angle.

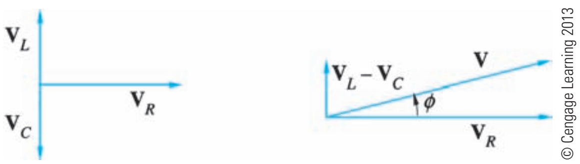


Figure 8.16

37. ***Electronics*** A resistor, capacitor, and inductor are connected in series to an ac power source. If the effective voltages across the circuit components are $V_R = 15.0$ V, $V_C = 17.0$ V, and $V_L = 8.0$ V, determine the effective voltage and the phase angle.

38. ***Electronics*** A resistor, capacitor, and inductor are connected in series to an ac power source. If the effective voltages across the circuit components are $V_R = 22.6$ V, $V_C = 15.2$ V, and $V_L = 28.3$ V, determine the effective voltage and phase angle.

39. ***Navigation*** A light airplane is flying due north at 145 mph. The plane is flying into a 45-mph headwind from the northwest at an angle of 35° as shown in Figure 8.17. **(a)** Draw the resultant vector for the plane's path. **(b)** Determine the magnitude for the resultant vector of the plane's path. This is the plane's actual velocity.

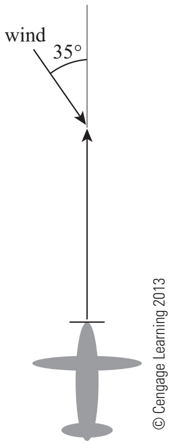


Figure 8.17

40. ***Aeronautics*** A passenger jet plane needs to change its altitude. The plane's speed (velocity) is 535.0 mph and it begins climbing at an angle of 13.5° with the horizontal. Find the horizontal, V_x, and vertical, V_y, components of the velocity vector **V**.

41. ***Fire science*** The water from a fire hose exerts a force of 200 lb on the person holding the hose. The nozzle of the hose weighs 20 pounds. **(a)** What force is needed to hold the hose horizontal? **(b)** At what angle to the horizontal will this force need to be applied?

42. ***Navigation*** A river 0.75 mi wide flows south with a current of 5 mph. What speed and heading should a motorboat assume to travel directly across the river from east to west in 10 min?

[IN YOUR WORDS]

43. **(a)** Distinguish between scalars and vectors. How are they alike and how are they different?

(b) What does it mean for two vectors to be equal?

44. **(a)** Describe how to use the components of a vector to determine the vector's magnitude and direction.

(b) Explain how to use the magnitude and direction of a vector to determine its component vectors.

8.2 ADDING AND SUBTRACTING VECTORS

In Section 8.1, we learned how to use diagrams to add and subtract vectors. We also learned how to use trigonometry to determine the horizontal and vertical components of a vector. In this section, we will learn how to use trigonometry and the Pythagorean theorem to add and subtract vectors.

We will begin by looking at two special cases. In the first case, both vectors are on the same axis. In the second case, the vectors are on different axes.

EXAMPLE 8.8

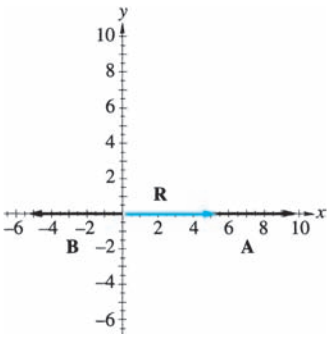


Figure 8.18

Add vectors **A** and **B**, where $A = 9.6$, $\theta_A = 0°$ and $B = 4.3$, $\theta_B = 180°$.

SOLUTION If we find the horizontal and vertical components of these two vectors we see that

$$\begin{aligned} A_x &= 9.6 \cos 0° = 9.6 \\ A_y &= 9.6 \sin 0° = 0 \\ B_x &= 4.3 \cos 180° = -4.3 \\ B_y &= 4.3 \sin 180° = 0 \end{aligned}$$

The resultant vector **R** has horizontal component $R_x = A_x + B_x = 9.6 + -4.3 = 5.3$ and a vertical component $R_y = 0 + 0 = 0$. The angle of the resultant vector θ_R is found using $\tan \theta_R = \dfrac{R_y}{R_x} = \dfrac{0}{5.3} = 0$. Since $R_x > 0$, we know that θ_R is 0° rather than 180° (see Figure 8.18).

In this first case, the component method required a little extra work. But, the example gave us a foundation for the next example:

EXAMPLE 8.9

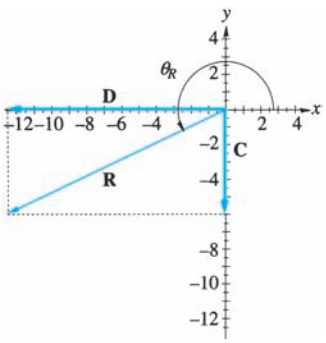


Figure 8.19

Find the resultant of **C** and **D**, when $C = 6.2$, $\theta_C = 270°$, $D = 12.4$, and $\theta_D = 180°$, as shown in Figure 8.19.

SOLUTION Since **C** and **D** are perpendicular, the length of the resultant vector can be found by using the Pythagorean theorem:

$$R = \sqrt{C^2 + D^2} = \sqrt{(6.2)^2 + (12.4)^2} \approx 13.9$$

We also know that $\tan \theta = \dfrac{C}{D} = \dfrac{6.2}{12.4} = \dfrac{1}{2}$ and that $\tan^{-1}\frac{1}{2} = 26.57°$. From Figure 8.19, we can see that **R** is in the third quadrant. So, $\theta_R = 206.57° \approx 207°$.

If we want to find the resultant of two vectors that are not at right angles, the method takes a bit longer. We first resolve each vector into its horizontal and vertical components and then add the horizontal and vertical components, as outlined in the following box.

ADDING VECTORS USING COMPONENTS

1. Resolve each vector into its horizontal and vertical components.
2. Add the horizontal components. This sum is R_x, the horizontal component of the resultant vector.
3. Add the vertical components. This sum is R_y, the vertical component of the resultant vector.
4. Use R_x and R_y to determine the magnitude R and direction θ of the resultant vector, where

$$R = \sqrt{R_x^2 + R_y^2} \quad \text{and} \quad \tan\theta = \frac{R_y}{R_x}$$

CAUTION Calculators yield the same answer for $\tan^{-1}\left(\frac{R_y}{R_x}\right)$ and $\tan^{-1}\left(\frac{-R_y}{-R_x}\right)$. Similarly, they also give the same answer to $\tan^{-1}\left(\frac{-R_y}{R_x}\right)$ and $\tan^{-1}\left(\frac{R_y}{-R_x}\right)$. You must be careful to check the quadrant in which **R** is located and then, if necessary, add π (or 180°) to the resultant angle.

EXAMPLE 8.10

Find the resultant of two vectors **E** and **F**, where $E = 109$, $\theta_E = 33.4°$, $F = 125$, and $\theta_F = 69.4°$ (see Figure 8.20a).

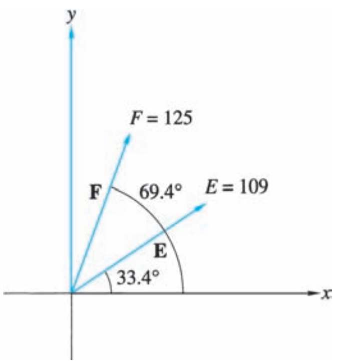


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Figure 8.20a

SOLUTION Resolving **E** into its horizontal and vertical components we get

$$E_x = 109 \cos 33.4° = 91$$
$$\text{and} \quad E_y = 109 \sin 33.4° = 60$$

The components for vector **F** are

$$F_x = 125 \cos 69.4° = 44$$
$$F_y = 125 \sin 69.4° = 117$$

These components are shown in Figure 8.20b. The components of the resultant vector are found by adding the horizontal and vertical components of **E** and **F**:

$$\begin{aligned} R_x &= E_x + F_x \\ &= 91 + 44 \\ &= 135 \\ R_y &= E_y + F_y \\ &= 60 + 117 \\ &= 177 \end{aligned}$$

These two component vectors and their resultant vector are shown in Figure 8.20c.

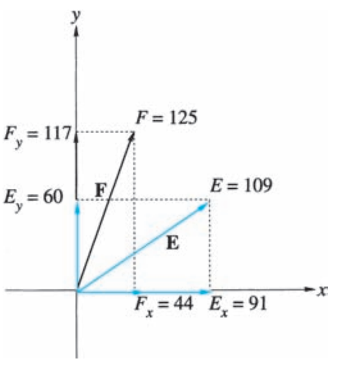

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Figure 8.20b

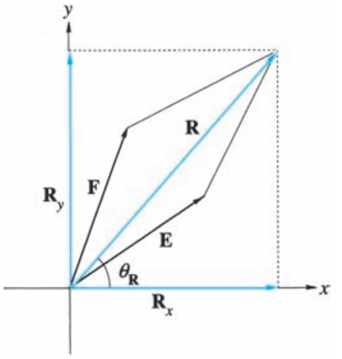


Figure 8.20c

$$R = \sqrt{R_x^2 + R_y^2}$$
$$= \sqrt{135^2 + 177^2}$$
$$\approx 222.61$$
$$\text{and} \quad \tan\theta_R = \frac{R_y}{R_x}$$
$$\tan\theta_R = \frac{177}{135}$$
$$\theta_R = \tan^{-1}\frac{177}{135}$$
$$\text{so } \theta_R = 52.7°$$

It is often helpful to use a table to help keep track of vectors and their components. It is an effective way to organize this information. A table for Example 8.10 follows. It lists the horizontal and vertical components of each vector and the resultant vector.

Vector	Horizontal component	Vertical component
E	$E_x = 109 \cos 33.4° = 91$	$E_y = 109 \sin 33.4° = 60$
F	$F_x = 125 \cos 69.4° = \underline{44}$	$F_y = 125 \sin 69.4° = \underline{117}$
R	R_x 135	R_y 177

From these values for R_x and R_y, we can determine $R \approx 223$ and $\theta_R = 52.7°$.

EXAMPLE 8.11

Find the resultant of two vectors **G** and **H**, where $G = 449$, $\theta_G = 128.6°$, $H = 521$, and $\theta_H = 327.6°$ (see Figure 8.21a).

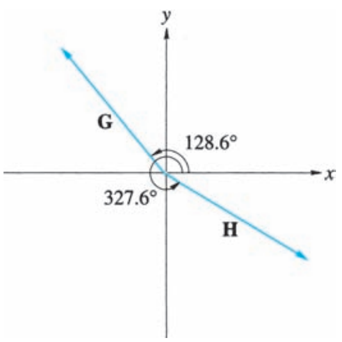


Figure 8.21a

SOLUTION Again, we will resolve each vector into its horizontal and vertical components.

$$G_x = 449 \cos 128.6°$$
$$= -280.1$$
$$G_y = 449 \sin 128.6°$$
$$= 350.9$$
$$H_x = 521 \cos 327.6°$$
$$= 439.9$$
$$H_y = 521 \sin 327.6°$$
$$= -279.2$$

Figure 8.21b shows each vector and its horizontal and vertical components.

Adding the horizontal components gives the horizontal component of the resultant vector.

$$R_x = G_x + H_x$$
$$= -280.1 + 439.9$$
$$= 159.8$$

EXAMPLE 8.11 (Cont.)

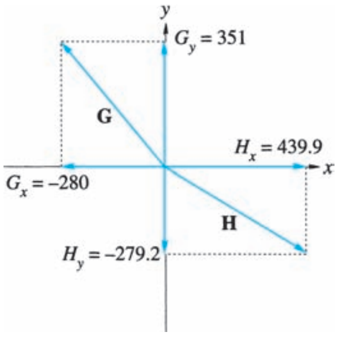


Figure 8.21b

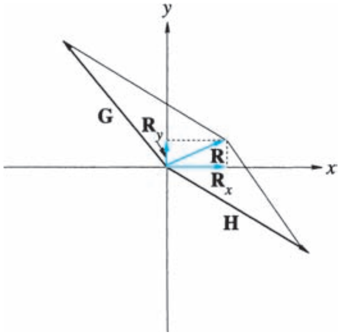


Figure 8.21c

Similarly, we can determine the vertical component of the resultant vector:

$$\begin{aligned} R_y &= G_y + H_y \\ &= 350.9 - 279.2 \\ &= 71.7 \end{aligned}$$

The resultant vector is shown in Figure 8.21c.

The magnitude of the resultant vector is

$$\begin{aligned} R &= \sqrt{R_x^2 + R_y^2} \\ &= \sqrt{159.8^2 + 71.7^2} \\ &\approx 175.1 \end{aligned}$$

and the direction of the resultant vector is found from

$$\begin{aligned} \tan\theta_R &= \frac{R_y}{R_x} \\ &= \frac{71.7}{159.8} \\ &= 0.4487 \end{aligned}$$

so

$$\begin{aligned} \theta_R &= \tan^{-1} 0.4487 \\ &= 24.2° \end{aligned}$$

The table method for Example 8.11 would appear as follows.

Vector	Horizontal component		Vertical component	
G	$G_x = 449 \cos 128.6° =$	-280.1	$G_y = 449 \sin 128.6° =$	350.9
H	$H_x = 521 \cos 327.6° =$	$\underline{439.9}$	$H_y = 521 \sin 327.6° =$	$\underline{-279.2}$
R	R_x	159.8	R_y	71.7

These values for R_x and R_y can be used as before to find $R = 175.1$ and $\theta_R = 24.2°$.

EXAMPLE 8.12

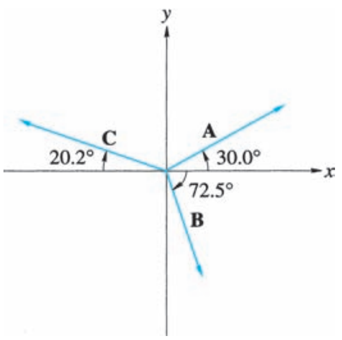


Figure 8.22

Find the resultant of the three vectors shown in Figure 8.22, if $A = 137$, $B = 89.4$, and $C = 164.6$.

SOLUTION The table that follows lists the horizontal and vertical components of each of the vectors and the resultant vector **R**.

Vector	Horizontal component		Vertical component	
A	$A_x = 137 \cos 30° =$	118.6	$A_y = 137 \sin 30° =$	68.5
B	$B_x = 89.4 \cos 287.5° =$	26.9	$B_y = 89.4 \sin 287.5° =$	-85.3
C	$C_x = 164.6 \cos 159.8° =$	$\underline{-154.5}$	$C_y = 164.6 \sin 159.8° =$	$\underline{56.8}$
R	R_x	-9.0	R_y	40.0

Once again, the values for R_x and R_y can be used to find

$$\begin{aligned} R &= \sqrt{R_x^2 + R_y^2} \\ &= \sqrt{(-9)^2 + 40^2} \\ &= 41 \end{aligned}$$

and

$$\theta_R = \tan^{-1}\frac{40}{-9}$$

$$\theta_R = 102.7^\circ$$

APPLICATION CONSTRUCTION

EXAMPLE 8.13

A sign has been lifted into position by two cranes, as shown in Figure 8.23a. If the sign weighs 420 lb, what is the tension in each of the three cables?

SOLUTION We will draw the coordinate axes and label the angles as shown in Figure 8.23b. The origin is placed at the ring where the three cables meet, because the tension on all three cables acts on this ring.

As usual, we make a table listing the horizontal and vertical components of each vector.

Vector	Horizontal component	Vertical component
A	$A_x = A\cos 145^\circ$	$A_y = A\sin 145^\circ$
B	$B_x = B\cos 40^\circ$	$B_y = B\sin 40^\circ$
C	$C_x = C\cos 270^\circ = \underline{0}$	$C_y = C\sin 270^\circ = -C = \underline{-420}$
R	R_x	R_y

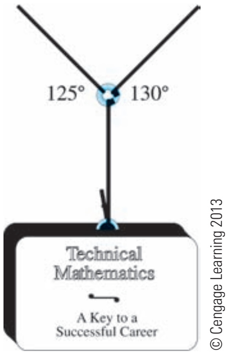


Figure 8.23a

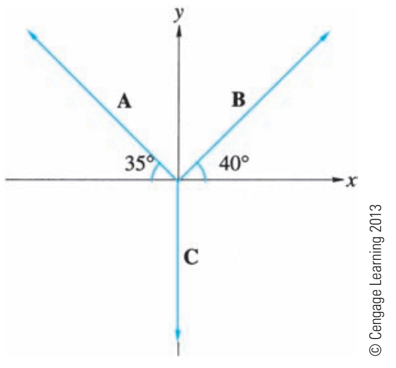


Figure 8.23b

The ring is at rest as a result of these three forces, so $R_x = 0$ and $R_y = 0$. Thus, we have

$$R_x = A\cos 145^\circ + B\cos 40^\circ = 0$$

$$R_y = A\sin 145^\circ + B\sin 40^\circ - 420 = 0$$

EXAMPLE 8.13 (Cont.)

Evaluating the trigonometric functions in these two equations, we are led to the following system of two equations in two variables:

$$\begin{cases} -0.81915A + 0.76604B = 0 \\ 0.57358A + 0.64279B = 420 \end{cases}$$

Solving this system of two equations in two variables (by Cramer's rule), we get

$$A \approx 333.0862 \text{ lb}$$

$$\text{and} \quad B \approx 356.1792 \text{ lb}$$

The tension in the cable on the left is about 333 lb and the tension in the cable on the right side is about 356 lb.

NOTE In flight terminology, the **heading** of an aircraft is the direction in which the aircraft is pointed. Usually the wind is pushing the aircraft so that it is actually moving in a different direction, called the **track** or **course**. The angle between the heading and the course is the **drift angle**. The **air speed** is the speed of the plane relative to the air. The **ground speed** is the speed of the aircraft relative to the ground.

NOTE In navigation, directions are usually given in terms of the size of the angle measured clockwise from true north. For example, the airplane in Figure 8.24 has a heading of 150°.

APPLICATION GENERAL TECHNOLOGY

EXAMPLE 8.14

An airplane is flying at 340.0 mph with a heading of 210°. If a 50-mph wind is blowing from 165°, find the ground speed, drift angle, and course of the airplane.

SOLUTION In Figure 8.25a, $\overrightarrow{OA}$ represents the airspeed of 340 mph with a heading of 210°, and $\overrightarrow{OW}$ represents a wind of 50 mph from 165°. In Figure 8.25b, we have completed the parallelogram to obtain the vector $\overrightarrow{OR} = \overrightarrow{OA} + \overrightarrow{OW}$. Notice that, while the wind is from 165°, it has a heading of $180° + 165° = 345°$.

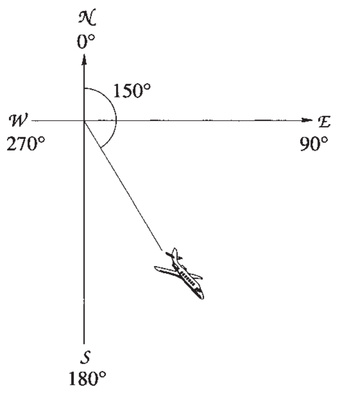


Figure 8.24

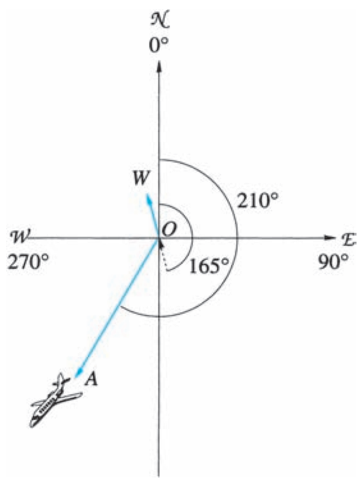


Figure 8.25a

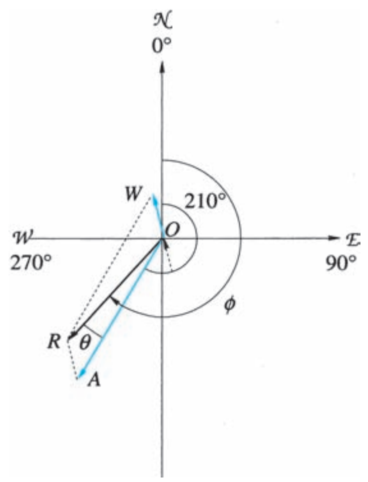


Figure 8.25b

The length $\overrightarrow{OR}$ represents the ground speed, θ is the drift angle, and ϕ is the track or course. The following table lists the horizontal and vertical components of each vector $\overrightarrow{OA}$ and $\overrightarrow{OW}$ and the resultant vector $\overrightarrow{OR}$.

Vector	Horizontal component		Vertical component	
$\overrightarrow{OA} = \mathbf{A}$	$A_x = 340 \cos 210° \approx$	-294.45	$A_y = 340 \sin 210° =$	-170.00
$\overrightarrow{OW} = \mathbf{W}$	$W_x = 50 \cos 345° \approx$	$\underline{48.30}$	$W_y = 50 \sin 345° \approx$	$\underline{-12.94}$
$\overrightarrow{OR} = \mathbf{R}$	R_x	-246.15	R_y	-182.94

Using these values for R_x and R_y, we see that the length of the resultant vector is $\overrightarrow{OR} = \sqrt{(-246.15)^2 + (-182.94)^2} \approx 306.69$ and $\phi \approx \tan^{-1}\left(\frac{-182.94}{-246.15}\right) = 36.62°$. Since ϕ is in the third quadrant, $\phi = 180° + 36.62° = 216.62°$.

So, the plane has a ground speed of 306.69 mph, a drift angle of 6.62°, and a course of 216.62°.

EXERCISE SET 8.2

*In Exercises 1–4, vectors **A** and **B** are both on the same axis. Find the magnitude and direction of the resultant vector.*

1. $A = 20.0, \theta_A = 0°, B = 32.5, \theta_B = 180°$

2. $A = 14.3, \theta_A = 90°, B = 7.2, \theta_B = 90°$

3. $A = 121.7, \theta_A = 270°, B = 86.9, \theta_B = 90°$

4. $A = 63.1, \theta_A = 180°, B = 43.5, \theta_B = 180°$

*In Exercises 5–8, vectors **C** and **D** are perpendicular. Find the magnitude and direction of the resultant vectors. It may help if you draw vectors **C**, **D**, and the resultant vector.*

5. $C = 55, \theta_C = 90°, D = 48, \theta_D = 180°$

6. $C = 65, \theta_C = 270°, D = 72, \theta_C = 180°$

7. $C = 81.4, \theta_C = 0°, D = 37.6, \theta_D = 90°$

8. $C = 63.4, \theta_C = 270°, D = 9.4, \theta_D = 0°$

In Exercises 9–16, find the magnitude and direction of the vector with the given components.

9. $A_x = 33, A_y = 56$

10. $B_x = 231, B_y = 520$

11. $C_x = 11.7, C_y = 4.4$

12. $D_x = 31.9, D_y = 36.0$

13. $E_x = 6.3, E_y = 1.6$

14. $F_x = 5.1, F_y = 14.0$

15. $G_x = 8.4, G_y = 12.6$

16. $H_x = 15.3, H_y = 9.2$

In Exercises 17–30, add the given vectors by using the trigonometric functions and the Pythagorean theorem.

17. $A = 4, \theta_A = 60°, B = 9, \theta_B = 20°$

18. $C = 12, \theta_C = 75°, D = 15, \theta_D = 37°$

19. $C = 28, \theta_C = 120°, D = 45, \theta_D = 210°$

20. $E = 72, \theta_E = 287°, F = 65, \theta_F = 17°$

21. $A = 31.2, \theta_A = 197.5°, B = 62.1, \theta_B = 236.7°$

22. $C = 53.1, \theta_C = 324.3°, D = 68.9, \theta_D = 198.6°$

23. $E = 12.52, \theta_E = 46.4°, F = 18.93, \theta_F = 315°$

24. $G = 76.2, \theta_G = 15.7°, H = 89.4, \theta_H = 106.3°$

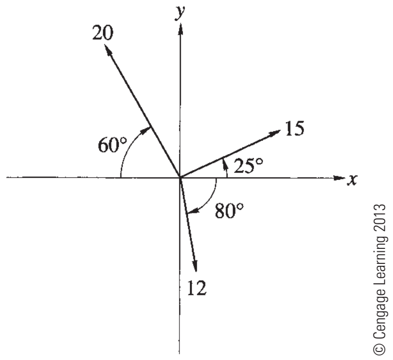
25. $A = 9.84, \theta_A = 215°30', B = 12.62, \theta_B = 105°15'$

26. $C = 79.63, \theta_C = 262°45', D = 43.72, \theta_D = 196°12'$

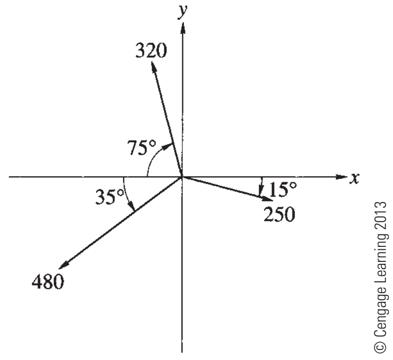
27. $E = 42.0, \theta_E = 3.4 \text{ rad}, F = 63.2, \theta_F = 5.3 \text{ rad}$

28. $G = 37.5, \theta_G = 0.25 \text{ rad}, H = 49.3, \theta_H = 1.92 \text{ rad}$

29.



30.



Solve Exercises 31–40.

31. ***Construction technology*** A sign is held in position by three cables, as shown in Figure 8.26. If the sign has a weight of 215 N, what is the tension in each of the three cables?

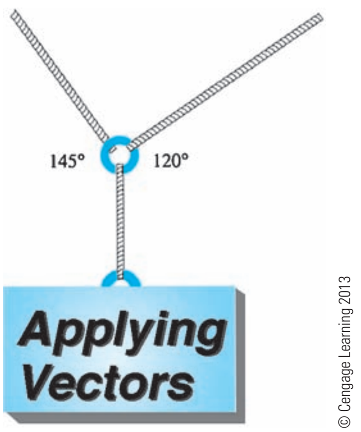


Figure 8.26

32. ***Construction technology*** A 175-lb sign is supported from a wall by a cable inclined 53° with the horizontal, and a brace perpendicular to the wall, as shown in Figure 8.27. Find the magnitudes of the forces in the cable and the brace that will keep the sign in equilibrium.

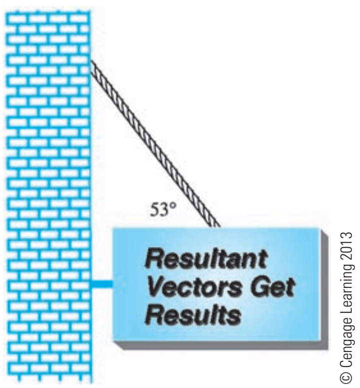


Figure 8.27

33. ***Construction technology*** A 235-lb sign is supported from a wall by a cable inclined 37° with the horizontal, and a brace perpendicular to the wall, similar to that shown in Figure 8.27. Find the tension in the cable and the compression in the boom.

34. ***Construction technology*** An 85-kg sign is held in position by three ropes, as shown in Figure 8.28. What is the tension in each of the three ropes?

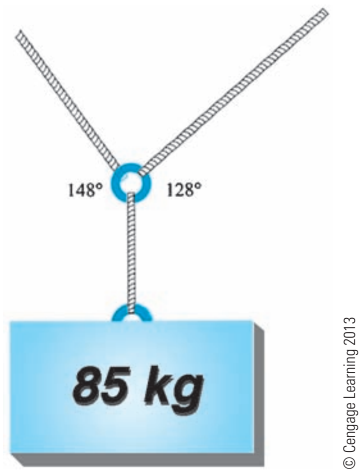


Figure 8.28

35. ***Navigation*** An airplane is flying at 480 mph with a heading of 63°. A 45.0-mph wind is blowing from 325°. Find the ground speed, course, and drift angle of the airplane.

36. ***Navigation*** An airplane is flying at 320.0 mph with a heading of 172°. A 72.0-mph wind is blowing from 137°. Find the ground speed, course, and drift angle of the airplane.

37. ***Electronics*** A circuit has two ac voltages that are 68° out of phase. Each voltage is 196 V. What is the total voltage?

38. ***Electronics*** A two-phase generator produces two voltages that are 120° out of phase. The first voltage is 86 V. The second voltage is 110 V. What is the total voltage?

39. ***Aeronautics*** An airplane takes off from an airport with a velocity of 225 mph at an angle of 12°.

(a) How fast is it rising?

(b) What is its speed relative to the ground?

40. ***Police science*** The *momentum* of an object is equal to its mass, m, times its velocity, v. Thus, momentum is a vector, and momentum $= mv$. If friction forces are very small, we may assume that the momentum before a collision is the same as the momentum after the collision. Two cars collide at an icy intersection, as shown in Figure 8.29. Car A has a mass of 2450 kg and was traveling at 10 m/s before the cars hit, and car B has a mass of 2100 kg and was traveling at 9 m/s. After the collision the cars are locked together and they slide on the flat icy road.

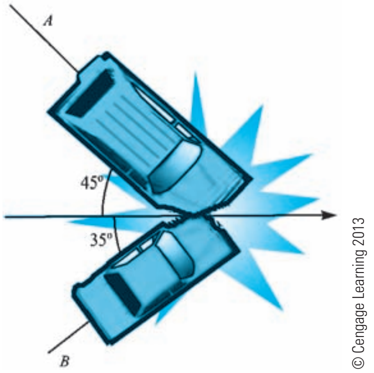


Figure 8.29

(a) What are the resultant vectors for the momentum of car A before the collision?

(b) What are the resultant vectors for the momentum of car B before the collision?

(c) What are the resultant vectors for the momentum of the cars after the collision?

(d) What was the speed and direction of the cars after the collision?

[IN YOUR WORDS]

41. Without looking in the text, describe how you can use a vector's components to add vectors.

42. Flight terminology uses several technical terms. Write a brief definition of each of the following terms. Draw a figure to illustrate your explanations.

(a) Heading

(b) Drift angle

(c) Course

(d) Ground speed

(e) Wind speed

8.3 APPLICATIONS OF VECTORS

Vectors are used in science and technology, as well as in mathematics. In this section, we will look at some of those applications.

APPLICATION GENERAL TECHNOLOGY

EXAMPLE 8.15

Two forces, F_1 and F_2, act on an object. If F_1 is 40 lb, F_2 is 75 lb, and the angle θ between them is 50°, find the magnitude and direction of the resultant force.

EXAMPLE 8.15 (Cont.)

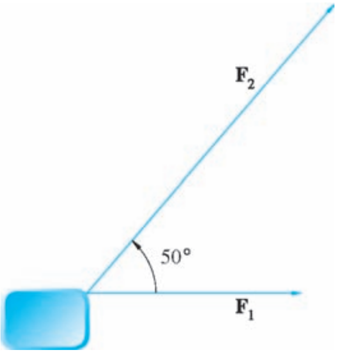


Figure 8.30

SOLUTION We sketch the two forces as vectors and place the object at the origin with $\mathbf{F}_1$ along the positive x-axis as in Figure 8.30. We will use a table similar to the one in the last example to find the components of $\mathbf{F}_1$, $\mathbf{F}_2$, and the resultant vector.

Vector	Horizontal component		Vertical component	
$\mathbf{F}_1$		40.0		0.0
$\mathbf{F}_2$	$75 \cos 50° =$	$\underline{48.2}$	$75 \sin 50° =$	$\underline{57.5}$
$\mathbf{R}$	R_x	88.2	R_y	57.5

So, $R_x = 88.2$ and $R_y = 57.5$. The magnitude of $\mathbf{R}$ is $R = \sqrt{88.2^2 + 57.5^2} = 105.3$ lb and $\theta_R = \tan^{-1} \frac{57.5}{88.2} = 33.1°$.

APPLICATION GENERAL TECHNOLOGY

EXAMPLE 8.16

A truck weighing 22,500 lb is on a 25° hill. Find the components of the truck's weight parallel and perpendicular to the road.

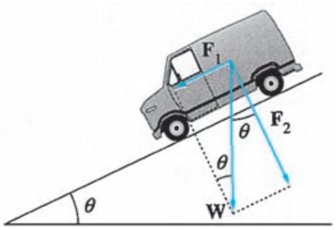


Figure 8.31

SOLUTION The weight of an object (truck, car, building, etc.) is the gravitational force with which earth attracts it. This force always acts vertically downward and is indicated by the vector $\mathbf{W}$ in Figure 8.31. The components of $\mathbf{W}$ have been labeled $\mathbf{F}_1$ and $\mathbf{F}_2$. Because $\mathbf{W}$ is vertical and $\mathbf{F}_2$ is perpendicular to the road, the angle θ between $\mathbf{W}$ and $\mathbf{F}_2$ is equal to the angle the road makes with the horizon. Using the trigonometric functions, we have

$$F_1 = W \sin \theta = 22{,}500 \sin 25° = 9{,}509 \text{ lb}$$

$$F_2 = W \cos \theta = 22{,}500 \cos 25° = 20{,}392 \text{ lb}$$

Thus, the components of the truck's weight are 9,509 lb parallel to the road and 20,392 lb perpendicular to the road.

APPLICATION GENERAL TECHNOLOGY

EXAMPLE 8.17

If the truck in Example 8.16 rolls down the hill at 15 mph, find the magnitudes of the horizontal and vertical components of the truck's velocity.

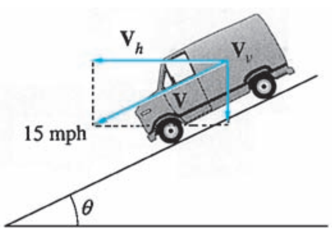


Figure 8.32

SOLUTION The velocity vector $\mathbf{V}$ is shown in Figure 8.32. The horizontal and vertical components of $\mathbf{V}$ are marked $\mathbf{V}_h$ and $\mathbf{V}_v$:

$$V_v = V \sin 25° = 15 \sin 25° = 6.3 \text{ mph}$$

$$V_h = V \cos 25° = 15 \cos 25° = 13.6 \text{ mph}$$

APPLICATION GENERAL TECHNOLOGY

EXAMPLE 8.18

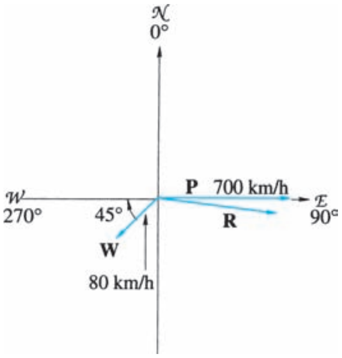


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Figure 8.33

A jet plane is traveling due east at an airspeed of 700 km/h. If a wind of 80 km/h is blowing due southwest, find the magnitude and direction of the plane's resultant velocity (see Figure 8.33).

SOLUTION A table of the components for the wind's vector **W** and the plane's vector **P** allows us to quickly find the resultant vector **R**.

Vector	Horizontal component	Vertical component
W	$80 \cos 225° = -56.6$	$80 \sin 225° = -56.6$
P	$700 \cos 0° = \underline{700.0}$	$\underline{0.0}$
R	643.4	-56.6

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The magnitude of **R** is $R = \sqrt{643.4^2 + (-56.6)^2} = 645.9$ km/h. This is the ground speed of the plane. The plane's direction is $\theta_R = \tan^{-1}\left(\frac{-56.6}{643.4}\right) = -5.03°$ or 5.03° south of east.

APPLICATION ELECTRONICS

EXAMPLE 8.19

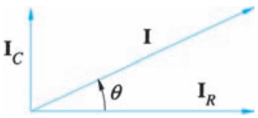

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Figure 8.34

In a parallel *RC* (resistance-capacitance) circuit, the current $\mathbf{I}_C$ through the capacitance leads the current $\mathbf{I}_R$ through the resistance by 90°, as shown in Figure 8.34. If $I_C = 0.5$ A and $I_R = 1.2$ A, find the total current **I** in the circuit and the phase angle θ of the circuit.

SOLUTION $I = \sqrt{I_R^2 + I_C^2} = \sqrt{(1.2)^2 + (0.5)^2} = 1.3$ A

$$\theta = \tan^{-1}\left(\frac{I_C}{I_R}\right)$$

$$\theta = \tan^{-1}\left(\frac{0.5}{1.2}\right) = 22.6°$$

APPLICATION GENERAL TECHNOLOGY

EXAMPLE 8.20

An airplane in level flight drops an object. The plane was traveling at 180 m/s at a height of 7 500 m. The vertical component of the dropped object is given by $V_y = -9.8t$ m/s. What is the magnitude of the velocity of the object after 8 s? At what angle with the ground is the object moving at this time?

SOLUTION The horizontal velocity of the object will have no effect on the vertical motion. Hence, V_x will remain at 180 m/s. We are told that $V_y = -9.8t$.

EXAMPLE 8.20 (Cont.)

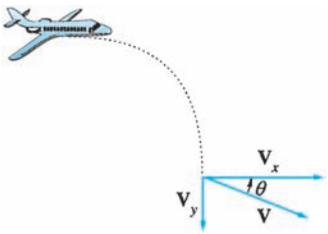

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Figure 8.35

So, when $t = 8$, $V_y = -78.4$. The magnitude of the velocity of the object when $t = 8$ is

$$V = \sqrt{(-78.4)^2 + 180^2} = 196.3 \text{ m/s}$$

If θ is the angle that the object makes with the ground, then

$$\theta = \tan^{-1}\frac{V_y}{V_x} = \tan^{-1}\left(\frac{-78.4}{180}\right) \approx -23.5°$$

or 23.5° below the horizontal (see Figure 8.35).

EXERCISE SET 8.3

Solve Exercises 1–24.

1. ***Physics*** Two forces act on an object. One force is 70 lb and the other is 50 lb. If the angle between the two forces is 35°, find the magnitude and direction of the resultant force.
2. ***Physics*** A person pulling a cart exerts a force of 35 lb on the cart at an angle of 25° above the horizontal. Find the horizontal and vertical components of this force.
3. ***Physics*** A person pushes a lawn mower with a force of 25 lb. The handle of the lawn mower is 55° above the horizontal. **(a)** How much downward force is being exerted on the ground? **(b)** How much horizontal forward force is being exerted? **(c)** How do these forces change if the handle is lowered to 40° with the horizontal?
4. ***Navigation*** Two tugboats are pulling a ship. As shown in Figure 8.36, the first tug exerts a force of 1 500 N on a cable, making an angle of 37° with the axis of the ship. The second tug pulls on a cable, making an angle of 36° with the axis of the ship. The resultant force vector is in line with the axis of the ship. What is the force being exerted by the second tug?

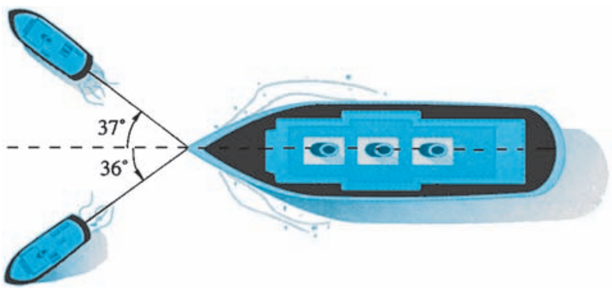

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Figure 8.36

5. ***Navigation*** A tugboat is turning a barge by pushing with a force of 1 700 N at an angle of 18° with the axis of the barge. What are the forward and sideward forces in newtons exerted by the tug on the barge? (See Figure 8.37)

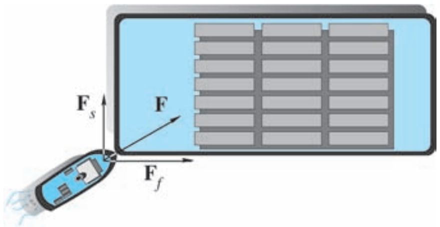

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Figure 8.37

6. ***Navigation*** A ship heads due northwest at 15 km/h in a river that flows east at 7 km/h. What is the magnitude and direction of the ship's velocity relative to earth's surface?
7. ***Navigation*** A plane is heading due south at 370 mph with a wind from the west at 40 mph. What are the ground speed and the true direction of the plane?
8. ***Navigation*** On a compass, due north is 0°, east is 90°, south 180°, and so on. A plane has a compass heading of 115° and is traveling at 420 mph. The wind is at 62 mph and blowing at 32°.

What are the ground speed and true direction of the plane?

9. ***Physics*** A car with a mass of 1 200 kg is on a hill that makes an angle of 22° with the horizon. Which components of the car's mass are parallel and perpendicular to the road?
10. ***Physics*** Find the force necessary to push a 30-lb ball up a ramp that is inclined 15° with the horizon. (You want to find the component parallel to the ramp.)
11. ***Construction*** A guy wire runs from the top of a utility pole 35 ft high to a point on the ground 27 ft from the base of the pole. The tension in the wire is 195 lb. What are the horizontal and vertical components?
12. ***Electronics*** In a parallel RC circuit, the current I_C through the capacitance leads the current through the resistance I_R by 90°. If $I_C = 2.4$ A and $I_R = 1.6$ A, find the magnitude of the total current in the circuit and the phase angle.
13. ***Electronics*** If the total current I in a parallel RC circuit is 9.6 A and I_R is 7.5 A, what are I_C and the phase angle?
14. ***Electronics*** In a parallel RL (resistance-inductance) circuit, the current I_L through the inductance lags the current I_R through the resistance by 90°, as shown in Figure 8.38. Find the magnitude of the total current I and the negative phase angle θ, when $I_L = 6.2$ A and $I_R = 8.4$ A.

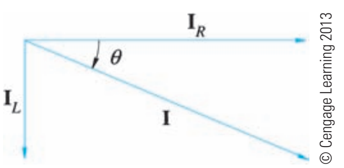

Figure 8.38

15. ***Electronics*** If the total current I in a parallel RL circuit is 12.4 A and I_R is 6.3 A, find I_L and the negative phase angle.
16. ***Electronics*** The total impedance Z of a series ac circuit is the resultant of the resistance R, the inductive reactance X_L, and the capacitive reactance X_C, as shown in Figure 8.39. Find Z and the phase angle, when $X_C = 50\ \Omega$, $X_L = 90\ \Omega$, and $R = 12\ \Omega$.

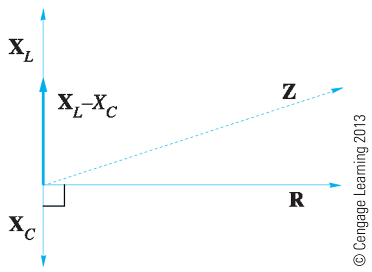

Figure 8.39

17. ***Electronics*** Find the total impedance Z and the phase angle of a series ac circuit, when $X_C = 60\ \Omega$, $X_L = 40\ \Omega$, and $R = 24\ \Omega$.
18. ***Electronics*** Find the total impedance and phase angle, when $X_L = 38\ \Omega$, $X_C = 265\ \Omega$, and $R = 75\ \Omega$.
19. ***Electronics*** In a synchronous ac motor, the current leads the applied voltage, and in an induction ac electric motor, the current lags the applied voltage. A circuit with a synchronous motor A connected in parallel with an induction motor B and a purely resistive load C has a current diagram as shown in Figure 8.40. If $I_A = 20$ A, $\theta_A = 35°$, $I_B = 15$ A, $\theta_B = -20°$, and $I_C = 25$ A, find the total current I and the phase angle between the total current and the applied voltage.

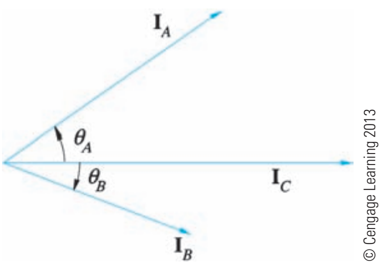

Figure 8.40

20. ***Navigation*** A ship is sailing at a speed of 12 km/h in the direction of 10°. A strong wind is exerting enough pressure on the ship's superstructure to move it in the direction of 270° at 2 km/h. A tidal current is flowing in the direction of 140° at the rate of 6 km/h. What is the ship's velocity and direction relative to the earth's surface?
21. ***Navigation*** An airplane in level flight drops an object. The plane was traveling at 120 m/s at a height of 5 000 m. The vertical component

of the dropped object is $V_y = -9.8t$ m/s. What is the magnitude of the velocity of the object after 4 s and at what angle with the ground is the object moving at this time?

22. ***Navigation*** At any time t the object in Exercise 21 will have fallen $4.9t^2$ m. How long will it take for it to strike the ground? What is its velocity at this time? At what angle will it strike the ground?

23. ***Construction*** A parallelogram with adjacent sides of lengths 1′9″ and 2′3″ is to be cut from a rectangular piece of plywood. The parallelogram contains a 40° angle. What are the dimensions of the smallest piece of plywood from which this parallelogram can be cut?

24. ***Construction*** A parallelogram with adjacent sides of lengths 15 and 32 cm is to be cut from a rectangular piece of plywood. The parallelogram contains a 35° angle. What are the dimensions of the smallest piece of plywood from which this parallelogram can be cut?

[IN YOUR WORDS]

25. Write a word problem in your technology area of interest that requires you to use vectors. On the back of the sheet of paper, write your name and explain how to solve the problem by using vectors. Give the problem you wrote to a friend and let him or her try to solve it. If your friend has difficulty understanding the problem or solving the problem, or if he or she disagrees with your solution, make any necessary changes in the problem or solution. When you have finished, give the revised problem and solution to another friend and see if he or she can solve it.

26. Write a word problem in your technology area of interest that requires you to use at least three vectors and their components in its solution. Follow the same procedures described in Exercise 25 for writing, sharing, and revising your problem.

8.4 OBLIQUE TRIANGLES: LAW OF SINES

The triangles we have examined until now have been right triangles. At first, trigonometry and the trigonometric functions dealt only with right triangles. Mathematicians quickly discovered that they needed to work with triangles that did not have a right angle. These triangles, the ones with no right angle, were named **oblique triangles**.

The trigonometric methods for solving right triangles do not work with oblique triangles. There are two methods that are usually used with oblique triangles. One of these, the Law of Sines, will be studied in this section. The other, the Law of Cosines, will be studied in Section 8.5.

The *Law of Sines* can be developed without too much difficulty. Consider the triangle in Figure 8.41. Select one of the vertices, in this case vertex C. Drop a perpendicular to the opposite side. From the resulting right triangles we have

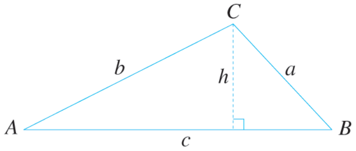

Figure 8.41

$$\sin A = \frac{h}{b} \quad \text{and} \quad \sin B = \frac{h}{a}$$

$$\text{or} \qquad h = b \sin A \quad \text{and} \quad h = a \sin B$$

Since both of these are equal to h, then $b \sin A = a \sin B$ and $\dfrac{a}{\sin A} = \dfrac{b}{\sin B}$. If we dropped the perpendicular from vertex B we would get $\dfrac{a}{\sin A} = \dfrac{c}{\sin C}$.

Putting these together, we get the Law of Sines.

THE LAW OF SINES

The **Law of Sines** or **Sine Law** is a continued proportion and states that if $\triangle ABC$ is a triangle with sides of lengths a, b, and c and opposite angles A, B, and C, then

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

You may recognize that this can also be written as the continued proportion $a:b:c = \sin A : \sin B : \sin C$.

The Law of Sines can be used to solve a triangle when the parts of a triangle are known in either of the following two cases:

Case 1 (SSA) The measure of two sides and the angle opposite one of them is known.

Case 2 (AAS) The measure of two angles and one side is known.

As is true with any continued proportion, you work with just two of the ratios at any one time. You should also remember that the angles of a triangle add up to 180° or π rad.

Let's look at an example that uses the Law of Sines. This example will fit the description of a Case 1 triangle or SSA.

EXAMPLE 8.21

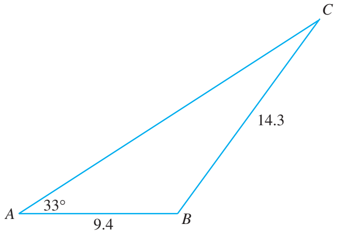

Figure 8.42

In triangle $\triangle ABC$ as shown in Figure 8.42, $A = 33°$, $a = 14.3$, and $c = 9.4$. Solve the triangle.

SOLUTION To solve this triangle means that we are to find the length of the third side and the sizes of the other two angles. The following data chart shows the parts that we know and those we are to determine.

Sides	Angles
$a =$ 14.3	$A =$ 33°
$b =$ ____	$B =$ ___
$c =$ 9.4	$C =$ ___

Since two of the known parts have the same letter, a, one of the ratios we should use is $\dfrac{a}{\sin A}$. The other known part has the letter c so the other ratio should be $\dfrac{c}{\sin C}$. By the Law of Sines

EXAMPLE 8.21 (Cont.)

$$\frac{a}{\sin A} = \frac{c}{\sin C}$$
$$\frac{14.3}{\sin 33°} = \frac{9.4}{\sin C}$$
$$\sin C = \frac{9.4(\sin 33°)}{14.3}$$
$$\approx 0.3580145$$
$$C \approx 20.98°$$

Since $C \approx 20.98°$ and $A = 33°$, then

$$B \approx 180 - 33° - 20.98°$$
$$= 126.02°$$

We can use this information to find the length of the third side of the triangle, b:

$$\frac{a}{\sin A} = \frac{b}{\sin B}$$
$$\frac{14.3}{\sin 33°} = \frac{b}{\sin 126.02°}$$
$$b = \frac{(14.3)(\sin 126.02°)}{\sin 33°}$$
$$\approx 21.24$$

We can now complete the data chart and show all the parts of this triangle:

Sides	Angles
$a = 14.3$	$A = 33°$
$b = 21.24$	$B = 126.02°$
$c = 9.4$	$C = 20.98°$

The last part of Example 8.21 was somewhat like the situation in Case 2. You knew two of the angles. Once you found the size of the third angle, you were ready to continue solving the problems.

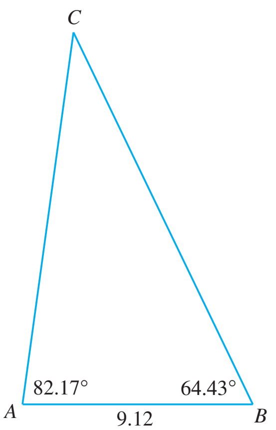

Figure 8.43

EXAMPLE 8.22

Solve the triangle ABC, if $A = 82.17°$, $B = 64.43°$, and $c = 9.12$.

SOLUTION A sketch of the triangle using the given information is in Figure 8.43. The beginning data chart is

Sides	Angles
$a =$ ___	$A = 82.17°$
$b =$ ___	$B = 64.43°$
$c = 9.12$	$C =$ ___

This is a triangle that satisfies Case 2. (It is an AAS triangle.) Since we know that $A + B + C = 180°$ and are given that $A + B = 82.17° + 64.43° = 146.60°$, we know that $C = 180° - 146.60° = 33.40°$.

We can now use the Sine Law to find either a or b. We will first find a:

$$\frac{a}{\sin A} = \frac{c}{\sin C}$$

$$\frac{a}{\sin 82.17°} = \frac{9.12}{\sin 33.40°}$$

$$a = \frac{(9.12)(\sin 82.17°)}{\sin 33.40°}$$

$$\approx 16.41$$

Now we use the Sine Law to find b:

$$\frac{b}{\sin B} = \frac{c}{\sin C}$$

$$\frac{b}{\sin 64.43°} = \frac{9.12}{\sin 33.40°}$$

$$b = \frac{(9.12)(\sin 64.43°)}{\sin 33.40°}$$

$$\approx 14.94$$

The completed data chart is

Sides	Angles
$a = 16.41$	$A = 82.17°$
$b = 14.94$	$B = 64.43°$
$c = 9.12$	$C = 33.40°$

It is possible to find 0, 1, or 2 correct solutions to the triangle if the given information includes two sides and one angle. When this **ambiguous case** occurs, carefully consider the practical application of the solution. We will now examine two situations that produce ambiguous results.

EXAMPLE 8.23

Solve $\triangle ABC$, if $a = 20$, $b = 24$, and $A = 55.4°$.

SOLUTION As usual, we begin with the data chart showing the given and unknown measurements.

Sides	Angles
$a = 20$	$A = 55.4°$
$b = 24$	$B =$ ___
$c =$ ___	$C =$ ___

EXAMPLE 8.23 (Cont.)

We will now find B.

$$\frac{a}{\sin A} = \frac{b}{\sin B}$$
$$\frac{20}{\sin 55.4°} = \frac{24}{\sin B}$$
$$\sin B = \frac{24(\sin 55.4°)}{20}$$
$$\approx 0.9877636$$
$$B = \sin^{-1}(0.9877636)$$

So, either $B = 81.03°$ or $B = 98.97°$. Remember, the sine is positive in both the first and second quadrants. Whenever you use the Sine Law and get an equation of the form $B = \sin^{-1} n$ or $\sin B = n$, where $0 < n < 1$, then there are two possible values for B.

Will both of these answers satisfy the given parts of the triangle? Let's call the two answers for angle B, B_1 and B_2. If $B_1 = 81.03°$, and since $A = 55.4°$, then $C_1 = 180° - 81.03° - 55.4° = 43.57°$. If $B_2 = 98.97°$, then $C_2 = 180° - 98.97° - 55.4° = 25.63°$.

We will now use these angles to find the length of side c. If $C_1 = 43.57°$, then

$$\frac{a}{\sin A} = \frac{c_1}{\sin C_1}$$
$$\frac{20}{\sin 55.4°} = \frac{c_1}{\sin 43.57°}$$
$$c_1 = \frac{20(\sin 43.57°)}{\sin 55.4°}$$
$$= 16.75$$

If we use $C_2 = 25.63°$, then we get

$$\frac{a}{\sin A} = \frac{c_2}{\sin C_2}$$
$$\frac{20}{\sin 55.4°} = \frac{c_2}{\sin 25.63°}$$
$$c_2 = \frac{20(\sin 25.63°)}{\sin 55.4°}$$
$$= 10.51$$

The data chart is now complete and written as two data charts.

Sides	Angles
$a = 20$	$A = 55.4°$
$b = 24$	$B_1 = 81.03°$
$c_1 = 16.75$	$C_1 = 43.57°$

Sides	Angles
$a = 20$	$A = 55.4°$
$b = 24$	$B_2 = 98.97°$
$c_2 = 10.51$	$C_2 = 25.63°$

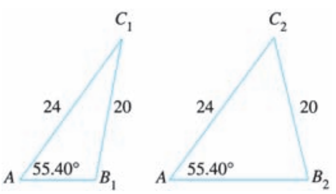

Figure 8.44

The triangles formed with these two solutions are shown in Figure 8.44. Both are correct.

The other problem that can develop with the ambiguous case is that there may be no solution. Consider the situation in Example 8.24.

EXAMPLE 8.24

Solve for triangle ABC, when $a = 20$, $b = 27$, and $A = 70°$.

SOLUTION The beginning data chart is:

Aides	Angles
$a = 20$	$A = 70°$
$b = 27$	$B =$ ___
$c =$ ___	$C =$ ___

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By the Sine Law $\dfrac{a}{\sin A} = \dfrac{b}{\sin B}$, so

$$\frac{20}{\sin 70°} = \frac{27}{\sin B}$$

$$\sin B = \frac{27(\sin 70°)}{20}$$

$$= 1.27$$

The sine of an angle is never larger than 1. This triangle is not possible. Perhaps if you look at Figure 8.45 you can get a better idea why this is not a legitimate triangle.

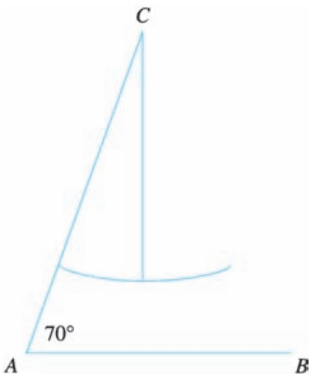

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Figure 8.45

As with all trigonometry, there are many applications of the Law of Sines in technical areas. The problem of the technician is to recognize when the application requires trigonometry and then to select the appropriate method to solve it. The next example and the problems in the exercise set will help you make the correct selection.

APPLICATION AUTOMOTIVE

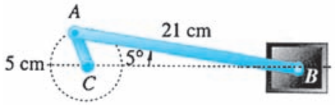
EXAMPLE 8.25

The crankshaft $\overline{CA}$ of an engine is 5 cm long and the connecting rod $\overline{AB}$ is 21 cm long. Find the size of $\angle ACB$ when the size of $\angle ABC$ is 5°.

SOLUTION A sketch of this crankshaft is in Figure 8.46. This problem falls into the SSA case. We know the lengths of two sides, AC and AB, and of the angle opposite one of them, $\angle B$. Since $AC = b$ and $AB = c$, we can use the Sine Law with

$$\frac{b}{\sin B} = \frac{c}{\sin C}$$

$$\frac{5}{\sin 5°} = \frac{21}{\sin C}$$



© Cengage Learning 2013
Figure 8.46

EXAMPLE 8.25 (Cont.)

$$\sin C = \frac{21 \sin 5°}{5} = 0.3660541$$

$$C = 21.47° \quad \text{or} \quad 158.53°$$

There are two possible solutions. By looking at Figure 8.46, you can see that both are acceptable.

EXERCISE SET 8.4

In Exercises 1–20, solve each triangle with the given parts. Check for the ambiguous cases.

1. $A = 19.4°$, $B = 85.3°$, $c = 22.1$
2. $a = 12.4$, $B = 62.4°$, $C = 43.9°$
3. $a = 14.2$, $b = 15.3$, $B = 97°$
4. $A = 27.42°$, $a = 27.3$, $b = 35.49$
5. $A = 86.32°$, $a = 19.19$, $c = 18.42$
6. $B = 75.46°$, $b = 19.4$, $C = 44.95°$
7. $B = 39.4°$, $b = 19.4$, $c = 35.2$
8. $A = 84.3°$, $b = 9.7$, $C = 12.7°$
9. $A = 45°$, $a = 16.3$, $b = 19.4$
10. $a = 10.4$, $c = 5.2$, $C = 30°$
11. $a = 42.3$, $B = 14.3°$, $C = 16.9°$
12. $A = 105.4°$, $B = 68.2°$, $c = 4.91$
13. $b = 19.4$, $c = 12.5$, $C = 35.6°$
14. $a = 121.4$, $A = 19.7°$, $c = 63.4$
15. $a = 19.7$, $b = 8.5$, $B = 78.4°$
16. $b = 9.12$, $B = 1.3$ rad, $C = 0.67$ rad
17. $b = 8.5$, $c = 19.7$, $C = 1.37$ rad
18. $b = 19.7$, $c = 36.4$, $C = 0.45$ rad
19. $A = 0.47$ rad, $b = 195.4$, $C = 1.32$ rad
20. $a = 29.34$, $A = 1.23$ rad, $C = 1.67$ rad

Solve Exercises 21–30.

21. ***Civil engineering*** Two high-tension wires are to be strung across a river. There are two towers, A and B, on one side of the river. These two towers are 360 m apart. A third tower, C, is on the other side of the river. If $\angle ABC$ is 67.4° and $\angle BAC$ is 49.3°, what are the distances between towers A and C and towers B and C?
22. ***Civil engineering*** A tunnel is to be dug between points A and B on opposite sides of a hill. A point C is chosen 250 m from A and 275 m from B. If $\angle BAC$ measures 43.62°, find the length of the tunnel.
23. ***Navigation*** A plane leaves airport A with a heading of 313°. Several minutes later the plane is spotted from airport B at a heading of 27°. Airport B is due west of airport A and the two airports are 37 mi apart. How far had the airplane flown?
24. ***Civil engineering*** From a point on the top of one end of a football stadium, the angle of depression to the 40-yard marker is 10.45°. The angle of depression to the 50-yard marker is 11.36°. How high is that end of the stadium above the playing field?
25. ***Civil engineering*** Two technicians release a balloon containing a radio-controlled camera. In order for the camera's photographs to cover enough territory, they plan to start the camera when the balloon reaches a certain height. The technicians are 400 m apart. One technician triggers the balloon when the angle of elevation at that spot is 47°. At that instant, the angle of elevation for the other technician is 67°. If the balloon is directly above the line connecting the two technicians, what is its height?

26. ***Automotive technology*** The angles between the three holes in a seat bracket are shown in Figure 8.47. If $AB = 15.6$ cm, determine AC and BC.

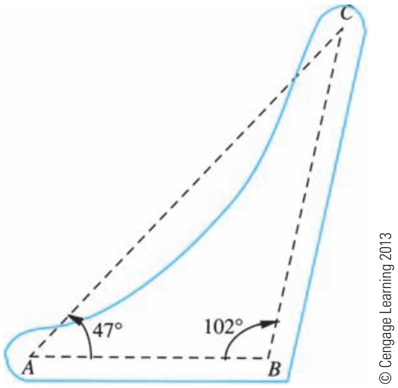


Figure 8.47

27. ***Civil engineering*** A 225.0-ft antenna mast stands on the edge of the roof of a building. A 6′0″-tall observer on the ground at some point away from the building determines that the angles of elevation to the top and bottom of the mast are 67.4° and 57.8°, respectively. How high is the building? (The observer's eyes are 5′7″ above the ground.)
28. ***Machine technology*** Holes are to be drilled in a metal plate at five equally spaced locations around a circle with a radius of 6.25 in. Find the distance between two adjacent holes.
29. ***Machine technology*** Holes are to be drilled in a metal plate at 12 equally spaced locations around a circle with a radius of 16.40 cm. Find the distance between two adjacent holes.
30. ***Civil engineering*** A guy wire to the top of a pole makes a 63.75° angle with level ground. At a point 8.2 m farther from the pole than the guy wire, the angle of elevation to the top of the pole is 42.5°. How long is the guy wire?
31. ***Industrial design*** Figure 8.48 is a drawing of three holes in a tooling plate. Determine the length of $\overline{AC}$.

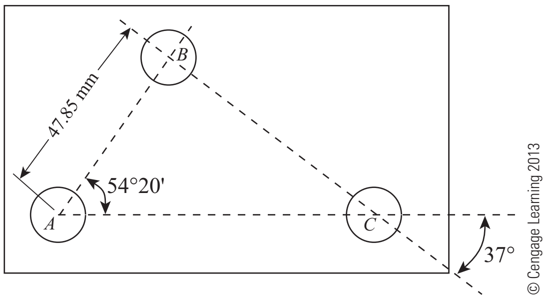


Figure 8.48

32. ***Machine design*** The side idler at C is used to maintain belt tightness in the bale drive shown in Figure 8.49. If $AC = 45.72$ cm, the distance from the center of the side idler to $\overline{AB}$ is 25.40 cm, and the size of $\angle A$ is twice the size of $\angle B$, find the lengths of $\overline{AB}$ and $\overline{BC}$.

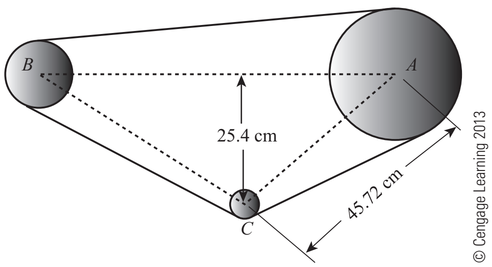


Figure 8.49

[IN YOUR WORDS]

33. Without looking in the text, write the Sine Law and describe how to use it.
34. Describe how you can tell when to use the Sine Law.
35. When using the Sine Law you have to be careful of the ambiguous case.
    (a) Explain what is meant by the ambiguous case.
    (b) How do you know that you might be working a problem that involves the ambiguous case?
    (c) What should you do if you are working a problem that may include the ambiguous case?

8.5 OBLIQUE TRIANGLES: LAW OF COSINES

In Section 8.4, we learned the Law of Sines and when to use it. We were able to use the Sine Law in two cases. In Case 1, we knew the measures for two sides and the angle opposite one of them. This we named the SSA case. Case 2 existed when we knew the measures of two angles and one side of a triangle. This we called the AAS case.

There are two other cases that lead to solvable triangles. We will learn to solve the following two cases in this section:

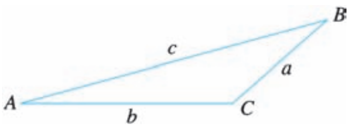


Figure 8.50

Case 3 (SAS) The measure of two sides and the included angle are known.

Case 4 (SSS) The measure of three sides is known.

The Law of Cosines is used to help solve both Cases 3 and 4. Using the general oblique triangle in Figure 8.50, the Law of Cosines can be stated as follows.

THE LAW OF COSINES

If $\triangle ABC$ is a triangle with sides of lengths a, b, and c, and opposite angles A, B, C, then by the **Law of Cosines** or **Cosine Law:**

$$a^2 = b^2 + c^2 - 2bc \cos A$$

$$b^2 = a^2 + c^2 - 2ac \cos B$$

$$\text{or} \quad c^2 = a^2 + b^2 - 2ab \cos C$$

Notice that there are three versions of the Cosine Law. Each version simply restates the law so that different parts of the triangle are used.

HINT In the Cosine Law, the side on the left-hand side of the equation has the same letter as the angle on the right-hand side of the equation.

EXAMPLE 8.26

If $b = 14.7$, $c = 9.3$, and $A = 46.3°$, solve the triangle.

SOLUTION A sketch of the triangle using the given information is in Figure 8.51. The data chart is

Sides	Angles
$a =$ ___	$A = 46.3°$
$b = 14.7$	$B =$ ___
$c = 9.3$	$C =$ ___

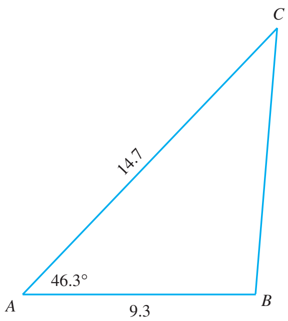


Figure 8.51

This is Case 3 or the SAS type of problem. Since we know the size of angle A, we first use the Law of Cosines to find the length of side a.

$$\begin{aligned} a^2 &= b^2 + c^2 - 2bc \cos A \\ &= (14.7)^2 + (9.3)^2 - 2(14.7)(9.3) \cos 46.3° \\ &= 216.09 + 86.49 - 273.42(0.6908824) \\ &= 216.09 + 86.49 - 188.90107 \\ &= 113.67893 \end{aligned}$$

So, $a = 10.662032$ or $a \approx 10.7$.

At this point, you have a choice as to which method to use to solve the remainder of the problem. You know the size of one angle so you could use the Sine Law. We will use an alternate version of the Cosine Law, which takes advantage of the fact that you know the lengths of all three sides. This alternate version of the Cosine Law is used for the SSS type of problem. It will be the next example, after we state the alternate version of the Cosine Law.

THE LAW OF COSINES (ALTERNATE VERSION)

If $\triangle ABC$ is a triangle with sides of lengths a, b, and c, and opposite angles A, B, C, then by the **Law of Cosines**:

$$\begin{aligned} \cos A &= \frac{b^2 + c^2 - a^2}{2bc} \\ \cos B &= \frac{a^2 + c^2 - b^2}{2ac} \\ \text{or } \cos C &= \frac{a^2 + b^2 - c^2}{2ab} \end{aligned}$$

You may notice that the alternate version of the Law of Cosines is just the original versions solved for $\cos A$, $\cos B$, and $\cos C$, respectively.

EXAMPLE 8.27

If $a = 10.7$, $b = 14.7$, and $c = 9.3$, find the sizes of the three angles.

SOLUTION A sketch of the triangle using the given information is in Figure 8.52. We will use the Cosine Law to find the size of angle B:

$$b^2 = a^2 + c^2 - 2ac \cos B$$

$$\cos B = \frac{(10.7)^2 + (9.3)^2 - (14.7)^2}{2(10.7)(9.3)}$$

$$\cos B = \frac{114.49 + 86.49 - 216.09}{199.02} \approx -0.07592201793$$

$$B = 94.354201 \approx 94.4°$$

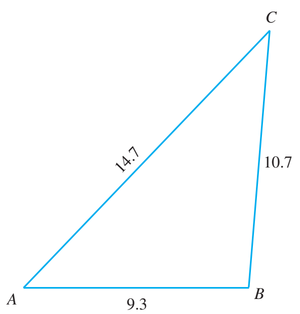


Figure 8.52

EXAMPLE 8.27 (Cont.)

Since this is a continuation of Example 8.26, we know that $A = 46.3°$, so $C = 180° - 46.3° - 94.4° = 39.3°$. The completed data chart is

Sides	Angles
$a = 10.7$	$A = 46.3°$
$b = 14.7$	$B = 94.4°$
$c = 9.3$	$C = 39.3°$

APPLICATION ELECTRONICS

EXAMPLE 8.28

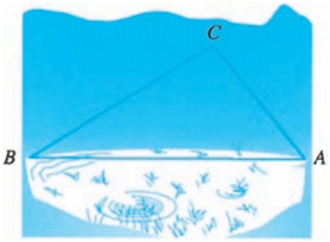


Figure 8.53

An electric transmission line is planned to go directly over a swamp. The power line will be supported by towers at points A and B in Figure 8.53. A surveyor measures the distance from B to C as 573 m, the distance from A to C as 347 m, and $\angle BCA$ as 106.63°. What is the distance from tower A to tower B?

SOLUTION $BC = a = 573$ m; $AC = b = 347$ m; $\angle BCA = \angle C = 106.63°$. This is an SAS type of problem. We want to find $AB = c$. Using the Cosine Law, we have

$$\begin{aligned} c^2 &= a^2 + b^2 - 2ab \cos C \\ &= (573)^2 + (347)^2 - 2(573)(347) \cos 106.63° \\ &= 562\,544.93 \\ c &\approx 750 \end{aligned}$$

The distance between the towers will be about 750 m.

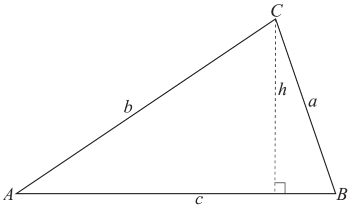


Figure 8.54

AREA OF TRIANGLES

Earlier, in Section 3.2, we saw that the area of a triangle is $A = \frac{1}{2}bh$, where b is the length of the base and h is the length of the altitude to that base.

There are situations in which you will need to compute the area of a triangle when you do not know the values of a base and a height. For example, in Figure 8.54, suppose you don't know h, but you do know the values of a, c, and angle B. You can use trigonometry to compute the height h as $h = a \sin B$. Therefore, we have the following formula for the area of a triangle if you know the lengths of two sides and the angle formed by those two sides.

AREA OF A TRIANGLE

$$A = \frac{1}{2}ac \sin B$$

Since the angles can be labeled in any order we also have the corresponding formulas:

$$A = \frac{1}{2}bc \sin A$$

$$A = \frac{1}{2}ab \sin C$$

The formula you use depends on which parts of the triangle are known.

EXAMPLE 8.29

If $\angle A = 37°$, $b = 1.2$ m, and $c = 2.5$ m, what is the area of $\triangle ABC$?

SOLUTION The area of the triangle is given by the formula:

$$\begin{aligned} A &= \frac{1}{2}bc \sin A \\ &= 0.5 \times 1.2 \times 2.5 \times \sin 37° \\ &\approx 0.90 \end{aligned}$$

The answer should contain only two significant figures, and so the area of this triangle is about 0.90 m^2.

APPLICATION ARCHITECTURE

EXAMPLE 8.30

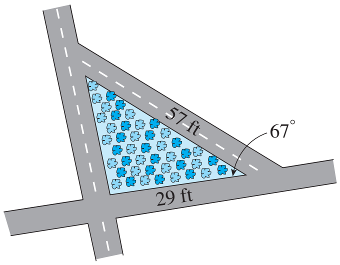


Figure 8.55

A landscape architect wants to plant flowers in a triangular section between two roads. The architect obtained the measures shown in Figure 8.55. **(a)** What is the area of the triangular region? **(b)** If each flower will eventually require a circle with a 4″ radius, how many flowers can be planted in this region?

SOLUTIONS

(a) Since we have the lengths, in feet, of two sides of the triangle and the size of the angle between these two sides, we can use the formula $A = \frac{1}{2}ac \sin B$ with $a = 29$, $c = 57$, and $B = 67°$. Thus,

$$\begin{aligned} A &= \frac{1}{2}ac \sin B \\ &\approx 760.797 \end{aligned}$$

The area is about 760.80 ft^2.

(b) If each flower will need a circle with a $4'' = \frac{1}{3}'$ radius, then each flower will need $\pi\left(\frac{1}{3}\right)^2 \approx 0.35$ ft^2. The number of flowers is determined by $760.80 \div 0.35 = 2173.71$. You can plant 2174 flowers in this triangular region.

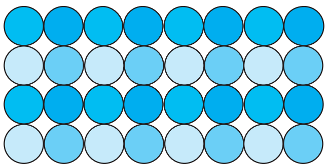


Figure 8.56a

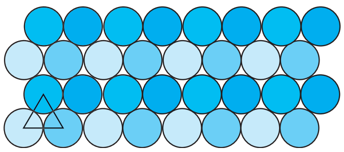


Figure 8.56b

Actually, the answer to Example 8.30(b) is too large. No matter how you arrange the flowers, there will have to be some unused space. One arrangement has all the flowers in rows and columns, as in Figure 8.56a. Another design puts three flowers at the vertices of an equilateral triangle as shown in the lower left-hand corner of Figure 8.56b. Which one will result in the most flowers for the triangular section in Example 8.30? We will leave that for you to answer in Exercise 40 of Exercise Set 8.5.

EXERCISE SET 8.5

In Exercises 1–20, solve each triangle with the given parts. The angles in Exercises 1–10 are in degrees and those in Exercises 11–20 are in radians.

1. $a = 9.3$, $b = 16.3$, $C = 42.3°$
2. $A = 16.25°$, $b = 29.43$, $c = 36.52$
3. $a = 47.85$, $B = 113.7°$, $c = 32.79$
4. $a = 19.52$, $b = 63.42$, $c = 56.53$
5. $a = 29.43$, $b = 16.37$, $c = 38.62$
6. $A = 121.37°$, $b = 112.37$, $c = 93.42$
7. $a = 63.92$, $B = 92.44°$, $c = 78.41$
8. $a = 19.53$, $b = 7.66$, $C = 32.56°$
9. $a = 4.527$, $b = 6.239$, $c = 8.635$
10. $A = 7.53°$, $b = 37.645$, $c = 42.635$
11. $a = 8.5$, $b = 15.8$, $C = 0.82$ rad
12. $A = 0.31$ rad, $b = 15.8$, $c = 38.47$
13. $a = 52.65$, $B = 1.98$ rad, $c = 35.8$
14. $a = 43.5$, $b = 63.4$, $c = 37.3$
15. $a = 36.27$, $b = 24.55$, $c = 44.26$
16. $A = 2.41$ rad, $b = 153.21$, $c = 87.49$
17. $a = 54.8$, $B = 1.625$ rad, $c = 38.33$
18. $a = 7.621$, $b = 3.429$, $C = 0.183$ rad
19. $a = 2.317$, $b = 1.713$, $c = 1.525$
20. $A = 0.09$ rad, $b = 40.75$, $c = 50.25$

In Exercises 21–24, find the area of each triangle.

21. $a = 3.7$ in., $b = 4.8$ in., $C = 39.2°$
22. $a = 9.72$ cm, $b = 3.84$ cm, $C = 117.5°$
23. $b = 34.7$ m, $c = 29.6$ m, $A = 87.5°$
24. $a = 12.875$ ft, $b = 15.250$ ft, $C = 17.2°$

Solve Exercises 25–36.

25. ***Space technology*** A tracking antenna is aimed 34.7° above the horizon. The distance from the antenna to a spacecraft is 12 325 km. If the radius of the earth is 6335 km, how high is the spacecraft above the surface of the earth? (See Figure 8.57) (Hint: Find the length of BD.)

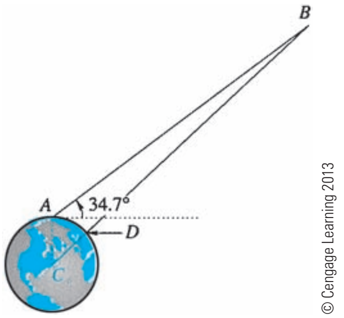


Figure 8.57

26. ***Navigation*** A ship leaves port at noon and travels due north at 21 km/h. At 3 p.m., the ship changes direction to a heading of 37°. How far from the port is the ship at 7 p.m.? What is the bearing of the ship from the port?
27. ***Construction*** A hill makes an angle of 12.37° with the horizontal. A 75-ft antenna is erected on the top of the hill. A guy wire is to be strung from the top of the antenna to a point on the hill that is 40 ft from the base of the antenna. How long is the guy wire?
28. ***Physics*** Two forces are acting on an object. The magnitude of one force is 35 lb and the magnitude of the second force is 50 lb. If the angle between the two forces is 32.15°, what is the magnitude of the resultant force?
29. ***Physics*** Figure 8.58 shows two forces represented by vector $\overrightarrow{AB}$ and $\overrightarrow{BC}$. If $AB = 12$ N, $BC = 23$ N, and $\angle ABC = 121.27°$, find the magnitude of $\mathbf{R}$ and the size of θ.

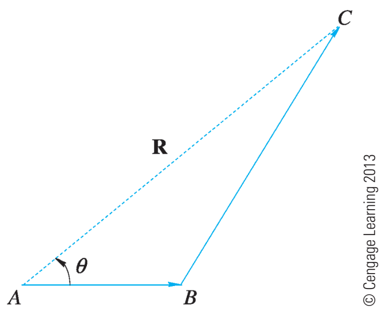


Figure 8.58

30. ***Physics*** Two forces of 15.5 and 36.4 lb are acting on an object. The resultant force is 30.1 lb. What is the size of the angle between the original two forces?

31. ***Electronics*** An inductive reactance, X_L, of 56 kΩ occurs in a circuit that has a resistance, R, of 38 kΩ. Power losses cause X_L to be 74° out of phase with R as shown in Figure 8.59. What is the impedance, Z, of the circuit?

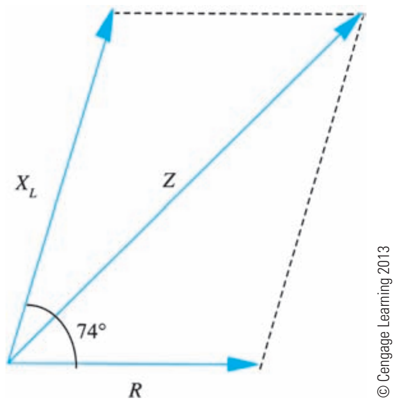


Figure 8.59

32. ***Civil engineering*** Find the length represented by x in the metal truss shown in Figure 8.60.

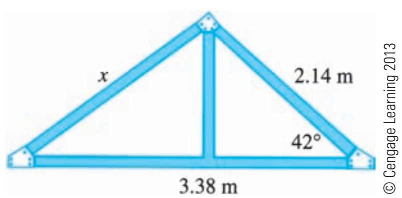


Figure 8.60

33. ***Civil engineering*** Figure 8.61 shows the drawing of a tower that is going to be erected vertically on sloping ground that is inclined 12.7° with the horizontal. Several cables will attach to a point 225 ft above the ground and be fastened to the ground. Two of the cables are shown in the figure. The downhill cable will be fastened at a point 75 ft from the base of the tower and the uphill cable will be located 40 ft from the base. What is the length of each cable?

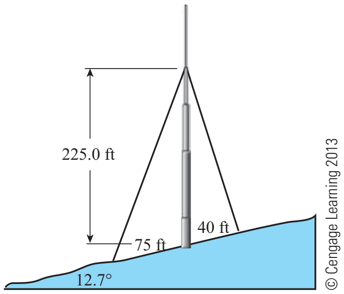


Figure 8.61

34. ***Industrial design*** Figure 8.62 is a drawing of three holes in a tooling plate. Determine the length of $\overline{BC}$.

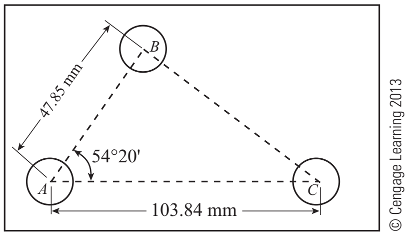


Figure 8.62

35. ***Land management*** A highway cuts a corner from a section of land, leaving the triangular piece shown in Figure 8.63. Determine the area in acres of the triangular lot. (1 acre = 43,560 ft^2)

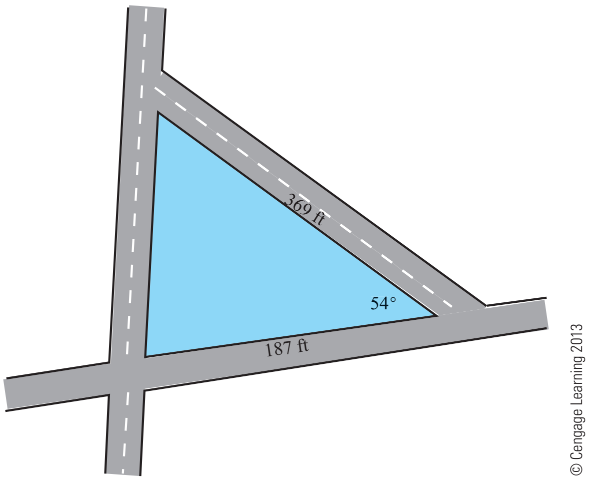


Figure 8.63

36. ***Transportation engineering*** Figure 8.64 shows a portion of a hill that must be removed in order to build a new highway. Determine the volume of the material that has to be removed.

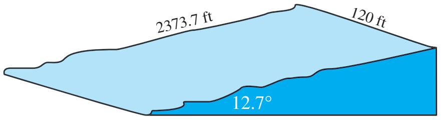


Figure 8.64

[IN YOUR WORDS]

37. Without looking in the text, write the Cosine Law and describe how to use it.

38. Describe the types of information about a triangle that you need in order to use the Cosine Law.

39. Explain how you decide whether to use the Sine Law or the Cosine Law.

40. The discussion following Example 8.30 showed two possible arrangements for placing flowers in the triangular section of Figure 8.55. Determine how many flowers can be planted in this section using each of the arrangements in Figures 8.56a and 8.56b. Decide if there might be a third arrangement that would result in more flowers.

(a) Write a description that explains how you arrived at each answer.

(b) Describe any other arrangements that you tried and the number of flowers that could be planted using each arrangement.

(c) Which arrangement results in the most flowers being planted?

(d) How many flowers can be planted using the arrangement you selected in Exercise 40(c)?

CHAPTER 8 REVIEW

IMPORTANT TERMS AND CONCEPTS

Adding vectors
- By adding components
- By the parallelogram method

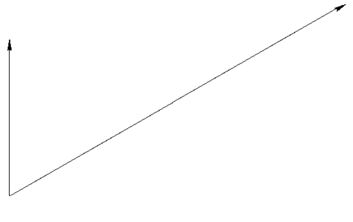
Component vectors
Cosine Law
Direction of a vector
Initial point
Magnitude of a vector
Oblique triangle
Parallelogram method
Position vector
Resultant vector
Scalar
Sine Law (Law of Sines)
- Ambiguous case

Terminal point
Vectors

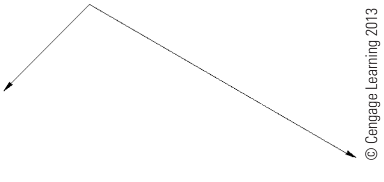
REVIEW EXERCISES

In Exercises 1–4, add the given vectors by drawing the resultant vector.

1.



2.



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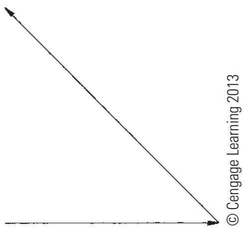
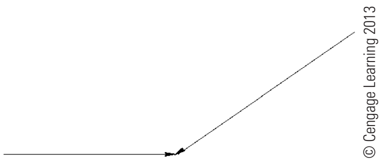


4.



In Exercises 5–8, find the horizontal and vertical components of the given vectors.

5. Magnitude 35, $\theta = 67°$

6. Magnitude 19.7, $\theta = 237°$

7. Magnitude 23.4, $\theta = 172.4°$

8. Magnitude 14.5, $\theta = 338°$

In Exercises 9 and 10, the horizontal and vertical components are given for a vector. Find each of the resultant vectors.

9. $A_x = 16, A_y = -8$

10. $B_x = -27, B_y = 32$

In Exercises 11–14, find the components of the indicated vectors.

11. $A = 38, \theta_A = 15°$

12. $B = 43.5, \theta_B = 127°$

13. $C = 19.4, \theta_C = 1.25$

14. $D = 62.7, \theta_D = 5.37$

In Exercises 15–18, add the given vectors by using the trigonometric functions and the Pythagorean theorem.

15. $A = 19, \theta_A = 32°, B = 32, \theta_B = 14°$

16. $C = 24, \theta_C = 57°, D = 35, \theta_D = 312°$

17. $E = 52.6, \theta_E = 2.53, F = 41.7, \theta_F = 3.92$

18. $G = 43.7, \theta_G = 4.73, H = 14.5, \theta_H = 4.42$

Solve each of the triangles in Exercises 19–26.

19. $a = 14, b = 32, c = 27$

20. $a = 43, b = 52, B = 86.4°$

21. $b = 87.4, B = 19.57°, c = 65.3$

22. $A = 121.3°, b = 42.5, c = 63.7$

23. $a = 127.35, A = 0.12, b = 132.6$

24. $b = 84.3, c = 95.4, C = 0.85$

25. $a = 67.9, b = 54.2, C = 2.21$

26. $a = 53.1, b = 63.2, c = 74.3$

Solve Exercises 27–32.

27. ***Physics*** Two tow trucks are attempting to right an overturned vehicle. One truck is exerting a force of 1650 kg. Its tow chain makes an angle of 68° with the axis of the vehicle. The other truck is exerting a force of 1325 kg. Its chain makes an angle of 76° with the axis of the vehicle. What is the magnitude and direction of the resultant force vector? (See Figure 8.65)

28. ***Civil engineering*** A highway engineer has to decide whether to go over or to cut through

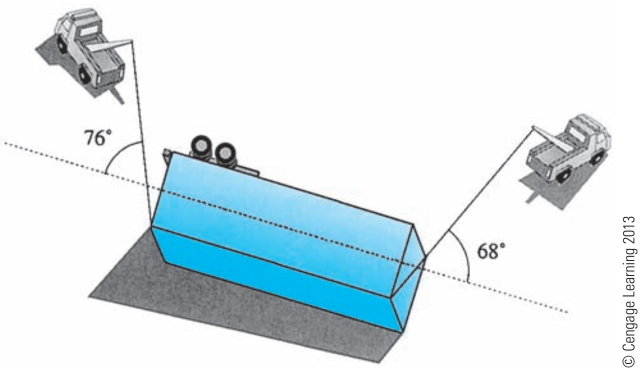


Figure 8.65

a hill. The top of the hill makes an angle of 72.4° with the sides. One side of the hill is 2,342 ft and the other side is 3,621 ft. It will cost 2.3 times as much per foot to cut through the hill and take the alternate route in Figure 8.66. How long is the alternate route? Which route is less expensive?

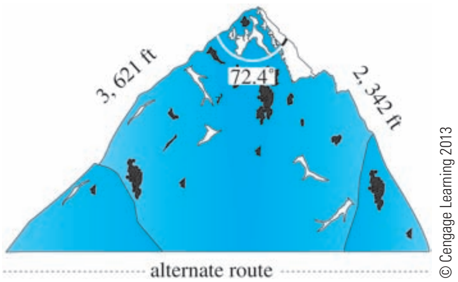


Figure 8.66

29. ***Physics*** A block is resting on a ramp that makes an angle of 31.7° with the horizontal. The block weighs 126.5 lb. Find the components of the block's weight that are parallel and perpendicular to the road. (See Figure 8.67)

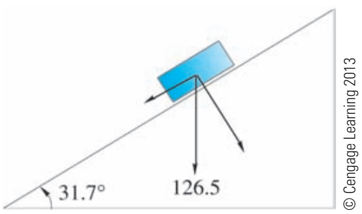


Figure 8.67

30. ***Electronics*** Find the total impedance Z and the phase angle of a series ac circuit when $X_C = 72\ \Omega$, $X_L = 52\ \Omega$, and $R = 35\ \Omega$.

31. ***Civil engineering*** A wire is to be strung across a valley. The wire will run from Tower A to Tower B. A surveyor is able to set up a position at a point C on the same side of the valley as Tower A, as shown in Figure 8.68. The distance from A to C is 73 m and $\angle BAC = 123.4°$ and $\angle ACB = 42.1°$. What is the distance from Tower A to Tower B?

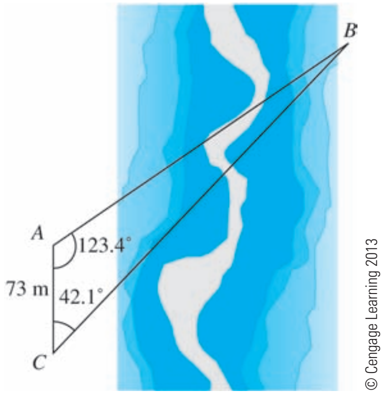


Figure 8.68

32. ***Surveying*** A surveyor needs to determine the distance across a swamp. From a point C in Figure 8.69, she locates a point B on one side of the swamp. The distance from B to C is 1235 m. Point A is directly across the swamp from B. The distance from A to C is 962 m and $\angle BCA$ is 52.57°. How far is it across the swamp from A to B?

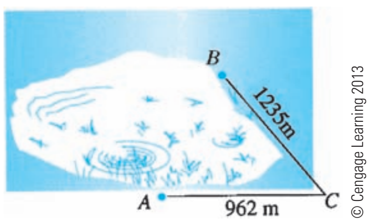


Figure 8.69

CHAPTER 8 TEST

1. Determine the horizontal and vertical components of a vector **V** of magnitude 47 and direction $\theta = 117°$.
2. If $A_x = 12.91$ and $A_y = -14.36$, determine the magnitude and direction of vector **A**.
3. Add the given vectors by using the trigonometric functions and the Pythagorean theorem: $A = 25, \theta_A = 64°, B = 40, \theta_B = 112°$.
4. Use the Sine Law to determine the length of side b in $\triangle ABC$, if $a = 9.42$, $\angle A = 35.6°$, and $\angle B = 67.5°$.
5. Use the Cosine Law to determine the length of side a of $\triangle ABC$, if $b = 4.95$, $c = 6.24$, and $\angle A = 113.4°$.
6. Solve $\triangle ABC$, if $A = 24°$, $b = 36.5$, and $C = 97°$.
7. Two walls meet at an angle of 97° to form the sides of a triangular corner cupboard. If the sides of the cupboard along each wall measure 30 and 36 in., what is the length of the front of the cupboard?

9 FRACTIONAL AND QUADRATIC EQUATIONS

Quadratic equations are needed to determine lengths or other dimensions. In Section 9.5, you will learn how to determine the lengths of the rafters in this solar collector.

In Chapter 7, we learned the special products, how to factor some algebraic expressions, and how to simplify and operate with fractions. In this chapter, we will use that knowledge to help solve problems.

You already know something about solving equations; but your knowledge has been limited to working with linear equations. In this chapter, we will venture into learning techniques for solving two new types of equations—fractional equations and quadratic equations.

OBJECTIVES

AFTER COMPLETING THIS CHAPTER, YOU WILL BE ABLE TO:

- Solve application problems involving direct, inverse, joint, or combined variation.
- Solve equations containing fractions, using algebraic and graphic methods.
- Solve quadratic equations using algebraic and graphic methods.
- Find the zeros of a quadratic or rational function.
- Solve application problems involving fractional and quadratic equations.

9.1 FRACTIONAL EQUATIONS

An equation in which one or more terms is a fraction is called a **fractional equation**. Solving a fractional equation requires a technique that we used in solving systems of linear equations. In order to add or subtract two linear equations in Section 6.2, you often had to multiply one or both equations by a nonzero number. To solve fractional equations, we will use that same technique—we will multiply the equation by a nonzero quantity. In particular, we will multiply the equation by the LCD, the lowest common denominator. This is often referred to as *clearing the equation*.

The easiest type of fractional equations to solve are those in which the variables occur only in the numerator.

EXAMPLE 9.1

Solve $\frac{2x}{3} - \frac{3x}{5} = \frac{1}{10}$ for x.

SOLUTION The LCD of $\frac{2x}{3}$, $\frac{3x}{5}$, and $\frac{1}{10}$ is 30, so we will multiply both sides of the equation by 30.

$$\begin{aligned} 30\left(\frac{2x}{3} - \frac{3x}{5}\right) &= 30\left(\frac{1}{10}\right) \\ 30\left(\frac{2x}{3}\right) - 30\left(\frac{3x}{5}\right) &= 30\left(\frac{1}{10}\right) \\ 20x - 18x &= 3 \\ 2x &= 3 \\ x &= \frac{3}{2} = 1.5 \end{aligned}$$

If we check our answer in the original problem, we see that $\frac{2(1.5)}{3} - \frac{3(1.5)}{5} = \frac{3}{3} - \frac{4.5}{5} = 1 - \frac{9}{10} = \frac{1}{10}$. The answer checks.

If the variables are in the denominator, we then need to use more caution.

CAUTION The original equation will not be defined for any values of the variable that give any of the denominators a value of 0. If you forget this, you may get an answer that does not satisfy the original problem. This type of answer is called an **extraneous solution** because it seems to be a solution but is not a valid one. For this reason, it is a good idea to study the equation first and to note any values that make the denominator 0.

In these next examples, we multiply the equation by the least common denominator for the fractions. Notice that we begin each solution by finding out which values make a denominator 0.

EXAMPLE 9.2

Solve $\frac{2}{x-5} = \frac{1}{4x-12}$ for x.

SOLUTION The LCD of $\frac{2}{x-5}$ and $\frac{1}{4x-12}$ is $4(x-5)(x-3)$. Since the LCD has a value of 0 when $x = 5$ or $x = 3$, neither of these values is a possible solution for this equation.

If we multiply both sides of the equation by the LCD, we obtain

$$\begin{aligned} 4(x-5)(x-3)\left(\frac{2}{x-5}\right) &= 4(x-5)(x-3)\left(\frac{1}{4x-12}\right) \\ 4(x-3)(2) &= x-5 \\ 8(x-3) &= x-5 \\ 8x-24 &= x-5 \\ 7x &= 19 \\ x &= \frac{19}{7} \end{aligned}$$

Thus, $x = \frac{19}{7}$ appears to be the solution. But, we should check our work to ensure that we have made no errors.

Check: The left-hand side of the equation becomes

$$\frac{2}{\frac{19}{7} - 5} = \frac{2}{\frac{19}{7} - \frac{35}{7}} = \frac{2}{\frac{-16}{7}} = -\frac{7}{8}$$

The value of the right-hand side is

$$\frac{1}{4\left(\frac{19}{7}\right) - 12} = \frac{1}{\frac{76}{7} - 12} = \frac{1}{\frac{76}{7} - \frac{84}{7}} = \frac{1}{\frac{-8}{7}} = -\frac{7}{8}$$

Both sides of the equation have a value of $-\frac{7}{8}$ when $x = \frac{19}{7}$, so $x = \frac{19}{7}$ must be the correct solution.

We could have used cross-multiplication to solve the equation in the last example. But, cross-multiplication can only be used when there is one term on each side of the equation. The next three examples show what to do when you cannot use cross-multiplication.

EXAMPLE 9.3

Solve $\frac{4}{x^2 - 1} = \frac{2}{x - 1} - \frac{3}{x + 1}$.

SOLUTION The LCD of $\frac{4}{x^2 - 1}$, $\frac{2}{x - 1}$, and $\frac{3}{x + 1}$ is $(x + 1)(x - 1) = x^2 - 1$. Notice that $x \neq 1$ and $x \neq -1$, because each of these values makes two of the denominators 0. Multiplying both sides of the given equation by $x^2 - 1$, we obtain

$$\begin{aligned} (x^2 - 1)\left(\frac{4}{x^2 - 1}\right) &= (x^2 - 1)\left(\frac{2}{x - 1}\right) - (x^2 - 1)\left(\frac{3}{x + 1}\right) \\ 4 &= (x + 1)2 - (x - 1)3 \\ 4 &= 2x + 2 - (3x - 3) \\ 4 &= 2x + 2 - 3x + 3 \\ 4 &= -x + 5 \\ -1 &= -x \\ \text{or} \qquad x &= 1 \end{aligned}$$

Since $x = 1$ is not an allowable solution, the "solution" $x = 1$ is extraneous. *This equation has no solution.*

EXAMPLE 9.4

Solve $\frac{3x}{x - 2} + 5 = \frac{7x}{x - 2}$.

SOLUTION The LCD is $x - 2$, so $x \neq 2$. Multiplying both sides by $x - 2$, we get

$$\begin{aligned} (x - 2)\left(\frac{3x}{x - 2} + 5\right) &= (x - 2)\left(\frac{7x}{x - 2}\right) \\ (x - 2)\left(\frac{3x}{x - 2}\right) + (x - 2)5 &= (x - 2)\left(\frac{7x}{x - 2}\right) \\ 3x + 5x - 10 &= 7x \\ 8x - 10 &= 7x \\ x &= 10 \end{aligned}$$

Substituting $x = 10$ into the original equation shows that it satisfies the equation.

APPLICATION GENERAL TECHNOLOGY

EXAMPLE 9.5

In a lens, if the object distance is p, the image distance is q, and the focal length is f, then the relation exists where $\frac{1}{f} = \frac{1}{p} + \frac{1}{q}$. Solve this equation for q.

EXAMPLE 9.5 (Cont.)

SOLUTION The LCD of $\frac{1}{f}$, $\frac{1}{p}$, and $\frac{1}{q}$ is fpq. Multiplying both sides of the equation by fpq, we obtain

$$fpq\left(\frac{1}{f}\right) = fpq\left(\frac{1}{p} + \frac{1}{q}\right)$$
$$pq = fpq\left(\frac{1}{p}\right) + fpq\left(\frac{1}{q}\right)$$

Multiplying further, we obtain

$$pq = fq + fp$$

To solve for q, we put the terms containing q on the left-hand side with all other terms on the right-hand side of the equation.

$$pq - fq = fp$$

At this point, we need to determine the coefficient of q. We do this by factoring.

$$q(p - f) = fp$$

We see that $p - f$ acts as the coefficient of q. Dividing by this coefficient, we obtain the desired solution.

$$q = \frac{fp}{p - f}$$

APPLICATION BUSINESS

EXAMPLE 9.6

A technician can assemble an instrument in 12.5 h. After working for 3 h on a job, the technician is joined by another technician, who is able to assemble the instrument alone in 9.5 h. How long does it take to assemble this instrument?

SOLUTION The problem is similar to the work problem we solved in Chapter 2. To solve this example, let h represent the number of hours that the technicians worked together on the instrument. The time to assemble the instrument will be $h + 3$ h because the first technician worked alone for 3 h.

The first technician, working alone, can complete the job in 12.5 h. So, each hour this technician works, $\frac{1}{12.5}$ of the instrument is assembled. Similarly, the second technician will assemble $\frac{1}{9.5}$ of the instrument for each hour worked. The first technician works $h + 3$ h and is able to complete $\frac{1}{12.5}(h + 3)$ of the work. The second technician works h hours and completes $\frac{1}{9.5}(h)$ of the work. Together, they assemble the entire instrument, so we get the equation

$$\frac{1}{12.5}(h + 3) + \frac{1}{9.5}(h) = 1$$

Multiplying by the common denominator (12.5)(9.5), we obtain

$$9.5(h + 3) + 12.5(h) = (9.5)(12.5)$$
$$9.5h + 28.5 + 12.5h = 118.75$$
$$22h = 90.25$$
$$h \approx 4.1$$

Thus, the two technicians will be able to completely assemble the instrument in about 7.1 h or 7 h 6 min. (Remember, the total time of 7.1 is $h + 3$ h.)

EXERCISE SET 9.1

In Exercises 1–30, solve the given equations and check the results.

1. $\frac{x}{2} + \frac{x}{3} = \frac{1}{4}$
2. $\frac{x}{3} - \frac{x}{4} = \frac{1}{2}$
3. $\frac{y}{2} + 3 = \frac{4y}{5}$
4. $\frac{y}{5} - 5\frac{1}{2} = \frac{3y}{4}$
5. $\frac{x - 1}{2} + \frac{x + 1}{3} = \frac{x - 1}{4}$
6. $\frac{x + 2}{3} - \frac{x + 4}{2} = \frac{x - 1}{6}$
7. $\frac{1}{x} + \frac{2}{x} = \frac{1}{3}$
8. $\frac{3}{x} - \frac{4}{x} = \frac{2}{5}$
9. $\frac{7}{w - 4} = \frac{1}{2w + 5}$
10. $\frac{5}{y + 1} = \frac{3}{y - 3}$
11. $\frac{2}{2x - 1} = \frac{5}{x + 5}$
12. $\frac{3}{4x + 2} = \frac{1}{x + 2}$
13. $\frac{4x}{x - 3} - 1 = \frac{3x}{x - 3}$
14. $7 - \frac{3x}{x + 2} = \frac{4x}{x + 2}$
15. $\frac{4}{x + 2} - \frac{3}{x - 1} = \frac{5}{(x - 1)(x + 2)}$
16. $\frac{3}{x - 3} + \frac{2}{2 - x} = \frac{5}{(x - 3)(x - 2)}$
17. $\frac{x + 1}{x + 2} + \frac{x + 3}{x - 2} = \frac{2x^2 + 3x - 5}{x^2 - 4}$
18. $\frac{x + 2}{x + 3} - \frac{x + 5}{x - 3} = \frac{2x - 1}{x^2 - 9}$
19. $\frac{3}{a + 1} + \frac{a + 1}{a - 1} = \frac{a^2}{a^2 - 1}$
20. $\frac{5}{x - 4} - \frac{x + 2}{x + 4} = \frac{x^2}{16 - x^2}$
21. $\frac{2}{x - 1} + \frac{5}{x + 1} = \frac{4}{x^2 - 1}$
22. $\frac{3x + 4}{x + 2} - \frac{3x - 5}{x - 4} = \frac{12}{x^2 - 2x - 8}$
23. $\frac{5x - 2}{x - 3} + \frac{4 - 5x}{x + 4} = \frac{10}{x^2 + x - 12}$
24. $\frac{2x}{x - 1} - \frac{3}{x + 2} = \frac{4x}{x^2 + x - 2} + 2$
25. $\frac{5}{x} + \frac{3}{x + 1} = \frac{x}{x + 1} - \frac{x + 1}{x}$
26. $\frac{y}{y + 2} + \frac{5}{y - 1} = \frac{3}{y + 2} + \frac{y}{y - 1}$

27. $\dfrac{2t-4}{2t+4}=\dfrac{t+2}{t+4}$

28. $\dfrac{3x+5}{x-5}=\dfrac{3x-1}{x+3}$

29. $\dfrac{3x+1}{x-1}-\dfrac{x-2}{x+3}=\dfrac{2x-3}{x+3}+\dfrac{4}{x-1}$

30. $\dfrac{7x+2}{x+2}+\dfrac{3x-1}{x+3}=\dfrac{6x+1}{x+3}+\dfrac{4x-3}{x+2}$

Solve each of Exercises 31–38 for the indicated variable.

31. $\dfrac{1}{r}+\dfrac{1}{s}=\dfrac{1}{t}$ for s

32. $\dfrac{P_1V_1}{T_1}=\dfrac{P_2V_2}{T_2}$ for T_1

33. $\dfrac{1}{R}=\dfrac{1}{R_1}+\dfrac{1}{R_2}+\dfrac{1}{R_3}$ for R

34. $P=\dfrac{E^2}{R+r}-\dfrac{E^2}{(R+r)^2}$ for E^2

35. $V=2\pi rh+2\pi r^2$ for h

36. $\frac{5}{9}(F-32)=C$ for F

37. $\dfrac{1}{f}=(n-1)\left(\dfrac{1}{R_1}+\dfrac{1}{R_2}\right)$ for R_2

38. $\dfrac{P_1}{g}+\dfrac{V_1^2}{2g}+h_1=\dfrac{P_2}{dg}+\dfrac{V_2^2}{2g}+h_2$ for g

Solve Exercises 39–56.

39. ***Electrical engineering*** The capacitance, C, of a spherical capacitor is given by the formula

$$\frac{d}{(9\times 10^9)C}=\frac{1}{R_2}-\frac{1}{R_1}$$

where R_2 is the outside radius of the sphere, R_1 is the inside radius, and d is the dielectric constant. Solve this equation for C.

40. ***Electrical engineering*** The capacitance, C, of a circuit containing three capacitances C_1, C_2, and C_3 in series is given by

$$\frac{1}{C}=\frac{1}{C_1}+\frac{1}{C_2}+\frac{1}{C_3}$$

Solve this equation for C.

41. ***Optics*** The *lensmaker's equation*

$$\frac{1}{f}=(n-1)\left(\frac{1}{r_1}-\frac{1}{r_2}\right)$$

gives the focal length, f, of a very thin lens. Solve this equation for f.

42. ***Optics*** An important equation in optics is

$$\frac{1}{f}=\frac{1}{p}+\frac{1}{q}$$

where f is the focal length of the lens, p is the distance to the object from the lens, and q is the distance to the image from the lens. Solve this equation for p.

43. ***Mechanical engineering*** A formula relating the depth, h, of a gear tooth to the major diameter, D, of the gear and the minor diameter, d, of the gear may be expressed as

$$\frac{h}{D-d}=2$$

Solve this equation for D.

44. ***Automotive engineering*** The formula for the efficiency of a diesel engine is given by

$$\text{Eff}=1-\frac{T_4-T_1}{\alpha(T_3-T_2)}$$

Solve this equation for T_1.

45. ***Business*** One computer can process a company's payroll in 10 h, while a newer computer can do the same job in 6 h. Working together at these rates, how long would it take to complete the payroll?

46. ***Wastewater technology*** Working alone, one pipe can fill a tank in 12 h, while a second pipe can fill the tank in 15 h. If both pipes are opened at the same time, how long does it take to fill the tank?

47. ***Wastewater technology*** Pipe A can fill a tank in 6 h and Pipe B can fill it in 4 h. If Pipe A is opened 1 h before Pipe B is opened, how long does it take to fill the tank?

48. ***Computer technology*** One microprocessor can process a set of data in 5 μs (microseconds) and a second microprocessor can process the same amount of data in 8 μs. If they process the data together, how many microseconds should it take?

49. ***Electricity*** A generator can charge a group of batteries in 18 h. It begins charging the batteries, and 4 h later a second generator starts charging the same set of batteries. If the second generator alone could charge the batteries in 12 h, how long will it take both to charge the batteries?

50. ***Transportation*** An airplane traveling against the wind travels 500 km in the same time it takes it to travel 650 km with the wind. If the wind speed is 20 km/h, find the speed of the airplane in still air.

51. ***Wastewater technology*** Working alone, it takes one large pipe 3 h to fill a tank. Two smaller pipes are used to drain the tank. Each of the smaller pipes requires 4 h to drain the tank. By mistake, one of the small pipes is left open when the tank is being filled. Assuming that no one notices the mistake, how long does it take to fill the tank?

52. ***Solar energy*** Solar collector A can absorb 12,000 Btu in 8 h. A second collector, B, is added. Together collectors A and B collect 45,000 Btu in 6 h. How long would it take collector B alone to collect 45,000 Btu?

53. ***Acoustics*** The apparent frequency f_a of sound when the source and listener are moving toward each other can be determined by solving the equation

$$\frac{V - V_s}{f} = \frac{V + V_L}{f_a}$$

for f_a. Solve the equation for f_a.

54. ***Automotive technology*** UH3D (UnderHood 3 Dimensional) is a computational fluid dynamics code used primarily in the application of prediction cooling system performance based on geometric data and performance maps of fans and heat exchangers. The volume flow coefficient ϕ used by UH3D is defined as the volumetric airflow, $\dot{Q}$, divided by the product of the fan area, A_{fan}, and the fan tip velocity, U_{tip}. That is,

$$\phi = \frac{\dot{Q}}{A_{\text{fan}} U_{\text{tip}}}$$

The fan tip velocity is defined as $U_{\text{tip}} = \pi d_{\text{tip}} \frac{N}{60}$, where d_{tip} is the diameter of the fan and N is the fan speed in rpm. Substitute this value of U_{tip} into the equation for ϕ and solve the resulting equation for d_{tip}.

55. ***Electronics*** Millman's theorem is used to determine the common voltage V across any network. It is written as

$$V = \frac{\frac{V_1}{R_1} + \frac{V_2}{R_2} + \frac{V_3}{R_3}}{\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}}$$

Solve this equation for R_1.

56. ***Optics*** Most of the discrepancy in focal points arises from approximations of the equivalency of sine and tangent values of respective angles made to the Gaussian lens equation for a spherical refracting surface:

$$\frac{n}{s} + \frac{n'}{s'} = \frac{n' - n}{r}$$

Solve this equation for r and simplify the answer.

[IN YOUR WORDS]

57. **(a)** What is an extraneous solution? **(b)** Are they good or bad? **(c)** How can you tell if you have an extraneous solution? **(d)** What should you do if you get an extraneous solution?

58. Describe what is meant by clearing an equation.

9.2 DIRECT AND INVERSE VARIATION

As we have seen, many scientific laws are given in terms of ratios or proportions. In this section, we will look at some relations between two or more variables.

DIRECT VARIATION

When two variables, x and y, are related so that their ratio $\frac{y}{x} = k$, where k is a constant, then y is **directly proportional** to x. Other ways of stating that y is directly proportional to x is to say that y is proportional to x, that y varies directly as x, or that y varies as x. The constant k is called the **constant of proportionality** or the **constant of variation**. The symbol α is sometimes used to indicate that y is directly proportional to x and is written $y \alpha x$. Note that $y \alpha x$ is not an equation, but that $y \alpha x$ means $y = kx$, where k is a constant.

DIRECT VARIATION

If x and y are variables and k is a constant, then the formula for direct variation is

$$y = kx \text{ or } \frac{y}{x} = k$$

NOTE Saying two variables are in direct variation means that if one variable increases, then the other variable increases in the same proportion. Similarly, when two variables are in direct variation, if one of the variables decreases, then the other decreases by the same amount proportionally.

EXAMPLE 9.7

The circumference of a circle is directly proportional to its diameter. Here $C = kd$, where $k = \pi$.

APPLICATION GENERAL TECHNOLOGY

EXAMPLE 9.8

The weight (w) of an object at the earth's surface is directly proportional to the mass (m) of the object. Here $w = mg$, where g is the constant acceleration of gravity. Notice that here g is used instead of k. We saw in Exercise 114 of Exercise Set 2.5 that $g \approx 9.8$ m/s^2 or $g \approx 32$ ft/s^2.

APPLICATION **GENERAL TECHNOLOGY**

EXAMPLE 9.9

When applied to a spring, *Hooke's law* states that the force F required to stretch a spring is proportional to the distance s the spring is stretched (provided the elastic limit of the spring is not exceeded). Thus $F = ks$, where k is called the *spring constant*.

APPLICATION **GENERAL TECHNOLOGY**

EXAMPLE 9.10

A force of 10 lb is required to stretch a spring from its natural length of 5 in. to a length of 8 in. Find the spring constant.

SOLUTION We know from Hooke's law that $F = ks$. Here, $F = 10$ lb and $s = 8 - 5 = 3$ in. So, $10 = k \cdot 3$ and $k = \frac{10}{3}$.

It is also possible to state that y varies as the square of x, $y = kx^2$, or y varies as the cube of x, $y = kx^3$, and so on.

EXAMPLE 9.11

The surface area of a sphere varies as the square of its radius. Since $A = 4\pi r^2$, $k = 4\pi$.

EXAMPLE 9.12

The volume of a sphere varies directly as the cube of its radius. Since $V = \frac{4}{3}\pi r^3$, the constant of proportionality is $\frac{4}{3}\pi$.

APPLICATION **GENERAL TECHNOLOGY**

EXAMPLE 9.13

The distance h that an object falls when dropped from rest is directly proportional to the square of the time t that the object has fallen. If a stone is dropped 30.625 m from a bridge and takes 2.5 s to strike the water, what is the constant of proportionality? How is the constant related to the acceleration of gravity?

SOLUTION The statement "the distance h that an object falls when dropped from rest is directly proportional to the square of the time t the object has fallen," can be written as $h = kt^2$. We are given $h = 30.625$ m and $t = 2.5$ s.

$$\text{Thus,} \qquad \begin{aligned} 30.625 &= k(2.5)^2 \\ &= k(6.25) \\ 4.9 &= k \end{aligned}$$

The constant is 4.9 m/s^2. Since the acceleration of gravity g is about 9.8 m/s^2, then $k = \frac{1}{2}g$.

If we have two pairs of values, x_1, y_1, and x_2, y_2, from a direct variation then we know that $\frac{y_1}{x_1} = k$ and $\frac{y_2}{x_2} = k$, so $\frac{y_1}{x_1} = \frac{y_2}{x_2}$, or $y_1x_2 = x_1y_2$.

APPLICATION GENERAL TECHNOLOGY

EXAMPLE 9.14

Charles' law states that at a constant pressure the temperature of a gas varies directly as its volume. This is often expressed as $\frac{V_1}{T_1} = \frac{V_2}{T_2}$, where V_1 represents the volume of the gas at an absolute temperature T_1, and similarly for V_2 and T_2. If a gas sample of 5 m^3 at 0°C is cooled until its volume is 2 m^3, what is the final temperature?

SOLUTION Charles' law is true only when we are working in degrees expressed in Kelvin (K). The official temperature unit in the SI metric system is Kelvin but people more often use the Celsius scale. If T_C is the temperature in degrees Celsius and T_K is the temperature in Kelvin, then $T_K = T_C + 273.18$.

In this problem, $0°C = 273.18$ K, and we have $V_1 = 5$ m^3, $V_2 = 2$ m^3, and $T_1 = 273.18$ K. According to Charles' law

$$\begin{aligned} \frac{V_1}{T_1} &= \frac{V_2}{T_2} \\ \frac{5}{273.18} &= \frac{2}{T_2} \\ 5T_2 &= 546.36 \\ T_2 &= 109.272\,\text{K} \end{aligned}$$

The temperature is 109.272 K or −163.908°C.

EXAMPLE 9.15

The area of a circle is directly proportional to the square of its radius. If the area of a circle with a radius of 2 cm is 4π cm^2, what is the area of a circle with a radius of 6 cm?

SOLUTION We could use the formula for the area of a circle. However, we will apply our knowledge of direct variation. If A_1, r_1, A_2, and r_2 are the areas and radii of two circles, then we know

$$\frac{A_1}{r_1^2} = \frac{A_2}{r_2^2}$$

Since $A_1 = 4\pi$, $r_1 = 2$, and $r_2 = 6$, then

$$\frac{4\pi}{2^2} = \frac{A_2}{6^2}$$

$$\text{or} \qquad \begin{aligned} 4A_2 &= (4\pi)(36) \\ A_2 &= 36\pi\ \text{cm}^2 \end{aligned}$$

INVERSE VARIATION

Another type of variation is **inverse variation**, where the product of two variables, x and y, is a constant. In inverse variation $xy = k$, or $y = \frac{k}{x}$, and we say that y is **inversely proportional** to x, or y *varies inversely* as x. Again k is called the constant of variation or the constant of proportionality.

INVERSE VARIATION

If x and y are variables and k is a constant, then the formula for inverse variation is

$$y = \frac{k}{x} \text{ or } yx = k$$

NOTE If two variables are inversely proportional, when one variable increases, the other variable decreases in the same proportion.

APPLICATION ELECTRONICS

EXAMPLE 9.16

The electrical resistance of a wire varies inversely as its cross-sectional area. If R is the resistance of a wire and A is the area of its cross-section, then $RA = k$ or $R = \frac{k}{A}$.

APPLICATION GENERAL TECHNOLOGY

EXAMPLE 9.17

Boyle's law states that at a constant temperature the volume of a sample of gas is inversely proportional to the absolute pressure applied to the gas. Thus, if p is the gas pressure and V its volume, then $pV = k$, or $V = \frac{k}{p}$.

APPLICATION GENERAL TECHNOLOGY

EXAMPLE 9.18

If 250 L of air in a cylinder exert a gauge pressure of 1 380 kilopascals (kPa), what is the constant of proportionality?

SOLUTION In Example 9.17, we learned that $pV = k$. In this example $p = 1\ 380$ kPa and $V = 250$ L, so $k = (1\ 380)(250) = 345\ 000$.

As was true with direct variation, it is also possible to state that y varies inversely as the square of x, thus $yx^2 = k\left(\text{or } y = \dfrac{k}{x^2}\right)$, or that y varies inversely as the cube of x, thus $yx^3 = k\left(\text{or } y = \dfrac{k}{x^3}\right)$, and so on.

APPLICATION ELECTRONICS

EXAMPLE 9.19

The resistance R of a wire varies inversely as its cross-sectional area A, and so $RA = k$. The area can be determined from the diameter, d. In fact, $A = \pi r^2 = \pi\left(\dfrac{d}{2}\right)^2 = \frac{\pi}{4}d^2$; thus, because $\frac{\pi}{4}$ is a constant, we could say that the resistance of a wire varies inversely as the square of its diameter. Thus, $Rd^2 = k$, or $R = \dfrac{k}{d^2}$. Note that this constant of variation is not the same constant as in Example 9.16.

If we have two pairs of values, x_1, y_1 and x_2, y_2, from the same inverse variation we then know that, since $x_1y_1 = k$ and $x_2y_2 = k$, then $x_1y_1 = x_2y_2$ or $\dfrac{x_1}{x_2} = \dfrac{y_2}{y_1}$.

APPLICATION GENERAL TECHNOLOGY

EXAMPLE 9.20

Boyle's law (see Example 9.17) is often given as $p_1V_1 = p_2V_2$, where p_1 is the gas pressure when the volume is V_1 and p_2 is its pressure when the volume is V_2. In Example 9.17, we had 250 L of air in a cylinder with a pressure of 1 380 kPa. If the volume of the cylinder is increased to 1 m^3, what is the pressure?

SOLUTION We must first change the units for one of the volumes so that both volumes are in the same unit. Since 1 m^3 = 1 000 L, we can let $V_2 = 1\,000$ L, and $V_1 = 250$ L, while $p_1 = 1\,380$ kPa. So,

$$\begin{aligned} p_1V_1 &= p_2V_2 \\ (1\,380)(250) &= p_2(1\,000) \\ 345 &= p_2 \end{aligned}$$

The pressure is 345 kPa when the volume is 1 m^3.

EXERCISE SET 9.2

In Exercises 1–6, express each of the given quantities as variations.

1. The resistance R of a wire is directly proportional to its length l.
2. The pressure P of a column of liquid varies directly as the height h of the liquid.

3. The area A of a geometric figure varies directly as the square of any dimension, d.

4. The energy E of a photon of light is directly proportional to its frequency, f.

5. The current I in a transformer is inversely proportional to the number of turns N in the winding.

6. The intensity of illumination E produced by a source of light varies inversely as the square of the distance d from the light source.

In Exercises 7–10, give the equation relating the variables and find the constant of proportionality for the given set of values.

7. r varies directly as t and $r = 4$ when $t = 6$.

8. a varies inversely as b and $a = 8$ when $b = 3$.

9. d varies directly as the square of r and $d = 8$ when $r = 4$.

10. p varies inversely as the cube of s and $p = 8$ when $s = 2$.

In Exercises 11–16, find the required value by setting up the equation and solving.

11. Find a when $d = 18$, if a varies directly as d and $a = 4$ when $d = 8$.

12. If r varies inversely with t, and $r = 6$ when $t = 3$, find the value of r when $t = 9$.

13. If v varies directly as t and $v = 60$ m/s when $t = 20$ s, what is v when $t = 1$ min?

14. If p varies inversely with q and $p = 30$ mm^3 when $q = 10$ kg/mm^2, what is p when $q = 40$ kg/mm^2?

15. If n varies directly as d^2 and $n = 10\,890$ neutrons when $d = 10\,\mu$m, what is n when $d = 2\,\mu$m?

16. If m varies inversely as r^3 and $m = 25$ kPa when $r = 30$ mm, what is m when $r = 5$ mm?

Solve Exercises 17–28.

17. ***Chemistry*** A balloon filled with 25 m^3 of air at 20°C is heated under pressure to 85°C. According to Charles' law, $V = kT$. Find **(a)** the value of the constant of proportionality and **(b)** the increase in volume. (Remember to change the temperatures from °C to K.)

18. ***Navigation*** If an airplane is traveling at a uniform rate, the distance traveled will vary directly as the time it has traveled. If a plane covers 475 mi in 1 h, how long will it take to travel 1,235 mi?

19. ***Electronics*** The resistance of an electrical wire varies directly with its length. If a wire 1,500 ft long has a resistance of 32 ohms, what is the resistance in 5,000 ft of the same wire?

20. ***Physics*** The distance d that an object falling from rest will travel varies directly as the square of the time t that the object has been falling. **(a)** If $d = 29.4$ m when $t = 3$ s, find the formula for d in terms of t. **(b)** What is d when $t = 8$ s? **(c)** What is t when $d = 150$ m?

21. ***Physics*** Using Boyle's law (see Example 9.17) and the fact that you know that the volume is 200 in.^{3} when the pressure is 40 psi, find the pressure when the volume is 50 in.^{3}.

22. ***Physics*** Use Exercise 6 and the fact that a given light source produces an illumination of 600 lux (1 lux = 1 lumen per m^2) on a surface 2 m away to determine how much illumination the same light source produces on a surface 6 m away.

23. ***Electronics*** The current in the winding of a transformer is inversely proportional to the number of turns. A transformer has 200 turns in its primary winding and 50 turns in its secondary winding. If the current in the secondary circuit is 0.3 A, what is the current in the primary circuit?

24. ***Electronics*** The voltage in a winding of a transformer is directly proportional to the number of turns in the winding. A 120-V ac power line has 300 turns in its primary winding. If there are 40 turns in its secondary winding, what is the voltage across its secondary winding?

25. ***Space technology*** The weight of a body in space varies inversely as the square of its distance from the center of earth. If a person weighs 180 lb on the surface of the earth, how much does he or she weigh 500 mi above the surface if the radius of the earth is 4,000 mi?

26. ***Recreation*** The length of time required to fill a swimming pool varies inversely as the square of the diameter of the pipe used to fill the pool. If a 2-in. pipe can fill the pool in 16 h, how long will it take for a 3-in. pipe to fill the pool?

27. ***Physics*** The mass at one end of a balanced lever varies inversely as its distance from the fulcrum. A 40-kg mass is 6 m from the fulcrum. Determine the mass needed at 15 m to balance the lever.

28. ***Machine technology*** A certain taper has a 0.516-in. taper per foot. Find the amount of taper in a 5-in. long taper if the amount of taper varies directly with the length.

[IN YOUR WORDS]

29. **(a)** Without looking at the definition in the book, explain direct variation.
(b) Find common examples, other than those in the book, that use direct variation.

30. **(a)** Without looking at the definition in the book, explain inverse variation.
(b) Find common examples, other than those in the book, that use inverse variation.

9.3 JOINT AND COMBINED VARIATION

In Section 9.2 we studied direct and inverse variation. In this section, we will study two more types of variation—joint and combined.

Both direct and inverse variation are functions of one variable. In each case we had $y = f(x)$. For direct variation, $y = kx$, and for inverse variation, $y = \frac{k}{x}$. The types of variation we will study in this section, joint and combined variation, are functions of two or more variables. Thus joint and combined variation are of the forms $y = f(x, z)$, $y = f(x, w, z)$, or $y = f(x, w, z, p)$.

JOINT VARIATION

Joint variation occurs when one quantity, y, varies directly as the product of two or more quantities. Thus, if y is jointly proportional to x and z, then $y = kxz$, or if y is jointly proportional to x, w, and z, then $y = kxwz$. The term "jointly" is sometimes omitted and we simply say that y varies as w and z or that y is proportional to w and z when $y = kwz$. Again, k is the constant of variation or the constant of proportionality.

JOINT VARIATION

If x, y, and z are variables and k is a constant, then the formula for joint variation of y with x and z is

$$y = kxz$$

EXAMPLE 9.21

The absolute temperature T of a perfect gas varies jointly as its pressure P and volume V. So, $T = kPV$.

EXAMPLE 9.22

The volume of a prism V varies jointly as the area of its base B and its height h. So, $V = kBh$. In this case, we know from our study of geometry in Chapter 3 that $k = 1$.

COMBINED VARIATION

If a quantity y varies with two or more variables in ways more complicated than that described in joint variation, it is then referred to as **combined variation**. Unlike direct, inverse, and joint variation, the term *combined variation* is seldom used in the description of the problem. In fact, a combined variation relationship will often use both "directly" and "inversely" in the description of the relationship.

GUIDELINES FOR SOLVING VARIATIONS

1. Write the appropriate variation formula.
2. Use the given information to find k.
3. Rewrite the variation formula using the value for k found in Step 2.
   If asked to find a particular value from other given information, then use Step 4.
4. Substitute this other information into the equation. Solve for the required value.

APPLICATION ELECTRONICS

EXAMPLE 9.23

The resistance R of a conductor is directly proportional to its length l, and inversely proportional to its cross-sectional area A. Thus, $R = \dfrac{kl}{A}$.

APPLICATION ELECTRONICS

EXAMPLE 9.24

The potential energy W of a capacitor is directly related to the square of its charge Q, and inversely related to its capacitance C. This relation is given by the formula $W = k\frac{Q^2}{C}$.

APPLICATION MECHANICAL

EXAMPLE 9.25

The volume of a cone V is directly related to the square of the radius of its base r and its height h. Here $V = kr^2h$ and we know that the constant of proportionality $k = \frac{1}{3}\pi$.

Just as was possible with direct and inverse variation, it is possible to use a given set of values from a joint or combined variation to find an unknown value from another set.

APPLICATION ELECTRONICS

EXAMPLE 9.26

The resistance R of an electrical wire varies directly as its length l, and inversely as the square of its diameter d. If a wire 600 m long with a diameter of 5 mm has a resistance of 32 ohms (Ω), determine the resistance in a 1 500-m wire made of the same material with a diameter of 10 mm.

SOLUTION We know that $R = \frac{kl}{d^2}$, where k is the constant of proportionality. From the given values of $R = 32\ \Omega$, $d = 5$ mm, and $l = 600$ m, we have $32 = \frac{k(600)}{5^2}$, or $k = \frac{4}{3}$. So

$$R = \frac{\frac{4}{3}(1500)}{10^2} = \frac{2000}{100} = 20$$

The resistance is 20 Ω.

APPLICATION ELECTRONICS

EXAMPLE 9.27

Express the resistance, R, of an electrical wire in terms of its length, l, and cross-sectional area, A.

SOLUTION Assuming that the cross-section is circular, then $A = \pi r^2 = \pi\left(\frac{d}{2}\right)^2 = \frac{\pi d^2}{4}$. Thus, $d^2 = \frac{4A}{\pi}$. Substituting this for d^2 in the formula from the previous example produces $R = \frac{kl}{d^2} = \frac{kl}{\frac{4A}{\pi}} = \frac{k\pi l}{4A} = \frac{\rho l}{A}$, where $\rho = \frac{k\pi}{4}$.

APPLICATION GENERAL TECHNOLOGY

EXAMPLE 9.28

Newton's law of universal gravitation states that the gravitational force of attraction F (in newtons, N) between two bodies is directly proportional to each of their masses m_1 and m_2 and inversely proportional to the square of the distance between them, d. Thus $F = \frac{km_1m_2}{d^2}$, where k is the *gravitational constant*. Two objects that each have a mass of 1 kg and are 1 m apart exert a mutual gravitational attraction of $F = \frac{k(1)(1)}{1^2} = 6.67 \times 10^{-11}$ N. So, $k \approx 6.67 \times 10^{-11}\ \text{N} \cdot \text{m}^2\ \text{kg}^2$.

The earth's mass is approximately 5.98×10^{24} kg. What is the earth's gravitational force on a space shuttle that has a mass of 29 500 kg and is in orbit 10^8 m from the center of the earth?

SOLUTION Since $k = 6.67 \times 10^{-11}$ N, we have $F = 6.67 \times 10^{-11}\frac{m_1m_2}{d^2}$, where $m_1 = 5.98 \times 10^{24}$ kg, $m_2 = 29\,500$ kg $= 2.95 \times 10^4$ kg, and $d = 10^8$ m.

$$\begin{aligned} F &= 6.67 \times 10^{-11}\frac{(5.98 \times 10^{24})(2.95 \times 10^4)}{(10^8)^2} \\ &= (6.67)(5.98)(2.95) \times 10^{-11+24+4-16} \\ &= 117.67 \times 10^1 = 1176.7 \text{ N} \end{aligned}$$

The earth's gravitational force on this space shuttle is about 1 176.7 N.

The previous examples have all used the guidelines for solving variation problems.

EXERCISE SET 9.3

In Exercises 1–6, express each of the given quantities as variations.

1. The kinetic energy K of a moving body is jointly proportional to its mass m and the square of its speed v.
2. The volume V of a parallelepiped varies jointly as its length l, width w, and height h.
3. The frequency f in hertz of a musical tone is directly proportional to the speed v of sound in air, and inversely proportional to the wavelength l of the sound wave.
4. The revolutions per minute r of a shaft being turned in a lathe varies directly as the cutting speed s of the shaft, and inversely as the diameter d of the shaft.

5. The voltage drop E in a wire varies directly as its length l and the current I, and inversely as the square of its diameter d.

6. The inductance of a solenoid L varies directly as the square of the number of turns of the coil N and the cross-sectional area of the solenoid, A, and inversely as its length l.

In Exercises 7–10, give the equation relating the variables and find the constant of proportionality for the given set of values.

7. a varies jointly as b and c, and $a = 20$, when $b = 4$ and $c = 2$.

8. p varies jointly as r, s, and t, and $p = 15$, when $r = 3$, $s = 10$, and $t = 25$.

9. u varies directly as v and inversely as w, and $u = 20$, when $v = 5$ and $w = 2$.

10. x varies directly as y and the cube of w, and inversely as the square of z, and $x = 175$, when $y = 3$, $w = 5$, and $z = 4$.

In Exercises 11–16, find the required value by setting up the equation and solving.

11. Find r when $s = 3$ and $t = 4$, if r varies jointly as s and t, and $r = 10$, when $s = 6$ and $t = 5$.

12. If f varies directly as r and inversely as l, and $f = 125$ when $r = 10$ and $l = 5$, then find the value of f when $r = 20$ and $l = 10$.

13. If a varies jointly as x and the square of y, and $a = 9$ when $x = 3$ and $y = 5$, then find a when $x = 6$ and $y = 15$.

14. Using the relationship from Exercise 13, find y when $x = 9$ and $a = 10$.

15. If p varies directly as r and the square root of t and inversely as w, and $p = 9$ when $r = 18$, $t = 9$, and $w = 6$, then find p when $r = 36$, $t = 4$, and $w = 2$.

16. Using the relationships from Exercise 15, find t when $p = 18$, $r = 9$, and $w = 4$.

Solve Exercises 17–26.

17. The area of a triangle varies jointly as the length of the base b and the height h of the triangle. If the area of a triangle is 12 when the base is 3 and the height is 8, what is **(a)** the constant of proportionality and **(b)** the area, when the base is 10 and the height is 5?

18. ***Electronics*** The energy W used by a device is directly related to the current I, the voltage V, and the length of time t, for which the energy is used. If a 240-V clothes drier draws a current of 15 A and in 45 min used 2.4 kWh (kilowatt hours) of energy, how much energy will a 120-V light bulb use, which draws a current of 1.25 A in 8 h of use?

19. ***Electronics*** The electric power consumed by a device is directly related to the square of the voltage and inversely related to the resistance. The 18 Ω filament of a tube is rated at 2 W and has an operating voltage of 6 V. What is the maximum voltage in a 50 Ω, 10-W resistor?

20. ***Electronics*** The capacitance of a parallel-plate capacitor varies directly as the area of either of its plates and inversely as the distance between them. If the plates are 0.5 mm apart and each is a square that measures 4 m on a side, the capacitance is 0.3 μF. What is the capacitance if the plates are moved so they are 0.1 mm apart?

21. ***Electronics*** The resistance of a copper wire varies directly as its length and inversely as its cross-sectional area. If a copper wire 80 m long with a cross-sectional area of 2.5 mm^2 has a resistance of 0.56 Ω, what length of 0.1-mm^2 copper wire is needed for a resistance of 3 Ω?

22. ***Electronics*** The specific resistances of most metals vary with temperature. The change in the resistance of a metal is jointly proportional to the resistance of the metal and the change in temperature. If a copper wire has a resistance of 2.25 Ω at 25°C and 2.09 Ω at 10°C, what is its resistance at 50°C?

23. ***Energy*** The rate of heat conduction through a flat plate varies jointly as the area of one face of the plate and the difference between the temperature of the opposite faces, and inversely as the thickness of the plate. A 3-in.-thick brick with a 12-in.^{2} area on one face loses heat at the rate of 50 Btu/h when the temperature is 72°F on one side and 32°F on the other side of the brick. What is the rate of conduction if the outside temperature of 32°F drops to 10°F?

24. ***Sound*** The frequency in hertz (Hz) of a guitar wire varies directly as the square root of the tension and inversely as the length. If a wire 500 mm long under 25 kg tension vibrates at 256 Hz, what is the frequency of a wire 450 mm long under a tension of 30 kg?

25. ***Physics*** The *ideal gas law* states that the pressure varies directly as the absolute temperature and inversely as the volume. A gas sample has a volume of 5 ft^3 at 70°F at a pressure of 15 lb/in.^{2} (psi). Find the volume at 200°F at a pressure of 75 lb/in.^{2} (The *Rankine scale* is the absolute temperature based on the Fahrenheit scale. If T_R represents a temperature on the Rankine scale and T_F represents the equivalent temperature on the Fahrenheit scale, then $T_R \approx T_F + 460°$. The freezing point of water is 32°F or about 492°R and the boiling point of water is 212°F or about 672°R.)

26. ***Construction*** The strength, S, of a horizontal beam supported at both ends varies jointly as the width, w, and square of the depth, d, and inversely as the length, l.
    (a) Express this relationship as a variation.
    (b) What is the effect on the strength of a beam if the width is doubled, the depth is halved, and the length remains the same?
    (c) What change in the depth will increase the strength by 75% if the width and length remain unchanged?

[IN YOUR WORDS]

27. **(a)** Describe joint variation.
    (b) Find common examples, other than those in the book, that use direct variation.
    (c) Explain how joint variation is alike and how it is different from direct variation.

28. **(a)** Without looking in the book, describe how you would solve a variation problem.
    (b) Compare your description in (a) with the guideline for solving variations given in the text.

9.4 QUADRATIC EQUATIONS AND FACTORING

Until now, all the equations we have solved have been first-degree, or linear, equations and systems of linear equations. Many technical problems require the ability to solve more complicated equations. In the remainder of this chapter, we will focus on second-degree, or quadratic, equations. As we continue through the book, we will learn how to solve more types of equations.

QUADRATIC EQUATIONS

We worked with quadratics, binomials, and trinomials in Chapter 7. A polynomial equation of the second degree is a **quadratic equation**.

QUADRATIC EQUATION

If a, b, and c are constants and $a \neq 0$, then

$$ax^2 + bx + c = 0$$

is the **standard quadratic equation**.

EXAMPLE 9.29

The following are all quadratic equations written in the standard form:

(a) $2x^2 - 3x + 5 = 0$	$a = 2, b = -3, c = 5$
(b) $4x^2 + 7x = 0$	$a = 4, b = 7, c = 0$
(c) $5x^2 - 125 = 0$	$a = 5, b = 0, c = -125$
(d) $(p + 3)x^2 + px - p + 2 = 0$	$a = p + 3, b = p, c = 2 - p$

EXAMPLE 9.30

The following are also quadratic equations, but are not in the standard form:

(a) $x^2 = 49$	$a = 1, b = 0, c = -49$
(b) $8 + 2x = \dfrac{7x^2}{2}$	$a = \dfrac{7}{2}, b = -2, c = -8$

NOTE Quadratic equations that contain fractions are often simplified by writing them without fractions and with $a > 0$. This simplification is achieved by multiplying the equation by the LCD of the coefficients. (We could write the equation in Example 9.30[b] as $-\frac{7}{2}x^2 + 2x + 8 = 0$, but it would be better to multiply the equation by -2 and obtain the equivalent equation $7x^2 - 4x - 16 = 0$, since integer coefficients are usually easier to use.)

EXAMPLE 9.31

The following are not quadratic equations:

(a) $2x^3 - x^2 + 5 = 0$	This equation has a term of degree 3. The highest degree of any term in a quadratic equation is 2.
(b) $4x + 5 = 0$	This does not have a term of degree 2. This means that $a = 0$, which contradicts part of the definition of a quadratic equation.

In order to solve a quadratic equation, we need another property for the real numbers. We have not needed the **zero-product rule** until now.

ZERO-PRODUCT RULE FOR REAL NUMBERS

If a and b are numbers and $ab = 0$, then $a = 0$, $b = 0$, or both a and b are 0.

This is a very simple but powerful statement, as shown by the next example.

EXAMPLE 9.32

Use the zero-product rule to solve $(x - 1)(x + 5) = 0$.

SOLUTION Here a has the value $x - 1$ and b has the value $x + 5$. According to the zero-product rule, either $x - 1 = 0$ or $x + 5 = 0$ (or both). If $x - 1 = 0$, then $x = 1$. If $x + 5 = 0$, then $x = -5$. These are the roots (also called solutions or zeros) of the equation.

Substitute 1 into the original equation. Did you get 0? Now substitute -5 and you will get 0 again. So, both of these answers check. The solutions are $x = -5$ and $x = 1$.

ROOTS OF QUADRATIC EQUATIONS

The zero-product rule indicates that if we can factor a quadratic equation, then we can find its roots or solutions. A quadratic equation will never have more than two roots. Normally, all quadratic equations are considered to have two roots. But, there are times when both of these roots are the same number. In this case, the roots are referred to as *double roots*. There are also times when there will be no real numbers that are roots of a quadratic equation. In this case, the roots will be imaginary numbers. (We will talk more about this later in this chapter and again in Chapter 14.)

FINDING ROOTS BY FACTORING

Let's begin by looking at the general idea behind the method of factoring to find roots of a quadratic equation. Suppose we have a general quadratic equation $ax^2 + bx + c = 0$ and that this equation can be factored as follows:

$$ax^2 + bx + c = (rx + t)(sx + v) = 0$$

From the zero-product rule, we know that $rx + t = 0$ or $sx + v = 0$. Solving each of these linear equations, we get $x = \frac{-t}{r}$ and $x = \frac{-v}{s}$, which are the roots of the quadratic equation. Now, let's look at some examples showing how to use this method.

EXAMPLE 9.33

Find the roots of $x^2 - x - 6 = 0$.

SOLUTION This quadratic equation factors to $(x - 3)(x + 2)$. So, from the zero-product rule, we have

$$\begin{aligned} x^2 - x - 6 &= 0 \\ (x - 3)(x + 2) &= 0 \\ x - 3 &= 0 \text{ and } x = 3 \\ \text{or} \qquad x + 2 &= 0 \text{ and } x = -2 \end{aligned}$$

EXAMPLE 9.34

Find the roots of $-x^2 + 6x - 9 = 0$.

SOLUTION Begin by factoring a -1 out of the left-hand side of the equation. The result is $-1(x^2 - 6x + 9) = 0$ or $-(x^2 - 6x + 9) = 0$. Factoring $x^2 - 6x + 9 = 0$, we get $(x - 3)^2$, so

$$\begin{aligned} -x^2 + 6x - 9 &= 0 \\ -(x^2 - 6x + 9) &= 0 \\ -(x - 3)(x - 3) &= 0 \\ x - 3 &= 0 \text{ and } x = 3 \end{aligned}$$

This is a double root. Both roots are the same: 3.

EXAMPLE 9.35

Find the roots of $6x^2 - 11x - 35 = 0$.

SOLUTION

$$\begin{aligned} 6x^2 - 11x - 35 &= 0 \\ (3x + 5)(2x - 7) &= 0 \\ 3x + 5 &= 0 \text{ and } x = -\frac{5}{3} \\ \text{or} \qquad 2x - 7 &= 0 \text{ and } x = \frac{7}{2} \end{aligned}$$

It is very important that one side of the equation is equal to 0.

CAUTION The property that we are using, the zero-product rule, says that if $ab = 0$, then $a = 0$ or $b = 0$ (or both). It is guaranteed to work *only* if the right-hand side of the equation is 0. For example, $ab = 1$ does *not* imply that $a = 1$ or $b = 1$.

Suppose you had a problem such as $(x - 2)(x + 4) = 16$, where the right-hand side of the equation is not 0, and you try to use the zero-product rule to solve the equation. If $(x - 2)(x + 4) = 16$, you might incorrectly say $x - 2 = 16$ or $x + 4 = 16$. In the first case, $x - 2 = 16$, we get a possible solution of $x = 18$. In the second case, $x + 4 = 16$, we obtain an answer of $x = 12$.

Now check these answers.

If $x = 18$, then $(x - 2)(x + 4) = (18 - 2)(18 + 4) = (16)(22) = 352$, which is certainly not 16. This answer does not check. Let's try the other solution.

If $x = 12$, then $(x - 2)(x + 4) = (12 - 2)(12 + 4) = (10)(16) = 160$. Again, we do not get an answer of 16.

This was intended to show you that it is important to make sure that the right-hand side of the equation is 0 before applying the zero-product rule. The next example will show you how you should have solved a problem such as this one when the right-hand side of the equation is not 0.

EXAMPLE 9.36

Solve $(x - 2)(x + 4) = 16$ for x.

SOLUTION Begin by expanding the left-hand side of the equation and then subtracting 16 from both sides.

$$\begin{aligned}(x-2)(x+4) &= 16\\ x^2+2x-8 &= 16\\ x^2+2x-24 &= 0\\ (x-4)(x+6) &= 0\\ x-4 &= 0 \text{ and } x = 4\\ \text{or} \qquad x+6 &= 0 \text{ and } x = -6\end{aligned}$$

Now check these answers.

If $x = 4$, then $(x - 2)(x + 4) = (4 - 2)(4 + 4) = (2)(8) = 16$. So, 4 is a solution.

If $x = -6$, then $(x - 2)(x + 4) = (-6 - 2)(-6 + 4) = (-8)(-2) = 16$. Again, we get a correct answer.

EXAMPLE 9.37

Find the roots of $4x^2 - 10 = 3x$.

SOLUTION Before we can factor this problem, we have to get all the terms on the left-hand side of the equation; then the right-hand side will be 0.

$$\begin{aligned}4x^2-10 &= 3x\\ 4x^2-3x-10 &= 0\\ (4x+5)(x-2) &= 0\\ 4x+5 &= 0 \text{ and } x = -\frac{5}{4}\\ \text{or} \qquad x-2 &= 0 \text{ and } x = 2\end{aligned}$$

So, the two roots of this equation are $x = -\frac{5}{4}$ and $x = 2$.

It is also possible to solve some fractional equations by factoring. You will need to first multiply the equation by the LCD, and then put all the nonzero terms on the left-hand side of the equation before you begin to factor the equation. The next example shows how to do this.

EXAMPLE 9.38

Find the roots of $\dfrac{6}{x(3-2x)} - \dfrac{4}{3-2x} = 1$.

SOLUTION The LCD is $x(3 - 2x)$, so $x \neq 0$ and $x \neq \frac{3}{2}$. Multiplying both sides of the equation by $x(3 - 2x)$ provides

EXAMPLE 9.38 (Cont.)

$$6 - 4x = x(3 - 2x)$$
$$6 - 4x - x(3 - 2x) = 0$$
$$6 - 4x - 3x + 2x^2 = 0$$
$$2x^2 - 7x + 6 = 0$$
$$(2x - 3)(x - 2) = 0$$
$$2x - 3 = 0 \text{ and } x = \frac{3}{2}$$
$$\text{or} \qquad x - 2 = 0 \text{ and } x = 2$$

Since $x \neq \frac{3}{2}$, the only root is $x = 2$.

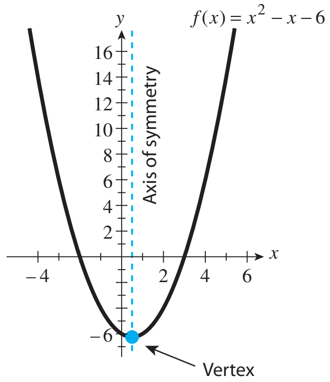


Figure 9.1

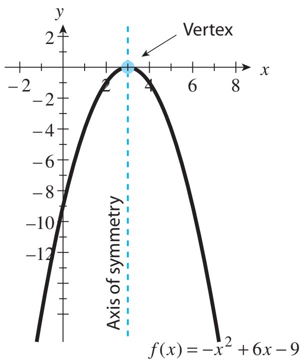


Figure 9.2

This section will not place much emphasis on graphing. But, you should realize that a quadratic function and its roots can be graphically represented. Figures 9.1 and 9.2 show the graphs for the quadratic functions $f(x) = x^2 - x - 6$ and $f(x) = x^2 + 6x + 9$, formed by the quadratic equations in Examples 9.33 and 9.34. As we will learn in Chapter 15, these are both examples of graphs called "parabolas."

Every **parabola** is symmetric about a line called the *axis of symmetry*. In Figure 9.1 the axis of symmetry is the line $x = 0.5$. In Figure 9.2 the axis of symmetry is the line $x = 3$. The point where the parabola crosses the axis of symmetry is the *vertex*. The x-coordinate of the vertex of a quadratic function $f(x) = ax^2 + bx + c$ is $x = \frac{-b}{2a}$. So, for the quadratic function in Figure 9.1, the vertex is when $x = \frac{-(-1)}{2(1)} = 0.5$. The y-coordinate of the vertex is $f(0.5) = -6.25$.

In Example 9.33, we found that the roots of $x^2 - x - 6 = 0$ were -2 and 3 and that these are the points where the graph crosses the x-axis as shown in Figure 9.1. In Example 9.34, the quadratic equation $-x^2 + 6x - 9 = 0$ had a double root, $x = 3$. The graph of the quadratic function for Example 9.34, $f(x) = -x^2 + 6x - 9$ as shown in Figure 9.2, intersects the x-axis at exactly one point, and that one point is $x = 3$.

EXAMPLE 9.39

Find the roots of $x^2 = 16$.

SOLUTION This equation is equivalent to $x^2 - 16 = 0$. Factoring, we obtain $(x - 4)(x + 4) = 0$ and $x = 4$ or $x = -4$. We could have solved this problem with less work if we had taken the square root of both sides.

$$x^2 = 16$$
$$x = \pm\sqrt{16}$$
$$x = \pm 4$$

In general, if $c \geq 0$ and $x^2 = c$, then $x = \pm\sqrt{c}$.

EXAMPLE 9.40

Find the roots of $9x^2 = 25$.

SOLUTION

$$9x^2 = 25$$
$$x^2 = \frac{25}{9}$$
$$x = \pm\sqrt{\frac{25}{9}} = \pm\frac{5}{3}$$

EXAMPLE 9.41

Find the roots of $(x - 4)^2 = 25$.

SOLUTION There are two ways to work this problem.

Method 1: First, we will square the left-hand side of the equation and collect like terms.

$$(x - 4)^2 = 25$$
$$x^2 - 8x + 16 = 25$$
$$x^2 - 8x + 16 - 25 = 0$$
$$x^2 - 8x - 9 = 0$$
$$(x - 9)(x + 1) = 0$$
$$x = 9 \text{ or } x = -1$$

Method 2: This is a shortcut. It will save time, but you have to be careful that you do not make errors. We will first take the square root of both sides. Once this is done, we will solve the linear equation for both values of x.

$$(x - 4)^2 = 25$$
$$\sqrt{(x - 4)^2} = \pm\sqrt{25}$$
$$x - 4 = \pm 5$$
$$x = 4 + 5 = 9$$

or

$$x = 4 - 5 = -1$$

This is the same answer we got using the first method.

APPLICATION MECHANICAL

EXAMPLE 9.42

A rectangular piece of aluminum is to be used to form a box. (See Figure 9.3a.) A 2-in. square is to be cut from each corner and the ends are to be folded up to form an open box. (See Figure 9.3b.) If the original piece of aluminum was twice as long as it was wide, and the volume of the box is 672 in.^{3}, what were the dimensions of the original rectangle?

SOLUTION If the width of the original rectangle is w, then the length is $2w$. The width of the box is $w - 4$ and the length of the box is $2w - 4$. The volume of the box is the product of the width, length, and height or

EXAMPLE 9.42 (Cont.)

$$
\begin{aligned}
wlh &= V \\
(w-4)(2w-4)2 &= 672 \\
4w^2 - 24w + 32 &= 672 && \text{Multiply.} \\
4w^2 - 24w - 640 &= 0 && \text{Put equation in standard form.} \\
w^2 - 6w - 160 &= 0 && \text{Divide both sides by 4.} \\
(w-16)(w+10) &= 0 && \text{Factor.}
\end{aligned}
$$

If $w - 16 = 0$, then $w = 16$, and if $w + 10 = 0$, then $w = -10$. The last answer does not make sense. We cannot have a rectangle with a width of -10 in. So, the width of the original rectangle is 16 in. and the length is 32 in.

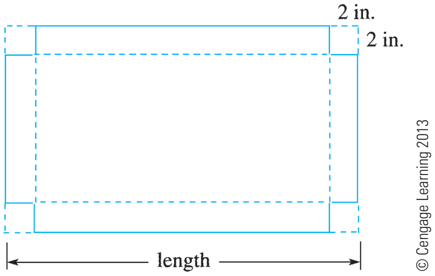


Figure 9.3a

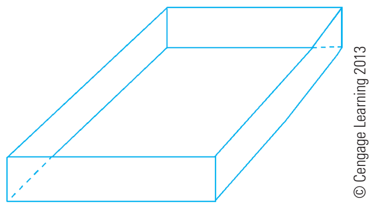


Figure 9.3b

EXERCISE SET 9.4

In Exercises 1–44, solve the quadratic equations by factoring.

1. $x^2 - 9 = 0$
2. $x^2 - 100 = 0$
3. $x^2 + x - 6 = 0$
4. $x^2 - 6x - 7 = 0$
5. $x^2 - 11x - 12 = 0$
6. $x^2 - 5x + 4 = 0$
7. $x^2 + 2x - 8 = 0$
8. $x^2 + 2x - 15 = 0$
9. $x^2 - 5x = 0$
10. $x^2 + 10x = 0$
11. $x^2 + 12 = 7x$
12. $x^2 = 7x - 10$
13. $2x^2 - 3x - 14 = 0$
14. $2x^2 + x - 15 = 0$
15. $2x^2 + 12 = 11x$
16. $2x^2 + 18 = 15x$
17. $3x^2 - 8x - 3 = 0$
18. $3x^2 - 4x - 4 = 0$
19. $4x^2 - 24x + 35 = 0$
20. $6x^2 - 13x + 6 = 0$
21. $6x^2 + 11x - 35 = 0$
22. $10x^2 + 9x - 9 = 0$
23. $10x^2 - 17x + 3 = 0$
24. $14x^2 - 29x - 15 = 0$
25. $6x^2 = 31x + 60$
26. $15x^2 = 23x - 4$
27. $(x - 1)^2 = 4$
28. $(x + 2)^2 = 9$
29. $(5x - 2)^2 = 16$
30. $(3x + 2)^2 = 64$
31. $\dfrac{x}{x+1} = \dfrac{x+2}{3x}$
32. $\dfrac{4x}{x-1} = \dfrac{7x+2}{x}$
33. $(x + 5)^3 = x^3 + 1385$
34. $(x - 3)^3 = x^3 - 63$
35. $(x - 4)^3 - x^3 = -316$
36. $(x + 2)^3 - x^3 = 56$
37. $\dfrac{1}{x-3} + \dfrac{1}{x+4} = \dfrac{1}{12}$
38. $\dfrac{1}{x-5} + \dfrac{1}{x+3} = \dfrac{1}{3}$
39. $\dfrac{1}{x-1} + \dfrac{1}{x-2} = \dfrac{7}{12}$
40. $\dfrac{1}{x+1} - \dfrac{1}{x+2} = \dfrac{1}{20}$
41. $\dfrac{2}{x-4} + \dfrac{1}{x-9} = \dfrac{1}{6}$
42. $\dfrac{x}{x+1} + \dfrac{1}{x} = \dfrac{13}{12}$
43. $\dfrac{x}{x+1} - \dfrac{2x}{x+3} = -\dfrac{1}{15}$
44. $\dfrac{3x}{x-1} - \dfrac{9x}{x+2} = 8.4$

Solve Exercises 45–58.

45. ***Dynamics*** A ball is thrown vertically upward into the air from the roof of a building 192 ft high. The height of the ball above the ground is a function of the time in seconds and the initial velocity of the ball. If the initial velocity is 64 ft/s, then the height is given by $h(t) = -16t^2 + 64t + 192$. How many seconds will it take for the ball to return to the roof? (That is, when will $h(t) = 192$?)

46. ***Dynamics*** If the ball in Exercise 45 misses the building when it comes down, how long will it take for the ball to hit the ground? (When will $h(t) = 0$?)

47. The length of a rectangle is 5 cm more than its width. Find the length and width, if the area is 104 cm^2.

48. ***Dynamics*** Figure 9.4 is a drawing of the Burj Khalifa in Dubai. At 828 m (2,716.5 ft), this is the tallest building in the world. Suppose that a ball is dropped from the top of the building and it can fall without hitting anything until it strikes the ground. Neglecting air resistance, the ball falls at 4.9 m/s^2. How long will it take for the ball to strike the ground? (Hint: Solve $4.9t^2 = 828$.)

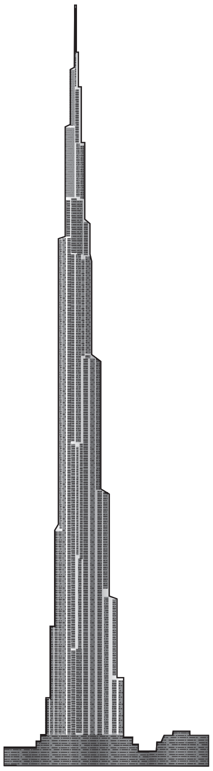


Figure 9.4

49. ***Solar Energy*** In order to support a solar collector at the correct angle, the roof trusses for a building are designed as right triangles, as shown in Figure 9.5. The rafter on the same side as the solar collector is 7 m shorter than the other rafter and the base of each truss is 13 m long. What are the lengths of the rafters?

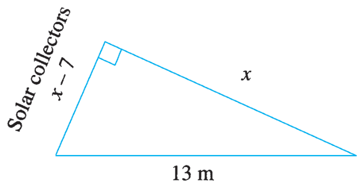


Figure 9.5

50. ***Computer technology*** Working alone, computer A can complete a data-processing job in 6 h less than computer B working alone. Together the two computers can complete the job in 4 h. How long would it take each computer by itself?

51. ***Construction*** A rectangular concrete pipe is constructed with a 10.00 in. by 16.00 in. interior channel, as shown in Figure 9.6. What uniform width w of concrete must be formed on all sides if the total cross-sectional area of the concrete must be 192 in.^{2}?

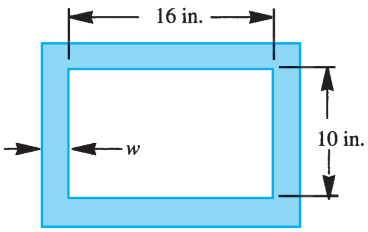


Figure 9.6

52. ***Construction*** A rectangular parking lot at a shopping mall is 50.0 m wide and 80.0 m long. The developers of the mall want to expand the parking area to 18 000 m^2. They plan to do this by adding an equal length, in meters, to the length and to the width. What are the dimensions of the new parking lot?

53. ***Recreation*** A swim club has a circular swimming pool with a diameter of 24 m. The club wants to build a deck of uniform width around the pool. Because of the financial condition of the club and the cost of materials, the club members can only afford to build a deck of area 432π m^2 around the pool. How wide should they build the deck?

54. ***Recreation*** A swim club has a rectangular swimming pool 30.0 ft long and 22 ft wide. The club wants to build a deck of uniform width around the pool. Because of the financial condition of the club and the cost of materials, the club members can only afford to build a deck of 480 ft^2 around the pool. How wide should they build the deck?

55. ***Wastewater technology*** Working together, two pipes, A and B, can fill a tank in 4 h. It takes pipe A 6 h longer than pipe B to fill the tank alone. How long would it take each pipe alone to fill the tank?

56. ***Electronics*** In a certain circuit, the power (in watts) is measured using current, I (in amperes, A), voltage E (in volts, V), and resistance, R (in ohms, Ω), by $RI^2 + EI = 18\,000$. If the resistance is 10.0 Ω and the voltage is 510 V, find the current in amperes.

57. ***Forestry*** Volume estimates, V, in ft^3, for shortleaf pine trees are based on D, the d.b.h. (diameter at breast height) in inches; top d.i.b. (the diameter inside the bark at the top of the tree) in inches; and H (the height of the tree in ft). One formula for trees with a 3-in. top d.i.b. is

$$V = 0.002837D^2H - 0.127248$$

Determine D for a 75-ft-high tree that has a volume estimate of 47.75 ft^3. (Note: This formula does not require any unit conversions.)

58. ***Forestry*** Volume estimates, V, in ft^3, for shortleaf pine trees are based on D, the d.b.h. (diameter at breast height) in inches; top d.i.b. (the diameter inside the bark at the top of the tree) in inches; and H (the height of the tree in ft). One formula for trees with a 4-in. top d.i.b. is

$$V = 0.002835D^2H - 0.337655$$

Determine D for an 85-ft-high tree that has a volume estimate of 61.35 ft^3.

[IN YOUR WORDS]

59. **(a)** Give an example of a quadratic equation.

(b) How do you know that your equation is quadratic?

60. **(a)** What is a root of a quadratic equation?

(b) What is the zero-product rule for real numbers?

(c) How does the zero-product rule help to find roots of a quadratic equation?

9.5 COMPLETING THE SQUARE AND THE QUADRATIC FORMULA

Factoring is one method that can be used to solve quadratic equations. However, it is very difficult to factor some equations. In fact, most equations cannot be factored using integers or rational numbers. We are going to introduce a technique called **completing the square**, which we can use to solve these quadratic equations. We will then use completing the square to develop a formula that will allow us to solve any quadratic equation.

COMPLETING THE SQUARE

Let's look at some special products—those that are perfect squares. The general form is $(x + k)^2 = x^2 + 2kx + k^2$. Notice that the constant, k^2, is the square of one-half of $2k$, the coefficient of x. If we combine the method for solving quadratic equations and the special product for perfect squares, we can develop the method of completing the square.

Suppose you had the equation $x^2 + 6x - 10 = 0$. A quick check of the discriminant $(b^2 - 4ac)$ shows that it is $6^2 - 4(1)(-10) = 36 + 40 = 76$. Since 76 is not a perfect square, we cannot factor this equation. Rewrite the equation so that the variables are on the left-hand side and the constant term is on the right-hand side. We want to add something to the left-hand side so that side is a perfect square. (We have placed an empty box on each side of the equation to show that we will add something to both sides when we complete the square.)

$$x^2 + 6x + \square = 10 + \square$$

Complete the square on the left-hand side by taking one-half of the coefficient of the x-term, squaring it, and adding this number to both sides of the equation. The coefficient of x is 6, half of that is 3, and $3^2 = 9$. This is added to both sides of the equation and placed inside the empty boxes.

$$x^2 + 6x + \boxed{9} = 10 + \boxed{9}$$

or $$x^2 + 6x + 9 = 19$$

The left-hand side is now a perfect square.

$$(x + 3)^2 = 19$$

We can solve this equation by using the second method from Example 9.41.

$$\begin{aligned}
\sqrt{(x + 3)^2} &= \pm\sqrt{19} \\
x + 3 &= \pm\sqrt{19} \\
x &= -3 \pm \sqrt{19} \\
x &= -3 + \sqrt{19} \text{ or } = -3 - \sqrt{19}
\end{aligned}$$

Check these answers. (The easiest way is to use your calculator.) You should see that they check when they are substituted in the original equation $x^2 + 6x - 10 = 0$.

EXAMPLE 9.43

Find the roots of $x^2 - 9x - 5 = 0$.

SOLUTION

$$x^2 - 9x - 5 = 0$$
$$x^2 - 9x + \boxed{} = 5 + \boxed{}$$
$$x^2 - 9x + \boxed{\left(\tfrac{9}{2}\right)^2} = 5 + \boxed{\left(\tfrac{9}{2}\right)^2}$$
$$= 5 + \frac{81}{4}$$
$$= \frac{101}{4}$$
$$\left(x - \frac{9}{2}\right)^2 = \frac{101}{4}$$
$$\sqrt{\left(x - \frac{9}{2}\right)^2} = \pm\sqrt{\frac{101}{4}} = \pm\frac{\sqrt{101}}{2}$$
$$x - \frac{9}{2} = \pm\frac{\sqrt{101}}{2}$$
$$x = \frac{9}{2} \pm \frac{\sqrt{101}}{2}$$
$$x = \frac{9 + \sqrt{101}}{2} \text{ or } x = \frac{9 - \sqrt{101}}{2}$$

Before you complete the square, the coefficient on the x^2-term must be 1. One way to do this is shown in the next example.

EXAMPLE 9.44

Find the roots of $2x^2 - 8x + 3 = 0$.

SOLUTION This is slightly complicated by the fact the coefficient of $2x^2$ is not 1. Our first step will be to divide the equation by 2 and then proceed as we have before.

$$2x^2 - 8x + 3 = 0$$
$$x^2 - 4x + \frac{3}{2} = 0$$
$$x^2 - 4x + \boxed{} = -\frac{3}{2} + \boxed{}$$
$$x^2 - 4x + \boxed{2^2} = -\frac{3}{2} + \boxed{2^2}$$
$$(x - 2)^2 = -\frac{3}{2} + 4 = \frac{5}{2}$$
$$x - 2 = \pm\sqrt{\frac{5}{2}}$$
$$x = 2 \pm \sqrt{\frac{5}{2}}$$

THE QUADRATIC FORMULA

We will now use completing the square to develop a general formula that can be used to find the roots of any quadratic equation.

Suppose we have a standard quadratic equation

$$ax^2 + bx + c = 0, \text{ with } a \neq 0$$

What are the roots of this quadratic equation? If we complete the square, we can find out.

$$ax^2 + bx + c = 0$$

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0 \qquad \text{Divide both sides by } a.$$

$$x^2 + \frac{b}{a}x = -\frac{c}{a} \qquad \text{Add } -\frac{c}{a} \text{ to both sides.}$$

$$x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 = -\frac{c}{a} + \left(\frac{b}{2a}\right)^2 \qquad \text{Complete the square by adding to } \left(\frac{b}{2a}\right)^2 \text{ both sides.}$$

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2}{4a^2} - \frac{4ac}{4a^2} \qquad \text{Factor the left-hand side; reverse terms on the right-hand side.}$$

$$= \frac{b^2 - 4ac}{4a^2} \qquad \text{Collect terms on the right-hand side.}$$

$$x + \frac{b}{2a} = \pm\sqrt{\frac{b^2 - 4ac}{4a^2}} \qquad \text{Take the square root of both sides.}$$

$$= \pm\frac{\sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a} \qquad \text{Solve for } x.$$

$$= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

This is the **quadratic formula**.

QUADRATIC FORMULA

The solutions of the equation $ax^2 + bx + c = 0$, $(a \neq 0)$ are given by

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

The expression $b^2 - 4ac$ is called the **discriminant**.

To solve a quadratic equation by using the quadratic formula, write the equation in the standard form, identify a, b, and c, and substitute these numbers into the equation. The quadratic formula is a very useful tool, but many equations are easier to solve by factoring.

You may recognize something we have used before in part of the quadratic formula. The quantity $b^2 - 4ac$ is the discriminant. We used it to help tell us when a quadratic equation can be factored. Now we can use it to tell us something else. Since the quadratic formula takes the square root of the discriminant, a quadratic equation will have only real numbers as roots when $b^2 - 4ac \geq 0$. In Chapter 14, when we study complex numbers, we will consider quadratic equations in which the discriminant is negative.

EXAMPLE 9.45

Solve $x^2 + 7x - 8 = 0$.

SOLUTION In this equation, $a = 1$, $b = 7$, and $c = -8$. Putting these values in the quadratic formula we get

$$\begin{aligned} x &= \frac{-7 \pm \sqrt{7^2 - 4(1)(-8)}}{2(1)} \\ &= \frac{-7 \pm \sqrt{49 + 32}}{2} \\ &= \frac{-7 \pm \sqrt{81}}{2} \\ &= \frac{-7 \pm 9}{2} \\ x &= \frac{-7 + 9}{2} = \frac{2}{2} = 1 \\ \text{or} \qquad x &= \frac{-7 - 9}{2} = \frac{-16}{2} = -8 \end{aligned}$$

The roots are 1 and -8. This is an equation that we could have solved by factoring.

EXAMPLE 9.46

Solve $2x^2 + 5x - 3 = 0$.

SOLUTION In this equation $a = 2$, $b = 5$, and $c = -3$. Putting these values in the quadratic formula we get

$$\begin{aligned} x &= \frac{-5 \pm \sqrt{5^2 - 4(2)(-3)}}{2(2)} \\ &= \frac{-5 \pm \sqrt{25 + 24}}{4} \\ &= \frac{-5 \pm \sqrt{49}}{4} \\ &= \frac{-5 \pm 7}{4} \end{aligned}$$

So,

$$x = \frac{-5+7}{4} = \frac{2}{4} = \frac{1}{2} \text{ or } x = \frac{-5-7}{4} = \frac{-12}{4} = -3$$

The roots are $\frac{1}{2}$ and -3.

EXAMPLE 9.47

Solve $9x^2 + 49 = 42x$.

SOLUTION This equation is not in the standard form. If we subtract $42x$ from both sides, we get $9x^2 - 42x + 49 = 0$, with $a = 9$, $b = -42$, and $c = 49$. Substituting these values for a, b, and c in the quadratic formula, we obtain

$$\begin{aligned} x &= \frac{42 \pm \sqrt{(-42)^2 - 4(9)(49)}}{2(9)} \\ &= \frac{42 \pm \sqrt{1764 - 1764}}{18} \\ &= \frac{42 \pm 0}{18} \\ &= \frac{7}{3} \end{aligned}$$

This is a double root; in this case both roots are $\frac{7}{3}$.

EXAMPLE 9.48

Find the roots of $3x^2 + 7x + 3 = 0$.

SOLUTION In this equation, $a = 3$, $b = 7$, and $c = 3$, so

$$\begin{aligned} x &= \frac{-7 \pm \sqrt{7^2 - 4(3)(3)}}{2(3)} \\ &= \frac{-7 \pm \sqrt{49 - 36}}{6} \\ &= \frac{-7 \pm \sqrt{13}}{6} \end{aligned}$$

So, $x = \dfrac{-7 + \sqrt{13}}{6}$ or $x = \dfrac{-7 - \sqrt{13}}{6}$

Notice that we really needed the quadratic formula to find the factors.

HINT Unless you quickly see that an equation can be factored, it may be best to use the quadratic formula.

APPLICATION MECHANICAL

EXAMPLE 9.49

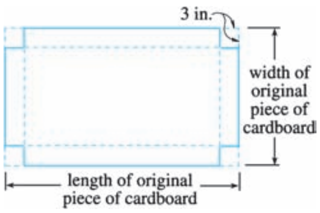


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Figure 9.7

The length of a rectangular piece of cardboard is 4 in. more than its width. A 3-in. square is removed from each corner as shown in Figure 9.7. The remaining cardboard is bent to form an open box. If the volume of the box is 420 in.^{3}, what are the dimensions of the original piece of cardboard?

SOLUTION We will begin by letting x be the width, in inches, of the original piece of cardboard. The length of this cardboard is $x + 4$ in. After the squares are removed and the sides have been folded up, the dimensions of the box are

$$\begin{aligned}\text{length} &= (x+4) - 2\cdot 3 = x - 2\\ \text{width} &= x - 6\\ \text{height} &= 3\end{aligned}$$

The volume, 420 in.^{3}, is: length × width × height, or $V = lwh$, and so we obtain

$$\begin{aligned}(x-2)(x-6)3 &= 420\\ (x-2)(x-6) &= 140\\ x^2 - 8x + 12 &= 140\\ x^2 - 8x - 128 &= 0\\ (x-16)(x+8) &= 0\\ x &= 16 \text{ or } x = -8\end{aligned}$$

It would make no sense for the width of a piece of cardboard to be -8 in., so we reject $x = -8$ as an answer.

Thus, the piece of cardboard must have a width of 16 in. and a length of $16 + 4 = 20$ in.

APPLICATION GENERAL TECHNOLOGY

EXAMPLE 9.50

A ball is thrown upward from the top of a building that is 555 ft high with an initial velocity of 64 ft/s. The height of the ball in feet above the ground at any time t is given by the formula $s(t) = 555 + 64t - 16t^2$. When will the ball hit the ground?

SOLUTION When the ball hits the ground, its height will be 0. So, we want to solve the equation

$$0 = 555 + 64t - 16t^2$$

for t. First write the equation in standard form and then use the quadratic formula to solve it. Multiplying the equation by -1 produces

$$16t^2 - 64t - 555 = 0$$

Here $a = 16$, $b = -64$, and $c = -555$. Substituting these into the quadratic formula we get

$$\begin{aligned} t &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-(-64) \pm \sqrt{(-64)^2 - 4(16)(-555)}}{2(16)} \\ &= \frac{64 \pm \sqrt{4096 + 35{,}520}}{32} \\ &= \frac{64 \pm \sqrt{39{,}616}}{32} \\ &\approx \frac{64 \pm 199.03768}{32} \\ &\approx 2 \pm 6.22 \end{aligned}$$

Thus, $t \approx 8.22$ s or $t \approx -4.22$ s. The second answer makes no sense, because this would mean that the ball struck the ground before it was thrown. The first answer, about 8.22 s, checks when it is substituted into the original equation, and so it is the correct answer.

APPLICATION ENVIRONMENTAL SCIENCE

EXAMPLE 9.51

The photograph in Figure 9.8a shows a solar collector. The roof trusses for a solar collector are often designed as right triangles, so the solar collector will be supported at the correct angle. Rafters form the legs of the right triangle and the base of the truss forms the hypotenuse. Suppose the rafter along the back of the solar collector is 3.5 m shorter than the other rafter and that the base of each truss is 6.5 m long. What is the length of each rafter?

SOLUTION A triangle has been drawn over the photograph of the solar collector in Figure 9.8a. The triangle is labeled with the given information and shown by itself in Figure 9.8b. Since this is a right triangle, we can use the Pythagorean theorem. Thus, we have

$$x^2 + (x - 3.5)^2 = 6.5^2$$

Squaring both sides produces

$$x^2 + (x^2 - 7x + 12.25) = 42.25$$

or

$$2x^2 - 7x - 30 = 0$$

EXAMPLE 9.51 (Cont.)

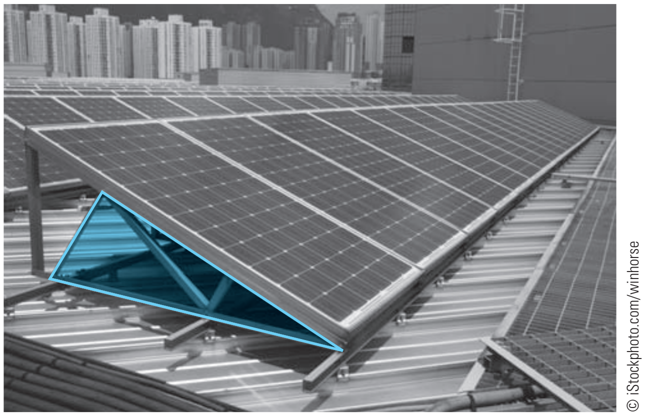

Figure 9.8a

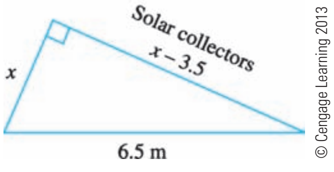

Figure 9.8b

Using the quadratic formula we get

$$x = \frac{-(-7) \pm \sqrt{(-7)^2 - 4(2)(-30)}}{2 \cdot 2}$$
$$= \frac{7 \pm \sqrt{49 + 240}}{4}$$
$$= \frac{7 \pm 17}{4}$$

Thus, $x = 6$ or $x = -2.5$.

It would not make sense for a rafter to be -2.5 m long. The lengths of the rafters must be 6 m and $6 - 3.5 = 2.5$ m long.

EXERCISE SET 9.5

Find the roots of each of the quadratic equations in Exercises 1–8 by completing the square.

1. $x^2 + 6x + 8 = 0$
2. $x^2 - 7x - 8 = 0$
3. $2x^2 - 3x = 14$
4. $3x^2 + 12x - 18 = 0$
5. $4x^2 - 12x - 18 = 0$
6. $3x^2 - 9x = 33$
7. $x^2 + 2kx + c = 0$
8. $px^2 + 2qx + r = 0$

In Exercises 9–46, use the quadratic formula to find the roots of each equation.

9. $x^2 + 3x - 4 = 0$
10. $x^2 - 8x - 33 = 0$
11. $3x^2 - 5x - 2 = 0$
12. $7x^2 + 5x - 2 = 0$
13. $7x^2 + 6x - 1 = 0$
14. $2x^2 - 3x - 20 = 0$
15. $2x^2 - 5x - 7 = 0$
16. $3x^2 + 4x - 7 = 0$
17. $3x^2 + 2x - 8 = 0$
18. $9x^2 - 6x + 1 = 0$
19. $9x^2 + 12x + 4 = 0$
20. $3x^2 + 3x - 7 = 0$
21. $2x^2 - 3x - 1 = 0$
22. $2x^2 - 5x + 1 = 0$
23. $x^2 + 5x + 2 = 0$
24. $3x^2 - 6x - 2 = 0$
25. $2x^2 + 6x - 3 = 0$
26. $5x^2 + 2x - 1 = 0$
27. $x^2 - 2x - 7 = 0$
28. $x^2 + 3 = 0$
29. $2x^2 - 3 = 0$
30. $2x^2 = 5$
31. $3x^2 + 4 = 0$
32. $3x^2 + 1 = 5x$
33. $\frac{2}{3}x^2 - \frac{1}{9}x + 3 = 0$
34. $\frac{1}{2}x^2 - 2x + \frac{1}{3} = 0$
35. $0.01x^2 + 0.2x = 0.6$
36. $0.16x^2 = 0.8x - 1$
37. $\frac{1}{4}x^2 + 3 = \frac{5}{2}x$
38. $\frac{3}{2}x^2 + 2x = \frac{7}{2}$
39. $1.2x^2 = 2x - 0.5$

40. $1.4x^2 + 0.2x = 2.3$

41. $3x^2 + \sqrt{3}x - 7 = 0$

42. $2x^2 - \sqrt{89}x + 5 = 0$

43. $\dfrac{x-3}{7} = 2x^2$

44. $\dfrac{x-5}{3} = 5x^2$

45. $\dfrac{2}{x-1} + 3 = \dfrac{-2}{x+1}$

46. $\dfrac{3x}{x+2} + 2x = \dfrac{2x^2-1}{x+1}$

Solve Exercises 47–68.

47. ***Dynamics*** The Petronas Towers are 452 m high. If a ball is dropped from the top of one of the towers, its height at time t is given by the formula $h(t) = -4.9t^2 + 452$. The ball will hit the ground when $h(t) = 0$. How long does it take for the ball to fall to the ground?

48. ***Dynamics*** If the ball in the previous problem is thrown downward with an initial velocity of 20 m/s, its height at time t is given by $h(t) = -4.9t^2 - 20t + 452$. When does it hit the ground?

49. ***Dynamics*** If the ball in the previous problem is thrown upward with an initial velocity of 20 m/s, its height at time t is given by $h(t) = -4.9t^2 + 20t + 452$. How long will it take to hit the ground?

50. ***Sheet metal technology*** An open box is to be made from a square piece of aluminum by cutting out a 4-cm square from each corner and folding up the sides. If the box is to have a volume of 100 cm^3, find the dimensions of the piece of aluminum that is needed.

51. ***Sheet metal technology*** An open box is to be made from a rectangular piece of aluminum. A 3-cm square is to be cut from each corner and the sides will be folded up. If the original piece of aluminum was 1.5 times as long as its width and the volume of the box is 578 cm^3, what were the dimensions of the original rectangle?

52. ***Construction*** An angle beam is to be constructed as shown in Figure 9.9. If the cross-sectional area of the angle beam is 81.25 cm^2, what is the thickness of the beam?

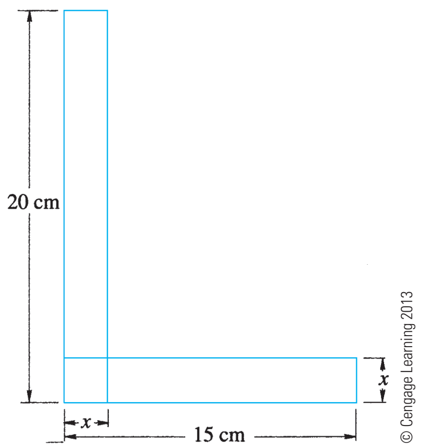

Figure 9.9

53. ***Sheet metal technology*** A gutter is to be made by folding up the edges of a strip of metal as shown in Figure 9.10. If the metal is 12 in. wide and the cross-sectional area of the gutter is to be $16\frac{7}{8}$ in.^{2}, what are the width and depth of the gutter?

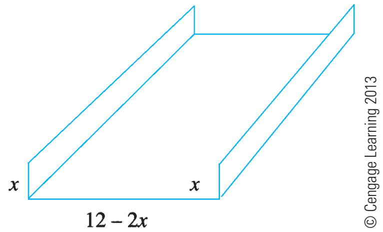

Figure 9.10

54. ***Package design*** A cylindrical container is to be made with a height of 10 cm. If the total surface area of the container is to be 245 cm^2, what is the radius of the base?

55. ***Business*** An oil distributor has 1,000 commercial customers who each pay a base rate of \$30 per month for oil. The distributor figures that for each \$1-per-month increase in the base rate, 5 customers will convert to coal. The distributor needs to increase its monthly income to \$45,000 and lose as few customers as possible. How much should the base rate be increased?

56. ***Electricity*** In an ac circuit that contains resistance, inductance, and capacitance in series, the applied voltage V can be found by solving $V^2 = V_R^2 + (V_L - V_C)^2$. If $V = 5.8$ V, $V_R = 5$ V, and $V_C = 10$ V, find V_L.

57. ***Electricity*** In an ac circuit that contains resistance, inductance, and capacitance in series, the impedance of the circuit Z is related to the resistance R, the inductive reactance X_L, and the capacitive reactance X_C, by the formula, $Z^2 = R^2 + (X_L - X_C)^2$. If a circuit has $Z = 610\ \Omega$, $R = 300\ \Omega$, and $X_C = 531\ \Omega$, find X_L.

58. ***Construction*** For a simply supported beam of length l having a distributed load of w kg/m, the binding moment M at any distance x from one end is given by

$$M = \frac{1}{2}wlx - \frac{1}{2}wx^2$$

At which locations is the binding moment zero?

59. ***Construction*** A rectangular concrete pipe is constructed with a 10.00 in. by 16.00 in. interior channel, as shown in Figure 9.11. What uniform width w of concrete must be formed on all sides if the total cross-sectional area of the concrete must be 245in.^{2}?

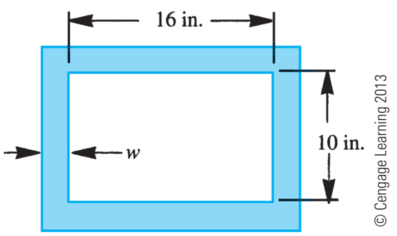

Figure 9.11

60. ***Construction*** A rectangular parking lot at a shopping mall is 50.0 m wide and 80.0 m long. The developers of the mall want to double the parking area. They plan to do this by adding an equal distance, in meters, to the length and to the width of the lot. What are the dimensions of the new parking lot?

61. ***Forestry*** The Scribner log-rule equation for 16-ft logs is $V = 0.79D^2 - 2D - 4$, where V is the volume, in board ft, of the log and D is the diameter, in in., of the small end of a log inside the bark. If a certain 16-ft log has a volume of 926.0 board ft, what is the diameter of the small end of the log inside the bark?

62. ***Forestry*** Use the Scribner log-rule equation (see Exercise 61), to determine the diameter of the small end of the log inside the bark of a 16.0-ft log that has a volume of 1078.0 board ft.

63. ***Physics*** When an object is dropped from a building that is 196 ft tall, its height h at any time t after it was dropped is given by the function $h(t) = 196 - 16t^2$. Find the time it takes for the object to strike the ground.

64. ***Business*** Total revenue from selling a certain object is the product of the price and the quantity of the object sold. If q represents the quantity sold and the price is \$2,520 $- 3q$, find the quantity that produces a total revenue of \$140,400.

65. ***Business*** Total revenue from selling a certain object is the product of the price and the quantity of the object sold. If q represents the quantity sold and the price of the object is \$1,560 $- 4q$, find the quantity that produces a total revenue of \$29,600.

66. ***Agriculture*** A farmer has 1,500 ft of fencing to enclose a rectangular field. Find the dimensions of the field so that the enclosed area will be 137,600 ft^2.

67. ***Agriculture*** A farmer has 2,700 ft of fencing to enclose two adjacent rectangular fields. Find the dimensions of the fields, so that the enclosed area will be 270,000 ft^2.

68. ***Electronics*** When two resistors, R_1 and R_2, are connected in parallel, their combined resistance, R, is given by $\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}$. Two certain resistors that are connected in parallel have a combined resistance of 6 Ω, and the resistance of one of them is 5 Ω more than that of the other. What is the resistance of each resistor?

[IN YOUR WORDS]

69. Without looking at the definition in the book, describe the technique called "completing the square."

70. Write a word problem in your technology area of interest that requires you to use a quadratic equation. On the back of the sheet of paper, write your name and explain how to solve the problem by using factoring or completing the square. Give the problem you wrote to a friend and let him or her try to solve it. If your friend has difficulty understanding the problem or solving the problem, or disagrees with your solution, make any necessary changes in the problem or solution. When you have finished, give the revised problem and solution to another friend and see if he or she can solve it.

71. **(a)** Use factoring to solve $3x^2 + 5x - 2 = 0$.

(b) Use the quadratic formula to solve $3x^2 + 5x - 2 = 0$. You should get the same answers you got in (a).

(c) Explain to a classmate how to use the quadratic formula. Check to see how well your classmate understands your explanation by asking him or her to solve $3x^2 + 5x - 2 = 0$.

(d) Under what conditions is the quadratic formula easier to use than factoring?

72. **(a)** What is the discriminant?

(b) What does the discriminant tell you if it is nonnegative?

9.6 MODELING WITH QUADRATIC FUNCTIONS

In Section 4.6 we learned how to use linear regression to determine the line that best approximates a set of points. But, when you look at a scatterplot of some data it does not always look linear. In this section we will see how to recognize when a set of data can be approximated by a quadratic function. In later chapters we will look at other kinds of nonlinear regression formulas.

QUADRATIC FUNCTIONS

A **quadratic function** is a function of the form $f(x) = ax^2 + bx + c$, where a, b, and c are constants and $a \neq 0$. A quadratic function has one of the two basic shapes shown in Figures 9.12a and 9.12b. A parabola opens up, as in Figure 9.12a, when a is positive, that is, when $a > 0$, and opens down, as in Figure 9.12b, when $a < 0$.

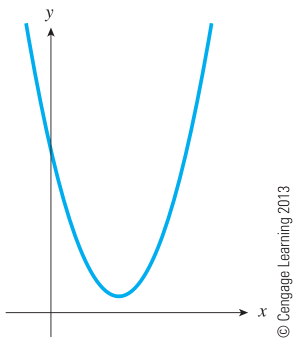

Figure 9.12a

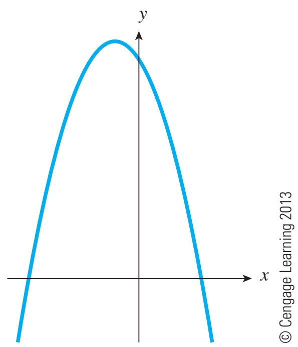

Figure 9.12b

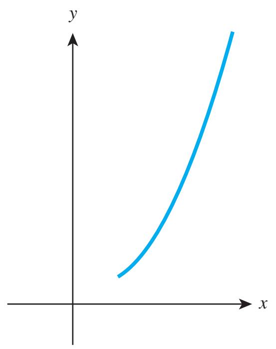

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Figure 9.13

It is possible that a function may be a parabola but you may only see part of it, as in Figure 9.13.

As with linear regression, you begin by graphing the data and then looking at the graph.

The following information in Example 9.52 is a little old, but serves as a good set of data. In Exercise 17 of Exercise Set 9.6 you will be asked to examine these data again plus the data from 2000 through 2008.

APPLICATION BUSINESS

EXAMPLE 9.52

The average price of residential electricity is shown in Table 9.1.

TABLE 9.1 U.S. Residential Electricity Prices, 1993–1999

Year	1993	1994	1995	1996	1997	1998	1999
Price ($ kWh)	8.32	8.38	8.40	8.36	8.43	8.26	8.16

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Using either a graphing calculator or a spreadsheet,

(a) Plot the points.

(b) Determine a quadratic regression curve that will fit the data.

(c) Use the regression curve to estimate the price of residential electricity in 2000.

SOLUTIONS

Using a graphing calculator:

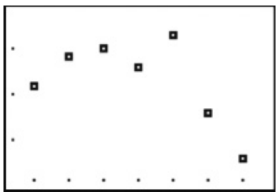

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Figure 9.14a

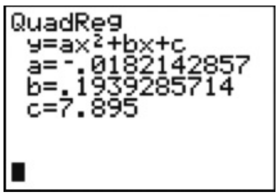

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Figure 9.14b

(a) We begin by plotting the points using the procedures outlined in Section 4.6. (To make things easier, the years were entered as 3, 4, 5,. . ., 9.) The result is shown in Figure 9.14a. As you can see, these points do not appear to lie on a line. In fact, except for the data for 1997, they look as if they could lie on a curve similar to the one in Figure 9.12b.

(b) Since these points seem to lie on a quadratic we will use **quadratic regression**. On a TI-83 or TI-84, press STAT ▶ [CALC] 5 [5:QuadReg] ENTER. The result is the quadratic regression information shown in Figure 9.14b.

This result gives the following information. Lines 1 and 2 indicate that a quadratic regression was performed and the results are of the form $y = ax^2 + bx + c$. Lines 3, 4, and 5 show the approximate values of a, b, and c. Thus, you can see that $a \approx -0.0182142857$, $b \approx 0.1939285714$, and $c = 7.895$.

If you want to graph this regression curve you have several options. Perhaps the easier is to tell the calculator to place the regression equation directly in Y=. There are two ways to do this, but perhaps the easier is to return to the direction for getting the regression line. (Press 2nd ENTRY and

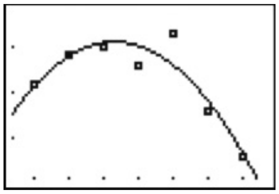

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Figure 9.14c

you should see QuadReg on the screen.) Now, press **2nd** **1** [L1] **,** **2nd** **2** [L2] **,** **VARS** **▶** **1** [1:Function] **ENTER** **ENTER**. This will place the regression equation on the Y1 line of the Y= screen. Press **GRAPH** and you should get the result in Figure 9.14c.

(c) We now need to use this quadratic regression equation to determine the price of residential electricity in the year 2000. Since the data is saved as Y1, we only need to return to the home screen and press **VARS** **▶** **1** [1:Function] **ENTER** **(** 10 **)** **ENTER**. Notice that a 10 was entered for the year 2000 because the years were entered as 3 for 1993 (that is, $3 = 1993 - 1990$), 4 for 1994 (here, $4 = 1994 - 1990$), and so on. Thus, we would use $2000 - 1990 = 10$ for the year 2000. This gives the result $Y1(10) \approx 8.0128$. Thus, according to this regression formula, residential electricity would have cost \$8.01/kWh in 2000. (The actual price of residential electricity in the United States in 2000 was \$8.24/kWh.)

Using a spreadsheet:

(a) Begin by plotting the points on a scatterplot using the procedures outlined in Section 4.6. When using large numbers like years, it is often best to change them to smaller numbers. For example, change 1992 to 2, and 1993 to 3. That way, the number of decimal places in the model will not be as critical. The result is shown in Figure 9.14d. As you can see, the points do not appear to lie on a line. In fact, except for the data for 1997, they look as if they could lie on a curve similar to the one in Figure 9.12b.

(b) Since the points seem to lie on a quadratic, we will use quadratic regression. Right click[1] on one of the data points and select Add Trendline. Then select Polynomial and Order 2, as shown in Figure 9.14e, which means that the model will be a quadratic. Near the bottom choose Display Equation on Chart. The result is shown in Figure 9.14f.

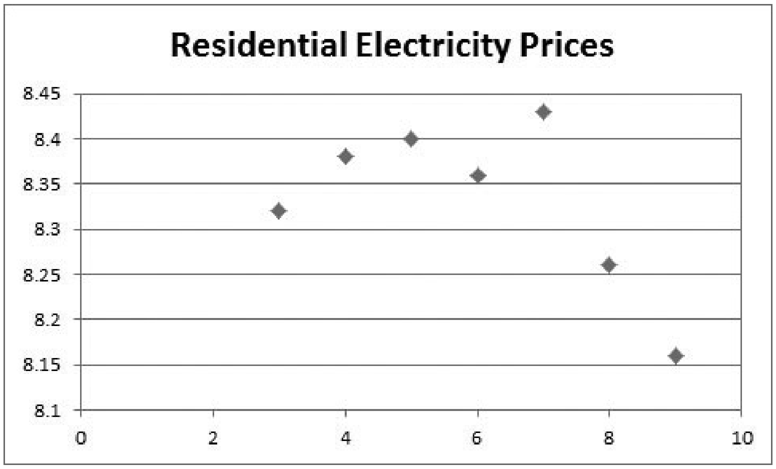

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Figure 9.14d

1 Remember, Macintosh computers do not come with a mouse that has a right button. Instead of right clicking, hold down the **CONTROL** key and click the mouse to achieve the same function. If the computer has a trackpad instead of a mouse, press the trackpad with two fingers.

EXAMPLE 9.52 (Cont.)

(c) We now need to use this quadratic regression equation to determine the price of residential electricity in the year 2000. To predict the price of residential electricity in the year 2000, we copy the equation for the model into Cell D4, see Figure 9.14g. Enter 10 (for 2000) in Cell D3 and the result is approximately \$8.01/kWh in 2000. (The actual price of residential electricity in the United States in 2000 was \$8.24/kWh.)

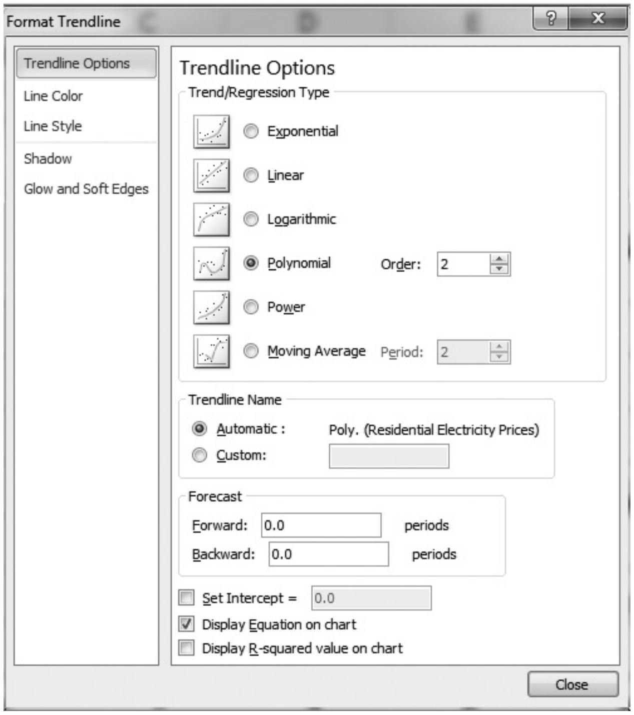

Figure 9.14e

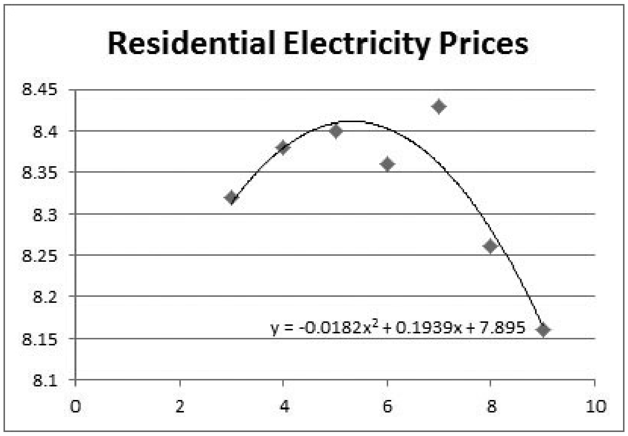

Figure 9.14f

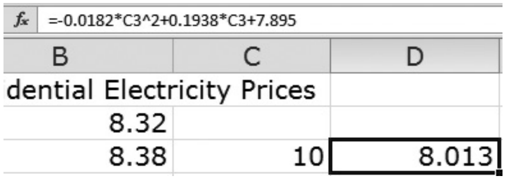

Figure 9.14g

PIECEWISE-DEFINED FUNCTIONS

Not every set of data can be approximated by a simple curve like a line or a parabola. Consider the data in Table 9.2 and the graph of these 12 points in Figure 9.15a.

TABLE 9.2 Piecewise Function Example

x	1	2	3	4	5	6	7	8	9	10	11	12
$f(x)$	13	11	9	7	5	3	3.25	4	5.25	7	9.25	12

These points clearly do not fall on a line, nor do they seem to form a parabola. What it actually looks like is a combination of a line and a parabola. We will draw a line through the first six points, as in Figure 9.15b, and then a parabola through the last seven points. The point (6, 3) will be on both graphs. The final result is in Figure 9.15c.

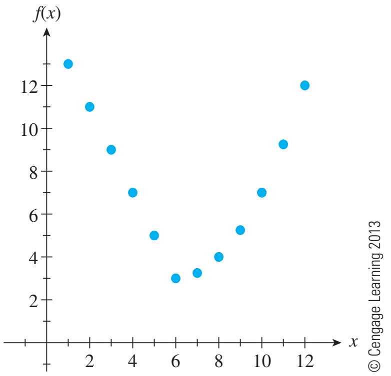

Figure 9.15a

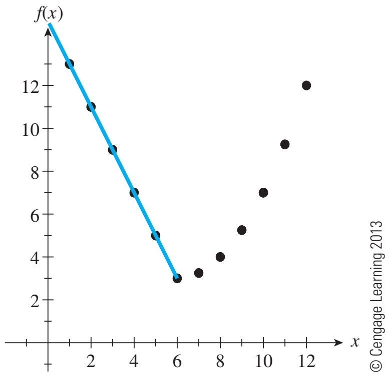

Figure 9.15b

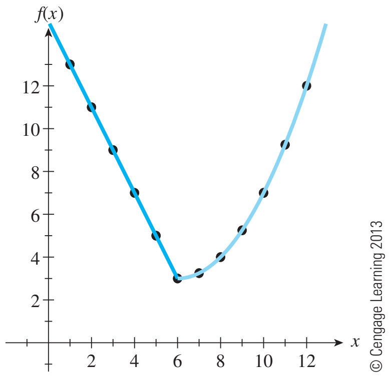

Figure 9.15c

How do you describe such a function? For this function

$$f(x) = \begin{cases} -2x + 15 & \text{if } x \le 6 \\ 0.25x^2 - 3x + 12 & \text{if } x \ge 6 \end{cases}$$

The first line says that when $x \le 6$ then the function uses the rule $f(x) = -2x + 15$. The second line says that for $x \ge 6$ the function uses the rule $f(x) = 0.25x^2 - 3x + 12$.

EXAMPLE 9.53

Consider the piecewise-defined function

$$g(x) = \begin{cases} |x + 5| & \text{if } -6 \le x < -3 \\ \sqrt{x + 3} + 2 & \text{if } -3 \le x < 1 \\ 2x - 4 & \text{if } 1 \le x \le 5 \end{cases}$$

Evaluate (a) $g(-4)$, (b) $g(-3)$, (c) $g(0)$, (d) $g(1)$, (e) $g(4)$, and (f) $g(7)$.

SOLUTIONS

(a) Since -4 is in the interval $-6 \le x < -3$ we use the first line of the definition, $g(x) = |x + 5|$. So, when $x = -4$ we see that $g(-4) = |-4 + 5| = |1| = 1$.

(b) Notice this -3 is not in the interval $-6 \le x < -3$ but is in the interval $-3 \le x < 1$. This means that we use the second line of the definition, $g(x) = \sqrt{x + 3} + 2$ and $g(-3) = \sqrt{-3 + 3} + 2 = \sqrt{0} + 2 = 2$.

(c) Here 0 is in the interval $-3 \le x < 1$ so we again use $g(x) = \sqrt{x + 3} + 2$ and $g(0) = \sqrt{0 + 3} + 2 = \sqrt{3} + 2 \approx 3.732$.

(d) We see that 1 is not in the interval $-3 \le x < 1$ but in the interval $1 \le x \le 5$ and so we use $g(x) = 2x - 4$. Thus, $g(1) = 2 \cdot 1 - 4 = -2$.

(e) We see that 4 is in the interval $1 \le x \le 5$ so we again use $g(x) = 2x - 4$ and $g(4) = 2 \cdot 4 - 4 = 4$.

(f) The value $x = 7$ is not in any of the three intervals defined for this function. Since 7 is not in the domain of g there is not a value for $g(7)$.

A graph of g is in Figure 9.16.

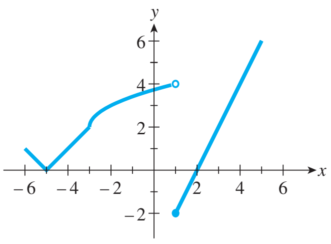

Figure 9.16

APPLICATION BUSINESS

EXAMPLE 9.54

The data in Table 9.3 shows the consumption of beef in the United States from 1940 through 2008.

(a) Plot the points.

(b) Determine a quadratic regression curve that will fit the data.

TABLE 9.3 Beef Consumption in the United States 1940–2008

Year	1940	1950	1960	1970	1980	1990
Beef consumption (million pounds)	7257	9525	15,490	23,390	23,560	24,031

Year	1995	2000	2005	2006	2007	2008
Beef consumption (million pounds)	25,534	27,338	27,754	28,137	28,144	28,127

SOLUTIONS

(a) The points are plotted in Figure 9.17a with a TI-84 and on a spreadsheet in Figure 9.17b.

(b) These points do not look as if they lie along a straight line. At this stage our only other choice is a parabola. If a quadratic regression is conducted on these points the result graphed on a TI-84 is shown in Figure 9.17c and with a spreadsheet in Figure 9.17d.

The results in Figures 9.17c and 9.17d do not seem very satisfactory since the parabola seems to be starting to level off on the right-hand side.

A better choice might be a piecewise-defined function. Looking at the points in Figures 9.17a and 9.17b we see that the first four points could be modeled with a parabola and the last eight points could be modeled with a different parabola. The next example will show how to construct such a parabola on a graphing calculator or a spreadsheet.

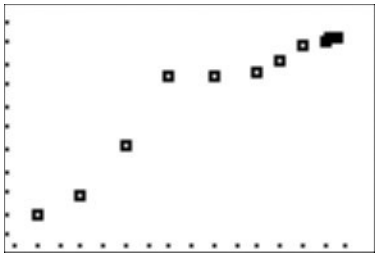

Figure 9.17a

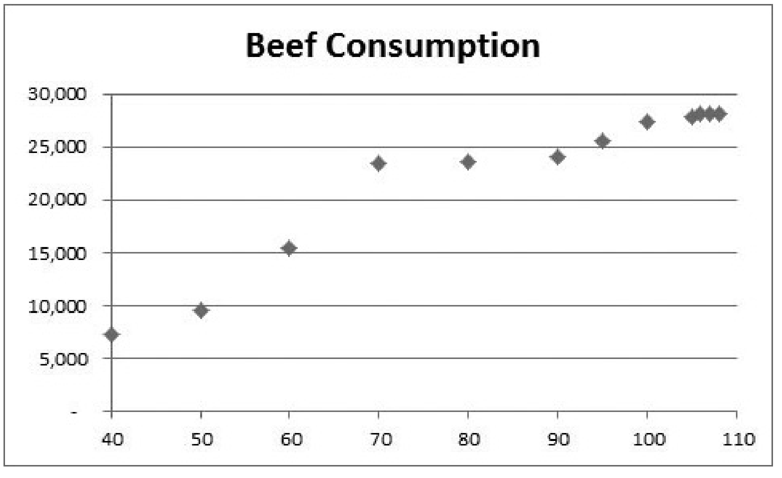

Figure 9.17b

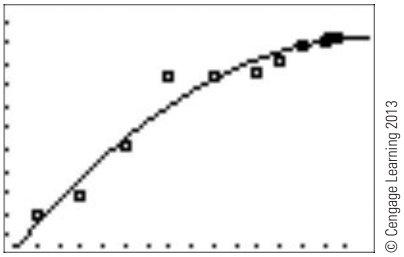

Figure 9.17c

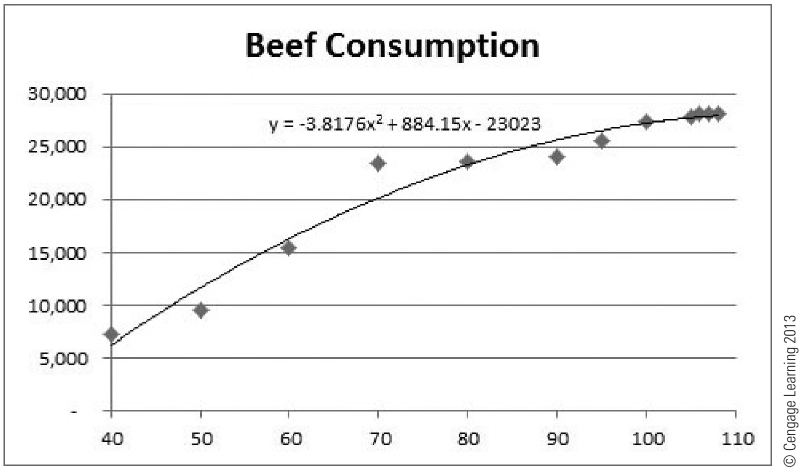

Figure 9.17d

APPLICATION BUSINESS

EXAMPLE 9.55 Use a piecewise-defined function to model the data in Table 9.3.

SOLUTION We will show how to do this first with a graphing calculator and then with a spreadsheet.

Using a graphing calculator

In order to have the calculator perform a regression on only some numbers in a list you will need to create a list with just those numbers. The easiest way to do this is to save an existing list with a second name and then edit each list to delete the unwanted entries.

For example, the list L1 contains all the *x*-values in Table 9.3 Press STAT 1 [1:EDIT] and then press ▶ until the cursor is in the L3 column. Press ▲ until the cursor is at the very top of the column with the L3 highlighted. Now press 2nd 1 [L1]. The result should look like Figure 9.18a. Notice at the very bottom of the screen is the equation L3=L1. This indicates that you want to copy the data from L1 to L3. Press ENTER and all the data from L1 is copied to L3, as shown in Figure 9.18b.

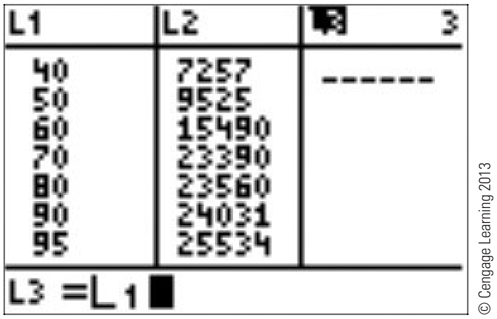

Figure 9.18a

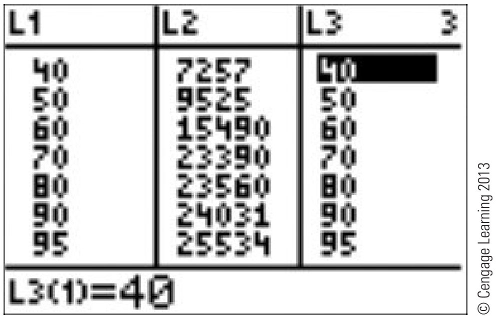

Figure 9.18b

EXAMPLE 9.55 (Cont.)

Press **▶** to move to column L4 and copy the entries from list L2 to list L4. You can edit each list by moving the cursor to an entry in a list and pressing **DEL**. You want to delete all the data after 1970 from the L1 and L2 lists and all the data before 1970 from the L3 and L4 lists.

To graph both pairs of lists on the same graph go to the STAT PLOT screen, select 2:Plot2 by pressing **2**, and change the settings to match those in Figure 9.18c. Notice that Xlist is set to L3, Ylist to L4, and Mark to +. Press **GRAPH** and you should get the graph in Figure 9.18d with a **2** marking the data in lists L1 and L2 and a + the data in lists L3 and L4.

Now, have the calculator compute the quadratic regression on each list pair. First, do

QuadReg L1, L2, Y1

Press **ENTER** and then compute the quadratic regression on the second list pair. Notice that this result is entered in Y2.

QuadReg L3, L4, Y2

Once you have finished the regressions you should get the following results:

$$b(t) \approx \begin{cases} 14.08t^2 - 1005.16t + 24847.3 & \text{if } t \leq 70 \\ 3.9645t^2 - 567.7356t + 43611.1497 & \text{if } t \geq 70 \end{cases}$$

where b is million pounds of beef consumption t years after 1900.

A graph of the two parabolas gives a result like the one in Figure 9.18e. With some fine tuning (see the *TI-83 Plus Graphing Calculator Handbook* or *TI-84 Plus Graphing Calculator Handbook*) you can get the result to look like the one in Figure 9.18f.

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Figure 9.18c

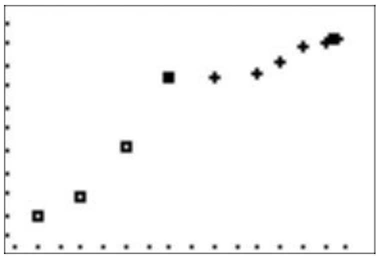

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Figure 9.18d

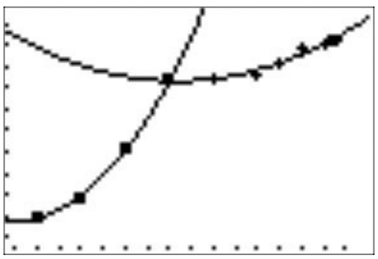

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Figure 9.18e

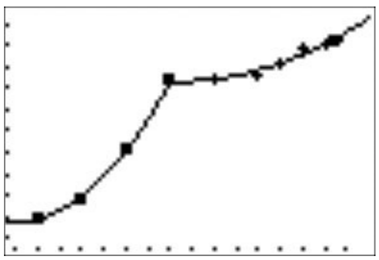

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Figure 9.18f

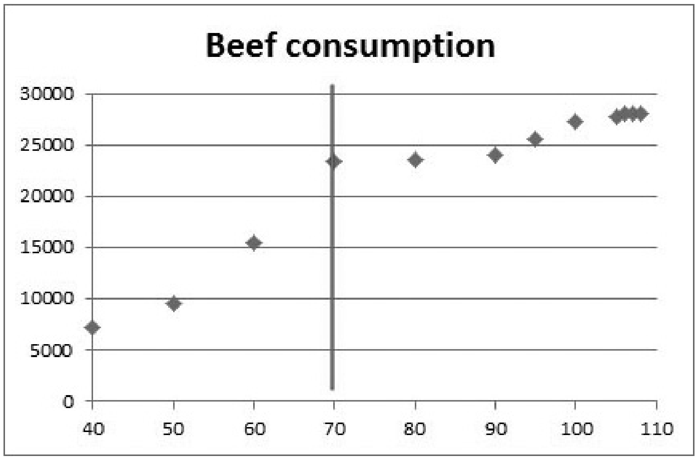


Figure 9.18g

	A	B
1	Year	Beef consumption
2	40	7257
3	50	9525
4	60	15400
5	70	23390
6	80	23560
7	90	24031
8	95	25534
9	100	27338
10	105	27754
11	106	28137
12	107	28144
13	108	28127

Figure 9.18h

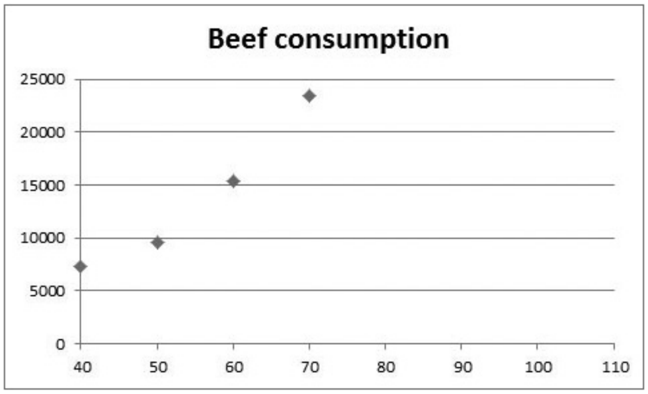


Figure 9.18i

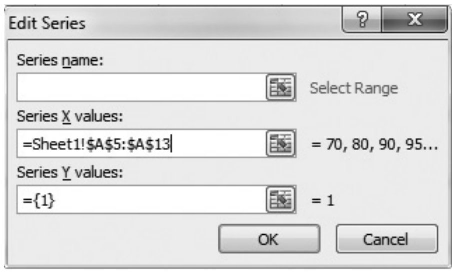


Figure 9.18j

Using a spreadsheet:

In order to perform a different regression on two parts of the same table, we must enter the data for the graph as two separate series. The obvious place to separate the graph into two functions is at the year 1970. This is indicated by the vertical line in Figure 9.18g.

Enter the data as shown in Figure 9.18h. The first step is to highlight only the data for the first series, the years 1940 through 1970. Using only those points, construct a scatter plot as shown in Figure 9.18i.

Then right click on any data point and select Select Data. Add Series 2 by clicking on Add. Place the cursor in the X-Values dialogue box and then move it up and click and drag along row 1, highlighting the cells that contain the years 1970 through 2008 (see Figure 9.18j).

Place the cursor in the Y-Values dialogue box and erase anything that is in the box. Move the cursor up to row 2 and click and drag along row 2, highlighting the cells that contain the beef consumption for the years 1970 through 2008 (see Figure 9.18k).

When you click OK, the result is a scatter plot with two series on the same graph. Each series should be displayed with a different color, as shown in Figure 9.18l.

EXAMPLE 9.55 (Cont.)

Right click on the any data point in the first series and insert a quadratic trendline. (See Figure 9.18m.)

Right click on any data point in the second series, and insert a quadratic trendline. (See Figure 9.18n.)

The regressions yield the following results:

$$b(t) \approx \begin{cases} 14.305t^2 - 1030.8t + 25527 & \text{if } t \leq 70 \\ 3.8994t^2 - 555.65t + 43074 & \text{if } 70 \leq t \end{cases}$$

where b is million pounds of beef consumption t years after 1900.

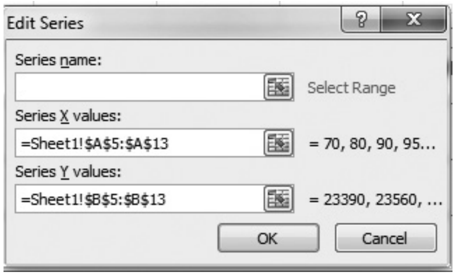


Figure 9.18k

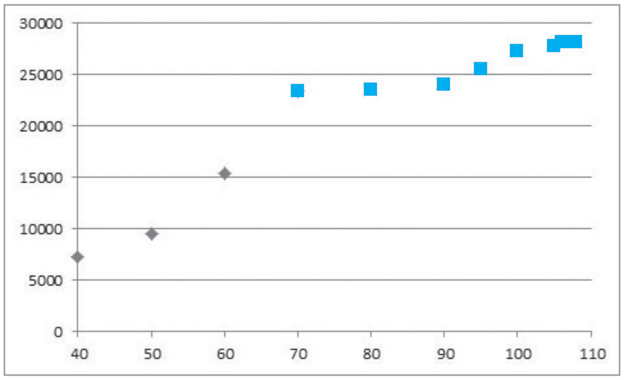


Figure 9.18l

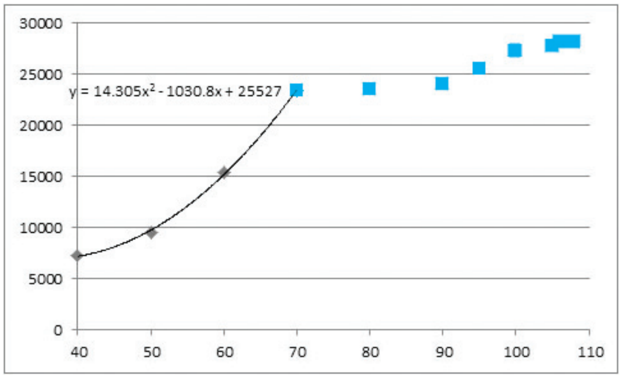


Figure 9.18m

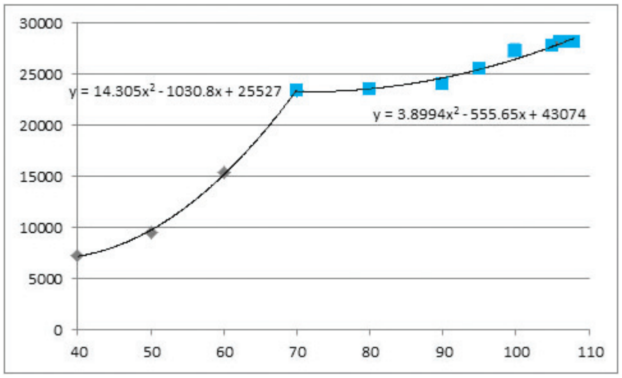


Figure 9.18n

EXERCISE SET 9.6

In Exercises 1–4 complete the table of values for each of the given piecewise-defined functions.

1. $f(x) = \begin{cases} 0.5x + 1 & \text{if } x \leq 0 \\ -x + 1 & \text{if } x > 0 \end{cases}$

x	-3	-2	-1	0	1	2	3	4
$f(x)$								

2. $g(x) = \begin{cases} -2x - 5 & \text{if } x \leq 1 \\ x + 6 & \text{if } x > 1 \end{cases}$

x	-3	-2	-1	0	1	2	3	4
$g(x)$								

3. $h(x) = \begin{cases} \sqrt{6-x} & \text{if } x \le 2 \\ x+6 & \text{if } x > 2 \end{cases}$

x	-3	-2	-1	0	1	2	3	4	5
$h(x)$									

4. $j(x) = \begin{cases} 4 & \text{if } x < -1 \\ x^2 & \text{if } x \ge -1 \end{cases}$

x	-3	-2	-1	0	1	2	3	4
$j(x)$								

Solve Exercises 5–16.

5. If $f(x) = \begin{cases} x^2 & \text{if } x < 0 \\ 5 & \text{if } x = 0 \\ x+1 & \text{if } x > 0 \end{cases}$

determine **(a)** $f(-3)$, **(b)** $f(0)$, and **(c)** $f(5)$.

6. If $g(x) = \begin{cases} x^3 & \text{if } x < 0 \\ 5x-1 & \text{if } x \ge 0 \end{cases}$

determine **(a)** $g(-2)$, **(b)** $g(0)$, and **(c)** $g(4)$.

7. If $h(x) = \begin{cases} 0.5x^2 - 2x & \text{if } x \le 0 \\ |7-2x| & \text{if } x > 0 \end{cases}$

determine **(a)** $h(-4)$, **(b)** $h(0)$, and **(c)** $h(4)$.

8. If $j(x) = \begin{cases} (x-2)^2 & \text{if } x \le -1 \\ \sqrt{x+4} & \text{if } x > 0 \end{cases}$

determine **(a)** $j(-4)$, **(b)** $j(0)$, **(c)** $j(2)$, and **(d)** $j(5)$.

9. ***Transportation*** Table 9.4 shows the average miles, in thousand miles, a passenger car was driven each year from 1990 to 2007.

TABLE 9.4 Average Passenger Car Distance Traveled: 1990–2007

Year	1990	1991	1992	1993	1994	1995	1996	1997	1998
Avg. miles per car ($\times$1000), $M(t)$	10.3	10.3	10.6	10.5	10.8	11.2	11.3	11.6	11.8

Year	1999	2000	2001	2002	2003	2004	2005	2006	2007
Avg. miles per car ($\times$1000), $M(t)$	11.9	11.9	11.8	12.2	12.3	12.5	12.5	12.5	12.3

Using either a graphing calculator or a spreadsheet,

(a) Plot the points.

(b) Determine a quadratic regression curve that will fit the data.

(c) Draw the quadratic regression curve on the scatter plot of the data.

10. ***Business*** Table 9.5 shows the number of franchised new car dealerships from 2000 to 2008.

TABLE 9.5 Number of Franchised New Car Dealerships: 2000–2008

Year	2000	2001	2002	2003	2004	2005	2006	2007	2008
New Car Dealerships, $C(t)$	22,250	21,800	21,725	21,650	21,640	21,495	21,200	20,770	20,010

Using either a graphing calculator or a spreadsheet,

(a) Plot the points.

(b) Determine a quadratic regression curve that will fit the data.

(c) Draw the quadratic regression curve on the scatter plot of the data.

11. ***Automotive technology*** For any particular car the distance it takes to stop depends on the speed of the car when the brakes are applied. The data in Table 9.6 shows the distance, in feet, that it took a particular car to stop at a given speed.

TABLE 9.6 Auto Stopping Distance

Speed (mph), s	10	20	30	40	50	60	70	80
Stopping distance (ft), $D(s)$	22	45	75	120	182	259	351	451

Using either a graphing calculator or a spreadsheet,

(a) Plot the points.

(b) Determine a quadratic regression curve that will fit the data.

(c) Draw the quadratic regression curve on the scatter plot of the data.

12. ***Automotive technology*** Table 9.7 shows the average fuel efficiency of gasoline-powered automobiles in the United States for selected years since 1940.

TABLE 9.7 Average Fuel Efficiency in mpg of U.S. Automobiles

Year	1940	1950	1960	1970	1980	1990	1995	2000	2005	2007
Avg. mpg, $E(t)$	14.8	13.9	13.4	13.5	16.0	20.3	21.1	21.9	22.1	22.5

Using either a graphing calculator or a spreadsheet,

(a) Plot the points.

(b) Determine a piecewise-defined regression curve that will fit the data.

(c) Draw the piecewise-defined curve on the scatter plot of the data.

13. ***Automotive technology*** Table 9.8 shows the average fuel efficiency of gasoline-powered SUVs in the United States for selected years since 1970.

TABLE 9.8 Average Fuel Efficiency in mpg of U.S. SUVs

Year	1970	1975	1980	1985	1990	1995	2000	2005	2007
Avg. mpg, $E(t)$	10.0	10.5	12.2	14.3	16.1	17.3	17.4	17.7	18.0

Using either a graphing calculator or a spreadsheet,

(a) Plot the points.

(b) Determine a piecewise-defined regression curve that will fit the data.

(c) Draw the piecewise-defined curve on the scatter plot of the data.

(d) Use your model to predict the year that the average fuel efficiency will be 25.0 mi per gallon.

14. ***Ecology*** Table 9.9 shows the total nitrous oxide emission in the United States for selected years since 1970. Emission values are given in thousands of metric tons.

TABLE 9.9 Total U.S. Nitrous Oxide Emissions: 1970–2007

Year	1970	1980	1990	1995	2000	2003	2004	2005	2006	2007
Emissions in tons $\times 10^6$, $E(t)$	26.9	27.1	25.5	25.0	22.6	20.3	19.5	18.7	17.7	17.0

Using either a graphing calculator or a spreadsheet,

(a) Plot the points.

(b) Determine a quadratic regression curve that will fit the data.

(c) Draw the quadratic regression curve on the scatter plot of the data.

15. ***Energy*** Table 9.10 shows the residential and commercial energy consumption in quadrillion Btus for various years from 1970 through 2008.

TABLE 9.10 Residential and Commercial Energy Consumption: 1970–2008

Year	1970	1975	1980	1985	1990	1995	2000	2005	2008
Energy consumption $\times 10^{12}$ Btu, $C(t)$	22.11	24.31	26.35	27.53	30.35	33.28	37.66	39.57	40.18

Use either a graphing calculator or a spreadsheet to answer each of the following:

(a) Plot the points.

(b) Determine a quadratic regression curve, $E_Q(t)$, that will fit the data.

(c) Determine a linear regression curve, $E_L(t)$, that will fit the data.

(d) Use each of your regression curves to estimate the amount of carbon emissions in the United States in 2015.

16. ***Agriculture*** Table 9.11 shows the relationship between Stewart's bacterial wilt and the yield on sweet corn hybrids. Stewart's wilt ratings ranged from 2.0 to 7.2 and percent yield of plants inoculated with *E. stewartii* ranged from 26% to 114% when compared to noninoculated plants.

TABLE 9.11 Stewart's Wilt Rating and Corn Yield

Wilt rating, r	2.0	4.0	2.7	2.3	3.1	3.4	4.4	3.8	3.2	3.9	4.4	4.6
Percent yield, $Y(r)$	107	96	95	98	95	114	74	72	90	107	92	80

Wilt rating, r	4.4	4.9	5.4	5.6	6.0	4.6	5.3	6.2	6.5	6.8	7.2	7.0
Percent yield, $Y(r)$	94	86	77	68	51	84	34	55	47	26	30	28

Using either a graphing calculator or a spreadsheet,

(a) Plot the points.

(b) Determine a quadratic regression curve, $Y(r)$, that will fit the data.

(c) Draw the quadratic regression curve on the scatter plot of the data.

(d) Write a statement about the meaning of your regression equation.

17. ***Construction*** The average price of residential electricity is shown in Table 9.12.

TABLE 9.12 U.S. Residential Electricity Prices, 1993–2008

Year, t	1993	1994	1995	1996	1997	1998	1999
Price (\$/kWh)	8.32	8.38	8.40	8.36	8.43	8.26	8.16

Year, t	2000	2003	2004	2005	2006	2007	2008
Price (\$/kWh)	8.24	8.70	8.97	9.45	10.40	10.65	11.36

Using either a graphing calculator or a spreadsheet,

(a) Plot the points.

(b) Determine a quadratic regression curve that will fit the data.

(c) Use the quadratic regression curve to estimate the price of residential electricity in 2009.

(d) Determine a piecewise-defined function that will fit the data.

(e) Use the piecewise-defined function to estimate the price of residential electricity in 2009.

18. ***Energy*** The amount of U.S. petroleum imports in quadrillion barrels for selected years from 1980 through 2008 is shown in Table 9.13.

TABLE 9.13 U.S. Petroleum Imports, 1980–2008

Year, t	1980	1990	1995	2000	2002	2003
Imports (quad. Btu), $I(t)$	14.66	17.12	18.88	24.53	24.67	26.22

Year, t	2004	2005	2006	2007	2008
Imports (quad. Btu), $I(t)$	28.20	29.25	29.16	28.76	27.56

Using either a graphing calculator or a spreadsheet,

(a) Plot the points.

(b) Determine a piecewise-regression curve that will fit the data.

(c) Draw the piecewise-defined curve on the scatter plot of the data.

(d) Use the piecewise-regression curve to estimate the amount of petroleum imports in 2009.

[IN YOUR WORDS]

19. Describe the basic shapes of the graph of a quadratic function.

20. Describe how to use your calculator or spreadsheet to obtain an equation for a quadratic regression curve.

CHAPTER 9 REVIEW

IMPORTANT TERMS AND CONCEPTS

Completing the square
Combined variation
Constant of proportionality
Constant of variation
Direct variation
Directly proportional
Discriminant
Extraneous solution
Fractional equation
Inverse variation
Inversely proportional
Joint variation
Parabola
Piecewise-defined function
Quadratic equation
Quadratic formula
Quadratic function
Quadratic regression
Standard quadratic equation
Zero-product rule

REVIEW EXERCISES

Solve each of Exercises 1–6 for x.

1. $\frac{x}{3} + \frac{x}{2} = 5$

2. $\frac{2}{x} - \frac{3}{x} = \frac{1}{5}$

3. $\frac{x-2}{4} - \frac{x+2}{5} = \frac{x}{2}$

4. $\frac{2}{x-1} + \frac{3}{x-2} = \frac{4}{x^2 - 3x + 2}$

5. $\frac{3x}{x-1} - \frac{3x+2}{x} = 3$

6. $\frac{4x}{2x-3} + \frac{1}{x-1} = \frac{2x+1}{x-1}$

Solve each of Exercises 7–10 for the indicated variable.

7. $A = 2lw + 2(l + w)h$, for h

8. $A = 2lw + 2(l + w)h$, for l

9. $\frac{1}{f} = \frac{1}{p} + \frac{1}{q}$, for f

10. $X = wL - \frac{1}{wc}$, for c

Use factoring to find the roots of each of the quadratic equations in Exercises 11–22.

11. $x^2 - 8x + 7 = 0$

12. $x^2 + 4x + 3 = 0$

13. $x^2 - 11x + 10 = 0$

14. $x^2 + 15x + 14 = 0$

15. $x^2 + 15x + 56 = 0$

16. $x^2 - 8x + 12 = 0$

17. $2x^2 - 5x + 3 = 0$

18. $3x^2 + 10x + 7 = 0$

19. $6x^2 + 7x - 10 = 0$

20. $8x^2 + 22x + 9 = 0$

21. $4x^2 - 9 = 0$

22. $9x^2 - 16 = 0$

Find the roots of each of the quadratic equations in Exercises 23–28 by completing the square.

23. $x^2 + 4x - 5 = 0$

24. $x^2 - 7x + 6 = 0$

25. $x^2 + 19x = 11$

26. $2x^2 + 7x = 15$

27. $3x^2 + 5x - 14 = 0$

28. $4x^2 + 2x - 5 = 0$

Use the quadratic formula to find the roots of each of the equations in Exercises 29–42.

29. $x^2 - 8x + 5 = 0$

30. $x^2 + 7x - 6 = 0$

31. $2x^2 + 3x - 5 = 0$

32. $2x^2 - 7x + 4 = 0$

33. $3x^2 + 2x - 4 = 0$

34. $4x^2 - 3x = 1$

35. $5x^2 + 2 = 8x$

36. $6x^2 + 2x = 3$

37. $3x^2 - 8x + 10 = 0$

38. $8x^2 = 4x + 3$

39. $\frac{x}{x - 1} + \frac{2}{x + 1} = 3$

40. $\frac{2}{x} - \frac{3}{x + 2} = 4$

41. If $f(x) = \begin{cases} x^2 - 1 & \text{if } x < 0 \\ \pi & \text{if } x > 0 \end{cases}$

determine **(a)** $f(-3)$, **(b)** $f(0)$, and **(c)** $f(5)$.

42. If $g(x) = \begin{cases} 2x^2 + 1 & \text{if } x < 0 \\ 5x - 2 & \text{if } 0 \leq x < 5 \\ -3x + 7 & \text{if } x \geq 5 \end{cases}$

determine **(a)** $g(-2)$, **(b)** $g(0)$, **(c)** $g(5)$, and **(d)** $g(10)$.

Solve Exercises 43–52.

43. ***Electricity*** Figure 9.19 shows two currents flowing in a single resistor R. The total current in the resistor is $i_1 + i_2$ and the power dissipated P is

$$P = (i_1 + i_2)^2 R$$

If $R = 50\ \Omega$ and $i_2 = 0.4$ A, find the current i_1 needed to produce a power of 12 W.

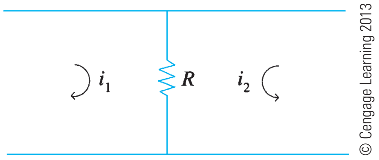


Figure 9.19

44. John can mow a field in 7 h and Matt can mow the same field in 8 h. If they work together, how long will it take to mow the field?

45. ***Dynamics*** A stone is dropped from the edge of a cliff that is 144 ft high. If the stone's height at time t is given by $L(t) = 144 - 16t^2$, how long does it take for the stone to reach the bottom of the cliff?

46. ***Industrial design*** A wedge is being designed in the shape of a right triangle. The hypotenuse will be 27 mm long and one leg will be 6 mm shorter than the other leg. What are the lengths of the sides of the triangle?

47. ***Physics*** The rate of flow of a liquid through a pipe is jointly proportional to its velocity and the cross-sectional area of the pipe. A hose with a 1-cm diameter has a liquid flow through it of 3 m/s.

(a) If a hose with a diameter of 5 mm is attached to one end of this hose, what is the velocity of the liquid through this smaller hose?

(b) What size hose should have been attached if the velocity was to be 30 m/s?

48. ***Energy*** The rate at which an object radiates energy, as given by the *Stefan–Boltzmann law*, varies directly as the fourth power of its absolute temperature. The sun radiates energy at the rate of 6.5×10^7 W/m^2 from its surface. If the surface temperature of the sun is 5 800 K, what is the constant of proportionality? (This is the *Stefan–Boltzmann constant*.)

49. ***Electricity*** The force that one charge exerts on another is given by *Coulomb's law* and is jointly proportional to the magnitudes of the charges and inversely proportional to the square of the distance between them. If a charge of 4×10^{-9} C is 5 cm from a charge of 5×10^{-8} C, and has a force of 7.192×10^{-4} N, what is the constant of proportionality (*Coulomb's constant*)? (Hint: Change 5 cm to m.)

50. ***Electricity*** The magnetic field of a straight current varies directly as the magnitude of the current and inversely as the distance from the current. A cable 5 m above the ground carrying a 100-A current has a magnetic field of 4×10^{-6} T. What is the magnetic field of a cable that carries a 150-A current and is 15 m above the ground?

51. ***Agriculture*** Table 9.14 shows the relationship between Stewart's bacterial wilt ratings and the percent of systematic infection of sweet corn hybrids. Stewart's wilt ratings ranged from 2.0 to 7.2 and the percent of infection ranged from 1% to 86%.

TABLE 9.14 Stewart's Wilt Infection Rating and Percent Corn Plants Infected

Wilt rating, r	2.0	4.0	2.7	2.3	3.1	3.4	4.4	3.8	3.2	3.9	4.4	4.6
Percent infected, $I(r)$	5	6	5	1	2	2	27	13	6	3	16	32

Wilt rating, r	4.4	4.9	5.4	5.6	6.0	4.6	5.3	6.2	6.5	6.8	7.0	7.2
Percent infected, $I(r)$	9	35	33	24	61	46	45	64	36	79	80	86

(a) Plot the points.

(b) Determine a quadratic regression curve, $I(r)$, that will fit the data.

(c) Draw the quadratic regression curve on the scatter plot of the data.

(d) Write a statement about the meaning of your regression equation.

52. ***Environmental science*** Table 9.15 shows the world-wide atmospheric concentration of carbon dioxide (CO_2) in parts per million (ppm) from 1960 to 1999.

TABLE 9.15 World Atmospheric Concentrations of Carbon Dioxide, 1960–1999

Year, t	1960	1965	1970	1975	1980	1985
CO_2 (ppm)	316.7	319.9	325.5	331.0	338.5	345.7
Year, t	1990	1995	1996	1997	1998	1999
CO_2 (ppm)	354.0	360.9	362.6	363.9	366.6	368.4

(a) Plot the points.

(b) Determine a quadratic regression curve, $C(t)$, that will fit the data.

(c) Draw the quadratic regression curve on the scatter plot of the data.

(d) Use your regression curve to estimate the world's atmospheric concentration of CO_2 in 2005.

CHAPTER 9 TEST

In Exercises 1–11, solve each equation for x.

1. $\frac{x}{8} + \frac{3x}{4} = \frac{7}{2}$

2. $\frac{12}{x^2 - 9} = \frac{x}{x - 3} - \frac{x}{x + 3}$

3. $x^2 - 4x - 5 = 0$

4. $3x^2 + x - 2 = 0$

5. $16x^2 + 9 = 24x$

6. $2x^2 + 3x + 3 = 0$

7. $x^2 + 3x + 1 = 0$

8. Solve $x^2 - 8x + 5 = 0$ by completing the square.

9. Solve $x^2 + 7x - 30 = 0$ by factoring.

10. Solve $3x^2 + 5x - 28 = 0$ by factoring.

11. If $f(x) = \begin{cases} \sqrt{4 - x^2} & \text{if } x \leq 2 \\ 3x + 5 & \text{if } x > 2 \end{cases}$

determine **(a)** $f(-2)$, **(b)** $f(9)$, **(c)** $f(2)$, and **(d)** $f(5)$.

Solve Exercises 12 and 13.

12. Solve $V = 3D^2H - 5$ for H.

13. Solve $V = 3D^2H - 5$ for D.

Solve Exercises 14–17.

14. A rectangular work area is to be 3 m longer than it is wide and will have an area of 46.75 m^2. What are the dimensions of the work area?

15. The position s at time t of an object moving rectilinearly with an initial velocity of v and an initial acceleration of a is

$$s = vt + \frac{at^2}{2}$$

Solve this equation for v in terms of a, t, and s.

16. The number of days d needed to erect a certain building varies inversely with the number of people working on the building p and the number of hours h they work each day. If a building requires 45 days to complete when 80 people work 8 h/day, how much time is needed if 60 people work 10 h/day?

17. A ball is thrown into the air at an inclination of 60° to the horizontal. Table 9.16 shows the height h in feet of the ball during the first five seconds of flight.

TABLE 9.16 Height of Ball t Seconds After Thrown

Time, t	0.5	1.0	1.5	2.0	2.5	3.0	3.5	4.0	4.5	5.0
Height, h, (ft)	39.3	70.6	93.9	109.2	116.5	115.8	107.1	90.1	65.7	33.0

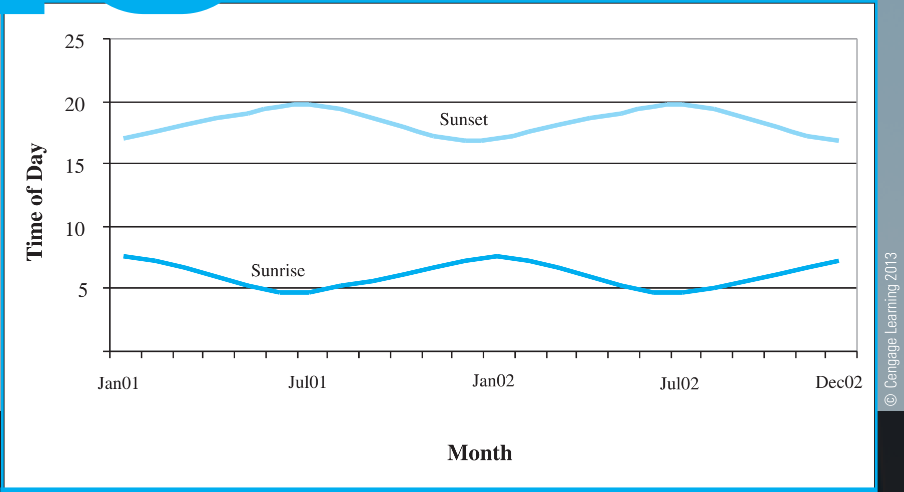
(a) Plot the points.

(b) Determine a quadratic regression curve, $h(t)$, that will fit the data.

(c) Draw the quadratic regression curve on the scatter plot of the data.

(d) What was the height of the ball 1.69 sec after it was tossed?

10 GRAPHS OF TRIGONOMETRIC FUNCTIONS



Many things in our lives occur periodically. One good example is the time of sunrise and sunset and the number of hours of daylight each day. In Section 10.2 we will see how to develop a formula for such a cycle.

Trigonometry originally helped people solve triangles. For many years this was the main use of trigonometry. When people began graphing equations, it became possible to obtain pictures or graphs of the trigonometric functions.

Later, when people discovered light and sound waves, they found that these waves repeated at regular intervals—just like the graphs of trigonometric functions.

In fact, light, sound, electricity, ocean waves, and spring action are all examples of things that can be displayed as a graph of one or more trigonometric functions.

OBJECTIVES

AFTER COMPLETING THIS CHAPTER, YOU WILL BE ABLE TO:

- Make a table of values and graph sine, cosine, and tangent functions.
- Use a spreadsheet and a graphing utility to graph sine, cosine, and tangent functions.
- Graph sine and cosine functions using amplitude, period, frequency, and shifts.
- Determine the amplitude, period, frequency, and shifts and model a given sinusoidal graph with an equation.
- Determine intercepts and asymptotes of secant, cosecant, and cotangent functions and graph these functions using a graphing utility.
- Classify applications as simple harmonic motion and model harmonic motion symbolically.
- Make a table of values and graph equations defined by parametric equations.
- Use a spreadsheet and a graphing utility to graph equations defined by parametric equations.
- Write a parametric set of equations as a function of *x*.
- Convert polar coordinates to rectangular coordinates and rectangular to polar.
- Make a table of values and graph equations defined by polar equations;
- Use a spreadsheet and a graphing utility to graph equations defined by polar equations.

10.1 SINE AND COSINE CURVES: AMPLITUDE AND PERIOD

As you know, the trigonometric functions repeat their values on a regular basis. Thus, $\sin 15° = \sin(360° + 15°) = \sin(720° + 15°)$, and so on. This means that the graphs of the trigonometric functions repeat. This repeating nature helps us apply these graphs to many physical and technical situations. In this chapter, we will use a graphics calculator to help draw the graphs, and show how to apply these graphs to your technical area.

SINE CURVES

Sine and cosine curves are used in applications of mathematics that involve circular and repetitive motion. In this chapter, we will first look at ways to graph the sine and cosine functions. Later in the chapter, we will look at some applications.

When we began graphing algebraic equations, we started by making a table of values. We shall adopt the same approach in graphing trigonometric functions. The following table gives the values of x in both degrees and radians of the function $y = \sin x$. The values of x only go from 0° to 360° (0 to 2π rad), since the values of y begin to repeat after 360° (or 2π rad). The graph of these values is shown in Figure 10.1.

x (degrees)	0	30	45	60	90	120	135	150	180
x (radians)	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π
$y = \sin x$	0	0.5	0.707	0.866	1	0.866	0.707	0.5	0

x (degrees)	180	210	225	240	270	300	315	330	360
x (radians)	π	$\frac{7\pi}{6}$	$\frac{5\pi}{4}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{7\pi}{4}$	$\frac{11\pi}{6}$	2π
$y = \sin x$	0	-0.5	-0.707	-0.866	-1	-0.866	-0.707	-0.5	0

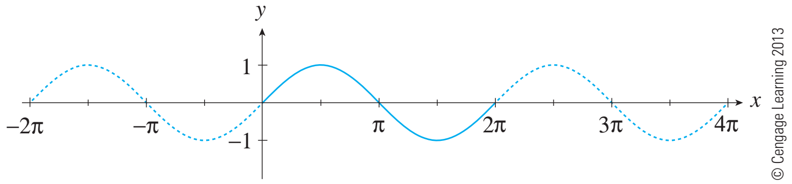


Figure 10.1

If we were to continue plotting the values for x larger than 2π rad or less than 0, we would get the points indicated by the dotted line in Figure 10.1. As you can see, the graph, like the values for $\sin x$, begins to repeat. The graph of the sine function has a very distinctive wave shape. A graph that has the general shape of the sine function is called a *sine wave*, a **sine curve**, or a **sinusoidal curve**.

AMPLITUDE

One term used to help describe some functions is amplitude.

AMPLITUDE

The **amplitude** A is one-half the difference between the maximum and minimum values for a periodic function.

For $y = \sin x$, the maximum value is 1 and the minimum value is -1. The difference between these two values is $1 - (-1) = 2$ and half of that is 1. So, the amplitude of $y = \sin x$ is 1.

Let's examine another trigonometric function. Suppose we want to graph $y = 3 \sin x$. This is the same as taking each of the values for $y = \sin x$ in the previous table and multiplying it by 3. The resulting table would be as follows and the graph would be like the one in Figure 10.2.

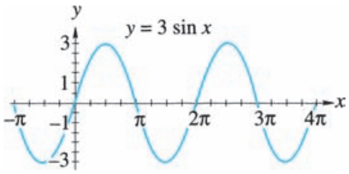


Figure 10.2

x	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{11\pi}{6}$	2π
$y = 3\sin x$	0	1.5	2.6	3	2.6	1.5	0	−1.5	−2.6	−3	−2.6	−1.5	0

Notice that the maximum value of $y = 3 \sin x$ is 3 and the minimum value is -3. The amplitude of $y = 3 \sin x$ is $\frac{3 - (-3)}{2} = 3$. In fact, you should be able to see that the amplitude of $y = A \sin x$ is $|A|$.

PERIOD AND FREQUENCY

The first two curves are examples of periodic functions. Each of them repeats its shape or completes one wave every 2π units, so we say these curves each have a period of 2π radians.

PERIODIC FUNCTION

For a function f, if p is the smallest positive constant where $f(x) = f(x + p)$ for all values of x, we say that f is a *periodic function* and that it has **period** p. The portion of a graph that falls within one period is called a *cycle*.

The sine function is a periodic function with period 2π (or 360°), because $\sin x = \sin(x + 2\pi)$ and 2π is the smallest positive number where this is true.

A cycle is the portion of the graph that falls within one period. In Figure 10.3 three different cycles have been indicated. Since the period of this function is 6 units each cycle is 6 units long.

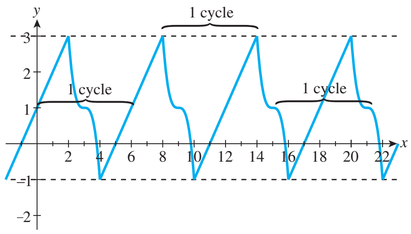


Figure 10.3

FREQUENCY

The **frequency** of a periodic function is defined as $\dfrac{1}{\text{period}}$.

EXAMPLE 10.1

Determine the (a) amplitude, (b) period, and (c) frequency of the function f graphed in Figure 10.4.

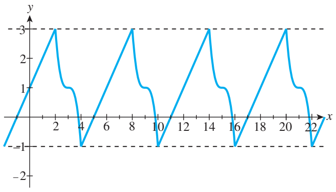


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Figure 10.4

SOLUTIONS

(a) The maximum value of this function is 3 and the minimum value is -1. Thus, the amplitude is $\frac{3-(-1)}{2} = 2$.

(b) Notice that $f(0) = f(6) = f(12) = f(18) = 1$. Similarly, $f(1) = f(7) = f(13) = f(19) = 2$ and $f(-1) = f(5) = f(11) = f(17) = 0$. This graph repeats every 6 units so it has a period of 6.

(c) Since the period is 6 the frequency is $\dfrac{1}{\text{period}} = \dfrac{1}{6}$ cycle/unit.

Frequency is measured in cycles per unit. If the x-axis unit is degrees, then the frequency is the number of cycles per degree. If the x-axis unit is radians, then the frequency is the number of cycles per radian. When using radians, the frequency is usually reported as the number of cycles. The metric unit of frequency is *hertz* (Hz) and

$$1 \text{ Hz} = 1 \text{ cycle/s} = 1 \text{ s}^{-1}$$

EXAMPLE 10.2

What is the frequency of $y = \sin x$ if x is measured in degrees?

SOLUTION Since the period of $y = \sin x$ is 360°, the frequency is

$$\frac{1 \text{ cycle}}{360^\circ} = \frac{1}{360} \text{ cycle/degree}$$

EXAMPLE 10.3

What is the frequency of $y = \sin x$ if x is measured in radians?

SOLUTION Since the period of $y = \sin x$ is 2π radians, the frequency is

$$\frac{1 \text{ cycle}}{2\pi} = \frac{1}{2\pi} \text{ cycle}$$

EXAMPLE 10.4

Find the period of a periodic wave with a frequency of 500 Hz.

SOLUTION Since frequency $= \dfrac{1}{\text{period}}$, we have period $= \dfrac{1}{\text{frequency}}$. We are given a frequency of 500 Hz, so the period is

$$\frac{1}{500 \text{ Hz}} = 0.002 \text{ s, or } 2 \text{ ms}$$

EXAMPLE 10.5

Determine the frequency of the periodic function in Figure 10.5.

SOLUTION Notice here that the horizontal axis is in units of time, in this case seconds. From the figures we can see that this function completes 10 cycles in 2 s (or 5 cycles in 1 s). Thus, the frequency is

EXAMPLE 10.5 (Cont.)

$$\frac{10\text{ cycles}}{2\text{ s}} = 5\text{ cycles/s} = 5\text{ Hz}$$

The frequency of this function is 5 Hz.

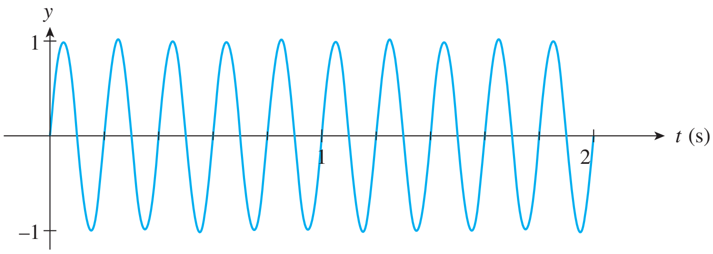

Figure 10.5

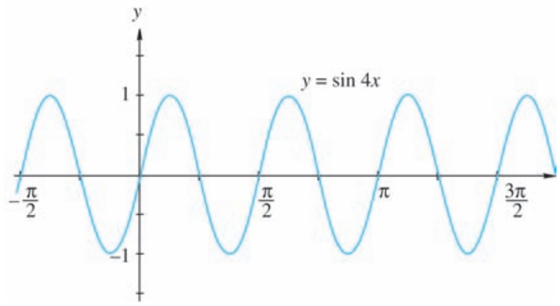

Figure 10.6

Now, let's graph a slightly different version of the sine curve, $y = \sin 4x$. Again, we will begin by using a table. This table will have values for x and $4x$, as well as for $y = \sin 4x$. The table follows, and the graph of $y = \sin 4x$ is in Figure 10.6.

x	0	$\frac{\pi}{24}$	$\frac{\pi}{12}$	$\frac{\pi}{8}$	$\frac{\pi}{6}$	$\frac{5\pi}{24}$	$\frac{\pi}{4}$	$\frac{7\pi}{24}$	$\frac{\pi}{3}$	$\frac{3\pi}{8}$	$\frac{5\pi}{12}$	$\frac{11\pi}{24}$	$\frac{\pi}{2}$
$4x$	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{11\pi}{6}$	2π
$y = \sin 4x$	0	0.5	0.866	1	0.866	0.5	0	−0.5	−0.866	−1	−0.866	−0.5	0

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This graph is interesting in that it looks like a sine curve, with an amplitude of 1, but it does not have a period of 2π. In fact, its period is $\frac{\pi}{2}$. Notice that $\frac{\pi}{2} = 2\pi \div 4$. In general, we have the following statement for both the sine and the cosine functions.

PERIOD AND FREQUENCY

The *period* and **frequency** of $y = \sin Bx$ and $y = \cos Bx$ are

Period	**Frequency**	
$\dfrac{2\pi}{\lvert B\rvert}$	$\dfrac{\lvert B\rvert}{2\pi}$	if measured in radians
$\dfrac{360°}{\lvert B\rvert}$	$\dfrac{\lvert B\rvert}{360°}$	if measured in degrees

COSINE CURVES

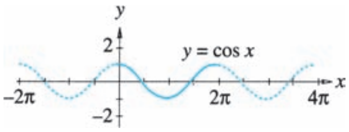


Figure 10.7

As yet, we have graphed only the sine curve. It is time to graph the cosine curve. A table of values follows and the graph of $y = \cos x$ is shown in Figure 10.7.

x	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{11\pi}{6}$	2π
$y = \cos x$	1	0.87	0.5	0	−0.5	−0.87	−1	−0.87	−0.5	0	0.5	0.87	1

Here, as with the sine curve in Figure 10.1, we have shown the values graphed from the table with a solid line. The dashed line indicates what the graph would look like if we continued to plot values for $x > 2\pi$ or $x < 0$. The solid line represents one cycle. As you can see, the cosine function is periodic; $y = \cos x$ has period of 2π and amplitude of 1. A careful examination of the curves in Figures 10.1 and 10.7 shows that the shapes of the curves are exactly the same but the curves are displaced along the horizontal axis relative to one another.

EXAMPLE 10.6

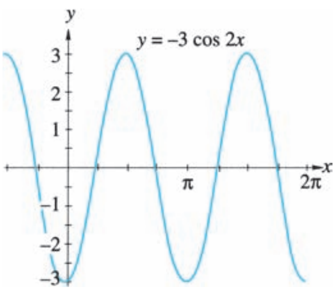


Figure 10.8

Sketch the graph of $y = -3 \cos 2x$.

SOLUTION The period of $y = \cos x$ is 2π, so the period of $y = -3 \cos 2x$ is $2\pi \div 2 = \pi$. The amplitude of $y = A \cos x$ is $|A|$, which means that the amplitude of $y = -3 \cos 2x$ is $|-3| = 3$. To sketch this curve, we need to find only a few values of x. We will use the x-values at the maximum and minimum y-values and at the x-intercepts (when $y = 0$). This table follows and the graph of $y = -3 \cos 2x$ is in Figure 10.8.

x	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π
$y = -3 \cos 2x$	−3	0	3	0	−3

Study the graph of $y = -3 \cos 2x$. This is the graph of $y = 3 \cos 2x$ only "flipped" over the x-axis.

NOTE The technical expression for indicating that a graph is "flipped over the x-axis" is that it is "reflected across the x-axis."

EXAMPLE 10.7

Sketch the graph of $y = 4 \sin 3x$.

SOLUTION We can see that the amplitude is 4 and the period is $\frac{2\pi}{3}$. The frequency is $\frac{1}{\frac{2\pi}{3}} = \frac{3}{2\pi}$ cycles. From this we know that $y = 0$ when $x = \frac{2\pi}{3}$, $x = \frac{\pi}{3}$, and $x = 0$. The maximum and minimum values will occur halfway between these values of x, or when $x = \frac{\frac{\pi}{3} + 0}{2} = \frac{\pi}{6}$ and $x = \frac{\frac{2\pi}{3} + \frac{\pi}{3}}{2} = \frac{\pi}{2}$. The table for these values follows:

EXAMPLE 10.7 (Cont.)

x	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$
$y = 4 \sin 3x$	0	4	0	-4	0

Based on these values and our knowledge of the shape of the sine curves, we get the graph shown in Figure 10.9. Notice how this graph is "compressed" or "shrunk" over a period of $\frac{2\pi}{3}$.

EXAMPLE 10.8

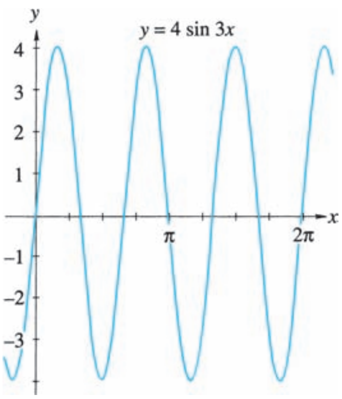


Figure 10.9

Sketch the graph of $y = 1.5 \cos \frac{x}{2}$.

SOLUTION We can see the amplitude is 1.5 and the period is $2\pi \div \frac{1}{2} = 4\pi$ or $720°$. Since the frequency is $\frac{1}{\text{period}}$, it is $\frac{1}{4\pi}$ cycles. The maximum values will occur when $x = 0$ and $x = 720°$. The minimum value will be halfway between these two points or when $x = 360°$. The zeros are at the midpoints between these three points or when $x = 180°$ and $x = 540°$. These findings are shown in following table:

x	$0°$	$180°$	$360°$	$540°$	$720°$
$y = 1.5 \cos \frac{x}{2}$	1.5	0	-1.5	0	1.5

Based on these values and our knowledge of the shape of the cosine curve, we get the graph in Figure 10.10. Notice how this curve is "stretched out" over a period of 4π or $720°$.

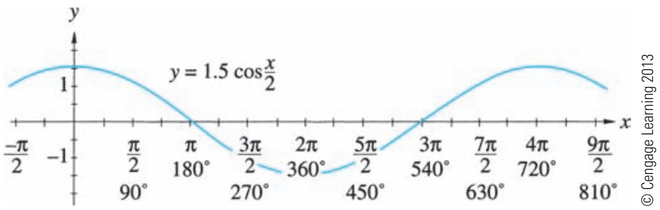


Figure 10.10

The following box summarizes this section's discussion on amplitude and period.

SUMMARY

A function of the form $y = A \sin Bx$ and $y = A \cos Bx$ has

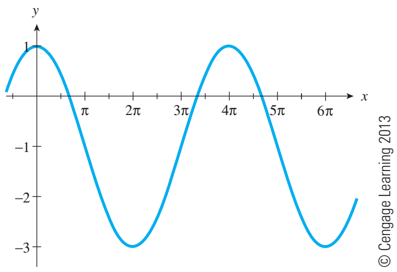
Amplitude	Period	Frequency	
$\lvert A \rvert$	$\frac{2\pi}{\lvert B \rvert}$	$\frac{\lvert B \rvert}{2\pi}$	if measured in radians
$\lvert A \rvert$	$\frac{360°}{\lvert B \rvert}$	$\frac{\lvert B \rvert}{360°}$	if measured in degrees

If $A < 0$, the graph is "flipped over" (or "reflected across") the x-axis.

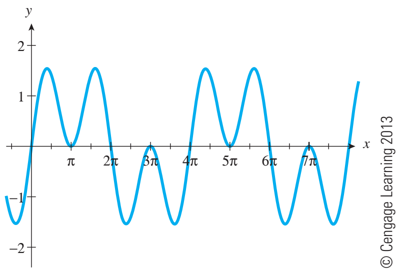
EXERCISE SET 10.1

In Exercises 1–4 estimate the (a) amplitude, (b) period, and (c) frequency for each of the given sinusoidal functions.

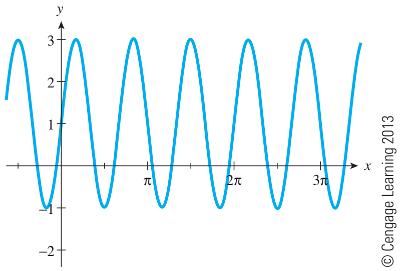
1.



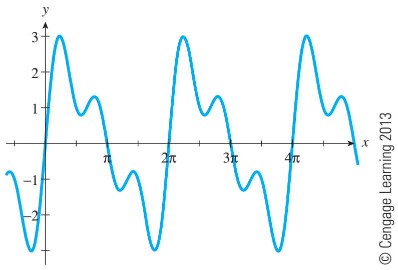
3.



2.

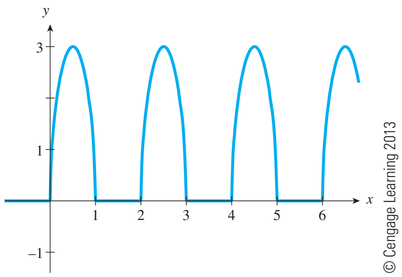


4.

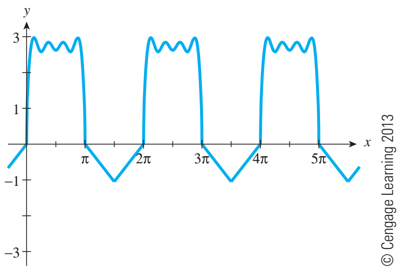


In Exercises 5–8 estimate the (a) amplitude, (b) period, and (c) frequency for each of the given nonsinusoidal funtions.

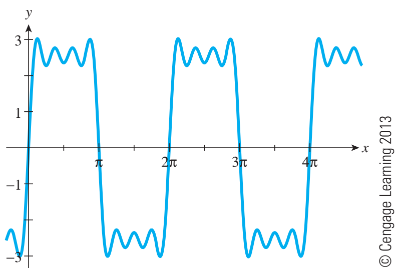
5.



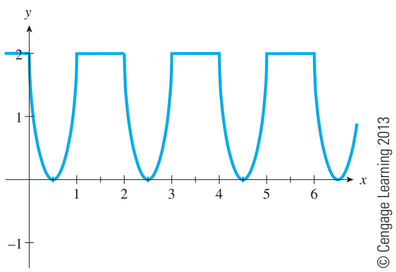
7.



6.



8.



In Exercises 9–22, find the period, amplitude, and frequency and sketch two cycles of the given functions.

9. $y = 3 \sin 2x$

10. $y = 5 \sin 6x$

11. $y = 2 \cos x$

12. $y = 7 \cos 3x$

13. $y = 8 \cos 2\pi x$

14. $y = 7 \cos \pi x$

15. $y = \frac{1}{2} \sin 4x$

16. $y = 5 \cos \frac{1}{3}x$

17. $y = \dfrac{\cos \frac{1}{2}x}{3}$

18. $y = \dfrac{\cos 3x}{5}$

19. $y = -3 \sin \dfrac{x}{4}$

20. $y = -5 \cos \dfrac{x}{3}$

21. $y = \dfrac{-\cos x}{2}$

22. $y = \dfrac{\sin 3x}{-2}$

Solve Exercises 23–28.

23. ***Electronics*** The current, I (in amperes), in an ac circuit is given by $I = 6.5 \sin 120\pi t$, where t is the time in seconds. **(a)** Find the period, amplitude, and frequency for this function. **(b)** Plot I as a function of t for $0 \le t \le 0.1$ s.

24. ***Electronics*** In a simple generator, the electromotive force, E (measured in volts), can be expressed as a function of time, t, as $E = 12 \sin 120\pi t$.

(a) Find the amplitude, period, and frequency for this function.

(b) Sketch the graph of E for two periods. Make sure that both axes are labeled correctly.

25. ***Solar energy*** When the receiving surface of a solar radiation detector is not perpendicular to the direction of the sun, the incident radiation per unit area, G_i, will be reduced by a factor of $\cos i$, where i is the angle between the rays of the sun and a line perpendicular to the receiving surface. The equation $G_i = G_n \cos i$ gives the incident radiation per unit area, where G_n is the radiant energy incident upon a surface on the earth placed normal (perpendicular) to the rays of the sun. Suppose $G_n = 1094\ \text{W/h/m}^2$.

(a) Find the amplitude, period, and frequency for this function.

(b) Sketch the graph of G_i for two periods. Make sure that both axes are labeled correctly.

26. ***Oceanography*** Under certain conditions, the height of the tide above its mean level is given by $y = 2.1 \cos 0.45t$, where y is in meters and t is in hours.

(a) Find the amplitude, period, and frequency for this function.

(b) Sketch the graph of y for two periods. Make sure that both axes are labeled correctly.

27. ***Medical technology*** An ultrasonic transducer used for medical diagnosis is oscillating at a frequency of 6.7 MHz (or 6.7×10^6 Hz). How much time does each oscillation take?

28. ***Aeronautics*** An airplane's electrical generator produces a time-varying output voltage described by the equation $V = 120 \sin 2513t$, where t is time in seconds and V is in volts.

(a) What is the amplitude?

(b) What is the period?

(c) What is the frequency?

[IN YOUR WORDS]

29. If you were shown the graphs of $y = \sin x$, $y = 2 \sin x$, and $y = \sin 2x$ on the same axes, describe how you could distinguish among the graphs.

30. **(a)** Describe how you can determine the amplitude of a sine or cosine function.

(b) How can you use the amplitude to help you draw the graph of $y = 5 \sin x$?

(c) Describe how you can determine the period and frequency of a sine or cosine function.

31. What is the horizontal distance between two consecutive maxima of the function **(a)** $f(x) = \sin 2x$ and **(b)** $g(x) = \sin 0.5x$? Explain your answers.

32. What is the horizontal distance between two consecutive zeros of the function **(a)** $f(x) = \sin 2x$ and **(b)** $g(x) = \sin 0.5x$? Explain your answers.

10.2 SINE AND COSINE CURVES: HORIZONTAL AND VERTICAL DISPLACEMENTS

The graphs of the sine and cosine functions that we have examined have all had one thing in common. That is, the sine curves all contained the point (0, 0) and the cosine curves all contained $(0, A)$, where $|A|$ was the amplitude. In this section, we will shift the graphs away from these points.

HORIZONTAL DISPLACEMENT

EXAMPLE 10.9

Sketch the cycle of the graph of $y = \sin\left(x + \frac{\pi}{3}\right)$.

SOLUTION As usual, we make a table of values. In this table, we show three rows. The first row contains values of x, the next row contains values of $x + \frac{\pi}{3}$, and the third contains the values for $y = \sin\left(x + \frac{\pi}{3}\right)$:

x	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π
$x + \frac{\pi}{3}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{4\pi}{3}$
$y = \sin\left(x + \frac{\pi}{3}\right)$	0.866	1	0.866	0.5	0	−0.5	−0.866

x	π	$\frac{7\pi}{6}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{11\pi}{6}$	2π
$x + \frac{\pi}{3}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{11\pi}{6}$	2π	$\frac{13\pi}{6}$	$\frac{8\pi}{3}$
$y = \sin\left(x + \frac{\pi}{3}\right)$	−0.866	−1	−0.866	−0.5	0	0.5	0.866

The graph of one cycle of this curve is given in Figure 10.11. Notice that this looks just like a sine curve except that it has to be shifted $\frac{\pi}{3}$ units to the left.

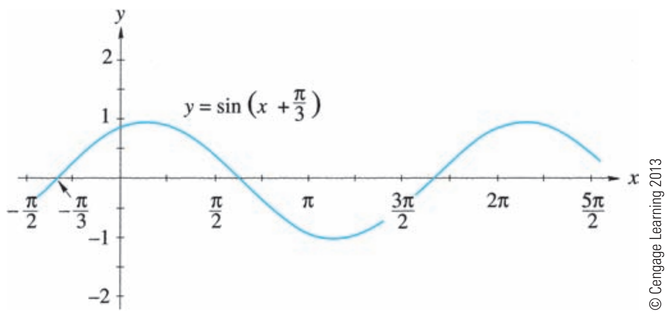


Figure 10.11

EXAMPLE 10.10

Sketch one cycle of the graph of $y = \sin\left(2x - \frac{\pi}{6}\right)$.

SOLUTION The following table contains three rows, x, $2x - \frac{\pi}{6}$, and $\sin\left(2x - \frac{\pi}{6}\right)$:

x	0	$\frac{\pi}{12}$	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{5\pi}{12}$	$\frac{\pi}{2}$
$2x - \frac{\pi}{6}$	$-\frac{\pi}{6}$	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$
$y = \sin\left(2x - \frac{\pi}{6}\right)$	−0.5	0	0.5	0.866	1	0.866	0.5

EXAMPLE 10.10 (Cont.)

x	$\frac{\pi}{2}$	$\frac{7\pi}{12}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	$\frac{11\pi}{12}$	π
$2x - \frac{\pi}{6}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{11\pi}{6}$
$y = \sin\left(2x - \frac{\pi}{6}\right)$	0.5	0	−0.5	−0.866	−1	−0.866	−0.5

One cycle of this graph is shown in Figure 10.12. Notice that it has a period of π and has been shifted $\frac{\pi}{12}$ units to the right.

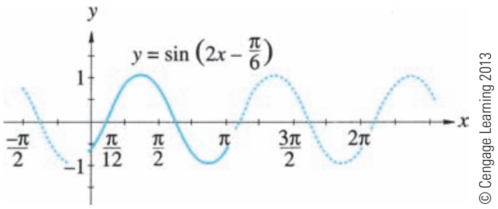


Figure 10.12

GRAPHS OF $y = A\sin(Bx + C)$ AND $y = A\cos(Bx + C)$

A function of the form $y = A\sin(Bx + C)$ or $y = A\cos(Bx + C)$ has

Amplitude	Period	Frequency	Horizontal displacement	If measured in
$\lvert A\rvert$	$\frac{2\pi}{\lvert B\rvert}$	$\frac{\lvert B\rvert}{2\pi}$	$-\frac{C}{B}$	radians
$\lvert A\rvert$	$\frac{360°}{\lvert B\rvert}$	$\frac{\lvert B\rvert}{360°}$	$-\frac{C}{B}$	degrees

If the horizontal displacement is negative, the entire curve will be shifted to the left. If the horizontal displacement is positive, the entire curve will be shifted to the right. The horizontal displacement gives you the x-coordinate at which to begin drawing one period of $y = A\sin(Bx + C)$ or $y = A\cos(Bx + C)$. If $A < 0$, the graph is "flipped over" (or "reflected across") the x-axis.

EXAMPLE 10.11

Sketch one cycle of the curve $y = \frac{1}{2}\cos(3x + \pi)$.

SOLUTION Using the information contained in the equation, $A = \frac{1}{2}$, $B = 3$, and $C = \pi$, so the amplitude is $\frac{1}{2}$, the period is $\frac{2\pi}{3}$, and the horizontal displacement is $-\frac{\pi}{3}$.

Since the horizontal displacement is $-\frac{\pi}{3}$, we start drawing the graph at $x = -\frac{\pi}{3}$. The period of $\frac{2\pi}{3}$ indicates that one complete cycle will end at $x = -\frac{\pi}{3} + \frac{2\pi}{3} = \frac{\pi}{3}$. For cosine, the maximum values occur at these two points. The minimum value will occur midway between the maximum values, or at $x = 0$. The zeros are midway between the maximum and minimum values, or at $x = -\frac{\pi}{6}$ and $x = \frac{\pi}{6}$. A sketch of $y = \frac{1}{2}\cos(3x + \pi)$ is in Figure 10.13 with the one complete cycle we just described in solid and additional cycles dashed.

EXAMPLE 10.12

Sketch one cycle of the graph of $y = -4\sin\left(\frac{1}{2}x - \frac{\pi}{8}\right)$.

SOLUTION Here $A = -4$, $B = \frac{1}{2}$, and $C = -\frac{\pi}{8}$. The function has an amplitude of $|-4| = 4$ and a period of $2\pi \div \frac{1}{2} = 4\pi$. The horizontal displacement is $-\left(-\frac{\pi}{8} \div \frac{1}{2}\right) = \frac{\pi}{4}$. We begin sketching the curve at $x = \frac{\pi}{4}$. The period of 4π indicates that one complete cycle will end at $x = \frac{\pi}{4} + 4\pi = \frac{17\pi}{4}$. The zeros of this sine function occur when $x = \frac{\pi}{4}$, $x = \frac{17\pi}{4}$, and midway between these two points, at $x = \frac{9\pi}{4}$. The minimum value is at $x = \frac{5\pi}{4}$ and the maximum when $x = \frac{13\pi}{4}$. The graph of $y = -4\sin\left(\frac{1}{2}x - \frac{\pi}{8}\right)$ is shown in Figure 10.14.

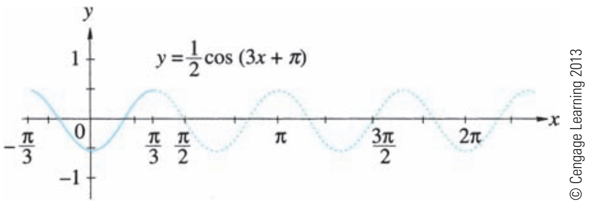


Figure 10.13

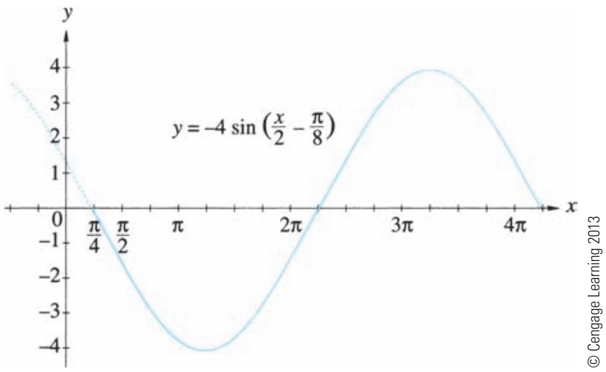


Figure 10.14

VERTICAL DISPLACEMENT

We have seen how graphs can be shifted horizontally. They can also be shifted vertically.

EXAMPLE 10.13

Sketch two cycles of the graph of $y = 3\cos 2x + 5$.

SOLUTION This equation is in the form $y = A\sin(Bx + C) + D$, with $A = 3$, $B = 2$, $C = 0$, and $D = 5$. The function has an amplitude of $|3| = 3$ and a period of $2\pi \div 2 = \pi$. Since $C = 0$, the horizontal shift is 0. What effect does D have on this graph? Let's look at a table of values:

x	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π	$\frac{5\pi}{4}$	$\frac{3\pi}{2}$	$\frac{7\pi}{4}$	2π
$y = 3\cos 2x$	3	0	−3	0	3	0	−3	0	3
$y = 3\cos 2x + 5$	8	5	2	5	8	5	2	5	8

The value of D in this example, 5, moves or displaces the graph 5 units vertically. This can be seen by the graphs in Figure 10.15. If you moved the graph of $y = 3\cos 2x$ up 5 units it would overlap the graph of $y = 3\cos 2x + 5$.

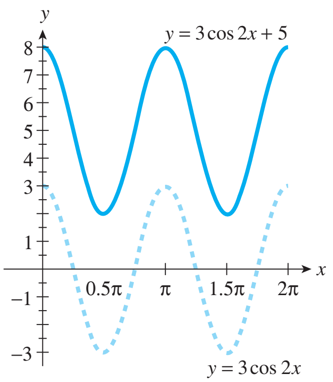


Figure 10.15

From Example 10.13 it seems as if the effect of D is to shift the graph vertically. When we add this information to the previous information about graphing, we get the following summary box. It includes everything we have discussed in the last two sections, so this one box can replace all of the other boxes in these two sections.

GRAPHS OF $y + A$ SIN $(Bx + C) + D$ AND $y + A$ COS $(Bx + C) + D$

A curve of the form $y = A\sin(Bx + C) + D$ or $y = A\cos(Bx + C) + D$ has

Amplitude	Period	Frequency	Horizontal displacement	Vertical displacement	If measured in
$\lvert A\rvert$	$\dfrac{2\pi}{\lvert B\rvert}$	$\dfrac{\lvert B\rvert}{2\pi}$	$-\dfrac{C}{B}$	D	radians
$\lvert A\rvert$	$\dfrac{360^\circ}{\lvert B\rvert}$	$\dfrac{\lvert B\rvert}{360^\circ}$	$-\dfrac{C}{B}$	D	degrees

If the horizontal displacement is negative, the entire curve will be shifted to the left. When the horizontal displacement is positive, the entire curve will be shifted to the right. The horizontal displacement gives you the x-coordinate at which to begin drawing one period of $y = A\sin(Bx + C) + D$ or $y = A\cos(Bx + C) + D$.

If $A < 0$, the graph is "flipped" over the line $y = D$.

EXAMPLE 10.14

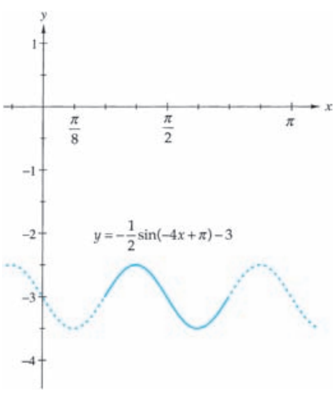


Figure 10.16

Sketch one cycle of the graph of $y = -\frac{1}{2}\sin(-4x + \pi) - 3$.

SOLUTION Here $A = -\frac{1}{2}$, $B = -4$, $C = \pi$, and $D = -3$. The function has an amplitude of $\left|-\frac{1}{2}\right| = \frac{1}{2}$ and a period of $2\pi \div |(-4)| = \frac{\pi}{2}$. Since $C = \pi$, the horizontal shift is $-(\pi \div (-4)) = \frac{\pi}{4}$. We begin sketching the curve at $x = \frac{\pi}{4}$. The period of $\frac{\pi}{2}$ indicates that one complete cycle will end at $x = \frac{\pi}{4} + \frac{\pi}{2} = \frac{3\pi}{4}$. The zeros of $y = -\frac{1}{2}\sin(-4x + \pi)$ will occur when $x = \frac{\pi}{4}$ and $x = \frac{3\pi}{4}$, and midway between these two points at $x = \frac{\pi}{2}$. The maximum value is at $x = \frac{3\pi}{8}$ and the minimum value is at $x = \frac{5\pi}{8}$. The vertical displacement is -3, which means that the graph will be shifted 3 units downward. The graph of $y = -\frac{1}{2}\sin(-4x + \pi) - 3$ is shown in Figure 10.16.

MODELING PERIODIC FUNCTIONS

In previous chapters we have seen how to use a calculator and spreadsheet to produce linear and quadratic regression equations. In this section we will use a calculator and spreadsheet to produce sinusoidal regression equations.

EXAMPLE 10.15

The data in Table 10.1 give the number of hours between sunrise and sunset in Seattle, Washington, on the 1st and 16th of each month for 2010.

TABLE 10.1 Length of Day in Seattle, WA, 2010

Month	January		February		March		April	
Date	1	16	1	16	1	16	1	16
Hours	8.52	8.92	9.58	10.35	11.07	11.93	12.85	13.68

Month	May		June		July		August	
Date	1	16	1	16	1	16	1	16
Hours	14.48	15.18	15.72	15.97	15.92	15.57	14.95	14.23

Month	September		October		November		December	
Date	1	16	1	16	1	16	1	16
Hours	13.37	12.52	11.68	10.82	9.97	9.27	8.27	8.43

Source: http://aa.usno.navy.mil/

Determine a sinusoidal regression for this data.

SOLUTION Before we enter the data in calculator we need to decide on a numbering system for the x-axis. One system would be to let January 1 = 1, January 16 = 16, February 1 = 32, February 28 = 59, . . . December 31 = 365. Using this procedure we get the data in Table 10.2.

TABLE 10.2 Length of Day in Seattle, WA, 2010

Month	January		February		March		April	
Date	1	16	1	16	1	16	1	16
Day	1	16	32	47	60	76	91	106
Hours	8.52	8.92	9.58	10.35	11.07	11.93	12.85	13.68

Month	May		June		July		August	
Date	1	16	1	16	1	16	1	16
Day	121	136	152	167	182	197	213	228
Hours	14.48	15.18	15.72	15.97	15.92	15.57	14.95	14.23

Month	September		October		November		December	
Date	1	16	1	16	1	16	1	16
Day	244	259	274	289	305	319	335	350
Hours	13.37	12.52	11.68	10.82	9.97	9.27	8.27	8.43

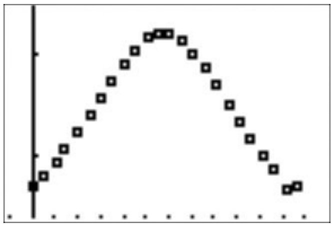


Figure 10.17a

Using a Calculator

The data is entered into a calculator using the same procedures we have used before. A graph of the data is in Figure 10.17a.

To determine the sinusoidal regression curve we first make sure the calculator is in radian mode. Then press STAT ► [CALC] ▲. The cursor should now be

EXAMPLE 10.15 (Cont.)

on the line C:SinReg. As before, if you add the name of the list with the x-values 2ND 1 [L1] , , the list with the y-values 2ND 2 [L2] , , VARS ▶ 1 1 . After you get the regression values press GRAPH to get the result in Figure 10.17b.

As we see, the regression equation for this data is

$$y \approx 3.7768 \sin(0.01652t - 1.2682) + 12.0707$$

hours of daylight on day t of 2010.

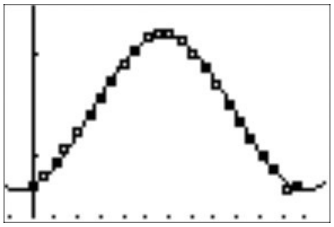


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Figure 10.17b

Using a Spreadsheet

The data is entered into the spreadsheet in the same format as the table. The data is graphed in the same way as earlier scatter plots with the result shown in Figure 10.17c.

Unlike many other functions, Excel does not have a built-in modeling tool for sine functions. However, a model can be found using SOLVER.

The model we will attempt to find is in the form $y = a \sin(bx + c) + d$ where a, b, c, and d are constants.

Construct a template similar to the one shown in Figure 10.17d. Column A and Column B should contain the points from the given information. Column C will contain the model using the four constants in cells D4, E4, F4, and G4: "= $D$4*SIN($E$4*A4+$F$4) + $G$4." Column H will be the squared error between the y-value in Column B and the Model in Column C: "= (B4 − C4)^2." Column I will contain the sum of the squared errors: "= SUM(H4 : H27)." (Note: it is helpful to have something in Cells D4, E4, F4, and G4.)

The idea is to make the "Sum of the Squared Errors" the very smallest number possible. You can experiment with different values for a, b, c, and d to see what happens to that sum.

Now, use Solver.[1] Place the cursor on Cell I4. We want to minimize the value in I4 by changing the values in D4, E4, F4, and G4 (see Figure 10.17f). Solver can produce errors if the values that are going to change are too far from the solution you desire. In this case, the data is sinusoidal with a period of 360 and an amplitude of about 4.5. The curve is shifted vertically by about 12. So, $2\pi/b = 360$ which means $b \approx .0175$.

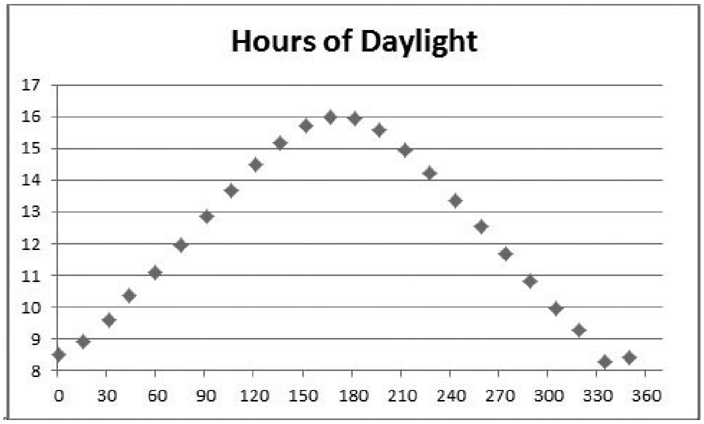

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Figure 10.17c

1 On recent versions of Microsoft Excel, Solver is an add-in program that is available when you install Microsoft Office or Excel. To use it in Excel, however, you need to load it first. Directions for loading the Solver add-in are in Excel help.

After several attempts, with different combinations of values for *a*, *b*, *c*, and *d*, Solver finds that an appropriate model (see Figure 10.17g) with the graph in Figure 10.17h. (A model is "appropriate" if the scatter plot of the data and the model-data is about the same.)

	A	B	C	D	E	F	G	H	I
1	Modeling the Sine Function								
2								Squared	Sum of Squares
3	Day	Hours	Model: y=	A	B	C	D	Errors	Errors
4	1	8.52	12.9092974	1	2	0	12	19.2659319	196.818266
5	16	8.92	12.5514267					13.1872597	177.5523341

Figure 10.17d

	A	B	C	D	E	F	G	H	I
1	Modeling the Sine Function								
2								Squared	Sum of Squares
3	Day	Hours	Model: y=	A	B	C	D	Errors	Errors
4	1	8.52	1.90929743	1	1	1	1	43.7013885	3182.207617
5	16	8.92	0.03860251					78.8792214	
6	32	9.58	1.99991186					57.4577362	
7	44	10.35	1.85090352					72.2346409	

Figure 10.17e

	A	B	C	D	E	F	G	H	I	J	K
1	Modeling the Sine Function										
2								Squared	Sum of Squares		
3	Day	Hours	Model: y=	A	B	C	D	Errors	Errors		
4	1	8.52	12.3628733	4	0.0175	0.04	12.133	14.7676751	320.2911452		
5	16	8.92	13.3912662								
6	32	9.58	14.3915699								
7	44	10.35	15.0301487								
8	60	11.07	15.6795077								
9	76	11.93	16.0526322								
10	91	12.85	16.1253877								
11	106	13.68	15.9246188								
12	121	14.48	15.4640804								
13	136	15.18	14.7753248								
14	152	15.72	13.8425195								
15	167	15.97	12.8455473								

Solver Parameters

Set Objective: $I$4

To: Max Min Value Of: 0

By Changing Variable Cells:

$D$4:$G$4

Figure 10.17f

	A	B	C	D	E	F	G	H	I
1	Modeling the Sine Function								
2								Squared	Sum of Squares
3	Day	Hours	Model: y=	A	B	C	D	Errors	Errors
4	1	8.52	8.48436113	3.7755	0.0165	-1.264	12.064	0.00127013	0.365817733
5	16	8.92	8.88736067					0.00106533	
6	32	9.58	9.52968692					0.00253141	

Figure 10.17g

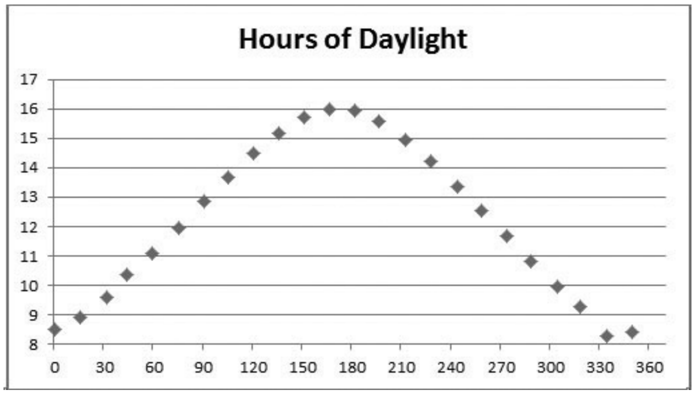


Figure 10.17h

EXERCISE SET 10.2

In Exercises 1–20, give the amplitude, period, horizontal displacement, and vertical displacement, and then graph the given functions.

1. $y = 2\sin(x + \frac{\pi}{4})$
2. $y = 1.5\cos(x - \frac{\pi}{3})$
3. $y = 2.5\sin(3x - \pi)$
4. $y = 3\cos(2x + \frac{\pi}{2})$
5. $y = 6\cos(1.5x + 180°)$
6. $y = 8\sin(3.5x - 90°)$
7. $y = -2\cos(3x - 90°)$
8. $y = -4\sin(5x + 600°)$
9. $y = 4.5\sin(6x - 8)$
10. $y = 9\cos(5x + 3)$
11. $y = 0.5\cos(\pi x + \frac{\pi}{8})$
12. $y = -0.25\sin\left(\frac{x}{\pi} - \frac{1}{4}\right)$
13. $y = -0.75\sin\left(\pi x + \frac{\pi^2}{3}\right)$
14. $y = \pi\cos\left(\frac{1}{\pi}x + \frac{2}{3}\right)$
15. $y = 2\cos(\pi x + \frac{1}{3}) + 4$
16. $y = 3.25\sin(3x - 4) - \frac{7}{2}$
17. $y = -\frac{1}{2}\sin(\pi x + 1) + \frac{3}{2}$
18. $y = -1.75\cos(2x - \pi) + 2.25$
19. $y = -2\cos(2x - 45°) + 3.5$
20. $y = 3\sin(6x + 510°) - 2.5$

Solve Exercises 21–31.

21. ***Electronics*** The current, I (in amperes), in an ac circuit is given by $I = 65\cos(120\pi t + \frac{\pi}{2})$, where t is the time in seconds. **(a)** Determine the amplitude, period, and horizontal displacement for this function. **(b)** Sketch one complete cycle of this function.

22. ***Oceanography*** At a certain point in the ocean, the vertical displacement of the water due to wave action is given by $y = 3\sin[\frac{\pi}{6}(t - 5)]$, where y is measured in meters and t is in seconds. **(a)** Determine the amplitude, period, and horizontal displacement for the graph of this wave action. **(b)** Sketch one complete cycle of this function.

23. ***Oceanography*** Tides rise and fall every 12.4 h. As a result of the tides, the depth in a certain harbor ranges from 6.2 to 11.8 m. The depth of the water at a given time can be approximated by a sine function.
    (a) Write an equation that gives the approximate depth of the harbor at any given time, t, if midnight is at $t = 0$ and high tide is at 4:00 a.m.
    (b) Sketch the graph of this function for 24 h, starting at midnight.

24. ***Electronics*** In an ac circuit with only a constant inductance, the voltage is given by

$$V = 12\sin(120\pi t + 0.5\pi)$$

    (a) Determine the amplitude, period, horizontal displacement, and vertical displacement of this function.
    (b) Sketch the graph of this function for one period. Make sure that you label the times on the x-axis where the graph crosses the axis and where it is at its highest and lowest points.

25. ***Meteorology*** The average weekly temperature at a particular location can usually be determined from the appropriate sine function. In most cases, the yearly average occurs around March 21. In Minneapolis, the maximum and minimum temperatures are 72°**F** and 11°**F**, respectively.
    (a) Write an equation that gives the approximate weekly temperature of any week in the year, starting with January 1. (March 21 will occur in the 12th week of the year.)
    (b) Sketch the graph of this function for 1 year, starting with the first week of January.

26. ***Meteorology*** The average weekly temperature at a particular location can usually be determined from the appropriate sine function. In most cases, the yearly average occurs around March 21. In Chattanooga, Tennessee, the

maximum and minimum temperatures are 89.3°F and 29.2°F, respectively.

(a) Write an equation that gives the approximate weekly temperature of any week in the year, starting with January 1. (March 21 will occur in the 12th week of the year.)

(b) Sketch the graph of this function for 1 year, starting with the first week of January.

27. ***Meteorology*** Use Figure 10.18 to determine the **(a)** amplitude, **(b)** period, **(c)** vertical displacement, and **(d)** horizontal displacement of the sunrise curve for Denver during the years 2000–2002. **(e)** Write a sinusoidal function that fits the data in the figure.

28. ***Meteorology*** Use Figure 10.18 to determine the **(a)** amplitude, **(b)** period, **(c)** vertical displacement, and **(d)** horizontal displacement of the sunset curve for Denver during the years 2000–2002. **(e)** Write a sinusoidal function that fits the data in the figure.

29. ***Meteorology*** Use Figure 10.19 to determine the **(a)** amplitude, **(b)** period, **(c)** vertical displacement, and **(d)** horizontal displacement of the hours of daylight curve for Denver during the years 2000–2002. **(e)** Write a sinusoidal function that fits the data in the figure.

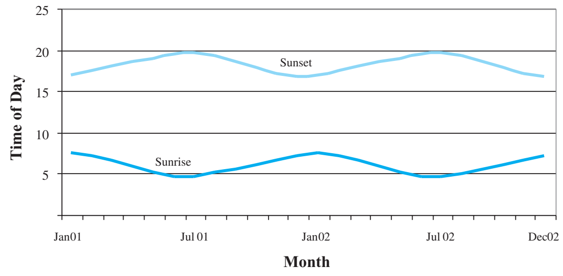


Figure 10.18

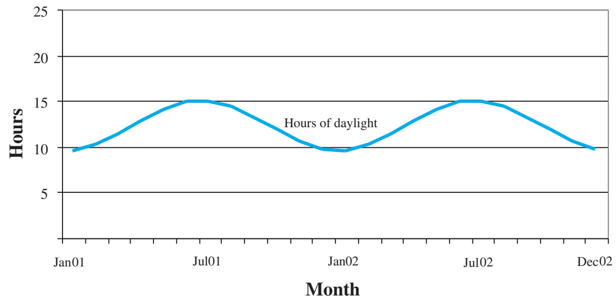


Figure 10.19

30. ***Meteorology*** Table 10.3 gives the normal daily mean temperature for the 30-year period 1971–2000 for Denver, Colorado.

TABLE 10.3 Average Temperature per Month, Denver, Colorado

Month	Jan	Feb	Mar	Apr	May	Jun	Jul	Aug	Sep	Oct	Nov	Dec
Mean Temp °F	29.2	33.2	39.6	47.6	57.2	67.6	73.4	71.7	62.4	51.0	37.5	30.3

(a) Plot these mean temperatures. Why do you expect this data to be cyclic?

(b) Use the data to determine the amplitude and vertical displacement.

(c) Approximate the period and horizontal displacement.

(d) Based on your answers in **(b)** and **(c)** (do not use technology), write a model for the data.

(e) Graph the data and graph the curve according to your answer to **(d)**. Is your answer a good fit? If not, why not?

(f) Use your calculator to determine the curve of best fit to this data. How close is this function to your answer in **(d)**?

31. ***Meteorology*** Table 10.4 gives the normal daily minimum temperature each month for the 30-year period 1971–2000 for Denver, Colorado.

TABLE 10.4 Average Minimum Temperature per Month, Denver, Colorado

Month	Jan	Feb	Mar	Apr	May	Jun	Jul	Aug	Sep	Oct	Nov	Dec
Daily Minimum Temp °F	15.2	19.1	25.4	34.2	43.8	53.0	58.7	57.4	47.3	35.9	23.5	16.4

(a) Plot these mean temperatures. Why do you expect this data to be cyclic?
(b) Use the data to determine the amplitude and vertical displacement.
(c) Approximate the period and horizontal displacement.
(d) Based on your answers in **(b)** and **(c)** (do not use technology), write a model for the data.
(e) Graph the data and the curve based on your answer to **(d)**. Is your answer a good fit? If not, why not?
(f) Use your calculator to determine the curve of best fit to this data. How close is this function to your answer in **(d)**?

[IN YOUR WORDS]

32. If you were shown the graphs of $y = \sin x$, $y = \sin x + 2$, and $y = \sin(x + 2)$ on the same axes, describe how you could distinguish among the graphs.

33. **(a)** Describe how you can determine the horizontal displacement of a sine or cosine function.
(b) Describe how you can determine the vertical displacement of a sine or cosine function.
(c) How can you use the amplitude and the vertical displacement to help you draw the graph of $y = 5 \sin x$?
(d) How can you use the amplitude and the vertical displacement to help you draw the graph of $y = 5 \sin x + 7$?

10.3 COMBINATIONS OF SINE AND COSINE CURVES

In earlier sections, we looked at adding and subtracting functions. At that time, the functions were all algebraic. For example, when $f(x) = x^2$ and $g(x) = x$, we had little difficulty graphing $(f + g)(x)$. In a case such as this, we simply graphed $x^2 + x$ as if it were one function. Graphing combinations of trigonometric curves is not as easy.

In this section, we will learn how to graph trigonometric curves that combine two or more functions, using the ability to graph that we have learned earlier in this book. After we are competent with this technique, we will begin graphing trigonometric functions with the help of a calculator and a spreadsheet. Later in this chapter, we will see how these curves are used in technical areas.

The easiest way to learn how to graph a combination or composite curve is through an example.

EXAMPLE 10.16

Sketch the graph of $y = \sin x + \cos x$.

SOLUTION We will begin with a table of values. Both of the functions $y_1 = \sin x$ and $y_2 = \cos x$ have amplitude of 1 and period of 2π. The period of a composite curve is an integral multiple of the least common multiple of the period of each of the individual curves. In this case, since y_1 and y_2 each have periods of 2π, the least common multiple is also 2π. Since the amplitudes of sin x and cos x are both 1, the largest possible amplitude for $y = \sin x + \cos x$ is 2. There is no vertical displacement. We can use this estimate for the amplitude and the amount of vertical displacement to help determine reasonable values for the y-axis. (It turns out that the amplitude is actually $\sqrt{2}$.)

The following table contains values for y_1, y_2, and $y = y_1 + y_2$. [Notice that $\sin\frac{\pi}{4} = \frac{\sqrt{2}}{2} \approx 0.71$ and $\cos\frac{\pi}{4} = \frac{\sqrt{2}}{2} \approx 0.71$, so $\sin\frac{\pi}{4} + \cos\frac{\pi}{4} = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} = \sqrt{2} \approx 1.41$. However, adding the values in the table for $\sin\frac{\pi}{4}$ and $\cos\frac{\pi}{4}$, we get $0.71 + 0.71 = 1.42$. The difference in these two answers is the result of adding exact values $\left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\right)$ or approximate values $(0.71 + 0.71)$. It helps to show why you should wait until you have finished all calculations before you round off any numbers.]

x	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π
$y_1 = \sin x$	0.0	0.5	0.71	0.87	1.0	0.87	0.71	0.5	0.0
$y_2 = \cos x$	1.0	0.87	0.71	0.5	0.0	−0.5	−0.71	−0.87	−1.0
$y = y_1 + y_2$	1.0	1.37	1.42	1.37	1.0	0.37	0.0	−0.37	−1.0

x	π	$\frac{7\pi}{6}$	$\frac{5\pi}{4}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{7\pi}{4}$	$\frac{11\pi}{6}$	2π
$y_1 = \sin x$	0.0	−0.5	−0.71	−0.87	−1.0	−0.87	−0.71	−0.5	0.0
$y_2 = \cos x$	−1.0	−0.87	−0.71	−0.5	0.0	0.5	0.71	0.87	1.0
$y = y_1 + y_2$	−1.0	−1.37	−1.42	−1.37	−1.0	−0.37	0.0	0.37	1.0

A sketch of all three curves is shown in Figure 10.20. At any point x, the value for y was obtained by adding the values for y_1 and y_2. The dotted line indicates how this was done graphically.

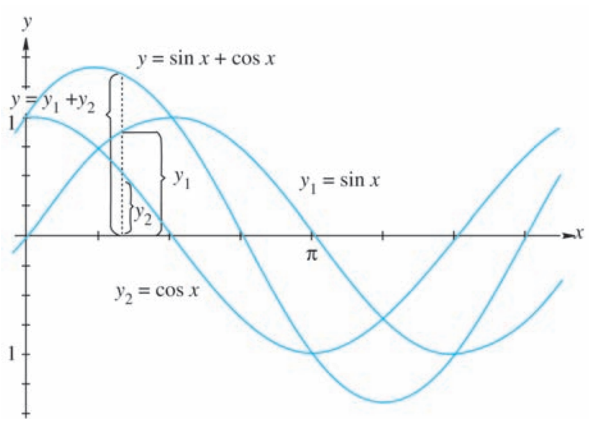


Figure 10.20

EXAMPLE 10.17 Sketch the graph of $y = 2 \sin x + \frac{1}{2} \sin 5x$.

SOLUTION Once again we begin with a table of values. The function $y_1 = 2 \sin x$ has amplitude 2 and period 2π. The function $y_2 = \frac{1}{2} \sin 5x$ has amplitude $\frac{1}{2}$ and period $\frac{2\pi}{5}$. Because the least common multiple of 2π and $\frac{2\pi}{5}$ is 2π, the period of y will be 2π . Since the amplitude of $2 \sin x$ is 2 and the amplitude of $\frac{1}{2} \sin 5x$ is $\frac{1}{2}$, the largest possible amplitude for $y = 2 \sin x + \frac{1}{2} \sin 5x$ is $2 + \frac{1}{2} = \frac{5}{2}$. There is no vertical displacement. We use this estimate for the amplitude and the amount of vertical displacement to help determine reasonable values for the y-axis. We plan for y-values of this graph to range from $-\frac{5}{2}$ to $\frac{5}{2}$.

The table contains values for y_1 and y_2 as well as for $y = y_1 + y_2$.

The sketch of the graphs for all three functions is in Figure 10.21. Notice at $\frac{\pi}{2}$, or 90°, the value y was obtained by adding the graphical heights of y_1 and y_2. This same method of adding the graphical distances or heights could be performed at any value of x to get the value of y at that point.

x	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π
$y_1 = 2 \sin x$	0	1.00	1.73	2.00	1.73	1.00	0
$y_2 = \frac{1}{2} \sin 5x$	0	0.25	−0.43	0.50	−0.43	0.25	0
$y = y_1 + y_2$	0	1.25	1.30	2.50	1.30	1.25	0

x	π	$\frac{7\pi}{6}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{11\pi}{6}$	2π
$y_1 = 2 \sin x$	0	−1.00	−1.74	−2.00	−1.74	−1.00	0
$y_2 = \frac{1}{2} \sin 5x$	0	−0.25	0.43	−0.50	0.43	−0.25	0
$y = y_1 + y_2$	0	−1.25	−1.31	−2.50	−1.31	−1.25	0

A different graph can be obtained by changing the two functions y_1 and y_2. If we shift y_1 to the left $\frac{\pi}{6}$ rad, we would get $y_1 = 2 \sin(x + \frac{\pi}{6})$. If y_2 remains the same, we have the graph of $y = 2 \sin(x + \frac{\pi}{6}) + \frac{1}{2} \sin 5x$. If you add y_1 and y_2 to get the value of y, you get the graph shown in Figure 10.22.

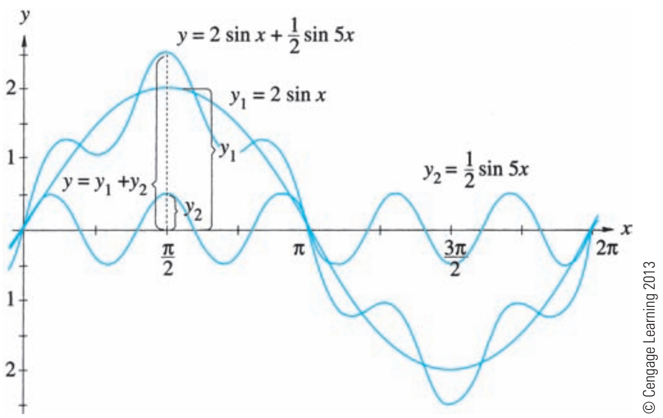


Figure 10.21

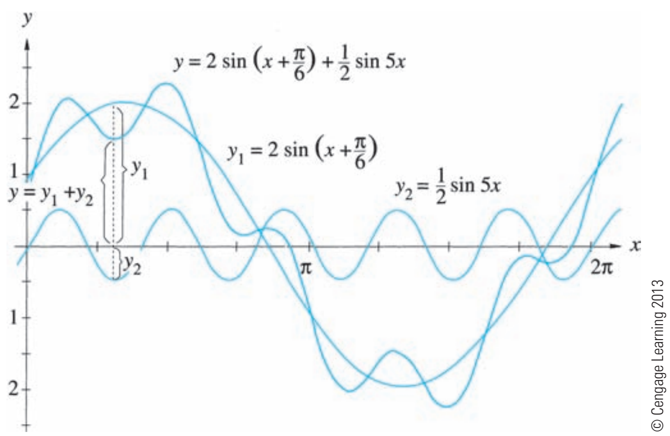


Figure 10.22

EXAMPLE 10.18

Sketch the graph of $y = y_1 y_2$, where $y_1 = \sin 2x$, and $y_2 = \dfrac{\sin x}{x}$.

SOLUTION The graph of y_2 is something like a sine curve. But you will notice that y_2 is not defined when $x = 0$. Also, since the values of $\sin x$ are always between $+1$ and -1, as the value of x gets larger and larger, the amplitude of y_2 will get smaller and smaller. The graph of $y_2 = \dfrac{\sin x}{x}$ is shown in Figure 10.23. If we multiply these two functions, we get a new function, y, which is $y = (y_1)(y_2) = (\sin 2x)\left(\dfrac{\sin x}{x}\right) = \dfrac{(\sin 2x)(\sin x)}{x}$. A table of values for y_1, y_2 and $y = y_1 \cdot y_2$ follows. The curves for all three functions are shown in Figure 10.24.

x	0	$\frac{\pi}{3}$	$\frac{2\pi}{3}$	π	$\frac{4\pi}{3}$	$\frac{5\pi}{3}$	2π	$\frac{7\pi}{3}$	$\frac{8\pi}{3}$	3π
$y_1 = \sin 2x$	0	0.87	-0.87	0.02	0.87	-0.87	0.02	0.87	-0.87	0.02
$y_2 = \frac{\sin x}{x}$	*	0.83	0.41	0	-0.21	-0.17	0	0.12	0.10	0
$y = y_1 \cdot y_2$	*	0.71	-0.36	0	-0.18	0.14	0	0.10	-0.09	0

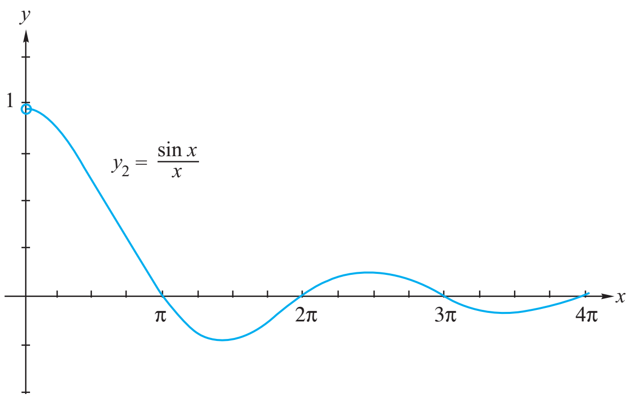


Figure 10.23

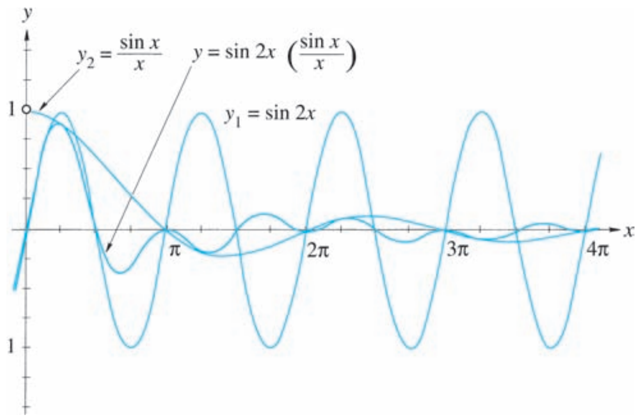


Figure 10.24

CALCULATOR AND SPREADSHEET GRAPHICS

Graphing these curves by hand takes a great deal of time. Using a calculator, spreadsheet, or a computer graphics program to draw the graphs of trigonometric curves would save time. We will briefly describe how to use a graphing calculator and a spreadsheet to graph trigonometric functions. All of the directions and illustrations are specific to a Texas Instruments TI-83 or TI-84 graphics calculator. If you have another brand or model then you may have to vary the directions slightly.

While you can graph trigonometric functions with your calculator in degree mode, it is easier if you first set it in radian mode.

EXAMPLE 10.19

Figure 10.25a

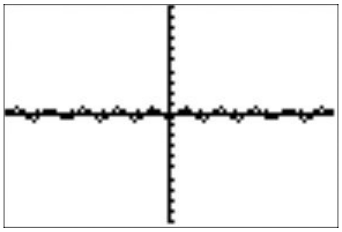


Figure 10.25b

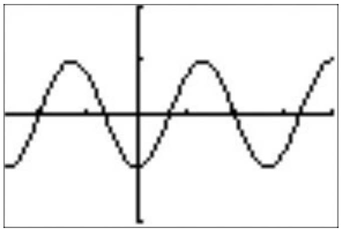


Figure 10.25c

Use a graphing calculator or spreadsheet to graph $y = \frac{1}{2}\cos(3x + \pi)$.

SOLUTION

Using a Calculator

Make sure that your calculator is in radian mode. Then, press Y= to enter the function you want to graph. Press the following sequence of keys:

0.5 COS 3 X, T, θ, n + 2ND π)

The calculator screen should look like the one shown in Figure 10.25a. (Note that we could have typed (1 ÷ 2) instead of 0.5.)

Press ZOOM 6 [6:ZStandard] for the standard viewing window and you should obtain the graph shown in Figure 10.25b. Compare this graph to the one in Figure 10.15.

This graph is very difficult to see. We need to change the window settings. We know that the amplitude of this function is $\frac{1}{2} = 0.5$ and that it has a period of $\frac{2\pi}{3} \approx 2.09$. Press WINDOW and set the scale as follows: Xmin $= -\frac{2\pi}{3}$, Xmax $= \pi$, Xscl $= \frac{\pi}{4}$, Ymin $= -1$, Ymax $= 1$, Yscl $= 0.5$. Why did we select these values for Xscl and Yscl? The x-values were selected to allow us to view two complete cycles of the graph. We selected Yscl $= 0.5$ to help us see where the highest and lowest values of the graph are. Now press GRAPH. The result is shown in Figure 10.25c.

Using a Spreadsheet

The first step in graphing any equation is to make a table of values. Notice that the input values will be radian values, not degrees. This is the default setting for trigonometric functions in Excel.

The first decision to make concerns the x-values. What values do we use? Your answer may not be exactly what you want the first time, but if it is close you will be able to adjust the scale on the x-axis after the first sketch.

We know the amplitude of this function is $\frac{1}{2} = 0.5$ and that it has a period of $\frac{2\pi}{3} \approx 2.09$. Since the period is in $\frac{2}{3}\pi$, we should scale the axis in $\frac{1}{6}\pi$, or $\frac{1}{12}\pi$, depending on how many values we wish to use.

Look at Figure 10.25d. The first column will be the multiples of $\frac{\pi}{12}$: 1, 2, 3, . . . , 24. The second column will be the x-values; obtained by multiplying $\frac{\pi}{12}$ by the number in Column A. In Cell B2 enter A2 * PI () /12 . This is a way to make filling in the x-values easier and quicker. Notice that π is entered as PI ().

Next enter the function in Cell C2:

= 0.5 * COS (3 * B2 + PI ())

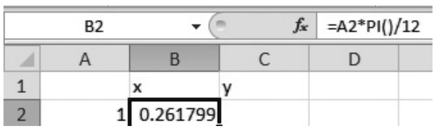


Figure 10.25d

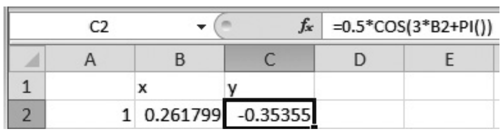


Figure 10.25e

Copy the two formulas in B2 and C2 down to match the values you placed in Column A. Figure 10.25e shows part of the finished table of values.

Now draw the graph using the connected option of the XY–Scatter. Figure 10.25g shows the graph using the data shown in Figure 10.25f.

If you want to show the graph beginning on the left side of the y-axis, then we can adjust the values in Column A as shown in Figure 10.25h. The revised graph is shown in Figure 10.25i.

	A	B	C
1		x	y
2	1	0.261799	-0.35355
3	2	0.523599	-9.2E-17
4	3	0.785398	0.353553
5	4	1.047198	0.5
6	5	1.308997	0.353553
7	6	1.570796	1.53E-16
8	7	1.832596	-0.35355
9	8	2.094395	-0.5
10	9	2.356194	-0.35355
11	10	2.617994	-2.1E-16
12	11	2.879793	0.353553
13	12	3.141593	0.5
14	13	3.403392	0.353553
15	14	3.665191	2.76E-16
16	15	3.926991	-0.35355

Figure 10.25f

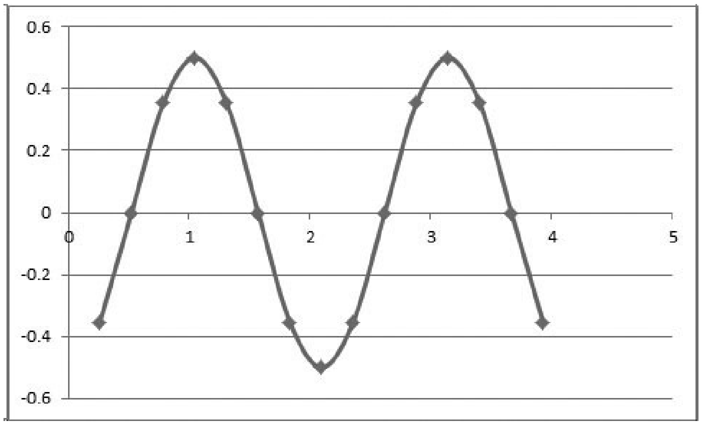


Figure 10.25g

	A	B	C
1		x	y
2	-4	-1.0472	0.5
3	-3	-0.7854	0.353553
4	-2	-0.5236	3.06E-17
5	-1	-0.2618	-0.35355
6	0	0	-0.5
7	1	0.261799	-0.35355
8	2	0.523599	-9.2E-17
9	3	0.785398	0.353553
10	4	1.047198	0.5
11	5	1.308997	0.353553
12	6	1.570796	1.53E-16
13	7	1.832596	-0.35355
14	8	2.094395	-0.5
15	9	2.356194	-0.35355
16	10	2.617994	-2.1E-16

Figure 10.25h

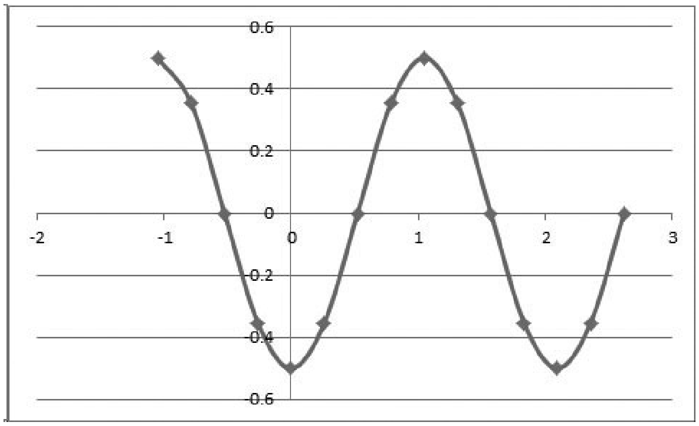


Figure 10.25i

EXAMPLE 10.20

Use a graphing calculator or spreadsheet to graph $y = (\sin 2x)\left(\frac{\sin x}{x}\right)$.

SOLUTION This is the same function we graphed in Example 10.18.

Using a Calculator

We will work this so we can separately graph $y_1 = \sin 2x$ and $y_2 = \frac{\sin x}{x}$ and then graph their product $y = y_1 y_2$.

Based on our work in Example 10.18, we choose window settings of Xmin $= 0$, Xmax $= 10$, Xscl $= \frac{\pi}{4}$, Ymin $= -1.1$, Ymax $= 1.1$, and Yscl $= 0.5$. Now press Y= SIN 2 X, T, θ, n) . This should appear on line Y1.

To enter $y_2 = \frac{\sin x}{x}$, first press ENTER to move the cursor to the next line, Y2. Now press

(SIN X, T, θ, n)) ÷ X, T, θ, n

We will put $y = y_1 y_2$ on line Y3. Rather than entering both y_1 and y_2 again, we will let the calculator do this for us. Press ENTER to move the cursor to line Y3 and press the following key sequence:

VARS ▶ 1 1 × VARS ▶ 1 2

Line Y3 now reads Y3 = Y1* Y2 and the calculator screen should look like that in Figure 10.26a. (It is very important that you NOT press ALPHA Y 1* Y 2 because the calculator identifies Y with the variable y and you would get y * $2y$ for whatever value of y is stored in the calculator rather than the functions stored in Y1 and Y2.)

Press GRAPH . You should see each of the functions y_1, y_2, and y graphed in sequence, with the final result like that shown in Figure 10.26b.

This graph is too cluttered. Let's change it so we see only the graph of $y = y_1 y_2$. Press the Y= ◀ to move the cursor over the =-sign in Y1 . Press ENTER . This changes the equals sign to black on a white background which means that the calculator will not graph Y1. Press ◀ to move the cursor to the Y2 line and press ENTER to tell the calculator not to graph Y2. Now press GRAPH . The result, shown in Figure 10.26c, is the graph of $y = y_1 y_2 = (\sin 2x)\left(\frac{\sin x}{x}\right)$.

Using a Spreadsheet

We will graph this function as the product of two functions: $y_1 = \sin 2x$ and $y_2 = \frac{\sin x}{x}$. Their product is $y = y_1 . y_2$.

Figure 10.26a

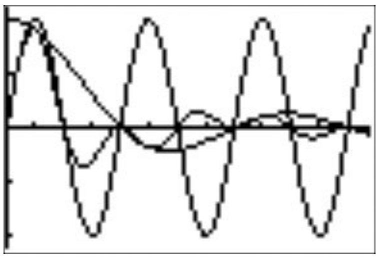


Figure 10.26b

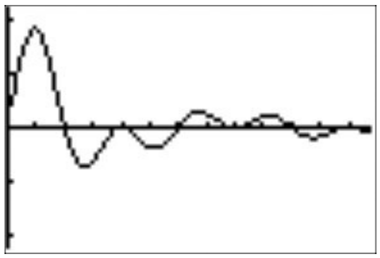


Figure 10.26c

Based on our work in Example 10.18, we choose x-values between 0 and 10 scaled by $\frac{\pi}{12}$. As we've done before, Column A will be the multiplier and Column B will contain the multiples of $\frac{\pi}{12}$.

Next, enter the functions. In Cell C2, enter = sin(2 * B2), enter = (sin(B2)) / B2 in Cell D2 (see Figure 10.26d), and = C2 * D2 in E2, as shown in Figure 10.26e.

Copy these values to complete the table.

The graph of all three functions is made using the source data from Column B (the x-values) and Columns C, D, and E (the y-values). The result is shown in Figure 10.26f.

But, we do not need the graphs of y_1 and y_2, so we only use the source data from Column B (the x-values) and Column E (the y-values). The result is shown in Figure 10.26g.

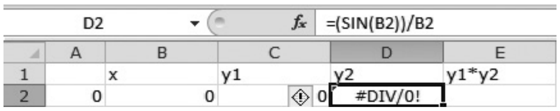


Figure 10.26d

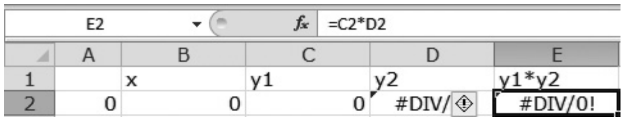


Figure 10.26e

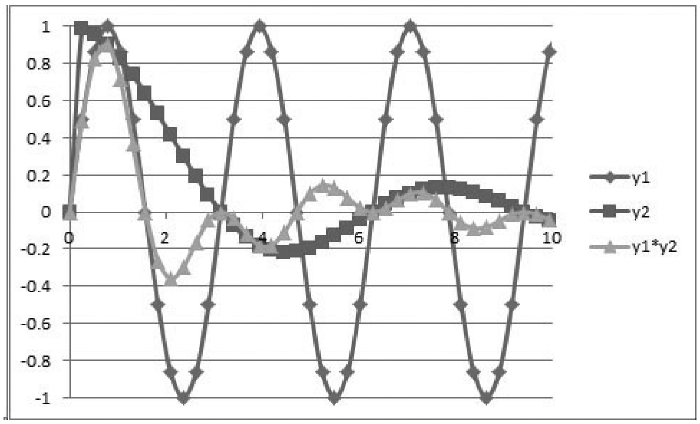


Figure 10.26f

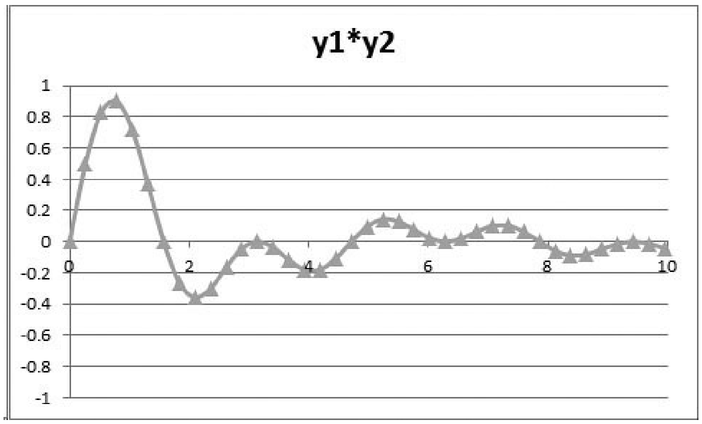


Figure 10.26g

EXAMPLE 10.21

Use a graphing calculator or spreadsheet to graph two cycles of $y = 3\cos 2x - \sin 4x + 5$.

SOLUTION We need to analyze this function to help determine the period and the range. To help our analysis we let $y = y_1 + y_2$ with $y_1 = 3\cos 2x$ and $y_2 = -\sin 4x + 5$.

The function $y_1 = 3\cos 2x$ has amplitude 3, period π, and vertical displacement 0. The function $y_2 = -\sin 4x + 5$ has amplitude 1, period $\frac{\pi}{2}$, and vertical displacement 5. Because the least common multiple of π and $\frac{\pi}{2}$ is π, the period for $y = y_1 + y_2$ is π . The amplitudes are 3 and 1, so the largest possible amplitude is 4. With a vertical displacement of 5, the range of this function is in the interval $[5 - 4, 5 + 4] = [1, 9]$. Note that this is probably *not* the actual range of the function. This range is just an estimate to help us determine what values we will need on the y-axis. This estimate will be very important when we do calculator graphing. (The actual range is about (1.515, 8.485).)

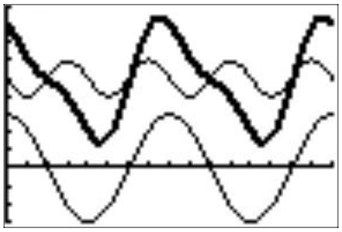


Figure 10.27a

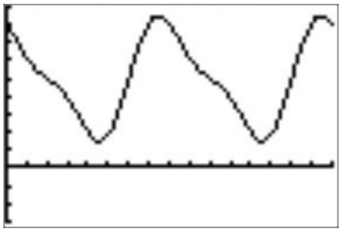


Figure 10.27b

Using a Calculator

Since the period is π, an interval from $x = 0$ to $x = 2\pi$ will include two cycles. We will set the viewing window to $[0, 2\pi, \frac{\pi}{8}] \times [0, 9, 1]$. Even though the range is no more than [1, 9], we set the y-values in the viewing window to [0, 9] so the x-axis would be included in the graph. The graphs of y_1, y_2, and $y_3 = y_1 + y_2$ are in Figure 10.27a. Notice that the graph of y_3 is thicker than the other two graphs. The graph of just $y = 3\cos 2x - \sin 4x + 5$, using the viewing window we selected, is in Figure 10.27b.

Using a Spreadsheet

Since the period is π, an interval from $x = 0$ to $x = 2\pi$ will include two cycles. The x-values will go from 0 to 2π by $\frac{\pi}{12}$. Next enter the functions; copy the formulas; and graph the result.

All three graphs are shown in Figure 10.27c and the graph of only the function is shown in Figure 10.27d.

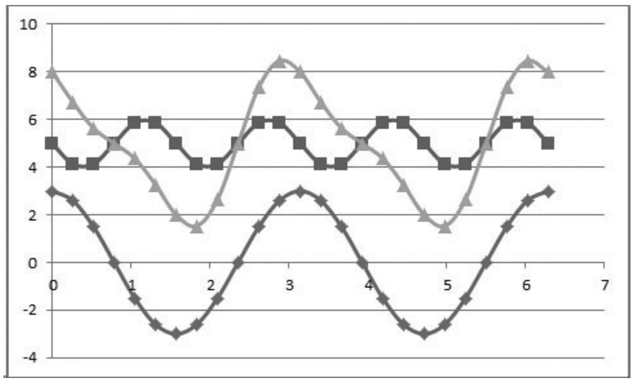


Figure 10.27c

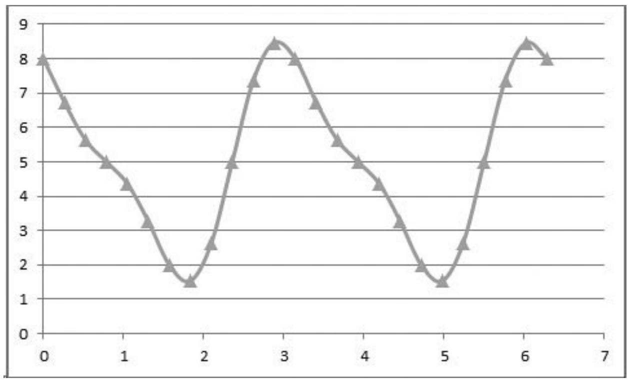


Figure 10.27d

EXERCISE SET 10.3

For each of the given functions in Exercises 1–24, (a) determine its period and maximum amplitude (if they exist), (b) determine a reasonable viewing window, and (c) sketch the graph with a graphing calculator, spreadsheet, or computer graphing software.

1. $y = \sin x - \cos x$

2. $y = \sin 2x + 2\cos x$

3. $y = \frac{1}{3}\sin 4x + \sin 2x$

4. $y = 3\sin\dfrac{x}{3} + \cos 2x$

5. $y = x + \sin x$

6. $y = \sin x - \sin 2x$

7. $y = \cos 2x - \cos x$

8. $y = x^2 + \cos x$

9. $y = \sin \pi x - \cos 2x$

10. $y = \cos \frac{\pi}{2}x + 2\sin 3x$

11. $y = \cos 2x + \sin\left(x - \frac{\pi}{3}\right)$

12. $y = \frac{1}{4}\cos\left(x - \frac{\pi}{6}\right) + 4\sin\left(x + \frac{\pi}{6}\right)$

13. $y = 4\cos 2\pi x + \sin\left(\dfrac{\pi x}{2} + \dfrac{\pi}{4}\right)$

14. $y = \dfrac{\cos x}{x^2 + 1} - \sin x$

15. $y = \dfrac{\sin x}{x^2 + 1} + 2\cos\left(\dfrac{x}{3} - \dfrac{\pi}{2}\right)$

16. $y = 2\sin\left(3x + \frac{\pi}{4}\right) + 5\cos\left(2x - \frac{\pi}{3}\right)$

17. $y = x\sin x$

18. $y = 3(x + \pi)\cos\left(x - \frac{\pi}{4}\right)$

19. $y = 3\sin(2x + 1) + 2\cos(4x - 1) + 2$

20. $y = 2\sin(3x - \pi) + 5\cos(3x + 1) - 4$

21. $y = -2\sin(3x - 2) + 4\cos(5x + 1) - 3$

22. $y = -3\sin\left(4x - \pi\right) + \frac{1}{2}\cos\left(5x + \frac{\pi}{2}\right) + 6$

23. $y = 2.5\sin\left(2\pi x + \frac{\pi}{4}\right) + 3\sin\left(x - \frac{\pi}{4}\right) - 3.5$

24. $y = 0.5\sin\left(x - \pi\right) + \frac{1}{3}\cos\left(x + \frac{\pi}{2}\right) + 2.1$

Solve Exercises 25–32.

25. ***Music*** Plucking a guitar string produces a triangular wave. It is possible to represent such a triangular wave by adding several sinusoidal functions, such as $y_1 = A\sin kx$, $y_2 = -\dfrac{A}{3^2}\sin 3kx$, and $y_3 = \dfrac{A}{5^2}\sin 5kx$. Graph $y = y_1 + y_2 + y_3$ with $A = 1$ and $k = 1$.

26. ***Electronics*** The equation for a half-wave rectified pulsating waveform can be approximated by

$$\begin{aligned} V = {} & 0.318\,V_m + 0.500V_m\cos\alpha \\ & + 0.212\,V_m\cos 2\alpha - 0.0424\,V_m\cos 4\alpha \end{aligned}$$

Graph V from $\alpha = 0$ to $\alpha = 4\pi$ with $V_{\mathrm{m}} = 1$.

27. ***Electronics*** The instantaneous power in an ac circuit containing an inductance is given by $P = vi$, where $v = 120\cos\omega t$ and $i = 5\sin\omega t$. Let ωt represent the dependent variable and

(a) Determine the period of P.

(b) Estimate the amplitude of P.

(c) Sketch two cycles of P.

(d) Sketch two cycles of $y = 300\sin 2\omega t$.

(e) Compare your graphs in **(c)** and **(d)**.

28. ***Automotive technology*** An automobile tire is driven over a nail. As a result the nail is embedded in the tire with only the head showing on the outside surface of the tire. As the tire moves down the highway the height of the nail above the road is given by $y = a(1 - \cos t)$, where a is the radius of the tire and t is measured in seconds. If the radius of the tire is 1.1 ft,

(a) Sketch the graph of the nail's height from the time it becomes stuck in the tire (when $t = 0$) until it hits the pavement three more times.

(b) What are the amplitude, period, and frequency of this function?

29. ***Automotive technology*** A function of the form $A_1\sin(Bt) + A_2\cos(Bt)$ may be used to define a vibrating system, such as an automobile suspension system. One such function is defined by $g(t) = 8\cos(5t) + 6\sin(5t)$.

(a) Graph two cycles of g.

(b) What is the amplitude of g?

(c) What is the period of g?

30. ***Automotive technology*** The acceleration of a piston is given by the equation $a = 1000(\cos 10t + 0.5\cos 20t)$.

(a) Graph two cycles of a.

(b) What is the maximum acceleration and when does it occur?

(c) Find three values of t where the acceleration reaches the maximum value.

31. ***Electronics*** A *triangular pulse* has applications in timer circuitry. You can approximate a triangular pulse with the following function, where x is expressed in radians:

$$y = \sin x - \frac{1}{3^2}\sin 3x + \frac{1}{5^2}\sin 5x - \frac{1}{7^2}\sin 7x + \cdots$$

The more terms you add, the closer the graph approaches a triangular pulse.

(a) Graph two complete cycles of the first three terms of this function. That is, graph $y_1 = \sin x - \frac{1}{3^2}\sin 3x + \frac{1}{5^2}\sin 5x$ with Xmin $= 0$ and Xmax $= 4\pi$.

(b) Graph two complete cycles of the first six terms of y using the same window settings.

32. ***Music*** The sounds of musical instruments are not simple sine waves but sums of different waves. A clarinet produces a **square wave** that can be approximated by the function

$$y = \sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \frac{1}{7}\sin 7x + \cdots$$

The more terms you add, the closer the graph approaches a square wave.

(a) Graph two complete cycles of the first three terms of this function. That is, graph $y_1 = \sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x$ with Xmin $= 0$ and Xmax $= 4\pi$.

(b) Graph two complete cycles of the first six terms of y using the same window settings.

[IN YOUR WORDS]

33. Explain how to use the amplitude and period to help determine a useful viewing window.

34. Write a word problem in your technology area of interest that requires you to add two or more trigonometric functions. Give the problem you wrote to a friend and let him or her try to solve it. If your friend has difficulty understanding or solving the problem, or disagrees with your solution, make any necessary changes in the problem or solution. When you have finished, give the revised problem and solution to another friend and see if he or she can solve it.

35. Sound waves are sinusoidal curves. Suppose that y_1 and y_2 as described in **(a)** and **(b)** below are sound waves.

(a) Sketch the graph of $y_1 = \cos 2x - 1$.

(b) Sketch the graph of $y_2 = 2\sin^2 x$. (*Note*: $\sin^2 x$ means $(\sin x)^2$.)

(c) Sketch the graph $y = y_1 + y_2$.

(d) Describe the graph of y. What implications, if any, does this have for reducing noise?

(e) What implications, if any, would this have if these were light waves rather than sound waves?

10.4 GRAPHS OF THE OTHER TRIGONOMETRIC FUNCTIONS

We have learned how to graph two of the trigonometric functions, $y = \sin x$ and $y = \cos x$. We can change the frequency, amplitude, and displacement of each of them. In this section, we will look at the graphs of the other four trigonometric functions.

The sin and cos functions that we have graphed were each bounded. That means that there was a limit, or bound, on how large or small the value of each function could get. For the sine and cosine functions, this bound was called the amplitude.

The four remaining trigonometric functions (tangent, cotangent, secant, cosecant) are not bounded. Thus, there is no limit to how large or small each function can become.

GRAPH OF THE TANGENT FUNCTION

We will begin with the tangent function. A table of values for the tangent function, $y = \tan x$, follows. A graph based on these values is in Figure 10.28.

x	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π
$y = \tan x$	0	0.58	1.73	*	−1.73	−0.58	0

x	π	$\frac{7\pi}{6}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{11\pi}{6}$	2π	$\frac{13\pi}{6}$	$\frac{7\pi}{3}$	$\frac{5\pi}{2}$
$y = \tan x$	0	0.58	1.73	*	−1.73	−0.58	0	0.58	1.73	*

*The tangent function at this point is not defined.

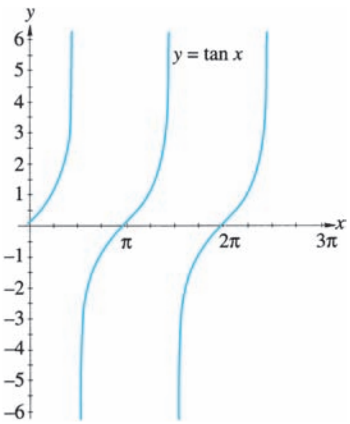


Figure 10.28

Notice that the function is not defined when $x = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}$, and so on. The vertical lines at each of these values of x that the graph of tangent approaches but never touches are called the *vertical asymptotes*. You might also note that the period of the tangent curve is π. (Remember, the sine and cosine have periods of 2π.)

GRAPHS OF COTANGENT, SECANT, AND COSECANT FUNCTIONS

In a similar procedure, we could graph the three remaining trigonometric functions: cotangent, secant, and cosecant. The graphs for each of these functions are given in Figures 10.29, 10.30, and 10.31, respectively. Notice that the period of the cotangent is also π. The period of the secant and cosecant functions is 2π. (Remember that $\csc x = \dfrac{1}{\sin x}$ and $\sec x = \dfrac{1}{\cos x}$, and both sine and cosine have period 2π.)

VARIATIONS IN THE GRAPHS OF TAN, COT, SEC, AND CSC

The terms *period* and *frequency* have the same meaning for these four functions as they did for the sine and cosine functions. The tangent and cotangent each have period π . The secant and cosecant have period 2π. Amplitude has

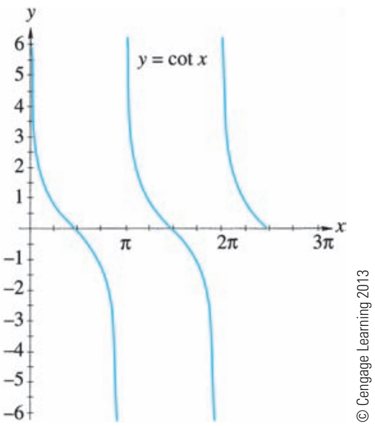


Figure 10.29

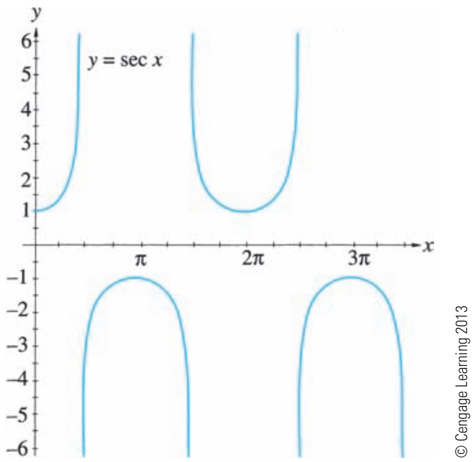


Figure 10.30

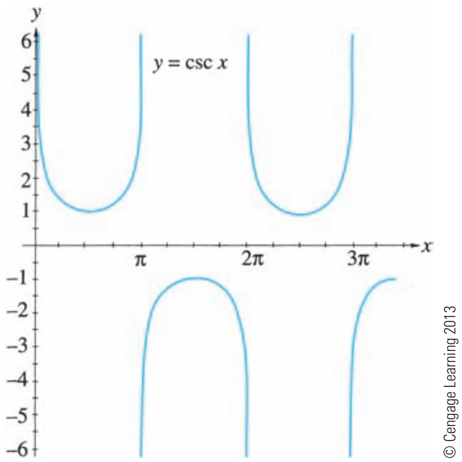


Figure 10.31

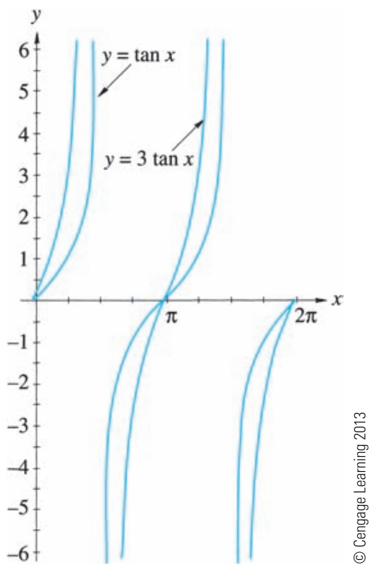


Figure 10.32

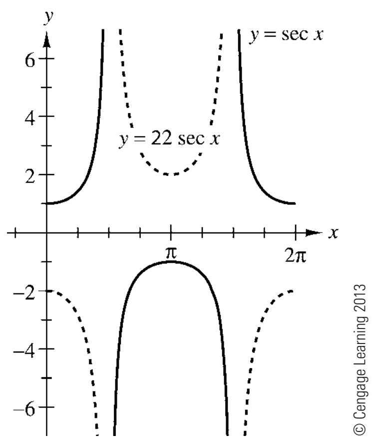


Figure 10.33

no meaning for these functions. Each of these graphs has vertical asymptotes where the function is not defined.

Sketching a function such as $y = 3 \tan x$ or $y = -2 \sec x$ is not difficult if you know the shapes of the six basic trigonometric functions. Knowing the shape means knowing where each graph crosses the x-axis (if it does), where its high and low points are, and where the function is not defined.

We will sketch both $y = 3 \tan x$ and $y = -2 \sec x$ to give you an idea of how to proceed. The graph for $y = 3 \tan x$ is shown in Figure 10.32. The graph for $y = -2 \sec x$ is shown in Figure 10.33. First, sketch the basic curve for each function. For $y = 3 \tan x$, sketch $y = \tan x$. For $y = -2 \sec x$, sketch $y = \sec x$. Now, multiply the y-values of each curve by the appropriate value and plot these points. For $y = \tan x$, multiply each y-value by 3 to get $y = 3 \tan x$. For $y = \sec x$, multiply each y-value by -2 to get $y = -2 \sec x$.

A summary of the period, amplitude, horizontal shift, and vertical asymptotes for all of the six trigonometric functions is given in Table 10.5.

TABLE 10.5 Summary for All Six Trigonometric Functions

Function	Period	Amplitude	Horizontal Shift	Vertical Asymptotes
$y = A\sin(Bx + C)$	$\frac{2\pi}{\lvert B\rvert}$	$\lvert A\rvert$	$-\frac{C}{B}$	None
$y = A\cos(Bx + C)$	$\frac{2\pi}{\lvert B\rvert}$	$\lvert A\rvert$	$-\frac{C}{B}$	None
$y = A\tan(Bx + C)$	$\frac{\pi}{\lvert B\rvert}$	*	$-\frac{C}{B}$	$Bx + C = n\pi + \frac{\pi}{2}$
$y = A\cot(Bx + C)$	$\frac{\pi}{\lvert B\rvert}$	*	$-\frac{C}{B}$	$Bx + C = n\pi$
$y = A\sec(Bx + C)$	$\frac{2\pi}{\lvert B\rvert}$	*	$-\frac{C}{B}$	$Bx + C = n\pi + \frac{\pi}{2}$
$y = A\csc(Bx + C)$	$\frac{2\pi}{\lvert B\rvert}$	*	$-\frac{C}{B}$	$Bx + C = n\pi$

* Not defined

CALCULATOR GRAPHICS

EXAMPLE 10.22

Use a graphing calculator or spreadsheet to sketch the graph of $y = -\frac{1}{3}\tan\left(2x + \frac{\pi}{4}\right)$.

SOLUTION

Using a Calculator

Make sure your calculator is in radian mode. From Table 10.5, we see that the period is $\frac{\pi}{2} \approx 1.57$. We would like to see at least two periods of this graph, so we set Xmin $= -\frac{\pi}{4}$, Xmax $= \frac{9\pi}{8}$, Xscl $= \frac{\pi}{8}$, Ymin $= -4$, Ymax $= 4$, and Yscl $= 1$. Now press Y= and clear any existing functions from the screen. Press the following sequence of keys:

(−) (1 ÷ 3) TAN 2 X, T, θ, n + (2ND π ÷ 4))

The calculator screen should now look like the one shown in Figure 10.34a. Press GRAPH and you should obtain the graph in Figure 10.34b.

Figure 10.34a

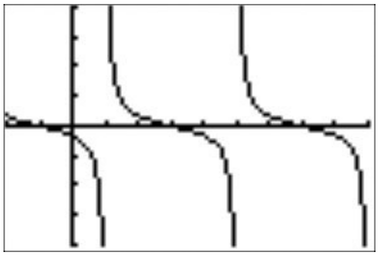


Figure 10.34b

EXAMPLE 10.22 (Cont.)

Some calculators will draw a graph like the one in Figure 10.34c where it seems to have drawn in the vertical asymptotes. Look closely. These are not vertical lines. The portion above the x-axis is not directly above the portion below the x-axis. These apparent vertical asymptotes are put in by some graphing calculators and computer graphing programs because the program is in "connected" mode. On a calculator this can be corrected by pressing MODE and selecting the "dot" option. The result, as shown in Figure 10.34d is not as nice, but at least it does not have the "false asymptotes."

Using a Spreadsheet

From Table 10.5, we see that the period is $\frac{\pi}{2} \approx 1.57$. We would like to see at least two periods of this graph, so we will want to use values of x between $-\frac{\pi}{4}$ and $\frac{9\pi}{8}$, between -0.8 and 4.0.

Spreadsheets aren't built to handle asymptotes. We expect to see asymptotes when we graph the tangent function, so we should know up front that we will have to use a great many more points than usual to construct an accurate graph. In the past few examples, we've used multiples of some fraction of π to make the table of values. In this case of the tangent function (and other functions that have asymptotes), it actually works out best not to put in the "exact" values. Figure 10.34e shows a graph of this function without connecting the points and using x-values incremented by 0.1 between -0.8 and 3.6.

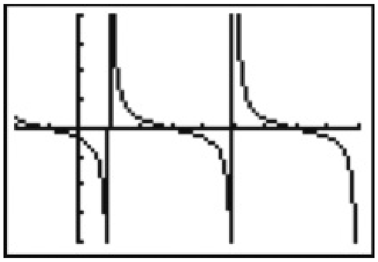


Figure 10.34c

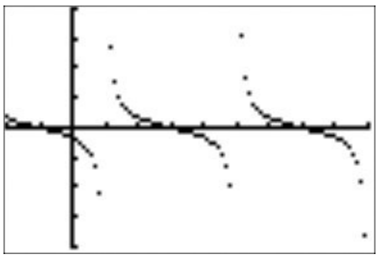
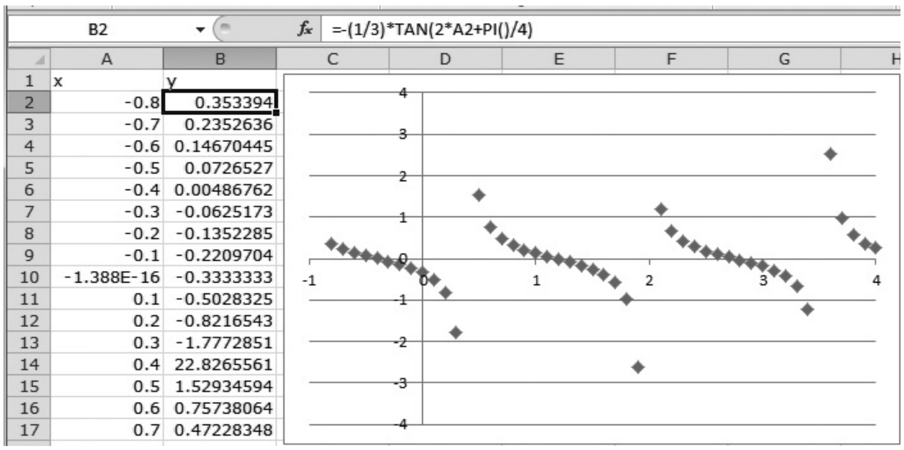


Figure 10.34d



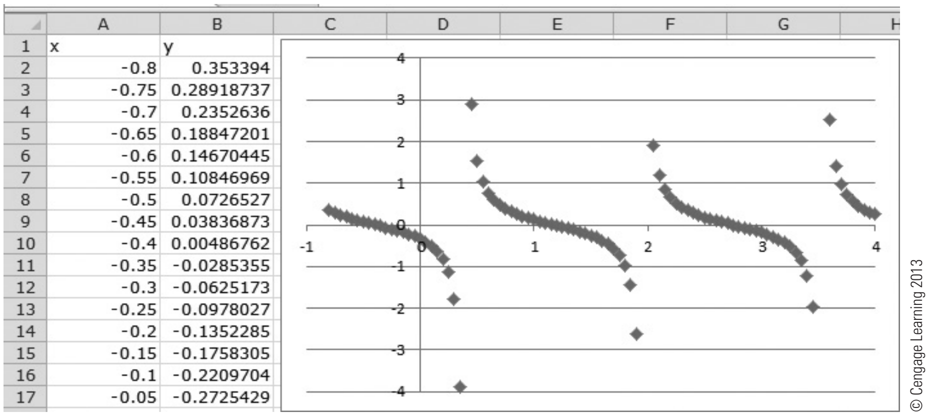
	A	B
1	x	y
2	-0.8	0.353394
3	-0.7	0.2352636
4	-0.6	0.14670445
5	-0.5	0.0726527
6	-0.4	0.00486762
7	-0.3	-0.0625173
8	-0.2	-0.1352285
9	-0.1	-0.2209704
10	-1.388E-16	-0.3333333
11	0.1	-0.5028325
12	0.2	-0.8216543
13	0.3	-1.7772851
14	0.4	22.8265561
15	0.5	1.52934594
16	0.6	0.75738064
17	0.7	0.47228348

Figure 10.34e

By making the increment for x smaller, and adding more points, the graph can be more accurate (see Figure 10.34f). Part of the table of values used for Figure 10.34e is shown in Figure 10.34f.

By connecting the dots a more recognizable representation of the graph of the function is created, as shown in Figure 10.34g.

Notice that when the dots are connected the computer seems to have drawn in the vertical asymptotes. Actually, these are not vertical lines. These apparent vertical asymptotes are put in by some graphing calculators and computer graphing programs as they try to connect the last point on the left side of an asymptote (a large negative number) to the first point on the right side of the asymptote (a large positive number).



	A	B
1	x	y
2	-0.8	0.353394
3	-0.75	0.28918737
4	-0.7	0.2352636
5	-0.65	0.18847201
6	-0.6	0.14670445
7	-0.55	0.10846969
8	-0.5	0.0726527
9	-0.45	0.03836873
10	-0.4	0.00486762
11	-0.35	-0.0285355
12	-0.3	-0.0625173
13	-0.25	-0.0978027
14	-0.2	-0.1352285
15	-0.15	-0.1758305
16	-0.1	-0.2209704
17	-0.05	-0.2725429

Figure 10.34f

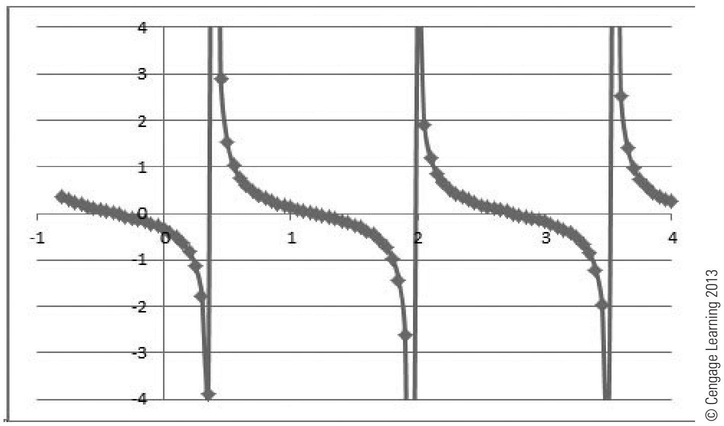


Figure 10.34g

EXAMPLE 10.23

Use a graphing calculator or spreadsheet to sketch $y = 1.5\sec(x + 2)$.

SOLUTION

Using a Calculator

We set Xmin $= -1$, Xmax $= 7$, Xscl $= 1$, Ymin $= -5$, Ymax $= 5$, and Yscl $= 1$. Press **Y=** **CLEAR** to erase the previously graphed function. A calculator does not

EXAMPLE 10.23 (Cont.)

have a secant key, so before you graph this function you will need to use the reciprocal identity $\sec x = \dfrac{1}{\cos x}$ to rewrite this function as $y = \dfrac{1.5}{\cos(x + 2)}$. Now, press the following sequence of keys:

$$1.5\ \boxed{\div}\ \boxed{(}\ \boxed{\text{COS}}\ \boxed{X, T, \theta, n}\ \boxed{+}\ 2\ \boxed{)}\ \boxed{)}$$

Press $\boxed{\text{GRAPH}}$ and you should obtain a graph like the one in Figure 10.35a. Depending on the calculator, you may see that the calculator seems to have drawn in the vertical asymptotes.

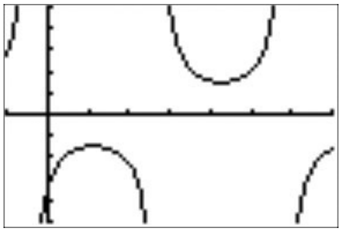


Figure 10.35a

Using a Spreadsheet

The x-values are between $x = -1$ and $x = 7$. Since we expect asymptotes in the graph of this function, we will set up the spreadsheet to allow us to change the increment for x until we are satisfied with the result as shown in Cells A1, A2, B1, and B2 of Figures 10.35b and 10.35c.

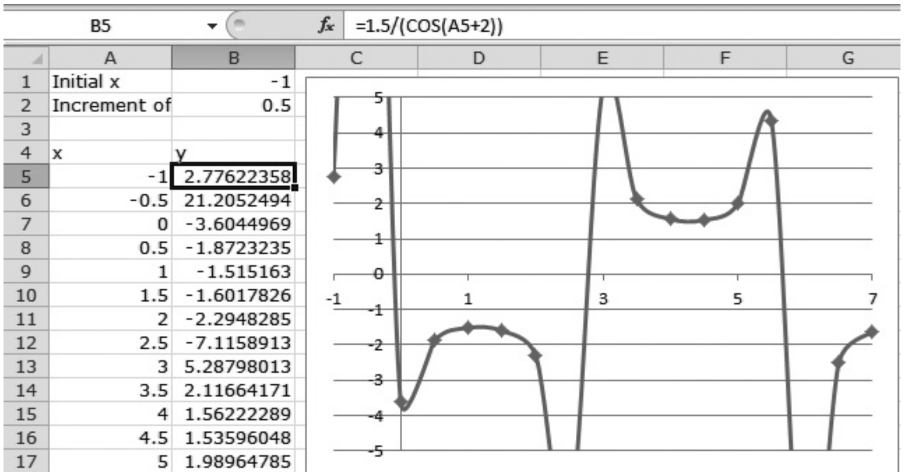


Figure 10.35b

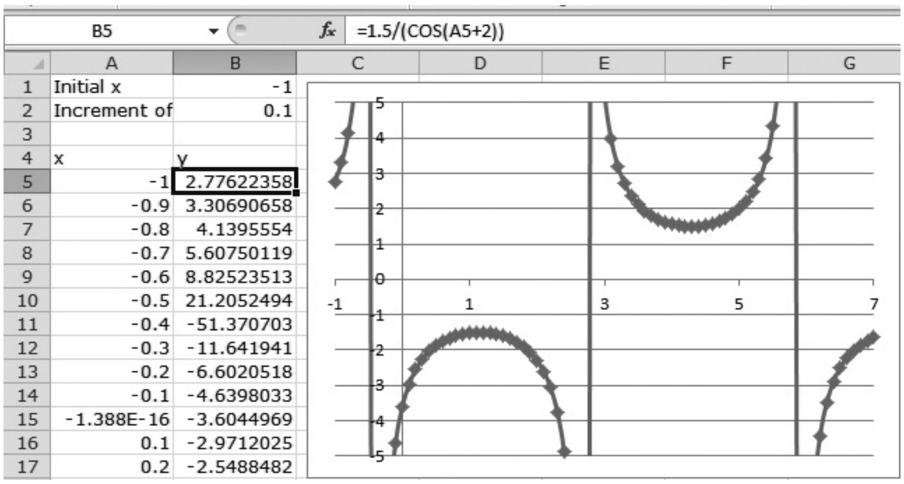


Figure 10.35c

The secant is not a built-in function so we will need to use the reciprocal identity $\sec x = \dfrac{1}{\cos x}$ to rewrite this function as

$$y = \frac{1.5}{\cos(x + 2)}$$

The first sketch is shown in Figure 10.35a. The graph is not as smooth as expected. Figure 10.35c shows the result after changing the increment to 0.1.

Once again, you see that the computer seems to have drawn in the vertical asymptotes. If you changed to an unconnected scatter plot you get the result shown in Figure 10.35d.

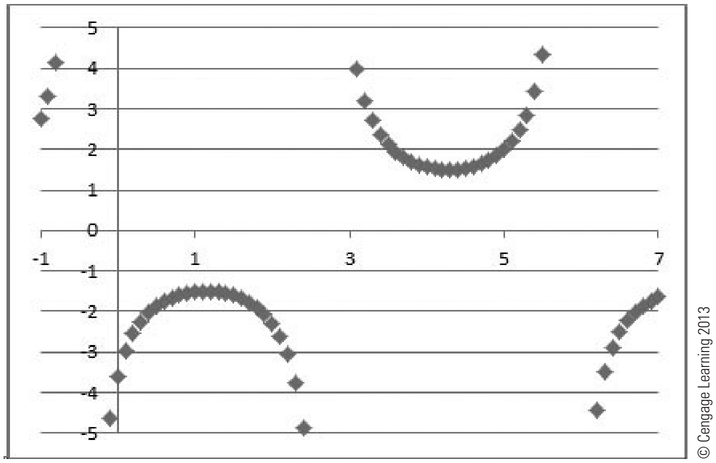


Figure 10.35d

EXERCISE SET 10.4

For each of the given functions in Exercises 1–20, (a) determine a reasonable viewing window, and (b) sketch the graph by hand, with a graphing calculator or with computer graphing software.

1. $y = 3 \tan x$

2. $y = -4 \tan x$

3. $y = \frac{1}{2} \sec x$

4. $y = 3 \csc x$

5. $y = -\frac{1}{2} \cot x$

6. $y = -2 \csc x$

7. $y = \tan(x + \frac{\pi}{4})$

8. $y = \sec(x - \frac{\pi}{3})$

9. $y = \cot(x + \frac{\pi}{6})$

10. $y = \csc(x - \frac{2\pi}{5})$

11. $y = \tan 2x$

12. $y = \sec 3x$

13. $y = \csc 4x$

14. $y = \cot 3x$

15. $y = 4 \tan(2x + \frac{\pi}{3})$

16. $y = -3 \cot(5x - \frac{\pi}{6})$

17. $y = \frac{1}{2} \sec(3x + \frac{\pi}{4})$

18. $y = -\frac{1}{3} \csc(2x + \frac{\pi}{6})$

19. $y = 2 \tan(0.5x + \frac{\pi}{4}) - 2$

20. $y = -2 \sec(\pi x - \frac{\pi}{6}) + 3$

Solve Exercises 21–24.

21. ***Machine technology*** The small-end diameter, d, of a tapered hole is a function of the angle, θ, of taper, and is given by

$$d = -2h \tan \theta + D$$

where h is the depth of the tapered hole and D is the large-end diameter. A certain tapered hole, measured in cm, has $h = 8$ and $D = 4.5$.

(a) What is the small-end diameter when $\theta = 10°$?

(b) What is the smallest angle that will produce a small-end diameter of 0?

(c) Sketch the graph of the equation for d when $0° \leq \theta < 30°$.

22. ***Broadcasting*** A television camera will be used to broadcast a parade. The camera, C, is located on a platform 75.0 ft from a point on the street, P, where $\overline{CP}$ is perpendicular to the street. Because of several obstructions, the camera can only show the parade between the angles of 40° to the left and 55° to the right.

(a) Determine the maximum distance to the left of P that this camera can televise.

(b) Determine the maximum distance to the right of P that this camera can televise.

(c) Sketch the graph of the distance from the camera to the parade over these viewing angles.

23. ***Machine technology*** The height h of a round workpiece of radius r resting in a V-block is related to the angle of the V cut by the equation

$$h = -(R - r) \csc\left(\frac{B}{2}\right) - R + y$$

where R is the radius of a standard plug gauge, y is the height of the standard plug gauge when resting in the V-block, and B is the angle of the V. Graph h as a function of B if $R = 3.81$ cm, $y = 5.72$ cm, and $r = 2.56$ cm.

24. ***Astronomy*** For a telescope the distance f, in cm, of an object from the objective is given by $f = 2.19 \cot \theta$, if θ is the angle subtended by an image 25 cm away from the lens and the object subtends an angle of 5° when placed 25 cm from the eye. Sketch the graph of f for $0° < \theta < 90°$.

[IN YOUR WORDS]

25. Explain how to graph a function such as $y = \csc(3x - 1)$ on your graphing calculator, spreadsheet, or with computer graphing software. Make sure that you show what keys need to be pressed.

26. What is a vertical asymptote? How can you tell when a trigonometric function will have a vertical asymptote?

10.5 APPLICATIONS OF TRIGONOMETRIC GRAPHS

There are many applications of trigonometric functions and their graphs. In this section we will examine some of them.

PERIODIC AND SIMPLE HARMONIC MOTIONS

One area in which period, frequency, amplitude, and displacement are applied is physics. **Periodic motion** is when an object repeats a certain motion indefinitely, always returning to its starting point after a constant time interval and then beginning a new cycle.

Simple harmonic motion is one type of periodic motion. One form of simple harmonic motion is a straight line. The acceleration is proportional to the displacement and in the opposite direction. Examples of simple harmonic motion include a weight bobbing on a spring, a simple pendulum, and a boat bobbing on water.

SIMPLE HARMONIC MOTION

If a point moves so that its displacement varies with time t, measured in seconds, the point is in equilibrium when $t = 0$. Its displacement at any time t is given by

$$y = A \sin 2\pi ft$$

$$\text{or} \quad y = A \sin \omega t$$

where $\omega = 2\pi f$. Then the point is in *simple harmonic motion*.

As you can see from these equations, the amplitude is $|A|$. The Greek letter omega, ω, represents the angular velocity in radians per second. The period is $\frac{2\pi}{\omega}$ and the frequency f is $\frac{\omega}{2\pi}$. The frequency is measured in cycles per second and the units are hertz (Hz), kilohertz (kHz), and megahertz (MHz). One *hertz* is equivalent to one cycle per second. Period is measured in seconds.

APPLICATION GENERAL TECHNOLOGY

EXAMPLE 10.24

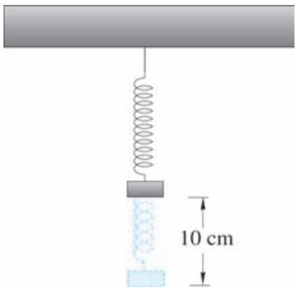


Figure 10.36

An object is suspended from a spring. The object is pulled down 10 cm and released. It oscillates in simple harmonic motion. If it takes 0.2 s for it to complete one oscillation, and if we let $t = 0$ when it is at rest, find the amplitude, period, frequency, and angular velocity. Write an equation to describe the motion (see Figure 10.36).

SOLUTION We are given $A = 10$ cm and period $= 0.2$ s. Since the frequency is $\frac{1}{\text{period}}$, we can determine that the frequency is $\frac{1}{0.2} = 5$ oscillations/s. Now, the angular velocity ω is $2\pi f = 2\pi(5) = 10\pi$ rad/s. Since $\omega = 10\pi$, and $A = 10$, the equation that describes this motion is $y = 10 \sin 10\pi t$.

Simple harmonic motion can also be described using the cosine function. If a point is at its maximum displacement when $t = 0$, and the rest of the previously mentioned conditions are met, then the displacement of the point at any time t can be expressed by

$$y = A \cos 2\pi ft$$

$$\text{or} \quad y = A \cos \omega t$$

If a point is in simple harmonic motion, but is not in equilibrium or at its maximum value when $t = 0$, the sine or cosine curves can be used with the appropriate phase shift. Thus, $y = A \sin(\omega t + \phi)$ could be the equation for an object in simple harmonic motion with a *phase shift* of ϕ.

ALTERNATING CURRENT

Another common use of trigonometric curves is in the study of alternating current (ac) in electricity. The voltage of an ac generator varies with time t. The curve of the output voltage for a generator is a sine curve. Voltage varies with time according to these formulas:

$$V = V_{max} \sin 2\pi ft$$

$$\text{or } V = V_{max} \sin \omega t$$

In these formulas, V_{max} is the maximum value of the voltage and corresponds to the amplitude of the sine wave. The *angular frequency* of the alternating current is represented by $\omega = 2\pi f$ and is in rad/s. As before, f represents the frequency of the voltage in hertz.

In a similar manner, the current in an ac circuit varies with time in the same manner as the voltage. If the maximum value of the current is I_{max}, then the current I is represented by

$$I = I_{max} \sin 2\pi ft$$

$$\text{or } I = I_{max} \sin \omega t$$

APPLICATION ELECTRONICS

EXAMPLE 10.25

Maximum voltage across a power line is 170 V at a frequency of 60 Hz. What is the amplitude, period, and angular frequency of this voltage?

SOLUTION The amplitude is $|V_{max}| = 170$ V. Since the frequency is 60 Hz or 60 cycles/s and the period is $1/f$, the period is $\frac{1}{60}$ s/cycle. The angular frequency is $\omega = 2\pi f = 2\pi(60) = 120\pi$ rad/s.

APPLICATION ELECTRONICS

EXAMPLE 10.26

If household current has a frequency of 60 Hz and a maximum of 2.5 A, find the amplitude of the current.

SOLUTION The amplitude is $|I_{max}| = 2.5$ A.

APPLICATION ELECTRONICS

EXAMPLE 10.27

In the household current of Example 10.26, what is the current when $t = 1.32$ s?

SOLUTION From Example 10.26 we know that $I_{max} = 2.5\ A$ and $f = 60$ Hz. So, $I = I_{max} \sin 2\pi ft = 2.5 \sin 120\pi t$ describes the current. When $t = 1.32$ s, $I = 2.378$ A.

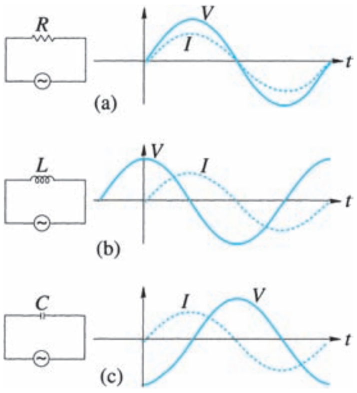


Figure 10.37

There are three different relationships that exist between a voltage V and a current I in an ac circuit. These three relationships are shown in Figure 10.37. In an ac circuit that contains only resistance, the current and voltage are *in phase* with each other. This means that both are 0 at the same time and both reach their maximum value at the same time. This is shown in Figure 10.37a.

In an ac circuit containing only inductance, the voltage leads the current by $\frac{1}{4}$ cycle, as shown in Figure 10.37b. In this case, the current and voltage are said to be 90° ($\frac{\pi}{2}$ rad) *out of phase*, since 90° is $\frac{1}{4}$ cycle. This is also referred to by stating that the current *lags* the voltage by 90°.

Figure 10.37c demonstrates the relationship between the current and the voltage in a circuit that contains only capacitance. The current *leads* the voltage by 90° ($\frac{\pi}{2}$ rad) or $\frac{1}{4}$ cycle.

In an ac circuit that contains resistance, inductance, and capacitance in series, the instantaneous voltages across the circuit elements are V_R, V_L, and V_C, respectively. At any moment, the applied voltage $V = V_R + V_L + V_C$. Because V_R, V_L, and V_C are out of phase with one another, this formula holds only for instantaneous voltages. The phase angle ϕ can be determined from the relationship $\tan \phi = \dfrac{V_{L_{max}} - V_{C_{max}}}{V_{R_{max}}}$ or $\phi = \tan^{-1}\left(\dfrac{V_{L_{max}} - V_{C_{max}}}{V_{R_{max}}}\right)$.

APPLICATION ELECTRONICS

EXAMPLE 10.28

A capacitor, inductor, and resistor are connected in series with a 120-V, 60-Hz power source. If $V_{R_{max}} = 135$ V, $V_{L_{max}} = 190$ V, and $V_{C_{max}} = 90$ V, graph the instantaneous voltage curve for the capacitance, inductance, resistance, and applied voltage and determine the phase angle.

SOLUTION The equation for the resistance is determined from the equation $V_R = V_{R_{max}} \sin \omega t$. Since $V_{R_{max}} = 135$ V and $\omega = 2\pi f = 120\pi$, we have $V_R = 135 \sin 120\pi t$. The inductance leads the current by 90° or $\frac{\pi}{2}$ rad, so $V_L = 190 \sin\left(120\pi + \frac{\pi}{2}\right)$.

Similarly, since the capacitance lags the current by 90°, or $\frac{\pi}{2}$, we have $V_C = 90 \sin(120\pi - \frac{\pi}{2})$. Each of these curves and $V = V_R + V_L + V_C$ are shown in Figure 10.38. The phase angle ϕ is determined from $\tan \phi = \dfrac{V_{L_{max}} - V_{C_{max}}}{V_{R_{max}}} = \dfrac{190 - 90}{135} = 0.74$, so $\phi = \tan^{-1} 0.74 = 36.5°$ or 0.638 rad.

PHASORS

In electronic circuits that contain capacitance or inductance, phase shifts occur between voltage and current and between resistance and reactance. Vectors, called **phasors**, are used to represent the phase relationships. As a phasor rotates counter-clockwise its vertical component traces out a sine wave. Figure 10.39 shows what happens during the first 180° of the rotation. As the phasor continues to rotate its vertical component traces out the remainder of the sine wave as in Figure 10.40. If there is a second phasor the same thing happens. But, what happens if one of the phasors does not start on the x-axis? Figure 10.41 shows phasors **A** and **B** both with length 2. Phasor **B** is at an angle of $-\frac{\pi}{6}$ to phasor **A**. As they rotate at the same angular velocity they maintain the separation of

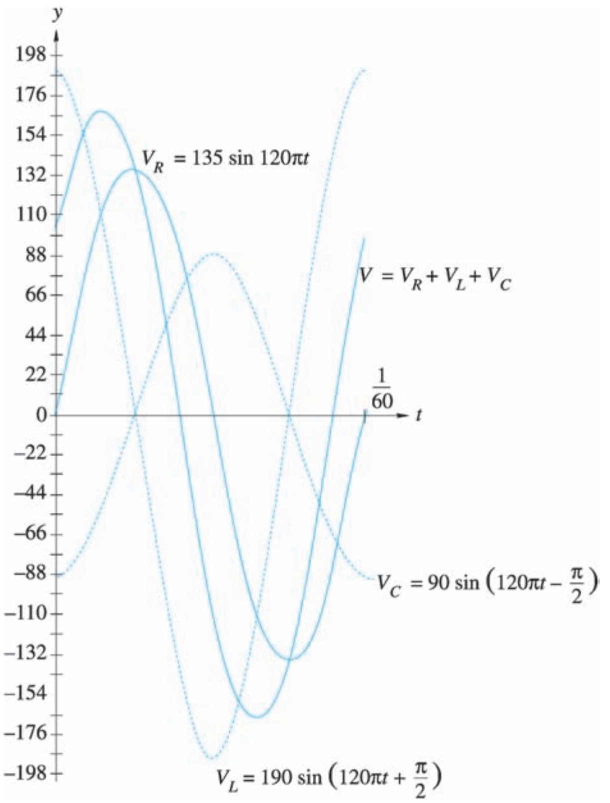


Figure 10.38

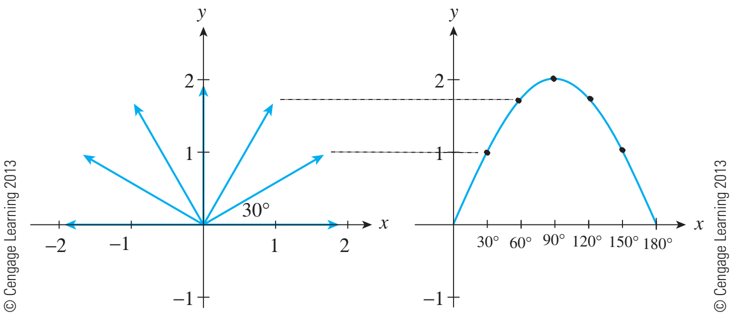


Figure 10.39

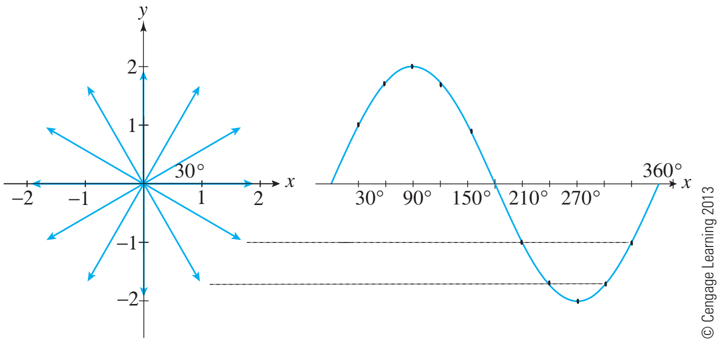


Figure 10.40

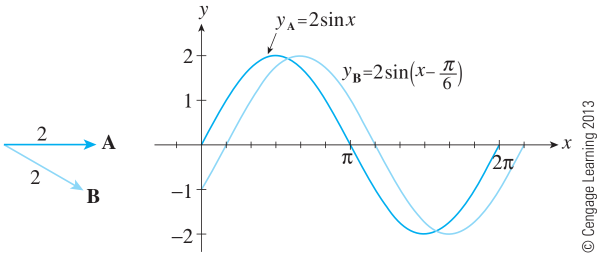


Figure 10.41

$\frac{\pi}{6}$ radians. Phasor **A** generates the wave $y_A = \sin x$ and phasor **B** generates the wave $y_B = \sin(x - \frac{\pi}{6})$.

In a similar way, Figure 10.42 shows phasors **A** and **C** both with length 2. Phasor **C** is at an angle of $\frac{\pi}{6}$ to phasor **A**. As they rotate at the same angular velocity they maintain the separation of $\frac{\pi}{6}$ radians. Here phasor **C** generates the wave $y_C = \sin(x + \frac{\pi}{6})$.

APPLICATION ELECTRONICS

EXAMPLE 10.29

Figure 10.43 shows three phasors of various magnitudes and phase shifts. On the same axes, sketch the sine waves they represent.

SOLUTION We will use phasor **A** as the reference. Since it has a peak amplitude of 2.5 it is graphed as $y_A = 2.5 \sin x$. Phasor **B** leads **A** by 60° and has a peak amplitude of 2.0 so it is graphed as $y_B = 2.0 \sin(x + 60°)$. Phasor **C** lags **A** by 45° and has a peak amplitude of 1.5 so it is graphed as $y_C = 1.5 \sin(x - 45°)$. The resulting graph is shown in Figure 10.44.

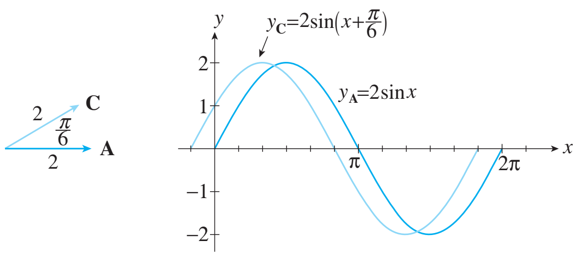


Figure 10.42

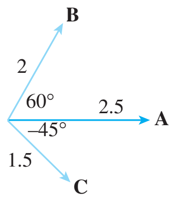


Figure 10.43

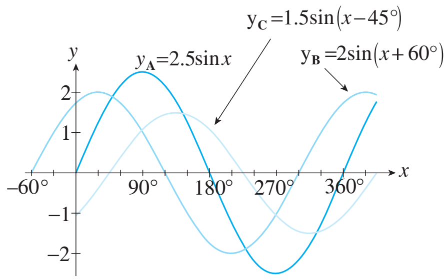
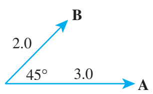


Figure 10.44

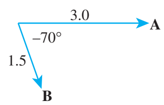
EXERCISE SET 10.5

In Exercises 1–4, sketch the sine waves represented by the given phasor diagram.

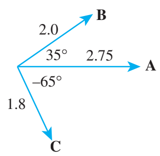
1.



2.



3.



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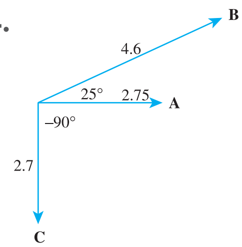


Solve Exercises 5–26.

5. ***Mechanics*** A weight vibrates on a spring in simple harmonic motion. The amplitude is 10 cm and the frequency is 4 Hz. Write an equation for the weight's position y, at any time t, if $y = 0$ when $t = 0$ s.
6. ***Mechanics*** Write an equation for the position of the weight in Exercise 5 at any time t if $y = 8$ when $t = 0.1$ s.
7. ***Oceanography*** A raft bobs on the water in simple harmonic motion. Write a formula for the raft's position if the amplitude is 0.8 m, $\omega = \frac{\pi}{3}$ rad/s, and $y = 0$ when $t = 0$ s.
8. ***Oceanography*** What is the frequency of the raft in Exercise 7?
9. ***Physics*** The acceleration a of a pendulum is given by $a = -g \sin \theta$, where g is gravitational acceleration and θ is the angular displacement from the vertical. What is the acceleration of a pendulum when $\theta = 5°$ and $g = 32$ ft/s^2?
10. ***Physics*** What is the acceleration of a pendulum when $\theta = 0.05$ rad and $g = 9.8$ m/s^2?
11. ***Mechanics*** A weight hanging from a spring vibrates in simple harmonic motion according to the equation $y = 8.5 \cos 2.8t$, where y is the position of the weight in centimeters and t is time in seconds.
    (a) What is the amplitude of the vibrating weight?
    (b) What is its period?
    (c) What is its frequency?
    (d) Sketch one complete cycle of this curve.
12. ***Automotive technology*** Each piston in a certain engine has stroke or total travel distance of 10.50 cm. **(a)** If the engine is operating at 3 000 rpm, what is the frequency of oscillation for each piston? **(b)** Write an equation to represent the position of the piston at any second t, if we assume the piston was at its maximum height (TDC), when $t = 0$. **(c)** If the engine speed is increased to 4 500 rpm, what equation would represent the position of the piston in **(b)** at any time t?
13. ***Electronics*** In an ac circuit, the current I is given by the relation $I = 10 \sin 120\pi t$. What are the amplitude, period, frequency, and angular velocity?
14. ***Electronics*** In an ac circuit, write the equation of the sine curve for the current when $I_{\max} = 6.8$ A and $f = 80$ Hz.
15. ***Electronics*** In the circuit in Exercise 14, what is the equation of the sine curve when the phase angle is $\frac{\pi}{3}$ rad?
16. ***Electronics*** What is the equation of the sine curve for the voltage in an ac circuit when $V_{\max} = 220$ V and $f = 40$ Hz?
17. ***Electronics*** What is the equation of the sine curve for the voltage in Exercise 16 when the phase angle is $-\frac{\pi}{3}$ rad?
18. ***Electronics*** What is the equation of the cosine curve for the voltage in Exercise 16?
19. ***Electronics*** What is the equation of the cosine curve for the voltage in Exercise 17?
20. ***Medical technology*** The voltage for an x-ray can have a maximum value of 250 000 V. If the frequency of an x-ray is 10^{19} Hz, write the equation of the sine curve for the voltage.
21. ***Physics*** A gamma ray can be described by the equation
$$y = 10^{-12} \sin 2\pi 10^{23} t$$
What are the frequency and amplitude of this gamma ray?
22. ***Electronics*** In an ac circuit containing only a constant capacitance, the current leads the voltage by $\frac{1}{4}$ cycle. If $V = V_{\max} \sin 2\pi ft$, $I_{\max} = 3.6$ A and $f = 60$ Hz, write the equation for the current. Sketch one cycle of the curve.
23. ***Electronics*** In a resistance-inductance circuit, $I = I_{\max} \sin(2\pi ft + \phi)$. If $I_{\max} = 1.2$ A, $f = 400$ Hz, and $\phi = 37°$, sketch I vs. t for one cycle.
24. ***Electronics*** A capacitor, inductor, and resistor are connected in series with a 120-V, 60-Hz power source. If $V_{R_{\max}} = 120$ V, $V_{L_{\max}} = 80$ V,

and $V_{C_{max}} = 130$ V, graph the instantaneous voltage curves for the capacitance, inductance, resistance, and applied voltage, and determine the phase angle.

25. ***Medical technology*** An x-ray is described by the equation

$$y = 10^{-10} \sin(2\pi \times 10^{18} t + \pi)$$

where y is in meters and t is in seconds.

(a) Determine the period and amplitude of this x-ray.

(b) Sketch one cycle of this function. Make sure that you label the axes correctly.

26. ***Ecology*** The population of a certain species of animal in a region can be determined at any given time by the equation

$$P = 25{,}000 + 7{,}250 \cos\left(\tfrac{\pi}{12} t\right)$$

where P is the population after t months.

(a) What are the maximum and minimum sizes in the population?

(b) What is the population after 1 year?

(c) What is a reasonable viewing window for a graph of this function on your calculator?

(d) Graph this function for a 3-year period.

[IN YOUR WORDS]

27. You may have had difficulty using a graphing calculator, spreadsheet, or computer graphing software to obtain a graph of the x-ray in Exercise 25. Explain why the calculator or software may have caused this problem.

28. Write a word problem in your technology area of interest that requires you to use a trigonometric function. On the back of the sheet of paper, write your name and explain how to solve the problem. Give the problem you wrote to a friend and let him or her try to solve it. If your friend has difficulty understanding or solving the problem, or disagrees with your solution, make any necessary changes in the problem or solution. When you have finished, give the revised problem and solution to another friend and see if he or she can solve it.

29. Write a word problem in your technology area of interest that requires you to use a trigonometric function different from the one you used in Exercise 28. Follow the same procedures described in Exercise 28 for writing, sharing, and revising your problem.

10.6 PARAMETRIC EQUATIONS

Graphing an equation in terms of two variables x and y is often done by setting up a table of values. We usually select a value for x or y and solve for the other variables. In the case of a function such as $y = x^2$ or $y = x^3 + 2x$, this has not been a problem. We have had difficulty with equations that were not functions, such as $x^2 + y^2 = 9$.

PARAMETRIC EQUATIONS

One solution is to rewrite this equation by expressing x and y as functions of a third variable, called a *parameter*. These equations are called **parametric equations**.

EXAMPLE 10.30

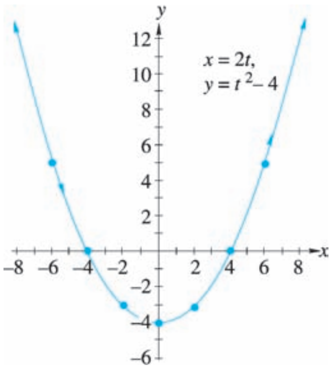


Figure 10.45

Describe and sketch the curve represented by the parametric equations:

$$x = 2t, \qquad y = t^2 - 4$$

SOLUTION The parameter for these equations is t. We will set up a table of values for t, x, and y. The graph is shown in Figure 10.45. We connect the points in order of the increasing values of t, as indicated by the arrows in Figure 10.45.

t	-4	-3	-2	-1	0	1	2	3	4
x	-8	-6	-4	-2	0	2	4	6	8
y	12	5	0	-3	-4	-3	0	5	12

Does the curve look familiar? It looks very much like some of the equations that we graphed earlier. It is possible to eliminate the parameter and write the equation in rectangular form. In these two equations, since $x = 2t$, then $t = \frac{x}{2}$. Substituting this value for t in the equation $y = t^2 - 4$, we get $y = \frac{x^2}{4} - 4$ or $y = \frac{1}{4}x^2 - 4$.

When eliminating the parameter, you must be very careful about the domain. Consider the following example.

EXAMPLE 10.31

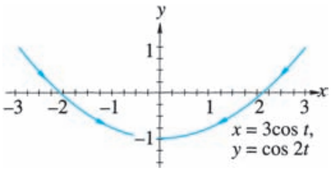


Figure 10.46

Describe and sketch the curve represented by the parametric equations $x = 3 \cos t$ and $y = \cos 2t$.

SOLUTION We will set up a table of values for t, x, and y:

t	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{5\pi}{4}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$
x	3	2.6	2.1	1.5	0	-1.5	-2.1	-2.6	-3	-2.6	-2.1	-1.5	0
y	1	0.5	0	-0.5	-1	-0.5	0	0.5	1	0.5	0	-0.5	-1

Notice that these values begin to repeat once we get to $t = \pi$. The sketch of this curve is shown in Figure 10.46. Again, we get a shape similar to the one in Example 10.29, except that the curve does not continue in each direction. It oscillates back and forth. Using techniques we will develop later, we can eliminate the parameter to form the rectangular equation $y = \frac{2x^2}{9} - 1$, where $-3 \le x \le 3$. Notice the restriction on the domain of x.

A simpler example demonstrating how the domain is often restricted when parametric equations are written in the rectangular form is shown by the next example.

EXAMPLE 10.32

Describe and sketch the curve represented by the parametric equations $x = t^2$ and $y = 2t^2$.

SOLUTION Since $x = t^2$ and $y = 2t^2$, we see that the rectangular form is $y = 2x$. This is the equation of a straight line. Notice, however, that for all values

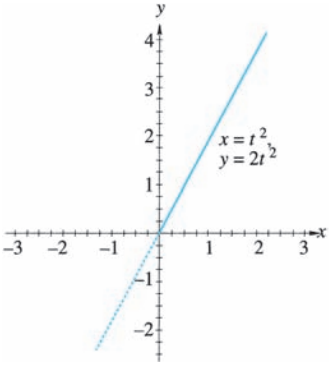


Figure 10.47

of t, $x \geq 0$ and $y \geq 0$. The graph formed by these parametric equations is shown in Figure 10.47 as a solid line. The remainder of the curve $y = 2x$ is indicated by the dashed line.

LISSAJOUS FIGURES

When the parametric equations of a point describe simple harmonic motion, the resulting curve is called a **Lissajous figure**. When voltages of different frequencies are applied to the vertical and horizontal plates of an oscilloscope, a Lissajous figure results.

EXAMPLE 10.33

Sketch the graph of the parametric equations $x = \cos t$ and $y = \sin 3t$.

SOLUTION The Lissajous curve for these parametric equations is shown in Figure 10.48.

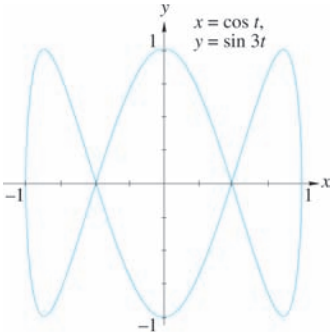


Figure 10.48

Examine the Lissajous figure in Figure 10.48. There are three loops along the top edge of the figure and one loop along the side. This is a ratio of 3:1. Now look at the frequencies of the parametric equations that generated this curve. The frequency of $x = \cos t$ is $\frac{1}{2\pi}$ and the frequency of $y = \sin 3t$ is $\frac{3}{2\pi}$. The ratio of the frequencies is 3:1. In Exercise Set 10.6 we will predict the number of loops on the top and side from the parametric equations and then graph the curve. While this is only an exercise in this text, it is a technique that can be used to calibrate signal generators.

USING A CALCULATOR TO GRAPH PARAMETRIC EQUATIONS

Before you can use a graphing calculator to draw the graphs of parametric equations, you need to put the calculator in parametric mode. On a TI-83 or TI-84 this is done by pressing MODE ▼ ▼ ▼ ▼ ▶ ENTER. The screen on a TI-83/84 should look like the one shown in Figure 10.49.

Figure 10.49

EXAMPLE 10.34

Use a graphing calculator to sketch the curve represented by the parametric equations $x = 2t$ and $y = t^2 - 4$.

SOLUTION These are the same parametric equations we graphed in Example 10.29. We will use the following table from that example to help graph these equations.

t	-4	-3	-2	-1	0	1	2	3	4
x	-8	-6	-4	-2	0	2	4	6	8
y	12	5	0	-3	-4	-3	0	5	12

To enter the window settings, first press WINDOW. On a TI-83 or TI-84, you are first asked for Tmin, Tmax, and Tstep. Based on the table, we will let Tmin $= -4$ and Tmax $= 4$. The value of Tstep determines how much the

EXAMPLE 10.34 (Cont.)

value of t should be increased before calculating new values for x and y. We will pick Tstep = 0.5. You may want to try different values. Based on this table, let Xmin = −8, Xmax = 8, Xscl = 1, Ymin = −4, Ymax = 12, and Yscl = 1.

Now press Y=. On the first line, enter the right-hand side of the parametric equation for x. Press 2 X, T, θ, n. Notice that pressing the X, T, θ, n key put a T on the screen.

Next, enter the parametric equation for y. Press ENTER to move the cursor to the line labeled Y1T and press the key sequence X, T, θ, n x^2 − 4. The result in displayed in Figure 10.50a. Now press GRAPH and you should see the result in Figure 10.50b.

Figure 10.50a

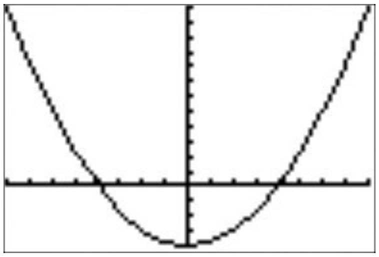


Figure 10.50b

EXAMPLE 10.35

Use a graphing calculator to sketch the curve represented by the parametric equations $x = 2 \cos t$ and $y = \sin t$.

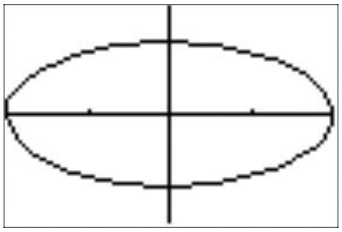


Figure 10.51

SOLUTION Because the periods of sin and cos are 2π, we will let Tmin = 0, Tmax = 6.3, and Tstep = 0.1. Since the range of $x = 2 \cos t$ is $[-2, 2]$, we choose Xmin = −2, Xmax = 2, and Xscl = 1. The range of $y = \sin t$ is $[-1, 1]$. We let Ymin = −1.5, Ymax = 1.5, and Yscl = 1. Now press Y= and enter the parametric equations by pressing

2 COS X, T, θ, n) ENTER

SIN X, T, θ, n) GRAPH

The result is displayed in Figure 10.51.

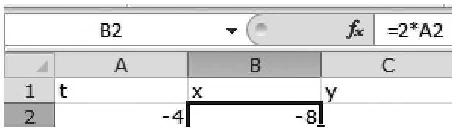
Using a Spreadsheet to Graph Parametric Equations

EXAMPLE 10.36

Use a spreadsheet to sketch the curve represented by the parametric equations $x = 2t$ and $y = t^2 - 4$.

SOLUTION Graphing parametric equations using a spreadsheet is relatively easy.

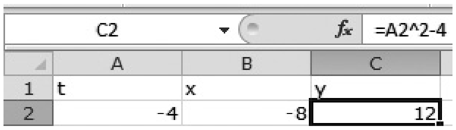
Create three columns, one for t, one for x, and one for y. Next enter the values for t from −4 to 4 by one, enter the equation for x in Cell B2, as shown in Figure 10.52a, and enter the equation for y in Cell C2 as shown in Figure 10.52b. Copy the two equations and graph the result using Columns B and C as your source data. You should get a graph like the one in Figure 10.52c.



	A	B	C
1	t	x	y
2	-4	-8	

B2 f_x =2*A2

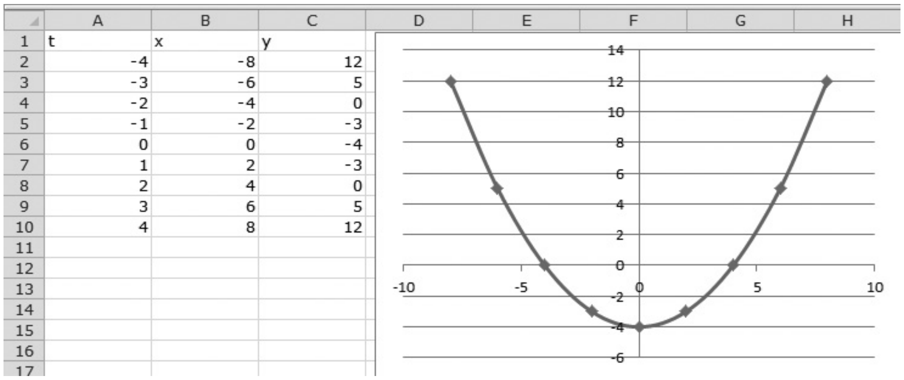
Figure 10.52a



	A	B	C
1	t	x	y
2	-4	-8	12

C2 f_x =A2^2-4

Figure 10.52b



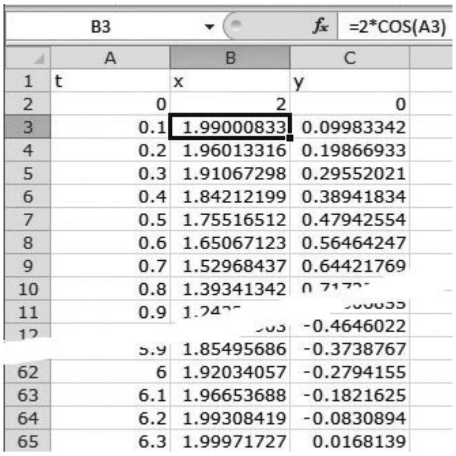
	A	B	C
1	t	x	y
2	-4	-8	12
3	-3	-6	5
4	-2	-4	0
5	-1	-2	-3
6	0	0	-4
7	1	2	-3
8	2	4	0
9	3	6	5
10	4	8	12

Figure 10.52c

EXAMPLE 10.37

Use a spreadsheet to sketch the curve represented by the parametric equations $x = 2 \cos t$ and $y = \sin t$.

SOLUTION Choose t values between 0 and 6.3 in increments of 0.1. You should type = 2 * cos (A2) in Cell B2 and = sin (A2) in Cell C2. The resulting table of values is partially shown in Figure 10.53a and the resulting graph is displayed in Figure 10.53b. This graph looks a little different because we used a scatter plot with a smooth line without showing the points that are in the table.



B3 f_x =2*COS(A3)

	A	B	C
1	t	x	y
2	0	2	0
3	0.1	1.99000833	0.09983342
4	0.2	1.96013316	0.19866933
5	0.3	1.91067298	0.29552021
6	0.4	1.84212199	0.38941834
7	0.5	1.75516512	0.47942554
8	0.6	1.65067123	0.56464247
9	0.7	1.52968437	0.64421769
10	0.8	1.39341342	0.717...
11	0.9	1.24...	...00633
12		...03	-0.4646022
	5.9	1.85495686	-0.3738767
62	6	1.92034057	-0.2794155
63	6.1	1.96653688	-0.1821625
64	6.2	1.99308419	-0.0830894
65	6.3	1.99971727	0.0168139

Figure 10.53a

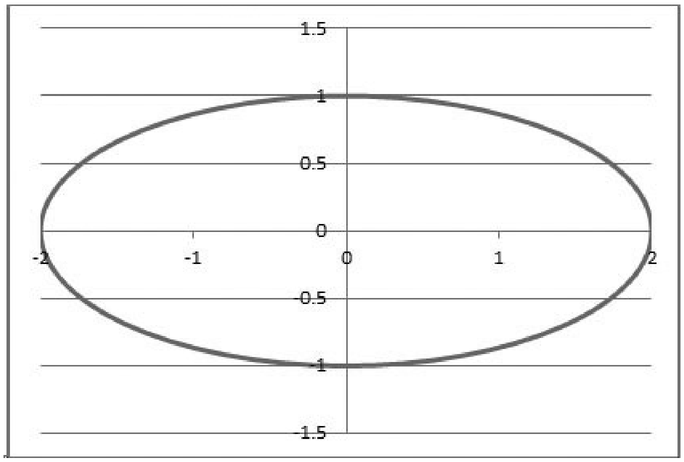


Figure 10.53b

GRAVITY-INFLUENCED TRAJECTORIES

When a communications satellite or a space shuttle is launched, it is given enough velocity to enable it to go into earth's orbit instead of falling back to earth. How much velocity is enough? The answer depends on the weight of the rocket and where you want the orbit to be. A space shuttle, for example, must be sent off at 17,500 mph in order to reach its orbit. If a projectile moves with an initial velocity that is below its *escape velocity*, it returns to earth without going into orbit. We'll now return to motion that does not reach orbit, and is therefore called *sub-orbital motion*.

SUMMARY OF SUB-ORBITAL PROJECTILE MOTION

The parametric equations that describe the position of an object thrown into the air with velocity less than the earth's escape velocity are

$$\begin{cases} x(t) = v_{x_0}t + x_0 \\ y(t) = -\frac{1}{2}gt^2 + v_{y_0}t + y_0 \end{cases}.$$

where t is the time after the toss; $x(t)$ the horizontal distance traveled; $y(t)$ the vertical distance traveled; v_{x_0} the initial horizontal component of velocity (v_{x_0} is constant during flight); x_0 the horizontal distance from zero point at release; $g = 32$ if the units are feet and seconds and $g = 9.81$ if the units are meters and seconds; v_{y_0} is the initial vertical component of velocity; and y_0 is the height above ground at release.

EXAMPLE 10.38

A ball is thrown upward at an angle of 50° from a height of 30 ft with a velocity of 92 ft/s.

(a) Write the parametric equations of the position of the ball at any time t.

(b) Plot the trajectory with your calculator or a spreadsheet.

(c) Estimate to two decimal places the time when the ball reaches the ground.

(d) Estimate to the nearest foot the maximum height the ball reaches.

(e) Estimate to the nearest foot the horizontal distance the ball traveled.

SOLUTIONS We are given that $y_0 = 30$, and we can set $x_0 = 0$. Since $v_0 = 92$, we have $v_{x_0} = 92 \cos 50° \approx 59$ ft/s, and $v_{y_0} = 92 \sin 50° \approx 70$ ft/s. Since y_0 and v_0 are both in terms of feet, we use $g = -32$.

(a) Using the above information, the equations are

$$\begin{cases} x = 59t \\ y = -16t^2 + 70t + 30 \end{cases}$$

(b) The trajectory is shown in Figures 10.54a and 10.54b.

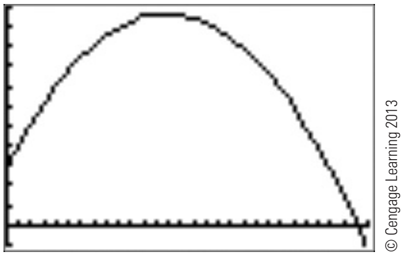


Figure 10.54a

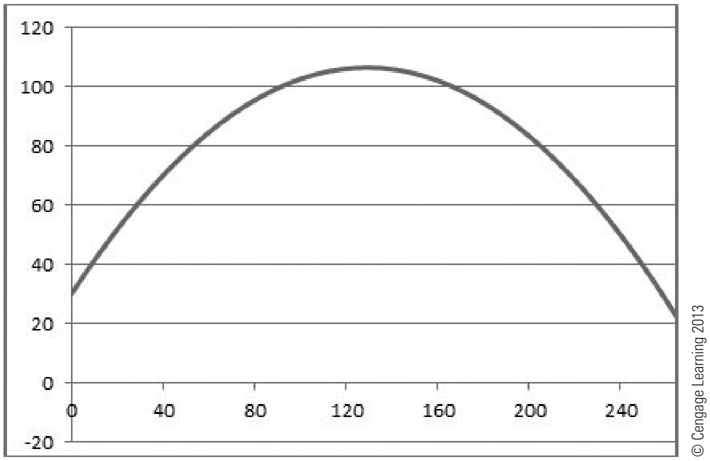


Figure 10.54b

	A	B	C
1	t	x	y
2	4.7	277.3	5.56
3	4.71	277.89	4.7544
4	4.72	278.48	3.9456
5	4.73	279.07	3.1336
6	4.74	279.66	2.3184
7	4.75	280.25	1.5
8	4.76	280.84	0.6784
9	4.77	281.43	-0.1464
10	4.78	282.02	-0.9744
11	4.79	282.61	-1.8056
12	4.8	283.2	-2.64

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Figure 10.54c

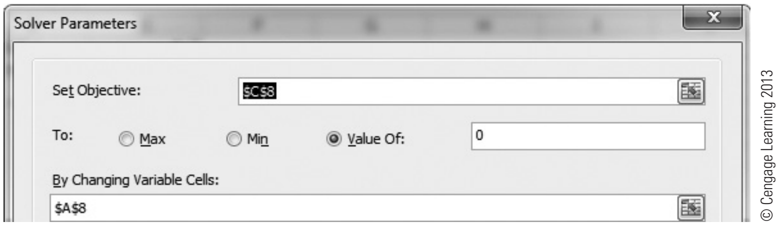

Figure 10.54d

(c) The ball will hit the ground when $y = 0$. Using the quadratic formula to solve $-16t^2 + 70t + 30 = 0$, we see that $t \approx -0.39$ s or $t \approx 4.77$ s. The first time is unacceptable since it was before the ball was thrown. Thus, it took about 4.77 s for the ball to hit the ground.

The table of values on a spreadsheet provides another way to estimate the time. In Figure 10.54c, we see that the y-value becomes zero sometime between $t = 4.76$ s and $t = 4.77$ s. A more exact answer can be found by choosing smaller increments of t.

A third way is to use Solver Figure 10.54d shows Solver being used to set cell C8 to a value of 0 by changing cell A8. The result indicates that the projectile will hit the ground in 4.76822785 s.

(d) In the last chapter we learned that the vertex of a parabola $f(t) = at^2 + bt + c$ occurs when $t = \dfrac{-b}{2a}$. Here we have $y(t) = -16t^2 + 70t + 30$, so the vertex is at $t = \dfrac{-70}{2(-16)} = 2.1875$. Using the TRACE feature of the calculator, we see in Figures 10.54e and 10.54f that when $t = 2.1875$, then $y = 106.5625$. So, the maximum height is about 107 ft.

By using $t = 2.1875$, a height of about 106.6 ft will be obtained on a calculator (see Figure 10.54e) or a spreadsheet (see Figure 10.54f).

EXAMPLE 10.38 (Cont.)

(e) At the time it hits the ground (when $t = 4.77$), the ball has traveled $x(4.8)$ ft horizontally. Since $x(4.77) = 59 \times 4.77 \approx 281.4$, the ball lands 281.4 ft away from the point where it was thrown.

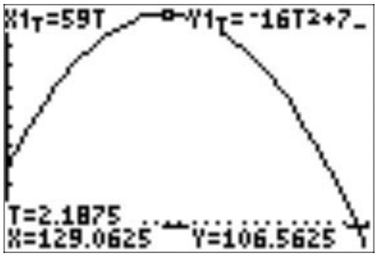

Figure 10.54e

	A	B	C
1	t	x	y
2	2.1875	129.0625	106.5625

Figure 10.54f

The next example uses the same ideas in a different setting.

APPLICATION CIVIL ENGINEERING

EXAMPLE 10.39

A plane is attempting to drop a package of food near a campsite in the wilderness. The plane is moving horizontally in level flight at 120 mph at a height of 1000 ft.

(a) Write the parametric equations that model the position of the plane and the package.

(b) Plot the trajectories on your calculator or a spreadsheet.

(c) Estimate to the nearest tenth of a second the time when the food package reaches the ground.

(d) Estimate to the nearest foot the horizontal distance from where the food was dropped to where it landed.

SOLUTIONS In this example $v_{y_0} = 0$ because the package is being dropped, not thrown. However $v_{x_0} = 120$ mph ≈ 176 ft/s. Finally, $y_0 = 1000$ ft.

(a) The equations for the position of the package are

$$\begin{cases} x = 176t \\ y = -16t^2 + 1000 \end{cases}$$

The equations for the position of the plane are

$$\begin{cases} x = 176t \\ y = 1000 \end{cases}$$

(b) With the settings Tmin = 0, Tmax = 12, Tstep = 0.1, Xmin = 0, Xmax = 1400, Xscl = 100, Ymin = 0, Ymax = 1200, and Yscl = 100, the two trajectories are plotted on a TI-84 in Figure 10.55a. A similar graph on a spreadsheet is in Figure 10.55b.

(c) Solving $y = -16t^2 + 1000 = 0$, we see that the package reaches the ground at about $t = 7.9$ s.

(d) The horizontal distance the food package traveled is approximately $176 \times 7.9 \approx 1390$ ft before it hit the ground.

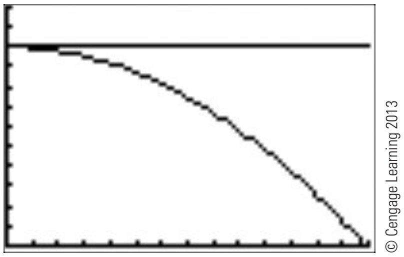

Figure 10.55a

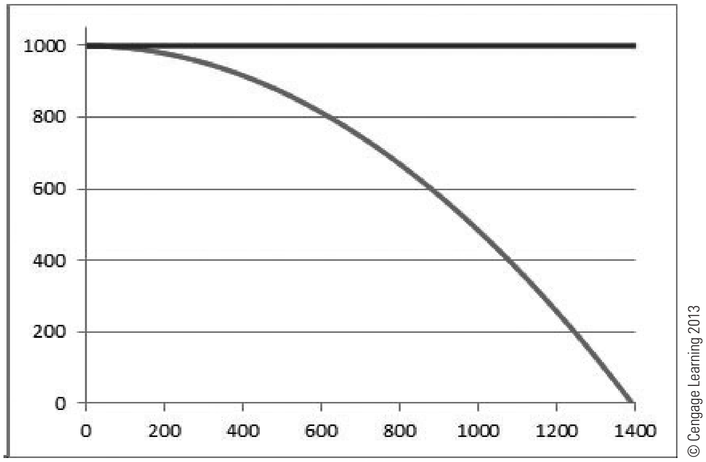

Figure 10.55b

EXERCISE SET 10.6

Graph each curve given by the parametric equations in Exercises 1–14. Make a table of values for at least six values of t. Show how the function behaves by drawing arrows on the graph. Eliminate the parameters in Exercises 1–6 and write the equation as a function of y.

1. $x = t, y = 3t$

2. $x = 2t, y = 4t + 1$

3. $x = t, y = \dfrac{1}{t}$

4. $x = t + 5, y = 3t - 2$

5. $x = 3 - t, y = t^2 - 9$

6. $x = t + 2, y = t^2 - t$

7. $x = 2 \sin t, y = 2 \cos t$

8. $x = 5 \sin t, y = 2 \cos t$

9. $x = 5 \sin t, y = 3 \sin t$

10. $x = 2 \cos t, y = 6 \cos t$

11. $x = t - \sin t, y = 1 - \cos t$

12. $x = \tan t, y = 6 \cot t$

13. $x = \sec t, y = 2 \csc t$

14. $x = 3 \sec t, y = \tan t$

Examine the pairs of parametric equations in Exercises 15–24. For each pair, (a) predict the ratio of the loops along the top to the number of loops along the side, then (b) graph each of these curves. If possible, use a graphing calculator, spreadsheet, or computer graphing program to graph the curves.

15. $x = \sin t, y = \cos t$

16. $x = \sin 2t, y = \cos t$

17. $x = \sin t, y = \cos 2t$

18. $x = \sin 3t, y = \cos t$

19. $x = \sin 4t, y = \cos t$

20. $x = \sin 4t, y = \cos 2t$

21. $x = \sin 3t, y = \cos 2t$

22. $x = \sin 4t, y = \cos 3t$

23. $x = \sin 5t, y = \cos 2t$

24. $x = \sin 5t, y = \cos 3t$

Exercises 25–30 are Lissajous curves. Graph each curve.

25. $x = 2 \sin t, y = \cos t$

26. $x = 4 \sin t, y = \cos t$

27. $x = 3 \sin 2t, y = 2 \cos t$

28. $x = 3 \sin 3t, y = \cos t$

29. $x = \sin 3t, y = 4 \cos t$

30. $x = 2 \sin 2t, y = 5 \cos 3t$

Solve Exercises 31–36.

31. ***Automotive technology*** The motion of a piece of gravel thrown back by a spinning rear tire at an angle α with speed V, in ft/s, can be described by

$$x = (V \cos \alpha)t$$
$$y = (V \sin \alpha)t - \frac{1}{2}gt^2$$

where $g \approx 32$ ft/s^2.

(a) Assume that a car is traveling 30 mph (44 ft/s) and that three pieces of gravel leave its rear tire, one at $\alpha = 30°$, one at $\alpha = 45°$, and one at $\alpha = 50°$. Graph the path of each of these three pieces of gravel on the same set of axes.

(b) Rewrite these parametric equations as one equation in rectangular form.

(c) Use your answer to (b) to determine how far the gravel travels before hitting the road.

32. ***Business*** After a new consumer electronics product is introduced, sales rise quickly and the price gradually decreases. Let t be the number of years since a product was introduced (so, $t = 0$ is when the product was introduced). Suppose the unit price at time t, in hundreds of dollars, is $p(t) = \dfrac{t^2 + 20}{t^2 + 5}$, and the monthly sales, in 100,000 units, are $s(t) = \dfrac{t^2 + 3t}{t^2 + 1}$.

(a) Graph this pair of parametric equations for 5 years with $s(t)$ on the horizontal axis and $p(t)$ on the vertical axis.

(b) What was the price when the product was introduced? After 1 year? After 5 years?

(c) What were the monthly sales during the 12th month? After 5 years?

33. ***Automotive technology*** An automobile tire is driven over a nail. As a result, the nail is embedded in the tire with only its head showing on the outside surface of the tire. As the tire moves down the highway the nail follows a path known as a *cycloid* and given by the parametric equations

$$\begin{cases} x = a(t - \sin t) \\ y = a(1 - \cos t) \end{cases}$$

where a is the radius of the tire.

If the tire has a radius of 0.9 ft and t is measured in seconds,

(a) Sketch the graph of the nail's path from the time it becomes stuck in the tire (when $t = 0$) until it hits the pavement three more times.

(b) What are the amplitude, period, and frequency of this periodic function?

34. ***Sports technology*** A major-league baseball leaves a pitcher's hand traveling horizontally, with initial speed 140 ft/s and initial height 7 ft 6 in. Ignoring wind resistance,

(a) Write a set of parametric equations that describes the path of the ball.

(b) Graph your parametric equations from the instant the ball leaves the pitcher's hand until it hits the ground.

(c) If the ball is released 58.0 ft from home plate, how long does it take for the ball to cross the plate?

(d) If the strike zone stretches from approximately 1.5 – 4.0 ft above the ground, is this pitch a strike?

35. ***Sports technology*** A major-league baseball leaves a pitcher's hand traveling horizontally, with initial speed 100 ft/s and initial height 7 ft 6 in. Ignoring wind resistance,

(a) Write a set of parametric equations that describe the path of the ball.

(b) Graph your parametric equations from the instant the ball leaves the pitcher's hand until it hits the ground.

(c) If the ball is released 58.0 ft from home plate, how long does it take for the ball to cross the plate?

(d) If the strike zone stretches from approximately 1.5 – 4.0 ft above the ground, is this pitch a strike?

36. ***Navigation*** A plane is attempting to drop a package of food near a campsite in the wilderness. The plane is moving at 150.0 mph level flight. When the plane reaches a point that is 2500 ft above the ground the package is dropped. Ignoring the effects of air resistance,

(a) Write the parametric equations that model the position of the plane and the package.

(b) Plot the trajectories on your calculator.

(c) Estimate to the nearest tenth of a second the time when the food package reaches the ground.

(d) Estimate to the nearest foot the horizontal distance from where the food was dropped to where it landed.

[IN YOUR WORDS]

37. Distinguish between rectangular equations and parametric equations.

38. Describe situations in which it is more helpful to use parametric equations than rectangular equations for graphing equations.

10.7 POLAR COORDINATES

Every time we have represented a point in a plane, we used the rectangular coordinate system. Each point has an x- and a y-coordinate. There is another type of coordinate system that is used to represent points. This system is called the **polar coordinate system**.

To introduce a system of polar coordinates in a plane, we begin with a fixed point O called the *pole* or *origin*. From the pole, we shall draw a half-line that has O as its endpoint. This half-line will be called the *polar axis*.

POLAR COORDINATES

Consider any point in the plane different from point O. Call this point P. Let the polar axis form the initial side of an angle and $\overrightarrow{OP}$ the terminal side. This angle has measure θ. If the distance from O to P is r, we can say that the **polar coordinates** of P are (r, θ), as shown in Figure 10.56.

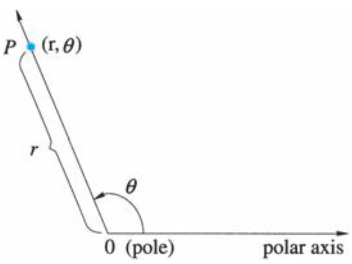

Figure 10.56

We have some of the same understanding about polar coordinates as we do about the angles from trigonometry. If the angle is generated by a counterclockwise rotation of the polar axis, the angle is positive. If it is generated by a clockwise rotation, θ is negative. If r is negative, the terminal side of the angle is extended in the opposite direction through the pole and is measured off $|r|$ units on this extended side.

A special type of graph paper, called *polar coordinate paper*, is used to graph polar coordinates. This paper has concentric circles with their centers at the pole. The distance between any two consecutive circles is the same. Lines

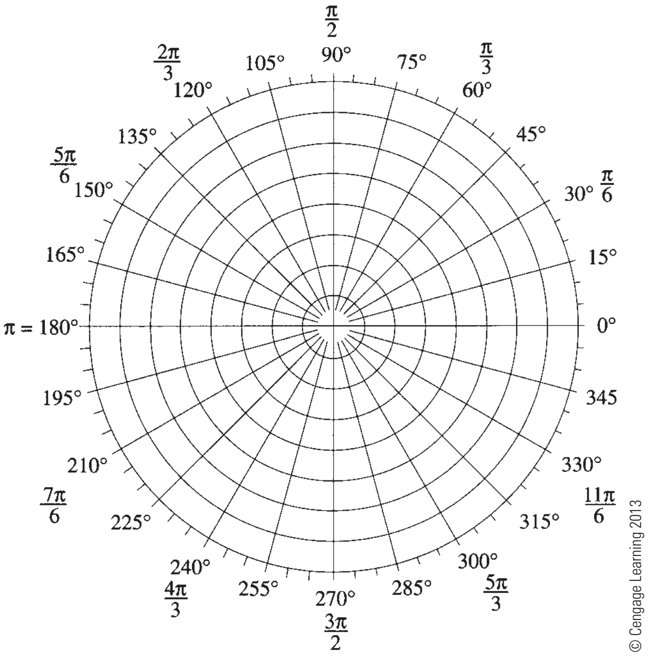

Figure 10.57

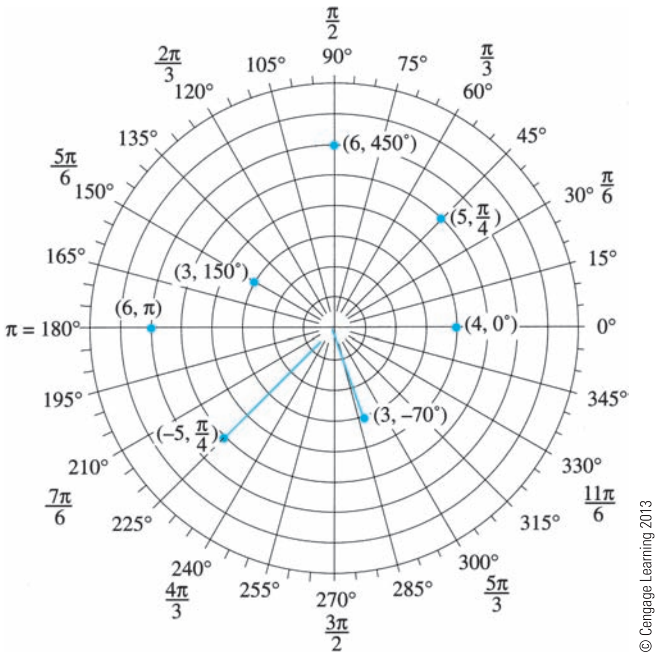

Figure 10.58

are drawn through the pole and correspond to some of the common angles. An example of polar coordinate paper is shown in Figure 10.57.

EXAMPLE 10.40

Plot the points with the following polar coordinates: $(5, \frac{\pi}{4})$, $(-5, \frac{\pi}{4})$, $(3, 150°)$, $(3, -70°)$, $(6, \pi)$, $(4, 0°)$, and $(6, 450°)$.

SOLUTION The points are plotted in Figure 10.58.

Notice that there is nothing unique about these points. For example, the point $(6, 450°)$ would be the same as the points $(6, 90°)$, $(6, -270°)$, or $(-6, 270°)$. The pole has the polar coordinate $(0, \theta)$, where θ can be any angle.

CONVERTING BETWEEN POLAR AND RECTANGULAR COORDINATES

Converting between the polar coordinate system and the rectangular coordinate system requires the use of trigonometry.

CONVERTING POLAR COORDINATES TO RECTANGULAR COORDINATES

If the point P has polar coordinates (r, θ) and rectangular coordinates (x, y), then

$$x = r\cos\theta$$

$$y = r\sin\theta$$

This converts the polar coordinates to rectangular coordinates.

EXAMPLE 10.41

Find the rectangular coordinates of the point with the polar coordinates $(8, \frac{5\pi}{6})$.

SOLUTION From the equations previously given, we have $x = r\cos\theta$ and $y = r\sin\theta$. In this example, $r = 8$ and $\theta = \frac{5\pi}{6}$. So, $x = 8\left(\frac{-\sqrt{3}}{2}\right) = -4\sqrt{3} \approx -6.93$ and $y = 8(\frac{1}{2}) = 4$. The rectangular coordinates are $(-4\sqrt{3}, 4)$.

To convert from rectangular to polar coordinates, we need to use the Pythagorean theorem.

CONVERTING RECTANGULAR COORDINATES TO POLAR COORDINATES

The equations

$$r^2 = x^2 + y^2 \quad \text{or} \quad r = \pm\sqrt{x^2 + y^2}$$

$$\tan\theta = \frac{y}{x}, x \neq 0$$

will convert rectangular coordinates to polar coordinates (see Figure 10.59).

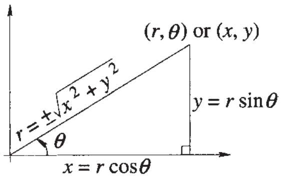

Figure 10.59

EXAMPLE 10.42

Find polar coordinates of $(-4\sqrt{3}, 4)$.

SOLUTION This is the reverse of Example 10.41, so we know that the answer should be $(8, \frac{5\pi}{6})$. Let's practice using the two conversion formulas $r^2 = x^2 + y^2$ and $\tan\theta = \frac{y}{x}$.

Here, $x = -4\sqrt{3}$ and $y = 4$.

$$\begin{aligned} r^2 &= x^2 + y^2 \\ &= \left(-4\sqrt{3}\right)^2 + 4^2 \\ &= 48 + 16 = 64 \\ \text{and} \quad r &= \pm 8 \\ \tan\theta &= \frac{4}{-4\sqrt{3}} = \frac{-1}{\sqrt{3}} \\ \theta &= \tan^{-1}\left(\frac{-1}{\sqrt{3}}\right) \\ &\approx -0.5236 \end{aligned}$$

Since $\tan(-\frac{\pi}{6}) = \frac{-1}{\sqrt{3}}$ we see that $-0.5236 \approx -\frac{\pi}{6}$

It looks as if the answer is $(8, -\frac{\pi}{6})$ or $(-8, -\frac{\pi}{6})$. Now $(-8, -\frac{\pi}{6})$ is correct, but $(8, -\frac{\pi}{6})$ is not correct. What happened? Plot $(-4\sqrt{3}, 4)$. It is in Quadrant II. Now plot $(8, \frac{-\pi}{6})$. It is in Quadrant IV. Remember that $\tan^{-1}$ will only give angles in

EXAMPLE 10.42 (Cont.)

Quadrants I and IV. You should first determine which quadrant a point is in. If it is in Quadrants II or III, you will need to add π (or 180°) to the answer you get using the conversion formula. (Because θ may be in Quadrant II or III, we write $\tan\theta = \frac{y}{x}$, rather than $\tan^{-1}\frac{y}{x} = \theta$.) If we add π to $\frac{-\pi}{6}$, we get $\frac{5\pi}{6}$. Thus, two possible answers are $\left(-8, -\frac{\pi}{6}\right)$ and $\left(8, \frac{5\pi}{6}\right)$. Both of these are polar coordinates of the point with Cartesian coordinates $(-4\sqrt{3}, 4)$.

POLAR EQUATIONS

Equations can also be written using the variables r and θ. These are called **polar equations**. A polar equation states a relationship between all the points (r, θ) that satisfy the equation. In the remaining part of this section, we will graph some polar equations.

As we usually do when we graph an equation, we will use a table of values, plot the points in the table, then connect these points in order as the values of θ increase.

EXAMPLE 10.43

Graph the function $r = 4\cos\theta$.

SOLUTION A table of values follows. The graph of the points is shown in Figure 10.60.

θ	0°	30°	60°	90°	120°	150°	180°	210°	240°	270°	300°	330°	360°
r	4	3.46	2	0	−2	−3.46	−4	−3.46	−2	0	2	3.46	4

Notice that this is a circle with radius 2 centered at (2, 0°).

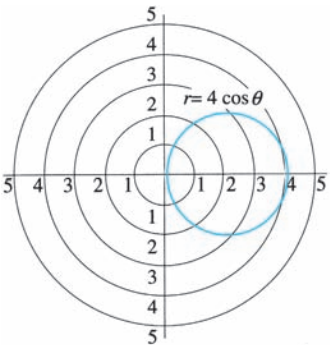

Figure 10.60

CIRCLES

The graph of any equation of the type $r = a\cos\theta$ is a circle of radius $\left|\frac{a}{2}\right|$ and center $\left(\frac{a}{2}, 0°\right)$. An equation of the type $r = a\sin\theta$ is a circle with radius $\left|\frac{a}{2}\right|$ and center $\left(\frac{a}{2}, 90°\right)$.

EXAMPLE 10.44

Graph the function $r = \sin 3\theta$.

SOLUTION The table of values follows. The graph of these points is in Figure 10.61.

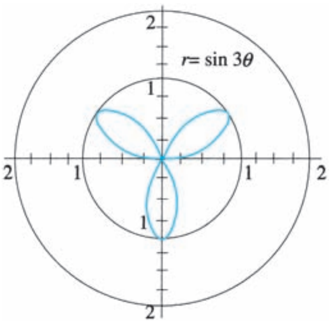

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Figure 10.61

θ	0°	10°	20°	30°	40°	50°	60°	70°	80°	90°
$r = \sin 3\theta$	0	0.5	0.87	1	0.87	0.5	0	−0.5	−0.87	−1

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θ	90°	100°	110°	120°	130°	140°	150°	160°	170°	180°
$r = \sin 3\theta$	−1	−0.87	−0.5	0	0.5	0.87	1	0.87	0.5	0

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A curve of this type is called a *rose*. This rose has three petals.

ROSES

Any polar equation of the form $r = \sin n\theta$ or $r = \cos n\theta$, where n is a positive integer, is a rose. If n is an odd number, the rose will have n petals. If n is an even number, the rose will have $2n$ petals.

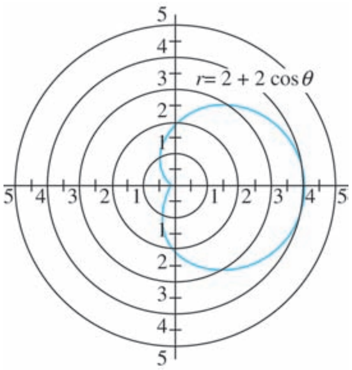

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Figure 10.62

For many functions, polar equations are much simpler to work with than are rectangular equations (and vice versa). A good example of this will be seen in a later chapter.

Another type of polar graph is demonstrated by the graph of the polar equation $r = 2 + 2\cos\theta$. The graph of this equation is shown in Figure 10.62.

The curve in Figure 10.62 is called a *cardioid* because of its heart shape. In Exercise Set 10.7, we will graph other polar equations. If the curve has a special name, we will give that name.

CONVERTING BETWEEN POLAR AND RECTANGULAR EQUATIONS

There are times when you may need to convert a polar equation to a rectangular equation.

You saw how to convert the polar form of a set of coordinates to the rectangular form in Example 10.41. In general, to convert a polar point (r, θ) to (x, y), use the conversions:

$$x = r\cos\theta$$

$$y = r\sin\theta$$

To convert an entire equation, you must convert *all* the points at once. That is not as difficult as it sounds. We must express r and θ in terms of x and y, so we need the conversions:

$$r = \sqrt{x^2 + y^2}$$

$$\theta = \tan^{-1}\left(\frac{y}{x}\right)$$

EXAMPLE 10.45

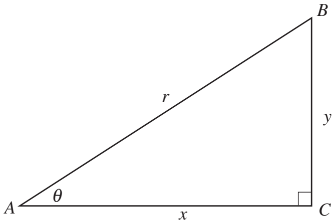

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Figure 10.63

Convert the polar equation $r = 5\cos\theta$ to a rectangular equation in x and y.

SOLUTION Replacing r and θ, we have

$$\sqrt{x^2 + y^2} = 5\cos\left(\tan^{-1}\left(\frac{y}{x}\right)\right)$$

That is certainly an equation in x and y. However, it should be simplified. The key to the simplification is to think about the triangle that includes x, y, r, and θ (Figure 10.63).

We started with $\cos\theta$, and the triangle reminds us that $\cos\theta = \frac{x}{r}$. But we know that $r = \sqrt{x^2 + y^2}$, so $\cos\theta = \frac{x}{\sqrt{x^2 + y^2}}$. Therefore the equation $r = 5\cos\theta$ becomes

$$\sqrt{x^2 + y^2} = 5\frac{x}{\sqrt{x^2 + y^2}}$$

which can be simplified by multiplying both sides of the equation by $\sqrt{x^2 + y^2}$ to give

$$x^2 + y^2 = 5x$$

This last equation is equivalent to the first equation in this solution, but it is easier to work with. However, in order to plot an equation like this on a graphing calculator, we would have to solve for y, which gives two equations

$$y = +\sqrt{5x - x^2} \text{ and } y = -\sqrt{5x - x^2}$$

If both of these are plotted on the calculator, the combination would be the same circle that is produced by $r = 5\cos\theta$ in polar coordinates.

EXAMPLE 10.46

Graph in polar coordinates the line whose equation is $y = 2x + 5$.

SOLUTION We know that $x = r\cos\theta$ and $y = r\sin\theta$. Substituting these expressions we have

$$r\sin\theta = 2r\cos\theta + 5$$

To graph such an equation we must solve it for r. This requires a few steps, including factoring out a common factor:

$$r\sin\theta - 2r\cos\theta = 5$$
$$r(\sin\theta - 2\cos\theta) = 5$$
$$r = \frac{5}{\sin\theta - 2\cos\theta}$$

This last equation produces the exact straight line you would expect, with a slope of 2 and a y-intercept of 5.

USING A CALCULATOR OR SPREADSHEET TO GRAPH POLAR EQUATIONS

You can use a graphing calculator, spreadsheet, or a computer to help you graph many of these curves. We will describe how it is done using a graphics calculator. As before, we will use a TI-86 for these examples.

EXAMPLE 10.47

Use a TI-83 or TI-84 graphing calculator to graph $r = \sin^2 3\theta$.

SOLUTION First, put your calculator in polar mode. On a TI-83 or TI-84 this is done by pressing MODE ▼ ▼ ▼ ▶ ▶ ENTER. When you have finished, the screen on a TI-83 or 84 should look like the one shown in Figure 10.64a.

Press WINDOW to set the size of the viewing window. The function $r = \sin^2 3\theta$ has a period of 2π, so it will take no more than from $\theta = 0$ to $\theta = 2\pi$ to completely sketch this graph, so set θMin $= 0$, θMax $= 2\pi$, and θstep $= \pi/24$. Since the function has an amplitude of 1, a window of $[-1.5, 1.5] \times [-1, 1]$ should be right. You are now ready to enter the function into the calculator.

Now press Y=. On the first line, enter the right-hand side of the parametric equation for r. Press

(SIN 3 X, T, θ, n)) x^2

Remember, that $\sin^2\theta$ is entered into a calculator or computer as $(\sin\theta)^2$. Did you notice that pressing the X, T, θ, n placed a θ on the screen? The result in displayed in Figure 10.64b. Now press GRAPH and you should see the result in Figure 10.64c.

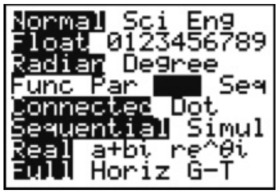


Figure 10.64a

Figure 10.64b

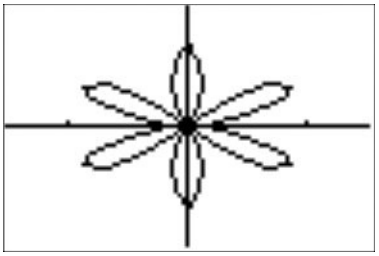


Figure 10.64c

EXAMPLE 10.48

Use a spreadsheet to graph $r = \sin^2 3\theta$.

SOLUTION To graph polar equations in Excel, we will plot the points (x, y) where $x = r\cos\theta$ and $y = r\sin\theta$. The table of values include the multiplier (in Column A), θ (in Column B), r (in Column C), x (in Column D), and y (in Column E). For your benefit, Row 2 of Figure 10.65a shows the formulas used in Row 3.

The function $r = \sin^2 3\theta$ has a period of 2π, so use values of θ from 0 to 2π and increment θ in steps of $\pi/24$. Notice that the function for r (shown in Cell C2), is written in the form $(\sin(3\theta))^2$. The graph that results is shown in Figure 10.65b.

	A	B	C	D	E
1		theta	r	x	y
2		=A3*PI()/24	=(SIN(3*B3))^2	=C3*COS(B3)	=C3*SIN(B3)
3	0	0	0	0	0
4	1	0.130899694	0.146446609	0.145193738	0.019115118
5	2	0.261799388	0.5	0.482962913	0.129409523

Figure 10.65a

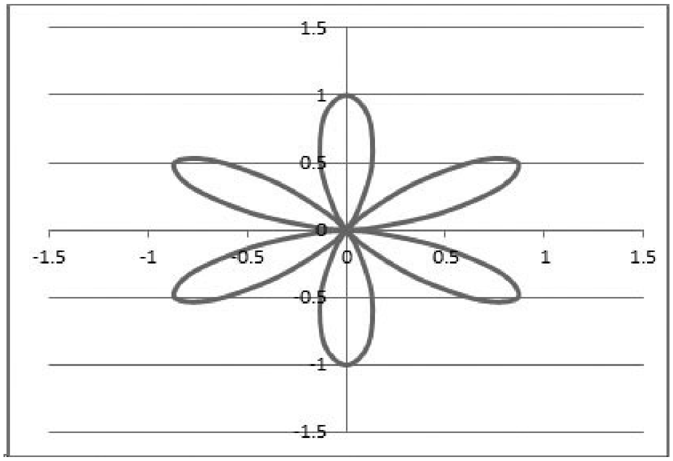


Figure 10.65b

EXAMPLE 10.49

Use a TI-83 or TI-84 graphing calculator to graph $r = 5 \sin 2\theta - 2\cos 3\theta$.

SOLUTION First, set the size of the viewing window. The period of $\sin 2\theta$ is π; the period of $\cos 3\theta$ is $\frac{2}{3}\pi$. Since the least common multiple of π and $\frac{2}{3}\pi$ is 2π, the period of r is a multiple of 2π. We will set θMin $= 0$, θMax $= 2\pi$. Since the ranges of sine and cosine are both $[-1, 1]$, we know that $-7 \leq r \leq 7$. Because the screen is longer horizontally than it is vertically, we set Xmin -10, Xmax $=$ 10, Xscl $= 1$, Ymin $= -6$, Ymax $= 6$, and Yscl $= 1$. Press Y= CLEAR and then the following:

5 SIN 2 X, T, θ, n) − 2 COS 3 X, T, θ, n) GRAPH

The result is a butterfly-shaped curve as shown in Figure 10.66.

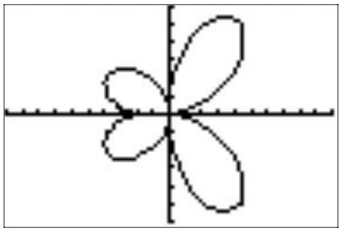


Figure 10.66

EXAMPLE 10.50

Use a spreadsheet to graph $r = 5 \sin 2\theta - 2 \cos 3\theta$

SOLUTION The period of r is a multiple of 2π since the period of $\sin 2\theta$ is π and the period of $\cos 3\theta$ is $2\pi/3$ and the least common multiple of π and $2\pi/3$ is 2π.

The spreadsheet is constructed with columns for r, θ, x, and y as we did in Example 10.48. The result, a butterfly-shaped curve, is shown in Figure 10.67.

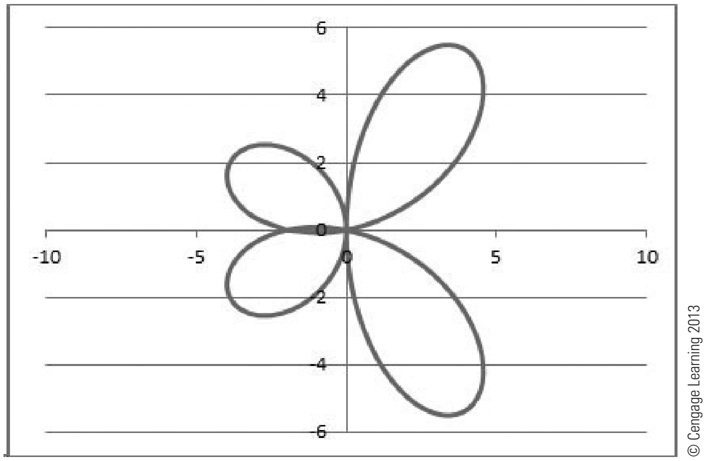


Figure 10.67

EXERCISE SET 10.7

In Exercises 1–12, plot the points with the given polar coordinates.

1. $(2, \frac{\pi}{4})$
2. $(4, 60°)$
3. $(3, 90°)$
4. $(5, \pi)$
5. $(-4, \frac{2\pi}{3})$
6. $(-6, 30°)$
7. $(-4, 0°)$
8. $(-2, \frac{5\pi}{6})$
9. $(5, -30°)$
10. $(3, -\frac{3\pi}{4})$
11. $(-7, \frac{11\pi}{6})$
12. $(-2, -270°)$

In Exercises 13–24, convert each polar coordinate to its equivalent rectangular coordinates.

13. $(4, \frac{\pi}{3})$
14. $(5, 75°)$
15. $(2, 135°)$
16. $(3, \frac{3\pi}{2})$
17. $(-6, 20°)$
18. $(-3, \frac{5\pi}{3})$
19. $(-2, 4.3)$
20. $(-5, 255°)$
21. $(3, -170°)$
22. $(4, -\frac{\pi}{8})$
23. $(-6, -2.5)$
24. $(-3, -195°)$

In Exercises 25–36, convert each rectangular coordinate to an equivalent polar coordinate with $r > 0$.

25. $(4, 4)$
26. $(3, 6)$
27. $(4, 3)$
28. $(5, 12)$
29. $(-20, 21)$
30. $(-12, 5)$
31. $(-3, 4)$
32. $(9, -5)$
33. $(-7, -10)$
34. $(-8, -3)$
35. $(2, 9)$
36. $(-6, 1)$

In Exercises 37–58, graph the polar equations.

37. $r = 4$
38. $r = -6$
39. $r = 3 \sin \theta$
40. $r = -5 \cos \theta$
41. $r = 4 - 4 \sin \theta$ (cardioid)
42. $r = 1 + 3 \cos \theta$ (limaçon)
43. $r = 3 \cos 5\theta$ (five-petaled rose)
44. $r = 3 \sin 2\theta$ (four-petaled rose)
45. $r = 5 \sec \theta$
46. $r = -7 \csc \theta$
47. $r = \theta$ (Let θ get larger than 4π.)
48. $r = 3^{\theta}$ (spiral)
49. $r = 4 + 4 \sec \theta$
50. $r = \frac{1}{\theta}, \theta > 0$
51. $r = 3 + \cos \theta$
52. $r^2 = 16 \sin 2\theta$
53. $r = 2 + 5 \sin \theta$

54. $r = 1 + 4 \sec \theta$

55. $r = \dfrac{6}{3 + 2 \sin \theta}$

56. $r = \dfrac{6}{1 + 3 \cos \theta}$

57. $r = \dfrac{3}{2 + 2 \cos \theta}$

58. $r = \dfrac{4 \sec \theta}{2 \sec \theta - 1}$

Solve Exercises 59–62.

59. ***Space technology*** The polar equation for a certain satellite's orbit is given by

$$r = \frac{6000}{1.4} - 0.25 \cos \theta$$

Sketch the graph of this satellite.

60. ***Architecture*** In the design of a geodesic dome, an architect uses the equation:

$$r^2 = \frac{E^2}{E^2 \cos^2 \theta + \sin^2 \theta}$$

where E is a constant.

(a) What is the rectangular form of this equation?

(b) Let $E = 0.5$ and graph this function on your calculator.

61. ***Electronics*** The field strength, r, in μV/m, of a broadcast station 1 mi from the antenna is given by

$$r = 2 + 5 \cos 2\theta + \sin \theta$$

Use your calculator to sketch the graph of this antenna pattern.

62. ***Robotics*** The path that an industrial robot must follow in a certain production process is described by

$$r = 1 - 3 \sin \theta$$

(a) Convert this equation into rectangular form.

(b) Use your calculator to sketch the graph of this robot's path.

[IN YOUR WORDS]

63. Distinguish among polar equations, rectangular equations, and parametric equations.

64. Describe situations in which polar equations are more helpful than rectangular equations for graphing equations.

CHAPTER 10 REVIEW

IMPORTANT TERMS AND CONCEPTS

Amplitude
Cosine curve
Displacement
Frequency
Harmonic motion
Horizontal displacement
Lissajous figures
Parametric equations
Period
Periodic motion
Phase shift
Phasors
Polar coordinates
Polar equations
Sine curves
Sinusoidal curve
Sinusoidal regression
Vertical displacement

REVIEW EXERCISES

In Exercises 1–10, find the period, amplitude, frequency, and displacement of the given functions and graph one cycle of each.

1. $y = 8\cos 4x$

2. $y = 3\sin 2x$

3. $y = 2\tan 3x$

4. $y = 3\sin\left(2x + \frac{\pi}{2}\right)$

5. $y = \frac{1}{2}\cos\left(3x - \frac{\pi}{3}\right)$

6. $y = \dfrac{1}{3}\sec\left(2x - \frac{\pi}{6}\right)$

7. $y = -4\cot\left(x + \frac{\pi}{4}\right)$

8. $y = -\frac{1}{2}\csc\left(\dfrac{x}{3} - \dfrac{\pi}{5}\right)$

9. $y = 2\sin\left(\frac{2}{3}x + \frac{\pi}{6}\right)$

10. $y = \frac{-1}{4}\cos\left(-\dfrac{x}{2} + \dfrac{2\pi}{3}\right)$

In Exercises 11–20, sketch the curves of the given equations.

11. $y = \sin 2x + \cos 3x$

12. $y = 4\sin\dfrac{x}{4} - \cos 2x$

13. $y = \cos 2x + \sin 3x$

14. $y = x + \sin\left(3x + \frac{\pi}{4}\right)$

15. $x = 3t, y = 5t - 2$

16. $x = t + 2, y = t^2 + t$

17. $x = 4\sin t, y = 3\cos t$

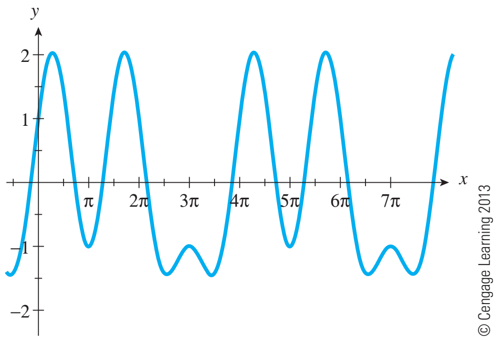
18. $x = \sin t, y = \cos 4t$

19. $r = \cos 7\theta$

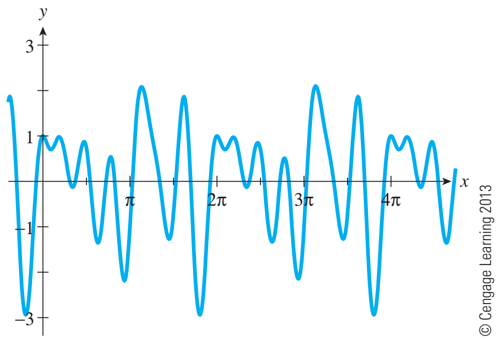
20. $r = 3 + 2\sin\theta$

In Exercises 21–22 estimate the (a) amplitude, (b) period, and (c) frequency for each of the given sinusoidal functions.

21.

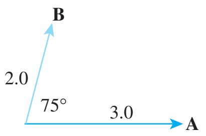


22.

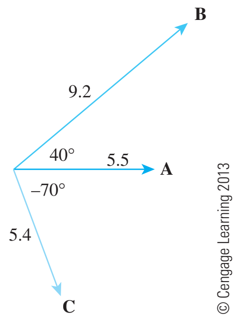


In Exercises 23–24, sketch the sine waves represented by the given phasor diagram.

23.



24.



Solve Exercises 25–29.

25. ***Mechanics*** A weight vibrates on a spring in simple harmonic motion. The amplitude is 85 mm and the frequency is 5 Hz. Write an equation for the weight's position y at any time t, if $y = 0$ when $t = 0$.

26. ***Oceanography*** A raft is bobbing on the water in simple harmonic motion according to the equation $y = 1.7 \sin 3.4t$, where y is the position of the raft in meters and t is the time in seconds. **(a)** What is the amplitude of the raft? **(b)** What is its period? **(c)** What is its frequency? **(d)** Sketch one complete cycle for this curve.

27. ***Electronics*** In an ac circuit containing only a constant capacitance, the current leads the voltage by $\frac{1}{4}$ cycle. If $V = V_{max} \sin 2\pi ft$ V, $I_{max} = 4.8$ A, and $f = 60$ Hz, write the equation for the current. Sketch one cycle of the curve.

28. ***Electronics*** In a resistance-inductance circuit, $I = I_{max} \sin(2\pi ft + \phi)$. If $I_{max} = 1.5$ A, $f = 360$ Hz, and $\phi = -31°$, sketch I vs. t for one cycle.

29. ***Meteorology*** Table 10.6 gives the observed tide levels at Bar Harbor, Maine, for the 24-h period of April 15, 2002. Note that an observation was not made every hour. Here hour 0 refers to midnight April 14/15.

TABLE 10.6 Hourly Tide Levels, Bar Harbor, Maine

Hour	0	1	2	3	4	5	6	7	8	9	10	11	12
Tide (ft)	10.61	11.15	10.34	8.35	5.62	2.91		0.05	0.74	2.06	4.80	7.43	9.50

Hour	13	14	15	16	17	18	19	20	21	22	23	24
Tide (ft)	10.52	10.36	8.91	6.61	4.10	2.11	1.17	1.16		4.87	7.57	9.67

(a) Plot these tide levels.

(b) Use the data to determine the amplitude and vertical displacement.

(c) Approximate the period and horizontal displacement.

(d) Based on your answers in **(b)** and **(c)** (do not use technology), write a model for the data.

(e) Graph the data and your answer to **(d)**. Is your answer a good fit? If not, why not?

(f) Use your calculator to determine the curve of best fit to this data. How close is this function to your answer in **(e)**?

CHAPTER 10 TEST

1. For the following trigonometric functions, find the period, amplitude, frequency, and displacement of the given functions.

(a) $y = -3 \sin 5x$

(b) $y = 2.4 \cos\left(3x - \frac{\pi}{4}\right)$

(c) $y = 1.5 \tan\left(2x + \frac{\pi}{5}\right)$

2. Sketch each of the given curves.

(a) $y = -3 \sin 5x$

(b) $y = \frac{1}{2} \sec\left(x + \frac{\pi}{2}\right)$

(c) $y = \sin 2x + 2 \cos 3x$

3. Sketch $x = 2t$, $y = t^2 + 1$.

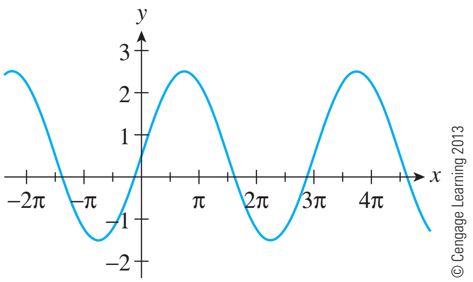
4. Sketch $r = 1 + 2 \cos \theta$.

5. Convert $(2, 55°)$ to rectangular coordinates.

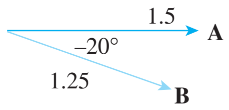
6. Convert $(5, -12)$ to polar coordinates.

7. In an ac electric circuit, write the equation of the sine curve for the current when $I_{max} = 5.7$ A and $f = 60$ Hz.

8. Estimate the **(a)** amplitude, **(b)** period, and **(c)** frequency for the given sinusoidal functions.



9. Sketch the sine waves represented by the following phasor diagram.



10. Table 10.7 gives the average daily temperature each month for Tampa, Florida. Use your calculator or spreadsheet to determine the curve of best fit to this data.

TABLE 10.7 Average Temperature per Month, Tampa, Florida

Month	Jan	Feb	Mar	Apr	May	Jun	Jul	Aug	Sep	Oct	Nov	Dec
Daily Minimum Temp (°F)	59.9	61.5	66.6	71.3	77.4	81.3	82.4	82.4	80.9	74.8	67.5	62.2

11 EXPONENTS AND RADICALS

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When an object is placed between two light sources so the illuminance is the same from each source, an equation involving radicals is used to determine the intensity of each light. In Section 11.4, we will learn how to determine this intensity.

Exponents and radicals were introduced in Chapter 1. In this chapter, we will explore the topic further and examine the close relationship between exponents and radicals.

OBJECTIVES

AFTER COMPLETING THIS CHAPTER, YOU WILL BE ABLE TO:

- Use basic rules for exponents to simplify exponential functions.
- Convert between radical form and exponential form.
- Use basic rules for radicals to simplify radical expressions.
- Add, subtract, multiply, and divide radical expressions.
- Solve equations involving radicals using both algebraic and graphical methods.

11.1 FRACTIONAL EXPONENTS

The basic rules for exponents were given in Section 1.3. They are repeated here to refresh your memory.

BASIC RULES FOR EXPONENTS

Rule 1: $b^m b^n = b^{m+n}$

Rule 2: $(b^m)^n = b^{mn}$

Rule 3: $(ab)^n = a^n b^n$

Rule 4: $\left(\dfrac{a}{b}\right)^m = \dfrac{a^m}{b^m}, b \neq 0$

Rule 5: $\dfrac{b^m}{b^n} = b^{m-n}, b \neq 0$

Rule 6: $b^0 = 1, b \neq 0$

Rule 7: $b^{-n} = \dfrac{1}{b^n}, b \neq 0$

When we studied these rules in Section 1.3, m and n had to be integers. In Section 1.5, we examined the meaning of $b^{1/n}$ and found that it meant $\sqrt[n]{b}$. This makes sense, since $(b^{1/n})^n = b^{n/n} = b$. This gives us a new rule:

RULE 8 FOR EXPONENTS

$$b^{1/n} = \sqrt[n]{b}$$

EXAMPLE 11.1

Evaluate (a) $27^{1/3}$, (b) $16^{1/2}$, and (c) $625^{1/4}$.

SOLUTIONS

(a) $27^{1/3} = \sqrt[3]{27} = 3$

(b) $16^{1/2} = \sqrt{16} = 4$

(c) $625^{1/4} = \sqrt[4]{625} = 5$

If we combine Rules 2 and 8, we obtain the following new rule.

RULE 9 FOR EXPONENTS

$$b^{m/n} = (b^m)^{1/n} = \sqrt[n]{b^m}$$

EXAMPLE 11.2

Evaluate (a) $8^{2/3}$, (b) $64^{5/2}$, (c) $81^{5/4}$, and (d) $9^{-3/2}$.

SOLUTIONS

(a) $8^{2/3} = \sqrt[3]{8^2} = \sqrt[3]{64} = 4$

(b) $64^{5/2} = \sqrt{64^5} = \sqrt{1,073,741,824} = 32,768$

(c) $81^{5/4} = \sqrt[4]{81^5} = \sqrt[4]{3,486,784,401} = 243$

(d) $9^{-3/2} = \dfrac{1}{\sqrt{9^3}} = \dfrac{1}{\sqrt{729}} = \dfrac{1}{27}$

It is often easier to find the root before raising the number to a power. Thus, we could rewrite Rule 9.

RULE 9 FOR EXPONENTS

$$b^{m/n} = (b^{1/n})^m = (\sqrt[n]{b})^m$$

We will rework the problems in Example 11.2 using this variation of Rule 9.

EXAMPLE 11.3

Evaluate (a) $8^{2/3}$, (b) $64^{5/2}$, (c) $81^{5/4}$, (d) $9^{-3/2}$, (e) $(-8)^{2/3}$, (f) $\left(\frac{8}{27}\right)^{-1/3}$, (g) $\left(\frac{1}{16}\right)^{-5/4}$, and (h) $\left(-\frac{64}{125}\right)^{-2/3}$.

SOLUTIONS

(a) $8^{2/3} = (\sqrt[3]{8})^2 = 2^2 = 4$

(b) $64^{5/2} = (\sqrt{64})^5 = 8^5 = 32,768$

(c) $81^{5/4} = (\sqrt[4]{81})^5 = 3^5 = 243$

(d) $9^{-3/2} = \dfrac{1}{(\sqrt{9})^3} = \dfrac{1}{3^3} = \dfrac{1}{27}$

(e) $(-8)^{2/3} = \left(\sqrt[3]{-8}\right)^2 = (-2)^2 = 4$

(f) $\left(\dfrac{8}{27}\right)^{-1/3} = \left(\dfrac{27}{8}\right)^{1/3} = \sqrt[3]{\dfrac{27}{8}} = \dfrac{3}{2}$

(g) $\left(\dfrac{1}{16}\right)^{-5/4} = 16^{5/4} = \left(\sqrt[4]{16}\right)^5 = 2^5 = 32$

(h) $\left(-\dfrac{64}{125}\right)^{-2/3} = \left(-\dfrac{125}{64}\right)^{2/3} = \left(\sqrt[3]{-\dfrac{125}{64}}\right)^2 = \left(-\dfrac{5}{4}\right)^2 = \dfrac{25}{16}$

All of the rules for integer exponents apply to fractional exponents. This is demonstrated in the following example.

EXAMPLE 11.4

(a) $x^{1/2}x^{2/3} = x^{1/2\,+\,2/3} = x^{7/6}$

(b) $(y^{2/3})^{4/5} = y^{2/3\cdot4/5} = y^{8/15}$

(c) $\left(\dfrac{x}{y}\right)^{4/5} = \dfrac{x^{4/5}}{y^{4/5}}$

(d) $\dfrac{x^{3/4}}{x^{5/3}} = x^{3/4\,-\,5/3} = x^{-11/12} = \dfrac{1}{x^{11/12}}$

(e) $\dfrac{4x^2}{x^{2/3}} = 4x^{2\,-\,2/3} = 4x^{4/3}$

(f) $(8y^3)^{5/3} = 8^{5/3}\,(y^3)^{5/3} = 32y^5$

(g) $\left(\dfrac{x^{15}}{y^9}\right)^{-1/3} = \left(\dfrac{y^9}{x^{15}}\right)^{1/3} = \dfrac{y^3}{x^5}$

(h) $\left(\dfrac{x^{1/3}y^{2/5}}{z^{3/5}}\right)^{15} = \dfrac{(x^{1/3})^{15}(y^{2/5})^{15}}{(z^{3/5})^{15}} = \dfrac{x^5y^6}{z^9}$

(i) $x^{2/5}\,x^{-1/3} = x^{2/5-1/3} = x^{1/15}$

EVALUATING EXPONENTS USING A CALCULATOR

You can use the x^2 key to calculate the square, or second power, of a number and the ^ key for the value of any power.

EXAMPLE 11.5

Evaluate $4.7^{3.42}$.

SOLUTION

PRESS	DISPLAY
4.7 ^ 3.42 ENTER	198.8724729

Some calculators have a $\sqrt[x]{\ }$ key that you access through a menu. To use this key to evaluate $\sqrt[3]{8}$ you press 3 $\sqrt[x]{\ }$ 8 ENTER. Check the manual for your calculator to see if it has such an option and which menu you use to access it.

EXAMPLE 11.6

Evaluate $\sqrt[7]{943.2}$.

SOLUTION

PRESS	DISPLAY
943.2 ^ (1 ÷ 7) ENTER	2.660378313
or 7 $\sqrt[x]{\ }$ 943.2 ENTER	2.660378313

HINT To evaluate $b^{m/n}$, where n is odd and b is a negative number, your calculator must "think" of this as $(b^m)^{1/n}$ or, equivalently as $(b^{1/n})^m$.

EXAMPLE 11.7

Evaluate $12^{5/3}$.

SOLUTION

PRESS	DISPLAY
12 ^ (5 ÷ 3) ENTER	62.89779351
or 3 $\sqrt[x]{\ }$ 12 ^ 5 ENTER	62.89779351

EXAMPLE 11.8

Evaluate $(-8)^{5/3}$.

SOLUTION You have to treat $(-8)^{5/3}$ as $[(-8)^5]^{1/3}$ or as $[(-8)^{1/3}]^5$.

PRESS	DISPLAY
((–) 8) ^ (1 ÷ 3) ^ 5 ENTER	-32
or ((–) 8) ^ 5 ^ (1 ÷ 3) ENTER	-32
or (3 $\sqrt[x]{\ }$ (–) 8) ^ 5 ENTER	-32

NOTE Remember, you cannot take an even root of a negative number. That is, you cannot take the square root of a negative number, you cannot take the fourth root of a negative number, you cannot take the sixth root of a negative number, and so on.

APPLICATION ENVIRONMENTAL SCIENCE

EXAMPLE 11.9

One study of a lake found that the light intensity was reduced 15% through a depth of 25 cm. The formula $I = (0.85)^{d/25}$ gives the approximate fraction of surface light intensity at a depth d, in centimeters. Find I at a depth of 60 cm.

SOLUTION Since $d = 60$ cm, we want to evaluate $I = (0.85)^{60/25}$. Using a calculator, we enter

0.85 ^ (60 ÷ 25) ENTER

and obtain 0.677026115965. So, the light intensity at 60 cm is about 67.7% of the surface light intensity.

EXERCISE SET 11.1

In Exercises 1–20, evaluate the given expression without the use of a calculator.

1. $25^{1/2}$
2. $27^{1/3}$
3. $64^{1/3}$
4. $125^{1/3}$
5. $25^{-1/2}$
6. $81^{-1/4}$
7. $32^{-1/5}$
8. $64^{-1/3}$
9. $27^{2/3}$
10. $81^{3/4}$
11. $125^{2/3}$
12. $32^{3/5}$
13. $16^{-3/4}$
14. $(-8)^{2/3}$
15. $(-8)^{-1/3}$
16. $(-27)^{-2/3}$
17. $\left(\frac{1}{8}\right)^{1/3}$
18. $\left(\frac{1}{25}\right)^{3/2}$
19. $\left(\frac{1}{16}\right)^{-5/4}$
20. $\left(\frac{-1}{27}\right)^{4/3}$

In Exercises 21–70, express each of the given expressions in the simplest form containing only positive exponents.

21. $3^2 \cdot 3^5$
22. $5^9 5^8$
23. $7^6 7^{-2}$
24. $11^9 11^{-6}$
25. $x^4 x^6$
26. $y^7 y^9$
27. $y^6 y^{-4}$
28. $x^8 x^{-2}$
29. $(9^5)^2$
30. $(11^8)^{-5}$
31. $(x^7)^3$
32. $(p^9)^{-5}$
33. $(xy)^5$
34. $(yt)^3$
35. $(ab)^{-5}$
36. $(xyz)^{-9}$
37. $\frac{x^{10}}{x^2}$
38. $\frac{p^9}{p^3}$
39. $\frac{x^2}{x^8}$
40. $\frac{a^3}{a^{12}}$
41. $x^{1/2}x^{3/2}$
42. $a^{1/3}a^{4/3}$
43. $r^{3/4}r$
44. $a^2a^{2/3}$
45. $a^{1/2}a^{1/3}$
46. $b^{2/3}b^{1/4}$
47. $d^{2/3}d^{-1/4}$
48. $x^{3/5}y^{-2/3}$
49. $\frac{a^2b^5}{a^5b^2}$
50. $\frac{x^3y^2}{x^7y}$
51. $\frac{r^5s^2t}{tr^3s^5}$
52. $\frac{a^2bc^3}{(abc)^3}$
53. $\frac{(xy^2z)^4}{x^4(yz^2)^2}$
54. $\frac{m^6n^7}{(m^2n)^3}$
55. $\left(\frac{a}{b^2}\right)^3\left(\frac{a}{b^3}\right)^2$
56. $\left(\frac{x^2}{y}\right)^4\left(\frac{x}{y^2}\right)^2$
57. $\frac{(xy^2b^3)^{1/2}}{(x^{1/4}b^4y)^2}$
58. $\frac{(x^{1/3}y^3)^3}{(y^{10}x^5)^{1/5}}$
59. $\left(\frac{2x}{p^2}\right)^{-2}\left(\frac{p}{4}\right)^{-1}$
60. $\left(\frac{5a^2}{6b}\right)^{-2}\left(\frac{6}{a}\right)^{-4}$
61. $\frac{(x^2y^{-1}z)^{-2}}{(xy)^{-4}}$
62. $\left\{\left[(2x^2)^3\right]^{-4}\right\}^{-1}$
63. $\frac{(27x^6y^3)^{-1/3}}{(16x^4y^{12})^{-1/4}}$
64. $\frac{(64x^6y^{12})^{-5/6}}{(9x^4y^2)^{-3/2}}$
65. $\left(\frac{x^2}{y^3}\right)^{-1/5}\left(\frac{y^2}{x^5}\right)^{1/3}$
66. $\left(\frac{a^2x^3}{b^5y}\right)^{-4/3}\left(\frac{axy^{-2}}{b}\right)^{5/3}$
67. $\left(\frac{9x^3}{t^5}\right)^{-1/2}\left(\frac{8t^2}{x^5}\right)^{-1/3}$
68. $\left(\frac{125a^6}{8b^3}\right)^{-2/3}\left(\frac{8b}{5a^2}\right)^{-1/2}$
69. $\frac{(9x^4y^{-6})^{-3/2}}{(8x^{-6}y^3)^{-2/3}}$
70. $\frac{(81x^5y^{-8}z)^{-3/4}}{(64xy^{-3}z^2)^{-5/3}}$

Use a calculator or computer to determine the value of each of the numbers in Exercises 71–80.

71. $8.3^{2/3}$

72. $7.3^{2/5}$

73. $92.47^{5/7}$

74. $81.94^{3/5}$

75. $(-81.52)^{2/7}$

76. $(-78.64)^{1/3}$

77. $432.61^{1/4}$

78. $(-537.15)^{2/3}$

79. $(-32.35)^{-3/7}$

80. $(-0.1439074804)^{-3/5}$

Solve Exercises 81–86.

81. ***Physics*** The distance in meters traveled by a falling body starting from rest is $9.8t^2$, where t is the time, in seconds, the object has been falling. In 4.75 s, the distance fallen will be $9.8(4.75)^2$ m. Evaluate this quantity.

82. The volume of a sphere is $\frac{4}{3}\pi r^3$. If the radius r of a sphere is 19.25 in., what is the volume?

83. ***Physics*** When 5 m^3 of helium at a temperature of 315 K and a pressure of 15 N/m^2 is adiabatically compressed to 0.5 m^3, its new pressure p and temperature T are given by

$$p = (5)^5\left(\frac{15}{0.5}\right)^{5/3}$$

$$\text{and } T = 315\left(\frac{15}{0.5}\right)^{5/3}$$

Evaluate p and T.

84. ***Nuclear technology*** Radium has a half-life of approximately 1,600 years. It decays according to the formula $q(t) = q_0 2^{-t/1,600}$, where q_0 is the original quantity of radium and $q(t)$ is the amount left at time t. In this problem, t is given in years. If you begin with 75 mg of pure radium, how much will remain after 2,000 years?

85. ***Aeronautical engineering*** The efficiency of a turbojet engine is given by the expression

$$E = 1 - \left(\frac{p_1}{p_2}\right)^{\alpha/(1-\alpha)}$$

where $\dfrac{p_1}{p_2}$ is the compression ratio. Simplify the expression for $\alpha = 1.4$.

86. ***Finance*** The periodic payment, R, for a debt, A, at the annual interest rate, r, with n payments per year for t years is given by

$$R = A\left[\frac{\frac{r}{n}}{1 - \left(1 + \frac{r}{n}\right)^{-nt}}\right]$$

A person borrowed \$15,000 for an automobile at an annual rate of 9%. How much is each monthly payment if she borrowed the money for 5 years?

[IN YOUR WORDS]

87. Describe how to use your calculator to evaluate $(1.44)^{3/2}$.

88. **(a)** Use your calculator to evaluate $\left(-\frac{1}{8}\right)^{-2/3}$, $-\dfrac{1}{8^{-2/3}}$, $-8^{2/3}$, $(-8)^{2/3}$, and $(-8)^{-2/3}$.

(b) Explain why some answers in (a) are the same and some are different.

11.2 LAWS OF RADICALS

In Section 1.3, we introduced the concept of roots. We used them in Section 11.1 for the discussion of fractional exponents. The more general name for roots is **radicals**. Fractional exponents can be used for any operation that requires radicals.

LAWS OF RADICALS

A *radical* is any number of the form $\sqrt[n]{b}$. The number under the radical b is called the **radicand**. The number indicating the root n is called the *order* or **index**.

In Section 1.3, we introduced four basic rules or laws of radicals. These rules are listed in the following box.

BASIC RULES FOR RADICALS

Rule 1: $\sqrt[n]{ab} = \sqrt[n]{a}\sqrt[n]{b}$

Rule 2: $\sqrt[n]{\dfrac{a}{b}} = \dfrac{\sqrt[n]{a}}{\sqrt[n]{b}}, b \neq 0$

Rule 3: $(\sqrt[n]{b})^n = b^{n/n} = b$

Rule 4: $\sqrt[n]{b} = b^{1/n}$

We assume in each of these rules that if n is even, neither a nor b is a negative real number.

EXAMPLE 11.10

Simplify (a) $\sqrt[6]{27}$, (b) $\sqrt[5]{96}$, (c) $\sqrt{x^6y^8}$, (d) $\sqrt[3]{\dfrac{125y^6}{x^9}}$, and (e) $\sqrt[3]{54x^8}$.

SOLUTIONS

(a) $\sqrt[6]{27} = \sqrt[6]{3^3} = 3^{3/6} = 3^{1/2} = \sqrt{3}$

(b) $\sqrt[5]{96} = \sqrt[5]{32 \cdot 3} = \sqrt[5]{32}\sqrt[5]{3} = \sqrt[5]{2^5}\sqrt[5]{3} = 2\sqrt[5]{3}$

(c) $\sqrt{x^6y^8} = (x^6y^8)^{1/2} = x^{6/2}y^{8/2} = x^3y^4$

(d) $\sqrt[3]{\dfrac{125y^6}{x^9}} = \sqrt[3]{\dfrac{5^3y^6}{x^9}} = \dfrac{5^{3/3}y^{6/3}}{x^{9/3}} = \dfrac{5y^2}{x^3}$

EXAMPLE 11.10 (Cont.)

$$\text{(e)}\quad \sqrt[3]{54x^8} = \sqrt[3]{2 \cdot 3^3x^6x^2}$$
$$= 2^{1/3}3^{3/3}x^{6/3}x^{2/3}$$
$$= 2^{1/3} \cdot 3 \cdot x^2x^{2/3}$$
$$= 3x^22^{1/3}x^{2/3}$$
$$= 3x^2(2x^2)^{1/3}$$
$$= 3x^2\sqrt[3]{2x^2}$$

Notice that several examples started with as many factors as possible with nth roots that could be easily found. Thus, in Example 11.10(b), we used $\sqrt[n]{ab} = \sqrt[n]{a}\sqrt[n]{b}$, when we wrote $\sqrt[5]{96} = \sqrt[5]{32} \cdot \sqrt[5]{3}$.

HINT In general, we try to express a radical so that the exponent of any factor in the radicand is less than the index of the radical. Thus, in Example 11.10(e), we wrote $\sqrt[3]{x^8}$ as $x^2\sqrt[3]{x^2}$.

RATIONALIZING DENOMINATORS

When a radicand is a fraction, a variation of Rule 2, $\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}$, is used to eliminate the radical in the denominator. This technique is called **rationalizing the denominator**.

RATIONALIZING THE DENOMINATOR

To rationalize a denominator of the form $\sqrt[n]{x^r}$, multiply the denominator by another radical with the same radicand and the same index, $\sqrt[n]{x^s}$, where $r + s$ is a multiple of n.

The process of rationalizing the denominator makes the denominator a perfect power of x and eliminates the radical in the denominator.

NOTE Remember that whenever you multiply the denominator by something other than 1, you must also multiply the numerator by the same quantity.

EXAMPLE 11.11

Rationalize the denominators of **(a)** $\sqrt{\frac{3}{5}}$, **(b)** $\frac{1}{\sqrt[3]{2}}$, and **(c)** $\sqrt[5]{\frac{3}{8x^2}}$.

SOLUTIONS

(a) Rewrite $\sqrt{\frac{3}{5}}$ as $\frac{\sqrt{3}}{\sqrt{5}}$.

Multiply both the numerator and denominator by $\sqrt{5}$ because $5 \cdot 5 = 5^2$.

$$\sqrt{\frac{3}{5}} = \frac{\sqrt{3}}{\sqrt{5}} = \frac{\sqrt{3}}{\sqrt{5}} \cdot \frac{\sqrt{5}}{\sqrt{5}}$$
$$= \frac{\sqrt{3 \cdot 5}}{\sqrt{5 \cdot 5}} = \frac{\sqrt{15}}{\sqrt{5^2}} = \frac{\sqrt{15}}{5}$$

(b) Multiply both the numerator and denominator by $\sqrt[3]{2^2}$ because $2 \cdot 2^2 = 2^3$, a perfect cube.

$$\frac{1}{\sqrt[3]{2}} = \frac{1}{\sqrt[3]{2}} \cdot \frac{\sqrt[3]{2^2}}{\sqrt[3]{2^2}}$$
$$= \frac{\sqrt[3]{2^2}}{\sqrt[3]{2^3}} = \frac{\sqrt[3]{4}}{2}$$

(c) Rewrite $\sqrt[5]{\dfrac{3}{8x^2}}$ as $\dfrac{\sqrt[5]{3}}{\sqrt[5]{8x^2}}$.

$\sqrt[5]{8x^2} = \sqrt[5]{2^3x^2}$, so we will multiply both the numerator and denominator by $\sqrt[5]{2^2x^3}$ because $(2^3x^2)(2^2x^3) = 2^5x^5$.

$$\sqrt[5]{\frac{3}{8x^2}} = \frac{\sqrt[5]{3}}{\sqrt[5]{8x^2}} = \frac{\sqrt[5]{3}}{\sqrt[5]{8x^2}} \cdot \frac{\sqrt[5]{2^2x^3}}{\sqrt[5]{2^2x^3}}$$
$$= \frac{\sqrt[5]{3 \cdot 2^2x^3}}{\sqrt[5]{8x^2 2^2x^3}} = \frac{\sqrt[5]{3 \cdot 2^2x^3}}{\sqrt[5]{2^5x^5}}$$
$$= \frac{\sqrt[5]{12x^3}}{2x}$$

Rationalizing the denominator was originally developed as a way to help computations, but the increased use of calculators and computers reduces its importance in calculations. Rationalizing the denominator, however, is often a useful way to write numbers.

Another helpful rule is used with radicals.

RULE 5 FOR RADICALS

$$\sqrt[m]{\sqrt[n]{b}} = \sqrt[mn]{b}$$

EXAMPLE 11.12

(a) $\sqrt[3]{\sqrt{27}} = \sqrt[6]{27}$, (b) $\sqrt[4]{\sqrt[5]{914}} = \sqrt[20]{914}$, (c) $\sqrt[5]{\sqrt[3]{-82}} = \sqrt[15]{-82}$, and (d) $\sqrt[4]{\sqrt[3]{x}} = \sqrt[12]{x}$

Sometimes it is possible to reduce the index of a radical. For example,

$$\sqrt[6]{y^2} = y^{2/6} = y^{1/3} = \sqrt[3]{y}$$

Here the index was reduced from 6 to 3. Another example can be seen from Example 11.12(a).

$$\sqrt[3]{\sqrt{27}} = \sqrt[3]{\sqrt{3^3}} = \sqrt[6]{3^3} = 3^{3/6} = 3^{1/2} = \sqrt{3}$$

The last version, $\sqrt{3}$, is a simpler version with which to work.

EXAMPLE 11.13

Reduce the index of **(a)** $\sqrt[8]{16}$, **(b)** $\sqrt[6]{16x^2}$, and **(c)** $\sqrt[12]{27x^6y^3}$.

SOLUTIONS

(a) $\sqrt[8]{16} = \sqrt[8]{2^4} = 2^{4/8} = 2^{1/2} = \sqrt{2}$

(b) $\sqrt[6]{16x^2} = \sqrt[6]{2^4x^2} = 2^{4/6}x^{2/6} = 2^{2/3}x^{1/3} = \sqrt[3]{4x}$

(c) $\sqrt[12]{27x^6y^3} = \sqrt[12]{3^3x^6y^3} = 3^{3/12}x^{6/12}y^{3/12} = 3^{1/4}x^{2/4}y^{1/4} = \sqrt[4]{3x^2y}$

Notice that, in the last example, we could have factored the x^2 out of the radical using Rule 1 and written $\sqrt[4]{3x^2y} = \sqrt[4]{3y}\sqrt[4]{x^2} = \sqrt[4]{3y}\sqrt{x}$.

SIMPLIFYING RADICALS

Simplifying a radical makes it easier to work with. The following three steps are used to simplify a radical.

STEPS FOR SIMPLIFYING RADICALS

A radical is simplified when all of the following steps are finished.

Step 1. All possible factors have been removed from the radicand.

Step 2. All denominators are rationalized.

Step 3. The index has been reduced as much as possible.

EXAMPLE 11.14

Simplify the following: (a) $\sqrt{\dfrac{x^3}{y}}$, (b) $\sqrt[4]{x^5y^{11}}$, (c) $\sqrt[5]{x^3y^{-7}}$, and (d) $\sqrt[3]{\dfrac{x}{y^2} + \dfrac{5y^2}{x}}$.

SOLUTIONS

(a) $\sqrt{\dfrac{x^3}{y}} = \sqrt{\dfrac{x^3}{y}}\sqrt{\dfrac{y}{y}} = \dfrac{\sqrt{x^3y}}{\sqrt{y^2}} = \dfrac{\sqrt{x^3y}}{|y|} = \dfrac{|x|\sqrt{xy}}{|y|}$

(b) $\sqrt[4]{x^5y^{11}} = \sqrt[4]{x^4xy^8y^3} = \sqrt[4]{x^4y^8} \cdot \sqrt[4]{xy^3} = |x|y^2\sqrt[4]{xy^3}$

(c) $\sqrt[5]{x^3y^{-7}} = \sqrt[5]{\dfrac{x^3}{y^7}} = \sqrt[5]{\dfrac{x^3}{y^7}} \cdot \sqrt[5]{\dfrac{y^3}{y^3}} = \dfrac{\sqrt[5]{x^3y^3}}{\sqrt[5]{y^{10}}} = \dfrac{\sqrt[5]{x^3y^3}}{y^2}$

(d) $$\sqrt[3]{\frac{x}{y^2}+\frac{5y^2}{x}} = \sqrt[3]{\frac{x\cdot x}{y^2x}+\frac{5y^2y^2}{xy^2}} = \sqrt[3]{\frac{x^2+5y^4}{xy^2}}$$
$$= \sqrt[3]{\frac{(x^2+5y^4)x^2y}{(xy^2)x^2y}} = \frac{\sqrt[3]{x^4y+5x^2y^5}}{\sqrt[3]{x^3y^3}}$$
$$= \frac{\sqrt[3]{x^4y+5x^2y^5}}{xy}$$

In Examples 11.14(a) and (b) we needed to use the absolute value symbol to denote the principal square root of an even root. Remember, $\sqrt{(-4)^2} = |-4| = 4$. Thus, $\sqrt{x^2} = |x|$ and $\sqrt[4]{x^{12}} = |x^3|$ because we do not know if x is positive or negative.

You do not need to use absolute values when finding odd roots. Thus, just as $\sqrt[3]{-64} = \sqrt[3]{-4^3} = -4$, we can write $\sqrt[3]{x^3} = x$.

NOTE $\sqrt{a^2b} = |a|\sqrt{b}$ but $\sqrt{a^2+b} \neq |a|\sqrt{1+b}$ and also $\sqrt{a^2+b} \neq |a| + \sqrt{b}$.

Notice also that in Example 11.14(d) we had to find a common denominator before we could add the fractions. After the fractions were added, we were able to rationalize the denominator.

APPLICATION GENERAL TECHNOLOGY

EXAMPLE 11.15

The frequency of oscillation f of a simple pendulum is given by

$$f = \frac{1}{2\pi}\sqrt{\frac{g}{L}}$$

where $g \approx 9.8$ m/s^2 is the acceleration due to gravity in the metric system and L is the length of the pendulum in meters. (a) Express f in simplest form, when $L = 0.35$ m. (b) Evaluate f to the nearest hundredth.

SOLUTIONS

(a) Given $g \approx 9.8$ m/s^2 and $L = 0.35$ m,

$$f = \frac{1}{2\pi}\sqrt{\frac{g}{L}}$$
$$= \frac{1}{2\pi}\sqrt{\frac{9.8}{0.35}}$$
$$= \frac{1}{2\pi}\sqrt{28}$$
$$= \frac{\sqrt{7}}{\pi}$$

EXAMPLE 11.15 (Cont.)

(b) Evaluating $\frac{\sqrt{7}}{\pi}$ with a calculator, we press 2nd √ 7 ÷ 2nd π ENTER and obtain 0.842168798696. Thus, the pendulum oscillates about once every 0.84 sec.

EXERCISE SET 11.2

Use the rules for radicals to express each of Exercises 1–60 in simplest radical form.

1. $\sqrt[3]{16}$
2. $\sqrt[3]{81}$
3. $\sqrt{45}$
4. $\sqrt[3]{40}$
5. $\sqrt[3]{y^{12}}$
6. $\sqrt[4]{p^8}$
7. $\sqrt[5]{a^7}$
8. $\sqrt[7]{b^{10}}$
9. $\sqrt{x^2y^7}$
10. $\sqrt[3]{x^5y^3}$
11. $\sqrt[4]{a^5b^3}$
12. $\sqrt[5]{p^{12}y^8}$
13. $\sqrt[3]{8x^4}$
14. $\sqrt[4]{81y^9}$
15. $\sqrt{27x^3y}$
16. $\sqrt[3]{32a^5b^2}$
17. $\sqrt[3]{-8}$
18. $\sqrt[5]{-243}$
19. $\sqrt[3]{a^2b^4}\sqrt[3]{ab^5}$
20. $\sqrt[5]{x^3y^2z^4}\sqrt[5]{x^2y^8z}$
21. $\sqrt[4]{p^3q^2r^6}\sqrt[4]{pq^6r}$
22. $\sqrt[6]{m^3n^2e^7}\sqrt[6]{m^2n^4e^5}$
23. $\sqrt[3]{\frac{8x^3}{27}}$
24. $\sqrt[4]{\frac{81y^8}{16}}$
25. $\sqrt[5]{\frac{x^5y^{10}}{z^5}}$
26. $\sqrt[3]{\frac{a^3b^9}{c^6}}$
27. $\sqrt[3]{\frac{16x^3y^2}{z^6}}$
28. $\sqrt{\frac{125a^5b^2}{c^4}}$
29. $\sqrt{\frac{64x^3y^4}{9z^4p^2}}$
30. $\sqrt[3]{\frac{8a^5b^3}{27r^6s^9}}$
31. $\sqrt{\frac{16}{3}}$
32. $\sqrt{\frac{4}{5}}$
33. $\sqrt[3]{\frac{27}{4}}$
34. $\sqrt[3]{\frac{16}{25}}$
35. $\sqrt{\frac{25}{2x}}$
36. $\sqrt[3]{\frac{8}{5y^2}}$
37. $\sqrt[4]{\frac{81}{32z^2}}$
38. $\sqrt[3]{\frac{-2}{25r^2}}$
39. $\sqrt[3]{\frac{16x^2y}{x^5}}$
40. $\sqrt[4]{\frac{25a^3b^5}{a^7b}}$
41. $\sqrt[3]{\frac{-8x^3yz}{27b^2z^4}}$
42. $\sqrt[4]{\frac{25a^2b^3}{16c^3b^6}}$
43. $\sqrt{4 \times 10^4}$
44. $\sqrt{9 \times 10^6}$
45. $\sqrt{25 \times 10^3}$
46. $\sqrt{16 \times 10^7}$
47. $\sqrt{4 \times 10^7}$
48. $\sqrt{9 \times 10^9}$
49. $\sqrt[3]{1.25 \times 10^{10}}$
50. $\sqrt[5]{3.2 \times 10^{14}}$
51. $\sqrt{\frac{x}{y} + \frac{y}{x}}$
52. $\sqrt{\frac{a}{b} - \frac{b}{a}}$
53. $\sqrt{a^2 + 2ab + b^2}$
54. $\sqrt{x^2 - 2xy + y^2}$
55. $\sqrt{\frac{1}{a^2} + \frac{1}{b}}$
56. $\sqrt{\frac{x}{y^2} + \frac{y}{x^2}}$
57. $\sqrt[6]{\sqrt[3]{27x^2}}$
58. $\sqrt[4]{\sqrt[6]{125x^3y^5}}$
59. $\sqrt[3]{\sqrt[7]{-624.2x^{15}y^{10}z}}$
60. $\sqrt{\sqrt[4]{9x^8y^3}}$

Solve Exercises 61–66.

61. ***Music*** Many musical instruments contain strings, which vibrate to produce music. The frequency of vibration f of a string of length L fixed at both ends and vibrating in its fundamental mode is given by

$$f = \frac{1}{2L}\sqrt{\frac{T}{\mu}}$$

where μ is the mass per unit length and T is the tension in the string. Rationalize the right-hand side of this equation.

62. ***Music*** In the equation in Exercise 61, what happens to the frequency when the tension is quadrupled?

63. ***Electricity*** The impedance Z of a certain circuit is given by the equation

$$Z = \frac{1}{\sqrt{\frac{1}{x^2} + \frac{1}{R^2}}}$$

Rationalize the denominator in order to simplify this equation.

64. ***Chemistry*** The distance between ion layers of a sodium chloride crystal is given by the expression

$$\sqrt[3]{\frac{M}{2\rho N}}$$

where M is the molecular weight, N is Avogadro's number, and ρ is the density. Express this in simplest form.

65. ***Mechanical engineering*** The formula

$$k = 1 + \sqrt{1 + \frac{2h}{m}}$$

is used to calculate the impact factor for dynamic loading. Rewrite the right-hand side in simplest radical form. Make sure you rationalize the denominator.

66. ***Civil engineering*** The frequency f, in Hz, of an object attached to the end of a cantilever beam of length l is given by

$$f = \frac{1}{2\pi}\sqrt{\frac{3EI}{ml^3}}$$

Express the formula in simplest radical form.

[IN YOUR WORDS]

67. Without looking in the text, explain the meaning of each of the following terms: **(a)** radical, **(b)** radicand, and **(c)** index.

68. Explain how to rationalize the denominator of the form $\sqrt[n]{x^r}$.

11.3 BASIC OPERATIONS WITH RADICALS

In this section, we will study the basic operations of addition, subtraction, multiplication, and division of radicals. Adding and subtracting radicals are similar to adding and subtracting algebraic expressions. However, there are two cases to consider when multiplying or dividing radicals. The first case is when the radicals have the same index; the second case is when they have different indices. Each of these cases will be considered in this section.

ADDITION AND SUBTRACTION OF RADICALS

When we learned how to add and subtract algebraic expressions, we found that only like terms can be added or subtracted. Addition and subtraction of radicals are very similar.

NOTE Radicals can only be added or subtracted if the radicands are identical and the indices are the same.

Trying to add or subtract two radicals such as $\sqrt{2}$ and $\sqrt{5}$ is similar to adding and subtracting x and y. Example 11.16 shows the similarity between combining radicals and combining algebraic expressions.

EXAMPLE 11.16

Combining Radicals	Combining Algebraic Expressions
(a) $\sqrt{2}+\sqrt{5}+\sqrt{2}$ $=2\sqrt{2}+\sqrt{5}$	$x+y+x=2x+y$, where $x=\sqrt{2}$ and $y=\sqrt{5}$
(b) $2\sqrt{3}+4\sqrt{7}+6\sqrt{3}$ $=8\sqrt{3}+4\sqrt{7}$	$2a+4b+6a=8a+4b$, where $a=\sqrt{3}$ and $b=\sqrt{7}$
(c) $3\sqrt[3]{6}-7\sqrt[3]{6}=-4\sqrt[3]{6}$	$3p-7p=-4p$, where $p=\sqrt[3]{6}$
(d) $\frac{5\sqrt{3}}{2}-\frac{\sqrt{3}}{2}=\frac{4\sqrt{3}}{2}=2\sqrt{3}$	$\frac{5x}{2}-\frac{x}{2}=\frac{4x}{2}=2x$, where $x=\sqrt{3}$

In each case, only the similar radicals are combined. Radicals that are not similar remain as separate terms and the addition or subtraction is only indicated. Sometimes it is possible to combine radicals that do not appear to be similar by first simplifying the radicals. For example $\sqrt{8}$ and $\sqrt{2}$ can be combined, since $\sqrt{8}=\sqrt{4}\sqrt{2}=2\sqrt{2}$.

EXAMPLE 11.17

Simplify and combine similar radicals.

(a) $\sqrt{2}+\sqrt{8}=\sqrt{2}+2\sqrt{2}=3\sqrt{2}$

(b) $3\sqrt{8}+5\sqrt{18}=6\sqrt{2}+15\sqrt{2}=21\sqrt{2}$

Note that $\sqrt{18}=\sqrt{9}\sqrt{2}=3\sqrt{2}$ and $5\sqrt{18}=5\sqrt{9}\sqrt{2}=5\cdot 3\sqrt{2}$ $=15\sqrt{2}$.

(c) $\sqrt{98x}+\sqrt{32x}=7\sqrt{2x}+4\sqrt{2x}=11\sqrt{2x}$

(d) $\sqrt{20x^3}+\sqrt{8x^2}-\sqrt{45x}=2x\sqrt{5x}+2x\sqrt{2}-3\sqrt{5x}$
$=(2x-3)\sqrt{5x}+2x\sqrt{2}$

Note that the difference $2x\sqrt{5x}-3\sqrt{5x}$ can only be simplified as $(2x-3)\sqrt{5x}$. We factored $\sqrt{5x}$ out of each term.

(e)
$$\sqrt{\frac{5}{3}}+\frac{\sqrt{3}}{\sqrt{125}}=\frac{\sqrt{5}}{\sqrt{3}}+\frac{\sqrt{3}}{5\sqrt{5}}=\frac{\sqrt{5}\sqrt{3}}{\sqrt{3}\sqrt{3}}+\frac{\sqrt{3}\sqrt{5}}{5\sqrt{5}\sqrt{5}}$$
$$=\frac{\sqrt{15}}{3}+\frac{\sqrt{15}}{25}=\frac{25\sqrt{15}}{75}+\frac{3\sqrt{15}}{75}$$
$$=\frac{28\sqrt{15}}{75}$$

Here, we rationalized the denominators and then added the fractions by finding the common denominator. We could have rationalized the second fraction by multiplying by $\sqrt{125}$, but it was easier to first simplify the denominator to $5\sqrt{5}$ then multiply by $\sqrt{5}$.

MULTIPLYING RADICALS WITH THE SAME INDEX

Multiplying and dividing radicals that have the same index use two of the rules we have discussed.

$$\sqrt[n]{a}\sqrt[n]{b} = \sqrt[n]{ab}$$

and
$$\frac{\sqrt[n]{a}}{\sqrt[n]{b}} = \sqrt[n]{\frac{a}{b}},\ b \neq 0$$

EXAMPLE 11.18

Multiply the following radical expressions: **(a)** $\sqrt{2}\sqrt{5}$, **(b)** $\sqrt[3]{5}\sqrt[3]{10}$, **(c)** $\sqrt{xy}\sqrt{2x}$, **(d)** $\sqrt[3]{\frac{3}{2}}\sqrt[3]{\frac{5x}{4}}$, **(e)** $\sqrt{5x}(\sqrt{5x} + \sqrt{10x^3})$, and **(f)** $(\sqrt{x} + \sqrt{y})(\sqrt{x} - \sqrt{y})$.

SOLUTIONS

(a) $$\sqrt{2}\sqrt{5} = \sqrt{2 \cdot 5} = \sqrt{10}$$

(b) $$\sqrt[3]{5}\sqrt[3]{10} = \sqrt[3]{5 \cdot 10} = \sqrt[3]{50}$$

(c) $$\sqrt{xy}\sqrt{2x} = \sqrt{(xy)(2x)} = \sqrt{2x^2y} = x\sqrt{2y}$$

(d) $$\sqrt[3]{\frac{3}{2}}\sqrt[3]{\frac{5x}{4}} = \sqrt[3]{\left(\frac{3}{2}\right)\left(\frac{5x}{4}\right)} = \sqrt[3]{\frac{15x}{8}} = \frac{\sqrt[3]{15x}}{2}$$

(e) $$\begin{aligned}\sqrt{5x}\left(\sqrt{5x} + \sqrt{10x^3}\right) &= \sqrt{5x}\sqrt{5x} + \sqrt{5x}\sqrt{10x^3}\\ &= \sqrt{25x^2} + \sqrt{50x^4}\\ &= 5x + 5x^2\sqrt{2}\end{aligned}$$

(f) $(\sqrt{x} + \sqrt{y})(\sqrt{x} - \sqrt{y}) = (\sqrt{x})^2 - (\sqrt{y})^2 = x - y$ using the special product for the difference of two squares.

MULTIPLYING RADICALS WITH DIFFERENT INDICES

Radicals with different indices can be multiplied if they are rewritten so that they have the same index. The easiest way to do this is with fractional exponents.

EXAMPLE 11.19

Multiply the following: (a) $\sqrt[3]{2}\sqrt{7}$, (b) $\sqrt[4]{5x^2}\sqrt[3]{2x}$, and (c) $\sqrt[3]{4ab^2}\sqrt[5]{16a^4b^2}$.

SOLUTIONS In each of these solutions, we first rewrite the radical expressions using fractional exponents. Then, if they need to be added as in Examples 11.19(b) and (c), we next find the common denominator.

(a) $$\begin{aligned}\sqrt[3]{2}\sqrt{7} = 2^{1/3}7^{1/2} = 2^{2/6}7^{3/6} &= (2^27^3)^{1/6}\\ &= \sqrt[6]{2^27^3}\\ &= \sqrt[6]{1{,}372}\end{aligned}$$

EXAMPLE 11.19 (Cont.)

(b) $\sqrt[4]{5x^2}\sqrt[3]{2x} = (5x^2)^{1/4}(2x)^{1/3} = (5x^2)^{3/12}(2x)^{4/12}$

$$= \sqrt[12]{(5x^2)^3}\,\sqrt[12]{(2x)^4}$$
$$= \sqrt[12]{(5x^2)^3(2x)^4}$$
$$= \sqrt[12]{5^3x^62^4x^4}$$
$$= \sqrt[12]{2{,}000x^{10}}$$

(c) $\sqrt[3]{4ab^2}\sqrt[5]{16a^4b^2} = (4ab^2)^{1/3}(16a^4b^2)^{1/5}$

$$= (4ab^2)^{5/15}(16a^4b^2)^{3/15}$$
$$= \sqrt[15]{(4ab^2)^5}\,\sqrt[15]{(16a^4b^2)^3}$$
$$= \sqrt[15]{(2^2ab^2)^5}\sqrt[15]{(2^4a^4b^2)^3}$$
$$= \sqrt[15]{2^{10}a^5b^{10}}\sqrt[15]{2^{12}a^{12}b^6}$$
$$= \sqrt[15]{2^{22}a^{17}b^{16}}$$
$$= 2ab\sqrt[15]{2^7a^2b}$$
$$= 2ab\sqrt[15]{128a^2b}$$

DIVISION OF RADICALS

Division of radicals with the same index is done using Rule 2 for radicals. The result is usually simplified by rationalizing the denominator.

EXAMPLE 11.20

Divide each of the following: (a) $\dfrac{\sqrt{10}}{\sqrt{3}}$, (b) $\dfrac{\sqrt[3]{4x}}{\sqrt[3]{2x}}$, (c) $\dfrac{\sqrt[3]{x^2y}}{\sqrt[3]{z}}$, and (d) $\dfrac{\sqrt{3xy}}{\sqrt{7x^5y}}$.

SOLUTIONS

(a) $\dfrac{\sqrt{10}}{\sqrt{3}} = \dfrac{\sqrt{10}}{\sqrt{3}}\cdot\dfrac{\sqrt{3}}{\sqrt{3}} = \dfrac{\sqrt{30}}{3}$

(b) $\dfrac{\sqrt[3]{4x}}{\sqrt[3]{2x}} = \sqrt[3]{\dfrac{4x}{2x}} = \sqrt[3]{2}$

(c) $\dfrac{\sqrt[3]{x^2y}}{\sqrt[3]{z}} = \dfrac{\sqrt[3]{x^2y}\sqrt[3]{z^2}}{\sqrt[3]{z}\,\sqrt[3]{z^2}} = \dfrac{\sqrt[3]{x^2yz^2}}{z}$

(d) $\dfrac{\sqrt{3xy}}{\sqrt{7x^5y}} = \dfrac{\sqrt{3xy}}{x^2\sqrt{7xy}} = \dfrac{1}{x^2}\cdot\sqrt{\dfrac{3xy}{7xy}} = \dfrac{1}{x^2}\sqrt{\dfrac{3}{7}}$

We now rationalize the denominator.

$$= \frac{1}{x^2}\sqrt{\frac{3}{7}}\sqrt{\frac{7}{7}} = \frac{\sqrt{21}}{7x^2}$$

CONJUGATE OF A RADICAL EXPRESSION

If a, b, $\sqrt{m}$, and $\sqrt{n}$ are real numbers, then radical expressions of the form $a\sqrt{m} + b\sqrt{n}$ and $a\sqrt{m} - b\sqrt{n}$ are **conjugates** of each other.

Sometimes the denominator of a fraction is the sum of the difference of two square roots. Examples of this are $\sqrt{2} + \sqrt{5}$ and $\sqrt{x} - \sqrt{y}$. In this case, the numerator and denominator can be multiplied by the conjugate in order to make the denominator the difference of two squares. For example, if the denominator is $\sqrt{2} + \sqrt{5}$, then multiply the numerator and denominator by $\sqrt{2} - \sqrt{5}$. If the denominator is $\sqrt{x} - \sqrt{y}$, then multiply both numerator and denominator by $\sqrt{x} + \sqrt{y}$. Notice that in each case the denominator is then in the form $a^2 - b^2$, where a or b is a radical.

EXAMPLE 11.21

Rationalize each of these denominators: (a) $\dfrac{1}{\sqrt{2} + \sqrt{5}}$, (b) $\dfrac{a}{\sqrt{x} - \sqrt{y}}$, and (c) $\dfrac{\sqrt{x + y}}{1 + \sqrt{x + y}}$.

SOLUTIONS (a) The conjugate of $\sqrt{2} + \sqrt{5}$ is $\sqrt{2} - \sqrt{5}$.

$$\begin{aligned}\frac{1}{\sqrt{2} + \sqrt{5}} &= \frac{1}{\sqrt{2} + \sqrt{5}} \cdot \frac{\sqrt{2} - \sqrt{5}}{\sqrt{2} - \sqrt{5}} \\ &= \frac{\sqrt{2} - \sqrt{5}}{(\sqrt{2})^2 - (\sqrt{5})^2} \\ &= \frac{\sqrt{2} - \sqrt{5}}{2 - 5} \\ &= \frac{\sqrt{2} - \sqrt{5}}{-3} = \frac{\sqrt{5} - \sqrt{2}}{3}\end{aligned}$$

(b) The conjugate of $\sqrt{x} - \sqrt{y}$ is $\sqrt{x} + \sqrt{y}$.

$$\begin{aligned}\frac{a}{\sqrt{x} - \sqrt{y}} &= \frac{a}{\sqrt{x} - \sqrt{y}} \cdot \frac{\sqrt{x} + \sqrt{y}}{\sqrt{x} + \sqrt{y}} \\ &= \frac{a(\sqrt{x} + \sqrt{y})}{(\sqrt{x})^2 - (\sqrt{y})^2} \\ &= \frac{a\sqrt{x} + a\sqrt{y}}{x - y}, x \neq y\end{aligned}$$

EXAMPLE 11.21 (Cont.)

(c) The conjugate of $1 + \sqrt{x+y}$ is $1 - \sqrt{x+y}$.

$$\frac{\sqrt{x+y}}{1+\sqrt{x+y}} = \frac{\sqrt{x+y}}{1+\sqrt{x+y}} \cdot \frac{1-\sqrt{x+y}}{1-\sqrt{x+y}}$$
$$= \frac{\sqrt{x+y}(1-\sqrt{x+y})}{(1)^2 - (\sqrt{x+y})^2}$$
$$= \frac{\sqrt{x+y} - (\sqrt{x+y})^2}{1-(x+y)}$$
$$= \frac{\sqrt{x+y} - x - y}{1-x-y}, x + y \neq 1$$

Finally, to find the quotient of two radicals with different indices, we use fractional exponents in the same manner as when we multiplied radicals with different indices.

EXAMPLE 11.22

Find the following quotients: (a) $\frac{\sqrt[3]{15}}{\sqrt{15}}$ and (b) $\frac{\sqrt{2x}}{\sqrt[4]{8x^3}}$.

SOLUTIONS

(a) $$\frac{\sqrt[3]{15}}{\sqrt{15}} = \frac{(15)^{1/3}}{(15)^{1/2}} = \frac{(15)^{2/6}}{(15)^{3/6}} = \frac{\sqrt[6]{15^2}}{\sqrt[6]{15^3}} = \frac{\sqrt[6]{15^2}}{\sqrt[6]{15^3}} \cdot \frac{\sqrt[6]{15^3}}{\sqrt[6]{15^3}}$$
$$= \frac{\sqrt[6]{15^5}}{15}$$

(b) $$\frac{\sqrt{2x}}{\sqrt[4]{8x^3}} = \frac{(2x)^{1/2}}{(8x^3)^{1/4}} = \frac{(2x)^{2/4}}{(8x^3)^{1/4}} = \frac{\sqrt[4]{(2x)^2}}{\sqrt[4]{8x^3}} = \frac{\sqrt[4]{4x^2}\sqrt[4]{2x}}{\sqrt[4]{8x^3}\sqrt[4]{2x}}$$
$$= \frac{\sqrt[4]{8x^3}}{2x}$$

EXERCISE SET 11.3

In Exercises 1–60, perform the indicated operations and express the answers in simplest form.

1. $2\sqrt{3} + 5\sqrt{3}$
2. $5\sqrt{6} - 3\sqrt{6}$
3. $\sqrt[3]{9} + 4\sqrt[3]{9}$
4. $\sqrt[4]{8} + 3\sqrt[4]{8}$
5. $2\sqrt{3} + 4\sqrt{2} + 6\sqrt{3}$
6. $5\sqrt{3} - 6\sqrt{5} - 9\sqrt{3}$
7. $\sqrt{5} + \sqrt{20}$
8. $\sqrt{8} + \sqrt{2}$
9. $\sqrt{7} - \sqrt{28}$
10. $\sqrt{8} - \sqrt{32}$
11. $\sqrt{60} - \sqrt{\frac{5}{3}}$
12. $\sqrt{84} + \sqrt{\frac{3}{7}}$
13. $\sqrt{\frac{1}{2}} - \sqrt{\frac{9}{2}}$
14. $\sqrt{\frac{4}{3}} - \sqrt{\frac{25}{3}}$
15. $\sqrt{x^3y} + 2x\sqrt{xy}$
16. $\sqrt{a^5b^3} - 3ab\sqrt{a^3b}$
17. $\sqrt[3]{24p^2q^4} + \sqrt[3]{3p^8q}$
18. $\sqrt[4]{16a^2b} - \sqrt[4]{81a^6b}$
19. $\sqrt{\frac{x}{y^3}} - \sqrt{\frac{y}{x^3}}$
20. $\sqrt{\frac{a^3}{b^3}} + \sqrt{\frac{b}{a^5}}$
21. $a\sqrt{\frac{b}{3a}} + b\sqrt{\frac{a}{3b}}$
22. $x\sqrt{\frac{y}{5x}} - y\sqrt{\frac{x}{5y}}$
23. $\sqrt{5}\sqrt{8}$
24. $\sqrt[5]{-7}\sqrt[5]{11}$
25. $\sqrt{3x}\sqrt{5x}$

26. $\sqrt[3]{7x^2}\sqrt[3]{3x}$
27. $(\sqrt{4x})^3$
28. $(\sqrt[3]{2x^2y})^4$
29. $\sqrt{\frac{5}{8}}\sqrt{\frac{9}{10}}$
30. $\sqrt{\frac{7}{6}}\sqrt{\frac{12}{3}}$
31. $\sqrt{2}(\sqrt{x} + \sqrt{2})$
32. $\sqrt{3}(\sqrt{12} - \sqrt{y})$
33. $(\sqrt{x} + \sqrt{y})^2$
34. $(\sqrt{a} + 3\sqrt{b})^2$
35. $\sqrt[3]{\frac{5}{2}}\sqrt[3]{\frac{2}{7}}$
36. $\sqrt{\frac{ab}{5c}}\sqrt{\frac{abc}{5}}$
37. $(\sqrt{a} + \sqrt{b})(\sqrt{a} - \sqrt{b})$
38. $(\sqrt{x} - 2\sqrt{y})(\sqrt{x} + 5\sqrt{y})$
39. $\sqrt[3]{x}\sqrt{x}$
40. $\sqrt{5x}\sqrt[3]{2x^2}$
41. $\sqrt[3]{49x^2}\sqrt[4]{3x}$
42. $\sqrt[3]{9y}\sqrt[5]{-8x^2}$
43. $\dfrac{\sqrt{32}}{\sqrt{2}}$
44. $\dfrac{\sqrt[3]{a^2}}{\sqrt[3]{4a}}$
45. $\dfrac{\sqrt[3]{4b^2}}{\sqrt[3]{16b}}$
46. $\dfrac{\sqrt{5a^3b}}{\sqrt{15ab^3}}$
47. $\dfrac{\sqrt{3x^2}}{\sqrt[3]{9x}}$
48. $\dfrac{\sqrt[3]{15x}}{\sqrt{5x}}$
49. $\dfrac{\sqrt[3]{16a^2b}}{\sqrt[4]{2ab^3}}$
50. $\dfrac{\sqrt[5]{75a^3x^2}}{\sqrt[3]{-15ax^2}}$
51. $\dfrac{\sqrt[3]{4}}{\sqrt{2}}$
52. $\dfrac{\sqrt[5]{-8}}{\sqrt[3]{4}}$
53. $\dfrac{1}{x + \sqrt{5}}$
54. $\dfrac{1}{x + \sqrt{3}}$
55. $\dfrac{\sqrt{5} - \sqrt{3}}{\sqrt{5} + \sqrt{3}}$
56. $\dfrac{\sqrt{5} + \sqrt{7}}{\sqrt{7} - \sqrt{5}}$
57. $\dfrac{\sqrt{x} + 1}{\sqrt{x} - 1} + \dfrac{\sqrt{x} - 1}{\sqrt{x} + 1}$
58. $\dfrac{\sqrt{x} + 3}{\sqrt{x} - 3} - \dfrac{\sqrt{x} - 3}{\sqrt{x} + 3}$
59. $\dfrac{\sqrt{x + y}}{\sqrt{x - y} - \sqrt{x}}$
60. $\dfrac{\sqrt{1 + y}}{\sqrt{1 - y} + \sqrt{y}}$

Solve Exercises 61–66.

61. Use the quadratic equation to find the roots of $ax^2 + bx + c = 0$. What is the sum of the two roots?

62. ***Oceanography*** The velocity v of a small water wave is given by

$$v = \sqrt{\frac{\pi}{4\lambda d}} + \sqrt{\frac{4\pi}{\lambda d}}$$

Simplify and combine this equation.

63. ***Electronics*** The equivalent resistance R of two resistors, R_1 and R_2, connected in parallel is expressed

$$R = \frac{R_1R_2}{R_1 + R_2}$$

In a given circuit, $R_1 = x^{3/2}$ and $R_2 = \sqrt{x}$. **(a)** Express R in terms of x. Make sure you simplify the answer. **(b)** If $x = 20\ \Omega$, what is R?

64. ***Sound*** The theory of waves in wires uses the equation

$$\frac{\sqrt{d_1} - \sqrt{d_2}}{\sqrt{d_1} + \sqrt{d_2}}$$

Simplify this expression.

65. What is the product of the two roots of the quadratic equation $ax^2 + bx + c = 0$?

66. ***Civil Engineering*** The potential energy in a truss is given by

$$C = \frac{3}{8\sqrt[3]{2}} \cdot \frac{KA}{\sqrt[3]{L}}$$

Simplify this expression.

[IN YOUR WORDS]

67. Explain how to add two radicals. What precautions do you need to take regarding the radicands and the indices?

68. Describe how to multiply two radicals
(a) With the same index.
(b) With different indices.

11.4 EQUATIONS WITH RADICALS

You should now know the laws of radicals. However, the laws of radicals are rather dry without some technical and algebraic applications. Because technical situations can involve solving equations that contain radicals, we now learn how to use radicals in equations and to solve those equations.

RADICAL EQUATIONS; EXTRANEOUS ROOTS

Working with radicals often means that we must work with radical equations. An equation in which the variable occurs under a radical sign or has a fractional exponent is called a **radical equation.** In order to solve radical equations, we need to eliminate the radicals or the fractional exponents. This is done by raising both sides of the equation to some power that will eliminate the radical. The equation that results may not be equivalent to the original equation. In fact, the new equation may have more roots than the original one. The roots of the new equation that are not roots of the original equation are called **extraneous roots.** Check all solutions to make sure they are actual roots. Reject any extraneous roots.

EXAMPLE 11.23

Solve the radical equation $\sqrt{x+4} - 9 = 0$.

SOLUTION This radical equation has only one radical term. So, isolate this radical term on one side of the equation.

$$\begin{aligned} \sqrt{x+4} &= 9 \\ (\sqrt{x+4})^2 &= 9^2 && \text{Square both sides.} \\ x + 4 &= 81 \\ x &= 77 \end{aligned}$$

Substituting 77 in the original equation, we get $\sqrt{77+4} - 9 = \sqrt{81} - 9 = 9 - 9 = 0$. The answer checks, so 77 is the solution.

EXAMPLE 11.24

Solve $\sqrt{x+5} - \sqrt{x-3} = 10$.

SOLUTION This equation contains two radicals. Rewrite the equation with one radical on each side of the equals sign.

$$\begin{aligned} \sqrt{x+5} &= \sqrt{x-3} + 10 \\ (\sqrt{x+5})^2 &= (\sqrt{x-3} + 10)^2 && \text{Square both sides.} \\ x + 5 &= (\sqrt{x-3})^2 + 20\sqrt{x-3} + 10^2 && \text{"FOIL"} \\ x + 5 &= x - 3 + 20\sqrt{x-3} + 100 \\ x + 5 &= x + 97 + 20\sqrt{x-3} \end{aligned}$$

This is now an equation with one radical. Isolate the term with the radical so it is on one side of the equation, and use the method of squaring both sides.

$$
\begin{aligned}
-92 &= 20\sqrt{x-3} \\
(-92)^2 &= \left(20\sqrt{x-3}\right)^2 \\
8{,}464 &= 400(x-3) \\
8{,}464 &= 400x - 1{,}200 \\
9{,}664 &= 400x \\
x &= \frac{9{,}664}{400} = \frac{604}{25} = 24.16
\end{aligned}
$$

Replacing $x = 24.16$ in the left-hand side of the original equation gives $\sqrt{29.16} - \sqrt{21.16} = 0.8$. This does not equal the right-hand side of the original equation, 10. Since we have not made any mistakes, we can only conclude that $x = 24.16$ is an extraneous root and that there is no solution to this problem.

If we had not wanted to show how you can get extraneous roots when you square both sides, we could have solved the previous example with less work. Notice the equation $-92 = 20\sqrt{x-3}$. Since $\sqrt{x-3}$ is never negative, the product of $20\sqrt{x-3}$ can never be negative. So, there is no solution to the equation $-92 = 20\sqrt{x-3}$. The next example also involves an extraneous root.

EXAMPLE 11.25

Solve $\sqrt{2x-3} - \sqrt{x+7} = 4$.

SOLUTION As in the last example, this equation contains two radicals, so we will first rewrite the problem with one radical on each side of the equal sign.

$$
\begin{aligned}
\sqrt{2x-3} &= \sqrt{x+7} + 4 \\
(\sqrt{2x-3})^2 &= (\sqrt{x+7} + 4)^2 && \text{Square both sides.} \\
(\sqrt{2x-3})^2 &= (\sqrt{x+7})^2 + 8\sqrt{x+7} + 4^2 \\
2x - 3 &= x + 7 + 8\sqrt{x+7} + 16 \\
x - 26 &= 8\sqrt{x+7} && \text{Simplify.} \\
(x-26)^2 &= (8\sqrt{x+7})^2 && \text{Square again.} \\
x^2 - 52x + 676 &= 64(x+7) \\
x^2 - 52x + 676 &= 64x + 448 \\
x^2 - 116x + 228 &= 0 \\
(x-2)(x-114) &= 0
\end{aligned}
$$

So, $x = 2$ or $x = 114$.

EXAMPLE 11.25 (Cont.)

Replacing 2 for x in the original equation, we get $\sqrt{2 \cdot 2 - 3} - \sqrt{2 + 7} = \sqrt{1} - \sqrt{9} = 1 - 3 = -2$. Since $-2 \neq 4$, the number 2 must be an extraneous root.

Next we replace 114 for x in the original equation, and we get $\sqrt{2 \cdot 114 - 3} - \sqrt{114 + 7} = \sqrt{225} - \sqrt{121} = 15 - 11 = 4$. This checks. The only solution to this problem is 114.

EXAMPLE 11.26

Solve $\sqrt{x + 3} = \sqrt[3]{x + 21}$.

SOLUTION This problem also involves two radicals, but the indices of the radicals are not the same. If these were written with fractional exponents, you would have $(x + 3)^{1/2} = (x + 21)^{1/3}$. Raise both sides of the equation to the least common denominator of these powers. In this problem, the LCD is 6, so we would have

$$\begin{aligned}
(\sqrt{x + 3})^6 &= (\sqrt[3]{x + 21})^6 \\
(x + 3)^3 &= (x + 21)^2 \\
x^3 + 9x^2 + 27x + 27 &= x^2 + 42x + 441 \\
x^3 + 8x^2 - 15x - 414 &= 0
\end{aligned}$$

As you will learn in Chapter 17, the only real number solution to this problem is $x = 6$. Check this in the original problem to see if it is a root.

EXAMPLE 11.27

Solve $\sqrt{(x + 1)^3} = 64$.

SOLUTION $\sqrt{(x + 1)^3}$ can be written as $(x + 1)^{3/2}$. Raising both sides to the $\frac{2}{3}$ power we get

$$\begin{aligned}
\left((x + 1)^{3/2}\right)^{2/3} &= 64^{2/3} \\
x + 1 &= 16 \\
x &= 15
\end{aligned}$$

Checking this in the original problem confirms that it is a root.

APPLICATION GENERAL TECHNOLOGY

EXAMPLE 11.28

At some point P between two light sources, the illuminance from each source is the same. If d_1 represents the distance from the light source with intensity of I_1 and d_2 is the distance from the source with intensity I_2, then

$$\frac{d_1}{d_2} = \frac{\sqrt{I_1}}{\sqrt{I_2}}$$

Find I_1 and I_2 in candelas (cd), when $d_1 = 15$ m, $d_2 = 6$ m, and $I_2 = 5I_1 - 11$.

SOLUTION Substituting the given values in the previous equation produces

$$\frac{d_1}{d_2} = \frac{\sqrt{I_1}}{\sqrt{I_2}}$$
$$\frac{15}{6} = \frac{\sqrt{I_1}}{\sqrt{5I_1 - 11}}$$

Squaring both sides, we obtain

$$\begin{aligned} \frac{225}{36} &= \frac{I_1}{5I_1 - 11} \\ \text{or } 225(5I_1 - 11) &= 36I_1 \\ 1125I_1 - 2475 &= 36I_1 \\ 1125I_1 - 36I_1 &= 2475 \\ 1089I_1 &= 2475 \\ I_1 &= \frac{2475}{1089} \approx 2.27 \end{aligned}$$

So, I_1 is approximately 2.27 cd.

Substituting the exact value $I_1 = \frac{2\,475}{1\,089}$ in the equation $I_2 = 5I_1 - 11$, we determine

$$I_2 = 5\left(\frac{2475}{1089}\right) - 11 = \left(\frac{12\,375}{1089}\right) - 11 \approx 0.36$$

The illuminance at I_2 is approximately 0.36 cd.

SOLVING RADICAL EQUATIONS WITH A CALCULATOR

As you might expect, you can use a graphing calculator to solve an equation with radicals. There are several procedures that you might follow. Example 11.29 shows one of them.

EXAMPLE 11.29

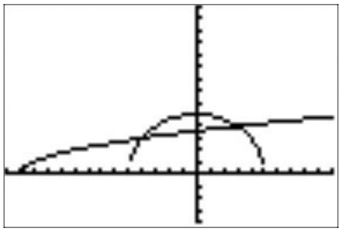

Figure 11.1a

Use a graphing calculator to solve $\sqrt{25 - x^2} = \sqrt{x + 13}$.

SOLUTION We will use the methods of Chapter 4 to solve this problem. Think of this equation as $y_1 = y_2$, where $y_1 = \sqrt{25 - x^2}$ and $y_2 = \sqrt{x + 13}$. We want to find the x-values that either make $y_1 = y_2$ or make $y_1 - y_2 = 0$. (You might want to look back at Example 4.28.)

We begin by entering the left-hand side of the equation as $Y1$ and the right-hand side of the equation as $Y2$. Although it is not necessary for the solution, we will graph these two functions so you can see, in Figure 11.1a, that they intersect in two different points.

Now, what we want to do is solve $Y3 = Y1 - Y2$. We "deselect" $Y1$ and $Y2$ so that they do not graph and enter $Y3 = Y1 - Y2$. The result is shown in Figure 11.1b and the graph of $Y3$ is shown in Figure 11.1c.

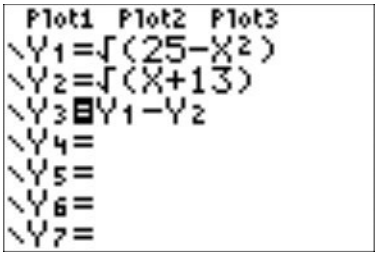

Figure 11.1b

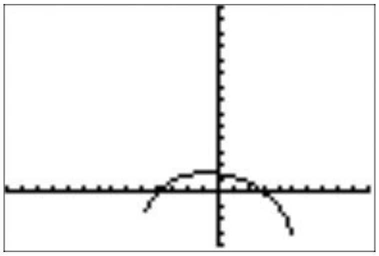

Figure 11.1c

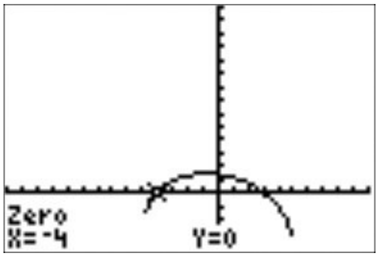

Figure 11.1d

EXAMPLE 11.29 (Cont.)

You need to find the points where $Y3 = 0$, that is, where $Y3$ intersects the x-axis. Here $Y3$ has the same x-values as the points where $Y1$ and $Y2$ intersect. Press **2nd** **TRACE** [CALC] **2** [2: zero].[1] Following the same procedures we used earlier we see, in Figure 11.4d, that the left-hand root is $x = -4$. Repeating the procedure will show that the right-hand root is at $x = 3$. Thus, the solutions are $x = -4$ and $x = 3$.

SOLVING RADICAL EQUATIONS WITH A SPREADSHEET

As you might expect, you can use a spreadsheet to solve an equation with radicals. There are several procedures that you might follow. Example 11.30 shows one of them.

EXAMPLE 11.30

Use a spreadsheet to solve $\sqrt{25 - x^2} = \sqrt{x + 13}$.

SOLUTION We will use the methods of Chapter 4 to solve this problem. Think of this equation as $y_1 = y_2$, where $y_1 = \sqrt{25 - x^2}$ and $y_2 = \sqrt{x + 13}$. We want to find the x-values that either make $y_1 = y_2$ or make $y_1 - y_2 = 0$. (You might want to look back at Example 4.28(b).)

Begin by constructing a table of values for x, y_1, and y_2. y_1 is entered as = (25−A5^2) ^ 0.5 and y_2 is entered as = (A5+13) ^ 0.5. The table of values will be constructed so that it is easy to change both the initial x and the increment for x as we search for the x-value that makes $y_1 = y_2$. See Figure 11.2a.

Although it is not necessary for the solution, we will graph these two functions so you can see, in Figure 11.2b, that they intersect in two different points. We'll review how to graph two functions at once. First, select source data that represents one function, the data in Column A and in Column B (see Figure 11.2c). Use this data to construct a graph. Once the graph is constructed, right click on the graph and select Select Data. To select new series click on **ADD** and select the data in Column A for the x-values and in Column C for the y-values.

Our attention now turns to the table of values. We know there are two points that make $y_1 = y_2$, so we attempt to find them by adjusting the initial x and the increment for x.

The first point is obvious from Figure 11.2c. At $x = 3$, both y_1 and y_2 equal four. Thus, $x = 3$ is one solution.

1 On some calculators this will be [2: root].

The next point seems to occur around $x = -4$ by looking at the graph. So, we change the increment to 0.1 to get a better look. Figure 11.2d shows that $x = -4$ is a solution since both y_1 and y_2 equal three.

Thus, the solutions are $x = -4$ and $x = 3$.

C5 | f_x =SQRT(A5+13)

	A	B	C	D
1	Initial x		-4.5	
2	Increment for x		0.75	
3				
4	x	Y1	Y2	
5	-4.5	2.17944947	2.91547595	
6	-3.75	3.30718914	3.04138127	

Figure 11.2a

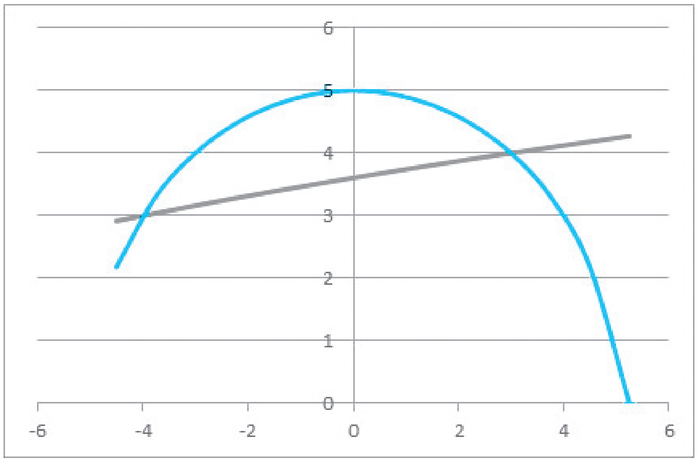

Figure 11.2b

	A	B	C
1	Initial x		-4.5
2	Increment for x		0.75
3			
4	x	Y1	Y2
5	-4.5	2.17944947	2.91547595
6	-3.75	3.30718914	3.04138127
7	-3	4	3.16227766
8	-2.25	4.46514277	3.27871926
9	-1.5	4.76969601	3.39116499
10	-0.75	4.94342998	3.5
11	0	5	3.60555128
12	0.75	4.94342998	3.70809924
13	1.5	4.76969601	3.80788655
14	2.25	4.46514277	3.90512484
15	3	4	4
16	3.75	3.30718914	4.09267639
17	4.5	2.17944947	4.18330013
18	5.25	#NUM!	4.27200187

Figure 11.2c

A10 | f_x =A9+$C$2

	A	B	C	D	E
1	Initial x		-4.5		
2	Increment for x		0.1		
3					
4	x	Y1	Y2		
5	-4.5	2.17944947	2.91547595		
6	-4.4	2.37486842	2.93257566		
7	-4.3	2.55147016	2.94957624		
8	-4.2	2.71293199	2.96647939		
9	-4.1	2.8618176	2.98328678		
10	-4	3	3		
11	-3.9	3.12889757	3.01662063		
12	-3.8	3.24961536	3.03315018		

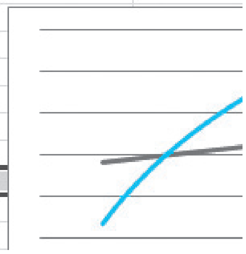

Figure 11.2d

EXERCISE SET 11.4

In Exercises 1–20, solve the radical equations. Check all roots.

1. $\sqrt{x+3} = 5$

2. $\sqrt{x-16} = 5$

3. $\sqrt{2x+4} - 7 = 0$

4. $\sqrt{3y-9} - 18 = 0$

5. $\sqrt{x^2+24} = x - 4$

6. $\sqrt{x^2-75} = x + 5$

7. $(y+12)^{1/3} = 3$

8. $(r+9)^{1/4} = 7$

9. $\left(\frac{x}{2}+1\right)^{1/2} = 3$

10. $\left(\frac{y}{3}+4\right)^{1/3} = \frac{5}{2}$

11. $(x-1)^{3/2} = 27$

12. $(x+1)^{3/4} = 8$

13. $\sqrt{x+1} = \sqrt{2x-1}$

14. $\sqrt{x^2-4} = \sqrt{2x-1}$

15. $\sqrt{x+1} + \sqrt{x-5} = 7$

16. $\sqrt{x-3} - \sqrt{x+2} = 19$

17. $\sqrt{x-1}+\sqrt{x+5}=2$

18. $\sqrt{x+3}-\sqrt{2x-4}=3$

19. $\sqrt{\frac{1}{x}}=\sqrt{\frac{4}{3x-1}}$

20. $\sqrt{\frac{2}{x-1}}=\sqrt{\frac{5}{x+1}}$

In Exercises 21–28, (a) sketch the graph of the following radical equations showing all solutions and (b) determine the solutions.

21. $\sqrt{x+7}=3$

22. $\sqrt{x^2-36}=-0.5x+3$

23. $\sqrt{x^2+16}=-x^2+4x+4$

24. $\sqrt[3]{x^2+4x}=x^2+6x+1$

25. $\sqrt[4]{x^2+3x-2}=x^2+6x-1$

26. $\sqrt[5]{18-5x}=5-\sqrt{x^2-16}$

27. $\sqrt{\frac{x}{x^2+1}}=\sqrt{\frac{x^2}{x+4}}$

28. $\sqrt{\frac{x^4}{x^2+1}}=\sqrt{\frac{x^2}{x+4}}$

Solve Exercises 29–34.

29. ***Civil engineering*** The maximum speed at which it is possible to negotiate a curve without slipping is given by

$$v=\sqrt{\mu_s gR}$$

Solve this for R.

30. ***Physics*** The velocity v of an object falling under the influence of gravity g is given by $v=\sqrt{v_0^2-2gh}$, where v_0 is the initial velocity and h is the height fallen. Solve this equation for h.

31. ***Lighting technology*** The illuminance from two light sources is the same at some point P between them. Suppose d_A is the distance to P from light A and d_B is the distance from P to light B. If the luminous intensity at A is I_A and at B is I_B, then we have the equation

$$\frac{d_A}{d_B}=\frac{\sqrt{I_A}}{\sqrt{I_B}}$$

If $d_A = 11$ m, $d_B = 20$ m, and $I_A = 3.5I_B - 15$, then find I_A and I_B.

32. ***Lighting technology*** The intensity of a light source can be determined with a photometer. The relationship between the luminous intensity I_x of an unknown source is found by comparing it with a standard source of known intensity I_y. The distances from each source are adjusted until a grease spot is equally illuminated by each source. If r_x is the distance to the unknown source and r_y is the distance to the standard, then the equation

$$\frac{\sqrt{I_x}}{r_x}=\frac{\sqrt{I_y}}{r_y}$$

can be used. **(a)** Solve this equation for I_x. **(b)** If $I_y = 30$ cd and $r_x = 2r_y + 3$, then solve for I_x.

33. ***Physics*** If an object is dropped, the time t it takes for it to fall s ft is given by $t=\frac{1}{4}\sqrt{s}$. An object is dropped from the top of a building. It takes 8.6 sec for the object to hit the ground. How tall is the building?

34. ***Electronics*** The impedance, Z, in an ac circuit can be determined by the formula $Z=\sqrt{R^2+X^2}$, where R is the resistance in Ω and X is the reactance in Ω. Find the reactance of a circuit with a resistance of 25 Ω and an impedance of 43 Ω.

[IN YOUR WORDS]

35. **(a)** What is an extraneous root?

(b) How can you tell if a root is extraneous?

36. Describe the general approach you should use to solve a radical equation.

CHAPTER 11 REVIEW

IMPORTANT TERMS AND CONCEPTS

Adding radicals
Conjugate
Dividing radicals
Extraneous roots
Fractional exponent
Index
Multiplying radicals
Radical
Radical equations
Radicand
Rationalizing denominators
Subtracting radicals

REVIEW EXERCISES

In Exercises 1–6, evaluate the given expression without the use of a calculator. Check your answers with a calculator.

1. $49^{1/2}$
2. $81^{1/4}$
3. $25^{3/2}$
4. $64^{2/3}$
5. $9^{-3/2}$
6. $125^{-4/3}$

In Exercises 7–18, express each of the given expressions in the simplest form containing only positive exponents.

7. $5^4 5^9$
8. $13^6 13^{-9}$
9. $(2^4)^8$
10. $(5^3)^6$
11. $(x^2y^3)^4$
12. $(y^6x)^4$
13. $\left(\dfrac{a^3}{ya^4}\right)^{-5}$
14. $\left(\dfrac{a^5}{ab^4}\right)^3$
15. $x^{1/2}x^{1/3}$
16. $y^{2/3}y^{3/4}$
17. $\dfrac{(xy^2a)^5}{x^4(ya^3)^2}$
18. $\dfrac{(ab^3c^2)^4}{(a^2bc)^3}$

Use the laws of radicals to express each of Exercises 19–42 in simplest radical form.

19. $\sqrt[3]{-27}$
20. $\sqrt[4]{81^3}$
21. $\sqrt{a^2b^5}$
22. $\sqrt[4]{a^3b^7}$
23. $\sqrt[3]{\dfrac{-8x^3y^6}{z}}$
24. $\sqrt[4]{\dfrac{32a^6b^8}{cd^2}}$
25. $\sqrt{\dfrac{a}{b^2} - \dfrac{b}{a^2}}$
26. $\sqrt{a^2 + 8a + 16}$
27. $2\sqrt{5} + 6\sqrt{5}$
28. $3\sqrt{7} - 6\sqrt{7}$
29. $3\sqrt{6} + 5\sqrt{6} - 2\sqrt{3}$
30. $\sqrt{6} + \sqrt{24}$
31. $\sqrt[4]{16a^3b} + \sqrt[4]{81a^7b}$
32. $\sqrt[4]{32x^5} - x^2\sqrt[4]{2x}$
33. $a\sqrt{\dfrac{a}{7b}} - b\sqrt{\dfrac{b}{7a}}$
34. $\sqrt{\dfrac{x-2}{x+2}} + \sqrt{\dfrac{x+2}{x-2}}$
35. $\sqrt{3}\sqrt{5}$
36. $\sqrt{7}\sqrt{6}$
37. $\sqrt{2x}\sqrt{7x}$
38. $\sqrt[3]{5y}\sqrt[3]{25y}$
39. $(\sqrt{a} + \sqrt{2b})(\sqrt{a} - \sqrt{2b})$
40. $(\sqrt{x} - 3\sqrt{y})(\sqrt{x} + 3\sqrt{y})$
41. $\dfrac{\sqrt[3]{2a^2}}{\sqrt[3]{16a}}$
42. $\dfrac{\sqrt[3]{5a^2b}}{\sqrt[3]{25ab^2}}$

In Exercises 43–50, solve the radical equations. Check your answers for extraneous roots.

43. $\sqrt{x-1} = 9$
44. $\sqrt{2x-7} - 5 = 0$
45. $\sqrt{x^2-7} = 2x - 4$
46. $\sqrt{\dfrac{3x}{4} - 1} = 5x - 3$
47. $\sqrt{x+1} + \sqrt{x-5} = 8$
48. $\sqrt{x+3} + \sqrt{x-2} = 20$
49. $\sqrt{\dfrac{1}{x}} = \sqrt{\dfrac{3x}{4} - 1}$
50. $\sqrt{\dfrac{3}{x-1}} = \sqrt{\dfrac{4}{x+1}}$

In Exercises 51–54, **(a)** *sketch the graph of the following radical equations showing all solutions and* **(b)** *determine the solutions.*

51. $\sqrt{x+7} = 5 - x$

52. $\sqrt{x^2 - 16} = \sqrt[3]{x^2 - 3x}$

53. $\sqrt{16 - x^2} + \sqrt{5 - x} = 10 - \sqrt{25 - x^2}$

54. $\sqrt{\dfrac{x^2}{x^2 + 4x + 5}} = 1.05 + \sqrt[3]{\dfrac{x - 4}{x^2 + 5}}$

Solve Exercises 55–58.

55. ***Nuclear technology*** The half-life of tritium is 12.5 years. It decays according to the formula $q(t) = q_0 2^{-t/12.5}$, where q_0 is the original amount of tritium and $q(t)$ is the amount left at time t, where t is given in years. If you begin with 50 mg of tritium, **(a)** how much will remain after 10 years? **(b)** after 20 years?

56. ***Nuclear technology*** The velocity of a proton can be described in terms of its kinetic energy, KE, and mass, m, with the formula

$$v = \sqrt{\frac{2\text{KE}}{m}}$$

(a) Express this in simplest radical form.

(b) Solve the equation for m.

57. ***Navigation*** Figure 11.3 shows the vector diagram for the resultant velocity $\mathbf{v}$ of a missile with horizontal component $\mathbf{v}_x$ and vertical component $\mathbf{v}_y$. If $\mathbf{v}_x = 3t + 2$ and $\mathbf{v}_y = 4t - 1$, express $\mathbf{v}$ in terms of t in simplest radical form.

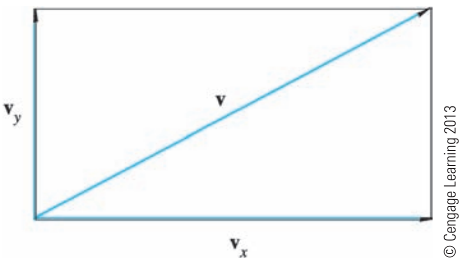

Figure 11.3

58. ***Energy technology*** Two sources, A and B, are 12 m apart as shown in Figure 11.4. Each emits waves of wavelength $\lambda = 5$ *m. Constructive interference* occurs when the joint effect of the two waves results in a wave of larger amplitude. Here, constructive interference of the waves will occur at point C, located x units from A when

$$BC - AC = \lambda$$

How far is point C from A? How far from B?

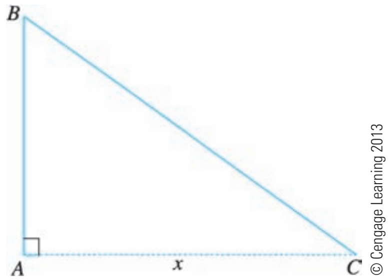

Figure 11.4

CHAPTER 11 TEST

In Exercises 1–4, express each of the given expressions in the simplest form containing only positive exponents.

1. $8^5 8^{-7}$

2. $(3^5)^4$

3. $\left(\dfrac{x^2}{xy^3}\right)^5$

4. $x^{1/4}x^{4/3}$

In Exercises 5–13, use the laws of radicals to express each of the following in simplest form.

5. $\sqrt[3]{-64}$
6. $\sqrt[4]{81}$
7. $\sqrt{x^4y^3}$
8. $\sqrt[3]{\dfrac{-27x^6y}{z^2}}$
9. $\sqrt{x^2 + 6x + 9}$
10. $3\sqrt{6} + 5\sqrt{24}$
11. $\sqrt{2x}\sqrt{6x}$
12. $(3\sqrt{x} + \sqrt{5y})(3\sqrt{x} - \sqrt{5y})$
13. $\dfrac{\sqrt[3]{4x^2y}}{\sqrt[3]{2x^2y^2}}$

In Exercises 14–18, solve the radical equations.

14. $\sqrt{x + 3} = 6$
15. $\sqrt{2x + 1} = 5$
16. $\sqrt{2x^2 - 7} = x + 3$
17. $\sqrt{\dfrac{5}{x + 3}} = \sqrt{\dfrac{4}{x - 3}}$
18. **(a)** Sketch the graph of the radical equation $\sqrt{16 + 3x - x^2} = 6 + 4x - x^2$ showing all solutions and **(b)** determine the solutions.

Solve the following exercise.

19. The diameter d of a cylindrical column necessary for it to resist a crushing force F is

$$d = (1.12 \times 10^{-2})\sqrt[3]{Fl}$$

where l is the length of the column. **(a)** Solve for F. **(b)** Determine F, when $d = 1.2$ m and $l = 5$ m.

12 EXPONENTIAL AND LOGARITHMIC FUNCTIONS

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Medical technologists and people working in microbiology often need to determine how many of a certain type of bacteria they can expect after a certain period of time. This type of growth tends to be exponential. In Section 12.2, you will learn how to determine exponential growth.

The first functions we studied in this text were algebraic functions. Then we introduced the trigonometric functions, examples of functions that were not algebraic. In this chapter, we will introduce two new functions. These functions are not algebraic functions, but they are very important in many technical areas, such as business, finance, nuclear technology, acoustics, electronics, and astronomy. Many of the applications in this chapter will involve growth or decay.

OBJECTIVES

AFTER COMPLETING THIS CHAPTER, YOU WILL BE ABLE TO:

- Recognize data that is either exponential or logarithmic.
- Graph the exponential function.
- Convert expressions between exponential and logarithmic forms.
- Evaluate expressions containing exponential and logarithmic functions.
- Use logarithms to re-express paired data.
- Evaluate, manipulate, and simplify logarithmic expressions.
- Solve exponential and logarithmic equations.
- Make graphs on logarithmic or semilogarithmic paper.
- Collect, organize, and graph data and determine an approximate equation to model the data.

12.1 EXPONENTIAL FUNCTIONS

We will begin this section with the definition of an exponential function and with some examples of exponential functions.

EXPONENTIAL FUNCTION

An **exponential function** is any function of the form:

$$f(x) = b^x$$

where $b > 0$, $b \neq 1$, and x is any real number. The number b is called the *base.*

EXAMPLE 12.1

Some examples of exponential functions are: $f(x) = 2^x$, $g(x) = 3^x$, $h(x) = \pi^x$, $j(x) = 4.2^x$, and $k(x) = (\sqrt{3})^x$. The following exponential functions contain constants, represented by a and b: $y(x) = (5b)^{-x}$, $h(x) = (2a)^{x+1}$, and $k(x) = 3a^{x/3}$.

The functions $f(x) = (-2)^x$, $g(x) = 0^x$, and $h(x) = 1^x$ are not exponential functions. In $f(x) = (-2)^x$ the base, -2, is less than zero. Therefore, it is not an exponential function. We can see that if $x = \frac{1}{2}$, then $(-2)^{1/2} = \sqrt{-2}$ is not a real number. The function $g(x) = 0^x$ is not an exponential function, because the base is 0. Also, $h(x) = 1^x$ is not an exponential function, since the base is 1.

GRAPHING EXPONENTIAL FUNCTIONS; ASYMPTOTES

Graphing an exponential function is done in the same manner in which we have graphed other functions. We will choose values for x and determine the corresponding values for $f(x)$. We will then plot these points and connect them in order to get the graph. Notice that, since the base b of an exponential function is positive, all powers of that base are also positive. Thus, an exponential function is positive for all values of x.

EXAMPLE 12.2

Graph the exponential function $f(x) = 2^x$.

SOLUTION The following table gives integer values of x from -3 to 5:

x	-3	-2	-1	0	1	2	3	4	5
$f(x) = 2^x$	0.125	0.25	0.5	1	2	4	8	16	32

The first seven of these points are graphed in Figure 12.1a.

Notice that as x gets larger, $f(x)$ increases at a faster and faster rate. As x gets smaller, $f(x)$ keeps getting closer to 0, but never reaches 0.

The graph of $f(x) = 2^x$ is shown in Figure 12.1b. Notice how the curve keeps getting closer to the negative x-axis. As a curve approaches a line when a variable approaches a certain value, the curve is said to be asymptotic to the line. The line is an *asymptote* of the curve. In this example, $y = 2^x$ is asymptotic to the negative x-axis.

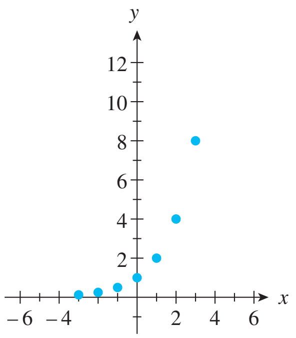

Figure 12.1a

Either a calculator or a computer is helpful for finding values of an exponential function. On a calculator, you will use the ^ key.

EXAMPLE 12.3

Determine $f(4.5)$, when $f(x) = 2^x$.

SOLUTION

PRESS	DISPLAY
2 ^ 4.5 ENTER	22.627417

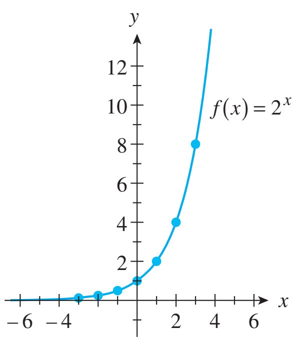

Figure 12.1b

The most common bases are those that are larger than 1. In fact, when the base is less than 1 (and greater than 0), you get an interesting effect, as shown in the next example.

EXAMPLE 12.4

Graph the exponential function $g(x) = \left(\frac{1}{2}\right)^x$.

SOLUTION Again, we will make a table of values:

x	-5	-4	-3	-2	-1	0	1	2	3
$g(x) = \left(\frac{1}{2}\right)^x$	32	16	8	4	2	1	0.5	0.25	0.125

The graph of the first seven of these points is shown in Figure 12.2a and the graph of the function is in Figure 12.2b. The graph of $g(x) = \left(\frac{1}{2}\right)^x$ would be identical to the graph of $f(x) = 2^x$, if it were reflected over the y-axis.

Study the function again. Remember that $\frac{1}{2} = 2^{-1}$, so $\left(\frac{1}{2}\right)^x = 2^{-x}$. These two examples show the differences between a function of the form b^x and one of the form b^{-x}. If $b > 1$, the graph of b^x rises and the graph of b^{-x} falls, as we move from left to right.

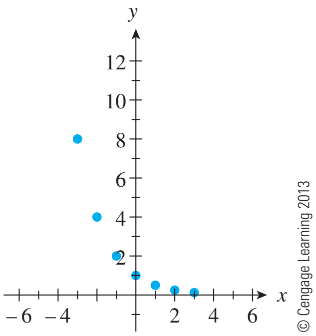

Figure 12.2a

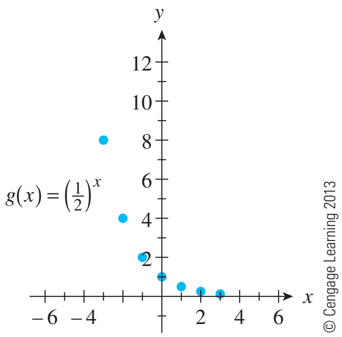

Figure 12.2b

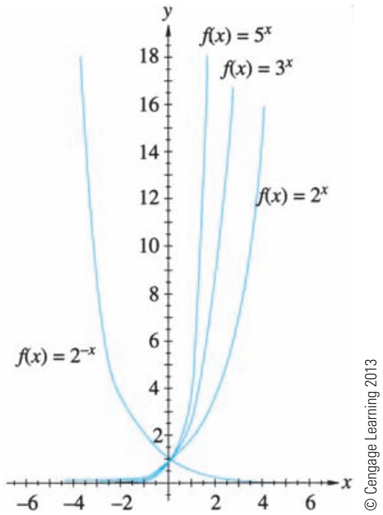

Figure 12.3

Figure 12.3 shows the graph of several exponential functions on the same set of coordinates. These functions, and all exponential functions, have the following three features in common.

COMMON FEATURES OF EXPONENTIAL FUNCTIONS

If $f(x) = b^x$, then

1. The y-intercept is 1.
2. If $b > 1$, the negative x-axis is an asymptote; if $b < 1$, the positive x-axis is an asymptote.
3. If $b > 1$, the curves all rise as the values of x increase; if $b < 1$, the curves all fall as the values of x increase.

APPLICATION BUSINESS

One application of exponential functions has to do with money. When an amount of money P, called the principal, is invested and interest is compounded annually (once a year) at the interest rate of r per year, then the amount of money at the end of t years is given by the formula $S = P(1 + r)^t$. In this formula, r is expressed as a decimal. So, at 5% interest, $r = 0.05$ and at $6\frac{1}{4}$% interest, $r = 0.0625$.

EXAMPLE 12.5

If \$800 is invested at 6% compounded annually, how much is the total value after 10 years?

SOLUTION Here $P = \$800$, $r = 0.06$, and $t = 10$. As a result we have

$$\begin{aligned} S &= P(1 + r)^t \\ &= 800(1 + 0.06)^{10} \\ &\approx 800(1.790847697) \\ &\approx 1{,}432.68 \end{aligned}$$

After 10 years, the total value of this investment is \$1,432.68.

If interest is compounded more than once a year, then the formula is changed. If interest is compounded semiannually, or twice a year, the formula becomes

$$S = P\left(1 + \frac{r}{2}\right)^{2t}$$

If interest is compounded quarterly, or four times a year, the formula is changed to

$$S = P\left(1 + \frac{r}{4}\right)^{4t}$$

In general, we have the following compound interest formula.

COMPOUND INTEREST FORMULA

If an amount of money P is invested and interest is compounded k times a year at an interest rate of r per year, then the amount of money S at the end of t years is

$$S = P\left(1 + \frac{r}{k}\right)^{kt}$$

EXAMPLE 12.6

If \$800 is invested in a savings account paying 6% interest compounded monthly, how much is this money worth after 10 years?

SOLUTION In this example, $P = \$800$, $r = 0.06$, $k = 12$, and $t = 10$, so

$$\begin{aligned} S &= P\left(1 + \frac{r}{k}\right)^{kt} \\ &= 800\left(1 + \frac{0.06}{12}\right)^{12\cdot 10} \\ &= 800(1.005)^{120} \end{aligned}$$

$$\approx 800(1.819396734)$$
$$\approx 1455.52$$

Compare this to the result we obtained in Example 12.5. Compounding monthly increased the total by \$22.84.

In Section 13.2, we will examine what happens if you compound interest daily or continuously. We will see that there is a limit to the amount of money you will receive.

MODELING WITH EXPONENTIAL FUNCTIONS

When you look at a graph of some data and it has the general shape of either of the graphs in Figures 12.1b and 12.2b, then the data may be modeled by an exponential function. Obtaining a regression formula is much the same as obtaining a linear, quadratic, or sinusoidal formula. The next example will outline the procedure and the one following will look at a variation.

APPLICATION ENVIRONMENTAL SCIENCE

EXAMPLE 12.7

The population of the United States from 1900 to 2010 is given in Table 12.1.

(a) Determine the exponential regression function that best fits this data.

(b) Use the regression function to predict the U.S. population in 2020.

TABLE 12.1 U.S. Population 1900–2010

Year (t)	1900	1910	1920	1930	1940	1950	1960	1970	1980	1990	2000	2010
Population ($\times 10^6$)	76.0	92.0	105.7	122.8	131.7	151.3	179.3	203.3	226.5	248.7	281.4	308.7

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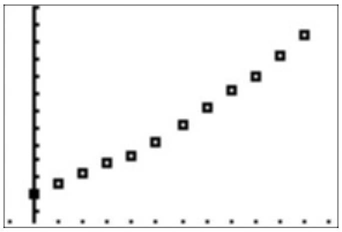

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Figure 12.4a

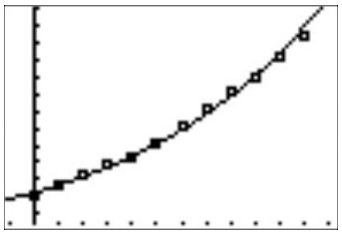

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Figure 12.4b

SOLUTIONS

(a) **Using a Calculator**

Enter the data into list L1 and L2 lists as before. However, in L1 enter the year after 1900, so instead of the values 1900, 1910, 1920, . . . , 2010, use 0, 10, 20, . . . , 110. A graph of these 12 points is in Figure 12.4a. If we ask the calculator to conduct an exponential regression using STAT ▶ 0 [ExpReg] ENTER, the result is $y \approx 80.9776 \times (1.0127)^x$. We can think of this as the function:

$$P(t) \approx 80.9776(1.0127)^t$$

where P is the population of the United States in millions of people in year t after 1900. When this regression equation is graphed with the original data, we get the result shown in Figure 12.4b.

EXAMPLE 12.7 (Cont.)

Using a spreadsheet

The data is entered in two columns as in the table. However, instead of using the values 1900, 1910, 1920, . . . , 2010, we use 0, 10, 20, . . . , 110. The graph of these 12 points is shown in Figure 12.4c.

An exponential regression is performed on this data. The result is shown in Figure 12.4d. The regression equation is $y = 80.978e^{0.0126t}$. Recall that $(a^m)^n = a^{m \times n}$, so $y = 80.978\,(e^{0.0126})^t \approx 80.978(1.0126797)^t$. Thus

$$P(t) \approx 80.978(1.0127)^t$$

where P is the population of the United States in millions of people in year t since 1900.

(b) The estimated population for 2020 is $P(120) \approx 366.8$ million people.

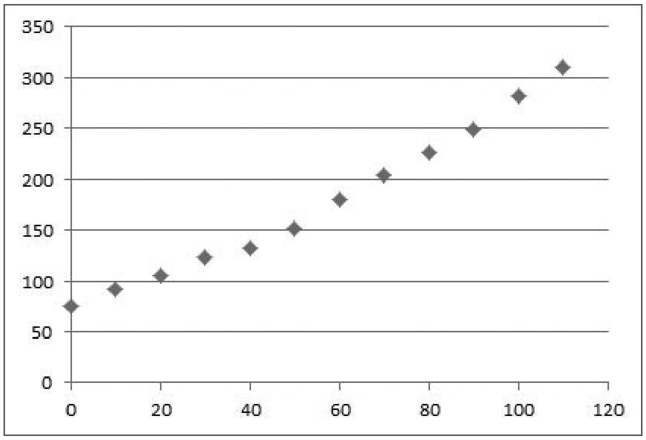

Figure 12.4c

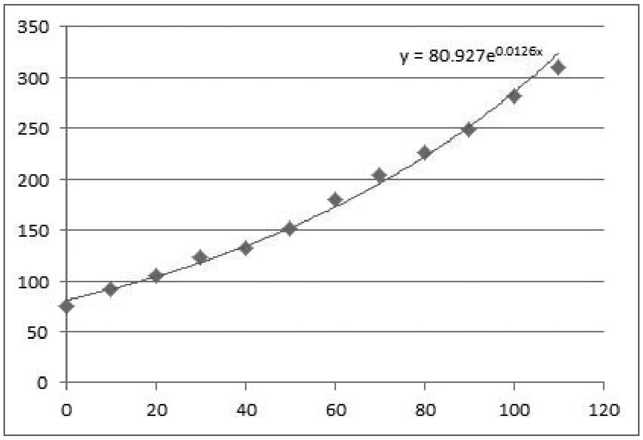

Figure 12.4d

Notice that both the calculator and the spreadsheet computed the regression formula $y = a * b^x$. This assumes that the data has a horizontal asymptote at the x-axis. This is not always the case. Just as the graph of a trigonometric function can have a vertical displacement (or vertical translation), so can an exponential function. Thus, if there is reason to believe that an exponential function has the line $y = c$ as a horizontal asymptote, then the regression formula would be

$$y = ab^x + c$$

The next example will show how this is accomplished with a calculator that does not take a vertical translation into account.

APPLICATION GENERAL TECHNOLOGY

EXAMPLE 12.8

In a room with a temperature of 72°F, hot coffee is poured into a cup and temperature readings are taken every 10 s for $1\frac{1}{2}$ min. The results of the reading are in Table 12.2. Determine the exponential regression function that best fits this data.

TABLE 12.2 Temperature of Coffee, °F

Time (s)	0	10	20	30	40	50	60	70	80	90
Temperature (°F)	180	164	153	141	130	121	116	111	106	102

SOLUTION

Using a Calculator

Enter the data into the L1 and L2 lists as before. A graph of these 10 points is in Figure 12.5a. The calculator produces the exponential regression $y \approx 172.9678(0.9937)^x$. When this regression equation is graphed with the original data, we get the result shown in Figure 12.5b.

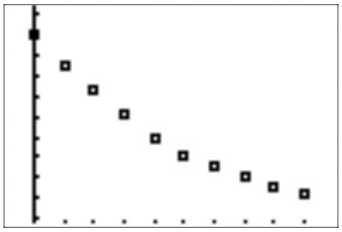

Figure 12.5a

While this does not look "too" bad, we can see that it is not realistic when we look at the same information for 180 s (3 min), as shown in Figure 12.5c. The graph of $y = 72$ has been added to the graph. Remember, the room was 72°F, so we would not expect the coffee to get cooler than 72°F. In fact, according to this regression formula, after 3 min the coffee has cooled to about 55.4°. This does not make sense and would seem to indicate that this is not a very good regression function.

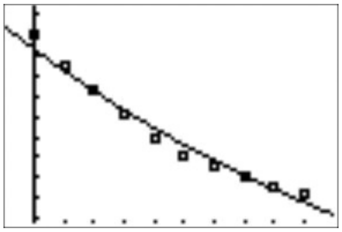

Figure 12.5b

What happened and how can we correct this? The exponential function produced by the calculator has a horizontal asymptote at the x-axis. If we want to use exponential regression with this data we will have to adjust the y-values so that they approach 0. In this case, we know that the coffee will never get below 72°F, so we will subtract 72 from each y-value and perform the exponential regression on the adjusted values.

Begin by saving the values in L2. Go the home screen, and press **2nd** **2** [L2] **STO▶** **2nd** **3** [L3] **ENTER**. You should see

{180 164 153 141 . . .

on the screen. This copies all the values in L2 into L3.

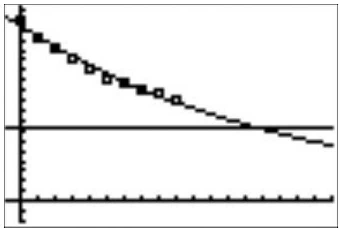

Figure 12.5c

To subtract 72 from each of the values in L2, press **2nd** **2** [L2] **–** 72 **STO▶** **2nd** **2** [L2] **ENTER**. You should now see

{108 92 81 69 58 . . .

on the screen. Now, compute the exponential regression by pressing **STAT** ▶ **0** [0:ExpReg] **ENTER**. The result is

$$y \approx 105.7977(0.9858)^x$$

This is *not* the final answer. Remember, this regression was performed on y-values that were 72 less than the values in Table 12.2. To get the regression equation we will add 72 to this equation and obtain the function:

$$T(t) \approx 105.7977(0.9858)^t + 72$$

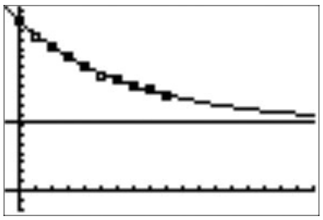

Figure 12.5d

When the function T is graphed with the original data (in L1 and L3) and the line $y = 72$, we get the graph in Figure 12.5d. Notice that this regression curve does not cross the line $y = 72$.

EXAMPLE 12.8 (Cont.)

Using a Spreadsheet

Enter the data and produce a graph as we've done before (see Figure 12.5e).

When the regression equation is graphed with the original data, we get the result shown in Figure 12.5f.

While this does not look "too" bad, we can see that it is not realistic when we look at the same information for 180 s (3 min), as shown in Figure 12.5g. The graph of $y = 72$ has been added to this graph. Remember, the room was 72°F, so we would not expect the coffee to get cooler than 72°F. In fact, according to this regression formula, after 3 min, the coffee has cooled to about 55°F. This does not make sense and would seem to indicate that this is not a very good regression function.

What happened and how can we correct this? The exponential function produced has a horizontal asymptote at the x-axis. If we want to use exponential regression with this data, we will have to adjust the y-values so that they approach 0. In this case, we know the coffee will never get below 72°F,

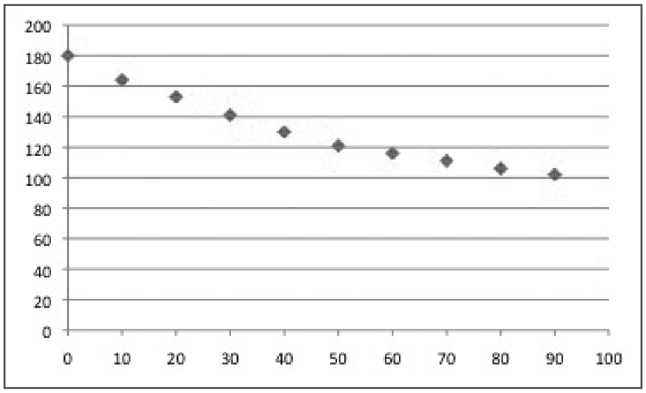

Figure 12.5e

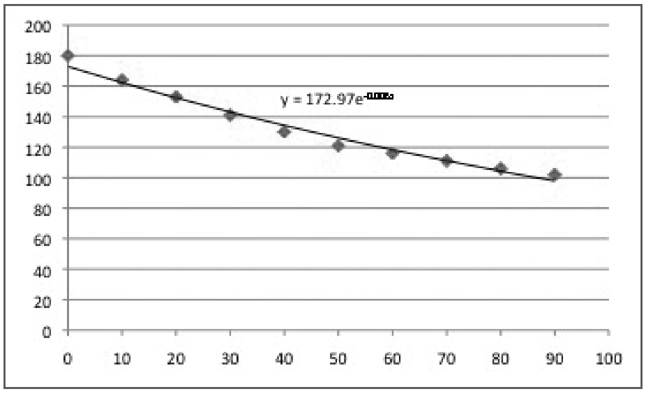

Figure 12.5f

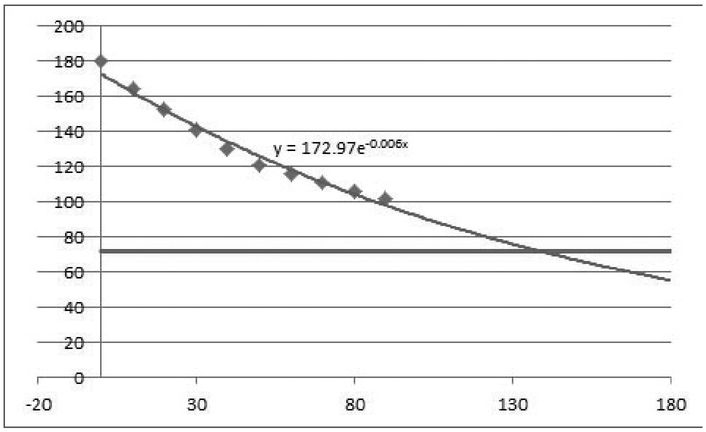

Figure 12.5g

so we will subtract 72 from each y-value and perform the exponential regression on the adjusted values. We first create a third column as shown in Figure 12.5h.

Now create a new scatter plot (see Figure 12.5i) and a new regression equation (see Figure 12.5j).

The regression equation is

$$\begin{aligned} y &= 105.8e^{-0.0143x} \\ &\approx 105.8(e^{-0.0143})^x \\ &\approx 105.8(0.9858)^x \end{aligned}$$

This is not the final answer. Recall that this regression was performed on y-values that were 72 less than the values in Table 12.2. To get a regression equation we will add 72 to this equation and obtain the function:

$$T(t) \approx 105.8(0.9858)^x + 72$$

When the function T is graphed with the original data, we get the graph in Figure 12.5k. Notice that this regression curve does not cross the line $y = 72$.

	A	B	C	D
1	Time	Temperature	Temp	
2	t	T	adjusted	
3	0	180	108	177.8
4	10	164	92	163.700944
5	20	153	81	151.480748
6	30	141	69	140.889032
7	40	130	58	131.708783
8	50	121	49	123.751907
9	60	116	44	116.855376
10	70	111	39	110.877885
11	80	106	34	105.696964
12	90	102	30	101.206459

Figure 12.5h

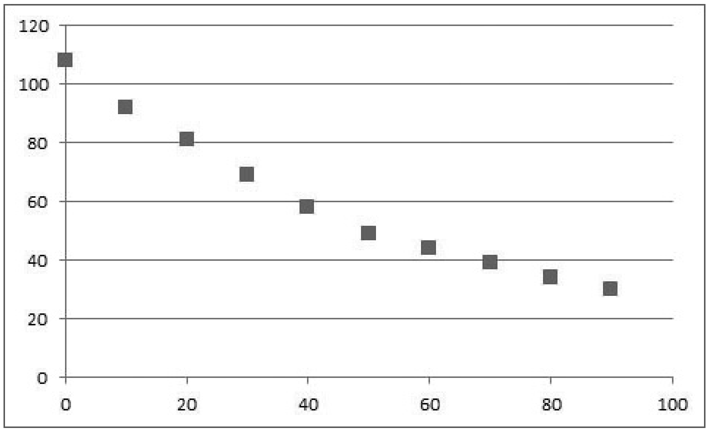

Figure 12.5i

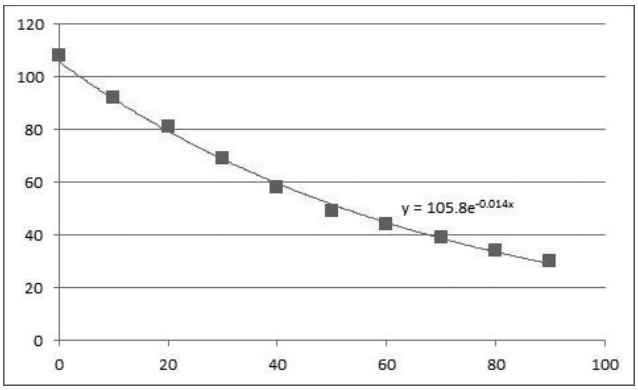

Figure 12.5j

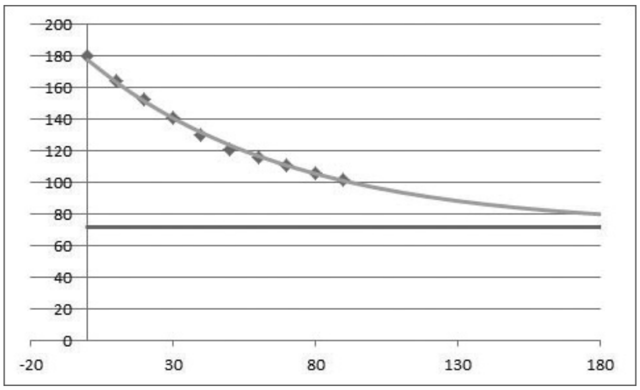

Figure 12.5k

EXERCISE SET 12.1

In Exercises 1–6, use a calculator to approximate the given numbers. Round off each answer to four decimal places.

1. $3^{\sqrt{2}}$
2. 4^{π}
3. π^3
4. $(\sqrt{3})^{\pi}$
5. $(\sqrt{3})^{\sqrt{5}}$
6. $(\sqrt{4})^{4/3}$

In Exercises 7–16, make a table of values and draw the graph of each function.

7. $f(x) = 4^x$
8. $g(x) = 3^x$
9. $h(x) = 1.5^x$
10. $k(x) = 2.5^x$
11. $f(x) = 3^{-x}$
12. $g(x) = 5^{-x}$
13. $h(x) = 2.4^{-x}$
14. $k(x) = (\sqrt{3})^x$
15. $f(x) = 3^{(x + 1/2)}$
16. $g(x) = 2^{(x - 1/4)}$

Solve Exercises 17–34.

17. ***Finance*** The sum of \$1 000 is placed in a savings account at 6% interest. If interest is compounded **(a)** annually, **(b)** semiannually, **(c)** quarterly, and **(d)** monthly, what is the total after 5 years?

18. ***Finance*** The sum of \$2 000 is placed in a savings account at 3.5% interest. If interest is compounded **(a)** annually, **(b)** semiannually, **(c)** quarterly, and **(d)** monthly, what is the total after 10 years?

19. ***Finance*** One bank offers 4.1% interest compounded semiannually. A second bank offers the same interest but compounded monthly. How much more income will result by depositing \$1 000 in the second account for 5 years than by depositing \$1 000 in the first account for 5 years?

20. ***Finance*** A bank offers 3.25% interest compounded annually. A second bank offers 3.25% interest compounded quarterly. How much more income will result by depositing \$2 000 in the second bank for 10 years than in the first?

21. ***Biology*** The number of a certain type of bacteria is given by the equation $Q = Q_0 2^t$, where Q_0 is the initial number of bacteria (that is, the number of bacteria when $t = 0$) and t is the time in hours since the initial count was taken.
    (a) If $Q = 200\,000$ when $t = 2.3$, find Q_0.
    (b) Find the number of bacteria present at the end of 4 h.
    (c) How long does it take for Q to become twice as large as Q_0?
    (d) How long does it take for Q to become eight times as large as Q_0?

22. ***Medical technology*** A pharmaceutical company is growing an organism to be used in a vaccine. The organism's growth is given by the equation $Q = Q_0 3^t$, where Q_0 is the initial number of bacteria (that is, the number of bacteria when $t = 0$) and t is the time in hours since the initial count was taken. When $t = 1$, we know that $Q = 2\,760$.
    (a) Find Q_0.
    (b) What is the number of organisms at the end of 5 h?
    (c) How long will it take for Q to become three times as large as Q_0?

23. ***Medical technology*** A pharmaceutical company is growing an organism to be used in a vaccine. The organism's growth is given by the equation $Q = Q_0 3^{kt}$, where Q_0 is the initial number of bacteria, t is the time in hours since the initial count was taken, and k is a constant. When $t = 0$, we know that $Q = 400$, and when $t = 2\frac{1}{2}$ h, $Q = 1{,}200$.
    (a) Find Q_0.
    (b) Find k.
    (c) What is the number of organisms present at the end of 5 h?

24. ***Ecology*** In 1995, the world population was growing at the rate of approximately 1.8% per year. This can be expressed mathematically as $P = P_0(1.018)^t$, where $P_0 = 5.7 \times 10^9$ and P is the population t years after 1995.

(a) What will the world's approximate population be in the year 2000?

(b) What will the world's approximate population be in the year 2050?

(c) Graph $P = P_0(1.018)^t = 5.7 \times 10^9(1.018)^t$.

(d) Use the graph from (c) to approximate the year that the world's population will reach $10{,}000{,}000{,}000 = 10^{10}$.

25. ***Lighting technology*** Each 1-mm thickness of a certain translucent material reduces the intensity of a light beam passing through it by 12%. This means that the intensity, I, is a function of the thickness, T, of the material as given by $I = 0.88^T$.

(a) What is the intensity when the thickness is 1.75 mm?

(b) Graph I as a function of T.

(c) Use the graph from (b) to approximate the thickness that produces an intensity of 0.50.

26. ***Electronics*** An important triode formula is

$$I_p + I_g = K\left(V_g + \frac{V_p}{\mu}\right)^{3/2}$$

where I_p is the plate current in amperes, I_g is the grid current in amperes, V_p is the plate voltage, V_g is the grid voltage, and μ is an amplification factor. Calculate $I_p + I_g$ if $K = 0.0005$, $V_p = 350$ V, $V_g = 8$ V, and $\mu = 14$.

27. ***Medical technology*** The amount D of medication, in mg, in the body t hours after taking a pill is given by $D(t) = 25(0.825^t)$.

(a) What was the initial dose of the medication?

(b) What percent of the medication leaves the body each hour?

(c) How much of the medication will remain in the body after 10 h?

(d) How long will it take before only 1 mg of the medication remains in the body?

28. ***Meteorology*** The relative humidity R is the ratio (expressed as a percent) of the amount of water vapor in the air to the maximum amount that the air can hold at a specific temperature and is given by the formula:

$$R = \left(10^{\left(\frac{37.5\,D - 6000}{1973.3 + 5\,D} - \frac{37.5\,T - 6000}{1973.3 + 5\,T}\right)}\right)100$$

where T is the air temperature (in °F) and D is the dew point temperature (in °F).

(a) Determine the relative humidity if the air temperature is 85°F and the dew point is 65°F.

(b) Determine the relative humidity if the air temperature is 75°F and the dew point is 55°F.

(c) What is the relative humidity if the air temperature and the dew point are the same?

29. ***Sports management*** Table 12.3 gives the average major league baseball player's salary on opening day each decade from 1970 through 2010.

TABLE 12.3 Average Major League Baseball Player's Salary

Year, t	1970	1980	1990	2000	2010
Average salary, $S(t)$	29,303	143,756	597,537	1,895,630	3,014,572

(a) Assuming that the average salary is increasing exponentially, write a model, $S_e(t)$, that expresses the average salary t years after 1970.

(b) Write a quadratic model, $S_q(t)$, that describes the average salary t years after 1970.

(c) Use each model to estimate the average salary in 2010.

(d) What is the percent error for each of the estimates in **(c)**?

(e) Use the model with the lower percent error to estimate to three significant figures the average baseball salary in 2015.

(f) Use the model with the lower percent error to estimate to three significant figures the average baseball salary in 2020.

30. ***Communications*** Table 12.4 gives the number of billion text messages in December from 2003 through 2008.

TABLE 12.4 Number of Text Messages in December

Year, t	2003	2004	2005	2006	2007	2008
No. of text messages in billions, $M(t)$	2.1	4.7	9.8	18.7	48.1	110.4

(a) Assuming that the number of text messages is increasing exponentially, write a model, $M(t)$, for the number of billion text messages t years after 2000.

(b) Use your model to estimate the number of billion text messages in 2008.

(c) What is the percent error for the estimate in **(b)**?

(d) Use your model to predict the number of text messages in 2010.

(e) If possible, use the Internet to determine the actual number of text messages in 2010.

(f) Use your model to predict the number of text messages in 2015.

31. ***Thermodynamics*** In a room with a temperature of 22°C, boiling water is removed from a heat source and allowed to cool. Temperature readings are taken every minute. The results of the reading are in Table 12.5.

(a) Determine the exponential regression function that best fits this data.

(b) Estimate the temperature after 1 h.

(c) How long will it take for the water to reach 23.0°C?

TABLE 12.5 Water Temperature, °C

Time (min)	2	4	6	8	10	12	14	16
Temp (° C)	89.2	85.4	79.9	75.6	71.8	70.2	67.4	64.7

Time (min)	18	20	22	24	26	28	30
Temp (° C)	61.2	59.0	57.3	55.5	53.9	52.4	50.9

32. ***Finance*** Table 12.6 shows the amount of U.S. consumer credit outstanding from 1970 through 2005.

TABLE 12.6 U.S. Consumer Credit

Year, t	1970	1975	1980	1985	1990	1995	2000	2005
Credit (billion dollars), $C(t)$	134	208	349	593	824	1,096	1,741	2,321

(a) Fit an exponential model to the data.

(b) Use your model to predict the consumer credit in 2008.

(c) Use your model to predict the consumer credit in 2015.

33. ***Communications*** Table 12.7 shows the number of cell phone subscribers, in millions, in the United States from 2000 through 2009.

TABLE 12.7 U.S. Cell Phone Subscribers

Year, t	2000	2003	2004	2005	2006	2007	2008	2009
Subscribers (millions), $S(t)$	109	159	182	208	233	256		287

(a) Fit an exponential model to the data.

(b) Use your model to predict the number of million subscribers in 2008.

(c) If the actual number of subscribers in 2008 was 270 million, was the percent error with your prediction in **(b)**?

(d) Use your model to predict the number of million subscribers in 2015.

(e) Is your answer in **(d)** realistic? Explain your answer.

34. ***Environmental science*** The concentration of carbon dioxide (CO_2), in parts per million (ppm), at the Mauna Loa, Hawaii, observatory is given in Table 12.8.

TABLE 12.8 Carbon Dioxide at Mauna Loa, Hawaii, Observatory

Year	1975	1980	1985	1990	1995	2000	2002	2005	2008
CO_2, $C(t)$	331	339	346	354	361	369	373	380	385

(a) Fit an exponential model to the data.

(b) Use your model to predict the concentration of carbon dioxide (CO_2), in ppm, at the Mauna Loa, Hawaii, observatory in 2010.

(c) If possible, test your prediction and calculate the percent error between your prediction and the actual value.

(d) Use your model to predict the concentration of carbon dioxide (CO_2), in ppm, at the Mauna Loa, Hawaii, observatory in 2015.

[IN YOUR WORDS]

35. The text listed three common features of exponential functions. Without looking in the text, list each of these features. You may want to draw some graphs to help your explanation.

36. (a) Graph the two functions $f(x) = x^2$ and $g(x) = 2^x$ on your calculator.

(b) Explain how the two graphs are alike and how they are different.

(c) Describe how you would help someone learn which was the graph of $f(x) = x^2$ and which was the graph of $g(x) = 2^x$.

37. Write a paragraph that describes how the functions $f(x) = a^x$, where $a > 1$, and $g(x) = b^x$, where $0 < b < 1$, are alike and how they are different.

38. In the definition of an exponential function $f(x) = b^x$ it states that $b \neq 0$. Explain why this is not allowed.

12.2 THE EXPONENTIAL FUNCTION e^x

In Section 12.1, we were working with an equation that determined the amount of interest gathered in an account. A variation of that formula is $\left(1 + \frac{1}{n}\right)^n$. Let's examine the values of $\left(1 + \frac{1}{n}\right)^n$ as n gets larger. Use your calculator to check these numerical values.

n	$\left(1+\frac{1}{n}\right)^n$
1	2
10	2.59374246
100	2.704813829
1,000	2.716923932
10,000	2.718145927
100,000	2.718268237
1,000,000	2.718280469
10,000,000	2.718281693
100,000,000	2.718281815
1,000,000,000	2.718281827

Mathematicians have been able to prove that as the value of n gets larger, the value of $\left(1 + \frac{1}{n}\right)^n$ also continues to get larger, but has a limit on how large it can get. This limit is such a special number that it has been given its own symbol, e. The first nine digits of e are the same ones that we got in the previous table, 2.71828182.

As we will see later in this section, the number e is a very important number. Some calculators have two special numbers marked on them—π and e. Later in this section, we will need to determine values for the exponential function $f(x) = e^x$. Both the calculator and the computer can be used to find these values.

FINDING e^x WITH A CALCULATOR

Some calculators have an e^x key. If you want to determine e^5, use this method.

PRESS	DISPLAY
2nd e^x 5 ENTER	148.4131591

FINDING e^x WITH A SPREADSHEET

On a spreadsheet, EXP() returns e raised to the power of the number in parentheses. For example, Figure 12.6 shows the value of e^5 in Cell A2.

	A
1	=exp(5)
2	148.4131591

Figure 12.6

BUSINESS AND FINANCE

Let's return to a problem involving compound interest. In Section 12.1, we learned that if a certain principal amount P is invested and interest is compounded k times a year at an interest rate of r, the amount after t year would be

$$S = P\left(1 + \frac{r}{k}\right)^{kt}$$

If we let $n = \frac{k}{r}$, then $k = nr$, and this formula becomes

$$\begin{aligned} S &= P\left(1 + \frac{r}{nr}\right)^{nrt} \\ &= P\left[\left(1 + \frac{1}{n}\right)^n\right]^{rt} \end{aligned}$$

Now, if interest is compounded continuously, then the expression inside the brackets, $\left(1 + \frac{1}{n}\right)^n$, is equal to the number represented by e. So, the amount accumulated after t year at the interest rate of r would be

$$S = Pe^{rt}$$

CONTINUOUS INTEREST FORMULA

If an amount of money P is invested at an interest rate of r per year and interest is compounded continuously, then the amount of money S at the end of t years is

$$S = Pe^{rt}$$

APPLICATION BUSINESS

EXAMPLE 12.9

If \$800 is invested in a savings account paying interest compounded continuously at 6%, how much has accumulated after 10 years?

SOLUTION Here $P = 800$, $r = 0.06$, and $t = 10$, so

$$\begin{aligned} S &= 800e^{(0.06)10} \\ &= 800e^{0.6} \\ &\approx 800(1.8221188) \\ &= 1\,457.70 \end{aligned}$$

After 10 years, this \$800 investment has increased to \$1 457.70.

When this is compared to the result in Example 12.6, we see that, after 10 years, continuous compounding of interest has provided an additional \$2.18.

EXPONENTIAL GROWTH AND DECAY

The number represented by e is used in the two areas known as **exponential growth** and **exponential decay**.

Exponential growth can be explained by letting y represent the size of a quantity at time t. If this quantity grows or decays exponentially, it obeys the exponential growth formula given here.

EXPONENTIAL GROWTH AND DECAY

The basic formula for the exponential growth or decay of a quantity is

$$y = ce^{kt}$$

where y represents the size of the quantity at time t, c is a positive real number constant, and k is a nonzero constant.

If $k > 0$, this is an *exponential growth* function.

If $k < 0$, this is an *exponential decay* function.

APPLICATION ENVIRONMENTAL SCIENCE

EXAMPLE 12.10

A culture of bacteria originally numbers 500. After 4 h, there are 8 000 bacteria in the culture. If we assume that these bacteria grow exponentially, how many will there be after 10 h?

SOLUTION Since the number of bacteria grows exponentially, their growth obeys the formula $y = ce^{kt}$. When $t = 0$, we are told that $y = 500$, and so $500 = ce^{k \cdot 0} = c$. (Remember, $k \cdot 0 = 0$, $e^{k \cdot 0} = e^0 = 1$.) The formula is now $y = 500e^{kt}$. When $t = 4$ h, we have $y = 8\,000$ and from the formula,

$$8\,000 = 500e^{k4}$$
$$16 = e^{k4}$$

However, the example asks us to determine y, when $t = 10$. It is possible to do this without determining k. (In Section 12.3, we will learn how to find the value of k.)

$$\begin{aligned} y &= 500e^{k \cdot 10} \\ &= 500(e^{k4})^{2.5}, \text{ since } 10 = (4)(2.5) \\ &= 500(16)^{2.5}, \text{ since } e^{k4} = 16 \\ &= 500(1{,}024) \\ &= 512\,000 \end{aligned}$$

After 10 h, the number of bacteria increased from 500 to 512 000.

Exponential decay is seen most often in radioactive substances. A common measure of the rate of decay is the *half-life* of a substance. The half-life is the amount of time needed for a substance to diminish to one-half its original size.

For example, 1000 g of radioactive material would be reduced to 500 g after one half-life. After a second half-life there would be half of the 500 g (or 250 g) remaining. Each new half-life cuts the amount of radioactive material in half as outlined in Table 12.9.

TABLE 12.9 Amount of Radioactive Material Remaining After Each Half-Life

No. of half-lives, n	0	1	2	3	4	5	6	7
Amount remaining, g	1000	500	250	125	62.5	31.25	15.625	7.8125

APPLICATION GENERAL TECHNOLOGY

EXAMPLE 12.11

The half-life of copper-67 is 62 h. How much of 100 g will remain after 15 days?

SOLUTION As in Example 12.10, we have some basic information. When $t = 0$, we know that $y = 100$ g, so

$$100 = ce^{k \cdot 0}$$
$$\text{or } 100 = c$$

and the exponential decay formula for copper-67 becomes

$$y = 100e^{kt}$$

Now, since the half-life is 62 h, we know that, when $t = 62$, there is only half as much copper-67, or $y = 50$ g, so

$$50 = 100e^{k(62)}$$
$$\tfrac{1}{2} = e^{k(62)}$$

We want to determine the amount when $t = 15$ days. Because the half-life is given in hours, we first convert 15 days to $15 \times 24 = 360$ h. Since $360 \div 62 \approx 5.8$, we can write the formula as

$$y = 100e^{k \cdot 360}$$
$$\approx 100\left(e^{k(62)}\right)^{5.8}$$

and since $e^{k(62)} = \frac{1}{2}$, we have

$$y \approx 100\left(\frac{1}{2}\right)^{5.8}$$
$$= 1.79 \text{ g}$$

So, of the original 100 g of copper-67, about 1.79 g remain after 15 days.

APPLICATION ELECTRONICS

EXAMPLE 12.12

When a charged capacitor is discharged through a resistance, as in Figure 12.7, the charge Q in coulombs (C) is given by the formula

$$Q = Q_0 e^{-t/T}$$

where $Q_0 = CV$, the initial charge, $T = RC$, and C is the capacitance, V is the battery voltage, and R is the resistance. The product RC is called the time constant of the circuit. If a 15-μF capacitor is charged by being connected to a 60-V battery through a circuit with a resistance of 12 000 Ω, what is the charge on the capacitor 9 s after the battery is disconnected? (*)

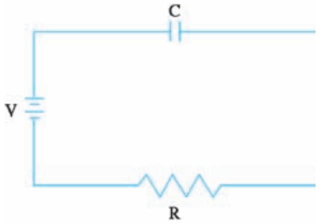

Figure 12.7

SOLUTION We are given $C = 15\ \mu\text{F} = 15 \times 10^{-6}$ F, $V = 60$ V, and $R = 12\,000\ \Omega$. Thus

$$\begin{aligned} Q_0 &= CV \\ &= 15 \times 10^{-6}\text{ F} \times 60\text{ V} \\ &= 900 \times 10^{-6}\text{ C} \\ &= 9 \times 10^{-4}\text{ C} \\ \text{and} \quad T &= RC \\ &= 12\,000\Omega \times 15 \times 10^{-6}\text{ F} \end{aligned}$$

EXAMPLE 12.12 (Cont.)

$$= 180\,000 \times 10^{-6}\text{ s}$$
$$= 0.18\text{ s}$$

Substituting these values for Q_0 and T into the formula $Q = Q_0e^{-t/T}$, we obtain

$$Q = 9 \times 10^{-4}\, e^{-t/0.18}$$

We want to find the value of Q when $t = 9$ s, or

$$\frac{t}{T} = \frac{9\text{ s}}{0.18\text{ s}} = 50$$

Thus, returning to equation with $Q_0 = 9 \times 10^{-4}$ C and $\frac{t}{T} = 50$, we get

$$Q = 9 \times 10^{-4}e^{-50}\text{ C}$$
$$= 1.74 \times 10^{-25}\text{ C}$$

So, the charge is about 1.74×10^{-25} C, 9 s after the battery is disconnected.

EXERCISE SET 12.2

Use a calculator or a computer to evaluate each of the numbers in Exercises 1–8.

1. e^3
2. e^7
3. $e^{4.65}$
4. $e^{5.375}$
5. e^{-4}
6. e^{-9}
7. $e^{-2.75}$
8. $e^{-0.25}$

Make a table of values and graph each of the functions in Exercises 9–12 over the given domains of x.

9. $f(x) = 4e^x$, $\{x: -2 \le x \le 4\}$
10. $g(x) = 3.5e^{5x}$, $\{x: -1 \le x \le 4\}$
11. $h(x) = 4e^{-x}$, $\{x: -4 \le x \le 2\}$
12. $k(x) = 8.5e^{-6x}$, $\{x: -4 \le x \le 2\}$

Solve Exercises 13–30.

13. ***Nuclear technology*** The number of milligrams of a radioactive substance present after t year is given by $Q = 125e^{-0.375t}$. **(a)** How many milligrams were present at the beginning? **(b)** How many milligrams are present after 1 year? **(c)** How many milligrams are present after 16 years?

14. ***Nuclear technology*** Radium decays exponentially and has a half-life of 1 600 years. How much of 100 mg will be left after 2 000 years?

15. ***Biology*** The number of bacteria in a certain culture increases from 5 000 to 15 000 in 20 h. If we assume these bacteria grow exponentially, **(a)** how many will be there after 10 h? **(b)** How many can we expect after 30 h? **(c)** How many can we expect after 3 days?

16. ***Finance*** If \$5 000 is invested in an account that pays 5% interest compounded continuously, how much can we expect to have after 10 years?

17. ***Biology*** The population of a certain city is increasing at the rate of 7% per year. The present population is 200 000. **(a)** What will be the population in 5 years? **(b)** What can the population be expected to reach in 10 years?

18. ***Medical technology*** A pharmaceutical company is growing an organism to be used in a vaccine. The organism grows at a rate of 4.5%

per hour. How many units of this organism must they begin with in order to have 1 000 units at the end of 7 days?

19. ***Thermodynamics*** According to *Newton's law of cooling*, the rate at which a hot object cools is proportional to the difference between its temperature and the temperature of its surroundings. The temperature T of the object after a period of time t is

$$T = T_m + (T_0 - T_m)e^{-kt}$$

where T_0 is the initial temperature and T_m is the temperature of the surrounding medium. An object cools from 180°F to 150°F in 20 min when surrounded by air at 60°F. What is the temperature at the end of 1 h of cooling?

20. ***Thermodynamics*** A piece of metal is heated to 150°C and is then placed in the outside air, which is 30°C. After 15 min the temperature of the metal is 90°C. What will its temperature be in another 15 min? (See Exercise 19.)

21. ***Thermodynamics*** You like your drinks at 45°F. When you arrive home from the store, the cans of drink you bought are 87°F. You place the cans in a refrigerator. The thermostat is set at 37°F. When you open the refrigerator 25 min later, the drinks are at 70°F. How long will it take for the drinks to get to 45°F? (See Exercise 19.)

22. ***Electronics*** The circuit in Figure 12.8 contains a resistance R, a voltage V, and an inductance L. The current I at t s after the switch is closed is given by

$$I = I_0\left(1 - e^{-t/T}\right)$$

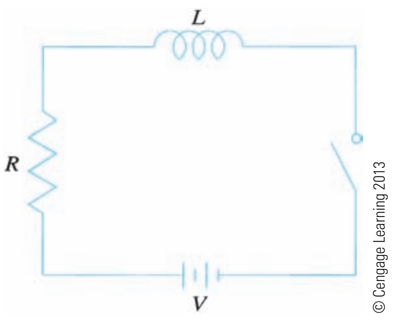

Figure 12.8

where I_0 is the steady state current $\frac{V}{R}$ and $T = \frac{L}{R}$. If a circuit has $V = 120$ V, $R = 40\ \Omega$, and $L = 3.0$ H (henrys), determine the current in the circuit **(a)** 0.01 s, **(b)** 0.1 s, and **(c)** 1.0 s after the connection is made.

23. ***Electronics*** A 130-μF capacitor is charged by being connected to a 120-V circuit. The resistance is 4 500 Ω. What is the charge on the capacitor 0.5 s after the circuit is disconnected?

24. ***Nuclear technology*** The half-life of tritium is 12.5 years. How much of 100 g will remain after 40 years?

25. ***Medical technology*** A radioactive material used in radiation therapy has a half-life of 5.4 days. This means that the radioactivity decreases by one-half each 5.4 days. A hospital gets a new supply that measures 1 200 microcuries (μCi). How much of this material will still be radioactive after 30 days?

26. ***Medical technology*** The average doubling time for a breast cancer cell is 100 days. It takes about 9 years before a lump can be seen on a mammogram and 10 years before a breast cancer is large enough to be felt as a lump. (Use 365.25 days = 1 year.)

(a) How many cancer cells are in a lump that is just large enough to be seen on a mammogram?

(b) How many cancer cells are in a lump that is just large enough to be felt?

(c) If a lump must be approximately 1 cm in diameter before it can be felt, how large is each cancer cell? (Assume the lump is a perfect sphere.)

27. ***Electronics*** In a capacitive circuit, the equation for the current in amperes is given by

$$i = \frac{V}{R}e^{-t/RC}$$

where t is any elapsed time in seconds after the switch is closed, V is the impressed voltage, C is the capacitance of the circuit in F, and R is the circuit resistance in Ω. A capacitance of 500 μF in series with 1 kΩ is connected across a 50-V generator.

(a) What is the value of the current at the instant the switch is closed?

(b) What is the value of the current 0.02 s after the switch is closed?

(c) What is the value of the current 0.04 s after the switch is closed?

28. ***Biology*** The length L of a shark as a function of its age t can be predicted by the *von Bertalanffy growth function*, $L(t) = M - (M - b)e^{-kt}$ where M is the mean maximum length of this species of shark, b is the mean length at birth, k is a growth rate constant per year, and t is year since birth. For one species of shark $M = 3.0$ m, $b = 0.5$ m, and $k = 0.13863$/year. **(a)** Sketch the graph of L. **(b)** What is the length of this shark after 5 years? **(c)** What is the length of this shark after 10 years? **(d)** How many years does it take this shark to reach a length of 2.7 m?

29. ***Medical technology*** Hospitals use the radioactive substance iodine-131 in the diagnosis of conditions of the thyroid gland. The half-life of iodine-131 is 8 days. Suppose that it takes two days from the time an order is placed with the distributer until it arrives. If 20 units of iodine-131 are needed on a certain day, how much should be ordered?

30. The number e can be approximated by the infinite series:

$$\begin{aligned} e &= 2 + \frac{1}{2} + \frac{1}{2 \cdot 3} + \frac{1}{2 \cdot 3 \cdot 4} + \frac{1}{2 \cdot 3 \cdot 4 \cdot 5} \cdots \\ &= 2 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} \end{aligned}$$

(Here $3! = 3 \cdot 2 \cdot 1 = 6$ and $4! = 4 \cdot 3 \cdot 2 \cdot 1 = 24$. In general, $n! = n \cdot (n-1) \cdot (n-2) \cdots 4 \cdot 3 \cdot 2 \cdot 1$. Some calculators have an x! key. On a TI-83/84 the ! symbol is produced by pressing MATH ◄ [PRB] 4 [4: !]. On a TI-86 you get the ! symbol by pressing 2nd MATH F2 [PROB] F1 [!]. The symbols $n!$, 3!, and 2! are read n factorial, 3 factorial, and 2 factorial.) Calculate e to four figures by adding the first six terms of this series.

[IN YOUR WORDS]

31. Suppose that you are shown the graphs of $y = e^x$, $y = 2^x$, $y = x^2$, and $y = x^3$, but the graphs are not labeled. Explain how you would distinguish among the graphs.

32. Explain what is meant by the term "half-life."

12.3 LOGARITHMIC FUNCTIONS

If you look at the graphs of exponential functions, such as the ones in Figures 12.1b, 12.2b, and 12.3, you can see that they would pass the horizontal line test for the inverse of a function. The inverse of the exponential function $f(x) = b^x$, $b > 0$, $b \neq 1$, is called the **logarithmic function**. The symbol for the logarithmic function is $\log_b x$. So, if $f(x) = b^x$, we have

$$f^{-1}(x) = \log_b x, \quad b > 0, b \neq 1$$

which is read "log to the base b of x." This provides for the following definition.

LOGARITHMIC FUNCTION

A *logarithmic function* is any function of the form:

$$f(x) = \log_b x$$

where $x > 0$, $b > 0$, and $b \neq 1$. If $y = \log_b x$, then $x = b^y$. The number represented by b is called the *base*.

Since $y = \log_b x$ is equivalent to $b^y = x$, we can express each logarithm in exponential form.

EXAMPLE 12.13

Use the fact that $y = \log_b x$ is equivalent to $b^y = x$ to rewrite each logarithm in exponential form.

	Logarithmic form	Exponential form
(a)	$\log_5 125 = 3$	$5^3 = 125$
(b)	$\log_7 49 = 2$	$7^2 = 49$
(c)	$\log_2 128 = 7$	$2^7 = 128$
(d)	$\log_5 \left(\frac{1}{25}\right) = -2$	$5^{-2} = \frac{1}{25}$
(e)	$\log_2 0.125 = -3$	$2^{-3} = 0.125 = \frac{1}{8}$
(f)	$\log_8 16 = \frac{4}{3}$	$8^{4/3} = 16$
(g)	$\log_b 1 = 0$	$b^0 = 1$

Remember that for any function f, with an inverse f^{-1}, we have the following relationship:

$$f(f^{-1}(x)) = x \quad \text{and} \quad f^{-1}(f(x)) = x$$

This leads to the following important properties.

TWO PROPERTIES OF LOGARITHMS

If $f(x) = b^x$ and $f^{-1}(x) = \log_b x$, then we see that

$$b^{\log_b x} = x \quad \text{and} \quad \log_b b^x = x$$

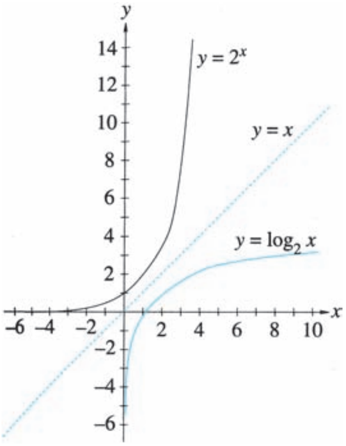

Figure 12.9

Figure 12.9 shows the graph of $y = 2^x$. The line $y = x$ is shown as a dashed line. The reflection of $y = 2^x$ in the line $y = x$ is shown by the colored curve and is the graph of $y = \log_2 x = f^{-1}(2^x)$.

This is a rather awkward method to graph the curve of $y = \log_b x$. Let's find another way to evaluate $\log_b x$ so that we can set up a table of values and plot points, and to connect the points to sketch the graph.

COMMON LOGS; NATURAL LOGS

There is another way to evaluate $\log_b x$, but the ease of doing it depends on the base of the logarithm. There are two bases that are used most often. These are 10 and e. Logarithms that have a base of 10 are called **common logs** and those with a base of e are called **natural logs**.

Because the bases 10 and e are used so often, they have special symbols. The symbol *log*, written with no indicated base, shows that common logs, or base 10 logs, are being used. If natural logs are involved, the symbol *ln* is used.

There are three major ways to find the values of a logarithm. These ways include a calculator, a computer, and a table. We will explain how to use a calculator and a spreadsheet to find the logarithm of a number. Tables are seldom used today. If, or when, you need to use a table of logarithms, you should carefully read the instructions for its use.

FINDING LOGS WITH A CALCULATOR

Look at your calculator. It probably has two keys on it that we have seldom used. One key is LOG, and the other is LN. Now, since $\log_{10} x$ means $\log x$, we can use the LOG key to determine $\log_{10} x$. Similarly, we can use the LN key to determine values of $\log_e x = \ln x$.

EXAMPLE 12.14

Use a calculator to evaluate (a) log 2, (b) log 100, (c) log 9.53, (d) ln 2, (e) ln 12.4, (f) ln 1, and (g) ln 32.4.

SOLUTIONS

	PRESS	DISPLAY
(a)	LOG 2 ENTER	0.3010299957
(b)	LOG 100 ENTER	2
(c)	LOG 9.53 ENTER	0.9790929006
(d)	LN 2 ENTER	0.6931471806
(e)	LN 12.4 ENTER	2.517696473
(f)	LN 1 ENTER	0
(g)	LN 32.4 ENTER	3.478158423

FINDING LOGS WITH A SPREADSHEET

On a spreadsheet, LOG() returns $\log_{10}$ and LN() returns $\log_e$ of the number in parentheses.

EXAMPLE 12.15

Use a spreadsheet to evaluate (a) log 2, (b) log 1000, (c) log 12.725, (d) ln 14.75, (e) ln 1, and (f) ln 2.71826.

SOLUTION The keystrokes and the results are shown in Figure 12.10.

	A	B	C	D
1		Function	Enter	Result
2	(a)	log 2	=LOG(2)	0.30103
3	(b)	log 1000	=LOG(1000)	3
4	(c)	log 12.725	=LOG(12.725)	1.104658
5	(d)	ln 14.75	=LN(14.75)	2.691243
6	(e)	ln 1	=LN(1)	0
7	(f)	ln 2.71826	=LN(2.71826)	0.999992

Figure 12.10

LOGS OF DIFFERENT BASES

Now we see that we can use a calculator or spreadsheet to help us get values of $\ln x$ and $\log x$. We can also use calculators and computers to help us find the value of $\log_b x$. To do this, we use the following relationship.

CHANGE OF BASE FORMULA

$$\log_b x = \frac{\ln x}{\ln b}$$

The relationship $\log_b x = \dfrac{\ln x}{\ln b}$ seems to be an unusual one. It uses some properties of logarithms that we will learn in Section 12.4. But, it works! In fact, it works for any base a of the logarithms on the right-hand side. So, it is true that $\log_b x = \dfrac{\log_a x}{\log_a b}$. Since $5^3 = 125$, we know $\log_5 125 = 3$. By this formula, $\log_5 125 = \dfrac{\ln 125}{\ln 5}$ should also be 3. Try it on your calculator.

PRESS	DISPLAY
125 **LN**	4.8283137
÷ 5 **LN**	1.6094379
=	3

With a spreadsheet LOG(a) returns $\log_{10} a$ and LOG(a, b) returns $\log_b a$.

EXAMPLE 12.16

Use a spreadsheet to evaluate (a) $\log_5 125$ and (b) $\log_9 12\,863.2$.

SOLUTION The keystrokes and the results are shown in Figure 12.11.

	A	B	C	D
1		Function	Enter	Result
2	(a)	log_5 125	=LOG(125,5)	3
3	(b)	lo_9 12,863.2	=LOG(12863.2,9)	4.306399

Figure 12.11

EXAMPLE 12.17

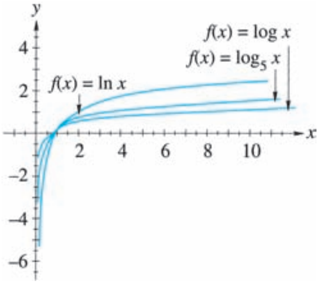

Figure 12.12

Plot the graphs of $\log x$, $\ln x$, and $\log_5 x$.

SOLUTION A table of values follows and the graphs of the functions are shown in Figure 12.12.

x	0.50	1	1.50	2.00	2.50	3.00	3.50	4.00	4.50	5.00	5.50	6.00
$\log x$	−0.30	0	0.18	0.30	0.40	0.48	0.54	0.60	0.65	0.70	0.74	0.78
$\ln x$	−0.69	0	0.41	0.69	0.92	1.10	1.25	1.39	1.50	1.61	1.70	1.79
$\log_5 x$	−0.43	0	0.25	0.43	0.57	0.68	0.78	0.86	0.93	1.00	1.06	1.11

Notice that all three curves cross the x-axis at the point (1, 0). If you look back at Example 12.14(f) or Example 12.15(e), you will see that $\log_b 1 = 0$. All logarithmic curves cross the x-axis at the point (1, 0).

EXERCISE SET 12.3

In Exercises 1–10, rewrite each logarithm in exponential form.

1. $\log_6 216 = 3$
2. $\log_9 6{,}561 = 4$
3. $\log_4 16 = 2$
4. $\log_7 16{,}807 = 5$
5. $\log_{1/7} \frac{1}{49} = 2$
6. $\log_{1/2} \frac{1}{64} = 6$
7. $\log_2 \frac{1}{32} = -5$
8. $\log_3 \frac{1}{243} = -5$
9. $\log_9 2{,}187 = \frac{7}{2}$
10. $\log_8 2{,}048 = \frac{11}{3}$

In Exercises 11–20, rewrite each exponential in logarithmic form.

11. $5^4 = 625$
12. $3^5 = 243$
13. $2^7 = 128$
14. $4^3 = 64$
15. $7^3 = 343$
16. $11^2 = 121$
17. $5^{-3} = \frac{1}{125}$
18. $3^{-5} = \frac{1}{243}$
19. $4^{7/2} = 128$
20. $125^{5/3} = 3{,}125$

In Exercises 21–32, use a calculator, spreadsheet, or computer to evaluate each of these logarithms.

21. $\ln 5$
22. $\ln 19$
23. $\ln 4.751$
24. $\ln 35.62$
25. $\log 4$
26. $\log 23$
27. $\log 12.67$
28. $\log 78.143$
29. $\log_5 8$
30. $\log_3 20$
31. $\log_{12} 16.4$
32. $\log_8 691.45$

In Exercises 33–39, make a table of values and sketch the graph of each function.

33. $f(x) = \ln x$
34. $g(x) = \log x$
35. $h(x) = \log_2 x$
36. $k(x) = \log_3 x$
37. $f(x) = \log_{12} x$
38. $g(x) = \log_{1/2} x$
39. $h(x) = \log_{1/4} x$

Solve Exercises 40–44.

40. ***Seismology*** The *Richter scale* used to be used to measure the magnitude of earthquakes. The formula for the Richter scale is $R = \log I$, where R is the Richter number, and I is the intensity of the earthquake. Express the Richter scale in exponential form.

41. ***Acoustical engineering*** The *decibel (dB) scale* is used for sound intensity. This scale is used because the response of the human ear to sound intensity is not proportional to the intensity. The intensity, $I_0 = 10^{-12}$ W/m^2 (watts/m^2), is just audible and so is given a value of 0 dB. A sound 10 times more intense is given the value 10 dB; a sound $100 = 10^2$ times more intense than 0 dB is given the value 20 dB. Continuing in this manner gives the formula:

$$\beta = 10 \log \tfrac{I}{I_0}$$

where β is the intensity in decibels and I is the intensity in W/m^2. **(a)** Express the decibel scale in exponential notation. **(b)** What is the intensity in decibels of a sound that measures 10^{-7} W/m^2? **(c)** What is the intensity of a heavy truck passing a pedestrian at the side of a road if the sound wave intensity of the truck I is 10^{-3} W/m^2?

42. ***Land management*** To plan for the future needs of a city, the city engineer uses the function:

$$P(t) = 47{,}000 + 9{,}000 \ln(0.7t + 1)$$

to estimate the population t years from now.

(a) What is the city's current population?

(b) What is the expected population in 10 years?

(c) How long will it be until the population reaches 75 000?

43. ***Forestry*** The yield, Y, in total ft^3/acre, of thinned stands of yellow poplar can be predicted by the equation:

$$\ln Y = 5.36437 - 101.16296S^{-1} - 22.00048A^{-1} + 0.97116 \ln \mathrm{BA}$$

where S is the site index, A is the current age of the trees, and BA is the basal area. What is the predicted yield of a 40-year-old stand growing on a site with an index of 110 and a basal area of 90 ft^2 per acre?

44. ***Computer technology*** In designing a PC board, the impedance of a trace depends on the trace width, conductor thickness, and PC board material in a strip-line configuration, as shown by the equation:

$$Z_0 = \frac{87}{\sqrt{\varepsilon' + 1.4}} \ln\left(\frac{6H}{0.8W + t}\right)$$

Solve this equation for t.

45. ***Seismology*** The Richter scale described in Exercise 40 has been replaced with the moment magnitude scale (M_W) that is given by $M_W = \dfrac{\log E - 9}{1.5}$, where E is the energy, in joules, released by the earthquake. Express the moment magnitude scale in exponential form.

46. ***Seismology*** The 2010 earthquake in Haiti was a 7.0-magnitude quake. The 2011 earthquake in Japan measured 9.0. How much stronger was the earthquake in Japan than the one in Haiti?

[IN YOUR WORDS]

47. Explain how the graph of a logarithmic function differs from that of an exponential function.

48. **(a)** What are common logarithms?

(b) What are natural logarithms?

(c) How are common logarithms and natural logarithms alike and how are they different?

12.4 PROPERTIES OF LOGARITHMS

In Section 12.3, we saw that logarithms were the inverses of exponentials. We also saw that we could write each logarithm as an exponential. It is not unexpected that the properties of logarithms must be related to the rules of exponents. We will examine three properties of logarithms and mention three others. These properties used to be very important in helping to calculate logarithms. Today, we use calculators and computers for these calculations, but the properties will be important later when we need to solve equations that involve logarithms or exponents.

PROPERTY 1

Let's consider two numbers, x and y, and consider $\log_b xy$. Suppose we know that $\log_b x = m$ and $\log_b y = n$, or, in exponential form, $b^m = x$ and $b^n = y$. Then,

$$xy = b^m b^n = b^{m+n}$$

Rewriting $xy = b^{m+n}$ in logarithmic form, we get $\log_b xy = m + n = \log_b x + \log_b y$. We have established the first property of logarithms.

LOGARITHMS: PROPERTY 1

If x and y are positive real numbers, $b > 0$, and $b \neq 1$, then

$$\log_b xy = \log_b x + \log_b y$$

EXAMPLE 12.18

Use the first property of logarithms to rewrite each of the following: (a) $\log_4 35$, (b) $\log 21$, (c) $\ln 18$, (d) $\log_3 5x$, (e) $\log_8 2 + \log_8 5$, (f) $\log 5 + \log 2 + \log 6$, (g) $\ln 3 + \ln 13$, and (h) $\log 5 + \log a$.

SOLUTIONS

(a) $\log_4 35 = \log_4 (5 \cdot 7) = \log_4 5 + \log_4 7$

(b) $\log 21 = \log (3 \cdot 7) = \log 3 + \log 7$

(c) $\ln 18 = \ln(2 \cdot 3 \cdot 3) = \ln 2 + \ln 3 + \ln 3$

(d) $\log_3 5x = \log_3 5 + \log_3 x$

(e) $\log_8 2 + \log_8 5 = \log_8 (2 \cdot 5) = \log_8 10$

(f) $\log 5 + \log 2 + \log 6 = \log (5 \cdot 2 \cdot 6) = \log 60$

(g) $\ln 3 + \ln 13 = \ln(3 \cdot 13) = \ln 39$

(h) $\log 5 + \log a = \log 5a$

PROPERTY 2

For the second property, let's consider $\log_b \frac{x}{y}$. Again, if we let $x = b^m$ and $y = b^n$, we have $\frac{x}{y} = \frac{b^m}{b^n} = b^{m-n}$. Now, $\log_b b^{m-n} = m - n = \log_b x - \log_b y$. This gives us the following property of logarithms.

LOGARITHMS: PROPERTY 2

If x and y are positive real numbers, $b > 0$, and $b \neq 1$, then

$$\log_b \frac{x}{y} = \log_b x - \log_b y$$

CAUTION The symbol $\frac{\log_a x}{\log_b y}$ is not the same as $\log_b \frac{x}{y}$. In particular, $\frac{\log_b x}{\log_b y} \neq \log_b x - \log_b y$.

EXAMPLE 12.19

Use the second property of logarithms to rewrite each of the following: (a) $\log \frac{3}{5}$, (b) $\ln \frac{5}{4}$, (c) $\log_5 \frac{7}{x}$, (d) $\ln 8 - \ln 2$, (e) $\log_3 5 - \log_3 2$, and (f) $\log 8x^2 - \log 2x$.

SOLUTIONS

(a) $\log \frac{3}{5} = \log 3 - \log 5$

(b) $\ln \frac{5}{4} = \ln 5 - \ln 4$

(c) $\log_5 \frac{7}{x} = \log_5 7 - \log_5 x$

(d) $\ln 8 - \ln 2 = \ln \frac{8}{2} = \ln 4$

(e) $\log_3 5 - \log_3 2 = \log_3 \frac{5}{2}$

(f) $\log 8x^2 - \log 2x = \log \frac{8x^2}{2x} = \log 4x$

PROPERTY 3

Now let's consider x^p. If $m = \log_b x$, then $x = b^m$ and $x^p = (b^m)^p = b^{mp}$. So, if $x^p = b^{mp}$, then $\log_b x^p = mp$, and since $m = \log_b x$, we have the following third property of logarithms.

LOGARITHMS: PROPERTY 3

If x and y are positive real numbers, $b > 0$, and $b \neq 1$, then

$$\log_b x^p = p \log_b x.$$

EXAMPLE 12.20

Use the third property of logarithms to rewrite each of the following: (a) $\log 5^3$, (b) $\log_2 16^5$, (c) $\log 100^{3.4}$, (d) $\log \sqrt[3]{25}$, and (e) $2 \log 5 + 3 \log 4 - 4 \log 2$.

SOLUTIONS

(a) $\log 5^3 = 3 \log 5$

(b) $\log_2 16^5 = 5 \log_2 16$

(c) $\log 100^{3.4} = \log (10^2)^{3.4} = \log 10^{6.8} = 6.8 \log 10 = 6.8$

(d) $\log \sqrt[3]{25} = \log 25^{1/3} = \frac{1}{3} \log 25$

(e)
$$\begin{aligned} 2 \log 5 + 3 \log 4 - 4 \log 2 &= \log 5^2 + \log 4^3 - \log 2^4 \\ &= \log 25 + \log 64 - \log 16 \\ &= \log \frac{25 \cdot 64}{16} = \log 100 \\ &= 2 \end{aligned}$$

PROPERTIES 4, 5, AND 6

In addition to these properties, there are three other properties of logarithms. These three properties are listed in the following box.

LOGARITHMS: PROPERTIES 4, 5, AND 6

If x and y are positive real numbers, $b > 0$, and $b \neq 1$, then

$\log_b 1 = 0$	Property 4
$\log_b b = 1$	Property 5
$\log_b b^n = n$	Property 6

The properties of logarithms allow us to simplify the logarithms of products, quotients, powers, and roots.

CAUTION There is no way to simplify logarithms of sums or differences. You cannot change $\log_b (x + y)$ to $\log_b x + \log_b y$. This is very tempting, but do not make this error. Remember, $\log_b (x + y) \neq \log_b x + \log_b y$.

Logarithms were originally developed to help people compute. Slide rules were developed as a computational tool based on logarithms. At one time, every technician had, and knew how to use, a slide rule. Electronic calculators and the microcomputers have replaced slide rules and reduced the importance of logarithms as a help in computing.

However, logarithms are still an important tool in working many problems. In the remainder of this section, we will demonstrate the properties of logarithms. Values are sometimes obtained from a table of logarithms. We will not expect you to get values from tables. These exercises will prepare you for the next section where you will solve equations that involve logarithms.

EXAMPLE 12.21

If $\log 2 = 0.3010$, $\log 3 = 0.4771$, and $\log 5 = 0.6990$, determine each of the following: (a) $\log 6$, (b) $\log 81$, (c) $\log 1.5$, (d) $\log \sqrt{5}$, and (e) $\log 50$.

SOLUTIONS

(a) $\log 6 = \log(2 \cdot 3) = \log 2 + \log 3 = 0.3010 + 0.4771 = 0.7781$

(b) $\log 81 = \log 3^4 = 4 \log 3 = 4(0.4771) = 1.9084$

(c) $\log 1.5 = \log \frac{3}{2} = \log 3 - \log 2 = 0.4771 - 0.3010 = 0.1761$

(d) $\log \sqrt{5} = \log 5^{1/2} = \frac{1}{2} \log 5 = \frac{1}{2}(0.6990) = 0.3495$

(e) $\log 50 = \log(5 \cdot 10) = \log 5 + \log 10 = 0.6990 + 1 = 1.6990$

Use a calculator to check each of these answers.

EXERCISE SET 12.4

In Exercises 1–12, write each logarithm as the sum or difference of two or more logarithms.

1. $\log \frac{2}{3}$
2. $\log \frac{5}{4}$
3. $\log 14$
4. $\log 55$
5. $\log 12$
6. $\log 28$
7. $\log \frac{150}{7}$
8. $\log \frac{588}{5x}$
9. $\log 2x$
10. $\log \frac{1}{5x}$
11. $\log \frac{2ax}{3y}$
12. $\log \frac{4bc}{3xy}$

Express each of Exercises 13–24 as a single logarithm.

13. $\log 2 + \log 11$
14. $\log 3 + \log 13$
15. $\log 11 - \log 3$
16. $\log 17 - \log 23$
17. $\log 2 + \log 2 + \log 3$
18. $\log 3 + \log 3 + \log 5$
19. $\log 4 + \log x - \log y$
20. $\log 5 + \log 7 - \log 11$
21. $5 \log 2 + 3 \log 5$
22. $7 \log 3 - 2 \log 8$
23. $\log \frac{2}{3} + \log \frac{6}{7}$
24. $\log \frac{3}{24} + \log \frac{4}{7} - \log \frac{1}{3}$

If $\log 2 = 0.3010$, $\log 3 = 0.4771$, and $\log 5 = 0.6990$, determine the value of each logarithm in Exercises 25–36.

25. $\log 8$
26. $\log 9$
27. $\log 12$
28. $\log 36$
29. $\log 15$
30. $\log 30$
31. $\log \frac{75}{2}$
32. $\log \frac{45}{8}$
33. $\log 200$
34. $\log 150$
35. $\log 5\,000$
36. $\log 4\,500$

If $\log_b 2 = 0.3869$, $\log_b 3 = 0.6131$, and $\log_b 5 = 0.8982$, determine the value of each logarithm in Exercises 37–40.

37. $\log_b 16$

38. $\log_b 15$

39. $\log_b \frac{5}{3}$

40. $\log_b \frac{24}{5}$

Solve Exercises 41–44.

41. Use the change of base formula to determine the base for the logarithms in Exercises 37–40. (*Hint:* The answer is an integer. You may not get an integer, because the values have been rounded off to four decimal places.)

42. ***Electronics*** The gain or loss of power can be determined by the formula:

$$N = 20(\log I_1 - \log I_2) + 10(\log R_1 - \log R_2)$$

Simplify this formula.

43. ***Acoustical engineering*** Decibel gain or loss, Δ dB, can be computed by the formula:

$$\Delta\text{dB} = 20\left(\log I_2 + \frac{1}{2}\log R_2 - \log I_1 - \frac{1}{2}\log R_1\right)$$

Simplify the right-hand side of this equation.

44. ***Ecology*** As a result of pollution, the population of fish in a certain river decreases according to the formula $\ln\left(\frac{P}{P_0}\right) = -0.0435t$, where P is the population after t years and P_0 is the original population.

(a) After how many years will there be only 50% of the original fish population remaining?

(b) After how many years will the original population be reduced by 90%?

(c) What percentage will die in the first year of pollution?

[IN YOUR WORDS]

45. Explain Properties 4, 5, and 6 in your own words. Why do you think these three properties are especially useful?

46. **(a)** Explain the differences between $\log_b x - \log_b y$, $\log_b (x - y)$, and $\log_b x - y$.

(b) Graph $y_1 = \ln x - \ln 4x$, $y_2 = \ln (x - 4x)$, and $y_3 = \ln x - 4x$.

(c) Do your graphs in (b) support your explanations in (a)? If not, examine your work in both parts (a) and (b) and make any necessary changes.

12.5 EXPONENTIAL AND LOGARITHMIC EQUATIONS

We mentioned in Section 12.4 that logarithms were originally developed as an aid for computation. The wide use of electronic calculators decreased the importance of logarithms as a help for calculation. In this section, we will focus on ways we can use logarithms to help solve equations.

EXPONENTIAL EQUATIONS

We will start with an equation that is relatively easy. For one thing, we already know the answer. This will help us see that the method works.

EXPONENTIAL EQUATIONS

An **exponential equation** is any equation in which the variable is an exponent.

STEPS FOR SOLVING EXPONENTIAL EQUATIONS

In solving an exponential equation, you should:

1. Use the properties of exponents to rewrite each side of the equation in terms of exponents with the same base.
2. Use the fact that $x = b^y$ is equivalent to $\log_b x = y$ to rewrite the equation.
3. Solve the resulting equation.

EXAMPLE 12.22

Solve $3^x = 81$.

SOLUTION Since $3^4 = 81$ we know that $x = 4$. Now, let's see how we can solve this equation algebraically using techniques that we can use to solve more difficult problems.

To solve this equation, we will take the logarithm of both sides. We can use any base for the logarithm, but it is easiest to use one of the bases that is on a calculator. We will first use base 10 and then solve the same equation using base e to show that it will work either way:

$$\begin{aligned} 3^x &= 81 \\ \log 3^x &= \log 81 \end{aligned}$$

Since $\log 3^x = x \log 3$, we have

$$\begin{aligned} x \log 3 &= \log 81 \\ x &= \frac{\log 81}{\log 3} \\ &= 4 \end{aligned}$$

NOTE Notice that $\log 81 \div \log 3 = \dfrac{\log 81}{\log 3} = 4$ and that $\log \frac{81}{3} = \log 81 - \log 3 \approx 1.431$. Since $4 \neq 1.431$, we see that $\log 81 \div \log 3 = \dfrac{\log 81}{\log 3} \neq \log \frac{81}{3}$.

Let's solve this same equation using natural logarithms:

$3^x = 81$	
$\ln 3^x = \ln 81$	Take ln of both sides.
$x \ln 3 = \ln 81$	Use Property 3 of logs.
$x = \dfrac{\ln 81}{\ln 3} = 4$	Divide.

You could have used the properties of logarithms in the last step of Example 12.22 and the above comment. Thus, you could have solved Example 12.22 as

$$x = \frac{\log 81}{\log 3} = \frac{\log 3^4}{\log 3} = \frac{4 \log 3}{\log 3} = 4$$

or, using natural logarithms:

$$x = \frac{\ln 81}{\ln 3} = \frac{\ln 3^4}{\ln 3} = \frac{4 \ln 3}{\ln 3} = 4$$

EXAMPLE 12.23

Solve $5^{2x+1} = 25^{4x-1}$.

SOLUTION Notice that $25 = 5^2$, so we can write this equation as

$$\begin{aligned} 5^{2x+1} &= 25^{4x-1} \\ &= (5^2)^{4x-1} \\ &= 5^{8x-2} \end{aligned}$$

Taking $\log_5$ of both sides produces

$$\begin{aligned} \log_5 (5^{2x+1}) &= \log_5 (5^{8x-2}) \\ \text{or} \quad (2x + 1)\log_5 5 &= (8x - 2)\log_5 5 \end{aligned}$$

which simplifies to

$$\begin{aligned} 2x + 1 &= 8x - 2 \\ 3 &= 6x \\ \tfrac{1}{2} &= x \end{aligned}$$

LOGARITHMIC EQUATIONS

Just as we used logarithms to solve equations that involved exponentials, we can use exponentials to solve problems that involve logarithms.

LOGARITHMIC EQUATIONS

A **logarithmic equation** is an equation that contains a logarithm of the variable.

STEPS FOR SOLVING LOGARITHMIC EQUATIONS

In solving a logarithmic equation, you should:

1. Use the properties of logarithms to combine all the logarithmic terms into one.
2. Use the fact that $\log_b x = y$ is equivalent to $x = b^y$ to rewrite the equation.
3. Solve the resulting equation.

EXAMPLE 12.24 Solve $\ln x = 7$.

SOLUTION The base in this equation is e, so $\ln x$ is the same as $\log_e x$ and this is equivalent to $\log_e x = 7$. Using the fact that $\log_e x = y$ is the same as $x = e^y$, we get

$$\begin{aligned} x &= e^7 \\ &\approx 1{,}096.6 \end{aligned}$$

It is important that you remember to use the correct base when you solve a logarithmic equation. The next example will show you how to proceed with a more complicated situation. It also uses a different base; in this case, the base is 10.

EXAMPLE 12.25 Solve $\log(3x - 5) + 2 = \log 4x$.

SOLUTION We will first combine this into one logarithm:

$$\begin{aligned} \log(3x - 5) + 2 &= \log 4x \\ \log(3x - 5) - \log 4x &= -2 \\ \log\left(\frac{3x - 5}{4x}\right) &= -2 \end{aligned}$$

We will now use the fact that $\log_b x = y$ is equivalent to $x = b^y$. Since we are using common logarithms, the base is 10:

$$\begin{aligned} \frac{3x - 5}{4x} &= 10^{-2} \\ \frac{3x - 5}{4x} &= 0.01 \end{aligned}$$

We solve this as we would any fractional equation:

$$\begin{aligned} \frac{3x - 5}{4x} &= 0.01 \\ 3x - 5 &= 0.04x \\ 2.96x &= 5 \\ x &= \frac{5}{2.96} \approx 1.69 \end{aligned}$$

NOTE The approximate answer 1.69 does not check, but the exact answer $\frac{5}{2.96}$ does check.

BUSINESS AND FINANCE

An interesting equation results from the work with exponentials and we can use logarithms to solve it. If a certain amount of money P was invested at rate r compounded continuously, the investment would be worth

$$S = Pe^{rt}$$

after t years. How long does it take for the money to double?

If it doubles, then it is worth $2P$ and we have

$$2P = Pe^{rt}$$

$$\text{or} \quad 2 = e^{rt}$$

Taking the natural logarithm of both sides, we obtain

$$\ln 2 = \ln e^{rt}$$

$$\text{or} \quad \ln 2 = rt$$

Solving for t results in

$$t = \frac{\ln 2}{r}$$

It will take $\frac{\ln 2}{r}$ year for the money to double. This same formula will apply to anything that is growing or decaying at an exponential rate.

APPLICATION BUSINESS

EXAMPLE 12.26

How long will it take for $1 000 to double at 5% compounded continuously?

SOLUTION We use the formula:

$$t = \frac{\ln 2}{r}$$

and since

$$r = 5\% = 0.05,$$

$$t = \frac{\ln 2}{0.05} \approx 13.86$$

APPLICATION GENERAL TECHNOLOGY

EXAMPLE 12.27

What is the half-life of a radioactive material that decays at the rate of 1% per year?

SOLUTION The same formula will apply; only here $r = 1\% = 0.01$.

$$t = \frac{\ln 2}{0.01} \approx 69.31 \text{ years}$$

EXERCISE SET 12.5

In Exercises 1–24, solve each equation for the indicated variable. You may want to use a calculator.

1. $5^x = 29$
2. $6^y = 32$
3. $4^{6x} = 119$
4. $3^{-7x} = 4$
5. $3^{y+5} = 16$
6. $2^{y-7} = 67$
7. $e^{2x-3} = 10$
8. $4^{3y+5} = 30$
9. $3^{4x+1} = 9^{3x-5}$
10. $2^{3-2x} = 8^{1+2x}$
11. $e^{3x-1} = 5e^{2x}$
12. $3^{2-5x} = 8(9^{x-1})$
13. $\log x = 2.3$
14. $\ln x = 5.4$
15. $\log (x-5) = 17$
16. $\ln (y+3) = 19$
17. $\ln 2x + \ln x = 9$
18. $2 \ln x = \ln 4$
19. $2 \ln 3x + \ln 2 = \ln x$
20. $2 \ln (x+3) = \ln x + 4$
21. $\log (x-1) = 2$
22. $\log (x-4) = \log x - 4$
23. $2 \log (x-4) = 2 \log x - 4$
24. $\ln x = 1 + 3 \ln x$

Solve Exercises 25–42.

25. ***Finance*** The amount of \$5000 is placed in a savings account, where interest is compounded continuously at the rate of 6% per year. How long will it take for this amount to be doubled?

26. ***Finance*** How long will it take the money in Exercise 25 to double if the rate is 8% per year?

27. ***Nuclear energy*** A radioactive substance decays at the rate of 0.5% per year. What is the half-life of this substance?

28. ***Nuclear energy*** Another radioactive substance decays at the rate of 3% per year. What is its half-life?

29. ***Nuclear energy*** The half-life of tritium is 12.5 years. What is its annual rate of decay?

30. ***Nuclear energy*** The half-life of the sodium isotope ${}^{24}_{11}\text{Na}$ against beta decay is 15 h. What is the rate of decay?

31. ***Finance*** A person has some money to invest and would like to double the investment in 8.5 years. What annual rate of interest, compounded continuously, will be needed in order for this to be accomplished?

32. ***Automotive technology*** In a chrome-electroplating process, the mass m in grams of the chrome plating increases according to the formula $m = 200 - 2^{t/2}$, where t is the time in minutes. How long does it take to form 100 g of plating?

33. ***Environmental science*** In chemistry, the pH of a substance is defined by $\text{pH} = -\log [\text{H}^+]$, where $[\text{H}^+]$ is the concentration of hydrogen ions in the substance measured in moles per liter. The pH of distilled water is 7. A substance with a pH less than 7 is known as an *acid*. A substance with a pH greater than 7 is a *base*. Rain and snow have a natural concentration of $[\text{H}^+] = 2.5 \times 10^{-6}$ moles per liter. What is the natural pH of rain and snow?

34. ***Environmental science*** The pH of some acid rain is 5.3. What is the concentration of hydrogen ions in acid rain?

35. ***Meteorology*** The barometric equation,

$$H = (30T + 8{,}000) \ln \frac{P_0}{P}$$

relates the height H in meters above sea level, the air temperature T in degrees Celsius, the atmospheric pressure P_0 in centimeters of mercury at sea level, and the atmospheric pressure P in centimeters of mercury at height H. Atmospheric pressure at the summit of Mt. Whitney in California on a certain day is 43.29 cm of mercury. The average air temperature is -5°C and the atmospheric pressure at sea level is 76 cm of mercury. What is the height of Mt. Whitney?

36. ***Environmental science*** In Exercise Set 12.3, Exercise 41, we saw that the loudness in

decibels β of a noise is given by the formula $\beta = 10 \log \frac{I}{I_0}$, where $I_0 = 10^{-12}$ W/m^2, and I is the intensity of the noise in W/m^2. At takeoff, a certain jet plane has a noise level of 105 dB. What is the intensity I of the sound wave produced by this airplane?

37. ***Electronics*** The formula for the exponential decay of electric current is given by the formula $Q = Q_0 e^{-t/T}$, where $T = RC$. (See Example 12.12) If $Q_0 = 0.40$ A, $R = 500\ \Omega$, and $C = 100\ \mu$F, what is t when $Q = 0.05$ A?

38. ***Thermodynamics*** The temperature T of an object after a period of time t is given by $T = T_0 + ce^{-kt}$, where T_0 is the temperature of the surrounding medium, and $c = T_I - T_0$, where T_I is the initial temperature of the object. A steel bar with a temperature of 1 200°C is placed in water with a temperature of 20°C. If the rate of cooling k is 8% per hour, how long will it take for the steel to reach a temperature of 40°C?

39. ***Electronics*** In an ac circuit, the current I at any time t, in seconds, is given by $I = I_0(1 - e^{-Rt/L})$, where I_0 is the maximum current, L is the inductance, and R the resistance. If a circuit has a 0.2-H inductor, a resistance of 4 Ω, and a maximum current of 1.5 A, at what instant does the current reach 1.4 A?

40. ***Forestry*** The yield, Y, in total ft^3/acre, of thinned stands of yellow poplar can be predicted by the equation:

$$\ln Y = 5.36437 - 101.16296\,S^{-1} - 22.00048\,A^{-1} + 0.97116 \ln \text{BA}$$

where S is the site index, A is the current age of the trees, and BA is the basal area. What is the basal area if the predicted yield of a 60-year-old stand growing on a site with an index of 110 is 4 720 ft^3/acre?

41. ***Medical technology*** An implantable pacemaker normally has a capacitive output. The amplitude of emitted pulses declines with time according to the equation:

$$E = \frac{U^2C}{2}(1 - e^{-2t/RC})$$

where C is the capacitance of the pacemaker's output capacitor in farads (F), delivering the impulse to the heart, U is the pulse voltage in volts (V), E is the energy in joules (J), I is the pulse current in amperes (A), and t is the pulse duration in seconds (s). Solve this equation for t.

42. ***Medical technology*** In Exercise 41, the following formula was given for the amplitude of emitted pulses of an implantable pacemaker:

$$E = \frac{U^2C}{2}(1 - e^{-2t/RC})$$

Solve this equation for R.

[IN YOUR WORDS]

43. Without looking in the text, describe the steps for solving logarithmic equations.

44. Explain the difference between a logarithmic equation and an exponential equation.

12.6 GRAPHS USING SEMILOGARITHMIC AND LOGARITHMIC PAPER

Exponential functions often produce some very large numbers. This makes graphing exponential functions difficult to show on normal graph paper. Look at the graph of $f(x) = 2^x$ in Figure 12.1b. By the time x is 5, y is 32; when $x = 6$,

$y = 64$. This presents a problem when trying to represent a large portion of an exponential graph. Semilogarithmic and logarithmic graph papers have been developed to allow for the plotting of a large range of values.

SEMILOGARITHMIC GRAPH PAPER

Semilogarithmic or **semilog graph paper** has one of the axes (usually the y-axis) marked off in distances proportional to the logarithms of numbers. Thus, the distances between lines on this axis are not equally spaced. The other axis has the distance between lines equally spaced. The result of all this is that the graph of an exponential function $y = b^x$ is a straight line when it is drawn on semilog paper.

EXAMPLE 12.28

Sketch the graph of $y = 2^x$ on semilog paper.

SOLUTION Since this is the same curve we graphed in Figure 12.1b, we will use the same table of values from that example. The table is reproduced here.

x	-3	-2	-1	0	1	2	3	4	5
$f(x) = 2^x$	0.125	0.25	0.5	1	2	4	8	16	32

Along the x-axis, we label the points -3 through 5. Along the y-axis, the points are labeled as indicated in Figure 12.13. Notice that each unit in the colored interval has been labeled 1, 2, 3, . . ., 10. The interval directly above it has each unit representing a multiple of 10 (10, 20, 30, etc.). In the next interval, the units would be multiples of 100 (100, 200, 300, etc.). The interval directly below the colored one has each unit represented in tenths (0.1, 0.2, 0.3, etc.), and the interval below that would be in hundredths (0.01, 0.02, 0.03, etc.).

EXAMPLE 12.29

Sketch the graph of $y = 5(4^x)$ on semilog paper.

SOLUTION A table of values from $x = -2$ to $x = 5$ is given here.

x	-2	-1	0	1	2	3	4	5
$f(x) = 5(4^x)$	0.31	1.25	5	20	80	320	1,280	5,120

The graph of this curve on semilog paper is shown in Figure 12.14. You should study the scale of the y-axis to be sure that you understand how the units are marked.

Study Figures 12.13 and 12.14. Both of these graphs are straight lines. Yet, in both cases we started with exponential equations of the form $y = b^x$.

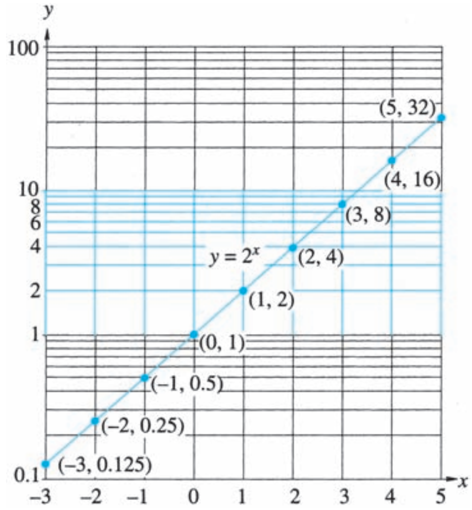

Figure 12.13

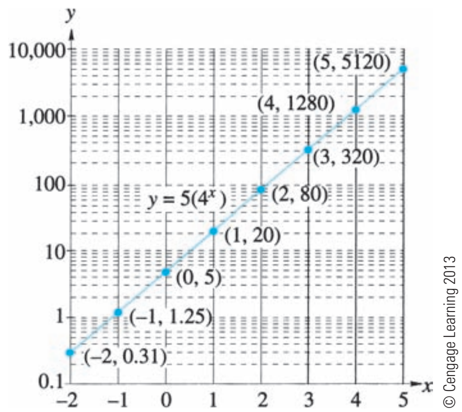

Figure 12.14

NOTE One of the reasons for using semilog paper is that graphs of exponential functions of the form $y = b^x$ will be straight lines when they are drawn on semilogarithmic paper.

SEMILOGARITHMIC GRAPHING WITH A SPREADSHEET

Most graphing calculators cannot be used to construct semilog graphs; but a spreadsheet can be used for semilog graphs. We will use a spreadsheet to draw the graphs with both rectangular and semilog axes of the data in Table 12.9.

EXAMPLE 12.30

In Table 12.9 we saw each new half-life of an original amount of 1000 g of radioactive material. The table is copied below. Use a spreadsheet to draw both rectangular and semilog graphs of the data in Table 12.9.

TABLE 12.9 Amount of Radioactive Material Remaining After Each Half-Life

No. of half-lives, n	0	1	2	3	4	5	6	7
Amount remaining, g	1000	500	250	125	62.5	31.25	15.625	7.8125

SOLUTION Figure 12.15a shows the data in that table graphed on a normal rectangular graph with an exponential trendline connecting the points. This was graphed in the same manner we have been using.

To change this to a semilog graph, right click on the axis you want in logarithmic scale, in this case the vertical axis. Then click on "Format Axis . . ." and under Axis Options check the box to the left of "Logarithmic scale," as shown in Figure 12.15b. Click **Close** and you should see a result like the one in Figure 12.15c.

Figure 12.15c graphically demonstrates that half-life is an exponential function.

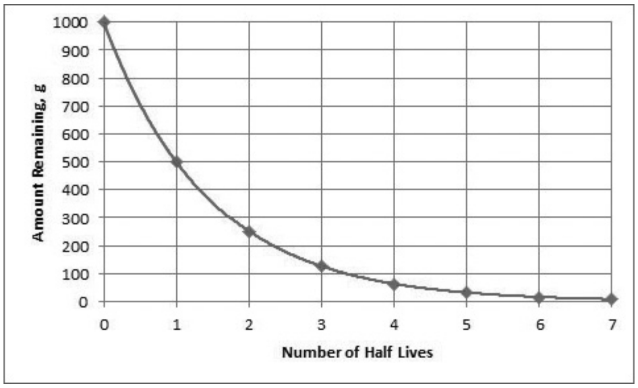

Figure 12.15a

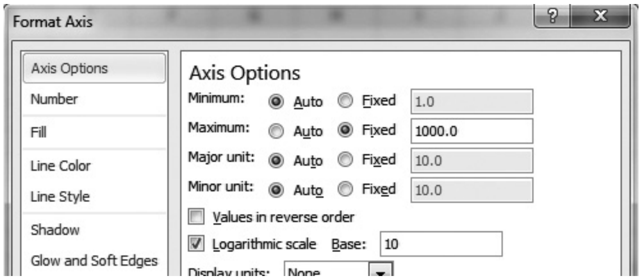

Figure 12.15b

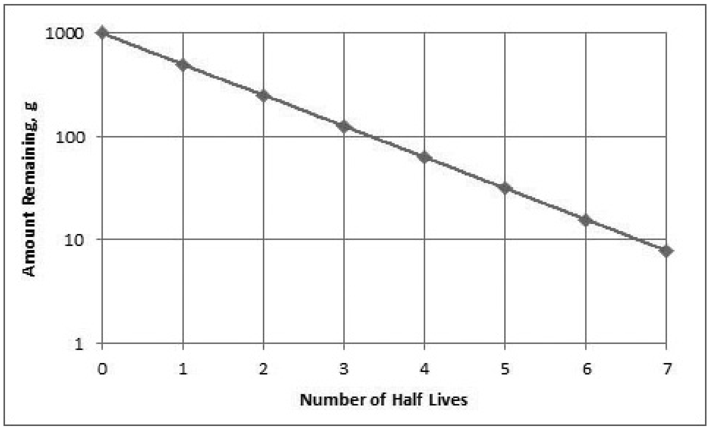

Figure 12.15c

LOGARITHMIC GRAPH PAPER

Use semilog paper when you want to indicate a large range of values for one of the variables. When this is needed for both variables, use **logarithmic** or **log-log graph paper**. Both axes on log-log paper are marked off in logarithmic scales. A function of the type $y^b = x^a$ or $y = x^{a/b}$ graphs as a straight line on log-log paper.

EXAMPLE 12.31

Sketch the graph of $y^3 = x^2$ on log-log paper.

SOLUTION The equation $y^3 = x^2$ is equivalent to $y = x^{2/3}$. If we select values of x that are perfect cubes, we then get integer values for y. The following table gives some values.

x	-1	0	$\frac{1}{8} = 0.125$	1	8	27	64	125	216	1000
$y = x^{2/3}$	1	0	$\frac{1}{4} = 0.25$	1	4	9	16	25	36	100

In Figures 12.16a and 12.16b, we show the sketch of this curve, with Figure 12.16b graphed using a spreadsheet. Notice the way in which both axes are labeled.

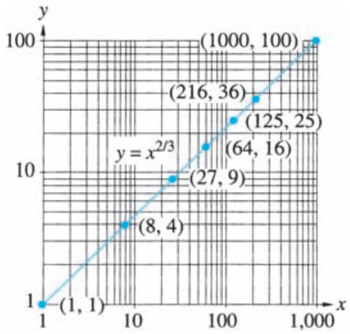

Figure 12.16a

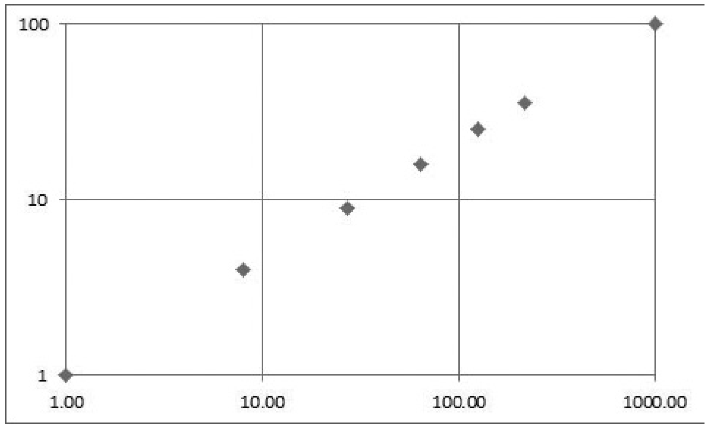

Figure 12.16b

NOTE Negative values of x and the point (0, 0) do not appear on the graph. Since a logarithmic scale contains only positive values, the coordinates that are not positive cannot be plotted.

Semilog and log-log paper can be used to graph functions that may not graph as a straight line. This happens when a large range of values needs to be graphed, particularly when the data are exponential in nature or when the data are gathered from an experiment.

BOTH SEMILOGARITHMIC AND LOGARITHMIC GRAPH PAPER

Sometimes it is necessary to graph the data on both semilog and log-log graph paper.

USING GRAPHS TO HELP DETERMINE AN EQUATION

If the graph of data is a straight line on semilog paper, then the data are related by a function of the form $y = ab^x$.

If the graph of data is a straight line with slope m on log-log paper, then the data are related by a function of the form $y = ax^m$. A function of this form, $y = ax^m$, is called a *power function.*

APPLICATION

EXAMPLE 12.32

Data from a certain experiment result in the following table.

x	1	2	3	4	5	6
y	5	40	135	320	625	1,080

Determine an equation that relates y as a function of x.

SOLUTION We will plot this data on both semilog graph paper and log-log graph paper. When the points are connected, we get the curves shown in Figures 12.17a and 12.17b.

The curve in Figure 12.17b is a straight line. This curve was drawn on log-log graph paper, so we conclude that these data satisfy a power function of the form $y = ax^m$.

The slope of this straight line is m, and if (x_1, y_1) and (x_2, y_2) are two points on the curve, we have

$$m = \frac{\log y_2 - \log y_1}{\log x_2 - \log x_1}$$

If we use the points (1, 5) and (2, 40), we will get

$$\begin{aligned} m &= \frac{\log 40 - \log 5}{\log 2 - \log 1} \\ &\approx \frac{1.602059991 - 0.698970004}{0.301029996 - 0} \\ &= \frac{0.903089987}{0.301029996} \\ &= 3 \end{aligned}$$

Thus, we know that these data are of the form $y = ax^3$.

To determine a, we select one pair of values from the data (3, 135) and get

$$\begin{aligned} y &= ax^3 \\ 135 &= a(3)^3 \\ &= 27a \\ a &= \frac{135}{27} \\ &= 5 \end{aligned}$$

The data in the table satisfy the equation $y = 5x^3$.

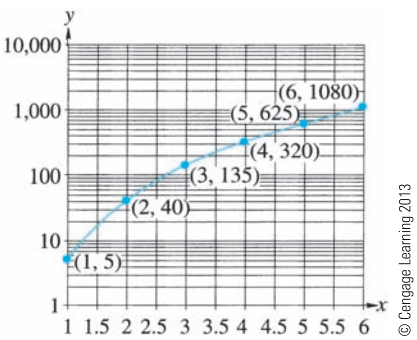

Figure 12.17a

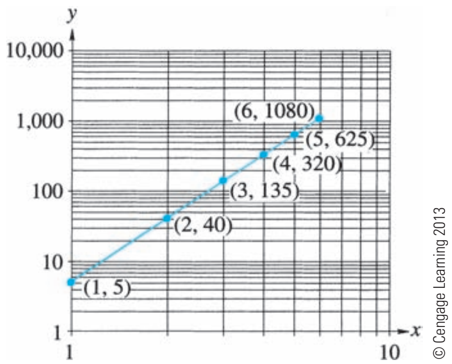

Figure 12.17b

Unlike Example 12.32, data from an actual experiment will seldom fit a curve exactly. In that case, we select points on a curve that seem to "best fit" the actual data points.

EXERCISE SET 12.6

In Exercises 1–12, sketch the graphs of the given functions on semilog graph paper or with a spreadsheet.

1. $y = 3^x$
2. $y = 5^x$
3. $y = 3(6^x)$
4. $y = 5(7^x)$
5. $y = 2^{-x}$
6. $y = 8^{-x}$
7. $y = x^4$
8. $y = x^5$
9. $y = 8x^3$
10. $y = 5x^4$
11. $y = 3x^2 + 4x$
12. $y = 8x^3 + 2x^2 + x$

In Exercises 13–24, sketch the graphs of the given functions on log-log graph paper or with a spreadsheet.

13. $y = x^{1/2}$
14. $y = x^{1/3}$
15. $y = 2x^3$
16. $y = 3x^5$
17. $y = x^2 + \sqrt{x}$
18. $y = x^3 + x$
19. $y = 5x^{-1}$
20. $y = 8x^{-3}$
21. $y^2 = 4x^3$
22. $y^4 = 6x^5$
23. $x^2 y^3 = 8$
24. $x^3 y = 100$

In Exercises 25–34, sketch the indicated graphs either on paper or with a spreadsheet.

25. ***Meteorology*** The atmospheric pressure P at a given altitude h, in feet, is given by $P = P_0e^{-kh}$, where P_0 and k are constants. On semilog paper or with a spreadsheet, plot P in atmospheres and h, the altitude, in feet, for $0 \le h \le 10^4$, $P_0 = 1$ atmosphere, and at 0°C, $k = 1.25 \times 10^{-4}$.

26. ***Meteorology*** The atmospheric pressure in kilopascals (kPa) is given approximately by $P = 100e^{-0.3h}$, where h is the altitude in kilometers. Graph P vs. h on semilog paper or with a spreadsheet.

27. ***Electricity*** The resistance R of a copper wire varies inversely as the square of its cross-sectional diameter D. Thus $RD^2 = k$. If $k = 0.9$ Ω·mm, graph R vs. D on log-log paper or with a spreadsheet, for $D = 0.1$–10 mm.

28. ***Physics*** Boyle's law states that at a constant temperature, the volume V of a sample of gas is inversely proportional to the absolute pressure applied to the gas P. Thus, $PV = C$, a constant. If $C = 5$ atm·ft^3, use log-log paper or a spreadsheet to plot the graph of P in atmospheres vs. V in cubic feet. Let values of P range from 0.1 to 10.

29. An experimenter gathered the data in the following table. Plot the data on semilog and log-log papers (or use a spreadsheet) and determine the equation that relates y as a function of x.

x	1	2	5	10
y	1,000	250	40	10

30. Repeat Exercise 29 for the data in this table.

x	1	2	3	4	5
y	16	80	400	2 000	10 000

31. ***Electronics*** For a certain electric circuit, the voltage is given by $V = e^{-0.25t}$. Graph V vs. t on semilog graph paper or with a spreadsheet for $0 \le t \le 5$.

32. ***Electronics*** For a certain electric circuit, the voltage is given by $V = e^{-0.35t}$. Graph V vs. t on semilog graph paper or with a spreadsheet for $0 \le t \le 10$.

33. ***Electronics*** The following data show the voltage across a capacitor during discharge.

Time, t (s)	6	10	18	26	34
Voltage, V (V)	106.8	67.0	26.4	10.4	4.1

Plot the data on semilog paper or use a spreadsheet and determine the equation that relates V as a function of t.

34. ***Dynamics*** A solid object is dropped from a tall building. The following data shows the distance the object has fallen after each second.

Time, t (s)	0	1	2	3	4	5
Distance, d (m)	0.0	4.9	19.6	44.1	78.4	122.5

Plot the data on log-log paper or with a spreadsheet and determine the equation that relates d as a function of t.

[IN YOUR WORDS]

35. Explain the difference between semilog and log-log graph paper.

36. Explain how a graph of a function drawn on semilog or log-log graph paper can help you determine the equation that describes that function.

CHAPTER 12 REVIEW

IMPORTANT TERMS AND CONCEPTS

Common logs
Exponential decay
Exponential equation
Exponential function
Exponential growth
Logarithmic equation
Logarithmic function
Logarithmic graph paper
Natural logs
Semilogarithmic graph paper

REVIEW EXERCISES

In Exercises 1–8, use a calculator to evaluate each number.

1. e^5
2. e^{-7}
3. $e^{4.67}$
4. $e^{-3.91}$
5. log 8
6. log 196.5
7. ln 81.3
8. ln 325.6

In Exercises 9–16, make a table of values and graph each of these functions on ordinary graph paper over the indicated domains.

9. $f(x) = 5e^x$ $(x = -2 \text{ to } 4)$
10. $g(x) = 9e^{-0.5x}$ $(x = -4 \text{ to } 2)$
11. $h(x) = 5^x$ $(x = -3 \text{ to } 2)$
12. $k(x) = 2.1^{-x}$ $(x = -4 \text{ to } 2)$
13. $f(x) = \log x$ $(0 \le x \le 10)$
14. $g(x) = 3 \log 2x$ $(0 \le x \le 10)$
15. $h(x) = \ln(2x + 1)$ $(-\frac{1}{2} \le x \le 10)$
16. $k(x) = 2 + 3 \ln x$ $(0 \le x \le 10)$

In Exercises 17–24, express each as a sum, difference, or multiple of logarithms.

17. $\log \frac{3}{4}$
18. $\log 77$
19. $\log 5x$
20. $\ln \dfrac{7x}{3}$
21. $\ln (4x)^3$
22. $\ln 4x^3$
23. $\log \sqrt{48}$
24. $\log \dfrac{\sqrt[3]{2x^5}}{4x}$

In Exercises 25–30, express each as a single logarithm.

25. $\log 5 + \log 9$
26. $\log 19 - \log 11$
27. $4 \log x + \log x$
28. $\log 3a + \log b - \log a$
29. $7 \log x + 2 \log x - \log x$
30. $\ln \frac{2}{3} + \ln \frac{15}{8}$

In Exercises 31–38, solve each equation for the indicated variable. Use a calculator if you feel it is necessary.

31. $4^x = 28$
32. $5^{3x} = 250$
33. $e^{5x} = 11$
34. $e^{5x-1} = 2e^x$
35. $\ln x = 9.1$
36. $4 \log x = 49$
37. $\log(2x + 1) = 50$
38. $2 \ln 4x = 100 + \ln 2x$

In Exercises 39–42, sketch the graphs of the given functions on semilog paper.

39. $y = 4(2^x)$
40. $y = 8x^4$
41. $y = 8(4^x)$
42. $y = 3(4^{x-1})$

In Exercises 43–46, sketch the graphs of the given functions on log-log paper.

43. $y = x^{1/4}$
44. $y = 5x^3$
45. $x^3y^4 = 2$
46. $y^3 = 10x^2$

Solve Exercises 47–56.

47. ***Finance*** One savings institution offers 3.5% interest compounded semiannually. A second offers the same interest rate, but it is compounded monthly. A third compounds that same interest continuously. How much income will result if \$1 000 is deposited in each institution for 10 years? How long will it take for the \$1 000 to double in the third bank?

48. ***Nuclear engineering*** If radium has a half-life of 1 600 years, how much of 100 mg will remain after 100 years? after 1 000 years?

49. ***Biology*** The amount of bacteria in a culture increases from 4 000 to 25 000 in 24 h. If we assume these bacteria grow exponentially, **(a)** how many will be there after 12 h? **(b)** How many can we expect there to be after 48 h?

50. ***Seismology*** The San Francisco earthquake of 1906 registered 8.3 on the Richter scale. If $R = \log I$, where I is the relative intensity of the shock, determine the number of times the 1906 earthquake was greater than **(a)** the San Francisco quake of 1979, which registered a 6.0, and **(b)** the San Francisco quake of 1989, which registered a 7.1.

51. Data from an experiment produced a straight line on semilog paper. Two of the data points were (2, 15.76) and (4, 620.84). Find an equation that approximates this scale.

52. Data from an experiment produced a straight line on log-log paper. Two of the data points were (2, 8.75) and (4, 2.1875). Find an equation that approximates this data.

53. ***Astronomy*** The brightness of a star perceived by the naked eye is measured in terms of a quantity called *magnitude*. The brightest stars are of magnitude 1 and the dimmest of magnitude 6. The magnitude M is given by $M = 6 - 2.5 \log \frac{I}{I_0}$, where I_0 is the intensity of light from a just-visible star, and I is the actual intensity from a star. Even though the sun is a star, it has a magnitude of -27, and the moon has a magnitude of -12.5. **(a)** Express this equation in exponential form by solving for I. **(b)** How many times more intense is the moon than Polaris, which has a magnitude of 2.0?

54. ***Biology*** When the size of a colony of bacteria is limited because of a lack of room or nutrients, the growth is described by the law of logistic growth:

$$Q = \frac{mQ_0}{Q_0 + (m - Q_0)e^{-kmt}}$$

where Q_0 is the initial quantity present, m the maximum size, and k a positive constant. If $Q_0 = 400$, $m = 2000$, and $Q = 800$ when $t = \frac{1}{2}$, find k.

55. ***Finance*** The following data show the amount of money, in billions of dollars, paid in Social Security for selected years.

Year	1990	1994	1995	1996	1997	1998	1999	2000
Annual benefits ($\$ \times 10^9$)	247.8	316.8	332.6	347.1	362.0	375.0	385.8	407.6

(a) Determine the exponential regression function that best fits the data.

(b) Use the exponential regression function from (a) to predict the Social Security payments in 2010.

(c) Determine the linear regression function that best fits the data.

(d) Use the linear regression function from (c) to predict the Social Security payments in 2010.

56. ***Food science*** As dry food is stored it loses its flavor (diacetyl). The table below gives the amount of diacetyl remaining, in ppm, in gum acacia over a 116-day period.

Storage time (days)	0	6	10	19	41	81	116
Diacetyl (ppm)	282	222	197	153	105	74	61

(a) Plot the data on normal rectangular, semilog, and log-log paper.

(b) Determine the exponential regression function that best fits the data.

CHAPTER 12 TEST

In Exercises 1–3, use a calculator to evaluate each number.

1. $e^{5.3}$ **2.** $\log 715.3$ **3.** $\ln 72.35$

In Exercises 4–7, express each as a sum, difference, or multiple of logarithms.

4. $\log \frac{9}{15}$ **5.** $\log 65x$ **6.** $\ln(7x)^{-2}$ **7.** $\log \dfrac{\sqrt[5]{5x^3}}{9x}$

In Exercises 8–10, express each as a single logarithm.

8. $\log 11 + \log 3$ **9.** $\log 17 - 2\log x$ **10.** $\ln 4 - \ln 5 + \ln 15 - \ln 8$

In Exercises 11–16, solve each equation for the indicated variable. Use a calculator if you feel it is necessary.

11. $\ln x = 17.2$

12. $3\log x = 55$

13. $3^x = 35$

14. $e^{4x+1} = 2e^x$

15. $\log(2x^2 + 4x) = 5 + \log 2x$

16. Make a table of values and graph $f(x) = 3^x$ on ordinary graph paper over the domain $x = -3$ to 3.

Solve Exercises 17–21.

17. Make a table of values and graph $y = 0.4(3^x)$ on semilog paper from $x = 1$ to 7.

18. Graph $y = 2x^{1/5}$ on log-log paper.

19. The yield, Y, in total ft^3/acre, of thinned stands of yellow poplar can be predicted by the equation:

$$\ln Y = 5.36437 - 101.16296S^{-1} - 22.00048A^{-1} + 0.97116 \ln \mathrm{BA}$$

where S is the site index, A is the current age of the trees, and BA is the basal area. What is the predicted yield of a 60-year-old stand growing on a site with an index of 120 and a basal area of 130 ft^2 per acre?

20. A bank offers 5.5% interest compounded monthly. If \$5 000 is deposited in this bank, how much will it be worth in 8 years?

21. A radioactive substance decays at the rate of 0.75% per year. What is the half-life of the substance?

22. Some money was deposited, and left, in a savings account. The table below shows the amount of money in the account at the end of each year.

Year (t)	0	1	2	4	6	8	10
Amount (\$)	527.00	547.11	567.98	612.14	659.74	711.04	766.33

(a) What was the initial amount of the deposit?

(b) What regression formula should you use for this data?

(c) Use your regression formula to predict the amount in the savings account at the end of year 20.

13 STATISTICS AND EMPIRICAL METHODS

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How many spaces will you need for a parking lot? How many staff will you need and what hours should they work? In this chapter we will see some ways in which statistics can be used to answer these questions.

In this chapter, we will begin exploring a new area of mathematics—the world of probability, statistics, and empirical methods. Probability theory has many applications in the physical sciences and technology. It is of basic importance in statistical mechanics. Probability is needed in any problem dealing with large numbers of variables where it is impossible or impractical to have complete information.

In many technological settings, information is needed about an operation. When it is not possible to gather information about the entire operation, information is gathered on a part, or sample, of the operation. This information is then analyzed and decisions are made as a result of this analysis. Statistics are the basis of this analysis, and in this chapter we will learn how to conduct some statistical analyses and how to use that information to help us make decisions.

OBJECTIVES

AFTER COMPLETING THIS CHAPTER, YOU WILL BE ABLE TO:

- Determine the probability of an event using the number of successes and trials.
- Determine the probability of an event based on the geometric area of the probability distribution or relative frequency distribution.
- Determine the mean, median, mode, range, and standard deviation for a set of data and make decisions about which statistic is best to use for an application.

13.1 PROBABILITY

The word "probably" is used often in everyday life. For example, you might tell someone that "the test will probably be hard" or that "it will probably rain today." The word "probably" indicates that we do not know what will happen and are predicting what we think *might* happen. The theory of probability tries to express more precisely just how much we do know.

In the theory of probability, a numerical value between 0 and 1 is given to the likelihood that some particular event will happen. When we do this, we assume that all events are equally likely to occur, unless we have some special information that indicates otherwise.

OUTCOME AND SAMPLE SPACE

In working with probability, we will be concerned with the possible outcomes of experiments. An experiment can be something as simple as tossing a coin or something more complicated such as determining the number of bad computer chips at a production station. A result of an experiment is called an **outcome**. The group of all possible outcomes for an experiment is the **sample space**. Each performance of an experiment is called a **trial**.

EXAMPLE 13.1

(a) If a coin is tossed, the sample space contains the two possible outcomes of heads (H) or tails (T) and can be written as $\{H, T\}$.

(b) If a die is rolled, the sample space of the number of dots on the upper face is $\{1, 2, 3, 4, 5, 6\}$. Each trial will produce one of these numbers.

(c) If two coins are tossed, the sample space has the following four possible outcomes $\{HH, HT, TH, TT\}$.

EVENT

An event is something that may or may not occur from the possible sample space. In particular, an **event** is some part (or all) of the sample space. For example, in rolling a die we might consider the event that an odd number is obtained. Another possible event would be that a 6 is obtained. A third possible event would be that a number larger than 2 is obtained. The outcomes for which an event occurs are said to be *favorable* to the event. For example, the outcomes 1, 3, and 5 are favorable to the event of "rolling an odd number."

THREE TYPES OF PROBABILITY

If E is an event from a sample space with n equally likely elements and k is the number of ways in which this event can happen, then the *probability of E*, $P(E)$, is

$$P(E) = \frac{k}{n}$$

This definition is known as the *classical* or *a priori* approach. In this approach, no experiment is conducted and the probability is based on our knowledge of the nature of the event.

EXAMPLE 13.2

Find the probability of drawing a 9 or 10 from a well-shuffled deck of cards.

SOLUTION A deck has 52 cards. Since this deck is well shuffled, the likelihood of getting any one card is the same as the likelihood of getting any other card. There are four 9s and four 10s in the deck, so there are 8 cards in this event. Thus, $P(9 \text{ or } 10) = \frac{8}{52} = \frac{2}{13}$. This is an example of the classical approach to probability.

A second approach is the *empirical* or *frequency* approach, and it is based on the results of an actual experiment or of previous experience. In this definition, a series of N trials are made and the event is observed to occur K times. The empirical definition says that the probability of this event occurring would be

$$P(E) = \frac{K}{N}$$

APPLICATION **MECHANICAL**

EXAMPLE 13.3

Samples have shown that, out of every 1 000 discs it copies, a certain machine will produce 75 discs with errors. The probability of a disc being bad if copied on this machine is $\frac{75}{1\,000} = \frac{3}{40}$. This is an example of the empirical approach to probability.

There is a third approach to probability called the *subjective* approach, and it is based on a person's belief that an event will occur. The subjective approach is normally employed when there is no past history to use for an empirical approach and no basis for using the classical approach.

APPLICATION GENERAL TECHNOLOGY

EXAMPLE 13.4

A software designer might say, "I believe that this software has a probability of 0.9 of becoming a best seller since it is something that I would be willing to buy."

As you can see, since $0 \leq K \leq N$, we can make the following five points about the probability of any event:

- The probability that a given event will happen ranges from 0 to 1 or $0 \leq P(E) \leq 1$.
- The higher the probability (the closer to 1), the more likely it is that the event will happen.
- If the probability is $\frac{1}{2}$, then it is equally likely that an event will happen or will not happen.
- A probability of 1 indicates that the event is certain to occur.
- A probability of 0 means that the event will not happen.

If E represents a certain event, then E' represents all the elements in the sample space that are not in E. So, if the probability that E will happen is $P(E)$, then the probability that E will not happen (or that E' will happen) is $P(E') = 1 - P(E)$.

APPLICATION MECHANICAL

EXAMPLE 13.5

What is the probability that the disc-copying machine in Example 13.3 will make a good copy?

SOLUTION The probability that a copy is bad is $\frac{3}{40}$, so the probability that a copy is good is $1 - \frac{3}{40} = \frac{37}{40}$.

Sometimes we are interested in the probability of an event when we know that another event has already occurred. If the probability that the second event will occur is not affected by what happens to the first event, we then say that the events are **independent**.

PROBABILITY OF TWO INDEPENDENT EVENTS

If events A and B are independent, then the probability that both A and B will occur is $P(A \text{ and } B) = P(A) \cdot P(B)$.

EXAMPLE 13.6

Two cards are to be drawn from a well-shuffled deck of cards. After the first card is drawn, it is replaced and the deck is shuffled again. What is the probability that we will draw two diamonds?

SOLUTION There are 13 diamonds in the deck, so the probability of getting a diamond is $\frac{13}{52} = \frac{1}{4}$. Since the first card is replaced and the deck is shuffled, the probability of getting a diamond on the second draw is independent of what was drawn on the first card. The probability of getting a diamond on the second card is $\frac{1}{4}$ and the probability of getting two diamonds is $\frac{1}{4} \cdot \frac{1}{4} = \frac{1}{16}$.

How would the results of Example 13.6 have been different if we had not replaced the first card? In this instance, we are working with **conditional probability**. If A is an event that has already occurred, and we want to know the probability that a second event B will occur, we call this the *probability of B given A*. This is often written $P(B|A)$. To determine $P(B|A)$, we will use the following reasoning: We know that event A has already occurred. This reduces the sample space to the elements in A. We now count the number of times that event B is in this reduced sample space. Then $P(B|A)$ are the number of elements in B that are also in A divided by the number of elements in A. Thus, we have the following.

PROBABILITY OF TWO EVENTS THAT ARE NOT INDEPENDENT

$$P(A \text{ and } B) = P(A) \cdot P(B|A) \quad \text{or} \quad P(B|A) = \frac{P(A \text{ and } B)}{P(A)}$$

EXAMPLE 13.7

Two cards are going to be drawn from a well-shuffled deck of cards. The first card will not be replaced after it is drawn. What is the probability that both cards will be diamonds?

SOLUTION Let A be the event that the first card is a diamond. Let B be the event that the second card is a diamond. We want the probability of A and B. We know from Example 13.6 that $P(A) = \frac{1}{4}$. Since the first card is not replaced,

EXAMPLE 13.7 (Cont.) the deck now has 51 cards. Of these 51 cards, 12 of them are diamonds. So, $P(B|A) = \frac{12}{51} = \frac{4}{17}$ and

$$P(A \text{ and } B) = P(A) \cdot P(B|A)$$
$$= \frac{1}{4} \cdot \frac{4}{17} = \frac{1}{17}$$

In this section, we have given you a very brief introduction to probability. As you might imagine, there are many more complicated situations that could happen, and determining their probabilities is much more involved. We have, however, given you a foundation for the work with statistics that is in the remainder of this chapter.

EXERCISE SET 13.1

In Exercises 1–10, let the sample space be an ordinary deck of well-shuffled playing cards. What is the probability of drawing each of the following?

1. a heart
2. a six
3. a queen
4. a two or a seven
5. a face card (jack, queen, or king)
6. a black card
7. two black cards on successive draws, if the first card is replaced before the second card is drawn
8. a black card and then a face card, if the first card is replaced before the second card is drawn
9. a red card and then a black card, if the first card is not replaced before the second draw
10. a face card and then an ace, if the first card is not replaced before the second draw

In Exercises 11–18, assume that you are rolling a single die. What is the probability of rolling the following?

11. 6
12. 1 or 6
13. even number
14. 4 or more
15. two successive 6s
16. not a 5
17. at least one 5 on 2 successive rolls (First, determine the probability of not getting a 5.)
18. three successive 6s

In Exercises 19–20, assume that you are rolling two dice. There are 36 possible ways for the dice to fall.

19. What is the sample space?
20. What is the probability of getting two 6s?

When rolling a pair of dice, what is the probability of obtaining each of the totals in Exercises 21–30?

21. 12

22. 2

23. 1

24. 3

25. 7

26. 7 or 11

27. less than 5

28. 7 on 2 successive rolls

29. 12 on 2 successive rolls

30. at least 7

In Exercises 31–34, use the fact that a loaded (or unfair) die has probabilities of $\frac{1}{21}, \frac{2}{21}, \frac{3}{21}, \frac{4}{21}, \frac{5}{21}$, *and* $\frac{6}{21}$ *on showing a 1, 2, 3, 4, 5, and 6, respectively.*

31. What is the probability of rolling two 6s in succession?

32. What is the probability of rolling a 3 and then a 4?

33. What is the probability that the number on the die is even?

34. What is the probability of a total of 12 when 2 dice are rolled?

Solve Exercises 35–42.

35. ***Medical technology*** A certain medication is known to cure a specific illness for 75% of the people who have the illness. If two people with the illness are selected at random and take the medicine, what is the probability that

(a) both will be cured?

(b) neither will be cured?

(c) only one will be cured?

36. ***Aeronautics*** On a two-engine jet plane, the probability of either engine failing is 0.001. If one engine fails, then the probability that the second will fail is 0.005. What is the probability that both engines will fail?

37. ***Insurance*** Insurance company tables show that for a married couple of a certain age group the probability that the husband will be alive in 25 years is 0.7. The probability that the wife will be alive in 25 years is 0.8.

(a) What is the probability that both are alive in 25 years?

(b) What is the probability that both are dead in 25 years?

(c) What is the probability that in 25 years only one is alive?

38. ***Industrial technology*** A machine produces 25 defective parts out of every 1 000. What is the probability of a defective part being produced?

39. ***Industrial technology*** What is the probability of testing two parts from the machine in Exercise 38 and finding that one of them is defective?

40. ***Industrial technology*** What is the probability of testing four parts from the machine in Exercise 38 and finding that all four are not defective?

41. ***Medical technology*** Blood groups for a certain sample of people are shown in the following table.

Blood Group	Frequency
O	110
A	64
B	20
AB	6

If one person from this sample of people is randomly selected, what is the probability that he or she has type B blood?

42. ***Insurance*** Among 800 randomly selected drivers in the 20–24 age bracket, 316 were in a car accident during the last year. If a driver of that age bracket is randomly selected, what is the probability that he or she will be in a car accident during the next year?

[IN YOUR WORDS]

43. What is an event?

44. What is meant by the probability of an event?

45. Explain how the classical or *a priori* approach to determine the probability of an event is different from the empirical or frequency approach.

46. What are independent events?

47. What is conditional probability?

13.2 MEASURES OF CENTRAL TENDENCY

In Section 13.1, we stated that there were three types of probability. One of these was the empirical or frequency approach. This approach is based on past experience or on the results of an actual experiment. The experiment might consist of a series of trials. For each of these trials, the person directing the experiment will collect data, analyze the data, and then interpret it. In the remainder of this chapter, we will look at some of the ways in which the experimenter might gather and analyze this data.

In statistics, we deal with numbers. Sometimes it is possible to deal with entire sets of numbers. For example, if your teacher wants to determine how much your class has learned about probability, a test could be given to the class. Businesses also are sometimes able to test an entire group of items. For example, a company might be able to test each television set to see that it works. These situations describe **populations**.

There are times when it is not possible to check each item and so a **sample** of items must be selected. For example, it is not possible to check every electric fuse to see that it provides the proper protection. It also might not be practical to check every part made on a machine that produces 10 000 parts a day. In these last two cases, a sample is selected and tested. They might check every 100th fuse or every 1 000th part made by a machine. No matter how the data is gathered, it needs to be organized in a way that makes it easier to understand.

NOTE A sample is a subset of the population and is used to provide *estimates* about the population.

FREQUENCY DISTRIBUTION

One way to organize the information is in a **frequency distribution**. In a frequency distribution, one line contains a list of possible values and a second line contains the number of times each value was observed in a particular time. The data is first separated into groups. Each group is called a *class* and the number of values in each class is the *frequency*.

APPLICATION **ENVIRONMENTAL SCIENCE**

EXAMPLE 13.8

People who live near an interstate highway have complained about traffic noise. A measuring instrument is placed along the highway and every 15 min it measures the noise level in decibels. A frequency distribution is made of the readings for the first day.

Decibels	50--54	55--59	60--64	65--69	70--74	75--79	80--84
Frequency	2	2	4	4	4	6	8

Decibels	85--89	90--94	95--99	100--104	105--109	110--114	115--119
Frequency	8	10	12	14	10	8	4

There are times when we want to know the *relative frequency* of a distribution. The relative frequency is the number of values in each class divided by the total number of observations. Although the relative frequency may be expressed as a fraction or a decimal, it is usually expressed as a percent. In Example 13.8, there were 96 samples. A noise level of 50–54 dB occurred twice, so the relative frequency for 50–54 dB is $\frac{2}{96} \approx 0.021 \approx 2.1\%$.

APPLICATION **ENVIRONMENTAL SCIENCE**

EXAMPLE 13.9

In Example 13.8 we made a frequency distribution of the day one reading of noise levels along an interstate highway. Determine the relative frequency of each of these readings.

SOLUTION As mentioned above, we get the relative frequency by dividing the frequency of each class by the total number of readings, 96. The result is reported as a percent. The relative frequencies have been added to the table from Example 13.8.

Decibels	50--54	55--59	60--64	65--69	70--74	75--79	80--84
Frequency	2	2	4	4	4	6	8
Relative frequency (%)	2.1	2.1	4.2	4.2	4.2	6.3	8.3

Decibels	85--89	90--94	95--99	100--104	105--109	110--114	115--119
Frequency	8	10	12	14	10	8	4
Relative frequency (%)	8.3	10.4	12.5	14.6	10.4	8.3	4.2

The relative frequency in this example adds up to 100.1% rather than 100.0% because each of the relative frequencies had to be rounded off to the nearest 0.1%.

GRAPHS

Many times it is difficult for us to understand the numbers in a table such as a frequency distribution. We find that a chart or graph helps us to get a better idea of these numbers. One type of graph is the **histogram**. A histogram displays each class and its frequency as a rectangle. The width of each rectangle is the *class width* and the height of the rectangle is the frequency of the class. Each rectangle is labeled at the center by the *class mark*.

Another type of graph is the *frequency polygon* or *broken-line graph*. In a broken-line graph the frequency of each class is represented by placing a dot at the y-value above the class mark. These points are then joined by line segments.

APPLICATION ENVIRONMENTAL SCIENCE

EXAMPLE 13.10

Construct a histogram and a broken-line graph for the data in the frequency table of Example 13.8 that shows the noise level along an interstate highway.

SOLUTION The class marks are determined by dividing the sum of the upper and lower values of each class by 2. Thus, the first class mark is $\frac{50 + 54}{2} = 52$. The histogram is in Figure 13.1 and the broken-line graph for the same information is shown in Figure 13.2.

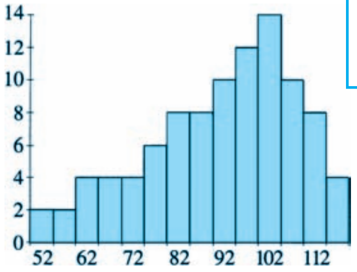

Figure 13.1

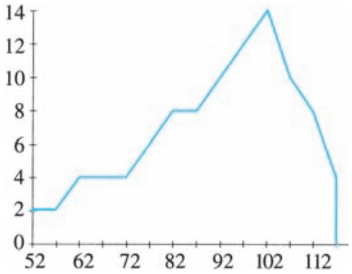

Figure 13.2

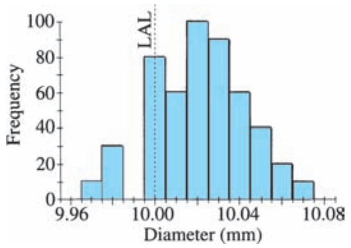

Figure 13.3

Sometimes a histogram can provide very important information. For example, the histogram in Figure 13.3 shows the results when inspectors measured the diameters in millimeters of 500 steel rods. The dotted line marked LAL is the lowest acceptable limit. Rods smaller than 10.00 mm are too loose in the bearing and would be thrown out or rejected in a later operation. When a rod is rejected, the company loses all the labor and material cost that went into making that rod.

The histogram in Figure 13.3 contains some very interesting information. We can see that 40 rods were rejected because they were below the LAL; but, there is a gap in the histogram at the interval just below the LAL. The gap was unexpected and is a result of inspectors passing rods that were barely below the LAL. This gap is one that the inspectors could have corrected.

MEDIAN, QUARTILES, AND BOX PLOTS

Tables and graphs can give us some general ideas about the data, but we often need more information. Among this information are some indications for the location of the center of the distribution. We would also like some measure of how the data is spread. This gives us some numerical descriptions of the data and helps us compare groups of data.

The first types of information that we will examine are the ones that provide some indication of the center. These are known as the **measures of central tendency** and there are three of them: the median, the mean, and the mode.

The first measure of central tendency we will examine is the median.

MEDIAN

The **median** is the middle number of the numbers in the distribution. One-half of the values are larger than the median, and one-half are smaller. To find the median,

1. Arrange the numbers in increasing (or decreasing) order.
2. If there is an odd number of items, the median is the value of the middle item.
3. If there is an even number of items, the median is the number half-way between the two middle items.

EXAMPLE 13.11

Given the numbers 9, 8, 3, 2, 4, determine the median.

SOLUTION First arrange the numbers in increasing order: 2, 3, 4, 8, 9. There are five numbers, so the middle number is the third number. The third number is 4, so the median is 4.

EXAMPLE 13.12

Find the median of 11, 12, 15, 18, 20, 20.

SOLUTION These number are already in numerical order. There are six numbers. The median will be half-way between the third and fourth numbers. The third number is 15 and the fourth is 18. Midway between these is 16.5; thus the median is 16.5.

The median divides the items into two equally sized parts. In the same way, the *quartiles* Q_1, Q_2, and Q_3 divide the numbers into four equally sized parts when the numbers are arranged in increasing (or decreasing) order. As we will see, quartiles provide a quick way to graphically see how the numbers are distributed.

FINDING QUARTILES

1. Arrange the numbers in increasing order.
2. Q_2 is the median. It divides the numbers into a lower half and an upper half.
3. Q_1 is the median of the lower half of the numbers.
4. Q_3 is the median of the upper half of the numbers.

EXAMPLE 13.13

Determine the quartiles of 12, 15, 42, 37, 61, 14, 14, 9, 25, 32, 27, and 30.

SOLUTION We begin by arranging the numbers in increasing order:

9, 12, 14, 14, 15, 25, 27, 30, 32, 37, 42, 61

There are 12 numbers, so the second quartile, Q_2 (or median), is midway between the sixth and seventh numbers. The sixth number is 25. The seventh number is 27.

$$Q_2 = \frac{25 + 27}{2} = 26$$

Q_1 is the median of the lower half, and so it is the median of the smallest six numbers. Q_1 is midway between the third and fourth numbers. These are both 14, so $Q_1 = 14$.

Q_3 is the median of the upper half. The upper half of the items is 27, 30, 32, 37, 42, and 61. The median of these is midway between 32 and 37, so

$$Q_3 = \frac{32 + 37}{2} = 34.5$$

The three quartiles, the lowest number, and the highest number are used to make a diagram called a **box plot** or a **box-and-whisker diagram**. The following steps describe how to make a box-and-whisker diagram.

MAKING A BOX-AND-WHISKER DIAGRAM

1. Arrange the numbers in increasing order.
2. Find the lowest and highest scores, Q_1, Q_2, and Q_3.
3. Draw a scale that will include the lowest and highest numbers.
4. Draw a box with the ends of the box at Q_1 and Q_3.
5. Draw a line through the box at Q_2.
6. Draw a whisker from the lowest number to the box and draw another whisker from the highest number to the box.

NOTE The box of a box-and-whisker diagram contains the middle 50% of the data, one whisker shows the bottom 25% of the data, and the other whisker shows the top 25% of the data.

APPLICATION BUSINESS

EXAMPLE 13.14

At an automobile engine plant, a quality control technician pulls crankshafts from the assembly line at regular intervals. The technician measures a critical dimension on each of these crankshafts. Even though the dimension is supposed to be 182.000 mm, some variation will occur during production. Here are the measurements for one morning's sample:

182.120	182.005	182.025	181.987	181.898	182.034
181.960	181.940	182.055	181.897	181.935	182.063
182.015	182.026	181.965	181.985	182.362	181.998
182.107	181.934	181.991	182.005	182.012	181.984

Make a box plot for these measurements.

SOLUTION We begin by listing the numbers in numerical order.

181.897	181.898	181.934	181.935	181.940	181.960
181.965	181.984	181.985	181.987	181.991	181.998
182.005	182.005	182.012	182.015	182.025	182.026
182.034	182.055	182.063	182.107	182.120	182.362

From this arrangement of the 24 numbers, we can see that the lowest is 181.897 mm and the highest is 182.362 mm. Notice that the numbers are arranged in four rows with six numbers in each row. Thus, Q_1 will be between the last number in the first row and the first number in the second row, or $Q_1 = \frac{181.960 + 181.965}{2} = 191.9625$. In a similar way, we find $Q_2 = \frac{181.998 + 182.005}{2} = 182.0015$; and $Q_3 = \frac{182.026 + 182.034}{2} = 182.03$. The box plot for this data is shown in Figure 13.4.

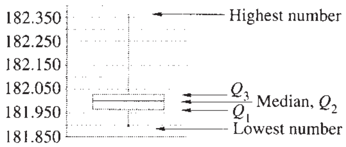

Figure 13.4

MEAN

The second measure of central tendency is the **mean**, sometimes called the *arithmetic mean*.

MEAN

To determine the mean, you add all the values and divide by the number of values. The symbol $\bar{x}$ is used to represent the mean. In symbols,

$$\bar{x} = \frac{\sum_{i=1}^{n} x_i}{n}$$

You may have noticed some new notation in this definition of the mean. This notation is referred to as *summation* or *sigma notation* because it uses the capital Greek letter sigma $(\sum)$. In general, the sigma notation is one way of telling you to add a group of numbers. Consider

$$\sum_{k=1}^{n} a_k = a_1 + a_2 + a_3 + \cdots + a_n$$

Here $\sum$ indicates a sum. The letter k is called an *index of summation*. The summation begins with $k = 1$ as is indicated below the $\sum$ and ends with $k = n$ as is indicated above the $\sum$.

EXAMPLE 13.15

Evaluate $\sum_{k=1}^{5} 2.3\,k$.

SOLUTION

$$\begin{aligned}\sum_{k=1}^{5} 2.3\,k &= (2.3)1 + (2.3)2 + (2.3)3 + (2.3)4 + (2.3)5\\ &= 2.3 + 4.6 + 6.9 + 9.2 + 11.5\\ &= 34.5\end{aligned}$$

The numbers below and above the $\sum$ are the *limits of summation*. Here they are 1 and 5.

In the definition of the mean, the notation $\sum_{i=1}^{n} x_i$ indicates that you are to add all n values from the first $i = 1$ to the last $i = n$.

EXAMPLE 13.16

Find the mean of the values in **(a)** Example 13.11 and **(b)** Example 13.12.

SOLUTIONS

(a) In Example 13.11, the five numbers were 9, 8, 3, 2, and 4, so

$$\bar{x} = \frac{9 + 8 + 3 + 2 + 4}{5} = \frac{26}{5} = 5.2$$

(b) In Example 13.12, the six numbers were 11, 12, 15, 18, 20, and 20. The mean of these six numbers is

$$\bar{x} = \frac{11 + 12 + 15 + 18 + 20 + 20}{6} = \frac{96}{6} = 16$$

If you want to find the mean of a large number of values, and if some appear more than once, then there is a quicker method to get the total. Consider the following frequency distribution in Table 13.1 for the sample of 500 steel rods in Figure 13.3.

TABLE 13.1

Diameter	9.97	9.98	9.99	10.00	10.01	10.02	10.03	10.04	10.05	10.06	10.07
Frequency	10	30	0	80	60	100	90	60	40	20	10

It would take a long time to add these 500 values. Instead, if we multiply each diameter by the frequency for that diameter, we reduce the 500 additions to 10 multiplications and 9 additions. Thus,

$$\begin{aligned}\bar{x} &= \frac{\begin{array}{c}9.97(10) + 9.98(30) + 9.99(0) + 10.00(80) + 10.01(60) + 10.02(100)\\ + 10.03(90) + 10.04(60) + 10.05(40) + 10.06(20) + 10.07(10)\end{array}}{10 + 30 + 80 + 60 + 100 + 90 + 60 + 40 + 20 + 10}\\ &= \frac{5{,}010.7}{500}\\ &= 10.0214\end{aligned}$$

Using sigma notation, the mean is given by the following formula. In this formula, since f_i denotes the frequency of each of the n values, we see that $\sum f_i = n$.

MEAN

$\bar{x} = \dfrac{\sum_{i=1}^{n} x_i f_i}{\sum_{i=1}^{n} f_i}$, where x_i represents the ith value and f_i the frequency for that value.

In Table 13.1, there were 11 values: $x_1 = 9.97$ and $f_1 = 10$, $x_2 = 9.98$ and $f_2 = 30$, and so on until $x_{11} = 10.07$ and $f_{11} = 10$.

MODE

The third, and final, measure of central tendency is the mode.

MODE

The **mode** is the value or class that has the greatest frequency. A set of numbers can have more than one mode. If there are two modes, the data are said to be *bimodal*.

EXAMPLE 13.17

What is the mode for the data in Example 13.12: 11, 12, 15, 18, 20, and 20?

SOLUTION The mode is 20, because that value occurs twice and all the other values occur once.

APPLICATION BUSINESS

EXAMPLE 13.18

What is the mode for the sample of 500 steel rods in Table 13.1?

SOLUTION The 500 steel rods had a mode of 10.02, because the 100 measures for that diameter were more than for any other.

USING A GRAPHING CALCULATOR

A graphing calculator can be a great help in finding many statistical measures. In most graphing calculators, there are two modes for working with statistics. One mode is for data entry. In this mode you enter the raw data, usually giving each score its frequency. The other mode is used to perform the calculations using the data stored during the data entry mode.

There are several types of statistical calculations that will be performed. All calculators will compute the mean and sample size from this section as well as the variance and standard deviation from the next section. Some calculators will also give the median, mode, and quartiles, and draw a histogram. All of these calculations are performed under the category of single-variable (or 1-var) statistics.

APPLICATION BUSINESS

EXAMPLE 13.19

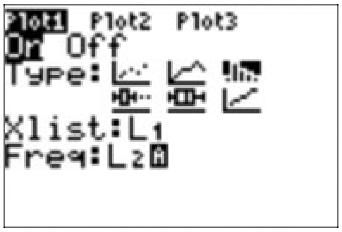

Figure 13.5a

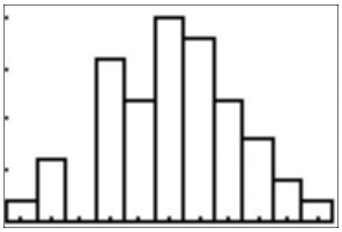

Figure 13.5b

Use a TI-83 or TI-84 and the data for the 500 steel rods in Table 13.1 to draw a histogram and box plot, and determine the mean, median, and quartiles.

SOLUTION We have reproduced the table below.

Diameter	9.97	9.98	9.99	10.00	10.01	10.02	10.03	10.04	10.05	10.06	10.07
Frequency	10	30	0	80	60	100	90	60	40	20	10

We begin by entering the diameters in one list (we used L1) and the frequencies in a second list (L2). Next, press 2nd Y = [STAT PLOT]. Turn on Plot1 and use ▼ to move to the line labeled Type:. On this line and the next, there are a total of six options: the second or middle option on the top line is for a broken-line graph, the third or last option on the top line is for a histogram, and the first two options on the second line are for box plots. We first select the histogram by pressing ► ► ENTER, as shown in Figure 13.5a. Now, we need to tell the calculator where to find the class values and the frequencies. Press ▼ 2nd 2 [L1] to indicate that the class values are in list L1. For the frequencies press ▼ 2nd 1 [L2].

Now, press WINDOW to make the window settings. Since the class width is 0.005 we will set xMin $= 9.97 - 0.005 = 9.965$, xMax $= 10.07 + 0.005 = 10.075$, xScl $= 0.01$, yMin $= 0$, yMax $= 105$, and yScl $= 25$. Now, press GRAPH to get the result in Figure 13.5b (compare this to Figure 13.3).

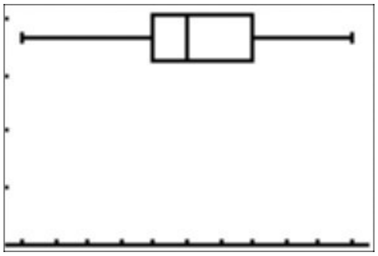

Figure 13.5c

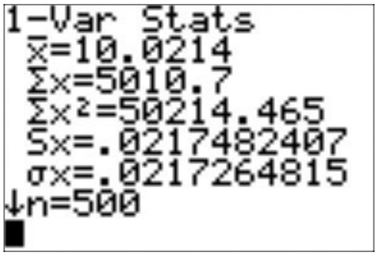

Figure 13.5d

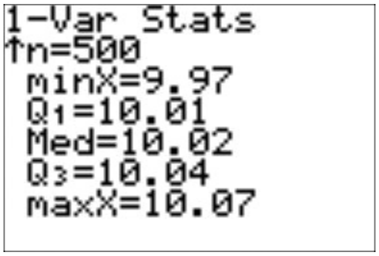

Figure 13.5e

Pressing 2nd Y = [STAT PLOT] 1, moving to the Type line, and selecting the fifth type and then pressing GRAPH produces the box plot in Figure 13.5c. Note that you do not have to change the window settings.

Pressing STAT ► [CALC] 1 [1-Var Stats] 2nd L1 , 2nd L2 ENTER produces results in Figure 13.5d, and if you hold down the ▼ key you will eventually get the results in Figure 13.5e.

From Figure 13.5d you see that the mean is $\bar{x} = 10.0214$. We will wait until Section 13.3 to discuss the rest of the information in Figure 13.5d. In Figure 13.5e we see that $Q_1 = 10.01$, Med $= Q_2 = 10.02$, and $Q_3 = 10.04$. Notice that we cannot get the mode with the calculator.

APPLICATION COMMUNICATIONS

EXAMPLE 13.20

Use a TI-83 or TI-84 calculator to draw a histogram and determine the mean of the average monthly cell phone bill, in dollars, from the data in Table 13.2.

TABLE 13.2 Average Monthly Cell Phone Bill, 1990–2009

Year	1990	1991	1992	1993	1994	1995	1996	1997	1998	1999
Bill ($)	80.90	72.74	68.68	61.48	56.21	51.00	47.70	42.78	39.43	41.24

Year	2000	2001	2002	2003	2004	2005	2006	2007	2008	2009
Bill ($)	45.27	47.37	48.40	49.91	50.64	49.98	50.56	49.79	50.07	48.16

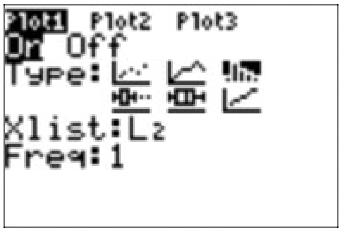

Figure 13.6a

SOLUTION The data in Table 13.2 is not grouped, but that is not necessary for the calculator to draw the histogram. Enter the years in L1 and the monthly bills in L2. Press 2nd Y = [STAT PLOT]. Turn on Plot1 and use ▼ to move to the line labeled Type: and select the histogram. Now, we need to tell the calculator where to find the class values and the frequencies. Press ▼ 2nd 2 [L2] to indicate that the class values are in list L2. The frequencies will be 1. If the Freq: line is not already selected, press ▼ ALPHA 1 ENTER. When you have finished, your screen should look like Figure 13.6a.

Press ZOOM 9 [9:ZoomStat] and you get the histogram in Figure 13.6b.

This is a histogram of the data but it is not very informative. A quick look at the data shows that the most expensive monthly payment was $80.90 in 1990

Figure 13.6b

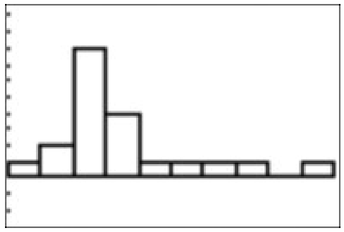

Figure 13.6c

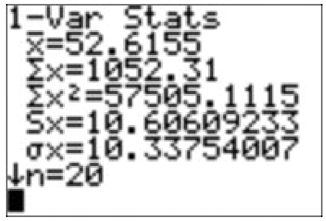

Figure 13.6d

and the least expensive was \$39.43 in 1998. If we change the window settings so that Xmin = 35, Xmax = 85, and Xscl = 5, we will get 10 subdivisions. With these changes, the first bar of the histogram in Figure 13.6c represents the interval [35, 40), the second interval is [40, 45), . . . , and the last interval is [80, 85). Pressing GRAPH results in the histogram in Figure 13.6c.

To determine the mean, press STAT ▶ [CALC] 1 [1-Var Stats] 2nd L2 ENTER. This produces the results in Figure 13.6d and we see that the mean is about 52.62. From this we conclude that the average (mean) cellular phone bill from 1900 through 2009 was \$52.62.

USING A SPREADSHEET

APPLICATION **BUSINESS**

EXAMPLE 13.21

Use a spreadsheet and the data for the 500 steel rods in Table 13.1 to draw a histogram and box plot, and determine the mean, median, and quartiles.

SOLUTION We have reproduced the table below.

Diameter	9.97	9.98	9.99	10.00	10.01	10.02	10.03	10.04	10.05	10.06	10.07
Frequency	10	30	0	80	60	100	90	60	40	20	10

To construct a histogram, enter the data in two columns similar to Table 13.1.

Construction of a histogram:

1. Highlight the frequencies. click on Insert, and then click on Chart to begin the process.
2. Choose Column and then the type of column graph that you want. We will select the first one under the "2-D Column" heading and the graph will appear, but it will not have the proper horizontal axis scale.
3. Right click on the horizontal axis and click on "Select Data. . . ." This will produce a popup window. On the right-hand side of that popup window under the heading "Horizontal (Category) Axis Labels," click on "Edit." This will produce another popup window with the heading "Axis Label Range."

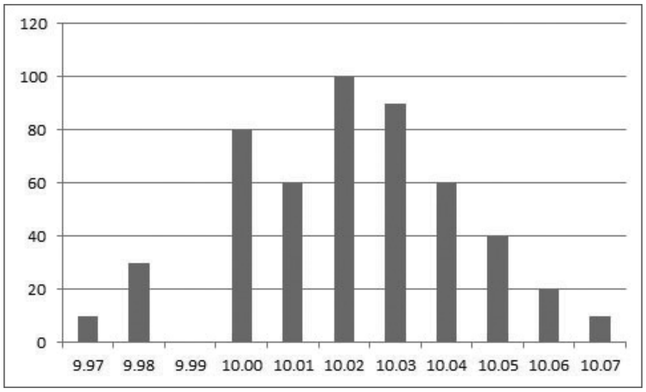

Figure 13.7a

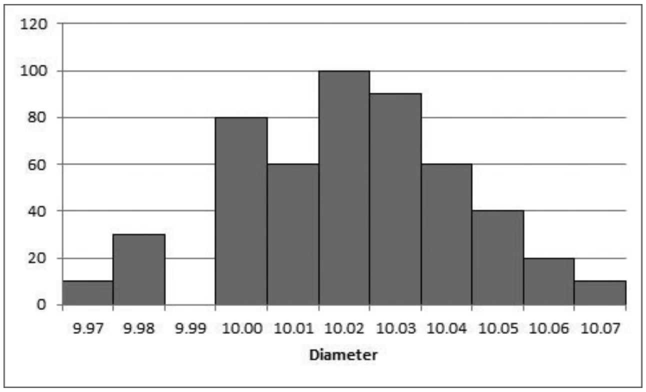

Figure 13.7b

	A	B	C
1	Diameter	Frequency	
2	9.97	10	99.7
3	9.98	30	299.4
4	9.99	0	0
5	10.00	80	800
6	10.01	60	600.6
7	10.02	100	1002
8	10.03	90	902.7
9	10.04	60	602.4
10	10.05	40	402
11	10.06	20	201.2
12	10.07	10	100.7
13		500	5010.7
14		Mean	10.0214

Figure 13.7c

	A	B	C	D
1	Diameter	Frequency		Cummulative Freq.
2	9.97	10	99.7	10
3	9.98	30	299.4	40
4	9.99	0	0	40
5	10.00	80	800	120
6	10.01	60	600.6	180
7	10.02	100	1002	280
8	10.03	90	902.7	370
9	10.04	60	602.4	430
10	10.05	40	402	470
11	10.06	20	201.2	490
12	10.07	10	100.7	500
13		500	5010.7	
14		Mean	10.0214	

Figure 13.7d

Highlight the column (or row) of the diameters. Click **OK** and then click on **OK** again and you should get a figure like the one in Figure 13.7a.

4. Technically this is not a histogram because there are gaps between the columns. To make it into a histogram, right click on a column and then click on "Format Data Series. . . ." Slide the Gap Width to No Gap and click close to produce a histogram like the one in Figure 13.7b.

5. When you were at "Format Data Series . . . " there were several other options you could have made. For example, you can change the color or type fill for the columns and the color and width of the column borders.

Determine the mean: There are two ways to find the mean of this data.

(a) To find the mean of this data, extend the table one column by adding a column representing the sum of all the entries in that row. For example, since there are 10 rods with a diameter of 9.97, then the sum of the diameters of those 10 rods is $10 \times 9.97 = 99.7$. The 30 rods with a diameter of 9.98 add up to $30 \times 9.98 = 299.4$, and so on. Adding those products gives us the sum of all the diameters from which we can obtain the mean.

The mean is found by first finding the sum of the frequencies in Column B (Cell B13: = SUM (B2 : B12)) and then dividing the total sum by the number of rods (Cell C14: = C13/B13). (See Figure 13.7c.)

EXAMPLE 13.21 (Cont.)

Minimum	9.97
1st Quartile	10.01
Median	10.02
3rd Quartile	10.04
Maximum	10.07

Figure 13.7e

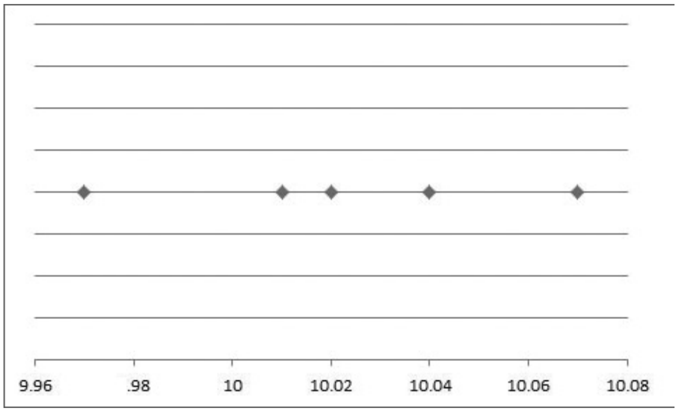

Figure 13.7f

(b) The second method uses two of the built-in functions of Excel: SUMPRODUCT and SUM. To use this type = sumproduct (and select the diameters of the rods, type a comma, select the frequencies, and type a right parenthsis ")". What you have should look like this: = SUMPRODUCT (A2:A12, B2:B12). This is to be divided by the sum, so in the same cell as the sumproduct type /SUM (, select the frequencies, type). When you have finished, the cell should look something like this:

= SUMPRODUCT (A2:A12, B2:B12)/SUM(B2:B12)

Press **Return** and the mean of 10.0214 should result.

Determine the median: The median is the middle number. A new column is again added to the original table. The fourth column, cumulative frequency, shows the total of the frequencies to that point. (See Figure 13.7d.)

We can see that the median is 10.02 since the 250th data entry will occur in that row.

Determine the mode: The mode is obtained by reviewing the data. It is apparent that the diameter that occurs most frequently is 10.02. Thus 10.02 is the mode.

Create a box plot: The closest a spreadsheet can come to easily producing a box plot is a scatterplot of the five quartile points used in a box plot: the minimum, the first quartile, the median, the third quartile, and the maximum. Figure 13.7e shows the data and Figure 13.7f the result.

APPLICATION COMMUNICATIONS

EXAMPLE 13.22

Use a spreadsheet to draw a histogram of the average monthly cell phone bill, in dollars, from the data in Table 13.2 in Example 13.20. The table is copied below.

Average Monthly Cell Phone Bill, 1990–2009

Year	1990	1991	1992	1993	1994	1995	1996	1997	1998	1999
Bill ($)	80.90	72.74	68.68	61.48	56.21	51.00	47.70	42.78	39.43	41.24

Year	2000	2001	2002	2003	2004	2005	2006	2007	2008	2009
Bill ($)	45.27	47.37	48.40	49.91	50.64	49.98	50.56	49.79	50.07	48.16

SOLUTION Begin by entering the years in Column A and the amount of the phone bill in Column B. This is partially shown in Figure 13.8a.

We next want to group the data in bins. A bin is a set of ranges of the variable. A histogram has one bar representing the count of the number of measurements that fall within a single bin. We want our histogram to look like the one in Figure 13.6c. In Figure 13.6c the bars had the intervals [35, 40), [40, 45), and so on. In Excel, if we put the first bin at 40 it will include all numbers 40 and below, so technically we should set the first bin limit at 39.99. Because it will be neater, we will set the first bin, in Cell D2, at 40. The next bin limit will be $5 higher or 45. We continue this until we have 10 bins in Cells D2 through D11.

You could count the number of measurements that fall within each of these bins, but an easier way is to use the Excel function FREQUENCY. Click in Cell E2 and then click on the Insert function button (it looks like *fx* and is just to the left of the Formula Bar). Pick the Statistical Function category and scroll down in the box choose FREQUENCY as the Function name. You should now see something like Figure 13.8b.

	A	B
1	Year	Phone bill
2	1990	80.90
3	1991	72.74
4	1992	68.68
5	1993	61.48
6	1994	56.21

Figure 13.8a

FREQUEN... fx =FREQUENCY()

	A	B	C	D	E
1	Year	Phone bill		Bin limit	Frequency
2	1990	80.90		40	ENCY()
3	1991	72.74		45	
4	1992	68.68		50	
5	1993	61.48		55	
6	1994	56.21			
7	1995	51.00			
8	1996	47.70			
9	1997	42.78			
10	1998	39.43			

Function Arguments
FREQUENCY
Data_array
Bins_array

Figure 13.8b

fx =FREQUENCY(B2:B21,D2:D11)

C	D	E
	Bin limit	Frequency
	39.99	1.00
	44.99	

Figure 13.8c

{=FREQUENCY(B2:B21,D2:D11)}

C	D	E
	Bin limit	Frequency
	40	1
	45	2
	50	8
	55	4
	60	1
	65	1
	70	1
	75	1
	80	0
	85	1

Figure 13.8d

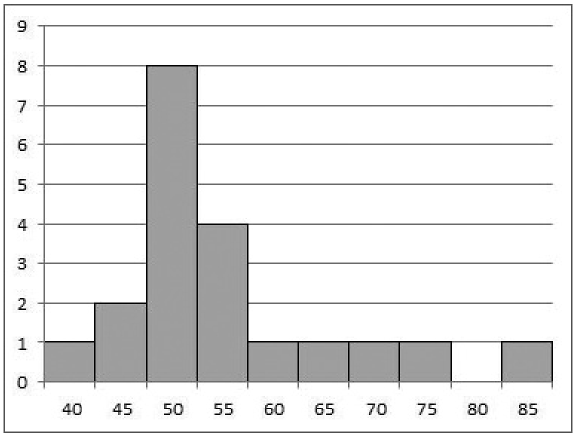

Figure 13.8e

EXAMPLE 13.22 (Cont.)

Now fill in the Function Arguments. Put the cursor in the Data_array box and click. Go to the spreadsheet page and highlight the data values (B2:B21). Next move the cursor to the Bins_array box and click. Then go to the spreadsheet and highlight the bin limits cells (D2:D11). Click OK. The completed formula is shown in the Formula Bar and the correct count value is shown in the Bin Limit 40 count cell (E2), as shown in Figure 13.8c.

We need to copy the array function to the other Frequency count cells. This is somewhat different than typical cell copying. Highlight the Frequency cells (E2:E11) and click on the FREQUENCY function in the Formula Bar (= FREQUENCY (B2:B21, D2:D11). Propagate the function by typing Control-Shift-Enter on a PC (type Command-Return on the Mac).

The frequency values should now fill the cells next to the bin increments, as shown in Figure 13.8d. Note that your first bin increment, 40, holds all the measurements at 40 and below.

To get the histogram, follow the procedures outlined in Example 13.21 using the frequencies in Cells E2 through E11. Use the bin values when you modify the horizontal axis. The final histogram should look something like Figure 13.8e. You might want to compare this to the calculator histogram in Figure 13.6c.

EXERCISE SET 13.2

Use the following four sets of numbers for Exercises 1–20.

A: 4, 2, 6, 3, 7, 4, 6, 2, 4, 4

B: 50, 52, 52, 54, 54, 54, 54, 56, 58, 58

C: 80, 77, 82, 73, 92, 89, 100, 96, 96, 94, 74, 94, 94, 96, 83, 84, 96, 87, 84, 96

D: 100, 98, 96, 94, 93, 90, 89, 85, 82, 78, 76, 66, 64, 64, 78, 89, 93, 96, 98, 96, 93, 64, 96

In Exercises 1–4, set up a frequency distribution and relative frequency table for the numbers in the indicated set.

1. Set A
2. Set B
3. Set C
4. Set D

In Exercises 5–8, set up a frequency distribution and relative frequency table for each of the given intervals in the indicated sets.

5. 1–2, 3–4, 5–6, 7–8 in Set A

6. 50–52, 53–55, 56–58 in Set B

7. 71–75, 76–80, 81–85, 86–90, 91–95, 96–100 in Set C

8. 61–65, 66–70, 71–75, 76–80, 81–85, 86–90, 91–95, 96–100 in Set D

In Exercises 9–12, draw histograms for the data in the given exercises.

9. Exercise 1
10. Exercise 2
11. Exercise 7
12. Exercise 8

In Exercises 13–16, draw broken-line graphs for the data in the indicated exercises.

13. Exercise 1 **14.** Exercise 2 **15.** Exercise 7 **16.** Exercise 8

In Exercises 17–20, determine the mean, median, mode, and quartiles of the given set. Then draw a box plot of the data.

17. Set A **18.** Set B **19.** Set C **20.** Set D

Solve Exercises 21–40.

21. ***Energy technology*** The following data are based on the energy consumption for one household's electric bills for 36 two-month periods.

Energy (kWh)	700–719	720–739	740–759	760–779	780–799
Frequency	2	2	4	5	3

Energy (kWh)	800–819	820–839	840–859	860–879	880–899
Frequency	4	7	5	2	2

Construct a histogram that corresponds to this frequency table.

22. ***Energy technology*** A sample of 100 batteries was selected from the day's production for a machine. The batteries were tested to see how long they would operate a flashlight, with these results:

Hours	211–215	216–220	221–225	226–230
Frequency	4	9	19	23

Hours	231–235	236–240	241–245	246–250
Frequency	16	14	10	5

Form a histogram for this data.

23. ***Energy technology*** Form a histogram for the data in Exercise 22, using the intervals 211–220, 221–230, etc.

24. ***Industrial technology*** A technician was measuring the thickness of a plastic coating on some pipe and obtained the following data:

Thickness (mm)	0.01	0.02	0.03	0.04	0.05
Frequency	1	5	40	50	36

Thickness (mm)	0.06	0.07	0.08	0.09
Frequency	30	25	10	3

Form a histogram of this data.

25. ***Industrial technology*** Draw a broken-line graph for the data in Exercise 24.

26. ***Industrial technology*** Form a histogram for the data in Exercise 24 over the intervals 0.01–0.02, 0.03–0.04, etc.

27. ***Industrial technology*** Determine the mean, median, mode, and quartiles for the data in Exercise 24.

28. ***Industrial technology*** Draw a box plot for the data in Exercise 24.

29. ***Electrical technology*** A technician tested an electric circuit and found the following values in milliamperes on successive trials:

5.24, 5.31, 5.42, 5.26, 5.31, 5.47, 5.27, 5.29, 5.35, 5.44,
5.35, 5.31, 5.45, 5.46, 5.39, 5.34, 5.35, 5.46, 5.26, 5.27,
5.47, 5.34, 5.28, 5.39, 5.34, 5.42, 5.43, 5.46, 5.34, 5.29

Form a frequency distribution for this data.

30. ***Electrical technology*** For the data in Exercise 29, draw a histogram for the intervals 5.21–5.25, 5.26–5.30, etc.

31. ***Electrical technology*** Determine the mean, median, and mode for the data in Exercise 29.

32. ***Electrical technology*** Determine the quartiles and draw a box plot for the data in Exercise 29.

33. ***Police science*** A patrol officer using a laser gun recorded the following speeds for motorists driving through a 55-mph speed zone:

52 57 62 59 67 54

55 64 65 59 63 72

(a) Determine the mean, median, mode, and quartiles for the given data.

(b) Draw a box plot for the given data.

34. ***Environmental science*** An environmental officer measured the carbon monoxide emissions (in g/m) for several vehicles. The results were as follows:

5.02 12.36 13.46 6.92 7.44 8.52 12.82

11.92 14.32 12.06 8.02 11.34 6.66 9.28

(a) Determine the mean, median, mode, and quartiles for the given data.

(b) Draw a box plot for the given data.

35. ***Energy science*** The carbon monoxide emissions (in g/m) were measured for several vehicles. The results were as follows:

892 673 534 437 449 524

627 735 892 923 1024 905

865 704 624 535 432 495

572 625 655 684 532 484

Determine the mean, median, mode, and quartiles for the given data.

36. ***Insurance*** The blood alcohol content of 15 drivers involved in fatal accidents and then convicted with jail sentences are given below:

0.14 0.16 0.21 0.10 0.13

0.19 0.26 0.22 0.13 0.09

0.11 0.18 0.12 0.24 0.27

Determine the mean, median, mode, and quartiles for the given data.

37. ***Business*** The daily sales in dollars for one 31-day month at a store are shown below:

24 562 38 646 43 988 15 122 14 321 17 479 19 478

25 625 39 476 45 353 15 972 13 793 17 457 18 681

20 562 38 606 53 788 15 122 10 321 13 037 17 038

25 625 37 036 45 353 15 732 13 373 13 053 18 681

21 903 41 775 52 117

Determine the mean, median, mode, and quartiles for the given data.

38. ***Food technology*** In a popcorn experiment, 100 kernels of different brands of popcorn were heated in oil for 1 min. At the end of that time, the number of popped kernels were counted and recorded in the following table.

23	77	20	12	19	54	15	44	41	15
73	31	41	31	79	70	80	69	79	83

(a) Sketch a frequency distribution and a histogram of the given data.

(b) Determine the mean, median, mode, and quartiles for the given data.

39. ***Information technology*** During a 24-h time period, a World Wide Web site kept track of the number of times its home page was accessed, or "hits" it received. The results are shown in the following table. Here hour 0 represents 12:00 midnight–1:00 AM, hour 1 represents 1:00 AM–2:00 AM, etc.

Hour	0	1	2	3	4	5	6	7
Number of "hits"	181	120	138	96	146	115	142	273

Hour	8	9	10	11	12	13	14	15
Number of "hits"	776	697	836	886	922	838	892	947

Hour	16	17	18	19	20	21	22	23
Number of "hits"	625	558	355	349	270	402	238	204

(a) Sketch a histogram, broken-line graph, and box plot of the given data.

(b) Determine the mean, median, mode, and quartiles for the given data.

40. ***Environmental science*** The table below shows the concentration of carbon tetrachloride (CCl_4) in the atmosphere for the period 1979–2008. Emission values are given in parts per trillion (ppt).

Year	1979	1980	1981	1982	1983	1984	1985	1986
CCl_4 (ppt)	88	90	90	92	94	95	97	98

Year	1987	1988	1989	1990	1991	1992	1993	1994
CCl_4 (ppt)	100	101	101	102	102	102	101	101

Year	1995	1996	1997	1998	1999	2000	2001	2002
CCI$_4$ (ppt)	100	99	98	97	96	95	94	93

Year	2003	2004	2005	2006	2007	2008
CCI$_4$ (ppt)	92	92	91	88	87	86

(a) Plot the given data.

(b) Sketch a histogram and box plot of the given data.

(c) Determine the mean, median, mode, and quartiles for the given data.

[IN YOUR WORDS]

41. What are the mean, median, and mode? What do they measure? How are they determined?

42. What are the quartiles? Describe how you find them.

43. Describe how to make a box plot.

44. What is a histogram? Describe how a histogram is constructed.

45. What information does a histogram give that is not given by a box plot?

46. What information does a box plot give that is not given by a histogram?

13.3 MEASURES OF DISPERSION

In Section 13.2, we looked at the three measures of central tendency. In this section, we will learn a technique that will tell us how close together the information is distributed. *Measures of dispersion* tell how close together or how spread out the data is.

VARIANCE AND STANDARD DEVIATION

There are several measures of dispersion. We will look at two of them: variance and standard deviation.

VARIANCE AND STANDARD DEVIATION

The **variance** v of a set of numbers is given by the formula

$$v = \frac{\sum_{i=1}^{n} (x_i - \bar{x})^2}{n - 1}$$

and the **standard deviation** s is the square root of the variance. Thus,

$$s = \sqrt{v}$$

EXAMPLE 13.23

Find the variance and standard deviation of the numbers 9, 8, 3, 2, 4.

SOLUTION This is the same set of numbers we used in Example 13.11. In Example 13.16, we determined that the mean was 5.2. We can set up a table to help with the calculations.

x	$x - \bar{x}$	$(x - \bar{x})^2$
9	3.8	14.44
8	2.8	7.84
3	−2.2	4.84
2	−3.2	10.24
4	−1.2	1.44
		38.80

$$v = \frac{38.8}{4} = 9.70$$

$$s = \sqrt{9.70} \approx 3.11$$

In some sets of data, some values occur more than once. The formula for the variance becomes

$$v = \frac{\sum f_i(x_i - \bar{x})^2}{n - 1}$$

What you would do is add two more columns to your table, one for f_i and the other for $f_i(x_i - \bar{x})^2$, as is shown by the next example. Note that $n = \sum f_i$.

EXAMPLE 13.24

Determine the mean and standard deviation for the following values: 39, 41, 39, 44, 39, 40, 39, 40, 37, 42, 37, 43, 44, 38, 38, 38, 43, 38, 41, 39.

SOLUTION We first make a frequency distribution table and then add columns for $x_i f_i$, $x_i - \bar{x}$, $(x_i - \bar{x})^2$, and $(x_i - \bar{x})^2 f_i$.

x_i	f_i	$x_i f_i$	$x_i - \bar{x}$	$(x_i - \bar{x})^2$	$(x_i - \bar{x})^2 f_i$
37	2	74	−2.95	8.7025	17.4050
38	4	152	−1.95	3.8025	15.2100
39	5	195	−0.95	0.9025	4.5125
40	2	80	0.05	0.0025	0.0050
41	2	82	1.05	1.1025	2.2050
42	1	42	2.05	4.2025	4.2025
43	2	86	3.05	9.3025	18.6050
44	2	88	4.05	16.4025	32.8050
Totals	20	799			94.95

The mean is $\bar{x} = \dfrac{\sum x_i f_i}{\sum f_i} = \dfrac{799}{20} = 39.95$ and the variance is $v = \dfrac{\sum (x_i - \bar{x})^2 f_i}{\left(\sum f_i\right) - 1} = \dfrac{94.95}{19} = 4.9974$. The standard deviation is $s = \sqrt{4.9974} \approx 2.24$.

STANDARD DEVIATION ON A CALCULATOR

Again, a calculator will do much of the work for you. Follow the procedures for finding the mean as discussed in the user's guide for your calculator. In most cases, a graphing calculator will calculate both the variance and standard deviation at the same time as it computes the mean.

CAUTION Some calculators report two standard deviation scores. For example, the TI graphing calculators report an Sx and a σx. The symbol *Sx*, or S_x, represents the *sample standard deviation* and the symbol σx, or σ_x, represents the *population standard deviation*. In most cases you will want the higher of the two scores, Sx or S_x, the sample standard deviation.

The importance of different values of the standard deviation is interpreted mostly from experience. But a small standard deviation indicates that the values are closely clustered about the mean. On the other hand, a large standard deviation indicates that the values are spread out widely from the mean.

EXAMPLE 13.25

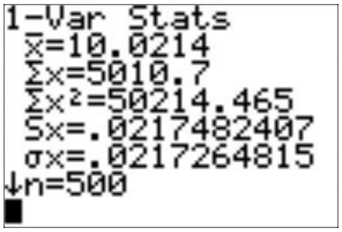

Figure 13.9

Use a TI-83 or TI-84 and the data for the 500 steel rods in Table 13.1 to find the mean and standard deviation.

SOLUTION We have reproduced the table below, as Table 13.3.

In Example 13.19 we let the calculator determine the mean for this data. At the time we got the result shown in Figure 13.5d and reproduced in Figure 13.9. Examine Figure 13.9 and you will see that it gives the sample standard deviation as $S_x = 0.021748241$ and the population standard deviation as $\sigma_x = 0.021726482$.

TABLE 13.3

Diameter	9.97	9.98	9.99	10.00	10.01	10.02	10.03	10.04	10.05	10.06	10.07
Frequency	10	30	0	80	60	100	90	60	40	20	10

STANDARD DEVIATION WITH A SPREADSHEET

A spreadsheet has built-in functions for variance, sample standard deviation, and population standard deviation. These calculations are rather straightforward if the data is not grouped by frequency. The next two examples will show how this is done, with the second example showing how to use grouped data.

EXAMPLE 13.26

Use a spreadsheet to find the variance and standard deviation of 9, 8, 3, 2, and 4.

SOLUTION This is the same data as Example 13.23, so we should get the same results. Enter the data in Column A of the spreadsheet. The command for the sample variance is VAR.S and for the sample standard deviation it is STDEV.S. In Figure 13.10 if we enter = VAR.S (A1:A5) and press return we get 9.70. Similarly, = STDEV.S (A1:A5) gives the standard deviation of 3.114482. Figure 13.10 shows what the spreadsheet should look like just before you finish the last step.

EXAMPLE 13.27

Use a spreadsheet to find the variance and standard deviation of the data from Example 13.24:

Number	37	38	39	40	41	42	43	44
Frequency	2	4	5	2	2	1	2	2

SOLUTION Unlike in Example 13.22, we will just have three columns in our spreadsheet. The first two columns are the number and frequency as shown in the above table. Below this we next determine the mean as in Example 13.20. In Figure 13.11a the numbers are in Cells A2 through A9 and the frequencies in Cells B2 through B9. So, the mean is entered in Cell B10 using

$$= \text{SUMPRODUCT (A2:A9, B2:B9)/SUM (B2:B9)}$$

In the third column we place the values of $(x_i - \bar{x})^2$ by using $=$ (A2 $-$ \$B\$10) ^2 in Cell C3 and then dragging this down the column to Cell C9. This is all the information we need to compute the variance $v = \dfrac{\sum(x_i - \bar{x})^2 f_i}{(\sum f_i) - 1}$ in Cell B11 with the command

$$= \text{SUMPRODUCT (C2:C9, B2:B9) / (SUM (B2:B9)} - 1)$$

From this we get $v \approx 4.997368$. The standard deviation is the square root of the variance, so in Cell B12 we use $=$ SQRT (B11) with the result $s \approx 2.235479$, as shown in Figure 13.11b. These are the same values we got in Example 13.22.

	A
1	9
2	8
3	3
4	2
5	4
6	9.70
7	=STDEV.S(A1:A5)

Figure 13.10

	A	B
1	Number	Frequency
2	37	2
3	38	4
4	39	5
5	40	2
6	41	2
7	42	1
8	43	2
9	44	2
10	Mean	39.95

Figure 13.11a

	A	B	C
1	Number	Frequency	(No. - Mean)^2
2	37	2	8.7025
3	38	4	3.8025
4	39	5	0.9025
5	40	2	0.0025
6	41	2	1.1025
7	42	1	4.2025
8	43	2	9.3025
9	44	2	16.4025
10	Mean	39.95	
11	Variance	4.997368	
12	St. Dev.	2.235479	

Figure 13.11b

NORMAL CURVE

When we make a large number of measurements, we usually expect that a majority of them are near the mean. One of the most important frequency curves is called the **normal curve** or the *normal distribution curve*. The curve is bell shaped and symmetric about the mean. The normal curve was first discovered by DeMoivre. You will study his theorem for complex numbers in Chapter 14.

The standard form of the normal curve considers the mean to be the origin $(\bar{x} = 0)$ and the standard deviation as 1, and has the equation

$$y = \frac{1}{\sqrt{2\pi}}e^{-x^2/2}$$

Some results of this curve are that 68% of the values should be within one standard deviation of the mean. (While 68.26% is closer, we will use 68%.) Thus, 68% of the values are in the interval from $\bar{x} - s$ to $\bar{x} + s$. A total of about 95% of the values (it is actually closer to 95.44%) should be within two standard deviations of the mean or from $\bar{x} - 2s$ to $\bar{x} + 2s$. (See Figure 13.12.) You may want to round off the values $\bar{x} - s$, $\bar{x} + s$, $\bar{x} - 2s$, and $\bar{x} + 2s$ before you count the values in each interval.

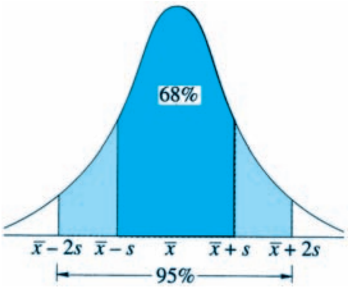

Figure 13.12

EXAMPLE 13.28

In Example 13.24, there were 20 values. The mean was 39.95 and the standard deviation was 2.18. Thus, the interval from $\bar{x} - s$ to $\bar{x} + s$ is from 37.77 to 42.13. This rounds off to the interval from 38 to 42 and there are 14 values in this interval. This is 70% of the values and is the closest you can get to 68% with a sample of 20 values.

A z-score is used to make it easier to tell whether the score is above or below the mean and how far the score is from the mean with respect to the standard deviation.

CHANGING AN X-VALUE TO A Z-SCORE

Population		Sample
$z = \dfrac{x - \mu}{\sigma}$	or	$z = \dfrac{x - \bar{x}}{s}$

EXAMPLE 13.29

For a set of data with $\bar{x} = 45$ and $s = 8$, find the z-score corresponding to (a) $x = 35$ and (b) $x = 51$.

SOLUTIONS

(a) $z = \dfrac{x - \bar{x}}{s} = \dfrac{35 - 45}{8} = \dfrac{-10}{8} = -1.25$

(b) $z = \dfrac{x - \bar{x}}{s} = \dfrac{51 - 45}{8} = \dfrac{6}{8} = 0.75$

APPLICATION HEALTHCARE

EXAMPLE 13.30

For men between the ages of 18 and 24 years, serum cholesterol levels, in mg/100 ml, have a mean of 178.1 and a standard deviation of 40.7. Find the z-score for a 22-year-old man who has a serum cholesterol level of 237.5.

SOLUTION $z = \dfrac{237.5 - 178.1}{40.7} = \dfrac{59.4}{40.7} \approx 1.48$

When a distribution is described in intervals, and you have no other information, then you must make some decisions before counting or not counting the points in an interval. In computing the mean and standard deviation, the midpoint of each interval is used as the "value" for that interval. The same criteria are used when counting the number of points within one or more standard deviations of the mean. For example, if $\bar{x} + s$ is at or past the midpoint of an interval, then include all the points of that interval. On the other hand, if $\bar{x} + s$ does not reach the midpoint of the interval, then do not count any of the points in the interval.

EXERCISE SET 13.3

In Exercises 1–12, use the following sets of numbers.

A. 4, 2, 6, 3, 7, 4, 6, 2, 4, 4

B. 50, 52, 52, 54, 54, 54, 54, 56, 58, 58

C. 80, 77, 82, 73, 92, 89, 100, 96, 96, 94, 74, 94, 94, 96, 83, 84, 96, 87, 84, 96

D. 100, 98, 96, 94, 93, 90, 89, 85, 82, 78, 76, 66, 64, 64, 78, 89, 93, 96, 98, 96, 93, 64, 96

In Exercises 1–4, find the variance v for the indicated set of numbers.

1. Set A **2.** Set B **3.** Set C **4.** Set D

In Exercises 5–8, find the standard deviation for the indicated set of numbers.

5. Set A **6.** Set B **7.** Set C **8.** Set D

In Exercises 9–12, determine the number of values within (a) one standard deviation and (b) two standard deviations of the mean.

9. Set A **10.** Set B **11.** Set C **12.** Set D

In Exercises 13–16, assume the given data is normally distributed, and (a) find the standard deviation for the indicated sets of numbers, (b) determine the number of data points within one standard deviation of the mean, and (c) determine the number of data points within two standard deviations of the mean.

13. ***Machine technology*** The frequency table for the 500 steel rods in Table 13.1 is reproduced below, as Table 13.4.

TABLE 13.4

Diameter	9.97	9.98	9.99	10.00	10.01	10.02
Frequency	10	30	0	80	60	100

Diameter	10.03	10.04	10.05	10.06	10.07
Frequency	90	60	40	20	10

14. ***Energy technology*** A sample of 100 batteries was selected from the day's production for a machine. The batteries were tested to see how long they would operate a flashlight, with these results. (Use the midpoint of each interval as the value for that interval.)

Hours	211–215	216–220	221–225	226–230
Frequency	4	9	19	23

Hours	231–235	236–240	241–245	246–250
Frequency	16	14	10	5

15. ***Machine technology*** A technician was measuring the thickness of a plastic coating on some pipe and obtained the following data:

Thickness (mm)	0.01	0.02	0.03	0.04	0.05
Frequency	1	5	40	50	36

Thickness (mm)	0.06	0.07	0.08	0.09
Frequency	30	25	10	3

16. ***Electrical technology*** A technician tested an electric circuit and found the following values in milliamperes on successive trials:

5.24, 5.31, 5.42, 5.26, 5.31, 5.47, 5.27, 5.29, 5.35, 5.44, 5.35, 5.31, 5.45, 5.46, 5.39, 5.34, 5.35, 5.46, 5.26, 5.27, 5.47, 5.34, 5.28, 5.39, 5.34, 5.42, 5.43, 5.46, 5.34, 5.29

In Exercises 17–24, (a) find the mean and standard deviation for each of the indicated sets of numbers, (b) determine the percent of the given data within one standard deviation of the mean, and (c) determine the percent of the given data within two standard deviations of the mean.

17. ***Police science*** A patrol officer using a laser gun recorded the following speeds for motorists driving through a 55-mph speed zone:

52	57	62	59	67	54
55	64	65	59	63	72

18. ***Environmental science*** An environmental officer measured the carbon monoxide emissions (in g/m) for several vehicles. The results were as follows:

5.02	12.36	13.46	6.92	7.44	8.52	12.82
11.92	14.32	12.06	8.02	11.34	6.66	9.28

19. ***Environmental science*** An environmental officer measured the carbon monoxide emissions (in g/m) for several vehicles. The results were as follows:

892	673	534	437	449	524
627	735	892	923	1024	905
865	704	624	535	432	495
572	625	655	684	532	484

20. ***Insurance*** The blood alcohol content of 15 drivers involved in fatal accidents and then convicted with jail sentences are given below:

0.14	0.16	0.21	0.10	0.13
0.19	0.26	0.22	0.13	0.09
0.11	0.18	0.22	0.24	0.16

21. ***Business*** The daily sales in dollars for one 31-day month at a store are shown below:

24 562 38 646 43 988 15 122 14 321 17 479 19 478

25 625 39 476 45 353 15 972 13 793 17 457 18 681

20 562 38 606 53 788 15 122 10 321 13 037 17 038

25 625 37 036 45 353 15 732 13 373 13 053 18 681

21 903 41 775 52 117

22. ***Energy technology*** The following data are based on the energy consumption for one household's electric bills for 36 two-month periods. (Hint: Use the midpoint of each class as the value for that class.)

Energy (kWh)	700–719	720–739	740–759	760–779	780–799
Frequency	2	2	4	5	3

Energy (kWh)	800–819	820–839	840–859	860–879	880–899
Frequency	4	7	5	2	2

23. ***Information technology*** During a 24-h time period a World Wide Web site kept track of the number of "hits" its home page received. The results are shown in the following table. Here hour 0 represents 12:00 midnight–1:00 AM, hour 1 represents 1:00 AM–2:00 AM, etc.

Hour	0	1	2	3	4	5	6	7
Number of "hits"	181	120	138	96	146	115	142	273

Hour	8	9	10	11	12	13	14	15
Number of "hits"	776	697	836	886	922	838	892	947

Hour	16	17	18	19	20	21	22	23
Number of "hits"	625	558	355	349	270	402	238	204

24. ***Environmental science*** The table below shows the concentration of carbon tetrachloride (CCl_4) in the atmosphere for the period 1979–2008. Emission values are given in parts per trillion (ppt).

Year	1979	1980	1981	1982	1983	1984	1985	1986
CCl_4 (ppt)	88	90	90	92	94	95	97	98

Year	1987	1988	1989	1990	1991	1992	1993	1994
CCl_4 (ppt)	100	101	101	102	102	102	101	101

Year	1995	1996	1997	1998	1999	2000	2001	2002
CCl_4 (ppt)	100	99	98	97	96	95	94	93

Year	2003	2004	2005	2006	2007	2008
CCl_4 (ppt)	92	92	91	88	87	86

[IN YOUR WORDS]

25. Explain the standard deviation and the variance. How are they alike? How are they different?

26. What is a normal curve?

13.4 STATISTICAL PROCESS CONTROL

For many years statistics were used for *statistical quality control* (**SQC**). In SQC statistical data is collected, analyzed, and interpreted to solve quality problems. One of the major difficulties with SQC is that it analyzes the data *after* production has been completed. For example, speaking of the 500 steel rods in Table 13.1, we looked at the statistics after all 500 rods had been produced.

People began to think that there was a need to try to determine problems *during* the production process rather than after it was finished. Prevention of defects by applying statistical methods during the process of production is known as **statistical process control (SPC)**. SPC emphasizes the prevention of defects.

No two manufactured parts are exactly alike. Designers translate the needs, requirements, and expectations of the customer into the part specifications. Specifications can be given as *nominal target dimensions or as tolerance limits*.

APPLICATION **BUSINESS**

EXAMPLE 13.31

What are some examples of nominal target dimensions?

SOLUTION A few nominal target dimensions are given in the following table:

Item	Nominal Dimension
Door height	6 ft 6 in.
Wheelbase length of a car (in customer catalog)	112 in.
Servings in a package	8
Structural timbers	2 in.
6060 glass style fabric	0.0018 in.
Adhesive dot diameter	14 mil

APPLICATION **BUSINESS**

EXAMPLE 13.32

What are some examples of specifications for tolerance limits?

SOLUTION Some specifications for tolerance limits include the following:

Item	Specifications
Car tire pressure	30–36 psi or 33 psi $\pm$ 3 psi
Wheelbase length of a car (during production)	112 in. $\pm$ 0.10 in.
100-μL variable volume pipette	100-μL $\pm$ 1.6%, or 1.6 μL
6060 glass style fabric	0.0018 in.
Diameter of a bolt	4.24–4.26 in. or 4.25 in. $\pm$ 0.01 in.
Adhesive dot diameter	14 mil

To manufacture parts within specifications, the process producing the parts needs to be stable and predictable. Small variations that occur during the production process are called *chance causes* or *common causes*. Common causes result from occasional voltage changes, small vibrations, or small variations in materials. A process is said to be *under control* or **in control** when the variability from one part to another is stable and predictable as a result of common causes.

Causes that cannot be classified as common causes are called *assignable causes*, and they often result in the process being out of control.

APPLICATION **BUSINESS**

EXAMPLE 13.33

The manufacturer of a DVD-R disc states that the thickness of a disk in the clamping area is 1.20 + 0.20/−0.10 mm. When is the production of these discs in control?

SOLUTION If the thickness in the clamping area of all the discs is in the range 1.10 mm–1.40 mm then the production process is in control. However, if some discs are less than 1.10 mm or thicker than 1.40 mm the process is out of control.

There has to be a reason for a process to be out of control. One reason might be that a machine needs adjusting. On the other hand, some impurities may have gotten in the process. Whatever the reason, the process will probably be halted until the cause is determined and corrected.

CONTROL CHARTS

The primary tool in SPC is the *control chart*. A control chart gives a continuous series of snapshots from small inspections taken at regular intervals. At each inspection, samples are pulled from the production line and measured. These measurements are graphed to make it easier to notice trends or abnormalities in the production process.

There are two general types of control charts.

GENERAL TYPES OF CONTROL CHARTS

Variables chart: A dimension or characteristic is measured and the result is a number.
Attribute chart: A dimension or characteristic is not measured in numbers but is classified as either "good" or "bad."

VARIABLES CHARTS: THE $\bar{x}$ AND R CHARTS

Perhaps the most common variable charts are the $\bar{x}$ (x-bar) and R charts. While these are separate charts, they are usually considered together. For each chart you need to compute, and then draw, the **central line** and the control limits. There are two control limits—the **upper control limit (UCL)** and the **lower control limit (LCL)**. The LCL and UCL are used to determine if the process is in control or out of control.

The following steps are used to prepare $\bar{x}$ and R charts for each variable you want to measure.

1. Choose a sample size n and how often a sample is selected. Sample sizes usually contain four to seven items.
2. Collect the data. Compute the mean and range of each sample.
3. Start the control charts.
4. Determine the central lines. The central lines are determined by computing the average mean of the samples $\bar{\bar{x}}$ and the average range $\bar{R}$. Thus, the central lines are $y = \bar{\bar{x}}$ and $y = \bar{R}$.
5. Determine the control limits.
6. Draw the central lines and the upper and lower control limits.
7. Interpret the chart.

Example 13.33 shows how this is done, using some data from the manufacturing of bearings.

The control limits for the $\bar{x}$ and R charts are supposed to be ± 3 standard deviations ($\pm 3\sigma$) from the central line in each chart. The problem is that σ is the population standard deviation and we normally do not have that figure. So, the upper and lower control limits for the $\bar{x}$ chart are

$$UCL_{\bar{x}} = \bar{\bar{x}} + A_2\bar{R}$$
$$LCL_{\bar{x}} = \bar{\bar{x}} - A_2\bar{R}$$

where A_2 is a number from Table 13.5.

For the R chart, the average of the sample ranges $\bar{R}$ is multiplied by the D_3 and D_4 factors in Table 13.5.

$$UCL_R = D_4\bar{R}$$
$$LCL_R = D_3\bar{R}$$

TABLE 13.5 Control Chart Factors

n	A	A_2	d_2	D_1	D_2	D_3	D_4
4	1.500	0.729	2.059	0	4.698	0	2.282
5	1.342	0.577	2.326	0	4.918	0	2.114
6	1.225	0.483	2.534	0	5.078	0	2.004
7	1.134	0.419	2.704	0.204	5.204	0.076	1.924

APPLICATION BUSINESS

EXAMPLE 13.34

Construct $\bar{x}$ and R charts for a diameter of a certain bearing.

SOLUTION

1. Choose a sample size n and how often a sample is selected. We will select 5 ball bearings in each sample and will pick a sample every 30 min.

2. Collect the data. Compute the mean and range of each sample. The data in Table 13.6 was collected during a 10-h period.

TABLE 13.6 Diameter of Ball Bearings (0.001 mm)

Sample	Diameters of 5 Ball Bearings (mm)					Mean $\bar{x}$	Range R
1	2.7	3.2	2.9	2.2	2.8	2.8	1.0
2	1.8	2.5	3.1	1.9	2.2	2.3	1.3
3	2.0	2.4	2.1	2.2	2.2	2.2	0.4
4	1.4	0.4	1.7	1.2	0.7	1.1	1.3
5	2.4	3.3	3.8	3.7	2.9	3.2	1.4
6	2.8	3.0	2.6	2.1	1.1	2.3	1.9
7	2.6	2.0	2.5	2.9	0.9	2.2	2.0
8	3.5	2.8	2.6	2.6	2.2	2.7	1.3
9	3.9	3.3	3.0	1.5	2.9	2.9	2.4
10	3.2	2.1	4.1	3.7	2.9	3.2	2.0
11	2.4	3.3	2.7	0.6	3.1	2.4	2.7
12	5.6	3.8	4.5	2.9	3.9	4.1	2.7
13	3.6	3.8	4.2	2.5	2.7	3.4	1.7
14	3.1	3.1	3.1	3.3	3.5	3.2	0.4
15	2.9	2.6	2.8	2.4	2.7	2.7	0.5
16	2.7	2.0	1.5	3.2	2.9	2.5	1.7
17	1.8	2.4	1.7	1.4	2.2	1.9	1.0
18	2.7	4.6	1.9	2.4	2.7	2.9	2.7
19	3.8	4.6	2.0	3.2	1.8	3.1	2.8
20	3.6	3.1	2.8	1.9	1.6	2.6	2.0

3. Start the control charts. We first plot the values for $\bar{x}$ and R using broken-line graphs as in Figure 13.13a.

4. Determine the central lines: $y = \bar{\bar{x}}$ and $y = \bar{R}$. Using a calculator or spreadsheet we find that the mean of the numbers in the $\bar{x}$ column of Table 13.6 is $\bar{\bar{x}} = 2.71$. We also find the mean of the ranges, the numbers in the R column, is $\bar{R} = 1.53$. Thus, the central line for the x-bar chart is $y = \bar{\bar{x}} = 2.71$ and the central line for the R chart is $y = \bar{R} = 1.53$.

5. Determine the control limits.
   For the $\bar{x}$ chart we get

$$\begin{aligned} UCL_{\bar{x}} &= \bar{\bar{x}} + A_2\bar{R} \\ &= 2.71 + (0.577)(1.53) \\ &\approx 3.593 \\ LCL_{\bar{x}} &= \bar{\bar{x}} - A_2\bar{R} \\ &= 2.71 - (0.577)(1.53) \\ &\approx 1.827 \end{aligned}$$

where A_2 is the number from Table 13.5 when $n = 5$.

For the R chart we get

$$\begin{aligned} UCL_R &= D_4\bar{R} \\ &= (2.114)(1.53) \\ &\approx 3.234 \\ LCL_R &= D_3\bar{R} \\ &= (0)(1.53) = 0 \end{aligned}$$

6. Draw the central line and the upper and lower control limits on each chart. When these lines are drawn on the control charts we get the charts in Figure 13.13b.
7. Interpret the chart. The process was out of control when the 4th and 12th samples were taken.

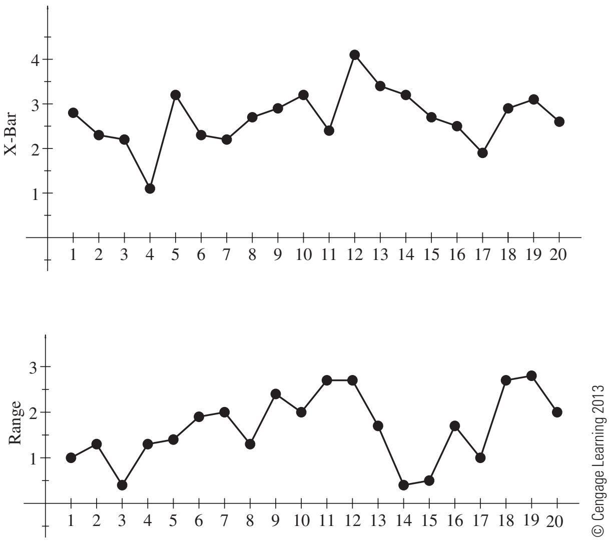

Figure 13.13a

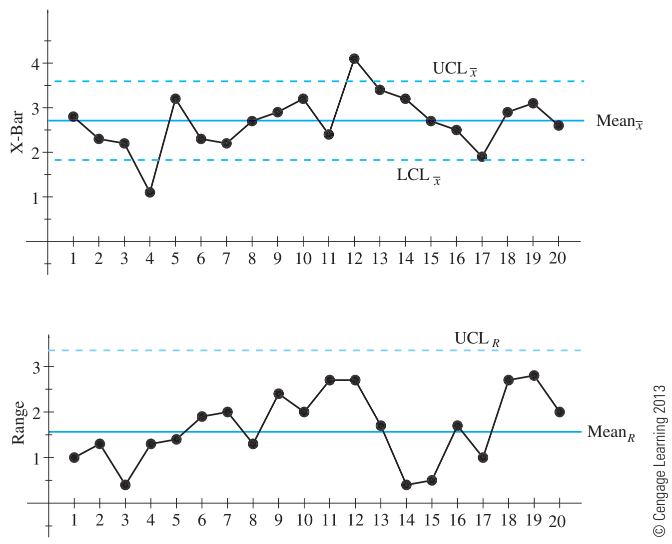

Figure 13.13b

Since the process was out of control when the 4th and 12th samples were taken and in control when the next samples were taken, an adjustment was probably made to the process.

ATTRIBUTE CHARTS: THE *P* CHART

Attributes are characteristics associated with a product or service. An attribute is normally not measured in numbers but is classified as either "good" or "bad." But, the number of times that the product is classified as "good" or "bad" can be counted. A condition or characteristic that keeps a product from being used is called *nonconforming*. When you want to study the proportion of the product that is nonconforming, then a *fraction nonconforming (p) chart*, a *number nonconforming (np)* chart, or a *percent nonconforming chart* is used. We will examine the fraction nonconforming, or p, chart.

The fraction nonconforming, or p, chart is used to study the proportion of nonconforming products or services. A p chart is constructed using the following steps:

1. Collect the data.
2. Calculate p, the proportion nonconforming.
3. Plot the proportion nonconforming on the control chart. The central line of the control chart is $y = \bar{p}$, where $\bar{p}$ is the proportion of the total group that was nonconforming. The upper and lower control limits for the p chart are

$$\text{UCL}_p = \bar{p} + \frac{3\sqrt{\bar{p}(1-\bar{p})}}{\sqrt{n}}$$

$$\text{LCL}_p = \bar{p} - \frac{3\sqrt{\bar{p}(1-\bar{p})}}{\sqrt{n}}$$

4. Determine the central lines and control limits.
5. Draw the central lines and control limits on the chart.
6. Interpret the chart.

Example 13.35 shows how this is done, using some data from the service department of an automobile dealership.

APPLICATION **BUSINESS**

EXAMPLE 13.35

An automobile dealership conducts telephone surveys of customers who have had their cars serviced at the dealership in the past week. Construct a p chart of the results.

SOLUTION

1. Collect the data. Each week customers who had their car serviced within the past week are selected at random and telephoned until 50 customers

have been surveyed. The data in Table 13.7 was collected during a 3-month (13-week) period.

TABLE 13.7 Survey of Customer Satisfaction

Week	Number Surveyed n	Nonconforming Number np	Nonconforming Proportion p
1	50	1	0.02
2	50	2	0.04
3	50	4	0.08
4	50	1	0.02
5	50	1	0.02
6	50	0	0.00
7	50	7	0.14
8	50	3	0.06
9	50	5	0.10
10	50	1	0.02
11	50	3	0.06
12	50	2	0.04
13	50	1	0.02
Total	650	31	

2. Calculate p, the proportion nonconforming. The proportion nonconforming each week is $p = \dfrac{np}{n}$. These values have been entered in the last column of Table 13.7.

3. Plot the proportion nonconforming on the control chart. The values of p are plotted using a broken-line graph in Figure 13.14a.

4. Determine the central lines and control limits. The central line of the control chart is $y = \bar{p}$, where $\bar{p}$ is the proportion of the total group that were nonconforming. There was a total of 650 people surveyed and 31 did not conform, so $\bar{p} = \frac{31}{650} \approx 0.048$. The upper and lower control limits for the p chart are

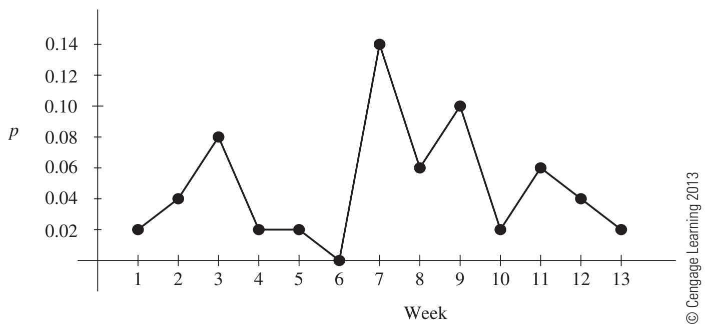

Figure 13.14a

EXAMPLE 13.35 (Cont.)

$$\mathrm{UCL}_p = \bar{p} + \frac{3\sqrt{\bar{p}(1-\bar{p})}}{\sqrt{n}}$$
$$= 0.048 + \frac{3\sqrt{0.048(1-0.048)}}{\sqrt{50}}$$
$$= 0.139$$
$$\mathrm{LCL}_p = \bar{p} - \frac{3\sqrt{\bar{p}(1-\bar{p})}}{\sqrt{n}}$$
$$= 0.048 - \frac{3\sqrt{0.048(1-0.048)}}{\sqrt{50}}$$
$$= -0.043$$

5. Draw the central lines and control limits on the chart. The central line and the UCL line are drawn in Figure 13.14b. The LCL line was not drawn because it was negative.

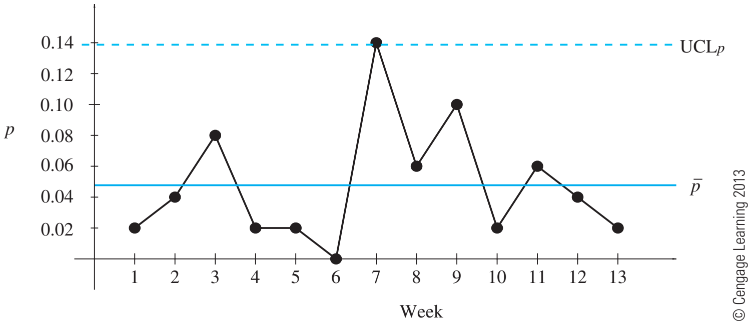

Figure 13.14b

6. Interpret the chart. In week 7 the proportion of customers who were not satisfied is above the upper control limit. The cause for this was investigated to determine if an assignable cause existed. The cause was determined and steps were taken to prevent future occurrences.

CONTROL CHARTS FOR POPULATIONS

As the production process continues and more data is gathered, you start to have good estimates of the population mean μ and standard deviation σ. In these cases we have modified versions of the formulas for the central lines and upper and lower control limits.

For the $\bar{x}$ chart: Central line: $y = \mu$

$$\mathrm{UCL}_{\bar{x}} = \mu - A\sigma$$
$$\mathrm{LCL}_{\bar{x}} = \mu - A\sigma$$

For the R chart: Central line: $y = d_2$

$$\mathrm{UCL}_R = D_2\sigma$$
$$\mathrm{LCL}_R = D_1\sigma$$

where A, d_2, D_1, and D_2 are in Table 13.5.

SIX SIGMA

In 1988, Motorola Corp. became one of the first companies to receive the Malcolm Baldrige National Quality Award. The award strives to honor companies that are worthy role models for other businesses. Motorola's **Six Sigma** (6σ) program was one innovation that attracted a great deal of attention. Six Sigma is, basically, a process quality goal—an approach for eliminating defects. At many organizations, Six Sigma means a measure of quality that strives for near perfection.

The traditional quality paradigm defined a process as capable if the process's natural spread, plus and minus three sigma ($\pm 3\sigma$), was less than the engineering tolerance. Under the assumption of normality, this translates to a process yield of 99.73%. Until Six Sigma, all quality calculations were based on this distribution.

A later refinement considered the process location as well as its spread and tightened the minimum acceptable so that the process was at least four sigma from the nearest engineering requirement. Motorola's Six Sigma asks that processes operate such that the nearest engineering requirement is at least plus or minus six sigma from the process mean.

The "official" Six Sigma literature states that a process operating at Six Sigma levels will produce no more than 3.4 defects per million opportunities. A Six Sigma defect is defined as anything outside of customer specifications.

APPLICATION **BUSINESS**

EXAMPLE 13.36

Specifications call for a steel rod to have a length of 25.78 cm $\pm$ 0.001 cm. What standard deviation would result in the specifications meeting Six Sigma quality standards?

SOLUTION The length of the rod is to fall between 25.78 cm $-$ 0.001 cm and 25.78 cm $+$ 0.001 cm or 25.779 $-$ 25.781 cm. If this is to meet Six Sigma quality standards, then $25.779 = \bar{x} - 6\sigma$ and $25.781 = \bar{x} + 6\sigma$. This also means that $6\sigma = 0.001$ cm and so $\sigma = \frac{0.001 \text{ cm}}{6} \approx 0.000167$ cm.

EXERCISE SET 13.4

Solve Exercises 1–8 using the formulas for population control charts.

1. In the production of silicon chips it was found that for the number of successful chips on a wafer $\mu = 254.37$ and $\sigma = 0.238$. If the sample size is 6, what are the central line, UCL, and LCL for the mean?
2. For the silicon chips in Exercise 1, find the central line, UCL, and LCL for the range.
3. A hospital is using $\bar{x}$ and R charts to record the time it takes to process account information. Based on several years of data the hospital has concluded that $\mu = 16.4$ min and $\sigma = 5.9$ min. For a sample size of 5, determine the central line, UCL, and LCL for the mean.
4. For the hospital in Exercise 3, find the central line, UCL, and LCL for the range.
5. A certain breakfast cereal is supposed to contain 453 g in each box of cereal. Every 15 min seven boxes are removed and the contents are weighed. If $\mu = 453.0$ g and $\sigma = 5.9$ g, find the central line, UCL, and LCL for the mean.

6. For the cereal in Exercise 5, find the central line, UCL, and LCL for the range.

7. A certain flashlight battery has a life of $\mu = 22.25$ h and $\sigma = 2.35$ h. Each day, four batteries are pulled out of production and tested. Find the central line, UCL, and LCL for the $\overline{x}$ chart.

8. For the batteries in Exercise 7, find the central line, UCL, and LCL for the range.

In Exercises 9–12, use the data in Table 13.8. The readings in Table 13.8 are dial indicator readings of a pin diameter. Samples were collected each half hour.

TABLE 13.8 Pin Diameters (mm)

Sample	Diameters					
1	2.49	2.51	2.52	2.47	2.50	2.50
2	2.53	2.50	2.47	2.46	2.47	2.55
3	2.54	2.4	2.56	2.47	2.48	2.47
4	2.47	2.55	2.53	2.46	2.55	2.45
5	2.45	2.53	2.57	2.51	2.52	2.51
6	2.50	2.46	2.6	2.56	2.45	2.55
7	2.47	2.54	2.50	2.52	2.50	2.47
8	3.5	2.50	2.47	2.47	2.47	2.49
9	3.9	2.53	2.46	2.44	2.52	2.52
10	2.51	2.56	2.53	2.51	2.52	2.51
11	2.45	2.53	2.49	2.60	2.47	2.49
12	2.56	2.57	2.57	2.52	2.58	2.55
13	2.54	2.53	2.52	2.50	2.49	2.50
14	2.47	2.47	2.47	2.53	2.52	2.51
15	2.52	2.47	2.50	2.45	2.49	2.49
16	2.49	2.54	2.44	2.51	2.52	2.50

9. ***Quality control*** Determine the central line, upper control limit, and lower control limit for the mean ($\overline{x}$).

10. ***Quality control*** Determine the central line, upper control limit, and lower control limit for the range.

11. ***Quality control*** Plot an x-bar chart.

12. ***Quality control*** Plot an R chart.

The readings in Table 13.9 are test results from checking the volume of 100-μL variable-volume pipettes. Samples were collected each half hour on four pipettes. Use the data in Table 13.9 to answer Exercises 13–16.

TABLE 13.9 Volume of 100-μL Variable-Volume Pipettes

Sample	Volume (μL)				Sample	Volume (μL)			
1	100.37	100.17	98.13	100.33	13	101.92	100.57	98.23	99.25
2	99.86	98.64	101.27	98.40	14	99.40	99.64	98.48	100.76
3	101.60	99.03	98.46	100.69	15	101.75	101.25	101.10	98.57
4	98.36	101.15	98.99	98.79	16	98.48	99.39	101.20	100.64
5	101.47	98.79	99.33	101.22	17	101.00	100.47	99.85	98.03
6	101.80	101.61	101.58	101.91	18	99.99	98.34	98.65	98.59
7	100.97	100.51	101.80	100.20	19	101.80	98.45	99.17	99.00
8	100.58	101.80	100.61	99.37	20	98.56	101.76	100.28	99.98
9	100.08	101.83	98.48	98.58	21	100.76	99.15	101.52	98.98
10	100.29	101.91	99.86	99.14	22	100.54	100.02	99.94	100.23
11	101.36	99.45	100.73	101.85	23	99.45	99.42	99.68	99.78
12	101.21	98.21	101.67	98.15	24	100.04	99.32	99.78	100.51

13. ***Quality control*** Determine the central line, upper control limit, and lower control limit for the mean ($\bar{x}$).

14. ***Quality control*** Determine the central line, upper control limit, and lower control limit for the range.

15. ***Quality control*** Plot an x-bar chart.

16. ***Quality control*** Plot an R chart.

The length of time a person is "on hold" before someone at a technical assistance help line answers the call is a very important indicator of customer satisfaction with the product. If a person is on hold for less than 60 sec, the call is listed as satisfactory. When person is on hold for 60 sec or more, the call is listed as unacceptable. A particular help line rates each call as either satisfactory or unacceptable and has recorded the results in Table 13.10. Use the data in Table 13.10 to answer Exercises 17–18.

TABLE 13.10 Survey of Customer Satisfaction

Sample	Number *n*	No. of Unaccep. Calls *np*	Sample	Number *n*	No. of Unaccep. Calls *np*
1	75	3	13	75	0
2	75	1	14	75	1
3	75	4	15	75	3
4	75	2	16	75	4
5	75	4	17	75	1
6	75	3	18	75	5
7	75	1	19	75	3
8	75	4	20	75	2
9	75	5	21	75	7
10	75	4	22	75	6
11	75	3	23	75	4
12	75	3	24	75	1

17. ***Quality control*** Determine the central line, upper control limit, and lower control limit for a p chart.

18. ***Quality control*** Plot a p chart.

A bottle of a particular nasal spray is supposed to contain 120 metered sprays of 50 mcl each. Each hour 6 bottles are removed from the production line and tested to see if each bottle gives 120 sprays and if each spray is 50 mcl. At the end of each day a total of 48 bottles have been tested. The data in Table 13.11 contains the results of testing the number of sprays for one month. Use the data in Table 13.11 to answer Exercises 19–20.

19. ***Quality control*** Determine the central line, upper control limit, and lower control limit for a p chart of the data.

20. ***Quality control*** Plot a p chart.

TABLE 13.11 Nasal Spray Tests

Sample	Number n	No. < 120 Sprays np	Sample	Number n	No. < 120 Sprays np
1	48	1	11	48	0
2	48	1	12	48	1
3	48	2	13	48	2
4	48	2	14	48	4
5	48	4	15	48	5
6	48	3	16	48	5
7	48	1	17	48	0
8	48	0	18	48	2
9	48	1	19	48	1
10	48	0	20	48	0

Solve Exercises 21–22.

21. ***Quality control*** The coating on a plastic pipe is to be 0.0485 ± 0.004 mm thick. What standard deviation in the coating thickness would result in the specifications meeting Six Sigma quality standards?

22. ***Quality control*** A particular silicon chip is to be a square measuring 0.5 ± 0.0015 cm on each side. What standard deviation in the length and width would result in the specifications meeting Six Sigma quality standards?

[IN YOUR WORDS]

23. Make a list of some assignable causes that could cause a process to be out of control.

24. Under what types of situations would you use $\bar{x}$ and R charts?

25. Under what types of situations would you use a p chart?

26. Why would you use a p chart rather than $\bar{x}$ and R charts?

CHAPTER 13 REVIEW

IMPORTANT TERMS AND CONCEPTS

Attribute chart
Box-and-whisker diagram
Box plot
Central line
Conditional probability
Dependent events
Event
Frequency distribution
Histogram
In control
Independent events
Lower control limit (LCL)
Measures of central tendency
- Mean
- Median
- Mode

Normal curve
Normal equations
Out of control
Outcome
p chart
Population
Probability
R chart
Sample
Sample space
Scatter diagram
Six Sigma
Standard deviation
Statistical process control (SPC)
Trial
Upper control limit (UCL)
Variables chart
Variance
$\bar{x}$ (x-bar) chart
z-score

REVIEW EXERCISES

In Exercises 1–8, find the probability of the event.

1. Getting a "tail" when a coin is tossed.
2. Drawing a joker from a deck of 54 cards.
3. Drawing a green ball, blindfolded, from a bag containing 6 red and 10 green balls.
4. Not drawing a queen from a deck of 52 cards.
5. Rolling a 5 or higher with a single die.
6. Rolling a sum of 4 or less with a pair of dice.
7. Rolling a 5 and then a 2 with a single die.
8. Drawing two aces from a well-shuffled deck of cards, if the first card is not replaced.

Solve Exercises 9–18.

9. ***Quality control*** A quality control engineer inspects a random sample of three disk drives from each lot of 50 that is ready to be shipped. If such a lot contains four drives with slight defects, find the probability that the inspector's sample will include only drives with no defects.
10. ***Quality control*** An inspector at a computer disk-drive manufacturing company has collected the following information on the revolutions per minute of the disk drives: 300.1, 300.3, 298.9, 299.6, 300.1, 300.4, 300.5, 297.8, 299.8, 300.7, and 301.2. Determine the median of these numbers.
11. ***Quality control*** Determine the quartiles of the numbers in Exercise 10.
12. ***Quality control*** Draw a box plot of the data in Exercise 10.
13. ***Quality control*** Determine the mode of the numbers in Exercise 10.
14. ***Quality control*** Determine the mean of the data in Exercise 10.
15. ***Quality control*** Determine the standard deviation of the data in Exercise 10.
16. ***Quality control*** What percent of the data is in the interval $\bar{x} - s, \bar{x} + s$ for the data in Exercise 10?
17. ***Quality control*** What percent of the data in Exercise 10 is in the interval $\bar{x} - 2s, \bar{x} + 2s$?
18. ***Machine technology*** Some forged alloy bars were twisted until they broke. The number of twists for the bars tested were 33, 24, 37, 48, 26, 52, 37, 45, 23, 43, 39, 42, 32, 50, and 38. Find **(a)** the mean, **(b)** the median, **(c)** the quartiles, and **(d)** the standard deviation for the data.

Use the following information and Table 13.12 to solve Exercises 19–32. Twelve-foot-long steel bars are purchased from a supplier and cut into 4.375-in. lengths. From the finished parts, an operator selects six samples 24 times during the day. The readings in Table 13.12 are from today's production for one cutting machine.

19. Draw a histogram of this data.
20. Determine the median of these numbers.
21. Determine the quartiles of the numbers in Table 13.12.
22. Draw a box plot of the data in Table 13.12.
23. Determine the mode of the numbers in Table 13.12.
24. Determine the mean of the data in Table 13.12.
25. Determine the standard deviation of the data in Table 13.12.
26. What percent of the data is in the interval $\bar{x} - s, \bar{x} + s$ for the data in Table 13.12?
27. What percent of the data in Table 13.12 is in the interval $\bar{x} - 2s, \bar{x} + 2s$?
28. Determine the central line, upper control limit, and lower control limit for the mean ($\bar{x}$).
29. Determine the central line, upper control limit, and lower control limit for the range.
30. Plot an x-bar chart.
31. Plot an R chart.
32. What standard deviation is needed if these shafts are to satisfy Six Sigma quality standards with lengths of 4.375 ± 0.015 in.?
33. The production of a certain part requires that a hole be made in a steel sheet. The diameter of the hole is 2.25 in. with a tolerance of ± 0.003 in. What standard deviation is needed if these holes are to satisfy Six Sigma quality standards?

TABLE 13.12 Shaft Lengths (in.)

Sample	Time	Length (in.)					
1	8:18	4.382	4.378	4.369	4.380	4.382	4.374
2	8:38	4.364	4.370	4.378	4.385	4.376	4.372
3	8:55	4.365	4.378	4.376	4.380	4.376	4.366
4	9:15	4.374	4.379	4.374	4.376	4.367	4.364
5	9:35	4.375	4.384	4.369	4.368	4.372	4.378
6	9:52	4.366	4.371	4.371	4.385	4.366	4.374
7	10:15	4.385	4.380	4.382	4.382	4.384	4.366
8	10:35	4.368	4.378	4.370	4.370	4.378	4.372
9	10:53	4.376	4.367	4.376	4.380	4.366	4.368
10	11:10	4.372	4.383	4.365	4.367	4.376	4.381
11	11:29	4.374	4.376	4.382	4.379	4.380	4.378
12	11:50	4.371	4.374	4.369	4.382	4.362	4.374
13	12:40	4.368	4.376	4.370	4.364	4.375	4.372
14	12:59	4.380	4.369	4.378	4.380	4.379	4.391
15	1:17	4.370	4.376	4.372	4.376	4.369	4.370
16	1:38	4.373	4.371	4.380	4.380	4.374	4.374
17	1:55	4.382	4.372	4.381	4.384	4.376	4.381
18	2:15	4.365	4.364	4.377	4.369	4.370	4.382
19	2:32	4.381	4.382	4.374	4.380	4.369	4.366
20	2:50	4.367	4.378	4.379	4.368	4.366	4.369
21	3:08	4.377	4.376	4.374	4.378	4.380	4.375
22	3:32	4.371	4.378	4.366	4.371	4.367	4.366
23	3:55	4.370	4.387	4.383	4.375	4.369	4.375
24	4:17	4.377	4.380	4.378	4.374	4.379	4.381

Each day 100 integrated circuits are removed from production and checked to electrical specifications. The data in Table 13.13 show the results of testing the circuits for 30 days. Use the data in Table 13.13 to answer Exercises 34–37.

TABLE 13.13 Number of Defective Integrated Circuits

Day	Number defective *n*	Day	Number defective *n*	Day	Number defective *n*
1	24	11	44	21	23
2	38	12	29	22	31
3	62	13	30	23	26
4	35	14	34	24	32
5	37	15	45	25	35
6	38	16	22	26	15
7	48	17	43	27	24
8	52	18	28	28	38
9	33	19	34	29	21
10	21	20	43	30	17

34. Determine the mean of the data in Table 13.13.

35. Determine the standard deviation of the data in Table 13.13.

36. Determine the central line, upper control limit, and lower control limit for a p chart of the data.

37. Plot a p chart.

CHAPTER 13 TEST

1. What is the probability of drawing a red queen from a deck of 52 cards?

2. A quality control engineer inspects a random sample of 5 carburetors from each lot of 100 that is ready to be shipped. If such a lot of 100 carburetors contains 2 with a defect, find the probability that the engineer's sample will include only carburetors with no defects.

Use the following information and Table 13.14 to solve Exercises 3–14. An inspector at a disk-drive manufacturing company has collected 5 samples of 4 disk drives and measured the number of revolutions per minute for each drive.

TABLE 13.14 Hard-Disk-Drive Test Results

Sample	Revolutions per Minute (rpm)			
1	3 600.1	3 600.4	3 599.9	3 599.2
2	3 598.9	3 601.2	3 600.8	3 598.7
3	3 599.6	3 600.4	3 598.2	3 600.1
4	3 599.6	3 600.4	3 599.2	3 601.6
5	3 598.6	3 600.4	3 600.2	3 601.4

3. Draw a histogram of this data.

4. Determine the median of these numbers.

5. Determine the quartiles of the numbers in Table 13.14.

6. Draw a box plot of the data in Table 13.14.

7. Determine the mode of the numbers in Table 13.14.

8. Determine the mean of the data in Table 13.14.

9. Determine the standard deviation of the data in Table 13.14.

10. What percent of the data is in the interval $\bar{x} - s, \bar{x} + s$ for the data in Table 13.14?

11. What percent of the data in Table 13.14 is in the interval $\bar{x} - 2s, \bar{x} + 2s$?

12. Determine the central line, upper control limit, and lower control limit for the mean ($\bar{x}$).

13. Determine the central line, upper control limit, and lower control limit for the range.

14. What standard deviation is needed if these hard disks are to satisfy Six Sigma quality standards with 3600 ± 0.03 rpm?

14 COMPLEX NUMBERS

Complex numbers have many important applications in electronics. In Section 14.6, we will see how to use complex numbers to solve problems that involve alternating current.

All the numbers we have used until now have been real numbers. Several times in the history of mathematics, problems developed that people could not solve with the number system they had. These problems led to the invention of new numbers. The first time this happened was when the number 0 had to be invented. Later, negative numbers were needed. Each time, new numbers were invented to allow people to solve new kinds of problems. At last, the set of real numbers was developed, and with it we can solve many problems.

Not all problems, however, can be solved with the real numbers. People learned that they could take the cube root of -1 or -8. We know that $\sqrt[3]{-1} = -1$ because $(-1)^3 = -1$ and that $\sqrt[3]{-8} = -2$ because $(-2)^3 = -8$. But, there is no real number for the square root of -1, or -4, or for the square root of any negative number. People had problems that could be worked only if it were possible to take the square root of a negative number. As a result, they invented the numbers we will begin using with this chapter—the complex numbers. Complex numbers have many important uses in technology. They make it much easier to work with vectors and problems that involve alternating current (ac). We will see many uses for complex numbers as we work through this chapter.

OBJECTIVES

AFTER COMPLETING THIS CHAPTER, YOU WILL BE ABLE TO:

- Simplify radicals with negative radicands.
- Write complex numbers in rectangular, polar, trigonometric, and exponential forms.
- Evaluate powers of j.
- Find sums, differences, products, quotients, powers, and roots of complex numbers.
- Solve quadratic equations that have complex solutions.
- Perform operations on complex numbers in polar form.
- Solve alternating current problems using complex numbers.

14.1 IMAGINARY AND COMPLEX NUMBERS

IMAGINARY UNIT

As we stated in the chapter introduction, the need for complex numbers arose because people had to solve problems that involved the square roots of negative numbers. In order to solve this dilemma, a new number was invented to correspond to the square root of -1. The name for this number is the **imaginary unit**, and it is represented by the symbol j. Thus, we have

$$j = \sqrt{-1}$$

Another popular name for the imaginary unit is the ***j** operator*.

NOTE Many mathematics books use the symbol i instead of j. But, because i is used to represent electrical current, people in science and technology use j for the imaginary unit.

One of the basic steps we learn in working with imaginary numbers allows us to represent the square root of a negative number as the product of a real number and the imaginary unit, j. The square root of a negative number is called a **pure imaginary number** and is defined in the following box. Remember, if b is a real number, the symbol $\sqrt{b}$ represents the *principal square root* of b and is never negative. Thus, $\sqrt{9} = 3$, $\sqrt{25} = 5$, and $\sqrt{\frac{16}{9}} = \frac{4}{3}$.

PURE IMAGINARY NUMBER

If b is a real number and $b > 0$, then $\sqrt{-b}$ is a *pure imaginary number* and we have

$$\sqrt{-b} = \sqrt{(-1)b} = \sqrt{-1}\sqrt{b} = j\sqrt{b}$$

where $j = \sqrt{-1}$.

We call $j\sqrt{b}$ or $\sqrt{b}j$ the *standard form for a pure imaginary number.*

EXAMPLE 14.1

Simplify and express each of the following radicals in the standard form for a pure imaginary number: (a) $\sqrt{-9}$, (b) $\sqrt{-0.25}$, (c) $\sqrt{-3}$, (d) $-\sqrt{-18}$, and (e) $\sqrt{\dfrac{-4}{9}}$.

SOLUTIONS

(a) $\sqrt{-9} = \sqrt{9}\sqrt{-1} = 3j$

(b) $\sqrt{-0.25} = \sqrt{0.25}\sqrt{-1} = 0.5j$

(c) $\sqrt{-3} = \sqrt{3}\sqrt{-1} = \sqrt{3}j$

(d) $-\sqrt{-18} = -\sqrt{18}\sqrt{-1} = -\sqrt{9}\sqrt{2}\sqrt{-1} = -3\sqrt{2}j = -3j\sqrt{2}$

(e) $\sqrt{\frac{-4}{9}} = \sqrt{\frac{4}{9}}\sqrt{-1} = \dfrac{\sqrt{4}}{\sqrt{9}}\sqrt{-1} = \frac{2}{3}j$

NOTE Many people write the symbol j in front of a radical sign in order to reduce the danger of thinking that it is under the radical. Thus, you might prefer to write the answers to **(c)** and **(d)** as $j\sqrt{3}$ and $-3j\sqrt{2}$.

Since $j = \sqrt{-1}$, we have some interesting relationships.

$$\begin{aligned} j^2 &= -1 \\ j^3 &= j^2 j = (-1)j = -j \\ j^4 &= j^2 j^2 = (-1)(-1) = 1 \end{aligned}$$

Any larger power of j can be reduced to one of these basic four. Thus,

$$\begin{aligned} j^5 &= j^{4+1} = j^4 j^1 = 1 \cdot j = j \\ j^{15} &= j^{4+4+4+3} = j^4 j^4 j^4 j^3 = 1 \cdot 1 \cdot 1 \cdot (-j) = -j \end{aligned}$$

NOTE The powers of j are cyclic, as can be seen above and in the table below.

$$\begin{array}{rcccccccccc} 1 &=& j^0 &=& j^4 &=& j^8 &=& j^{12} &=& \cdots \\ j &=& j^1 &=& j^5 &=& j^9 &=& j^{13} &=& \cdots \\ -1 &=& j^2 &=& j^6 &=& j^{10} &=& j^{14} &=& \cdots \\ -j &=& j^3 &=& j^7 &=& j^{11} &=& j^{15} &=& \cdots \end{array}$$

This cyclic nature of imaginary numbers and of the trigonometric functions allow us to connect imaginary and complex numbers to cyclic applications, such as alternating electrical current.

We need to be careful when we work with imaginary numbers. Consider the problem $\sqrt{-9}\sqrt{-4}$. We know that $\sqrt{-9} = 3j$ and $\sqrt{-4} = 2j$, so $\sqrt{-9}\sqrt{-4} = (3j)(2j) = 6j^2 = -6$. But, we have gotten used to using the property $\sqrt{a}\sqrt{b} = \sqrt{ab}$. What if we tried to use it on this problem: $\sqrt{-9}\sqrt{-4} = \sqrt{(-9)(-4)} = \sqrt{36} = 6$? If we are going to have a successful set of numbers, we cannot get two different answers when we multiply imaginary numbers.

CAUTION Remember that whenever you work with square roots of negative numbers, express each number in terms of j before you proceed.

Thus, the correct answer to $\sqrt{-9}\sqrt{-4}$ is -6.

MULTIPLICATION OF RADICALS

If a and b are real numbers, then

$$\sqrt{a}\sqrt{b} = \sqrt{ab} \quad \text{if } a \geq 0 \text{ and } b \geq 0$$

If either $a < 0$ or $b < 0$ (or both a and b are negative), then convert the radical to "j form" before multiplying.

EXAMPLE 14.2

Simplify the following: (a) $(\sqrt{-4})^2$, (b) $\sqrt{-3}\,\sqrt{-12}$, (c) $\sqrt{2}\sqrt{-8}$, (d) $\sqrt{-0.5}\sqrt{-7}$, and (e) $(2\sqrt{-5})(\sqrt{-7})(3\sqrt{-14})$.

SOLUTIONS

(a) $$\begin{aligned}(\sqrt{-4})^2 &= (2j)^2 \\ &= 4j^2 \\ &= -4\end{aligned}$$

(b) $$\begin{aligned}\sqrt{-3}\sqrt{-12} &= (j\sqrt{3})(2j\sqrt{3}) \\ &= 2j^2\sqrt{3^2} \\ &= 2(-1)(3) = -6\end{aligned}$$

(c) $$\begin{aligned}\sqrt{2}\sqrt{-8} &= (\sqrt{2})(j\sqrt{8}) \\ &= j\sqrt{16} \\ &= 4j\end{aligned}$$

EXAMPLE 14.2 (Cont.)

(d) $\sqrt{-0.5}\sqrt{-7} = (j\sqrt{0.5})(j\sqrt{7})$
$$= j^2\sqrt{(0.5)(7)}$$
$$= -\sqrt{3.5}$$

(e) $(2\sqrt{-5})(\sqrt{-7})(3\sqrt{-14}) = (2j\sqrt{5})(j\sqrt{7})(3j\sqrt{14})$
$$= 6j^3\sqrt{5 \cdot 7 \cdot 14}$$
$$= 42j^3\sqrt{10}$$
$$= -42j\sqrt{10}$$

Note that $\sqrt{5 \cdot 7 \cdot 14} = 7\sqrt{10}$ and that $j^3 = -j$.

COMPLEX NUMBERS

When an imaginary number and a real number are added, we get a complex number. A **complex number** is of the form $a + bj$, where a and b are real numbers. When $a = 0$ and $b \neq 0$, we have a number of the form bj, which is a *pure imaginary number*. When $b = 0$, we get a number of the form a, which is a real number.

RECTANGULAR FORM OF A COMPLEX NUMBER

The form $a + bj$ is known as the **rectangular form** of a complex number, where a is the **real part** and b is the **imaginary part**.

Two complex numbers are equal if both the real parts are equal and the imaginary parts are equal. Symbolically, we express this as follows.

EQUALITY OF COMPLEX NUMBERS

If $a + bj$ and $c + dj$ are two complex numbers, then $a + bj = c + dj$, if and only if $a = c$ and $b = d$.

HINT Two complex numbers, $a + bj$ and $c + dj$, are equal if the real parts are equal, that is, if $a = c$, **and** if the imaginary parts are equal, that is, if $b = d$.

EXAMPLE 14.3

Solve $4 + 3j = 7j + x + 2 + yj$ for x and y.

SOLUTION Here we need to determine both x and y. The best way is to rearrange the terms so that the known values are on one side of the equation and the variables are on the other.

$$4 + 3j = 7j + x + 2 + yj$$
$$4 + 3j - (2 + 7j) = x + yj$$
or
$$x + yj = 4 + 3j - (2 + 7j)$$
$$x + yj = 2 - 4j$$

So, $x = 2$ and $y = -4$, since the real parts must be equal and the imaginary parts must also be equal.

EXAMPLE 14.4

Simplify and express in the form $a + bj$: (a) $7(3 + 2j)$, (b) $j(5 - 3j)$, and (c) $\dfrac{4 - \sqrt{-12}}{2}$.

SOLUTIONS

(a) $7(3 + 2j) = 21 + 14j$

(b) $j(5 - 3j) = 5j - 3j^2 = 5j - 3(-1) = 3 + 5j$

(c) $\dfrac{4 - \sqrt{-12}}{2} = \dfrac{4 - 2j\sqrt{3}}{2} = \dfrac{4}{2} - \dfrac{2j\sqrt{3}}{2} = 2 - j\sqrt{3}$

Notice that, in this last example, we had to divide *each* term of the numerator by 2 in order to get the final answer in the form $a + bj$.

CONJUGATES OF COMPLEX NUMBERS

Every complex number has a conjugate. As you will see in Section 14.2, conjugates are particularly useful when you are dividing by a complex number.

CONJUGATE OF A COMPLEX NUMBER

The **conjugate of a complex number** $a + bj$ is the complex number $a - bj$.

To form the conjugate of a complex number, you need to change only the sign of the imaginary part of the complex number.

EXAMPLE 14.5

(a) The conjugate of $3 + 4j$ is $3 - 4j$.

(b) The conjugate of $5 - 2j$ is $5 + 2j$.

(c) The conjugate of $-7j$ is $7j$, since $-7j = 0 - 7j$ and its conjugate is $0 + 7j = 7j$.

(d) The conjugate of 15 is 15, since $15 = 15 + 0j$ and its conjugate is $15 - 0j = 15$.

NOTE The conjugate of $a + bj$ is $a - bj$ and the conjugate of $a - bj$ is $a + bj$. Thus, each number is the conjugate of the other.

EXAMPLE 14.6

Use the quadratic formula to solve $x^2 + 2x + 10 = 0$.

SOLUTION The discriminant is $b^2 - 4ac$. Here $a = 1$, $b = 2$, and $c = 10$, so the discriminant is $2^2 - 4(1)(10) = 4 - 40 = -36$. Since the discriminant is negative, this trinomial has no real number roots.

The quadratic formula states that the solutions of $ax^2 + bx + c = 0$ are $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$. Use the values of $a = 1$, $b = 2$, and $c = 10$ with the quadratic formula. From above we know that $b^2 - 4ac = -36$, so

$$\begin{aligned} x &= \frac{-2 \pm \sqrt{-36}}{2} \\ &= \frac{-2}{2} \pm \frac{\sqrt{36}}{2}j \\ &= -1 \pm \frac{6}{2}j = -1 \pm 3j \end{aligned}$$

Thus, the roots are $x = -1 + 3j$ and $x = -1 - 3j$. Notice that these roots are complex conjugates of each other.

EXAMPLE 14.7

Use the quadratic formula to solve $2x^2 + 3x + 5 = 0$.

SOLUTION Here $a = 2$, $b = 3$, and $c = 5$, and the discriminant is -31. Thus, this quadratic equation has no real number solutions. Using the quadratic formula, we obtain

$$x = \frac{-3 \pm \sqrt{-31}}{4} = \frac{-3 \pm j\sqrt{31}}{4}$$

Writing these in the form $a + bj$, we get $x = \dfrac{-3}{4} + \dfrac{\sqrt{31}}{4}j$ and $x = \dfrac{-3}{4} - \dfrac{\sqrt{31}}{4}j$.

As you can see, these roots are complex conjugates of each other.

APPLICATION ELECTRONICS

EXAMPLE 14.8

The susceptance, B, in siemens (S) of an ac circuit that contains R resistance and X reactance, both in Ω, is given by $B = \dfrac{X}{R^2 + X^2}$. Find the value of X when $B = 0.1$ S and $R = \sqrt{34} \approx 5.831\,\Omega$.

SOLUTION Substituting the values for B and R in the given equation, we get

$$0.1 = \frac{X}{(\sqrt{34})^2 + X^2} = \frac{X}{34 + X^2}$$

Multiplying both sides by the denominator produces the equation:

$$\begin{aligned} 0.1(34 + X^2) &= X \\ 0.1(34 + X^2) - X &= 0 \\ 3.4 + 0.1X^2 - X &= 0 \\ 0.1X^2 - X + 3.4 &= 0 \end{aligned}$$

This is a quadratic equation with $a = 0.1$, $b = -1$, and $c = 3.4$. Substituting these values in the quadratic formula, we obtain

$$\begin{aligned} X &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-(-1) \pm \sqrt{(-1)^2 - 4(0.1)(3.4)}}{2(0.1)} \\ &= \frac{1 \pm \sqrt{-0.36}}{0.2} \\ &= \frac{1 \pm \sqrt{0.36}j}{0.2} \\ &= \frac{1}{0.2} \pm \frac{\sqrt{0.36}j}{0.2} \\ &= \frac{1}{0.2} \pm \frac{0.6j}{0.2} \\ &= 5 \pm 3j \end{aligned}$$

The reactance is either $5 + 3j\,\Omega$ or $5 - 3j\,\Omega$.

In Chapter 17, we will show that if a polynomial has only real number coefficients and has one complex root, then the conjugate of that complex number is also a root.

EXERCISE SET 14.1

In Exercises 1–12, simplify and express each radical in terms of j.

1. $\sqrt{-25}$

2. $\sqrt{-81}$

3. $\sqrt{-0.04}$

4. $\sqrt{-1.44}$

5. $\sqrt{-75}$

6. $-\sqrt{-72}$

7. $-3\sqrt{-20}$

8. $5\sqrt{-30}$

9. $\sqrt{-\frac{9}{16}}$

10. $\sqrt{-\frac{25}{36}}$

11. $-4\sqrt{-\frac{9}{16}}$

12. $-3\sqrt{-\frac{10}{81}}$

In Exercises 13–38, simplify each problem.

13. $(\sqrt{-11})^2$
14. $(\sqrt{-7})^2$
15. $(3\sqrt{-2})^2$
16. $(-2\sqrt{-3})^2$
17. $\sqrt{-4}\sqrt{-25}$
18. $\sqrt{-9}\sqrt{-16}$
19. $(\sqrt{-49})(2\sqrt{-9})$
20. $(3\sqrt{-16})(\sqrt{-36})$
21. $(\sqrt{-5})(-\sqrt{5})$
22. $(-\sqrt{-7})(\sqrt{7})$
23. j^7
24. j^{26}
25. $\sqrt{-\frac{1}{4}}\sqrt{\frac{16}{9}}$
26. $\sqrt{\frac{4}{25}}\sqrt{-\frac{49}{9}}$
27. $\sqrt{-\frac{3}{25}}\sqrt{\frac{3}{16}}$
28. $\sqrt{\frac{75}{36}}\sqrt{-\frac{49}{5}}$
29. $\sqrt{-\frac{25}{36}}\sqrt{-\frac{9}{16}}$
30. $\sqrt{-\frac{16}{81}}\sqrt{-\frac{9}{64}}$
31. $(\sqrt{-0.5})(\sqrt{0.5})$
32. $(\sqrt{0.8})(\sqrt{-0.2})$
33. $(-2\sqrt{2.7})(\sqrt{-3})$
34. $(-4\sqrt{1.6})(2\sqrt{-0.4})$
35. $(\sqrt{-5})(\sqrt{-6})(\sqrt{-2})$
36. $(\sqrt{-3})(\sqrt{-9})(\sqrt{-15})$
37. $(-\sqrt{-7})^2(\sqrt{-2})^2j^3$
38. $(\sqrt{-3})^2(\sqrt{-5})^2j^2(\sqrt{-2})$

In Exercises 39–48, solve each problem for the variables x and y.

39. $x + yj = 7 - 2j$
40. $x + yj = -9 + 2j$
41. $x + 5 + yj = 15 - 3j$
42. $6j - x + yj = 4 + 2j$
43. $x - 5j + 2 = 4 - 3j + yj$
44. $2x - 4j = 6j + 4 - yj$
45. $\frac{1}{2}x + \frac{3}{4}j = 2j - \frac{1}{4} + yj$
46. $\frac{2}{5}x - \frac{2}{3}j + 1 = 5 - \frac{4}{3}j - yj$
47. $1.2x + 3 + yj = 7.2 - 4.3j$
48. $3.7j - 1.5x = -4.8 + 2.4j + 0.5yj$

In Exercises 49–70, simplify each problem and express it in the form a + bj.

49. $2(4 + 5j)$
50. $3(2 - 4j)$
51. $-5(2 + j)$
52. $-3(4 + 7j)$
53. $j(3 - 2j)$
54. $j(5 + 4j)$
55. $2j(4 + 3j)$
56. $-3j(2 - 5j)$
57. $\frac{1}{2}(6 - 8j)$
58. $\frac{2}{3}(6 + 9j)$
59. $\dfrac{5 - 10j}{5}$
60. $\dfrac{6 + 12j}{3}$
61. $\dfrac{6 + \sqrt{-18}}{3}$
62. $\dfrac{7 - \sqrt{-98}}{7}$
63. $\dfrac{8 - \sqrt{-24}}{4}$
64. $\dfrac{9 + \sqrt{-27}}{6}$
65. $\frac{2}{3}(\frac{3}{4} - \frac{1}{3}j)$
66. $\frac{1}{2}(\frac{8}{5} + \frac{9}{4}j)$
67. $-\frac{5}{3}(-\frac{3}{8} + \frac{6}{15}j)$
68. $-\frac{7}{4}(\frac{8}{21} - \frac{16}{35}j)$
69. $1.5(2.4 - 3j)$
70. $-7.2(-0.25 + 1.75j)$

In Exercises 71–80, write the conjugate of the given numbers.

71. $7 + 2j$
72. $9 + \frac{1}{2}j$
73. $6 - 5j$
74. $\frac{1}{2} - 9j$
75. 19
76. $7j$
77. $-8j$
78. -11
79. $\sqrt{2} + 7.3j$
80. $-4.5 - \sqrt{19}j$

In Exercises 81–88, use the quadratic formula to solve each of the problems. Express your answers in the form a + bj.

81. $x^2 + x + 2.5 = 0$
82. $x^2 + 2x + 5 = 0$
83. $x^2 + 9 = 0$
84. $x^2 + 25 = 0$
85. $2x^2 + 3x + 7 = 0$
86. $2x^2 + 7x + 9 = 0$
87. $5x^2 + 2x + 5 = 0$
88. $3x^2 + 2x + 10 = 0$

Solve Exercises 89–92.

89. ***Electronics*** The susceptance, B, in siemens (S) of an ac circuit that contains R resistance and X reactance, both in Ω, is given by $B = \frac{X}{R^2 + X^2}$. Find the value of X when $B = 0.08$ S and $R = 50\ \Omega$.

90. ***Electronics*** The susceptance, B, in siemens (S) of an ac circuit that contains R resistance and X reactance, both in Ω, is given by $B = \frac{X}{R^2 + X^2}$. Find the value of X when $B = 0.05$ S and $R = 12\ \Omega$.

91. ***Electronics*** The susceptance, B, in siemens (S) of an ac circuit that contains R resistance and X reactance, both in Ω, is given by $B = \frac{X}{R^2 + X^2}$. Solve the equation for R.

92. ***Electronics*** The susceptance, B, in siemens (S) of an ac circuit that contains R resistance and X reactance, both in Ω, is given by $B = \frac{X}{R^2 + X^2}$. Find the general formula in terms of X.

[IN YOUR WORDS]

93. Explain what it means for two complex numbers to be equal.

94. Describe how to determine the conjugate of a complex number.

14.2 OPERATIONS WITH COMPLEX NUMBERS

As with all number systems, we want to be able to perform the four basic operations of addition, subtraction, multiplication, and division. These operations are performed after all complex numbers have been expressed in terms of j.

ADDITION AND SUBTRACTION

We will begin by giving the definitions for addition and subtraction of complex numbers. After each definition, we will provide several examples showing how to use the definition.

ADDITION OF COMPLEX NUMBERS

If $a + bj$ and $c + dj$ are any two complex numbers, then their sum is defined as

$$(a + bj) + (c + dj) = (a + c) + (b + d)j$$

In words, this says to add the real parts of the complex numbers and add their imaginary parts.

EXAMPLE 14.9

Find each of these sums: (a) $(9 + 2j) + (8 + 6j)$, (b) $(6 + 3j) + (5 - 7j)$, (c) $(-2\sqrt{3} + 4j) + (5 - 6j)$, and (d) $(-4 + 3j) + (-1 - \sqrt{-4})$.

SOLUTIONS

(a) $$\begin{aligned}(9 + 2j) + (8 + 6j) &= (9 + 8) + (2 + 6)j \\ &= 17 + 8j\end{aligned}$$

(b) $$\begin{aligned}(6 + 3j) + (5 - 7j) &= (6 + 5) + (3 - 7)j \\ &= 11 - 4j\end{aligned}$$

(c) $$\begin{aligned}(-2\sqrt{3} + 4j) + (5 - 6j) &= (-2\sqrt{3} + 5) + (4 - 6)j \\ &= 5 - 2\sqrt{3} - 2j\end{aligned}$$

Notice that the real part of this complex number is $5 - 2\sqrt{3}$ and the imaginary part is $-2j$.

(d) $$\begin{aligned}(-4 + 3j) + (-1 - \sqrt{-4}) &= (-4 + 3j) + (-1 - 2j) \\ &= (-4 - 1) + (3 - 2)j \\ &= -5 + j\end{aligned}$$

APPLICATION ELECTRONICS

EXAMPLE 14.10

In an ac circuit, if two sections are connected in series and have the same current in each section, the voltage V is given by $V = V_1 + V_2$. Find the total voltage in a given circuit if the voltages in the individual sections are $8.9 - 2.4j$ and $11.2 + 6.3j$.

SOLUTION To find the total voltage in this circuit, we need to add the voltages in the individual sections.

$$\begin{aligned}V &= V_1 + V_2 \\ &= (8.9 - 2.4j) + (11.2 + 6.3j) \\ &= (8.9 + 11.2) + (-2.4 + 6.3)j \\ &= 20.1 + 3.9j\end{aligned}$$

The voltage in this circuit is $20.1 + 3.9j$ V.

SUBTRACTION OF COMPLEX NUMBERS

If $a + bj$ and $c + dj$ are complex numbers, then their difference is defined as

$$(a + bj) - (c + dj) = (a - c) + (b - d)j$$

EXAMPLE 14.11

Find each of the following differences: (a) $(3+4j)-(2+j)$, (b) $(5+7j)-(3-10j)$, (c) $(-8+4j)-(3+10j)$, and (d) $(9+\sqrt{-18})-(6+\sqrt{-2})$.

SOLUTIONS

(a) $$\begin{aligned}(3+4j)-(2+j) &= (3-2)+(4-1)j\\ &= 1+3j\end{aligned}$$

(b) $$\begin{aligned}(5+7j)-(3-10j) &= (5-3)+[7-(-10)]j\\ &= 2+17j\end{aligned}$$

(c) $$\begin{aligned}(-8+4j)-(3+10j) &= (-8-3)+(4-10)j\\ &= -11-6j\end{aligned}$$

(d) $$\begin{aligned}(9+\sqrt{-18})-(6+\sqrt{-2}) &= (9+3j\sqrt{2})-(6+j\sqrt{2})\\ &= (9-6)+(3\sqrt{2}-\sqrt{2})j\\ &= 3+2j\sqrt{2}\end{aligned}$$

NOTE Notice that in Examples 14.9(d) and 14.11(d), we had to first write some of the numbers in the $a+bj$ form. In Example 14.9(d), we changed $\sqrt{-4}$ to $2j$, and in Example 14.11(d) we changed $\sqrt{-18}$ to $3j\sqrt{2}$ and $\sqrt{-2}$ to $j\sqrt{2}$. Do not overlook this step.

MULTIPLICATION

Multiplication of complex numbers uses the FOIL method, which was introduced in Chapter 2. As you can see, you will have to replace j^2 with -1 to obtain a simplified answer.

$$\begin{aligned}(a+bj)(c+dj) &= ac+adj+bcj+bdj^2\\ &= ac+adj+bcj-bd\\ &= (ac-bd)+(ad+bc)j\end{aligned}$$

MULTIPLICATION OF COMPLEX NUMBERS

If $a+bj$ and $c+dj$ are any two complex numbers, then their product $(a+bj)(c+dj)$ is defined as

$$(a+bj)(c+dj) = (ac-bd)+(ad+bc)j$$

EXAMPLE 14.12

Multiply and write each answer in the form $a+bj$: (a) $(2+5j)(3-4j)$, (b) $(5+3j)^2$, (c) $(7+3j)(7-3j)$, and (d) $(a+bj)(a-bj)$.

SOLUTIONS We have used the FOIL method rather than the definition to determine these products. By using the FOIL method, you do not have to remember the rule. However, remember that $j^2=-1$.

EXAMPLE 14.12 (Cont.)

(a) $$\begin{aligned}(2+5j)(3-4j) &= 2\cdot 3 + 2(-4)j + 5j(3) + 5(-4)j^2\\ &= 6 - 8j + 15j - 20(-1)\\ &= 6 - 8j + 15j + 20\\ &= 26 + 7j\end{aligned}$$

(b) $$\begin{aligned}(5+3j)^2 &= (5+3j)(5+3j)\\ &= 5^2 + 2(5)(3)j + 9j^2\\ &= 25 + 30j + 9(-1)\\ &= 25 + 30j - 9\\ &= 16 + 30j\end{aligned}$$

(c) Here we can use the difference of squares.

$$\begin{aligned}(7+3j)(7-3j) &= 7^2 - 3^2j^2\\ &= 49 - 9(-1)\\ &= 49 + 9\\ &= 58\end{aligned}$$

(d) $$\begin{aligned}(a+bj)(a-bj) &= a^2 + abj - abj - b^2j^2\\ &= a^2 - b^2j^2\\ &= a^2 + b^2\end{aligned}$$

APPLICATION ELECTRONICS

EXAMPLE 14.13

In an ac circuit, the formula $V = ZI$ relates the voltage V to impedance Z and current I. Find the voltage in a given circuit if the impedance is $8 - 2j\,\Omega$ and the current is $11 + 6j$ A.

SOLUTION To find the voltage, we need to multiply the given values of Z and I. As before, we will use the FOIL method.

$$\begin{aligned}V &= ZI\\ &= (8-2j)(11+6j)\\ &= 8\cdot 11 + 8(6j) + (-2j)(11) + (-2j)6j\\ &= 88 + 48j - 22j + 12\\ &= 100 + 26j\end{aligned}$$

The voltage in this circuit is $100 + 26j$ V.

DIVISION

In Examples 14.12(c) and (d) we multiplied a complex number and its conjugate. The following note will be helpful when we divide complex numbers.

NOTE The product of a complex number and its conjugate is a real number.

DIVISION OF COMPLEX NUMBERS

If $a + bj$ and $c + dj$ are complex numbers, then the quotient $(a + bj) \div (c + dj) = \dfrac{a + bj}{c + dj}$ is defined as

$$\frac{a + bj}{c + dj} = \frac{a + bj}{c + dj} \cdot \frac{c - dj}{c - dj} = \frac{(ac + bd) + (bc - ad)j}{c^2 + d^2}$$
$$= \frac{ac + bd}{c^2 + d^2} + \frac{bc - ad}{c^2 + d^2}j$$

This looks very complicated, but the following hint gives an important idea to remember whenever you divide by a complex number.

HINT In division of complex numbers, you multiply both the numerator and the denominator by the conjugate of the denominator.

The three problems in Example 14.14 show how to use the conjugate to divide.

EXAMPLE 14.14

Divide and express each answer in the form $a + bj$: (a) $\dfrac{8 + 6j}{2 - j}$, (b) $\dfrac{3 - 2j}{4 + 2j}$, and (c) $\dfrac{0.5 + j\sqrt{3}}{2.4j}$.

SOLUTIONS

(a)
$$\frac{8 + 6j}{2 - j} = \frac{8 + 6j}{2 - j} \cdot \frac{2 + j}{2 + j}$$
$$= \frac{16 + 8j + 12j + 6j^2}{4 + 1}$$
$$= \frac{16 + 8j + 12j - 6}{5}$$
$$= \frac{10 + 20j}{5} = 2 + 4j$$

(b)
$$\frac{3 - 2j}{4 + 2j} = \frac{3 - 2j}{4 + 2j} \cdot \frac{4 - 2j}{4 - 2j}$$
$$= \frac{12 - 6j - 8j - 4}{16 + 4}$$
$$= \frac{8 - 14j}{20} = \frac{8}{20} - \frac{14}{20}j$$
$$= \frac{2}{5} - \frac{7}{10}j = 0.4 - 0.7j$$

EXAMPLE 14.14 (Cont.)

(c) $$\frac{0.5 + j\sqrt{3}}{2.4j} = \frac{0.5 + j\sqrt{3}}{2.4j} \cdot \frac{-2.4j}{-2.4j}$$
$$= \frac{(0.5)(-2.4j) + (j\sqrt{3})(-2.4j)}{(2.4j)(-2.4j)}$$
$$= \frac{-1.2j + 2.4\sqrt{3}}{5.76}$$
$$= \frac{2.4\sqrt{3} - 1.2j}{5.76}$$
$$= \frac{2.4\sqrt{3}}{5.76} - \frac{1.2}{5.76}j = \frac{\sqrt{3}}{2.4} - \frac{1}{4.8}j$$

APPLICATION ELECTRONICS

EXAMPLE 14.15

In an ac circuit, use the formula $V = ZI$ to find the impedance in a given circuit if the voltage is $95 + 9j$ V and the current is $7 - 3j$ A.

SOLUTION We want to find the impedance Z, given V and I. Using the formula $V = ZI$, we see that $Z = \dfrac{V}{I}$.

$$Z = \frac{V}{I}$$
$$= \frac{95 + 9j}{7 - 3j}$$
$$= \frac{95 + 9j}{7 - 3j} \cdot \frac{7 + 3j}{7 + 3j}$$
$$= \frac{95(7) + 95(3j) + (9j)7 + (9j)(3j)}{7^2 - (3j)^2}$$
$$= \frac{665 + 285j + 63j + 27j^2}{49 - 9j^2}$$
$$= \frac{638 + 348j}{58}$$
$$= 11 + 6j$$

The impedance in this circuit is $11 + 6j\,\Omega$.

USING CALCULATORS WITH COMPLEX NUMBERS

A TI-83 or TI-84 has two modes for complex numbers. The rectangular complex mode displays numbers in the form $a + bi$, and we will use that form here. To put the calculator in the rectangular complex mode, press **MODE** and scroll down and over until $a + bi$ is highlighted, as in Figure 14.1a, press **ENTER**, and then press **2nd** **QUIT**.

To enter a complex number in rectangular form, enter the value of a (the real component), press either + or −, enter the value of b (the imaginary component), and press 2nd . [i], as demonstrated in Figure 14.1b. Notice that the TI-83 and TI-84 calculators display the answers in the $a + bi$ from rather than $a + bj$.

Figure 14.1a

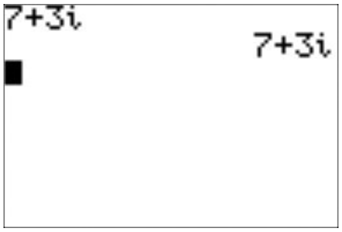

Figure 14.1b

EXAMPLE 14.16

Enter (a) $9 + 7j$ and (b) $12 - \frac{3}{4}j$ into a TI-84 calculator.

SOLUTIONS: First, make sure the calculator is in rectangular complex mode.

PRESS	DISPLAY
(a) 9 + 7 2nd . [i]	$9 + 7i$
(b) 12 − 3 ÷ 4 2nd . [i]	$12 - .75i$

Figure 14.2 shows these results as they appear on the screen of a TI-84 calculator.

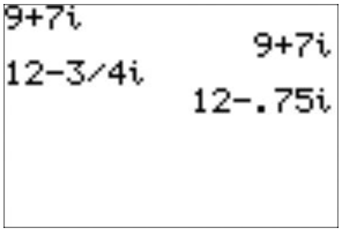

Figure 14.2

Calculations with a calculator are done in the same manner in which they are done with real numbers. The following example shows how each of these is done with a TI-83 or 84. However, other graphing calculators that allow computations with non-real complex numbers use similar procedures.

EXAMPLE 14.17

Use a calculator to perform each of the following: (a) $(3 + 5j) + (7 - 8j)$, (b) $(2 + j)(-6 - 3j)$, and (c) $\dfrac{10 - 4j}{1 + j}$.

SOLUTIONS

PRESS	DISPLAY
(a) (3 + 5 2nd . [i])	
+ (7 − 8 2nd . [i]) ENTER	$10 - 3i$
(b) (2 + 2nd . [i])	
((−) 6 − 3 2nd . [i]) ENTER	$-9 - 12i$
(c) (10 − 4 2nd . [i]) ÷	
(1 + 2nd . [i]) ENTER	$3 - 7i$

Figure 14.3 shows these results as the appear on the screen of a TI-84 calculator.

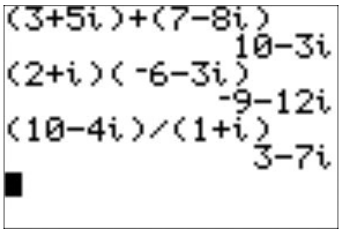

Figure 14.3

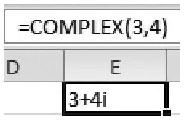

Figure 14.4a

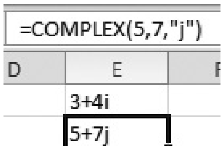

Figure 14.4b

USING SPREADSHEETS WITH COMPLEX NUMBERS

A complex number can be entered in a spreadsheet as either $a + bi$ or $a + bj$ by using the COMPLEX command. For example, Figure 14.4a shows that if you enter = COMPLEX (3, 4) you will get $3 + 4i$. On the other hand, entering = COMPLEX (5, 7, "j") produces $5 + 7j$, as shown in Figure 14.4b.

However, we seldom want to just enter a complex number into a spreadsheet. The command IMSUM (_, _) is used to add two complex numbers. Similarly, IMSUB (_, _), IMPRODUCT (_, _), and IMDIV (_, _) are used to subtract, multiply, and divide two complex numbers, respectively.

EXAMPLE 14.18

Use a spreadsheet to calculate each of the following: (a) $(9 - 7j) + (14 + 6j)$, (b) $(8 - 7j) - (-2 + 5j)$, (c) $(2 + j)(-6 - 3j)$, and (d) $\dfrac{10 - 4j}{1 + j}$.

SOLUTIONS: The key strokes and results are shown in Figure 14.5. Notice that each complex number is placed between quotation marks.

	A	B	C	D
1		Operation	Enter	Result
2	(a)	(9-7j) + (14 +6j)	=IMSUM("9-7j","14+6j")	23-j
3	(b)	(8-7j)-(-2+5j)	=IMSUB("8-7j","-2+5j")	10-12j
4	(c)	(2+j)(-6-3j)	=IMPRODUCT("2+j","-6-3j")	-9-12j
5	(d)	(10-4j)/(1+j)	=IMDIV("10-4j","1+j")	3-7j

Figure 14.5

EXERCISE SET 14.2

In Exercises 1–48, perform the indicated operations. Express all answers in the form $a + bj$. If possible, use a calculator to check your answers.

1. $(5 + 2j) + (-6 + 5j)$
2. $(9 - 7j) + (6 - 8j)$
3. $(11 - 4j) + (-6 + 2j)$
4. $(21 + 3j) + (-7 - 6j)$
5. $(4 + \sqrt{-9}) + (3 - \sqrt{-16})$
6. $(-11 + \sqrt{-4}) + (9 + \sqrt{-36})$
7. $(2 + \sqrt{-9}) + (8j - \sqrt{5})$
8. $(3 + \sqrt{-8}) + (3 - \sqrt{8})$
9. $(14 + 3j) - (6 + j)$
10. $(-8 + 3j) - (4 - 3j)$
11. $(9 - \sqrt{-4}) - (\sqrt{-16} + 6)$
12. $(\sqrt{-25} - 3) - (3 - \sqrt{-25})$
13. $(4 + 2j) + j + (3 - 5j)$
14. $(2 - 3j) - j - (6 + \sqrt{-81})$
15. $(2 + j)3j$
16. $(5 - 3j)2j$
17. $(9 + 2j)(-5j)$
18. $(11 - 4j)(-3j)$
19. $(2 + j)(5 + 3j)$
20. $(3 - 2j)(4 + 5j)$
21. $(6 - 2j)(5 + 3j)$
22. $(4 - 2j)(7 - 3j)$
23. $(2\sqrt{-9} + 3)(5\sqrt{-16} - 2)$
24. $(6\sqrt{-25} - 4)(-3 - 2\sqrt{-49})$
25. $(\sqrt{-3})^4$
26. $(\sqrt{-9})^3$
27. $(1 + 2j)^2$
28. $(3 + 4j)^2$
29. $(7 - j)^2$
30. $(4 - 3j)^2$
31. $(5 + 2j)(5 - 2j)$
32. $(7 + 3j)(7 - 3j)$
33. $\dfrac{6 - 4j}{1 + j}$
34. $\dfrac{4 - 8j}{2 - 2j}$

35. $\dfrac{6 - 3j}{1 + 2j}$

36. $\dfrac{5 - 10j}{1 - 2j}$

37. $\dfrac{4 + 2j}{1 - 2j}$

38. $\dfrac{9 + 5j}{3 + j}$

39. $\dfrac{2j}{5 + j}$

40. $\dfrac{5j}{6 - j}$

41. $\dfrac{\sqrt{3} - \sqrt{-6}}{\sqrt{-3}}$

42. $\dfrac{\sqrt{5} + \sqrt{-10}}{\sqrt{-5}}$

43. $\dfrac{(5 + 2j)(3 - j)}{4 + j}$

44. $\dfrac{(6 - j)(2 + 3j)}{-1 + 3j}$

45. $(1 + j)^4$

46. $(1 - j)^4$

47. $\left(\dfrac{1}{2} + \dfrac{\sqrt{3}}{2}j\right)^2$

48. $\dfrac{4 + j}{(3 - 2j) + (4 - 3j)}$

Solve Exercises 49–63.

49. Show that the sum of a complex number and its conjugate is a real number.

50. Show that the difference of a complex number $a + bj$, with $b \neq 0$ and its conjugate, is a pure imaginary number.

51. Show that the product of a complex number and its conjugate is a real number.

52. ***Electronics*** In an ac circuit, if two sections are connected in series and have the same current in each section, the voltage V is given by $V = V_1 + V_2$. Find the total voltage in a given circuit if the voltages in the individual sections are $9.32 - 6.12j$ and $7.24 + 4.31j$.

53. ***Electronics*** Find the total voltage in an ac series circuit that has the same current in each section, if the voltages in the individual sections are $6.21 - 1.37j$ and $4.32 - 2.84j$.

54. ***Electronics*** If two sections of an ac series circuit have the same current in each section, what is the voltage in one section if the total voltage is $19.2 - 3.5j$ and the voltage in the other section is $12.4 + 1.3j$?

55. ***Electronics*** If two sections of an ac series circuit have the same current in each section, what is the voltage in one section if the total voltage is $7.42 + 1.15j$ and the voltage in the other section is $2.34 - 1.73j$?

56. ***Electronics*** The total impedance Z of an ac circuit containing two impedances Z_1 and Z_2 in series is $Z = Z_1 + Z_2$. If $Z_1 = 0.25 + 0.20j\,\Omega$ and $Z_2 = 0.15 - 0.25j\,\Omega$, what is Z?

57. ***Electronics*** If the total impedance of an ac series circuit containing two impedances is $Z = 9.13 - 4.27j\,\Omega$ and one of the impedances is $3.29 - 5.43j\,\Omega$, what is the impedance of the other circuit?

58. ***Electronics*** If an ac circuit contains two impedances Z_1 and Z_2 in parallel, then the total impedance Z is given by

$$Z = \frac{Z_1 Z_2}{Z_1 + Z_2}$$

What is Z when $Z_1 = 6\,\Omega$ and $Z_2 = 3j\,\Omega$?

59. ***Electronics*** What is the total impedance in an ac circuit that contains two impedances Z_1 and Z_2 in parallel, if $Z_1 = 20 + 10j\,\Omega$ and $Z_2 = 10 - 20j\,\Omega$?

60. ***Electronics*** In an ac circuit the voltage V, current I, and impedance Z are related by $V = IZ$. If $I = 12.3 + 4.6j$ A and $Z = 16.4 - 9.0j\,\Omega$, what is the voltage?

61. ***Electronics*** What is the current when $V = 5.2 + 3j$ V and $Z = 4 - 2j\,\Omega$? (See Exercise 60.)

62. ***Electronics*** What is the impedance when $V = 10.6 - 6.0j$ V and $I = 4 + j$ A?

63. ***Programming*** If your calculator cannot be used to calculate with complex numbers, write a program for your calculator that will **(a)** allow you to enter a complex number and **(b)** add, subtract, multiply, and divide two complex numbers.

[IN YOUR WORDS]

64. Describe how to multiply two complex numbers.

65. Describe how to divide two complex numbers.

66. If your calculator can be used to calculate with complex numbers, explain how to enter a complex number into your calculator and how to divide a complex number by another complex number. Give your explanation to a classmate and ask him or her to follow your directions. Rewrite your directions to clarify places where your classmate had difficulty.

67. Describe how to enter a complex number into a spreadsheet and how to divide a complex number by another complex number. Give your explanation to a classmate and ask him or her to follow your directions. Rewrite your directions to clarify places where your classmate had difficulty.

14.3 GRAPHING COMPLEX NUMBERS; POLAR FORM OF A COMPLEX NUMBER

We have been able to graph real numbers since Chapter 4. It would be helpful if we could also represent complex numbers as points in a plane. The fact that each complex number has a real part and an imaginary part makes it possible to graph complex numbers.

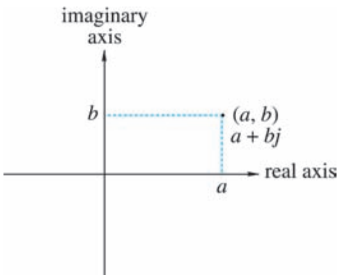

Figure 14.6

COMPLEX PLANE

Each complex number can be written in the form $a + bj$. We can represent this as the point (a, b) in the plane, as shown in Figure 14.6. Notice that the origin corresponds to the point $(0, 0)$, or $0 + 0j$. Since the points in the plane are representing complex numbers, this is called the **complex plane**. It is also referred to as the *Argand plane*, after the French mathematician Argand (1768–1822). In the complex plane, the horizontal axis acts as the real axis and the vertical axis as the *imaginary axis*.

EXAMPLE 14.19

Graph the following complex numbers: (a) $4 + 2j$, (b) $-3 + 5j$, (c) $-5 - 10j$, (d) $9 - 4j$, (e) 12, and (f) $-3j$.

SOLUTIONS The solutions are shown in Figure 14.7.

The complex number $a + bj$ can also be represented in the plane by the position vector **OP** from the origin to the point $P(a, b)$. We now have three correct and interchangeable ways to refer to the complex number: $a + bj$, the point (a, b) on the complex (Argand) plane, and the vector $\mathbf{a} + \mathbf{bj}$ (see Figure 14.8.)

If we have two complex numbers on a graph, their sum or difference can be represented in the same way in which we add or subtract vectors. To add two complex numbers graphically, locate the point corresponding to one of them

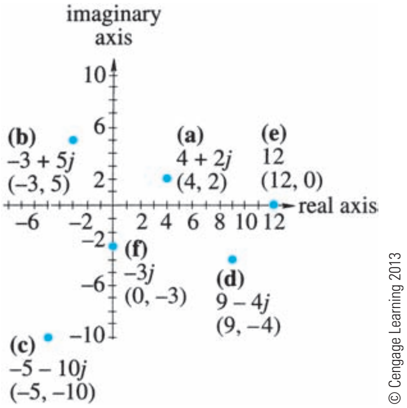

Figure 14.7

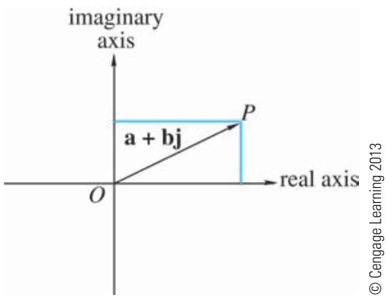

Figure 14.8

and draw the position vector for that point. Repeat the process for the second point. Finally, use the parallelogram method to add these two vectors. The sum will be the diagonal of the parallelogram that has the origin as an endpoint.

EXAMPLE 14.20

Graphically add the complex numbers $-9 + 5j$ and $4 - 7j$.

SOLUTION The solution is shown in Figure 14.9.

As you can see, the graphical solution agrees with the method given in Section 14.2.

$$\begin{aligned}(-9 + 5j) + (4 - 7j) &= (-9 + 4) + (5 - 7)j \\ &= -5 - 2j\end{aligned}$$

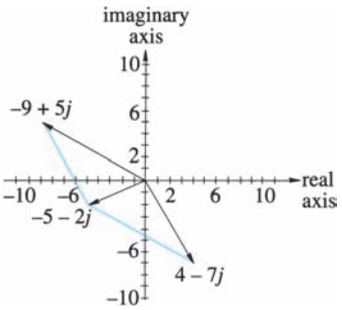

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Figure 14.9

POLAR FORM OF A COMPLEX NUMBER

The rectangular form is not the only way to represent complex numbers. In Figure 14.10, the angle θ that the vector **OP** makes with the positive real axis is called the *argument* or *amplitude* of the complex number $a + bj$. The length r of **OP** is called the **absolute value** or *modulus* of $a + bj$. The absolute value is a real number and is never negative. The absolute value is always positive or 0. Using the Pythagorean theorem, we see that the length of r is $\sqrt{a^2 + b^2}$.

Since $a + bj$ can be considered a vector in the complex plane, $|a + b\,j|$ is the magnitude of the vector.

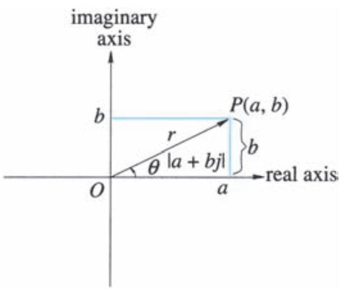

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Figure 14.10

ABSOLUTE VALUE OF A COMPLEX NUMBER

The *absolute value of a complex number* $a + bj$ is denoted $|a + bj|$ and has the value

$$|a + bj| = \sqrt{a^2 + b^2}$$

A careful examination of Figure 14.10 reveals four useful relationships. First, from our definitions of the trigonometric functions, we see that $\cos\theta = \frac{a}{r}$ and $\sin\theta = \frac{b}{r}$. From these, we see that

$$a = r\cos\theta \tag{1}$$

$$\text{and} \quad b = r\sin\theta \tag{2}$$

The other two relationships require not only our knowledge of trigonometry but of the Pythagorean theorem. These two relationships state that

$$\tan\theta = \frac{b}{a} \tag{3}$$

$$\text{and} \qquad r = \sqrt{a^2 + b^2} \tag{4}$$

These four equations will be very valuable to us. From the first two, we see that

$$a + bj = r\cos\theta + jr\sin\theta = r(\cos\theta + j\sin\theta)$$

The expression $r(\cos\theta + j\sin\theta)$ is often abbreviated as r cis θ or $r\underline{/\theta}$. In the abbreviation r cis θ, the c represents cosine, the s represents sine, and the i represents the mathematician's symbol for j. The symbol $r\underline{/\theta}$ is read "r at angle θ." The right-hand side of the previous equation, $r(\cos\theta + j\sin\theta)$, is called the *polar* or **trigonometric form** of a complex number.

CHANGING COMPLEX NUMBERS FROM POLAR TO RECTANGULAR FORM

A complex number written in **polar form** as

$$r\underline{/\theta} \qquad \text{or} \qquad r\text{ cis }\theta \qquad \text{or} \qquad r(\cos\theta + j\sin\theta)$$

has the rectangular coordinates $a + bj$, where

$$a = r\cos\theta \qquad \text{and} \qquad b = r\sin\theta$$

EXAMPLE 14.21

Locate the point $5(\cos 120° + j\sin 120°)$ in the complex plane and convert the number to rectangular form.

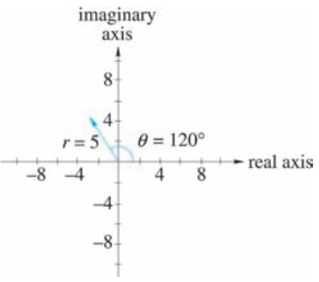

Figure 14.11

SOLUTION The graphical representation is given in Figure 14.11.

In this example, $r = 5$ and $\theta = 120°$, and so

$$a = r\cos\theta = 5\cos 120° = 5(-0.5) = -2.5$$
$$b = r\sin\theta = 5\sin 120° \approx 5(0.8660) = 4.3301$$

Thus, $5 \text{ cis } 120° \approx -2.5 + 4.3301j$.

The previous example was worked using a calculator. The exact value for b would have been represented by $\sin\theta = \frac{\sqrt{3}}{2}$, and so $b = \frac{5\sqrt{3}}{2}$. Even though a calculator gives values to more than four decimal places, we will give degrees to the nearest tenth and trigonometric functions to four decimal places.

CAUTION Do not round off numbers until all calculations have been finished. Rounding off numbers before you complete the problem can make your final results differ from the degree of accuracy you are seeking.

EXAMPLE 14.22

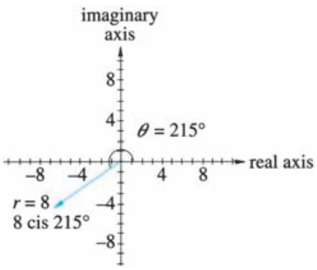

Figure 14.12

Locate the point $8\underline{/215°}$ in the complex plane and convert the number to rectangular form.

SOLUTION The graphical representation is given in Figure 14.12.

In this example, $r = 8$ and $\theta = 215°$, so

$$a = r\cos\theta = 8\cos 215° \approx 8(-0.8192) = -6.5532$$

$$b = r\sin\theta = 8\sin 215° \approx 8(-0.5736) = -4.5886$$

Thus, $8\underline{/215°} \approx -6.5532 - 4.5886j$.

CHANGING COMPLEX NUMBERS FROM RECTANGULAR TO POLAR FORM

A complex number written in rectangular form as

$$a + bj$$

has the polar coordinate forms

$$r\underline{/\theta}, \qquad r\text{ cis }\theta, \qquad \text{or} \qquad r(\cos\theta + j\sin\theta),$$

where

$$r = \sqrt{a^2 + b^2} \qquad \text{and} \qquad \tan\theta = \frac{b}{a}$$

NOTE While there may be many values for θ that satisfy the given conditions, we will normally select the smallest *positive* value.

EXAMPLE 14.23

Represent $5 - 12j$ graphically and convert it to polar form.

SOLUTION Graphically, this point is shown as a vector in Figure 14.13.

Here $a = 5$ and $b = -12$, so

$$\begin{aligned} r &= \sqrt{a^2 + b^2} \\ &= \sqrt{5^2 + (-12)^2} = \sqrt{25 + 144} \\ &= \sqrt{169} = 13 \end{aligned}$$

EXAMPLE 14.23 (Cont.)

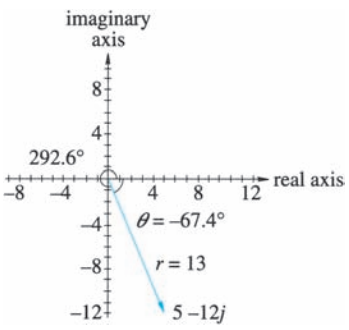

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Figure 14.13

From the graph in Figure 14.13, we can see that this point is in Quadrant IV.

$$\tan\theta = \frac{b}{a}$$
$$= \frac{-12}{5}$$
$$\text{so} \qquad \theta = \tan^{-1}\left(\frac{-12}{5}\right)$$
$$\approx -67.4^\circ$$

Notice that we used the arctan function. We can let θ be either -67.4° or $-67.4^\circ + 360^\circ = 292.6^\circ$. So, $5 - 12j \approx 13\angle 292.6^\circ = 13\angle -67.4^\circ$. But, as we just stated, we will usually use the smallest positive value for θ that satisfies the given conditions. In this case, $\theta = 292.6^\circ$.

EXAMPLE 14.24

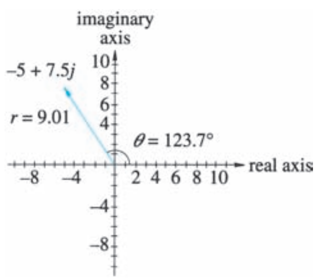

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Figure 14.14

Locate the point $-5 + 7.5j$ in the complex plane and convert it to polar form.

SOLUTION The point $-5 + 7.5j$ is shown in the graph in Figure 14.14 and we see that it is in Quadrant II.

In this example $a = -5$ and $b = 7.5$, so

$$r = \sqrt{(-5)^2 + 7.5^2}$$
$$= \sqrt{25 + 56.25}$$
$$= \sqrt{81.25} \approx 9.01$$
$$\tan\theta = \frac{7.5}{-5} = -1.5$$
$$\text{so} \qquad \theta \approx 123.7^\circ$$

Thus, $-5 + 7.5j \approx 9.01\angle 123.7^\circ$.

CAUTION Remember that using a calculator to work with inverse trigonometric functions may not give the desired angle. You must determine which quadrant contains the point.

In Example 14.24, we had $\tan\theta = -1.5$. If you use a calculator (in degree mode) to determine $\theta = \tan^{-1}(-1.5)$, you get

PRESS	DISPLAY
2nd TAN (−) 1.5	−56.30993247

This value for θ, -56.3°, is an angle in the fourth quadrant. We can see, from Figure 14.14, that the point is in Quadrant II. So, $\theta = -56.3^\circ + 180^\circ = 123.7^\circ$

The polar form of a real number or a pure imaginary number is on one of the axes. This makes finding them relatively easy, as shown in Example 14.25.

EXAMPLE 14.25

Express each of the following complex numbers in polar form: (a) 8, (b) -9, (c) $3j$, and (d) $-4j$.

SOLUTIONS

(a) $8 = 8 + 0j$ lies on the positive real axis, so $\theta = 0°$ and $r = 8$; thus, $8 = 8(\cos 0° + j \sin 0°) = 8 \text{ cis } 0°$.

(b) $-9 = -9 + 0j$ lies on the negative real axis so, $\theta = 180°$ and $r = 9$; thus, $-9 = 9 \text{ cis } 180°$.

(c) $3j = 0 + 3j$ lies on the positive imaginary axis, so $\theta = 90°$ and $r = 3$; $3j = 3\underline{/90°}$.

(d) $-4j = 0 - 4j$ lies on the negative imaginary axis, so $\theta = 270°$ and $r = 4$; $-4j = 4(\cos 270° + j \sin 270°) = 4 \text{ cis } 270°$.

NOTE Many graphing calculators can convert a complex number from polar form to rectangular form or from rectangular form to polar form. Since each make and model does this in a different way, consult the user's manual for your calculator to see how this is done.

APPLICATION ELECTRONICS

EXAMPLE 14.26

The voltage in an ac circuit is represented by the complex number $V = 28.4 - 65.7j$ V. Express this complex number in polar form.

SOLUTION First, we find r. Here, $a = 28.4$ and $b = -65.7$, so

$$r = \sqrt{28.4^2 + (-65.7)^2} = \sqrt{806.56 + 4{,}316.49} = \sqrt{5{,}123.05} \approx 71.58$$

Next, we find the angle θ.

$$\tan\theta = \frac{-65.7}{28.4}$$
$$\theta = \tan^{-1}\left(\frac{-65.7}{28.4}\right)$$
$$\theta \approx -66.6°$$

Since V is in the fourth quadrant, $\theta \approx -66.6°$. If we want to express θ as an angle in the interval $[0, 360°)$, then $\theta = -66.6° + 360° = 293.4°$.

So, in polar form, voltage $V = 28.4 - 65.7j \approx 71.6\underline{/-66.6°} = 71.6\underline{/293.4°}$ V.

APPLICATION ELECTRONICS

EXAMPLE 14.27 Express current $i = 4.5\underline{/40^\circ}$ A in rectangular form.

SOLUTION Here, $r = 4.5$ and $\theta = 40^\circ$, so $a = r\cos\theta = 4.5\cos 40^\circ \approx 3.45$ and $b = r\sin\theta = 4.5\sin 40^\circ \approx 2.89$. In rectangular form the current in amps, A, is $i = 4.5\underline{/40^\circ} \approx 3.45 + 2.89j$.

EXERCISE SET 14.3

For each number in Exercises 1–16, locate the point in the complex plane and express the number in rectangular form.

1. $4(\cos 30^\circ + j\sin 30^\circ)$
2. $10(\cos 45^\circ + j\sin 45^\circ)$
3. $5(\cos 135^\circ + j\sin 135^\circ)$
4. $8(\cos 305^\circ + j\sin 305^\circ)$
5. $7 \text{ cis } 260^\circ$
6. $3 \text{ cis } 340^\circ$
7. $2 \text{ cis } 115^\circ$
8. $5 \text{ cis } 285^\circ$
9. $3\underline{/25^\circ}$
10. $4\underline{/240^\circ}$
11. $5\underline{/340^\circ}$
12. $6\underline{/90^\circ}$
13. $4.5\underline{/245^\circ}$
14. $6.8\underline{/10^\circ}$
15. $2.5\underline{/180^\circ}$
16. $5.9\underline{/270^\circ}$

For each number in Exercises 17–32, locate the point in the complex plane and express the number in polar form.

17. $5 + 5j$
18. $6 + 3j$
19. $4 - 8j$
20. $8 - 2j$
21. $-4 + 7j$
22. $-9 + 3j$
23. $-6 - 2j$
24. $-10 - 2j$
25. 6
26. $1.2 + 7.3j$
27. $4.2 - 6.3j$
28. $9j$
29. $-5.8 + 0.2j$
30. $-7j$
31. -2.7
32. $-4.7 - 1.1j$

In Exercises 33–40, perform the indicated operations graphically.

33. $(5 + j) + (3 + 2j)$
34. $(3 - 4j) + (4 - 2j)$
35. $(-4 + 2j) + (2 - 8j)$
36. $(8 - j) + (9 + 6j)$
37. $(-8 + 7j) + (-5 - 3j)$
38. $(4 + 2j) + (-3 - 9j)$
39. $(-5 + 3j) + (4 - 8j)$
40. $(-3 + 2j) + (-8 - 5j)$

Solve Exercises 41–46.

41. ***Electronics*** The current in amps, A, of an ac circuit is given by the complex number $4.7 - 6.5j$. Write this number in polar form.
42. ***Electronics*** The current in amps, A, of an ac circuit is given by the complex number $8.2 + 5.4j$. Write this number in polar form.
43. ***Electronics*** Convert the voltage given by $V = -110.4 + 46.1j$ V to polar form.
44. ***Electronics*** Convert the voltage given by $V = 108.5 - 57.6j$ V to polar form.
45. ***Electronics*** Express the current $i = 2.5\underline{/-50^\circ}$ A in rectangular form.
46. ***Electronics*** Express the impedance $Z = 8.5\underline{/2\pi/3}\ \Omega$ in rectangular form.

[IN YOUR WORDS]

47. **(a)** What is the absolute value or modulus of a complex number?

(b) What is the amplitude or argument of a complex number?

48. Describe how to change a complex number in rectangular form to its equivalent complex number in polar form.

49. Describe how to change a complex number in polar form to its equivalent complex number in rectangular form.

50. Explain how to use your calculator to change a complex number from rectangular form to polar form, or vice versa.

51. Consult the help menu for your spreadsheet and describe how to change a complex number from rectangular form to polar form, or vice versa. Give your explanation to a classmate and ask him or her to follow your directions. Rewrite your directions to clarify places where your classmate had difficulty.

14.4 EXPONENTIAL FORM OF A COMPLEX NUMBER

There is yet another way in which complex numbers are often represented. It is called the **exponential form of a complex number**, because it involves exponents of the number e. (Remember from Section 12.2 that $e \approx 2.718281828$.) If $z = r\underline{/\theta}$ is a complex number, then we know that $z = r(\cos\theta + j\sin\theta)$.

EXPONENTIAL FORM OF A COMPLEX NUMBER

The *exponential form* of a complex number uses *Euler's formula*, $e^{j\theta} = \cos\theta + j\sin\theta$, and states that

$$z = re^{j\theta}$$

where θ is in radians.

While θ can have any value, we will express answers with $0 \le \theta < 2\pi$.

CAUTION When using the exponential form, you must remember that θ is in radians.

EXAMPLE 14.28

Write the complex number $6(\cos 180° + j\sin 180°)$ in exponential form.

SOLUTION In this example, $r = 6$ and $\theta = 180°$. The exponential form requires that θ be expressed in radians: $180° = \pi$ rad and so $\theta = \pi$. Thus $6(\cos 180° + j\sin 180°) = 6e^{j\pi}$.

EXAMPLE 14.29

Write the complex number $8\underline{/225^\circ}$ in exponential form.

SOLUTION In this example, $r = 8$ and $\theta = 225^\circ = \frac{5\pi}{4} \approx 3.927$ rad.

$$8\underline{/225^\circ} = 8e^{\frac{5\pi}{4}j} = 8e^{5j\pi/4} \approx 8e^{3.927j}$$

All of the last three versions are correct. Because you will probably be using a calculator to convert from degrees to radians, the last version is the one that you will most likely use. But remember, $8e^{3.927j}$ is rounded off and therefore is the least accurate.

EXAMPLE 14.30

Express $-4 + 3j$ in exponential form.

SOLUTION This example is in the form $a + bj$. We must first determine r and θ. Now $r = \sqrt{a^2 + b^2} = \sqrt{(-4)^2 + 3^2} = 5$ and $\tan\theta = \frac{3}{-4} = -0.75$. With a graphing calculator in radian mode, we can determine θ.

PRESS	DISPLAY
2nd tan (−) 0.75 + π ENTER	2.498091545

The last two steps were needed because the arctan of a negative number produces an angle in Quadrant IV. The point $-4 + 3j$ is in Quadrant II, so we added π to the original answer. Thus, we can see that $\theta \approx 2.4981$ and $-4 + 3j \approx 5e^{2.4981j}$.

EXAMPLE 14.31

Express $-5 - 8j$ in exponential form.

SOLUTION $r = \sqrt{a^2 + b^2} = \sqrt{(-5)^2 + (-8)^2} = \sqrt{25 + 64} = \sqrt{89} \approx 9.4340$. $\tan\theta = \frac{-8}{-5} = 1.6$. Since $-5 - 8j$ is in Quadrant III, $\theta = \tan^{-1} 1.6 + \pi \approx 4.1538$. Thus, we see that $-5 - 8j = 9.4340e^{4.1538j}$.

EXAMPLE 14.32

Express $4.6e^{5.7j}$ in polar and rectangular form.

SOLUTION Here, $r = 4.6$ and $\theta = 5.7$, so we write the given expression in polar form as $4.6e^{5.7j} = 4.6(\cos 5.7 + j \sin 5.7)$. With a calculator in radian mode, we get

$$a = 4.6 \cos 5.7 \approx 3.8397$$
$$b = 4.6 \sin 5.7 \approx -2.5332$$

We have determined that $4.6e^{5.7j} = a + bj \approx 3.8397 - 2.5332j$.

APPLICATION ELECTRONICS

EXAMPLE 14.33

The voltage in an ac circuit is represented by the complex number $V = -85.6 + 72.3j$ V. Express this complex number in exponential form.

SOLUTION First, we find r.

$$r = \sqrt{(-85.6)^2 + 72.3^2} = \sqrt{7327.36 + 5227.29} \approx 112.0$$

Next, we find the angle θ.

$$\tan\theta = \frac{72.3}{-85.6}$$
$$\theta \approx -0.70137$$

Since V is in the second quadrant, $\theta = -0.70137 + \pi \approx 2.4402$. So, in exponential form, we find that voltage $V = -85.6 + 72.3j \approx 112e^{2.4402j}$ V.

APPLICATION ELECTRONICS

EXAMPLE 14.34

Express the current $I = 2.5\underline{/-50^\circ}$ A in exponential and rectangular forms.

SOLUTION Here, $r = 2.5$ and $\theta = -50^\circ \approx -0.8727$ rad, so in exponential form we have $I = re^{j\theta} = 2.5e^{-0.8727j}$.

To convert the given number to rectangular form, we have $x = 2.5\cos(-50^\circ) \approx 1.6070$ and $y = 2.5\sin(-50^\circ) \approx -1.9151$. Thus, in rectangular form, the current is $I = 1.6070 - 1.9151j$ A.

MULTIPLYING AND DIVIDING IN EXPONENTIAL FORM

One advantage of using the exponential form for complex numbers is that complex numbers written in exponential form obey the laws of exponents. There are three properties of exponents that are of interest. These three properties concern multiplication, division, and powers, and all of them use four basic rules introduced in Section 1.4.

$$b^m b^n = b^{m+n}$$
$$\frac{b^m}{b^n} = b^{m-n}$$
$$(b^m)^n = b^{mn}$$
$$(ab)^m = a^m b^m$$

If we have two complex numbers $z_1 = r_1e^{j\theta_1}$ and $z_2 = r_2e^{j\theta_2}$, we can then multiply, divide, or take powers of them using the preceding rules. For example,

$$z_1z_2 = (r_1e^{j\theta_1})(r_2e^{j\theta_2}) = r_1r_2e^{j(\theta_1+\theta_2)}$$

and

$$\frac{z_1}{z_2} = \frac{r_1e^{j\theta_1}}{r_2e^{j\theta_2}} = \frac{r_1}{r_2}e^{j(\theta_1-\theta_2)}$$

EXAMPLE 14.35

Multiply $7e^{4.2j}$ and $2e^{1.5j}$.

SOLUTION $(7e^{4.2j})(2e^{1.5j}) = 7 \cdot 2e^{(4.2+1.5)j} = 14e^{5.7j}$

If we want to divide, then

$$\frac{z_1}{z_2} = \frac{r_1 e^{j\theta_1}}{r_2 e^{j\theta_2}} = \frac{r_1}{r_2} e^{j(\theta_1 - \theta_2)}$$

EXAMPLE 14.36

Divide $9e^{3.2j}$ by $2e^{4.3j}$.

SOLUTION $\dfrac{9e^{3.2j}}{2e^{4.3j}} = \frac{9}{2} e^{(3.2 - 4.3)j} = 4.5e^{-1.1j}$

If you want to express the angle in the answer between 0 and 2π $(0 \le \theta < 2\pi)$, then let $\theta = 2\pi - 1.1$, or, using a calculator,

PRESS	DISPLAY
2 × π −	6.2831853
1.1 =	5.1831853

Thus, $4.5e^{-1.1j}$ could be expressed as $4.5e^{5.2j}$.

The last property involves raising a complex number to a power and the properties $(b^m)^n = b^{mn}$ and $(ab)^m = a^m b^m$.

$$z^n = (re^{j\theta})^n = r^n e^{jn\theta}$$

EXAMPLE 14.37

Calculate $(4e^{2.3j})^5$.

SOLUTION
$$\begin{aligned}(4e^{2.3j})^5 &= 4^5 e^{5(2.3)j}\\ &= 1{,}024e^{11.5j}\\ &\approx 1{,}024e^{5.2j}\end{aligned}$$

Since 11.5 is greater than 2π, we subtracted 2π to get an angle of 5.2, which is between 0 and 2π.

APPLICATION ELECTRONICS

EXAMPLE 14.38

Given that $I = 5.8e^{-0.4363j}$ A and $Z = 8.5e^{1.047j}\,\Omega$, calculate $V = IZ$.

SOLUTION Since $V = IZ$, we want to determine the product of the given numbers.

$$\begin{aligned}V = (5.8e^{-0.4363j})(8.5e^{1.047j}) &= (5.8)(8.5)e^{-0.4363j + 1.047j}\\ &= 49.3e^{0.6107j}\end{aligned}$$

The voltage is $49.3e^{0.6107j}$ V.

APPLICATION ELECTRONICS

EXAMPLE 14.39 If $V_C = 78.3e^{-.3725j}$ V and $X_C = 87.0e^{-1.6500j}\,\Omega$, and $V_C = IX_C$, determine I.

SOLUTION Since $V_C = IX_C$, then $I = \dfrac{V_C}{X_C}$, and so we need to divide.

$$\begin{aligned} I = \frac{V_C}{X_C} &= \frac{78.3e^{-.3725j}}{87.0e^{-1.6500j}} \\ &= \frac{78.3}{87.0}e^{-.3725j-(-1.6500j)} \\ &= 0.9e^{1.2775j} \end{aligned}$$

So, we have determined that $I = 0.9e^{1.2775j}$ A.

EXERCISE SET 14.4

In Exercises 1–12, express each complex number in exponential form.

1. $3\left(\cos \frac{3\pi}{2} + j \sin \frac{3\pi}{2}\right)$
2. $7(\cos 1.4 + j \sin 1.4)$
3. $2(\cos 60° + j \sin 60°)$
4. $11(\cos 320° + j \sin 320°)$
5. $1.3(\cos 5.7 + j \sin 5.7)$
6. $9.5(\cos 2.1 + j \sin 2.1)$
7. $3.1(\cos 25° + j \sin 25°)$
8. $10.5(\cos 195° + j \sin 195°)$
9. $8 + 6j$
10. $12 - 5j$
11. $-9 + 12j$
12. $-8 - 12j$

In Exercises 13–16, express each number in rectangular and polar form.

13. $5e^{0.5j}$
14. $8e^{1.9j}$
15. $2.3e^{4.2j}$
16. $4.5e^{7j\pi/6}$

In Exercises 17–34, perform each of the indicated operations.

17. $2e^{3j} \cdot 6e^{2j}$
18. $3e^{j} \cdot 4e^{2j}$
19. $e^{1.3j} \cdot 2.4e^{4.6j}$
20. $1.5e^{4.1j} \cdot 0.2e^{1.7j}$
21. $7e^{4.3j} \cdot 4e^{5.7j}$
22. $3.6e^{5.4j} \cdot 2.5e^{6.1j}$
23. $8e^{3j} \div 2e^{j}$
24. $28e^{5j} \div 7e^{2j}$
25. $17e^{4.3j} \div 4e^{2.8j}$
26. $8.5e^{3.4j} \div 2e^{5.3j}$
27. $(3e^{2j})^4$
28. $(4e^{3j})^5$
29. $(2.5e^{1.5j})^4$
30. $(7.2e^{2.3j})^5$
31. $(4e^{6j})^{1/2}$
32. $(16e^{6j})^{1/4}$
33. $(6.25e^{4.2j})^{1/2}$
34. $(1.728e^{2.1j})^{1/3}$

Solve Exercises 35–44.

35. ***Electronics*** The voltage in an ac circuit is represented by the complex number $V = 56.5 + 24.1j$ V. Express this complex number in exponential form.
36. ***Electronics*** Express the current $I = 4.90 - 4.11j$ A in exponential form.
37. ***Electronics*** Express the impedance $Z = 135 \times \underline{/-52.5°}\,\Omega$ in exponential and rectangular forms.
38. ***Electronics*** Express the capacitive reactance $X_C = 40.5\underline{/-\pi/2}\ \Omega$ in exponential and rectangular forms.
39. ***Electronics*** Given that $I = 12.5e^{-0.7256j}$ A and $Z = 6.4e^{1.4285j}\,\Omega$, find $V = IZ$.
40. ***Electronics*** Given that $I = 4.24e^{0.5627j}$ A and $X_L = 28.5e^{-1.5708j}\,\Omega$, find $V_L = IX_L$.

41. ***Electronics*** If $V = 115e^{-0.2145j}$ V and $Z = 2.5e^{0.5792j}\,\Omega$, find I, given that $V = IZ$.

42. ***Electronics*** If $V_R = 35.1e^{1.3826j}$ V and $I = 0.78e^{1.3826j}$ A, find R, given that $V_R = IR$.

43. ***Electronics*** If $V = 122.4e^{0.2551j}$ V and $I = 36e^{-0.8189j}$ A, find Z, given that $V = IZ$.

44. ***Electronics*** If $V_L = IX_L$ with $V_L = 119.7e^{0.7254j}$ V and $I = 4.2e^{-0.1246j}$ A, find X_L.

[IN YOUR WORDS]

45. Describe how to change a complex number in rectangular form to its equivalent complex number in exponential form.

46. Describe how to change a complex number in exponential form to its equivalent complex number in rectangular form.

14.5 OPERATIONS IN POLAR FORM; DeMOIVRE'S FORMULA

Multiplication, division, and powers of complex numbers are easily performed when the numbers are written in exponential form. They are just as easily performed when the numbers are in polar form. Remember the relationship between the exponential and polar forms.

$$re^{j\theta} = r(\cos\theta + j\sin\theta)$$

MULTIPLICATION

Again, we will let $z_1 = r_1e^{j\theta_1}$ and $z_2 = r_2e^{j\theta_2}$ be two complex numbers. We know that

$$z_1z_2 = r_1r_2e^{j(\theta_1+\theta_2)}$$

so in polar form, this would be

$$r_1(\cos\theta_1 + j\sin\theta_1)\cdot r_2(\cos\theta_2 + j\sin\theta_2) = r_1r_2[\cos(\theta_1+\theta_2) + j\sin(\theta_1+\theta_2)]$$

Using the alternative way of writing the polar form of a complex number, we have

$$r_1\angle\theta_1 \cdot r_2\angle\theta_2 = r_1r_2\angle\theta_1+\theta_2$$

Notice that the angles do not have to be written in radians. Angles can be in either degrees or radians.

PRODUCT OF TWO COMPLEX NUMBERS

The product of two complex numbers $z_1 = r_1 \operatorname{cis}\theta_1 = r_1\angle\theta_1 = r_1e^{j\theta_1}$ and $z_2 = r_2\operatorname{cis}\theta_2 = r_2\angle\theta_2 = r_2e^{j\theta_2}$ is

$$\begin{aligned} z_1z_2 &= (r_1\operatorname{cis}\theta_1)(r_2\operatorname{cis}\theta_2) = r_1r_2\operatorname{cis}(\theta_1+\theta_2) \\ &= r_1\angle\theta_1 r_2\angle\theta_2 = r_1r_2\angle\theta_1+\theta_2 \\ &= r_1r_2e^{j(\theta_1+\theta_2)} \end{aligned}$$

EXAMPLE 14.40

Find each of the following products.

(a) $2(\cos 15^\circ + j\sin 15^\circ)\cdot 5(\cos 80^\circ + j\sin 80^\circ)$, (b) $6\underline{/30^\circ}\cdot 3\underline{/60^\circ}$, (c) $4\underline{/2.3}\cdot 1.5\underline{/0.5}$, and (d) $(3e^{1.2j})(5e^{0.3j})$.

SOLUTIONS

(a)
$$\begin{aligned} 2(\cos 15^\circ + j\sin 15^\circ)\cdot 5(\cos 80^\circ + j\sin 80^\circ) &= 2\cdot 5[\cos(15^\circ + 80^\circ) + j\sin(15^\circ + 80^\circ)] \\ &= 10(\cos 95^\circ + j\sin 95^\circ) \end{aligned}$$

(b)
$$\begin{aligned} 6\underline{/30^\circ}\cdot 3\underline{/60^\circ} = 6\cdot 3\underline{/30^\circ + 60^\circ} &= 18\underline{/90^\circ} \\ &= 18(\cos 90^\circ + j\sin 90^\circ) \\ &= 18(0 + j\cdot 1) = 18j \end{aligned}$$

(c) $4\underline{/2.3}\cdot 1.5\underline{/0.5} = 4(1.5)\underline{/2.3 + 0.5} = 6\underline{/2.8}$

(d) $(3e^{1.2j})(5e^{0.3j}) = 3(5)e^{(1.2+0.3)j} = 15e^{1.5j}$

NOTE When complex numbers are written in exponential form, $z = re^{j\theta}$, you *must* have θ in radians. Complex numbers written in polar form, $z = r\operatorname{cis}\theta$, may have θ in either degrees or radians.

DIVISION

Division uses a similar process. Again, we will use the exponential form from Section 14.4 to show that

$$\frac{z_1}{z_2} = \frac{r_1 e^{j\theta_1}}{r_2 e^{j\theta_2}} = \frac{r_1}{r_2}e^{j(\theta_1 - \theta_2)}$$

The results for both the exponential and polar forms are summarized as follows.

QUOTIENT OF TWO COMPLEX NUMBERS

The quotient of two complex numbers $z_1 = r_1\operatorname{cis}\theta_1 = r_1\underline{/\theta_1} = r_1e^{j\theta_1}$ and $z_2 = r_2\operatorname{cis}\theta_2 = r_2\underline{/\theta_2} = r_2e^{j\theta_2}$ is

$$\begin{aligned} \frac{z_1}{z_2} &= \frac{r_1(\cos\theta_1 + j\sin\theta_1)}{r_2(\cos\theta_2 + j\sin\theta_2)} \\ &= \frac{r_1}{r_2}[\cos(\theta_1 - \theta_2) + j\sin(\theta_1 - \theta_2)] \\ \text{or} \quad &= \frac{r_1\underline{/\theta_1}}{r_2\underline{/\theta_2}} \\ &= \frac{r_1}{r_2}\underline{/\theta_2 - \theta_2} \\ \text{or} \quad &= \frac{r_1e^{j\theta_1}}{r_2e^{j\theta_2}} = \frac{r_1}{r_2}e^{j(\theta_1 - \theta_2)} \end{aligned}$$

EXAMPLE 14.41

Find each of the following quotients.

(a) $\dfrac{12(\cos 45^\circ + j\sin 45^\circ)}{2(\cos 15^\circ + j\sin 15^\circ)}$, (b) $\dfrac{15\angle 135^\circ}{3\angle 75^\circ}$, and (c) $4e^{2.1j} \div 8e^{1.7j}$.

SOLUTIONS

(a) $$\frac{12(\cos 45^\circ + j\sin 45^\circ)}{2(\cos 15^\circ + j\sin 15^\circ)} = \frac{12}{2}[\cos(45^\circ - 15^\circ) + j\sin(45^\circ - 15^\circ)]$$
$$= 6(\cos 30^\circ + j\sin 30^\circ)$$

(b) $$\frac{15\angle 135^\circ}{3\angle 75^\circ} = \frac{15}{3}\angle 135^\circ - 75^\circ = 5\angle 60^\circ$$

(c) $$4e^{2.1j} \div 8e^{1.7j} = \tfrac{4}{8}e^{(2.1-1.7)j} = \tfrac{1}{2}e^{0.4j}$$

APPLICATION ELECTRONICS

EXAMPLE 14.42

Ohm's law for ac circuits is $V = IZ$. If $V = 15\angle 39^\circ$ V and $Z = 8\angle 26^\circ\ \Omega$, what is the current?

SOLUTION Substituting the given values into the formula $V = IZ$, we obtain $15\angle 39^\circ = I \cdot 8\angle 26^\circ$. Solving for I produces

$$\begin{aligned} I &= \frac{V}{Z} \\ &= \frac{15\angle 39^\circ}{8\angle 26^\circ} \\ &= \frac{15}{8}\angle 39^\circ - 26^\circ \\ &= \frac{15}{8}\angle 13^\circ \end{aligned}$$

The current is $\frac{15}{8}\angle 13^\circ = 1.875\angle 13^\circ$ A.

POWERS, ROOTS, AND DeMOIVRE'S FORMULA

Finding powers of complex numbers in polar form also uses the same process we developed for powers in the exponential form.

DeMOIVRE'S FORMULA

For any complex number $z = r\,\text{cis}\,\theta = r\angle\theta = re^{j\theta}$

$$z^n = (re^{j\theta})^n = r^n e^{j\theta n} \text{ or } [r(\cos\theta + j\sin\theta)]^n = r^n(\cos n\theta + j\sin n\theta)$$

This formula is known as **DeMoivre's formula**.

EXAMPLE 14.43

Use DeMoivre's formula to find each of the following powers. Convert your answers to rectangular form.
(a) $[3(\cos 30° + j\sin 30°)]^6$, (b) $(2\underline{/135°})^5$, (c) $(16\underline{/225°})^{1/4}$, and (d) $(2e^{0.4j})^7$.

SOLUTIONS

(a) $$\begin{aligned}[3(\cos 30° + j\sin 30°)]^6 &= 3^6(\cos 6\cdot 30° + j\sin 6\cdot 30°)\\ &= 729(\cos 180° + j\sin 180°)\\ &= -729\end{aligned}$$

(b) $$\begin{aligned}(2\underline{/135°})^5 &= 2^5\underline{/5\cdot 135°} = 32\underline{/675°}\\ a &= 32\cos 675° \approx 22.6274\\ b &= 32\sin 675° \approx -22.6274\end{aligned}$$

so, $(2\underline{/135°})^5 = 22.6274 - 22.6274j$

(c) $$\begin{aligned}(16\underline{/225°})^{1/4} &= 16^{1/4}\underline{/\tfrac{225°}{4}} = 2\underline{/56.25°}\\ a &= 2\cos 56.25° \approx 1.1111\\ b &= 2\sin 56.25° = 1.6629\end{aligned}$$

so, $(16\underline{/225°})^{1/4} \approx 1.1111 + 1.6629j$

There are three other fourth roots of $16\underline{/255°}$. We will soon see how to use DeMoivre's formula to find these other three roots.

(d) $$\begin{aligned}(2e^{0.4j})^7 &= (2)^7e^{(0.4)7j} = 128e^{2.8j}\\ &\approx -120.6045 + 42.8785j\end{aligned}$$

DeMoivre's formula can be used to help find all of the roots of a complex number. For example, the equation $z^3 = -1$ has three roots. One of the roots is -1. What are the other two? DeMoivre's formula can be used to find those roots.

First, we will write -1 in polar form.

$$-1 = 1(\cos 180° + j\sin 180°)$$

Using DeMoivre's formula with $n = \frac{1}{3}$, we get

$$\begin{aligned}(-1)^{1/3} &= 1^{1/3}\left(\cos\frac{180°}{3} + j\sin\frac{180°}{3}\right)\\ &= 1(\cos 60° + j\sin 60°)\\ &= 0.5000 + \frac{\sqrt{3}}{2}j\\ &\approx 0.5000 + 0.8660j\end{aligned}$$

We can see that this is not -1, the answer that we had before. A check would verify that $(0.5 + 0.8660j)^3 = -1$. So, this is a correct answer.

Why did we get a different answer? If you divide any number between 0° and 1,080° (or 0 and 6π) by 3, you find an angle between 0° and 360° (or between 0 and 2π). Now, 180°, 540°, and 900° all have the same terminal side. So, we could have written -1 as $1(\cos 540° + j \sin 540°)$ or as $1(\cos 900° + j \sin 900°)$. Let's find the cube root of each of these numbers.

$$\begin{aligned}[1(\cos 540° + j \sin 540°)]^{1/3} &= 1^{1/3}\left(\cos \frac{540°}{3} + j \sin \frac{540°}{3}\right)\\ &= 1(\cos 180° + j \sin 180°)\\ &= -1\end{aligned}$$

$$\begin{aligned}[1(\cos 900° + j \sin 900°)]^{1/3} &= 1^{1/3}\left(\cos \frac{900°}{3} + j \sin \frac{900°}{3}\right)\\ &= 1(\cos 300° + j \sin 300°)\\ &= 0.5000 - 0.8660j\end{aligned}$$

We have found three different cube roots of -1. The first and last are conjugates of each other. We have also given an example of a process we can use to find all of the nth roots of a number, where n is a positive integer.

ROOTS OF A COMPLEX NUMBER

If $z = r(\cos \theta + j \sin \theta)$, then the nth roots of z are given by the formula

$$w_k = \sqrt[n]{r}\left[\cos\left(\frac{\theta}{n} + \frac{360° \cdot k}{n}\right) + j \sin\left(\frac{\theta}{n} + \frac{360° \cdot k}{n}\right)\right] \qquad (*)$$

where $k = 0, 1, 2, \ldots, n-1$.

If θ is in radians, then substitute 2π for 360°.

EXAMPLE 14.44

Find the five fifth roots of $32j$.

SOLUTION The five fifth roots will be called w_0, w_1, w_2, w_3, and w_4. Using $32j = 32(\cos 90° + j \sin 90°)$ and applying formula $(*)$, we obtain the following.

$$\begin{aligned}w_0 &= \sqrt[5]{32}\left[\cos\left(\frac{90°}{5} + \frac{360° \cdot 0}{5}\right) + j \sin\left(\frac{90°}{5} + \frac{360° \cdot 0}{5}\right)\right]\\ &= 2(\cos 18° + j \sin 18°)\\ &\approx 1.9021 + 0.6180j\end{aligned}$$

$$\begin{aligned}w_1 &= \sqrt[5]{32}\left[\cos\left(\frac{90°}{5} + \frac{360° \cdot 1}{5}\right) + j \sin\left(\frac{90°}{5} + \frac{360° \cdot 1}{5}\right)\right]\\ &= 2(\cos 90° + j \sin 90°)\\ &= 2j\end{aligned}$$

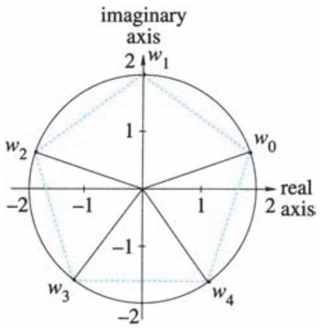


Figure 14.15

$$w_2 = \sqrt[5]{32}\left[\cos\left(\frac{90°}{5} + \frac{360° \cdot 2}{5}\right) + j\sin\left(\frac{90°}{5} + \frac{360° \cdot 2}{5}\right)\right]$$
$$= 2(\cos 162° + j\sin 162°)$$
$$\approx -1.9021 + 0.6180j$$
$$w_3 = \sqrt[5]{32}\left[\cos\left(\frac{90°}{5} + \frac{360° \cdot 3}{5}\right) + j\sin\left(\frac{90°}{5} + \frac{360° \cdot 3}{5}\right)\right]$$
$$= 2(\cos 234° + j\sin 234°)$$
$$\approx -1.1756 - 1.6180j$$
$$w_4 = \sqrt[5]{32}\left[\cos\left(\frac{90°}{5} + \frac{360° \cdot 4}{5}\right) + j\sin\left(\frac{90°}{5} + \frac{360° \cdot 4}{5}\right)\right]$$
$$= 2(\cos 306° + j\sin 306°)$$
$$\approx 1.1756 - 1.6180j$$

The five roots are shown in Figure 14.15. Notice the symmetry of these five roots around the circle. Any time you graph the n roots of a number, they should be equally spaced around a circle with the center at the origin in the complex plane.

EXAMPLE 14.45

Find the three cube roots of $2\sqrt{11} + 10j$.

SOLUTION We first write this number in polar form. $r = \sqrt{(2\sqrt{11})^2 + 10^2} = \sqrt{144} = 12$ and $\tan\theta = \frac{10}{2\sqrt{11}} \approx 1.5075567$, so $\theta \approx 56.4°$. We will find the three cube roots w_0, w_1, and w_2.

$$w_0 = \sqrt[3]{12}\left[\cos\left(\frac{56.4°}{3} + \frac{360° \cdot 0}{3}\right) + j\sin\left(\frac{56.4°}{3} + \frac{360° \cdot 0}{3}\right)\right]$$
$$= \sqrt[3]{12}(\cos 18.8° + j\sin 18.8°)$$
$$\approx 2.1673 + 0.7378j$$
$$w_1 = \sqrt[3]{12}\left[\cos\left(\frac{56.4°}{3} + \frac{360° \cdot 1}{3}\right) + j\sin\left(\frac{56.4°}{3} + \frac{360° \cdot 1}{3}\right)\right]$$
$$= \sqrt[3]{12}(\cos 138.8° + j\sin 138.8°)$$
$$\approx -1.7230 + 1.5076j$$
$$w_2 = \sqrt[3]{12}\left[\cos\left(\frac{56.4°}{3} + \frac{360° \cdot 2}{3}\right) + j\sin\left(\frac{56.4°}{3} + \frac{360° \cdot 2}{3}\right)\right]$$
$$= \sqrt[3]{12}(\cos 258.8° + j\sin 258.8°)$$
$$\approx -0.4447 - 2.2459j$$

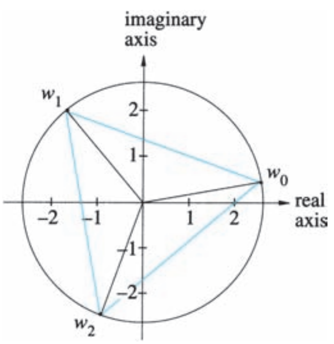


Figure 14.16

Geometrically, these three points are shown on the graph in Figure 14.16.

EXAMPLE 14.46

Find the three cube roots of $(-8)^5$.

SOLUTION We first write -8 in polar form. Since $r = 8$ and $\theta = 180°$, we see that $-8 = 8 \text{ cis } 180°$. Using DeMoivre's formula to raise this to the fifth power, we obtain

$$\begin{aligned}(-8)^5 &= 8^5 \text{ cis }(5 \cdot 180°)\\ &= 32{,}768 \text{ cis } 900°\\ &= 32{,}768 \text{ cis } 180°\end{aligned}$$

We will find the three cube roots, w_0, w_1, and w_2, of 32,768 cis 180°.

$$\begin{aligned}w_0 &= \sqrt[3]{32{,}768}\left[\cos\left(\frac{180°}{3} + \frac{360° \cdot 0}{3}\right) + j \sin\left(\frac{180°}{3} + \frac{360° \cdot 0}{3}\right)\right]\\ &= 32(\cos 60° + j \sin 60°)\\ &\approx 16 + 27.71j\\ w_1 &= \sqrt[3]{32{,}768}\left[\cos\left(\frac{180°}{3} + \frac{360° \cdot 1}{3}\right) + j \sin\left(\frac{180°}{3} + \frac{360° \cdot 1}{3}\right)\right]\\ &= 32(\cos 180° + j \sin 180°)\\ &= -32\\ w_2 &= \sqrt[3]{32{,}768}\left[\cos\left(\frac{180°}{3} + \frac{360° \cdot 2}{3}\right) + j \sin\left(\frac{180°}{3} + \frac{360° \cdot 2}{3}\right)\right]\\ &= 32(\cos 300° + j \sin 300°)\\ &\approx 16 - 27.71j\end{aligned}$$

Thus, the three cube roots of $(-8)^5$ are $16 + 27.71j$, -32, and $16 - 27.71j$.

EXERCISE SET 14.5

In Exercises 1–20, perform the indicated operations and give the answers in polar form.

1. $3(\cos 46° + j \sin 46°) \cdot 5(\cos 23° + j \sin 23°)$
2. $4(\cos 135° + j \sin 135°) \cdot 5(\cos 63° + j \sin 63°)$
3. $2.5(\cos 1.43 + j \sin 1.43) \cdot 4(\cos 2.67 + j \sin 2.67)$
4. $6.4(\cos 0.25 + j \sin 0.25) \cdot 3.5(\cos 1.1 + j \sin 1.1)$
5. $\dfrac{8(\cos 85° + j \sin 85°)}{2(\cos 25° + j \sin 25°)}$
6. $\dfrac{6(\cos 273° + j \sin 273°)}{3(\cos 114° + j \sin 114°)}$
7. $\dfrac{9\angle 137°}{2\angle 26°}$
8. $\dfrac{18\angle 3.52}{5\angle 2.14}$
9. $(3\angle 2.7)(4\angle 5.3)$
10. $[3(\cos 20° + j \sin 20°)]^4$
11. $[5(\cos 84° + j \sin 84°)]^6$
12. $[2.5(\cos 118° + j \sin 118°)]^3$
13. $[10.4(\cos 3.42 + j \sin 3.42)]^3$
14. $(2\angle 1.38)^5$
15. $(3.4\angle 5.3)^4$
16. $(4.41\angle 124°)^{1/2}$
17. $(4e^{2.1j})(3e^{1.7j})$
18. $6e^{1.5j} \div 4e^{0.9j}$
19. $(0.5e^{0.3j})^3$
20. $(0.0625e^{4.2j})^{1/2}$

In Exercises 21–28, use DeMoivre's formula to find the indicated roots. Give the answers in rectangular form.

21. cube roots of 1

22. cube roots of j

23. cube roots of $-8j$

24. fourth roots of -16

25. fourth roots of $-16j$

26. fourth roots of $1 - j$

27. fifth roots of $1 + j$

28. sixth roots of $-1 - j$

In Exercises 29–32, solve the given equations. Express your answers in rectangular form.

29. $x^3 = -j$

30. $x^3 = 125j$

31. $x^6 - 64j = 0$

32. $x^6 - 1 = j$

Solve Exercises 33–40.

33. ***Electronics*** Ohm's law for alternating current states that for a current with voltage V, current I, and impedance Z, $V = IZ$. If the current is $12\underline{/-23^\circ}$ and the impedance is $9\underline{/42^\circ}$, what is the voltage?

34. ***Electronics*** If an ac circuit has a voltage of $20\underline{/30^\circ}$ and a current of $5\underline{/40^\circ}$, what is the impedance?

35. ***Electronics*** The voltage divider rule in ac circuits is

$$V_x = \frac{Z_x E}{Z_T}$$

where V_x is the voltage across one or more elements in series that have total impedance Z_x, E is the total voltage appearing across the series circuit, and Z_T is the total impedance of the series circuit. Find the voltage across the element V_x given that $Z_x = 4\underline{/-90^\circ}\,\Omega$, $E = 100\underline{/0^\circ}\,V$, and $Z_T = 4\underline{/-90^\circ} + 3\underline{/0^\circ}\,\Omega$.

36. ***Electronics*** Use the voltage divider rule in ac circuits to find the voltage across the element V_x given that $Z_x = 6\underline{/0^\circ}\,\Omega$, $E = 50\underline{/30^\circ}\,V$, and $Z_T = 6\underline{/0^\circ} + 9\underline{/90^\circ} + 17\underline{/-90^\circ}\,\Omega$.

37. ***Electronics*** The impedance in a series RLC circuit is given by

$$Z = \frac{(1 + j)^2(1 - j)^2}{(3 - 4j)^2}$$

Evaluate Z by changing the expression to polar form.

38. ***Electronics*** The admittance Y in an ac circuit is measured in siemens (S) and is given by $Y = Z^{-1}$, where Z is the impedance. If $Z = 12 - 5j$, find Y by using

(a) polar form and DeMoivre's formula. (Give the answer in polar form.)

(b) rectangular form.

39. ***Electronics*** The admittance, Y, in siemens (S) of an ac circuit is given by $Y = Z^{-1}$, where Z is the impedance. If $Z = 4.68\underline{/20.56^\circ}\,\Omega$,

(a) find Y by using DeMoivre's formula. (Give the answer in polar form.)

(b) convert your answer in (a) to rectangular form.

40. ***Electronics*** The admittance Y in an ac circuit is given by $Y = Z^{-1}$, where Z is the impedance. If $Y = 0.087 - 0.034j$ S,

(a) find Z by using DeMoivre's formula. (Give the answer in polar form.)

(b) convert your answer in (a) to rectangular form.

[IN YOUR WORDS]

41. Without looking in the text, describe how to multiply or divide two complex numbers in **(a)** exponential form and **(b)** polar form.

42. Explain how to use DeMoivre's formula to find all the n nth roots of the complex number z.

14.6 COMPLEX NUMBERS IN AC CIRCUITS

In direct current (dc) circuits, the basic relation between voltage and current is given by Ohm's law. If V is the voltage across a resistance R, and I is the current flowing through the resistor, then Ohm's law states that

$$V = I R$$

In alternating current (ac) circuits, there is a very similar equation. If V is the voltage across an impedance Z, and I is the current flowing through the impedance, then we have the relationship

$$V = I Z$$

The main difference in these equations is that the dc circuits are expressed as real numbers. Using complex numbers with the ac equations allows them to take the same simple form as the dc equations, except that all quantities are complex numbers. We will examine some of those relationships in this section.

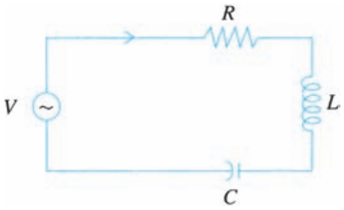


Figure 14.17

Impedance is the opposition to alternating current produced by a resistance R, an inductance L, a capacitance C, or any combination of these. When a sinusoidal voltage V of a given frequency f is applied to a circuit of constant resistance R, constant capacitance C, and constant inductance L, the circuit, like the one in Figure 14.17, is an RLC circuit.

In Figure 14.17, the opposition to the current produced by the inductance is called the *inductive reactance* X_L and the opposition to the current produced by the capacitance is the *capacitive reactance* X_C. The *reactance* X is a measure of how much the capacitance and inductance retard the flow of current in an ac circuit and is the difference between the capacitive reactance and the inductive reactance. Thus,

$$X = X_L - X_C$$

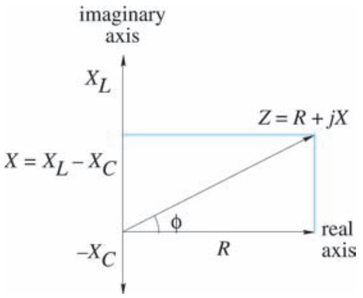


Figure 14.18

The impedance, resistance, and reactance can be represented by a *vector impedance triangle*. The angle ϕ between Z and R is the *phase angle*. In the complex plane, we can represent the resistance along the real axis and the reactance along the imaginary axis as shown in Figure 14.18 Thus,

$$\begin{aligned} Z &= R + jX = R + j(X_L - X_C) \\ &= Z\underline{/\phi} \end{aligned}$$

From our study of complex numbers, you can see that the *magnitude* of the impedance is $|Z| = \sqrt{R^2 + X^2}$ and $\tan\phi = \dfrac{X}{R}$.

When the common circuit components of resistor, inductor, and capacitor are expressed as complex numbers they can be treated in much the same way as pure resistances are treated in Ohm's law. In such a case, Ohm's law is stated as $i = \dfrac{v}{Z}$ where i is the effective current, v is the applied voltage, and Z is the impedance. These circuit elements include normal resistance of a resistor, the "resistance" of a capacitor as it opposes the change in current, and the "resistance" of a capacitor as it opposes the change in voltage. The common circuit elements and their impedances are shown in Table 14.1.

TABLE 14.1

Circuit Element	Symbol and Units	Resistance or Reactance	Impedance Expressed in	
			Rectangular Form	Polar Form
Resistor	 R (ohm)	Resistance: $R = R$	$Z_R = R = R + 0j$	$Z_R = R\angle 0°$
Inductor	L (henry)	Inductive reactance: $X_L = \omega L$	$Z_L = jX_L = 0 + jX_L$	$Z_L = X_L\angle 90°$
Capacitor	 C (farad)	Capacitive reactance: $X_C = \dfrac{1}{\omega C}$	$Z_C = -jX_C = 0 - jX_C$	$Z_C = X_C\angle -90°$

APPLICATION ELECTRONICS

EXAMPLE 14.47

A circuit has a resistance of 8 Ω in series with a reactance of 5 Ω. What are the magnitude of the impedance and its phase angle?

SOLUTION Using a vector impedance triangle, we can see that the impedance can be represented by

$$Z = 8 + 5j$$

The magnitude of the impedance is

$$|Z| = \sqrt{R^2 + X^2} = \sqrt{8^2 + 5^2} = \sqrt{89} \approx 9.43\ \Omega$$

The phase angle ϕ is given by

$$\phi = \arctan \tfrac{5}{8} = 32° \text{ or } 0.56 \text{ rad}$$

APPLICATION ELECTRONICS

EXAMPLE 14.48

In the *RLC* circuit in Figure 14.18, $R = 60\ \Omega$, $X_L = 75\ \Omega$, $X_C = 30\ \Omega$, and $I = 2.25$ A. Find (a) the magnitude and phase angle of Z, and (b) the voltage across the circuit.

EXAMPLE 14.48 (Cont.)

SOLUTIONS

(a) The reactance $X = X_L - X_C = 75 - 30 = 45\ \Omega$. The impedance $Z = 60 + 45j$, so the magnitude of the impedance is

$$|Z| = \sqrt{R^2 + X^2} = \sqrt{60^2 + 45^2} = 75\,\Omega$$

The phase angle ϕ is given by

$$\phi = \arctan \tfrac{45}{60} \approx 36.87° \text{ or } 0.64 \text{ rad}$$

(b) Since the current is 2.25 A and the impedance is 75 Ω, the voltage $V = IZ = (2.25)(75) = 168.75$ V, or approximately 169 V.

An alternating current is produced by a coil of wire rotating through a magnetic field. If the angular velocity of the wire is ω, the capacitive reactance X_C and the inductive reactance X_L are given by the formulas

$$X_C = \frac{1}{\omega C} \qquad \text{and} \qquad X_L = \omega L$$

Since $\omega = 2\pi f$, where f is the frequency of the current, these are also expressed as

$$X_C = \frac{1}{2\pi f C} \qquad \text{and} \qquad X_L = 2\pi f L$$

From these formulas you can see that if C, L, and either ω or f are known, the reactance of the circuit may be determined.

APPLICATION ELECTRONICS

EXAMPLE 14.49

If $R = 40\ \Omega$, $L = 0.1$ H, $C = 50\ \mu$F, and $f = 60$ Hz, determine the impedance and phase difference between the current and voltage.

SOLUTION Converting $C = 50\ \mu$F to farads produces $C = 50 \times 10^{-6}$ F.

$$X_C = \frac{1}{2\pi f C} = \frac{1}{2\pi(60)(50 \times 10^{-6})} \approx 53\ \Omega$$

$$X_L = 2\pi f L = 2\pi(60)(0.1) \approx 38\ \Omega$$

$$Z = R + jX = R + j(X_L - X_C) = 40 + (38 - 53)j = 40 - 15j$$

$$|Z| = \sqrt{R^2 + X^2} = \sqrt{40^2 + (-15)^2} \approx 42.7\ \Omega$$

$$\phi = \arctan \tfrac{-15}{40} \approx -20.6°$$

So, the impedance is 42.7 Ω and the phase difference is −20.6°.

In the study of dc circuits, you learn that if several resistors are connected in series, their total resistance is the sum of the individual resistances. Thus, if two resistors R_1 and R_2 are connected in series, their total resistance, R, equals $R_1 + R_2$. If R_1, R_2, and R_3 are connected in series, then $R = R_1 + R_2 + R_3$.

If the resistors are connected in parallel, the relationship is more complicated. The total resistance is the reciprocal of the sum of the reciprocals of the resistances. What this means is that if two resistors R_1 and R_2 are connected in parallel, then the total resistance $R = \dfrac{1}{\dfrac{1}{R_1} + \dfrac{1}{R_2}}$. This can be simplified to $R = \dfrac{R_1R_2}{R_1 + R_2}$. If three resistors, R_1, R_2, and R_3, are connected in parallel, then $R = \dfrac{1}{\dfrac{1}{R_1} + \dfrac{1}{R_2} + \dfrac{1}{R_3}}$. This can be rewritten as $R = \dfrac{R_1R_2R_3}{R_1R_2 + R_1R_3 + R_2R_3}$

Corresponding formulas hold for complex impedances. Thus, if two impedances Z_1 and Z_2 are connected in series, the total impedance is $Z = Z_1 + Z_2$. If there are three impedances in series, Z_1, Z_2, and Z_3, then $Z = Z_1 + Z_2 + Z_3$.

If complex impedances are connected in parallel, we then have the more complicated formulas. If two impedances are connected in parallel, then the total impedance $Z = \dfrac{1}{\dfrac{1}{Z_1} + \dfrac{1}{Z_2}} = \dfrac{Z_1Z_2}{Z_1 + Z_2}$. If three impedances are connected in parallel, then the total impedance is $Z = \dfrac{1}{\dfrac{1}{Z_1} + \dfrac{1}{Z_2} + \dfrac{1}{Z_3}} = \dfrac{Z_1Z_2Z_3}{Z_1Z_2 + Z_1Z_3 + Z_2Z_3}$.

APPLICATION ELECTRONICS

EXAMPLE 14.50

If $Z_1 = 2 + 3j\,\Omega$ and $Z_2 = 1 - 6j\,\Omega$, what is the total impedance if these are connected (a) in series and (b) in parallel?

SOLUTIONS

(a) If they are connected in series,

$$\begin{aligned} Z &= Z_1 + Z_2 \\ &= (2 + 3j) + (1 - 6j) \\ &= 3 - 3j\,\Omega \end{aligned}$$

EXAMPLE 14.50 (Cont.)

(b) If they are connected in parallel,

$$\begin{aligned} Z &= \frac{1}{\dfrac{1}{Z_1} + \dfrac{1}{Z_2}} = \frac{Z_1 Z_2}{Z_1 + Z_2} \\ &= \frac{(2 + 3j)(1 - 6j)}{(2 + 3j) + (1 - 6j)} \\ &= \frac{20 - 9j}{3 - 3j} && \text{multiply by } \frac{3 + 3j}{3 + 3j} \\ &= \frac{87 + 33j}{18} \\ &= \frac{87}{18} + \frac{33}{18} j\,\Omega = \frac{29}{6} + \frac{11}{6} j\,\Omega \end{aligned}$$

APPLICATION ELECTRONICS

EXAMPLE 14.51

If $Z_1 = 3.16\angle 18.4^\circ\,\Omega$ and $Z_2 = 4.47\angle 63.4^\circ\,\Omega$, what is the total impedance if these are connected (a) in series and (b) in parallel?

SOLUTIONS

(a) If they are connected in series,

$$Z = Z_1 + Z_2$$

Since we cannot add complex numbers in polar form, we need to change these to rectangular form.

$$\begin{aligned} Z_1 &= 3.16(\cos 18.4^\circ + j \sin 18.4^\circ) \approx 2.998 + 0.997j \\ Z_2 &= 4.47(\cos 63.4^\circ + j \sin 63.4^\circ) \approx 2.001 + 3.997j \\ Z &= (2.998 + 0.997j) + (2.001 + 3.997j) \\ &= 4.999 + 4.994j\,\Omega \\ &\approx 7.07\angle 45.0^\circ\,\Omega \end{aligned}$$

(b) If they are in parallel,

$$Z = \frac{Z_1 Z_2}{Z_1 + Z_2} = \frac{(3.16\angle 18.4^\circ)(4.47\angle 63.4^\circ)}{7.07\angle 45.0^\circ}$$

Using our knowledge of multiplying and dividing complex numbers in polar form, we get

$$\begin{aligned} Z &= \frac{(3.16)(4.47)}{7.07}\angle 18.4^\circ + 63.4^\circ - 45.0^\circ \\ &= 2.0\angle 36.8^\circ\,\Omega \end{aligned}$$

As you can see, some problems are easier to work if you use the rectangular form and some are easier if you use the polar form.

EXERCISE SET 14.6

In Exercises 1–12, find the total impedance if the given impedances are connected (a) in series and (b) in parallel.

1. $Z_1 = 2 + 3j$, $Z_2 = 1 - 5j$

2. $Z_1 = 4 - 7j$, $Z_2 = -3 + 4j$

3. $Z_1 = 1 - j$, $Z_2 = 3j$

4. $Z_1 = 2 + j$, $Z_2 = 4 - 3j$

5. $Z_1 = 2.19\angle 18.4^\circ$, $Z_2 = 5.16\angle 67.3^\circ$

6. $Z_1 = 2\sqrt{3}\angle 30^\circ$, $Z_2 = 2\angle 120^\circ$

7. $Z_1 = 3\sqrt{5}\angle \frac{\pi}{7}$, $Z_2 = 1.5\angle 0.45$

8. $Z_1 = 2.57\angle 0.25$, $Z_2 = 1.63\angle 1.38$

9. $Z_1 = 4 + 3j$, $Z_2 = 3 - 2j$, $Z_3 = 5 + 4j$

10. $Z_1 = 3 - 4j$, $Z_2 = 1 + 5j$, $Z_3 = -2j$

11. $Z_1 = 1.64\angle 38.2^\circ$, $Z_2 = 2.35\angle 43.7^\circ$, $Z_3 = 4.67\angle -39.6^\circ$

12. $Z_1 = 0.15\angle 0.95$, $Z_2 = 2.17\angle 1.39$, $Z_3 = 1.10\angle 0.40$

In Exercises 13–18, use the formula $V = IZ$ to determine the missing unit.

13. $I = 4 - 3j$ A, $Z = 8 - 15j\,\Omega$

14. $V = 5 + 5j$ V, $I = 4 + 3j$ A

15. $Z = 1 - j\,\Omega$, $I = 1 + j$ A

16. $V = 3 + 4j$ V, $Z = 5 - 12j\,\Omega$

17. $V = 7\angle 36.3^\circ$ V, $I = 2.5\angle 12.6^\circ$ A

18. $V = 3\angle 1.37$ V, $Z = 4\angle 0.16\ \Omega$

In Exercises 19–24, determine the inductive reactance, capacitive reactance, impedance, and phase difference between the current and voltage.

19. $R = 38\ \Omega$, $L = 0.2$ H, $C = 40\ \mu$F, and $f = 60$ Hz

20. $R = 35\ \Omega$, $L = 0.15$ H, $C = 80\ \mu$F, and $f = 60$ Hz

21. $R = 20\ \Omega$, $L = 0.4$ H, $C = 60\ \mu$F, and $f = 60$ Hz

22. $R = 12\ \Omega$, $L = 0.3$ H, $C = 250\ \mu$F, and $\omega = 80$ rad/s

23. $R = 28\ \Omega$, $L = 0.25$ H, $C = 200\ \mu$F, and $\omega = 50$ rad/s

24. $R = 2\,000\ \Omega$, $L = 3.0$ H, $C = 0.5\ \mu$F, and $\omega = 1000$ rad/s

In the RLC circuits in Exercises 25–28, find (a) the magnitude and phase angle of Z and (b) the voltage across the circuit.

25. $R = 75\ \Omega$, $X_L = 60\ \Omega$, $X_C = 40\ \Omega$, and $I = 3.50$ A

26. $R = 40\ \Omega$, $X_L = 30\ \Omega$, $X_C = 60\ \Omega$, and $I = 7.50$ A

27. $R = 3.0\ \Omega$, $X_L = 6.0\ \Omega$, $X_C = 5.0\ \Omega$, and $I = 2.85$ A

28. $R = 12.0\ \Omega$, $X_L = 11.4\ \Omega$, $X_C = 2.4\ \Omega$, and $I = 0.60$ A

Solve Exercises 29–34.

29. ***Electronics*** Figure 14.19 indicates part of an electrical circuit. Kirchhoff's law implies that $I_2 = I_1 + I_3$. If $I_1 = 7 + 2j$ A and $I_2 = 9 - 7j$ A, find I_3.

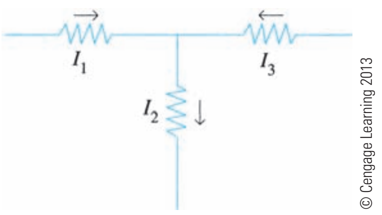


Figure 14.19

30. ***Electronics*** *Resonance* in an *RLC* circuit occurs when $X_L = X_C$. Under these conditions, the current and voltage are in phase. Resonance is required for the tuning of radio and television receivers. If $\omega = 100$ rad/s and $L = 0.500$ H, what is the value of C if the system is in resonance?

31. ***Electronics*** What is the frequency in hertz for a circuit in resonance if $L = 2.5$ H and $C = 20.0\ \mu$F?

32. ***Electronics*** The *admittance* Y is the reciprocal of the impedance. If $Z = 3 - 2j\ \Omega$, what is the admittance?

33. ***Electronics*** If the admittance is $4 + 3j\ \Omega$, what is the impedance?

34. ***Electronics*** The *susceptance* B of an ac circuit of reactance X and impedance Z is defined as $B = \frac{X}{Z^2}j$. If the reactance is 4 Ω and $Z = 8 + 7j$, what is the susceptance?

[IN YOUR WORDS]

35. Write an electrical application that requires you to use complex numbers and some of the calculation techniques of this chapter. Give your problem to a classmate and see if he or she understands and can solve your problem. Rewrite the problem as necessary to remove any difficulties encountered by your classmate.

36. Suppose that you are applying for a job with an electronics company. One question on the application form asks you to explain how to perform a specific type of calculation involving complex numbers. It then asks you to give a specific example that uses this type of calculation. Write your answer using complete sentences.

CHAPTER 14 REVIEW

IMPORTANT TERMS AND CONCEPTS

Changing complex numbers from
- Polar form to rectangular form
- Rectangular form to polar form

Complex numbers
- Absolute value
- Addition
- Conjugate
- Division
- Exponential form
- Imaginary part
- Multiplication
- Polar form
- Power
- Real part
- Rectangular form
- Rectangular part
- Roots
- Subtraction
- Trigonometric form

Complex plane
DeMoivre's formula
Imaginary unit
Pure imaginary number

REVIEW EXERCISES

In Exercises 1–6, simplify each number in terms of j.

1. $\sqrt{-49}$

2. $\sqrt{-36}$

3. $\sqrt{-54}$

4. $(2j^3)^3$

5. $\sqrt{-2}\sqrt{-18}$

6. $\sqrt{-9}\sqrt{-27}$

In Exercises 7–14, perform the indicated operation. Express each answer in the form a + bj.

7. $(2 - j) + (7 - 2j)$
8. $(9 + j)(4 + 7j)$
9. $(5 + j) - (6 - 3j)$
10. $\dfrac{1}{11 - j}$
11. $(6 + 2j)(-5 + 3j)$
12. $\dfrac{2 - 5j}{6 + 3j}$
13. $(4 - 3j)(4 + 3j)$
14. $\dfrac{-4}{\sqrt{3} + 2j}$

In Exercises 15–20, graph each complex number and change each number from rectangular form to polar form, or vice versa.

15. $9 - 6j$
16. $-8 + 2j$
17. $4 - 4j$
18. $4 \text{ cis } 60°$
19. $6.5\underline{/2.3}$
20. $10\underline{/20°}$

In Exercises 21–28, perform the indicated operation and express each answer in rectangular form.

21. $(2 \text{ cis } 30°)(5 \text{ cis } 150°)$
22. $\dfrac{3 \text{ cis } 20°}{6 \text{ cis } 80°}$
23. $\left(3 \text{ cis } \frac{5\pi}{4}\right)^{14}$
24. $(324 \text{ cis } 225°)^{1/5}$
25. $\left(3\underline{/\tfrac{\pi}{4}}\right)\left(9\underline{/\tfrac{2\pi}{3}}\right)$
26. $44\underline{/125°} \div 4\underline{/97°}$
27. $\left(2\underline{/\tfrac{\pi}{6}}\right)^{12}$
28. $\left(2048\underline{/330°}\right)^{1/11}$

For Exercises 29–32, find all roots and express in rectangular form.

29. $\sqrt[3]{-j}$
30. $\sqrt[4]{16}$
31. $\sqrt{16 \text{ cis } 120°}$
32. $\sqrt[3]{27j}$

In Exercises 33–36, change each number to the exponential form and perform the indicated operation.

33. $(3 + 2j)(5 - j)$
34. $\dfrac{4 - 7j}{3 + j}$
35. $(5 + 3j)^5$
36. $(-7 - 2j)^{1/3}$

Solve Exercises 37–40.

37. ***Physics*** A force vector in the complex plane is given by $7.3 - 1.4j$. What is the magnitude and direction (argument) of this vector?

38. ***Electronics*** Given an *RLC* circuit with $R = 3.0\ \Omega$, $X_L = 7.0\ \Omega$, $X_C = 4.5\ \Omega$, and $I = 1.5$ A, **(a)** what is the magnitude and phase angle of Z, and **(b)** what is the voltage across the circuit?

39. ***Electronics*** What are the inductive reactance, capacitive reactance, impedance, and phase difference between the current and voltage, if $R = 55\ \Omega$, $L = 0.3$ H, $C = 50\ \mu$F, and $f = 60$ Hz?

40. ***Electronics*** If $Z_1 = 3 + 5j$ and $Z_2 = 6 - 3j$, then what is Z, if Z_1 and Z_2 are connected **(a)** in series and **(b)** in parallel?

CHAPTER 14 TEST

1. Write $\sqrt{-80}$ in terms of j.

2. Change $7 - 2j$ from rectangular form to polar form.

3. Change $8 \text{ cis } 150°$ from polar form to rectangular form.

In Exercises 4–11, perform the indicated operation. Express each answer in the form $a + bj$.

4. $(5 + 2j) + (8 - 6j)$

5. $(-5 + 2j) - (8 - 6j)$

6. $(2 + 3j)(4 - 5j)$

7. $\dfrac{6 + 5j}{-3 - 4j}$

8. $(7 \text{ cis } 75°)(2 \text{ cis } 105°)$

9. $\dfrac{4 \text{ cis } 115°}{3 \text{ cis } 25°}$

10. $(9 \text{ cis } \frac{2\pi}{3})^{5/2}$

11. $(27\underline{/129°})^{1/3}$

Solve Exercises 12–16.

12. Find all four roots of $\sqrt[4]{j}$.

13. If $Z_1 = 4 + 2j$ and $Z_2 = 5 - 3j$, what is Z, if Z_1 and Z_2 are connected in **(a)** series and **(b)** parallel?

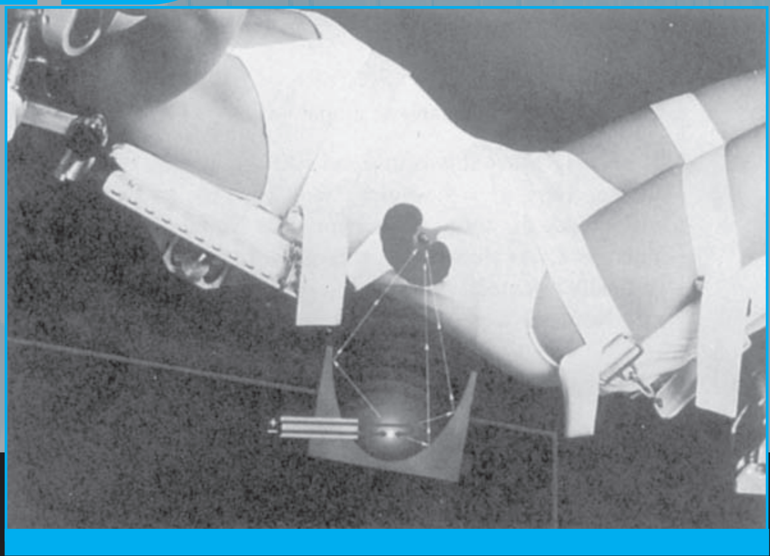
14. The admittance, Y, in an ac circuit is given by $Y = Z^{-1}$, where Z is the impedance. If $Z = 9 - 3j\,\Omega$, what is the admittance?

15. The voltage in a certain ac circuit is $5 + 2j$ V and the current is $3 - 4j$ A. Use Ohm's law, $V = IZ$, to determine the impedance Z.

16. Two ac circuits with impedances Z_1 and Z_2 have total impedance Z, where $Z = Z_1 + Z_2$ if they are connected in series, and $Z = \dfrac{1}{\dfrac{1}{Z_1} + \dfrac{1}{Z_2}}$ if they are connected in parallel.

If $Z_1 = 9 + 3j\,\Omega$ and $Z_2 = 7 - 2j\,\Omega$, determine the total impedance if they are connected **(a)** in series and **(b)** in parallel.

15 AN INTRODUCTION TO PLANE ANALYTIC GEOMETRY



Courtesy of Dornier Medical Systems Inc.

A lithotripter is a medical instrument. Its design is based on an ellipse. In Section 15.4, we will see how the ellipse makes this instrument effective.

Many scientific and technical applications make use of four special curves: the circle, parabola, ellipse, and hyperbola. These are four of the seven conic sections that can be formed by a plane intersecting a cone. (The others are a point, a line, and two intersecting lines.)

One medical instrument that uses an ellipse is a lithotripter, an instrument that uses sound waves to break up kidney stones. An ellipse has two foci. (Foci is the plural of focus.) A sound wave transmitter is placed at one focus and the patient's kidney is placed at the other. The elliptical shape of the lithotripter causes sound waves to be reflected off the ellipse and focused on the kidney stones.

In Chapter 4, the rectangular coordinate system was introduced. We began by graphing equations on the Cartesian coordinate system. The process of using algebra to describe geometry is called analytic geometry. In this chapter, we first review the line and then look at circles, parabolas, ellipses, and hyperbolas.

OBJECTIVES

AFTER COMPLETING THIS CHAPTER, YOU WILL BE ABLE TO:

- Calculate the distance between two points.
- Find the midpoint between two points.
- Determine the slope of a line given two points on the line.
- Determine the slope of a line given its angle of inclination.
- Find the angle of inclination of a line given its slope.
- Determine the slope of a line perpendicular to a given line.
- Write the equation of a line using the slope-intercept form or the point-slope form.
- Solve applications involving linear relationships.
- Write the equation of a circle, ellipse, parabola, or hyperbola from given information.
- Write an equation in standard form given the equation of any of the conic sections.
- Graph any conic section and determine all features of interest.
- Identify by inspection whether a given second-degree equation represents a circle, ellipse, parabola, or hyperbola.
- Write polar equations of conic sections from given information.
- Solve applied problems involving any of the conic sections.

15.1 BASIC DEFINITIONS AND STRAIGHT LINES

We introduced graphing in Chapter 4 and have worked with graphs in almost every chapter since then. In this section, we will introduce some new ideas and review some of those that were introduced earlier.

If we have two points in a plane, we often want to know some things about them. Besides the location of the points, some of the things we might want to know include the distance between them, the point halfway between them, and the equation of the line through those two points, including its slope and intercepts. Let's look at these and a few other ideas, one at a time.

THE DISTANCE FORMULA

For most of this discussion we will be using two points, P_1 and P_2. Point P_1 has coordinates (x_1, y_1) and P_2 has coordinates (x_2, y_2). Using the Pythagorean theorem, we can find the distance d between these points.

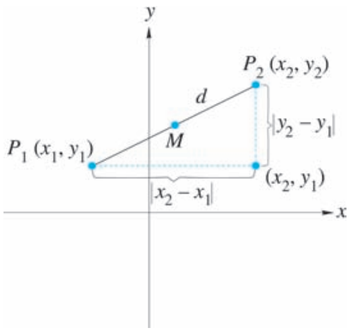


Figure 15.1

THE DISTANCE FORMULA

The distance d between any two points $p_1(x_1, y_1)$ and $p_2(x_2, y_2)$ in a plane is given by the **distance formula**:

$$d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

If there are several points, the symbol $d(P_1, P_2)$ would represent the two points that are being used (see Figure 15.1).

EXAMPLE 15.1

Find the lengths of the sides of the triangle in Figure 15.2. Its vertices are $A(-1, -3)$, $B(6, 1)$, and $C(2, -7)$.

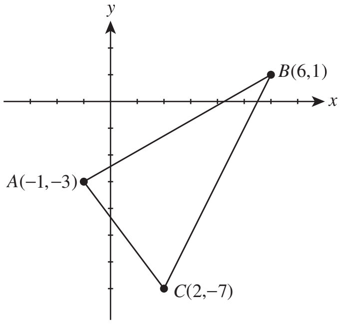


Figure 15.2

SOLUTIONS

$$\begin{aligned}
d(C,A) &= \sqrt{(-1-2)^2 + (-3+7)^2} = \sqrt{(-3)^2 + 4^2} \\
&= \sqrt{9 + 16} = \sqrt{25} = 5 \\
d(B,A) &= \sqrt{(-1-6)^2 + (-3-1)^2} = \sqrt{(-7)^2 + (-4)^2} \\
&= \sqrt{49 + 16} = \sqrt{65} \\
d(C,B) &= \sqrt{(6-2)^2 + (1+7)^2} = \sqrt{4^2 + 8^2} \\
&= \sqrt{16 + 64} = \sqrt{80} = 4\sqrt{5}
\end{aligned}$$

THE MIDPOINT FORMULA

If you want to find the coordinates of the point M halfway between two points P_1 and P_2, then you use the *midpoint formula*.

THE MIDPOINT FORMULA

The coordinates of the midpoint M between any two points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ are given by the **midpoint formula:**

$$M = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

The midpoint formula gives you the coordinates of the point on the segment $\overline{P_1P_2}$ halfway between P_1 and P_2.

EXAMPLE 15.2

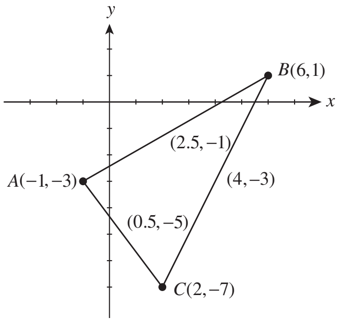


Figure 15.3

Find the midpoints of the sides of the triangle with vertices $A(-1, -3)$, $B(6, 1)$, and $C(2, -7)$. Note that this is the same triangle we used in Example 15.1.

SOLUTIONS

$$M(A, B) = \left(\frac{-1+6}{2}, \frac{-3+1}{2}\right) = \left(\frac{5}{2}, \frac{-2}{2}\right) = \left(\frac{5}{2}, -1\right)$$

$$M(A, C) = \left(\frac{-1+2}{2}, \frac{-3-7}{2}\right) = \left(\frac{1}{2}, \frac{-10}{2}\right) = \left(\frac{1}{2}, -5\right)$$

$$M(B, C) = \left(\frac{6+2}{2}, \frac{1-7}{2}\right) = \left(\frac{8}{2}, \frac{-6}{2}\right) = (4, -3)$$

The midpoints are shown in Figure 15.3.

LENGTHS OF IRREGULAR CURVES

In Section 3.4 we learned how to use the trapezoidal rule and Simpson's rule to approximate the area of irregular shapes. We can use similar techniques with the distance formula to approximate the lengths of irregular curves. This technique involves many calculations and can be made easier if you use a spreadsheet.

Consider a curve like the one in Figure 15.4a. You need to place the curve on a coordinate system and identify the coordinates of several points. In Figure 15.4a the points are $P_0(x_0, y_0)$, $P_1(x_1, y_1)$, $P_2(x_2, y_2)$, $P_3(x_3, y_3)$, . . . , $P_{n-1}(x_{n-1}, y_{n-1})$, and $P_n(x_n, y_n)$.

Next, find the distance between each consecutive pairs of points. Notice that this is the length of each of the line segments in Figure 15.4b:

$$d_1 = \sqrt{(x_1 - x_0)^2 + (y_1 - y_0)^2}$$

$$d_2 = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

$$d_3 = \sqrt{(x_3 - x_2)^2 + (y_3 - y_2)^2}$$

$$\vdots$$

$$d_n = \sqrt{(x_n - x_{n-1})^2 + (y_n - y_{n-1})^2}$$

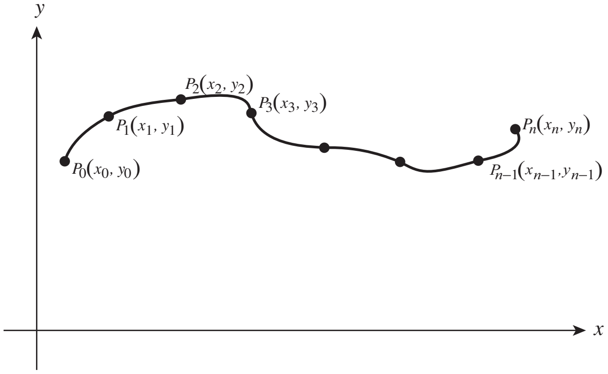


Figure 15.4a

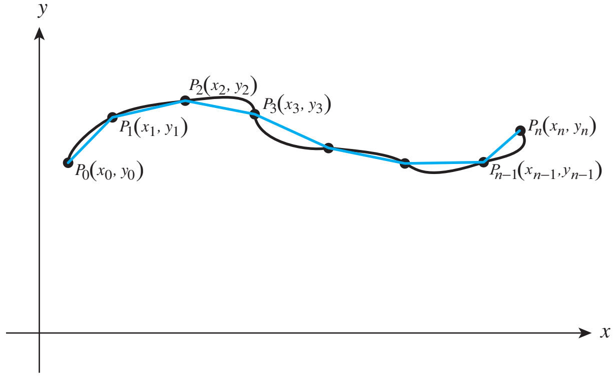


Figure 15.4b

The sum $\sum_{i=1}^{n} d_i = d_1 + d_2 + d_3 + \cdots + d_{n-1} + d_n$ is an approximation of the length L of this curve.

APPLICATION CONSTRUCTION

EXAMPLE 15.3

In Example 3.24 we determined the amount of asphalt that was needed for the parking lot in Figure 15.5a. Now we need to (a) determine the amount of concrete needed for putting a curb along the lake and (b) how much time it will take to extrude the curb.

SOLUTIONS

(a) To determine the amount of concrete needed for the curb we first need to determine the linear length of the curb. The actual curb will be a prism with a base that is $6'' \times 6'' = 36$ in.$^2 = \frac{36}{144}$ ft^2. For the linear length we will put a sketch of the parking lot on a coordinate system with the lower left-hand corner of the lot at the origin as in Figure 15.5b. The vertical lines are 50 ft apart and the coordinates at the lake end of each line segment are given in the figure.

The approximate length is $L = d_1 + d_2 + d_3 + \cdots + d_{12}$ where

$$d_1 = \sqrt{(50 - 0)^2 + (150 - 123)^2} = \sqrt{50^2 + 27^2} = \sqrt{3229}$$
$$d_2 = \sqrt{(100 - 50)^2 + (175 - 150)^2} = \sqrt{50^2 + 25^2} = \sqrt{3125}$$
$$d_3 = \sqrt{(150 - 100)^2 + (190 - 175)^2} = \sqrt{50^2 + 15^2} = \sqrt{2725}$$
$$d_4 = \sqrt{(200 - 150)^2 + (196 - 190)^2} = \sqrt{50^2 + 6^2} = \sqrt{2536}$$
$$d_5 = \sqrt{(250 - 200)^2 + (196 - 196)^2} = \sqrt{50^2 + 0^2} = \sqrt{2500}$$
$$d_6 = \sqrt{(300 - 250)^2 + (190 - 196)^2} = \sqrt{50^2 + (-6)^2} = \sqrt{2536}$$
$$d_7 = \sqrt{(350 - 300)^2 + (180 - 190)^2} = \sqrt{50^2 + (-10)^2} = \sqrt{2600}$$
$$d_8 = \sqrt{(400 - 350)^2 + (179 - 180)^2} = \sqrt{50^2 + (-1)^2} = \sqrt{2501}$$

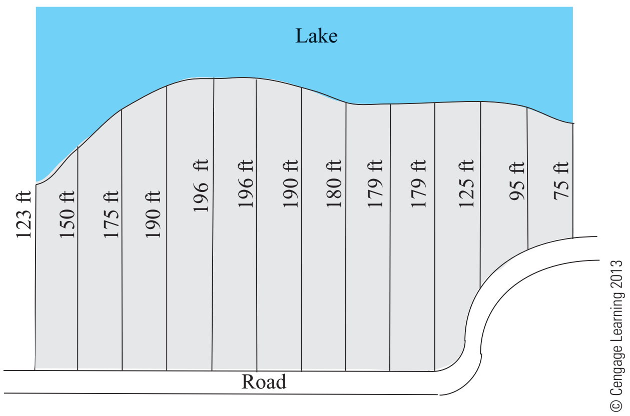


Figure 15.5a

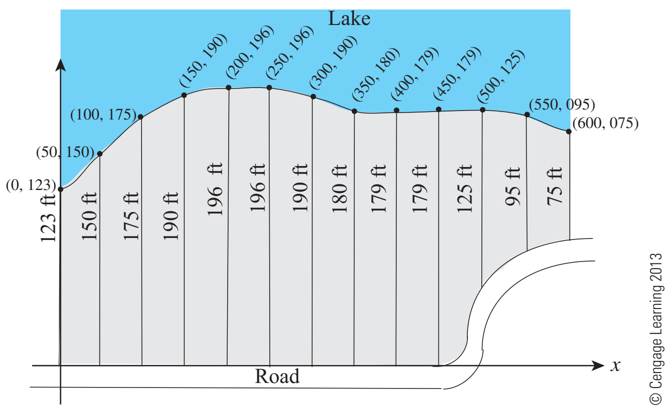


Figure 15.5b

EXAMPLE 15.3 (Cont.)

$$d_9 = \sqrt{(450-400)^2+(179-179)^2} = \sqrt{50^2+(0)^2} = \sqrt{2500}$$
$$d_{10} = \sqrt{(500-450)^2+(125-179)^2} = \sqrt{50^2+(-54)^2} = \sqrt{5416}$$
$$d_{11} = \sqrt{(550-500)^2+(95-125)^2} = \sqrt{50^2+(-30)^2} = \sqrt{3400}$$
$$d_{12} = \sqrt{(600-550)^2+(75-95)^2} = \sqrt{50^2+(-20)^2} = \sqrt{2900}$$

Thus,

$$\begin{aligned} L &= \sqrt{3229}+\sqrt{3125}+\sqrt{2725}+\sqrt{2536}+\sqrt{2500}+\sqrt{2536} \\ &\quad + \sqrt{2600}+\sqrt{2501}+\sqrt{2500}+\sqrt{5416}+\sqrt{3400}+\sqrt{2900} \\ &\approx 652.40 \end{aligned}$$

We will need about 652 linear feet of curbing.
The total volume of the curb will be $\frac{36}{144} \times 652.40 \approx 163.10 \text{ ft}^3 \approx 6.04 \text{ yd}^3$.

(b) An experienced crew operating a curb extrusion machine can cover 250–300 linear feet per day. You should budget for 2.5 days to get this curb installed.

If you are trying to approximate the length of a curve described by a function f the process is somewhat easier since the y-values are given by $f(x)$. Here the values of $d_1, d_2, \ldots, d_{n-1}$ become

$$d_1 = \sqrt{(x_1-x_0)^2+(f(x_1)-f(x_0))^2}$$
$$d_2 = \sqrt{(x_2-x_1)^2+(f(x_2)-f(x_1))^2}$$
$$\vdots$$
$$d_n = \sqrt{(x_n-x_{n-1})^2+(f(x_n)-f(x_{n-1}))^2}$$

EXAMPLE 15.4

Find the length of the curve $f(x) = 2x^3 - 10x^2 + 14x + 1$ from $x = 0$ to $x = 3$.

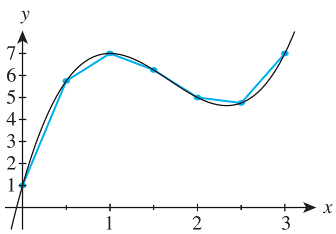


Figure 15.6

SOLUTION We will use six line segments to approximate the length of this curve. If we pick the x-values of these endpoints equally spaced from $x = 0$ to $x = 3$ we have $x_0 = 0, x_1 = 0.5, x_2 = 1.0, \cdots, x_6 = 3$. The curve and the six segments are shown in Figure 15.6.

The approximate length is $L = d_1 + d_2 + d_3 + \cdots + d_6$, where

$$\begin{aligned} d_1 &= \sqrt{(x_1-x_0)^2+(f(x_1)-f(x_0))^2} = \sqrt{(0.5-0)^2+(5.75-1)^2} \\ &= \sqrt{22.8125} \\ d_2 &= \sqrt{(x_2-x_1)^2+(f(x_2)-f(x_1))^2} = \sqrt{(1.0-0.5)^2+(7-5.75)^2} \\ &= \sqrt{1.8125} \\ &\vdots \\ d_6 &= \sqrt{(x_6-x_5)^2+(f(x_6)-f(x_5))^2} = \sqrt{(3.0-2.5)^2+(7-4.75)^2} \\ &= \sqrt{5.3125} \end{aligned}$$

Thus,

$$L = \sqrt{22.8125} + \sqrt{1.8125} + \sqrt{0.8125} + \sqrt{1.8125} + \sqrt{0.3125} + \sqrt{5.3125}$$
$$\approx 11.23$$

The graph of $f(x) = 2x^3 - 10x^2 + 14x + 1$ from $x = 0$ to $x = 3$ is about 11.23 units long.

SLOPE

As we have discussed before, the slope of a line measures the steepness of the line. The slope also indicates whether, as values of x increase, a line is rising (positive slope), falling (negative slope), or constant (slope of 0). A formula for determining the slope of the line through two points follows.

SLOPE OF A LINE

If a line through two points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ is not vertical, then the **slope**, m, of this line is

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

All lines except vertical lines have a slope. The slope of a vertical line is undefined. The slope of a horizontal line is 0.

EXAMPLE 15.5

What is the slope of the line through the points $A(2, -5)$ and $B(-4, 7)$?

SOLUTION

$$m = \frac{7 - (-5)}{-4 - 2} = \frac{7 + 5}{-4 - 2} = \frac{12}{-6} = -2$$

Any line that is not parallel to the x-axis must eventually cross the x-axis. The angle measure in a positive direction from the x-axis to a line is called the **angle of inclination** of the line. The inclination of a line parallel to the x-axis is defined as 0. The inclination provides us with an alternative definition for the slope. If α is the inclination that a line makes with the x-axis, as shown in Figures 15.7a and 15.7b, then the slope is

$$m = \tan \alpha, \qquad 0^\circ \le \alpha < 180^\circ \qquad \text{or} \qquad 0 \le \alpha < \pi$$

Of course, since $\tan 90^\circ$ and $\tan \frac{\pi}{2}$ are undefined, the inclination of a vertical line is undefined.

NOTE Many technical areas have their own terms that refer to the slope of a line or surface. For example, construction workers talk about the *pitch* of a roof, truck drivers refer to the *grade* of a hill, and, in some fields, *declination* refers to a line with negative slope.

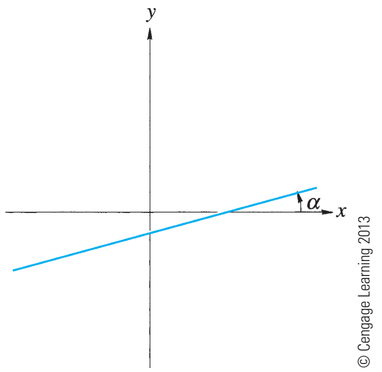


Figure 15.7a

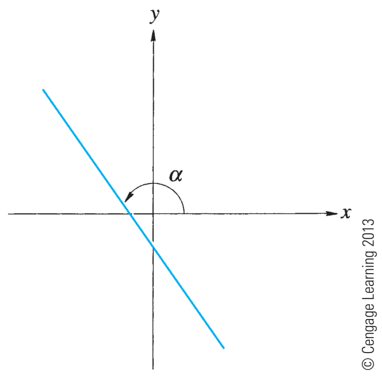


Figure 15.7b

EXAMPLE 15.6

What is the slope of a line with an inclination of (a) 30°, (b) 0.85 rad, and (c) 115°?

SOLUTIONS

(a) $m = \tan 30° \approx 0.5773503$

(b) $m = \tan 0.85 \text{ rad} \approx 1.1383327$

(c) $m = \tan 115° \approx -2.1445069$

EXAMPLE 15.7

The line in Example 15.5 has a slope of -2. What is the angle of inclination of this line?

SOLUTION

$$m = \tan\alpha = -2 \quad \text{so } \alpha = \tan^{-1}(-2) \approx 116.56505°$$

[If you used a calculator, $\tan^{-1}(-2) \approx -63.43498°$. To obtain the inclination, you need to add 180°, since the inclination must be between 0° and 180°, with the result $180° + (-63.43498°) = 116.56505°$.]

If two lines are parallel, they have the same inclination, and so parallel lines have the same slope. If two lines are perpendicular, they intersect at a 90° angle. This means that their inclinations must differ by 90°. It also means that the slope of one of these lines is the negative reciprocal of the other.

SLOPES OF PERPENDICULAR LINES

If m_1 is the slope of a line and m_2 is the slope of a line perpendicular to the first line, then

$$m_1 = -\frac{1}{m_2}$$

This can also be written as $m_1 m_2 = -1$.

CAUTION The above rule does not apply if one of the lines is horizontal because the other line would then be vertical and would not have a slope.

EXAMPLE 15.8

What is the slope of a line that is perpendicular to a line with a slope of -2?

SOLUTION If $m_1 = -2$, then the slope of the perpendicular line is $m_2 = -\frac{1}{-2} = \frac{1}{2}$.

In Chapter 6, we introduced several formulas for the equation of a line. If b represents the y-intercept of a line, then the line crosses the y-axis at the point $(0, b)$. If the slope of this line is m, then the *slope-intercept form for the equation of a line* is

$$y = mx + b$$

EXAMPLE 15.9

Write an equation for a line that has a slope of $\frac{1}{4}$ and a y-intercept of 5.

SOLUTION We have $m = \frac{1}{4}$ and $b = 5$, so $y = \frac{1}{4}x + 5$. Writing this without fractions, we get $4y = x + 20$.

EXAMPLE 15.10

What are the slope and y-intercept of the line $4x + 6y - 3 = 0$?

SOLUTION We need to rewrite this equation in the form $y = mx + b$:

$$\begin{aligned} 4x + 6y - 3 &= 0 \\ 6y &= -4x + 3 \\ y &= -\frac{4}{6}x + \frac{3}{6} \\ &= -\frac{2}{3}x + \frac{1}{2} \end{aligned}$$

The slope is $-\frac{2}{3}$ and the y-intercept is $\frac{1}{2}$.

If you know the slope of a line is m and $P(x_1, y_1)$ is a point on the line, then the equation for the line can be written in the *point-slope form for the equation of a line:*

$$y - y_1 = m(x - x_1)$$

EXAMPLE 15.11

What is the equation of the line through the point $(2, -3)$ and with a slope of 5?

SOLUTION In this example, $x_1 = 2$, $y_1 = -3$, and $m = 5$. Substitute these values into the equation $y - y_1 = m(x - x_1)$:

$$\begin{aligned} y - (-3) &= 5(x - 2) \\ y + 3 &= 5x - 10 \\ y &= 5x - 13 \end{aligned}$$

Every straight line can be written in the *general form for the equation of a line*. This form is represented by the equation:

$$Ax + By + C = 0$$

where A, B, and C represent constants and A and B are not both 0.

EXAMPLE 15.12

What are the slope and intercepts of the line $4x + 3y - 24 = 0$?

SOLUTION We know that the intercepts are the points where the line crosses the axes. The line crosses the x-axis at $(a, 0)$ and the y-axis at $(0, b)$.

If we let $y = 0$, we then get $x = 6$, and if $x = 0$, then $y = 8$; so the x-intercept is $(6, 0)$ and the y-intercept is $(0, 8)$. We will use these two points to determine the following slope:

$$m = \frac{8 - 0}{0 - 6} = \frac{8}{-6} = -\frac{4}{3}$$

The three forms for the equation of a line are summarized in the following box.

DIFFERENT FORMS FOR THE EQUATION OF A LINE

If a line has a slope of m, a y-intercept of $(0, b)$, and $P(x_1, y_1)$ is a point on the line, then the equation for the line can be written in any of these three forms:

Type of Equation	Equation for Line
Point-slope form	$y - y_1 = m(x - x_1)$
Slope-intercept form	$y = mx + b$
General form	$Ax + By + C = 0$

APPLICATION BUSINESS

EXAMPLE 15.13

A mechanic charges \$96 for a job that takes 2 h to finish and \$135 if the job is completed in 5 h. (a) Find a linear equation that describes how much the mechanic should charge for a job of x hours. (b) Use your equation to determine how much should be charged for a job that takes 7 h 15 min.

SOLUTIONS (a) We begin by thinking of the given information as points on a line. We give each point as an ordered pair of the form (x, C), where x represents the number of hours a job takes to complete and C stands for the amount the mechanic charges for this job.

A 2-h job with a charge of \$96 can be thought of as the point $(2, 96)$, and the 5-h job as the point $(5, 135)$. From this information we determine that the slope of this line is

$$m = \frac{135 - 96}{5 - 2} = \frac{39}{3} = 13$$

Using this value for the slope and selecting the point (2, 96), we can use the point-slope form for the equation of a line to determine the needed equation:

$$C - 96 = 13(x - 2)$$
$$C - 96 = 13x - 26$$
or
$$C = 13x + 70$$

Thus the desired equation is $C = 13x + 70$.

(b) A job of 7 h 15 min can be thought of as a 7.25-h job. Substituting 7.25 for x in the equation $C = 13x + 70$, we obtain $C = 13(7.25) + 70 = 94.25 + 70 = 164.25$. The mechanic should charge $164.25 for a job that will take 7 h and 15 min.

APPLICATION ELECTRONICS

EXAMPLE 15.14

The resistance of a circuit element varies directly with its temperature. The resistance at 0°C is found to be 5.00 Ω. Further tests show that the resistance increases by 1.5 Ω for every 1°C increase in temperature. (a) Determine the equation for the resistance R in terms of the temperature T. (b) Use the equation from part (a) to find the resistance at 27°C.

SOLUTIONS

(a) In order to find R in terms of T, we should think of R as the dependent variable. Since resistance varies directly with temperature, the two quantities have a linear relationship. From the given information, we notice that since $R = 5.00$ when $T = 0$, we have the R-intercept. As a result, we shall use the slope-intercept form for the equation of a line:

$$R = mT + b$$
$$R = mT + 5.00$$

To find the slope m, we will use

$$m = \frac{\text{Change in } R}{\text{Change in } T} = \frac{1.5\,\Omega}{1°\text{C}} = 1.5\,\Omega/°C$$

Substituting 1.5 for m in the slope-intercept equation, we are able to complete the equation:

$$R = 1.5T + 5.00$$

(b) To find the resistance at 27°C, we will substitute 27°C for T in the equation from part (a):

$$R = 1.5(27) + 5.00 = 45.5\,\Omega$$

EXERCISE SET 15.1

In Exercises 1–8, find the distance between the given pairs of points.

1. (2, 4) and (7, −9)
2. (−3, 5) and (0, 1)
3. (5, −6) and (−3, −5)
4. (−8, −4) and (6, 10)
5. (12, 1) and (3, −13)
6. (11, −2) and (4, −8)
7. (−2, 5) and (5, 5)
8. (−4, −4) and (−4, 7)

In Exercises 9–16, find the midpoints of the given pairs of points in Exercises 1–8.

In Exercises 17–20, a table gives the coordinates of several points along an irregular curve. Determine the length of each curve. Assume that all measures are in feet.

17.

x	0	5	10	15	20	25	30	35	40	45	50
y	7.2	8.7	9.6	12.8	25.6	34.8	33.7	27.3	19.5	12.4	7.5

18.

x	0	4	8	12	16	20	24	28	32
y	25.3	32.9	39.5	28.7	35.3	46.7	52.1	46.8	39.1

19.

x	0	0.5	1.0	1.5	2.0	2.5	3.0	3.5	4.0
y	23.5	32.9	30.3	27.6	25.3	26.7	21.6	19.8	21.8

20.

x	3	5	7	9	11	13	15	17	19
y	53.4	51.7	50.4	49.3	52.5	54.3	53.2	51.7	50.9

In Exercises 21–24, estimate the length of the curve $y = f(x)$ on the given interval using $n = 4$ and $n = 8$ line segments.

21. $f(x) = x^2 + 3, \quad 0 \le x \le 2$
22. $g(x) = \sqrt{x}, \quad 0 \le x \le 4$
23. $h(x) = \sin x, \quad 0 \le x \le \pi$
24. $j(x) = \frac{10}{x^2}, \quad 1 \le x \le 5$

In Exercises 25–32, find the slopes of the lines through the given pairs of points.

25. (2, 4) and (7, −9)
26. (−3, 5) and (0, 1)
27. (5, −6) and (−3, −5)
28. (−8, −4) and (6, 10)
29. (12, 1) and (3, −13)
30. (11, −2) and (4, −8)
31. (−2, 5) and (5, 5)
32. (−4, −4) and (−4, 7)

In Exercises 33–40, use the point-slope form of the equation for the line to write the equation of the lines through the given points.

33. (2, 4) and (7, −9)
34. (−3, 5) and (0, 1)
35. (5, −6) and (−3, −5)
36. (−8, −4) and (6, 10)
37. (12, 1) and (3, −13)
38. (11, −2) and (4, −8)
39. (−2, 5) and (5, 5)
40. (−4, −4) and (−4, 7)

In Exercises 41–44, find the slopes of the lines with the given inclinations.

41. $45°$ **42.** $175°$ **43.** 0.15 rad **44.** 1.65 rad

In Exercises 45–48, find the inclinations of the lines with the given slopes.

45. 2.5 **46.** 0.30 **47.** -0.50 **48.** -1.475

In Exercises 49–52, determine the slope of a line that is perpendicular to a line of the given slope.

49. 3 **50.** $\frac{2}{5}$ **51.** $-\frac{1}{2}$ **52.** -5

In Exercises 53–64, find the equation of each line with the given properties.

53. Passes through $(2, -5)$ with a slope of 6

54. Passes through $(6, -2)$ with a slope of $-\frac{1}{2}$

55. Passes through $(2, 3)$ and $(4, -2)$

56. Passes through $(-5, 2)$ and $(-3, -4)$

57. Passes through $(-2, -4)$ with an inclination of $60°$

58. Passes through $(5, 1)$ with an inclination of 0.75 rad

59. Has an inclination of $20°$ and a y-intercept of $(0, 3)$

60. Has an inclination of 2.5 rad and a y-intercept of $(0, -2)$

61. Passes through $(2, 5)$ and is parallel to $2x - 3y + 4 = 0$

62. Passes through $(-3, 2)$ and is parallel to $3x + 4y = 12$

63. Passes through $(4, -1)$ and is perpendicular to $2x + 5y = 20$

64. Passes through $(-1, -6)$ and is perpendicular to $8x - 3y = 24$

In Exercises 65–68, determine the slope and intercepts of each line.

65. $3x + 2y = 12$ **66.** $5x - 3y = 15$ **67.** $x - 3y = 9$ **68.** $6x + y = 9$

Solve Exercises 69–78.

69. ***Physics*** The instantaneous velocity v of an object under constant acceleration a during an elapsed time t is given by $v = v_0 + at$, where v_0 is its initial velocity. If an object has an initial velocity of 2.6 m/s and a velocity of 5.8 m/s after 8 s of constant acceleration, write the equation relating velocity to time.

70. ***Physics*** The *coefficient of linear expansion* α is the change in length of a solid due to the change in temperature.

(a) Determine the coefficient of linear expansion of a copper rod that is 1.000 000 cm at 10°C and expands to 1.000 084 cm at 15°C.

(b) Determine the coefficient of linear expansion of a copper rod that is 4.000 000 cm at 10°C and is 4.000 336 cm at 15°C.

(c) The coefficient of linear expansion for any specific solid is a constant. This means that the answers for parts (a) and (b) should have been the same. Assume that the answer in part (a) is correct. What changes need to be made in the way you determine the coefficient of linear expansion?

(d) Determine the coefficient of linear expansion of a glass rod that expands from 72.000 024 cm at 10°C to 72.005 208 cm at 18°C.

71. ***Electronics*** In a dc circuit, when the internal resistance of the voltage source is taken into account, the voltage E is a linear function of the current I and is given by $E = IR + Ir$, where R is the circuit resistance and r is the internal resistance. If the resistance of the circuit $R = 4.0$ Ω and $I = 2.5$ A when $E = 12.0$ V, find r.

72. ***Electronics*** The resistance of a circuit element is found to increase by 0.006 Ω for every 1°C increase in temperature over a wide range of temperatures. If the resistance is 7.000 Ω at 0°C, **(a)** write the equation relating the resistance R to the temperature T, and **(b)** find R, when $T = 17°$C.

73. ***Industrial management*** The production of a certain computer component has fixed costs of \$1,225 and additional costs of \$1.25 per component manufactured.

(a) Write an equation relating the total cost, C, to the number of components produced, n.

(b) What is the cost of producing 20 000 components?

74. ***Accounting*** The linear method for determining the depreciated value, V, of a deductible item is given by

$$V = C\left(1 - \frac{n}{N}\right)$$

where C is the original cost, n is the number of years since the depreciation began, and N is the usable life of the item. Graph V as a function of C when $n = 5$ and $N = 8$.

75. ***Environmental science*** One estimate of the concentration, C, of carbon dioxide (CO_2) in parts per million (ppm) in year t is given by

$$C = 1.44t + 282.88$$

where t is the number of years after 1940.

(a) What was the CO_2 concentration in 1995?

(b) Predict the CO_2 concentration for the year 2050.

(c) Predict the CO_2 concentration for the current year.

76. ***Energy technology*** The width, W, and height, H, of a roof overhang and the latitude, L, of a north-south-facing room determine whether the room will get any summer sunshine. The overhang dimensions that give no summer sunshine are $W = H \tan B$, where $B = L - 23.5°$ is the angle of inclination of the sun at summer solstice. (For houses at either latitude 37° N or 37° S, let $L = 37$.)

(a) Chattanooga, Tennessee is at latitude 31°2′ N. What overhang width on an 8′-high room would ensure that it gets no summer sunshine?

(b) The latitude of Vancouver, British Columbia, is 49° 16′ N. What overhang width on a 2.7-m-high room would ensure that it gets no summer sunshine?

(c) The latitude of Anchorage, Alaska, is 61° 13′ N. Will a room with an overhang that is 8′6″ high and 6′3″ wide get any summer sunshine?

77. ***Environmental science*** In Example 3.21 you found the area of a spill caused by a malfunction in a sewage treatment plant. Because of breezes and the current, only the grey portion of the lake in Figure 15.8 is polluted. To keep the spill from spreading, a series of booms will be placed around the spill. Determine the total length of the booms that will be needed if all distances are in meters.

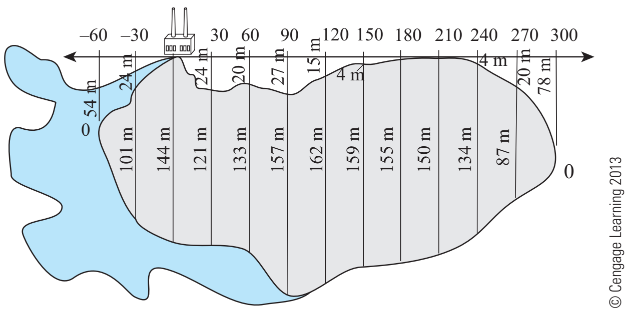


Figure 15.8

78. ***Construction*** A suspension bridge has its roadbed supported by vertical cables that are connected to a cable that is strung between two towers and from each tower to the ground. Each cable from the top of the tower to the ground fits the equation $y = \frac{2}{75}x^{3/2}$. Find the length of one of the cables, if it meets the ground 225 m from the base of the tower.

[IN YOUR WORDS]

79. Describe how to determine if two lines are perpendicular.

80. Compare and contrast the point-slope, slope-intercept, and general forms for the equation of a line. How are they alike and how are they different?

81. Explain the conditions that are needed for the method that is used in this section to approximate the length of a curve to produce the exact length of the curve.

82. Explain why the method used in this section to approximate the length of a curve always produces an answer that is less than the actual length of the curve.

15.2 THE CIRCLE

In Section 15.1, we developed a general equation for a straight line. In this section, we will work with the circle.

A **circle** is the set of all points in a plane that are at some fixed distance from a fixed point in the plane. The fixed point is called the **center** of the circle and the fixed distance is called the **radius**. In general, suppose we have a circle with radius r and center (h, k). An seen in Figure 15.9, if (x, y) is any point on the circle, then the distance from (x, y) to (h, k) must be r. Using the distance formula, we have the equation:

$$\sqrt{(x-h)^2+(y-k)^2}=r$$

Squaring both sides, we have

$$(x-h)^2+(y-k)^2=r^2$$

Thus, we have established the following standard equation of a circle.

STANDARD EQUATION OF A CIRCLE

The *standard equation of a circle* with center at (h, k) and radius r is

$$(x-h)^2+(y-k)^2=r^2$$

EXAMPLE 15.15

The equation $(x-5)^2+(y+3)^2=36$ is a circle with center at $(5, -3)$ and a radius of 6. Remember, since the equation states that $y+3=y-k$, you obtain $k=-3$. Note very carefully the signs of the numbers in the equation and the signs of the coordinates of the center. This circle is shown in Figure 15.10.

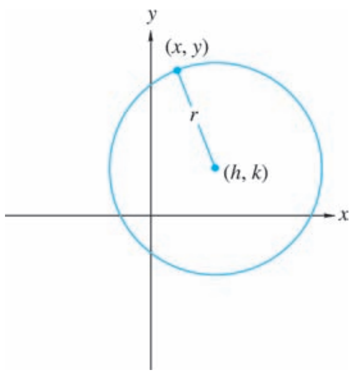


Figure 15.9

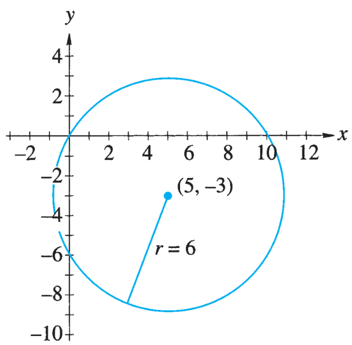


Figure 15.10

EXAMPLE 15.16

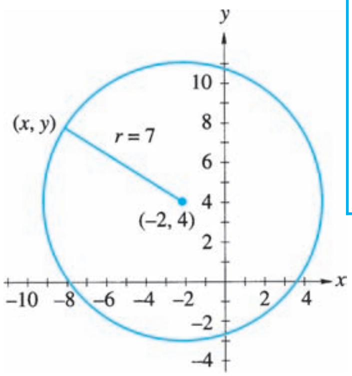


Figure 15.11

Write the equation for the circle with center at $(-2, 4)$ and radius 7.

SOLUTION The circle is shown in Figure 15.11. Since the center is $(-2, 4)$, we have $h = -2$ and $k = 4$, and since the radius is 7, then $r = 7$. So,

$$(x - h)^2 + (y - k)^2 = r^2$$
$$[x - (-2)]^2 + (y - 4)^2 = 7^2$$
$$(x + 2)^2 + (y - 4)^2 = 49$$

Notice that if the center is at the origin, then $h = 0$ and $k = 0$. The equation becomes

$$x^2 + y^2 = r^2$$

If we expand the general equation for a circle, we get

$$(x - h)^2 + (y - k)^2 = r^2$$
$$x^2 - 2hx + h^2 + y^2 - 2ky + k^2 = r^2$$

Since h^2, k^2, and r^2 are all constants, we will let $h^2 + k^2 - r^2 = F$. If we let the constants $-2h = D$ and $-2k = E$, then the equation for a circle can be written as the following general equation.

GENERAL EQUATION OF A CIRCLE

If A, D, E, and F are any real number constants, $A \neq 0$, then the *general equation for a circle* is

$$Ax^2 + Ay^2 + Dx + Ey + F = 0$$

EXAMPLE 15.17

Write $x^2 + y^2 + 4x - 8y + 4 = 0$ in the standard form for the equation of a circle and determine the center and radius.

SOLUTION To convert the equation to the standard form, we need to complete the square. We first write the constant on the right-hand side and group the terms containing x together and then group those containing y. The ○ and □ show the missing constants that we must determine in order to complete the square (see Section 9.5):

$$x^2 + y^2 + 4x - 8y + 4 = 0$$
$$(x^2 + 4x + \bigcirc) + (y^2 - 8y + \square) = -4 + \bigcirc + \square$$

The coefficient of the x-term is 4. If we take half of that and square it, we can add the result, 4, to both sides of the equation. Similarly, half of -8 is -4, and we also add $(-4)^2 = 16$ to both sides. The equation then becomes

$$(x^2 + 4x + 4) + (y^2 - 8y + \boxed{16}) = -4 + 4 + 16$$
$$(x + 2)^2 + (y - 4)^2 = 16$$

Since $16 = 4^2$, the radius is 4, and the center is $(-2, 4)$.

EXAMPLE 15.18

Write the equation $x^2 + y^2 + x - 5y + 2 = 0$ in the standard form for the equation of a circle and determine the center and radius.

SOLUTION Again we will complete the square and use a ○ and a □ to show where the constants need to be placed:

$$x^2 + y^2 + x - 5y + 2 = 0$$
$$(x^2 + x + \bigcirc) + (y^2 - 5y + \square) = -2 + \bigcirc + \square$$
$$\left(x^2 + x + \frac{1}{4}\right) + \left(y^2 - 5y + \frac{25}{4}\right) = -2 + \frac{1}{4} + \frac{25}{4}$$
$$\left(x + \frac{1}{2}\right)^2 + \left(y - \frac{5}{2}\right)^2 = \frac{18}{4}$$

The center is $\left(-\frac{1}{2}, \frac{5}{2}\right)$ and the radius is $\sqrt{\frac{18}{4}} = \dfrac{3\sqrt{2}}{2}$.

EXAMPLE 15.19

Write the equation $4x^2 + 4y^2 - 8x - 24y - 9 = 0$ in the standard form for the equation of a circle and determine the center and radius.

SOLUTION Again, we will complete the square. First, move the constant to the right-hand side and group the x-terms and the y-terms:

$$4x^2 - 8x + 4y^2 - 24y = 9$$

Next, factor the 4 out of the x-terms and the y-terms:

$$4(x^2 - 2x) + 4(y^2 - 6y) = 9$$

We now use a ○ and a □ to indicate the missing constants. Notice that a 4○ and a 4□ are added on the right-hand side of the equation to show where the constants need to be placed:

$$4(x^2 - 2x + \bigcirc) + 4(y^2 - 6y + \square) = 9 + 4\bigcirc + 4\square$$
$$4(x^2 - 2x + 1) + 4(y^2 - 6y + 9) = 9 + 4(1) + 4(9)$$
$$4(x - 1)^2 + 4(y - 3)^2 = 49$$
$$(x - 1)^2 + (y - 3)^2 = \frac{49}{4}$$

The center is $(1, 3)$ and the radius $r = \sqrt{\frac{49}{4}} = \dfrac{7}{2}$.

While we could have started by dividing this entire equation by 4, we waited until the end in order to delay working with fractions. This example uses a process that we will need in the sections on ellipses and hyperbolas.

A circle with an equation of the form $(x - h)^2 + y^2 = r^2$ has its center on the x-axis [at $(h, 0)$] and thus is symmetrical with the x-axis. In the same way, a circle with an equation of the form $x^2 + (y - k)^2 = r^2$ has its center on the y-axis [at $(0, k)$] and is symmetrical with the y-axis.

If $(x - h)^2 + (y - k)^2 = 0$, the circle has a radius of 0. This is sometimes called a *point circle*. If the radius is 1, the circle is often referred to as a *unit circle*.

Finally, if $r^2 < 0$, then the equation does not define a circle and this equation would have no graph on the Cartesian plane.

APPLICATION CONSTRUCTION

EXAMPLE 15.20

A square concrete post is reinforced axially with eight rods arranged symmetrically around a circle, much like those shown in the photo in Figure 15.12a. The cross-section in Figure 15.12b provides a schematic drawing of the post and rods. (a) Determine the equation of the circle using the upper left-hand corner of the post as the origin. (b) Find the coordinates of each rod's location.

SOLUTIONS (a) While it is not specifically stated, we will assume that the eight rods are placed around a circle that shares the center of the square post. This would place the center at the point $(15, -15)$. The radius of the circle is $2.925 + 7.075$ cm $= 10$ cm. Thus, the equation for this circle is $(x - 15)^2 + (y + 15)^2 = 100$. (b) The coordinates of rods 1, 3, 5, and 7 should be fairly easy to determine from the given information. Determining the coordinates of the four remaining rods is a little more difficult. We will show how to determine the coordinates for Rod 2 and then use the symmetry of the rod's placement to determine the coordinates of the other three rods.

The radius of the circle on which the rods are placed is 10 cm. Thus, each rod is located 10 cm from the center of the post. Rod 2 is located directly above a point that is 7.075 cm from the center of the post. Using the Pythagorean theorem, we find that this rod is placed $\sqrt{10^2 - 7.075^2} = \sqrt{100 - 50.055625} = \sqrt{49.944375} \approx 7.067$ cm above the line through Rods 7 and 3. Then the

Figure 15.12a

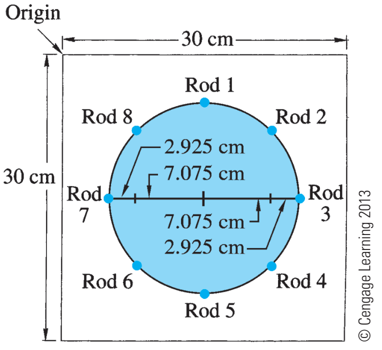


Figure 15.12b

y-coordinate of Rod 2 is $-15 + 7.067 = -7.933$. This places Rod 2 at the coordinates $(22.075, -7.933)$. The eight rods have the coordinates $(15, -5)$, $(22.075, -7.933)$, $(25, -15)$, $(22.075, -22.067)$, $(15, -25)$, $(7.925, -22.067)$, $(5, -15)$, and $(7.925, -7.933)$.

EXERCISE SET 15.2

In Exercises 1–8, find the standard and general form of an equation for the circle with the given center C and radius r.

1. $C = (2, 5), r = 3$
2. $C = (4, 1), r = 8$
3. $C = (-2, 0), r = 4$
4. $C = (0, -7), r = 2$
5. $C = (-5, -1), r = \frac{5}{2}$
6. $C = (-4, -5), r = \frac{5}{4}$
7. $C = (2, -4), r = 1$
8. $C = (5, -2), r = \sqrt{3}$

In Exercises 9–14, give the center and radius of the circle described by each equation. Sketch the circle.

9. $(x - 3)^2 + (y - 4)^2 = 9$
10. $(x - 7)^2 + (y + 5)^2 = 25$
11. $(x + \frac{1}{2})^2 + (y + \frac{13}{4})^2 = 7$
12. $(x + \frac{7}{2})^2 + (y - \frac{7}{3})^2 = \frac{11}{6}$
13. $x^2 + (y - \frac{7}{3})^2 = 6$
14. $(x + 3)^2 + y^2 = 1.21$

In Exercises 15–26, describe the graph of each equation. If it is a circle, give the center and radius.

15. $x^2 + y^2 + 4x - 6y + 4 = 0$
16. $x^2 + y^2 - 10x + 2y + 22 = 0$
17. $x^2 + y^2 + 10x - 6y - 47 = 0$
18. $x^2 + y^2 + 2x - 12y - 27 = 0$
19. $x^2 + y^2 - 2x + 2y + 3 = 0$
20. $x^2 + y^2 + 2x + 2y - 3 = 0$
21. $x^2 + y^2 + 6x - 16 = 0$
22. $x^2 + y^2 - 8y - 9 = 0$
23. $x^2 + y^2 + 5x - 9y = 9.5$
24. $x^2 + y^2 - 7x + 3y + 2.5 = 0$
25. $9x^2 + 9y^2 + 18x - 15y + 27 = 0$
26. $25x^2 + 25y^2 - 10x + 30y + 1 = 0$

Solve Exercises 27–34.

27. ***Physics*** When a particle with mass m and charge q enters a magnetic field of induction β with a velocity v at a right angle to β, the path of the particle is a circle. The radius of the path is $\sigma = \dfrac{mv}{\beta q}$. If a proton of mass 1.673×10^{-27} kg and charge 1.5×10^{-19} C enters a magnetic field with induction 4×10^{-4} T (tesla = kg/C · s) with a velocity of 1.186×10^7 m/s, find the equation of the path of the electron.

28. ***Astronomy*** The earth's orbit around the sun is approximately a circle of radius 1.495×10^8 km. The moon's orbit around earth is approximately a circle of radius 3.844×10^5 km. If the sun is placed at the center of a coordinate system, what is the equation of the earth's orbit?

29. ***Astronomy*** If the sun is placed at the center of a coordinate system and the earth on the positive x-axis, what is the equation of the moon's orbit around the earth?

30. ***Industrial design*** A drafter is drawing two gears that intermesh. They are represented by two intersecting circles, as shown in Figure 15.13. The first circle has a radius of 6 in. and the second has a radius of 5 in. The section of intersection has a maximum depth of 1 in. What is the equation of each circle if the center of the

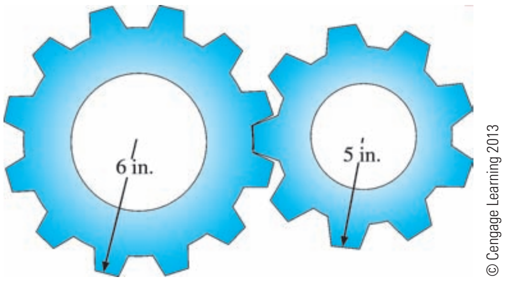


Figure 15.13

first circle is at the origin and the positive x-axis passes through the center of the second circle?

31. ***Industrial design*** In designing a tool, an engineer calls for a hole to be drilled with its center 0.9 cm directly above a specified origin. If the hole must have a diameter of 1.1 cm, find the equation of the hole.

32. ***Industrial engineering*** In designing dies for blanking a sheet metal piece, the configuration shown in Figure 15.14 is used. Write the equation of the small circle with its center at B, using the point marked A as the origin. Use the following values: $r = 3$ in., $a = 3.25$ in., $b = 1.0$ in., and $c = 0.25$ in.

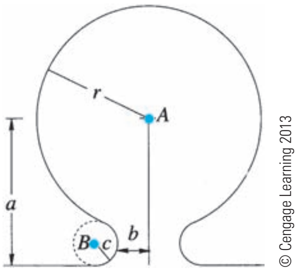


Figure 15.14

33. ***Industrial engineering*** A flywheel that is 26 cm in diameter is to be mounted so that its shaft is 5 cm above the floor, as shown in Figure 15.15. **(a)** Write an equation for the path followed by a point on the rim. Use the surface of the floor as the horizontal axis and the perpendicular line through the center as the vertical axis. **(b)** Find the width of the opening in the floor, allowing a 2-cm clearance on both sides of the wheel.

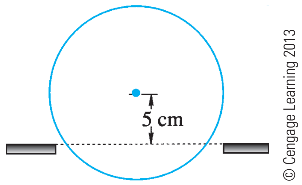


Figure 15.15

34. ***Electricity*** The impedance, Z, in an ac circuit is given by

$$Z = \sqrt{R^2 + (X_L - X_C)^2}$$

where R is the resistance, X_L the inductive reactance, and X_C the inductive capacitance.

(a) If $R = 6\,\Omega$, $X_L = 10\,\Omega$, and $X_C = 12\,\Omega$, determine Z.

(b) If $R = 8\,\Omega$ and $X_C = 12.5\,\Omega$, what are the possible values for Z?

(c) If $Z = 40\,\Omega$ and $X_C = 50\,\Omega$, what are the possible values for X_L

(d) Use your result from **(c)** to graph the relationship between resistance and inductive reactance when $Z = 40\,\Omega$ and $X_C = 50\,\Omega$. (Put R on the horizontal axis.)

[IN YOUR WORDS]

35. Without looking in the text, write the standard equation of a circle. Explain what each constant represents.

36. Describe the procedures you would use to change the general equation of a circle to the standard equation, and vice versa.

15.3 THE PARABOLA

The second curve we will study in this chapter is the parabola. A **parabola** is the set of points in a plane for which the distance from a fixed line is equal to its distance from a fixed point not on the line. The given line is called the **directrix** and the given point is the **focus**.

In Figure 15.16, we have indicated that the line ℓ is the directrix, and F is the focus. The line through F perpendicular to the directrix is called the **axis** of the parabola. Examination of the parabola in Figure 15.16 indicates that the axis of the parabola is the only axis of symmetry for the parabola. The point V on the axis that is halfway between the focus and the directrix is the *vertex*.

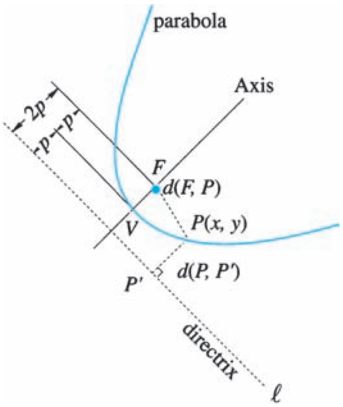


Figure 15.16

Let P be any point on the parabola. Draw a line through P that is perpendicular to the directrix at a point P'. According to the definition, the distance from P to P' is the same as the distance from P to F.

Let's set up a similar drawing on a coordinate system so that we can develop a formula for a parabola. We will let the vertex be at the origin and the focus F at $(0, p)$. Since the vertex is midway between the focus and the directrix, the directrix is the line $y = -p$, as shown in Figure 15.17. If $P(x, y)$ is any point on the parabola, then the distance from F to P is, by the distance formula:

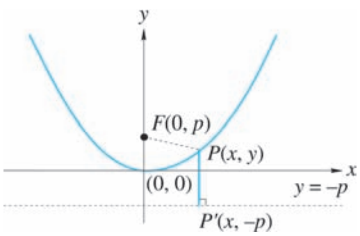


Figure 15.17

$$\sqrt{(x - 0)^2 + (y - p)^2}$$

Since P' is on the directrix, its coordinates are $(x, -p)$. So the distance from P to P' is

$$\sqrt{(x - x)^2 + (y + p)^2}$$

Since these two distances are equal, we have

$$\sqrt{(x - 0)^2 + (y - p)^2} = \sqrt{(x - x)^2 + (y + p)^2}$$

Squaring both sides and simplifying, we obtain

$$\begin{aligned} x^2 + (y - p)^2 &= (y + p)^2 \\ x^2 + y^2 - 2py + p^2 &= y^2 + 2py + p^2 \\ x^2 &= 4py \end{aligned}$$

We have established the following standard equation of a vertical parabola. The term *vertical parabola* is used because the axis is a vertical line. Notice that $|p|$ is the distance from the focus to the vertex.

STANDARD EQUATION OF A VERTICAL PARABOLA

The standard equation of a vertical parabola with focus F at $(0, p)$ and directrix $y = -p$ is

$$x^2 = 4py$$

If $p > 0$, the parabola opens upward, as shown in Figure 15.17.
If $p < 0$, the parabola opens downward.

EXAMPLE 15.21

Find the focus and directrix of the parabola with the equation $x^2 = -10y$. Sketch its graph.

SOLUTION Since $x^2 = 4py$, then $4p = -10$ and $p = -\frac{5}{2}$. The focus is at $\left(0, -\frac{5}{2}\right)$. The equation of the directrix is $y = -p$ or $y = \frac{5}{2}$. The graph is sketched in Figure 15.18. Notice that $p < 0$ and that this parabola opens downward.

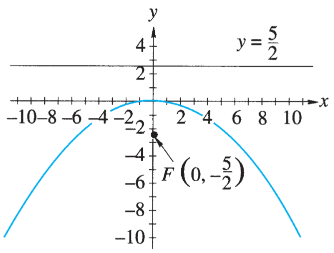


Figure 15.18

EXAMPLE 15.22

Find the equation of the parabola that has its vertex at the origin, opens upward, and passes through $(-5, 9)$.

SOLUTION The general form of the equation is $x^2 = 4py$. Since it opens upward, $p > 0$, and since $(-5, 9)$ is on the parabola, then $(-5)^2 = 4p(9)$. Solving for p, we get $p = \frac{25}{36}$. Substituting this value for p in the equation, we get

$$x^2 = 4\left(\frac{25}{36}\right)y$$

$$x^2 = \frac{25}{9}y$$

$$\text{or} \quad 9x^2 = 25y$$

If we had used the x-axis as the axis of the parabola, and the vertex was still at the origin, the focus would then have been at $(p, 0)$, and the directrix would have had the equation $x = -p$. Using the same method that we used earlier, we would get the following standard equation for a horizontal parabola. It is called a horizontal parabola because the axis is horizontal.

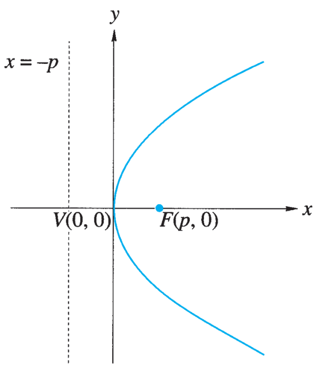


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Figure 15.19

STANDARD EQUATION OF A HORIZONTAL PARABOLA

The standard equation of a horizontal parabola with focus F at $(p, 0)$ and directrix $x = -p$ is

$$y^2 = 4px$$

If $p > 0$, the parabola opens to the right, as shown in Figure 15.19.
If $p < 0$, the parabola opens to the left.

EXAMPLE 15.23

Find the equation of the parabola with its vertex at the origin and focus at (5, 0).

SOLUTION Since the focus is at (5, 0), $p = 5$. A parabola with focus on the x-axis and vertex at the origin is of the form $y^2 = 4px$. Since $p = 5$, the equation is $y^2 = 20x$.

EXAMPLE 15.24

Sketch the graphs of one horizontal and one vertical parabola, each of which has its vertex at the origin and passes through the point $(3, -6)$. Determine the equation for each parabola.

SOLUTION The graphs are in Figure 15.20. One graph is of the form $x^2 = 4py$ and the other is of the form $y^2 = 4px$. Substituting 3 for x and -6 for y in each equation, we get

$$\begin{aligned} x^2 &= 4py & \qquad y^2 &= 4px \\ 3^2 &= 4p(-6) & (-6)^2 &= 4p(3) \\ 9 &= -24p & 36 &= 12p \\ p &= -\frac{9}{24} = -\frac{3}{8} & p &= 3 \\ x^2 &= 4\left(-\frac{3}{8}\right)y & y^2 &= 4(3)x \\ x^2 &= -\frac{3}{2}y & y^2 &= 12x \\ \text{or} \quad 2x^2 &= -3y & & \end{aligned}$$

The vertical parabola has the equation $x^2 = -\frac{3}{2}y$ and the horizontal parabola has the equation $y^2 = 12x$.

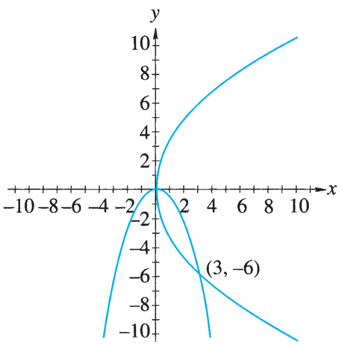

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Figure 15.20

REFLECTIVE PROPERTIES OF PARABOLAS

A three-dimensional object called a *paraboloid of revolution* is formed when a parabola is revolved around its axis of symmetry (see Figure 15.21). Paraboloids of revolution have many uses based on the following two facts:

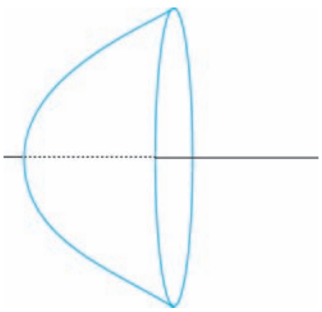

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Figure 15.21

1. Rays entering a parabola along lines parallel to its axis are all reflected through its focus. Many examples exist for the different types of energy rays. Radio telescopes, radar antennae, and satellite television dishes used as downlinks are all examples of paraboloids of revolution. Parabolic reflectors are sometimes used at sports events to pick out specific sounds from background noises. In each of these cases, a ray or wave that is directed toward the dish is reflected off the sides of the paraboloid through its focus.
2. Rays drawn through a parabola's focus are all reflected along lines parallel to the parabola's axis. This uses rays in the reverse of the first property. Applications include flashlights, automobile headlights, search lights, and satellite dishes used as uplinks.

Paraboloids are sometimes used to both send and receive information. For example, a radar transmitter alternately sends and receives waves.

APPLICATION GENERAL TECHNOLOGY

EXAMPLE 15.25

The television satellite dish shown in Figure 15.22a is in the shape of a paraboloid of revolution measuring 0.4 m deep and 2.5 m across at its opening. Where should the receiver be placed in order to pick up the incoming signals?

SOLUTION A cross-section through the axis of this satellite dish has been drawn over the photograph of the dish in Figure 15.22a. Because the cross-section is a parabola, the receiver should be placed at the parabola's focus. We will next determine the location of that focus.

© iStockphoto.com/Christopher Nolan

Figure 15.22a

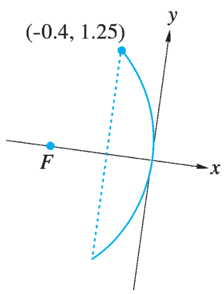

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Figure 15.22b

If we place the drawing of this cross-section on a coordinate system so that the vertex is at the origin and the focus is placed on the x-axis, we get Figure 15.22b. We know that the parabola in Figure 15.22b has an equation of the form $y^2 = 4px$. From the given data, the point $(-0.4, 1.25)$ is on the parabola. This means that $p = \frac{y^2}{4x} = \frac{-1.25^2}{-1.6} \approx -0.977$. Since the focus is at the point $(0, -p)$, this means that we can place the focus 0.977 m from the vertex.

USING A GRAPHING CALCULATOR

Graphing calculators can be used to sketch the graph of any of the curves in this chapter. You must remember, however, that these calculators will only sketch the graphs of functions. This means that you will first need to solve the equation algebraically for y and then graph the one or two equations that result. The next two examples will demonstrate how this is done.

EXAMPLE 15.26

Use a graphing calculator to sketch the graphs of the parabolas (a) $y = \frac{3}{4}x^2$ and (b) $y = \frac{3}{4}x^2 - 7$.

SOLUTIONS As usual, after you turn on the calculator, clear the screen of any previous graphs.

Now you are ready to graph $y = \frac{3}{4}x^2$. Since this equation has already been solved for y, we need only to press the following key sequence on a TI-83:

Y= (3 ÷ 4) X, T, θ, n x^2 [ZOOM] 6 [ZStandard]

The result is the graph in Figure 15.23a.

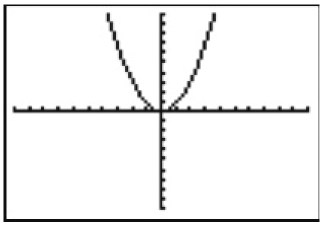


Figure 15.23a

If you want to graph $y = \frac{3}{4}x^2 - 7$, enter the following commands on the $Y2$ line:

Y= (3 ÷ 4) X, T, θ, n x^2 − 7 [GRAPH]

The result is the graph in Figure 15.23b. (If you did not clear the screen after you graphed $y = \frac{3}{4}x^2$, you will see both graphs on your screen.) Notice how the graph of $y = \frac{3}{4}x^2 - 7$ is simply the graph of $y = \frac{3}{4}x^2$ shifted down 7 units.

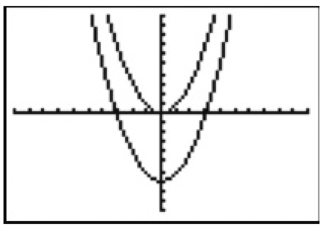


Figure 15.23b

EXAMPLE 15.27

Use a TI-83 or TI-84 graphing calculator to sketch the graph of the circle $x^2 + y^2 + 4x - 8y + 4 = 0$.

SOLUTION This is the same equation we used in Example 15.17. This equation can be simplified to

$$(x + 2)^2 + (y - 4)^2 = 16$$

and represents the equation of a circle with center $(-2, 4)$ and radius $r = 4$.

EXAMPLE 15.26 (Cont.)

In order to use the graphing calculator, we need to solve the equation for y:

$$\begin{aligned}(x+2)^2+(y-4)^2&=16\\(y-4)^2&=16-(x+2)^2\\y-4&=\pm\sqrt{16-(x+2)^2}\\y&=4\pm\sqrt{16-(x+2)^2}\end{aligned}$$

Since this result has two solutions, we will have to graph the two equations $y = 4 + \sqrt{16-(x+2)^2}$ and $y = 4 - \sqrt{16-(x+2)^2}$.

Now, we know that since this circle has its center at $(-2, 4)$ and a radius of 4, the x-values will be from $x = -2 - 4 = -6$ to $x = -2 + 4 = 2$ and the y-values will be from $y = 4 - 4 = 0$ to $y = 4 + 4 = 8$. We will want to set the calculator screen's "range" values so that both graphs appear on the screen. We will let x be over the interval $[-8, 4]$ and y be over the interval $[-2, 10]$.

The graph of $y = 4 + \sqrt{16-(x+2)^2}$ is obtained by first pressing **Y =** and then pressing the key sequence

4 **+** **√** 16 **−** **(** **X, T, θ, n** **+** 2 **)** **x^2** **)** **GRAPH**

The result is the graph in Figure 15.24a. Now graph $y = 4 - \sqrt{16-(x+2)^2}$ by pressing **Y =**, moving the cursor to Y2, and pressing the sequence:

4 **−** **√** 16 **−** **(** **X, T, θ, n** **+** 2 **)** **x^2** **)** **GRAPH**

This graph is displayed over the earlier graph as shown in Figure 15.24b.

This does not look like a circle. First, the top and bottom parts do not touch, and it does not have a circular shape. The fact that parts do not touch is a common error of graphing calculators and computer graphing programs. The noncircular shape is caused by the fact that the calculator screen's ranges for both the x- and y-values are 12 units long. But, the calculator does not have a square screen. In fact, the ratio of the screen's horizontal to vertical size on calculators such as the TI-83 or TI-84 is 3:2. So, if we want this graph of a circle to *look* like a circle, we need to have the ratio of the calculator's screen range of x-values to the range of y-values be 3:2.

Let's leave the y-values at $[-2, 10]$. This is 12 units, so the x-values should cover 18 units. We will change them to $[-10, 8]$. Now press **GRAPH** and the graphs should be redrawn as shown in Figure 15.24c.

The following procedure makes the units on the axes the same length. After you have graphed the two functions and get a graph like the one in Figure 15.24b, press the **ZOOM** button on the top row of the calculator. You should see a screen similar to the one in shown in Figure 15.24d. This zoom menu presents you with nine options. Pressing a **5** selects option 5:ZSquare. This option adjusts the current range values so the width of the dots on the x- and y-axes are equalized.

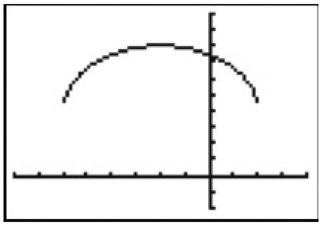


Figure 15.24a

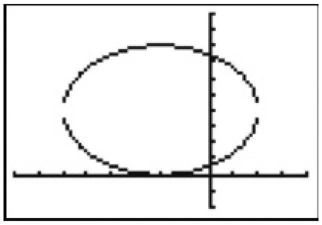


Figure 15.24b

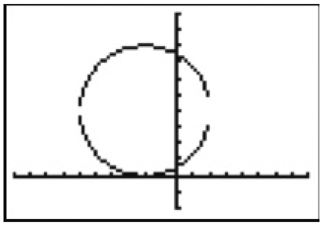


Figure 15.24c

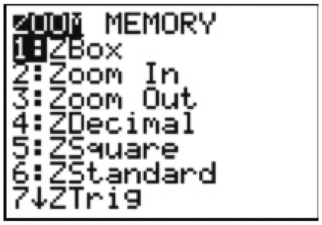


Figure 15.24d

EXAMPLE 15.28

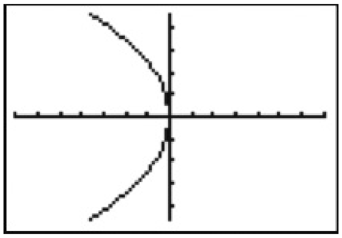


Figure 15.25

Use a graphing calculator to draw the parabola:

$$y^2 = -12x$$

SOLUTION In order to put this in the calculator, we will have to solve for y, getting $y = \pm\sqrt{-12x}$. This will have to be graphed as two separate functions, $y_1 = \sqrt{-12x}$ and $y_2 = -\sqrt{-12x}$. Make sure you graph both y_1 and y_2. The graph of y_1 is the "top half" of the parabola, and the graph of y_2 is the "bottom half." The result is shown in Figure 15.25.

USING A SPREADSHEET

Spreadsheets can be used to sketch the graph of any of the curves in this chapter. However, spreadsheets only sketch the graphs of functions. This means that in some cases you will first need to solve the equation algebraically for y and then graph the one or two equations that result. The next two examples will demonstrate how this is done.

EXAMPLE 15.29

Use a spreadsheet to sketch the graphs of the parabolas (a) $y = \frac{3}{4}x^2$ and (b) $y = \frac{3}{4}x^2 - 7$.

SOLUTIONS As usual, construct a table of values. Column A will be the x-values. Enter `=0.75*A5^2` in Cell B5 and `=0.75*A5^2-7` in Cell C5.

Column A and Column B are graphed as the first series and Column A and Column C are added as the second series. The result is shown in Figure 15.26.

Notice that the graph of $y = \frac{3}{4}x^2 - 7$ is simply the graph of $y = \frac{3}{4}x^2$ shifted down 7 units.

C5 =0.75*A5^2-7

	A	B	C
1	Initial x:		-5
2	Increment for x:		1
3			
4	x	y1	y2
5	-5	18.75	11.75
6	-4	12	5
7	-3	6.75	-0.25
8	-2	3	-4
9	-1	0.75	-6.25
10	0	0	-7
11	1	0.75	-6.25
12	2	3	-4
13	3	6.75	-0.25
14	4	12	5
15	5	18.75	11.75

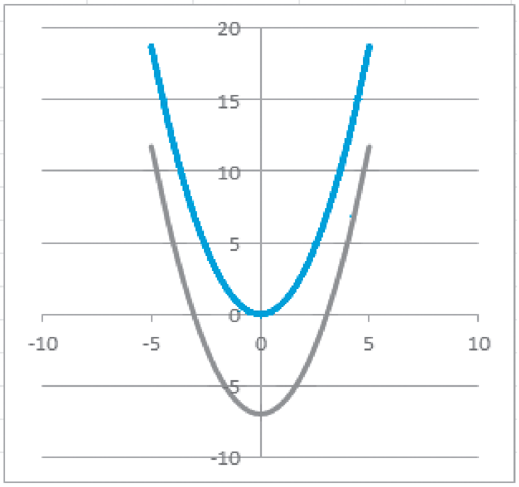


Figure 15.26

EXAMPLE 15.30

Use a spreadsheet to sketch the graph of the circle $x^2 + y^2 + 4x - 8y + 4 = 0$.

SOLUTION This is the same equation we used in Example 15.17. This equation can be simplified to

$$(x + 2)^2 + (y - 4)^2 = 16$$

and represents the equation of a circle with center $(-2, 4)$ and radius $r = 4$. In order to make a table of values, we need to solve the equation for y:

$$\begin{aligned} (x + 2)^2 + (y - 4)^2 &= 16 \\ (y - 4)^2 &= 16 - (x + 2)^2 \\ y - 4 &= \pm\sqrt{16 - (x + 2)^2} \\ y &= 4 \pm \sqrt{16 - (x + 2)^2} \end{aligned}$$

Since this result has two solutions, we will construct a table of values with three columns. Column A will be the x-values. Enter `=4+SQRT(16-(A5+2)^2)` in Cell B5 and `=4-SQRT(16-(A5+2)^2)` in Cell C5. We adjust the initial x and increment to fit the domain of the two functions. The result and the graph of these two curves is shown in Figure 15.27a.

C6 =4-SQRT(16-(A6+2)^2)

	A	B	C
1	Initial x:		-7
2	Increment for x:		1
3			
4	x	y1	y2
5	-7	#NUM!	#NUM!
6	-6	4	4
7	-5	6.645751	1.354249
8	-4	7.464102	0.535898
9	-3	7.872983	0.127017
10	-2	8	0
11	-1	7.872983	0.127017
12	0	7.464102	0.535898
13	1	6.645751	1.354249
14	2	4	4
15	3	#NUM!	#NUM!

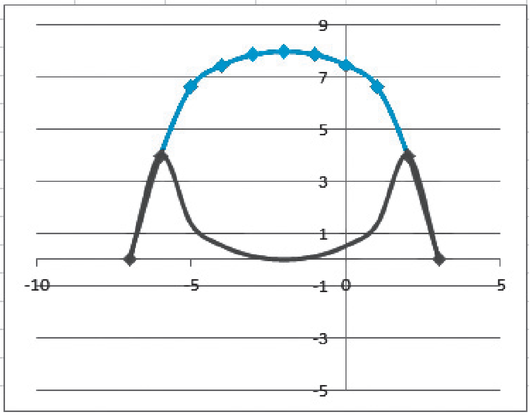


Figure 15.27a

	A	B	C
1	Initial x:		-6
2	Increment for x:		0.25
3			
4	x	y1	y2
5	-6	4	4
6	-5.75	5.391941	2.608059
7	-5.5	5.936492	2.063508
8	-5.25	6.331845	1.668155
9	-5	6.645751	1.354249
10	-4.75	6.904738	1.095262
11	-4.5	7.122499	0.877501
12	-4.25	7.307189	0.692811
13	-4	7.464102	0.535898
14	-3.75	7.596874	0.403126

Figure 15.27b

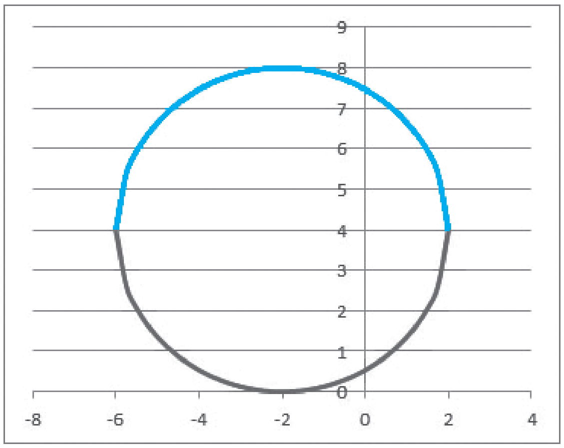


Figure 15.27c

The graph in Figure 15.27a does not look much like a circle! There are at least three reasons. One, there are not enough data points to "smooth" the curve. Using only nine points, it is like trying to build a circular fence with nine 10-ft sections. Second, the scale of the x- and y-axes are not the same and, as a result, the circle looks more like an ellipse. Finally, we attempted to graph some points that are not in the domain of these functions.

Figure 15.27b shows the top portion of another table of values that is more extensive. The graph constructed from that table is more to our liking (see Figure 15.27c).

EXAMPLE 15.31

Use a spreadsheet to draw the parabola:

$$y^2 = -12x$$

SOLUTION In order to put this in a spreadsheet, we will have to solve for y, getting $y = \pm\sqrt{-12x}$.

The table of values is shown in Figure 15.28a. Enter `=(-12*A5)^0.5` in Cell B5 and `=-(-12*A5)^0.5` in Cell C5.

The graph is shown in Figure 15.28b.

	A	B	C
1	Initial x:		-6
2	Increment for x:		0.3
3			
4	x	y1	y2
5	-6	8.485281	-8.48528
6	-5.7	8.270429	-8.27043
7	-5.4	8.049845	-8.04984
8	-5.1	7.823043	-7.82304
9	-4.8	7.589466	-7.58947

Figure 15.28a

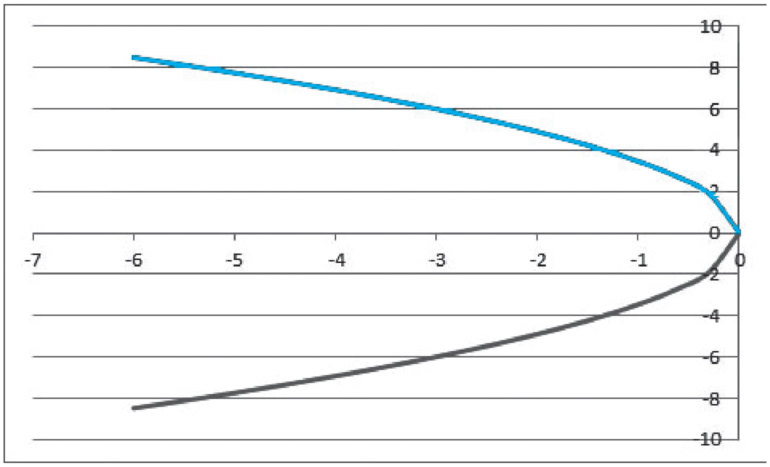


Figure 15.28b

EXERCISE SET 15.3

In Exercises 1–12, determine the coordinates of the focus and the equation of the directrix of each parabola. State the direction in which each parabola opens and sketch each curve.

1. $x^2 = 4y$
2. $x^2 = 12y$
3. $y^2 = -4x$
4. $y^2 = -20x$
5. $x^2 = -8y$
6. $x^2 = -16y$
7. $y^2 = 10x$
8. $y^2 = -14x$
9. $x^2 = 2y$
10. $x^2 = -17y$
11. $y^2 = -21x$
12. $y^2 = 5x$

In Exercises 13–20, determine the equation of the parabola satisfying the given conditions.

13. Focus $(0, 4)$, directrix $y = -4$

14. Focus $(0, -5)$, directrix $y = 5$

15. Focus $(-6, 0)$, directrix $x = 6$

16. Focus $(3, 0)$, directrix $x = -3$

17. Focus $(2, 0)$, vertex $(0, 0)$

18. Focus $(0, -7)$, vertex $(0, 0)$

19. Vertex $(0, 0)$, directrix $x = -\frac{3}{2}$

20. Vertex $(0, 0)$, directrix $y = \frac{3}{4}$

Solve Exercises 21–28.

21. ***Civil engineering*** In a suspension bridge, the main cables are in a parabolic shape. This is because a parabola is the only shape that will bear the total weight load evenly. The twin towers of a certain bridge extend 90 m above the road surface and are 360 m apart, as shown in Figure 15.29. The cables are suspended from the tops of the towers and are tangent to the road surface at a point midway between the towers. What is the height of the cable above the road surface at a point 100 m from the center of the bridge?

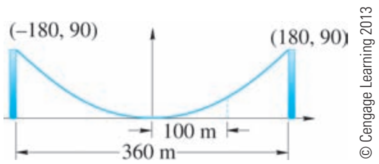


Figure 15.29

22. ***Optics*** A parabolic reflector is a mirror formed by rotating a parabola around its axis. If a light is placed at the focus of the mirror, all the light rays starting from the focus will be reflected off the mirror in lines parallel to the axis. A radar antenna is constructed in the shape of a parabolic reflector. The receiver is placed at the focus. If the reflector has a diameter of 2 m and a depth of 0.4 m, what is the location of the receiver?

23. ***Communications technology*** When rays from a distant source strike a parabolic reflector, they will be reflected to a single point—the focus. A parabolic antenna is used to catch the television signals from a satellite. The antenna is 5 m across and 1.5 m deep. If the receiver is located at the focus, what is its location?

24. ***Energy*** The rate at which heat is dissipated in an electric current is referred to as the power loss. This loss, P, is given by the relationship $P = I^2R$, where I is the current and R is the resistance. If the resistance is 10 Ω, sketch a graph of the power loss as a function of the current.

25. ***Electronics*** For a simple linear resistance R in Ω, the power, P, in W dissipated in the circuit depends on the current, I, in A according to the equation $P = RI^2$. Graph P as a function of I for $0 \leq I \leq 6.0$ when $R = 0.25\,\Omega$.

26. ***Thermodynamics*** The heat, H, in joules (J) produced by a voltage, V, across a heating coil is given by $H = \dfrac{V^2}{R}t$, where t is the time and R the resistance. If $t = 2$ h and $R = 15\,\Omega$, sketch the graph of H as a function of V from $V = 0$ to $V = 120$ V.

27. ***Broadcasting*** A cable television "dish" is 2 ft deep and measures 10 ft across at its opening. A technician must place the receiver at the focus. How far is this from the vertex?

28. ***Astronomy*** Astronomers and optical experts have been exploring the possibility of making mirrors from liquids. When a liquid is spun in a container, the surface of the liquid takes on a parabolic shape. If a liquid metal, such as mercury, is used, then the optical quality is about as good as that of more expensive ground-glass mirrors. The parabolic shape is kept as long as the mirror is kept spinning. The focal length of the mirror is proportional to the square of the rotation period, p. (The rotation period is the reciprocal of the rotational frequency.) If a mirror spinning at 45 rpm has a focal length of 22.1 cm, then

(a) determine the focal length of a mirror spinning at $33\frac{1}{3}$ rpm

(b) determine the focal length of a mirror spinning at 15 rpm

[IN YOUR WORDS]

29. Describe how you can tell by looking at a standard equation for a parabola whether it is a horizontal or a vertical parabola.

30. Explain how to tell from the equation if a parabola opens upward, downward, left, or right.

15.4 THE ELLIPSE

The third curve we will study in this chapter is the ellipse. An **ellipse** is the set of points in a plane that have the sum of their distances from two fixed points a constant. Each of the fixed points is called a **focus** (plural **foci**). The **major axis** of an ellipse is the line segment through the two foci with endpoints on the ellipse. The endpoints of the major axis are called the **vertices**. In Figure 15.30, the foci are labeled F and F' and the vertices are V and V'. The **center** C of the ellipse is the midpoint of the segment joining the foci. The segment through the center and perpendicular to the major axis is called the **minor axis**. The endpoints of the minor axis are on the ellipse. In Figure 15.30, the endpoints of the minor axis are M and M'. The major axis is always longer than the minor axis.

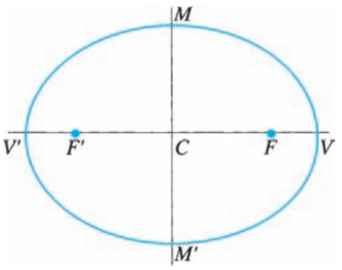


Figure 15.30

If we were to draw an ellipse with its center at the origin and its major axis along the x-axis, we would get a figure like the one in Figure 15.31. We will let the vertices have the coordinates $V(a, 0)$ and $V'(-a, 0)$. The foci will have the coordinates $F(c, 0)$ and $F'(-c, 0)$. If $P(x, y)$ is any point on the ellipse, then the distance from P to F, d_1, plus the distance from P to F', d_2, is a constant, which we will call $2a$. The sum of these distances is $d_1 + d_2 = (a - c) + (a + c) = 2a$.

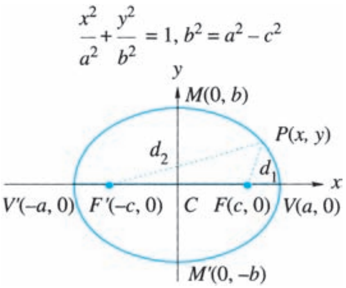


Figure 15.31

The points F, F', and M form an isosceles triangle with the lengths of two equal sides, $\overline{FM}$ and $\overline{F'M}$, being a. Thus, F, M, and C form a right triangle with legs of lengths c and b and hypotenuse of length a. As a result, we see that $a^2 = b^2 + c^2$.

Applying the distance formula, we see that $d_1 = \sqrt{(x-c)^2 + y^2}$ and $d_2 = \sqrt{(x+c)^2 + y^2}$. Since $d_1 + d_2 = 2a$, we have $\sqrt{(x-c)^2 + y^2} + \sqrt{(x+c)^2 + y^2} = 2a$. Using the techniques from Chapter 11 for solving radical equations, we get

$$\sqrt{(x-c)^2 + y^2} = 2a - \sqrt{(x+c)^2 + y^2}$$

$$\left[\sqrt{(x-c)^2 + y^2}\right]^2 = \left[2a - \sqrt{(x+c)^2}\right. \quad \text{Square both sides.}$$

$$(x-c)^2 + y^2 = (2a)^2 - 4a\sqrt{(x+c)^2 + y^2} + (x+c)^2 + y^2$$

$$4a\sqrt{(x+c)^2 + y^2} = 4a^2 + 4cx$$

$$a\sqrt{(x+c)^2 + y^2} = a^2 + cx$$

$$\left[a\sqrt{(x+c)^2+y^2}\right]^2 = \left[a^2+cx\right]^2 \qquad \text{Square again.}$$

$$a^2\left[(x+c)^2+y^2\right] = a^4+2a^2cx+c^2x^2$$

$$a^2x^2+2a^2cx+a^2c^2+a^2y^2 = a^4+2a^2cx+c^2x^2$$

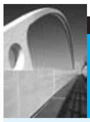
$$(a^2-c^2)x^2+a^2y^2 = a^2(a^2-c^2)$$

But, $a^2 = b^2 + c^2$, so $a^2 - c^2 = b^2$. The equation becomes

$$b^2x^2+a^2y^2 = a^2b^2$$

Dividing both sides by a^2b^2 and simplifying results in one of the standard equations for an ellipse:

$$\frac{x^2}{a^2}+\frac{y^2}{b^2} = 1$$



STANDARD EQUATION: ELLIPSE WITH A HORIZONTAL MAJOR AXIS

The standard equation of an ellipse with center at (0, 0) and the major axis on the x-axis is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2} = 1$$

where $a > b$.
The vertices are $(-a, 0)$ and $(a, 0)$.
The endpoints of the minor axis are $(0, -b)$ and $(0, b)$.
The foci are at $(-c, 0)$ and $(c, 0)$, where $c^2 = a^2 - b^2$.

EXAMPLE 15.32

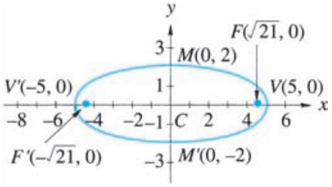


Figure 15.32

Sketch the ellipse $4x^2 + 25y^2 = 100$. Give the coordinates of the vertices, foci, and endpoints of the minor axis.

SOLUTION This equation is not in the standard form, since the right-hand side is not 1. To get the equation into the standard form, we must divide both sides of the equation by 100. This produces the equation

$$\frac{x^2}{25}+\frac{y^2}{4} = 1$$

From this, we see that $a^2 = 25$, so $a = 5$ and $b^2 = 4$ or $b = 2$. Since $c^2 = a^2 - b^2 = 25 - 4 = 21$, we see that $c = \sqrt{21}$. So, the vertices are $V(5, 0)$ and $V'(-5, 0)$, and the foci are $F(\sqrt{21}, 0)$ and $F'(-\sqrt{21}, 0)$. The endpoints of the minor axis are $M(0, 2)$ and $M'(0, -2)$. A sketch of this ellipse is shown in Figure 15.32.

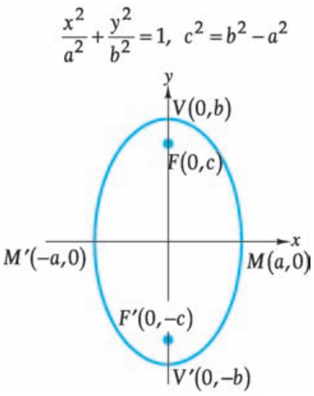


Figure 15.33

Not all ellipses have their major axis along the x-axis. For some situations, it is more convenient if the center is at the origin and the major axis is along the y-axis. When that is the case, the vertices are $V(0, b)$ and $V'(0, -b)$, the foci are $F(0, c)$ and $F'(0, -c)$, and the endpoints of the minor axis are $M(a, 0)$ and $M'(-a, 0)$. In this case, we have the following standard form of the equation for an ellipse. A typical graph of an ellipse with center at the origin and a vertical major axis is shown in Figure 15.33.

STANDARD EQUATION: ELLIPSE WITH A VERTICAL MAJOR AXIS

The standard equation of an ellipse with center at (0, 0) and the major axis on the y-axis is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

where $a < b$.
The vertices are $(0, -b)$ and $(0, b)$.
The endpoints of the minor axis are $(-a, 0)$ and $(a, 0)$.
The foci are at $(0, -c)$ and $(0, c)$, where $c^2 = b^2 - a^2$.

If we put these two types of ellipses together, we see that the equation of an ellipse with center at the origin and foci on a coordinate axis can always be written in the form:

$$\frac{x^2}{p^2} + \frac{y^2}{q^2} = 1 \quad \text{or} \quad q^2x^2 + p^2y^2 = p^2q^2$$

where p and q are both positive. If $p^2 > q^2$, then the major axis is along the x-axis. If $p^2 < q^2$, then the major axis is along the y-axis.

NOTE The x-axis is the major, or longer, axis if the denominator of x^2 is the larger denominator, and the y-axis is the major axis if the denominator of y^2 is the larger denominator.

HINT Remember, c^2 is found by subtracting the smaller denominator from the larger. You may want to remember this as $c^2 = |a^2 - b^2|$.

EXAMPLE 15.33

Discuss and graph the equation $16x^2 + 9y^2 = 144$.

SOLUTION Dividing both sides of the equation by 144, we get the standard equation:

$$\frac{16x^2}{144} + \frac{9y^2}{144} = \frac{144}{144}$$

$$\frac{x^2}{9} + \frac{y^2}{16} = 1$$

EXAMPLE 15.33 (Cont.)

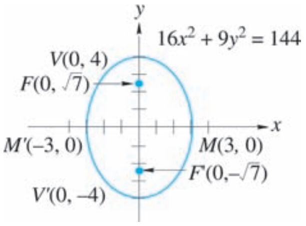


Figure 15.34

So, $a^2 = 9$, $b^2 = 16$, and $c^2 = b^2 - a^2 = 7$. Since $a = 3$ and $b = 4$, we see that $a < b$, so the major axis is along the y-axis. The vertices are $V(0, 4)$ and $V'(0, -4)$, the foci are $F(0, \sqrt{7})$ and $F'(0, -\sqrt{7})$, and the endpoints of the minor axis are $M(3, 0)$ and $M'(-3, 0)$. The sketch of this ellipse is shown in Figure 15.34.

EXAMPLE 15.34

Find the equation of the ellipse with center (0, 0), one focus at (5, 0), and a vertex at (−8, 0).

SOLUTION The second focus is at (−5, 0) and the other vertex is at (8, 0). Since the major axis is horizontal (along the x-axis), the equation is of the form:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

where $c^2 = a^2 - b^2, a > b$.

Since one focus is at (5, 0), $c = 5$ and the fact that one vertex is at (8, 0) tells us that $a = 8$. So, $5^2 = 8^2 - b^2$ or $b^2 = 64 - 25 = 39$, and $b = \sqrt{39}$. The equation of this ellipse is $\dfrac{x^2}{64} + \dfrac{y^2}{39} = 1$.

REFLECTIVE PROPERTIES OF ELLIPSES

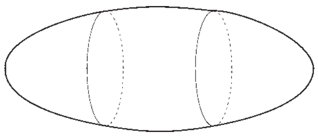


Figure 15.35

A three-dimensional object called a *ellipsoid of revolution* is formed when an ellipse is revolved around one of its axes of symmetry. In Figure 15.35, the ellipse has been revolved around its longer axis.

These ellipsoids of revolution have uses based on the fact that rays leaving or passing through one focus of an ellipse are all reflected by the ellipse so that they pass through the other focus of the ellipse. One architectural application of these ellipsoids of revolution is in the construction of whispering galleries, such as the one in Statuary Hall of the Capitol in Washington, D.C., or St. Paul's Cathedral in London, England. If you position yourself at one focus in a whispering gallery, you will be able to hear anything said at the other focus, regardless of the direction in which the speakers address their remarks.

APPLICATION HEALTHCARE

EXAMPLE 15.35

A lithotripter is a medical device that is used to break up kidney stones with shock waves through water. An ellipsoid is cut in half and careful measurements are used to place the patient's kidney stones at one focus of the ellipse. A source that produces ultra-high-frequency shock waves is placed at the other focus (see Figures 15.36a and 15.36b). The shock waves are reflected by the ellipsoid to the other focus, where they break up the kidney stones.

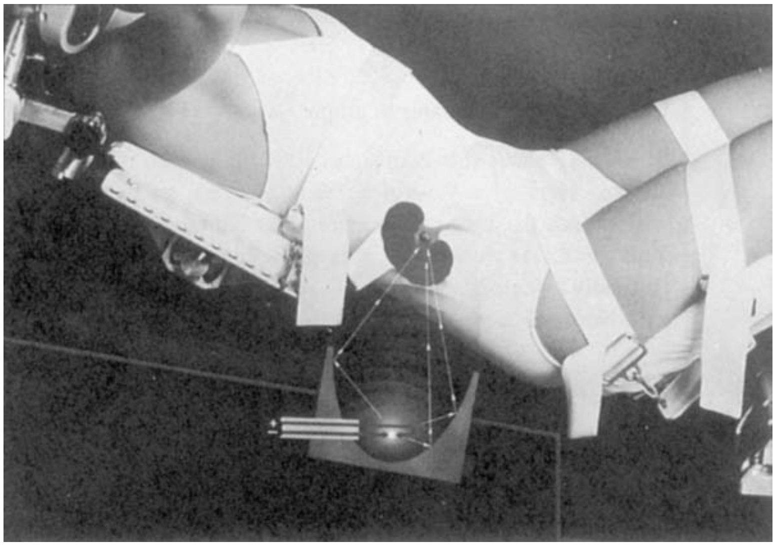


Figure 15.36a

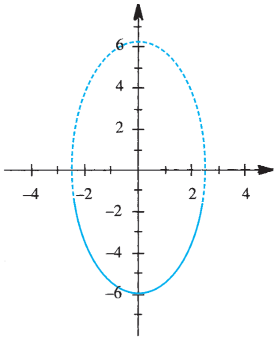


Figure 15.36b

EXAMPLE 15.35 (Cont.) If the endpoints of the major axis are located 6 in. from the center of the ellipse that is rotated to make a lithotripter and one end of the minor axis is 2.5 in. from the center, what are the locations of the lithotripter's foci?

SOLUTION This appears to be a vertically oriented ellipse. If we place the drawing of the cross-section of the ellipse in Figure 15.36a on a coordinate system so that the center is at the origin and the major axis is on the y-axis, we get Figure 15.36b. The solid curve in Figure 15.36b represents the lithotripter. The dotted curve represents the portion of the ellipse that has been removed. From the given information, we know that the ellipse in Figure 15.36b has endpoints on the major axis at $(0, -6)$ and $(0, 6)$, and that the endpoints of the minor axis are $(-2.5, 0)$ and $(2.5, 0)$. From the given data, we know that the desired ellipse has the equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{x^2}{2.5^2} + \frac{y^2}{6^2} = 1$. Thus, $a = 2.5$ and $b = 6$ and, since $c^2 = b^2 - a^2$, we have $c^2 = 6^2 - 2.5^2 = 29.75$ and so $c = \sqrt{29.75} \approx 5.45$. Thus, the foci are at $(0, -5.45)$ and $(0, 5.45)$.

USING A GRAPHING CALCULATOR TO GRAPH AN ELLIPSE

When using a graphing calculator, choose a viewing window that places the endpoints of the major and minor axes in the window. Again, since the calculator will graph only a function, you will need to solve the equation algebraically for y and then graph the two equations that result.

EXAMPLE 15.36 Use a graphing calculator to graph $\frac{x^2}{9} + \frac{y^2}{64} = 1$.

SOLUTION Since this is in standard form, we see that this is a vertically oriented ellipse. Here $a^2 = 9$, so $a = 3$ and $b^2 = 64$, so $b = 8$. A viewing window that will include the entire ellipse would set Xmin $= -3$, Xmax $= 3$, Ymin $= -8$,

EXAMPLE 15.36 (Cont.)

and Y max = 8. As we will see, this does not graph the ellipse so that it appears to be vertically oriented.

Now, solve the equation for y:

$$\frac{x^2}{9} + \frac{y^2}{64} = 1$$
$$\frac{y^2}{64} = 1 - \frac{x^2}{9}$$
$$y^2 = 64\left(1 - \frac{x^2}{9}\right)$$
$$y = \pm\sqrt{64\left(1 - \frac{x^2}{9}\right)}$$
$$= \pm 8\sqrt{1 - \frac{x^2}{9}}$$

We will graph $y_1 = 8\sqrt{1 - \frac{x^2}{9}}$ and $y_2 = -8\sqrt{1 - \frac{x^2}{9}}$. This result is shown in Figure 15.37a.

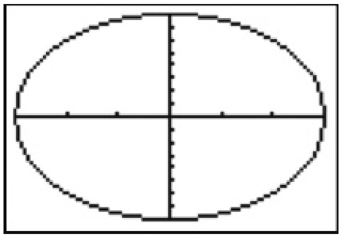


Figure 15.37a

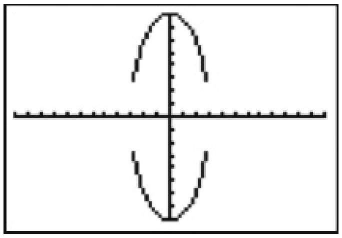


Figure 15.37b

This ellipse does not look as if it is a vertical ellipse. If we use the Zoom Square feature of the calculator by pressing ZOOM 5 [5:ZSquare], we get the result in Figure 15.37b, which does indeed look like a vertical ellipse.

USING A SPREADSHEET TO GRAPH AN ELLIPSE

When using a spreadsheet to graph an ellipse, choose values for x that are in the domain, starting and ending at the endpoints of the horizontal axis. Again, it will be necessary to solve the equation algebraically for y and then graph the two equations that result.

EXAMPLE 15.37

Use a spreadsheet to graph $\frac{x^2}{9} + \frac{y^2}{64} = 1$.

SOLUTION This is the same ellipse that was graphed in Example 15.36. Since this is in standard form, we see that this is a vertically oriented ellipse. Here $a^2 = 9$, so $a = 3$ and $b^2 = 64$, so $b = 8$. The x-axis will need to span the region from $x = -3$ to $x = 3$.

Solving the equation for y we get

$$\frac{x^2}{9} + \frac{y^2}{64} = 1$$
$$y^2 = 64\left(1 - \frac{x^2}{9}\right)$$
$$y = \pm\sqrt{64\left(1 - \frac{x^2}{9}\right)}$$
$$= \pm 8\sqrt{1 - \frac{x^2}{9}}$$

=-8*SQRT(1-A5^2/9)

	A	B	C
1	Initial x:		-3
2	Increment for x:		0.5
3			
4	x	y1	y2
5	-3	0	0
6	-2.5	4.422166	-4.42217
7	-2	5.962848	-5.96285
8	-1.5	6.928203	-6.9282
9	-1	7.542472	-7.54247
10	-0.5	7.888106	-7.88811
11	0	8	-8
12	0.5	7.888106	-7.88811
13	1	7.542472	-7.54247
14	1.5	6.928203	-6.9282
15	2	5.962848	-5.96285
16	2.5	4.422166	-4.42217
17	3	0	0

Figure 15.38a

Enter the first function in Cell B5 as `=8*(1-A5^2/9)^0.5` or `=8*SQRT(1-A5^2/9)` and the second function `=-8*(1-A5^2/9)^0.5` or `=-8*SQRT(1 - A5^2/9)` in Cell C5 (see Figure 15.38a).

Again, the first attempt at a graph, as shown in Figure 15.38b, is not a nice ellipse and does not seem to demonstrate that the graph is vertically oriented. With some modifications, a more accurate representation of the ellipse is shown in Figure 15.38c.

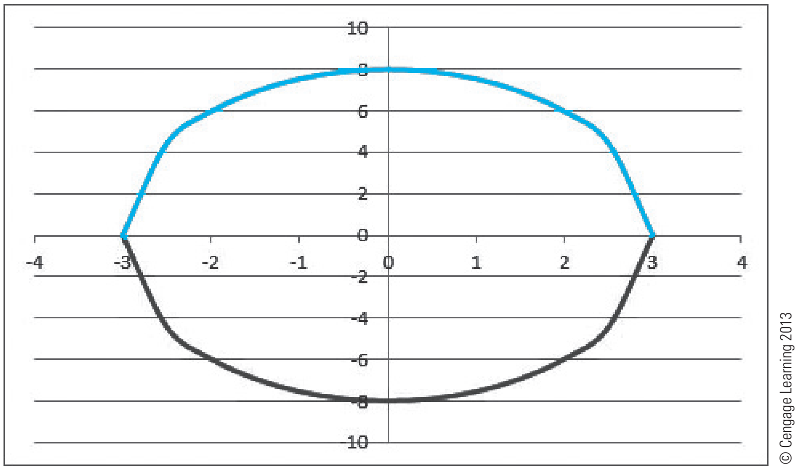


Figure 15.38b

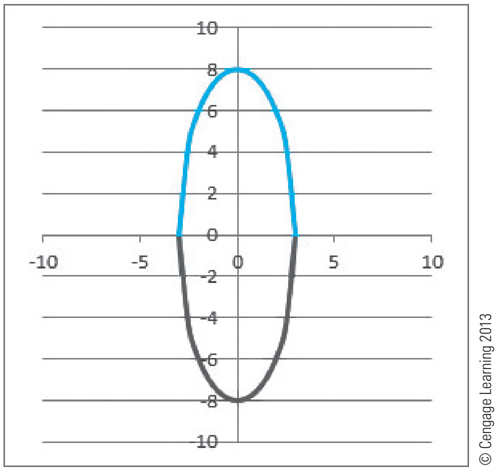


Figure 15.38c

EXERCISE SET 15.4

In Exercises 1–10, find the equation of the ellipse with the stated properties. Each ellipse has its center at (0, 0).

1. Focus at (4, 0), vertex at (6, 0)

2. Focus at (9, 0), vertex at (12, 0)

3. Focus at (0, 2), vertex at (0, -4)

4. Focus at (0, -3), vertex at (0, 5)

5. Focus at (3, 0), vertex at (5, 0)

6. Focus at (0, -2), vertex at (0, -3)

7. Length of the minor axis 6, vertex at (4, 0)

8. Length of the minor axis 10, vertex at (0, -6)

9. Length of major axis 8, length of minor axis 6, and it is a vertical ellipse.

10. Length of major axis 10, length of minor axis 5, and it is a horizontal ellipse.

In Exercises 11–18, give the coordinates of the vertices, foci, and endpoints of the minor axis, and sketch each curve.

11. $\frac{x^2}{4} + \frac{y^2}{9} = 1$

12. $\frac{x^2}{9} + \frac{y^2}{4} = 1$

13. $\frac{x^2}{4} + \frac{y^2}{1} = 1$

14. $\frac{x^2}{16} + \frac{y^2}{36} = 1$

15. $4x^2 + y^2 = 4$

16. $9x^2 + 4y^2 = 36$

17. $25x^2 + 36y^2 = 900$

18. $9x^2 + y^2 = 18$

Solve Exercises 19–26.

19. ***Astronomy*** The orbit of the earth is an ellipse with the sun at one focus. The major axis has a length of 2.992×10^8 km and the ratio of $\frac{c}{a} = \frac{1}{60}$. What is the length of the minor axis? (The ratio $\frac{c}{a}$ is called the **eccentricity**.)

20. ***Civil engineering*** In order to support a bridge, an arch in the shape of the upper half of an ellipse is built. As shown in Figure 15.39, the bridge is to span a river 80 ft wide and the center of the arch is to be 24 ft above the water. Use the water level as the x-axis and the center of the bridge as the y-axis. Write an equation for the ellipse that forms the arch of the bridge.

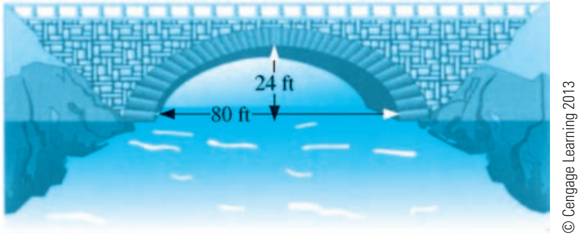


Figure 15.39

21. ***Mechanical engineering*** A circular pipe has an inside diameter of 10 in. One end of the pipe is cut at an angle of 45°, as shown in Figure 15.40. What are the lengths of the major and minor axes of the elliptical opening?

Figure 15.40

22. ***Mechanical engineering*** Elliptical gears are used in some machinery to provide a quick-return mechanism or a slow power stroke (for heavy cutting) in each revolution. Figure 15.41 shows two congruent gears that remain in contact as they rotate around the indicated foci. If the distance between the foci used at the centers of rotation is 6 cm and the shortest distance from a focus to the edge of a gear is 1 cm, what is the length of the minor axis?

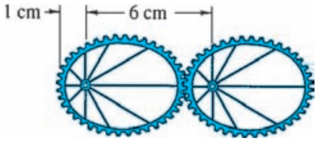


Figure 15.41

23. ***Astronomy*** The comet Kahoutek has major and minor axes of lengths 3 600 and 44 astronomical units. One astronomical unit is about 1.495×10^8 km. What is the eccentricity of the comet's orbit?

24. ***Space technology*** A satellite orbits the earth in an elliptical path with focus at the center of the earth. The altitude of the satellite ranges from 764.4 to 3 017.5 km. If the radius of the earth is approximately 6 373 km, what is the equation of the path of the orbit?

25. ***Mechanical engineering*** Two circular pipes that are each 8.0 in. in diameter are joined at a 45° angle as shown in Figure 15.42. Find the length of the major axis for the elliptical intersection.

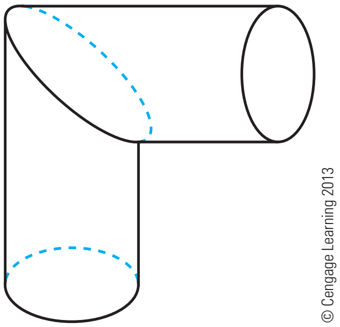


Figure 15.42

26. ***Transportation engineering*** A road passes through a tunnel whose cross-section is a semiellipse 52 ft wide and 12.5 ft high at the center. How tall is the tallest vehicle that can pass under the tunnel at a point 14 ft from the center?

[IN YOUR WORDS]

27. Describe how you can tell by inspecting the standard equation of a ellipse whether it is a horizontal or a vertical ellipse.

28. Explain how to use the endpoints of the axes to locate the foci of an ellipse.

15.5 THE HYPERBOLA

The fourth and last conic section that we will study is the hyperbola. A **hyperbola** is the set of points in a plane for which the difference of the distances of the points from two fixed points is a constant. As with the ellipse, each of the fixed points is called a *focus*. The **transverse axis** is the line segment through the two foci with its endpoints on the hyperbola. The endpoints of the transverse axis are called the **vertices**.

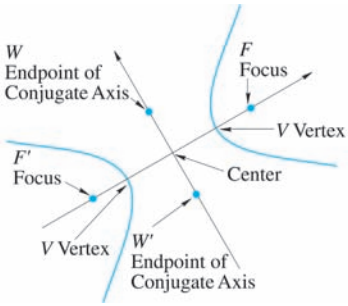


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Figure 15.43

A hyberbola is shown by the blue curve in Figure 15.43. In Figure 15.43, the foci are labeled F and F' and the vertices V and V'. The **center** C of the hyperbola is the midpoint of the foci. The line segment through the center that is perpendicular to the transverse axis is called the **conjugate axis**. The endpoints of the conjugate axis, labeled W and W' in Figure 15.43, are not on the hyperbola. We will learn how to determine the length of the conjugate axis later in this section.

If we draw a hyperbola with its center at the origin and its transverse axis along the x-axis, we get a figure like the one in Figure 15.44. If $P(x, y)$ is any point on the hyperbola, the distance from P to F, d_1, minus the distance from P to F', d_2, is a constant. The difference of the distances is $|d_1 - d_2| = |(c - a) - (c + a)| = 2a, a > 0$.

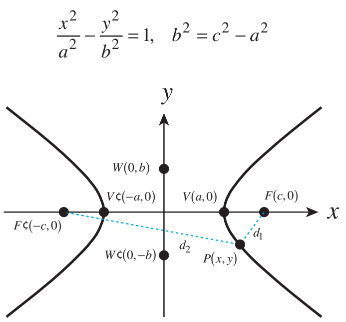

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Figure 15.44

Here we will use the distance formulas $d_1 = \sqrt{(x - c)^2 + y^2}$ and $d_2 = \sqrt{(x + c)^2 + y^2}$. As a result, we compute the difference of the distances as

$$\left|\sqrt{(x - c)^2 + y^2} - \sqrt{(x + c)^2 + y^2}\right| = 2a$$

$$\text{or} \qquad \sqrt{(x - c)^2 + y^2} - \sqrt{(x + c)^2 + y^2} = \pm 2a$$

Using the same technique we used on the ellipse, we get

$$c^2x^2 - a^2x^2 - a^2y^2 = a^2c^2 - a^4$$

$$\text{or} \qquad x^2(c^2 - a^2) - a^2y^2 = a^2(c^2 - a^2)$$

If we let $b^2 = c^2 - a^2$, this can be written as $b^2x^2 - a^2y^2 = a^2b^2$. Dividing both sides by a^2b^2, the equation becomes

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

which is one of the standard equations for a hyperbola. It can be shown that the lines $y = \frac{b}{a}x$ and $y = -\frac{b}{a}x$ are the hyperbola's asymptotes.

STANDARD EQUATION: HYPERBOLA WITH A HORIZONTAL MAJOR AXIS

The standard equation of a hyperbola with center at (0, 0) and the major axis on the x-axis is

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

The vertices are $(-a, 0)$ and $(a, 0)$.
The endpoints of the conjugate axis are $(0, -b)$ and $(0, b)$.
The foci are at $(-c, 0)$ and $(c, 0)$, where $c^2 = a^2 + b^2$.

The lines $y = \frac{b}{a}x$ and $y = -\frac{b}{a}x$ are **asymptotes** of this hyperbola.

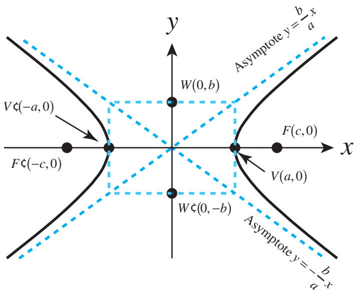

Figure 15.45

The asymptotes provide convenient guidelines for drawing hyperbolas. Asymptotes are easily drawn as the diagonals of a rectangle. The midpoints of the sides of the rectangles are the vertices and the endpoints of the conjugate axis, as shown in Figure 15.45.

If the transverse axis is along the y-axis and the conjugate axis is along the x-axis, the hyperbola has a second standard form.

STANDARD EQUATION: HYPERBOLA WITH A VERTICAL MAJOR AXIS

The standard equation of a hyperbola with center at (0, 0) and the major axis on the y-axis is

$$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$$

The vertices are $(0, -a)$ and $(0, a)$.
The endpoints of the conjugate axis are $(-b, 0)$ and $(b, 0)$.
The foci are at $(0, -c)$ and $(0, c)$, where $c^2 = a^2 + b^2$.

The lines $y = \frac{a}{b}x$ and $y = -\frac{a}{b}x$ are asymptotes of this hyperbola.

CAUTION The formulas relating a, b, and c for the hyperbola are different from the corresponding formulas for the ellipse.

For the ellipse

$c^2 = |a^2 - b^2|$ (See Figure 15.46a) $a > c$ (See Figure 15.46b)

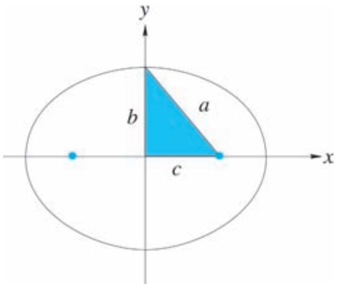

Figure 15.46a

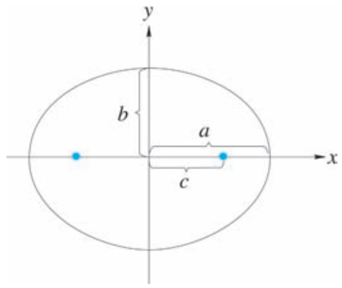

Figure 15.46b

For the hyperbola

$c^2 = a^2 + b^2$ (See Figure 15.47a) $c > a$ (Figure 15.47b)

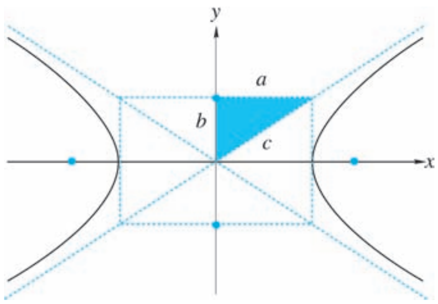

Figure 15.47a

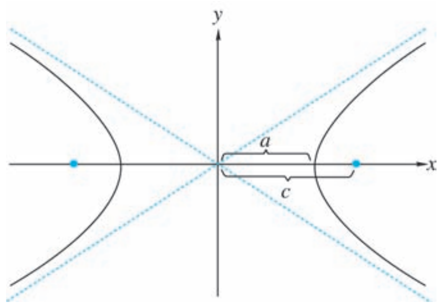

Figure 15.47b

EXAMPLE 15.38

Discuss and graph the equation $16x^2 - 9y^2 = 144$.

SOLUTION To get the equation in standard form, we divide both sides by 144 and obtain the equation:

$$\frac{x^2}{9} - \frac{y^2}{16} = 1$$

This is the equation of a hyperbola with center (0, 0). The transverse axis is the x-axis. Also, $a^2 = 9$ and $b^2 = 16$, so $c^2 = a^2 + b^2 = 9 + 16 = 25$. The foci are at $F(5, 0)$ and $F'(-5, 0)$, the vertices at $V(3, 0)$ and $V'(-3, 0)$, and the endpoints of the conjugate axis at $W(0, 4)$ and $W'(0, -4)$. If we form the rectangle with midpoints of its sides V, V', W, and W' and draw its diagonals, we get the asymptotes $y = \frac{4}{3}x$ and $y = -\frac{4}{3}x$ shown by the dotted lines in Figure 15.48. The sketch of the graph is shown by the solid curve in Figure 15.48.

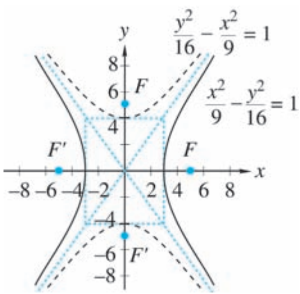

Figure 15.48

EXAMPLE 15.39

Discuss and graph the equation $9y^2 - 16x^2 = 144$.

SOLUTION Again, we divide both sides by 144 to get

$$\frac{y^2}{16} - \frac{x^2}{9} = 1$$

This is the equation of a hyperbola with center (0, 0). The transverse axis is the y-axis. Also, $a^2 = 16$ and $b^2 = 9$, so $c^2 = 25$. The foci are at $F(0, 5)$ and $F'(0, -5)$, the vertices at $V(0, 4)$ and $V'(0, -4)$, and the endpoints of the conjugate axis at $W(3, 0)$ and $W'(-3, 0)$. This hyperbola has the same asymptotes as the hyperbola in Example 15.38. The sketch of this hyperbola is shown by the dotted curves in Figure 15.48. The hyperbolas in Examples 15.38 and 15.39 are called *conjugate hyperbolas*, because they have the same set of asymptotes.

EXAMPLE 15.40

Find the equation of a hyperbola with center (0, 0), a focus at (6, 0), and a vertex at $(-3, 0)$.

SOLUTION The second focus is at $(-6, 0)$, and the second vertex at (3, 0). Since the foci and vertices are on the x-axis, it is the transverse axis. The equation is

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

Since $a = 3$ and $c = 6$, then $b^2 = c^2 - a^2 = 36 - 9 = 27$. The equation of this hyperbola is

$$\frac{x^2}{9} - \frac{y^2}{27} = 1$$

or $27x^2 - 9y^2 = 243$.

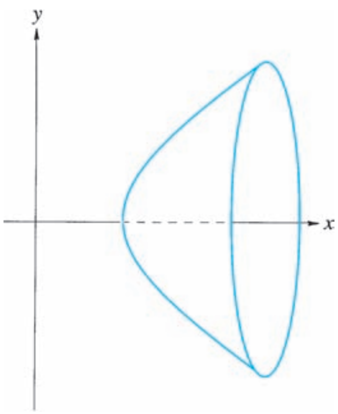

Figure 15.49

REFLECTIVE PROPERTIES OF HYPERBOLAS

Like the other conic sections, hyperbolas are used in many of today's technological applications. A three-dimensional object called a *hyperboloid of revolution* is formed when a hyperbola is revolved around its transverse axis of symmetry (see Figure 15.49). Notice that a hyperboloid of revolution normally uses only one branch of the hyperbola. Uses of these hyperboloids of revolution are based on the fact that rays aimed at one focus of a hyperbolic reflector are reflected so that they pass through the other focus.

Hyperboloids are frequently used to locate the source of a sound or radio signal. In Exercise Set 15.5, Exercise 18 describes how a system called LORAN (for LOng RAnge Navigation) uses hyperbolas as aids for navigation.

A telescope can use both a hyperbolic lens and a parabolic lens. The main lens is parabolic with its focus at F_1 and vertex at F_2. A second lens is hyperbolic with its foci at F_1 and F_2, as shown in Figure 15.50. The eye is positioned at F_2. The key to this arrangement is that the same point, F_1, is both the focus of the parabola and a focus of the hyperbola.

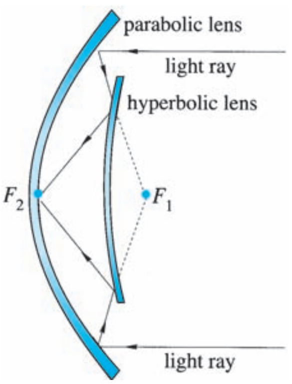

Figure 15.50

APPLICATION GENERAL TECHNOLOGY

EXAMPLE 15.41

As we will see in Section 15.6, many telescopes use both a parabolic reflector and a hyperbolic reflector. In this example, we shall examine a simple hyperbolic reflector. In Section 15.6, we shall consider an application that involves both parabolic and hyperbolic reflectors.

A hyperbolic reflector is designed to fit in a confined area. It is possible to get the foci so they are 8 ft apart and one vertex is 3 ft from a focus. What is the equation for this hyperbolic reflector?

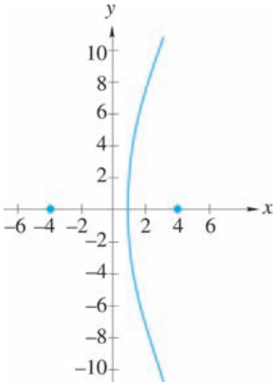

Figure 15.51

SOLUTION We shall arrange this reflector on a coordinate system, so that the transverse axis is on the x-axis and the center is at the origin. A cross-section through the axis of this hyperbolic reflector has been drawn on the desired coordinate system in Figure 15.51.

Since the foci are 8 ft apart, this means that the foci are at $(-4, 0)$ and $(4, 0)$. The given vertex is 3 ft from one of the foci (and hence, 5 ft from the other), so it is at $(-1, 0)$ or $(1, 0)$. We have $a = 1$ and $c = 4$, so $b^2 = c^2 - a^2 = 4^2 - 1^2 = 15$. Thus, the desired equation is $\frac{x^2}{1} - \frac{y^2}{15} = 1$.

USING A GRAPHING CALCULATOR

When using a graphing calculator, choose a viewing window that places the endpoints of the transverse and conjugate axes in substantially from the edge of the window. The next example will give a way to estimate where to put the viewing window. Again, since the calculator will graph only a function, you will need to solve the equation algebraically for y and then graph the two equations that result.

EXAMPLE 15.42

Use a graphing calculator to graph $\frac{x^2}{25} - \frac{y^2}{81} = 1$.

SOLUTION Since this is in standard form, we see that this is a horizontally oriented hyperbola. Here $a^2 = 25$, so $a = 5$ and $b^2 = 81$, so $b = 9$. A viewing window that sets Xmin $= -5$, Xmax $= 5$, Ymin $= -9$, and Ymax $= 9$ would only show the vertices of the hyperbola. Thus, we need a wider window. As a rough rule, we will use $2a$ and $2b$ to set the window settings. For this hyperbola, we will use a viewing window that sets Xmin $= -10$, Xmax $= 10$, Ymin $= -18$, and Ymax $= 18$.

Next, solve the equation for y:

$$\frac{x^2}{25} - \frac{y^2}{81} = 1$$

$$\frac{y^2}{81} = \frac{x^2}{25} - 1$$

$$y^2 = 81\left(\frac{x^2}{25} - 1\right)$$

EXAMPLE 15.42 (Cont.)

$$y = \pm\sqrt{81\left(\frac{x^2}{25} - 1\right)}$$

$$y = \pm 9\sqrt{\frac{x^2}{25} - 1}$$

We will graph $y_1 = 9\sqrt{\frac{x^2}{25} - 1}$ and $y_2 = -9\sqrt{\frac{x^2}{25} - 1}$. This result is shown in Figure 15.52.

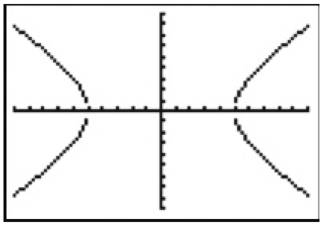

Figure 15.52

EXAMPLE 15.43

Use a spreadsheet to graph $\frac{x^2}{25} - \frac{y^2}{81} = 1$.

SOLUTION This is the same equation that we used in Example 15.42. Since it is in standard form, we see that this is a horizontally oriented hyperbola, with $a^2 = 25$, (and $a = 5$) and $b^2 = 81$, which means $b = 9$. Using x-values between -5 and 5 will not show any of the hyperbola, so values of x must extend far enough to the left of -5 and to the right of 5.

Next, solve the equation for y:

$$\frac{x^2}{25} - \frac{y^2}{81} = 1$$

$$y^2 = 81\left(\frac{x^2}{25} - 1\right)$$

$$y = \pm\sqrt{81\left(\frac{x^2}{25} - 1\right)}$$

$$y = \pm 9\sqrt{\frac{x^2}{25} - 1}$$

Enter either `=9*(A5^2/25-1)^0.5` or `=9*SQRT(A5^2/25-1)` in Cell B5 and `=-9*(A5^2/25-1)^0.5` or `=-9*SQRT(A5^2/25-1)` in Cell C5.

As shown in Figure 15.53, the graph is actually constructed using four arcs, one for each portion of the hyperbola, by first graphing the values from $x = -10$ to $x = -5$ and then adding the parts from $x = 5$ to $x = 10$.

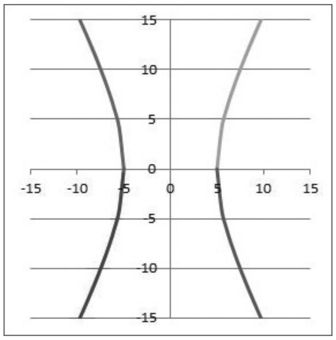

Figure 15.53

EXERCISE SET 15.5

In Exercises 1–8, find the equation of the hyperbola with the stated properties. Each hyperbola has its center at (0, 0).

1. Focus at (6, 0), vertex at (4, 0)
2. Focus at (12, 0), vertex at (9, 0)
3. Focus at (0, 5), vertex at (0, -3)
4. Focus at (0, -4), vertex at (0, 2)
5. Focus at (5, 0), vertex at (3, 0)
6. Focus at (0, -3), vertex at (0, -2)
7. Focus at (4, 0), length of conjugate axis, 6
8. Focus at (0, -6), length of conjugate axis, 10

In Exercises 9–16, give the coordinates of the vertices, foci, and endpoints of the conjugate axis and sketch each curve.

9. $\frac{x^2}{4} - \frac{y^2}{9} = 1$

10. $\frac{x^2}{9} - \frac{y^2}{4} = 1$

11. $\frac{y^2}{4} - \frac{x^2}{1} = 1$

12. $\frac{y^2}{36} - \frac{x^2}{16} = 1$

13. $4x^2 - y^2 = 4$

14. $9x^2 - 4y^2 = 36$

15. $36y^2 - 25x^2 = 900$

16. $y^2 - 9x^2 = 18$

Solve Exercises 17–26.

17. The **eccentricity** e of the hyperbola is defined as $\frac{c}{a}$. Since $c > a$, then $e > 1$. (Remember, for the ellipse, $0 < e < 1$.) What is the equation of a hyperbola with center at the origin, vertex at $(7, 0)$, and $e = 1.5$?

18. ***Navigation*** Hyperbolas are used in long-range navigation as part of the LORAN system of navigation. A transmitter is located at each focus and radio signals are sent to the navigator simultaneously from each station. The difference in time at which the signals are received allows the navigator to determine the position. Suppose the transmitting towers are 1 000 km apart on an east-west line. A ship on the line between two towers determines its position to be 250 km from the west tower.

(a) Sketch the signal hyperbola on which the ship is located. Place the center at $(0, 0)$ and let the x-axis be the transverse axis.

(b) Find the equation of the hyperbola.

19. A hyperbola for which $a = b$ is called an *equilateral hyperbola*. Find the eccentricity of an equilateral hyperbola.

20. ***Civil engineering*** A proposed design for a cooling tower at a nuclear power plant is a branch of a hyperbola rotated about a conjugate axis. Find the equation of the hyperbola passing through the point $(238, -616)$ with center $(0, 0)$ and one vertex at $(153, 0)$.

21. ***Electronics*** For any given voltage the product of current I and resistance R in a simple circuit is constant. If the current in a circuit is 3.2 A when the resistance is 18 Ω, sketch the graph of current as a function of resistance for this value of the voltage.

22. ***Civil engineering*** The silhouette of the cooling tower of a nuclear reactor forms a hyperbola, like the one in Figure 15.54. If the asymptotes are the lines given by $y = 1.25x$ and $y = -1.25x$, find the equation of the hyperbola.

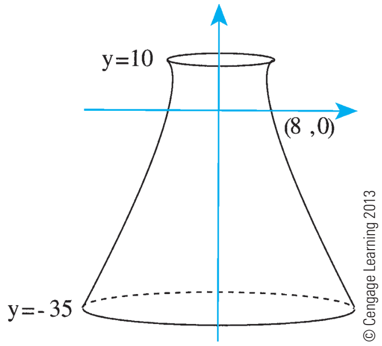

Figure 15.54

23. ***Aeronautical engineering*** A supersonic jet plane flying at a constant speed produces a shock wave in the shape of a cone. If the plane's path is parallel to the ground, the intersection of the cone with the ground is a hyperbola. Find the equation of the hyperbola if the center is at the origin of a coordinate system, one vertex is at $(-42, 0)$, and the hyperbola passes through the point $(-126, 30\sqrt{3})$.

24. ***Electronics*** Ohm's law states that voltage V, current I, and resistance R are related by the formula $V = IR$. If $V = 14.0$ V, sketch the graph of I as a function of positive values of R.

25. ***Wastewater technology*** The formula $A = 8.34FC$ is used to determine the amount, A, of chlorine (in lb) to add to a basin with a flow rate of F million gallons per day to achieve a chlorine concentration of C parts per million (ppm). If 2250 lb of chlorine is added to a certain reservoir, sketch the graph of flow rate as a function of chlorine concentration for this amount of added chlorine.

26. ***Wastewater technology*** The formula $A = 8.34FC$ is used to determine the amount, A, of chlorine in kg to add to a basin with a flow rate of F million liters per day to achieve a chlorine concentration of C parts per million (ppm). If 750 kg of chlorine is added to a certain reservoir, sketch the graph of flow rate as a function of chlorine concentration for this amount of added chlorine.

[IN YOUR WORDS]

27. Describe how you can tell by inspecting the standard equation of a hyperbola whether it is a horizontal or a vertical hyperbola.

28. Explain how to use the endpoints of the axes to locate the foci of an hyperbola.

29. In Exercise 19, it was stated that an equilateral hyperbola is one in which $a = b$. Why is this called an equilateral hyperbola?

15.6 TRANSLATION OF AXES

So far, the equations that we have used for the ellipse and the hyperbola have all been centered at the origin. You may remember from our study of circles that some circles had their center at the origin and some did not. In this section, we will look at cases where one of the axes of an ellipse is parallel to one of the coordinate axes. To do this, we will use translation of axes.

The method we will use to translate axes will be very similar to the one we used when we were solving the equation for a circle. We took each equation that was not centered at the origin and completed the square. This told us the location of the center and the radius.

The equations that we will be working with are all in the form

$$Ax^2 + Cy^2 + Dx + Ey + F = 0$$

where A and C cannot both be zero at the same time. The process can best be explained by following the next example.

EXAMPLE 15.44

Discuss and sketch the graph of $9x^2 + 4y^2 - 36x + 40y + 100 = 0$.

SOLUTION As we just mentioned, we will complete the square. First we will group the terms, and use ○ and □ to indicate the missing constants that we must determine in order to complete the square:

$$(9x^2 - 36x) + (4y^2 + 40y) + 100 = 0$$
$$9(x^2 - 4x + \bigcirc) + 4(y^2 + 10y + \square) = -100 + 9\bigcirc + 4\square$$

Next, we complete the square of each expression within parentheses by adding 4 to the x terms and 25 to the y terms. Remember that on the right-hand side of the equation, each of these numbers (4 and 25) must be multiplied by the number outside the parentheses (9 and 4):

$$9(x^2 - 4x + 4) + 4(y^2 + 10y + 25) = -100 + 9(4) + 4(25)$$

or
$$9(x - 2)^2 + 4(y + 5)^2 = 36$$

Dividing both sides by 36, we get

$$\frac{(x-2)^2}{4} + \frac{(y+5)^2}{9} = 1$$

This looks like the equation for an ellipse. If we let $x' = x - 2$ and $y' = y + 5$, then the equation can be written as

$$\frac{(x')^2}{4} + \frac{(y')^2}{9} = 1$$

This is an ellipse with center at $(x', y') = (0', 0')$. If we draw a new coordinate system, the $x'y'$-system, with its origin at $(2, -5)$, we can draw our ellipse on this new coordinate system. Thus, this ellipse is centered at $(0', 0')$, the vertices are at $V(0', 3')$ and $V'(0', -3')$, the endpoints of the minor axis at $M(2', 0')$ and $M'(-2', 0')$, and the foci are at $F(0', \sqrt{5}')$ and $F'(0', -\sqrt{5}')$. Notice that we have used $3'$, $0'$, $2'$, and so on, to show that these are points on the $x'\,y'$-axes. A sketch of this graph is shown in Figure 15.55. The $x'y'$-axes have been traced over the xy-system.

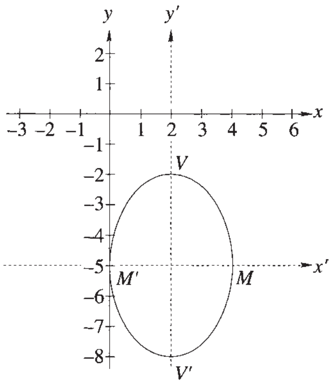

Figure 15.55

We can use a table to show the coordinates in both systems. Since we know the coordinates in the $x'y'$-system and we know that $x' = x - 2$ and $y' = y + 5$, to find the coordinates in the xy-system, we solve these equations for x and y: $x = x' + 2$ and $y = y' - 5$. The table follows.

	$x'y'$-system	xy-system
center	$(0', 0')$	$(2, -5)$
vertices	$(0', \pm 3')$	$(2, -2), (2, -8)$
foci	$(0', \pm\sqrt{5}')$	$(2, \sqrt{5} - 5), (2, -\sqrt{5} - 5)$
endpoints of minor axis	$(\pm 2', 0')$	$(4, -5), (0, -5)$

In general, if we want the point (h, k) in the xy-coordinate system to become identified as the origin of the $x'y'$-coordinate system, we use the substitution:

$$x' = x - h \quad \text{and} \quad y' = y - k$$

This procedure is called a **translation of axes**. As a result of a translation of axes, new axes are formed that are parallel to the old ones.

EXAMPLE 15.45

Discuss and sketch the graph of $x^2 - 18y^2 + 6x - 36y - 45 = 0$.

SOLUTION Again, we will complete the square:

$$(x^2 + 6x) + (-18y^2 - 36y) = 45$$

$$(x^2 + 6x + \bigcirc) - 18(y^2 + 2y + \square) = 45 + \bigcirc - 18\,\square$$

We add 9 to complete the square of $x^2 + 6x$ and 1 to complete $y^2 + 2y$:

$$(x^2 + 6x + 9) - 18(y^2 + 2y + 1) = 45 + 9 - 18(1)$$

$$(x + 3)^2 - 18(y + 1)^2 = 36$$

EXAMPLE 15.45 (Cont.)

Dividing both sides by 36, we get

$$\frac{(x+3)^2}{36} - \frac{(y+1)^2}{2} = 1$$

If $x' = x + 3$ and $y' = y + 1$, then the equation becomes

$$\frac{x'^2}{36} - \frac{y'^2}{2} = 1$$

which is a hyperbola with center at $(0', 0')$ and vertices at $(\pm 6', 0')$ of the $x'y'$-system. The origin of the $x'\,y'$-system is found at $(-3, -1)$ of the xy-system. The sketch of this hyperbola on both coordinate systems is shown in Figure 15.56. Corresponding coordinates in the two systems are shown in the following table.

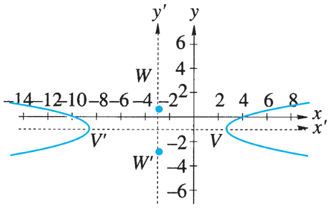

Figure 15.56

	$x'y'$-system	xy-system
center	$(0', 0')$	$(-3, -1)$
vertices	$(\pm 6', 0')$	$(-9, -1), (3, -1)$
foci	$(\pm\sqrt{38}', 0')$	$(-3+\sqrt{38}, -1)$, $(-3-\sqrt{38}, -1)$
endpoints of minor axis	$(0', \pm\sqrt{2}')$	$(-3, -1+\sqrt{2}), (-3, -1-\sqrt{2})$

EXAMPLE 15.46

Discuss and sketch the graph of $y^2 + 12x + 2y - 23 = 0$.

SOLUTION There is no x^2-term, so this is the equation for a parabola. We will complete the square on just the y-terms. We will also put the y-terms on one side of the equation and all the other terms on the other side:

$$\begin{aligned} y^2 + 2y &= -12x + 23 \\ y^2 + 2y + 1 &= -12x + 23 + 1 \\ (y+1)^2 &= -12(x-2) \end{aligned}$$

If $y' = y + 1$ and $x' = x - 2$, the equation becomes

$$y'^2 = -12x'$$

which is a parabola with vertex at $(0', 0')$, a horizontal axis, and that opens to the left. Since $4p = -12$, $p = -3$ and the directrix is the line $x = 3'$. The focus is at $(-3', 0')$. The sketch of this curve is shown in Figure 15.57.

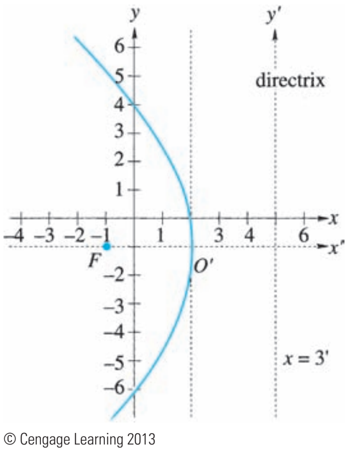

Figure 15.57

EXAMPLE 15.47

Find the equation of the hyperbola with vertices $(3, 1)$ and $(-5, 1)$, and focus $(5, 1)$.

SOLUTION The center of a hyperbola is at the midpoint between the vertices. Using the midpoint formula, we can determine that the center is at $\left(\frac{3+(-5)}{2}, \frac{1+1}{2}\right) = (-1, 1)$. The distance from the vertex to the center is 4, so $a = 4$. The distance from the focus to the center is 6, so $c = 6$. This is a hyperbola, so $b^2 = c^2 - a^2 = 6^2 - 4^2 = 20$. Finally, this is a hyperbola with foci on the x'-axis and the equation is

$$\frac{x'^2}{16} - \frac{y'^2}{20} = 1$$

Since $x' = x - h$ and $y' = y - k$, we have $x' = x + 1$ and $y' = y - 1$, so the equation becomes

$$\frac{(x+1)^2}{16} - \frac{(y-1)^2}{20} = 1$$

APPLICATION GENERAL TECHNOLOGY

EXAMPLE 15.48

A telescope is shown in Figure 15.58a. The focus of one branch of the hyperbolic lens is 20.4 cm from the vertex of the other branch of this same lens. The parabolic lens is 63 cm deep and measures 168 cm across. Determine (a) the location of the focus for the parabolic lens, (b) an equation for the parabola, and (c) an equation for the hyperbola.

SOLUTION A cross-section through the axis of this telescope has been drawn on a coordinate system in Figure 15.58b. Because the focus of one branch of the hyperbolic lens is also the vertex of the parabolic lens, we have placed the given focus at the origin. The given vertex of the other branch has been located at (20.4, 0). The parabolic lens is 63 cm deep and 168 cm across and so the point (63, 84) is on this lens.

This is a parabola. Its equation is of the form $y^2 = 4px$. We know that (63, 84) is a point on the parabola, hence we have $p = \frac{y^2}{4x} = \frac{84^2}{4(63)} = 28$. Thus, the parabola's focus is at (28, 0) or 28 cm from the vertex and the equation of the parabola is $y^2 = 4(28)x = 112x$.

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Figure 15.58a

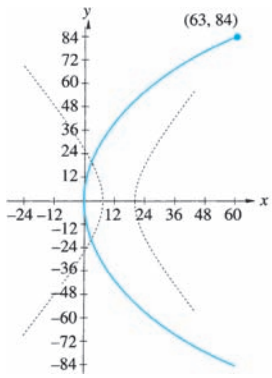

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Figure 15.58b

EXAMPLE 15.48 (Cont.)

The foci of the hyperbola are at (0, 0) and (28, 0). Thus, the center of the hyperbola is at (14, 0), and so $c = 28 - 14 = 14$. Since one of the hyperbola's vertices is at (20.4, 0), $a = 20.4 - 14 = 6.4$. Using these values of a and c, we can determine that $b^2 = c^2 - a^2 = 14^2 - 6.4^2 = 155.04$. Thus, the standard equation of the hyperbola for this lens is

$$\frac{(x-14)^2}{6.4^2} - \frac{y^2}{155.04} = 1$$

USING A GRAPHING CALCULATOR

When using a graphing calculator, choose a viewing window that puts the center of the conic section at the center of the window. Then use the guidelines presented earlier for determining the viewing window. The next example shows how to do this with a translated ellipse. Again, since the calculator will graph only a function, you will need to solve the equation algebraically for y and then graph the two equations that result.

EXAMPLE 15.49

Use a graphing calculator to graph $9x^2 + 4y^2 - 36x + 40y + 100 = 0$.

SOLUTION This is the same curve we graphed in Example 15.44. From that example, we know that the conic is an ellipse with standard equation $\frac{(x-2)^2}{4} + \frac{(y+5)^2}{9} = 1$, center at $(2, -5)$, $a = 2$, and $b = 3$. For the viewing window we will use the following settings, where c_x and c_y represent the x- and y-coordinates of the center:

$$\begin{aligned}
\text{Xmin} &= c_x - a &&= 2-2 &&= 0\\
\text{Xmax} &= c_x + b &&= 2+2 &&= 4\\
\text{Ymin} &= c_y - b &&= -5-3 &&= -8\\
\text{Ymax} &= c_y + b &&= -5+3 &&= -2
\end{aligned}$$

Next, solve the equation for y:

$$\begin{aligned}
\frac{(x-2)^2}{4} + \frac{(y+5)^2}{9} &= 1\\
\frac{(y+5)^2}{9} &= 1 - \frac{(x-2)^2}{4}\\
(y+5)^2 &= 9\left(1 - \frac{(x-2)^2}{4}\right)\\
y + 5 &= \pm\sqrt{9\left(1 - \frac{(x-2)^2}{4}\right)}\\
y &= -5 \pm \sqrt{9\left(1 - \frac{(x-2)^2}{4}\right)}\\
y &= -5 \pm 3\sqrt{1 - \frac{(x-2)^2}{4}}
\end{aligned}$$

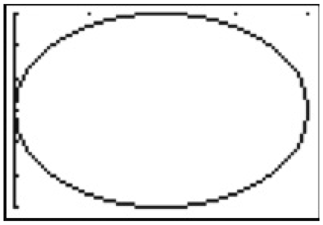


Figure 15.59a

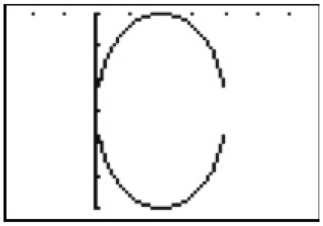


Figure 15.59b

we will graph $y_1 = -5 + 3\sqrt{1 - \frac{(x-2)^2}{4}}$ and $y_2 = -5 - 3\sqrt{1 - \frac{(x-2)^2}{4}}$.

This result is shown in Figure 15.59a. Pressing ZOOM 5, we obtain the proportionally correct graph shown in Figure 15.59b.

USING A SPREADSHEET

When using a spreadsheet, choose a viewing window that puts the center of the conic section at the center of the window. Then use the guidelines presented earlier for determining the viewing window. The next example shows how to do this with a translated ellipse. Again, since the spreadsheet will graph only a function, you will need to solve the equation algebraically for y and then graph the two equations that result.

EXAMPLE 15.50

Use a spreadsheet to graph $9x^2 + 4y^2 - 36x + 40y + 100 = 0$.

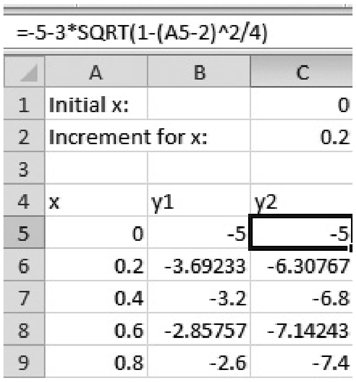
SOLUTION This is the same curve we graphed in Examples 15.48 and 15.49. From those examples, we know that the conic is an ellipse with standard equation:

$$\frac{(x-2)^2}{4} + \frac{(y+5)^2}{9} = 1$$

The ellipse is centered at $(2, -5)$, $a = 2$, and $b = 3$. Solving the equation for y we get

$$y = -5 \pm 3\sqrt{1 - \frac{(x-2)^2}{4}}$$

To construct the table of values, enter `=-5+3*SQRT(1-(A5-2)^2/4)` in Cell B5 and `=-5-3*SQRT(1-(A5-2)^2/4)` in Cell C5, as shown in Figure 15.60a. The result is shown in Figure 15.60b.



=-5-3*SQRT(1-(A5-2)^2/4)

	A	B	C
1	Initial x:		0
2	Increment for x:		0.2
3			
4	x	y1	y2
5	0	-5	-5
6	0.2	-3.69233	-6.30767
7	0.4	-3.2	-6.8
8	0.6	-2.85757	-7.14243
9	0.8	-2.6	-7.4

Figure 15.60a

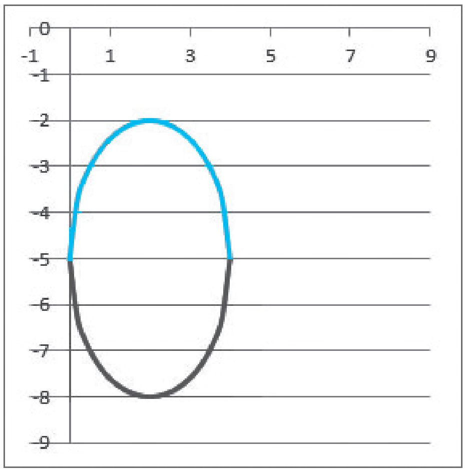


Figure 15.60b

EXERCISE SET 15.6

Sketch the graphs in Exercises 1–12, after making suitable translations of axes.

1. $\frac{(x-4)^2}{9} + \frac{(y+3)^2}{4} = 1$
2. $\frac{(x+5)^2}{16} - \frac{(y+3)^2}{25} = 1$
3. $\frac{(y-3)^2}{36} - (x+4)^2 = 1$
4. $(y+5)^2 = 12(x+1)$
5. $100(x+5)^2 - 4y^2 = 400$
6. $(y+3)^2 = 8(x-2)$
7. $16x^2 + 4y^2 + 64x - 12y + 57 = 0$
8. $x^2 + y^2 - 2x + 2y - 2 = 0$
9. $25x^2 + 4y^2 - 250x - 16y + 541 = 0$
10. $100x^2 - 180x - 100y + 81 = 0$
11. $2x^2 - y^2 - 16x + 4y + 24 = 0$
12. $9x^2 - 36x + 16y^2 - 32y - 524 = 0$

In Exercises 13–22, determine the equation of each of the curves described by the given information.

13. Parabola, vertex at $(2, -3)$, $p = 8$, axis parallel to y-axis
14. Hyperbola, vertex at $(2, 1)$, foci at $(-6, 1)$ and $(8, 1)$
15. Ellipse, center at $(4, -3)$, focus at $(8, -3)$, vertex at $(10, -3)$
16. Ellipse, center at $(-2, 0)$, focus at $(6, 0)$, vertex at $(9, 0)$
17. Hyperbola, center at $(-3, 2)$, focus at $(-3, 7)$, transverse axis 6 units
18. Ellipse, center at $(3, 5)$, focus at $(3, 8)$, minor axis 2 units
19. Parabola, vertex at $(-5, 1)$, $p = -4$, axis parallel to x-axis
20. Parabola, vertex at $(-3, -6)$, $p = -12$, axis parallel to y-axis
21. Ellipse, center at $(-4, 1)$, focus at $(-4, 9)$, minor axis 12 units
22. Hyperbola, center at $(2, 8)$, vertex at $(6, 8)$, conjugate axis 10 units

Solve Exercises 23–24.

23. ***Physics*** The height s of a ball thrown vertically upward is given by the equation $s = 29.4t - 4.9t^2$, where s is in meters and t is the elapsed time in seconds. Graph this curve. Discuss the curve. Determine the maximum height of the ball.
24. ***Astronomy*** Satellites often orbit earth in an elliptical path with the center of the earth at one of the foci. For a certain satellite, the maximum altitude is 140 mi above the surface of the earth and the minimum is 90 mi. If the radius of earth is approximately 4,000 mi, find the equation of the orbit.
25. ***Navigation*** Two navigational transmitting towers A and B are 1 000 km apart along an east-west line. Radio signals are sent (traveling at 300 m/μs) simultaneously from both towers. An airplane is located somewhere north of the line joining the two towers. The signal from A arrives at the plane 600 μs after the signal from B. The signal sent from B and reflected by the plane takes a total of 8 000 μs to reach A. What is the location of the plane?
26. ***Astronomy*** The orbit of Halley's comet is an ellipse with the sun at one focus. The major and minor semi-axes of this orbit are 18.09 and 4.56 A.U., respectively. (A *semi-axis* is half an axis. One A.U. is one astronomical unit or about 1.496×10^8 km.) What is the equation of this orbit, if the sun is at the origin and the major axis is along the x-axis? What are the

maximum and minimum distances from the sun to Halley's comet?

27. ***Mechanical engineering*** A cantilever beam is a beam fixed at one end. Under a uniform load, the beam assumes a parabolic curve with the fixed end as the vertex. For a cantilever beam 2.00 m long, the equation of the load parabola is approximately $x^2 + 200y - 4.00 = 0$, where x is the distance in meters from the fixed end and y is the displacement (see Figure 15.61).

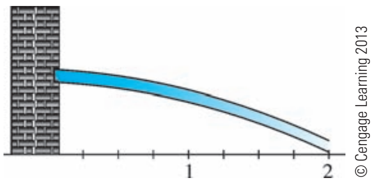


Figure 15.61

(a) Change this equation to the standard equation for a parabola.

(b) How far is the free end of the beam displaced from its no-load position?

28. ***Mechanical engineering*** For a cantilever beam 8.0 ft long, the equation of the load parabola is approximately $x^2 + 100y - 4.00 = 0$, where x is the distance in feet from the fixed end and y is the displacement.

(a) Change this equation to the standard equation.

(b) How far is the free end of the beam displaced from its no-load position?

29. ***Air traffic control*** An airplane starts at the origin and flies in a straight line northeast, that is, up and to the right along the line $y = x$. Transmitters at coordinates (in miles) (0, 0) and (0, 250) send out synchronized radio signals, and instruments in the plane measure the difference in their arrival times. If the speed of the radio signal is known, then it is possible to locate the set of points where a plane is 50 mi farther from (0, 0) than from (0, 250) as one branch of a hyperbola.

(a) Find the equation of the hyperbola and identify the correct branch.

(b) Find the location of the airplane when it is 50 mi farther from (0, 0) than from (0, 250).

30. ***Air traffic control*** In the same setting as Exercise 29, a second plane flies along the line $y = 2x$. Comparison of radio signals shows that it is 50 mi farther from (0, 0) than from (0, 250). Find the location of the airplane.

31. ***Automotive technology*** The design of an automobile cam consists of the graph of half the circle $0.25x^2 + 0.25y^2 = 0.0625$ connected to half of the ellipse $0.4x^2 + 0.16y^2 = 0.1$.

(a) Solve each of these equations so their graphs, when drawn on the same pair of axes, will produce this cam.

(b) Use your equations from (a) to sketch this cam.

32. ***Police science*** The stopping distance, d, in ft of a car moving at v mph is given approximately by the equation $d = v + \frac{1}{20}v^2$. Plot d as a function of v for $0 \le v \le 80$.

33. ***Industrial management*** A factory normally packages items in lots of 24. It costs \$0.90/item to package a lot of exactly 24 items. However, the cost per item is reduced by \$0.01 for each item over 24.

(a) Determine a function for the cost in terms of the number of items produced.

(b) Graph your function from (a).

(c) Using your graph in (b), determine the lot size that will produce the highest packaging cost.

(d) What is the highest packaging cost?

34. ***Electronics*** For a certain temperature-sensitive electronic device, the voltage, V, is given by $V = 5.0T - 0.025T^2$, where T is the ambient temperature in °C.

(a) Plot V as a function of t for values of $V \ge 0$.

(b) At what temperature is $V = 0$?

[IN YOUR WORDS]

35. Explain how you can tell by looking at the equation $Ax^2 + Cy^2 + Dx^2 + Ey + F = 0$ whether the graph of the equation is a circle, parabola, ellipse, or hyperbola.

36. Describe how to change an equation of the form $Ax^2 + Cy^2 + Dx + Ey + F = 0$ into the standard form for its conic section.

15.7 ROTATION OF AXES: THE GENERAL SECOND-DEGREE EQUATION

The conic sections that we have considered so far have all had their axes on a coordinate axis or parallel to a coordinate axis. All of these could be written as second-degree equations of the form:

$$Ax^2 + Cy^2 + Dx + Ey + F = 0$$

Notice that this equation does not contain an xy-term.

In this section, we will work with curves in which an axis is not parallel to a coordinate axis. However, the axes are still perpendicular to each other. These equations will all have an xy-term. In order to recognize its graph, we will change to a new coordinate system. In Section 15.6, we changed to a new coordinate system through a translation of axes. In this section, we will use a rotation of axes and we will work with the general second-degree equation.

GENERAL FORM OF A SECOND-DEGREE EQUATION

The *general form of a second-degree equation* is

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

where A, B, and C are not all 0.

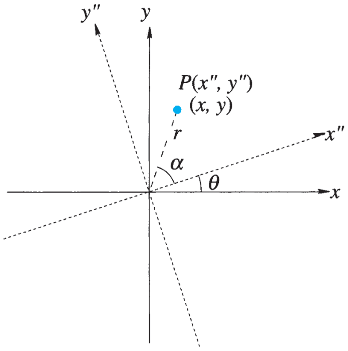


Figure 15.62

In a rotation, the origin remains fixed, while the x-axis and y-axis are rotated through a positive acute angle θ. These rotated axes will be labeled the x''-axis and the y''-axis (see Figure 15.62). If $P(x, y)$ is any general point in the xy-coordinate system, then that same point would have the coordinates $P(x'', y'')$ in the $x''y''$-coordinate system.

We can convert from one coordinate system to the other with the help of a pair of equations that express x and y in terms of x'' and y'':

$$x = x'' \cos\theta - y'' \sin\theta$$
$$y = y'' \cos\theta + x'' \sin\theta$$

These appear to be terrible equations to work with, but the following examples should show that they are not as difficult to use as they seem.

EXAMPLE 15.51

Transform the equation $xy = 1$ by rotating the axes through an angle of 45°.

SOLUTION Since $\theta = 45°$, the conversion equations become

$$x = x'' \cos 45° - y'' \sin 45° = 0.7071x'' - 0.7071y''$$
$$y = y'' \cos 45° + x'' \sin 45° = 0.7071y'' + 0.7071x''$$

Substituting these into the original equation, $xy = 1$, we get

$$[0.7071(x'' - y'')] \cdot [0.7071(y'' + x'')] = 1$$
$$0.5(x''^2 - y''^2) = 1$$
$$\frac{x''^2}{2} - \frac{y''^2}{2} = 1$$

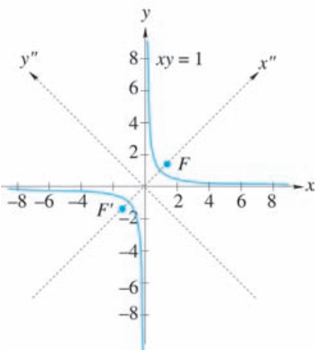


Figure 15.63

This is an equation of a hyperbola with center at the origin of the $x''y''$-coordinate system and transverse axis along the x''-axis. Here $a = b = \sqrt{2}$ and $c = 2$. The asymptotes of this hyperbola happen to be the original xy-axes. A graph of this curve is shown in Figure 15.63.

Any equation of the type in Example 15.51 is called an *equilateral hyperbola* or a *rectangular hyperbola* because the asymptotes are perpendicular. Equilateral hyperbolas are of the form $xy = k$, where k is a constant.

Now, suppose we have an equation that is a general second-degree equation and $B \neq 0$. What we want to do is rotate the axes so that we will get an equation of the form:

$$A''(x^2) + B''xy + C''(y^2) + D''x + E''y + F'' = 0$$

where $B'' = 0$. Then we will be able to put the equation in the $x''y''$-coordinate system in a recognizable form of one of the conic sections.

We can cause B'' to be 0 if we let θ be the unique acute angle where

$$\cot 2\theta = \frac{A - C}{B} \qquad 0° < \theta < 90° \qquad 0 < \theta < \frac{\pi}{2}$$

If you use a calculator, you have to be careful. First there is no COT key on your calculator, so you will have to take the reciprocal of the value. Since $\frac{1}{\cot \theta} = \tan \theta$, you can then use the INV TAN keys and obtain $\theta = 0.5 \tan^{-1}\left(\frac{B}{A - C}\right)$. But, the arctan function will give answers between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$. If you get a negative angle, you will have to add π rad or 180°. Let's see how it works in the next example.

EXAMPLE 15.52

Determine the graph of the equation $29x^2 + 24xy + 36y^2 - 54x - 72y - 135 = 0$.

SOLUTION We will first determine the angle of rotation using the formula $\cot 2\theta = \frac{A - C}{B}$. Here $A = 29$, $B = 24$, and $C = 36$. The following description shows how to use a graphing calculator to determine θ.

EXAMPLE 15.52 (Cont.)

PRESS	DISPLAY	
(29 − 36) ÷ 24 ENTER	−0.2916666667	
x^{-1} ENTER	−3.428571429	Changes from $\cot 2\theta$ to $\tan 2\theta$
2nd TAN 2nd ANS ENTER	−73.73979529	2θ in degrees
+ 180	106.2602047	Add 180° because 2θ is negative
÷ 2	53.13010235	This is θ

So, using our conversion formulas and the fact that $\sin\theta = 0.8$ and $\cos\theta = 0.6$, we get

$$x = x''\cos\theta - y''\sin\theta = 0.6x'' - 0.8y''$$
$$y = y''\cos\theta + x''\sin\theta = 0.6y'' + 0.8x''$$

Substituting these values for x and y in the given equation, $29x^2 + 24xy + 36y^2 - 54x - 72y - 135 = 0$, we obtain

$$\begin{aligned}
&29(0.6x'' - 0.8y'')^2 + 24(0.6x'' - 0.8y'')(0.6y'' + 0.8x'') \\
&\quad + 36(0.6y'' + 0.8x'')^2 - 54(0.6x'' - 0.8y'') \\
&\quad - 72(0.6y'' + 0.8x'') - 135 = 0 \\
&29(0.36x''^2 - 0.96x''y'' + 0.64y''^2) \\
&\quad + 24(0.48x''^2 + 0.36x''y'' - 0.48y''^2 - 0.64x''y'') \\
&\quad + 36(0.36y''^2 + 0.96x''y'' + 0.64x''^2) \\
&\quad -54(0.6x'' - 0.8y'') - 72(0.6y'' + 0.8x'') - 135 = 0 \\
&10.44x''^2 - 27.84x''y'' + 18.56y''^2 + 11.52x''^2 + 8.64x''y'' - 11.52y''^2 \\
&\quad - 15.36x''y'' + 12.96y''^2 + 34.56x''y'' + 23.04x''^2 - 32.4x'' \\
&\quad + 43.2y'' - 43.2y'' - 57.6x'' - 135 = 0 \\
&45x''^2 + 0x''y'' + 20y''^2 - 90x'' + 0y'' - 135 = 0 \\
&45x''^2 + 20y''^2 - 90x'' - 135 = 0
\end{aligned}$$

Completing the square on the x''-terms, we get

$$45(x''^2 - 2x'' + 1) + 20y''^2 = 135 + 45$$
$$45(x'' - 1)^2 + 20y''^2 = 180$$

Dividing both sides by 180 we obtain

$$\frac{(x'' - 1)^2}{4} + \frac{y''^2}{9} = 1$$

This is the equation of an ellipse with a major axis of 6 ($a = 3$) and a minor axis of 4 ($b = 2$) with the major axis along the vertical axis. The center of this ellipse is at $(1'', 0'')$. The graph of this ellipse is shown in Figure 15.64.

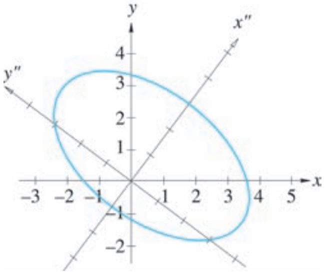


Figure 15.64

THE DISCRIMINANT

There is an easy method for determining the nature of the curve described by a general second-degree equation. It turns out that the *discriminant*, $B^2 - 4AC$, will tell the type of curve described by the equation.

CLASSIFYING THE GRAPH OF A GENERAL QUADRATIC EQUATION

The graph of a general quadratic equation

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

is either a conic or a degenerate conic. The discriminant, $B^2 - 4AC$, can be used to classify the graph of the equation by using the following:

Discriminant	Type of Curve
$B^2 - 4AC > 0$	A hyperbola or two intersecting lines (degenerate hyperbola).
$B^2 - 4AC = 0$	A parabola or a line (degenerate parabola).
$B^2 - 4AC < 0$	An ellipse, circle, or point (degenerate ellipse).

There are a few other possibilities, but you are not likely to meet them. For example, the quadratic equation $x^2 + y^2 + 9 = 0$ has discriminant $0^2 - 4(1)(1) = -4$, and its graph is empty because all its solutions are complex numbers. A similar thing happens when you try to graph $x^2 + 10 = 0$. This equation has a discriminant of 0. A different result occurs when you try to graph the solution to $y^2 - 9 = 0$. This also has a discriminant of $0^2 - 4(0)(1) = 0$. However, the solution to this equation is $y = \pm 3$, which has a graph of two parallel lines.

The majority of the problems we will work are not as long as the last example. We are more likely to get problems similar to the one in the next example.

APPLICATION ELECTRONICS

EXAMPLE 15.53

When the power P in an electric circuit is constant, the voltage V is inversely, proportional to the current I, as indicated by the equation $P = IV$. If the power is 120 W, sketch the relationship of I versus V.

SOLUTION The equation is $IV = 120$. A table of values is

I	1	2	4	6	8	10	12	15	20	30	60	120
V	120	60	30	20	15	12	10	8	6	4	2	1

Negative values do not apply. The graph is one branch of an equilateral hyperbola, as shown in Figure 15.65.

EXAMPLE 15.53 (Cont.)

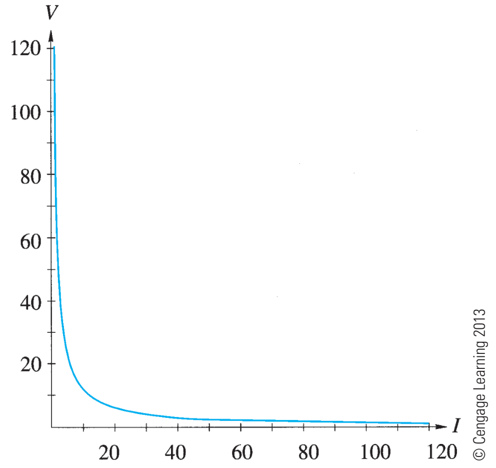


Figure 15.65

USING A GRAPHING CALCULATOR

The most difficult part about using a graphing calculator to graph a rotated conic is getting the equation for the conic in a form that the calculator can use. This requires completing the square in a way you have probably not seen.

EXAMPLE 15.54

Use a graphing calculator to graph $29x^2 + 24xy + 36y^2 - 54x - 72y - 135 = 0$.

SOLUTION This is the same curve we graphed in Example 15.52. From that example, we know that the conic is an ellipse rotated about 53.13° and with standard equation on the rotated axes $\frac{(x'' - 1)^2}{4} + \frac{y''^2}{9} = 1$.

We will use the original equation and complete the square in order to solve the equation for y:

$$29x^2 + 24xy + 36y^2 - 54x - 72y - 135 = 0$$

$$24xy + 36y^2 - 72y = 135 - 29x^2 + 54x$$

$$36y^2 + 24xy - 72y = 135 + 54x - 29x^2$$

$$36y^2 + 24y(x - 2) = 135 + 54x - 29x^2$$

$$36\left[y^2 + \frac{2}{3}y(x - 2)\right] = 135 + 54x - 29x^2$$

$$y^2 + \frac{2}{3}y(x - 2) = \frac{1}{36}(135 + 54x - 29x^2)$$

$$y^2 + \frac{2}{3}y(x - 2) + \left[\frac{1}{3}(x - 2)\right]^2 = \frac{1}{36}(135 + 54x - 29x^2) + \left[\frac{1}{3}(x - 2)\right]^2$$

$$\left[y + \frac{1}{3}(x - 2)\right]^2 = \frac{1}{36}(151 + 38x - 25x^2)$$

$$y + \frac{1}{3}(x - 2) = \pm\frac{1}{6}\sqrt{(151 + 38x - 25x^2)}$$

$$y = -\frac{1}{3}(x - 2) \pm \frac{1}{6}\sqrt{(151 + 38x - 25x^2)}$$

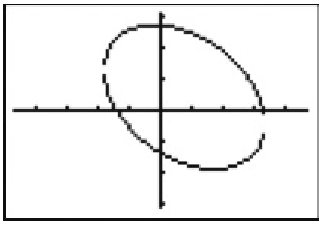


Figure 15.66

We will graph the equations $y_1 = -\frac{1}{3}(x - 2) + \frac{1}{6}\sqrt{(151 + 38x - 25x^2)}$ and $y_2 = -\frac{1}{3}(x - 2) - \frac{1}{6}\sqrt{(151 + 38x - 25x^2)}$. This result is shown in Figure 15.66. Compare this to the earlier result in Figure 15.64.

USING A SPREADSHEET

The most difficult part about using a spreadsheet to graph a rotated conic is getting the equation for the conic in a form that the computer can use. This requires completing the square in a way you have probably not seen.

EXAMPLE 15.55

Use a spreadsheet to graph $29x^2 + 24xy + 36y^2 - 54x - 72y - 135 = 0$.

SOLUTION This is the same curve we graphed in Example 15.54. From that example, we know that the conic is an ellipse rotated about 53.13° and with standard equation on the rotated axes $\frac{(x'' - 1)^2}{4} + \frac{y''^2}{9} = 1$.

Solving this equation for y produces

$$y = -\frac{1}{3}(x - 2) \pm \frac{1}{6}\sqrt{(151 + 38x - 25x^2)}$$

As before, construct a table of values for the functions. In Cell B5 enter

```
=-1/3*(A5 - 2)+(1/6)*SQRT(151+38*A5-25*A5^2)
```

and in Cell C5 enter

```
=-1/3*(A5-2)-(1/6)*SQRT(151+38*A5-25*A5^2)
```

(See Figure 15.67a.) The graph is shown in Figure 15.67b.

	A	B	C
1	Initial x:		-1.8
2	Increment for x:		0.3
3			
4	x	y1	y2
5	-1.8	1.477485	1.055848
6	-1.5	2.190684	0.14265
7	-1.2	2.455111	-0.32178

Figure 15.67a

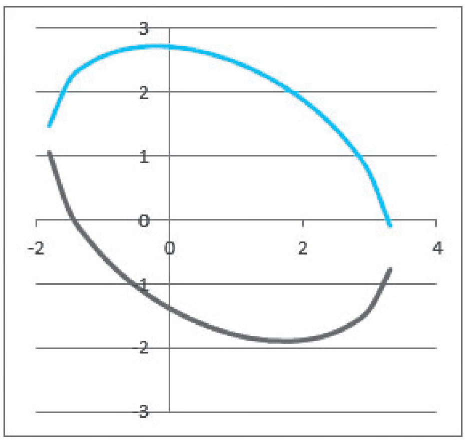


Figure 15.67b

EXERCISE SET 15.7

In Exercises 1–12, (a) use the discriminant to identify the graph of the given equation, (b) determine the angle θ needed to rotate the coordinate axes to remove the xy-term, (c) rotate the axes through the angle θ, (d) determine the equation of the conic in the $x''y''$-coordinate system, and (e) sketch the graph.

1. $xy = -9$
2. $xy = 9$
3. $x^2 - 6xy + y^2 - 8 = 0$
4. $x^2 + 4xy - 2y^2 - 6 = 0$
5. $52x^2 - 72xy + 73y^2 - 100 = 0$
6. $11x^2 + 10\sqrt{3}xy + y^2 - 4 = 0$
7. $x^2 - 2xy + y^2 + x + y = 0$
8. $3x^2 + 2\sqrt{3}xy + y^2 - 2x + 2\sqrt{3}y = 0$
9. $2x^2 + 12xy - 3y^2 - 42 = 0$
10. $7x^2 - 20xy - 8y^2 + 52 = 0$
11. $6x^2 - 5xy + 6y^2 - 26 = 0$
12. $9x^2 - 6xy + y^2 - 12\sqrt{10}x - 36\sqrt{10}y = 0$

Solve Exercises 13–16.

13. ***Physics*** Boyle's law states that the volume of a gas is inversely proportional to its pressure, provided that the mass and temperature are constant. Thus, if P is the pressure and V the volume, $PV = k$, where k is a constant. Draw the graph of P versus V, if 15 mm^3 of gas is under a pressure of 400 kPa at a constant temperature.
14. ***Electronics*** For a given alternating current circuit, the capacitance C and the capacitive reactance X_C are related by the equation $X_C = \frac{1}{\omega C}$, where ω is the angular frequency. What kind of curve is this? Sketch a graph of the equation if $\omega = 280$ rad/s.
15. ***Thermodynamics*** When a cross-section of a pipe is suddenly enlarged, as shown in Figure 15.68, the loss of heat of the fluid through the pipe is related by the formula $19.62h_L = (\overline{v}_1 - \overline{v}_2)^2$, where h_L is the heat loss and $\overline{v}_1$ and $\overline{v}_2$ are the average velocities in the two pipes. If h_L is to be held to less than 5°C, describe this curve. Sketch a graph of the curve with $(\overline{v}_1 - \overline{v}_2)$ on one axis and h_L on the other.

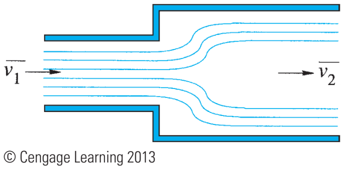


Figure 15.68

16. The sides of a rectangle are $3x$ and y and the diagonal is $x + 10$. What kind of curve is represented by the equation relating x and y? Sketch the curve.

[IN YOUR WORDS]

17. **(a)** What is the discriminant of the general quadratic equation

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

(b) Explain how to use the discriminant to classify the graph of a general quadratic equation.

18. Describe how you can graph an equation of the form $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$ on your graphing calculator.

15.8 CONIC SECTIONS IN POLAR COORDINATES

In Section 10.7, we studied graphs in the polar coordinate system. Some of the special curves we were able to draw were the rose and the cardioid. In this section, we will return to the polar coordinate system and see how we can graph the conic sections using polar coordinates.

As we progressed through this chapter, you saw that the equations of the conic sections have very simple forms if the center or vertex is at the origin. But, we had to learn a different form of each conic section. Polar coordinates will allow us to use one equation to represent each conic except the circle. Each of these conics will have a focus at the origin and one axis will be a coordinate axis.

When we defined the parabola, we said that it was the set of points that were an equal distance from the focus and the directrix. The ratio of these two distances, since the distances are the same, is 1. This ratio is called the *eccentricity*. In the problems for the ellipse and the hyperbola, we also used the eccentricity. Each conic can be defined in terms of a point on the curve and the ratio of its distance from a focus and a line called the directrix. This ratio is the eccentricity, e.

CAUTION Do not confuse the e used to denote the eccentricity with the number $e < 2.71828182846$.

Each type of conic is determined by the eccentricity and leads to the general definition of a conic section that follows.

GENERAL DEFINITION OF A CONIC SECTION

Let ℓ be a fixed line (the directrix) and F a fixed point (*focus*) not on ℓ. A *conic section* is the set of all points P in the plane such that

$$\frac{d(P, F)}{d(P, \ell)} = e$$

where $d(P, F)$ is the distance from P to F, and $d(P, \ell)$ is the perpendicular distance from P to ℓ. The constant e is called the *eccentricity* and if

$0 < e < 1$, the conic is an ellipse

$e = 1$, the conic is a parabola

$e > 1$, the conic is a hyperbola

NOTE Since $d(P, F)$ and $d(P, l)$ are distances, they are not negative, and so e cannot be negative.

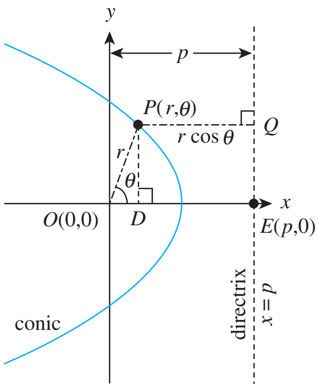


Figure 15.69a

Now, let $P(r, \theta)$ be a point on a conic with focus O located at the origin, directrix $x = p (p > 0)$ on the right of the origin, and eccentricity e. The polar coordinates of P are (r, θ) (see Figure 15.69a) and

$$\begin{aligned} d(P, Q) &= d(O, E) - d(O, D) \\ &= p - r\cos\theta \end{aligned}$$

where Q is a point on the directrix.
Then

$$\begin{aligned} e = \frac{d(P, O)}{d(P, Q)} &= \frac{r}{p - r\cos\theta} \\ e(p - r\cos\theta) &= r \\ ep - er\cos\theta &= r \\ ep &= r + er\cos\theta \\ ep &= r(1 + e\cos\theta) \\ \frac{ep}{1 + e\cos\theta} &= r \end{aligned}$$

If we had chosen the directrix as $x = -p (p > 0)$, then the directrix is on the left of the origin and the polar equation is

$$r = \frac{pe}{1 - e\cos\theta}$$

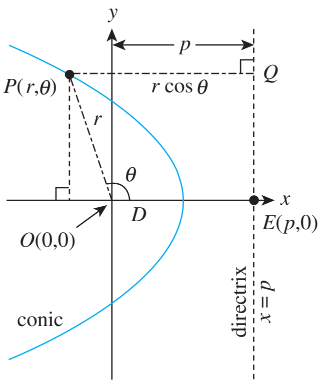


Figure 15.69b

Even if we had let θ be in the second quadrant, as in Figure 15.69b, we would have gotten the same result. Here, because θ is in the second quadrant, $r\cos\theta$ represents a negative number and $d(P, Q) = p - r\cos\theta$.

If the directrix is $y = \pm p (p > 0)$, the equations would be

$$r = \frac{pe}{1 \pm e\sin\theta}$$

Thus, we have two sets of equations, and by determining e, we can determine the type of curve. The previous results are summarized in the following box.

POLAR EQUATIONS OF CONIC SECTIONS

A polar equation that has one of the four forms

$$r = \frac{pe}{1 \mp e\cos\theta} \qquad r = \frac{pe}{1 \pm e\sin\theta}$$

is a conic section. The conic is a parabola if $e = 1$, an ellipse if $0 < e < 1$, or a hyperbola if $e > 1$.

EXAMPLE 15.56

Describe and sketch the graph of the equation:

$$r = \frac{10}{5 + 3\cos\theta}$$

SOLUTION Since the constant term in the denominator must be 1, we divide both numerator and denominator by 5. The equation then becomes

$$r = \frac{2}{1 + \frac{3}{5}\cos\theta}$$

From this, we see that $e = \frac{3}{5}$. Since $\frac{3}{5} < 1$, the conic is an ellipse. The denominator contains the cosine function and so the major axis of this conic section is horizontal. The vertices can be determined by setting θ equal to 0 and π. When $\theta = 0$, $r = \dfrac{2}{8/5} = \dfrac{10}{8} = 1.25$ and when $\theta = \pi$, $r = \dfrac{2}{2/5} = 5$. So $2a = 5 + 1.25 = 6.25$, or $a = 3.125$. The eccentricity $e = \dfrac{c}{a}$, so $\dfrac{3}{5} = \dfrac{c}{3.125}$ and we get $c = 1.875$. Finally, $b^2 = a^2 - c^2 = 6.25$, thus $b = 2.5$. The sketch of this ellipse is given in Figure 15.70.

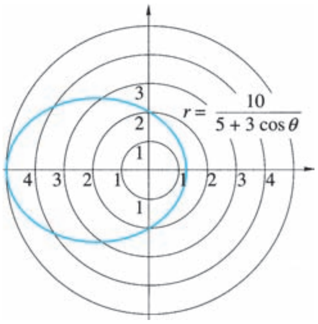


Figure 15.70

EXAMPLE 15.57

Describe and sketch the graph of the equation:

$$r = \frac{8}{2 + 2\sin\theta}$$

SOLUTION We divide the numerator and denominator by 2, obtaining

$$r = \frac{4}{1 + 1\sin\theta}$$

From this we see that $e = 1$, and the curve is a parabola. Also, since $pe = 4$, $p = 4$. This curve has a vertical axis, so the directrix is $y = 4$. If we plot the points that correspond to the x- and y-intercepts, we get the following table and the curve in Figure 15.71.

θ	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$
r	4	2	4	Not defined

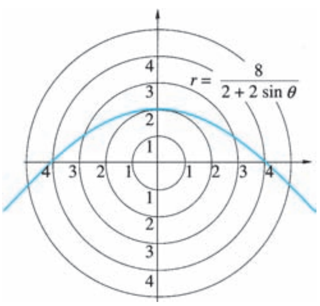


Figure 15.71

In Section 15.7, we learned a very complicated process for rotating conic sections on a rectangular coordinate system. The polar coordinate equation of a rotated conic is quite simple. Consider the ellipse in Figure 15.72. It has been rotated through a positive angle α about its focus at the origin O. In the rotated system with polar coordinates (r', θ'), the ellipse has the equation:

$$r' = \frac{pe}{1 - e\cos\theta'}$$

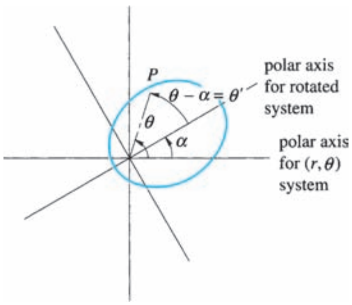


Figure 15.72

But, $r' = r$ and $\theta' = \theta - \alpha$, so the equation of this ellipse in the original (unrotated) polar coordinate system is

$$r = \frac{pe}{1 - e\cos(\theta - \alpha)}$$

Since, in this example, the conic is an ellipse, $0 < e < 1$. But if $e = 1$, the conic would be a parabola, and if $e > 1$, the conic would be a hyperbola.

EXAMPLE 15.58

Discuss and sketch the graph of the equation:

$$r = \frac{1.30}{1 + 0.65\cos\left(\theta + \frac{\pi}{6}\right)}$$

SOLUTION This equation has a denominator that begins with the number 1, so we can immediately see that $e = 0.65$. Since $e < 1$, the conic is an ellipse. The angle is $\theta + \frac{\pi}{6}$, so the major axis is rotated $\alpha = -\frac{\pi}{6}$. The denominator contains the cosine function, thus the major axis is the horizontal axis, which has been rotated $-\frac{\pi}{6}$ rad. Again, the vertices can be determined by setting $\theta = 0$ and $\theta = \pi$. This gives $V = 0.83$ and $V' = 2.97$, so $2a = 3.80$ and $a = 1.90$. The eccentricity $e = \frac{c}{a}$, so $c = ea = (0.65)(1.90) = 1.235$. Thus, $b^2 = a^2 - c^2 = 2.085$ or $b \approx 1.444$. The sketch of this ellipse is shown in Figure 15.73.

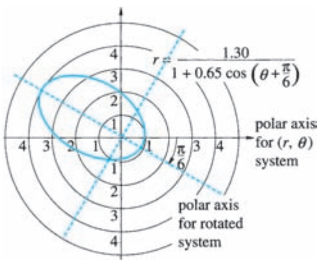


Figure 15.73

We will see some applications of this polar equation for a rotated conic section in the following exercise set and again in Chapter 21.

USING A GRAPHING CALCULATOR

In order to graph a conic section in polar coordinates, you must make sure that your calculator is in polar mode. (Not all graphing calculators have a polar mode, so check your user's manual.) Once your calculator is in polar mode, you graph a conic section by entering the function $r(\theta)$ into the calculator, setting an appropriate viewing window, and graphing the function.

EXAMPLE 15.59

Use a graphing calculator to graph $r = \dfrac{1.30}{1 + 0.65\cos\left(\theta + \dfrac{\pi}{6}\right)}$.

SOLUTION This is the same function we graphed in Example 15.58, so the result should look much like the graph in Figure 15.73.

Make sure that the calculator is in both radian and polar modes. On a TI-83 or TI-84, press Y= and enter the right-hand side of the equation, as shown in Figure 15.74a. Using the window settings $\theta\min = 0$, $\theta\max = 2\pi$, $\theta\text{step} = 0.1$, $X\min = -4.7$, $X\max = 4.7$, $X\text{scl} = 1$, $Y\min = -3.1$, $Y\max = 3.1$, $Y\text{scl} = 1$, you should get the result in Figure 15.74b.

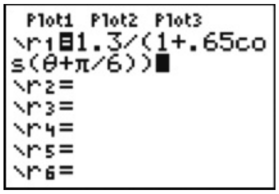


Figure 15.74a

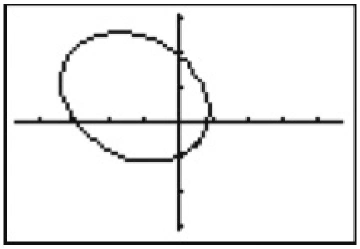


Figure 15.74b

USING A SPREADSHEET

In order to graph a function written in the form $r(\theta)$, you must first convert the polar coordinates to rectangular coordinates.

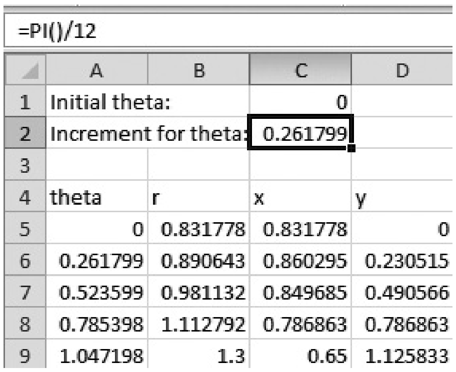
EXAMPLE 15.60

Use a spreadsheet to graph $r = \dfrac{1.30}{1 + 0.65\cos\left(\theta + \dfrac{\pi}{6}\right)}$.

SOLUTION This is the same function we graphed in Examples 15.58 and 15.59, so the result should look much like the graph in Figures 15.73 and 15.74b.

The table of values will have a θ column, an r column, and then an x and y column. θ needs to go from 0 to 2π.

Enter `=1.3/(1+0.65*COS(A5+PI()/6))` in Cell B5, `=B5*COS(A5)` in Cell C5, and `=B5*SIN(A5)` in Cell D5 (see Figure 15.75a). The graph is shown in Figure 15.75b.


=PI()/12

	A	B	C	D
1	Initial theta:		0	
2	Increment for theta:		0.261799	
3				
4	theta	r	x	y
5	0	0.831778	0.831778	0
6	0.261799	0.890643	0.860295	0.230515
7	0.523599	0.981132	0.849685	0.490566
8	0.785398	1.112792	0.786863	0.786863
9	1.047198	1.3	0.65	1.125833

Figure 15.75a

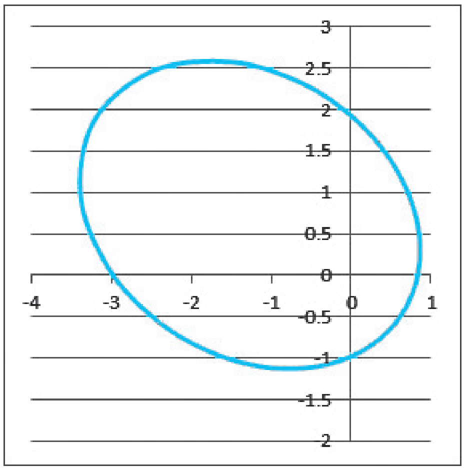


Figure 15.75b

EXERCISE SET 15.8

In Exercises 1–10, identify and sketch the conic section with the given equation.

1. $r = \dfrac{6}{1 + 3\cos\theta}$
2. $r = \dfrac{8}{1 - 2\sin\theta}$
3. $r = \dfrac{12}{3 - \cos\theta}$
4. $r = \dfrac{15}{3 + 9\sin\theta}$
5. $r = \dfrac{12}{4 - 4\cos\theta}$
6. $r = \dfrac{8}{6 + 6\sin\theta}$
7. $r = \dfrac{12}{3 + 2\cos\theta}$
8. $r = \dfrac{12}{2 - 3\cos\theta}$
9. $r = \dfrac{12}{2 - 4\cos\left(\theta + \frac{\pi}{3}\right)}$
10. $r = \dfrac{6}{2 + \sin\left(\theta + \frac{\pi}{6}\right)}$

In Exercises 11–16, find a polar equation of the conic with focus at the origin and the given eccentricity and directrix.

11. Directrix $x = 4$; $e = \frac{3}{2}$
12. Directrix $x = -2$; $e = \frac{3}{4}$
13. Directrix $y = -5$; $e = 1$
14. Directrix $y = 3$; $e = 2$
15. Directrix $x = 1$; $e = \frac{2}{3}$
16. Directrix $x = 5$; $e = 1$

Solve Exercises 17–20.

17. ***Astronomy*** The planet Mercury travels around the sun in an elliptical orbit given approximately by

$$r = \frac{3.442 \times 10^7}{1 - 0.206\cos\theta}$$

where r is measured in miles and the sun is at the pole. Determine Mercury's greatest and shortest distance from the sun.

18. ***Mechanical engineering*** An engine gear is tested for dynamic balance by rotating it and tracing a polar graph of a point on its rim. Draw the graph given by the equation $r = \dfrac{4}{1 - 0.05\sin\theta}$ and determine the state of balance of the gear.

19. ***Mechanical engineering*** A cam is shaped such that the equation of the upper half is given by $r = 2 + \cos\theta$ and the equation of the lower half is given by $r = \dfrac{3}{2 - \cos\theta}$. Sketch the shape of this cam.

20. ***Astronomy*** A certain comet is following a parabolic path, with the sun as the focus. When the comet is 50 000 000 km from the center of the sun, the line from the comet to the sun makes an angle of 45° with the axis of the parabola.
    (a) Write an equation that describes the path of this comet.
    (b) Sketch the path of this comet.
    (c) How close will the comet come to the center of the sun; that is, how far is the vertex of this parabola from the sun?

[IN YOUR WORDS]

21. Describe how to tell if a polar equation of a conic section is a parabola, ellipse, or hyperbola.
22. Explain the differences between the e used to denote the eccentricity and the number $e \approx 2.71828182846$.

CHAPTER 15 REVIEW

IMPORTANT TERMS AND CONCEPTS

Angle of inclination
Circle
  Center
  Radius
Degenerate conics
Distance formula
Eccentricity
Ellipse
  Center
  Foci
  Major axis
  Minor axis
  Vertices
Hyperbola
  Asymptotes
  Center
  Conjugate axis
  Foci
  Transverse axis
  Vertices
Length of a curve
Midpoint formula
Parabola
  Axis
  Directrix
  Focus
  Vertices
Polar equations for conic sections
Rotation of axes
Slope
Translation of axes
x-intercept
y-intercept

REVIEW EXERCISES

For each pair of points in Exercises 1–4, find (a) the distance between them, (b) their midpoint, (c) the slope of the line through the points, and (d) the equation of the line through the points.

1. $(2, 5)$ and $(-1, 9)$
2. $(-2, -5)$ and $(10, -10)$
3. $(1, -4)$ and $(3, 6)$
4. $(2, -5)$ and $(-6, 3)$

Solve Exercises 5–8.

5. For each line in Exercises 1–4, find the slope of one of its perpendiculars.

6. For each pair of points in Exercises 1–4, write the equation for the line passing through the midpoint of each pair and perpendicular to the line through them.

7. What is the equation of the line that passes through the point $(-3, 5)$ and is parallel to $2y + 4x = 9$?

8. What is the equation of the line through $(2, -7)$ with a slope of 4?

Sketch each of the conic sections in Exercises 9–28.

9. $x^2 + y^2 = 16$
10. $x^2 - y^2 = 16$
11. $x^2 + 4y^2 = 16$
12. $y^2 = 16x$
13. $(x - 2)^2 + (y + 4)^2 = 16$
14. $(x - 2)^2 - (y + 4)^2 = 16$
15. $(x - 2)^2 + 4(y + 4)^2 = 16$
16. $(y + 4)^2 = 16(x - 2)$
17. $x^2 + y^2 + 6x - 10y + 18 = 0$
18. $x^2 - 4y^2 + 6x + 40y - 107 = 0$
19. $x^2 + 4y^2 + 6x - 40y + 93 = 0$
20. $x^2 + 6x - 4y + 29 = 0$
21. $2x^2 + 12xy - 3y^2 - 42 = 0$
22. $5x^2 - 4xy + 8y^2 - 36 = 0$

23. $3x^2 + 2\sqrt{3}xy + y^2 + 8x - 8\sqrt{3}y = 32$

24. $2x^2 - 4xy - y^2 = 6$

25. $r = \dfrac{16}{5 - 3\cos\theta}$

26. $r = \dfrac{9}{3 - 5\cos\theta}$

27. $r = \dfrac{9}{3 + 3\sin\theta}$

28. $r = \dfrac{2}{1 + \cos\theta}$

Solve Exercises 29 and 30.

29. **(a)** Graph $y^2 + 6y = 16x + 13$.
(b) Identify the graph.
(c) Specify all the "important" points, such as foci, vertices, etc.

30. **(a)** Graph $16x^2 - 9y^2 = 144$.
(b) Identify the graph.
(c) Specify all the "important" points, such as foci, vertices, etc.

Solve Exercises 31–35.

31. ***Civil engineering*** A concrete bridge for a highway overpass is constructed in the shape of half an elliptic arch, as shown in Figure 15.76. The arch is 100 m long. In order to have enough clearance for tall vehicles, the arch must be 6.1 m high at a point 5 m from the end of the arch. What is the equation of the ellipse for this arch, if the origin is at the midpoint of the major axis?

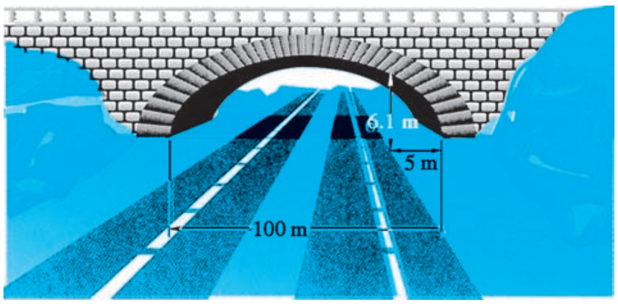


Figure 15.76

32. ***Civil engineering*** The main cables on a suspension bridge approximate a parabolic shape. The twin towers of a suspension bridge are to be 120 m above the road surface and are 400 m apart. If the cable's lowest point is 10 m above the road surface, what is the equation of the parabola for the main cable?

33. ***Navigation*** An airplane sends out an impulse that travels at the speed of sound (320 m/μs). The plane is 50 km south of the line connecting two receiving stations. The stations are on an east-west line with station A 400 km west of station B. Station A receives the signal from the plane 500 μs after station B. What is the location of the plane?

34. ***Astronomy*** A map of the solar system is drawn so that the surface of earth is represented by the equation $x^2 + y^2 - 2x + 4y - 6361 = 0$. A satellite orbits earth in a circular orbit 0.8 units above earth. What is the equation of the satellite's orbit on this map?

35. ***Landscaping*** Some edging needs to be placed around the fish pond in Figure 15.77. What is the length of the edging that will be needed?

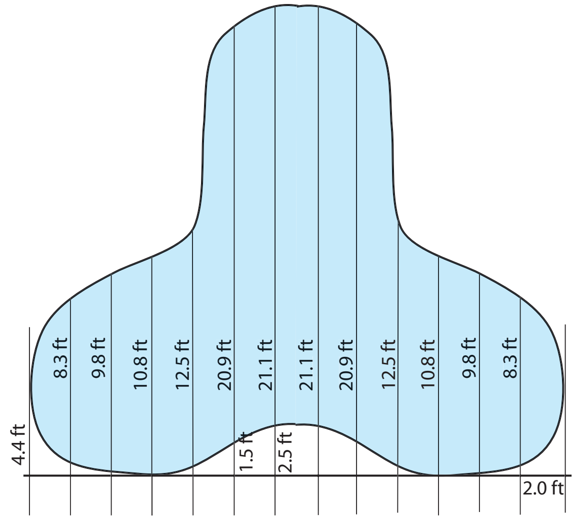


Figure 15.77

CHAPTER 15 TEST

1. Find the focus and the directrix of the parabola $y^2 = -18x$, and sketch the graph.
2. Graph the hyperbola $9x^2 - 4y^2 = 36$. Specify the foci, vertices, and endpoints of the conjugate axis.
3. **(a)** Graph $16x^2 + 9y^2 = 144$.
   (b) Identify the graph.
   (c) Specify all the "important" points, such as foci, vertices, etc.
4. Determine the distance between the points $(-2, 4)$ and $(5, -6)$.
5. What is the equation of the line that is perpendicular to the line through the points $(4, -5)$ and $(2, -8)$ and passes through their midpoint?
6. The x-intercept of a line is 3, and its angle of inclination is 30°. Write the equation of the line in slope-intercept form.
7. **(a)** Graph $14(x + 1)^2 - 9(y - 2)^2 = 36$.
   (b) Identify the graph.
   (c) Specify all the "important" points, such as foci, vertices, etc.
8. **(a)** Write the equation $3x^2 + 4y^2 - 6x + 16y + 7 = 0$ in the appropriate standard form. **(b)** Determine the type of conic section described by this equation. **(c)** Determine the coordinates of all "significant" points, such as vertices and foci. **(d)** Graph the equation.
9. **(a)** Determine the angle needed to rotate the coordinate axes so that the transformed equation of $x^2 + xy + y^2 = 4$ has no xy-term. **(b)** Identify the graph.
10. Identify and sketch the graph of the polar equation

$$r = \frac{2}{1 + 4\cos\theta}.$$

11. A cable hangs in a parabolic curve between two vertical supports that are 120 ft apart. At a distance 48 ft in from each support, the cable is 3.0 ft above its lowest point. How high up is the cable attached on each support?

16 COMPUTER NUMBER SYSTEMS

© iStockphoto.com/Bonita Hein.
"Binary Code" by Bonita Hein. istockphoto Stock photoFile #: 405671

Binary numbers like the ones in this photo consist only of 1s and 0s. In Section 16.2 we will learn how to add and subtract binary numbers to prepare us to program computers.

There are many programming languages. Among the ones you may have heard of are BASIC, FORTRAN, PASCAL, COBOL, Lisp, Ada, and C++. All of these languages have to be converted to numbers in order to communicate with the computer. Each number in a computer is represented by a binary switch. Much like a light bulb, a binary switch is either on or off.

Because a binary switch is either on or off, or a diode is either conducting or not, it is convenient to represent these options by the binary numbers 0 and 1. It requires many binary digits, or *bits*, to represent large numbers, and so octal and hexadecimal numbers are often used because they require much less space. In this chapter you will learn the basics of binary, octal, and hexadecimal numbers, how to convert numbers from one system to the other and to the decimal system, how to perform addition and subtraction in these systems and some ways in which these numbers are used in information and computer technology.

OBJECTIVES

AFTER COMPLETING THIS CHAPTER, YOU WILL BE ABLE TO:

- Express binary numbers as decimal numbers.
- Express decimal numbers as binary numbers.
- Determine the 2's complement of a binary number.
- Express decimal numbers as BCD numbers.
- Add and subtract binary numbers.
- Express octal numbers as decimal numbers.
- Express decimal numbers as octal numbers.
- Determine the 8's complement of an octal number.
- Add and subtract octal numbers.
- Express hexadecimal numbers as decimal numbers.
- Express decimal numbers as hexadecimal numbers.
- Determine the 16's complement of a hexadecimal number.
- Add and subtract hexadecimal numbers.

16.1 THE BINARY NUMBER SYSTEM

In order to understand the binary number system we must first briefly review decimal numbers.

DECIMAL NUMBERS

The decimal number system uses 10 symbols called **digits** to form numbers. The digits in the decimal number system are 0, 1, 2, 3, 4, 5, 6, 7, 8, and 9. The number of digits in a system is called the **radix**. So, the radix of the decimal numbers is 10.

In Section 1.1 we learned about the place values of the decimal number system. The place values, and the place names, most often used with the decimal numbers were shown in Figure 1.3, and are duplicated below as Table 16.1. You might notice that as we move from right to left each place value is 10 times the value of the place to its immediate right.

TABLE 16.1

Place name		Ten thousands	Thousands	Hundreds	Tens	Units or ones		Tenths	Hundredths	Thousandths	Ten thousandths	
Place value		10^4	10^3	10^2	10^1	10^0	.	10^{-1}	10^{-2}	10^{-3}	10^{-4}	...
	...	10000	1000	100	10	1	.	0.1	0.01	0.001	0.0001	...

Decimal point

Any decimal number other than the 10 digits is expressed using these digits in combination with the power of 10 associated with its place value. For example, the number 4597 is expressed in **expanded form** as

$$\begin{aligned} 4597 &= 4000 + 500 + 90 + 7 \\ &= (4 \times 1000) + (5 \times 100) + (9 \times 10) + (7 \times 1) \\ &= (4 \times 10^3) + (5 \times 10^2) + (9 \times 10^1) + (7 \times 10^0) \end{aligned}$$

and the number 37.582 is expressed in expanded form as

$$\begin{aligned} 37.582 &= 30 + 7 + 0.5 + 0.08 + 0.002 \\ &= (3 \times 10) + (7 \times 1) + (5 \times 0.1) + (8 \times 0.01) + (2 \times 0.001) \\ &= (3 \times 10^1) + (7 \times 10^0) + (5 \times 10^{-1}) + (8 \times 10^{-2}) + (2 \times 10^{-3}) \end{aligned}$$

EXAMPLE 16.1

Write the decimal number 784.39 in expanded form.

SOLUTION First expand the number and then use the place values from Table 16.1 to write the number in expanded form:

$$\begin{aligned} 784.39 &= 700 + 80 + 4 + 0.3 + 0.09 \\ &= (7 \times 100) + (8 \times 10) + (4 \times 1) + (3 \times 0.1) + (9 \times 0.01) \\ &= (7 \times 10^2) + (8 \times 10^1) + (4 \times 10^0) + (3 \times 10^{-1}) + (9 \times 10^{-2}) \end{aligned}$$

EXAMPLE 16.2

Write the decimal number 406.37 in expanded form.

SOLUTION First expand the number and then use the place values from Table 16.1 to write the number in expanded form:

$$\begin{aligned} 406.37 &= 400 + 00 + 6 + 0.3 + 0.07 \\ &= (4 \times 100) + (0 \times 10) + (6 \times 1) + (3 \times 0.1) + (7 \times 0.01) \\ &= (4 \times 10^2) + (0 \times 10^1) + (6 \times 10^0) + (3 \times 10^{-1}) + (7 \times 10^{-2}) \end{aligned}$$

But, we do not have to use the zero digits in the final answer:

$$= (4 \times 10^2) + (6 \times 10^0) + (3 \times 10^{-1}) + (7 \times 10^{-2})$$

BINARY NUMBERS

The **binary number system** has just two digits: 0 and 1, so its radix is 2. Place values in the binary system are expressed in terms of powers of two, as shown in Table 16.2.

TABLE 16.2

Place value	2^6	2^5	2^4	2^3	2^2	2^1	2^0	.	2^{-1}	2^{-2}	2^{-3}
Decimal value	64	32	16	8	4	2	1	↑	0.5	0.25	0.125
								Binary point			

A **binary number** is a sequence of binary digits. For example, 11001101 is a **binary integer** because it has no fractional part. On the other hand, the number 10010.1101 does have a fractional part, as indicated by the **binary point** in the number.

BITS, NIBBLES, AND BYTES

Each binary digit is called a **bit**, for *binary digit*. A group of eight bits is called a **byte**, and a **nibble** is half a byte, or four bits. The largest string of bits that a computer can handle in one operation is a **word** and the **word length** is the number of bits in a word.

Different computers use different word lengths. The original personal computers used 8 bits but most desktop computers today use 32 or 64 bits.

A **kilobyte** (KB) is 1024 (2^{10}) bytes. A **megabyte** (MB) is 1 048 576 or 2^{20} bytes, a **gigabyte** (GB) is 1024 megabytes or 1 073 741 824 bytes $= 2^{30}$ bytes, and a **terabyte** (TB) is 1024 gigabytes or $= 2^{40}$ bytes.

NOTE Even though the kilo in its name indicates a kilobyte should have 1000 bytes, it has 1024.

WRITING BINARY NUMBERS

A binary number is usually written with a subscript of "2" or "two" so it is clear that it is not a decimal number. The **base** of the binary numbers is two. Thus, the binary number 11001101 should be written either as 11001101_2 or 11001101_{two}. Long binary numbers are often written in nibbles to make them easier to read. Thus, 11001101_2 would be written in nibbles as $1100\ 1101_2$.

NOTE A number without an indicated base is assumed to be a decimal number.

EXPANDED NOTATION

The value of any bit is the product of the bit and its place value. The decimal value of the entire number is the sum of these products. Table 16.2 gave a partial list of the place values. A more complete list is in Table 16.3.

TABLE 16.3

Place value	2^{10}	2^9	2^8	2^7	2^6	2^5	2^4	2^3	2^2	2^1	2^0	.	2^{-1}	2^{-2}	2^{-3}	2^{-4}	2^{-5}
Decimal value	1024	512	256	128	64	32	16	8	4	2	1	↑ Binary point	0.5	0.25	0.125	0.0625	0.03125

EXAMPLE 16.3

Write the binary number $1100\ 1101_2$ in expanded form.

SOLUTION Expand the number using the place values from Table 16.3:

$$\begin{aligned}1100\ 1101_2 &= (1 \times 128) + (1 \times 64) + (0 \times 32) + (0 \times 16) + (1 \times 8) + \\ &\quad (1 \times 4) + (0 \times 2) + (1 \times 1) \\ &= (1 \times 2^7) + (1 \times 2^6) + (0 \times 2^5) + (0 \times 2^4) + (1 \times 2^3) + \\ &\quad (1 \times 2^2) + (0 \times 2^1) + (1 \times 2^0) \\ &= (1 \times 2^7) + (1 \times 2^6) + (1 \times 2^3) + (1 \times 2^2) + (1 \times 2^0)\end{aligned}$$

Notice that you only need to add the place values that have digits of 1. You do not need to consider the zero digits.

BINARY-TO-DECIMAL CONVERSION

To convert a binary number to its decimal number equivalent, write the number in expanded form and add the result.

EXAMPLE 16.4

Convert the binary number $1100\ 1101_2$ to its decimal equivalent.

SOLUTION In Example 16.2 we found the expanded form of $1100\ 1101_2$:

$$\begin{aligned}1100\ 1101_2 &= (1 \times 2^7) + (1 \times 2^6) + (1 \times 2^3) + (1 \times 2^2) + (1 \times 2^0) \\ &= (1 \times 128) + (1 \times 64) + (1 \times 8) + (1 \times 4) + (1 \times 1) \\ &= 128 + 64 + 8 + 4 + 1 = 205\end{aligned}$$

The decimal equivalent of $1100\ 1101_2$ is 205.

EXAMPLE 16.5

Convert the binary number $110\ 1011_2$ to its decimal equivalent.

SOLUTION

$$\begin{aligned}110\ 1011_2 &= (1 \times 2^6) + (1 \times 2^5) + (1 \times 2^3) + (1 \times 2^1) + (1 \times 2^0) \\ &= (1 \times 64) + (1 \times 32) + (1 \times 8) + (1 \times 2) + (1 \times 1) \\ &= 64 + 32 + 8 + 2 + 1 = 107\end{aligned}$$

The decimal equivalent of $110\ 1011_2$ is 107.

The process is the same even if the number contains a binary point and is not a binary integer.

EXAMPLE 16.6

Convert the binary number 1110.101_2 to its decimal equivalent.

SOLUTION

$$\begin{aligned}1110.101_2 &= (1 \times 2^3) + (1 \times 2^2) + (1 \times 2^1) + (1 \times 2^{-1}) + (1 \times 2^{-3}) \\ &= (1 \times 8) + (1 \times 4) + (1 \times 2) + (1 \times 0.5) + (1 \times 0.125) \\ &= 8 + 4 + 2 + 0.5 + 0.125 = 14.625\end{aligned}$$

The decimal equivalent of 1110.101_2 is 14.625.

DECIMAL-TO-BINARY CONVERSION

Converting decimal integer to a binary number requires a different method than converting a decimal fraction to a binary number. We will examine each in turn.

CONVERTING DECIMAL INTEGERS TO BINARY INTEGERS

To convert a decimal integer to a binary number we use **repeated division** and a method called the **method of remainders**. Begin by dividing the decimal number by the base, 2. Record the remainder and then divide 2 into the quotient. Continue until the quotient is zero.

EXAMPLE 16.7

Convert the decimal number 37 to its binary equivalent.

SOLUTION Begin by dividing 37 by 2.

$$\begin{array}{rl} 18 & \\ 2\overline{)37} & \\ \underline{36} & \\ 1 & \qquad \text{Remainder} = 1 \end{array}$$

We will use a different method for showing our work. Because we will next divide 2 into the quotient, in this case 18, we will write the quotient below and the remainder to the right. This makes the previous division look like this:

$$\begin{array}{rc} & \text{Remainders} \\ 2\underline{)\,37} & \\ 18 & 1 \end{array}$$

In the next step, divide 2 into 18. The quotient is 9 and the remainder is 0. We show this by placing the division under the previous division, as shown here:

$$\begin{array}{rc} & \text{Remainders} \\ 2\underline{)\,37} & \\ 2\underline{)\,18} & 1 \\ 2\quad 9 & 0 \end{array}$$

Continue in this manner until you get a quotient of zero. This should produce the following:

$$\begin{array}{rcc} & \text{Remainders} & \\ 2\underline{)\;\;37} & & \\ 2\underline{)\;\;18} & 1 & \uparrow \\ 2\underline{)\;\;\;9} & 0 & | \\ 2\underline{)\;\;\;4} & 1 & | \\ 2\underline{)\;\;\;2} & 0 & | \\ 2\underline{)\;\;\;1} & 0 & | \\ 0 & 1 & | \end{array}$$

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The binary digits, when written from left to right, are the remainders reading from bottom to top. Thus, $37 = 10\ 0101_2$.

EXAMPLE 16.8

Use the method of remainders to convert the decimal number 91 to its binary equivalent.

SOLUTION

	Remainders
2) 91	
2) 45	1
2) 22	1
2) 11	0
2) 5	1
2) 2	1
2) 1	0
0	1

Reading the remainders from bottom to top, we see that $91 = 101\ 1011_2$.

CONVERTING DECIMAL FRACTIONS TO BINARY FRACTIONS

To convert a decimal fraction to a binary fraction we use **repeated multiplication**. Begin by multiplying the decimal fraction by 2. Record the integral digit in the product that is to the left of the decimal point (either a 0 or 1) and then multiply the fractional result by 2. Repeat the process until the fractional part is zero or repeats. The integral digits, in order, represent the binary fraction.

EXAMPLE 16.9

Use the method of repeated multiplication to convert the decimal fraction 0.8125 to its binary equivalent.

SOLUTION Begin by multiplying 2×0.8125. The answer is 1.625. Record the integer to the left of the decimal point in the product, here a 1, and then multiply the fractional result, in this case 0.625, by 2:

Fractional Part	Integral Digits
$2(0.8125) = 1.\overbrace{625}$	1
$2(0.625) = 1.25$	1
$2(0.25) = 0.5$	0
$2(0.5) = 1.0$	1

Reading the integral digits from top to bottom, we see that $0.8125 = 0.1101_2$.

Some decimal fractions cannot be written as terminating decimals but must be written as repeating decimals. For example, $\frac{1}{3} = 0.3333\ldots = 0.\overline{3}$ and $\frac{3}{110} = 0.0272727\ldots = 0.0\overline{27}$. A horizontal bar is placed over the part that repeats.

Converting a decimal fraction to a binary fraction may result is a repeating binary fraction. You can tell if it will repeat when a fractional part repeats after multiplying by 2. This is shown by the next example.

EXAMPLE 16.10

Use the method of repeated multiplication to convert the decimal fraction 0.675 to its binary equivalent.

SOLUTION

	Integral Digits
$2(0.675) = 1.35$	1
$2(0.35) = 0.7$	0
$2(0.7) = 1.4$	1
$2(0.4) = 0.8$	0
$2(0.8) = 1.6$	1
$2(0.6) = 1.2$	1
$2(0.2) = 0.4$	0

Since the fractional part, 4, is a value we have seen before, we know that the fractional part repeats.

Reading the integral digits from top to bottom, we see that $0.675 = 0.11\overline{0110}_2$.

EXAMPLE 16.11

Convert 23.125 to its binary equivalent.

SOLUTION We have to use both methods for this conversion. We use the method of repeated remainders on 23, the part to the left of the decimal point, and repeated multiplication on 0.125, the part to the right of the decimal point:

	Remainders
2) 23	
2) 11	1
2) 05	1
2) 02	1
2) 1	0
0	1

	Integral Digits
$2(0.125) = 0.25$	0
$2(0.25) = 0.5$	0
$2(0.5) = 1.0$	1

Reading the remainders from bottom to top and the integral digits from top to bottom, we see that $23.125 = 10111.001_2$.

SUBNET MASKING

Information technology uses **subnet masking** to allow the TCP/IP protocol stack to determine where to send packets of data. It is a method used to segment a network and give each segment a network ID so that other networks can still communicate with it. Segmenting the network greatly reduces traffic, because not all computers are trying to use the same bandwidth. It often becomes necessary to send packets of information from one network segment to another, and in such a case, a bridge or router must be used to combine the networks.

A **router** is a device that can distinguish the destination network ID of a packet sent on the network by using the destination IP address and subnet mask, and can route that packet accordingly, without having to send it to all the other network segments. If a packet is not intended for a computer on a specific segment, the router will filter out that packet to reduce traffic on the network, and only send it to the segment that contains the destination computer's IP address.

An **IP address** is a binary address consisting of 16 eight-bit numbers, or **octets**. These octets can be converted back and forth between dot-decimal and binary notation. You may be accustomed to an IP address in dot-decimal form. Such an address is 192.168.5.1.37.54.121.2.252.187.212.5.12.91.246.4. The computer sees this address as 11000000.10101000.00000101.00000001.00100101.00110110.01111001.00000010.11111100.10111011.11010100.00000101.00001100.01011011.11110110.00000100.

Starting in 2011, all IP addresses have a total of 128 bits, or digits that can assume either a 1 or 0 value. Even if all eight bits were ones, an octet can only add up to 255. An IP octet never go above 255 because $1111\ 1111_2 = 255$.

What we have been discussing is the Internet Protocol version 6 (IPv6), which began being used around 2010. Before that, Internet Protocol version 4 (IPv4) was used. It had four eight-bit numbers and the dot-decimal form of an IPv4 address looked like 192.168.5.1, which the computer saw as 11000000.10101000.00000101.00000001. There were $2^{32} = 4{,}294{,}967{,}296$ possible different addresses with IPv4, and in the early 2010s these were all used up. With IPv6, there are $2^{128} \approx 3.4 \times 10^{38}$ different addresses. Because the IPv6 address is so long, we will look at examples of IPv4 addresses in this section and in Section 16.5, we will look at IPv6 addresses.

APPLICATION ELECTRONICS

EXAMPLE 16.12

What is the dot-decimal version of the IPv4 address 11110110.10001001.10100111.10100110?

SOLUTION We will work with each octet or byte separately, beginning with the octet on the left.

Converting 11110110_2 to base 10, we get 246. Next, we see that $10001001_2 = 137$, then $10100111_2 = 167$, and finally $10100110_2 = 166$.

Thus, we see that the IP address 11110110.10001001.10100111.10100110 has the dot-decimal equivalent of 246.137.167.166.

EXERCISE SET 16.1

In Exercises 1–8, write each decimal number in expanded form.

1. 7259
2. 329
3. 12,953
4. 137,489
5. 28.137
6. 752.49
7. 321.794
8. 123.7814

In Exercises 9–12, write each binary number in expanded form.

9. 1101_2
10. $11\ 0110_2$
11. $1\ 1101.011_2$
12. 1001.0101_2

In Exercises 13–24, convert the given binary number to its decimal equivalent.

13. 1101_2
14. 101_2
15. 1111_2
16. $1111\ 1111_2$
17. $101\ 0011_2$
18. $1111\ 1101_2$
19. $10\ 1101\ 0111_2$
20. $101\ 0111\ 1100_2$
21. $1\ 1101.011_2$
22. $10\ 1101.0101_2$
23. $101\ 1110.01101_2$
24. 1001.0101_2

In Exercises 25–40, convert the given decimal number to its binary equivalent.

25. 97
26. 64
27. 54
28. 17
29. 35
30. 275
31. 0.25
32. 0.625
33. 0.95
34. 0.275
35. 12.75
36. 42.375
37. 144.875
38. 257.4375
39. 196.59375
40. 512.90625

Solve Exercises 41–42, using the following information

The **BCD code** or **binary-coded decimal code** is one method used by computers to store data. With the BCD code each nibble represents a decimal digit. For example, since $0101_2 = 5$ and $1001_2 = 9$, the decimal number 59 would be represented in BCD code as 0101 1001.

41. ***Computer technology*** Write the decimal number that represents the following BCDs.
(a) 0110 0011
(b) 0100 0111
(c) 0001 0111 0100
(d) 0001 0000.0101

42. ***Computer technology*** Write each of the following decimal numbers in BCD.
(a) 98
(b) 7
(c) 632
(d) 54.7

Solve Exercises 43–50.

43. ***Information technology*** Determine the dot-decimal version of the IPv4 address 10110110.00001001.10100101.10101110.

44. ***Information technology*** Determine the dot-decimal version of the IPv4 address 00101001.11101111.00011111.00101010.

45. ***Information technology*** Determine the dot-decimal version of the IPv4 address 01101001.00011001.10110101.10100010.

46. ***Information technology*** Determine the dot-decimal version of the IPv4 address 00110000.00010010.00100101.00001110.

47. ***Information technology*** What is the binary version of the IPv4 dot-decimal address 150.72.180.140?

48. ***Information technology*** Determine the binary version of the IPv4 dot-decimal address 75.95.125.165.

49. ***Information technology*** Determine the binary version of the IPv4 dot-decimal address 107.207.57.71.

50. ***Information technology*** What is the binary version of the IPv4 dot-decimal address 254.1.8.171?

[IN YOUR WORDS]

51. List and describe two differences between the decimal number system and the binary number system.

52. Describe how to change a binary number to a decimal number.

53. Describe how to change a decimal integer to a binary integer.

54. Describe how to change a decimal fraction to a binary fraction.

16.2 BINARY ARITHMETIC

The rules for binary arithmetic are essentially the same as for decimal numbers. But, because the binary system has only two digits, addition is much easier.

BINARY ADDITION

Table 16.4 is an addition table for binary numbers.

TABLE 16.4 Binary Addition

+	0	1
0	0	1
1	1	10

EXAMPLE 16.13

Determine the sum $1110_2 + 101_2$.

SOLUTION We will place the numbers in columns and start adding with the rightmost column and work to the left using the sums in Table 16.4:

$$\begin{array}{r} 1\,1\,1\,0_2 \\ 1\,0\,1_2 \\ \hline 1_2 \end{array} \Rightarrow \begin{array}{r} 1\,1\,1\,0_2 \\ 1\,0\,1_2 \\ \hline 1\,1_2 \end{array} \Rightarrow \begin{array}{r} \mathit{1} \\ 1\,1\,1\,0_2 \\ 1\,0\,1_2 \\ \hline 0\,1\,1_2 \end{array} \Rightarrow \begin{array}{r} \mathit{1} \\ 1\,1\,1\,0_2 \\ 1\,0\,1_2 \\ \hline 1\,0\,0\,1\,1_2 \end{array} \quad \text{carried digits}$$

Notice in the third column that $1_2 + 1_2 = 10_2$. A 1 was carried to the fourth column as shown in italics at the top.

Thus, we see that $1110_2 + 101_2 = 10011_2$.

EXAMPLE 16.14

Determine the sum $110.1_2 + 11.11_2$.

SOLUTION Again, we place the numbers in columns according to their place value. You might find it helpful to rewrite 1101.1_2 as 1101.10_2. Start adding with the rightmost column, and work to the left using the sums in Table 16.4:

$$\begin{array}{r} \\ 1\,1\,0.\,1_2 \\ 1\,1.\,1\,1_2 \\ \hline \\ \end{array} \Rightarrow \begin{array}{r} \text{Rewrite} \\ 1\,1\,0.\,1\,0_2 \\ 1\,1.\,1\,1_2 \\ \hline 1_2 \end{array} \Rightarrow \begin{array}{r} \mathit{1} \\ 1\,1\,0.\,1\,0_2 \\ 1\,1.\,1\,1_2 \\ \hline 0\,1_2 \end{array}$$

$$\Rightarrow \begin{array}{r} \mathit{1}\,\mathit{1} \\ 1\,1\,0.\,1\,0_2 \\ 1\,1.\,1\,1_2 \\ \hline 0.\,0\,1_2 \end{array} \Rightarrow \begin{array}{r} \mathit{1}\,\mathit{1}\,\mathit{1} \\ 1\,1\,0.\,1\,0_2 \\ 1\,1.\,1\,1_2 \\ \hline 1\,0.\,0\,1_2 \end{array} \Rightarrow \begin{array}{r} \mathit{1}\,\mathit{1}\,\mathit{1} \\ 1\,1\,0.\,1\,0_2 \\ 1\,1.\,1\,1_2 \\ \hline 1\,0\,1\,0.\,0\,1_2 \end{array}$$

Notice in the fifth step that we added $1_2 + 1_2 + 1_2 = 11_2$. A 1 was carried to the fifth column as shown in italics at the top.

Thus, we see that $110.1_2 + 11.11_2 = 1010.01_2$.

BINARY SUBTRACTION

We begin by reviewing some terms that are used in subtraction. Consider the subtraction problem $7340 - 986 = 6354$:

$$\begin{array}{rll} 7340 & \leftarrow & \text{minuend} \\ -\ \ 986 & \leftarrow & \text{subtrahend} \\ \hline 6354 & \leftarrow & \text{difference} \end{array}$$

In decimal subtraction it is often necessary to use the fact that $10 - 1 = 9$. We use this fact to borrow from the column to the immediate left. For example, consider $7340 - 986$:

$$\begin{array}{r} 7\,3\,\overset{3}{\cancel{4}}\,{}^{1}0 \\ -\quad 9\,8\,6 \\ \hline 4 \end{array} \Rightarrow \begin{array}{r} 7\,\overset{2}{\cancel{3}}\,\overset{{}^{1}3}{\cancel{4}}\,{}^{1}0 \\ -\quad 9\,8\,6 \\ \hline 5\,4 \end{array} \Rightarrow \begin{array}{r} \overset{6}{\cancel{7}}\,\overset{{}^{1}2}{\cancel{3}}\,\overset{{}^{1}3}{\cancel{4}}\,{}^{1}0 \\ -\quad 9\,8\,6 \\ \hline 3\,5\,4 \end{array} \Rightarrow \begin{array}{r} \overset{6}{\cancel{7}}\,\overset{{}^{1}2}{\cancel{3}}\,\overset{{}^{1}3}{\cancel{4}}\,{}^{1}0 \\ -\quad 9\,8\,6 \\ \hline 6\,3\,5\,4 \end{array}$$

Thus, we see that $7340 - 986 = 6354$.

Binary subtraction may also require you to borrow across columns. This requires the use of the following subtraction fact:

$$\begin{array}{r} 10_2 \\ -\ \ 1_2 \\ \hline 1_2 \end{array}$$

Let us use this to subtract one binary number from another.

EXAMPLE 16.15

Determine the difference $11101_2 - 1110_2$.

SOLUTION Again, we place the numbers in columns according to their place value and start subtracting with the rightmost column, and work to the left using Table 16.4:

$$\begin{array}{r} 1\,1\,1\,0\,1_2 \\ -\quad 1\,1\,1\,0_2 \\ \hline 1_2 \end{array} \Rightarrow \begin{array}{r} 1\,1\,\overset{0}{\cancel{1}}\,{}^{1}0\,1_2 \\ -\quad 1\,1\,1\,0_2 \\ \hline 1\,1_2 \end{array} \Rightarrow \begin{array}{r} 1\,\overset{0}{\cancel{1}}\,\overset{{}^{1}0}{\cancel{1}}\,{}^{1}0\,1_2 \\ -\quad 1\,1\,1\,0_2 \\ \hline 1\,1\,1_2 \end{array} \Rightarrow \begin{array}{r} \overset{0}{\cancel{1}}\,\overset{{}^{1}0}{\cancel{1}}\,\overset{{}^{1}0}{\cancel{1}}\,{}^{1}0\,1_2 \\ -\quad 1\,1\,1\,0_2 \\ \hline 1\,1\,1\,1_2 \end{array}$$

This shows that $11101_2 - 1110_2 = 1111_2$.

As with decimal subtraction, you can check you answer by adding the difference and subtrahend. If the subtraction (and addition) are correct, the sum should be the minuend of the subtraction problem.

EXAMPLE 16.16

Check your answer to Example 16.15.

SOLUTION To show that $11101_2 - 1110_2 = 1111_2$, we add $1110_2 + 1111_2$. If the answer is correct, we should get a sum of 11101_2:

$$\begin{array}{r} 1\,1\,1\,0_2 \\ +\ 1\,1\,1\,1_2 \\ \hline 1_2 \end{array} \Rightarrow \begin{array}{r} 1\,{}^{1}1\,1\,0_2 \\ +\ 1\,1\,1\,1_2 \\ \hline 0\,1_2 \end{array} \Rightarrow \begin{array}{r} {}^{1}1\,{}^{1}1\,1\,0_2 \\ +\ 1\,1\,1\,1_2 \\ \hline 1\,0\,1_2 \end{array} \Rightarrow \begin{array}{r} {}^{1}1\,{}^{1}1\,1\,0_2 \\ +\quad 1\,1\,1\,1_2 \\ \hline 1\,1\,1\,0\,1_2 \end{array}$$

Since 11101_2 was the minuend in Example 16.15, the answer checks.

DECIMAL SUBTRACTION USING COMPLEMENTS

Most people find it easier to add than subtract. Subtraction in computers is usually performed using **radix complements**. This method is used because it replaces the subtraction operation with an addition operation. Once you have computed the radix complement of the subtrahend, only the addition operation is needed.

To find the radix complement, first find the **diminished radix complement** and then add one. People often refer to the diminished radix complement in base r as the $r - 1$'s complement and the radix complement as the r's complement. The r's complement is found by adding 1 to the $r - 1$'s complement.

FINDING THE 9'S AND 10'S COMPLEMENTS

If the decimal number N has n digits, then subtract N from a number with n 9s and with the decimal point in the same location as in N. To find the 10's complement, add a 1 in the least significant position of the original number.

This process for finding the 9's and 10's complements is demonstrated in the next two examples.

EXAMPLE 16.17

Determine the diminished radix, or 9's, complement of 48,739.

SOLUTION Since no base is indicated, this is a decimal number and the base or radix is $r = 10$. Here $N = 48{,}739$ and the number of digits is $n = 5$. Thus, we subtract N from a number with five 9s, or 99,999. The subtraction is $99{,}999 - 48{,}739 = 51{,}260$.

Thus, the 9's complement of 48,739 is 51,260.

EXAMPLE 16.18

Determine the 10's complement of 48,739.

SOLUTION In the previous example, we found the 9's complement of 48,739 to be 51,260. The 10's complement is $51{,}260 + 1 = 51{,}261$.

EXAMPLE 16.19

Determine the 9's and 10's complements of 37.214.

SOLUTION Here $N = 37.214$ has five digits, with three digits to the right of the decimal point. Thus, we will subtract N from a number with five 9s that has three of the 9s to the right of the decimal point, 99.999.

Thus, the 9's complement is $99.999 - 37.214 = 62.785$. The least significant digit in the original number was the 0.004, so we add 0.001 to get the 10's complement. We find that the 10's complement is $62.785 + 0.001 = 62.786$.

To use complements to subtract, add the minuend to the complement of the subtrahend and discard the leading 1. Both the minuend and subtrahend must have digits in the same place value locations. If they do not, add 0s to the left or right until they are the same.

EXAMPLE 16.20

Use complements to determine $73{,}106 - 48{,}739$.

SOLUTION In Example 16.18 we found that the 10's complement of 48,739 is 51,261.

	Replace subtrahend with 10's complement	Add and drop the leading 1
$\begin{array}{r} 73{,}196 \\ -\ \underline{48{,}739} \end{array}$	$\begin{array}{r} 73{,}196 \\ +\ \underline{51{,}261} \end{array}$	$\begin{array}{r} 73{,}196 \\ +\ \underline{51{,}261} \\ \cancel{1}24{,}367 \end{array}$

Dropping the leading 1 gives the subtraction result 24,367. Thus, we have used complements to determine that $73{,}106 - 48{,}739 = 24{,}367$.

What if the minuend and subtrahend do not have digits in the same place value locations? Then, add zeros until there are the same number of digits. Let's use complements and rework the subtraction problem 7340 − 986 that was shown just before Example 16.15.

EXAMPLE 16.21

Use complements to determine 7340 − 986.

SOLUTION The number 7340 has four digits, and 986 has just three digits. We will rewrite 986 as 0986. Then the 10's complement is 9999 − 0986 + 1 = 9014.

	Replace subtrahend with 10's complement	Add and drop the leading 1
$\begin{array}{r} 7340 \\ -\ \ 986 \\ \hline \end{array}$	$\begin{array}{r} 7340 \\ +\ 9014 \\ \hline \end{array}$	$\begin{array}{r} 7340 \\ +\ \ 9014 \\ \hline \cancel{1}6354 \end{array}$

Thus, we have used complements to determine that 7340 − 986 = 6354. This is the same answer we got earlier.

The next example shows how to use complements to subtract decimal fractions.

EXAMPLE 16.22

Use complements to determine 592.637 − 37.214.

SOLUTION In Example 16.19 we found that the 10's complement of 37.214 is 62.786. Because the minuend has more place values, we need the 10's complement of 037.214, which is 962.786.

	Replace subtrahend with 10's complement	Add and drop the leading 1
$\begin{array}{r} 592.637 \\ -\ \ 37.214 \\ \hline \end{array}$	$\begin{array}{r} 592.637 \\ +\ 962.786 \\ \hline \end{array}$	$\begin{array}{r} 592.637 \\ +\ \ 962.786 \\ \hline \cancel{1}555.423 \end{array}$

Thus, we have used complements to determine 592.637 −37.214 = 555.423.

BINARY SUBTRACTION USING COMPLEMENTS

In binary subtraction we will add the 2's complement. To find the 2's complement, we find the 1's complement and add one. To find the 1's complement, change each 0 to 1 and each 1 to 0.

EXAMPLE 16.23

Determine the diminished radix, or 1's, complement of $10\ 1110_2$.

SOLUTION To find the 1's complement, change each 0 to 1 and each 1 to 0. Thus, the 1's complement of $10\ 1110_2$ is $01\ 0001_2$.

EXAMPLE 16.24

Determine the 2's complement of $10\ 1110_2$.

SOLUTION In Example 16.23, we found the 1's complement of $10\ 1110_2$ to be $01\ 0001_2$. The 2's complement is $01\ 0001_2 + 1_2 = 01\ 0010_2$.

EXAMPLE 16.25

Determine the 1's and 2's complement of 1101.01_2.

SOLUTION The 1's complement of 1101.01_2 is 0010.10_2, so the 2's complement is 0010.11_2.

EXAMPLE 16.26

Use complements to determine $11\ 0110_2 - 10\ 1110_2$.

SOLUTION Notice that both binary integers have six digits.

In the previous example, we found that the 2's complement of $10\ 1110_2$ is $01\ 0010_2$.

	Replace subtrahend with 2's complement	Add and drop the leading 1
$\begin{array}{r} 11\ 0110_2 \\ -\ \underline{10\ 1110_2} \end{array}$	$\begin{array}{r} 11\ 0110_2 \\ +\ \underline{01\ 0010_2} \end{array}$	$\begin{array}{r} 11\ 0110_2 \\ +\ \underline{01\ 0010_2} \\ \cancel{1}00\ 1000_2 \end{array}$

Dropping the leading 1 gives the subtraction result $00\ 1000_2$, or, as it would usually be written, 1000_2. Thus, we have used complements to determine that $11\ 0110_2 - 10\ 1110_2 = 1000_2$.

EXAMPLE 16.27

Use complements to determine $1011\ 0110_2 - 10\ 1110_2$.

SOLUTION The binary integer $1011\ 0110_2$ has eight digits, whereas $10\ 1110_2$ has just six digits. We will rewrite $10\ 1110_2$ as $0010\ 1110_2$ so that it has eight digits. Then the 2's complement is $1101\ 0001_2 + 1 = 1101\ 0010_2$.

	Replace subtrahend with 2's complement	Add and drop the leading 1
$\begin{array}{r} 1011\ 0110_2 \\ -\ \underline{10\ 1110_2} \end{array}$	$\begin{array}{r} 1011\ 0110_2 \\ +\ \underline{1101\ 0010_2} \end{array}$	$\begin{array}{r} 1011\ 0110_2 \\ +\ \underline{1101\ 0010_2} \\ \cancel{1}\ 1000\ 1000_2 \end{array}$

Thus, we have used complements to determine that $1011\ 0110_2 - 10\ 1110_2 = 1000\ 1000_2$.

EXAMPLE 16.28

Use complements to determine $101.1101_2 - 10.011_2$.

SOLUTION The minuend has three places to the left of the binary point and four to the right of the binary point, whereas the subtrahend has only two digits to the left of the binary point and three to the right.

EXAMPLE 16.28 (Cont.)

We rewrite the subtrahend as 010.0110_2 so that it has the same number of digits on each side of the binary point as the minuend. The 2's complement of the subtrahend is 101.1010_2.

$$\begin{array}{rrr} & \text{Replace subtrahend} & \text{Add and drop} \\ & \text{with 2's complement} & \text{the leading 1} \\ 101.1101_2 & 101.1101_2 & 101.1101_2 \\ -\ \underline{10.011_2} & +\ \underline{101.1010_2} & +\ \underline{101.1010_2} \\ & & \cancel{1}011.0111_2 \end{array}$$

Dropping the leading 1 we see that $101.1101_2 - 10.011_2 = 11.0111_2$.

EXAMPLE 16.29

Use complements to determine $1101.01_2 - 10.101_2$.

SOLUTION Since the minuend has two places to the right of the binary point while the subtrahend has three, we add a zero at the right of the minuend to make it 1101.010_2. We add two zeros at the left of the subtrahend to make it 0010.101_2. These changes make the subtraction now read $1101.010_2 - 0010.101_2$.

The 1's complement of the subtrahend is 1101.010_2, so its 2's complement is 1101.011_2.

$$\begin{array}{rrr} & \text{Replace subtrahend} & \text{Add and drop} \\ & \text{with 2's complement} & \text{the leading 1} \\ 1101.010_2 & 1101.010_2 & 1101.010_2 \\ -\ \underline{0010.101_2} & +\ \underline{1101.011_2} & +\ \underline{1101.011_2} \\ & & \cancel{1}1010.101_2 \end{array}$$

Dropping the leading 1, we see that $1101.1101_2 - 10.101_2 = 1010.101_2$.

NEGATIVE NUMBERS

Until now, all the numbers we have considered have been positive numbers. In base 10 a negative number is indicated with a "negative sign" placed in front of the number. A similar technique is not feasible in the binary system if we want to represent every number with a combination of 0s and 1s.

One solution is to use a "sign digit." That is, if the number begins with a 0 it is positive and if it begins with a 1 it is negative. In this way, the number 5 would be represented as 0101_2 and the number -5 as 1101_2. But, this presents another problem. How do we tell if 1101_2 represents -5 or $+13$? The representation of negative binary numbers is too complicated for us to consider at this time. So, you can relax a little, since we will only be considering positive binary numbers.

EXERCISE SET 16.2

In Exercises 1–12, determine each of the binary sums.

1. $\begin{array}{r} 101_2 \\ +\ \underline{01_2} \end{array}$

2. $\begin{array}{r} 1011_2 \\ +\ \underline{101_2} \end{array}$

3. $\begin{array}{r} 11\,0011_2 \\ +\ \underline{1\,1101_2} \end{array}$

4. $\begin{array}{r} 11\,1011_2 \\ +\ \ 1\,1011_2 \\ \hline \end{array}$

5. $110\,1011_2 + 1\,1101_2$

6. $111\,0010_2 + 10\,0101_2$

7. $1010\,1011_2 + 11\,1101_2$

8. $10\,1111\,0010_2 + 1101\,0011_2$

9. $10\,1111\,0010_2 + 1101\,0011_2$

10. $11\,1100_2 + 1011_2 + 110\,1011_2$

11. $1101.001_2 + 101.1011_2 + 10\,0111.11_2$

12. $10\,1111.0010_2 + 110.1001_2 + 1101.101_2$

In Exercises 13–18, determine the 10's complement of each of the decimal numbers.

13. 763

14. 2564

15. 12,729

16. 37,148

17. 52.38

18. 147.216

In Exercises 19–30, (a) determine the 10's complement of each subtrahend and (b) use the 10's complement to solve each of the following subtraction exercises.

19. $\begin{array}{r} 3217 \\ -\ 2958 \\ \hline \end{array}$

20. $\begin{array}{r} 471{,}923 \\ -\ 256{,}358 \\ \hline \end{array}$

21. $\begin{array}{r} 2{,}376 \\ -\ 827 \\ \hline \end{array}$

22. $\begin{array}{r} 4{,}364 \\ -\ 358 \\ \hline \end{array}$

23. $\begin{array}{r} 52{,}453 \\ -\ 3{,}657 \\ \hline \end{array}$

24. $\begin{array}{r} 627{,}582 \\ -\ 48{,}219 \\ \hline \end{array}$

25. $\begin{array}{r} 397.16 \\ -\ 127.28 \\ \hline \end{array}$

26. $\begin{array}{r} 5{,}135.372 \\ -\ 2{,}947.718 \\ \hline \end{array}$

27. $\begin{array}{r} 594.376 \\ -\ 36.27 \\ \hline \end{array}$

28. $\begin{array}{r} 732.518 \\ -\ 9.76 \\ \hline \end{array}$

29. $\begin{array}{r} 473.27 \\ -\ 94.764 \\ \hline \end{array}$

30. $\begin{array}{r} 1{,}235.734 \\ -\ 473.2193 \\ \hline \end{array}$

In Exercises 31–36, determine the 2's complement of each of the binary numbers.

31. 1101_2

32. 10110_2

33. $101\,1001_2$

34. $101\,0100_2$

35. 101.011_2

36. 1101.101_2

In Exercises 37–52, (a) determine the 2's complement of each subtrahend and (b) use the 2's complement to solve each of the following subtraction exercises.

37. $\begin{array}{r} 1101_2 \\ -\ 1001_2 \\ \hline \end{array}$

38. $\begin{array}{r} 1\,1101_2 \\ -\ 1\,0010_2 \\ \hline \end{array}$

39. $\begin{array}{r} 11\,0101_2 \\ -\ 10\,0100_2 \\ \hline \end{array}$

40. $\begin{array}{r} 110\,0110_2 \\ -\ 100\,1001_2 \\ \hline \end{array}$

41. $\begin{array}{r} 101\,0110_2 \\ -\ 10\,1101_2 \\ \hline \end{array}$

42. $\begin{array}{r} 1011\,1010_2 \\ -\ 110\,1011_2 \\ \hline \end{array}$

43. $\begin{array}{r} 1101\,0111_2 \\ -\ 101\,1110_2 \\ \hline \end{array}$

44. $\begin{array}{r} 11\,0111\,1100_2 \\ -\ 1\,0100\,1101_2 \\ \hline \end{array}$

45. $\begin{array}{r} 1111.101_2 \\ -\ 1101.011_2 \\ \hline \end{array}$

46. $\begin{array}{r} 1011.011_2 \\ -\ 1001.101_2 \\ \hline \end{array}$

47. $\begin{array}{r} 1011.1011_2 \\ -\ 101.011_2 \\ \hline \end{array}$

48. $\begin{array}{r} 1001.0101_2 \\ -\ 10.101_2 \\ \hline \end{array}$

49. $\begin{array}{r} 1010.01_2 \\ -\ 101.011_2 \\ \hline \end{array}$

50. $\begin{array}{r} 1101.101_2 \\ -\ 10.0011_2 \\ \hline \end{array}$

51. $\begin{array}{r} 10101.101_2 \\ -\ 100.1101_2 \\ \hline \end{array}$

52. $\begin{array}{r} 1011.011_2 \\ -\ 1000.1011_2 \\ \hline \end{array}$

[IN YOUR WORDS]

53. Write a definition of the 1's complement of a binary number and explain how it is determined.

54. Explain how the 2's complement of a binary number differs from its 1's complement.

55. Write a short explanation telling how to use complements to subtract.

56. List as many reasons as you can of the advantages of using complements to subtract.

16.3 THE OCTAL NUMBER SYSTEM

The next group of numbers we will study is the base eight, or **octal number system**. As its name indicates, an octal number system has eight digits: 0, 1, 2, 3, 4, 5, 6, and 7. The place values for the octal system are powers of eight, as shown in Table 16.5.

TABLE 16.5

Place value	8^6	8^5	8^4	8^3	8^2	8^1	8^0	.	8^{-1}	8^{-2}	8^{-3}
Decimal value	262,144	32,768	4,096	512	64	8	1	↑	$\frac{1}{8} = 0.125$	$\frac{1}{64} = 0.015625$	$\frac{1}{512}$
								Octal point			

OCTAL-TO-DECIMAL CONVERSION

As we did with the binary numbers, to convert a octal number to its decimal-number equivalent, write the number in expanded form and then compute the result.

EXAMPLE 16.30

Convert the octal number 754_8 to its decimal equivalent.

SOLUTION We begin with the expanded form of 754_8:

$$\begin{aligned} 754_8 &= (7 \times 8^2) + (5 \times 8^1) + (4 \times 8^0) \\ &= (7 \times 64) + (5 \times 8) + (4 \times 1) \\ &= 448 + 40 + 4 = 492 \end{aligned}$$

The octal equivalent of 754_8 is 492.

EXAMPLE 16.31

Determine the decimal equivalent of the octal number $21{,}375_8$.

SOLUTION As usual, we begin by expanding $21{,}375_8$:

$$\begin{aligned} 21{,}375_8 &= (2 \times 8^4) + (1 \times 8^3) + (3 \times 8^2) + (7 \times 8^1) + (5 \times 8^0) \\ &= (2 \times 4096) + (1 \times 512) + (3 \times 64) + (7 \times 8) + (5 \times 1) \\ &= 8192 + 512 + 192 + 56 + 5 = 8957 \end{aligned}$$

The decimal equivalent of $21{,}375_8$ is 8957.

DECIMAL-TO-OCTAL CONVERSION

To convert a decimal number to its octal equivalent, we use methods similar to those we used when we converted decimal numbers to their binary equivalents.

CONVERTING DECIMAL INTEGERS TO OCTAL INTEGERS

We will use repeated division and the method of remainders with a divisor of 8.

EXAMPLE 16.32

Convert the decimal number 3576 to its octal equivalent.

SOLUTION Begin by dividing 3576 by 8:

		Remainder
8)	3576	
8)	447	0
8)	55	7
8)	6	7
	0	6

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So, we have shown that $3576 = 6770_8$.

CONVERTING DECIMAL FRACTIONS TO OCTAL FRACTIONS

To convert a decimal fraction to an octal fraction, we use repeated multiplication with the radix, 8, keeping track of the integral part of the product, until the fractional part is zero or repeats. Then we read the integral parts of the answer in ascending order. This is exactly the same technique we used with the binary number system.

EXAMPLE 16.33

Use the method of repeated multiplication to convert the decimal fraction 0.921875 to its octal equivalent.

SOLUTION

Fractional Part	Integral Digits
$8(0.921875) = 7.375$	7
$8(0.375) = 3.0$	3

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Reading the integral digits from top to bottom, we see that $0.921875 = 0.73_8$.

EXAMPLE 16.34

Use the method of repeated multiplication to convert the decimal fraction 0.0375 to its octal equivalent.

SOLUTION

Fractional Part	Integral Digits
$8(0.0375) = 0.3$	0
$8(0.3) = 2.4$	2
$8(0.4) = 3.2$	3
$8(0.2) = 1.6$	1
$8(0.6) = 4.8$	8
$8(0.8) = 6.4$	6

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We have seen the fractional part 0.4 before, so we know that this is a repeating octal fraction. Reading the integral digits from top to bottom, we see that $0.0375 = 0.0\overline{23186}_8$.

EXAMPLE 16.35

Convert the number 3576.0375 to its octal equivalent.

SOLUTION We have already done the work for this problem. We just need to combine the answer to Examples 16.32 and 16.34 to get $6770.0\overline{2318}_8$.

OCTAL TO BINARY CONVERSION

In order to change an octal number to its binary equivalent, change each octal digit to its binary equivalent. It may be necessary to add leading zeros. Table 16.6 shows the binary value of each octal digit. It also shows how each octal number is written as a three-digit binary number.

TABLE 16.6

Octal no.	0	1	2	3	4	5	6	7
Binary value	0	1	10	11	100	101	110	111
Three-digit binary value	000	001	010	011	100	101	110	111

EXAMPLE 16.36

Convert the octal number 5237_8 to its binary equivalent.

SOLUTION Writing the three-digit binary value of each octal digit, we get

$$\overbrace{101}^{5}\quad\overbrace{010}^{2}\quad\overbrace{011}^{3}\quad\overbrace{111}^{7}$$

This indicates that $5237_8 = 101\ 010\ 011\ 111_2$. Long binary numbers are usually written in nibbles, and in that form we see that $5237_8 = 1010\ 1001\ 1111_2$.

BINARY-TO-OCTAL CONVERSION

In order to convert a binary number to an octal number, write the binary number in clusters of three moving away from the binary point. It may be necessary to add 0s so that each cluster has three digits. Then replace each three-digit cluster with its octal equivalent.

EXAMPLE 16.37

Convert the binary number $10\ 1001\ 1101_2$ to its octal equivalent.

SOLUTION As is customary, this binary number is written in nibbles. We begin by rewriting it in clusters of three digits. But, there are only 10 digits in this number and 10 is not a multiple of three. The next largest multiple of three is 12, so we will first add two 0s to the left of this binary number and then group the 12-digit binary number in clusters of three:

$$10\ 1001\ 1101_2 = 0010\ 1001\ 1101_2$$

Rewrite the number in three-digit clusters:

$$= 001\ 010\ 011\ 101_2$$

Write the octal value of each three-digit cluster:

$$= 1\quad 2\quad 3\quad 5_8$$

This shows that the binary number $10\ 1001\ 1101_2$ has the octal equivalent of 1235_8.

EXAMPLE 16.38

Convert the binary number $110\ 0101\ 1111.1100\ 101_2$ to its octal equivalent.

SOLUTION As before, we begin by rewriting this number in clusters of three digits. This time there are 11 digits to the left of the binary point and 7 digits to the right. Neither 11 nor 7 is a multiple of 3. For 11, the next largest multiple of 3 is 12, so we will first add one 0 to the left of the integer part of this binary number. For 7, the next largest multiple of 3 is 9, so we will add two 0s to the right of the number:

$$110\ 0101\ 1111.1100\ 101_2 = 0110\ 0101\ 1111.1100\ 1010\ 0$$

Rewrite the number in three-digit clusters:

$$= 011\ \ 001\ \ 011\ \ 111.\ \ 110\ \ 010\ \ 100_2$$

Write the octal value of each three-digit cluster:

$$= 3\quad 1\quad 3\quad 7.\quad 6\quad 2\quad 4_8$$

So, the binary number $110\ 0101\ 1111.1100\ 101_2$ has the octal equivalent of 3137.624_8.

ASCII: AN APPLICATION OF BINARY AND OCTAL NUMBERS

The words you are reading were typed on a computer. How did the computer know that pressing an A key should produce an image of an A on the computer screen? It all began in the 1870s when a Frenchman Jean-Maurice-Émile Baudot began developing a telegraph system that would print a message in ordinary alphabetic characters on a strip of paper. His method eventually evolved into the **American Standard Code for Information Interchange,** or **ASCII**. Computers can only understand numbers, so the ASCII code is a numerical representation of a character such as an "A" or a "?" or an action such as pressing the space key.

The original ASCII code used seven-bit words and consisted of 128 characters. As people required computers to understand additional characters, the code was expanded to eight-bit words and 256 characters. Some of the characters are shown in Table 16.7. Notice that it starts with the 32nd character. The first 31 are nonprinting and not of interest to this discussion. We will also not consider those after the first 128.

Here is a sentence written using octal code: 124 150 151 163 041 151 163 041 101 123 103 111 111 041 143 157 144 145 056. If you use Table 16.7 to translate this, you find that it is the sentence "This is ASCII code." Notice that each octal number represents a single character. This same sentence is binary code is: 01010100 01101000 01101001 01110010 00100000 01101001 01110010 00100000 01000001 01010011 01000011 01001001 01001001 00100000 01100011 01101110 01100100 01100101 00101110. Here each byte represents one character. Fortunately, we seldom have to write our sentences in either binary or octal code and the computer does the work for us.

TABLE 16.7 Partial ASCII Table

Dec	Binary	Oct	Char	Dec	Binary	Oct	Char	Dec	Binary	Oct	Char	Dec	Binary	Oct	Char
32	0010 0000	040	Space	55	0011 0111	067	7	78	0100 1110	116	N	101	0110 0101	145	e
33	0010 0001	041	!	56	0011 1000	070	8	79	0100 1111	117	O	102	0110 0110	146	f
34	0010 0010	042	"	57	0011 1001	071	9	80	0101 0000	120	P	103	0110 0111	147	g
35	0010 0011	043	#	58	0011 1010	072	:	81	0101 0001	121	Q	104	0110 1000	150	h
36	0010 0100	044	$	59	0011 1011	073	;	82	0101 0010	122	R	105	0110 1001	151	i
37	0010 0101	045	%	60	0011 1100	074	<	83	0101 0011	123	S	106	0110 1010	152	j
38	0010 0110	046	&	61	0011 1101	075	=	84	0101 0100	124	T	107	0110 1011	153	k
39	0010 0111	047	’	62	0011 1110	076	>	85	0101 0101	125	U	108	0110 1100	154	l
40	0010 1000	050	(	63	0011 1111	077	?	86	0101 0110	126	V	109	0110 1101	155	m
41	0010 1001	051	)	64	0100 0000	100	@	87	0101 0111	127	W	110	0110 1110	156	n
42	0010 1010	052	*	65	0100 0001	101	A	88	0101 1000	130	X	111	0110 1111	157	o
43	0010 1011	053	+	66	0100 0010	102	B	89	0101 1001	131	Y	112	0111 0000	160	p
44	0010 1100	054	,	67	0100 0011	103	C	90	0101 1010	132	Z	113	0111 0001	161	q
45	0010 1101	055	–	68	0100 0100	104	D	91	0101 1011	133	[	114	0111 0010	162	r
46	0010 1110	056	.	69	0100 0101	105	E	92	0101 1100	134	\	115	0111 0011	163	s
47	0010 1111	057	/	70	0100 0110	106	F	93	0101 1101	135	]	116	0111 0100	164	t
48	0011 0000	060	0	71	0100 0111	107	G	94	0101 1110	136	^	117	0111 0101	165	u
49	0011 0001	061	1	72	0100 1000	110	H	95	0101 1111	137	–	118	0111 0110	166	v
50	0011 0010	062	2	73	0100 1001	111	I	96	0110 0000	140	‘	119	0111 0111	167	w
51	0011 0011	063	3	74	0100 1010	112	J	97	0110 0001	141	a	120	0111 1000	170	x
52	0011 0100	064	4	75	0100 1011	113	K	98	0110 0010	142	b	121	0111 1001	170	y
53	0011 0101	065	5	76	0100 1100	114	L	99	0110 0011	143	c	122	0111 1010	172	z
54	0011 0110	066	6	77	0100 1101	115	M	100	0110 0100	144	d	123	0111 1011	173	{

EXERCISE SET 16.3

In Exercises 1–4, write each octal number in expanded form.

1. 324_8 **2.** 572_8 **3.** 6304_8 **4.** 71035_8

In Exercises 5–12, convert the given octal number to its decimal equivalent.

5. 324_8 **7.** 6304_8 **9.** 6024.3_8 **11.** 7456.73_8

6. 572_8 **8.** $71\,035_8$ **10.** 25603.7_8 **12.** 3070.37_8

In Exercises 13–20, convert the given decimal number to its octal equivalent.

13. 153 **15.** 880 **17.** 0.734375 **19.** 675.453125

14. 437 **16.** 963 **18.** 0.54296875 **20.** 1763.8671875

In Exercises 21–28, convert the given octal number to its binary equivalent.

21. 527_8 **23.** 2504_8 **25.** 735.02_8 **27.** 5034.352_8

22. 736_8 **24.** 32154_8 **26.** 1356.47_8 **28.** 7702.5503_8

In Exercises 29–36, convert the given binary number to its octal equivalent.

29. 1011_2

30. $1011\ 1001\ 0111_2$

31. $10\ 1100\ 1111\ 0011\ 1100_2$

32. $110\ 1011\ 1101_2$

33. $1\ 0110\ 1011.1101\ 11_2$

34. $101\ 1010\ 1001.1110\ 101_2$

35. $1101\ 0101\ 0111.0110\ 0110\ 1_2$

36. $1\ 0111\ 0101\ 1010.1010\ 1110\ 101_2$

Solve Exercises 37–40.

37. ***Information technology*** Translate the following octal ASCII code into English: 104 157 040 170 157 165 040 155 141 164 150 040 150 157 154 145 167 157 162 153 056.

38. ***Information technology*** Translate the following octal ASCII code into English: 111 155 160 162 157 166 145 040 170 157 165 162 163 145 154 146 054 040 163 164 165 144 170 040 155 141 164 150 041.

39. ***Information technology*** Translate the following sentence into octal ASCII code: The study of matrices is in Chapter 19.

40. ***Information technology*** Translate the following sentence into octal ASCII code: Have you read a good book?

[IN YOUR WORDS]

41. List and describe two differences between the decimal number system and the octal number system.

42. List and describe two differences between the octal number system and the binary number system.

43. Describe how to change an octal number to a decimal number.

44. Describe how to change a decimal integer to an octal integer.

45. Describe how to change a decimal fraction to an octal fraction.

16.4 OCTAL ARITHMETIC

The rules for octal arithmetic are essentially the same as those for binary and decimal numbers.

OCTAL ADDITION

Table 16.8 is an addition table for binary numbers.

TABLE 16.8 Octal Addition

+	0	1	2	3	4	5	6	7
0	0	1	2	3	4	5	6	7
1	1	2	3	4	5	6	7	10
2	2	3	4	5	6	7	10	11
3	3	4	5	6	7	10	11	12
4	4	5	6	7	10	11	12	13
5	5	6	7	10	11	12	13	14
6	6	7	10	11	12	13	14	15
7	7	10	11	12	13	14	15	16

EXAMPLE 16.39

Determine the sum $573_8 + 645_8$.

SOLUTION As with decimal and binary addition, we place the numbers in columns according to their place value, start adding with the rightmost column, and work to the left using the sums in Table 16.8:

$$\begin{array}{r} 5\ {}^{1}7\ 3_8 \\ +\ 4\ \ 6\ 5_8 \\ \hline 0_8 \end{array} \Rightarrow \begin{array}{r} {}^{1}5\ {}^{1}7\ 3_8 \\ +\ 4\ \ 6\ 5_8 \\ \hline 6\ 0_8 \end{array} \Rightarrow \begin{array}{r} {}^{1}5\ {}^{1}7\ 3_8 \\ +\ 4\ \ 6\ 5_8 \\ \hline 1\ 2\ \ 6\ 0_8 \end{array}$$

Thus, we see that $573_8 + 645_8 = 1260_8$.

EXAMPLE 16.40

Determine the sum $246.75_8 + 67.342_8$.

SOLUTION Again, place the numbers in columns according to their place value. We rewrite 246.75_8 as 246.750_8 so that both addends have three digits to the right of the octal point. Start adding with the rightmost column, and work to the left using the sums in Table 16.8:

$$\begin{array}{r} 2\,4\,6.\,7\,5\,0_8 \\ +\ 6\,7.\,3\,4\,2_8 \\ \hline 2_8 \end{array} \Rightarrow \begin{array}{r} 2\,4\,6.\,{}^{1}7\,5\,0_8 \\ +\ 6\,7.\ 3\,4\,2_8 \\ \hline 1\,2_8 \end{array} \Rightarrow \begin{array}{r} 2\,4\,{}^{1}6.\,{}^{1}7\,5\,0_8 \\ +\ 6\ 7.\ 3\,4\,2_8 \\ \hline 3\,1\,2_8 \end{array} \Rightarrow \begin{array}{r} {}^{1}2\,{}^{1}4\,{}^{1}6.\,{}^{1}7\,5\,0_8 \\ +\ \ 6\ 7.\ 3\,4\,2_8 \\ \hline 3\,3\,6.\ 3\,1\,2_8 \end{array}$$

Thus, we see that $246.75_8 + 67.342_8 = 336.312_8$.

OCTAL COMPLEMENTS

Octal subtraction is most easily done using complements. To determine the diminished radix, or 7's, complement, subtract each digit from 7. The radix, or 8's, complement is obtained by adding one to the 7's complement.

EXAMPLE 16.41

Determine the diminished radix, or 7's, complement of 4572_8.

SOLUTION To find the 7's complement of 4572_8 we subtract it from 7777_8. We can use Table 16.8 to help with the answers or we can just use our knowledge of subtraction in the decimal number system:

$$\begin{array}{r} 7777_8 \\ -\ 4572_8 \\ \hline 3205_8 \end{array}$$

Thus, the 7's complement of 4572_8 is 3205_8.

EXAMPLE 16.42

Determine the 8's complement of 4572_8.

SOLUTION In the previous example, we found that the 7's complement of 4572_8 is 3205_8. The 8's complement is $3205_8 + 1_8 = 3206_8$.

OCTAL SUBTRACTION

In octal subtraction we add the 8's complement and drop the leading 1.

EXAMPLE 16.43

Use complements to determine $7324_8 - 4572_8$.

SOLUTION Notice that the both numbers have four digits.

In the previous example, we found that the 8's complement of 4572_8 is 3206_8:

	Replace subtrahend with 8's complement	Add and drop the leading 1
$\begin{array}{r} 7324_8 \\ -\ \underline{4572_8} \end{array}$	$\begin{array}{r} 7324_8 \\ +\ \underline{3206_8} \end{array}$	$\begin{array}{r} 7324_8 \\ +\ \underline{3206_8} \\ \cancel{1}2532_8 \end{array}$

Dropping the leading 1 gives the subtraction result 2532_8. We have used complements to determine that $7324_8 - 4572_8 = 2532_8$.

EXAMPLE 16.44

Use complements to determine $54\,3213_8 - 5726_8$.

SOLUTION The number $54\,3213_8$ has six digits, whereas 5726_8 has just four digits. We will rewrite 5726_8 as $00\,5726_8$ so that it has six digits. The 7's complement of $00\,5726_8$ is $77\,2051_8$, and so the 8's complement is $77\,2052_8$:

	Replace subtrahend with 8' complement	Add and drop the leading 1
$\begin{array}{r} 54\,3213_8 \\ -\ \underline{\quad 5726_8} \end{array}$	$\begin{array}{r} 54\,3213_8 \\ +\ \underline{77\,2052_8} \end{array}$	$\begin{array}{r} 54\,3213_8 \\ +\ \underline{77\,2052_8} \\ \cancel{1}53\,5265_8 \end{array}$

By dropping the leading 1 we have used complements to determine that $54\,3213_8 - 5726_8 = 53\,5265_8$.

EXAMPLE 16.45

Use complements to determine $102.4236_8 - 35.275_8$.

SOLUTION The minuend has three places to the left of the octal point and four to the right of the octal point, whereas the subtrahend has only two digits to the left of the octal point and three to the right.

We rewrite the subtrahend as 035.2750_8 so that it has the same number of digits on each side of the hex point as does the minuend. The 8's complement of the subtrahend is 742.5030_8:

	Replace subtrahend with 8's complement	Add and drop the leading 1
$\begin{array}{r} 102.4236_8 \\ -\ \underline{35.275_8} \end{array}$	$\begin{array}{r} 102.4236_8 \\ +\ \underline{742.5030_8} \end{array}$	$\begin{array}{r} 102.4236_8 \\ +\ \underline{742.5030_8} \\ \cancel{1}045.1266_8 \end{array}$

Dropping the leading 1, we see that $102.4236_8 - 35.275_8 = 045.1266_8$. This answer would normally be written without the leading zero, 45.1266_8.

EXERCISE SET 16.4

In Exercises 1–8, determine each of the octal sums.

1. $\begin{array}{r} 503_8 \\ +\ \ 75_8 \\ \hline \end{array}$

2. $\begin{array}{r} 3045_8 \\ +\ \ 763_8 \\ \hline \end{array}$

3. $\begin{array}{r} 56\ 1327_8 \\ +\ \ 4\ 3726_8 \\ \hline \end{array}$

4. $\begin{array}{r} 605\ 3271_8 \\ +\ 172\ 4507_8 \\ \hline \end{array}$

5. $52.316_8 + 324.527_8$

6. $541.32_8 + 36.215_8$

7. $4105.316_8 + 326.25_8$

8. $27\ 5213.64_8 + 7\ 3422.327_8 + 4372.6254_8$

In Exercises 9–14, determine the 8's complement of each of the octal numbers.

9. 532_8

10. 7053_8

11. $2\ 7324_8$

12. 625.37_8

13. 52.370_8

14. 5371.672_8

In Exercises 15–22, (a) determine the 8's complement of each subtrahend and (b) use the 8's complement to solve each of the following subtraction exercises.

15. $\begin{array}{r} 3217_8 \\ -\ 2653_8 \\ \hline \end{array}$

16. $\begin{array}{r} 47\ 1523_8 \\ -\ 25\ 6354_8 \\ \hline \end{array}$

17. $\begin{array}{r} 15\ 6322_8 \\ -\ \ 7\ 6444_8 \\ \hline \end{array}$

18. $\begin{array}{r} 1523.473_8 \\ -\ \ 456.425_8 \\ \hline \end{array}$

19. $\begin{array}{r} 1643.24_8 \\ -\ \ 625.513_8 \\ \hline \end{array}$

20. $\begin{array}{r} 5027.3264_8 \\ -\ \ 32.05_8 \\ \hline \end{array}$

21. $\begin{array}{r} 14\ 0263.25_8 \\ -\ \ 67.5521_8 \\ \hline \end{array}$

22. $\begin{array}{r} 537.77_8 \\ -\ \ 74.564_8 \\ \hline \end{array}$

[IN YOUR WORDS]

23. Write a definition of the 7's complement of an octal number and explain how it is determined.

24. Explain how the 8's complement of an octal number differs from its 7's complement.

16.5 THE HEXADECIMAL NUMBER SYSTEM

Binary numbers are nice because they use only two digits. But, because we are not a computer, we find the lengths of the number to be quite awkward to work with. A binary number is generally more than three times as long as a decimal number. Number systems with a larger base can require less room and, if the base is a power of two, can be easily converted to the binary system. Two of the most common of these bases are the octal (base eight) and hexadecimal (base 16) number systems. Because the octal base is not used as frequently as it was in the earliest computers, we will focus on the hexadecimal system.

HEXADECIMAL NUMBERS

The **hexadecimal (or hex) number system** has 16 digits, so its radix is 16. The first 10 digits are the same as the ones in the decimal system. Rather than create new symbols for the other six digits, the first six letters of the alphabet are used. Thus, the hexadecimal number system has the 16 digits shown in Table 16.9.

TABLE 16.9

Hexadecimal number	0	1	2	3	4	5	6	7	8	9	A	B	C	D	E	F
Decimal value	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15

The place values of the hexadecimal digits are powers of 16 as shown in Table 16.10.

TABLE 16.10

Place value	16^6	16^5	16^4	16^3	16^2	16^1	16^0	.	16^{-1}	16^{-2}	16^{-3}
Decimal value	16,777,216	1,048,576	65,536	4096	256	16	1	↑ Hexadecimal point	$\frac{1}{16} = 0.0625$	$\frac{1}{256} = 0.00390625$	$\frac{1}{4096}$

HEXADECIMAL-TO-DECIMAL CONVERSION

As with the binary numbers, to convert a hexadecimal number to its decimal-number equivalent, write the number in expanded form and then compute the result.

EXAMPLE 16.46

Convert the hexadecimal number $A3B_{16}$ to its decimal equivalent.

SOLUTION We begin with the expanded form of $A3B_{16}$:

$$A3B_{16} = (A \times 16^2) + (3 \times 16^1) + (B \times 16^0)$$

Next, we convert each hexadecimal digit to its decimal equivalent:

$$\begin{aligned} &= (10 \times 16^2) + (3 \times 16^1) + (11 \times 16^0) \\ &= (10 \times 256) + (3 \times 16) + (11 \times 1) \\ &= 2560 + 48 + 11 = 2619 \end{aligned}$$

The decimal equivalent of $A3B_{16}$ is 2619.

EXAMPLE 16.47

Determine the decimal equivalent of the hexadecimal number $2F8C.7_{16}$.

SOLUTION As before, we start with the expanded form of $2F8C.7_{16}$:

$$\begin{aligned} 2F8C.7_{16} &= (2 \times 16^3) + (F \times 16^2) + (8 \times 16^1) + (C \times 16^0) + (7 \times 16^{-1}) \\ &= (2 \times 16^3) + (15 \times 16^2) + (8 \times 16^1) + (12 \times 16^0) + (7 \times 16^{-1}) \\ &= (2 \times 4096) + (15 \times 256) + (8 \times 16) + (12 \times 1) + (7 \times 0.0625) \\ &= 8{,}192 + 3{,}840 + 128 + 12 + 0.4375 = 12{,}172.4375 \end{aligned}$$

The decimal equivalent of $2F8C.7_{16}$ is 12,172.4375.

DECIMAL-TO-HEXADECIMAL CONVERSION

Converting a decimal number to its hexadecimal equivalent uses procedures similar to the ones we used when we converted decimal numbers to their binary equivalents.

CONVERTING DECIMAL INTEGERS TO HEXADECIMAL INTEGERS

We will use repeated division and the method of remainders with a divisor of 16.

EXAMPLE 16.48

Convert the decimal number 3576 to its hexadecimal equivalent.

SOLUTION Begin by dividing 3576 by 16. Notice that there are two remainder columns. The first remainder column, on the left, gives the remainder in base 10 and the second gives the remainder in base 16:

	Remainders	Remainders in Hex
16) 3576		
16) 223	8	8
16) 13	15	F
0	13	D

So, we have shown that $3576 = DF8_{16}$.

EXAMPLE 16.49

Convert the decimal number 51 837 to its hexadecimal equivalent.

SOLUTION

	Remainders	Remainders in Hex
16) 51837		
16) 3239	13	D
16) 202	7	7
16) 12	10	A
0	12	C

And so, we see that $51\,837 = CA7D_{16}$.

CONVERTING DECIMAL FRACTIONS TO HEXADECIMAL FRACTIONS

To convert a decimal fraction to a hexadecimal fraction, we use repeated multiplication with a multiplier of 16.

EXAMPLE 16.50

Use the method of repeated multiplication to convert the decimal fraction 0.362793 to its hexadecimal equivalent.

SOLUTION

	Fractional Part	Integral Digits
$16(0.362793) =$	5.80469	5
$16(0.80469) =$	12.875	C
$16(0.875) =$	14.0	E

Reading the integral digits from top to bottom, we see that $0.362793 = 0.5CE_{16}$.

EXAMPLE 16.51

Use the method of repeated multiplication to convert the decimal fraction 0.8175 to its hexadecimal equivalent.

SOLUTION

	Fractional Part	Integral Digits
$16(0.8175) =$	13.08	D
$16(0.08) =$	1.28	1
$16(0.28) =$	4.48	4
$16(0.48) =$	7.68	7
$16(0.68) =$	10.88	A
$16(0.88) =$	14.08	E

We have seen the fractional part 0.08 before, so we know that this is a repeating hexadecimal fraction. Reading the integral digits from top to bottom, we see that $0.8175 = 0.D\overline{147AE}_{16}$.

HEXADECIMAL-TO-BINARY CONVERSION

In order to change a hexadecimal number to its binary equivalent, change each hexadecimal digit to its binary equivalent. It may be necessary to add leading zeros. Table 16.11 shows the decimal and binary value of each hexadecimal digit.

TABLE 16.11

Hexadecimal no.	0	1	2	3	4	5	6	7	8	9	A	B	C	D	E	F
Decimal value	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
Binary value	0	1	10	11	100	101	110	111	1000	1001	1010	1011	1100	1101	1110	1111

EXAMPLE 16.52

Convert the hexadecimal number $6E_{16}$ to its binary equivalent.

SOLUTION Writing the binary values of the hexadecimal digits as nibbles, we get

$$\overbrace{0110}^{6} \qquad \overbrace{1110}^{E}$$

This indicates that $6E_{16} = 0110\ 1110_2 = 110\ 1110_2$.

BINARY-TO-HEXADECIMAL CONVERSION

To convert a binary number to its hexadecimal equivalent, arrange the binary number in nibbles. Again, add leading zeros if necessary. Then, use Table 16.11 to change each nibble into a hexadecimal digit.

EXAMPLE 16.53

Convert the binary number $11\ 1010\ 1101_2$ to its hexadecimal equivalent.

SOLUTION We add two zeros at the left so the left nibble has four digits. Next, we write the hexadecimal digit of each binary nibble:

$$\overbrace{0011}^{3} \qquad \overbrace{1010}^{A} \qquad \overbrace{1101}^{D}$$

This shows that $11\ 1010\ 1101_2 = 3AD_{16}$.

IPv6 ADDRESSES

As mentioned in Section 16.1, an IP address is a binary address consisting of 16 eight-bit numbers. IPv6 addresses are written with eight groups of hexadecimal quartets separated by colons. For example, the dot-decimal version of an IPv6 might be 192.168.5.1.37.54.121.2.252.187.212.5.12.91.246.4. This would be written in hexadecimal notation as c0a8:0501:2536:7902:fcbb:d405:0c5b:f604 and the computer would see this as

11000000.10101000.00000101.00000001.00100101.00110110.01111001.00000010. 11111100.10111011.11010100.00000101.00001100.01011011.11110110.00000100.

Starting in 2011, all IP addresses were IPv6.

APPLICATION ELECTRONICS

EXAMPLE 16.54

What is the dot-decimal version of the IPv6 address ab03:4519:f2ff:0807:9018:18d0:0ea0:499a?

SOLUTION Each quartet consists of two hexadecimal numbers. We begin with the quartet on the left: ab03. This consists of the hexadecimal numbers ab and 03. The hex number ab had the decimal value 171 and the hex number 03 the decimal number 03. Thus, the quartet ab03 has the dot-decimal value of 171.3. Notice that the 0s were left out of the dot-decimal number 03. The next quartet is 4519. The hex number 45 has the decimal value of 69 and the hex number 19 is 25. So, the quartet 4519 has the dot-decimal equivalent of 69.25. Proceeding with the next quartet f2ff would be $f2_{12} = 242$ and $ff_{12} = 255$, so f2ff would be 242.255. The next quartet 0807 has the dot-decimal value 8.7; then 9018 is 144.24; 18d0 is 24.208; 0ea0 is 15.160; and 499a is 73.154.

Putting these all together, we see that the IPv6 address ab03:4519:f2ff:0807:9018:18d0:0ea0:499a has the dot-decimal equivalent of 171.3.69.25.242.255.8.7.144.24.24.208.15.160.73.154.

EXERCISE SET 16.5

In Exercises 1–4, write each hexadecimal number in expanded form.

1. $A5D_{16}$ **2.** $C07E_{16}$ **3.** $2A037.B_{16}$ **4.** $30\,9D4B.3F_{16}$

In Exercises 5–16, convert the given hexadecimal number to its decimal equivalent.

5. 123_{16}
6. AB_{16}
7. $A5D_{16}$
8. $9A7_{16}$
9. $2FC5_{16}$
10. $73CE4_{16}$
11. $B3.F_{16}$
12. $F21.B_{16}$
13. $3A4.BC_{16}$
14. $5FC.9D_{16}$
15. $D7E2.8E_{16}$
16. $7A8C6.F7_{16}$

In Exercises 17–32, convert the given decimal number to its hexadecimal equivalent.

17. 395
18. 736
19. 2038
20. 7609
21. 52,706
22. 320,784
23. 0.3125
24. 0.5
25. 0.8125
26. 0.84375
27. 0.703125
28. 0.6884765625
29. 0.255
30. 0.7025
31. 725.6075
32. 3854.67578125

In Exercises 33–36, convert the given hexadecimal number to its binary equivalent.

33. 395_{16} **34.** $F0E_{16}$ **35.** $A73C_{16}$ **36.** $B4D2_{16}$

In Exercises 37–40, convert the given binary number to its hexadecimal equivalent.

37. $1011\,0100_2$ **38.** $101\,1101_2$ **39.** $1011\,1101\,0001_2$ **40.** $11\,1101\,0010_2$

Solve Exercises 41–42.

41. ***Information technology*** Determine the dot-decimal version of the IPv6 address 2001:cbba:0000:427a:abcd:4321:3257:9652.

42. ***Information technology*** Determine the dot-decimal version of the IPv6 address 2011:ffaa:0309:100e:dcba:5678:9abc:4321.

[IN YOUR WORDS]

43. Describe three differences between the binary number system and the hexadecimal system.

44. Why do you think that programmers prefer the hexadecimal system to the binary system?

45. Write a short description explaining how to either (a) change a binary number to a hexadecimal number or (b) change a hex number to a binary number.

46. What are some advantages of the hexadecimal system over the binary system?

16.6 HEXADECIMAL ARITHMETIC

The rules for hexadecimal arithmetic are essentially the same as for binary, octal, and decimal numbers. But, because the hexadecimal system has 16 digits, addition can seem more complicated.

HEXADECIMAL ADDITION

Table 16.12 is an addition table for binary numbers.

TABLE 16.12 Hexadecimal Addition

+	0	1	2	3	4	5	6	7	8	9	A	B	C	D	E	F
0	0	1	2	3	4	5	6	7	8	9	A	B	C	D	E	F
1	1	2	3	4	5	6	7	8	9	A	B	C	D	E	F	10
2	2	3	4	5	6	7	8	9	A	B	C	D	E	F	10	11
3	3	4	5	6	7	8	9	A	B	C	D	E	F	10	11	12
4	4	5	6	7	8	9	A	B	C	D	E	F	10	11	12	13
5	5	6	7	8	9	A	B	C	D	E	F	10	11	12	13	14
6	6	7	8	9	A	B	C	D	E	F	10	11	12	13	14	15
7	7	8	9	A	B	C	D	E	F	10	11	12	13	14	15	16
8	8	9	A	B	C	D	E	F	10	11	12	13	14	15	16	17
9	9	A	B	C	D	E	F	10	11	12	13	14	15	16	17	18
A	A	B	C	D	E	F	10	11	12	13	14	15	16	17	18	19
B	B	C	D	E	F	10	11	12	13	14	15	16	17	18	19	1A
C	C	D	E	F	10	11	12	13	14	15	16	17	18	19	1A	1B
D	D	E	F	10	11	12	13	14	15	16	17	18	19	1A	1B	1C
E	E	F	10	11	12	13	14	15	16	17	18	19	1A	1B	1C	1D
F	F	10	11	12	13	14	15	16	17	18	19	1A	1B	1C	1D	1E

EXAMPLE 16.55

Determine the sum $57B_{16} + 46A8_{16}$.

SOLUTION As with decimal and binary addition, we place the numbers in columns according to their place value, start adding with the rightmost column, and work to the left using the sums in Table 16.12:

$$\begin{array}{r} {\scriptstyle 1} \phantom{B_{16}} \\ 5\,7\,B_{16} \\ +\ 4\,6\,A\,8_{16} \\ \hline 3_{16} \end{array} \Rightarrow \begin{array}{r} {\scriptstyle 1\ 1} \phantom{B_{16}} \\ 5\,7\,B_{16} \\ +\ 4\,6\,A\,8_{16} \\ \hline 2\,3_{16} \end{array} \Rightarrow \begin{array}{r} {\scriptstyle 1\ 1} \phantom{B_{16}} \\ 5\,7\,B_{16} \\ +\ 4\,6\,A\,8_{16} \\ \hline C\,2\,3_{16} \end{array} \Rightarrow \begin{array}{r} {\scriptstyle 1\ 1} \phantom{B_{16}} \\ 5\,7\,B_{16} \\ +\ 4\,6\,A\,8_{16} \\ \hline 4\,C\,2\,3_{16} \end{array}$$

Thus, we see that $57B_{16} + 46A8_{16} = 4C23_{16}$.

EXAMPLE 16.56

Determine the sum $97A.D_{16} + 826.7C_{16}$.

SOLUTION Once again the numbers are placed in columns according to their place value. We rewrite $97A.D_{16}$ as $97A.D0_{16}$ so that both addends have two digits to the right of the hex point. Start adding with the rightmost column, and work to the left using the sums in Table 16.12:

$$\begin{array}{r} 97A.D_{16} \\ +\,826.7C_{16} \\ \hline \end{array} \overset{\text{Rewrite}}{\Rightarrow} \begin{array}{r} 97A.D0_{16} \\ +\,826.7C_{16} \\ \hline C_{16} \end{array} \Rightarrow \begin{array}{r} 97\overset{1}{A}.D0_{16} \\ +\,826.7C_{16} \\ \hline 4C_{16} \end{array}$$

$$\Rightarrow \begin{array}{r} 9\overset{1}{7}\overset{1}{A}.D0_{16} \\ +\,826.7C_{16} \\ \hline 1.4C_{16} \end{array} \Rightarrow \begin{array}{r} 9\overset{1}{7}\overset{1}{A}.D0_{16} \\ +\,826.7C_{16} \\ \hline A1.4C_{16} \end{array} \Rightarrow \begin{array}{r} 9\overset{1}{7}\overset{1}{A}.D0_{16} \\ +\,826.7C_{16} \\ \hline 11A1.4C_{16} \end{array}$$

We have shown that $97A.D_{16} + 826.7C_{16} = 11A1.4C_{16} + 826.7C_{16}$.

HEXADECIMAL COMPLEMENTS

Hexadecimal subtraction is most easily done using complements. To determine the diminished radix, or 15's, complement, subtract each digit from 15, or F. The radix, or 16's, complement is obtained by adding one to the 15's complement.

EXAMPLE 16.57

Determine the diminished radix, or 15's, complement of $4B37_{16}$.

SOLUTION To find the 15's complement of $4B37_{16}$, we subtract it from $FFFF_{16}$. We use Table 16.12 to help with the answers. For example, to determine $F - 7$, we look at the "7" row of the table to find the solution to $7 + x = F$. In this case, we see that $x = 8$. Continuing, we determine the 15's complement of $4B37_{16}$:

$$\begin{array}{r} FFFF_{16} \\ -4B37_{16} \\ \hline B4C8_{16} \end{array}$$

Thus, the 15's complement of $4B37_{16}$ is $B4C8_{16}$.

EXAMPLE 16.58

Determine the 16's complement of $4B37_{16}$.

SOLUTION In the previous example, we found that the 15's complement of $4B37_{16}$ is $B4C8_{16}$. The 16's complement is $B4C8_{16} + 1 = B4C9_{16}$.

HEXADECIMAL SUBTRACTION

In hexadecimal subtraction, we add the 16's complement and drop the leading 1.

EXAMPLE 16.59

Use complements to determine $AB52_{16} - 4B37_{16}$.

SOLUTION Notice that the both numbers have four digits.

In the previous example, we found that the 16's complement of $4B37_{16}$ is $B4C9_{16}$:

	Replace subtrahend with 16's complement	Add and drop the leading 1
$\begin{array}{r} AB52_{16} \\ -\ \underline{4B37_{16}} \end{array}$	$\begin{array}{r} AB52_{16} \\ +\ \underline{B4C9_{16}} \end{array}$	$\begin{array}{r} AB52_{16} \\ +\ \underline{B4C9_{16}} \\ \cancel{1}601B_{16} \end{array}$

Dropping the leading 1 gives the subtraction result $601B_{16}$. We have used complements to determine that $AB52_{16} - 4B37_{16} = 601B_{16}$.

EXAMPLE 16.60

Use complements to determine $F2E\ 4C97_{16} - A23F_{16}$.

SOLUTION The number $F2E\ 4C97_{16}$ has seven digits, whereas $A23F_{16}$ has just four digits. We will rewrite $A23F_{16}$ as $000\ A23F_{16}$ so that it has seven digits. The 15's complement of $000\ A23F_{16}$ is $FFF\ 5DC0_{16}$, and so the 16's complement is $FFF\ 5DC1_{16}$:

	Replace subtrahend with 16's complement	Add and drop the leading 1
$\begin{array}{r} F2E\ 4C97_{16} \\ -\ \underline{A23F_{16}} \end{array}$	$\begin{array}{r} F2E\ 4C97_{16} \\ +\ \underline{FFF\ 5DC1_{16}} \end{array}$	$\begin{array}{r} F2E\ 4C97_{16} \\ +\ \underline{FFF\ 5DC1_{16}} \\ \cancel{1}\ F2D\ AA58_{16} \end{array}$

By dropping the leading 1 we have used complements to determine that $F2E\ 4C97_{16} - A23F_{16} = F2D\ AA58_{16}$.

EXAMPLE 16.61

Use complements to determine $109.AA53_{16} - 3A.0B8_{16}$.

SOLUTION The minuend has three places to the left of the binary point and four to the right of the hexadecimal point, whereas the subtrahend has only two digits to the left of the hex point and three to the right.

We rewrite the subtrahend as $03A.0B80_{16}$ so that it has the same number of digits on each side of the hex point as does the minuend. The 16's complement of the subtrahend is $FC5.F480_{16}$:

	Replace subtrahend with 16's complement	Add and drop the leading 1
$\begin{array}{r} 109.AA53_{16} \\ -\ \underline{3A.0B8_{16}} \end{array}$	$\begin{array}{r} 109.AA53_{16} \\ +\ \underline{FC5.F480_{16}} \end{array}$	$\begin{array}{r} 109.AA53_{16} \\ +\ \underline{FC5.F480_{16}} \\ \cancel{1}0CF.9ED3_{16} \end{array}$

Dropping the leading 1, we see that $109.AA53_{16} - 3A.0B8_{16} = 0CF.9ED3_{16}$. This answer would normally be written without the leading zero, $CF.9ED3_{16}$.

EXAMPLE 16.62

Use complements to determine $9BD5.02_{16} - 75.70F_{16}$.

SOLUTION Since the minuend has two places to the right of the binary point and the subtrahend has three, we add a zero at the right of the minuend to make it $9BD5.020_{16}$. We add two zeros at the left of the subtrahend to make it $0075.70F_{16}$. These changes make the subtraction now read $9BD5.020_{16} - 0075.70F_{16}$.

The 15's complement of the subtrahend is $FF8A.8F0_{16}$, so its 16's complement is $FF8A.8F1_{16}$:

$$\begin{array}{rr} & 9BD5.020_{16} \\ - & \underline{0075.70F_{16}} \end{array} \qquad \begin{array}{c}\text{Replace subtrahend}\\ \text{with 16's complement}\end{array} \quad \begin{array}{rr} & 9BD5.020_{16} \\ + & \underline{FF8A.8F1_{16}} \end{array} \qquad \begin{array}{c}\text{Add and drop}\\ \text{the leading 1}\end{array} \quad \begin{array}{rr} & 9BD5.020_{16} \\ + & \underline{FF8A.8F1_{16}} \\ & \cancel{1}9B5F.911_{16} \end{array}$$

Dropping the leading 1, we see that $9BD5.02_{16} - 75.70F_{16} = 9B5F.911_{16}$.

THE RGB COLOR MODEL

The standard color model used on the Internet is the RGB model. RGB stands for red, green, and blue. Each of these three colors is represented by a decimal value from 0 to 255 or the hexadecimal values 00 to FF. Each color is normally expressed in the form #RRGGBB, where RR, GG, and BB represent the hexadecimal digits for the amount of red, green, and blue. Zero indicates the absence of a particular color, and 255 (or the hex value FF) indicates the maximum use of that color.

APPLICATION ELECTRONICS

EXAMPLE 16.63

What are the RGB color representations for red, blue, and green?

SOLUTION The color red would use the maximum amount of red color and none of the colors green and blue. Thus, red is indicated by #FF0000.

Blue uses the maximum amount of blue color and none of the colors red and green. Thus, blue is indicated by #0000FF.

You would expect that green would use the maximum amount of green color and none of the colors red and blue. But, the color indicated by #00FF00 is called lime. Green is indicated by #008000.

EXERCISE SET 16.6

In Exercises 1–8, determine each of the hexadecimal sums.

1. $\begin{array}{rr} & 509_{16} \\ + & \underline{201_{16}} \end{array}$

2. $\begin{array}{rr} & B2A7_{16} \\ + & \underline{903_{16}} \end{array}$

3. $\begin{array}{r} \text{CB 9755}_{16} \\ +\ \ \text{8 E3A4}_{16} \\ \hline \end{array}$

4. $\begin{array}{r} \text{FA BCAB}_{16} \\ +\ \ \text{7BEAB}_{16} \\ \hline \end{array}$

5. $\text{7F6 248A}_{16} + \text{96 DB87}_{16}$

6. $\text{753 1ECA}_{16} + \text{DF 9758}_{16}$

7. $\text{53 E782}_{16} + \text{72 10C5}_{16} + \text{95 217D}_{16}$

8. $\text{19 735C}_{16} + \text{7 DD1B}_{16} + \text{97B 834A}_{16} + \text{531A 95E7}_{16}$

In Exercises 9–12, determine the 16's complement of each of the hexadecimal numbers.

9. DA75_{16} **10.** 7 F006_{16} **11.** 345 1289_{16} **12.** 1101.101_{16}

In Exercises 13–28, (a) determine the 16's complement of each subtrahend and (b) use the 16's complement to solve each of the following subtraction exercises.

13. $\begin{array}{r} \text{A101}_{16} \\ -\ \ \text{1901}_{16} \\ \hline \end{array}$

14. $\begin{array}{r} \text{6 7324}_{16} \\ -\ \ \text{2 8107}_{16} \\ \hline \end{array}$

15. $\begin{array}{r} \text{47 0101}_{16} \\ -\ \ \text{18 2D59}_{16} \\ \hline \end{array}$

16. $\begin{array}{r} \text{A04 DF57}_{16} \\ -\ \ \text{70B E379}_{16} \\ \hline \end{array}$

17. $\begin{array}{r} \text{1C5 077A}_{16} \\ -\ \ \text{F8 5B27}_{16} \\ \hline \end{array}$

18. $\begin{array}{r} \text{53CA 9537}_{16} \\ -\ \ \text{EC9 96BE}_{16} \\ \hline \end{array}$

19. $\begin{array}{r} \text{759B 0D37}_{16} \\ -\ \ \text{A7 3679}_{16} \\ \hline \end{array}$

20. $\begin{array}{r} \text{32E 85FA}_{16} \\ -\ \ \text{FD 972C}_{16} \\ \hline \end{array}$

21. $\begin{array}{r} \text{756E.573}_{16} \\ -\ \ \text{3C49.48A}_{16} \\ \hline \end{array}$

22. $\begin{array}{r} \text{C472.A73}_{16} \\ -\ \ \text{A7B1.95D}_{16} \\ \hline \end{array}$

23. $\begin{array}{r} \text{50AC.8B23}_{16} \\ -\ \ \text{10E.07D}_{16} \\ \hline \end{array}$

24. $\begin{array}{r} \text{1069.5683}_{16} \\ -\ \ \text{78.579}_{16} \\ \hline \end{array}$

25. $\begin{array}{r} \text{75E8.91}_{16} \\ -\ \ \text{457.2B5}_{16} \\ \hline \end{array}$

26. $\begin{array}{r} \text{3DB1.753}_{16} \\ -\ \ \text{EF.5846}_{16} \\ \hline \end{array}$

27. $\begin{array}{r} \text{5B734.AF2}_{16} \\ -\ \ \text{8E9.7BA1}_{16} \\ \hline \end{array}$

28. $\begin{array}{r} \text{D377.12A}_{16} \\ -\ \ \text{D368.10A5}_{16} \\ \hline \end{array}$

[IN YOUR WORDS]

29. ***Information technology HTML*** specifies 16 color names that can be used to define a color in an ***HTML*** attribute. These colors were originally picked because they were the standard 16 colors supported by the Windows VGA palette. What are these 16 colors and the hexadecimal code for each color?

30. Write a definition of the 16's complement of a hexadecimal number and explain how it is determined.

31. Explain how the 16's complement of a hexadecimal number differs from its 15's complement.

32. Write a short explanation telling how hexadecimal numbers differ from decimal numbers.

CHAPTER 16 REVIEW

IMPORTANT TERMS AND CONCEPTS

Base
BCD or binary-coded decimal
Binary
- Integer
- Number
- Number system
- Point

Bit
Byte
Digit
Diminished radix complement
Gigabyte
Hexadecimal (or hex) number system
Information technology

IP address
Kilobyte
Megabyte
Method of remainders
Nibble
Octal number system
Octet
Radix
Radix complement
Repeated division
Repeated multiplication
Router
Subnet masking
Terabyte
Word
Word length

REVIEW EXERCISES

In Exercises 1–4, write each decimal number in expanded form.

1. 8531 **2.** 27,478 **3.** 53.194 **4.** 3.08

In Exercises 5–8, write each binary number in expanded form.

5. $1\ 0011_2$ **6.** $11\ 1001_2$ **7.** 11.0101_2 **8.** 101.0011_2

In Exercises 9–12, write each hexadecimal number in expanded form.

9. 7452_8 **10.** 3206_8 **11.** 502.16_8 **12.** 4271.35_8

In Exercises 13–16, write each hexadecimal number in expanded form.

13. $A72_{16}$ **14.** $B5E3_{16}$ **15.** $7F.0D_{16}$ **16.** $30B.80C_{16}$

In Exercises 17–20, convert each decimal number to (a) its binary equivalent, (b) its octal equivalent, and (c) its hexadecimal equivalent.

17. 5,716 **18.** 927 **19.** 32.8 **20.** 185.40625

In Exercises 21–24, convert (a) each binary number to its hexadecimal equivalent, or (b) each hexadecimal number to its binary equivalent.

21. $1010\ 1101.0110\ 1_2$ **22.** $111\ 0011.111_2$ **23.** $F4.3C_{16}$ **24.** $2D9.10B_{16}$

In Exercises 25–32, determine each of the indicated sums.

25. $\begin{array}{r} 1101_2 \\ +\ \ 101_2 \\ \hline \end{array}$

26. $\begin{array}{r} 11\ 0101_2 \\ +\ \ 1\ 1001_2 \\ \hline \end{array}$

27. $\begin{array}{r} 1011.0101_2 \\ +\ \ 101.111_2 \\ \hline \end{array}$

28. $\begin{array}{r} 11\ 1111.0111_2 \\ 11.1011_2 \\ +\ \ 1101.1101_2 \\ \hline \end{array}$

29. $\begin{array}{r} 7905_{16} \\ +\ \ 8867_{16} \\ \hline \end{array}$

30. $\begin{array}{r} A7\ 09BE_{16} \\ +\ \ 69\ FC4C_{16} \\ \hline \end{array}$

31. $\begin{array}{r} A4B.83F7_{16} \\ +\ \ DF.B99_{16} \\ \hline \end{array}$

32. $\begin{array}{r} 7A3.45C8_{16} \\ ED7.47BE_{16} \\ +\ \ 4FD8.736_{16} \\ \hline \end{array}$

In Exercises 33–36, (a) determine the 2's complement of each subtrahend, and (b) use the 2's complement to solve each of the following subtraction exercises.

33. $\begin{array}{r} 1101_2 \\ -\ \underline{1010_2} \end{array}$

34. $\begin{array}{r} 11\ 0101_2 \\ -\ \underline{1\ 1010_{16}} \end{array}$

35. $\begin{array}{r} 10\ 1011.0101_2 \\ -\ \underline{1\ 1001.101_2} \end{array}$

36. $\begin{array}{r} 110\ 1001.011_{16} \\ -\ \underline{1\ 1101.1011_{16}} \end{array}$

In Exercises 37–40, (a) determine the 16's complement of each subtrahend, and (b) use the 16's complement to solve each of the following subtraction exercises.

37. $\begin{array}{r} \text{A392}_{16} \\ -\ \underline{\text{7B9C}_{16}} \end{array}$

38. $\begin{array}{r} \text{52 D2C1}_{16} \\ -\ \underline{\text{E F5C7}_{16}} \end{array}$

39. $\begin{array}{r} \text{93 A425.B1}_{16} \\ -\ \underline{\text{C D796.A7}_{16}} \end{array}$

40. $\begin{array}{r} \text{A2 2734.0B5}_{16} \\ -\ \underline{\text{C95D.7E}_{16}} \end{array}$

Solve Exercises 41–43.

41. ***Information technology*** Determine the dot-decimal version of the IPv4 address 11110100.01001001.10101101.10001100.

42. ***Information technology*** Determine the binary version of the IPv4 dot-decimal address 250.3.27.133.

43. ***Information technology*** Translate the following sentence into octal ASCII code: Mathematics: The queen of the sciences.

CHAPTER 16 TEST

1. Write the decimal number 3057.26 in expanded form.

2. Write the number $101\,1011.01_2$ in expanded form.

3. Convert the decimal number 5701.625 to (a) its binary equivalent, (b) its octal equivalent, and (c) its hexadecimal equivalent.

4. Convert the binary number $101\,1011.01_2$ to its hexadecimal equivalent.

5. Convert the binary number $111\,1001.001_2$ to its octal equivalent.

6. Convert the hexadecimal number 2F8.C7_{16} to its binary equivalent.

7. What is the 2's complement of 1011.011_2?

8. What is the 16's complement of 13F4.3C_{16}?

Solve Exercises 9–14.

9. $\begin{array}{r} 11101_2 \\ +\ \underline{101_2} \end{array}$

10. $\begin{array}{r} \text{A37B}_2 \\ +\ \text{4 BF58}_2 \end{array}$

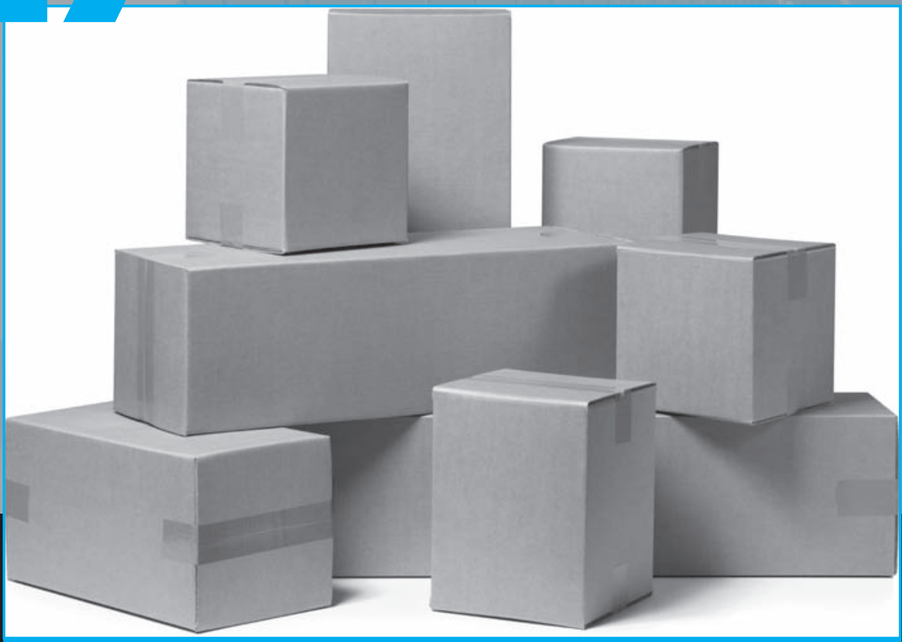
11. $\begin{array}{r} 1011.0101_2 \\ -\ \underline{101.111_2} \end{array}$

12. $\begin{array}{r} \text{B3 2B5C}_{16} \\ -\ \underline{\text{6 FC7E}_{16}} \end{array}$

13. $\begin{array}{r} \text{5312.B7}_{16} \\ -\ \underline{\text{FD3.E92}_{16}} \end{array}$

14. Determine the binary version of the IPv4 dot-decimal address 155.13.207.33.

17 HIGHER-DEGREE EQUATIONS



Package designers often need to design a box that will hold a specific volume. In Section 17.3, we will see how to design such a box made from a given sheet of material.

Most of the equations that we have solved have been linear or quadratic equations, or systems of linear equations. We were able to solve higher-degree equations when they could be factored. It seems, however, that most equations do not factor. When you get to equations of degree higher than two, there is no easy formula, such as the quadratic formula, to help solve them. In this chapter, we will explore some methods for solving polynomial equations of a degree higher than two.

OBJECTIVES

AFTER COMPLETING THIS CHAPTER, YOU WILL BE ABLE TO:

- Use synthetic division to find roots of polynomials.
- Reduce a polynomial to linear and quadratic factors using synthetic division to find all the roots of a polynomial.
- Determine the possible rational roots and test them using synthetic division.
- Find irrational roots using a numerical method and technology, model applications with polynomial functions, and use the function to solve a problem.
- Find vertical and horizontal asymptotes of rational functions.
- Solve equations that involve rational functions.

17.1 THE REMAINDER AND FACTOR THEOREM

We will begin this section with the definition of a polynomial.

POLYNOMIAL

A **polynomial** is a function of the form

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0$$

where $a_0, a_1, a_2, \ldots, a_n$ are numbers, n is a nonnegative integer, and $a_n \neq 0$. This polynomial has *degree n*.

a_n is called the *leading coefficient* of the polynomial.

EXAMPLE 17.1

The following are some examples of polynomials:

$$P(x) = 3x^2 + 2x - 1 \text{ has degree 2.}$$

$$P(x) = 5x^6 + \frac{7}{3}x^3 - 4x + 1 \text{ has degree 6.}$$

$$P(x) = -\sqrt{7} \text{ has degree 0.}$$

$$P(x) = \frac{2}{3}x^4 - \frac{1}{2}x^3 + x^2 \text{ has degree 4.}$$

EXAMPLE 17.2

The following are not polynomials.

$P(x) = 4x^{5/2} + 2x^2 - 3$, because one of the exponents, $\frac{5}{2}$, is not an integer.

$P(x) = 5x^{-3}$, because the exponent is not positive.

The third polynomial in Example 17.1, $P(x) = -\sqrt{7}$, is an example of a constant polynomial.

CONSTANT AND ZERO POLYNOMIALS

A **constant polynomial** function is of the form $P(x) = c$, where c is a real number and $c \neq 0$. The degree of a constant polynomial function is zero.

$P(x) = 0$ is called the **zero polynomial**. It has no degree.

The roots, solutions, or zeros of a polynomial function $P(x)$ are the values of x for which $P(x) = 0$. In earlier chapters, we learned that if $P(x) = ax + b$, then $P(x) = 0$, when $x = \frac{-b}{a}$. We also learned the quadratic formula, which states that if $P(x)$ is of degree 2 and $P(x) = ax^2 + bx + c$, then $P(x) = 0$, when $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

EXAMPLE 17.3

Find the roots of (a) $P(x) = 7x - \sqrt{2}$ and (b) $P(x) = 6x^2 - 3x - 5$.

SOLUTIONS

(a) The roots occur when $P(x) = 7x - \sqrt{2} = 0$. That is, when $7x = \sqrt{2}$ or $x = \frac{1}{7}\sqrt{2}$.

(b) Here, we want the solution of $6x^2 - 3x - 5 = 0$. Using the quadratic formula, we get

$$x = \frac{-(-3) \pm \sqrt{(-3)^2 - 4(6)(-5)}}{2(6)}$$

$$= \frac{3 \pm \sqrt{9 + 120}}{12} = \frac{3 \pm \sqrt{129}}{12}$$

The roots are $x = \frac{1}{4} + \frac{1}{12}\sqrt{129}$ and $x = \frac{1}{4} - \frac{1}{12}\sqrt{129}$.

We will now look at some ways we can solve polynomial functions of a degree larger than two.

REMAINDER THEOREM

Whenever we divide a polynomial $P(x)$ by $x - a$ we get a quotient $Q(x)$ that is a polynomial and a remainder R that is a constant. In fact, when $P(x)$ is divided by $x - a$, the remainder $R = P(a)$. This is the **remainder theorem**.

REMAINDER THEOREM

If a polynomial $P(x)$ is divided by $x - a$ until a remainder that does not contain x is obtained, then the remainder $R = P(a)$.

EXAMPLE 17.4

Determine the remainder when $P(x) = x^3 - 4x^2 + 2x + 5$ is divided by $x - 3$ and $x + 2$.

SOLUTION When $P(x) = x^3 - 4x^2 + 2x + 5$ is divided by $x - 3$, then $a = 3$. So, $P(3) = 3^3 - 4(3)^2 + 2(3) + 5 = 27 - 36 + 6 + 5 = 2$. When $P(x)$ is divided by $x - 3$, the remainder is 2.

When we divide $P(x)$ by $x + 2$, we have $a = -2$, so $P(-2) = (-2)^3 - 4(-2)^2 + 2(-2) + 5 = -23$. So, when $P(x)$ is divided by $x + 2$, the remainder is -23.

We can check to see if these are true by dividing $P(x)$ by $x - 3$ and by $x + 2$:

$$
\begin{array}{r}
x^2 - x - 1 \\
x - 3\overline{)\,x^3 - 4x^2 + 2x + 5} \\
\underline{x^3 - 3x^2}\phantom{{}+2x+5} \\
-x^2 + 2x + 5 \\
\underline{-x^2 + 3x}\phantom{{}+5} \\
-x + 5 \\
\underline{x + 3} \\
2
\end{array}
\qquad
\begin{array}{r}
x^2 - 6x - 14 \\
x + 2\overline{)\,x^3 - 4x^2 + 2x + 5} \\
\underline{x^3 + 2x^2}\phantom{{}+2x+5} \\
-6x^2 + 2x + 5 \\
\underline{-6x^2 - 12x}\phantom{{}+5} \\
14x + 5 \\
\underline{14x + 28} \\
-23
\end{array}
$$

From this we see that when $P(x) = x^3 - 4x^2 + 2x + 5$ is divided by $x - 3$, we get a quotient $Q(x) = x^2 - x - 1$ and a remainder of $R = 2$. When $P(x)$ is divided by $x + 2$, the quotient is $Q(x) = x^2 - 6x + 14$ with a remainder of $R = -23$. These remainders are the same ones we found in Example 17.4.

FACTOR THEOREM

In some cases, $R = 0$, and we see that $P(x) = Q(x)(x - a) + R$ can be rewritten as

$$P(x) = Q(x)(x - a)$$

That is, $x - a$ is a factor of $P(x)$. This leads to our second theorem for this section.

FACTOR THEOREM

The **factor theorem** states that a polynomial $P(x)$ contains $x - a$ as a factor if and only if $P(a) = 0$.

The factor theorem means that a is a root of $P(x)$ if x - a is a factor of $P(x)$.

EXAMPLE 17.5

Use the factor theorem to determine if $x + 1$ and $x - 2$ are factors of $P(x) = x^4 + x^3 - 3x^2 - 4x - 1$.

SOLUTION To see if $x + 1$ is a factor, we evaluate $P(x)$ at $x = -1$. Since $P(-1) = 0$, then $x + 1$ is a factor of P. [In fact, $P(x) = (x + 1)(x^3 + 3x - 1)$.]

To see if $x - 2$ is a factor of P, we evaluate $P(x)$ at $x = 2$. Since $P(2) = 3$, we know that $x + 2$ is not a factor of $x^4 + x^3 - 3x^2 - 4x - 1$.

SYNTHETIC DIVISION

What we need is a shorter method to determine the quotient and remainder of a polynomial. Both of the divisions we worked earlier took a lot of time and a lot of space. A process has been developed that allows us to shorten the division. This process is called **synthetic division**.

Consider the problem we worked earlier when $x^3 - 4x^2 + 2x + 5$ was divided by $x + 2$. We had the following work:

$$
\begin{array}{r}
x^2 - 6x + 14 \\
x + 2\overline{)x^3 - 4x^2 + 2x + 5} \\
\underline{x^3 + 2x^2} \qquad\qquad \\
-6x^2 + 2x + 5 \\
\underline{-6x^2 - 12x} \quad\;\; \\
14x + 5 \\
\underline{14x + 28} \\
-23
\end{array}
$$

In synthetic division, we do not write the x's and we do not repeat terms. So, if we erase all the x's and the terms that repeat, we get a group of coefficients that look like this:

$$
\begin{array}{rrrrr}
 & & 1 & -6 & 14 \\
\cline{2-5}
2) & 1 & -4 & 2 & 5 \\
 & & 2 & & \\
\cline{3-5}
 & & -6 & & \\
 & & & -12 & \\
\cline{3-5}
 & & & 14 & \\
 & & & & 28 \\
\cline{5-5}
 & & & & -23
\end{array}
$$

We can write all the numbers below the dividend in two lines and get

$$
\begin{array}{rrrrr}
 & & 1 & -6 & 14 \\
\cline{2-5}
2) & 1 & -4 & 2 & 5 \\
 & & 2 & -12 & 28 \\
\cline{2-5}
 & & -6 & 14 & -23
\end{array}
$$

If we repeat the leading coefficient on the bottom line, we see that the numbers on the third line represent the quotient and the remainder. Finally, we will change the sign of the divisor. Changing the sign of the divisor forces us to change the signs in the second row. This allows us to add the first two rows:

$$\begin{array}{r|rrrr}
-2 & 1 & -4 & 2 & 5 \\
 & & -2 & 12 & -28 \\
\hline
 & 1 & -6 & 14 & -23
\end{array}$$
$$\underbrace{1 \quad -6 \quad 14}_{\text{quotient}} \qquad \underbrace{-23}_{\text{remainder}}$$

The next example shows you how to use synthetic division.

EXAMPLE 17.6

Use synthetic division to determine the quotient and remainder when $x^4 - 3x^2 + 10x - 5$ is divided by $x + 3$.

SOLUTION

$$\begin{array}{r|rrrrr}
-3 & 1 & 0 & -3 & 10 & -5
\end{array}$$

Step 1: If the divisor is $x - a$, write a in the box. Arrange the coefficients of the dividend by descending powers of x. Use a zero coefficient when a power is missing. In this example, there is no x^3 term and $a = -3$.

$$\begin{array}{r|rrrrr}
-3 & 1 & 0 & -3 & 10 & -5 \\
 & & & & & \\
\hline
 & 1 & & & &
\end{array}$$

Step 2: Copy the leading coefficient in the third row.

$$\begin{array}{r|rrrrr}
-3 & 1 & 0 & -3 & 10 & -5 \\
 & & -3 & & & \\
\hline
 & 1 & -3 & & &
\end{array}$$

Step 3: Multiply the last entry in the third row by the number in the box and write the result in the second row under the second coefficient. Add the numbers in that column.

$$\begin{array}{r|rrrrr}
-3 & 1 & 0 & -3 & 10 & -5 \\
 & & -3 & 9 & & \\
\hline
 & 1 & -3 & 6 & &
\end{array}$$

Step 4a: Repeat the process from Step 3, but write the results under the third coefficient.

$$\begin{array}{r|rrrrr}
-3 & 1 & 0 & -3 & 10 & -5 \\
 & & -3 & 9 & -18 & 24 \\
\hline
 & 1 & -3 & 6 & -8 & 19
\end{array}$$
$$\underbrace{1 \quad -3 \quad 6 \quad -8}_{\text{quotient}} \qquad \underbrace{19}_{\text{remainder}}$$

Step 4b: Repeat the process from Step 3 until there are as many entries in row 3 as there are in row 1. The last number in row 3 is the remainder. The other numbers are the coefficients of the quotient.

From this we see that the quotient is $Q(x) = x^3 - 3x^2 + 6x - 8$ and the remainder is $R = 19$.

EXAMPLE 17.7

Use synthetic division to determine the quotient and remainder when $2x^6 - 11x^4 + 17x^2 - 20$ is divided by $x - 2$.

SOLUTION Since there are no x^5, x^3, or x terms, we will replace them with 0 coefficients. The synthetic division looks like this:

$$\begin{array}{r|rrrrrrr} 2 & 2 & 0 & -11 & 0 & 17 & 0 & -20 \\ & & 4 & 8 & -6 & -12 & 10 & 20 \\ \hline & \underbrace{2 \quad 4 \quad -3 \quad -6 \quad 5 \quad 10}_{\text{quotient}} & & & & & & \underbrace{0}_{\text{remainder}} \end{array}$$

The quotient is $2x^5 + 4x^4 - 3x^3 - 6x^2 + 5x + 10$ and the remainder is 0. So, $x - 2$ is a factor of the polynomial $2x^6 - 11x^4 + 17x^2 - 20$. In fact, this means that $2x^6 - 11x^4 + 17x^2 - 20 = (x - 2)(2x^5 + 4x^4 - 3x^3 - 6x^2 + 5x + 10)$.

EXAMPLE 17.8

Use synthetic division to determine if $3x - 2$ and $x - 1$ are factors of $18x^4 - 12x^3 - 45x^2 + 57x - 18$.

SOLUTION The remainder and factor theorems both refer to division by $x - a$. We are to check $3x - 2$. But $3x - 2 = 3(x - \frac{2}{3})$, and if $x - \frac{2}{3}$ divides the polynomial, then $3x - 2$ will also divide it. Our synthetic division follows:

$$\begin{array}{r|rrrrr} \frac{2}{3} & 18 & -12 & -45 & 57 & -18 \\ & & 12 & 0 & -30 & 18 \\ \hline & 18 & 0 & -45 & 27 & 0 \end{array}$$

Since the remainder is zero, $3x - 2$ is a factor of $18x^4 - 12x^3 - 45x^2 - 57x - 18$. We will now see if $x - 1$ is a factor. Instead of checking the original equation, we will use the result from the synthetic division.

When we divided $18x^4 - 12x^3 - 45x^2 + 57x - 18$ by $x - \frac{2}{3}$, we got 18 0 −45 27 in the third row. This told us that $18x^4 - 12x^3 - 45x^2 + 57x - 18 = (x - \frac{2}{3})(18x^3 - 45x + 27)$. The factor $18x^3 - 45x + 27$ is called a **depressed equation** of the original equation. If $x - 1$ is a factor of the depressed equation $18x^3 - 45x + 27$, it is a factor of the original equation. Thus we can use the third row from our earlier synthetic division as the first row when we check $x - 1$.

$$\begin{array}{r|rrrrr} \frac{2}{3} & 18 & -12 & -45 & 57 & -18 \\ & & 12 & 0 & -30 & 18 \\ \hline 1 & 18 & 0 & -45 & 27 & 0 \\ & & 18 & 18 & -27 & \\ \hline & 18 & 18 & -27 & 0 & \end{array}$$

Thus, $x - 1$ is a factor of the depressed equation $18x^3 - 45x + 27$, so $x - 1$ is also a factor of $18x^4 - 12x^3 - 45x^2 + 57x - 18$.

EXAMPLE 17.8 (Cont.)

So,

$$18x^4 - 12x^3 - 45x^2 + 57x - 18$$
$$= \left(x - \frac{2}{3}\right)(x-1)(18x^2 + 18x - 27)$$
$$= 9\left(x - \frac{2}{3}\right)(x-1)(2x^2 + 2x - 3)$$

We can use the quadratic formula on the last depressed equation to find that the remaining roots are $x = \dfrac{-1 + \sqrt{7}}{2}$ and $x = \dfrac{-1 - \sqrt{7}}{2}$.

EXERCISE SET 17.1

In Exercises 1–8, find the value of P(x) for the given value of x.

1. $P(x) = 3x^2 - 2x + 1; x = 2$
2. $P(x) = 2x^3 - 4x + 5; x = -1$
3. $P(x) = x^4 + x^3 + x^2 - x + 1; x = -1$
4. $P(x) = x^4 - 2x^2 + x; x = -2$
5. $P(x) = 5x^3 - 4x + 7; x = 3$
6. $P(x) = 7x^4 - 5x^2 + x - 7; x = -2$
7. $P(x) = x^5 - x^4 + x^3 + x^2 - x + 1; x = -1$
8. $P(x) = 3x^5 + 4x^2 - 3; x = -3$

In Exercises 9–16, use the remainder theorem to find the remainder R, when P(x) is divided by x − a.

9. $P(x) = x^3 + 2x^2 - x - 2; x - 1$
10. $P(x) = x^3 + 2x^2 - 12x - 9; x - 3$
11. $P(x) = x^3 - 3x^2 + 2x + 5; x - 3$
12. $P(x) = x^3 - 9x^2 + 23x - 15; x - 1$
13. $P(x) = 4x^4 + 13x^3 - 13x^2 - 40x + 12; x + 2$
14. $P(x) = 2x^4 - 2x^3 - 6x^2 - 14x - 7; x - 7$
15. $P(x) = 3x^4 - 12x^3 - 60x + 4; x - 5$
16. $P(x) = 4x^3 - 4x^2 - 10x + 8; x - \frac{1}{2}$

In Exercises 17–24, use the factor theorem to determine whether or not the second factor is a factor of the first.

17. $x^3 + 2x^2 - 12x - 9; x - 3$
18. $x^4 - 9x^3 + 18x^2 - 3; x + 1$
19. $2x^5 - 6x^3 + x^2 + 4x - 1; x + 1$
20. $2x^5 - 6x^3 + x^2 + 4x - 1; x - 1$
21. $3x^5 + 3x^4 - 14x^3 + 4x^2 - 24x; x + 3$
22. $3x^5 + 3x^4 - 14x^3 + 4x^2 - 24x; x - 2$
23. $6x^4 - 15x^3 - 8x^2 + 20x; 2x - 5$
24. $20x^4 + 12x^3 + 10x + 9; 5x + 3$

In Exercises 25–32, use synthetic division to determine the quotient and remainder when each polynomial is divided by the given x − a.

25. $x^5 - 17x^3 + 75x + 9; x - 3$
26. $2x^5 - x^2 + 8x + 44; x + 2$
27. $5x^3 + 7x^2 + 9; x + 3$
28. $x^3 + 3x^2 - 2x - 4; x - 2$
29. $8x^5 - 4x^3 + 7x^2 - 2x; x - \frac{1}{2}$
30. $9x^5 + 3x^4 - 6x^3 - 2x^2 + 6x + 1; x + \frac{1}{3}$
31. $4x^4 - 12x^3 + 9x^2 - 8x + 12; 2x - 3$
32. $4x^3 + 7x^2 - 3x - 15; 4x - 5$

[IN YOUR WORDS]

33. Suppose that $P(x)$ is a polynomial and you determine that $P(3) = -5$. What does this mean?

34. Explain what it means for $P(r) = 0$ for some polynomial $P(x)$.

35. What precautions must you take when using synthetic division?

17.2 ROOTS OF AN EQUATION

In Section 17.1, we learned the factor theorem can help us determine if a number is a root of a polynomial. We also learned how to use synthetic division to quickly find the quotient and remainder when a polynomial is divided by a first-degree polynomial, $x - a$. In this section, we shall learn some theorems that determine the number of roots of the equation $P(x) = 0$.

In working with first-degree polynomials, we were always able to find one root. With second-degree polynomials, we could find two roots. At times, as in $P(x) = x^2 + 6x + 9$, both of these roots were the same. (Both roots of $x^2 + 6x + 9 = 0$ were -3.) As you might expect, every polynomial of degree n has exactly n roots. The **fundamental theorem of algebra** states that every polynomial equation of degree $n > 0$ has at least one (real or complex) root.

Combining the fundamental theorem of algebra with the factor theorem leads to the **linear factorization theorem**, which states the following:

LINEAR FACTORIZATION THEOREM

If $P(x)$ is a polynomial function of degree $n > 1$, then there is a non zero number a and there are numbers, $r_1, r_2, \ldots, r_n$, such that

$$P(x) = a(x - r_1)(x - r_2) \cdots (x - r_n)$$

The proof of this theorem is fairly easy. If $P(x)$ is a polynomial and $P(x) = 0$, then by the fundamental theorem there is a number r_1 such that $P(r_1) = 0$. From the factor theorem, we know $P(x) = (x - r_1)P_1(x)$, where $P_1(x)$ is a polynomial.

Again, the fundamental theorem states that there is a number r_2 such that $P_1(r_2) = 0$, so $P_1(x) = (x - r_2)P_2(x)$ and $P(x) = (x - r_1)(x - r_2)P_2(x)$.

We continue until one of the quotients is a constant a. At that time, we have

$$P(x) = a(x - r_1)(x - r_2) \cdots (x - r_n)$$

A linear factor appears each time a root is found. Since the degree of $P(x)$ is n, there are n linear factors and n roots. These roots are $r_1, r_2, r_3, \ldots, r_n$. Now, as we have seen, these roots may not all be different. Yet, even if they are not distinct, each root is counted.

The next example, Example 17.9, will apply the linear factorization theorem to a polynomial with distinct roots. The two examples after that, Examples 17.10 and 17.11, show how the theorem works when the roots are not all different.

EXAMPLE 17.9

Use the linear factorization theorem to determine the roots of the polynomial $P(x) = 3x^4 - 8x^3 - 11x^2 + 28x - 12$.

SOLUTION

$$\begin{aligned} 3x^4 - 8x^3 - 11x^2 + 28x - 12 &= (x-1)(3x^3 - 5x^2 - 16x + 12) \\ &= (x-1)(x-3)(3x^2 + 4x - 4) \\ &= (x-1)(x-3)(x+2)(3x-2) \\ &= 3(x-1)(x-3)(x+2)\left(x - \frac{2}{3}\right) \end{aligned}$$

The roots are 1, 3, -2, and $\frac{2}{3}$.

Since $P(x)$ was of degree 4, it should have four roots. It does. Notice that the constant factor is the leading coefficient.

Just as it is not necessary that all roots be distinct, it is not necessary that all roots are real numbers. Remember that complex numbers can be roots to quadratic equations.

EXAMPLE 17.10

Determine the roots of $P(x) = x^2 + 6x + 25$.

SOLUTION Using the quadratic formula, we determine that $P(x) = x^2 + 6x + 25$ has the following roots:

$$\begin{aligned} x &= \frac{-6 \pm \sqrt{6^2 - 4(25)}}{2} \\ &= \frac{-6 \pm \sqrt{36 - 100}}{2} \\ &= \frac{-6 \pm \sqrt{-64}}{2} \\ &= \frac{-6 \pm 8j}{2} \\ &= -3 \pm 4j \end{aligned}$$

The roots are $r_1 = -3 + 4j$ and $r_2 = -3 - 4j$.

EXAMPLE 17.11

Determine the roots of $P(x) = (x-2)^3(x^2 + 6x + 25)$.

SOLUTION From the previous example we know that the roots of $x^2 + 6x + 25$ are $r_1 = -3 + 4j$ and $r_2 = -3 - 4j$. Thus, the roots of $P(x) = (x-2)^3(x^2 + 6x + 25)$ would be 2, 2, 2, $-3 + 4j$, and $-3 - 4j$. Notice that there are five roots, but three of them are the same.

One basic property of complex roots follows.

COMPLEX ROOTS THEOREM

If $P(x)$ is a polynomial with real coefficients and $a + bj$ is a root of $P(x)$, then its conjugate, $a - bj$, is also a root.

This would not be true if $P(x)$ was a polynomial with complex coefficients, but it is true when all the coefficients of P are real numbers.

HINT Remember, when solving polynomials, if you find enough roots so the remaining factor is quadratic, you can always find the last two roots by using the quadratic formula.

EXAMPLE 17.12

Solve the equation $4x^3 - 9x^2 - 25x - 12$, given the fact that $-\frac{3}{4}$ is a root.

SOLUTION Using synthetic division, we get

$$\begin{array}{r|rrrr} -\frac{3}{4} & 4 & -9 & -25 & -12 \\ & & -3 & 9 & 12 \\ \hline & 4 & -12 & -16 & 0 \end{array}$$

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So,

$$\begin{aligned} 4x^3 - 9x^2 - 25x - 12 &= \left(x + \frac{3}{4}\right)(4x^2 - 12x - 16) \\ &= 4\left(x + \frac{3}{4}\right)(x^2 - 3x - 4) \end{aligned}$$

We can factor $x^2 - 3x - 4$ as $(x - 4)(x + 1)$, and so

$$4x^3 - 9x^2 - 25x - 12 = 4\left(x + \frac{3}{4}\right)(x - 4)(x + 1)$$

The roots of $P(x) = 4x^3 - 9x^2 - 25x - 12$ are $-\frac{3}{4}$, 4, and -1.

EXAMPLE 17.13

Solve $P(x) = x^4 + 5x^3 + 10x^2 + 20x + 24$ if you know that $2j$ is a root.

SOLUTION Since $2j$ is a root and the coefficients of P are all real, its conjugate $-2j$ is also a root. So, two of the linear factors are $x - 2j$ and $x + 2j$. We can use synthetic division twice or divide the original polynomial by $(x - 2j)(x + 2j) = x^2 + 4$. We will use synthetic division twice:

$$\begin{array}{r|rrrrr} 2j & 1 & 5 & 10 & 20 & 24 \\ & & +2j & -4+10j & -20+12j & -24 \\ \hline -2j & 1 & (5+2j) & (6+10j) & (0+12j) & 0 \\ & & -2j & -10j & -12j & \\ \hline & 1 & 5 & 6 & 0 & \end{array}$$

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EXAMPLE 17.13 (Cont.)

Thus,

$$x^4 + 5x^3 + 10x^2 + 20x + 24 = (x - 2j)(x + 2j)(x^2 + 5x + 6)$$
$$= (x - 2j)(x + 2j)(x + 3)(x + 2)$$

The four roots are $2j$, $-2j$, -3, and -2.

Notice that the second time we performed synthetic division it was done on the depressed equation that resulted from the first synthetic division.

EXAMPLE 17.14

Solve $2x^4 - 17x^3 + 49x^2 - 51x + 9$ if 3 is a double root.

SOLUTION Since 3 is a double root, we know that two of the linear factors are $x - 3$ and $x - 3$. Using synthetic division twice, we get

$$\begin{array}{r|rrrrr}
3 & 2 & -17 & 49 & -51 & 9 \\
 & & 6 & -33 & 48 & -9 \\
\hline
3 & 2 & -11 & 16 & -3 & 0 \\
 & & 6 & -15 & 3 & \\
\hline
 & 2 & -5 & 1 & 0 &
\end{array}$$

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The last factor is $2x^2 - 5x + 1$. Using the quadratic formula, we see that its roots are

$$x = \frac{5 \pm \sqrt{(-5)^2 - 4(2)(1)}}{2(2)}$$
$$= \frac{5 \pm \sqrt{17}}{4}$$

Thus, the roots are 3, 3, $\frac{5 + \sqrt{17}}{4}$, and $\frac{5 - \sqrt{17}}{4}$.

EXERCISE SET 17.2

In Exercises 1–22, solve the equations using synthetic division and the given roots.

1. $5x^3 - 8x + 3$, $r_1 = 1$
2. $2x^3 + 5x^2 - 11x + 4$, $r_1 = -4$
3. $9x^3 - 3x^2 - 81x + 27$, $r_1 = \frac{1}{3}$
4. $10x^3 - 4x^2 - 40x + 16$, $r_1 = \frac{2}{5}$
5. $x^4 - 3x^2 - 4$, $r_1 = j$
6. $3x^4 + 6x^2 - 189$, $r_1 = -3j$
7. $x^4 + 2x^3 - 4x^2 - 18x - 45$, $r_1 = -1 + 2j$
8. $2x^4 + 4x^3 + 2x^2 - 16x - 40$, $r_1 = -1 - 2j$
9. $x^4 - 3x^3 - 3x^2 + 7x + 6$, $r_1 = 2$, $r_2 = 3$
10. $x^4 - 3x^3 - 12x^2 + 52x - 48$, $r_1 = 2$, $r_2 = -4$
11. $3x^4 + 12x^3 + 6x^2 - 12x - 9$, $r_1 = r_2 = -1$ (a double root)
12. $2x^4 + 6x^3 - 12x^2 - 24x + 16$, $r_1 = 2$, $r_2 = -2$
13. $6x^4 + 25x^3 + 33x^2 + x - 5$, $r_1 = -\frac{1}{2}$, $r_2 = \frac{1}{3}$
14. $12x^4 - 47x^3 + 55x^2 + 9x - 5$, $r_1 = \frac{1}{4}$, $r_2 = -\frac{1}{3}$
15. $3x^4 - 2x^3 - 3x + 2$, $r_1 = \frac{2}{3}$, $r_2 = 1$
16. $4x^4 + 3x^3 - 32x - 24$, $r_1 = 2$, $r_2 = -\frac{3}{4}$
17. $x^5 - x^4 + x^3 - 7x^2 + 10x - 4$, $r_1 = r_2 = r_3 = 1$ (a triple root)
18. $x^5 - 5x^4 + 7x^3 - 2x^2 + 4x - 8$, $r_1 = r_2 = r_3 = 2$

19. $3x^5 - 2x^4 - 24x^3 + x^2 + 28x - 12$, $r_1 = 3$, $r_2 = -2$, $r_3 = \frac{2}{3}$

20. $3x^5 - 4x^4 + 5x^3 - 18x^2 - 28x - 8$, $r_1 = -\frac{2}{3}$, $r_2 = 2j$

21. $x^6 - 6x^5 + 3x^4 - 60x^3 - 61x^2 - 54x - 63$, $r_1 = j$, $r_2 = -3j$

22. $6x^6 - 19x^5 + 63x^4 - 152x^3 + 216x^2 - 304x + 240$, $r_1 = r_2 = 2j$

[IN YOUR WORDS]

23. The polynomial $P(x) = x^3 + ix^2 - 4x - 4i$ has two real roots, 2 and -2, and one nonreal complex root, i. Explain why this does not violate the complex roots theorem.

24. Explain how to create a polynomial if you know its leading coefficient and its roots.

17.3 FINDING ROOTS OF HIGHER-DEGREE EQUATIONS

In Section 17.2, we determined the number of roots of a polynomial. We also learned that complex roots come in pairs and that once we are able to reduce the nonlinear factor to degree 2, we can use the quadratic formula.

SOLVING EQUATIONS WITH A CALCULATOR

The TI-83 and TI-84 graphing calculators allow you to solve any polynomial of degrees 2–30. This is done by using the program `PlySmlt2`.[1] The following directions are for `PlySmlt2`.

To use this program, which TI refers to as an application, press **APPS** and use the **▼** to scroll down until the cursor is on `PlySmlt2` and press **ENTER**.[2] You will see a screen that shows the Texas Instruments logo and the name and version of the program. Press any key and you will get the `MAIN MENU` screen with several options. Select option `1:PolyRootFinder`

You will be presented with the heading "`POLY ROOT FINDER MODE`" and then several lines of options, and you should enter the degree (order) of the polynomial, whether you want real or complex roots, and if you want the roots as decimals (dec) or fractions (frac). Other options include representing numbers in normal, scientific, or engineering mode and the number of decimal points in the answer. The screen should look something like the one in Figure 17.1.

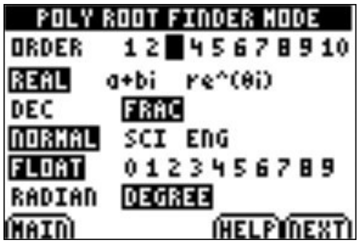

Figure 17.1

1 This is an updated version of the program PolySmlt and may need to be downloaded to your calculator. To download this software you need to go to the http://education.ti.com/ webpage. You must use `TI`™ Connect or `TI-GRAPH LINK`™ software and a `TI-GRAPH LINK` cable to install the application.

2 A faster method is to press **APPS** **ALPHA** **P**. This goes to all the apps that begin with the letter P. Then use the **▼** to scroll down until the cursor is on `PlySmlt2`.

Press the **GRAPH** button to advance to the next screen. You will then be asked to enter the coefficients starting with a_n, the coefficient of the x^n term, through a_0, the constant term. Then press the `SOLVE` by pressing the **GRAPH** button, and after a pause the n roots will be displayed. To solve another polynomial, press the **Y=** to select the `MAIN` option.

EXAMPLE 17.15

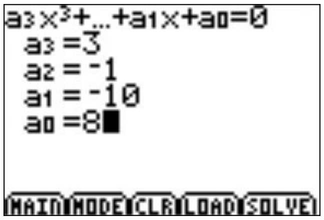

Figure 17.2a

Find all the roots of $P(x) = 3x^3 - x^2 - 10x + 8$.

SOLUTION In this polynomial, $P(x) = 3x^3 - x^2 - 10x + 8$, $n = 3$, so the degree of the polynomial is 3. We also have $a_3 = 3$, $a_2 = -1$, $a_1 = -10$, and $a_0 = 8$. The data is entered as shown in Figure 17.2a and the solution looks like that in Figure 17.2b. From Figure 17.2b we see that the roots are $x_1 = -2$, $x_2 = 4/3 = \frac{4}{3}$, and $x_3 = 1$. If you want to save any of these solutions, access the `STO` (Store) feature by pressing the **TRACE** key and selecting from the options you are given.

The solutions are $x_1 = -2$, $x_2 = 4/3 = \frac{4}{3}$, and $x_3 = 1$, and using the factor theorem, we can write the polynomial as $P(x) = 3(x + 2)(x - \frac{4}{3})(x - 1)$.

EXAMPLE 17.16

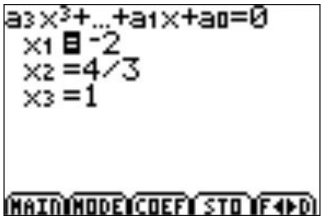

Figure 17.2b

Find all the roots of $P(x) = 4x^3 - 5x^2 - 2x + 3$.

SOLUTION In this polynomial, $P(x) = 4x^3 - 5x^2 - 2x + 3$ with $n = 3$, so the order of the polynomial is 3. We also have $a_3 = 4$, $a_2 = -5$, $a_1 = -2$, and $a_0 = 3$. The data is entered as in Example 17.15 and the solution looks like that in Figure 17.3a, with $x_1 = 1$, $x_2 = 1$, and $x_3 = -\frac{3}{4}$.

If we look at the graph of this polynomial in the neighborhood of these roots we get Figure 17.3b. We can see that $x_1 = -\frac{3}{4} = -0.75$ is one place where the graph crosses the x-axis and it seems as if the graph just touches, but does not cross, the x-axis at $x = 1$. However, the resolution of the graphing calculator makes it difficult to tell whether the graph touches the x-axis.

We will check to see if $x = 1$ is a solution. Store 1 as x by pressing **1** **STO▶** **X,T,θ,n** **ENTER**. Now, key in $4x^3 - 5x^2 - 2x + 3$ **ENTER**. The result is 0, which means that $P(1) = 0$. So, $x_1 = 1$ is a solution and so is $x_2 = 1$. Thus, 1 is a double root and this polynomial has three real roots. In factored form, $P(x) = (x + 0.75)(x - 1)^2 = (4x + 3)(x - 1)^2$.

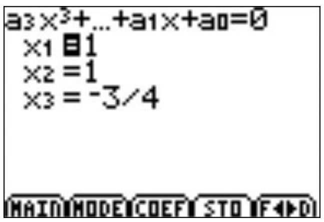

Figure 17.3a

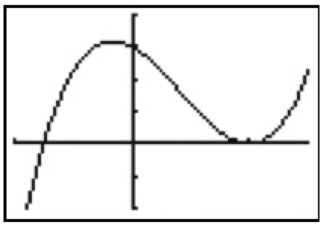

Figure 17.3b

EXAMPLE 17.17

Determine all the roots of $P(x) = 5x^6 - 4x^5 - 41x^4 + 32x^3 + 43x^2 - 28x - 7$.

SOLUTION In this polynomial $n = 6$. Entering the coefficients $a_6 = 5$, $a_5 = -4$, $a_4 = -41, \ldots, a_1 = -28$, and $a_0 = -7$ into the calculator, we get the following solutions:

$$\begin{aligned} x_1 &= -2.645751311 \\ x_2 &= 2.645751311 \\ x_3 &= -1 \\ x_4 &= 1 + 3.844892716\text{E} - 7i \\ x_5 &= 1 - 3.844892716\text{E} - 7i \\ x_6 &= -0.2 \end{aligned}$$

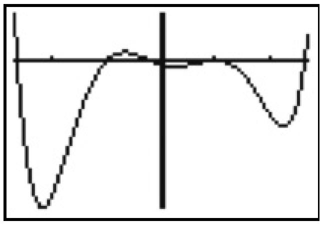

Figure 17.4

All the roots seem to be between -2.7 and 2.7. If we graph P on this interval and use ZFIT, we obtain the graph in Figure 17.4. Here we can clearly see that there are real roots at $x_1 \approx -2.6458$, $x_2 \approx 2.6458$, $x_3 = -1$, and $x_6 = -0.2$. It appears that $x_4 = x_5 = 1$ is a double root, and if we check we see that these are the roots.

Thus,

$$P(x) \approx 5(x + 2.6458)(x - 2.6458)(x + 1)(x - 1)^2(x + 0.2)$$

In most technical areas the decimal approximations of x_1 and x_2 given by the calculator are sufficient. If you needed to determine the exact values of the roots $x_1 \approx -2.6458$ and $x_2 \approx 2.6458$ of the polynomial $P(x) = 5x^6 - 4x^5 - 41x^4 + 32x^3 + 43x^2 - 28x - 7$, you could use synthetic division using the first four roots and then use the quadratic formula on the resulting quadratic factor:

1	5	−4	−41	32	43	−28	−7
		5	1	−40	−8	35	7
1	5	1	−40	−8	35	7	0
		5	6	−34	−42	−7	
−1	5	6	−34	−42	−7	0	
		−5	−1	35	7		
−0.2	5	1	−35	−7	0		
		−1	0	7			
	5	0	−35	0			

Thus, we see that

$$\begin{aligned} P(x) &= (x + 1)(x - 1)^2(x + 0.2)(5x^2 - 35) \\ &= 5(x + 1)(x - 1)^2(x + 0.2)(x^2 - 7) \end{aligned}$$

Since the solution to $x^2 - 7 = 0$ is $x = \pm\sqrt{7}$, we see that the exact values of x_1 and x_2 in Example 17.17 are $x_1 = -\sqrt{7}$ and $x_2 = \sqrt{7}$.

	A	B	C
1		Initial x:	-8
2		Increment for x:	1
3			
4	x	P(x)	
5	-8		
6	-7		
7	-6		
8	-5		

Figure 17.5

SOLVING EQUATIONS WITH A SPREADSHEET

To find the roots of a polynomial with a spreadsheet, it is best to sketch the curve first to get an approximation of the roots. A template like the one partially shown in Figure 17.5 can be constructed to make graphing polynomials easy. The polynomial is typed in Cell B5 and copied down Column B.

EXAMPLE 17.18

Find all the roots of $P(x) = 3x^3 - x^2 - 10x + 8$.

SOLUTION The degree of this polynomial is 3. The polynomial is entered in Cell B5 and a graph is constructed (see Figure 17.6a).

It appears that there are three real roots since the curve crosses the axis in three distinct places. The table of values identifies two roots since the values are integral (see Figure 17.6b). The two roots are $x = -2$ and $x = 1$. To find the third root, we will use `Solver`.

We copied a small portion of the table of values to another location where we'll keep track of the zeros we know. Figure 17.6c shows the two zeros we already know. Enter an estimate for the third root in Cell I7 (see Figure 17.6d). Place the cursor in Cell J7 and then use `Solver`. Change the value of Cell J7 to 0 by changing Cell I7. The result is shown in Figure 17.6e.

This third solution appears to actually be $\frac{4}{3}$ and this is verified by placing 4/3 in Cell I7 and seeing that Cell J7 has a value of 0.

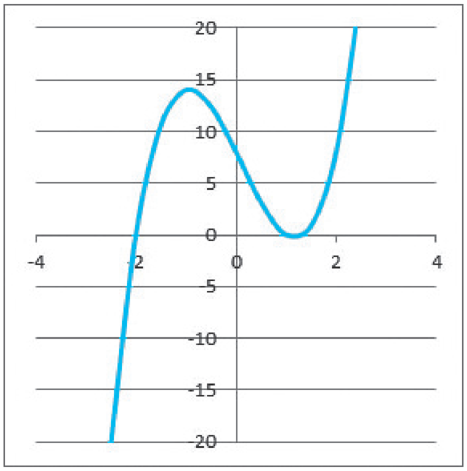

Figure 17.6a

4	x	P(x)
5	-3	-52
6	-2.5	-20.125
7	-2	0
8	-1.5	10.625
9	-1	14
10	-0.5	12.375
11	0	8
12	0.5	3.125
13	1	0
14	1.5	0.875
15	2	8

Figure 17.6b

I	J
ROOTS	
x	P(x)
-2	0
1	0
	8

Figure 17.6c

I	J
ROOTS	
x	P(x)
-2	0
1	0
1.20	-0.256

Figure 17.6d

I	J
ROOTS	
x	P(x)
-2	0
1	0
1.333333	0.000000

Figure 17.6e

EXAMPLE 17.19

Find all the roots of $P(x) = 4x^3 - 5x^2 - 2x + 3$.

SOLUTION Construct a table of values using the template and then sketch the curve. The degree of the polynomial is 3. After adjusting the initial x and increment for x, *a* reasonable sketch is obtained (see Figures 17.7a and 17.7b).

The graph, supported by the table of values, identifies one root at $x = -0.75$ and a double root at $x = 1$.

We can verify that $x = 1$ is a double root and this polynomial has three real roots. In factored form $P(x) = (x + 0.75)(x - 1)^2 = (4x + 3)(x - 1)^2$.

	A	B	C
1		Initial x:	-1.5
2		Increment for x:	0.25
3			
4	x	P(x)	
5	-1.5	-18.75	
6	-1.25	-10.125	
7	-1	-4	
8	-0.75	0	
9	-0.5	2.25	
10	-0.25	3.125	
11	0	3	
12	0.25	2.25	
13	0.5	1.25	
14	0.75	0.375	
15	1	0	
16	1.25	0.5	

Figure 17.7a

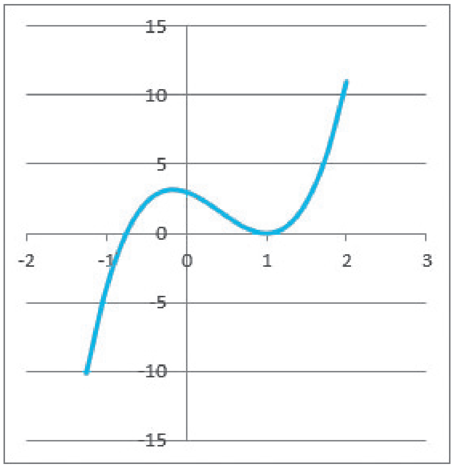

Figure 17.7b

EXAMPLE 17.20

Determine all the roots of $P(x) = 5x^6 - 4x^5 - 41x^4 + 32x^3 + 43x^2 - 28x - 7$.

SOLUTION The degree of this polynomial is 6. A sketch (with the table of values) is shown in Figure 17.8a.

	A	B	C
1		Initial x:	-1.15
2		Increment for x:	0.15
3			
4	x	P(x)	
5	-1.15	-18.699	
6	-1	1.78E-14	
7	-0.85	10.47381	
8	-0.7	14.11043	
9	-0.55	12.67146	
10	-0.4	8.04384	
11	-0.25	2.032471	
12	-0.1	-3.80606	
13	0.05	-8.28876	
14	0.2	-10.6906	
15	0.35	-10.7876	
16	0.5	-8.85938	
17	0.65	-5.65028	
18	0.8	-2.2896	
19	0.95	-0.17092	
20	1.1	-0.79033	

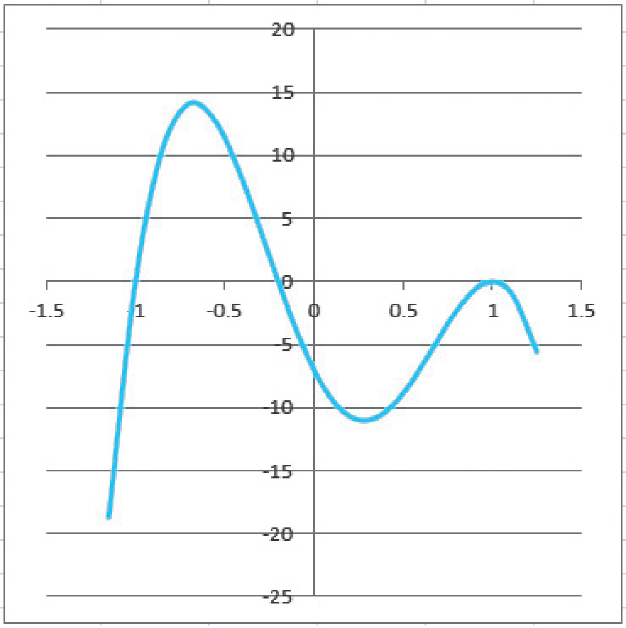

Figure 17.8a

EXAMPLE 17.20 (Cont.)

K	L
ROOTS	
x	P(x)
-1	0
1	0
-0.2	0

Figure 17.8b

It appears that there are roots at $x = 1$ (a double root) and $x = -1$. These two roots are verified in Figure 17.8b. In addition, we are able to find (using `Solver` or trial and error) that -0.2 is a root. The other two roots must be found using synthetic division. Figure 17.8c shows the synthetic division using a spreadsheet template.

Thus, we see that

$$\begin{aligned} P(x) &= (x+1)(x-1)^2(x+0.2)(5x^2-35) \\ &= 5(x+1)(x-1)^2(x+0.2)(x^2-7) \end{aligned}$$

Since the solution to $x^2 - 7 = 0$ is

$$x = \pm\sqrt{7}$$

we see that the exact values of the other two roots are

$$x_5 = \sqrt{7} \qquad \text{and} \qquad x_6 = -\sqrt{7}$$

This is verified in Figure 17.8d.

C3 =$A$2*B4

	A	B	C	D	E	F	G	H
1								
2	1	5	-4	-41	32	43	-28	-7
3			5	1	-40	-8	35	7
4	1	5	1	-40	-8	35	7	0
5			5	6	-34	-42	-7	
6	-1	5	6	-34	-42	-7	0	
7			-5	-1	35	7		
8	-0.2	5	1	-35	-7	0		
9			-1	0	7			
10		5	0	-35	0			

Figure 17.8c

K	L
ROOTS	
x	P(x)
-1	0
1	0
-0.2	0
2.645751	2.13E-13
-2.64575	3.55E-13

Figure 17.8d

APPLICATION BUSINESS

EXAMPLE 17.21

A company needs a box like the one shown in Figure 17.9a. The box is to be made from a sheet of metal that measures 45 cm by 30 cm by cutting a square from each corner and bending up the sides. If the box is to hold 3 500 cm^3, what are the lengths of the sides of the squares that are cut out of each corner?

SOLUTION A sketch of this situation is shown in Figure 17.9b. The lengths of the sides of the squares have been labeled x. Once these squares have been removed, the part of the remaining sheet that will form the length of the box is $45 - 2x$ cm. Similarly, the box's width is $30 - 2x$ cm. The width of the box must be at least 0 and less than 30 cm. So, $0 < 30 - 2x < 30$ or $0 < x < 15$.

When the metal is bent to form the sides of the box, a box like the one in Figure 17.9c is obtained. This box is a rectangular prism, so its volume is $V = (45 - 2x)(30 - 2x)x$. We are told that this box is to have a volume of 3 500 cm^3, so

$$(45 - 2x)(30 - 2x)x = 3\,500$$

Multiplying the factors on the left-hand side produces

$$1350x - 150x^2 + 4x^3 = 3500$$
$$\text{or } 4x^3 - 150x^2 + 1350x - 3500 = 0$$

Using the PlySmlt2 feature of the calculator, we see that the solutions are

$$x_1 \approx 25.6873$$
$$x_2 \approx 6.8127$$
$$x_3 \approx 5$$

One of these possible solutions, $x \approx 25.7$ is too large since $x < 15$.

Thus, there are two possible ways to cut this metal to obtain a box with the desired volume. If $x = 5$ cm, the box has a length of $45 - 2(5) = 35$ cm and a width of $30 - 2(5) = 20$ cm. Checking, we see that $(35)(20)5 = 3\,500$.

If $x \approx 6.8127$ cm, the length is approximately 31.3746 cm and the width is about 16.3746 cm. Multiplying, we get a volume of $(31.3746)(16.3746)(6.8127) = 3\,500.000\,952 \approx 3\,500\ \text{cm}^3$.

This problem has two correct solutions. The solution the company uses will depend on other factors, such as which is easier (or less expensive) to make, which one does a better job of holding the product, and which shape is more appealing to the customer.

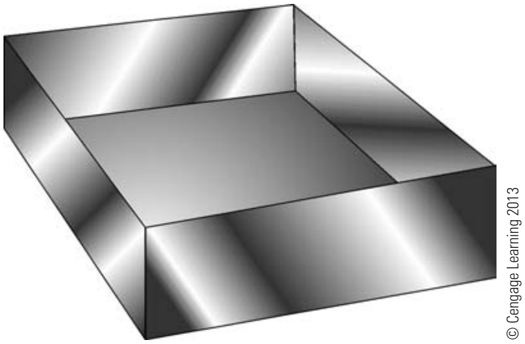

Figure 17.9a

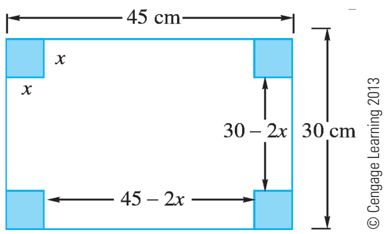

Figure 17.9b

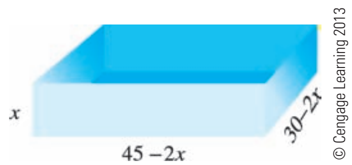

Figure 17.9c

APPLICATION GENERAL TECHNOLOGY

EXAMPLE 17.22

The stability of a molecule can be determined by solving its characteristic polynomial, $P(x)$. The roots of the equation $P(x) = 0$ determine the molecule's pi electrons. The characteristic polynomial of naphthalene $C_{10}H_8$ is

$$P(x) = x^{10} - 11x^8 + 41x^6 - 65x^4 + 43x^2 - 9$$

Determine the energies of naphthalene's 10 pi electrons.

EXAMPLE 17.22 (Cont.)

SOLUTION P is an even function and so it is symmetric about the y-axis. Thus, for each positive root, its additive inverse is also a root.

Using the PlySmlt2 feature of the calculator we obtain the roots, which are approximately ± 2.3028, ± 1.6180, ± 1.3028, ± 1, and ± 0.6180.

EXERCISE SET 17.3

In Exercises 1–16, find to four decimal places all rational and irrational roots of the given polynomial equation.

1. $x^3 + 5x - 3 = 0$

2. $x^3 - 3x + 1 = 0$

3. $x^4 + 2x^3 - 5x^2 + 1 = 0$

4. $x^4 - 5x^3 + 2x^2 + 1 = 0$

5. $x^3 + x^2 - 7x + 3 = 0$

6. $x^3 - 4x^2 - 7x + 3 = 0$

7. $x^4 - x - 2 = 0$

8. $x^5 - 2x^2 + 4 = 0$

9. $x^4 + x^3 - 2x^2 - 7x - 5 = 0$

10. $x^4 - 2x^3 - 3x + 4 = 0$

11. $2x^4 + 3x^3 - x^2 - 2x - 2 = 0$

12. $3x^4 + 3x^3 + x^2 + 4x + 3 = 0$

13. $2x^5 - 5x^3 + 2x^2 + 4x - 1 = 0$

14. $x^5 + 3x^4 - 5x^3 - 2x^2 + x + 2 = 0$

15. $8x^4 + 6x^3 - 15x^2 - 12x - 2 = 0$

16. $9x^4 + 15x^3 - 20x^2 - 20x + 16 = 0$

Solve Exercises 17–30.

17. ***Petroleum engineering*** The pressure drop P in pounds per square inch (psi) in a particular oil reservoir is a function of the number of years t that the reservoir has been in operation. The pressure drop is approximated by the equation:

$$P = 150t - 20t^2 + t^3$$

How many years will it take for the pressure to drop 400 psi?

18. ***Industrial design*** A cylindrical storage tank 12 ft high contains 674 ft^3. Determine the thickness of the tank, if the outside radius is 4.5 ft and the sides, top, and bottom have the same thickness.

19. ***Nuclear technology*** A cylindrical container for storing radioactive waste is to be constructed from lead. The sides, top, and bottom of the cylinder of the container must be at least 15.5 cm thick. (**a**) If the volume of the outside cylinder is $1\,000\,000\pi$ cm^3, and the height of the inside cylinder is twice the radius of the inside cylinder (as shown in Figure 17.10), determine the radius of the inside cylinder. (**b**) what is the volume of the inside cylinder?

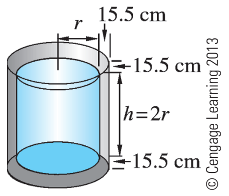

Figure 17.10

20. ***Mechanical engineering*** The characteristic polynomial for a certain material is

$$S^3 - 6S^2 - 78S + 108 = 0$$

Find the approximate value(s) of the stress that lies between 1 and 2 psi.

21. ***Chemistry*** The reference polynomial for a certain molecule is given by

$$R(x) = x^{10} - 11x^8 + 41x^6 - 61x^4 + 31x^2 - 3$$

Solve this reference polynomial.

22. ***Medical technology*** A pharmaceutical company is growing an organism to be used in a vaccine. The number of bacteria in millions at any given time t, in hours, is given by $Q(t) = -0.009t^5 + t^3 + 5.5$. Determine to the nearest tenth of a minute the value of t for which $Q(t) = 32.6$.

23. ***Sheet metal technology*** A rectangular sheet of metal was made from a box by cutting identical squares from the four corners and bending up the sides. If the piece of sheet metal originally measured 8.0 in. by 10.0 in. and the volume of the box is 48 in.^{3}, what was the length of each side of the squares that were removed?

24. ***Drafting*** A rectangular box is constructed so that its width is 2.5 cm longer than its height and the length is 4 cm longer than the width. If the box has a volume of 210 cm^3, what are its dimensions?

25. ***Agriculture*** A grain silo has the shape of a right circular cylinder with a hemisphere on top, as shown in Figure 17.11. The total height of the silo is 34 ft. Determine the radius of the cylinder if the total volume is $2{,}511\pi$ ft^3.

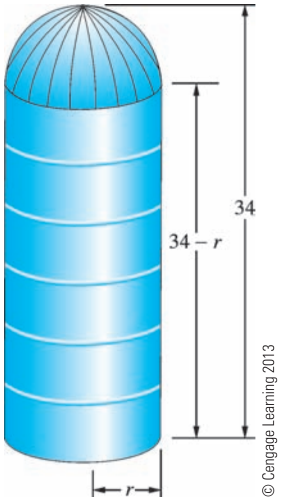

Figure 17.11

26. ***Petroleum engineering*** A propane gas storage tank is in the shape of a right circular cylinder of height 6 m with a hemisphere attached at each end. Determine the radius r so that the volume of the tank is 18π m^3.

27. ***Electricity*** Three electric resistors are connected in parallel. The second resistor is 4 Ω greater than the first and the third resistor is 1 Ω larger than the first. The total resistance is 1 Ω. In order to find the first resistance R, we must solve the equation:

$$\frac{1}{R} + \frac{1}{R+4} + \frac{1}{R+1} = 1$$

What are the values of the resistances?

28. ***Electronics*** Suppose that the resistors in Exercise 27 had been related such that the second was 4 Ω larger than the first and the third 9 Ω larger than the first. If the total resistance was 3 Ω, then

$$\frac{1}{R} + \frac{1}{R+4} + \frac{1}{R+9} = \frac{1}{3}$$

What are the values of these resistances?

29. ***Industrial technology*** A rectangular box is made from a piece of metal 12 cm by 19 cm by cutting a square from each corner and bending up the sides and welding the seams. If the volume of the box is 210 cm^3, what is the size of the square that is cut from each corner?

30. ***Architecture*** The bending moment of a beam is given by $M(d) = 0.1d^4 - 2.2d^3 + 15.2d^2 - 32d$, where d is the distance in meters from one end. Find the values of d, where the bending moment is zero. (Hint: First multiply by 10 to eliminate the decimals.)

[IN YOUR WORDS]

31. Some people think that the wording of Exercise 24 is not clear. Rewrite the exercise so that you think it is easier to understand. Hand it to a classmate for his or her comments. Did your classmate really think your wording was easier to understand than the wording in the textbook?

32. Without looking in the textbook or at your notes, write the hints for finding roots of polynomials.

17.4 RATIONAL FUNCTIONS

When you multiply two polynomials, you get another polynomial. In this section, we will study what happens when we divide two polynomials.

RATIONAL FUNCTION

If $P(x)$ and $Q(x)$ are two polynomials and if $Q(x) \neq 0$, then a function of the form

$$R(x) = \frac{P(x)}{Q(x)}$$

is called a **rational function**.

EXAMPLE 17.23

Each of the following are rational functions. Notice the restrictions placed on x so that the denominator is not zero.

(a) $R(x) = \dfrac{2x^2 + 3x - 5}{x + 2}, \qquad x \neq -2$

(b) $f(x) = \dfrac{4x^3 - 7x + 3}{3x - 4}, \qquad x \neq \dfrac{4}{3}$

(c) $g(x) = \dfrac{4}{x + 3}, \qquad x \neq -3$

(d) $h(x) = \dfrac{9x^2 + 4x - 5}{x^2 + 6x + 5}, \qquad x \neq -1, x \neq -5$

Let's first look at solving these functions and then at graphing them. In order to solve a rational function, we need to remember that any fraction $\dfrac{a}{b} = 0$, if and only if $a = 0$ and $b \neq 0$. Using this rule, we will determine when each of the functions in Example 17.23 is zero.

EXAMPLE 17.24

Solve: $\dfrac{2x^2 + 3x - 5}{x + 2} = 0$.

SOLUTION This equation will be true only when $2x^2 + 3x - 5 = 0$ and $x + 2 \neq 0$. The left-hand equation factors to $2x^2 + 3x - 5 = (2x + 5)(x - 1) = 0$, and so $x = 1$, or $x = -\frac{5}{2}$. Neither of these make $x + 2 = 0$, so 1 and $-\frac{5}{2}$ are both roots of the original equation.

EXAMPLE 17.25

Solve: $\dfrac{4x^3 - 7x + 3}{3x - 4} = 0$.

SOLUTION This equation will be true only when $4x^3 - 7x + 3 = 0$ and $3x - 4 \neq 0$. The left-hand equation has a root at 1, so $4x^3 - 7x + 3 = (x - 1)(4x^2 + 4x - 3) = (x - 1)(2x - 1)(2x + 3)$; thus $x = 1, \frac{1}{2}, -\frac{3}{2}$. None of these make $3x - 4 = 0$, so all three are roots of the original equation.

EXAMPLE 17.26

Solve: $\dfrac{4}{x + 3} = 0$.

SOLUTION Since the numerator, 4, is never zero, this equation has no solution. That is, there are no numbers that make it true.

EXAMPLE 17.27

Solve: $\dfrac{9x^2 + 4x - 5}{x^2 + 6x + 5} = 0$.

SOLUTION This equation will be true when $9x^2 + 4x - 5 = 0$, if $x^2 + 6x + 5 \neq 0$ for the same values of x. The numerator factors to $9x^2 + 4x - 5 = (9x - 5)(x + 1) = 0$, so the numerator is zero when $x = -1$ or $x = \frac{5}{9}$. The denominator is zero when $x^2 + 6x + 5 = (x + 5)(x + 1) = 0$, or $x = -5$ or $x = -1$. Notice that -1 makes both the numerator and the denominator zero. Since the denominator cannot be zero, -1 is not a solution. The only solution of the original equation is $\frac{5}{9}$.

GRAPHING RATIONAL FUNCTIONS

Some interesting things occur when we graph rational functions. Let's graph some of the examples we just worked.

EXAMPLE 17.28

Sketch the graph of $g(x) = \dfrac{4}{x + 3}$.

SOLUTION We already know that this is defined everywhere except when $x = -3$. We also know that it has no real roots, so it does not cross the x-axis. Some of the values for $g(x)$ are shown in this table.

x	-11	-10	-9	-8	-7	-6	-5	-4
$g(x)$	-0.5	-0.57	$-\frac{2}{3}$	-0.8	-1	$-\frac{4}{3}$	-2	-4

x	-2	-1	0	1	2	3	4	5
$g(x)$	4	2	$\frac{4}{3}$	1	0.8	$\frac{2}{3}$	0.57	0.5

Notice that $x = -3$ is not in the table. Let's see what happens when we get values of x closer to -3, as shown in this table.

EXAMPLE 17.28 (Cont.)

x	-4	-3.8	-3.6	-3.4	-3.2	-3.1
$g(x)$	-4	-5	$-6\frac{2}{3}$	-10	-20	-40

x	-2.9	-2.8	-2.6	-2.4	-2.2	-2.0
$g(x)$	40	20	10	$6\frac{2}{3}$	5	4

From this second table, you can see that as x gets closer to -3 from the left, $x + 3$ gets closer to zero and $g(x)$ gets smaller. As x gets closer to -3 from the right, $x + 3$ again gets closer to zero, but this time $g(x)$ gets larger. Because $g(x)$ is not defined at $x = -3$, the graph never crosses the line $x = -3$. Thus $x = -3$ is a **vertical asymptote** for this graph.

In the same way, the x-axis is a **horizontal asymptote** for the graph. The graph of $g(x)$ is shown in Figure 17.12.

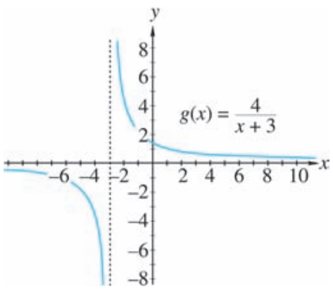

Figure 17.12

VERTICAL ASYMPTOTES

The graph of any rational function $R(x) = \dfrac{P(x)}{Q(x)}$, $Q(x) \neq 0$, where P and Q have no common factors, will have vertical asymptotes at the values of x where $Q(x) = 0$.

In Example 17.28, the denominator is 0 when $x = -3$ and the line $x = -3$ is a vertical asymptote.

EXAMPLE 17.29

Determine the vertical asymptotes, if any, for the graph of $R(x) = \dfrac{2x^2 + 3x - 5}{x + 2}$.

SOLUTION The denominator is zero when $x = -2$ and this is the vertical asymptote, as shown in Figure 17.13.

EXAMPLE 17.30

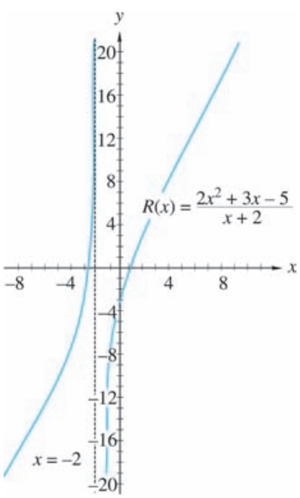

Figure 17.13

Sketch the graph of $h(x) = \dfrac{9x^2 + 4x - 5}{x^2 + 6x + 5}$.

SOLUTION The denominator is zero when $x = -5$ or $x = -1$. These are *not* both vertical asymptotes. Both the numerator and denominator have a factor of $x + 1$, so, $\dfrac{9x^2 + 4x - 5}{x^2 + 6x + 5} = \dfrac{9x - 5}{x + 5}$, except when $x = -1$. The graph of these functions will be the same, except one is defined at $x = -1$ and the other is not. Thus, the only vertical asymptote is $x = -5$. The following is a table of values and the graph is given in Figure 17.14.

x	-10	-9	-8	-7	-6	-4	-3	-2	-1
$h(x)$	19	21.5	$25\frac{2}{3}$	34	59	-41	-16	$-7\frac{2}{3}$	$\boxed{-3.5}$
x	0	1	2	3	4	5	10		
$h(x)$	-1	$\frac{2}{3}$	1.86	2.75	3.44	4	$5\frac{2}{3}$		

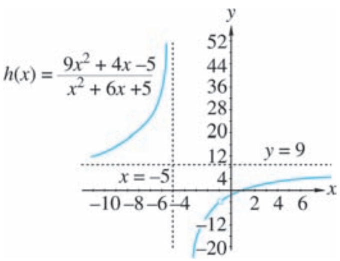

Figure 17.14

The value of $h(x) = -3.5$ is boxed because the function is not defined at $x = -1$. The value of -3.5 is the value of the simplified version of the function, $h(x) = \dfrac{9x-5}{x+5}$, when $x \neq -1$. In Figure 17.14, an open circle at the point $(-1, -3.5)$ emphasizes the fact that $h(x)$ is not defined at $x = -1$. This is called a **point of discontinuity**.

It appears as if the function h shown in Figure 17.14 has another asymptote, a horizontal one; and, in fact, it does. This asymptote is $y = 9$.

HORIZONTAL ASYMPTOTE

- If two polynomials, $P(x)$ and $Q(x)$, $Q(x) \neq 0$, are of the same degree, then the rational function $R(x) = \dfrac{P(x)}{Q(x)}$ has a *horizontal asymptote* at $y = \dfrac{a}{b}$, where a is the leading coefficient of $P(x)$ and b is the leading coefficient of $Q(x)$.
- If the degree of P is less than the degree of Q, then $y = 0$ is a horizontal asymptote.

What you are really doing when you find a horizontal asymptote is examining what happens to "y" as $|x|$ gets very large. That is, what happens to "y" as the positive values to x get very large, often written as $x \longrightarrow \infty$, and as the negative values of x get very small, written $x \longrightarrow -\infty$.

In Example 17.30, $P(x) = 9x^2 + 4x - 5$ with a leading coefficient of 9 and $Q(x) = x^2 + 6x + 5$ with a leading coefficient of 1. So, $y = \frac{9}{1} = 9$ is the horizontal asymptote.

EXAMPLE 17.31

Determine the asymptotes of $j(x) = \dfrac{2x+3}{x-1}$.

SOLUTION The numerator and denominator both have degree 1, so there is a horizontal asymptote. The leading coefficient of the numerator, $2x + 3$, is 2; the leading coefficient of the denominator, $x - 1$, is 1. The horizontal asymptote is $y = \frac{2}{1} = 2$.

The denominator is zero when $x = 1$. Since that value does not make the numerator zero it is a vertical asymptote. A graph of j is in Figure 17.15, with the asymptotes indicated by dashed lines.

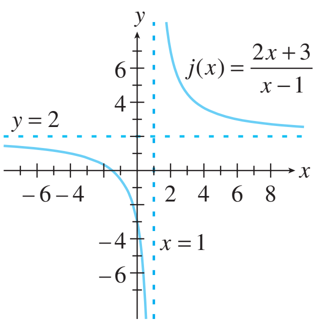

Figure 17.15

EXAMPLE 17.32

Determine the asymptotes of $f(x) = \dfrac{8x^2 - 22x - 6}{3x^2 - 2x - 1}$.

SOLUTION The numerator and denominator have the same degree, 2, so there is a horizontal asymptote. The leading coefficient of the numerator, $8x^2 - 22x - 6$, is 8; the leading coefficient of the denominator, $3x^2 - 2x - 1$, is 3. The horizontal asymptote is $y = \frac{8}{3}$.

The denominator factors into $(3x + 1)(x - 1)$, so it is zero when $x = \frac{1}{3}$ or $x = 1$. Since neither of these makes the numerator zero, they are both vertical asymptotes. The graph of f is shown in Figure 17.16, with the asymptotes indicated by dashed lines.

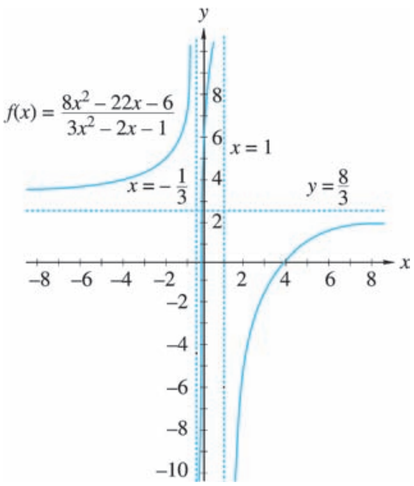

Figure 17.16

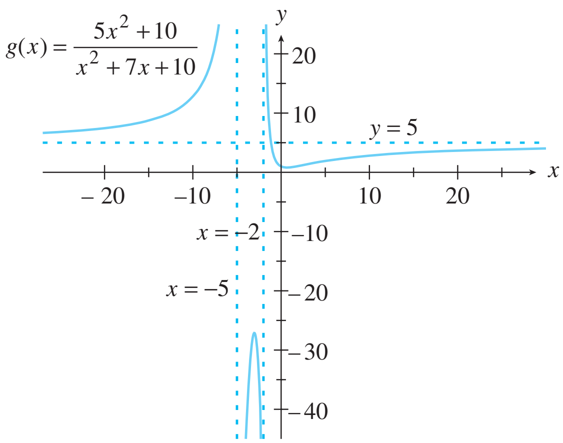

Figure 17.17

EXAMPLE 17.33

Determine the asymptotes and sketch the graph of $g(x) = \dfrac{5x^2 + 10}{x^2 + 7x + 10}$.

SOLUTION The numerator and denominator are both degree 2, so there is a horizontal asymptote at $y = 5$.

The denominator factors as $x^2 + 7x + 10 = (x + 5)(x + 2)$, so it is zero when $x = -5$ and $x = -2$. Neither of these makes the numerator zero, so they are vertical asymptotes. The graph of g is shown in Figure 17.17, with the asymptotes indicated by dashed lines. Notice that the graph actually crosses the asymptote $y = 5$ at $x \approx -1.143$.

EXAMPLE 17.34

Determine the asymptotes of $h(x) = \dfrac{2x^2 + 3x - 20}{x^2 + 3x - 4}$.

SOLUTION The numerator and denominator both have degree 2, so there is a horizontal asymptote. The leading coefficient of the numerator, $2x^2 + 3x - 20$, is 2; the leading coefficient of the denominator, $x^2 + 3x - 4$, is 1. The horizontal asymptote is $y = \frac{2}{1} = 2$.

The denominator factors as $(x + 4)(x - 1)$ and is zero when $x = -4$ and $x = 1$. The numerator factors as $(x + 4)(2x - 5)$. Notice that the numerator and denominator have a common factor of $x + 4$. Thus, there will be a "hole" in the

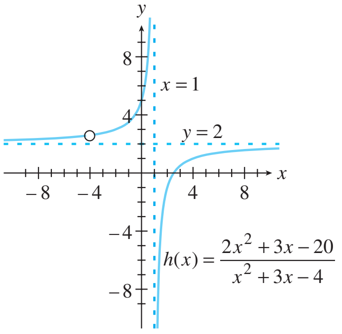

Figure 17.18

graph when $x = -4$. Since the numerator is not zero when $x = 1$, the graph has a vertical asymptote at $x = 1$.

To determine the y-coordinate of the "hole" we need to evaluate the simplified version of h. Simplifying h produces $h(x) = \frac{2x^2 + 3x - 20}{x^2 + 3x - 4} = \frac{(x + 4)(2x - 5)}{(x + 4)(x - 1)} = \frac{2x - 5}{x - 1}$. When $x = -4$, we see that $\frac{2x - 5}{x - 1}$ is 2.6. So, the "hole" is at the point $(-4, 2.6)$. A graph of h is in Figure 17.18 with the asymptotes indicated by dashed lines and the "hole" by an empty circle.

SOLVING RATIONAL EQUATIONS

Now let's return to solving rational equations. We want to solve an equation of the form $R(x) = T(x)$, where R and T are both rational functions. We solve this in the same way that we work with fractions. First we write $R(x) - T(x)$ as a fraction and then solve $R(x) - T(x) = 0$.

EXAMPLE 17.35

Solve $\frac{6}{x + 1} = \frac{4}{x + 2}$.

SOLUTION First, we get both rational functions on the same side of the equal sign:

$$\frac{6}{x + 1} = \frac{4}{x + 2}$$

or

$$\frac{6}{x + 1} - \frac{4}{x + 2} = 0$$

The lowest common denominator (LCD) of these two rational functions is $(x + 1)(x + 2)$. Writing each of these rational functions with this common denominator produces

$$\frac{6}{x + 1} = \frac{6(x + 2)}{(x + 1)(x + 2)}$$

and

$$\frac{4}{x + 2} = \frac{4(x + 1)}{(x + 1)(x + 2)}$$

so,

$$\begin{aligned}\frac{6}{x + 1} - \frac{4}{x + 2} &= \frac{6(x + 2)}{(x + 1)(x + 2)} - \frac{4(x + 1)}{(x + 1)(x + 2)}\\ &= \frac{(6x + 12) - (4x + 4)}{(x + 1)(x + 2)}\\ &= \frac{2x + 8}{(x + 1)(x + 2)} = 0\end{aligned}$$

This is zero when $2x + 8 = 0$ or $x = -4$. Since $x = -4$ does not make the denominator zero, the solution is $x = -4$.

You might be asking why we did not use a calculator or a spreadsheet to solve the equation $\frac{6}{x+1} = \frac{4}{x+2}$. We will do that in the next examples. As you will see, you need to use calculators or spreadsheets along with algebra to get the answer to many of these equations.

EXAMPLE 17.36

Use a graphing calculator to solve $\frac{6}{x+1} = \frac{4}{x+2}$.

SOLUTION If you begin by graphing Y1 $= \frac{6}{x+1}$ and Y2 $= \frac{4}{x+2}$ on the same set of axes, you get a result something like the one in Figure 17.19a or Figure 17.19b, depending on your calculator. As you can see, Figure 17.19a is a mess: it is very difficult to read and will not be of much use in finding a solution.

Graphing Y1 − Y2 on the standard viewing window produces Figure 17.19c or Figure 17.19d. This is still a difficult figure to read, but it has one major advantage over Figures 17.19a and 17.19b. In Figures 17.19a and 17.19b we wanted to find where the graphs of Y1 $= \frac{6}{x+1}$ and Y2 $= \frac{4}{x+2}$ intersected. In Figures 17.19c and 17.19d we are looking for the places where Y1 − Y2 crosses the x-axis.

If we zoom in a little by just changing the y-values of the viewing window to `Ymin` = − 2 and `Ymax` = 2, we get Figure 17.19e or Figure 17.19f. In Figure 17.19e it looks as if this graph crosses the x-axis three times, near $x = -4$, $x = -2$, and $x = -1$. But, the latter two, $x = -2$ and $x = -1$, are vertical asymptotes and cannot be solutions.

Here the `PlySmlt2` application of the calculator will not help. We need to resort to the `zero` feature we first explored in Section 4.5. Press **2nd** **TRACE** **2** [2: zero]. After selecting the left bound, right bound, and a guess, we see, in Figures 17.19g and 17.19h, that the solution is $x = -4$.

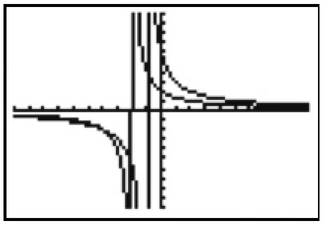

Figure 17.19a

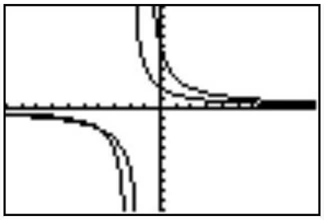

Figure 17.19b

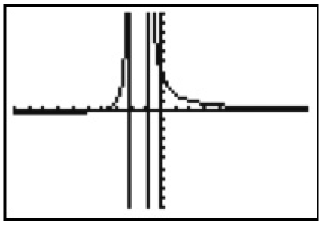

Figure 17.19c

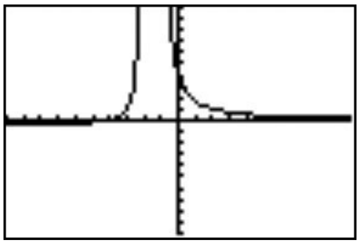

Figure 17.19d

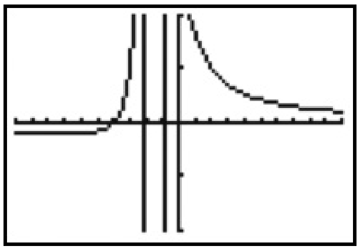

Figure 17.19e

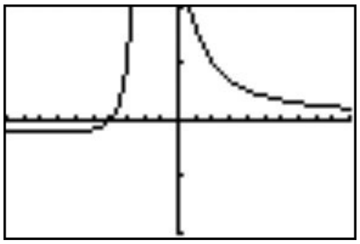

Figure 17.19f

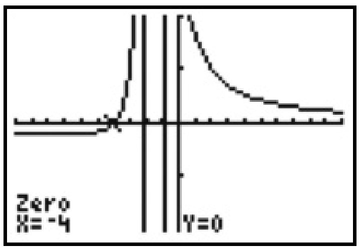

Figure 17.19g

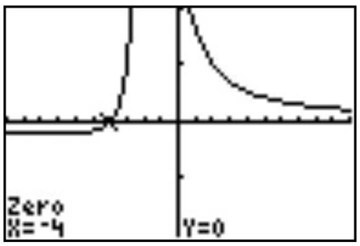

Figure 17.19h

EXAMPLE 17.37

Use a spreadsheet to solve $\dfrac{6}{x+1} = \dfrac{4}{x+2}$.

SOLUTION We will solve the equation $y1 = y2$ where $y1 = \dfrac{6}{x+1}$ and $y2 = \dfrac{4}{x+2}$.

The graph will give us an estimate of the solutions. Those estimates will be used to find a more accurate solution using the table.

A table of values is constructed and is partially shown in Figure 17.20a. The graphs of $y1$ and $y2$ are shown in Figure 17.20b with the graph of y1 in blue.

The graph is not easy to read since the two curves have asymptotes that add lines that aren't part of either graph. A graph of the difference, Column D, is easier to read and appears in Figure 17.20c.

This graph appears to have 3 zeros, x-intercepts: $x = -1$, $x = -2$, and $x = -4$. However, the first two values are actually asymptotes and so can't be solutions. Looking at the table of values, $x = -4$ is a solution since the value in Column C, $y_1 - y_2$, is zero.

	A	B	C	D
1		Initial x:	-10	
2		Increment for x:	1	
3				
4	x	y1	y2	y1 - y2
5	-10	-0.667	-0.500	-0.167
6	-9	-0.750	-0.571	-0.179
7	-8	-0.857	-0.667	-0.190
8	-7	-1.000	-0.800	-0.200
9	-6	-1.200	-1.000	-0.200
10	-5	-1.500	-1.333	-0.167
11	-4	-2.000	-2.000	0.000
12	-3	-3.000	-4.000	1.000
13	-2	-6.000	#DIV/0!	#DIV/0!
14	-1	#DIV/0!	4.000	#DIV/0!
15	0	6.000	2.000	4.000

Figure 17.20a

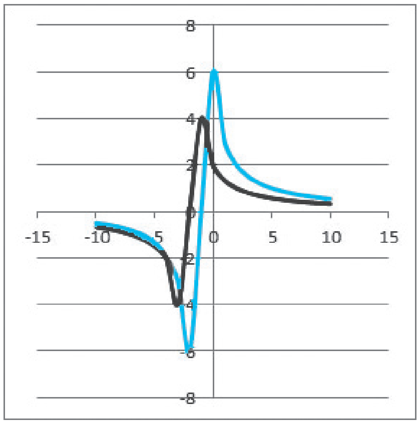

Figure 17.20b

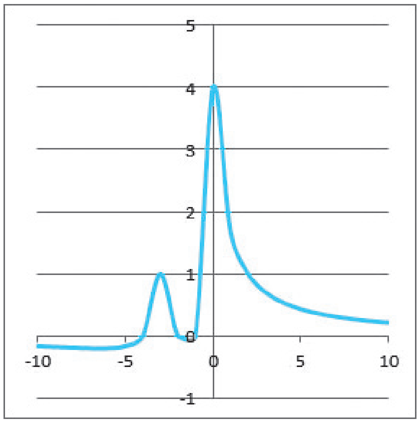

Figure 17.20c

EXERCISE SET 17.4

In Exercises 1–10, determine the horizontal and vertical asymptotes, if any, of the function. Sketch the graph.

1. $f(x) = \dfrac{3}{x+2}$

2. $g(x) = \dfrac{4}{x^2-9}$

3. $g(x) = \dfrac{2x}{x+4}$

4. $k(x) = \dfrac{5x}{x^2+x-6}$

5. $j(x) = \dfrac{3x^2}{x^2+6x+8}$

6. $f(x) = \dfrac{x^2-4}{x^2-16}$

7. $g(x) = \dfrac{x(x+1)}{(x+1)(x+2)}$

8. $h(x) = \dfrac{x^2+1}{2x^2}$

9. $j(x) = \dfrac{4x^2-12x-27}{x^2-6x+8}$

10. $k(x) = \dfrac{4x^2-9}{2x^2+7x+3}$

In Exercises 11–20, solve the given equation.

11. $\frac{(x+2)(x-3)}{(x+5)(x+4)} = 0$

12. $\frac{x^2 - 2x - 1}{x(x+1)} = 0$

13. $\frac{x^2 - 1}{x^2 + 2x + 1} = 0$

14. $\frac{3x - 4}{6x^2 - 5x - 4} = 0$

15. $\frac{4}{x-3} = \frac{2}{x+1}$

16. $\frac{5}{x+7} = \frac{2}{x-8}$

17. $\frac{5}{2x} - \frac{3}{4x} = \frac{2}{3}$

18. $\frac{2x - 3}{x + 5} = 2$

19. $\frac{3}{x-2} + \frac{5}{x+2} = \frac{20}{x^2 - 4}$

20. $\frac{4}{x+3} - \frac{6}{x-3} = \frac{10}{x^2 - 9}$

Solve Exercises 21–24.

21. ***Medical technology*** When medicine is injected into the bloodstream, its concentration t h after injection is given by

$$C(t) = \frac{3t^2 + t}{t^3 + 50}$$

The concentration is greatest when $3t^4 + 2t^3 - 300t - 50 = 0$. Approximate this time to the nearest hundredth of an hour.

22. ***Medical technology*** One of Poiseuille's laws states that the resistance, R, encountered by blood flowing through a blood vessel is given by $R(r) = C\frac{L}{r^4}$, where C is a positive constant determined by the viscosity of the blood, L is the length of the blood vessel, and r is the radius of the blood vessel. If $C = 1$, $L = 100$ mm, and $R = 6.25/\text{mm}^3$, determine r.

23. ***Business*** The average cost in dollars of producing x sets of golf clubs is given by $A(x) = 73 + \frac{7250}{x}$. What is the lowest average cost that the manufacturer can expect?

24. ***Business*** It has been determined that after t weeks of training, a certain student in a word processing class can keyboard at the rate of $W(t) = 65 + \frac{70t^2}{t^2 + 15}$ words per minute. What is the most words per minute that we can expect this student to keyboard?

[IN YOUR WORDS]

25. **(a)** Explain how to determine if a rational function has a horizontal asymptote.
    (b) Explain how to determine the horizontal asymptote of a rational function.

26. **(a)** Explain how to determine if a rational function has a vertical asymptote.
    (b) Explain how to determine the vertical asymptotes of a rational function.

CHAPTER 17 REVIEW

IMPORTANT TERMS AND CONCEPTS

Complex roots theorem
Constant polynomial
Factor of a polynomial
Factor theorem
Fundamental theorem of algebra
Horizontal asymptote
Imaginary root
Irrational root
Linear factorization theorem
Multiple roots
Point of discontinuity
Polynomial function

Rational function
Remainder theorem
Root of an equation
Synthetic division
Vertical asymptote
Zero of a function
Zero polynomial

REVIEW EXERCISES

In Exercises 1–4, find the value of $P(x)$ for the given value of x and give the degree of $P(x)$.

1. $P(x) = 9x^2 - 4x + 2, x = 2$

2. $P(x) = 2x^3 - 4x^2 + 7, x = 3$

3. $P(x) = 4x^3 - 5x^2 + 7x + 3, x = -1$

4. $P(x) = 5x^2 + 6x - 3, x = -3$

In Exercises 5–8, find the quotient and remainder when $P(x)$ is divided by $x - a$.

5. $P(x) = 4x^2 - 6x - 3; x + 3$

6. $P(x) = 6x^3 - 2x + 1; x - 1$

7. $P(x) = 9x^4 - 5x^3 + 2x^2 - 7x + 1; x - 1$

8. $P(x) = 7x^3 - 3x^2 - 12; x + 2$

In Exercises 9–16, find all the roots of the given equation. Approximate irrational roots to three decimal places.

9. $x^3 - 3x^2 + 3x - 1 = 0$

10. $x^4 - 5x^3 + 9x^2 - 7x + 2 = 0$

11. $x^4 - 6x^2 - 8x - 3 = 0$

12. $x^4 + 10x^3 + 25x^2 - 16 = 0$

13. $x^4 + x^3 - 6x^2 + x + 1 = 0$

14. $2x^5 - 3x^4 - 18x^3 + 75x^2 - 104x + 48 = 0$

15. $x^6 + 2x^5 + 3x^4 + 4x^3 + 3x^2 + 2x + 1 = 0$

16. $x^6 - 2x^5 - 4x^4 + 6x^3 - x^2 + 8x + 4 = 0$

In Exercises 17–20, determine the horizontal and vertical asymptotes, if any, and solve the equation.

17. $\dfrac{5x}{x^2 + 3x - 4} = 0$

18. $\dfrac{7x^3}{x^3 + 1} = 0$

19. $\dfrac{4}{x - 2} = \dfrac{5}{x + 2}$

20. $\dfrac{3}{x - 1} + \dfrac{2}{x + 1} = \dfrac{4}{x^2 - 1}$

Solve Exercises 21 and 22.

21. ***Medical technology*** Psychologists have developed mathematical models, called *learning curves*, that are used to predict the performance as a function of the number of trials n for a certain task. One learning curve is

$$P(n) = \frac{0.5 + 0.8(n - 1)}{1.5 + 0.8(n - 1)}, \quad n > 0$$

where P is the percentage, expressed as a decimal, of the correct responses after n trials. According to this model, what is the limiting percentage of correct responses as n increases; that is, what is the horizontal asymptote?

22. ***Medical technology*** An experiment on learning retention is conducted by the training people at a certain company. Each trainee is given 1 day to memorize a list of 30 rules. At the end of the day, the list is turned in. Each day after that, each trainee is asked to turn in a written list of as many rules as he or she can remember. It is found that

$$N(t) = \frac{5t^2 + 6t + 15}{t^2}, \quad t > 0$$

provides a close approximation of the average number of rules, $N(t)$, remembered after t days. Determine the limiting number of rules that the trainees remembered; that is, find the horizontal asymptote of N.

CHAPTER 17 TEST

1. What is the degree of $P(x) = 7x^5 - 2x^4 - 5$?
2. If $P(x) = 4x^3 - 5x^2 + 2x - 8$, what is $P(-2)$?
3. Find the quotient and remainder when $P(x) = x^4 - 8x^3 + 3x^2 + 44x + 75$ is divided by $x - 6$.
4. Find all the roots of $P(x) = 3x^4 - 7x^2 + 2x + 1 = 0$.
5. Find all the roots of $3x^3 + 5x^2 - 4x - 4 = 0$
6. Find all the roots of $x^6 + 2x^5 - 4x^4 - 10x^3 - 41x^2 - 72x - 36 = 0$.
7. Determine the horizontal and vertical asymptotes, if any, and solve the equation:
$$\frac{3x^2 - 2x - 1}{x^2 + 5x + 6} = 0$$
8. A rectangular piece of cardboard 12.0 in. by 22 in. is formed into a box by cutting a square from each corner and folding up the sides. If the volume of the box is 175 in.^3, what is the length of each side of the squares that were cut out?
9. The pressure drop in lb/in.^2 in a certain oil reservoir is given by $P(t) = 205t - 22t^2 + t^3$, where t is the time in years. After how much time does the pressure drop by exactly 500 lb/in.^2?

18 SYSTEMS OF EQUATIONS AND INEQUALITIES

Every week, an electronics company manufactures CD players and television sets. In Section 18.5, we will learn how to analyze variables, such as cost of materials and the profit on each manufactured item, to determine at which point the company will make the most profit.

In Chapter 6, we solved systems of linear equations using four methods: graphing, elimination by substitution, elimination by addition and subtraction, and Cramer's rule. In this chapter, we will use some of these same methods to solve systems where one or both equations are of the second degree.

We will also begin to explore inequalities. Many problems in technology, industry, business, science, engineering, and mathematics require the use of inequalities. Some of the same techniques we used on equations will be used to solve inequalities. Finally, we will study linear programming and use this powerful, technique in applied situations.

OBJECTIVES

AFTER COMPLETING THIS CHAPTER, YOU WILL BE ABLE TO:

- Use a graph to determine the number of real solutions to a system of nonlinear equations.
- Find an approximate solution to a system of nonlinear equations using graphing techniques.
- Solve linear and nonlinear inequalities in one variable (simple and compound) and identify the solution set algebraically and graphically.
- Graph the solution set of an inequality in two variables.
- Graph the solution set of a system of linear inequalities, and find the coordinates of the corner points of the solution set.
- Solve a linear programming problem.

18.1 SOLUTIONS OF NONLINEAR SYSTEMS OF EQUATIONS

We will look at two types of nonlinear systems of equations in this section. A **nonlinear system** is a system of equations in which one or more of the equations is not linear. The first nonlinear system we will look at contains one linear and one quadratic equation. Next, we will look at a nonlinear system of two quadratic equations. With each type, we will begin with a graphical method of solution and then consider algebraic techniques.

For our first system of nonlinear equations, we will use the following:

$$\begin{cases} x^2 + y^2 = 25 \\ x - y = 1 \end{cases}$$

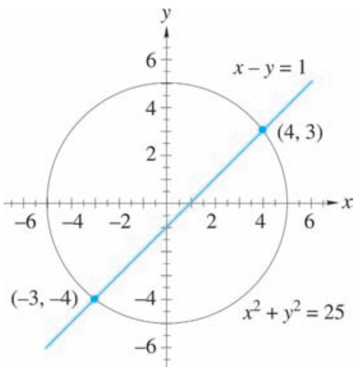

Figure 18.1

Graphically, we can see in Figure 18.1 that these two curves seem to intersect at the points (4, 3) and (−3, −4). In the next example, we will see that most graphical solutions are not solved as accurately as this one.

We can use elimination by substitution to algebraically solve this system. When you have a nonlinear system of equations that contains one linear equation, first solve the linear equation for one of the variables and substitute this solution into the other equation.

EXAMPLE 18.1

Use the substitution method to solve the system

$$\begin{cases} x^2 + y^2 = 25 & (1) \\ x - y = 1 & (2) \end{cases}$$

SOLUTION Solve equation (2) for one of the variables. If you solve equation (2) for x, you get

$$x = y + 1 \tag{3}$$

Substitute the value of x from equation (3) into equation (1) and then solve for y:

$$\begin{aligned} (y+1)^2 + y^2 &= 25 \\ y^2 + 2y + 1 + y^2 &= 25 \\ 2y^2 + 2y &= 24 \\ y^2 + y - 12 &= 0 \\ (y+4)(y-3) &= 0 \\ \text{So,} \qquad y &= -4 \text{ or } y = 3 \end{aligned}$$

Substituting these values for y in equation (3), we get the corresponding values for x:

$$\begin{aligned} \text{If } y = -4, \text{ then } x &= -4 + 1 \\ x &= -3 \\ \text{and if } y = 3, \text{ then } x &= 3 + 1 \\ x &= 4 \end{aligned}$$

The solutions are the points $(-3, -4)$ and $(4, 3)$. These are the same solutions we got when we graphed this system of equations.

Now, let's try this technique on another system of nonlinear equations that contains one linear equation.

EXAMPLE 18.2

Solve

$$\begin{cases} x^2 - 4x - 4y = 16 & (1) \\ x - 2y + 5 = 0 & (2) \end{cases}$$

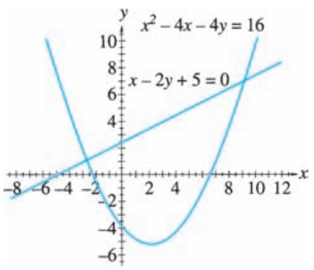

Figure 18.2

SOLUTION The graphs of these two curves are shown in Figure 18.2. We need to determine where they intersect.

We solve equation (2) for x, and obtain

$$x = 2y - 5$$

Substituting this value of x into equation (1), we get

$$\begin{aligned} (2y-5)^2 - 4(2y-5) - 4y &= 16 \\ 4y^2 - 20y + 25 - 8y + 20 - 4y &= 16 \\ 4y^2 - 32y + 29 &= 0 \end{aligned}$$

Using the quadratic formula, $y = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, we find that

EXAMPLE 18.2 (Cont.)

$$y = \frac{32 \pm \sqrt{32^2 - 4(4)(29)}}{2(4)}$$
$$= \frac{32 \pm \sqrt{1024 - 464}}{8}$$
$$= \frac{32 \pm \sqrt{560}}{8}$$
$$\approx \frac{32 \pm 23.66}{8}$$

So, $y \approx \dfrac{32 + 23.66}{8} \approx 6.96$

and $y \approx \dfrac{32 - 23.66}{8} \approx 1.04$

Substituting these approximate values of y into equation (2) we get

$$x - 2(6.96) + 5 = 0 \text{ or } x = 8.92$$

and $$x - 2(1.04) + 5 = 0 \text{ or } x = -2.92$$

The approximate solutions are $(-2.92, 1.04)$ and $(8.92, 6.96)$.

When neither of the equations is a linear equation, you may then use either of the elimination methods: substitution, or addition and subtraction.

EXAMPLE 18.3

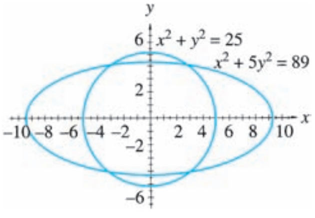

Figure 18.3

Solve

$$\begin{cases} x^2 + 5y^2 = 89 & (1) \\ x^2 + y^2 = 25 & (2) \end{cases}$$

SOLUTION The graphical solution of these two equations is shown in Figure 18.3.

Algebraically, we will use elimination by addition and subtraction. Since both equations have an x^2 term, we will subtract equation (2) from equation (1) to get

$$4y^2 = 64 \qquad (3)$$
$$y^2 = 16$$
$$y = \pm 4$$

Since $y = \pm 4$, we will substitute these values in equation (2) to determine x. For both substitutions we get

$$x^2 + 16 = 25$$
$$x^2 = 9$$
$$x = \pm 3$$

This gives a total of four solutions: $(3, 4)$, $(3, -4)$, $(-3, 4)$, and $(-3, -4)$, and these are the four points in Figure 18.3 where the two curves intersect.

In Example 18.3, a circle and an ellipse intersected in four points. It is also possible that a circle and an ellipse do not intersect or that they intersect at one point, at two points, or at three points.

EXAMPLE 18.4

Solve

$$\begin{cases} x^2 - 3y^2 = 22 & (1) \\ xy = -5 & (2) \end{cases}$$

SOLUTION The graphical solution is shown in Figure 18.4.

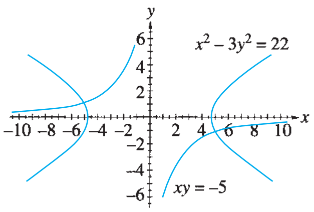

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Figure 18.4

Algebraically, we will use elimination by substitution. We will solve equation (2) for y and substitute this value into equation (1):

$$y = \frac{-5}{x} \qquad (3)$$

$$x^2 - 3\left(\frac{-5}{x}\right)^2 = 22$$

$$x^2 - \frac{75}{x^2} = 22$$

Multiplying both sides of the equation by x^2, we get

$$x^4 - 75 = 22x^2$$

$$\text{or} \quad x^4 - 22x^2 - 75 = 0$$

which factors into

$$(x^2 - 25)(x^2 + 3) = 0$$

Now, $x^2 - 25 = 0$ means that $x = \pm 5$ and $x^2 + 3 = 0$ means that $x = \pm j\sqrt{3}$. Substituting these values of x into equation (3), we get $y = \pm 1$ (when $x = \pm 5$) and $y = \pm\dfrac{5}{j\sqrt{3}} = \pm\frac{5}{3}j\sqrt{3}$ (when $x = \pm j\sqrt{3}$).

The two real solutions are $(5, -1)$ and $(-5, 1)$. These are shown on the graph in Figure 18.4. There are two imaginary roots $(j\sqrt{3}, \frac{5}{3}j\sqrt{3})$ and $(-j\sqrt{3}, -\frac{5}{3}j\sqrt{3})$. These two solutions are not on the graph, because this graph is in the real number plane and not the complex plane.

APPLICATION GENERAL TECHNOLOGY

EXAMPLE 18.5

Two satellites are in orbit in the same plane. One orbit is described by $5(x + 10)^2 + 3y^2 = 3\,000$. The other is described by $5x^2 + 4y^2 = 4\,000$. What are the points where the two orbits intersect?

EXAMPLE 18.5 (Cont.)

SOLUTION We want to solve the system that contains the equations of two ellipses:

$$\begin{cases} 5(x + 10)^2 + 3y^2 = 3000 & (1) \\ 5x^2 + 4y^2 = 4000 & (2) \end{cases}$$

We will multiply equation (1) by 4 and equation (2) by 3. This will give the y^2-terms the coefficient of 12:

$$\begin{cases} 20(x + 10)^2 + 12y^2 = 12000 \\ 15x^2 + 12y^2 = 12000 \end{cases}$$

Subtracting, we get

$$\begin{aligned} 20(x + 10)^2 - 15x^2 &= 0 \\ \text{or} \quad 5x^2 + 400x + 2000 &= 0 \\ x^2 + 80x + 400 &= 0 \end{aligned}$$

Using the quadratic formula, we get

$$\begin{aligned} x = \frac{-80 \pm \sqrt{80^2 - 4(400)}}{2} &= \frac{-80 \pm \sqrt{4800}}{2} \\ &= -40 \pm 20\sqrt{3} \end{aligned}$$

Substituting $x = -40 + 20\sqrt{3} \approx -5.36$ into equation (2), we get

$$\begin{aligned} 143.65 + 4y^2 &= 4000 \\ y^2 &= 964.09 \\ y &\approx \pm 31.05 \end{aligned}$$

Substituting $x = -40 - 20\sqrt{3} \approx -74.64$ into equation (2), we get

$$\begin{aligned} 27855.65 + 4y^2 &= 4000 \\ y^2 &\approx -5963.91 \end{aligned}$$

Since $y^2 < 0$, these roots are imaginary.

The satellites will intersect at two points: $(-5.36, 31.05)$ and $(-5.36, -31.05)$.

EXERCISE SET 18.1

In Exercises 1–10, graph the systems of equations and then solve each system algebraically for x and y. Be sure to include real and complex roots.

1. $\begin{cases} x - 2y = 5 \\ x^2 - 4y^2 = 45 \end{cases}$

2. $\begin{cases} x^2 + y^2 = 13 \\ 2x - y = 4 \end{cases}$

3. $\begin{cases} x^2 + 4y^2 = 32 \\ x + 2y = 0 \end{cases}$

4. $\begin{cases} x - y = 4 \\ x^2 - y^2 = 32 \end{cases}$

5. $\begin{cases} x^2 + y^2 = 4 \\ x^2 - 2y = 1 \end{cases}$

6. $\begin{cases} x^2 - y^2 = 4 \\ 2x - 3y = 10 \end{cases}$

7. $\begin{cases} x^2 + y^2 = 7 \\ y^2 = 6x \end{cases}$

8. $\begin{cases} x^2 - y^2 + 6 = 0 \\ y^2 = 5x \end{cases}$

9. $\begin{cases} xy = 3 \\ 2x^2 - 3y^2 = 15 \end{cases}$

10. $\begin{cases} y = x^2 - 4 \\ x^2 + 3y^2 + 4y - 6 = 0 \end{cases}$

Solve Exercises 11–24.

11. ***Transportation*** A truck driver travels the first 100 mi of a 120-mi trip in light traffic. For the last 20 mi, traffic is heavy enough that the average speed is reduced by 10 mph. The total time for the trip is 3 h. What was the average speed for each part of the trip?

12. ***Agricultural technology*** The perimeter of a rectangular field is 160 m and the area is 1 200 m^2. What are the dimensions of the field?

13. ***Electronics*** In a direct current (dc) circuit, the equivalent resistance R of two resistors R_1 and R_2 connected in series is $R = R_1 + R_2$. If the resistors are connected in parallel, then $R = \dfrac{R_1R_2}{R_1 + R_2}$. When two resistors are connected in series, the equivalent resistance is 100 Ω. When they are connected in parallel, $R = 24\,\Omega$. Find R_1 and R_2.

14. ***Electronics*** When two resistors are connected in series, the equivalent resistance is 10 Ω. When they are connected in parallel, the equivalent resistance is 2.1 Ω. What are each of the resistances? (See Exercise 13)

15. ***Physics*** When a 40-kg object traveling at a velocity of v_1 m/s collides with a 60-kg object on a frictionless surface traveling in the same direction at v_2 m/s, the final total momentum is 450 kg · m/s and is given by the formula

$$40\,v_1 + 60\,v_2 = 450$$

The relationship between the initial and final kinetic energy $\left(KE = \frac{1}{2}mv^2\right)$ is given by the equation

$$20v_1{}^2 + 30v_2{}^2 = 1\,087.5\text{ J}$$

where J stands for joule. Determine the velocity of each object.

16. ***Space technology*** Two satellites are in orbits in the same plane. One orbit is described by $20x^2 + 40y^2 = 8\,000$. The other is described by $30x^2 + 20y^2 = 6\,000$. What are the points where these two orbits intersect?

17. ***Physics*** A 200-kg object 0.8 m from the fulcrum of a lever can be lifted by a certain minimum force at the other end of the lever. If that same force is applied 1 m closer to the fulcrum, an additional 50 kg are needed to lift the object. If the minimum force is F and its original distance from the fulcrum is ℓ, then we know that

$$F\ell = (200)(0.8)$$
$$\text{and} \qquad (F + 50)(\ell - 1) = (200)(0.8)$$

Find F and ℓ.

18. ***Physics*** The vertical distance in meters that an object falls is given by $y = 15t - 4.9t^2$ m. When the horizontal distance equals twice the vertical distance in meters, then $y = 10t$. What values of y and t satisfy these conditions?

19. ***Industrial design*** One of the surfaces of a surgical tool is a circle with a radius of x cm. A square hole y cm on each side is cut from the center of the tool, leaving 303.22 mm^2 of metal. If $x - y = 5.59$ cm, find x and y.

20. ***Electrical engineering*** In determining the Thomson electromotive force of a certain thermocouple, it is necessary to solve the system of equations

$$0.02(T_2{}^2 - T_1{}^2) = 4280$$
$$T_2 - T_1 = 250$$

Solve this system of equations for T_1 and T_2 in degrees Kelvin.

21. ***Transportation engineering*** In the track for a rapid transit system, the total length of the merge and demerge region is related to the radius, r, of the curve and the offset of the distance, x, by the equation $L = 4rx$. The radius is also related to the velocity of the vehicle by the equation $r = 0.334v^2$. Combine these two equations in order to express L in terms of r and v only.

22. ***Space technology*** A space probe is designed to separate into two parts when a spring-loaded connecting bolt is fired. The motion of the two parts, with mass $m_1 = 52$ kg and $m_2 = 78$ kg, satisfies the energy equation $(\frac{1}{2})(52)(v_1^2) + (\frac{1}{2})(78)(v_2^2) = 23\,400$ and the momentum equation $52v_1 + 78v_2 = 0$, where v_1 and v_2 are the velocities of the parts in m/s. Find v_1 and v_2.

23. ***Business*** In business, the break-even point for a product is where the revenue from selling the product equals the cost of producing the item. A company sells two cold remedies, one a generic remedy and the other a "name" brand. The company has determined that the cost of producing the two brands is given by $4g^2 + 15n = 4291$. Meanwhile, the revenue is given by $8g^2 + 15g - 5n = 8607$. In both equations, g is the number of items (in millions) of the generic brand and n is the number of items (in millions) of the name brand. What is the break-even point?

24. ***Business*** In business, the break-even point for a product is where the revenue from selling the product equals the cost of producing the item. A company produces and sells x units of product A and y units of product B. The company has determined that the cost of producing the two products is given by $x^2 + 1.5y^2 - xy = 2{,}891{,}189$. Meanwhile, the revenue is given by $80x + 100y = 252{,}640$. What is the break-even point?

[IN YOUR WORDS]

25. In Example 18.3, a circle and an ellipse intersected in four points. It is also possible that a circle and an ellipse can intersect in 0, 1, 2, or 3 points. Describe how each of these can happen.

26. Explain how you decide whether to use the substitution method or the addition and subtraction method to solve a system of nonlinear equations.

18.2 PROPERTIES OF INEQUALITIES; LINEAR INEQUALITIES

In Section 1.1 we introduced the inequality symbols. Since that time, we have not used them except to indicate when conditions were being placed on a group of numbers. The fact that we have not used the inequality symbols should not indicate that they are seldom used. In fact, in some parts of mathematics, inequalities are used as often as equations. In the remainder of this chapter, we will focus on inequalities.

An *inequality* is formed whenever two expressions are separated by one of the inequality symbols: $>$, $<$, $\geq$, and $\leq$. The two expressions are called the *sides*, or *members* of the inequality. A *compound inequality* has more than two expressions separated by inequality symbols.

EXAMPLE 18.6

$$2x < 5,$$
$$3x + 7 \geq 8,$$
$$16x - 8 \leq 5x - 2,$$
$$\text{and} \quad 5x^2 - 3x - 2 > x - 7$$

are all inequalities.

$$3x < 4x - 5 < 9,$$
$$x < y \leq 2z,$$
$$-8 \leq 3x + 1 \leq 4,$$
$$\text{and} \quad 1 < 2x - 3 \leq 17$$

are all compound inequalities.

A compound inequality can always be written as a combination of simple inequalities.

EXAMPLE 18.7

The compound inequality $3x < 4x - 5 < 9$ is the same as the two simple inequalities $3x < 4x - 5$ and $4x - 5 < 9$.

$-8 \leq 3x + 1 \leq 4$ is the same as the two simple inequalities $-8 \leq 3x + 1$ and $3x + 1 \leq 4$.

There are three types of inequalities. Most of the inequalities we will use are conditional inequalities. A *conditional inequality* is true for some real numbers and false for others. For example, $x \geq 5$ is a conditional inequality. It is true if x is 5, $6\frac{1}{2}$, 19, or any other number larger than 5. On the other hand, it is false for values of x such as 4, -3, $2\frac{1}{2}$, or any number smaller than 5.

An *absolute inequality* is true for all real numbers. An example of an absolute inequality is $x + 1 > x$. No matter which real number you select, this is a true statement. The opposite of an absolute inequality is a contradictory inequality. A *contradictory inequality* is false for all real numbers. For example, $x + 1 < x$ is a contradictory inequality, because it is never true.

The *solution* of an inequality consists of those values of the variable that make the inequality true. To *solve an inequality* is to determine the solutions of the inequality.

PROPERTIES OF INEQUALITIES

There are six basic properties or rules of inequalities that are used to solve an inequality. In all of the explanations and examples of these properties, we will use the $<$ symbol. We could just as easily have used any of the other three inequality symbols.

PROPERTY 1 OF INEQUALITIES

If a, b, and c are real numbers, with $a < b$, then $a + c < b + c$.

In words, this property says that the same algebraic expression can be added to or subtracted from both sides of an inequality without changing the direction of the inequality symbol.

EXAMPLE 18.8

(a) $x + 3 < 7$ is equivalent to $x < 4$. — Subtract 3 from both sides.

(b) $3x + 4 < 6 - 2x$ is equivalent to $5x < 2$. — Add $2x$ to both sides, and subtract 4 from both sides.

PROPERTY 2 OF INEQUALITIES

If a, b, and c are real numbers, with $a < b$ and $c > 0$, then $ac < bc$ or $\frac{a}{c} < \frac{b}{c}$.

In words, this states that you can multiply or divide both sides of an inequality by the same *positive* expression without changing the direction of the inequality symbol.

EXAMPLE 18.9

(a) $\frac{x}{3} - \frac{4}{3} < 2x + 1$ is equivalent to $x - 4 < 6x + 3$. — Multiply both sides by 3.

(b) $3x < 21$ is equivalent to $x < 7$. — Divide both sides by 3.

PROPERTY 3 OF INEQUALITIES

If a, b, and c are real numbers, with $a < b$ and $c < 0$, then $ac > bc$ or $\frac{a}{c} > \frac{b}{c}$.

NOTE If you multiply or divide both sides of an inequality by the same *negative* expression, then the direction of the inequality symbol is reversed.

EXAMPLE 18.10

(a) $-3 < 7$ is equivalent to $6 > -14$. Multiply both sides by -2.

(b) $\dfrac{-x}{3} < 5$ is equivalent to $x > -15$. Multiply both sides by -3.

(c) $-4x < 20$ is equivalent to $x > -5$. Divide both sides by -4.

PROPERTY 4 OF INEQUALITIES

If a, b, and n are positive real numbers and $a < b$, then $a^n < b^n$ and $\sqrt[n]{a} < \sqrt[n]{b}$.

If both sides of an inequality are positive and n is a positive number, then the nth power or root of both sides retains the direction of the inequality.

EXAMPLE 18.11

(a) If $3 < 5$ and $n = 6$, then $3^6 < 5^6$.

(b) If $2 < 10$ and $n = 8$, then $\sqrt[8]{2} < \sqrt[8]{10}$.

PROPERTY 5 OF INEQUALITIES

If x and a are real numbers, $a > 0$, and $|x| < a$, then $-a < x < a$.

EXAMPLE 18.12

(a) $|x + 3| < 5$ is equivalent to $-5 < x + 3 < 5$.

(b) $|2x - 7| < 10$ is equivalent to $-10 < 2x - 7 < 10$.

These absolute value inequalities can be written as one compound inequality.

PROPERTY 6 OF INEQUALITIES

If x and a are real numbers, $a > 0$, and $|x| > a$, then $x > a$ or $x < -a$.

EXAMPLE 18.13

(a) $|x - 5| > 2$ is equivalent to $x - 5 > 2$ or $x - 5 < -2$.

(b) $|7x + 4| > 9$ is equivalent to $7x + 4 > 9$ or $7x + 4 < -9$.

These absolute value inequalities have to be written as *two* simple inequalities.

SOLVING LINEAR INEQUALITIES

We can use these six properties to help us solve problems that use inequalities. As is true when you solve an equation, you often use several of the properties in each problem. Each of the problems is an example of a linear inequality.

LINEAR INEQUALITY

A *linear inequality* in one variable, x, is an inequality of the form $ax + b < 0$, where a and b are constants and $a \neq 0$.

First, we will solve each of the linear inequalities in the following examples algebraically and then graph the solution. Since these are inequalities in one variable, each solution will be graphed on a number line.

EXAMPLE 18.14

Solve $6x + 3 \geq 21$.

SOLUTION $6x + 3 \geq 21$

$6x \geq 18$ Subtract 3 from both sides.

$x \geq 3$ Divide both sides by 6.

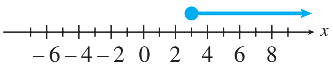

Figure 18.5

The solution to $6x + 3 \geq 21$ is all numbers greater than or equal to 3. This is shown by the colored line above the number line in Figure 18.5. (Technically, the colored line should be placed on the number line, but it is too difficult to see when this is done.) The solid circle at 3 indicates that 3 is included in the solution.

EXAMPLE 18.15

Solve $\frac{3}{4} - \frac{5}{6}x > \frac{-1}{2}x - \frac{11}{12}$.

SOLUTION $\dfrac{3}{4} - \dfrac{5}{6}x > \dfrac{-1}{2}x - \dfrac{11}{12}$

$9 - 10x > -6x - 11$ Multiply by 12, the LCD of the denominators.

$9 - 4x > -11$ Add $6x$ to both sides.

$-4x > -20$ Subtract 9 from both sides.

$x < 5$ Divide both sides by -4 and reverse the inequality sign.

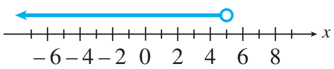

Figure 18.6

The solution is $x < 5$. This is shown graphically in Figure 18.6. The open circle at 5 indicates that the solution does not include 5.

EXAMPLE 18.16

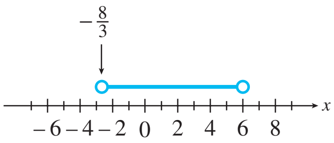

Figure 18.7

Solve $|3x - 5| < 13$.

SOLUTION $|3x - 5| < 13$

$$-13 < 3x - 5 < 13 \qquad \text{Apply Property 5.}$$
$$-8 < 3x < 18 \qquad \text{Add 5 to all three expressions.}$$
$$-\frac{8}{3} < x < 6 \qquad \text{Divide by 3.}$$

This solution is shown graphically in Figure 18.7. Notice that $-\frac{8}{3} < x < 6$ is equivalent to $-\frac{8}{3} < x$ and $x < 6$.

EXAMPLE 18.17

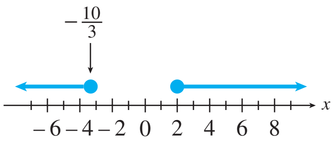

Figure 18.8

Solve $|3x + 2| \geq 8$.

SOLUTION $|3x + 2| \geq 8$

$$3x + 2 \geq 8 \quad \text{or} \quad 3x + 2 \leq -8 \qquad \text{Apply Property 6.}$$
$$3x \geq 6 \quad \text{or} \quad 3x \leq -10 \qquad \text{Subtract 2.}$$
$$x \geq 2 \quad \text{or} \quad x \leq -\frac{10}{3} \qquad \text{Divide by 3.}$$

The solution is the numbers less than or equal to $-\frac{10}{3}$ or those greater than or equal to 2. This is shown graphically in Figure 18.8.

APPLICATION ELECTRONICS

EXAMPLE 18.18

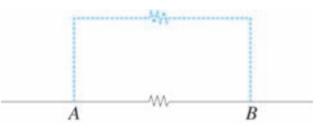

Figure 18.9

A technician determines that an electronic circuit fails to operate because the resistance between A and B, 600 Ω, exceeds the specifications (see Figure 18.9). The specifications state that the resistance must be between 200 and 500 Ω. If a shunt resistor of $R\,\Omega$, $R > 0$, is added, the circuit will satisfy the specifications. The equivalent resistance for the parallel connection is $\dfrac{1}{\dfrac{1}{600} + \dfrac{1}{R}}$ or $\dfrac{600R}{600 + R}\,\Omega$.

What are the possible values for R?

SOLUTION To satisfy the specifications, the following inequality must be true:

$$200 < \frac{600R}{600 + R} < 500$$

Since $R > 0$, $600 + R > 0$, we can multiply the inequality by $600 + R$ without changing the direction of the inequality sign:

$$200(600 + R) < 600R < 500(600 + R)$$
$$120\,000 + 200R < 600R < 300\,000 + 500R$$

EXAMPLE 18.18 (Cont.)

We will now write this compound inequality as two simple inequalities:

$$120\,000 + 200R < 600R \quad \text{and} \quad 600R < 300\,000 + 500R$$
$$120\,000 < 400R \quad \text{and} \quad 100R < 300\,000$$
$$300 < R \quad \text{and} \quad R < 3000$$

Thus, $300 < R < 3\,000$, so the shunt resistance must be between 300 and 3 000 Ω.

EXERCISE SET 18.2

In Exercises 1–8, using the inequality $x < 10$, state the inequality that results when the operations given are performed on each side. Assume $x > 0$.

1. Add 5.
2. Subtract 7.
3. Multiply by 3.
4. Multiply by -4.
5. Divide by -2.
6. Divide by 5.
7. Square both.
8. Take the square root of both.

In Exercises 9–32, solve each of the given inequalities algebraically and graphically.

9. $3x > 6$
10. $4x < -8$
11. $2x + 5 > -7$
12. $3x - 4 \leq 8$
13. $2x - 5 < x$
14. $3x + 4 \geq x$
15. $4x - 7 \geq 2x + 5$
16. $7 - 3x \leq 3 + 5x$
17. $\frac{2}{3}x - 4 < \frac{1}{3}x + 2$
18. $\frac{1}{5}x - \frac{2}{5} > \frac{3}{5}x + \frac{4}{5}$
19. $\dfrac{x-2}{4} \leq \dfrac{3}{8}$
20. $\dfrac{x+2}{3} \geq \dfrac{5}{6}$
21. $\dfrac{x-3}{4} \leq \dfrac{2x}{3}$
22. $\dfrac{2x+5}{3} > \dfrac{3x-1}{2}$
23. $|x + 1| < 5$
24. $|2x - 1| < 7$
25. $|x + 4| > 6$
26. $|3x + 9| \geq 6$
27. $|x + 5| < -3$
28. $|3x + 2| < 12$
29. $-7 \leq 3x + 5 < 26$
30. $-6 < 5 - 3x \leq 17$
31. $3x + 1 < 5 < 2x - 3$
32. $4x - 6 \leq 11 < 9x + 1$

For Exercises 33–44, express the answer in terms of an inequality.

33. ***Astronomy*** The radius of the earth at the equator is 6 378 km and the polar radius is 6 357 km. What is the range of the radius r?
34. ***Astronomy*** The moon has a maximum distance of 407 000 km from the earth and a minimum distance of 357 000 km. What is the range of the moon's distance, d, from the earth?
35. ***Automotive technology*** The company specifications state that the camber angle c of the wheels should be $0.60° \pm 0.50°$. Express the range for c as an inequality.
36. ***Automotive technology*** The amount of R-12 in an automotive air-conditioning system may vary from 0.9 to 1.8 kg (2–4 lb). Knowing that there are 16 oz./lb, express the range of R-12 in ounces.
37. ***Electronics*** If the resistance between A and B in Example 18.18 had been 800 Ω, what are the possible values for the shunt resistance?
38. ***Machine technology*** A certain welding operation must be performed at temperatures between 1 800°C and 2 200°C. What is the temperature range in degrees Fahrenheit? [Use $C = \frac{5}{9}(F - 32)$.]
39. ***Business*** The weekly cost of manufacturing x microcomputers of a certain type is given by

$C = 1{,}500 + 25x$. The revenue from selling these is given by $R = 60x$. At least how many microcomputers must be made and sold each week to produce a profit?

40. ***Energy technology*** A rectangular solar collector is to have a height of 1.5 m. The collector will supply 600 W per m^2 and is to supply a total between 2400 and 4000 W. What range of values can the length of the collector have in order to provide this voltage?

41. ***Electronics*** The instrument error, ε, of a wattmeter is less than 0.6 W. When the pointer is exactly on the 9-W mark, a technician records the power, P, as 9.0 W.

(a) Express the error as an inequality, using absolute value.

(b) Write an absolute value inequality that indicates the possible true power the technician should have recorded.

(c) What is the possible range of values for P?

42. ***Medical technology*** The medical dosage for a very young child is sometimes calculated by the formula $c = \frac{Ad}{150}$, where d is the adult dose, c is the child's dose, and A is the child's age in months. For what ages is the child's dosage between 25% and 50% of an adult's dose?

43. ***Business*** A publishing company has found that the cost of publishing each copy of a certain magazine is \$0.38. The company gets \$0.35/copy from magazine dealers. Advertising revenue is 10% of the revenue received from dealers after the first 10,000 copies have been sold. What is the least number of copies that must be sold in order for the company to make a profit on this magazine?

44. ***Computer science*** A certain computer diskette must be kept within a temperature range given by $|x - 29.4° C| \leq 3.7° C$. What are the minimum and maximum temperatures of the range?

[IN YOUR WORDS]

45. Properties 2 and 3 of inequalities concern multiplying or dividing both sides of an inequality by a nonzero expression. Explain each property, emphasizing how they are alike and how they are different.

46. Properties 5 and 6 of inequalities concern when an absolute value is less than or greater than a positive expression. Explain each property, emphasizing how they are alike and how they are different.

18.3 NONLINEAR INEQUALITIES

In this section, we will learn how to solve nonlinear inequalities in one variable. There are two generally accepted methods for solving these kinds of inequalities. We will briefly show you one of those methods and then concentrate on the other method, which uses technology. You can use the one that seems easier.

Let's consider the inequality $x^2 + x - 6 > 0$. This is not linear, since it has an x^2-term. It is an inequality in one variable, x.

The first method for solving this inequality is to graph the equation $y = f(x) = x^2 + x - 6$, as shown in Figure 18.10. Look at the graph. When is $y > 0$? That

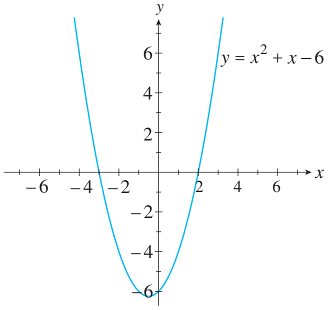

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Figure 18.10

is, when is the graph of $y = x^2 + x - 6$ above the x-axis? By using the quadratic formula or factoring we see that $x^2 + x - 6 = 0$ when $x = -3$ or $x = 2$. These are the two places where the graph of $y = x^2 + x - 6$ crosses the x-axis. You can see that $y > 0$ when $x < -3$ or $x > 2$. Thus, the solution of $x^2 + x - 6 > 0$ is $x < -3$ or $x > 2$. This is a fairly easy approach. The fact that you have to graph the function requires time. While a graphing calculator can speed up the graphing, it is not always easy to determine *exactly* where the graph crosses the x-axis, although the methods from Chapter 17 will help.

Technology can be very helpful is solving nonlinear inequalities. We will first show how to use a calculator and then a spreadsheet to solve nonlinear inequalities.

SOLVING NONLINEAR INEQUALITIES WITH A CALCULATOR

EXAMPLE 18.19

Solve $(x - 2)(x + 3)(x - 5)(x + 1) \le 0$.

SOLUTION This is already factored. The roots are 2, −3, 5, and −1. From the graph of $y = (x - 2)(x + 3)(x - 5)(x + 1)$ in Figure 18.11a, with window settings $[-4, 6] \times [-10,10]$, it appears as if the solution is $-3 \le x \le -1$ or $2 \le x \le 5$. (Remember, the graph will cross the x-axis at each of the roots.) Figure 18.11b was obtained by using `ZoomFit` and it gives a view of the graph that makes it easier to see when the graph is below the x-axis and confirms that the solution is $-3 \le x \le -1$ or $2 \le x \le 5$.

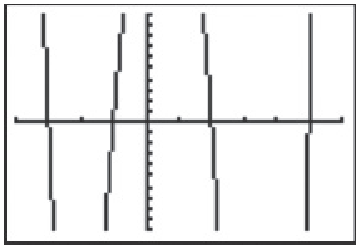
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Figure 18.11a

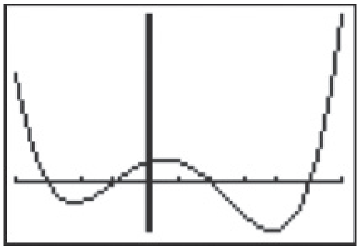
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Figure 18.11b

EXAMPLE 18.20

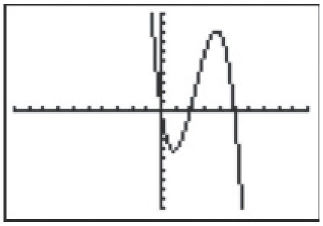
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Figure 18.12

Solve $7x^2 - x^3 - 10x > 0$.

SOLUTION We will first write the inequality with the left-hand side in descending order, $-x^3 + 7x^2 - 10x > 0$, and then factor out a $-x$ to get $-x(x^2 - 7x + 10) > 0$. Continuing, we see that the left-hand side of this inequality factors as $-x(x - 2)(x - 5)$, which means that the graph of $y = 7x^2 - x^3 - 10x$ will cross the x-axis at 0, 2, and 5. The graph of y, on the standard viewing window in Figure 18.12, shows when the curve is above the x-axis and we can see that the solution is $x < 0$ or $2 < x < 5$.

The procedure for finding the solutions to a division problem can be determined the same way. You must check to ensure that any numbers that make the quotient zero are not in the solutions.

EXAMPLE 18.21

Solve $\dfrac{x^2 + 6x + 5}{x + 2} \leq 0$.

SOLUTION When you look at the graph of $y = \dfrac{x^2 + 6x + 5}{x + 2}$ in Figure 18.13a in the `ZStandard` viewing window, you see that it appears as if the graph crosses the x-axis at three points. Remember, the vertical line in the middle, when $x = -2$, is a vertical asymptote and is not a line on the graph but is caused by the manner in which the calculator draws graphs. One way to see this is to look at the graph of the same function in Figure 18.13b with `Xmin` $= -9.4$ and `Xmax` $= 9.4$.

The graph of y is below the x-axis when $x \leq -5$ and when $-2 < x < -1$. The graph crosses the x-axis when $x = -5$ or $x = -1$. Combining these properties, when the graph is below the x-axis and when it crosses the x-axis, we get the solution: $x \leq -5$ or -2 , $x \leq -1$.

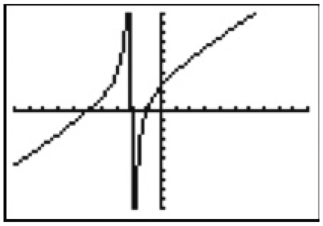


Figure 18.13a

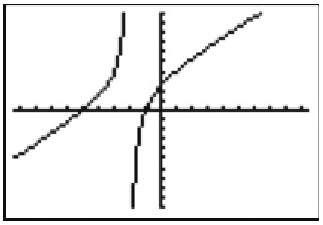


Figure 18.13b

SOLVING INEQUALITIES WITH A GRAPHING CALCULATOR

1. Determine the real roots and the domain of the equation associated with the inequality.
2. Graph the function associated with the equation or inequality.
3. (a) If the given inequality is of the form $f(x) > 0$, the solutions are the intervals on the x-axis where the graph is above the x-axis.
   (b) If the given inequality is of the form $f(x) < 0$, the solutions are the intervals on the x-axis where the graph is below the x-axis.

EXAMPLE 18.22

Solve $\dfrac{x^3 - 5x^2 - 3x + 15}{x^2 + 5x - 6} \geq 0$.

SOLUTION The roots of the numerator are 5, $-\sqrt{3}$, and $\sqrt{3}$ and the roots of the denominator are -6 and 1. We have to examine the graph over each of the intervals between these points, that is, the intervals $(-\infty, -6)$, $(-6, -\sqrt{3})$, $(-\sqrt{3}, 1)$, $(1, \sqrt{3})$, $(\sqrt{3}, 5)$ and $(5, \infty)$. Figures 18.14a and 18.14b show the graph of $y = \dfrac{x^3 - 5x^2 - 3x + 15}{x^2 + 5x - 6}$ in the standard viewing window. The actual figure you get will depend on your calculator. From the graph we can see that $y \geq 0$ when $x \geq 5$ and when $-6 < x \leq -\sqrt{3}$. We cannot tell what is happening when $x < -6$, and it is difficult to see what is happening when $1 < x < \sqrt{3}$.

In Figure 18.14c we have changed the y-values of the viewing window in order to get a better idea of what is happening when $x < -6$. As you can see from the figure, the graph is below the x-axis when $x < -6$, so this is not part of the solution.

$$[-10, 10, 1] \times [-50, 50, 5]$$

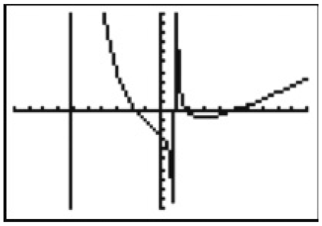


Figure 18.14a

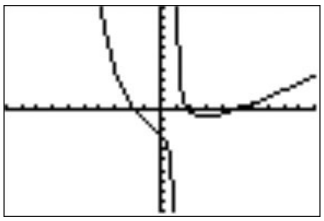


Figure 18.14b

EXAMPLE 18.22 (Cont.)

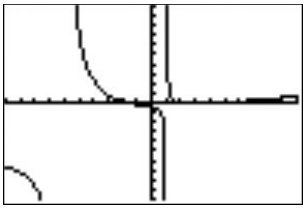


Figure 18.14c

Figure 18.14d

Figure 18.14d contains a closer look when $0 < x < 6$, and we can tell that the graph of y is above the x-axis over the interval $1 < x < \sqrt{3}$:

$$[0, 7, 1] \times [-5, 5, 1]$$

Putting all these facts together, we see that the solution to the inequality

$$\frac{x^3 - 5x^2 - 3x + 15}{x^2 + 5x - 6} \geq 0$$

is when $-6 < x \leq -\sqrt{3}$ or $1 < x < \sqrt{3}$ or $x \geq 5$.

Not every inequality asks that an expression be compared to zero. When one function is compared to another, it is often to think of the left-hand side of the inequality as y_1, the right-hand side as y_2, and then look at $y_1 - y_2$. The next two examples demonstrate how this can be done.

EXAMPLE 18.23

Solve the inequality $x^2 - 3x < 5 - 2x^2 - x^3$.

SOLUTION Begin by graphing this as two separate functions. Let $y_1 = x^2 - 3x$ and $y_2 = 5 - 2x^2 - x^3$. As you can see from the graphs of these two functions in Figure 18.15a, they appear to intersect at two points.

Now we will graph $y_1 - y_2$ as shown in Figure 18.15b. This has the effect of changing the original inequality to

$$x^2 - 3x - (5 - 2x^2 - x^3) < 0$$

That means that finding the roots of $x^2 - 3x - (5 - 2x^2 - x^3) = 0$ is the same as finding where y_1 and y_2 intersect.

When we graph $y_1 - y_2$, as in Figure 18.15c, we just graph $y_1 - y_2$ and not y_1 and y_2. This keeps the screen less cluttered and easier to read. The graph in Figure 18.15c has a surprise. The graph of $y_1 - y_2$ intersects the x-axis at three points, which means that the graphs of y_1 and y_2 must have intersected at three points.

Using the `zero` feature of the calculator (begin by pressing **2nd** **TRACE** [CALC] **2** [2: zero]), you can find that the left root is about $x \approx -3.4495$. Repeating the procedure, we determine that the other two roots are -1 and $x \approx 1.4495$. An examination of either Figure 18.15c or Figure 18.15d indicates that the solution to $x^2 - 3x - (5 - 2x^2 - x^3) < 0$ is $x < -3.4495$ or $-1 < x < 1.4495$. Thus, the solution to the original inequality, $x^2 - 3x < 5 - 2x^2 - x^3$, is $x < -3.4495$ or $-1 < x < 1.4495$.

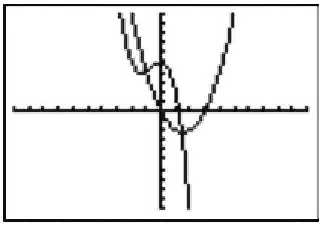


Figure 18.15a

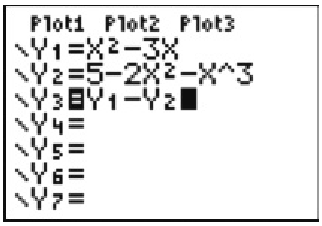


Figure 18.15b

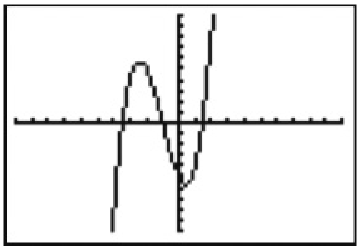


Figure 18.15c

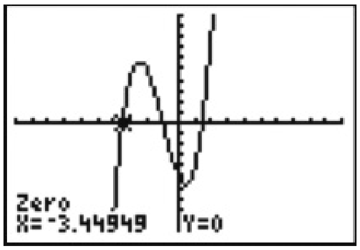


Figure 18.15d

EXAMPLE 18.24

Solve the inequality $|3x - 2| \leq |x + 4|$.

SOLUTION Again, we begin by letting y_1 represent the left-hand side, so $y_1 = |3x - 2|$. y_2 represents the right-hand side, so $y_2 2 = |x + 4|$. Graph $y_1 - y_2 = |3x - 2| - |x + 4|$. We are interested in when $y_1 - y_2 \leq 0$. From Figures 18.16a and 18.16b, we see that the solution is $-0.5 < x \leq 3$.

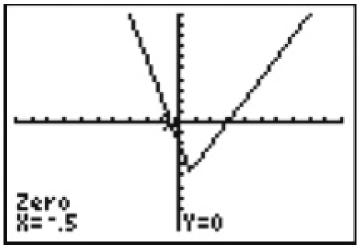


Figure 18.16a

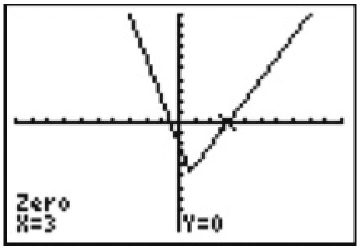


Figure 18.16b

APPLICATION PHYSICS

EXAMPLE 18.25

From the top of a 120-ft building an object is thrown upward with an initial velocity of 68 ft/s. Its distance d above the ground at any time t is given by the equation $d = 120 + 68t - 16t^2$. For what period of time is the object higher than the building?

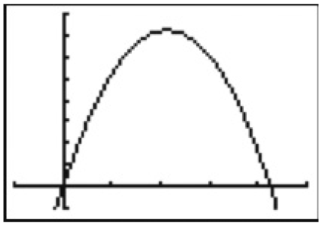


Figure 18.17

SOLUTION Since the building is 120 ft high, this is really asking when

$$120 + 68t - 16t^2 > 120$$

or

$$-16t^2 + 68t > 0$$

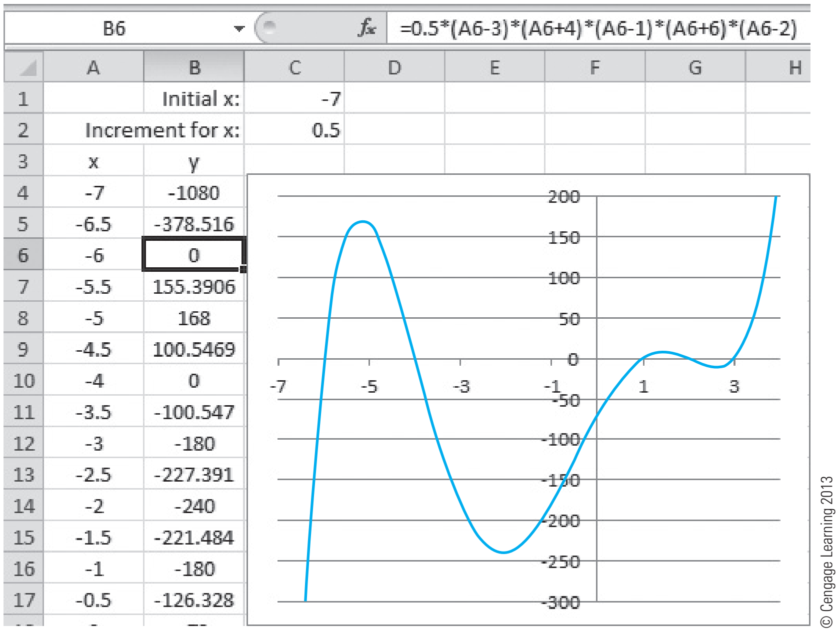
This factors into $-16t(t - 4.25) > 0$. This inequality is zero at $t = 0$ and $t = 4.25$. The graph in Figure 18.17 shows that $16t^2 + 68t > 0$ when $0 < t < 4.25$. Thus, the object is above the building for the first 4.25 s after it is thrown.

USING A SPREADSHEET TO SOLVE NONLINEAR INEQUALITIES

EXAMPLE 18.26

Solve $0.5(x - 3)(x + 4)(x - 1)(x + 7)(x - 2) \geq 0$.

SOLUTION This is already factored. The roots are 3, -4, 1, -7, and 2. From the graph of $y = 0.5(x - 3)(x + 4)(x - 1)(x + 7)(x - 2)$ in Figure 18.18, it appears as if the solution is $-7 \leq x \leq -4$ or $1 \leq x \leq 2$ or $x \geq 3$. (Remember, the graph will cross the x-axis at each of the roots and we want to know when the graph is above the x-axis.)



B6 | f_x =0.5*(A6-3)*(A6+4)*(A6-1)*(A6+6)*(A6-2)

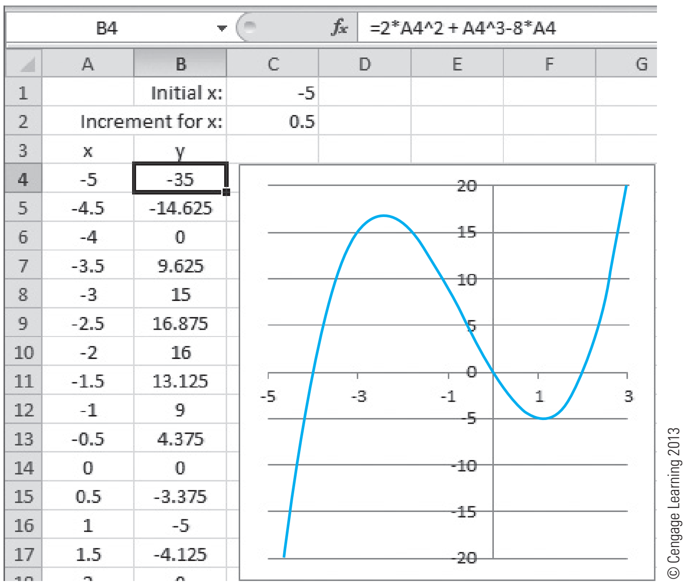
	A	B	C
1		Initial x:	-7
2	Increment for x:		0.5
3	x	y	
4	-7	-1080	
5	-6.5	-378.516	
6	-6	0	
7	-5.5	155.3906	
8	-5	168	
9	-4.5	100.5469	
10	-4	0	
11	-3.5	-100.547	
12	-3	-180	
13	-2.5	-227.391	
14	-2	-240	
15	-1.5	-221.484	
16	-1	-180	
17	-0.5	-126.328	

Figure 18.18

EXAMPLE 18.27 Solve $2x^2 + -8x + x^3 < 0$.

SOLUTION First, write the inequality with the left side in descending order, $x^3 + 2x^2 - 8x < 0$, and then factor out x to get $x(x^2 + 2x - 8) < 0$. Continuing, we see that the left-hand side of this inequality factors as $x(x - 2)(x + 4)$, which means that the graph of $y = 2x^2 + -8x + x^3$ will cross the x-axis at 0, -4, and 2. The graph of y in Figure 18.19 shows when the curve is below the x-axis, and we can see that the solution is $x < -4$ or $0 < x < 2$.

The procedure for finding the solutions to a division problem can be determined the same way. You must check to ensure that any numbers that make the quotient zero are not in the solutions.



B4 | f_x =2*A4^2 + A4^3-8*A4

	A	B	C
1		Initial x:	-5
2	Increment for x:		0.5
3	x	y	
4	-5	-35	
5	-4.5	-14.625	
6	-4	0	
7	-3.5	9.625	
8	-3	15	
9	-2.5	16.875	
10	-2	16	
11	-1.5	13.125	
12	-1	9	
13	-0.5	4.375	
14	0	0	
15	0.5	-3.375	
16	1	-5	
17	1.5	-4.125	

Figure 18.19

EXAMPLE 18.28

Solve $\dfrac{x^2 + 6x + 8}{x + 1} \leq 0.$

SOLUTION When you look at the graph of $y = \dfrac{x^2 + 6x + 8}{x + 1}$ in Figure 18.20a, it appears as if the graph crosses the x-axis three times. However, since there is a denominator that can be zero, we should expect an asymptote at $x = -1$. The line that appears to cross at -1, then, is the computer's attempt to connect the curve across the asymptote. By adjusting the scale and reformatting the markers (with no connecting line), we see in Figure 18.20b that the curve will cross the x-axis in only two places.

The graph of y is below the x-axis when $x < -4$ and when $-2 < x < -1$. The graph crosses the x-axis when $x = -4$ or $x = -2$. Combining these properties, when the graph is below the x-axis and when it crosses the x-axis, we get the solution: $x \leq -4$ or $-2 \leq x < -1$.

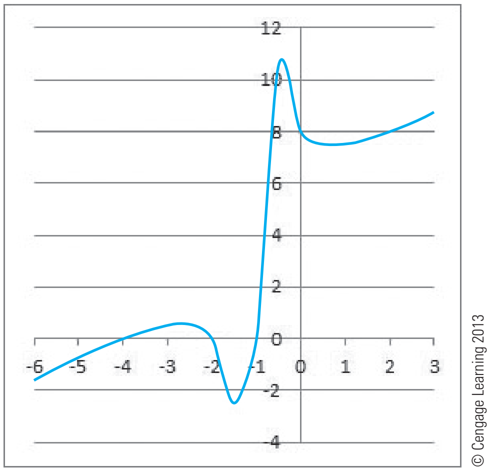


Figure 18.20a

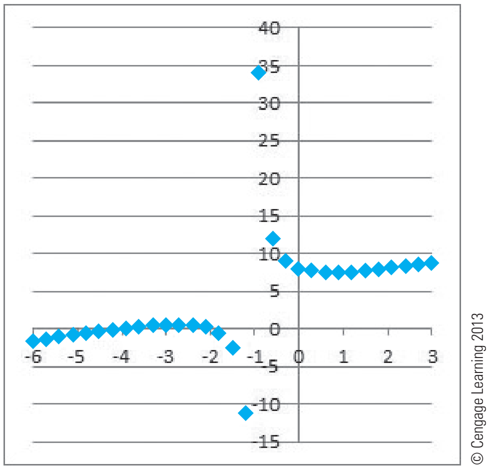


Figure 18.20b

SOLVING INEQUALITIES WITH A SPREADSHEET

1. Determine the real roots and the domain of the equation associated with the inequality.
2. Graph the function associated with the equation or inequality.
3. (a) If the given inequality is of the form $f(x) > 0$, the solutions are the intervals on the x-axis where the graph is above the x-axis.
   (b) If the given inequality is of the form $f(x) < 0$, the solutions are the intervals on the x-axis where the graph is below the x-axis.

EXAMPLE 18.29

Solve $\dfrac{x^3 - 5x^2 - 3x + 15}{x^2 + 5x - 6} \geq 0.$

SOLUTION The roots of the numerator are $5, -\sqrt{3}$, and $\sqrt{3}$ and the roots of the denominator are -6 and 1. We have to examine the graph over each of the

EXAMPLE 18.29 (Cont.) intervals between these points, that is, the intervals $(-\infty, -6)$, $(-6, -\sqrt{3})$, $(-\sqrt{3}, 1)$, $(1, \sqrt{3})$, $(\sqrt{3}, 5)$, and $(5, \infty)$. Figure 18.21a shows the graph of $y = \dfrac{x^3 - 5x^2 - 3x + 15}{x^2 + 5x - 6}$. (You might want to compare the graph in Figure 18.21a with the ones in Figures 18.14a and 18.14b.) From the graph we can see that $y \geq 0$ when $-6 < x \leq -\sqrt{3}$. We cannot tell what is happening when $x < -6$, and it is difficult to see what is happening when $1 < x < \sqrt{3}$ or when $x > 5$.

Figure 18.21b provides more information, as does Figure 18.21c. These figures are obtained by changing the scale on the x-axis. These figures indicate that the graph is below the x-axis when $x < -6$, so this is not part of the solution. They also indicate that the graph is above the x-axis over the interval $1 < x < \sqrt{3}$.

Another way to approach this solution on a spreadsheet is to test a value in each interval as defined earlier. The calculations are done by the spreadsheet and a logical test can be employed to show which regions are in the solution.

Figure 18.21d shows the same result as earlier

$$\frac{x^3 - 5x^2 - 3x + 15}{x^2 + 5x - 6} \geq 0$$

when $-6 < x \leq -\sqrt{3}$ or $1 < x < \sqrt{3}$ or $x \geq 5$.

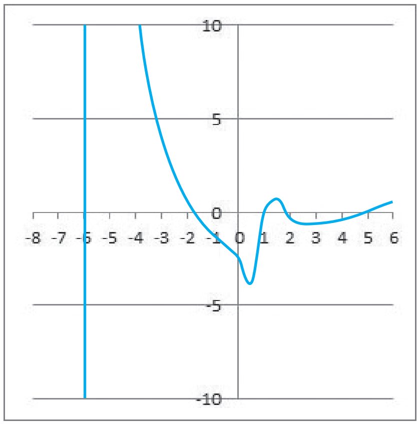


Figure 18.21a

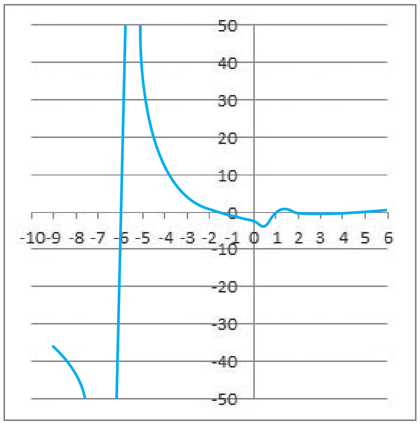


Figure 18.21b

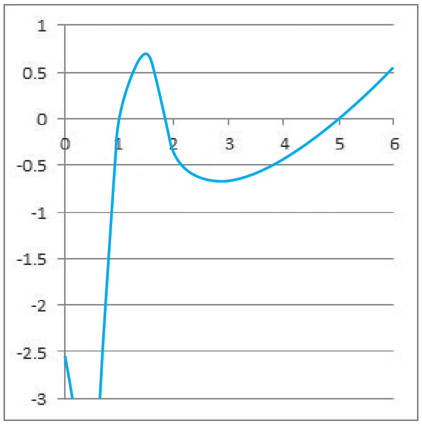


Figure 18.21c

C5 f_x =IF((C3^3-5*C3^2-3*C3+15)/(C3^2+5*C3-6)<0,"No","Solution")

	A	B	C	D	E	F	G	H	I	J	K	L	M	N
1	Critical Points	-INF		-6		-SQRT(3)		1		SQRT(3)		5		INF
2														
3	Test Point		-20		-4		0		1.2		4		8	
4														
5	Logical Test		No		Solution		No		Solution		No		Solution	

Figure 18.21d

Not every inequality asks that an expression be compared to zero. When one function is compared to another, it is often to think of the left-hand side of the inequality as y_1, the right-hand side as y_2, and then look at $y_1 - y_2$. The next two examples demonstrate how this can be done.

EXAMPLE 18.30

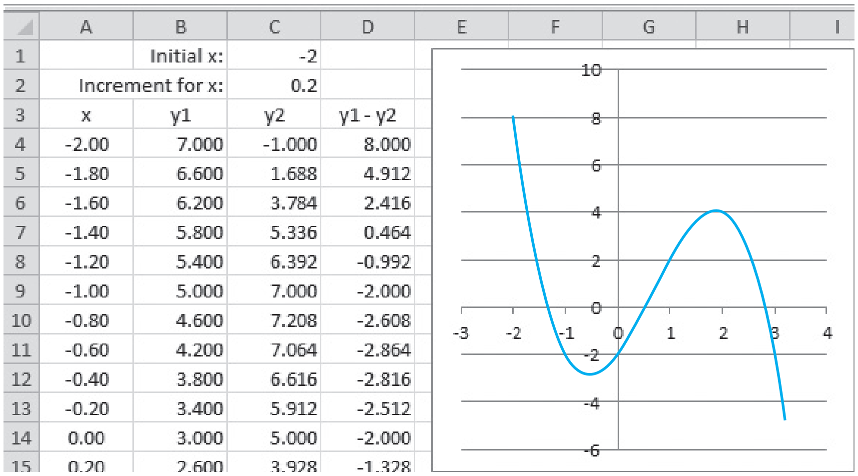
Solve the inequality $3 - 2x < x^3 - 2x^2 - 5x + 5$.

SOLUTION Begin by constructing a table of values to determine when $y_1 < y_2$ where $y_1 = 3 - 2x$ and $y_2 = x^3 - 2x^2 - 5x + 5$. The table of values and graph of $y_1 - y_2$ are shown in Figure 18.22a.

We must identify the x-intercepts of this curve. We can do this with `Goal Seek`. Figure 18.22b shows `Goal Seek` being used on the intercept near -1.4. The result is $x = -1.34$.

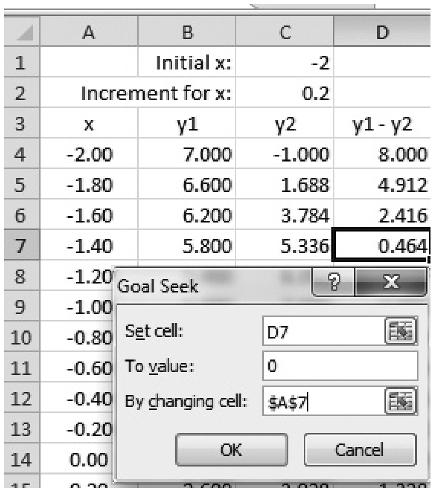
Using `Goal Seek` twice more, we find that the other two x-intercepts of $y_1 - y_2$ are approximately $x = 0.53$ and $x = 2.81$. Those three points means that there are four intervals to test: $(-\infty, -1.34)$, $(-1.34, 0.53)$, $(0.53, 2.81)$, and $(2.81, \infty)$. Figure 18.22c shows the results of testing points in each interval.

The solution to the original inequality, $3 - 2x < x^3 - 2x^2 - 5x + 5$, is $-1.34 < x < 0.53$ or $x > 2.81$.



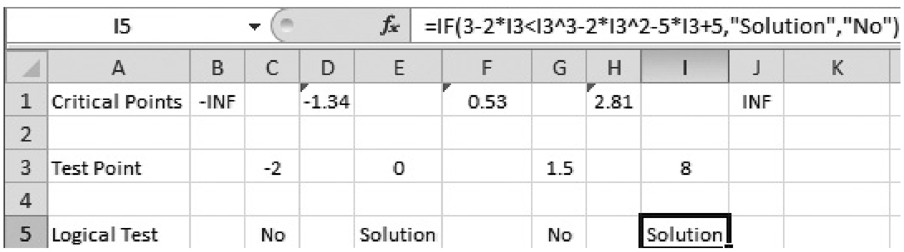
	A	B	C	D
1		Initial x:	-2	
2		Increment for x:	0.2	
3	x	y1	y2	y1 - y2
4	-2.00	7.000	-1.000	8.000
5	-1.80	6.600	1.688	4.912
6	-1.60	6.200	3.784	2.416
7	-1.40	5.800	5.336	0.464
8	-1.20	5.400	6.392	-0.992
9	-1.00	5.000	7.000	-2.000
10	-0.80	4.600	7.208	-2.608
11	-0.60	4.200	7.064	-2.864
12	-0.40	3.800	6.616	-2.816
13	-0.20	3.400	5.912	-2.512
14	0.00	3.000	5.000	-2.000
15	0.20	2.600	3.928	-1.328

Figure 18.22a



	A	B	C	D
1		Initial x:	-2	
2		Increment for x:	0.2	
3	x	y1	y2	y1 - y2
4	-2.00	7.000	-1.000	8.000
5	-1.80	6.600	1.688	4.912
6	-1.60	6.200	3.784	2.416
7	-1.40	5.800	5.336	0.464
8	-1.20			
9	-1.00			
10	-0.80			
11	-0.60			
12	-0.40			
13	-0.20			
14	0.00			

Figure 18.22b



I5 =IF(3-2*I3<I3^3-2*I3^2-5*I3+5,"Solution","No")

	A	B	C	D	E	F	G	H	I	J	K
1	Critical Points	-INF		-1.34		0.53		2.81		INF	
2											
3	Test Point		-2		0		1.5		8		
4											
5	Logical Test		No		Solution		No		Solution		

Figure 18.22c

EXAMPLE 18.31

Solve the inequality $|x^2 - 7x + 6| \leq |x + 4|$.

SOLUTION The process is the same as in the last example.

Let $y_1 = |x^2 - 7x + 6|$ and $y_2 = |x + 4|$. Enter y_1 as `= ABS (A4^2−7*A4+6)` and y_2 as `=ABS(A4+4)`. Figure 18.23 shows the graph of y_1, y_2, and, $y_1 - y_2$, which is the dotted line.

The graph of $y_1 - y_2$ crosses the x-axis at about $x = 0.258$ and $x = 7.742$, and from the graph we see that the solution is $0.258 < x < 7.742$.

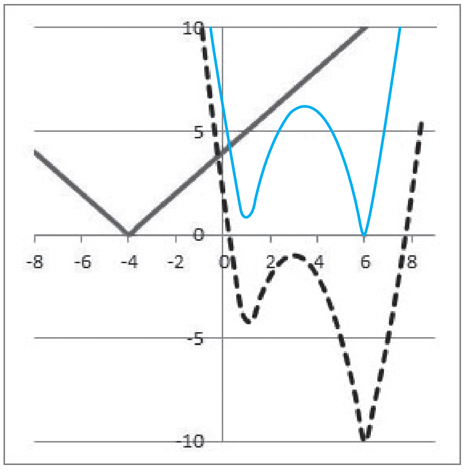


Figure 18.23

In each of the calculator and spreadsheet examples, we were concerned with solutions in the real numbers. If the nonlinear equation does not have any real roots, the problem will be an absolute inequality and every real number will satisfy the inequality, or it will be a contradictory inequality and no real number will satisfy the inequality. The next example illustrates one of these situations.

EXAMPLE 18.32

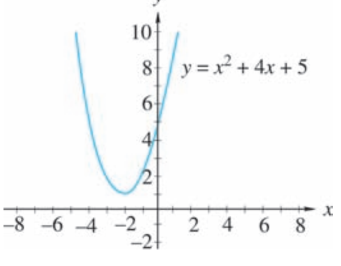


Figure 18.24

Solve $x^2 + 4x + 5 > 0$.

SOLUTION Using the quadratic formula, we see that the roots are $\dfrac{-4 \pm \sqrt{16 - 20}}{2} = -2 \pm j$. Both of these are complex numbers. Since there are only two roots to $x^2 + 4x + 5 = 0$, and neither of them is a real number, this inequality is either absolute or contradictory. If it is absolute, any real number will satisfy it. Select any real number, say 0, and we get $0^2 + 4(0) + 5 = 5 > 0$. So, 0 is a solution. This is an absolute inequality and all real numbers satisfy it.

The graph of $y = x^2 + 4x + 5$ is shown in Figure 18.24. From this graph, you can see that y is always larger than 0. This confirms our solution.

EXAMPLE 18.33

Solve $x^2 + 4x + 5 < 0$.

SOLUTION Check the graph of $y = x^2 + 4x + 5$ in Figure 18.24. As we saw in the previous example, y is always positive. This is a contradictory inequality and no real numbers will make it true.

We could have also solved this in the same manner as described in Example 18.32. Select a real number, say 2. Since $2^2 + 4(2) + 5 = 17$, this inequality has no real-number solutions.

The techniques described in Examples 18.32 and 18.33 depend on first determining that there are no real roots.

EXERCISE SET 18.3

Solve the inequalities in Exercises 1–40.

1. $(x + 1)(x - 3) > 0$

2. $(x - 4)(x + 2) \geq 0$

3. $(x - 1)(x + 4) \leq 0$

4. $(x - 2)(x + 5) < 0$

5. $x^2 - 1 < 0$

6. $x^2 \leq 16$

7. $x^2 - 2x - 15 \leq 0$

8. $x^2 + x - 2 > 0$

9. $x^2 - x - 2 \geq 0$

10. $x^2 - 3x - 10 < 0$

11. $x^2 - 5x > -6$

12. $x^2 + 7x \leq -12$

13. $2x^2 + 7x + 3 < 0$

14. $2x^2 - 5x + 2 \geq 0$

15. $2x^2 - x < 1$

16. $2x^2 - x \leq 3$

17. $4x^2 + 2x > x^2 - 1$

18. $2x^2 + 5x < 2 - x^2$

19. $(x + 1)(x - 2)(x + 3) < 0$

20. $(x - 1)(x + 2)(x - 3) \leq 0$

21. $(x + 3)(2x - 5)(x + 4) > 0$

22. $(x + 4)(3x - 1)(x - 5) \geq 0$

23. $(x - 2)^2 (x + 4) < 0$

24. $(x + 3)^2 (x - 5) \geq 0$

25. $x^3 - 4x > 0$

26. $x^4 - 9x^2 \leq 0$

27. $x^2 + 2x + 3 \leq 0$

28. $x^2 + x + 1 > 0$

29. $\dfrac{(x - 2)(x - 5)}{x + 1} < 0$

30. $\dfrac{(x + 2)(x - 4)}{x - 2} \geq 0$

31. $\dfrac{x}{(x + 1)(x - 2)} > 0$

32. $\dfrac{x(x - 3)}{(x + 1)(x - 4)} \leq 0$

33. $\dfrac{(3x + 1)(x - 3)}{2x - 1} \leq 0$

34. $\dfrac{(x - 1)(x + 6)}{(x + 1)(x - 3)} > 0$

35. $\dfrac{4}{x - 1} < \dfrac{5}{x + 1}$

36. $\dfrac{-3}{x + 2} \geq \dfrac{6}{x - 3}$

37. $\dfrac{x^2 + 2x + 3}{x - 1} \geq 0$

38. $\dfrac{x^2 + x + 1}{x + 2} < 0$

39. $|x^2 + 3x + 2| \leq 4$

40. $|x^2 - 3x + 2| > 5$

Solve Exercises 41–48.

41. ***Lighting technology*** The intensity I in candelas (cd) of a certain light is $I = 75d^2$, where d is the distance in meters from the source. For what range of distances will the intensity be between 75 and 450 cd?

42. ***Physics*** An object is thrown straight upward from the ground with an initial velocity of 34.3 m/s. Its distance d above the ground at any time t is given by the equation $d = 34.3t - 4.9t^2$. For what time period is the object more than 49 m above the ground?

43. ***Architecture*** The deflection of a beam d is given by $x^2 - 1.1x + 0.2$, where x is the distance from one end. For what values of x is $d > 0.08$?

44. ***Architecture*** The load L that can be safely supported by a wooden beam of length ℓ with a rectangular cross-section of width w and depth d is given by the formula

$$L = \frac{kwd^2}{\ell}$$

where k is a constant, depending on the type of wood. If $k = 90$ and the beam is to be 18 ft long and 6 in. wide, what are the acceptable values for d, if the beam must support at least 3,000 lb and $d < 12$ in.? (Do not change the length measurements to the same units.)

45. ***Dynamics*** A ball is thrown vertically upward into the air from the roof of a 452-ft building. If the initial velocity of the ball is 64 ft/s, the height of the ball above the ground is given by the function $h(t) = -16t^2 + 64t + 452$.

(a) For what times t will the height be greater than 500 ft?

(b) For what times t will the height be less than 395 ft?

46. ***Forestry*** Volume estimates V, in cubic feet, for shortleaf pine trees are based on D, the d.b.h. (diameter at breast height) in inches; top d.i.b., the diameter, in inches, inside the bark at the top of the tree; and H, the height of the tree in feet. One formula for trees with a 3-in. top d.i.b. is

$$V = 0.002837D^2H - 0.127248$$

Determine D for a 75-ft tree that has a volume estimate greater than 47.75 ft^3.

47. ***Forestry*** The Scribner log-rule equation for 16-ft logs is $V = 0.79D^2 - 2D - 4$, where V is the volume, in board feet, of the log and D is the diameter, in inches, of the small end of a log inside the bark. What diameter of the small end of the log inside the bark is needed for a 16-ft log to have a volume of at least 926.0 board ft?

48. ***Material science*** The range of temperatures, T (in degrees Kelvin), for 100.00 g of silver when the silver receives 2.00×10^4 cal of heat is given by $|-2.8T^2 + 820T - 44\,000| < 10\,000$. Solve for T to the nearest 0.1 K.

[IN YOUR WORDS]

49. Describe how you can use your calculator to solve a nonlinear inequality.

50. Explain how to use a graph to solve an inequality.

18.4 INEQUALITIES IN TWO VARIABLES

One variable is not enough to describe some real problems. As a result, we often work with equations of two or more variables. In this section, we will consider inequalities with two variables.

Solutions to equations and inequalities in one variable are represented on a number line. Solutions to equations are represented by points, and solutions to inequalities by intervals.

Equations in two variables are represented by a graph in a plane. Inequalities in two variables are also represented graphically in a plane. The graph, however, is not a line or a curve.

We will begin by looking at linear inequalities in two variables.

LINEAR INEQUALITIES IN TWO VARIABLES

A *linear inequality in two variables*, x and y, is an inequality of the form $ax + by + c < 0$, where a, b, and c are constants and a and b are not both zero.

NOTE The use of the $<$ symbol in this definition is not meant to restrict this to only inequalities that use $<$. This same definition applies for $>$, $\leq$, and $\geq$.

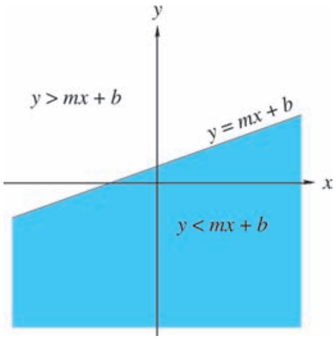


Figure 18.25

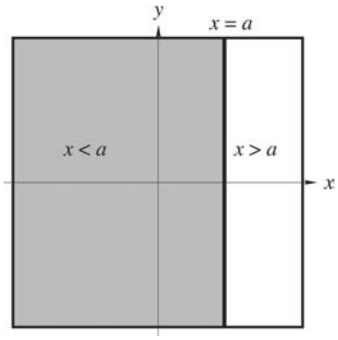


Figure 18.26

Every nonvertical line can be written in the slope-intercept form, $y = mx + b$. This line divides the plane into three separate regions, as shown in Figure 18.25.

1. The points that satisfy $y = mx + b$ (the line)
2. The region that satisfies $y < mx + b$ (the points below the line)
3. The region that satisfies $y > mx + b$ (the points above the line)

A vertical line is of the form $x = a$, where a is a constant. Vertical lines also divide the plane into three separate regions, as shown in Figure 18.26: the line $(x = a)$, the points to the right of the line $(x > a)$, and the points to the left of the line $(x < a)$.

To illustrate how this works, we will solve $2x + y < 4$. We begin by solving the inequality for y, getting $y < -2x + 4$. Next, we graph the line $y = -2x + 4$. This inequality does not include the points where $y = -2x + 4$, so we will make this a dotted line rather than a solid line. The plane is now divided into three regions. We want the region below the line. This area has been shaded in Figure 18.27.

It is always a good idea to check your results. Select any point in the shaded region. For example, select the point $(-1, 4)$. Substitute these values in the original inequality and, on the left-hand side, you get $2(-1) + 4 = -2 + 4 = 2$. This is certainly less than 4, so this point checks. Obviously checking one value does not mean that you have not made a mistake. This answer has an infinite number of solutions and we checked only one. But, checking one value at least *helps* us see if we may be correct.

NOTE Notice that if the inequality does not include equality, the curve dividing the region will be a broken line. A solid line will indicate that the points on the line are included in the solution.

EXAMPLE 18.34

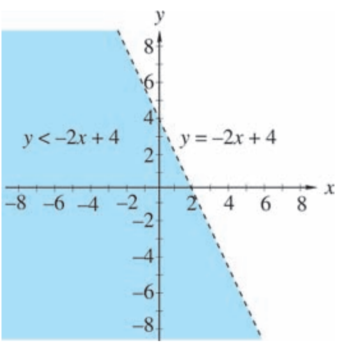


Figure 18.27

Find the region described by $2x - 3y \geq 6$.

SOLUTION We first solve the inequality for y:

$$2x - 3y \geq 6$$
$$-3y \geq -2x + 6$$
$$y \leq \frac{2}{3}x - 2$$

Notice that, when we divided by -3, the direction of the inequality symbol reversed.

Next, graph the line $y = \frac{2}{3}x - 2$, as shown in Figure 18.28. Because this problem involves equality $(\leq)$, we make it a solid line. When we solved the inequality for y, we got $y \leq \frac{2}{3}x - 2$, and this indicates that we want the region below the line, as is shaded in Figure 18.28. This shaded region and the line are the solutions for this problem.

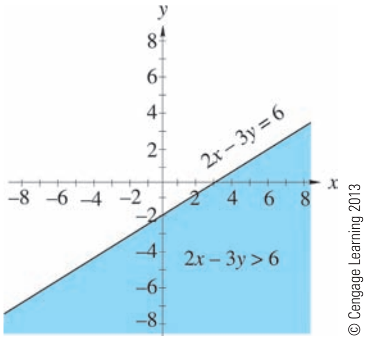


Figure 18.28

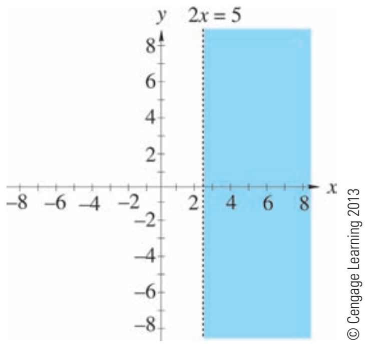


Figure 18.29

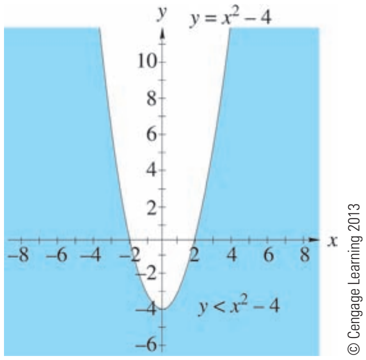


Figure 18.30

EXAMPLE 18.35

Find the solution to $2x > 5$.

SOLUTION If you solve this for x, you get $x > \frac{5}{2}$. Graph the line $x = \frac{5}{2}$ with a dotted line. The desired region is to the right of this line. The solution is the shaded region in Figure 18.29.

Not all equations in two variables are equations of a line. In the same way, not all inequalities in two variables are of regions separated by a line.

EXAMPLE 18.36

Find the solution to $y \leq x^2 - 4$.

SOLUTION The graph of $y = x^2 - 4$ is the parabola in Figure 18.30. All points on the parabola or below it satisfy the given inequality, as indicated in Figure 18.30.

APPLICATION BUSINESS

EXAMPLE 18.37

A company has determined that it can make a profit on its microcomputers if the production cost C satisfies the inequality $C(n) < 2.1 + 0.4n$, where n is the number of computers produced. Graph this inequality.

SOLUTION We will let n be on the horizontal axis and C the vertical axis. We graph the line $C(n) = 2.1 + 0.4n$, and make the line dashed to show that the solution does not include the points on the line. Since we want $C(n) < 2.1 + 0.4n$, we shade the points below the line. We only shade points in the first quadrant, because $n > 0$. (After all, the company cannot make less than 0 microcomputers.) We have also $C(n) > 0$, because it will cost something to produce the computers. The solution is shown by the graph in Figure 18.31.

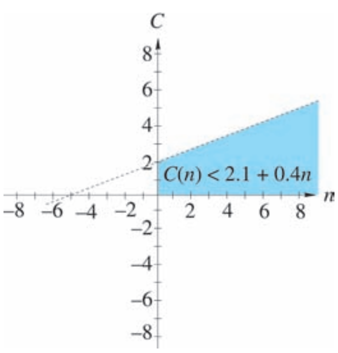


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Figure 18.31

EXERCISE SET 18.4

Graph each inequality in Exercises 1–30.

1. $y > 2$
2. $y \le -3$
3. $x \ge 5$
4. $y < 2$
5. $x + y > 3$
6. $y - x \le 4$
7. $2x + y < 4$
8. $3x + y > -4$
9. $x + 2y \le 4$
10. $2y - x > -3$
11. $2x + 3y < 6$
12. $3x + 2y \ge 6$
13. $4x - 3y \le -6$
14. $3y + 4x \ge 9$
15. $0.5x + 1.5y > 2.5$
16. $0.3x - 0.9y \le 1.2$
17. $1.4x - 0.7y < 1.0$
18. $0.2x + 0.6y \le 2.0$
19. $y \le x^2 + 2$
20. $y \ge x^2 - 3$
21. $y \ge 3x^2 - 1$
22. $y \le 4x^2 - 5$
23. $y < x^2 + 2x + 1$
24. $y \ge x^2 - 4x + 4$
25. $x^2 + y^2 \le 16$
26. $x^2 + y^2 > 9$
27. $4x^2 + 9y^2 \le 36$
28. $9x^2 + y^2 > 9$
29. $|x + 1| < 4$
30. $|x - 5| \le 3$

Solve Exercises 31–34.

31. ***Chemistry*** The temperature (°C) and pressure (kPa) of a controlled chemical reaction must satisfy $1.8P + T < 270$. Graph this inequality for $T > 0$ and $P \ge 0$.

32. ***Business*** In order to make a profit, the cost C of producing a computer chip in a certain factory must satisfy the inequality $C(n) < n^2 + 10n + 35$, where n is the number of chips produced. Graph this inequality. Remember that $n \ge 0$.

33. ***Business*** A certain bank's return on a home loan is 8.5% and 6.75% on a commercial loan. The manager of the loan department wants to earn a return of at least \$3.6 million on these loans. Let x represent the amount (in million dollars) for home loans and y the amount for commercial loans. Graph this inequality.

34. ***Nutrition*** A brand-X multivitamin tablet contains 15 mg of iron. One brand-Y multivitamin tablet contains 9 mg of iron. A nutritionist advises a patient to take at least 90 mg of iron each day. Graph this inequality.

[IN YOUR WORDS]

35. Describe how you can decide which region of the plane should be shaded.

36. What determines whether the curve dividing the region will be a broken line or a solid line?

18.5 SYSTEMS OF INEQUALITIES; LINEAR PROGRAMMING

Several times in this book we have worked with systems of equations. In fact, the first section of this chapter was devoted to solving systems of quadratic equations. You can just as easily have systems of inequalities. In this section, you will learn how to solve linear inequalities and then learn how to apply this to a technique called linear programming.

The *graph of a system of inequalities* consists of all the points in the plane that satisfy every inequality in the system. To get this graph you will graph on the same coordinate system the region determined by each of the inequalities. The region where all these regions overlap is the graph of the system of inequalities.

EXAMPLE 18.38

Graph the solution of the following system:

$$\begin{cases} x + y \leq 5 \\ x - 3y < 3 \end{cases}$$

SOLUTION This system is equivalent to

$$\begin{cases} y \leq -x + 5 \\ y > \dfrac{x}{3} - 1 \end{cases}$$

As we did in Section 18.4, we first sketch the lines $y = -x + 5$ and $y = \dfrac{x}{3} - 1$. Use broken lines when you sketch them. Next, lightly shade the regions that satisfy each inequality, as shown in Figures 18.32a and 18.32b.

The part where both of the shaded regions overlap is part of the solution. Shade this overlapped region more heavily. The first inequality used the $\leq$ symbol, so the part of the line $y = -x + 5$ that borders the answer region should be made solid. The area you have shaded is the solution and should look like the one in Figure 18.32c. Remember to work with broken lines until you can determine which section of the line is included in the solution.

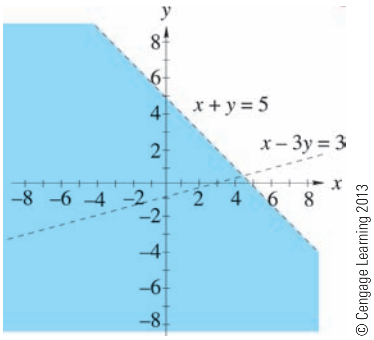

Figure 18.32a

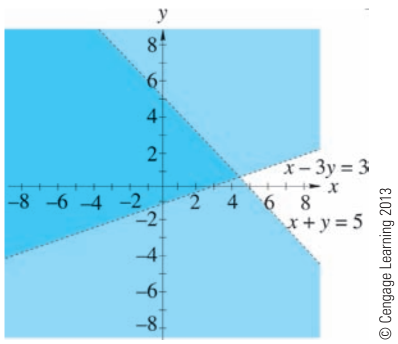

Figure 18.32b

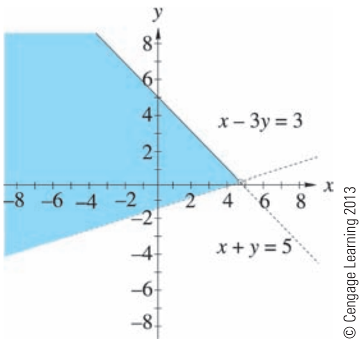

Figure 18.32c

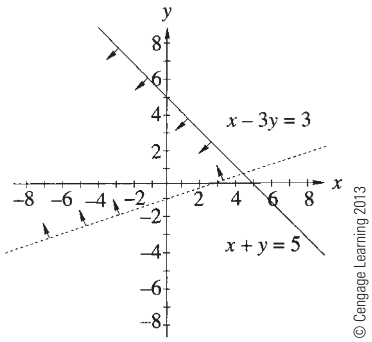

Figure 18.32d

Notice the "hole" where the two boundaries intersect. This point of intersection (4.5, 0.5) does not satisfy the second inequality and so it is not part of the solution for this linear system. In essence, the intersection of a solid line and a broken line results in a "hole."

Another approach is just to place arrows on the boundaries, with the arrows pointing toward the region that satisfies the inequality. This is shown in Figure 18.32d.

EXAMPLE 18.39

Graph the solution of this system:

$$\begin{cases} y + x^2 < 5 \\ x \geq -1 \\ x + 2y \geq -2 \end{cases}$$

SOLUTION This system is equivalent to

$$\begin{cases} y < -x^2 + 5 \\ x \geq -1 \\ y \geq -\frac{1}{2}x - 1 \end{cases}$$

Each curve is sketched using broken lines. The region defined by each inequality is lightly shaded and the region common to all three is then shaded more darkly. The borders of this common region that were formed by the last two inequalities are changed from broken lines to solid lines. The final region is shown in Figure 18.33. If you want to verify that this is the correct region, then select a point in the region and verify that it satisfies all of these inequalities. A good point to test is (0, 0), and you can readily see that it satisfies all three inequalities.

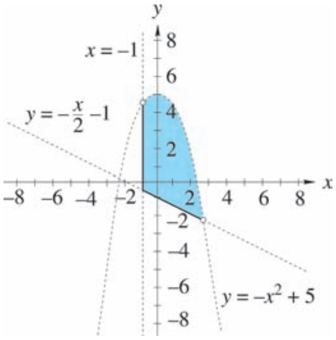

Figure 18.33

One very important area of discipline that uses graphs of systems of inequalities is called **linear programming**. With the use of linear programming it is possible to solve problems in which a quantity is to be maximized or minimized, subject to certain limitations called *constraints*.

Linear programming is used with functions of two or more variables. We will work only with functions or equations of two variables. In linear programming, you establish a system of linear inequalities and graph its solution. This region is the set of **feasible solutions** to the problem and is called the **feasible region**. Any point in this region is called a *feasible point*. Finally, the **objective function** is formed, which describes the quantity to be maximized or minimized. The maximum and minimum values of the objective function will occur on the boundary of the feasible region. This will always be a vertex or a segment that connects two vertices of the feasible region.

APPLICATION BUSINESS

EXAMPLE 18.40

A company manufactures two types of laptop computers: Model A with 256 MB RAM and Model B with 512 MB RAM. The company can make a maximum of 75 model A laptops per day and 50 Model B laptops per day. It takes 8 h to manufacture a Model A computer and 3 h to manufacture a Model B machine. The number of employees can provide a total of 630 h of work each day. The profit on each Model A computer is \$20 and the profit on each Model B is \$27. How many of each type should be made each day to give the maximum profit?

SOLUTION If x is the number of Model A laptop computers made each day and y the number of Model B, we then have the following constraints:

$0 \le x \le 75$	They can make no more than 75 of Model A.
$0 \le y \le 50$	They can make a maximum of 50 Model B.
$8x + 3y \le 630$	They can only work 630 h a day.

The profit P is given by $P = 20x + 27y$. This is the objective function.

The set of feasible points is the shaded region in Figure 18.34. While it is too lengthy to prove here, it can be shown that the solution to the problem lies on the boundary of the feasible region. Thus, since this region is bordered by a polygon, the solution will be a vertex or one side of the polygon.

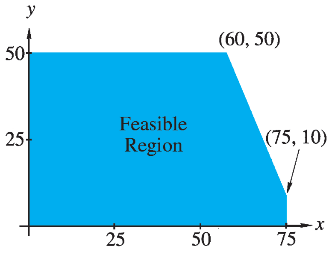

Figure 18.34

There are two ways to approach this solution. We can test the object value at each vertex or we can select a value for P and draw its graph, a test line, for that equation. The answer will be found by drawing a line parallel to the test line so that every point in the feasible region is either on the line or on one side of the line. We will begin with the first method.

The feasible region has five vertices. If we evaluate the objective function $P = 20x + 27y$, at each of these vertices we get the following table of value:

Vertex	P
(0, 0)	0
(0, 50)	1350
(60, 50)	2550
(75, 10)	1770
(75, 0)	1500

From this table, we can clearly see that the most profit will occur when the company makes 60 Model A and 50 Model B microcomputers each day.

With the second method, you can select a value for P, say 1080, and graph $1080 = 20x + 27y$, as shown in Figure 18.35. Now find a line parallel to this, where the entire feasible region is on the same side of the line except for the point, or points, where the line intersects the region. (We know that this occurs when $P = 2550$.) This happens when $y = 50$ and $8x + 3y = 630$ intersect. Solving these two simultaneous equations gives us the same solution, $x = 60$ and $y = 50$, and the maximum daily profit of \$2550.

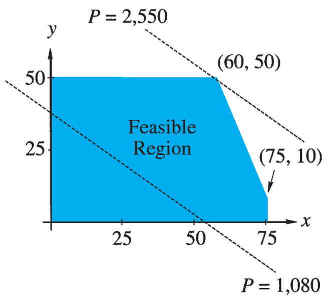

Figure 18.35

APPLICATION BUSINESS

EXAMPLE 18.41

Each week, an electronics company makes home entertainment sets. Among those produced are compact disc players (CD players) and television sets (TVs). The number of hours it takes to manufacture each set, the cost of the materials, and the profit are shown in the following table. The last line of the table shows the maximum number of each item that is available each week. How many of each type of set should the company make in order to make the largest profit?

	Manufacturing Time (h)	Material Cost ($)	Profit ($)
CD Player	3.5	112	67
TV	4.0	76	58
Available	488.0	13,500	

SOLUTION Since the company makes a profit of \$67 on each CD player and only \$58 on each TV, you might suggest that they make only CD players. Since there are 488 h of work time available each week, and since each CD player takes 3.5 h to make, the company could make a total of

$$488 \div 3.5 = 139.4 \text{ CD players}$$

It requires \$112 of materials to make each CD player so a total of 139 CD players would require $\$112 \times 139 = \$15{,}568$. Unfortunately, there is only \$13,500 available for materials each week.

Using a similar argument, we can see that it would not be wise to make as many CD players as possible with the available money for materials. Since each CD player requires \$112 worth of materials and we have \$13,500 available, then why not make

$$\$13{,}500 \div \$112 = 120.5 \text{ CD players}$$

Thus, we could make 120 CD players each week (and make a weekly profit of $120 \times \$67 = \$8{,}040$). It would require $120 \times 3.5 = 420$ h to make these sets. That would leave 68 h when no manufacturing was taking place, but no more money to make any TV sets. The company is committed to making both CD players and TV sets, so this solution is not acceptable.

Now, let's use linear programming to see how many CD players and TVs should be made in order to make the largest profit.

If we let c represent the number of CD players and t the number of TVs we can make in 1 week, then our weekly profit P will be

$$P = 67c + 58t$$

This equation, $P = 67c + 58t$, is the objective function.

The manufacturing time cannot be more than 488 h, so

$$3.5c + 4t \leq 488$$

The cost of materials cannot be more than \$13,500, so

$$112c + 76t \leq 13{,}500$$

EXAMPLE 18.41 (Cont.)

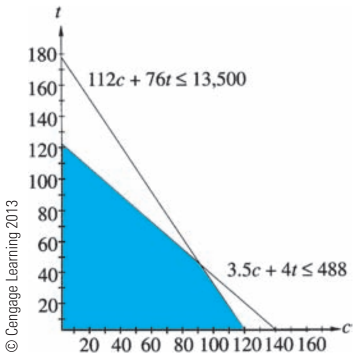

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Figure 18.36

Finally, we assume that $c \geq 0$ and $t \geq 0$. Both of these last two inequalities contain the equal signs, because we want to allow for the possible (but unexpected) answer that we should make all CD players and no TVs or all TVs and no CD players. We now have our objective function and four constraints. The constraints are graphed in Figure 18.36.

Three vertices of the feasible region are (0, 0), (0, 120), and (120, 0). We find the fourth vertex of the feasible region by simultaneously solving the systems of equations

$$\begin{cases} 3.5c + 4t = 488 \\ 112c + 76t = 13{,}500 \end{cases}$$

Using the techniques from Chapter 6, such as Cramer's rule, we see that $c \approx 92.92$ and $t \approx 40.69$. These are not integers, so the solution is still not clear. If we round both of these numbers off to the nearest whole number, we would conclude that we should make 93 CD players and 41 TVs. But if we do, this would require 489.5 h of manufacturing time and \$13,532 of materials a week. Both of these are more than we are allowed.

The alternative is to round one of these numbers up and the other one down. This gives the following two possibilities:

$$\begin{aligned} & c = 92 \text{ and } t = 41 \\ \text{or}\quad & c = 93 \text{ and } t = 40 \end{aligned}$$

In order to determine the answer, we substitute each pair of values in the objective function $P = 67c + 58t$ and see which pair of numbers produces the higher profit. When

$$\begin{aligned} c = 92 \text{ and } t = 41, P &= 67(92) \text{ and } 58(41) \\ &= 8{,}542 \\ \text{and when } c = 93 \text{ and } t = 40, P &= 67(93) \text{ and } 58(40) \\ &= 8{,}551 \end{aligned}$$

Thus, the company will make the most profit, \$8,551, when it manufactures 93 CD players and 40 TVs.

For many companies, other factors might have entered into this example. One factor is the size of the company's warehouse. Suppose that the company in Example 18.41 only ships once a week. That means that the company must store all the work from 1 week in the warehouse. Further suppose that there was only enough room for 125 boxes in the warehouse (assuming that the TV and CD player boxes are the same size). Then our answer of 93 CD players and 40 TVs, or 133 total products, might not be acceptable. Other factors might be considered, such as the amount of time available to test each new product or the number of people trained to assemble each type of equipment.

EXERCISE SET 18.5

In Exercises 1–14, give the graphical solution to each system of inequalities.

1. $\begin{cases} x - y > 0 \\ y - 2x < 4 \end{cases}$

2. $\begin{cases} x + y \geq 6 \\ y - 3x < 6 \end{cases}$

3. $\begin{cases} x + y \geq 5 \\ y > 1 \end{cases}$

4. $\begin{cases} x + 2y \leq 4 \\ y < 1 \end{cases}$

5. $\begin{cases} 2x + 3y \leq 7 \\ 3x - y \leq 5 \end{cases}$

6. $\begin{cases} -2x + y > 6 \\ x + 3y \leq 9 \end{cases}$

7. $\begin{cases} x + 2y - 3 \leq 0 \\ 2x + y - 4 > 0 \end{cases}$

8. $\begin{cases} -2x - 3y < 6 \\ -4x + 3y \geq 12 \end{cases}$

9. $\begin{cases} 3x + y \geq 4 \\ y - 2x \geq -1 \end{cases}$

10. $\begin{cases} -4x + 7y \leq 28 \\ 2x + 3y < 6 \\ y > -3 \end{cases}$

11. $\begin{cases} x + 2y < 4 \\ x - 2y < 4 \\ y \leq 3 \end{cases}$

12. $\begin{cases} x > 1 \\ y < 2 \\ 8x + 3y \geq 24 \end{cases}$

13. $\begin{cases} -x + 2y \leq 6 \\ 3x + y \leq 9 \\ x > -1 \\ y \geq 1 \end{cases}$

14. $\begin{cases} x - 2y > -10 \\ 4x + 3y \geq 26 \\ 3x - 4y < 7 \\ x + y \leq 14 \end{cases}$

In Exercises 15–20, find the maximum or minimum value (as specified) of the objective function that is subject to the given constraints.

15. Maximum $P = 2x + y$ and $\begin{cases} 3x + 4y \leq 24 \\ x \geq 2 \\ y \geq 3 \end{cases}$

16. Minimum $C = 9x + 5y$ and $\begin{cases} 3x + 4y \geq 25 \\ x + 3y \geq 15 \\ x \geq 0 \\ y \geq 0 \end{cases}$

17. Maximum $F = 15x + 10y$ and $\begin{cases} 3x + 2y \leq 80 \\ 2x + 3y \leq 70 \\ x \geq 0 \\ y \geq 0 \end{cases}$

18. Minimum $F = 6x + 9y$ and $\begin{cases} 2x + 5y \leq 50 \\ x + y \leq 12 \\ 2x + y \leq 20 \\ y - x \geq 2 \\ x \geq 0 \end{cases}$

19. Maximum $P = 8x + 10y$ and $\begin{cases} 5x + 10y \leq 180 \\ 10x + 5y \leq 180 \\ x \geq 0 \\ y \geq 3 \end{cases}$

20. Minimum $M = 10x - 8y + 15$ and $\begin{cases} y \geq -3 \\ x - y \geq -5 \\ x + y \geq -5 \\ x \leq 2 \end{cases}$

Solve Exercises 21–28.

21. ***Business*** A company manufactures two products. Each product must pass two inspection points, A and B. Each unit of Product X requires 30 min at A and 45 min at B. Product Y requires 15 min at A and 10 min at B. There are enough trained people to provide 100 h at A and 80 h at B. The company makes a profit of \$10 and \$8 on each of X and Y, respectively. What numbers of each should be manufactured to make the most profit? How much is the most profit? If more people could be added at A or B (but not both), where should they be added to increase profits?

22. ***Business*** An electronics company manufactures two models of computer chips. Model A requires 1 unit of labor and 4 units of parts. Model B requires 1 unit of labor and 3 units of parts. If 120 units of labor and 390 units of

parts are available, and if the company makes a profit of $7.00 on each model *A* chip and $5.50 on each model *B*, how many should it manufacture to maximize its profits?

23. ***Business*** The company in Exercise 22 raises its prices so that it makes a profit of $8.00 on each model *A* chip and $9.50 on each model *B*. Now how many should it manufacture to maximize profits?

24. ***Business*** A computer company manufactures a personal computer (PC) and a business computer (BC). Each computer uses two types of chips, an AB chip and an EP chip. The number of chips needed for each computer and the profit for each are given in the following table.

Computer	AB	EP	Profit
PC	2	3	145
BC	3	8	230

Because of a shortage in chips, the company has 200 AB chips and 450 EP chips in stock. **(a)** How many of each computer should be made in order to maximize profits? **(b)** What is that maximum profit?

25. ***Business*** The loan department of a bank has set aside $50 million for commercial and home loans. The bank's policy is to allocate at least five times as much money to home loans as to commercial loans. The policy also states that it must make some commercial loans. A certain bank's return on a home loan is 8.5% and 6.75% on a commercial loan. The manager of the loan department wants to earn a return of at least $3.6 million on these loans. What is the fewest of each kind of loan the bank should make in order to reach its goal of at least $3.6 million dollars?

26. ***Nutrition*** A brand-X multivitamin tablet contains 15 mg of iron and 10 mg of zinc. One brand-Y multivitamin tablet contains 9 mg of iron and 12 mg of zinc. A nutritionist advises a patient to take at least 90 mg of iron and 90 mg of zinc each day. If brand-X pills cost 5 cents each and brand-Y pills cost 3.5 cents each, how many of each should the patient take per day to fulfill the prescription at minimum cost?

27. ***Industrial management*** The chemistry department of a company decides to stock at least 8000 small test tubes and 5000 large test tubes. It can take advantage of a special price if it buys at least 15000 test tubes. The small tubes are broken twice as often as the large tubes, so the company plans to order at least twice as many small tubes as large ones. If the small test tubes cost 15 cents each and the large ones, made of cheaper glass, cost 12 cents each, how many of each size should the company order to minimize cost?

28. ***Industrial management*** Two types of industrial machines are produced by a certain manufacturer. Machine A requires 3 h of labor for the body and 1 h for wiring. Machine B requires 2 h of labor for the body and 2 h for wiring. The profit on machine A is $32, and the profit on machine B is $45. The body shop can provide 120 h of time per week, and the wiring area can furnish 80 h. How many of each type of machine should be manufactured each week in order to maximize profit?

[IN YOUR WORDS]

29. Explain what is meant by a feasible region and a feasible point? How are they alike? How are they different?

30. Describe how to use the objective function to determine the solution to a linear system.

CHAPTER 18 REVIEW

IMPORTANT TERMS AND CONCEPTS

Feasible region
Feasible solution
Inequalities
  Linear
  Nonlinear
  Properties
Linear programming
Linear system
Nonlinear system of equations
Objective function
Substitution method

REVIEW EXERCISES

In Exercises 1–8, solve each inequality both algebraically and graphically.

1. $3x < -12$

2. $4x + 5 < 6$

3. $2x - 7 \geq 15$

4. $7 - 5x < 2 + 3x$

5. $\dfrac{2x + 5}{4} < \dfrac{4x - 1}{3}$

6. $|2x - 1| \leq 5$

7. $|3x + 2| > 7$

8. $2x + 3 < 13 \leq 3x - 9$

In Exercises 9–13, solve each inequality graphically.

9. $-2x + y < 5$

10. $2x + 3y \geq 6$

11. $4x - 3y < 3$

12. $y > 2x^2 - 5$

13. $y \leq x^2 - 6x + 9$

In Exercises 14–21, graph each system of equations or inequalities.

14. $\begin{cases} x + y = 8 \\ x^2 + y^2 = 49 \end{cases}$

15. $\begin{cases} x^2 + y^2 = 9 \\ x^2 = 5y \end{cases}$

16. $\begin{cases} 2x^2 - y^2 = 8 \\ x^2 + 2y^2 = 4 \end{cases}$

17. $\begin{cases} 2x + 4y^2 = 16 \\ xy = 10 \end{cases}$

18. $\begin{cases} x + 3y < 5 \\ x - 4y \leq 8 \end{cases}$

19. $\begin{cases} 4x - y \leq 4 \\ y + x > -2 \\ y - x < 1 \end{cases}$

20. $\begin{cases} 2x + y - 3 < 0 \\ x - 2y - 4 \geq 0 \end{cases}$

21. $\begin{cases} 3x - y \leq -2 \\ 4 - x - y \geq 0 \\ x > -3 \\ 2x + y > -4 \end{cases}$

In Exercises 22–23, find the maximum or minimum value (as specified) of the objective function that is subject to the given constraints.

22. Maximum $P = 3x + 5y$ and $\begin{cases} x + y \leq 5 \\ y - x \geq -2 \\ x \geq -1 \\ x \leq 3 \end{cases}$

23. Minimum $F = 4x - 3y$ and $\begin{cases} 2x + y \leq 6 \\ y \geq \dfrac{x}{2} - 3 \\ x \geq -1 \\ y \leq 4x - 1 \end{cases}$

Solve Exercises 24–27.

24. The length of a computer chip is given as $7.0 \text{ mm} < \ell < 7.5 \text{ mm}$ and the width as $1.4 \text{ mm} < w < 1.6 \text{ mm}$. What is the range of the area?

25. ***Electronics*** Two resistances R_1 and R_2 connected in parallel must have a total resistance R of at least 4 Ω. If $R_1R_2 = 20$ Ω, what are acceptable values for $R_1 + R_2$? Remember $R = \dfrac{R_1R_2}{R_1 + R_2}$.

26. ***Transportation*** A trucker was delayed 30 min because of trouble loading the shipment. To make up for the delay, the trucker increased the speed and took advantage of interstate highways. The speed was increased by an average of 4 mph, and as a result, the trucker was on time, 7 h and 30 min after leaving. Find the usual average speed of the trucker and the distance the truck traveled.

27. ***Business*** A robotics manufacturing company makes two types of robots: a cylindrical coordinate robot (Model C) and a spherical (polar) coordinate robot (Model S). Each robot has two assembly stations. The number of hours needed at each station and the profit of each robot are given in the following table.

Robot	Station 1 (hours)	Station 2 (hours)	Profit ($)
C	12	15	1,250
S	18	10	1,620

During the next month, the workers will take their vacations. As a result, the company will only have enough workers for 220 h at Station 1 and 180 h at Station 2. How many of each robot should be manufactured in order to make the most profit?

CHAPTER 18 TEST

1. Solve the inequality $5x > 30$, both algebraically and graphically.

2. Solve $|4x + 5| < 17$, both algebraically and graphically.

3. Solve $\dfrac{4x - 5}{2} \leq \dfrac{7x - 2}{3}$, both algebraically and graphically.

4. Solve $(x - 2)(x - 5) > 0$, both algebraically and graphically.

5. Solve $5x^2 + 3x \leq 2x^2 - x - 1$, both algebraically and graphically.

6. Graph this system of inequalities:

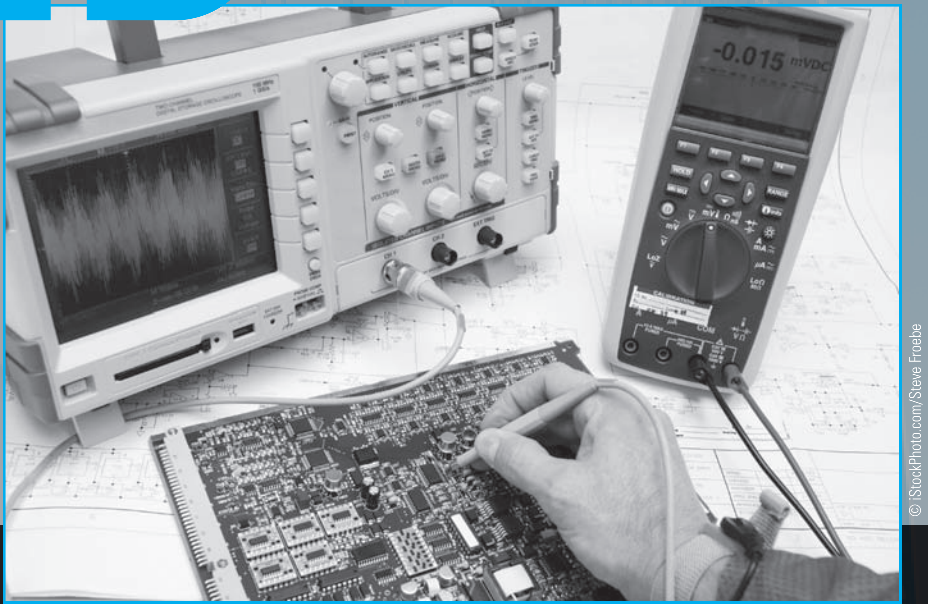
$$\begin{cases} x + 2y > 4 \\ 3x - y \leq 3 \end{cases}$$

7. A brand-X multivitamin tablet contains 12 mg of iron and 10 mg of zinc. Each brand-Y tablet contains 5 mg of iron and 8 mg of zinc. A nutritionist suggests that a patient take at least 80 mg of iron and 90 mg of zinc each day. The brand-X tablets cost 4 cents each and brand-Y pills are 6 cents each.

(a) How many of each pill should the patient take each day to satisfy the nutritionist's suggestion at the minimum cost?

(b) What is the minimum cost?

19 MATRICES



Kirchhoff's laws can be used to write a system of linear equations. In Section 19.4, we will learn how to use matrices to solve this linear system and determine the currents in each electric circuit's path.

In Chapter 6, we studied systems of equations and we learned that determinants can be used to help solve systems of linear equations. In this chapter, another technique for solving systems of equations will be introduced. This technique uses matrices. The introduction of computers has led to increased applications of matrices in engineering, physics, biology, information science, transportation, and other technical areas. We will use calculators to help us work with matrices.

OBJECTIVES

AFTER COMPLETING THIS CHAPTER, YOU WILL BE ABLE TO:

- Identify various types of arrays, vectors, and matrices, and recognize their dimensions.
- Write the transpose of a matrix.
- Add, subtract, and multiply matrices.
- Use technology to perform operations on matrices.
- Employ matrix operations to solve applications.
- Use technology to find the inverse of a nonsingular matrix.

19.1 MATRICES

DIMENSION

A **matrix** is a rectangular array of numbers arranged in rows and columns. A matrix with m rows and n columns is an $m \times n$ matrix. ($m \times n$ is read m by n.) The **dimensions** of a matrix are given in the form $m \times n$, where m represents the number of rows and n the number of columns.

A matrix is often used for presenting numerical data in a condensed form.

EXAMPLE 19.1

$\begin{bmatrix} 4 & 3 & 7 \\ 8 & 19 & -11 \end{bmatrix}$ is a matrix with dimension 2×3.

$\begin{bmatrix} 5 & 8 & -9 & 12 \\ 4 & 0 & -3 & 5.7 \\ 9 & 3 & 21 & -6 \end{bmatrix}$ is a 3×4 matrix.

$\begin{bmatrix} 2 & 4 & 6 & 8 & 9 \end{bmatrix}$ is a 1×5 matrix.

$\begin{bmatrix} 10 \\ 8 \\ 7 \\ 6 \\ 3 \end{bmatrix}$ is a 5×1 matrix.

Suppose a company makes two types of robots, C and S. Model C requires 78 h to assemble and contains 14 Type I computer chips, 7 Type II chips, and 16 m of wiring. Model S requires 65 h to assemble and contains 12 Type I chips, 9 Type II chips, and 11 m of wiring. The company makes a profit of $1,250 on each Model C it sells and $1,500 on each Model S. This information can be presented in a rectangular array or matrix like the following.

$$\begin{array}{l} \\ \\ \text{Model } C \\ \text{Model } S \end{array}\begin{array}{c} \begin{array}{ccccc} \text{Hours} & \text{Type I chip} & \text{Type II chip} & \text{Wiring (m)} & \text{Profit (\$)} \end{array} \\ \begin{bmatrix} 78 & 14 & 7 & 16 & 1{,}250 \\ 65 & 12 & 9 & 11 & 1{,}500 \end{bmatrix} \end{array}$$

The numbers in the first row indicate the data for the Model C robot and those in the second row indicate the Model S data. This is a simple example of a matrix that has two rows and five columns.

ROW AND COLUMN VECTORS

A matrix of dimension $1 \times n$ is a **row vector** and a matrix of dimension $m \times 1$ is a **column vector**.

EXAMPLE 19.2

Some row vectors are:

$$\begin{bmatrix} 2 & 4 & 6 \end{bmatrix} \qquad \begin{bmatrix} 1 & -3 & 2 & 8 \end{bmatrix}$$
$$\begin{bmatrix} 4 & 6 & 0 & -7 & 0 \end{bmatrix} \qquad \begin{bmatrix} 9 \end{bmatrix}$$

Some column vectors are:

$$\begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix} \quad \begin{bmatrix} 1 \\ -3 \\ 2 \\ 8 \end{bmatrix} \quad \begin{bmatrix} 4 \\ 6 \\ 0 \\ -7 \\ 0 \end{bmatrix} \quad \begin{bmatrix} 9 \end{bmatrix}$$

Notice that [9] is both a row vector and a column vector.

ZERO MATRIX

A matrix is a **zero matrix** if all the elements are zero. The matrix

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

is a zero matrix of dimension 3×2.

SQUARE MATRIX

A matrix with the same number, n, of rows and columns is a **square matrix of order n**.

EXAMPLE 19.3

The matrix

$$\begin{bmatrix} 2 & 1 \\ 5 & 7 \end{bmatrix}$$

is a square matrix of order 2 and the matrix

$$\begin{bmatrix} 5 & -7 & 8 \\ 2 & -1 & 0 \\ 0 & 3 & 5 \end{bmatrix}$$

is a square matrix of order 3.

A double-subscript notation has been developed to allow us to refer to specific elements. The first numeral in the subscripts refers to the row in which the element lies and the second numeral refers to the column. An example of this notation is matrix A.

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}$$

The elements a_{11}, a_{22}, a_{33}, a_{44} are the *main diagonal* of this square matrix of order 4.

COEFFICIENT MATRIX

When working with a system of linear equations, a **coefficient matrix** can be formed from the coefficients of the equations. For example, the system

$$\begin{cases} 2x + 3y - 5z = 8 \\ 4x - 7y + 22z = 15 \end{cases}$$

has the coefficient matrix

$$\begin{bmatrix} 2 & 3 & -5 \\ 4 & -7 & 22 \end{bmatrix}$$

AUGMENTED MATRIX

If the column vector formed by the constants to the right of the equal symbols, $\begin{bmatrix} 8 \\ 15 \end{bmatrix}$, is adjoined to the right of the coefficient matrix, we get a new matrix called the **augmented matrix** of the system of linear equations. For example, for the system used previously,

$$\begin{cases} 2x + 3y - 5z = 8 \\ 4x - 7y + 22z = 15 \end{cases}$$

the augmented matrix would be

$$\left[\begin{array}{rrr:r} 2 & 3 & -5 & 8 \\ 4 & -7 & 22 & 15 \end{array}\right]$$

The dashed line between columns 3 and 4 is not needed but is often used to indicate an augmented matrix.

Two matrices are *equal* if they have the same dimension and their corresponding elements are equal.

EXAMPLE 19.4

If $\begin{bmatrix} 6 & 4 & 3 & x \\ 2 & 9 & y & -3 \end{bmatrix} = \begin{bmatrix} z & 4 & 3 & -11 \\ 2 & w & 7 & -3 \end{bmatrix}$, then $x = -11$, $y = 7$, $z = 6$, and $w = 9$.

EXAMPLE 19.5

For the matrices $A = \begin{bmatrix} 1 & 2 & 6 \\ 7 & 8 & 9 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 2 & 6 \\ 7 & 8 & 5 \end{bmatrix}$, and $C = \begin{bmatrix} 1 & 2 \\ 7 & 8 \end{bmatrix}$, $A \neq B$ because the corresponding elements $a_{23} = 9$ and $b_{23} = 5$ are not equal. $A \neq C$ because they have different dimensions.

ADDITION AND SUBTRACTION OF MATRICES

If A and B are two matrices, each of dimension $m \times n$, then their sum (or difference) is defined to be another matrix, C, also of dimension $m \times n$, where every element of C is the sum (or difference) of the corresponding elements A and B.

EXAMPLE 19.6

(a) $$\begin{bmatrix} 2 & 3 & 5 \\ 6 & 7 & 8 \end{bmatrix} + \begin{bmatrix} 10 & 11 & 12 \\ 14 & 16 & 22 \end{bmatrix} = \begin{bmatrix} 2+10 & 3+11 & 5+12 \\ 6+14 & 7+16 & 8+22 \end{bmatrix} = \begin{bmatrix} 12 & 14 & 17 \\ 20 & 23 & 30 \end{bmatrix}$$

(b) $$\begin{bmatrix} 4 & 9 \\ -2 & 8 \\ 6 & 1 \end{bmatrix} + \begin{bmatrix} 16 & 21 \\ 2 & -8 \\ 9 & 14 \end{bmatrix} = \begin{bmatrix} 4+16 & 9+21 \\ -2+2 & 8+(-8) \\ 6+9 & 1+14 \end{bmatrix} = \begin{bmatrix} 20 & 30 \\ 0 & 0 \\ 15 & 15 \end{bmatrix}$$

(c) $$\begin{bmatrix} 1 & -5 \\ 4 & 3 \\ 2 & -1 \end{bmatrix} - \begin{bmatrix} 4 & -6 \\ 2 & 1 \\ -3 & 9 \end{bmatrix} = \begin{bmatrix} 1-4 & -5-(-6) \\ 4-2 & 3-1 \\ 2-(-3) & -1-9 \end{bmatrix} = \begin{bmatrix} -3 & 1 \\ 2 & 2 \\ 5 & -10 \end{bmatrix}$$

CAUTION You can only add or subtract two matrices if they have the same dimension.

EXAMPLE 19.7

If $A = \begin{bmatrix} 4 & 2 & 7 & 9 \\ 5 & 4 & 6 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 5 & 3 & 8 & 1 \\ 4 & 6 & 2 & 4 \end{bmatrix}$, then determine $A + B$, $A - B$, and $B - A$.

SOLUTIONS

$$A + B = \begin{bmatrix} 4+5 & 2+3 & 7+8 & 9+1 \\ 5+4 & 4+6 & 6+2 & 1+4 \end{bmatrix}$$
$$= \begin{bmatrix} 9 & 5 & 15 & 10 \\ 9 & 10 & 8 & 5 \end{bmatrix}$$
$$A - B = \begin{bmatrix} 4-5 & 2-3 & 7-8 & 9-1 \\ 5-4 & 4-6 & 6-2 & 1-4 \end{bmatrix}$$
$$= \begin{bmatrix} -1 & -1 & -1 & 8 \\ 1 & -2 & 4 & -3 \end{bmatrix}$$
$$B - A = \begin{bmatrix} 5-4 & 3-2 & 8-7 & 1-9 \\ 4-5 & 6-4 & 2-6 & 4-1 \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 1 & 1 & -8 \\ -1 & 2 & -4 & 3 \end{bmatrix}$$

APPLICATION HEALTHCARE

EXAMPLE 19.8

At the beginning of a laboratory experiment, two groups of four mice were timed to see how long it took them to go through a maze. The mice in the group that went through the maze at night took 8, 6, 5, and 7 sec. The mice in the group that went through the maze during the day took 22, 14, 18, and 12 sec. (a) Write a 2×4 matrix using this information.

A week later, the mice were sent through the maze again. They went in the same order as they did the week before. This time the night group completed the maze in 4, 3, 5, and 3 sec and the day group took 9, 8, 10, and 8 sec. (b) Write this information as a 2×4 matrix and use matrix subtraction to write a matrix that shows the amount of change each mouse made in its time to complete the maze.

SOLUTIONS

(a) We will put the night group in the top row and the day group of mice in the bottom row. The result is the following matrix:

$$\begin{array}{l} \text{Night group} \\ \text{Day group} \end{array} \begin{bmatrix} 8 & 6 & 5 & 7 \\ 22 & 14 & 18 & 12 \end{bmatrix}$$

(b) The matrix for the data at the end of the week is

$$\begin{array}{l} \text{Night group} \\ \text{Day group} \end{array} \begin{bmatrix} 4 & 3 & 5 & 3 \\ 9 & 8 & 10 & 8 \end{bmatrix}$$

To find the change in the amount of time each mouse took going through the maze, we subtract the last matrix from the matrix in (a).

$$\begin{array}{l} \text{Night group} \\ \text{Day group} \end{array} \begin{bmatrix} 8 & 6 & 5 & 7 \\ 22 & 14 & 18 & 12 \end{bmatrix} - \begin{bmatrix} 4 & 3 & 5 & 3 \\ 9 & 8 & 10 & 8 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 3 & 0 & 4 \\ 13 & 6 & 8 & 4 \end{bmatrix}$$

SCALAR MULTIPLICATION

If A is an $m \times n$ matrix and k a real number, then kA is an $m \times n$ matrix B, where $b_{ij} = ka_{ij}$ for each element of A. This is referred to as **scalar multiplication** and the real number k is called a **scalar**.

EXAMPLE 19.9

If $A = \begin{bmatrix} 3 & 5 \\ 9 & 2 \end{bmatrix}$ and $k = 4$, then

$$kA = 4\begin{bmatrix} 3 & 5 \\ 9 & 2 \end{bmatrix} = \begin{bmatrix} 4 \cdot 3 & 4 \cdot 5 \\ 4 \cdot 9 & 4 \cdot 2 \end{bmatrix}$$

$$= \begin{bmatrix} 12 & 20 \\ 36 & 8 \end{bmatrix}$$

EXAMPLE 19.10

If $B = \begin{bmatrix} 2 & 4 \\ -3 & 7 \end{bmatrix}$ and $k = -\frac{1}{2}$, then

$$kB = -\frac{1}{2}\begin{bmatrix} 2 & 4 \\ -3 & 7 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} \cdot 2 & -\frac{1}{2} \cdot 4 \\ -\frac{1}{2}(-3) & -\frac{1}{2} \cdot 7 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & -2 \\ \frac{3}{2} & -\frac{7}{2} \end{bmatrix}$$

These two examples help show that $A + A = 2A$, $B + B + B = 3B$, and so on.

APPLICATION BUSINESS

EXAMPLE 19.11

A computer retailer sells three types of computers: the personal computer (PC), the business computer (BC), and the industrial computer (IC). The retailer has two stores. The matrix below shows the number of each computer model in stock at each store.

EXAMPLE 19.11 (Cont.)

$$\begin{array}{c} \\ \text{Store A} \\ \text{Store B} \end{array} \begin{array}{c} \begin{array}{ccc} \text{PC} & \text{BC} & \text{IC} \end{array} \\ \begin{bmatrix} 6 & 18 & 12 \\ 10 & 22 & 28 \end{bmatrix} \end{array}$$

The store plans to have a sale in the near future. In order to have enough computers in stock at each store when the sale begins, it plans to order 1.5 times the current number. How many of each computer should be ordered for each store?

SOLUTION Here the scalar is 1.5, so we want to multiply each element of this matrix by 1.5. The result is

$$1.5\left(\begin{array}{c} \\ \text{Store A} \\ \text{Store B} \end{array} \begin{array}{c} \begin{array}{ccc} \text{PC} & \text{BC} & \text{IC} \end{array} \\ \begin{bmatrix} 6 & 18 & 12 \\ 10 & 22 & 28 \end{bmatrix} \end{array} \right) = \begin{array}{c} \\ \text{Store A} \\ \text{Store B} \end{array} \begin{array}{c} \begin{array}{ccc} \text{PC} & \text{BC} & \text{IC} \end{array} \\ \begin{bmatrix} 9 & 27 & 18 \\ 15 & 33 & 42 \end{bmatrix} \end{array}$$

Thus, we see that the retailer needs to order 9 PCs for Store A and 15 for Store B, 27 BCs for Store A and 33 for Store B, and 18 ICs for Store A and 42 for Store B.

During this section we have used the following properties of matrices.

MATRIX PROPERTIES

If A, B, and C are three matrices, all of the same dimension, 0 is the zero matrix, and k is a real number, then

$A + B = B + A$ (commutative law)
$A + 0 = 0 + A = A$ (identity for addition)
$A + (B + C) = (A + B) + C$ (associative law)
$k(A + B) = kA + kB$

EXERCISE SET 19.1

Give the dimension of the matrices in Exercises 1–6. Solve Exercises 7–10.

1. $\begin{bmatrix} 2 & 4 & 5 \\ 3 & 2 & 1 \end{bmatrix}$

2. $\begin{bmatrix} 3 & 4 & 6 \\ 2 & 5 & 9 \\ 8 & 7 & 2 \end{bmatrix}$

3. $\begin{bmatrix} 2 & 1 & 0 & 7 & 9 & 6 \\ 3 & 2 & 4 & 8 & 7 & 2 \\ 9 & 6 & 4 & 5 & 0 & 2 \end{bmatrix}$

4. $\begin{bmatrix} 3 \\ 2 \\ 1 \\ 4 \end{bmatrix}$

5. $\begin{bmatrix} 3 & 2 \\ 4 & 1 \\ 5 & 3 \\ 7 & 9 \end{bmatrix}$

6. $\begin{bmatrix} 1 & 2 & 4 & 6 & 9 & 11 & 12 \end{bmatrix}$

7. Given matrix A, determine the values of elements a_{11}, a_{24}, a_{21}, and a_{32}.

$$A = \begin{bmatrix} 1 & 2 & 5 & 7 \\ 8 & 9 & 11 & 13 \\ 4 & 6 & 12 & 15 \end{bmatrix}$$

8. Given matrix B, determine the values of elements b_{12}, b_{21}, b_{23}, and b_{32}.

$$B = \begin{bmatrix} -5 & 0 & 2 \\ 4 & 7 & 9 \\ 5 & -8 & 11 \end{bmatrix}$$

9. Given that

$$\begin{bmatrix} 4 & 3 & 2 & x \\ 5 & 9 & 7 & 11 \\ y & 8 & 3 & 4 \end{bmatrix} = \begin{bmatrix} 4 & 3 & z & 12 \\ w & 9 & 7 & 11 \\ -6 & 8 & 3 & 4 \end{bmatrix},$$

determine the values of x, y, z, and w.

10. Given that

$$\begin{bmatrix} 3 & 2 & x-2 \\ 4 & y+5 & 9 \\ w+6 & 11 & 10 \end{bmatrix} = \begin{bmatrix} 3z & 2 & 7 \\ 4 & 3 & 9 \\ 7 & 11 & 2p \end{bmatrix},$$

determine the values of x, y, z, w, and p.

In Exercises 11–14, find the indicated sum or difference.

11. $\begin{bmatrix} 3 & 2 & 1 & -4 \\ 9 & 7 & 6 & -1 \\ 4 & 3 & 5 & 2 \end{bmatrix} + \begin{bmatrix} -6 & 4 & 5 & -4 \\ 3 & 2 & 11 & 1 \\ 5 & -3 & 0 & 8 \end{bmatrix}$

12. $\begin{bmatrix} 2 & 4 \\ -5 & 1 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} 5 & -6 \\ 2 & 9 \\ 8 & 2 \end{bmatrix}$

13. $\begin{bmatrix} 3 & 2 & -1 \\ 9 & 12 & 4 \end{bmatrix} - \begin{bmatrix} 1 & 3 & -5 \\ 4 & 3 & 6 \end{bmatrix}$

14. $\begin{bmatrix} 3 & 2 & -1 & 5 \\ 4 & 3 & 11 & 9 \\ 16 & 4 & 3 & -8 \\ 12 & 5 & 4 & 6 \\ 2 & 9 & 18 & 7 \end{bmatrix} - \begin{bmatrix} 1 & 2 & -4 & 2 \\ 1 & 3 & 2 & 8 \\ 2 & -4 & 3 & 6 \\ 2 & -5 & 2 & -4 \\ 2 & 3 & -5 & 3 \end{bmatrix}$

In Exercises 15–22, use matrices A, B, and C to determine the indicated matrix, where

$$A = \begin{bmatrix} 4 & 3 & 2 \\ 5 & 0 & 7 \end{bmatrix},\ B = \begin{bmatrix} -5 & 4 & 2 \\ 3 & 1 & 8 \end{bmatrix},\ \textit{and } C = \begin{bmatrix} 5 & 8 & 4 \\ 6 & -2 & 9 \end{bmatrix}.$$

15. $A + B$

16. $B - C$

17. $3A$

18. $4B$

19. $2B + C$

20. $4A - 3C$

21. $2B - 3A$

22. $4C + 2B$

In Exercises 23–30, use matrices D, E, and F to determine the indicated matrix, where

$$D = \begin{bmatrix} 4 \\ 2 \\ -1 \\ 5 \end{bmatrix},\ E = \begin{bmatrix} 7 \\ 9 \\ -4 \\ 10 \end{bmatrix},\ \textit{and } F = \begin{bmatrix} 14 \\ 18 \\ -8 \\ 5 \end{bmatrix}.$$

23. $D - E$

24. $E + F$

25. $3E$

26. $F - 2E$

27. $3D + 2E$

28. $2F - 7D$

29. $7D - 2F$

30. $5E - 2F$

Solve Exercises 31–36.

31. ***Business*** In keeping inventory records, a company uses a matrix. One storage location has 18 of computer chip A, 7 of computer chip B, 9 EPROMS, 7 keyboards, 11 motherboards, and 4 disk drives. This is represented by the matrix $\begin{bmatrix} 18 & 7 & 9 \\ 7 & 11 & 4 \end{bmatrix}$. At a second storage point, the inventory is $\begin{bmatrix} 9 & 5 & 6 \\ 11 & 4 & 12 \end{bmatrix}$. How many of each item are on hand?

32. ***Business*** Wrecker's Auto Supply Company has three stores, each of which carries five sizes of a certain tire model. The following matrix represents the present inventories:

$$\begin{array}{c|c} \text{Store} & \begin{array}{ccccc} \text{195/70SR-14} & \text{205/70SR-14} & \text{185/70SR-15} & \text{205/70SR-15} & \text{215/70SR-15} \end{array} \\ \begin{array}{c} A \\ B \\ C \end{array} & \begin{bmatrix} 12 & 15 & 28 & 7 & 16 \\ 25 & 19 & 11 & 40 & 11 \\ 15 & 29 & 21 & 17 & 17 \end{bmatrix} \end{array}$$

At the beginning of the next month, they would like to have the following inventory:

$$\begin{array}{c|c} \text{Store} & \begin{array}{ccccc} \text{195/70SR-14} & \text{205/70SR-14} & \text{185/70SR-15} & \text{205/70SR-15} & \text{215/70SR-15} \end{array} \\ \begin{array}{c} A \\ B \\ C \end{array} & \begin{bmatrix} 20 & 48 & 60 & 24 & 44 \\ 36 & 64 & 24 & 40 & 32 \\ 44 & 72 & 48 & 60 & 40 \end{bmatrix} \end{array}$$

Write a matrix that represents the number of each size of tire that must be ordered for each store, so as to achieve their goal. (Assume that no more tires are sold.)

33. ***Medical technology*** A drug company tested 400 patients to see if a new medicine is effective. Half of the patients received the new drug and half received a placebo. The results for the first 200 patients are shown in the following matrix:

$$\begin{array}{cc} & \begin{array}{cc} \text{New drug} & \text{Placebo} \end{array} \\ \begin{array}{c} \text{Effective} \\ \text{Not effective} \end{array} & \begin{bmatrix} 70 & 40 \\ 30 & 60 \end{bmatrix} \end{array}$$

Using the same matrix format, the results for a second 200 patients were $\begin{bmatrix} 65 & 42 \\ 35 & 58 \end{bmatrix}$. What were the results for the entire test group?

34. ***Business*** A computer supply company has its inventory of four types of computer chips in three warehouses. At the beginning of the month it had the following inventory:

$$\begin{array}{cc} \text{Chip type:} & \begin{array}{cccc} \text{2 GHz} & \text{2.5 GHz} & \text{2.8 GHz} & \text{3.1 GHz} \end{array} \\ \begin{array}{c} \text{Warehouse A} \\ \text{Warehouse B} \\ \text{Warehouse C} \end{array} & \begin{bmatrix} 1200 & 3200 & 4800 & 900 \\ 1650 & 4580 & 7200 & 700 \\ 1120 & 5100 & 6200 & 1200 \end{bmatrix} \end{array}$$

The sales for the month were

$$\begin{array}{cc} \text{Chip type:} & \begin{array}{cccc} \text{2 GHz} & \text{2.5 GHz} & \text{2.8 GHz} & \text{3.1 GHz} \end{array} \\ \begin{array}{c} \text{Warehouse A} \\ \text{Warehouse B} \\ \text{Warehouse C} \end{array} & \begin{bmatrix} 270 & 2130 & 3210 & 265 \\ 1120 & 4230 & 3124 & 75 \\ 320 & 3126 & 2743 & 1012 \end{bmatrix} \end{array}$$

Write a matrix that shows the inventory at the end of the month.

35. ***Business*** The computer supply company in Exercise 34 expects sales to increase by 10% during the next month. What are next month's projected sales? (Round off any fractional answers.) Remember, last month the company sold

$$\begin{array}{cc} \text{Chip type:} & \begin{array}{cccc} \text{2 GHz} & \text{2.5 GHz} & \text{2.8 GHz} & \text{3.1 GHz} \end{array} \\ \begin{array}{c} \text{Warehouse A} \\ \text{Warehouse B} \\ \text{Warehouse C} \end{array} & \begin{bmatrix} 270 & 2130 & 3210 & 265 \\ 1120 & 4230 & 3124 & 75 \\ 320 & 3126 & 2743 & 1012 \end{bmatrix} \end{array}$$

36. ***Business*** The following matrix represents the normal monthly order of a retail store for three models of exercise suits in four different sizes:

$$\begin{array}{l} \\ \text{Jogging} \\ \text{Sweating} \\ \text{Walking} \end{array} \begin{array}{c} \begin{array}{cccc} \text{S} & \text{M} & \text{L} & \text{XL} \end{array} \\ \begin{bmatrix} 5 & 4 & 3 & 4 \\ 7 & 12 & 15 & 7 \\ 4 & 8 & 12 & 14 \end{bmatrix} \end{array}$$

During its spring sale, the store expects to do four times the usual volume of business. Determine the matrix that represents the store's order for the month of the sale.

[IN YOUR WORDS]

37. Describe how to enter a matrix into your calculator. (Hint: Review the procedures in Section 6.4.)

38. Describe how to use your calculator to (**a**) add or subtract two matrices and (**b**) multiply a matrix by a scalar.

39. Explain what is needed for two matrices to be equal.

40. What is an augmented matrix?

19.2 MULTIPLICATION OF MATRICES

Learning to add and subtract matrices was straightforward. We had to be careful that each matrix had the same number of rows and the same number of columns. Multiplying matrices is more involved.

MULTIPLYING MATRICES

If A and B are two matrices, then in order to define the product AB, the number of columns in matrix A must be the same as the number of rows in matrix B. Thus, matrix A must be of dimension $m \times n$ and matrix B dimension $n \times p$. The product will have dimension $m \times p$.

We will begin by multiplying a row vector and a column vector.

PRODUCT OF A ROW VECTOR AND A COLUMN VECTOR

If $A = [a_{11} \quad a_{12} \quad a_{13} \quad \cdots \quad a_{1n}]$ is a row vector, and $B = \begin{bmatrix} b_{11} \\ b_{21} \\ b_{31} \\ \vdots \\ b_{n1} \end{bmatrix}$ is a column vector, then

$$AB = [a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31} + \cdots + a_{1n}b_{n1}]$$

EXAMPLE 19.12

If $A = \begin{bmatrix} 3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 5 \\ 1 \end{bmatrix}$, then

$$AB = [3 \cdot 5 + 4 \cdot 1] = [15 + 4] = [19]$$

Since A is 1×2 and B is 2×1, the product AB is 1×1.

EXAMPLE 19.13

If $C = \begin{bmatrix} 1 & 2 & 3 & x \end{bmatrix}$, $D = \begin{bmatrix} 3 \\ -1 \\ 1 \\ 2 \end{bmatrix}$, and $CD = [22]$, find x.

SOLUTION

$$\begin{aligned} CD &= [1(3) + 2(-1) + 3(1) + 2x] \\ &= [3 - 2 + 3 + 2x] \\ &= [4 + 2x] = [22] \end{aligned}$$

These two matrices are equal only if $4 + 2x = 22$ or $2x = 18$, so $x = 9$.

We will use the following method of multiplying a row vector and a column vector to multiply larger matrices.

MATRIX MULTIPLICATION

If A is an $m \times n$ matrix and B is an $n \times p$ matrix, then the product matrix $C = AB$ is an $m \times p$ matrix, where element c_{ij} in the ith row and jth column is formed by multiplying the elements in the ith row of A by the corresponding elements in the jth column of B and adding the results. Thus,

$$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj}$$

We will illustrate this definition by showing how you get an element in the following product. Suppose. $A = \begin{bmatrix} 1 & 2 & -1 & 0 \\ 3 & 5 & 2 & -3 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 1 & 3 \\ 0 & 4 & -2 \\ 1 & 2 & 0 \\ 1 & 3 & -2 \end{bmatrix}$.

You can see that A is a 2×4 matrix and B is a 4×3 matrix, so AB will have dimension 2×3. If we let $C = AB$, we will show how to find c_{12} or the element in row 1, column 2 of matrix C. To get this element, we multiply row 1 of matrix A and column 2 of matrix B. This is the same as multiplying a row vector and a column vector. Thus,

$$\begin{aligned} c_{12} &= \begin{bmatrix} 1 & 2 & -1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 4 \\ 2 \\ 3 \end{bmatrix} \\ &= 1(1) + 2(4) - 1(2) + 0(3) \\ &= 7 \end{aligned}$$

EXAMPLE 19.14

Multiply $\begin{bmatrix} 1 & 2 & -1 & 0 \\ 3 & 5 & 2 & -3 \end{bmatrix} \begin{bmatrix} 2 & 1 & 3 \\ 0 & 4 & -2 \\ 1 & 2 & 0 \\ 1 & 3 & -2 \end{bmatrix}$.

SOLUTION

$$\begin{bmatrix} 1 & 2 & -1 & 0 \\ 3 & 5 & 2 & -3 \end{bmatrix} \begin{bmatrix} 2 & 1 & 3 \\ 0 & 4 & -2 \\ 1 & 2 & 0 \\ 1 & 3 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} 1(2) + 2(0) - 1(1) + 0(1) & 1(1) + 2(4) - 1(2) + 0(3) & 1(3) + 2(-2) - 1(0) + 0(-2) \\ 3(2) + 5(0) + 2(1) - 3(1) & 3(1) + 5(4) + 2(2) - 3(3) & 3(3) + 5(-2) + 2(0) - 3(-2) \end{bmatrix}$$

$$= \begin{bmatrix} 2 + 0 - 1 + 0 & 1 + 8 - 2 + 0 & 3 - 4 - 0 + 0 \\ 6 + 0 + 2 - 3 & 3 + 20 + 4 - 9 & 9 - 10 + 0 + 6 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 7 & -1 \\ 5 & 18 & 5 \end{bmatrix}$$

NOTE In general, $AB \neq BA$. In fact, if AB is defined, then BA may not be defined. The only time that both AB and BA are defined is when both A and B are square matrices with the same dimensions. Thus, the commutative law does not hold for multiplication of matrices.

The following properties will hold, assuming that the dimensions of the matrices allow the products or sums to be defined.

$$\begin{aligned} A(BC) &= (AB)C & &\text{associative law} \\ A(B + C) &= AB + AC & &\text{left distributive law} \\ (B + C)A &= BA + CA & &\text{right distributive law} \end{aligned}$$

Because the commutative law does not hold, the left and right distributive laws usually give different results.

APPLICATION BUSINESS

EXAMPLE 19.15

A computer supply company has its inventory of four types of computer chips in three warehouses. The sales of each chip from each warehouse are shown in matrix S.

$$\begin{array}{rl} \text{Chip type:} & \begin{array}{cccc} \;\;286 & \;\;386 & \;\;486 & \;\;586 \end{array} \\ \begin{array}{r} \text{Warehouse A} \\ S = \text{Warehouse B} \\ \text{Warehouse C} \end{array} & \begin{bmatrix} 270 & 2130 & 3210 & 265 \\ 1120 & 4230 & 3124 & 75 \\ 320 & 3126 & 2743 & 1012 \end{bmatrix} \end{array}$$

EXAMPLE 19.15 (Cont.) Matrix P below shows the selling price and the profit for each type of chip.

$$P = \begin{array}{c} \text{Chip} \\ 286 \\ 386 \\ 486 \\ 586 \end{array} \begin{array}{c} \begin{array}{cc} \text{Selling Price} & \text{Profit} \end{array} \\ \begin{bmatrix} 25 & 10 \\ 35 & 13 \\ 52 & 17 \\ 97 & 33 \end{bmatrix} \end{array}$$

(a) How much money in sales did each warehouse generate during this month?

(b) How much profit did each warehouse make?

SOLUTION Since matrix S is a 3×4 matrix and P is a 4×2, we can determine the solution from the 3×2 matrix that results when we multiply SP.

$$SP = \begin{bmatrix} 270 & 2130 & 3210 & 265 \\ 1120 & 4230 & 3124 & 75 \\ 320 & 3126 & 2743 & 1012 \end{bmatrix} \cdot \begin{bmatrix} 25 & 10 \\ 35 & 13 \\ 52 & 17 \\ 97 & 33 \end{bmatrix}$$

$$SP = \begin{array}{c} \\ \text{Warehouse A} \\ \text{Warehouse B} \\ \text{Warehouse C} \end{array} \begin{array}{c} \begin{array}{cc} \text{Sales} & \text{Profit} \end{array} \\ \begin{bmatrix} 273{,}925 & 93{,}705 \\ 345{,}773 & 121{,}773 \\ 358{,}210 & 123{,}865 \end{bmatrix} \end{array}$$

From this matrix we can see, for example, that Warehouse A sold \$273,925 in chips for a profit of \$93,705.

EXERCISE SET 19.2

In Exercises 1–10, find the indicated products.

1. $\begin{bmatrix} 2 & 3 \\ 1 & -4 \end{bmatrix} \begin{bmatrix} 4 & -3 \\ 6 & 5 \end{bmatrix}$

2. $\begin{bmatrix} 1 & -2 & 4 \end{bmatrix} \begin{bmatrix} 3 \\ 9 \\ -7 \end{bmatrix}$

3. $\begin{bmatrix} 5 & 1 & 6 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ 9 \\ -8 \\ 3 \end{bmatrix}$

4. $\begin{bmatrix} 2 & 3 & 1 \\ -4 & 2 & 0 \end{bmatrix} \begin{bmatrix} 4 & 1 \\ -5 & 2 \\ 3 & 0 \end{bmatrix}$

5. $\begin{bmatrix} 1 & 2 & 4 \\ 2 & 3 & 6 \end{bmatrix} \begin{bmatrix} 5 & -7 & 4 \\ 4 & 6 & 0 \\ -4 & 2 & 1 \end{bmatrix}$

6. $\begin{bmatrix} 4 & 7 \\ 2 & 3 \\ 6 & 2 \\ 9 & 1 \end{bmatrix} \begin{bmatrix} 3 & 7 & 6 & 1 & -5 \\ 4 & -3 & 2 & 0 & 10 \end{bmatrix}$

7. $\begin{bmatrix} 1 & 2 & 3 \\ 6 & 5 & 4 \\ -1 & 0 & 2 \end{bmatrix} \begin{bmatrix} 9 & 8 & 7 \\ -2 & 3 & -1 \\ 0 & 1 & 0 \end{bmatrix}$

8. $\begin{bmatrix} 9 & 8 & 7 \\ -2 & 3 & -1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 6 & 5 & 4 \\ -1 & 0 & 2 \end{bmatrix}$

9. $\begin{bmatrix} 3 & 4 & -2 \\ 7 & 8 & 10 \end{bmatrix} \begin{bmatrix} 0 & 4 & 1 \\ 1 & 0 & 2 \\ 8 & -1 & 0 \end{bmatrix}$

10. $\begin{bmatrix} 1 & 2 & -1 \\ 2 & -1 & 3 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 2 \\ 1 & 0 & 2 & -1 \\ 3 & 4 & -1 & 0 \end{bmatrix}$

In Exercises 11–20, use matrices A, B, and C to determine the indicated answer, where $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 3 & 6 \\ 2 & 5 \end{bmatrix}$, *and* $C = \begin{bmatrix} 4 & -2 \\ -6 & 2 \end{bmatrix}$.

11. AB
12. BA
13. AC
14. BC
15. $A(BC)$
16. $(AB)C$
17. $A(B + C)$
18. $(B + C)A$
19. $(2A)(3B)$
20. $(2A + \frac{1}{2}C)B$

Solve Exercises 21–29.

21. Suppose $A = \begin{bmatrix} 2 & 10 \\ 3 & 15 \end{bmatrix}$ and $B = \begin{bmatrix} -10 & 25 \\ 2 & -5 \end{bmatrix}$. What is AB? Can you conclude that if $AB = 0$, then either $A = 0$ or $B = 0$?

22. If $\begin{bmatrix} 2 & x \\ y & 5 \end{bmatrix} \begin{bmatrix} 4 & 6 \\ 9 & -1 \end{bmatrix} = \begin{bmatrix} 35 & 9 \\ 49 & 1 \end{bmatrix}$, determine x and y.

23. ***Business*** A computer supply company has its inventory of four types of computer processors in three warehouses. One month it had the following sales:

$$\begin{array}{r} \text{Chip type:} \\ \text{Warehouse A} \\ \text{Warehouse B} \\ \text{Warehouse C} \end{array} \begin{array}{c} \begin{array}{cccc} \text{2 GHz} & \text{2.5 GHz} & \text{2.8 GHz} & \text{3.1 GHz} \end{array} \\ \begin{bmatrix} 270 & 2{,}130 & 3{,}210 & 265 \\ 1{,}120 & 4{,}230 & 3{,}124 & 75 \\ 320 & 3{,}126 & 2{,}743 & 1{,}012 \end{bmatrix} \end{array}$$

The 2-GHz processors sell for \$85, the 2.5-GHz processors sell for \$180, the 2.8-GHz processors sell for \$375, and the 3.1-GHz processors sell for \$725. How much money in sales did each warehouse generate during this month?

24. ***Construction*** A contractor builds three sizes of houses (ranch, bi-level, and two story) in two different models, A and B. The contractor plans to build 100 new homes in a subdivision. Matrix P shows the number of each type of house planned for the subdivision.

$$P = \begin{array}{r} \\ \text{Model A} \\ \text{Model B} \end{array} \begin{array}{c} \begin{array}{ccc} \text{Ranch} & \text{Bi-level} & \text{Two story} \end{array} \\ \begin{bmatrix} 30 & 20 & 15 \\ 10 & 10 & 15 \end{bmatrix} \end{array}$$

The amounts of each type of exterior material needed for each house are shown in matrix A. Here, concrete is in cubic yards, lumber in 1,000 board feet, bricks in 1,000s, and shingles in units of 100 ft^2.

$$A = \begin{array}{r} \\ \text{Ranch} \\ \text{Bi-level} \\ \text{Two story} \end{array} \begin{array}{c} \begin{array}{cccc} \text{Concrete} & \text{Lumber} & \text{Bricks} & \text{Shingles} \end{array} \\ \begin{bmatrix} 10 & 2 & 2 & 3 \\ 15 & 3 & 4 & 4 \\ 25 & 5 & 6 & 3 \end{bmatrix} \end{array}$$

The cost of each of the units for each kind of material is given by matrix C.

$$C = \begin{array}{l} \\ \text{Concrete} \\ \text{Lumber} \\ \text{Brick} \\ \text{Shingles} \end{array} \begin{array}{c} \text{Cost per unit} \\ \begin{bmatrix} 25 \\ 210 \\ 75 \\ 40 \end{bmatrix} \end{array}$$

(a) How much of each type of material will the contractor need for each house model?

(b) What will it cost to build each size of house?

(c) What will it cost to build each house model?

(d) What will it cost to build the entire subdivision?

25. ***Physics*** The *Pauli spin matrices* in quantum mechanics are $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$, and $C = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, where $i = \sqrt{-1}$. Show that $A^2 = B^2 = C^2 = I$, where $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

26. ***Physics*** Show that the Pauli spin matrices *anticommute*. That is, show that $AB = -BA$, $AC = -CA$, and $BC = -CB$.

27. ***Physics*** The *commutator* of two matrices P and Q is $PQ - QP$. For the Pauli spin matrices, show that **(a)** the commutator of A and B is $2iC$, **(b)** the commutator of A and C is $-2iB$, and **(c)** the commutator of B and C is $2iA$, where $i = \sqrt{-1}$.

28. Show, by multiplying the matrices, that the following equation represents an ellipse:

$$\begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} 5 & -7 \\ 7 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 30 \end{bmatrix}$$

29. ***Cartography*** A figure defined by the four points $P_1(x_1, y_1)$, $P_2(x_2, y_2)$, $P_3(x_3, y_3)$, and $P_4(x_4, y_4)$ is rotated through an angle θ by using the product

$$\begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ x_3 & y_3 \\ x_4 & y_4 \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

(a) Find the resulting matrix if the points $P_1(1, 2)$, $P_2(5, 4)$, $P_3(-2, 7)$, and $P_4(-1, -5)$ are rotated through an angle $\theta = \pi$.

(b) Find the resulting matrix if the points $P_1(1, 2)$, $P_2(5, 4)$, $P_3(-2, 7)$, and $P_4(-1, -5)$ are rotated through an angle $\theta = \frac{\pi}{2}$.

(c) What product would be used if five points $P_1(x_1, y_1)$, $P_2(x_2, y_2)$, $P_3(x_3, y_3)$, $P_4(x_4, y_4)$, and $P_5(x_5, y_5)$ were rotated through an angle θ?

[IN YOUR WORDS]

30. The text states that the commutative law does not hold for the multiplication of matrices.

(a) If $A = \begin{bmatrix} 1 & 2 & 5 \\ 6 & 8 & 9 \end{bmatrix}$ and $B = \begin{bmatrix} 5 & 4 & 2 \\ 0 & 1 & 2 \\ 7 & 2 & 0 \end{bmatrix}$, does $AB = BA$? Explain why or why not.

(b) If $C = \begin{bmatrix} 2 & 5 \\ 3 & 4 \end{bmatrix}$ and $D = \begin{bmatrix} 5 & 4 \\ -2 & 0 \end{bmatrix}$, does $CD = DC$? Explain why or why not.

31. Describe how to multiply two matrices on your calculator.

19.3 INVERSES OF MATRICES

One use of matrices is to help solve a system of linear equations. For example, if you had the system of linear equations

$$\begin{cases} a_1x_1 + a_2x_2 + a_3x_3 = k_1 \\ b_1x_1 + b_2x_2 + b_3x_3 = k_2 \\ c_1x_1 + c_2x_2 + c_3x_3 = k_3 \end{cases}$$

you could write these as the matrices:

$$\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix}$$

Another way to look at this would be to let the coefficients of the linear equations be represented by the matrix

$$A = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}$$

If X represents the column vector of variables, and K represents the column vector of constants, you have

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \text{ and } K = \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix}$$

and the system of equations could be represented by $AX = K$.

With normal equations you would divide both sides by A to get the values of X, but these are matrices and there is no commutative law for multiplication. We will return to this dilemma after we introduce a new matrix.

In Section 19.2, we introduced matrix multiplication. A square $n \times n$ matrix with a 1 in each position of the main diagonal and zeros elsewhere is called the $n \times n$ **identity matrix** and is denoted as I_n. Thus,

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \text{ and } I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

are both identity matrices.

If the size of the identity matrix is understood, we simply write I.

Now, back to our problem of $AX = K$. What we need is a matrix A^{-1}, where $A^{-1}A = I$. We would then find that

$$X = A^{-1}K$$

When A is an $n \times n$ matrix, then A is an **invertible** or **nonsingular matrix**, if there exists another $n \times n$ matrix, A^{-1}, where

$$A^{-1}A = AA^{-1} = I$$

If A^{-1} exists, it is called the **inverse of matrix A**. A matrix is called a **singular matrix** if it does not have an inverse.

There are two procedures for finding the inverse of a matrix. The first method may be easier on 2×2 matrices.

STEPS FOR FINDING THE INVERSE OF A 2 × 2 MATRIX

If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is a 2×2 matrix, then use the following four steps to determine A^{-1}:

1. Interchange the elements on the main diagonal.
2. Change the signs of the elements that are not on the main diagonal.
3. Find the determinant of the original matrix.
4. Divide each element at the end of Step 2 by the determinant of the original matrix.

NOTE Symbolically, we can summarize the four steps for finding the inverse of a 2×2 matrix as

$$A^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

EXAMPLE 19.16

Find the inverse of the matrix $A = \begin{bmatrix} 1 & 2 \\ 4 & 10 \end{bmatrix}$.

SOLUTION

Step 1: $\begin{bmatrix} 10 & 2 \\ 4 & 1 \end{bmatrix}$ Interchange elements on main diagonal.

Step 2: $\begin{bmatrix} 10 & -2 \\ -4 & 1 \end{bmatrix}$ Change signs of elements not on main diagonal.

Step 3: $\begin{vmatrix} 1 & 2 \\ 4 & 10 \end{vmatrix} = 10 - 8 = 2$ Find the determinant of the original matrix.

Step 4: $\begin{bmatrix} \frac{10}{2} & \frac{-2}{2} \\ \frac{-4}{2} & \frac{1}{2} \end{bmatrix}$ Divide each element from Step 2 by the determinant (Step 3).

The inverse is $A^{-1} = \begin{bmatrix} 5 & -1 \\ -2 & \frac{1}{2} \end{bmatrix}$. Check your answer. Is $A^{-1}A = I$?

$$\begin{bmatrix} 5 & -1 \\ -2 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 4 & 10 \end{bmatrix} = \begin{bmatrix} 5 - 4 & 10 - 10 \\ -2 + 2 & -4 + 5 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

The second method for determining the inverse of a matrix is slightly more complicated for a 2×2 matrix, but it will work for a matrix of any size.

With the second method, we are going to transform the given matrix into the identity matrix. At the same time, we will change an identity matrix into the inverse of the given matrix.

STEPS FOR FINDING THE INVERSE FOR ANY $n \times n$ MATRIX

The idea in this method for finding A^{-1} is to begin with the matrix $[A \;\vdots\; I]$ and use the following two valid matrix operations as necessary to change it to $[I \;\vdots\; A^{-1}]$.

1. Multiply or divide all elements in a row by a nonzero constant.
2. Add a constant multiple of the elements of one row to the corresponding elements of another row.

We demonstrate this method in Example 19.17 with the same matrix used in Example 19.16.

EXAMPLE 19.17

Find the inverse of $A = \begin{bmatrix} 1 & 2 \\ 4 & 10 \end{bmatrix}$.

SOLUTION First, we form the augmented matrix $[A \;\vdots\; I]$.

$$\left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 4 & 10 & 0 & 1 \end{array}\right]$$

In the following demonstration, we let R_1 mean Row 1 and R_2 mean Row 2 for this 2×4 matrix. The arrows point to the row that we changed and that row

EXAMPLE 19.17 (Cont.)

is always listed last. Everything we do is designed to change the left half of the matrix to the identity matrix.

$$\left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 4 & 10 & 0 & 1 \end{array}\right]$$

There is already a 1 in the a_{11} position. We need to get a 0 below it. First, multiply R_1 by -4 and add that to R_2 to get a new Row 2.

$$\left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & 2 & -4 & 1 \end{array}\right] \leftarrow -4R_1 + R_2$$

Now that the first column is $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$, we move to the second column to get it in the form $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

Next, multiply R_2 by -1 and add that to R_1. This is the new row 1.

$$\left[\begin{array}{cc|cc} 1 & 0 & 5 & -1 \\ 0 & 2 & -4 & 1 \end{array}\right] \begin{array}{l} \leftarrow -R_2 + R_1 \\ \end{array}$$

Now, all that is left is to divide Row 2 by 2 (or multiply it by $\frac{1}{2}$).

$$\left[\begin{array}{cc|cc} 1 & 0 & 5 & -1 \\ 0 & 1 & -2 & \frac{1}{2} \end{array}\right] \begin{array}{l} \leftarrow \frac{1}{2}R_2 \\ \end{array}$$

The left half of the matrix is the identity matrix and the right half is the inverse that we wanted.

$$A^{-1} = \begin{bmatrix} 5 & -1 \\ -2 & \frac{1}{2} \end{bmatrix}$$

This is the same answer we got for Example 19.16.

We will work another 2×2 example and then work a 3×3 example. Before beginning, you should always check to see if the matrix is invertible. It will not be invertible if every element in any row or column is zero. Thus

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 4 & 5 & 6 \end{bmatrix} \text{ and } \begin{bmatrix} 1 & 2 & 0 \\ 4 & 5 & 0 \\ 7 & 8 & 0 \end{bmatrix}$$

cannot be inverted. A matrix cannot be inverted if every element in one row (or column) is a constant multiple of every corresponding element in another row (or column). Thus $\begin{bmatrix} 4 & 2 \\ 12 & 6 \end{bmatrix}$ cannot be inverted, because each element in row 2 is 3 times its corresponding element in row 1. Notice that a matrix that cannot be inverted has a determinant of 0.

EXAMPLE 19.18

Find the inverse of $B = \begin{bmatrix} 8 & 4 \\ 3 & 2 \end{bmatrix}$.

SOLUTION

$$\left[\begin{array}{cc|cc} 8 & 4 & 1 & 0 \\ 3 & 2 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{cc|cc} 8 & 4 & 1 & 0 \\ 0 & 4 & -3 & 8 \end{array}\right] \begin{array}{l} \\ \leftarrow -3R_1 + 8R_2 \end{array}$$

$$\left[\begin{array}{cc|cc} 8 & 0 & 4 & -8 \\ 0 & 4 & -3 & 8 \end{array}\right] \begin{array}{l} \leftarrow -R_2 + R_1 \\ \\ \end{array}$$

$$\left[\begin{array}{cc|cc} 1 & 0 & \frac{1}{2} & -1 \\ 0 & 1 & -\frac{3}{4} & 2 \end{array}\right] \begin{array}{l} \leftarrow R_1 \div 8 \\ \leftarrow R_2 \div 4 \end{array}$$

The inverted matrix is $B^{-1} = \begin{bmatrix} \frac{1}{2} & -1 \\ -\frac{3}{4} & 2 \end{bmatrix}$.

EXAMPLE 19.19

Find the inverse of $C = \begin{bmatrix} 7 & -8 & 5 \\ -4 & 5 & -3 \\ 1 & -1 & 1 \end{bmatrix}$.

SOLUTION

$$\left[\begin{array}{ccc|ccc} 7 & -8 & 5 & 1 & 0 & 0 \\ -4 & 5 & -3 & 0 & 1 & 0 \\ 1 & -1 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 7 & -8 & 5 & 1 & 0 & 0 \\ -4 & 5 & -3 & 0 & 1 & 0 \\ 0 & -1 & -2 & 1 & 0 & -7 \end{array}\right] \begin{array}{l} \\ \\ \leftarrow R_1 - 7R_3 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 7 & -8 & 5 & 1 & 0 & 0 \\ 0 & 3 & -1 & 4 & 7 & 0 \\ 0 & -1 & -2 & 1 & 0 & -7 \end{array}\right] \begin{array}{l} \\ \leftarrow 4R_1 + 7R_2 \\ \\ \end{array}$$

$$\left[\begin{array}{ccc|ccc} 7 & -8 & 5 & 1 & 0 & 0 \\ 0 & 3 & -1 & 4 & 7 & 0 \\ 0 & 0 & -7 & 7 & 7 & -21 \end{array}\right] \begin{array}{l} \\ \\ \leftarrow R_2 + 3R_3 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 7 & -8 & 5 & 1 & 0 & 0 \\ 0 & -21 & 0 & -21 & -42 & -21 \\ 0 & 0 & -7 & 7 & 7 & -21 \end{array}\right] \begin{array}{l} \\ \leftarrow R_3 - 7R_2 \\ \\ \end{array}$$

$$\left[\begin{array}{ccc|ccc} 7 & -8 & 5 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & -1 & -1 & 3 \end{array}\right] \begin{array}{l} \\ \leftarrow R_2 \div -21 \\ \leftarrow R_3 \div -7 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 7 & -8 & 0 & 6 & 5 & -15 \\ 0 & 1 & 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & -1 & -1 & 3 \end{array}\right] \begin{array}{l} \leftarrow -5R_3 + R_1 \\ \\ \\ \end{array}$$

EXAMPLE 19.19 (Cont.)

$$\left[\begin{array}{ccc|ccc} 7 & 0 & 0 & 14 & 21 & -7 \\ 0 & 1 & 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & -1 & -1 & 3 \end{array}\right] \leftarrow 8R_2 + R_1$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & 3 & -1 \\ 0 & 1 & 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & -1 & -1 & 3 \end{array}\right] \leftarrow R_1 \div 7$$

The inverse of C is $C^{-1} = \begin{bmatrix} 2 & 3 & -1 \\ 1 & 2 & 1 \\ -1 & -1 & 3 \end{bmatrix}$.

INVERTING A MATRIX WITH A GRAPHING CALCULATOR

Using a graphing calculator to invert a matrix is much simpler than the process used in Examples 19.16 through 19.19.

USING A GRAPHING CALCULATOR TO INVERT A MATRIX

1. Define the matrix as described in Section 6.4
2. Enter the name of the matrix
3. Press x^{-1} ENTER

The next example shows how to find the inverse of a matrix by using a graphing calculator.

EXAMPLE 19.20

Use a graphing calculator to find the inverse of $B = \begin{bmatrix} 8 & 4 \\ 3 & 2 \end{bmatrix}$.

SOLUTION Enter matrix B as described in the user's guide for your calculator, or as described in Section 6.4. Let this matrix be named B.

Quit the edit matrix mode and return to the home screen. Now press 2nd MATRIX 2 x^{-1} ENTER. The result should be similar to the one shown in Figure 19.1a.

If your calculator can convert decimals to fractions, you might want to convert B^{-1} to fractional form. For example, on a TI-83 or TI-84, press the ▶Frac key. The result, shown in Figure 19.1b, matches the result we got in Example 19.18.

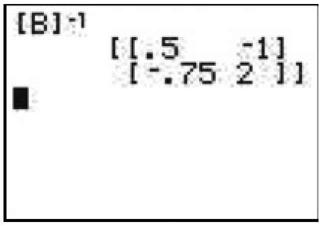


Figure 19.1a

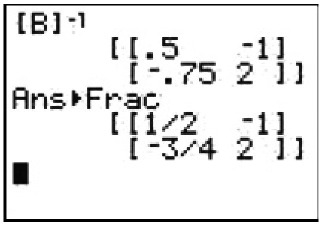


Figure 19.1b

INVERSES OF MATRICES WITH A SPREADSHEET

EXAMPLE 19.21

Use a spreadsheet to find the inverse of $B = \begin{bmatrix} 8 & 4 \\ 3 & 2 \end{bmatrix}$.

SOLUTION Enter matrix B in Cells C1 through D2 as shown in Figure 19.2a.

In order to display each cell of the inverse matrix, Excel requires two functions. The `MINVERSE` function returns the inverse matrix as an array of numbers. The `INDEX` function returns the contents of the cell defined by rows and columns.

In Cell C4, enter=`INDEX(MINVERSE(C1:D2),1,1)`. The C1:D2 portion of the formula identifies the matrix B in Cells C1 through D2. The 1, 1 portion of the formula identifies the ROW and COLUMN of the cell in the inverse matrix we want to write in this cell. (See Figure 19.2b.)

In Cell D4, enter =`INDEX(MINVERSE(C1:D2),1,2)`. (See Figure 19.2c.)

The 1, 2 portion of the formula identifies the first row and second column as the cell in the inverse we wish to write. The process is repeated for the other two cells (see Figures 19.2d and 19.2e.)

To convert all the entries to fractions, select the array and, under `Format`, select `Cells`. Select the `Number` tab and select `Fraction` to convert the decimal representations to fractions. The result is shown in Figure 19.2f.

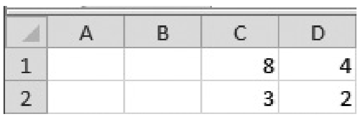


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Figure 19.2a

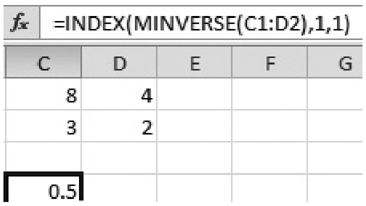

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Figure 19.2b

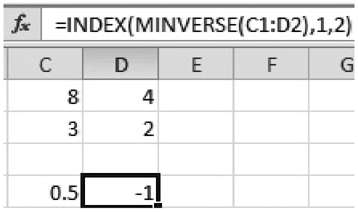

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Figure 19.2c

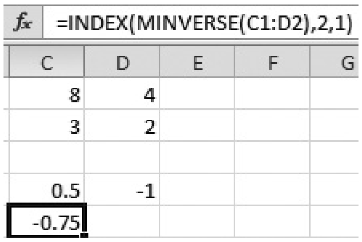

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Figure 19.2d

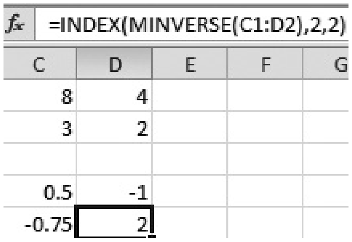

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Figure 19.2e

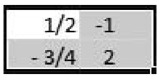

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Figure 19.2f

EXERCISE SET 19.3

In Exercises 1–20, if the given matrix is invertible, find its inverse.

1. $\begin{bmatrix} 6 & 1 \\ 5 & 1 \end{bmatrix}$ **2.** $\begin{bmatrix} 5 & 1 \\ 9 & 2 \end{bmatrix}$ **3.** $\begin{bmatrix} 10 & 4 \\ 8 & 3 \end{bmatrix}$ **4.** $\begin{bmatrix} 9 & 6 \\ 4 & 3 \end{bmatrix}$

5. $\begin{bmatrix} 8 & -6 \\ -6 & 4 \end{bmatrix}$

6. $\begin{bmatrix} 12 & 9 \\ -9 & -7 \end{bmatrix}$

7. $\begin{bmatrix} 15 & 10 \\ 4 & 3 \end{bmatrix}$

8. $\begin{bmatrix} 15 & 10 \\ 3 & 4 \end{bmatrix}$

9. $\begin{bmatrix} 0 & -3 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 4 \end{bmatrix}$

10. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

11. $\begin{bmatrix} 1 & 2 & 6 \\ 0 & 0 & 2 \\ -3 & -6 & -9 \end{bmatrix}$

12. $\begin{bmatrix} 4 & 5 & 1 \\ 1 & 0 & 1 \\ 4 & 5 & 1 \end{bmatrix}$

13. $\begin{bmatrix} 8 & 7 & -1 \\ -5 & -5 & 1 \\ -4 & -4 & 1 \end{bmatrix}$

14. $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 7 \\ 1 & 1 & 1 \end{bmatrix}$

15. $\begin{bmatrix} 3 & -1 & 0 \\ -6 & 2 & 0 \\ 1 & 0 & 5 \end{bmatrix}$

16. $\begin{bmatrix} 1 & -1 & 1 \\ 7 & -8 & 5 \\ -4 & 5 & -3 \end{bmatrix}$

17. $\begin{bmatrix} 2 & 0 & 0 \\ 2 & 2 & 0 \\ 2 & 2 & 2 \end{bmatrix}$

18. $\begin{bmatrix} 3 & 1 & -1 \\ 1 & -2 & 0 \\ 0 & 3 & 1 \end{bmatrix}$

19. $\begin{bmatrix} 1 & -1 & 1 \\ 0 & 2 & -1 \\ 2 & 3 & 0 \end{bmatrix}$

20. $\begin{bmatrix} 1 & 2 & -1 \\ 2 & -2 & 1 \\ 6 & 4 & 3 \end{bmatrix}$

Solve Exercise 21.

21. Check your answers to 1–20 by multiplying the given matrix and its inverse. Remember, if the matrix in the problem is called A and your answer A^{-1}, then $AA^{-1} = I$.

[IN YOUR WORDS]

22. Not all matrices have an inverse. List the conditions that will tell you if a matrix is not invertible.

23. If your calculator can work with complex numbers,

(a) describe how to find the inverse of the matrix $A = \begin{bmatrix} i & 4+3i \\ 3 & 2-2i \end{bmatrix}$, and **(b)** compute A^{-1}.

19.4 MATRICES AND LINEAR EQUATIONS

Earlier in this chapter, we said that a system of equations could be represented by matrices. For example, the system

$$\begin{cases} 2x + 3y = 5 \\ 3x + 5y = 9 \end{cases}$$

would be represented by three matrices A, X, and K, where $AX = K$. To do this, let A be the matrix formed by the coefficients of the variables, let X be a column vector of the variables, and let K be a column vector of the constants. Thus, for this system, we have

$$A = \begin{bmatrix} 2 & 3 \\ 3 & 5 \end{bmatrix} \qquad X = \begin{bmatrix} x \\ y \end{bmatrix} \qquad K = \begin{bmatrix} 5 \\ 9 \end{bmatrix}$$

Now that we can invert a matrix, we can find A^{-1}. Since $A^{-1}A = I$ then $A^{-1}(AX) = (A^{-1}A)X = IX = X$. With this knowledge, we can find the solution to our system of equations.

$$AX = K$$
$$A^{-1}(AX) = A^{-1}K$$
$$X = A^{-1}K$$

EXAMPLE 19.22

Use the inverse of the coefficient matrix to solve the linear system

$$\begin{cases} 2x + 3y = 5 \\ 3x + 5y = 9 \end{cases}$$

SOLUTION We can think of this as the matrix equation $AX = K$, with $A = \begin{bmatrix} 2 & 3 \\ 3 & 5 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \end{bmatrix}$, and $K = \begin{bmatrix} 5 \\ 9 \end{bmatrix}$.

Using a calculator

On a TI-83 or TI-84 you cannot name a matrix K, so we will use C, for constant. Figure 19.3a shows that the solution to this matrix is $X = A^{-1}C = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$

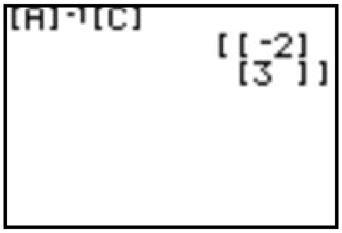

Figure 19.3a

Using a spreadsheet

We can think of this as the matrix equation $AX = K$ with $A = \begin{bmatrix} 2 & 3 \\ 3 & 5 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \end{bmatrix}$, and $K = \begin{bmatrix} 5 \\ 9 \end{bmatrix}$, as shown in Figure 19.3b.

To multiply two matrices, we use the `MMULT` function. The `MMULT` function returns an array (or list) of numbers, so again we must use the `INDEX` function to obtain each individual cell.

We know the result of the multiplication is a 2×1 matrix. So, move the cursor to Cell C7 and enter the appropriate function (see Figure 19.3c). Repeating this is Cell C8, we get the second value in the answer.

The result, shown in Figure 19.3d, means that $x = -2$ and $y = 3$.

B	C	D	E	F	G
Matrix A:	2	3			
	3	5			
Matrix A^-1:	5	-3		Matrix K:	5
	-3	2			9

Figure 19.3b

f_x =INDEX(MMULT(C4:D5,G4:G5),1,1)

B	C	D	E	F
X=A^(-1)K:	-2			

Figure 19.3c

f_x =INDEX(MMULT(C4:D5,G4:G5),2,1)

B	C	D	E	F
X=A^(-1)K:	-2			
	3			

Figure 19.3d

Let's try this method for solving a system of linear equations on a system of three equations with three variables.

EXAMPLE 19.23

Use the inverse of the coefficient matrix to solve the linear system

$$\begin{cases} 7x - 8y + 5z = 18 \\ -4x + 5y - 3z = -11 \\ x - y + z = 1 \end{cases}$$

SOLUTION This linear system can be represented by the three matrices A, X, and K, where $AX = K$. Again, A is the matrix formed by the coefficients, X is the column vector of the variables, and K is the column vector of the constants. From this system we have

$$A = \begin{bmatrix} 7 & -8 & 5 \\ -4 & 5 & -3 \\ 1 & -1 & 1 \end{bmatrix} \qquad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \qquad K = C\begin{bmatrix} 18 \\ -11 \\ 1 \end{bmatrix}$$

Using a calculator

We enter matrices A and C in the calculator. As you can see in Figure 19.4a, the solution to this equation is $X = A^{-1}C = \begin{bmatrix} 2 \\ -3 \\ -4 \end{bmatrix}$.

Using a spreadsheet

Figure 19.4b shows matrix A, X, and K.

(The dollar signs in the formula are used to "anchor" the cells from which we take the inverse [C1:E3] so that when the formula is copied from Cell C5 to the other eight cells, the reference will remain valid.)

The result, shown in Figure 19.4c, shows that $x = 2$, $y = -3$, and $z = -4$.

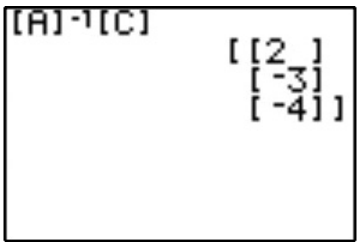

Figure 19.4a

C5 | f_x =INDEX(MINVERSE($C$1:$E$3),1,1)

	A	B	C	D	E	F	G	H
1		Matrix A:	7	-8	5			
2			-4	5	-3			
3			1	-1	1			
4								
5		Matrix A^-1:	2	3	-1		Matrix K:	18
6			1	2	1			-11
7			-1	-1	3			1

Figure 19.4b

C9 f_x =INDEX(MMULT($C$5:$E$7,$H$5:$H$7),1,1)

	A	B	C	D	E	F	G	H	I
1		Matrix A:	7	-8	5				
2			-4	5	-3				
3			1	-1	1				
4									
5		Matrix A^-1:	2	3	-1		Matrix K:	18	
6			1	2	1			-11	
7			-1	-1	3			1	
8									
9		X=A^(-1)K:	2						
10			-3						
11			-4						

Figure 19.4c

APPLICATION ELECTRONICS

EXAMPLE 19.24

Apply Kirchhoff's laws to the circuit in Figure 19.5 and determine the currents I_1, I_2, and I_3.

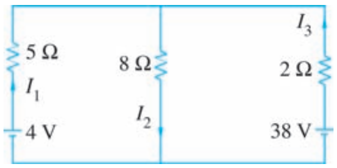

Figure 19.5

SOLUTION If we apply Kirchhoff's laws to the circuit in Figure 19.5, we obtain the following system of linear equations:

$$\begin{cases} I_1 - I_2 + I_3 = 0 \\ 5I_1 + 8I_2 = 4 \\ 8I_2 + 2I_3 = 38 \end{cases}$$

The coefficient matrix for this system is $A = \begin{bmatrix} 1 & -1 & 1 \\ 5 & 8 & 0 \\ 0 & 8 & 2 \end{bmatrix}$. (Notice that the coefficients of the missing variables are 0.) If the constant matrix is $B = \begin{bmatrix} 0 \\ 4 \\ 38 \end{bmatrix}$ and $X = \begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix}$, then

$$X = A^{-1}B$$

As before, we enter matrices A and B in a calculator or spreadsheet. As you can see from the calculator screen in Figure 19.6, the solution to this equation is $X = A^{-1}B = \begin{bmatrix} -4 \\ 3 \\ 7 \end{bmatrix}$. (A spreadsheet gives the same result.) Thus, the three currents are $I_1 = -4$ A, $I_2 = 3$ A, and $I_3 = 7$ A.

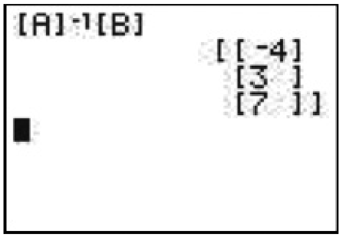

Figure 19.6

EXERCISE SET 19.4

In Exercises 1–10, solve the given systems of equations by using the inverse of the coefficient matrix.

1. $\begin{cases} 6x + y = -4 \\ 5x + y = -3 \end{cases}$

2. $\begin{cases} 10x + 4y = 8 \\ 8x + 3y = 7 \end{cases}$

3. $\begin{cases} 8x - 6y = -27 \\ -6x + 4y = 19 \end{cases}$

4. $\begin{cases} 15x + 10y = -5 \\ 4x + 3y = 0 \end{cases}$

5. $\begin{cases} x + 2y + 6z = 12 \\ 2z = 2 \\ -3x - 6y - 9z = -27 \end{cases}$

6. $\begin{cases} 8x + 7x - z = 9 \\ -5x - 5y + z = -1 \\ -4x - 4y + z = 0 \end{cases}$

7. $\begin{cases} x + 2y + 3z = 4 \\ 2x + 5y + 7z = 7.5 \\ x + y + z = 1 \end{cases}$

8. $\begin{cases} x - y + z = -6.6 \\ 7x - 8y + 5z = -43 \\ -4x + 5y - 3z = 26.9 \end{cases}$

9. $\begin{cases} 2x = 11 \\ 2x + 2y = 4 \\ 2x + 2y + 2z = -9 \end{cases}$

10. $\begin{cases} x - y + z = 22 \\ 2y - z = -23 \\ 2x + 3y = -11.2 \end{cases}$

In Exercises 11–20, solve each system of equations by using the inverse of the coefficient matrix.

11. $\begin{cases} 2x + y = 1 \\ -3x + 2y = 16 \end{cases}$

12. $\begin{cases} 4x + 5y = 2 \\ 3x - 2y = 13 \end{cases}$

13. $\begin{cases} 2x + 2y = 4 \\ 4x + 3y = 1 \end{cases}$

14. $\begin{cases} 3x + y = -5 \\ 4x + 3y = 2 \end{cases}$

15. $\begin{cases} 1.5x + 2.5y = 0.3 \\ 3.2x + 2.6y = 7.2 \end{cases}$

16. $\begin{cases} 7x + 2y + z = 2 \\ 3x - 2y + 4z = 13 \\ 4x + 5y - z = 1 \end{cases}$

17. $\begin{cases} 5x + 2y + 3z = 1 \\ -3x + 2y - 8z = 6 \\ 4x - 2y + 9z = -7 \end{cases}$

18. $\begin{cases} 2x + 4y + z = 10 \\ 4x + 2y + z = 8 \\ 6x + 4y + 7z = -2 \end{cases}$

19. $\begin{cases} x + 2y + 4z = 7 \\ 3x + y + 4z = -2 \\ 2x + 9y - 2z = 10 \end{cases}$

20. $\begin{cases} 2x - y + z = 7 \\ 4x - 2y + z = 5 \\ 6x - 3y + 5z = -3 \end{cases}$

Solve Exercises 21–30.

21. ***Electronics*** If Kirchhoff's laws are applied to the circuit in Figure 19.7, the following equations are obtained. Determine the indicated currents.

$$\begin{aligned} I_A - I_B - I_C &= 0 \\ 16I_A + 4I_C &= 100 \\ 12I_B - 4I_C &= 60 \end{aligned}$$

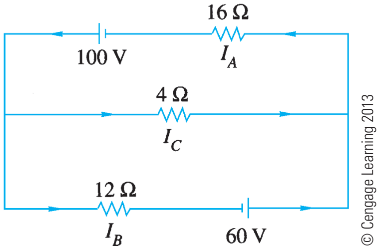

Figure 19.7

22. ***Electronics*** Applying Kirchhoff's laws to the circuit in Figure 19.8 results in the following equations. Determine the indicated currents.

$$\begin{aligned} I_3 - I_1 - I_2 &= 0 \\ 20I_1 + 0.5I_1 - 15I_2 - 0.4I_2 &= 120 - 80 \\ 15I_2 + 0.4I_2 + 10I_3 + 0.6I_3 &= 140 \end{aligned}$$

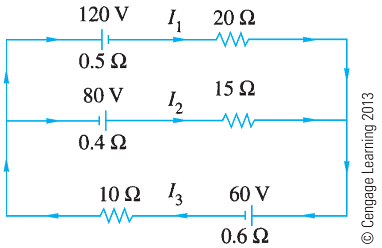

Figure 19.8

23. ***Electronics*** The currents through the resistors in Figure 19.9 produce the following equations. Determine the currents.

$$\begin{aligned} 7.18I_1 - I_2 + 2.2I_3 &= 10 \\ -I_1 + 5.8I_2 + 1.5I_3 &= 15 \\ 2.2I_1 + 1.5I_2 + 8.4I_3 &= 20 \end{aligned}$$

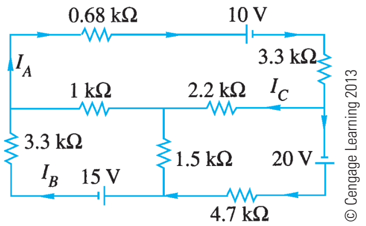

Figure 19.9

24. ***Business*** A company manufactures robotic controls. Its current models are the *RC*-1 and *RC*-2. Each *RC*-1 unit requires eight transistors and four integrated circuits. Each *RC*-2 unit uses nine transistors and five integrated circuits. Each day the company receives 1,596 transistors and 860 integrated circuits. How many units of each model can be made if all parts are used?

25. ***Metallurgy*** An alloy is composed of three metals, A, B, and C. The percentages of each metal are indicated by the following system of equations:

$$\begin{cases} A + B + C = 100 \\ A - 2B = 0 \\ -4A + C = 0 \end{cases}$$

Determine the percentage of each metal in the alloy.

26. ***Automotive technology*** A petroleum engineer was testing three different gasoline mixtures, A, B, and C, in the same car and under the same driving conditions. She noticed that the car traveled 90 km farther when it used mixture B than when it used mixture A. Using fuel C, the car traveled 130 km more than when it used fuel B. The total distance traveled was 1,900 km. Find the distance traveled on the three fuels.

27. ***Construction technology*** If three cables are joined at a point and three forces are applied so the system is in equilibrium, the following system of equations results.

$$\begin{cases} \frac{6}{7}F_B - \frac{2}{3}F_C = 2{,}000 \\ -F_A - \frac{3}{7}F_B + \frac{1}{3}F_C = 0 \\ \frac{2}{7}F_B + \frac{2}{3}F_C = 1{,}200 \end{cases}$$

Determine the three forces, F_A, F_B, and F_C, measured in newtons (N).

28. ***Environmental science*** To control ice and protect the environment, a certain city determines that the best mixture to be spread on roads consists of 5 units of salt, 6 units of sand, and 4 units of a chemical inhibiting agent. Three companies, A, B, and C, sell mixtures of these elements according to the following table:

	Salt	Sand	Inhibiting agent
Company A	2	1	1
Company B	2	2	2
Company C	1	5	1

(a) In what proportion should the city purchase from each company in order to spread the best mixture? (Assume that the city must buy, complete truckloads.)

(b) If the city expects to need 3,630,000 units for the winter, how many units should be bought from each company?

29. ***Automotive technology*** The relationship between the velocity, v, of a car (in mph) and the distance, d (in ft), required to bring it to a complete stop is known to be of the form $d = av^2 + bv + c$, where a, b, and c are constants. Use the following data to determine the values of a, b, and c. When $v = 20$, then $d = 40$; when $v = 55$, then $d = 206.25$; and when $v = 65$, then $d = 276.25$.

30. ***Metallurgy*** An alloy is composed of four metals, A, B, C, and D. The percentages of each metal are indicated by the following system of equations:

$$\begin{cases} A + B + C + D = 100 \\ A + B - C = 0 \\ -1.64A + D = 0 \\ 3A - 2C + 2D = -1 \end{cases}$$

Determine the percentage of each metal in the alloy.

[IN YOUR WORDS]

31. What conditions are necessary in order to use matrices to solve a system of equations?

32. Describe how to use your calculator to solve the system of linear equations $AX = K$, where A, X, and K are matrices.

CHAPTER 19 REVIEW

IMPORTANT TERMS AND CONCEPTS

Augmented matrix
Coefficient matrix
Column vector
Dimension
Identity matrix
Inverse of a matrix
Invertible matrix
Matrix
  Addition
  Multiplication
  Scalar multiplication
  Subtraction
Nonsingular matrix
Row vector
Scalar
Scalar multiplication
Singular matrix
Square matrix
Zero matrix

REVIEW EXERCISES

In Exercises 1–6, use the following matrices to determine the indicated matrix:

$$A = \begin{bmatrix} 4 & 3 & 2 & 5 \\ 6 & 7 & -1 & 4 \\ 9 & 10 & -8 & 3 \end{bmatrix} \quad B = \begin{bmatrix} 3 & -2 & 1 & 0 \\ 5 & -1 & 2 & 0 \\ 4 & 3 & -2 & 0 \end{bmatrix} \quad C = \begin{bmatrix} 0 & 1 & 0 & 2 \\ 3 & 0 & 4 & 0 \\ 0 & -5 & 0 & -6 \end{bmatrix}$$

1. $A + C$

2. $B + C$

3. $A - B$

4. $C - B$

5. $2A - 3C$

6. $4A + B - 2C$

In Exercises 7–10, find the indicated products.

7. $\begin{bmatrix} 3 & 2 \\ 4 & 5 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$

8. $\begin{bmatrix} 1 & -4 \\ 5 & 1 \end{bmatrix}\begin{bmatrix} 3 & 4 & 1 \\ 2 & 5 & 0 \end{bmatrix}$

9. $\begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix}\begin{bmatrix} 4 & 5 & 1 \end{bmatrix}$

10. $\begin{bmatrix} 4 & 5 & 1 \end{bmatrix}\begin{bmatrix} -2 \\ 7 \\ -3 \end{bmatrix}$

In Exercises 11–14, if the given matrix is invertible, find its inverse.

11. $\begin{bmatrix} 2 & 3 \\ -4 & -5 \end{bmatrix}$

12. $\begin{bmatrix} 2 & -1 \\ 0 & 4 \end{bmatrix}$

13. $\begin{bmatrix} -2 & 1 & 0 \\ 0 & 4 & 0 \\ 1 & 0 & 1 \end{bmatrix}$

14. $\begin{bmatrix} 2 & 3 & -1 \\ 1 & 2 & 1 \\ -1 & -1 & 3 \end{bmatrix}$

In Exercises 15–18, solve the system of equations by using the inverse of the coefficient matrix.

15. $\begin{cases} 12x + 5y = -2 \\ 3x + y = 1.1 \end{cases}$

16. $\begin{cases} 4x + y = -4 \\ -3x + 2y = 14 \end{cases}$

17. $\begin{cases} 2x + 3y + 5z = 20 \\ -2x + 3y + 5z = 12 \\ 5x - 3y - 2z = 9 \end{cases}$

18. $\begin{cases} x + y + 6z = -3 \\ -2x + 2y + 4z = -2 \\ 3x + 2y + 4z = 14 \end{cases}$

Solve Exercises 19–24.

19. ***Physics*** Masses of 9 and 11 kg are attached to a cord that passes over a frictionless pulley. When the masses are released, the acceleration a of each mass, in meters per second squared (m/s^2), and the tension T in the cord, in newtons (N), are related by the system

$$\begin{cases} T - 75.4 = 9a \\ 100.0 - T = 11a \end{cases}$$

Find a and T.

20. ***Business*** A computer company makes three types of computers—a personal computer (PC), a business computer (BC), and a technical computer (TC). There are three parts in each computer that it has difficulty getting: RAM chips, EPROMS, and transistors. The number of each part needed by each computer is shown in this table.

	RAM	EPROM	Transistor
PC	4	2	7
BC	8	3	6
TC	12	5	11

If the company is guaranteed 1,872 RAM chips, 771 EPROMS, and 1,770 transistors each week, how many of each computer can be made?

21. Given the equations

$$\begin{cases} x' = \frac{1}{2}(x + y\sqrt{3}) \\ y' = \frac{1}{2}(-x\sqrt{3} + y) \end{cases}$$

and

$$\begin{cases} x'' = \frac{1}{2}(-x' + y'\sqrt{3}) \\ y'' = -\frac{1}{2}(x'\sqrt{3} + y') \end{cases}$$

write each set as a matrix equation and solve for x'' and y'' in terms of x and y by multiplying matrices.

22. The equations in Exercise 21 represent rotations of axes in two directions. In Section 15.7, we found that, if the angle of rotation is θ, then

$$\begin{cases} x'' = x\cos\theta + y\sin\theta \\ y'' = -x\sin\theta + y\cos\theta \end{cases}$$

What was the rotation angle for the equations in Exercise 21?

23. ***Optics*** The following matrix product is used in discussing two thin lenses in air:

$$M = \begin{bmatrix} 1 & -\frac{1}{f_2} \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ d & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & -\frac{1}{f_1} \\ 0 & 1 \end{bmatrix}$$

where f_1 and f_2 are the focal lengths of the lenses and d is the distance between them. Evaluate M.

24. ***Optics*** In Exercise 23, element M_{12} of M is $-\frac{1}{f}$, where f is the focal length of the combination. Determine $\frac{1}{f}$.

CHAPTER 19 TEST

1. Given $A = \begin{bmatrix} 8 & 0 & -4 \\ 16 & -6 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 5 & -3 \\ 3 & 0 & 4 \end{bmatrix}$, find
(a) $A + B$
(b) $3A - 2B$

2. Calculate the product $\begin{bmatrix} 1 & -2 & 3 \end{bmatrix}\begin{bmatrix} -4 \\ -6 \\ 8 \end{bmatrix}$.

3. If $C = \begin{bmatrix} 4 & 6 \\ -10 & 4 \end{bmatrix}$ and $D = \begin{bmatrix} \frac{1}{2} & 0 \\ -\frac{3}{2} & 4 \end{bmatrix}$, calculate **(a)** CD and **(b)** DC.

4. If $E = \begin{bmatrix} 2 & -3 \\ 7 & 9 \end{bmatrix}$, **(a)** find E^{-1}. **(b)** Use E^{-1} to solve $EX = F$, where $F = \begin{bmatrix} -31 \\ 28 \end{bmatrix}$.

5. Solve the following system of equations by using the inverse of the coefficient matrix:

$$\begin{cases} x + 3y + z = -2 \\ 2x + 5y + z = -5 \\ x + 2y + 3z = 6 \end{cases}$$

6. Three machine parts cost a total of \$60. The first part costs as much as the other two together. The cost of the second part is \$3 more than twice the cost of the third part. How much does each part cost?

7. Find the currents of the system in Figure 19.10.

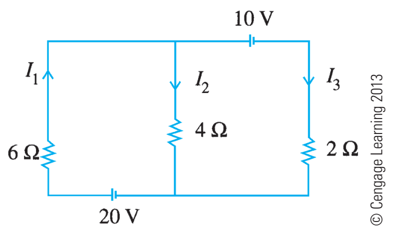

Figure 19.10

20 SEQUENCES, SERIES, AND THE BINOMIAL FORMULA

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How much insulation is needed to reduce fuel consumption by 20%? The solution to this question requires knowledge of geometric sequences, a topic we will study in Section 20.2.

Sequences have played an important role in advanced mathematics. Many natural and physical patterns can be described by a sequence of numbers. In this chapter, we will study two specific kinds of sequences—arithmetic and geometric. We will also study the sum of a sequence, which is called a series. Both sequences and series are studied in more detail in calculus.

OBJECTIVES

AFTER COMPLETING THIS CHAPTER, YOU WILL BE ABLE TO:

- Identify various types of sequences and series.
- Write the general term or a recursion formula for a given series.
- Compute any term or the sum of any number of terms of an arithmetic progression or a geometric progression.
- Compute any term of a harmonic progression.
- Compute the sum of an infinite geometric progression.
- Solve applications involving series.
- Raise a binomial to a power using the binomial theorem.
- Find any term in a binomial expansion.

20.1 SEQUENCES

A **sequence** is a set of numbers arranged in some order. Each number in the sequence is labeled with a variable, such as a. The variable is indexed with a natural number that tells its position in the sequence. The numbers a_1, a_2, a_3, and so on are the *terms* of the sequence. So, the first term in the sequence is a_1, the second term a_2, the third term a_3, and so on.

EXAMPLE 20.1

The sequence 3, 7, 11, 15, 19, 23 has $a_1 = 3$, $a_2 = 7$, $a_3 = 11$, $a_4 = 15$, $a_5 = 19$, and $a_6 = 23$. This sequence has six terms.

Many sequences follow some sort of pattern. The pattern is usually described by the nth term of the sequence. This term, a_n, is called the *general term* of the sequence. A **finite sequence** has a specific number of terms and so it has a last term. An **infinite sequence** does not have a last term. The notation $\{a_n\}$ is often used to present the nth term of a sequence. The { } indicate that it is a sequence.

EXAMPLE 20.2

Find the first six terms of the sequence $a_n = 4n + 1$.

SOLUTION

$$a_1 = 4(1) + 1 = 5$$
$$a_2 = 4(2) + 1 = 9$$
$$a_3 = 4(3) + 1 = 13$$
$$a_4 = 4(4) + 1 = 17$$

$$a_5 = 4(5) + 1 = 21$$
$$a_6 = 4(6) + 1 = 25$$

The first six terms of this sequence are 5, 9, 13, 17, 21, and 25.

EXAMPLE 20.3

Find the first five terms of the sequence $\{n^2 - 3\}$.

SOLUTION

$$a_1 = 1^2 - 3 = 1 - 3 = -2$$
$$a_2 = 2^2 - 3 = 4 - 3 = 1$$
$$a_3 = 3^2 - 3 = 9 - 3 = 6$$
$$a_4 = 4^2 - 3 = 16 - 3 = 13$$
$$a_5 = 5^2 - 3 = 25 - 3 = 22$$

The first five terms of this sequence are -2, 1, 6, 13, and 22.

EXAMPLE 20.4

Find the first seven terms of the sequence $\left\{\frac{n}{n+2}\right\}$.

SOLUTION

$$a_1 = \frac{1}{1+2} = \frac{1}{3}$$
$$a_2 = \frac{2}{2+2} = \frac{2}{4} = \frac{1}{2}$$
$$a_3 = \frac{3}{3+2} = \frac{3}{5}$$
$$a_4 = \frac{4}{4+2} = \frac{4}{6} = \frac{2}{3}$$
$$a_5 = \frac{5}{5+2} = \frac{5}{7}$$
$$a_6 = \frac{6}{6+2} = \frac{6}{8} = \frac{3}{4}$$
$$a_7 = \frac{7}{7+2} = \frac{7}{9}$$

The first seven terms of this sequence are $\frac{1}{3}, \frac{1}{2}, \frac{3}{5}, \frac{2}{3}, \frac{5}{7}, \frac{3}{4}$, and $\frac{7}{9}$.

SEQUENCES ON A CALCULATOR

There are two ways that you can use a TI-83 or TI-84 calculator to display the terms of a sequence. One method uses the **LIST** key and the other uses the **TABLE** feature.

USING THE LIST KEY

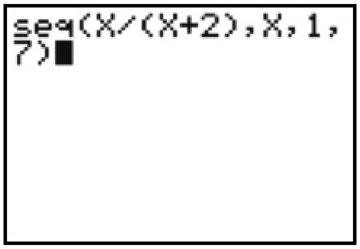


Figure 20.1a

The following steps show how to use the LIST key to display the first seven terms of the sequence in Example 20.4: $\left\{\frac{n}{n+2}\right\}$. To do this, use the keystrokes 2nd LIST ▶ [OPS] 5 [5: seq(] x/(x + 2) , X, T, θ, n , 1 , 7). Your calculator screen should like Figure 20.1a and when you press ENTER you should get the result shown in Figure 20.1b

While this is the same result we obtained in Example 20.4, it does not look the same. Let's convert it to fraction notation by pressing MATH ENTER or 1). This produces the result in Figure 20.1c.

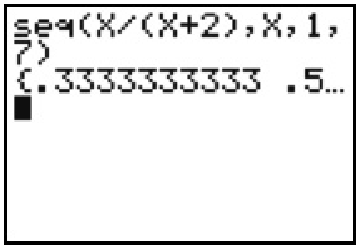


Figure 20.1b

THE TABLE FEATURE:

Before you can use this method you have to set the calculator in "sequence" mode by pressing MODE, moving the cursor to the right end of the fourth line, and pressing ENTER. Next you have to define the sequence in the Y = Editor, so press Y= and enter the sequence definition. When you press the X, T, θ, n an n will be displayed. The result should look like the one in Figure 20.1d.

Next, press 2nd TBLSET and set `TblStart` and $\triangle$`Tbl` to 1. When this is done, press 2nd TABLE and you should see the result in Figure 20.1e. While this is the same result we obtained in Example 20.4, it does not look the same because it is in decimal rather than fraction form.

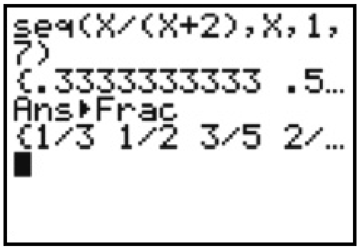


Figure 20.1c

One other nice feature of the calculator is that you can insert a "step" size in the calculator command. Thus, you could ask it to calculate every third term of the sequence. You do this by placing a comma and the step size before the last parenthesis in the command.

SEQUENCES ON A SPREADSHEET:

In Figure 20.1 the number of each term is displayed in Column A. The sequence is entered using A1 for n. The result is shown in Figure 20.1f, with decimal answers given in Column B and fraction versions in Column C.

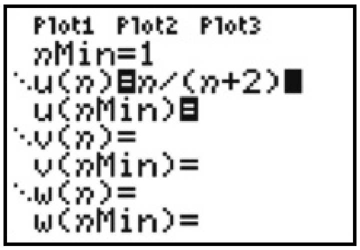


Figure 20.1d

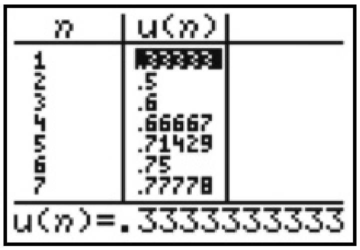


Figure 20.1e

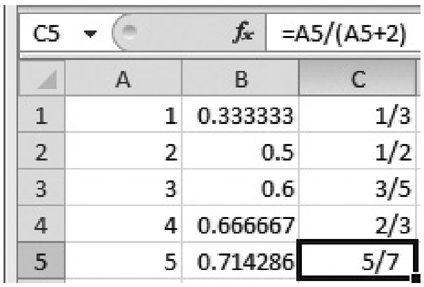


Figure 20.1f

EXAMPLE 20.5

Beginning with the first term, determine the values of the odd-numbered terms of the sequence $\left\{\dfrac{n}{n+1}\right\}$.

SOLUTION

TI-83/4 calculator solution:

What we want are terms 1, 3, 5, and 7. Since $3 - 1 = 2$, $5 - 3 = 2$, etc., the step size is 2. We use the same commands as above but add a comma 2 at the end. Thus, we would use the keystrokes **2nd** **LIST** **▶** [OPS] **5** [5: seq(] x/(x + 1) **,** **X, T, θ, n** **,** 1 **,** 7 **,** 2 **)** **ENTER**. When this is converted to fraction form, we see from Figure 20.2a that the terms are $\frac{1}{2}$, $\frac{3}{4}$, $\frac{5}{6}$, and $\frac{7}{8}$.

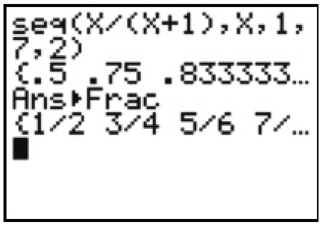


Figure 20.2a

With the TABLE feature the only difference is in the TblSet where you set $\triangle$`Tbl` = 2. The result is shown in Figure 20.2b.

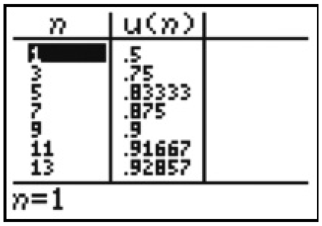


Figure 20.2b

Spreadsheet solution

Use the formula = `A1 + 2` in Cell A2, and copy it down Column A to obtain a list of odd numbers.

Enter the formula = `A1/(A1 + 1)` in Cell B1 and copy it down Column B. The result in shown in Figure 20.2c.

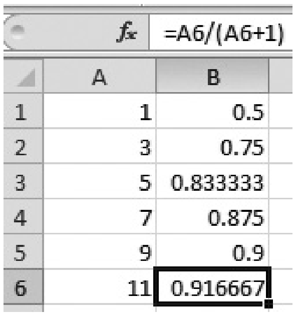


Figure 20.2c

It is not necessary to use the variable x when entering a sequence into a calculator. We usually do it because it is easier to hit the **X, T, θ, n** key rather than **ALPHA** **N** for the variable n.

EXAMPLE 20.6

Use your calculator to determine the odd-numbered terms of the first nine terms of the sequence $\left\{\dfrac{n}{n^2+1}\right\}$.

SOLUTION

TI-83/4 calculator solution

What we want are the first, third, fifth, seventh, and ninth terms of this sequence. We will use n as the variable. Thus, we use the keystrokes **2nd** **LIST** **▶** [OPS] **5** [5: seq(] **ALPHA** **N** **÷** **(** **ALPHA** **N** **x²** **+** 1 **)** **,** **ALPHA** **N** **,** 1 **,** 9 **,** 2 **)** **ENTER**. The result is shown in Figure 20.3a.

When this is converted to fraction form we see from Figure 20.3b that the terms are $\frac{1}{2}$, $\frac{3}{10}$, $\frac{5}{26}$, $\frac{7}{50}$, and $\frac{9}{82}$. In order to see the last two terms we have to use the **▶** key and scroll the screen to the right.

Spreadsheet solution

The result is shown in Figure 20.3c with the decimal representations in Column B and the fractions results in Column C.

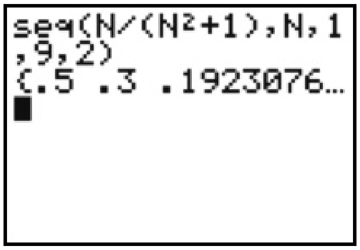


Figure 20.3a

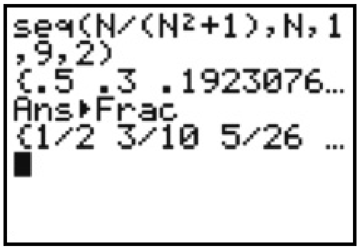
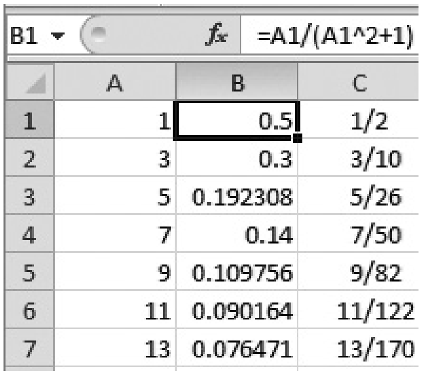


Figure 20.3b


B1 f_x =A1/(A1^2+1)

	A	B	C
1	1	0.5	1/2
2	3	0.3	3/10
3	5	0.192308	5/26
4	7	0.14	7/50
5	9	0.109756	9/82
6	11	0.090164	11/122
7	13	0.076471	13/170

Figure 20.3c

RECURSION FORMULA

A **recursion formula** defines a sequence in terms of one or more previous terms. A sequence that is specified by giving the first term, or the first few terms, and a recursion formula is said to be *defined recursively***.**

EXAMPLE 20.7

Find the first five terms of the sequence defined recursively by $a_1 = 2$ and $a_n = na_{n-1}$.

SOLUTION We are told that $a_1 = 2$. By the recursion formula

$$\begin{aligned} a_2 &= 2(a_1) = 2(2) = 4 \\ a_3 &= 3(a_2) = 3(4) = 12 \\ a_4 &= 4(a_3) = 4(12) = 48 \\ a_5 &= 5(a_4) = 5(48) = 240 \end{aligned}$$

The first five terms are 2, 4, 12, 48, and 240.

EXAMPLE 20.8

Give the first six terms of the sequence defined by $a_1 = 1$, $a_n = -2a_{n-1}$.

SOLUTION

$$\begin{aligned} a_1 &= 1 \\ a_2 &= -2(a_1) = -2(1) = -2 \\ a_3 &= -2(a_2) = -2(-2) = 4 \\ a_4 &= -2(a_3) = -2(4) = -8 \\ a_5 &= -2(a_4) = -2(-8) = 16 \\ a_6 &= -2(a_5) = -2(16) = -32 \end{aligned}$$

The first six terms are 1, -2, 4, -8, 16, and -32.

APPLICATION CONSTRUCTION

EXAMPLE 20.9

A contractor is preparing a bid for constructing an office building. The first floor will cost \$275,000. Each floor above the first will cost \$15,000 more than the floor below it. (a) How much will the fifth floor cost? (b) What is the total cost for the first five floors?

SOLUTIONS

(a) Notice that this is a recursive sequence with $a_1 = 275{,}000$, $a_n = a_{n-1} + 15{,}000$. The first five terms of the sequence are

$$\begin{aligned}
a_1 &= 275{,}000 \\
a_2 &= a_1 + 15{,}000 = 275{,}000 + 15{,}000 = 290{,}000 \\
a_3 &= a_2 + 15{,}000 = 290{,}000 + 15{,}000 = 305{,}000 \\
a_4 &= a_3 + 15{,}000 = 305{,}000 + 15{,}000 = 320{,}000 \\
a_5 &= a_4 + 15{,}000 = 320{,}000 + 15{,}000 = 335{,}000
\end{aligned}$$

The cost of the fifth floor is \$335,000.

(b) The total cost for the first five floors is

$$\begin{aligned}
a_1 + a_2 + a_3 + a_4 + a_5 &= 275{,}000 + 290{,}000 + 305{,}000 + 320{,}000 + 335{,}000 \\
&= 1{,}525{,}000
\end{aligned}$$

The cost for the first five floors in \$1,525,000.

EXERCISE SET 20.1

In Exercises 1–12, find the first six terms with the given specified general term.

1. $a_n = \dfrac{1}{n+1}$

2. $b_n = \dfrac{(-1)^n}{n}$

3. $a_n = (n+1)^2$

4. $a_n = \dfrac{1}{n(n+1)}$

5. $a_n = (-1)^n n$

6. $a_n = \dfrac{1 + (-1)^n}{1 + 4n}$

7. $\left\{\dfrac{2n-1}{2n+1}\right\}$

8. $\left\{\left(\frac{-1}{2}\right)^n\right\}$

9. $\left\{\left(\frac{2}{3}\right)^{n-1}\right\}$

10. $\{n \cos n\pi\}$

11. $\left\{\left(\dfrac{n-1}{n+1}\right)^2\right\}$

12. $\left\{\dfrac{n^2-1}{n^2+1}\right\}$

In Exercises 13–24, find the first six terms of the recursively defined sequence.

13. $a_1 = 1, a_n = na_{n-1}$

14. $a_1 = 3, a_n = a_{n-1} + n$

15. $a_1 = 5, a_n = a_{n-1} + 3$

16. $a_1 = 2, a_n = (a_{n-1})^n$

17. $a_1 = 2, a_n = (a_{n-1})^{n-1}$

18. $a_1 = 1, a_n = \left(\dfrac{-1}{n}\right)a_{n-1}$

19. $a_1 = 1, a_n = n^{a_{n-1}}$

20. $a_1 = \frac{1}{2}, a_n = (a_{n-1})^{-n}$

21. $a_1 = 1, a_2 = \frac{1}{2}, a_n = (a_{n-1})(a_{n-2})$

22. $a_1 = 1, a_2 = 1, a_n = a_{n-2} + a_{n-1}$

23. $a_1 = 0, a_2 = 2, a_n = a_{n-1} - a_{n-2}$

24. $a_1 = 1, a_2 = 2, a_n = (a_{n-1})(a_{n-2})$

Solve Exercises 25–28.

25. ***Environmental science*** An ocean beach is eroding at the rate of 3 in. per year. If the beach is currently 25 ft wide, how wide will the beach be in 25 y? (Hint: $a_1 = 25$ ft $-$ 3 in.; find a_{25}.)

26. ***Construction*** A contractor is preparing a bid for constructing an office building. The foundation and basement will cost \$750,000. The first floor will cost \$320,000. The second floor will cost \$240,000. Each floor above the second will cost \$12,000 more than the floor below it. **(a)** How much will the 10th floor cost? **(b)** How much will the 20th floor cost?

27. ***Business*** You are offered a job with a starting salary of \$26,000 per year with a guaranteed raise of \$1,100 per year. What can you expect as an annual salary in your **(a)** 5th year? **(b)** 10th year?

28. ***Architecture*** The first row of an auditorium has 60 seats. Each row after the first has four more seats than the row in front of it. How many seats are in the 10th row?

[IN YOUR WORDS]

29. What is a recursion formula? Support your explanation by giving an example different from those in the text.

30. What is a sequence? How does a finite sequence differ from an infinite sequence?

20.2 ARITHMETIC AND GEOMETRIC SEQUENCES

In Section 20.1, we began our study of sequences. In this section, we will learn about two special sequences, the arithmetic and geometric sequences.

ARITHMETIC SEQUENCE

The first special sequence we will study is the arithmetic sequence. Its definition depends on a recursion formula.

ARITHMETIC SEQUENCE

An **arithmetic sequence**, or *arithmetic progression*, is a sequence where each term is obtained from the preceding term by adding a fixed number called the **common difference**. If the common difference is d, then an arithmetic sequence follows the recursion formula:

$$a_n = a_{n-1} + d$$

The terms of an arithmetic sequence follow the pattern

$$\begin{aligned} &a_1 \\ &a_2 = a_1 + d \\ &a_3 = a_2 + d = (a_1 + d) + d = a_1 + 2d \end{aligned}$$

$$\begin{aligned}
a_4 &= a_3 + d = (a_1 + 2d) + d = a_1 + 3d \\
a_5 &= a_4 + d = (a_1 + 3d) + d = a_1 + 4d \\
&\ \ \vdots \qquad\qquad\qquad\qquad \vdots \\
a_n &= a_{n-1} + d = a_1 + (n - 1)d
\end{aligned}$$

We have developed a formula for finding the nth term of an arithmetic sequence.

TERMS OF AN ARITHMETIC SEQUENCE

If a_1 is the first term of an arithmetic sequence, a_n the nth term, and d the common difference, then

$$a_n = a_1 + (n - 1)d$$

If a_{n-1} and a_n are consecutive terms of an arithmetic sequence, then

$$d = a_n - a_{n-1}$$

EXAMPLE 20.10

Find the 15th term of 3, 7, 11, 15. . . .

SOLUTION Since $a_1 = 3$ and $a_2 = 7$, then $d = 7 - 3 = 4$. We want the 15th term, so $n = 15$, and we have

$$\begin{aligned}
a_{15} &= a_1 + (15 - 1)d \\
&= 3 + 14 \cdot 4 \\
&= 59
\end{aligned}$$

The 15th term of this sequence is 59.

EXAMPLE 20.11

If the 20th term is 122 and the first term is 8, what is the common difference?

SOLUTION Since we know the 20th term and the first term, we let $n = 20$. Then

$$\begin{aligned}
a_n &= a_1 + (n - 1)d \\
a_{20} &= a_1 + (20 - 1)d \\
122 &= 8 + (19)d \\
114 &= 19d \\
6 &= d
\end{aligned}$$

The common difference is 6.

EXAMPLE 20.12

If the first term is 5, the last (nth) term is -139, and $d = -6$, how many terms are there?

SOLUTION

$$\begin{aligned}
a_n &= a_1 + (n - 1)d \\
-139 &= 5 + (n - 1)(-6)
\end{aligned}$$

EXAMPLE 20.12 (Cont.)

$$-144 = -6n + 6$$
$$-150 = -6n$$
$$25 = n$$

There are 25 terms.

GEOMETRIC SEQUENCE

The second special sequence that we will study is the geometric sequence. Like the arithmetic sequence, it follows a recursion formula.

GEOMETRIC SEQUENCE

A **geometric sequence**, or *geometric progression*, is a sequence where each term is obtained by multiplying the preceding term by a fixed number called the **common ratio**. If the common ratio is r, then a geometric sequence follows the recursion formula:

$$a_n = ra_{n-1}$$

The terms of a geometric sequence follow the pattern

$$\begin{aligned}
&a_1\\
&a_2 = ra_1\\
&a_3 = ra_2 = r(ra_1) = r^2a_1\\
&a_4 = ra_3 = r(r^2a_1) = r^3a_1\\
&a_5 = ra_4 = r(r^3a_1) = r^4a_1\\
&\ \vdots \quad \vdots\\
&a_n = ra_{n-1} = r^{n-1}a_1
\end{aligned}$$

TERMS OF A GEOMETRIC SEQUENCE

If a_1 is the first term of a geometric sequence, a_n the nth term, and r the common ratio, then

$$a_n = r^{n-1}a_1$$

If a_{n-1} and a_n are consecutive terms of a geometric sequence, then

$$r = \frac{a_n}{a_{n-1}}$$

EXAMPLE 20.13

Find the eighth term of the geometric sequence 4, 20, 100, ...

SOLUTION Since we know that this is a geometric sequence, the common ratio is

$$r = \frac{a_2}{a_1} = \frac{20}{4} = 5$$

and since $a_1 = 4$, then

$$\begin{aligned} a_8 &= r^{8-1}a_1 \\ &= 5^7 \cdot 4 \\ &= 312{,}500 \end{aligned}$$

The eighth term is 312,500.

EXAMPLE 20.14

If the 12th term is $\frac{-3}{2{,}048}$ and the first term is -3, what is the common ratio?

SOLUTION Here we let $n = 12$, and so

$$\begin{aligned} a_{12} &= r^{11}a_1 \\ \frac{-3}{2{,}048} &= r^{11}(-3) \\ \frac{1}{2{,}048} &= r^{11} \\ r &= \sqrt[11]{\frac{1}{2{,}048}} \\ &= \frac{1}{2} \end{aligned}$$

The common ratio is $\frac{1}{2}$.

EXAMPLE 20.15

In a geometric sequence, if the first term is -2, the last (nth) term is $-1{,}062{,}882$, and $r = 3$, how many terms are there?

SOLUTION

$$\begin{aligned} a_n &= r^{n-1}a_1 \\ -1{,}062{,}882 &= 3^{n-1}(-2) \\ 531{,}441 &= 3^{n-1} \\ \ln 531{,}441 &= (n-1)\ln 3 \\ n - 1 &= \frac{\ln 531{,}441}{\ln 3} \\ &= 12 \\ n &= 13 \end{aligned}$$

There are 13 terms in this sequence.

APPLICATION CONSTRUCTION

EXAMPLE 20.16

Each 2-in. layer of insulation like that shown in the photograph in Figure 20.4, reduces energy consumption by 4%. How many layers would be needed to reduce consumption by 20%, from 850 to 680 kW?

SOLUTION This is a geometric sequence with $a_1 = 850$ and $a_n = 680$. Each 2-in. layer reduces energy consumption by 4%, so $r = 1 - 4\% = 1 - 0.04 = 0.96$. We need to determine n, the number of 2-in. layers of insulation.

Substituting the given values into the formula

$$a_n = a_1 r^{n-1}$$

we get

$$680 = 850(0.96)^{n-1}$$

or

$$0.80 = (0.96)^{n-1}$$

Using logarithms, we obtain

$$\begin{aligned} \ln 0.80 &= \ln\left[(0.96)^{n-1}\right] \\ &= (n-1)\ln 0.96 \\ \frac{\ln 0.80}{\ln 0.96} &= n - 1 \\ 5.47 &\approx n - 1 \\ \text{so,}\quad n &\approx 6.47 \end{aligned}$$

This means that 14 in. of insulation (7 layers of 2-in. insulation) are needed to reduce energy consumption by 20%.

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Figure 20.4

You should notice in Example 20.16 that even though the answer (6.47) would round off to 6, we would have to use 7 layers of insulation. The actual energy used after adding 7 layers of insulation is

$$\begin{aligned} a_7 &= a_1 r^6 \\ &= 850(0.96)^6 \\ &\approx 665.34 \text{ kW} \end{aligned}$$

This represents a savings of $850 - 665.34 = 184.66$ kW, which is $\frac{184.66}{850} \approx 0.217$ or about 21.7%.

You may have noticed in the last two solutions that we used a technique that we have not used for several chapters—the technique of taking the logarithm of both sides of an equation.

EXERCISE SET 20.2

In Exercises 1–20, determine whether the given sequence is an arithmetic sequence, a geometric sequence, or neither. For the arithmetic and geometric sequences, find the common difference or ratio and the indicated term.

1. 1, 9, 17, . . . (9th term)

2. 3, -5, -13, . . . (7th term)

3. 8, -4, 2, . . . (10th term)

4. 4, 1, $\frac{1}{4}$, . . . (8th term)

5. -1, 4, -16, . . . (6th term)

6. -1, 4, 9, . . . (12th term)

7. 3, -2, -7, . . . (8th term)

8. 1, -3, 9, . . . (7th term)

9. 3, -2, 1, . . . (10th term)

10. 1, $\sqrt{2}$, 2, . . . (12th term)

11. 0.4, 0.8, 1.2, . . . (9th term)

12. 4, 2, $\sqrt{2}$, . . . (11th term)

13. $\frac{2}{3}, \frac{1}{2}, \frac{1}{3}$, . . . (8th term)

14. $\frac{2}{3}, \frac{1}{3}, \frac{1}{6}$, . . . (8th term)

15. 5, 0.5, 0.05, . . . (6th term)

16. 6, -2, $\frac{2}{3}$, . . . (7th term)

17. $1 + \sqrt{3}$, $3 + \sqrt{3}$, $5 + \sqrt{3}$, . . . (9th term)

18. 1.02, 10.2, 102, . . . (10th term)

19. 4.3, -3.2, 2.1, . . . (6th term)

20. 3.4, -6.8, 13.6, . . . (9th term)

Solve Exercises 21–36.

21. For an arithmetic sequence $a_5 = 9$ and $a_6 = 24$, what is a_1?

22. For an arithmetic sequence $a_7 = 11$ and $a_9 = 15$, what is a_1?

23. For a geometric sequence $a_4 = 9$ and $a_5 = 3$, what is a_1?

24. For a geometric sequence $a_8 = \frac{1}{3}$ and $a_9 = 5$, what is a_1?

25. In an arithmetic sequence $a_1 = 5$, $a_n = 81$, and $d = 4$, what is n?

26. In an arithmetic sequence $a_1 = 20$, $a_n = -31$, and $d = -3$, what is n?

27. In a geometric sequence $a_1 = 3$, $a_n = \frac{1}{559{,}872}$, and $r = \frac{1}{6}$, find n.

28. In a geometric sequence $a_1 = -4$, $a_n = \frac{-1}{4{,}096}$, and $r = \frac{1}{2}$, what is n?

29. If the first term of an arithmetic sequence is -5 and the 7th term is 4, what is the second term?

30. If the first term of a geometric sequence is $\frac{1}{3}$ and the 8th term is 729, what is the third term?

31. ***Medical technology*** A group of people make an exercise program. The first day they will jog $\frac{1}{2}$ mi. After that they will jog a certain amount more each day until on the 61st day, 8 mi are jogged. How much was the distance increased each day?

32. ***Physics*** A ball is dropped to the ground from a height of 80 m. Each time it bounces it goes $\frac{7}{8}$ as high as the previous bounce. How high does it go on the sixth bounce?

33. ***Physics*** A pendulum swings 15 cm on its first swing. Each subsequent swing is reduced by 0.3 cm. How far is the 10th swing? How many times will the pendulum swing before it comes to rest?

34. ***Aeronautical engineering*** The atmospheric pressure at sea level is approximately 100 kPa and decreases 12.5% for each km increase in altitude. What is the atmospheric pressure at the top of Mt. Everest, which is about 8.8 km high?

35. ***Environmental science*** A chemical spill pollutes a river. A monitor located 1 mi downstream from the spill measures 940 parts of the chemical for every million parts of water. (This is written as 940 ppm.) The readings decrease 16.2% for each mile farther downstream. **(a)** How far downstream from the spill will it be before the concentration is reduced to 100 ppm? **(b)** The water is considered safe for human consumption at 1.5 ppm. How far downstream from the spill will you have to go before the water is considered safe for humans to drink?

36. ***Environmental science*** The level of a particular metal pollutant in a certain lake was found to be increasing geometrically. In January, there were 3.50 parts per billion (ppb) and in April there were 18.76 ppb.

(a) By what ratio is the level increasing each month?

(b) What will be the pollution level in December?

[IN YOUR WORDS]

37. Explain how an arithmetic sequence and a geometric sequence are alike and how they are different.

38. In Exercises 1–20, you were asked to determine if each given sequence is an arithmetic sequence, a geometric sequence, or neither. Describe how you made your decision.

20.3 SERIES

There are many times when we need to add the terms of a sequence. The sum of the terms of a sequence is called a **series**. The series

$$a_1 + a_2 + a_3 + a_4 + a_5$$

is a **finite series** with five terms. A finite series can have 1, 2, 5, 19, 437, or any number of terms as long as the number is a natural number. A series of the form

$$a_1 + a_2 + a_3 + a_4 + \cdots$$

is an **infinite series**. An infinite series has an infinite or endless number of terms.

SUMMATION NOTATION

A more compact notation is often used to indicate a series. This notation is referred to as **summation** or *sigma notation* because it uses the capital Greek letter sigma $(\sum)$. In general, the sigma notation means

$$\sum_{k=1}^{n} a_k = a_1 + a_2 + a_3 + \cdots + a_n$$

Here $\sum$ indicates a sum. The letter k is called an *index of summation*. The summation begins with $k = 1$ as is indicated below the $\sum$ and ends with $k = n$ as is indicated above the $\sum$. You saw this notation in Section 13.2.

Sometimes it is useful to indicate that a series begins with the zeroth, second, or other term. If it starts with the zeroth term, then

$$\sum_{k=0}^{n} a_k = a_0 + a_1 + a_2 + \cdots + a_n$$

and if it starts with the second term,

$$\sum_{k=2}^{n} a_k = a_2 + a_3 + a_4 + \cdots + a_n$$

The numbers below and above the $\sum$ are the *limits of summation*. Here they are 2 and n.

EXAMPLE 20.17

Evaluate $\sum_{k=1}^{5} (-3)k$.

SOLUTION

$$\begin{aligned}\sum_{k=1}^{5} (-3)k &= (-3)1 + (-3)2 + (-3)3 + (-3)4 + (-3)5 \\ &= -3 + (-6) + (-9) + (-12) + (-15) \\ &= -45\end{aligned}$$

ADDING SEQUENCES WITH A CALCULATOR

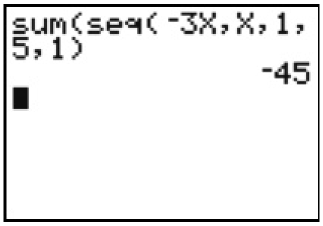
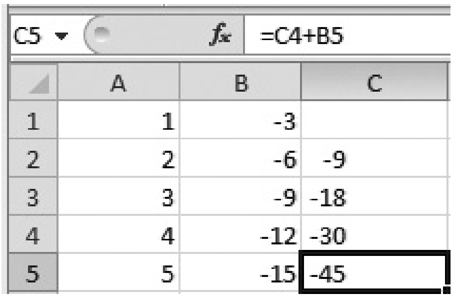


Figure 20.5a

You can use some calculators to add sequences. For instance, you can use a TI-83 to evaluate $\sum_{k=1}^{5} (-3)k$ by using the keystrokes 2nd LIST ◄ [MATH] 5 [5:sum(] 2nd LIST ► [OPS] 5 [5:seq(] (−) 3 X, T, θ, n , X, T, θ, n , 1 , 5 , 1) ENTER. The result, −45, as shown in Figure 20.5a, is the same result we obtained in Example 20.17.

ADDING SEQUENCES WITH A SPREADSHEET

You can add the terms in a sequence by using an additional column in a spreadsheet. For example, to evaluate $\sum_{k=1}^{5} (-3)k$ using a spreadsheet, enter the numerical values 1 through 5 in Column A. Next, enter the formula `=-3*A1` in Cell B1 and copy it down Column B. In Cell C2, enter the formula `=B2+B1` to find the sum of the first two terms. In Cell C3, enter `=C2+B3` to find the sum of the first three terms. Copy that formula down Column C as shown in Figure 20.5b.



C5 f_x =C4+B5

	A	B	C
1	1	-3	
2	2	-6	-9
3	3	-9	-18
4	4	-12	-30
5	5	-15	-45

Figure 20.5b

EXAMPLE 20.18

Evaluate $\sum_{k=1}^{4}(2k - 3)$.

SOLUTION

$$\begin{aligned}\sum_{k=1}^{4}(2k-3) &= (2\cdot 1 - 3) + (2\cdot 2 - 3) + (2\cdot 3 - 3) + (2\cdot 4 - 3)\\ &= (2-3) + (4-3) + (6-3) + (8-3)\\ &= -1 + 1 + 3 + 5\\ &= 8\end{aligned}$$

EXAMPLE 20.19

Evaluate $\sum_{k=1}^{4}(2k - 3)$ using a different method than the one you used in Example 20.18.

SOLUTION This is the same problem we just worked in Example 20.18. This time we will add the numbers in a different order.

$$\begin{aligned}\sum_{k=1}^{4}(2k-3) &= (2\cdot 1 - 3) + (2\cdot 2 - 3) + (2\cdot 3 - 3) + (2\cdot 4 - 3)\\ &= (2\cdot 1) + (2\cdot 2) + (2\cdot 3) + (2\cdot 4) + (-3) +\\ &\quad\ (-3) + (-3) + (-3)\\ &= 2(1 + 2 + 3 + 4) + 4(-3)\\ &= 2(10) + 4(-3)\\ &= 20 - 12 = 8\end{aligned}$$

This is the same answer we got in Example 20.18. However, the solution demonstrates three properties of $\sum$ notation.

PROPERTIES OF SUMMATION NOTATION

If a is a constant then we have the following properties of summation notation:

$$\sum_{k=1}^{n} ak = a\sum_{k=1}^{n} k$$

$$\sum_{k=1}^{n} a = na$$

$$\sum_{k=1}^{n} (x + y) = \sum_{k=1}^{n} x + \sum_{k=1}^{n} y$$

EXAMPLE 20.20

Evaluate $\displaystyle\sum_{k=0}^{3} \frac{2^k}{3k - 5}$.

SOLUTION

$$\begin{aligned}\sum_{k=0}^{3} \frac{2^k}{3k-5} &= \frac{2^0}{3\cdot 0 - 5} + \frac{2^1}{3\cdot 1 - 5} + \frac{2^2}{3\cdot 2 - 5} + \frac{2^3}{3\cdot 3 - 5}\\ &= \frac{1}{0-5} + \frac{2}{3-5} + \frac{4}{6-5} + \frac{8}{9-5}\\ &= \frac{1}{-5} + \frac{2}{-2} + \frac{4}{1} + \frac{8}{4}\\ &= 4\tfrac{4}{5}\end{aligned}$$

PARTIAL SUMS

Informally speaking, the expression $\displaystyle\sum_{k=0}^{n} a_k$ directs us to find the "sum" of the terms $a_1, a_2, a_3, \ldots, a_n$. To carry out this process, we proceed as follows. We let S_n denote the **sum of the first n terms** of the series. Thus,

$$\begin{aligned}S_1 &= a_1\\ S_2 &= a_1 + a_2\\ S_3 &= a_1 + a_2 + a_3\\ &\vdots\\ S_n &= a_1 + a_2 + a_3 + \cdots + a_n = \sum_{k=1}^{n} a_k\end{aligned}$$

The number S_n is called the *nth* **partial sum** of the series. The sequence $S_1, S_2, S_3, \ldots, S_n$ is called the *sequence of partial sums*.

EXAMPLE 20.21

Find the first five partial sums of the series $\sum_{k=1}^{n} k^2$.

SOLUTION

$$S_1 = 1^2 = 1$$
$$S_2 = 1^2 + 2^2 = 5$$
$$S_3 = 1^2 + 2^2 + 3^2 = 14$$
$$S_4 = 1^2 + 2^2 + 3^2 + 4^2 = 30$$
$$S_5 = 1^2 + 2^2 + 3^2 + 4^2 + 5^2 = 55$$

ARITHMETIC SERIES

Now let's look at the series formed from the two special sequences that we studied in Section 20.2. An arithmetic sequence has a first term of a_1 and a common difference d. The sum of the first n terms of an arithmetic sequence is the nth partial sum of the sequence:

$$S_n = a_1 + (a_1 + d) + (a_1 + 2d) + (a_1 + 3d) + \cdots + [a_1 + (n-1)d] \qquad \textbf{(1)}$$

If we write the nth term a_n first, then S_n could be written as

$$S_n = a_n + (a_n - d) + (a_n - 2d) + (a_n - 3d) + \cdots + [a_n - (n-1)d] \qquad \textbf{(2)}$$

If we add the corresponding terms of equations (1) and (2), we get

$$2S_n = (a_1 + a_n) + (a_1 + a_n) + (a_1 + a_n) + (a_1 + a_n) + \cdots + (a_1 + a_n)$$

The right-hand side has n terms that are all the same, $a_1 + a_n$. Thus, we have the following sum S_n of the first n terms of an arithmetic sequence.

SUM OF FIRST *N* TERMS OF AN ARITHMETIC SEQUENCE

The sum S_n of the first n terms of an arithmetic sequence is

$$S_n = \frac{n(a_1 + a_n)}{2}$$

where a_1 is the first term and a_n is the nth term.

EXAMPLE 20.22

Find the sum of the first eight terms of the arithmetic sequence 3, 8, 13, 18, 23, 28, 33, 38, ...

SOLUTION We will use the formula $S_n = \dfrac{n(a_1 + a_n)}{2}$ with $n = 8$, $a_1 = 3$, and $a_8 = 38$.

$$S_8 = \frac{8(3 + 38)}{2} = 164$$

EXAMPLE 20.23

Find the sum of the first 14 terms of the arithmetic sequence 9, 3, $-3, \ldots$.

SOLUTION In this sequence, $d = -6$, $a_1 = 9$, and $n = 14$. From Section 20.2, we know that

$$\begin{aligned} a_{14} &= a_1 + (14 - 1)d \\ &= 9 + 13(-6) \\ &= -69 \end{aligned}$$

We can now find the sum of the first 14 terms.

$$\begin{aligned} S_{14} &= \frac{14(a_1 + a_n)}{2} \\ &= \frac{14(9 - 69)}{2} \\ &= -420 \end{aligned}$$

The sum of the first 14 terms of this sequence is -420.

GEOMETRIC SERIES

Now let's see if we can develop a similar formula for the first n terms of a geometric series. The sum of the first n terms of a geometric sequence is

$$S_n = a_1 + a_1 r + a_1 r^2 + a_1 r^3 + \cdots + a_1 r^{n-1} \tag{3}$$

Multiplying both sides of equation (3) by r, we get

$$rS_n = a_1 r + a_1 r^2 + a_1 r^3 + a_1 r^4 + \cdots + a_1 r^{n-1} + a_1 r^n \tag{4}$$

If we subtract equation (4) from equation (3), we get

$$S_n - rS_n = a_1 - a_1 r^n$$

Factoring each side $S_n(1 - r) = a_1(1 - r^n)$ and solving for S_n, we have the following formula for the first n terms of a geometric sequence.

SUM OF FIRST N TERMS OF A GEOMETRIC SEQUENCE

The sum S_n of the first n terms of a geometric sequence is

$$S_n = a_1 \frac{1 - r^n}{1 - r}$$

where a_1 is the first term and r is the common ratio.

EXAMPLE 20.24

Find the sum of the first 10 terms of the geometric sequence 4, -8, 16, $-32, \ldots$.

SOLUTION In this sequence, $a_1 = 4$, $r = -2$, and $n = 10$.

$$S_{10} = \alpha_1 \frac{1 - r^{10}}{1 - r}$$

EXAMPLE 20.24 (Cont.)

$$\begin{aligned} &= 4\left(\frac{1-(-2)^{10}}{1-(-2)}\right) \\ &= 4\left(\frac{1-1{,}024}{3}\right) \\ &= 4\left(\frac{-1{,}023}{3}\right) \\ &= -1{,}364 \end{aligned}$$

The sum of the first 10 terms of $4 - 8 + 16 - 32 + \cdots$ is $-1{,}364$.

EXAMPLE 20.25

Find the sum of the geometric series $\frac{1}{2} + \frac{1}{3} + \frac{2}{9} + \cdots + \frac{1}{2}\left(\frac{2}{3}\right)^8$.

SOLUTION In this series, $a_1 = \frac{1}{2}$, $r = \frac{2}{3}$, and $n = 9$.

$$\begin{aligned} S_9 &= \frac{1}{2}\left(\frac{1-\left(\frac{2}{3}\right)^9}{1-\frac{2}{3}}\right) = \frac{1}{2}\left(\frac{1-\frac{512}{19{,}683}}{1-\frac{2}{3}}\right) = \frac{1}{2}\left(\frac{\frac{19{,}171}{19{,}683}}{\frac{1}{3}}\right) = \frac{57{,}513}{39{,}366} \\ &\approx 1.46 \end{aligned}$$

AN APPLICATION OF THE GEOMETRIC SERIES

One application of the geometric series deals with compound interest. Suppose you deposit \$100 in a savings account that pays 4% compounded annually. The amount of money after 1 year would be

$$\begin{aligned} 100 + 100(0.04) &= 100 + 4 \\ &= \$104 \end{aligned}$$

After 2 years the amount would be

$$\begin{aligned} 104 + 104(0.04) &= 104 + 4.16 \\ &= \$108.16 \end{aligned}$$

As you can see, the amount after each year forms a geometric sequence

$$100 + 104 + 108.16 + \cdots$$

where $a_1 = 100$ and $r = 1.04$. Thus $r = 1 + i$, where i is the interest rate. From this, we get the formula for compound interest.

COMPOUND INTEREST

$$A = P(1+i)^n$$

where A is the amount after n interest periods, P is the principal or initial amount invested, and i is the interest rate per interest period expressed as a decimal.

Thus A is similar to S_n. Since the initial investment is P, this formula begins with $n = 0$.

The interest rate is given for the interest period. This means that in order to determine i, you must divide the annual interest rate by the number of interest periods. An 8% interest rate compounded quarterly (four times a year) would have $i = \frac{0.08}{4} = 0.02$. (Remember 8% = 0.08.) If the 8% interest rate was compounded monthly, or 12 times a year, then $i = \frac{0.08}{12} = 0.0067$.

NOTE In Section 12.1, we gave a compound interest formula $A = P\left(1 + \frac{r}{k}\right)^{kt}$. The formula above, $A = P(1 + i)^n$, is the same formula with $i = \frac{r}{k}$ and $n = kt$. Here we used $A = P(1 + i)^n$ to make it easier to see it as a geometric series.

APPLICATION BUSINESS

EXAMPLE 20.26

If \$100 is deposited in a savings account paying 6% annually, what is the amount after 1 year if interest is compounded (a) quarterly, or (b) monthly?

SOLUTIONS We will use $A = P(1 + i)^n$ in both parts with $P = 100$.

(a) Since interest is compounded quarterly

$$i = \frac{0.06}{4} = 0.015 \text{ and } n = 4$$

$$\begin{aligned} A &= 100(1 + 0.015)^4 \\ &= 106.14 \end{aligned}$$

The total is \$106.14 at the end of 1 year.

(b) In this example, the interest is compounded monthly, so

$$i = \frac{0.06}{12} = 0.005 \text{ and } n = 12$$

$$\begin{aligned} A &= 100(1.005)^{12} \\ &= 106.17 \end{aligned}$$

The total is \$106.17 at the end of 1 year.

APPLICATION BUSINESS

EXAMPLE 20.27

Suppose the money in Example 20.26 were left in the savings account for 4 more years (a total of 5). What would it then be worth?

SOLUTIONS

(a) At quarterly compounding, i is still 0.015, but n is now 4×5 (4 times a year for 5 years).

$$\begin{aligned} A &= 100(1.015)^{20} \\ &= 134.69 \end{aligned}$$

EXAMPLE 20.27 (Cont.) (b) At monthly compounding, $i = 0.005$ and $n = 12 \times 5 = 60$.

$$A = 100(1.005)^{60}$$
$$= 134.89$$

The totals are \$134.69 if the money is compounded quarterly for 5 years and \$134.89 if it is compounded monthly.

EXERCISE SET 20.3

In Exercises 1–4, write the first four terms of the indicated series.

1. $\sum_{k=1}^{20} 3\left(\frac{1}{2}\right)^k$

2. $\sum_{n=1}^{50} [4 + (n - 1)3]$

3. $\sum_{n=0}^{20} (-1)^n \frac{3^n}{n + 1}$

4. $\sum_{k=1}^{60} (-1)^k$

In Exercises 5–12, evaluate the given sum.

5. $\sum_{k=1}^{5} (k - 3)$

6. $\sum_{k=0}^{4} (2k + 1)$

7. $\sum_{k=1}^{6} k^2$

8. $\sum_{k=2}^{8} \frac{k + 2}{(k - 1)k}$

9. $\sum_{n=1}^{5} n^3$

10. $\sum_{n=1}^{8} \frac{(-1)^n(n^2 + 1)}{n}$

11. $\sum_{i=3}^{8} \frac{2^i}{3i + 1}$

12. $\sum_{i=0}^{7} \frac{(-1)^{i+1}(i + 1)^2}{2i + 1}$

In Exercises 13–28, determine whether the terms of the given series form an arithmetic or geometric series and find the indicated sum.

13. $3 + 6 + 9 + 12 + \cdots; S_{10}$

14. $5 + 1 - 3 + \cdots; S_{12}$

15. $1 + 2 + 4 + \cdots; S_8$

16. $\frac{1}{2} + 2 + 8 + \cdots; S_9$

17. $-6 - 2 + 2 + \cdots; S_{12}$

18. $1 - \frac{1}{2} + \frac{1}{4} + \cdots; S_{10}$

19. $0.5 + 0.75 + 1.0 + \cdots; S_{20}$

20. $0.5 + 0.2 + 0.08 + \cdots; S_{15}$

21. $\frac{3}{4} - \frac{1}{4} + \frac{1}{12} + \cdots; S_{16}$

22. $0.4 + 0.04 + 0.004 + \cdots; S_6$

23. $0.4 + 1.6 + 2.8 + \cdots; S_{14}$

24. $16 + 8 + 4 + \cdots; S_{10}$

25. $2\sqrt{3} + 6 + 6\sqrt{3} + \cdots; S_{15}$

26. $\sqrt{5} + 10 + 20\sqrt{5} + \cdots; S_{12}$

27. $3 + 9 + 27 + \cdots; S_8$

28. $5 - 25 + 125 + \cdots; S_{10}$

Solve Exercises 29–36.

29. ***Physics*** A ball is dropped to the ground from a height of 80 m, and each time it bounces 80% as high as it did on the previous bounce. How far has it traveled when it hits the ground the fourth time? The 10th time?

30. ***Physics*** A pendulum swings a distance of 50 m initially from one side to the other. After the first swing, each swing is only 0.8 of the distance of the previous swing. What is the total distance covered by the pendulum in 10 swings?

31. ***Construction*** The end of a gable roof is in the shape of a triangle. If the span is 32 ft (as shown in Figure 20.6), the height of the gable is 8 ft, and the studs are placed every 16 in., what is the total length of the studs?

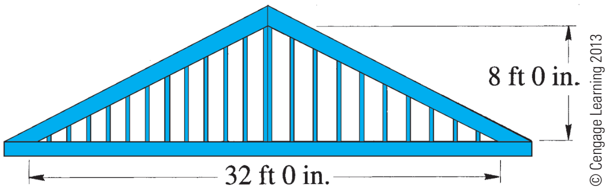


Figure 20.6

32. ***Finance*** How much will \$2,000 earn in 5 years if it is invested in a savings account that pays 3.5% compounded quarterly?

33. ***Finance*** You have the opportunity to invest the \$2,000 in Exercise 32 in a savings account that pays 3.5% compounded monthly. How much would this total after 5 years? How much more was this than the amount in Exercise 32?

34. ***Finance*** Your final chance is to invest the money in Exercise 32 at 3.5% compounded continuously. What would this total after 5 years? (See Section 12.2.)

35. ***Construction*** In Exercise 26, Exercise Set 20.1, a contractor was preparing a bid for the construction of an office building. The foundation and basement will cost \$750,000 and the first floor will cost \$320,000. The second floor will cost \$240,000 and each floor above the second will cost \$12,000 more than the floor below it. **(a)** If the 10th floor will cost \$336,000, how much will a 10-story building cost? **(b)** If the 20th floor will cost \$456,000, how much will it cost to build a 20-story building?

36. ***Finance*** At the beginning of each year, the owner of a automotive repair shop deposits \$3,000 into a retirement account paying 4.25% compounded annually. How much money will be in the account when the owner retires 25 years later? (Note: The owner retires at the end of the 25th year.)

[IN YOUR WORDS]

37. Explain how a series and a sequence are different.

38. What is a partial sum? How you determine the sequence of partial sums?

39. Describe how to determine the sum of the first n terms of an arithmetic sequence.

40. Describe how to determine the sum of the first n terms of a geometric sequence.

41. Three properties of summation notation were listed in the box following Example 20.19. Each property was used in Example 20.19. Tell where each property was used and describe how it was used.

20.4 INFINITE GEOMETRIC SERIES

In this section, we will continue discussing geometric series. All of the series that we discussed in Section 20.3 were finite series. A finite series has a first term and a last term. In this section, we will discuss series that are not finite. In particular, those that do not have a last term. A series that does not have a last term is called an *infinite series*.

If we have a sequence of the form

$$a_1, a_2, a_3, \ldots, a_n, \ldots$$

then this is an *infinite sequence* because it does not have a last term. The corresponding infinite series is designated by

$$\sum_{k=1}^{\infty} a_k = a_1 + a_2 + a_3 + \cdots + a_n + \cdots$$

The symbol ∞ is read, "infinity."

We have seen examples of infinite numbers before and have always found ways of working with these numbers. But now we want to find a way to determine the sum of an infinite series.

One way that we can determine the sum of an infinite series is with our earlier definition of partial sum. Remember, each partial sum S_n is the sum of the first n term in the series. This means that each partial sum is the sum of a finite number of terms. So,

$$\begin{aligned} S_1 &= a_1 \\ S_2 &= a_1 + a_2 \\ S_3 &= a_1 + a_2 + a_3 \\ &\vdots \\ S_n &= a_1 + a_2 + a_3 + \cdots + a_n \end{aligned}$$

S_n is called the *nth partial sum* of the series $\sum_{k=1}^{\infty} a_k$. The *sequence of partial sums* is $S_1, S_2, S_3, \ldots, S_n$. If it happens that, as n gets larger and larger, S_n seems to be approaching some number that we will call S, we then say that S is the sum of the infinite series.

The idea of the number S is much the same as when we studied asymptotes. If you remember, a curve kept getting closer and closer to an asymptote, but never reached it. In the same way, as n gets larger, S_n will get closer and closer to S, but never reach it. We say that S is the **limit** of the sequence S_n and write it symbolically as

$$S = \lim_{n\to\infty} S_n = \sum_{n=1}^{\infty} a_n$$

The symbol $\lim_{n\to\infty} S_n$ is read "the limit of S_n as n goes to infinity." Limits are studied in detail in calculus.

Now, let's consider a couple of infinite geometric sequences. The first one we will consider is

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^{n-1}} + \cdots$$

The partial sums are

$$\begin{aligned} S_1 &= 1 \\ S_2 &= 1 + \tfrac{1}{2} = \tfrac{3}{2} \\ S_3 &= 1 + \tfrac{1}{2} + \tfrac{1}{4} = \tfrac{7}{4} \\ S_4 &= 1 + \tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{8} = \tfrac{15}{8} \\ S_5 &= 1 + \tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{8} + \tfrac{1}{16} = \tfrac{31}{16} \end{aligned}$$

From Section 20.3 we know that

$$S_n = a_1\left(\frac{1 - r^n}{1 - r}\right) = 1\left[\frac{1 - (\frac{1}{2})^n}{1 - \frac{1}{2}}\right] = 2\left[1 - \left(\frac{1}{2}\right)^n\right]$$

The sequence of partial sums

$$1, \frac{3}{2}, \frac{7}{4}, \frac{15}{8}, \frac{31}{16}, \ldots, 2\left[1 - \left(\frac{1}{2}\right)^n\right], \ldots$$

seems to be approaching 2 as a limit. Since $(\frac{1}{2})^n$ keeps getting closer and closer to 0 as n gets larger, it seems that

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} 2\left[1 - \left(\frac{1}{2}\right)^n\right] = 2(1 - 0) = 2$$

Not all partial sums of infinite series approach a finite limit. If the partial sums of an infinite series approach a finite limit, we say that the series *converges* or is a **convergent series**. A series that does not converge is said to *diverge* or to be a **divergent series**. All arithmetic series diverge.

In general, an infinite geometric series of the form

$$\sum_{n=1}^{\infty} a_1 r^{n-1} = a_1 + a_1 r + a_1 r^2 + a_1 r^3 + \cdots$$

converges under certain circumstances. We know from Section 20.3 that the nth partial sum of this series is

$$S_n = \frac{a_1(1 - r^n)}{1 - r}$$

When does this have a limit? The key is in the term r^n. Pick a number, such as 2, and take larger powers of 2. What happens? $2^2 = 4$, $2^4 = 16$, $2^8 = 256, \ldots$. Notice that as n gets larger, 2^n keeps getting larger. Now select a smaller number such as 0.5. $(0.5)^2 = 0.25$, $(0.5)^4 = 0.0625$, $(0.5)^8 = 0.00390625$. As n gets larger, $(0.5)^n$ gets smaller and smaller. In fact, for any value r that is between -1 and 1, as n gets larger r^n gets closer to zero. In symbols, using our new limit notation, this is

$$\text{If } |r| < 1, \text{ then } \lim_{n\to\infty} r^n = 0$$

This means that for an infinite geometric series, if $|r| < 1$, then

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} a_1\left(\frac{1 + r^n}{1 - r}\right) = \frac{a_1}{1 - r}$$

This provides us with a formula for the sum of an infinite geometric series.

SUM OF AN INFINITE GEOMETRIC SERIES

If $|r| < 1$, then the infinite geometric series

$$a_1 + a_1 r + a_1 r^2 + a_1 r^3 + \cdots + a_1 r^n + \cdots$$

has the sum $S = \dfrac{a_1}{1 - r}$.

If $|r| \geq 1$, then the series diverges.

EXAMPLE 20.28

Show that the series $\sum_{n=1}^{\infty} \left(\frac{-2}{5}\right)^n$ converges and find its sum.

SOLUTION This is a geometric series with $r = \frac{-2}{5}$. Since $|r| = \frac{-2}{5} = \frac{2}{5}$, and $|r|$ is less than one, the sum of this geometric series is

$$\frac{a_1}{1 - r} = \frac{-\frac{2}{5}}{1 - \frac{-2}{5}} = \frac{\frac{-2}{5}}{\frac{7}{5}} = \frac{-2}{7}$$

Always remember that we can never reach $-\frac{2}{7}$ exactly, no matter how many terms in the series we add; but, by definition, $-\frac{2}{7}$ is the sum of this series.

EXAMPLE 20.29

Show that each of the following series is divergent:

(a) $1 + \frac{4}{3} + \frac{16}{9} + \cdots$; (b) $\sum_{k=1}^{\infty}(-1)^k$

SOLUTIONS

(a) Since $r = \frac{4}{3}$, we can see that $r > 1$ and so this series is divergent.

(b) The expanded series is $-1 + 1 - 1 + 1 - 1 + 1 - 1 + \cdots$ which has partial sums

$$S_1 = -1, S_2 = 0, S_3 = -1, S_4 = 0, S_5 = -1, \ldots$$

These partial sums do not get arbitrarily large, but they also do not approach some specific limit. Since $r = -1$, we see that $|r| = |-1| = 1$, and so this series diverges.

EXAMPLE 20.30

Find the fraction that has the repeating decimal form 0.232323 . . . .

SOLUTION This decimal can be thought of as the series $0.23 + 0.0023 + 0.000023 + \cdots$, which is the geometric series $\sum_{n=0}^{\infty} 0.23(0.01)^n$. In this series $a_1 = 0.23$ and $r = 0.01$, so

$$S = \frac{a_1}{1-r} = \frac{0.23}{1-0.01} = \frac{0.23}{0.99} = \frac{23}{99}$$

Thus, the decimal 0.232323 . . . is equivalent to the fraction $\frac{23}{99}$.

EXAMPLE 20.31

Express 4.2315315315 . . . as a rational number.

SOLUTION The number 4.2315315315 . . . repeats the digits 315. We can write the number as $4.2 + (0.0315 + 0.0000315 + 0.0000000315 + \cdots)$. The portion inside the parentheses is an infinite geometric series with $a_1 = 0.0315$ and $r = 0.001$. So, $4.2315315\ldots = 4.2 + S$, where

$$S = \frac{0.0315}{1-0.001} = \frac{0.0315}{0.999} = \frac{315}{9{,}990} = \frac{35}{1{,}110}$$

Since $4.2 = \frac{42}{10}$, we have

$$4.2315315\cdots = \frac{42}{10} + \frac{35}{1{,}110} = \frac{4{,}662 + 35}{1{,}110} = \frac{4{,}697}{1{,}110}.$$

EXERCISE SET 20.4

Determine which of the infinite geometric series in Exercises 1–22 converge and which diverge. For those that converge, find the sum.

1. $\frac{1}{2} - \frac{1}{4} + \frac{1}{8} - \frac{1}{16} + \cdots$

2. $1 + \frac{3}{4} + \frac{9}{16} + \cdots$

3. $1 - \frac{2}{3} + \frac{4}{9} - \frac{8}{27} + \cdots$

4. $1.5 + 2.25 + 3.375 + \cdots$

5. $3 + \frac{3}{4} + \frac{3}{16} + \cdots$

6. $4 - 1 + \frac{1}{4} + \cdots$

7. $\frac{5}{4} + \frac{1}{8} + \frac{1}{80} + \cdots$

8. $\frac{3}{4} + \frac{3}{4^2} + \frac{3}{4^3} + \cdots$

9. $0.03 + 0.003 + 0.0003 + \cdots$

10. $1.4 + 0.014 + 0.00014 + \cdots$

11. $\sum_{n=1}^{\infty} (-0.8)^{n-1}$

12. $\sum_{n=1}^{\infty} \left(\frac{3}{4}\right)^n$

13. $\sum_{n=1}^{\infty} \left(\frac{5}{4}\right)^{n+1}$

14. $\sum_{n=1}^{\infty} (-1.1)^n$

15. $\sum_{n=1}^{\infty} 0.3(10)^n$

16. $\sum_{n=1}^{\infty} 2^{-n}$

[Hint: Let $2^{-n} = (2^{-1})^n$.]

17. $\sum_{n=1}^{\infty} \left(\frac{3}{2}\right)^{-n}$

18. $\sum_{n=1}^{\infty} 6\left(-\frac{2}{3}\right)^n$

19. $\sum_{n=1}^{\infty} \left(\frac{1}{\sqrt{2}}\right)^n$

20. $\sum_{n=1}^{\infty} \left(\frac{\sqrt{3}}{3}\right)^n$

21. $\sum_{n=1}^{\infty} \left(\sqrt{5}\right)^n$

22. $\sum_{n=1}^{\infty} \left(\sqrt{7}\right)^{1-n}$

In Exercises 23–30, use infinite series to find the rational number corresponding to the given decimal number.

23. 0.444 . . .

24. 0.777 . . .

25. 0.575757 . . .

26. 0.848484 . . .

27. 1.352135213521 . . .

28. 4.12341234123 . . .

29. 6.3021021021 . . .

30. 2.1906906906 . . .

Solve Exercises 31–34.

31. ***Physics*** A ball is dropped from a height of 5 m. After the first bounce, the ball reaches a height of 4 m, after the second, 3.2 m, and so on. What is the total distance traveled by the ball before it comes to rest?

32. ***Physics*** An object suspended on a spring is oscillated up and down. The first oscillation was 100 mm and each oscillation after that was $\frac{9}{10}$ that of the preceding one. What is the total distance that the object traveled?

33. ***Machine technology*** When a motor is turned off, a flywheel attached to the motor coasts to a stop. In the first second, the flywheel makes 250 revolutions. Each of the following seconds, it revolves $\frac{8}{10}$ of the number in the preceding second. What are the total number of revolutions made by the flywheel when it stops?

34. ***Physics*** A pendulum swings a distance of 50 cm initially from one side to the other. After the first swing, each swing is 0.85 the distance of the previous swing. What is the total distance covered by the pendulum when it comes to a rest?

[IN YOUR WORDS]

35. What is the difference between a convergent series and a divergent series?

36. Describe how to determine if an infinite geometric series converges.

20.5 THE BINOMIAL THEOREM

In Section 7.1, we learned several special products. Among them were

$$(x + y)^2 = x^2 + 2xy + y^2$$
$$(x + y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$$
$$(x - y)^2 = x^2 - 2xy + y^2$$
$$\text{and} \quad (x - y)^3 = x^3 - 3x^2y + 3xy^2 - y^3$$

To find the larger powers of $x + y$ or $x - y$ would require repeated multiplication by $x + y$ or $x - y$. In this section, we will develop the binomial theorem. This theorem allows us to expand $x + y$ or $x - y$ to any power without direct multiplication.

EXPANSIONS OF $(X + Y)^N$ AND PASCAL'S TRIANGLE

By direct multiplication, we can obtain the following expansions of $x + y$:

$$(x + y)^0 = 1$$
$$(x + y)^1 = x + y$$
$$(x + y)^2 = x^2 + 2xy + y^2$$
$$(x + y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$$
$$(x + y)^4 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$$
$$(x + y)^5 = x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + y^5$$
$$(x + y)^6 = x^6 + 6x^5y + 15x^4y^2 + 20x^3y^3 + 15x^2y^4 + 6xy^5 + y^6$$

If you look at these expansions you may notice some patterns.

1. There are $n + 1$ terms in a binomial raised to the nth power.
2. The powers of x decrease by 1, each term beginning with x^n and ending with x^0.
3. The powers of y increase by 1, each term beginning with y^0 and ending with y^n.
4. The first term is x^n and the last term is y^n.
5. In each term, the sum of the exponents of x and y is n. For example, the third term of $(x + y)^6$ contains x^4y^2 and $4 + 2 = 6$, and the fifth term contains $xy^5 = x^1y^5$ and $1 + 5 = 6$.
6. The coefficients of terms equidistant from the ends are equal.
7. If the coefficient of any term is multiplied by the exponent of x in that term, and this product is divided by the exponent of y in the next term, we get the coefficient of the next term.
8. The coefficients form a pattern known as **Pascal's triangle** in which each coefficient is the sum of the two nearest coefficients in the row above.

$$
\begin{array}{lccccccccccccccc}
n = 0 & & & & & & & & 1 & & & & & & & \\
n = 1 & & & & & & & 1 & & 1 & & & & & & \\
n = 2 & & & & & & 1 & & 2 & & 1 & & & & & \\
n = 3 & & & & & 1 & & 3 & & 3 & & 1 & & & & \\
n = 4 & & & & 1 & & 4 & & 6 & & 4 & & 1 & & & \\
n = 5 & & & 1 & & \multicolumn{3}{c}{\underbrace{5 \quad + \quad 10}} & & 10 & & 5 & & 1 & & \\
n = 6 & & 1 & & 6 & & 15 & & 20 & & \multicolumn{3}{c}{\underbrace{15 \quad + \quad 6}} & & 1 & \\
n = 7 & 1 & & 7 & & 21 & & 35 & & 35 & & 21 & & 7 & & 1
\end{array}
$$

You can see that the first and last number in each row is 1 and the second and next-to-last numbers are n. The pattern in Pascal's triangle can be continued forever. It would take a long time to develop row 26 if you needed the coefficients for $(x + y)^{26}$. That is why pattern number 7 is easier to use.

EXAMPLE 20.32

Use Pascal's triangle to expand $(3a - 2b)^6$.

SOLUTION Here $x = 3a$ and $y = -2b$. The coefficients from row 6 of Pascal's triangle are 1, 6, 15, 20, 15, 6, and 1, and so we get

$$
\begin{aligned}
&1(3a)^6 + 6(3a)^5(-2b) + 15(3a)^4(-2b)^2 + 20(3a)^3(-2b)^3 \\
&\qquad + 15(3a)^2(-2b)^4 + 6(3a)(-2b)^5 + 1(-2b)^6 \\
&= 729a^6 - 2{,}916a^5b + 4{,}860a^4b^2 - 4{,}320a^3b^3 + 2{,}160a^2b^4 \\
&\quad - 576ab^5 + 64b^6
\end{aligned}
$$

Notice that the signs alternate when the second term in the binomial is negative.

BINOMIAL FORMULA

If we were to apply pattern number 7 to a general binomial expansion $(x + y)^n$, we would get the following expansion known as the **binomial formula**:

$$
\begin{aligned}
(x + y)^n &= x^n + nx^{n-1}y + \frac{n(n-1)}{2}x^{n-2}y^2 \\
&\quad + \frac{n(n-1)(n-2)}{2 \cdot 3}x^{n-3}y^3 + \cdots + y^n
\end{aligned}
$$

Mathematicians have developed a way to abbreviate these coefficients. They use $\binom{n}{r}$ to represent $\frac{n!}{r!(n-r)!}$, where $n!$, read "n factorial," is the product

$$
\begin{aligned}
& n! = n(n-1)(n-2)(n-3)\cdots(3)(2)(1) \\
\text{and} \quad & 0! = 1
\end{aligned}
$$

EXAMPLE 20.33

Determine 5! and 8!.

SOLUTIONS $5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120$

$8! = 8 \cdot 7 \cdot 6 \cdot (5!) = 40{,}320$

FACTORIALS WITH A CALCULATOR

Some calculators have a key marked $n!$ or $x!$. On some of these calculators you have to use the **2nd** key in order to use the $x!$. As you can see from comparing 5! and 8! in the last example, factorials get very large, very quickly. To determine 10! with a calculator you would

PRESS	DISPLAY
10 $n!$	3628800

On a TI-83/4 a factorial can be evaluated using the **!** key. To access the **!** press **MATH** ◄ **PRB** **4** [4: !]. For example, 7! would be 7 **MATH** ◄ **PRB** **4** [4: !] **ENTER**, with a result of 5040.

FACTORIALS WITH A SPREADSHEET

Excel has a built-in function `FACT` for evaluating factorials. To determine 9!, enter = `FACT(9)`, with a result of 362880.

Now,

$$\begin{aligned}\binom{n}{2} &= \frac{n!}{2!(n-2)!}\\ &= \frac{n(n-1)(n-2)\cdots(3)(2)(1)}{(2)(1)(n-2)(n-3)\cdots(2)(1)}\\ &= \frac{n(n-1)}{2\cdot 1}\end{aligned}$$

This is the coefficient we got for the $x^{n-2}y^2$ term. A similar pattern holds for each term. This allows us to rewrite the binomial formula as follows.

BINOMIAL FORMULA

$$(x+y)^n = x^n + \binom{n}{1}x^{n-1}y + \binom{n}{2}x^{n-2}y^2 + \binom{n}{3}x^{n-3}y^3 + \cdots$$

$$+ \binom{n}{n-1}xy^{n-1} + y^n$$

$$= \sum_{r=0}^{n}\binom{n}{r}x^{n-r}y^r$$

where the general term is $\binom{n}{r}x^{n-r}y^r$ and

$$\binom{n}{r} = \frac{n!}{(n-r)!r!} = \frac{n(n-1)(n-2)\ \cdots\ (n-r+1)}{r!}$$

EXAMPLE 20.34

Find the first four terms $(x + y)^{15}$.

SOLUTION The first two coefficients are 1 and 15. The next two terms are

$$\binom{15}{2} = \frac{15!}{2!13!} = \frac{15 \cdot 14 \cdot 13 \cdots 3 \cdot 2 \cdot 1}{(2 \cdot 1)(13 \cdot 12 \cdots 3 \cdot 2 \cdot 1)} = \frac{15 \cdot 14}{2} = 105$$

and

$$\binom{15}{3} = \frac{15!}{312!} = \frac{15 \cdot 14 \cdot 13 \cdots 3 \cdot 2 \cdot 1}{(3 \cdot 2 \cdot 1)(12 \cdot 11 \cdots 2 \cdot 1)} = \frac{15 \cdot 14 \cdot 13}{3 \cdot 2 \cdot 1} = 455$$

So, the first four terms of $(x + y)^{15}$ are

$$x^{15} + 15x^{14}y + 105x^{13}y^2 + 455x^{12}y^3$$

BINOMIAL COEFFICIENTS WITH A CALCULATOR

You could have used the **!** key on your calculator to help determine these coefficients. On a TI-83/4 the **!** is obtained by pressing **MATH** **◄** [PRB] **4** [4:!]. For example, to determine $\binom{15}{3}$ on a TI-83

PRESS	DISPLAY
15 **MATH** **◄** [PRB] **4** [4:!]	15!
÷ **(** 12 **MATH** **◄** [PRB] **4** [4:!]	15!/(12!
***** 3 **MATH** **◄** [PRB] **4** [4:!] **)**	15!/(12!*3!)
ENTER	455

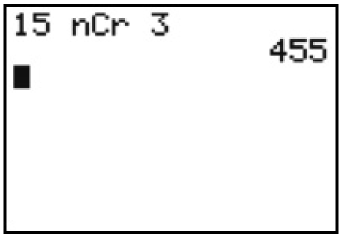


Figure 20.7a

Some calculators have a **$C_{y,x}$** or **$_nC_r$** key that can be used to determine this value. For example, on a TI-83/4, you first press the top number in this expression, 15, then press **MATH** to access that Math menu. Next press **◄** so that the PRB at the top of the Math menu is highlighted. (PRB stands for probability.) There are several operations listed on the PRB screen. The third operation is listed as "3: nCr." Press "3" (or **▼** three times) and the second number, 3. Press **ENTER**. The screen now displays the answer, 455, as shown in Figure 20.7a.

BINOMIAL COEFFICIENTS WITH A SPREADSHEET

The coefficients of the binomial formula can be found using a built-in function in Excel. To find $\binom{15}{3}$, use the function `COMBIN (x, y)`, where $x = 15$ and $y = 3$, as shown in Figure 20.7b.

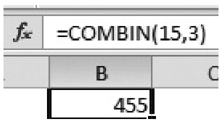


Figure 20.7b

EXAMPLE 20.35

Find the eighth term of $(x - 2)^{17}$.

SOLUTION The eighth term will be

$$\begin{aligned}\binom{17}{7}x^{10}(-2)^7 &= \frac{17!}{7!10!}x^{10}(-2)^7\\ &= 19{,}448x^{10}(-128)\\ &= -2{,}489{,}344x^{10}\end{aligned}$$

BINOMIAL SERIES

If we rewrite the binomial in the form $1 + x$, then we obtain the **binomial series**

$$(1 + x)^n = 1 + nx + \binom{n}{2}x^2 + \binom{n}{3}x^3 + \cdots + \binom{n}{n-1}x^{n-1} + x^n$$

It can be shown that the binomial series is valid for any real number n if $|x| < 1$. If n is negative or a rational number, we then get an infinite series.

$$(1 + x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 \cdots$$

Normally, with an infinite binomial series, we calculate as many terms as are needed. One application of the binomial series is to find numerical approximations, particularly if you want more accuracy than you can get from a calculator or computer.

EXAMPLE 20.36

Find the value of $\sqrt[7]{0.875}$ to three significant digits.

SOLUTION We can rewrite $\sqrt[7]{0.875} = (1 - 0.125)^{1/7}$. We will use the binomial series with $n = \frac{1}{7}$ and $x = -0.125$.

$$\sqrt[7]{0.875} = 1 + \frac{1}{7}(-0.125) + \frac{\frac{1}{7}\left(\frac{-6}{7}\right)}{2}(-0.125)^2 + \frac{\frac{1}{7}\left(\frac{-6}{7}\right)\left(\frac{-13}{7}\right)}{6}(-0.125)^3 + \cdots$$

We can omit the terms after the fourth one, because they are not significant.

$$\sqrt[7]{0.875} \approx 1 - 0.0179 - 0.0010 - 0.0001 = 0.981.$$

Checking this on a calculator, you get 0.9811 . . . , which is the same to three significant digits.

EXAMPLE 20.37

Approximate $\lim_{n\to\infty}\left(1 + \frac{1}{n}\right)^n$ and show that it is the number e.

SOLUTION This is a binomial series and

$$\left(1 + \frac{1}{n}\right)^n = 1 + n\left(\frac{1}{n}\right) + \frac{n(n-1)}{2!}\left(\frac{1}{n}\right)^2 + \frac{n(n-1)(n-2)}{3!}\left(\frac{1}{n}\right)^3 + \cdots$$

$$= 1 + 1 + \frac{1}{2!}\left(\frac{n-1}{n}\right) + \frac{1}{3!}\left(\frac{n-1}{n}\right)\left(\frac{n-2}{n}\right) + \cdots$$
$$= 1 + 1 + \frac{1}{2!}\left(1 - \frac{1}{n}\right) + \frac{1}{3!}\left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right) + \cdots$$

Since $\lim_{n\to\infty} \frac{1}{n} = 0$, this is equivalent to

$$1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots$$

If you calculate these values, you will get 2.7083333. Including more values will show that

$$\lim_{n\to\infty} 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!} + \cdots = 2.71828\cdots = e.$$

EXERCISE SET 20.5

In Exercises 1–8, expand and simplify the given expression by use of Pascal's triangle.

1. $(a + 1)^4$
2. $(2x + b)^5$
3. $(3x - 1)^4$
4. $(x - 2y)^5$
5. $\left(\frac{x}{2} + d\right)^6$
6. $\left(xy - \frac{a}{3}\right)^6$
7. $\left(\frac{a}{2} - \frac{4}{b}\right)^5$
8. $\left(\frac{x}{3} + \frac{2}{y}\right)^4$

In Exercises 9–16, expand and simplify the given expression by use of the binomial formula.

9. $(a + b)^7$
10. $(a + b)^9$
11. $(t - a)^8$
12. $(x - 2a)^7$
13. $(2a - 1)^6$
14. $(a^2 - 3)^9$
15. $\left(x^2y + \frac{a}{2}\right)^7$
16. $\left(\frac{a}{3} - \frac{xy}{2}\right)^6$

In Exercises 17–22, find the first four terms of each binomial expansion.

17. $(x + y)^{12}$
18. $(x - 5)^{10}$
19. $(1 - a)^{-2}$
20. $(1 + x)^{1/2}$
21. $(1 + b)^{-1/3}$
22. $(1 - y)^{-1/4}$

In Exercises 23–30, approximate the values of the given expression to three decimal places using the binomial series. Check your result with a calculator.

23. $(1.1)^4 = (1 + 0.1)^4$
24. $(0.85)^5 = (1 - 0.15)^5$
25. $\sqrt{1.1}$
26. $\sqrt[3]{1.01}$
27. $\sqrt[5]{1.04}$
28. $\sqrt[6]{0.98}$
29. $\sqrt[3]{0.95}$
30. $\sqrt[4]{0.925}$

In Exercises 31–36, find the indicated term of the given binomial expression.

31. The sixth term of $(x + y)^{15}$

32. The eighth term of $(a + b)^{20}$

33. The fifth term of $(2x - y)^{12}$

34. The fourth term of $(3x - 2y)^{10}$

35. The term involving b^4 in $(a + b)^{14}$

36. The term involving x^5 in $(x + y)^{15}$

Solve Exercises 37–40.

37. ***Nuclear physics*** The energy of an electron traveling at speed v in special relativity theory is $mc^2\left(1 - \frac{v^2}{c^2}\right)^{-1/2}$, where m is the electron mass and c is the speed of light. The factor mc^2 is called the *rest mass energy* (or the energy when $v = 0$). **(a)** Find the first three terms of the expansion of $\left(1 - \frac{v^2}{c^2}\right)^{-1/2}$. **(b)** Multiply the result of **(a)** by mc^2 to get the energy at speed v.

38. ***Nuclear physics*** The velocity v of electrons from a high-energy accelerator is very near the velocity of light, c. Given the voltage V of the accelerator, we often need to calculate the ratio v/c. This ratio can be calculated from the formula $\frac{v}{c} = \sqrt{1 - \frac{1}{4V^2}}$, V = number of million volts

(a) Find the first two terms of the expansion of

$$\sqrt{1 - \frac{1}{4V^2}}$$

(b) Use your answer from **(a)** to determine $1 - \frac{v}{c}$, given the following values of V. (Remember, V is the number of million volts.)

(i) 100 million volts

(ii) 500 MV (1 MV = 1 megavolt = 1 million volts)

(iii) 125 billion volts

(iv) 250 GV (1 GV = 1 gigavolt = 1 billion volts)

39. ***Physics*** At a point in a magnetic field, the field strength is given by

$$H = \frac{2m\ell}{(r^2 + \ell^2)^{3/2}}$$

Use the binomial series to find the first three terms of the expression $(r^2 + \ell^2)^{3/2}$ by expressing it as

$$(r^2)^{3/2}\left(1 + \frac{\ell^2}{r^2}\right)^{3/2}$$

40. In the formula for a derivative in calculus, the following expression may occur:

$$\frac{(x + h)^4 - x^4}{h}$$

Expand and simplify this expression.

[IN YOUR WORDS]

41. Describe the binomial formula. What is the binomial formula used for?

42. What is Pascal's triangle and what does it have to do with the binomial formula?

CHAPTER 20 REVIEW

IMPORTANT TERMS AND CONCEPTS

Arithmetic sequence
- Common difference
- Sum of first n terms

Binomial formula
Binomial theorem
Geometric sequence
- Common ratio
- Sum of first n terms

Limit
Partial sum
Pascal's triangle
Recursion formula
Sequence
- Arithmetic
- Finite
- Geometric
- Infinite

Series
- Arithmetic
- Binomial
- Convergent
- Divergent
- Finite
- Geometric
- Infinite

Summation notation

REVIEW EXERCISES

In Exercises 1–4, find the first six terms with the specified general term.

1. $a_n = \dfrac{1}{n+2}$ **2.** $b_n = \dfrac{3}{2n-1}$ **3.** $\left\{\dfrac{(-1)^n}{n(2n+1)}\right\}$ **4.** $\left\{\dfrac{n^2+1}{3n-1}\right\}$

In Exercises 5–8, find the first six terms of the recursively defined sequence.

5. $a_1 = 1, a_n = \dfrac{a_{n-1}}{n}$

6. $a_1 = 5, a_n = a_{n-1} - n$

7. $a_1 = 1, a_2 = 3, a_n = a_{n-1} + a_{n-2}$

8. $a_1 = 1, a_2 = 2, a_n = (a_{n-1})(a_{n-2})$

In Exercises 9–14, determine whether the given sequence is an arithmetic sequence, a geometric sequence, or neither. For the arithmetic and geometric sequences, find the common difference or ratio and the indicated term.

9. 10, 7, 4, . . . (10th term)

10. $8, 2, \frac{1}{2}, \cdots$ (8th term)

11. $-1, 3, 7, \ldots$ (7th term)

12. $7, 4, 0, -5, \ldots$ (9th term)

13. $3, -\frac{1}{2}, \frac{1}{12}, \cdots$ (10th term)

14. 2.05, 20.5, 205, . . . (7th term)

Solve Exercises 15–17.

15. For an arithmetic sequence, $a_7 = 12$ and $a_8 = 18$. What is a_1?

16. For a geometric sequence, $a_6 = 9$ and $a_7 = \frac{9}{7}$. What is a_8?

17. Write the first four terms of

(a) $\displaystyle\sum_{k=1}^{30} 4\left(\frac{1}{3}\right)^k$ and **(b)** $\displaystyle\sum_{k=0}^{25} \frac{k+2}{k+1}$.

In Exercises 18–20, evaluate the given sum.

18. $\displaystyle\sum_{k=1}^{4} (k+1)$ **19.** $\displaystyle\sum_{k=0}^{5} (3k-1)$ **20.** $\displaystyle\sum_{n=1}^{4} \frac{(-1)^n n}{2^n+1}$

In Exercises 21–24, determine whether the terms of the given series form an arithmetic or geometric series and find the indicated sum.

21. $4 + 9 + 14 + \cdots; S_{10}$

22. $2 - 5 - 12, \cdots; S_{12}$

23. $1 + \frac{1}{3} + \frac{1}{9} + \cdots; S_{14}$

24. $\sqrt{5} + 1 + 2\sqrt{5} + 2 + 3\sqrt{5} + 3 \cdots; S_8$

In Exercises 25–30, determine which of the infinite geometric series converge and which diverge. For those that converge, find the sum.

25. $\frac{1}{3} + \frac{1}{6} + \frac{1}{12} + \cdots$

26. $8 + 6 + 4.5 + \cdots$

27. $0.03 + 0.3 + 3 + \cdots$

28. $1.5 - 0.15 + 0.015 + \cdots$

29. $\sum_{n=1}^{\infty} \left(\frac{1}{\sqrt{3}}\right)^n$

30. $\sum_{n=0}^{\infty} (\sqrt{2})^{-n}$

In Exercises 31 and 32, use infinite series to find the rational numbers corresponding to the given decimal number.

31. 0.185185 . . .

32. 0.611111 . . .

In Exercises 33–36, expand and simplify the given binomial expression.

33. $(a + 2)^5$

34. $(3x - y)^6$

35. $\left(\frac{x}{2} - 3y^2\right)^6$

36. $\left(\frac{2a^3}{5} + \frac{5b}{2}\right)^5$

In Exercises 37–40, find the first four terms of the given binomial expansion.

37. $(2x + y)^{15}$

38. $(1 + ax^2)^{10}$

39. $(1 - x)^{-5}$

40. $(1 + b)^{-1/4}$

Solve Exercises 41–50.

41. What is the seventh term of $(x + 2y)^{20}$?

42. What is the fifth term of $(a - 3b)^{12}$?

43. Approximate $\sqrt[5]{1.02}$ to three decimal places using the binomial expansion. Check your work with a calculator.

44. Approximate $\sqrt[7]{0.98}$.

45. ***Finance*** If you invest \$400 at 6% compounded monthly for 10 years, how much will you have at the end of that time?

46. Some copying machines make reduced copies. Suppose that you copy a page 20 cm wide and it comes out $\frac{3}{4}$ as wide. Then you copy this, and so on. How wide is the fifth copy?

47. If you lay the original and five copies from Exercise 46 side by side on the floor, how far will they extend?

48. Suppose that you continued the copying process that you began in Exercise 46 indefinitely. If you lay the original and all the copies side by side on the floor, how far would they extend?

49. ***Physics*** A ball starting from rest rolls down a uniform incline so that it covers 10 in. during the first second, 25 in. during the second second, 40 in. during the third second, and so on. How long will it take until it covers 250 ft in 1 sec?

50. ***Business*** A company had sales of \$250,000 during its first year of operation. Sales increased by \$40,000 per year during each successive year. What were the sales of the company in the 10th year? What were the total sales of the company for the first 10 years?

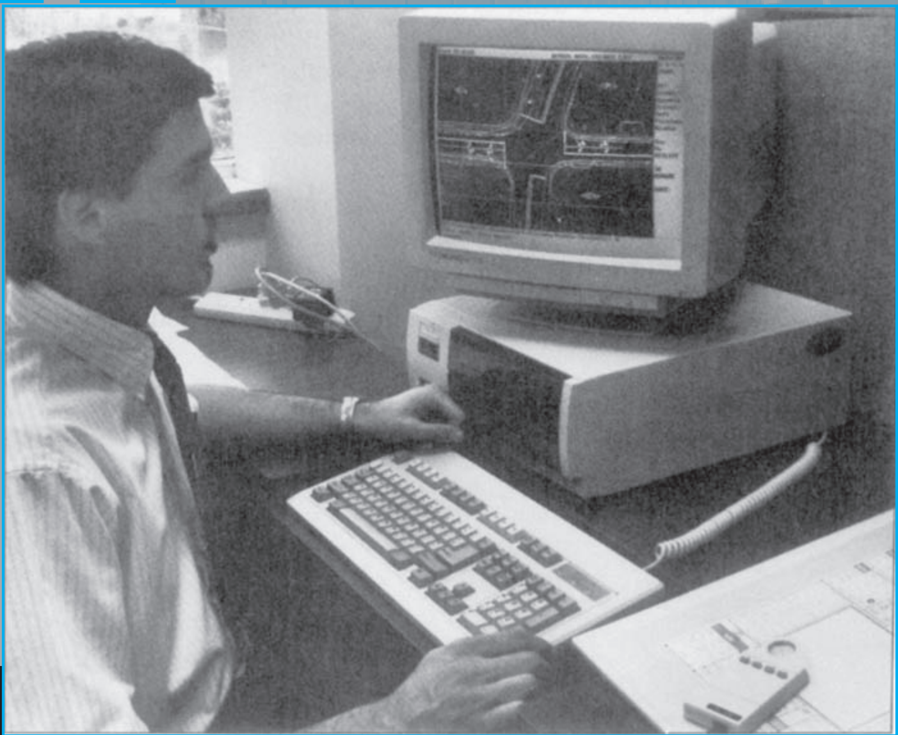
CHAPTER 20 TEST

1. Determine the first six terms of $a_n = \dfrac{5}{2n - 3}$.
2. Determine the first six terms of $a_1 = -2$, $a_n = 3 + na_{n-1}$.
3. Determine whether the given sequence is an arithmetic sequence, geometric sequence, or neither. If appropriate, find the common difference or ratio and the indicated term.

   $$15, 12.5, 10, \ldots \text{ (10th term)}$$
4. For an arithmetic sequence with $a_5 = 10$ and $a_6 = 14$, what is a_1?
5. Evaluate $\sum_{k=1}^{5}(2k - 1)$.
6. Determine whether this infinite geometric series converges or diverges. If it converges, find its sum.

   $$\frac{1}{5} - \frac{1}{10} + \frac{1}{20} + \cdots$$
7. Determine the rational number that corresponds to $0.435435\ldots$.
8. Find the first four terms of $(3x - 2y)^{12}$.
9. A computer software company predicts that it will sell 2,000 copies of a new type of software during the first month and that each month sales will be 1,500 copies more than the sales of the previous month. How many copies can it expect to sell during the first year?
10. A transistor's leakage current increases 1% for each increase of °C. If a certain transistor's leakage is 1.40 nA at 20°C, how much will it be at 30°C? (Hint: An increase of 1% means that $r = 1.01$.)

21 TRIGONOMETRIC FORMULAS, IDENTITIES, AND EQUATIONS


Courtesy of Ruby Gold

A highway engineer is designing the curve at an intersection where two highways intersect at angle θ. In Section 21.3, we will learn how to use trigonometry to help design this curve.

In earlier chapters, we discussed the fundamental reciprocal and quotient identities of trigonometry. We also learned how to use trigonometry to solve both right and oblique triangles and how to graph trigonometric functions. In this chapter, we will return to the study of trigonometry, establishing the remaining standard trigonometric identities. Among these will be identities for the sums, differences, and multiples of angles. The identities you will learn in this chapter are used in advanced mathematics, particularly calculus, and in engineering, physics, and technical areas to simplify complicated expressions and to help solve equations that involve trigonometry.

OBJECTIVES

AFTER COMPLETING THIS CHAPTER, YOU WILL BE ABLE TO:

- Verify identities using the basic eight identities.
- Use sum, difference, double-angle, and half-angle formulas to simplify expressions and verify identities.
- Solve trigonometric equations using trigonometric identities.

21.1 BASIC IDENTITIES

An *identity* is an equation that is true for all values of the variable. In Section 5.2, we introduced two groups of trigonometric identities. One group was called **reciprocal identities** and consisted of

$$\csc\theta = \frac{1}{\sin\theta} \qquad \sec\theta = \frac{1}{\cos\theta} \qquad \cot\theta = \frac{1}{\tan\theta}$$

The second group was the **quotient identities**.

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \qquad \cot\theta = \frac{\cos\theta}{\sin\theta}$$

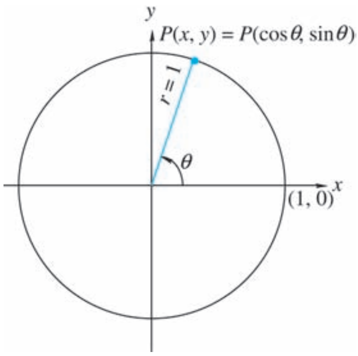


Figure 21.1

PYTHAGOREAN IDENTITY

Now, suppose that we have a unit circle as shown in Figure 21.1. (Remember, a unit circle is a circle with radius 1.) If this circle is centered at the origin and $P(x, y)$ is any point on the circle on the terminal side of an angle θ in standard position, then $x = \cos\theta$ and $y = \sin\theta$. By the Pythagorean theorem, $x^2 + y^2 = r^2$, and since $r = 1$, we get the first of the **Pythagorean identities**.

$$\sin^2\theta + \cos^2\theta = 1$$

NOTE The term $\sin^2\theta$ is an abbreviation for $(\sin\theta)^2$. Similarly, $\cos^2\theta$ is an abbreviation for $(\cos\theta)^2$ and $\tan^2\theta$ is an abbreviation for $(\tan\theta)^2$. While we write $\sin^2\theta$, $\cos^2\theta$, or $\tan^2\theta$, you must enter them in a calculator as $(\sin\theta)^2$, $(\cos\theta)^2$, or $(\tan\theta)^2$.

If we divide both sides of this identity by $\cos^2\theta$, we get

$$\frac{\sin^2\theta}{\cos^2\theta} + \frac{\cos^2\theta}{\cos^2\theta} = \frac{1}{\cos^2\theta}$$

or $$\tan^2\theta + 1 = \sec^2\theta$$

This is the second Pythagorean identity. The third, and last, Pythagorean identity is produced by dividing $\sin^2\theta + \cos^2\theta = 1$ by $\sin^2\theta$, with the result

$$1 + \cot^2\theta = \csc^2\theta$$

Thus, there are three Pythagorean identities.

PYTHAGOREAN IDENTITIES

$$\sin^2\theta + \cos^2\theta = 1$$
$$\tan^2\theta + 1 = \sec^2\theta$$
$$1 + \cot^2\theta = \csc^2\theta$$

PROVING IDENTITIES

The eight basic identities, the reciprocal, quotient, and Pythagorean identities, can be used to develop and prove other identities. Your ability to prove an identity depends greatly on your familiarity with the eight basic identities.

To prove that an identity is true, you change either side, or both sides, until the sides are the same. Each side must be worked separately. Since you do not know that the two sides are equal (which is what you are trying to prove), you cannot transpose terms from one side to the other. Some people draw a vertical line between the two sides until they can show that the sides are equal. The vertical line acts as a reminder that you should work on each side separately.

EXAMPLE 21.1

Prove the identity $\csc\theta = \dfrac{\cot\theta}{\cos\theta}$.

SOLUTION We will change the right-hand side of the identity until it looks like the left-hand side.

$$\begin{array}{r|ll}
\csc\theta & \dfrac{\cot\theta}{\cos\theta} & \\
 & \dfrac{\dfrac{\cos\theta}{\sin\theta}}{\cos\theta} & \text{Change } \cot\theta \text{ to } \dfrac{\cos\theta}{\sin\theta}. \\
 & \dfrac{\cos\theta}{\sin\theta}\cdot\dfrac{1}{\cos\theta} & \text{Change the division problem to a multiplication problem.} \\
 & \dfrac{1}{\sin\theta} & \text{Multiply.} \\
 & \csc\theta & \text{Reciprocal identity.}
\end{array}$$

So, $\csc\theta = \dfrac{\cot\theta}{\cos\theta}$.

EXAMPLE 21.2

Prove the identity: $|\sin x| = \dfrac{|\tan x|}{\sqrt{1 + \tan^2 x}}$.

SOLUTION The right-hand side is more complicated than the left-hand side, so we will simplify the right-hand side until it matches the left-hand side.

$\lvert\sin x\rvert$	$\dfrac{\lvert\tan\rvert}{\sqrt{1+\tan^2 x}}$	
	$\dfrac{\lvert\tan x\rvert}{\sqrt{sec^2 x}}$	Use the Pythagorean identity to replace $1+\tan^2 x$ with $\sec^2 x$.
	$\dfrac{\lvert\tan x\rvert}{\lvert\sec x\rvert}$	Take the square root. Notice that $\sqrt{\sec^2 x} = \lvert\sec x\rvert$.
	$\dfrac{\left\lvert\dfrac{\sin x}{\cos x}\right\rvert}{\left\lvert\dfrac{1}{\cos x}\right\rvert}$	Express $\tan x$ and $\sec x$ in terms of $\sin x$ and $\cos x$.
	$\left\lvert\dfrac{\sin x}{\cos x}\cdot\dfrac{\cos x}{1}\right\rvert$	Change the division problem to a multiplication problem.
	$\lvert\sin x\rvert$	Multiply.

Thus, we have shown that $\lvert\sin x\rvert = \dfrac{\lvert\tan x\rvert}{\sqrt{1+\tan^2 x}}$.

Notice that we had to use the properties of absolute value to prove this identity.

EXAMPLE 21.3

Prove the identity $\sec\theta - \sec\theta\sin^2\theta = \cos\theta$.

SOLUTION In this example, we start with the more complicated left-hand side and simplify it until it matches the right-hand side.

$\sec\theta - \sec\theta\sin^2\theta$	$\cos\theta$	
$\sec\theta\,(1-\sin^2\theta)$		Factor.
$\sec\theta\,(\cos^2\theta)$		Pythagorean identity.
$\dfrac{1}{\cos\theta}(\cos^2\theta)$		Reciprocal identity.
$\cos\theta$		Multiply.

And so, $\sec\theta - \sec\theta\sin^2\theta = \cos\theta$.

In this example, we used a different version of a Pythagorean identity. We used the identity $\sin^2\theta + \cos^2\theta = 1$. You should also recognize the two variations of this identity: $\sin^2\theta = 1 - \cos^2\theta$ and $\cos^2\theta = 1 - \sin^2\theta$. There are two variations of each of the other Pythagorean identities.

EXAMPLE 21.4

Prove the identity: $\sec^2\theta\csc^2\theta = \sec^2\theta + \csc^2\theta$.

SOLUTION Here we simplify the right-hand side until it matches the left-hand side.

$\sec^2\theta\csc^2\theta$	$\sec^2\theta + \csc^2\theta$	
	$\dfrac{1}{\cos^2\theta} + \dfrac{1}{\sin^2\theta}$	Reciprocal identities.
	$\dfrac{\sin^2\theta}{\cos^2\theta\sin^2\theta} + \dfrac{\cos^2\theta}{\cos^2\theta\sin^2\theta}$	Rewrite with a common denominator.
	$\dfrac{\sin^2\theta + \cos^2\theta}{\cos^2\theta\sin^2\theta}$	Add.
	$\dfrac{1}{\cos^2\theta\sin^2\theta}$	Pythagorean identity.
	$\dfrac{1}{\cos^2\theta}\cdot\dfrac{1}{\sin^2\theta}$	Factor.
	$\sec^2\theta\csc^2\theta$	Reciprocal identities.

So, $\sec^2\theta\csc^2\theta = \sec^2\theta + \csc^2\theta$.

APPLICATION GENERAL TECHNOLOGY

EXAMPLE 21.5

Malus's law concerns light incident on a polarizing plate and describes the amount of light transmitted, I, in terms of the angle of incidence, θ, and the, maximum intensity of light transmitted, M. Malus's law can be written as

$$I = M - M\tan^2\theta\cos^2\theta$$

Express the right-hand side of this equation in terms of $\cos\theta$.

SOLUTION We have

$$\begin{aligned} I &= M - M\tan^2\theta\cos^2\theta \\ &= M - M\left(\frac{\sin^2\theta}{\cos^2\theta}\right)\cos^2\theta \\ &= M - M\sin^2\theta \\ &= M(1 - \sin^2\theta) \\ &= M\cos^2\theta \end{aligned}$$

So, Malus's law can be more simply expressed as

$$I = M\cos^2\theta$$

USING GRAPHS TO HELP VERIFY IDENTITIES

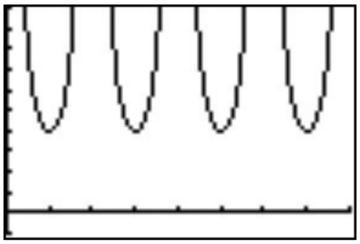
A graphing calculator, spreadsheet, or graphing software can also be used to prove or disprove an identity. With this technique you separately graph each side of the proposed identity over an interval that contains at least one complete period. If both graphs are identical, then you can conclude that the identity is true whenever the functions are defined. A major difficulty arises because neither calculators nor computers define the cot, sec, and csc functions.

EXAMPLE 21.6

Use a graphing calculator to "prove" the identity $\sec^2\theta\csc^2\theta = \sec^2\theta + \csc^2\theta$.

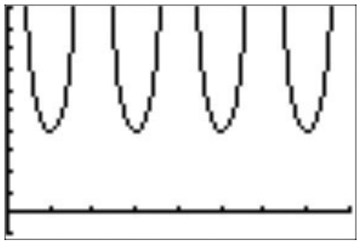
SOLUTION We know that this identity is true because it is the same identity that we proved in Example 21.4.

The graph of the left-hand side of this identity is shown in Figure 21.2a and the graph of the right-hand side is shown in Figure 21.2b. You can see that these two graphs are identical; hence this is a valid identity.



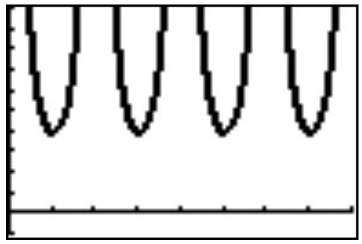
$\left[0, 2\pi, \frac{\pi}{4}\right] \times [0, 10, 1]$

Figure 21.2a



$\left[0, 2\pi, \frac{\pi}{4}\right] \times [0, 10, 1]$

Figure 21.2b



$\left[0, 2\pi, \frac{\pi}{4}\right] \times [0, 10, 1]$

Figure 21.2c

CAUTION In practice, when you graph the two sides of an identity, they both appear on the same screen. If you graph the second function using the calculator's "thick" style, as in Figure 21.2c, you can easily see if the new graph completely traces over the first function.

EXAMPLE 21.7

Use a spreadsheet to "prove" the identity $\sec^2\theta\csc^2\theta = \sec^2\theta + \csc^2\theta$.

SOLUTION We know that this identity is true because it is the same identity that we proved in Example 21.4.

Let's examine the graphs of the left- and right-hand sides. First, starting with Cell A2, enter 0 through 24 down Column A. This will be the multipliers for some fraction of π used in Column B. Enter `=A2*pi()/12` in Cell B2 and copy this down Column B (see Figure 21.3a). This gives us values between 0 and 2π, covering at least one period of the functions.

In Cell C2, enter `=(1/(COS(B2))^2)*(1/(SIN(B2))^2)` and copy this formula down Column C.

fx =A6*PI()/12

	A	B
1		t (theta)
2	0	0
3	1	0.261799
4	2	0.523599
5	3	0.785398
6	4	1.047198

Figure 21.3a

EXAMPLE 21.7 (Cont.)

Next, in Cell D2, enter `=(1/(COS(B2))^2)+(1/(SIN(B2))^2)` and copy this formula down Column D, as shown in Figure 21.3b.

It is obvious, from the table, that the values of each function are exactly the same. So, we expect the graphs to be the same, as shown in Figure 21.3c.

Looking at Figure 21.3c, we cannot tell that there are two graphs. A second way to use a spreadsheet to prove or disprove an identity is to subtract the two sides of the proposed identity and examine the difference. If the difference is zero over at least one period, then you can conclude that the identity is true whenever the functions are defined.

Column E of the spreadsheet contains the difference between the value in Column C and the value in Column D. We expect the values in this column to be zero. As you can see in Figure 21.2d, our expectations are fulfilled and we can conclude that the identity is "proved."

fx =(1/COS(B8))^2+(1/SIN(B8))^2

	A	B	C	D
1		t (theta)	(sec t)^2*(csc t)^2	(sec t)^2+(csc t)^2
2	0	0	#DIV/0!	#DIV/0!
3	1	0.261799	16	16
4	2	0.523599	5.333333333	5.333333333
5	3	0.785398	4	4
6	4	1.047198	5.333333333	5.333333333
7	5	1.308997	16	16
8	6	1.570796	2.66491E+32	2.66491E+32

Figure 21.3b

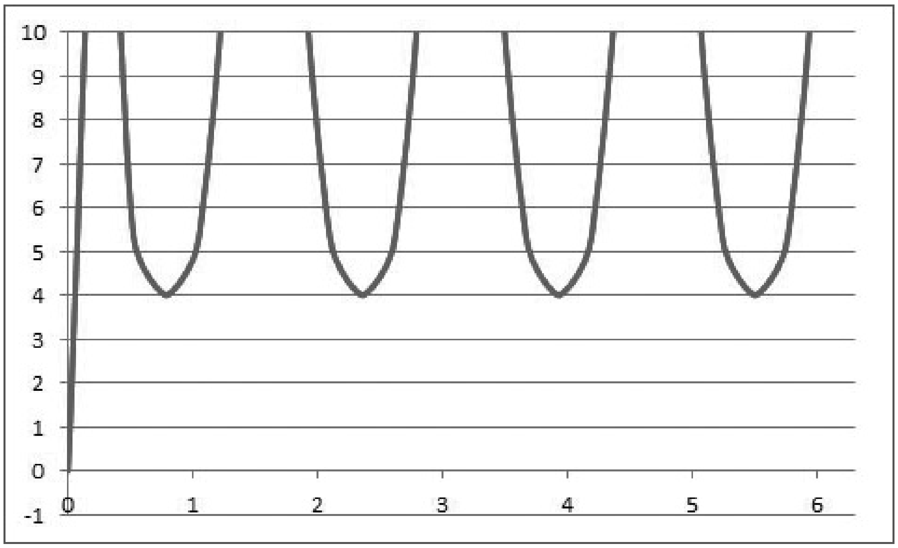


Figure 21.3c

f_x =C8-D8

	A	B	C	D	E
1		t (theta)	(sec t)^2*(csc t)^2	(sec t)^2+(csc t)^2	Difference
2	0	0	#DIV/0!	#DIV/0!	#DIV/0!
3	1	0.261799	16	16	0
4	2	0.523599	5.333333333	5.333333333	0
5	3	0.785398	4	4	0
6	4	1.047198	5.333333333	5.333333333	0
7	5	1.308997	16	16	0
8	6	1.570796	2.66491E+32	2.66491E+32	0

Figure 21.3d

EXERCISE SET 21.1

1. Prove the Pythagorean identity $1 + \cot^2\theta = \csc^2\theta$ from the identity $\sin^2\theta + \cos^2\theta = 1$.

Prove each of the identities in Exercises 2–26.

2. $\tan x \cot x = 1$

3. $\sin\theta \sec\theta = \tan\theta$

4. $\cos\theta(\tan\theta + \sec\theta) = \sin\theta + 1$

5. $\dfrac{\sin\theta}{\cot\theta} = \sec\theta - \cos\theta$

6. $\tan x = \dfrac{\sec x}{\csc x}$

7. $(1 - \sin^2\theta)(1 + \tan^2\theta) = 1$

8. $\dfrac{\sin A}{\csc A} + \dfrac{\cos A}{\sec A} = 1$

9. $1 - \dfrac{\sin A}{\csc A} = \cos^2 A$

10. $(1 + \tan\theta)(1 - \tan\theta) = 2 - \sec^2\theta$

11. $(1 + \cos x)(1 - \cos x) = \sin^2 x$

12. $\sec^4 x - \sec^2 x = \tan^4 x + \tan^2 x$

13. $2\csc\theta = \dfrac{\sin\theta}{1 + \cos\theta} + \dfrac{1 + \cos\theta}{\sin\theta}$

14. $\cos x = \sin x \cot x$

15. $(\sin\theta + \cos\theta)^2 = 1 + 2\sin\theta\cos\theta$

16. $\csc^2 x(1 - \cos^2 x) = 1$

17. $\dfrac{\tan\theta + \cot\theta}{\tan\theta - \cot\theta} = \dfrac{\tan^2\theta + 1}{\tan^2\theta - 1}$

18. $\dfrac{1 - \sin x}{\cos x} = \dfrac{\cos x}{1 + \sin x}$

19. $\dfrac{\sec\theta - \csc\theta}{\sec\theta + \csc\theta} = \dfrac{\tan\theta - 1}{\tan\theta + 1}$

20. $(\sin A + \cos A)^2 + (\sin A - \cos A)^2 = 2$

21. $\tan^2 x\cos^2 x + \cot^2 x\sin^2 x = 1$

22. $\tan\theta + \dfrac{\cos\theta}{1 + \sin\theta} = \sec\theta$

23. $\sec^4 x - \sec^2 x = \tan^2 x\sec^2 x$

24. $\cos^2 A - \sin^2 A = 2\cos^2 A - 1$

25. $\dfrac{\tan\theta - \sin\theta}{\sin^3\theta} = \dfrac{\sec\theta}{1 + \cos\theta}$

26. $\dfrac{\sin x - \cos x + 1}{\sin x + \cos x - 1} = \dfrac{\sin x + 1}{\cos x}$

In Exercises 27–34, use your graphing calculator or a spreadsheet to prove or disprove each of the following identities.

27. $\tan^2\theta\csc^2\theta\cot^2\theta\sin^2\theta = 1$

28. $\tan x\sin x + \cos x = \sec x$

29. $\dfrac{\sec A + \csc A}{\tan A + \cot A} = \sin A + \cos A$

30. $\dfrac{\sin^3 x + \cos^3 x}{\sin x + \cos x} = 1 - \sin x\cos x$

31. $2\csc 2x = \sec x\csc x$

32. $\cos 2x + 1 = 2\cos^2 x$

33. $\sin \frac{1}{2}x = \frac{1}{2}\sin x$

34. $1 - \cos 2x = \sin^2 x$

To show that something is not an identity, all you need is one counterexample. A counterexample is an example that shows that something is not true. In Exercises 35–39, use the indicated angles as a counterexample to show that the relation is not an identity.

35. $2\sin\theta \neq \sin 2\theta$; $\theta = 90°$

36. $\dfrac{\tan A}{2} \neq \tan\left(\dfrac{A}{2}\right)$; $A = 60°$

37. $\cos(\theta^2) \neq (\cos\theta)^2$; $\theta = \pi$

38. $\sin(x - y) \neq \sin x - \sin y$; $x = 60°$, $y = 30°$

39. $\sin x \neq \dfrac{\tan x}{\sqrt{1 + \tan^2 x}}$; $x = 120°$

Solve Exercises 40–44.

40. In finding the rate of change of $\cot x$, you get the expression $\dfrac{(\sin x)(-\sin x) - (\cos x)(\cos x)}{\sin^2 x}$. Show that this is equal to $-\csc^2 x$.

41. In finding the rate of change of $\cot^2 x$, you get the expression $-2\cot x\csc^2 x$. Show that this is equal to $-2\cos x\csc^3 x$.

42. In calculus, in order to determine the integral of $\sin^5 x$, we need to show that it is identical to $(1 - 2\cos^2 x + \cos^4 x)\sin x$. Prove this identity.

43. ***Electricity*** In electric circuit theory, we use the expression

$$\frac{(1.2\sin\omega t - 1.6\cos\omega t)^2 + (1.6\sin\omega t + 1.2\cos\omega t)^2}{2L}$$

Show that this is identical to $\dfrac{2.0}{L}$.

44. ***Physics*** The range, R, of a projectile fired at an acute angle θ with the horizontal at an initial velocity v is given by

$$R = \frac{2v^2\cos\theta\sin\theta}{g}$$

where g is the constant of gravitational acceleration. Rewrite the right-hand side using the sine function but not the cosine function.

[IN YOUR WORDS]

45. Develop the other two Pythagorean identities from the identity $\sin^2 x + \cos^2 x = 1$.

46. Explain the process you use to prove an identity is true.

21.2 THE SUM AND DIFFERENCE IDENTITIES

We saw in Exercises 35 and 38 in Exercise Set 21.1 that $2\sin\theta \neq \sin 2\theta$ and that $\sin(x - y) \neq \sin x - \sin y$. In this section, we will develop some identities for the sum and differences of the trigonometric functions. These identities are important for further studies in mathematics, such as in calculus. They are also important in wave mechanics, electric circuit theory, and in theory for other technical areas.

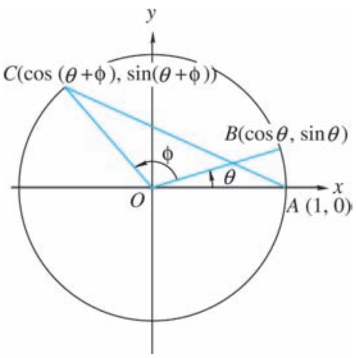


Figure 21.4a

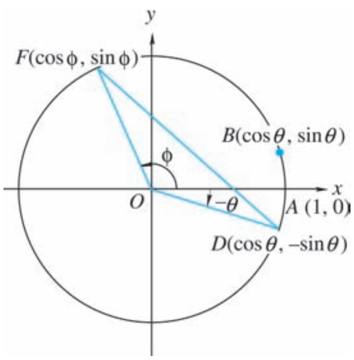


Figure 21.4b

COS($\theta + \phi$)

We will begin with a rather lengthy development of the identity for the $\cos(\theta + \phi)$. Even this lengthy proof does not include all cases, but it will serve our purposes.

In Figures 21.4a and 21.4b, we have drawn two unit circles. In Figure 21.4a, $\angle AOB$ is θ and $\angle BOC$ is ϕ, so $\angle AOC$ is $\theta + \phi$. The coordinates of A, B, and C are also given. Since A is on the x-axis, its coordinates are $(1, 0)$. The coordinates of B are given in terms of θ and those of C are given in terms of $\theta + \phi$.

In Figure 21.4b, we have rotated $\triangle AOC$ through the angle $-\theta$ to get $\triangle DOF$. The coordinates of D and F are given in terms of θ and ϕ. Now, since $\triangle AOC$ is congruent to $\triangle DOF$, the distance from A to C must be the same as the distance from D to F. According to the distance formula from Section 15.1, the distance from A to C is

$$d(A, C) = \sqrt{(\cos(\theta + \phi) - 1)^2 + (\sin(\theta + \phi) - 0)^2}$$

Squaring both sides, we get

$$\begin{aligned}
[d(A, C)]^2 &= (\cos(\theta + \phi) - 1)^2 + (\sin(\theta + \phi) - 0)^2 \\
&= \cos^2(\theta + \phi) - 2\cos(\theta + \phi) + 1 + \sin^2(\theta + \phi) \\
&= 2 - 2\cos(\theta + \phi)
\end{aligned}$$

In a similar manner, the distance from D to F is

$$d(D, F) = \sqrt{(\cos\phi - \cos\theta)^2 + (\sin\phi + \sin\theta)^2}$$

Again, squaring both sides, we get

$$\begin{aligned}
[d(D, F)^2] &= (\cos\phi - \cos\theta)^2 + (\sin\phi + \sin\theta)^2 \\
&= \cos^2\phi - 2\cos\theta\cos\phi \\
&\quad + \cos^2\theta + \sin^2\phi + 2\sin\theta\sin\phi + \sin^2\theta \\
&= (\cos^2\phi + \sin^2\phi) + (\cos^2\theta + \sin^2\theta) \\
&\quad - 2\cos\theta\cos\phi + 2\sin\theta\sin\phi \\
&= 2 - 2\cos\theta\cos\phi + 2\sin\theta\sin\phi.
\end{aligned}$$

Since $d(A, C) = d(D, F)$, we have

$$\begin{aligned}
2 - 2\cos(\theta + \phi) &= 2 - 2\cos\theta\cos\phi + 2\sin\theta\sin\phi \\
\text{or} \qquad \cos(\theta + \phi) &= \cos\theta\cos\phi - \sin\theta\sin\phi
\end{aligned}$$

COS($\theta - \phi$)

If we substitute $-\phi$ for ϕ in the previous formula and remember the two identities $\cos(-\phi) = \cos\phi$ and $\sin(-\phi) = -\sin\phi$, we could show that

$$\cos(\theta - \phi) = \cos\theta\cos\phi + \sin\theta\sin\phi$$

Identities for the sum and difference of the sine and tangent functions can also be developed in a similar manner. The result is a total of six sum and difference identities.

SUM AND DIFFERENCE IDENTITIES

$$\sin(\theta + \phi) = \sin\theta\cos\phi + \cos\theta\sin\phi$$
$$\sin(\theta - \phi) = \sin\theta\cos\phi - \cos\theta\sin\phi$$
$$\cos(\theta + \phi) = \cos\theta\cos\phi - \sin\theta\sin\phi$$
$$\cos(\theta - \phi) = \cos\theta\cos\phi + \sin\theta\sin\phi$$
$$\tan(\theta + \phi) = \frac{\tan\theta + \tan\phi}{1 + \tan\theta\tan\phi}$$
$$\tan(\theta - \phi) = \frac{\tan\theta - \tan\phi}{1 + \tan\theta\tan\phi}$$

EXAMPLE 21.8

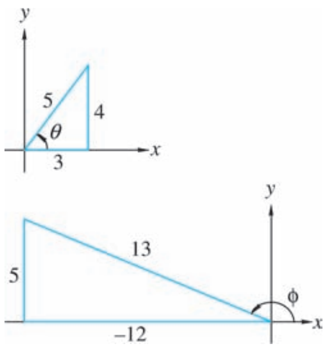


Figure 21.5

If $\sin\theta = \frac{4}{5}$, $\cos\phi = \frac{-12}{13}$, θ is in Quadrant I, and ϕ is in Quadrant II, find (a) $\sin(\theta + \phi)$, (b) $\cos(\theta - \phi)$, and (c) $\tan(\theta + \phi)$.

SOLUTIONS If we draw reference triangles for θ and ϕ, and use the Pythagorean theorem to determine the missing side, we can determine the values of $\cos\theta$, $\tan\theta$, $\sin\phi$, and $\tan\phi$. These triangles are shown in Figure 21.5. From them, we can determine that $\cos\theta = \frac{3}{5}$, $\tan\theta = \frac{4}{3}$, $\sin\phi = \frac{5}{13}$, and $\tan\phi = \frac{-5}{12}$. We are now ready to apply the formulas.

(a)
$$\begin{aligned}\sin(\theta + \phi) &= \sin\theta\cos\phi + \cos\theta\sin\phi\\ &= \left(\frac{4}{5}\right)\left(\frac{-12}{13}\right) + \left(\frac{3}{5}\right)\left(\frac{5}{13}\right)\\ &= \frac{-48}{65} + \frac{15}{65} = -\frac{33}{65}\end{aligned}$$

(b)
$$\begin{aligned}\cos(\theta - \phi) &= \cos\theta\cos\phi + \sin\theta\sin\phi\\ &= \left(\frac{3}{5}\right)\left(\frac{-12}{13}\right) + \left(\frac{4}{5}\right)\left(\frac{5}{13}\right)\\ &= \frac{-36}{65} + \frac{20}{65} = -\frac{16}{65}\end{aligned}$$

(c)
$$\begin{aligned}\tan(\theta + \phi) &= \frac{\tan\theta + \tan\phi}{1 - \tan\theta\tan\phi}\\ &= \frac{\frac{4}{3} + \frac{-5}{12}}{1 - \left(\frac{4}{3}\right)\left(\frac{-5}{12}\right)}\\ &= \frac{\frac{16}{12} - \frac{5}{12}}{\frac{56}{36}} = \frac{\frac{11}{12}}{\frac{56}{36}}\\ &= \frac{11}{12}\cdot\frac{36}{56} = \frac{33}{56}\end{aligned}$$

So, $\sin(\theta + \phi) = -\frac{33}{65}$, $\cos(\theta - \phi) = -\frac{16}{65}$, and $\tan(\theta + \phi) = \frac{33}{56}$. Since $\sin(\theta + \phi)$ is negative and $\tan(\theta + \phi)$ is positive, $\theta + \phi$ is in the third quadrant.

EXAMPLE 21.9

If $\sin \alpha = 0.25$ and $\cos \beta = 0.65$, α is in Quadrant II, and β is in Quadrant I, find (a) $\sin(\alpha - \beta)$ and (b) $\tan(\alpha + \beta)$.

SOLUTIONS Using the Pythagorean theorem and the fact that α is in Quadrant II, we determine that $\cos \alpha = -\sqrt{1 - 0.25^2} \approx -0.97$. Similarly, we find that $\sin \beta \approx 0.76$. We can now apply the formulas.

$$\begin{aligned}
\text{(a)}\ \sin(\alpha - \beta) &= \sin \alpha \cos \beta - \cos \alpha \sin \beta \\
&\approx (0.25)(0.65) - (-0.97)(0.76) \\
&= 0.1625 + 0.7372 \\
&= 0.8997 \\
&\approx 0.90
\end{aligned}$$

$$\begin{aligned}
\text{(b)}\ \tan(\alpha + \beta) &= \frac{\tan \alpha + \tan \beta}{1 - \tan \alpha \tan \beta} \\
&\approx \frac{\frac{0.25}{-0.97} + \frac{0.76}{0.65}}{1 - \left(\frac{0.25}{-0.97}\right)\left(\frac{0.76}{0.65}\right)} \\
&\approx \frac{-0.2577 + 1.1692}{1 - (-0.2577)(1.1692)} \\
&\approx 0.7005
\end{aligned}$$

EXAMPLE 21.10

Find the exact value of $\sin 75°$ by using the trigonometric values for 30° and 45°.

SOLUTION Since $75° = 30° + 45°$, we will use $\sin 75° = \sin(30° + 45°)$. Now $\sin 30° = \frac{1}{2}$, $\cos 30° = \frac{\sqrt{3}}{2}$, and $\sin 45° = \cos 45° = \frac{\sqrt{2}}{2}$. This gives

$$\begin{aligned}
\sin 75° &= \sin 30° \cos 45° + \cos 30° \sin 45° \\
&= \frac{1}{2} \cdot \frac{\sqrt{2}}{2} + \frac{\sqrt{3}}{2} \cdot \frac{\sqrt{2}}{2} \\
&= \frac{\sqrt{2}}{4} + \frac{\sqrt{6}}{4} \\
&= \frac{\sqrt{2} + \sqrt{6}}{4}
\end{aligned}$$

NOTE We realize that the use of calculators makes it very unlikely that you will use such procedures to evaluate a given trigonometric value. We did these examples and have included exercises to give you practice using the sum and difference identities with numbers that you can verify on your calculator. This practice will also help you remember the identities later when you need them.

EXAMPLE 21.11

Simplify $\sin(x + \frac{\pi}{2})$.

SOLUTION

$$\begin{aligned}\sin\left(x + \frac{\pi}{2}\right) &= \sin x \cos\frac{\pi}{2} + \cos x \sin\frac{\pi}{2}\\ &= \sin x \cdot 0 + \cos x \cdot 1\\ &= \cos x\end{aligned}$$

EXAMPLE 21.12

Verify that $\sin(\alpha + \beta) + \sin(\alpha - \beta) = 2 \sin \alpha \cos \beta$.

SOLUTION

$\sin(\alpha + \beta) + \sin(\alpha - \beta)$	$2 \sin \alpha \cos \beta$	
$\sin \alpha \cos \beta + \cos \alpha \sin \beta$ $+ \sin \alpha \cos \beta - \cos \alpha \sin \beta$		Expand $\sin(\alpha + \beta)$ and $\sin(\alpha - \beta)$.
$2 \sin \alpha \cos \beta$		Collect terms.

So, $\sin(\alpha + \beta) + \sin(\alpha - \beta) = 2 \sin \alpha \cos \beta$.

CAUTION Remember, $\sin(\alpha + \beta) \neq \sin \alpha + \sin \beta$. Make sure that you rewrite $\sin(\alpha + \beta)$ as $\sin \alpha \cos \beta + \cos \alpha \sin \beta$. In a similar way, you can show that $\cos(\alpha + \beta) \neq \cos \alpha + \cos \beta$ and $\tan(\alpha + \beta) \neq \tan \alpha + \tan \beta$.

APPLICATION GENERAL TECHNOLOGY

EXAMPLE 21.13

The displacement d of an object oscillating in simple harmonic motion can be determined by the expression

$$d = a \sin 2\pi f t \cos \beta + a \cos 2\pi f t \sin \beta$$

Express the right-hand side as a single term.

SOLUTION If we factor an a out of both terms, then

$$d = a(\sin 2\pi f t \cos \beta + \cos 2\pi f t \sin \beta)$$

If we let $\alpha = 2\pi f t$, then the expression in parentheses is in the form $\sin \alpha \cos \beta + \cos \alpha \sin \beta = \sin(\alpha + \beta)$. So, the desired expression is

$$d = a \sin(2\pi f t + \beta)$$

EXERCISE SET 21.2

In Exercises 1–8, use the fact that $\sin 30^\circ = \cos 60^\circ = \frac{1}{2}$, $\sin 60^\circ = \cos 30^\circ = \frac{\sqrt{3}}{2}$, *and* $\sin 45^\circ = \cos 45^\circ = \frac{\sqrt{2}}{2}$, *along with the other facts you know from trigonometry, to determine the following.*

1. $\sin 15^\circ$

2. $\cos 75^\circ$

3. $\sin 120^\circ$

4. $\cos(-15^\circ)$

5. $\tan 15^\circ$

6. $\tan 135^\circ$

7. $\sin 150^\circ$

8. $\cos 105^\circ$

In Exercises 9–16, simplify the given expression.

9. $\sin(x + 90^\circ)$

10. $\cos(x + \pi)$

11. $\cos(x + \frac{\pi}{2})$

12. $\sin(\frac{\pi}{2} - x)$

13. $\cos(\pi - x)$

14. $\tan(x - \frac{\pi}{4})$

15. $\sin(180^\circ - x)$

16. $\tan(180^\circ + x)$

If α *and* β *are first-quadrant angles,* $\sin \alpha = \frac{3}{4}$, *and* $\cos \beta = \frac{7}{8}$, *evaluate the given expressions in Exercises 17–24.*

17. $\sin(\alpha + \beta)$

18. $\cos(\alpha + \beta)$

19. $\tan(\alpha + \beta)$

20. $\sin(\alpha - \beta)$

21. $\cos(\alpha - \beta)$

22. $\tan(\alpha - \beta)$

23. In what quadrant is $\alpha + \beta$?

24. In what quadrant is $\alpha - \beta$?

If α *is a second-quadrant angle, and* β *is a third-quadrant angle with* $\sin \alpha = \frac{3}{4}$, *and* $\cos \beta = \frac{-7}{8}$, *determine each of the given expressions in Exercises 25–32.*

25. $\sin(\alpha + \beta)$

26. $\cos(\alpha + \beta)$

27. $\tan(\alpha + \beta)$

28. $\sin(\alpha - \beta)$

29. $\cos(\alpha - \beta)$

30. $\tan(\alpha - \beta)$

31. In what quadrant is $\alpha + \beta$?

32. In what quadrant is $\alpha - \beta$?

Simplify the given expression in Exercises 33–40.

33. $\sin 47^\circ \cos 13^\circ + \cos 47^\circ \sin 13^\circ$

34. $\sin 47^\circ \sin 13^\circ + \cos 47^\circ \cos 13^\circ$

35. $\cos 32^\circ \cos 12^\circ - \sin 32^\circ \sin 12^\circ$

36. $\dfrac{\tan 40^\circ + \tan 15^\circ}{1 - \tan 40^\circ \tan 15^\circ}$

37. $\cos(\alpha + \beta) \cos \beta + \sin(\alpha + \beta) \sin \beta$

38. $\sin(x - y) \cos y + \cos(x - y) \sin y$

39. $\cos(x + y) \cos(x - y) - \sin(x + y) \sin(x - y)$

40. $\sin A \cos(-B) + \cos A \sin(-B)$

Prove each of the identities in Exercises 41–46.

41. $\sin(x + y) \sin(x - y) = \sin^2 x - \sin^2 y$

42. $(\sin A \cos B - \cos A \sin B)^2 + (\cos A \cos B + \sin A \sin B)^2 = 1$

43. $\cos \theta = \sin(\theta + 30^\circ) + \cos(\theta + 60^\circ)$

44. $\dfrac{\sin(x + y)}{\cos(x - y)} = \dfrac{\tan x + \tan y}{1 + \tan x \tan y}$

45. $\tan x - \tan y = \dfrac{\sin(x - y)}{\cos x \cos y}$

46. $\cos(A + B) \cos(A - B) = 1 - \sin^2 A - \sin^2 B$

In Exercises 47–50, use a calculator or a spreadsheet to show that the statements are true.

47. $\sin(20° + 37°) = \sin 20° \cos 37° + \cos 20° \sin 37°$

48. $\cos(15° + 63°) = \cos 15° \cos 63° - \sin 15° \sin 63°$

49. $\tan(0.2 + 1.3) = \dfrac{\tan 0.2 + \tan 1.3}{1 - (\tan 0.2)(\tan 1.3)}$

50. $\sin(2.3 - 1.1) = (\sin 2.3)(\cos 1.1) - (\cos 2.3)(\sin 1.1)$

In Exercises 51–54, with the help of a calculator or a spreadsheet, use the given angles as a counterexample to show that each relation is not an identity.

51. $\sin(x + y) \neq \sin x + \sin y$; $x = 55°$, $y = 37°$

52. $\cos(x - y) \neq \cos x - \cos y$; $x = 68°$, $y = 24°$

53. $\cos(x + y) \neq \cos x + \cos y$; $x = 40°$, $y = 35°$

54. $\tan(x - y) \neq \tan x - \tan y$; $x = 76°$, $y = 37°$

Solve Exercises 55–60.

55. In Chapter 14, we learned that when two complex numbers are written in polar form their product is

$$[r_1(\cos\theta_1 + j\sin\theta_1)][r_2(\cos\theta_2 + j\sin\theta_2)] = r_1r_2[\cos(\theta_1 + \theta_2) + j\sin(\theta_1 + \theta_2)]$$

Prove this formula.

56. ***Physics*** A spring vibrating in harmonic motion described by the equation $y_1 = A_1\cos(\omega t + \pi)$ is subjected to another harmonic motion described by $y_2 = A_2\cos(\omega t - \pi)$. Show that

$$y_1 + y_2 = -(A_1 + A_2)\cos\omega t$$

57. ***Optics*** When a light beam passes from one medium through another medium and exists in a third medium of the same density as the first, the displacement d of the light beam is $d = \dfrac{h}{\cos\theta_r}\sin(\theta_i - \theta_r)$, where θ_r is the angle of refraction, θ_i is the angle of incidence, and h is the thickness of the medium. Show that $d = h(\sin\theta_i - \cos\theta_i\tan\theta_r)$.

58. ***Electronics*** The angle between voltage and current in an RC circuit is 45°. Develop an expression in terms of ω for $i(t)$ in milliamperes (mA) using $i(t) = I_p\sin(\theta - \omega t)$, if $I_p = 14.8$ mA.

59. ***Meteorology*** One way to measure cloud height at night is to place a light at point B so that the direction the light is pointing makes a 70° angle with the ground. An observer moves 1000 m away from the light to point A and measures the angle θ of elevation to the place where the light shines on the cloud, as shown in Figure 21.6. The height of the cloud is given by

$$h = \frac{1000}{\cot 70° + \cot\theta}$$

Show that this formula is identical to

$$h = \frac{1000\sin 70°\sin\theta}{\sin(70° + \theta)}$$

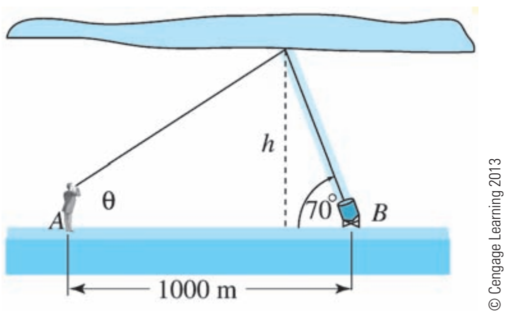


Figure 21.6

60. ***Electronics*** The potential difference in a certain three-phase alternator is given by

$$V = -E\left[\sin\left(2\pi f t - \frac{5\pi}{6}\right) + \sin\left(2\pi f t - \frac{7\pi}{6}\right)\right]$$

Show that $V = E\sqrt{3}\sin(2\pi f t)$.

[IN YOUR WORDS]

61. Describe how to develop $\sin(\theta - \phi)$ if you know that $\sin(\theta + \phi) = \sin\theta\cos\phi + \cos\theta\sin\phi$.

62. Describe how to develop the $\tan(\theta + \phi)$ identity from the identities for $\sin(\theta + \phi)$ and $\cos(\theta + \phi)$.

21.3 THE DOUBLE- AND HALF-ANGLE IDENTITIES

In Section 21.2, we studied the sum and difference identities. We can use these identities to develop double-angle identities. The double-angle identities can then be used to develop some half-angle identities.

DOUBLE-ANGLE IDENTITIES

The identity for $\sin(\theta + \phi)$ can be used to develop an identity for $\sin 2\theta$. To do this, calculate $\sin(\theta + \theta)$.

$$\begin{aligned}\sin 2\theta &= \sin(\theta + \theta)\\ &= \sin\theta\cos\theta + \cos\theta\sin\theta\\ &= 2\sin\theta\cos\theta\end{aligned}$$

In the same manner, we can develop $\cos 2\theta$.

$$\begin{aligned}\cos 2\theta &= \cos(\theta + \theta)\\ &= \cos\theta\cos\theta - \sin\theta\sin\theta\\ &= \cos^2\theta - \sin^2\theta\end{aligned}$$

This last identity has two other forms. If we use the Pythagorean identity, $\sin^2\theta + \cos^2\theta = 1$, we get

$$\begin{aligned}\cos 2\theta &= \cos^2\theta - \sin^2\theta\\ &= (1 - \sin^2\theta) - \sin^2\theta\\ &= 1 - 2\sin^2\theta\end{aligned}$$

We can replace $\sin^2\theta$ with $1 - \cos^2\theta$ and get a third version of this formula.

$$\begin{aligned}\cos 2\theta &= \cos^2\theta - \sin^2\theta\\ &= \cos^2\theta - (1 - \cos^2\theta)\\ &= 2\cos^2\theta - 1\end{aligned}$$

Once again, if we evaluate $\tan(\theta + \theta)$, we get the third double-angle identity:

$$\begin{aligned}\tan 2\theta &= \tan(\theta + \theta)\\ &= \frac{\tan\theta + \tan\theta}{1 - \tan\theta\tan\theta}\\ &= \frac{2\tan\theta}{1 - \tan^2\theta}\end{aligned}$$

This completes the list of double-angle identities.

DOUBLE-ANGLE IDENTITIES

$$\sin 2\theta = 2 \sin \theta \cos \theta$$

$$\begin{aligned} \cos 2\theta &= \cos^2 \theta - \sin^2 \theta \\ &= 2 \cos^2 \theta - 1 \\ &= 1 - 2 \sin^2 \theta \end{aligned}$$

$$\tan 2\theta = \frac{2 \tan \theta}{1 - \tan^2 \theta}$$

EXAMPLE 21.14

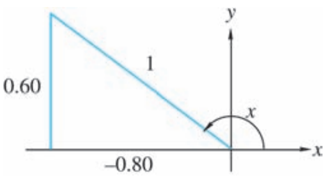


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Figure 21.7

If $\sin x = 0.60$ and x is in the second quadrant, then determine (a) $\sin 2x$, (b) $\cos 2x$, and (c) $\tan 2x$.

SOLUTIONS We will first determine the value of $\cos x$. Since $\sin x = 0.60$, $\cos^2 x = 1 - \sin^2 x = 1 - (0.60)^2 = 0.64$ and $\cos x = \pm\sqrt{0.64} = \pm 0.80$. Since x is in Quadrant II (see Figure 21.7), $\cos x = -0.80$.

(a) $$\begin{aligned} \sin 2x &= 2 \sin x \cos x \\ &= 2(0.60)(-0.80) = -0.96 \end{aligned}$$

(b) $$\begin{aligned} \cos 2x &= \cos^2 x - \sin^2 x \\ &= (-0.8)^2 - (0.6)^2 \\ &= 0.64 - 0.36 = 0.28 \end{aligned}$$

(c) Since $\sin x = 0.60$, $\cos x = -0.8$, and $\tan x = \dfrac{\sin x}{\cos x}$, we have $\tan x = \frac{0.60}{-0.80} = -\frac{3}{4}$. Thus,

$$\begin{aligned} \tan 2x &= \frac{2 \tan x}{1 - \tan^2 x} \\ &= \frac{2(-\frac{3}{4})}{1 - (-\frac{3}{4})^2} \\ &= \frac{-\frac{3}{2}}{1 - \frac{9}{16}} \\ &= \frac{-\frac{3}{2}}{\frac{7}{16}} \\ &= \frac{-24}{7} \\ &\approx -3.43 \end{aligned}$$

CAUTION Don't forget that $\sin 2\alpha \neq 2 \sin \alpha$ and $\cos 2\alpha \neq 2 \cos \alpha$. We now know that

$$\sin 2\alpha = 2 \sin \alpha \cos \alpha$$

and that

$$\cos 2\alpha = \cos^2 \alpha - \sin^2 \alpha$$

EXAMPLE 21.15

Rewrite $\cos 4x$ in terms of $\cos x$.

SOLUTION Using the double-angle identity for $\cos 2\theta$, if we let $\theta = 2x$, then we get $\cos 4x = 2 \cos^2 2x - 1$. Now using the double-angle identity $\cos 2x = 2 \cos^2 x - 1$, we get

$$\begin{aligned}\cos 4x &= 2(2 \cos^2 x - 1)^2 - 1 \\ &= 2(4 \cos^4 x - 4 \cos^2 x + 1) - 1 \\ &= 8 \cos^4 x - 8 \cos^2 x + 1\end{aligned}$$

HALF-ANGLE IDENTITIES

If we solve $\cos 2x = 2 \cos^2 x - 1$ for the $\cos x$, we get another identity—a half-angle identity.

$$\cos 2x = 2 \cos^2 x - 1$$

or $$\cos^2 x = \frac{1 + \cos 2x}{2}$$

and, taking the square root of both sides,

$$\cos x = \pm\sqrt{\frac{1 + \cos 2x}{2}}$$

If we let $x = \frac{\theta}{2}$, then

$$\cos \frac{\theta}{2} = \pm\sqrt{\frac{1 + \cos \theta}{2}}$$

If we solve $\cos^2 x = 1 - 2 \sin^2 x$ for $\sin x$, we obtain

$$\sin^2 x = \frac{1 - \cos 2x}{2}$$

or $$\sin x = \pm\sqrt{\frac{1 - \cos 2x}{2}}$$

Again, if $x = \frac{\theta}{2}$, then

$$\sin \frac{\theta}{2} = \pm\sqrt{\frac{1 - \cos \theta}{2}}$$

Since $\tan x = \frac{\sin x}{\cos x}$, we can show that

$$\tan\frac{\theta}{2} = \pm\sqrt{\frac{1-\cos\theta}{1+\cos\theta}}$$
$$= \frac{\sin\theta}{1+\cos\theta} = \frac{1-\cos\theta}{\sin\theta}$$

We have developed the following three half-angle identities.

HALF-ANGLE IDENTITIES

$$\sin\frac{\theta}{2} = \pm\sqrt{\frac{1-\cos\theta}{2}}$$
$$\cos\frac{\theta}{2} = \pm\sqrt{\frac{1+\cos\theta}{2}}$$
$$\tan\frac{\theta}{2} = \pm\sqrt{\frac{1-\cos\theta}{1+\cos\theta}} = \frac{\sin\theta}{1+\cos\theta}$$
$$= \frac{1-\cos\theta}{\sin\theta}$$

EXAMPLE 21.16

If $\cos\theta = \frac{-5}{13}$ and θ is in the third quadrant, find the values of (a) $\sin\frac{\theta}{2}$, (b) $\cos\frac{\theta}{2}$, and (c) $\tan\frac{\theta}{2}$.

SOLUTIONS We need to determine which quadrant $\frac{\theta}{2}$ is in. We know θ is in Quadrant III, or $\pi < \theta < \frac{3\pi}{2}$, and so $\frac{\pi}{2} < \frac{\theta}{2} < \frac{1}{2}\left(\frac{3\pi}{2}\right)$ or $\frac{\pi}{2} < \frac{\theta}{2} < \frac{3\pi}{4}$. This means that $\frac{\theta}{2}$ is in Quadrant II. In Quadrant II, the sine is positive, cosine is negative, and tangent is negative.

(a) $$\sin\frac{\theta}{2} = +\sqrt{\frac{1-\cos\theta}{2}} = +\sqrt{\frac{1-\left(-\frac{5}{13}\right)}{2}} = \sqrt{\tfrac{9}{13}} \approx 0.832$$

(b) $$\cos\frac{\theta}{2} = -\sqrt{\frac{1+\cos\theta}{2}} = -\sqrt{\frac{1+\left(-\frac{5}{13}\right)}{2}} = -\sqrt{\tfrac{4}{13}} \approx -0.555$$

(c) $$\tan\frac{\theta}{2} = -\sqrt{\frac{1-\cos\theta}{1+\cos\theta}} = -\sqrt{\frac{1-\left(-\frac{5}{13}\right)}{1+\left(-\frac{5}{13}\right)}} = -\sqrt{\frac{\frac{18}{13}}{\frac{8}{13}}} = -\sqrt{\frac{9}{4}}$$
$$= -\frac{3}{2} = -1.500$$

We could have determined $\tan \frac{\theta}{2}$ by using $\dfrac{\sin \frac{\theta}{2}}{\cos \frac{\theta}{2}}$. With the values above, we would have gotten -1.499. The error of 0.001 was caused by the use of approximations in parts (a) and (b).

EXAMPLE 21.17

Prove the identity $2 \sin \frac{x}{2} \cos \frac{x}{2} = \sin x$.

SOLUTION

$2 \sin \frac{x}{2} \cos \frac{x}{2}$	$\sin x$	
$2\left(\pm\sqrt{\frac{1 - \cos x}{2}}\right)\left(\pm\sqrt{\frac{1 + \cos x}{2}}\right)$		Replace with half-angle identities.
$2\sqrt{\frac{1 - \cos^2 x}{4}}$		Multiply.
$2\sqrt{\frac{\sin^2 x}{4}}$		Pythagorean identity.
$2\left(\frac{\sin x}{2}\right)$		Take the square root.
$\sin x$		Simplify.

CAUTION Be careful when you use the double- and half-angle formulas. Begin by calculating the necessary values of θ, 2θ, and. $\frac{\theta}{2}$ before they are substituted into the formula.

APPLICATION CIVIL ENGINEERING

EXAMPLE 21.18

A highway engineer is designing the curve at an intersection like the one shown by the photograph in Figure 21.8a. These two highways intersect at an angle θ. The edge of the highway is to join the two points A and B with an arc or a circle that is tangent to the highways at these two points. Determine the relationship between the radius of the arc r, the distance d of A and B from the intersection, and angle θ.

SOLUTION We begin by noticing in Figure 21.8b, that $\angle BCA$ and θ are supplementary angles. So, $m\angle BCA = 180° - \theta$. If the center of the circle is at P, then

EXAMPLE 20.18 (Cont.)

$\overline{PA} \perp \overline{AC}$, because a tangent to a circle, in this case $\overline{AC}$, is perpendicular to a radius at the point of tangency. Now, $\overline{PC}$ bisects $\angle BCA$, so $m\angle PCA = \frac{1}{2}m\angle BCA = 90° - \frac{\theta}{2}$. Since $\triangle PAC$ is a right triangle with right angle at A, $\tan \angle PCA = \frac{r}{d}$; so $d = \frac{r}{\tan \angle PCA} = r \cot \angle PCA = r \cot\left(90° - \frac{\theta}{2}\right) = r \tan \frac{\theta}{2}$. Thus, we have shown that $d = r \tan \frac{\theta}{2}$.

Figure 21.8a

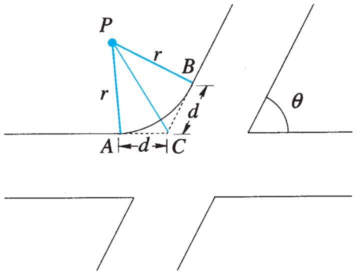


Figure 21.8b

APPLICATION CIVIL ENGINEERING

EXAMPLE 21.19

Two highways meet at an angle of 34°. The curb is to join points A and B located 45 ft from the beginning of the intersection. (a) Approximate the radius of the arc joining A and B. (b) Determine the length of the arc.

SOLUTIONS (a) From Example 21.18, we have the formula

$$d = r \tan \frac{\theta}{2}$$

In this example, $d = 45$ ft and $\theta = 34°$. We are to determine r.

$$r = \frac{d}{\tan \frac{\theta}{2}}$$

$$= \frac{45}{\tan \left(\frac{34}{2}\right)°}$$

$$= \frac{45}{\tan 17^\circ}$$
$$\approx \frac{45}{0.3057}$$
$$\approx 147.19 \text{ ft}$$

(b) We want the length of the arc that forms the curb. As we saw in Section 5.6, the arc length s of a circle with radius r, formed by an angle θ, is

$$s = r\theta$$

provided that θ is in radians.

In this example, $\theta = 34^\circ = \frac{34\pi}{180} = \frac{17\pi}{90}$ rad. So,

$$s = r\theta$$
$$= (147.19)\left(\frac{17\pi}{90}\right)$$
$$\approx 87.34 \text{ ft}$$

EXERCISE SET 21.3

In Exercises 1–14, if $\sin 30^\circ = \cos 60^\circ = \frac{1}{2}$, $\sin 60^\circ = \cos 30^\circ = \dfrac{\sqrt{3}}{2}$, and $\sin 45^\circ = \cos 45^\circ = \dfrac{\sqrt{2}}{2}$, determine the exact values of the given trigonometric function.

1. $\cos 15^\circ$

2. $\sin 75^\circ$

3. $\sin 15^\circ$

4. $\cos 105^\circ$

5. $\sin 105^\circ$

6. $\cos 210^\circ$

7. $\cos 7\frac{1}{2}^\circ$

8. $\tan 15^\circ$

9. $\tan 22\frac{1}{2}^\circ$

10. $\cos 67\frac{1}{2}^\circ$

11. $\cos 75^\circ$

12. $\sin 37\frac{1}{2}^\circ$

13. $\sin 127\frac{1}{2}^\circ$

14. $\tan(-15^\circ)$

Use the information given in each of Exercises 15–20 to determine $\sin 2x$, $\cos 2x$, $\tan 2x$, $\sin \dfrac{x}{2}$, $\cos \dfrac{x}{2}$, and $\tan \dfrac{x}{2}$.

15. $\sin x = \frac{7}{25}$, x in Quadrant II

16. $\cos x = \frac{8}{17}$, x in Quadrant IV

17. $\sec x = \frac{29}{20}$, x in Quadrant I

18. $\csc x = \frac{-41}{9}$, x in Quadrant III

19. $\tan x = \frac{35}{12}$, x in Quadrant III

20. $\cot x = \frac{-45}{28}$, x in Quadrant II

Prove the identities in Exercises 21–32 for all angles in the domains of the functions.

21. $\cos^2 x = \sin^2 x + \cos 2x$

22. $\cos^2 4x - \sin^2 4x = \cos 8x$

23. $\cos 4x = 1 - 8 \sin^2 x \cos^2 x$

24. $\cos 3x = 4 \cos^3 x - 3 \cos x$

25. $\dfrac{1 + \tan^2 \alpha}{1 - \tan^2 \alpha} = \sec 2\alpha$

26. $\sin 2x \cos 2x = \frac{1}{2} \sin 4x$

27. $1 - 2 \sin^2 3x = \cos 6x$

28. $\dfrac{2\tan 3x}{1-\tan^2 3x} = \tan 6x$

29. $\sin^2 x \cos^2 x = \frac{1}{4}\sin^2 2x$

30. $1 + \tan\beta \tan\dfrac{\beta}{2} = \dfrac{1}{\cos\beta}$

31. $\tan\left(\dfrac{\alpha+\beta}{2}\right)\cot\left(\dfrac{\alpha-\beta}{2}\right) = \dfrac{(\sin\alpha+\sin\beta)^2}{\sin^2\alpha - \sin^2\beta}$

32. $\cot\theta = \frac{1}{2}\left(\cot\dfrac{\theta}{2} - \tan\dfrac{\theta}{2}\right)$

In Exercises 33–36, use a calculator or a spreadsheet and the given angles as counterexamples to show that the following are not identities.

33. $\cos 2\theta \neq 2\cos\theta;\ \theta = 45°$

34. $\tan\dfrac{\theta}{2} \neq \dfrac{\tan\theta}{2};\ \theta = 80°$

35. $\cot 2\alpha \neq 2\cos\alpha;\ \alpha = 150°$

36. $\sin\dfrac{x}{2} \neq \dfrac{\sin x}{2};\ x = 210°$

Solve Exercises 37–42.

37. ***Optics*** The index of refraction n of a prism whose apex angle is α and whose angle of minimum deviation is ϕ is given by

$$n = \frac{\sin\left(\dfrac{\alpha+\phi}{2}\right)}{\sin\dfrac{\alpha}{2}} \text{ with } n > 0$$

Show that

$$n = \sqrt{\frac{1-\cos\alpha\cos\phi + \sin\alpha\sin\phi}{1-\cos\alpha}}$$

38. ***Optics*** Show that an equivalent expression for the index of refraction described in Exercise 37 is

$$n = \sqrt{\frac{1+\cos\phi}{2}} + \left(\cot\frac{\alpha}{2}\right)\sqrt{\frac{1-\cos\phi}{2}}$$

39. ***Electronics*** In an ac circuit containing reactance, the instantaneous power is given by

$$P = V_{max}I_{max}\cos\omega t\sin\omega t$$

Show that $P = \dfrac{V_{max}I_{max}}{2}\sin 2\omega t$.

40. ***Physics*** A cable vibrates with a decreased amplitude that is given by $A = \sqrt{e^{-2x}(1+\sin 2x)}$. Show that $A = e^{-x}(\sin x + \cos x)$.

41. ***Transportation*** Engineers use the equation

$$x = 2r\sin^2\left(\frac{\theta}{2}\right)$$

to determine the width of the merging region of a highway. Solve for x in terms of $\cos\theta$ only.

42. ***Electronics*** The electron beam that forms the picture on a television or computer screen is controlled basically by a sawtooth function. This function can be approximated by sinusoidal curves of varying periods and amplitudes. A first approximation to the sawtooth curve is given by

$$y = \pi - 2\sin(2x) - \sin(4x)$$

Show that this can be rewritten as

$$y = \pi - 4\sin(2x)\cos^2 x$$

[IN YOUR WORDS]

43. One of the identities for $\cos 2\theta$ is $\cos 2\theta = \cos^2\theta - \sin^2\theta$. There are two other identities for $\cos 2\theta$. Describe how you would develop each of them from the one above.

44. How would you establish the identity for $\tan 2\theta$?

45. The only difference between the formula for $\sin\dfrac{\theta}{2}$ and $\cos\dfrac{\theta}{2}$ is a + or − sign. Explain how you can tell which identity is for $\sin \frac{\theta}{2}$ and which is for $\cos \frac{\theta}{2}$.

46. The identities for $\sin \frac{\theta}{2}$ and $\cos \frac{\theta}{2}$ can both be developed from the identities for $\cos 2\theta$. Describe how you would do this.

21.4 TRIGONOMETRIC EQUATIONS

In Sections 21.1 through 21.3, we studied different types of trigonometric identities. Many people find proving and developing trigonometric identities to be very interesting. Our main interest in them, however, was to give you some skills for solving equations that involve trigonometric functions.

A **trigonometric equation** is an equation involving trigonometric functions of unknown angles. If these equations have been true for all angles, then we have called them identities. A trigonometric equation that is not an identity is a *conditional equation*. A conditional equation is true for some values for the angle and not true for others. To *solve* a conditional trigonometric equation means to find all values of the angle for which the equation is true. To solve a trigonometric equation, you must use both algebraic and trigonometric identities.

Solving a trigonometric equation of the type $2 \tan x = 1$ would produce an infinite number of answers. As you remember from our earlier study, the trigonometric functions are periodic. Thus, the solution to this equation would not only be true when $x = 26.565°$ but also for $x = 26.565° + 180°n$, where n is any integer. Usually, it is sufficient to give only the *primary solutions* or *principal values*, which are the solutions for x, where $0° \leq x < 360°$ or $0 \leq x < 2\pi$.

EXAMPLE 21.20

Solve $2 \tan x = 1$.

SOLUTION

$$2 \tan x = 1$$
$$\tan x = \frac{1}{2}$$

We know that $x = \arctan\frac{1}{2} = \tan^{-1}(\frac{1}{2})$. Using a calculator, we see that $x \approx 26.565°$. But, we know that the tangent function is also positive in Quadrant III, so $x = 26.565° + 180° = 206.565°$. The primary solutions are $26.565°$ and $206.565°$.

EXAMPLE 21.21

Solve $\cos 4x = \frac{\sqrt{2}}{2}$, where $0 \leq x < 2\pi$.

SOLUTION A natural way to proceed would be to use the double-angle identities to rewrite this as an equation in x.

A little foresight will save a lot of work. We will let $\theta = 4x$ and solve for θ.

EXAMPLE 21.21 (Cont.)

Once we have the value for θ we can then solve for x. But, since $x = \frac{\theta}{4}$ and $0 \le x < 2\pi$, we must solve for $0 \le \theta < 8\pi$.

$$\cos\theta = \frac{\sqrt{2}}{2}$$
$$\theta = \cos^{-1}\left(\frac{\sqrt{2}}{2}\right) = \frac{\pi}{4}$$

Since the cosine is also positive in Quadrant IV, we see that $\theta = \frac{7\pi}{4}$. If we keep adding 2π to each of these answers until we exceed 8π, we will get the other solutions for θ:

$$\frac{\pi}{4} + 2\pi = \frac{9\pi}{4},$$
$$\frac{\pi}{4} + 4\pi = \frac{17\pi}{4},$$
$$\frac{\pi}{4} + 6\pi = \frac{25\pi}{4};$$

and we also get

$$\frac{7\pi}{4} + 2\pi = \frac{15\pi}{4},$$
$$\frac{7\pi}{4} + 4\pi = \frac{23\pi}{4},$$
$$\text{and} \quad \frac{7\pi}{4} + 6\pi = \frac{31\pi}{4}$$

The solutions then for $0 \le \theta < 8\pi$ are $4x = \theta = \frac{\pi}{4}, \frac{7\pi}{4}, \frac{9\pi}{4}, \frac{15\pi}{4}, \frac{17\pi}{4}, \frac{23\pi}{4}, \frac{25\pi}{4}$, and $\frac{31\pi}{4}$, and so the values of x are $x = \frac{\pi}{16}, \frac{7\pi}{16}, \frac{9\pi}{16}, \frac{15\pi}{16}, \frac{17\pi}{16}, \frac{23\pi}{16}, \frac{25\pi}{16}$, and $\frac{31\pi}{16}$.

EXAMPLE 21.22

Solve $\sin\theta\tan\theta = \sin\theta$ for $0 \le \theta < 360°$.

SOLUTION We begin by collecting terms and factoring:

$$\sin\theta\tan\theta = \sin\theta$$
$$\sin\theta\tan\theta - \sin\theta = 0$$
$$\sin\theta(\tan\theta - 1) = 0$$

We now determine when each of these factors can be 0.

$$\text{So} \quad \sin\theta = 0 \quad \text{or} \quad \tan\theta - 1 = 0$$
$$\sin\theta = 0 \qquad \tan\theta - 1 = 0$$
$$\theta = 0^\circ, 180^\circ \qquad \tan\theta = 1$$
$$\theta = 45^\circ, 225^\circ$$

The solutions are 0°, 45°, 180°, and 225°.

GRAPHICAL SOLUTIONS TO TRIGONOMETRIC EQUATIONS

Earlier we used calculators and spreadsheets to help solve equations and inequalities. These techniques can also be applied to trigonometric equations. One of the major difficulties with using a TI-83 or TI-84 calculator is that it does not always give exact answers. We will rework Example 21.22 using a TI-84. Later we will use a spreadsheet to work Example 21.22.

USING A CALCULATOR TO SOLVE A TRIG EQUATION

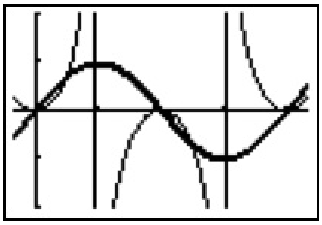
EXAMPLE 21.23

Use a calculator to solve $\sin\theta\tan\theta = \sin\theta$ for $0 \le \theta < 360°$.

SOLUTION There are two ways in which use graphs to solve this equation. Both methods begin by letting $y_1 = \sin\theta\tan\theta$ and $y_2 = \sin\theta$. One method uses the `intersect` feature of the calculator and the other method uses the `zero` method discussed previously.

Using `Intersect`**:**

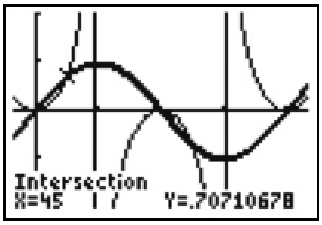
As you might be able to tell from the title, the intersect method locates where the two curves intersect. Figure 21.9a seems to indicate that the curves intersect in four points in the given interval. (The graph of $y_2 = \sin\theta$ is drawn using the "thick" style to make it easier to distinguish between the two graphs.)



$[-30°, 390°, 90°] \times [-2, 2]$

Figure 21.9a

Press **2nd** **CALC** **5** [5:intersect]. Move the cursor close to one of the points of intersection. We will select the second point from the left. Press **ENTER** twice. This tells the calculator which are the two curves that are intersecting. If there were more than two curves on the calculator screen, you could use ▲ or ▼ to move the cursor to the correct graphs. Make sure a cursor is near the point of intersection and press **ENTER**. After a few seconds you should see something like Figure 21.9b. This indicates that the two curves intersect near the point (45°, 0.7071). Notice that this gives both the x- and y-values where the curves intersect.



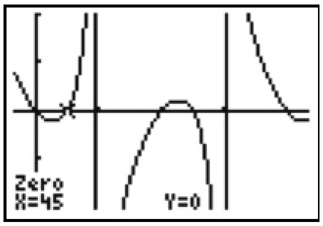
$[-30°, 390°, 90°] \times [-2, 2]$

Figure 21.9b

Repeat the procedure for each of the other three intersection points. You should get (0°, 0), (180°,0), and (225°, −0.7071).

Using `zero`**:**

We have used the zero technique before. Graph $y_1 - y_2$ and determine where the curve crosses the x-axis. In Figure 21.9c we see that one solution is $x = 45°$. The other solutions are $x = 0°$, $x = 180°$, and $x = 225°$. Notice that the zero method does not give the y-values of the points of intersection.



$[-30°, 390°, 90°] \times [-2, 2]$

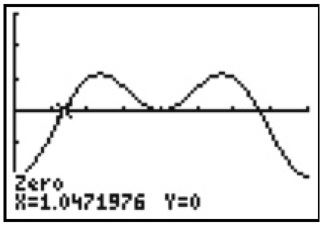
Figure 21.9c

EXAMPLE 21.24

Use a calculator to solve $2\sin^2 x - \cos x - 1 = 0$ for $0 \le x < 2\pi$.

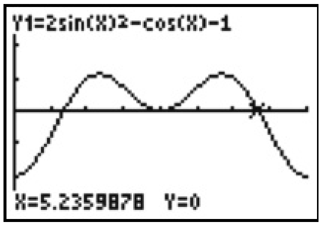
SOLUTION Since the right-hand side of this equation is zero, both the intersection method and the root method will produce the same result. From Figure 21.10a we see that one solution is $x \approx 1.0472$. (The actual value is $\frac{\pi}{3}$.)

EXAMPLE 21.24 (Cont.)



$[0, 2\pi, \frac{\pi}{4}] \times [-3, 3]$

Figure 21.10a



$[0, 2\pi, \frac{\pi}{4}] \times [-3, 3]$

Figure 21.10b

It looks as if the third solution (counting from the left) is approximately $2\pi - 1.0472$. We can verify that quickly by pressing 2 π − X, T, θ, n ENTER. Since the calculator automatically stores the root as x, you should get the result $x \approx 5.2360$, as in Figure 21.10b.

The second solution, the one in the middle, appears to be π. To verify this press π ENTER.

Thus, the solutions of $2 \sin^2 x - \cos x - 1 = 0$ for $0 \le x < 2\pi$ are $x \approx 1.0472$, $x = \pi$, and $x \approx 5.2360$.

SPREADSHEET SOLUTIONS FOR TRIGONOMETRIC EQUATIONS

Earlier, we used spreadsheets to help solve equations and inequalities. The same techniques used then can be applied to trigonometric equations.

EXAMPLE 21.25

Use a spreadsheet to solve $\sin\theta \tan\theta = \sin\theta$ for $0 \le \theta < 360°$.

SOLUTION There are two ways in which to use graphs to solve this equation. Both methods begin by letting $y_1 = \sin\theta \tan\theta$ and $y_2 = \sin\theta$.

Using a table and graph

Figure 21.11a shows a portion of the table and the graph used to find solutions to this equation.

Figures 21.11b to 21.11d show the portions of the table that identify close approximations of the four solutions.

Using Goal Seek

Figure 21.11e shows a portion of the table and the `Goal Seek` menu. Since the solutions to this equation were multiples of π, we were able to find exact solutions (in terms of π). However, Goal Seek may need to be employed on other equations.

EXAMPLE 21.26

Use a spreadsheet to solve $2 \sin^2 x - \cos x - 1 = 0$ for $0 \le x < 2\pi$.

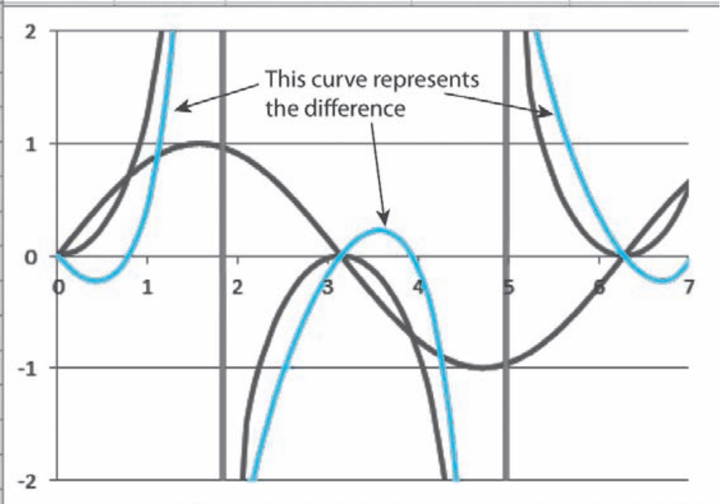
SOLUTION We will use Goal Seek to solve this equation. The graph in Figure 21.12a shows there are at least three solutions between 0 and 2π.

The first solution, shown in Figure 21.12b, is 1.0472.

Figure 21.12c shows a second solution at 3.131. However, since the y-value isn't zero, even after another try with Goal Seek, we might want to try something else here. 3.131 is very close to π. Perhaps Excel just isn't accurate enough to get the solution using the process in Goal Seek for a curve shaped like this one. Place π in Cell A33 and see if we getter a value closer to zero. Figure 21.12d shows that π is a solution. Figure 21.12e shows the third solution.

The solutions are $x \approx 1.0472$, $x = \pi$, and $x \approx 5.2358$.

	A	B	C	D	E
1		t (theta)	(sin t)*(tan t)	sin t	Difference
2	0	0	0	0	0
3	1	0.261799	0.069350354	0.258819045	-0.189468691
4	2	0.523599	0.288675135	0.5	-0.211324865
5	3	0.785398	0.707106781	0.707106781	0



	A	B	C	D	E
20	18	4.712389	-5.44152E+15	-1	-5.44152E+15
21	19	4.974188	3.60488426	-0.965925826	4.570810086
22	20	5.235988	1.5	-0.866025404	2.366025404
23	21	5.497787	0.707106781	-0.707106781	1.414213562
24	22	5.759587	0.288675135	-0.5	0.788675135

Figure 21.11a

f_x =A8*PI()/24

	A	B	C	D	E
8	6	0.785398	0.707106781	0.707106781	0

$$x = \frac{6\pi}{24} = \frac{\pi}{4} = 45^\circ$$

Figure 21.11b

f_x =A26*PI()/24

	A	B	C	D	E
26	24	3.141593	-1.50099E-32	1.22515E-16	-1.22515E-16

$$x = \frac{24\pi}{24} = \pi = 180^\circ$$

Figure 21.11c

f_x =A32*PI()/24

	A	B	C	D	E
32	30	3.926991	-0.707106781	-0.707106781	8.88178E-16

$$x = \frac{30\pi}{24} = \frac{5\pi}{4} = 225^\circ$$

Figure 21.11d

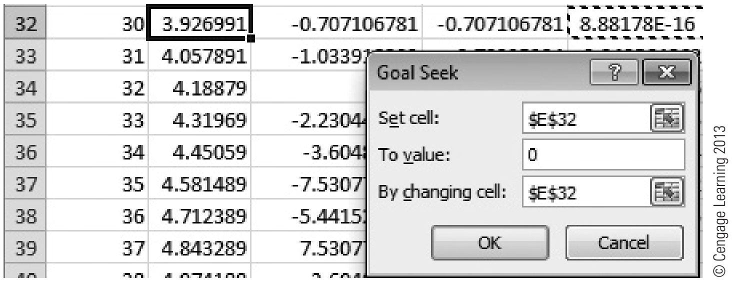


Figure 21.11e

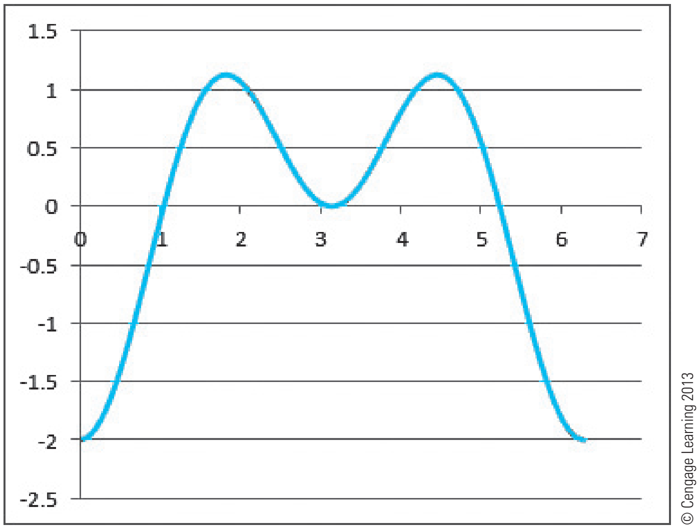


Figure 21.12a

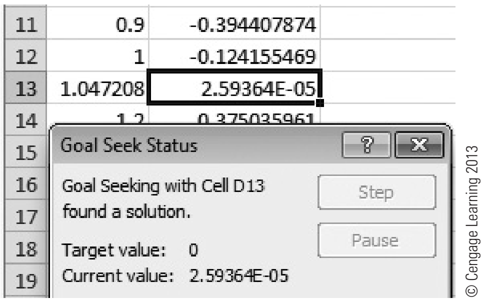


Figure 21.12b

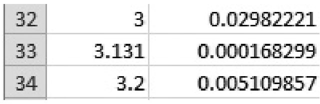


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Figure 21.12c

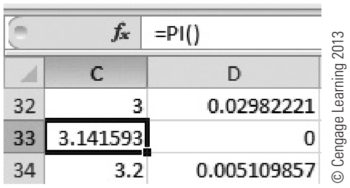

Figure 21.12d

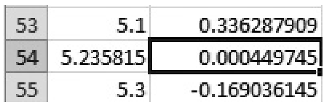

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Figure 21.12e

APPLICATION GENERAL TECHNOLOGY

EXAMPLE 21.27

The range r of a projectile thrown at an angle of elevation θ at a velocity v is given by

$$r = \frac{2v^2 \cos\theta \sin\theta}{g}$$

EXAMPLE 21.27 (Cont.)

If v is in ft/s, then g is 32 ft/s^2, and if v is in m/s, then g is 9.8 m/s^2. A projectile is fired with a velocity of 750 m/s with the purpose of hitting an object 20 000 m away. Determine the angle θ at which the projectile should be fired.

SOLUTION Here $v = 750$ m/s, $g = 9.8$ m/s^2, and $r = 20\,000$ m. Substituting these values in the given equation, we obtain

$$20\,000 = \frac{2(750)^2 \cos\theta \sin\theta}{9.8}$$

Now, $2\cos\theta\sin\theta = 2\sin\theta\cos\theta = \sin 2\theta$, and the equation becomes

$$20\,000 = \frac{(750)^2 \sin 2\theta}{9.8}$$

$$\text{or} \quad \sin 2\theta = \frac{20\,000(9.8)}{(750)^2} \approx 0.3484$$

so

$$2\theta \approx 20.39° \text{ or } 159.61°$$

$$\text{and} \quad \theta \approx 10.195° \text{ or } 79.805°$$

So, the projectile should be fired at an angle of 10.195° or 79.805°.

EXERCISE SET 21.4

Solve each equation in Exercises 1–30 for nonnegative angles less than 360° or 2π. You may want to use a calculator or a spreadsheet.

1. $2\cos\theta = 0$

2. $2\sin\theta = -1$

3. $\sqrt{3}\tan x = 1$

4. $\sqrt{3}\sec x = -2$

5. $4\sin\theta = -3$

6. $2\cos x = 3$

7. $4\tan\alpha = 5$

8. $3\csc x = 1$

9. $\cos 2x = -1$

10. $\sin 2x = \frac{1}{2}$

11. $\tan\frac{\theta}{4} = 1$

12. $\cos\frac{\theta}{3} = -1$

13. $\sin^2\alpha = \sin\alpha$

14. $\cos^2\beta = \frac{1}{2}\cos\beta$

15. $\sin x\cos x = 0$

16. $\frac{\sec\theta}{\csc\theta} = -1$

17. $3\tan^2 x = 1$

18. $\sec^2\theta = 2$

19. $4\sin\alpha\cos\alpha = 1$

20. $\sin^2\beta = \frac{1}{2}\sin\beta$

21. $\sin\theta - \cos\theta = 0$

22. $\tan\theta = \csc\theta$

23. $\sin 6\theta + \sin 3\theta = 0$

24. $4\tan^2 x = 3\sec^2 x$

25. $2\cos^2 x - 3\cos 2x = 1$

26. $\sin^2 4\alpha = \sin 4\alpha + 2$

27. $\sec^2\theta + \tan\theta = 1$

28. $\tan 2x + \sec 2x = 1$

29. $\sin 2x = \cos x$

30. $\csc^2\theta - \cot\theta = 1$

Solve Exercises 31–36.

31. ***Optics*** The second law of refraction (Snell's law) states that as a light ray passes from one medium to a second, the ratio of the sine of the angle of incidence θ_i to the sine of the angle of refraction θ_r is a constant, μ, called the *index of refraction*, with respect to the two mediums (see Figure 21.13). Thus, we have

$$\frac{\sin\theta_i}{\sin\theta_r} = \mu$$

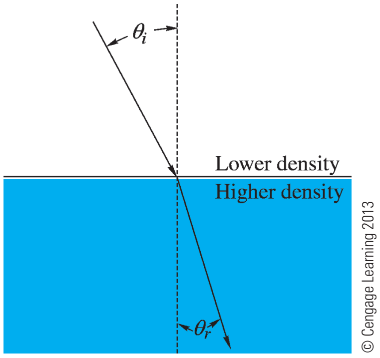

Figure 21.13

The index of refraction of general epoxy relative to air is $\mu = 1.61$. Determine the angle of refraction θ_r of a ray of light that strikes some general epoxy with an angle of incidence $\theta_i = 35°$.

32. ***Optics*** The index of refraction of glass silicone relative to air is 1.43. If the angle of incidence is 27°, what is the angle of refraction?

33. ***Electronics*** An oscillating signal voltage is given by $E = 125\cos(\omega t - \phi)$ millivolts, where the angular frequency is $\omega = 120\pi$, phase angle is $\phi = \frac{\pi}{2}$, and t is time in seconds. The triggering mechanism of an oscilloscope starts the sweep when $E = 60$ mV. What is the smallest positive value of t for which the triggering occurs?

34. ***Agriculture*** As shown in Figure 21.14, an irrigation ditch has a cross-section in the shape of an isosceles trapezoid with the smaller base on the bottom. The area A of the trapezoid is given by

$$A = a\sin\theta(b + a\cos\theta)$$

If $a = 3$ m, $b = 3.4$ m, and $A = 10\text{ m}^2$, find θ to the nearest tenth of a degree if $\sin\theta\cos\theta \approx 0.4675$.

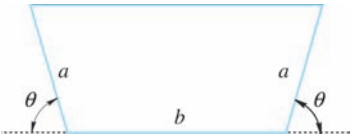

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Figure 21.14

35. ***Automotive technology*** The displacement d of a piston is given by

$$d = \sin\omega t + \frac{1}{2}\sin 2\omega t$$

For what primary solutions of ωt less than 2π is $d = 0$?

36. ***Optics*** Refraction causes a submerged object in a liquid to appear closer to the surface than it actually is. The relation between the true depth a and the apparent depth b is

$$\frac{a}{b} = \sqrt{\frac{\mu^2 - \sin^2\theta_i}{\cos^2\theta_i}}$$

where μ is the index of refraction for the two media and θ_i is the angle of incidence. An object that is 14 ft under water appears to be only 10 ft under water. If the index of refraction of water is 1.333, what is the angle of incidence?

[IN YOUR WORDS]

37. Equations with multiple angles have more than two solutions. Explain how you can determine that you have found all solutions.

38. Describe how you would use the methods for solving a trigonometric equation to solve a trigonometric inequality.

CHAPTER 21 REVIEW

IMPORTANT TERMS AND CONCEPTS

Double-angle identities
Half-angle identities
Pythagorean identities
Quotient identities
Reciprocal identities
Sum and difference identities
Trigonometric equations

REVIEW EXERCISES

Prove the identities in Exercises 1–10.

1. $\dfrac{\sin(x+y)}{\cos x \cos y} = \tan x + \tan y$

2. $(\sin x + \cos x)^2 = 1 + \sin 2x$

3. $\dfrac{\sin 3x}{\sin x} - \dfrac{\cos 3x}{\cos x} = 2$

4. $\cos(\theta + \phi)\cos(\theta - \phi) = \cos^2\phi - \sin^2\theta$

5. $\sin(\alpha - \beta)\cos\beta - \cos(\alpha + \beta)\sin\beta = \sin\alpha$

6. $\tan 2x = \dfrac{2\cos x}{\csc x - 2\sin x}$

7. $\cos^4 x - \sin^4 x = \cos 2x$

8. $\dfrac{\sin 2x - \sin x}{\cos 2x + \cos x} = \tan\dfrac{x}{2}$

9. $\sin 3\theta = 2\sin\theta\cos 2\theta + \sin\theta$

10. $\dfrac{\cos 3x - \cos 5x}{\sin 3x + \sin 5x} = \tan x$

(Hint: Let $3x = 4x - x$ and $5x = 4x + x$.)

Solve the equations in Exercises 11–20 for nonnegative values less than 360° or 2π.

11. $2\tan x = -\sqrt{3}$

12. $3\sin x = -2$

13. $\cos 2x + \cos x = -1$

14. $\cos x - \sin 2x - \cos 3x = 0$

15. $\sin 4x - 2\sin 2x = 0$

16. $\sin(30° + x) - \cos(60° + x) = -\dfrac{\sqrt{3}}{2}$

17. $2\sin\theta = \sin 2\theta$

18. $\sin^2\alpha + 5\cos^2\alpha = 3$

19. $\sin^2 x = 1 + \sin x$

20. $\sin\theta - 2\csc\theta = -1$

In Exercises 21–24, use the facts that $\sin x = \frac{5}{13}$, $\cos x = -\frac{12}{13}$, and x is in Quadrant II to determine the exact value of the given function.

21. $\sin 2x$

22. $\cos\dfrac{x}{2}$

23. $\sin\dfrac{x}{2}$

24. $\tan 2x$

In Exercises 25–30, use the facts that x is in Quadrant III, $\sin x = \frac{-5}{13}$, and $\cos x = \frac{-12}{13}$, and y is in Quadrant II, $\sin y = \frac{8}{17}$, and $\cos y = \frac{-15}{17}$ to determine the exact value of the given function.

25. $\sin(x + y)$

26. $\cos(x - y)$

27. $\cos(x + y)$

28. $\tan(x + y)$

29. $\cos(y - x)$

30. $\sin(x - y)$

Solve Exercises 31–33.

31. ***Automotive technology*** The acceleration of a piston is given by

$$a = 5.0(\sin\omega t + \cos 2\omega t)$$

For what primary solutions of ωt does $a = 0$?

32. ***Acoustical engineering*** When two sinusoidal sound waves that are close together in frequency are superimposed, the resultant disturbance exhibits beats. The two waves are represented by $y_1 = A_1\cos\omega_1 t$ and $y_2 = A_2\cos(\omega_2 t$

$+ \phi)$, where ω_2 is slightly larger than ω_1 and ϕ is a phase constant. If $\phi = 0$,

$$\alpha = \frac{\omega_1 + \omega_2}{2}, \text{ and } \beta = \frac{\omega_2 - \omega_1}{2}, \text{ show that}$$

$$y = y_1 + y_2 = A_1 \cos(\alpha - \beta)t + A_2 \cos(\alpha + \beta)t$$

33. ***Mechanics*** Consider a machine that is mounted on four springs with a known stiffness and on four dampers with a known damping constant. If this system is initially at rest and a certain force is applied, then under certain conditions the system has a time-displacement equation of $x = 0.01e^{-6t}(\cos 8t + \sin 8t)$. Another version gives the time-displacement equation at $x = \frac{\sqrt{2}}{100}e^{-6t}\cos\left(8t - \frac{\pi}{4}\right)$. Show that these two equations are identical.

CHAPTER 21 TEST

In Exercises 1–8, use the fact that $\sin \alpha = \frac{4}{5}$ and α is in Quadrant II and that $\cos \beta = -\frac{12}{13}$, and β is in Quadrant III to determine the exact value of the given function.

1. $\sin(\alpha + \beta)$

2. $\cos(\alpha + \beta)$

3. $\sin(\alpha - \beta)$

4. $\cos(\alpha - \beta)$

5. $\sin 2\alpha$

6. $\cos 2\beta$

7. $\sin \frac{\alpha}{2}$

8. $\cos \frac{\beta}{2}$

Solve Exercises 9–13.

9. Write $8 \cos 6x \sin 6x$ using a single trigonometric function.

10. Prove the identity $\tan x = \frac{\sec x}{\csc x}$

11. Prove the identity $\frac{\sin x}{1 - \cos x} + \frac{\sin x}{1 + \cos x} = 2 \csc x$.

12. Solve $6 \cos^2 x + \cos x = 2$ for x with either $0° \leq x < 360°$ or $0 \leq x < 2\pi$.

13. The equation $x = 4r \sin^2\left(\frac{\theta}{2}\right)$ arises when engineers design the merging region for a rapid transit system. Solve this equation for x in terms of $\cos \theta$.

22 AN INTRODUCTION TO CALCULUS

Courtesy of Michael A. Gallitelli, Metroland Photo Inc.

In order to pave this parking lot, we must figure out how much asphalt is needed. In this chapter, we will see how calculus can be used to find this answer.

In the middle of the seventeenth century, scientists began to study speed, motion, and rates of change. From this study evolved a new branch of mathematics called calculus. For the next 250 years, most of the important developments in mathematics and science were connected with calculus. Two men, Isaac Newton (1642–1727) and Gottfried Wilhelm Leibniz (1646–1716), are given credit for its discovery.

The development of calculus was a result of attempts to answer several questions in geometry. One question dealt with the slope of the tangent line to a curve, which led to differential calculus. A second question, concerning the area enclosed by the graph of a function and the x-axis, led to integral calculus. The solutions to these questions, the tangent question, and the area question will be explored in Sections 22.1 and 22.2.

OBJECTIVES

AFTER COMPLETING THIS CHAPTER, YOU WILL BE ABLE TO:

- Find the average slope of a curve over an indicated interval.
- Determine approximate areas under curves by the midpoint method.
- Evaluate limits using numerical, graphical, and algebraic methods.
- Evaluate one-sided limits and limits at infinity.
- Discuss the continuity of a function.

22.1 THE TANGENT QUESTION

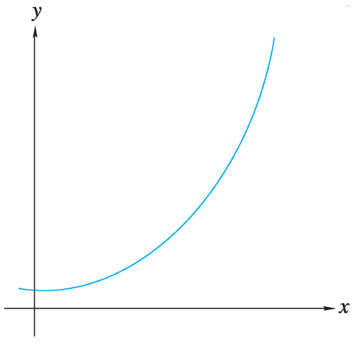


Figure 22.1

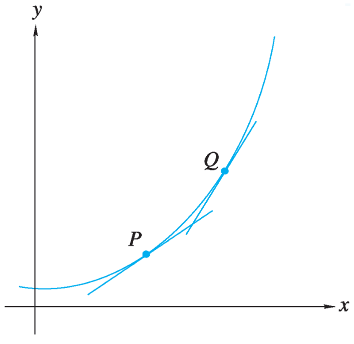


Figure 22.2

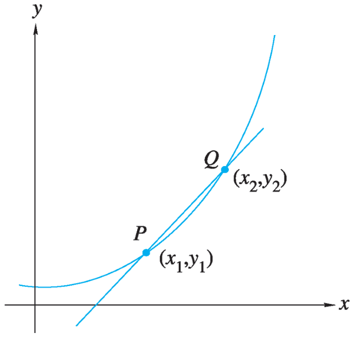


Figure 22.3

In Chapter 15, we studied the slope of a straight line. In particular, we found that if $P(x_1, y_1)$ and $Q(x_2, y_2)$ are two points on a line, then the slope of the line is given by the formula

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

But, what about curves that are not straight lines?

Consider the curve in Figure 22.1. As you can see, this is not a straight line, but it is the graph of a function. Slope is an indication of the "steepness" of a line. The slope of a straight line is the same at every point on the line. The curve in Figure 22.1 gives the impression that it gets steeper as the values of x increase. We might expect that the slope of a nonlinear curve would be different at different points on the curve. We would like a way to measure the steepness, or slope, of a nonlinear curve at any particular point on that curve.

One way to think of the slope of a curve at some point is to draw the tangent line to the curve at that point. Look at the curve in Figure 22.2. We have drawn the tangent to this curve at point P. We have also drawn the tangent to the curve at point Q. We can tell at a glance that the slope of the tangent at Q is greater than the slope of the tangent at P. Thus, we can let the slope of the tangent to a curve at some point be used for the slope of the curve at that point.

This is a helpful idea. Now, all we have to do is figure out some way to determine the slope of the tangent to a curve at any point. At present we do not have the background to determine the slope of a tangent line to a curve. We will substitute another idea until we have the necessary background.

Look at the graph of the function in Figure 22.3. This is the same curve we used in the previous example. As in Figure 22.2, P and Q are different points on the curve. P has the coordinates (x_1, y_1) and Q has the coordinates (x_2, y_2). The line that passes through points P and Q is called a *secant line*. The slope of the secant line through points P and Q is given by

$$m_{PQ} = \frac{y_2 - y_1}{x_2 - x_1}$$

AVERAGE SLOPE

This is an important idea. We have found that the slope of the secant line is a way to approximate the slope of the tangent line. What the secant line really gives is the *average rate of change* or **average slope** of the curve over the interval from (x_1, y_1) to (x_2, y_2). As we did in Chapter 13, we will use a bar to indicate an average. Thus, the symbol for the average slope will be $\overline{m}$, the symbol for the average velocity $\overline{v}$, and so on. The average rate of change through any two points, such as $P(x_1, y_1)$ and $Q(x_2, y_2)$, is the same as the slope of the line through those two points, and is given by

$$\overline{m} = \frac{y_2 - y_1}{x_2 - x_1}$$

or, since $y_1 = f(x_1)$ and $y_2 = f(x_2)$,

$$\overline{m} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

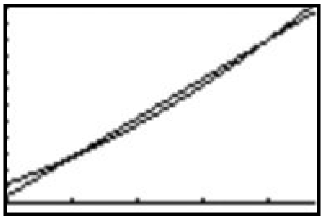
EXAMPLE 22.1

Consider the function $f(x) = x^2 + 3x - 4$. Find the average slope $\overline{m}$ of the curve over the interval from $x = 3$ to $x = 6$.

SOLUTION If we let $x_1 = 3$ and $x_2 = 6$, then, since $f(x) = x^2 + 3x - 4$, we have $f(3) = 14$ and $f(6) = 50$. Figure 22.4 shows a portion of the graph of $f(x) = x^2 + 3x - 4$ and the secant line through the points (3, 14) and (6, 50). Thus, the average slope of $\overline{m}$ is derived as follows:

$$\begin{aligned}\overline{m} &= \frac{f(x_2) - f(x_1)}{x_2 - x_1}\\ &= \frac{f(6) - f(3)}{6 - 3}\\ &= \frac{50 - 14}{6 - 3}\\ &= \frac{36}{3}\\ &= 12\end{aligned}$$

The average slope of $f(x) = x^2 + 3x - 4$ from $x = 3$ to $x = 6$ is 12.



$[2, 6.7, 1] \times [-2, 60, 5]$

Figure 22.4

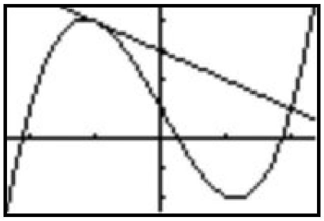
EXAMPLE 22.2

Find the average slope of the function $g(x) = x^3 - 4x + 1$ from $x = -1$ to $x = 2$.

SOLUTION We begin by letting $x_1 = -1$ and $x_2 = 2$, and evaluating $g(-1)$ and $g(2)$:

$$\begin{aligned}g(-1) &= (-1)^3 - 4(-1) + 1\\ &= 4\\ \text{and } g(2) &= 2^3 - 4(2) + 1\\ &= 1\end{aligned}$$

EXAMPLE 22.2 (Cont.)



$[-2.35, 2.35] \times [-2.5, 4.5]$

Figure 22.5

Figure 22.5 shows the graph of $g(x) = x^3 - 4x + 1$ over the interval $[-2.35, 2.35]$ and the secant line through the points $(-1, 4)$ and $(2, 1)$. Thus,

$$\begin{aligned}\overline{m} &= \frac{g(2) - g(-1)}{2 - (-1)} \\ &= \frac{1 - 4}{2 + 1} \\ &= -1\end{aligned}$$

The average slope of $g(x) = x^3 - 4x + 1$ from $x = -1$ to $x = 2$ is -1.

EXAMPLE 22.3

Find the average slope of the curve $h(x) = 2x^2 + 5x - 6$ from $x = -4$ to $x = x_1$, where $x_1 \neq -4$.

SOLUTION This is a more general version of the average slope problem. When $x = -4, h(x) = h(-4) = 6$, and when $x = x_1, h(x_1) = 2x_1^2 + 5x_1 - 6$. Figure 22.6 shows the graph of h and a line through the points $(-4, 6)$ and $(x_1, h(x_1))$. The average slope is

$$\begin{aligned}\overline{m} &= \frac{h(x_1) - h(-4)}{x_1 - (-4)} \\ &= \frac{(2x_1^2 + 5x_1 - 6) - 6}{x_1 + 4} \\ &= \frac{2x_1^2 + 5x_1 - 12}{x_1 + 4} \\ &= \frac{(2x_1 - 3)(x_1 + 4)}{x_1 + 4} \\ &= 2x_1 - 3\end{aligned}$$

since $x_1 \neq -4$.

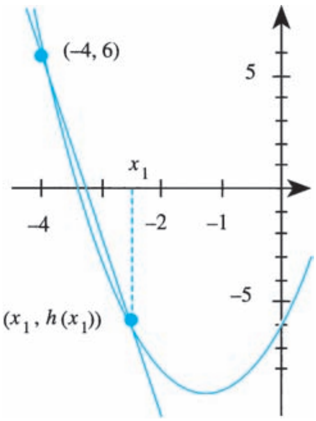


Figure 22.6

This last example provides a general formula for finding the average slope from $x = -4$ to any other point on the curve. For example, if we want the average slope from -4 to -5, we can use this formula with $x_1 = -5$ and get $\overline{m} = 2x_1 - 3 = 2(-5) - 3 = -13$.

The idea for finding the average slope can be applied to finding any type of average change. This can best be shown by the following example.

APPLICATION GENERAL TECHNOLOGY

EXAMPLE 22.4

A tank is filled with water by opening a valve on an inlet pipe. The volume V in liters of water in the tank t min after the valve is opened is given by the formula $V(t) = 5t^2 + 4t$. (a) What is the average rate of increase in the volume during the second minute? (b) What is the average rate of increase in the volume during the next 30 s?

SOLUTIONS We want to find the average rate of change in the volume.

(a) When $t = 2$, then $V(t) = V(2) = 28$ L, and when $t = 1$, we see that $V(t) = V(1) = 9$ L. If R_{Vol} is the rate of change in the volume, then

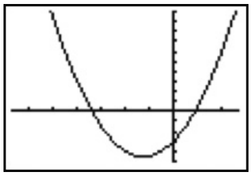
$$\begin{aligned}\overline{R}_{\text{Vol}} &= \frac{V(2) - V(1)}{2 \text{ min} - 1 \text{ min}} \\ &= \frac{28 \text{ L} - 9 \text{ L}}{2 \text{ min} - 1 \text{ min}} \\ &= \frac{28 - 9 \text{ L}}{2 - 1 \text{ min}} \\ &= 19 \text{ L/min}\end{aligned}$$

The volume changed at the rate of 19 L/min from the first to the second minute.

(b) The next 30 s will be from $t = 2.0$ min to $t = 2.5$ min. When $t = 2.5$, we determine that $V(t) = V(2.5) = 41.25$ L. Thus,

$$\begin{aligned}\overline{R}_{\text{Vol}} &= \frac{V(2.5) - V(2)}{2.5 \text{ min} - 2.0 \text{ min}} \\ &= \frac{41.25 \text{ L} - 28 \text{ L}}{2.5 \text{ min} - 2.0 \text{ min}} \\ &= \frac{41.25 - 28 \text{ L}}{2.5 - 2.0 \text{ min}} \\ &= 26.5 \text{ L/min}\end{aligned}$$

Thus, the volume changed at the rate of 26.5 L/min from 2 to 2.5 minutes.



$[-6.7, 2.7, 1] \times [-10, 20, 2]$

Figure 22.7a

USING A GRAPHING CALCULATOR

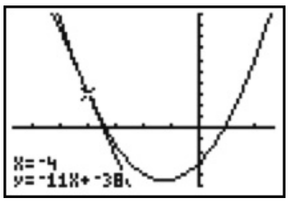
Many graphing calculators can be used to approximate the slope of a tangent line at a particular point on a curve. Example 22.5 will show how this can be done on the function used in Example 22.3.

EXAMPLE 22.5

Use a graphing calculator to find the slope of the tangent line to $h(x) = 2x^2 + 5x - 6$ at $x = -4$.

SOLUTION Begin by graphing h. The result of graphing h on a TI-83 or TI-84 as $y_1 = 2x^2 + 5x - 6$ is shown in Figure 22.7a.

Next, press **TRACE** and move the cursor to $x = -4$ and then press **2nd** **DRAW** 5 **ENTER**. (Note that pressing **2nd** **DRAW** 5 accesses the "`Draw-Tangent`" option on the calculator.) The result is shown in Figure 22.7b. In the lower left-hand corner of the calculator screen shown in Figure 22.7b, you should see $dy/dx = -11x + -38$. This is the equation of the tangent line and indicates that the slope of this tangent line is -11. In Example 22.3, we found that the average slope of the tangent line to $h(x) = 2x^2 + 5x - 6$ near $x = -4$ was $\overline{m} = 2x_1 - 3$. If you let $x_1 = -4$, then $\overline{m} = 2(-4) - 3 = -11$, the same value we got with the calculator.



$[-6.7, 2.7, 1] \times [-10, 20, 2]$

Figure 22.7b

APPLICATION BUSINESS

EXAMPLE 22.6

Table 22.1 shows the value of exports from the United States (in billions of dollars) from 2000 through 2008.

TABLE 22.1 Value of U.S. Exports, 2000–2008

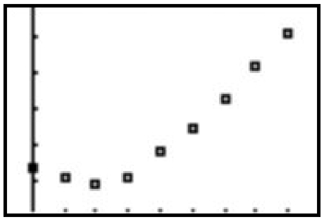
Year	2000	2001	2002	2003	2004	2005	2006	2007	2008
Exports (billions of dollars)	1071	1005	977	1020	1159	1281	1452	1643	1827

(a) Use the data to find the change in the value of U.S. exports from 2000 through 2008.

(b) Use the data to find the average rate of change in the value of U.S. exports from 2000 through 2008.

(c) Fit a model to the data. Use the model to find the change and the average rate of change in the value of U.S. exports from 2000 through 2008.

SOLUTIONS



$[-0.8, 8.8, 1] \times [-830, 1970, 200]$

Figure 22.8a

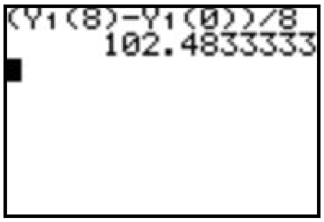


Figure 22.8b

(a) In 2000 exports were \$1,071 billion and in 2008 the value was \$1,827 billion. The total change is \$1,827 − \$1,071 = \$756 billion. Thus, from 2000 through 2008 exports increased by \$756 billion.

(b)
$$\text{Average rate of change} = \frac{\$1{,}827 \text{ billion} - 1{,}071 \text{ billion}}{2008 - 2000} = \frac{\$756 \text{ billion}}{8 \text{ years}} = \$94.5 \text{ billion dollars per year}$$

(c) A graph of these eight points is in Figure 22.8a. A quadratic regression produces the equation $E(t) \approx 19.42749t^2 - 52.93658t + 1041.94546$ billion dollars t years after 2000.

Storing this in the calculator as y_1 and evaluating the function, as in Figure 22.8b, we see that, according to the model, the average rate of change in the value of U.S. exports from 2000 through 2008 was an increase of about \$102.48 billion per year.

USING A SPREADSHEET

The slope of a tangent line to any point on a curve can be approximated using a spreadsheet and the fact that $\lim\limits_{\Delta x \to 0} \dfrac{\Delta y}{\Delta x} = \dfrac{dy}{dx}$. Example 22.7 will show how this can be done on the function used in Example 22.3.

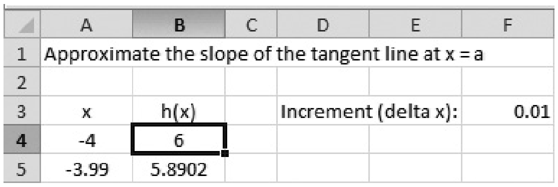
EXAMPLE 22.7

Use a spreadsheet to find the slope of the tangent line to $h(x) = 2x^2 + 5x - 6$ at $x = -4$.

SOLUTION We will construct a table of values with only two points—one being the x-value at which we want the tangent line to intersect the curve (x_t), and a second x-value only slightly larger ($x_t + \Delta x$, $\Delta x > 0$).

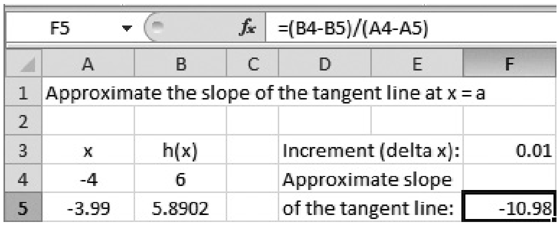
In Figure 22.9a, $x_t = -4$ (Cell A4) and $\Delta x = 0.01$ (Cell F3). Now we will add a cell to calculate the slope (see Figure 22.9b). As we change the value of Cell F3 to make Δx smaller, the approximation gets better (see Figure 22.9c).

As we continue to reduce the increment, we should recognize that the value in Cell F5 is approaching some limit. That limiting value is the slope of the tangent line (see Figure 22.9d).



	A	B	C	D	E	F
1	Approximate the slope of the tangent line at x = a					
2						
3	x	h(x)		Increment (delta x):		0.01
4	-4	6				
5	-3.99	5.8902				

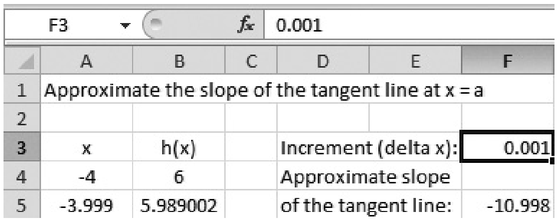
Figure 22.9a



F5 =(B4-B5)/(A4-A5)

	A	B	C	D	E	F
1	Approximate the slope of the tangent line at x = a					
2						
3	x	h(x)		Increment (delta x):		0.01
4	-4	6		Approximate slope		
5	-3.99	5.8902		of the tangent line:		-10.98

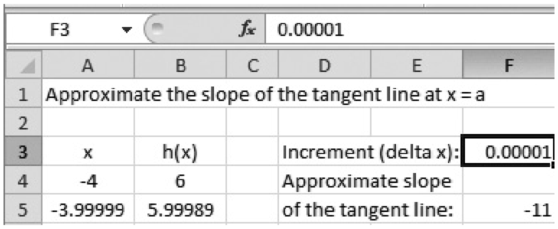
Figure 22.9b



F3 0.001

	A	B	C	D	E	F
1	Approximate the slope of the tangent line at x = a					
2						
3	x	h(x)		Increment (delta x):		0.001
4	-4	6		Approximate slope		
5	-3.999	5.989002		of the tangent line:		-10.998

Figure 22.9c



F3 0.00001

	A	B	C	D	E	F
1	Approximate the slope of the tangent line at x = a					
2						
3	x	h(x)		Increment (delta x):		0.00001
4	-4	6		Approximate slope		
5	-3.99999	5.99989		of the tangent line:		-11

Figure 22.9d

APPLICATION ENERGY

EXAMPLE 22.8

Table 22.2 shows the consumption of wind energy in the United States (in trillions of Btus) from 2000 through 2008.

(a) Use the data to find the change in the wind energy consumption from 2000 through 2008.

(b) Use the data to find the average rate of change in the wind energy consumption from 2000 through 2008.

(c) Fit a model to the data. Use the model to find the change and the average rate of change in U.S. wind energy consumption from 2000 through 2008.

TABLE 22.2 Consumption of Wind Energy in the United States, 2000–2008

Year	2000	2001	2002	2003	2004	2005	2006	2007	2008
Wind Energy Consumption (trillions of Btus)	60	70	448	110	110	140	180	260	340

EXAMPLE 22.8 (Cont.)

SOLUTIONS

(a) Figure 22.10a shows that the change in consumption from 2000 through 2008 is found by subtracting the energy consumed in 2000 from the amount consumed in 2008.

(b) The average rate of change over the years 2000 to 2008 is found by using the slope formula to find the change in y over the change in x. The average rate of change is the slope of the line through the two points (2000, 60) and (2008, 340) (see Figure 22.10b).

(c) A graph of these nine points (using x-values in years since 2000) is in Figure 22.10c. A quadratic regression (polynomial of order 2) produces the equation $E(t) \approx 8.8095t^2 - 20.81t + 81.333$, as shown in Figure 22.10d.

This model produces the two ordered pairs: $(0, 81.333)$, $(8, 578.66)$. The average rate of change, according to this model, is about 49.666 trillion Btus per year (see Figure 22.10e).

B6 f_x =J4-B4

	A	B	C	D	E	F	G	H	I	J
1	Table 22.2									
2										
3	Year	2000	2001	2002	2003	2004	2005	2006	2007	2008
4	Wind Consumption	60	70	110	110	140	180	260	340	510
5										
6	(a)	450	trillion Btu							

Figure 22.10a

B7 f_x =(J4-B4)/(J3-B3)

	A	B	C	D	E	F	G	H	I	J
1	Table 22.2									
2										
3	Year	2000	2001	2002	2003	2004	2005	2006	2007	2008
4	Wind Consumption	60	70	110	110	140	180	260	340	510
5										
6	(a)	450	trillion Btu							
7	(b)	56.25	trillion Btu/year							

Figure 22.10b

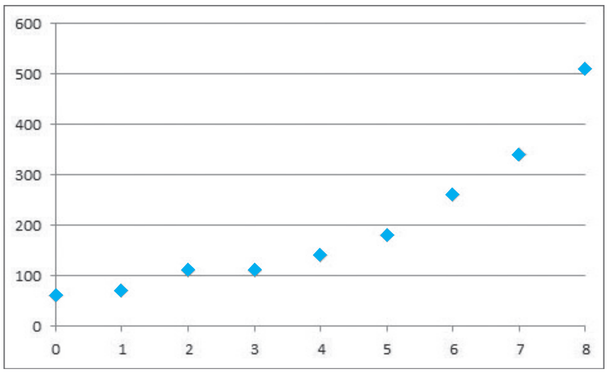


Figure 22.10c

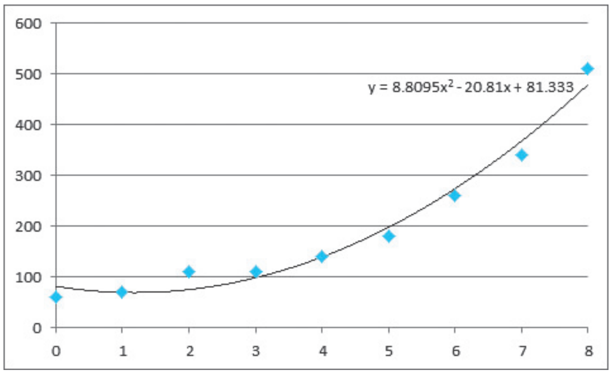


Figure 22.10d

x	0	8	Average rate of change:
Model	81.333	478.66	49.666

Figure 22.10e

EXERCISE SET 22.1

Find the average slope over the indicated intervals for the functions in Exercises 1–14.

1. $f(x) = x^2 + 5x$ from $x = 1$ to $x = 3$
2. $g(x) = 5x - 3$ from $x = 4$ to $x = 7$
3. $h(x) = 7x + 1$ from $x = -3$ to $x = -1$
4. $j(x) = 4x^2 - 5$ from $x = -2$ to $x = 0$
5. $k(x) = 3x^3 + 7x - 1$ from $x = -2$ to $x = -1$
6. $m(x) = 2x^4 - 5x^2 + x - 1$ from $x = 0$ to $x = 1$
7. $f(x) = \dfrac{5}{x}$ from $x = 2$ to $x = 5$
8. $g(x) = \dfrac{8}{x+2}$ from $x = -6$ to $x = -3$
9. $h(x) = x^2$ from $x = 1$ to $x = 0$
10. $j(x) = x^2 + 3x + 5$ from $x = -2$ to $x = b, b \neq -2$
11. $k(x) = x^2 + x - 7$ from $x = 1$ to $x = x_1, x_1 \neq 1$
12. $m(x) = x^3 + 4$ from $x = 1$ to $x = x_1, x_1 \neq 1$
13. $f(x) = x^2 + 1$ from $x = x_1$ to $x = x_1 + h, h \neq 0$
14. $g(x) = x^2 + 4x - 1$ from $x = x_1$ to $x = x_1 + h$, $h \neq 0$

Solve Exercises 15–24.

15. ***Machine technology*** What was the average rate of change in the volume for the tank in Example 22.4 **(a)** during the third minute, **(b)** during the fourth minute, that is, from the third to the fourth minute, and **(c)** from the second to the fourth minute?
16. ***Physics*** A stone is dropped into a pool of water and causes a ripple that travels in the shape of a circle out from the point of impact at a rate of 2 m/s. What is the average change in the area within this circle from the third to the fourth second?
17. Let $f(x) = 3x^2 - 5$. Find the average slope of the curve from $x_1 = 4$ to **(a)** $x_2 = 6$, **(b)** $x_3 = 5$, **(c)** $x_4 = 4.5$, **(d)** $x_5 = 4.25$, **(e)** $x_6 = 4.1$, and **(f)** $x_7 = 4.05$.
18. For the same function in Exercise 17, $f(x) = 3x^2 - 5$, find the average slope of the curve from $x_1 = 4$ to **(a)** $x_2 = 2$, **(b)** $x_3 = 3$, **(c)** $x_4 = 3.5$, **(d)** $x_5 = 3.75$, **(e)** $x_6 = 3.9$, and **(f)** $x_7 = 3.95$.

In Exercises 19–14, the average speed is the ratio $\dfrac{\text{distance traveled}}{\text{time to travel the distance}}$.

19. ***Transportation*** Table 22.3 shows the distance in feet that a Porsche 911 Carrera traveled as it accelerated from a standing start.

TABLE 22.3 Distance Traveled by Porsche 911 Carrera

Time (s)	0	2.0	3.0	4.0	5.0	6.0	7.0	8.0	9.0	10.0
Distance (ft)	0	47	99	166	246	335	435	544	660	784

(a) Determine the average speed of the Carrera from 8.0 s to 9.0 s.
(b) What units should be applied to your answer to (a)?
(c) Find a model for the distance traveled, in feet, of the Carrera as a function of the number of seconds after starting from 0.
(d) Use your model from (c) to determine the average speed, in feet per second, of the Carrera from 8.5 s to 9.0 s.
(e) Use your model from (c) to determine the average speed, in feet per second, of the Carrera from 8.9 s to 9.0 s.

20. ***Transportation*** Table 22.4 shows the distance in feet that a Dodge Viper GTS traveled as it accelerated from a standing start.

TABLE 22.4 Distance Traveled by Dodge Viper GTS

Time (s)	0	2.0	3.0	4.0	5.0	6.0	7.0	8.0	9.0	10.0
Distance (ft)	0	48	104	179	271	377	497	628	768	915

(a) Determine the average speed of the Viper from 8.0 to 9.0 s.

(b) Find a model for the distance traveled, in feet, of the Viper as a function of the number of seconds after starting from 0.

(c) Use your model from (b) to determine the average speed, in feet per second, of the Viper from 8.5 to 9.0 s.

(d) Use your model from (b) to determine the average speed, in feet per second, of the Viper from 8.9 to 9.0 s.

21. ***Environmental science*** Table 22.5 shows the atmospheric concentration (in ppm) of carbon dioxide (CO_2) every 5 years from 1960 to 2000.

TABLE 22.5 Atmospheric Concentration of Carbon Dioxide (CO_2) 1960–2000

Year	1960	1965	1970	1975	1980	1985	1990	1995	2000
Concentration (ppm)	317	320	326	331	339	346	354	361	369

(a) Determine the average annual change of the atmospheric concentration of carbon dioxide (CO_2) from 1990 to 2000.

(b) What units should be applied to your answer to (a)?

(c) Find a model that predicts the atmospheric concentration (in ppm) of carbon dioxide (CO_2) for year t, where t is the number of years since 1900.

(d) Use your model from (c) to determine the average annual change of the atmospheric concentration of carbon dioxide (CO_2) from 1995 to 2000.

(e) Use your model from (c) to determine the average annual change of the atmospheric concentration of carbon dioxide (CO_2) from 1999 to 2000.

22. ***Medical technology*** Consider the following sentence:
The child's temperature has been rising for the last 2 h, but not as rapidly since we gave the antibiotic an hour ago.

(a) With this statement in mind, sketch a graph of the child's temperature as a function of time.

(b) How did you (or how can you) use tangents to your graph to show that your graph is consistent with the statement?

23. ***Transportation*** On a 50-min trip, a car travels for 15 min with an average velocity of 20 mph and then 35 min with an average velocity of 54 mph. Determine **(a)** the total distance traveled and **(b)** the average velocity for the entire trip.

24. ***Agriculture*** The graph in Figure 22.11 gives the number of $f(t)$ farms in a certain Iowa county t years after 1980.

(a) Calculate the average rate of change in the number of farms from 1980 to 1985.

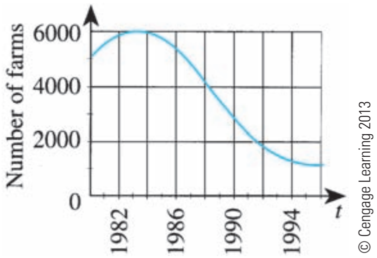


Figure 22.11

(b) Calculate the average rate of change in the number of farms from 1990 to 1995.

(c) During what 2-year period was the number of farms decreasing most rapidly?

(d) What was the rate of change in the number of farms for the 2 years you answered in (c)?

[IN YOUR WORDS]

25. You are in an automobile that is traveling from your home to school. Explain how you can determine the average rate that the car is traveling.

26. What is the difference between a secant line and a tangent line?

22.2 THE AREA QUESTION

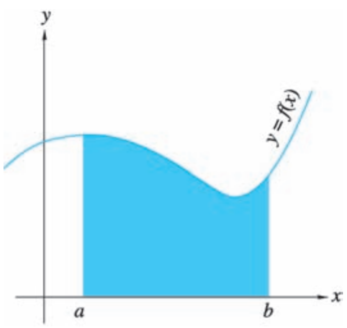


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Figure 22.12

In Section 22.1, we examined one of the basic questions that led to the development of calculus. In this section, we will look at the other basic question—the area question.

The area question concerns the area between the graph of a function and the x-axis. There are some restrictions on this problem that can best be described by examining Figure 22.12. The function $f(x)$ should be nonnegative over a closed interval; that is, it should not fall below the x-axis in some closed interval. If the closed interval is $[a, b]$, then the desired area lies between the graph of $f(x)$, the x-axis, and the vertical lines $x = a$ and $x = b$. This is indicated by the shading in Figure 22.12.

We have used several different formulas for the area. Most of these were introduced in Chapter 3. We will use them to help determine the shaded area in Figure 22.12.

RECTANGULAR METHOD FOR APPROXIMATING AREA

We will approximate the area by adding the areas of several rectangles. To begin, we will divide the interval $[a, b]$ into n segments. While each segment could be a different length, we will make them all the same length because it makes calculations easier. If we let $a = x_0$ and $b = x_n$, the first segment is $[x_0, x_1]$; the second segment $[x_1, x_2]$; the third segment $[x_2, x_3]$; and so on. The last segment is $[x_{n-1}, x_n]$. In general, the ith segment would be $[x_{i-1}, x_i]$. At each of these points, $x_0, x_1, x_2, \ldots, x_n$, we will erect a line perpendicular to the x-axis. This creates a group of n strips or bands, as shown in Figure 22.13.

We want to draw a line segment parallel to the x-axis across the top of each of these strips. Where we draw these segments is up to us. We would

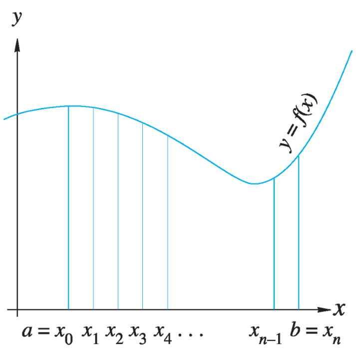

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Figure 22.13

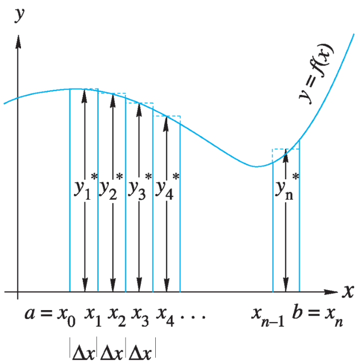


Figure 22.14

like to draw them so that we can get a close approximation to the area under the curve. What we will do is use an approach that should give a reasonable approximation.

Look at Figure 22.14. We have selected a point in the middle of each strip and drawn the line segment so that it went through the y-coordinate at that point. For example, in the first strip, the midpoint between x_0 and x_1 is the point we will call x_1^*. The height of the curve at x_1^* is $f(x_1^*)$, which we will call y_1^*. Now, since each segment is the same width, the width of each rectangle is the same. This width, which we will call Δx, is equal to $\dfrac{x_n - x_0}{n}$. This means that the area of the first rectangle is $(\Delta x)(y_1^*)$. In the same way, the area of the second rectangle would be $(\Delta x)(y_2^*)$ and the area of the last rectangle would be $(\Delta x)(y_n^*)$.

We can now get an idea of the area under this curve by adding all the areas of these rectangles. Thus, the area under the curve, A, can be written as

$$\begin{aligned} A &\approx (y_1^*)(\Delta x) + (y_2^*)(\Delta x) + (y_3^*)(\Delta x) + \cdots + (y_n^*)(\Delta x) \\ &= (y_1^* + y_2^* + y_3^* + \cdots + y_n^*)(\Delta x) \\ &= \left(\sum_{i=1}^{n} y_i^*\right)\Delta x \end{aligned}$$

EXAMPLE 22.9

Suppose $f(x) = x^2 + 3$. Find an approximation of the area between this curve and the x-axis from $x = 1$ to $x = 6$.

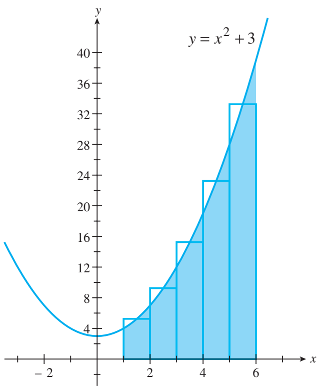


Figure 22.15

SOLUTION We will divide the interval [1, 6] into 5 equal segments. (The number of segments is purely arbitrary and we selected 5 because it means that each segment has a width of 1, or $\Delta x = 1$.) The midpoints of the segments are at 1.5, 2.5, 3.5, 4.5, and 5.5, as can be seen in Figure 22.15. The value of the function at each of these points can be seen in the following table.

x^*	1.50	2.50	3.50	4.50	5.50
$f(x^*) = y^*$	5.25	9.25	15.25	23.25	33.25

The area then is found using our new formula for area:

$$\begin{aligned} A &\approx (y_1^*)(\Delta x) + (y_2^*)(\Delta x) + (y_3^*)(\Delta x) + \cdots + (y_n^*)(\Delta x) \\ &\approx (y_1^* + y_2^* + y_3^* + \cdots + y_n^*)(\Delta x) \\ &= (5.25 + 9.25 + 15.25 + 23.25 + 33.25)(1) \\ &= 86.25 \end{aligned}$$

The area is approximately 86.25 square units.

We will work another example, only this time we will make the widths of the segments something other than 1.

EXAMPLE 22.10

Approximate the area under the curve $g(x) = 4x^3 + 2x - 1$ from $x = 2$ to $x = 5$.

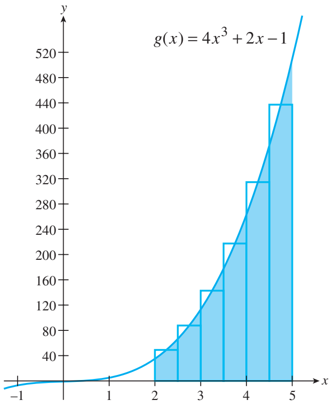


Figure 22.16

SOLUTION The curve is shown in Figure 22.16 with the shaded region indicating the area we want to approximate. The total width of the interval is 3 units. We will divide this into six segments, so each segment will be 0.5 units long, or $\Delta x = 0.5$. The following table gives the values of the function at the midpoints of each of these intervals.

x^*	2.25	2.7500	3.2500	3.7500	4.2500	4.7500
y^*	49.0625	87.6875	142.8125	217.4375	314.5625	437.1875

Again, using our formula for approximating the area, we get

$$\begin{aligned} A &\approx (y_1^* + y_2^* + y_3^* + \cdots + y_n^*)(\Delta x) \\ &= (49.0625 + 87.6875 + 142.8125 + 217.4375 \\ &\quad + 314.5625 + 437.1875)(0.5) \\ &= (1{,}248.75)(0.5) \\ &= 624.375 \end{aligned}$$

The approximate area under this curve from $x = 2$ to $x = 5$ is 624.375 square units.

APPLICATION CONSTRUCTION

EXAMPLE 22.11

The following table gives the results of a series of drillings to determine the depth of the bedrock at a building site. These drillings were taken along a straight line down the middle of the lot where the building will be placed. In the table, x is the distance from the front of the parking lot and y is the corresponding depth. Both x and y are given in feet.

x	0	20	40	60	80	100	120	140	160
y	33	35	40	45	42	38	46	40	48

Approximate the area of this cross-section.

SOLUTION The total length of the interval is 160 ft. This has been divided into 8 segments, with each segment 20 ft long, so $\Delta x = 20$. Since we do not know the function, we will have to approximate the depth at the midpoints of the intervals. We will do this by taking the average of the values at each end of an interval. Thus, the length at the midpoint of the first interval is $y_1^* = \frac{y_1 + y_2}{2} = \frac{33 + 35}{2} = 34$.

The next table gives the approximate values at the midpoints of these intervals.

x^*	10	30	50	70	90	110	130	150
y^*	34	37.5	42.5	43.5	40	42	43	44

EXAMPLE 22.11 (Cont.)

Once again, using our formula for approximating the area, we get

$$\begin{aligned} A &\approx (y_1^* + y_2^* + y_3^* + \cdots + y_8^*)\Delta x \\ &= (34 + 37.5 + 42.5 + 43.5 + 40 + 42 + 43 + 44)(20) \\ &= (326.5)(20) \\ &= 6{,}530 \text{ ft}^2 \end{aligned}$$

Example 22.11 has been worked twice before in this text. In Example 3.22 we used the trapezoidal rule and got the same result, 6,530 ft^2, and in Example 3.23 we used Simpson's rule and obtained 6,460 ft^2. Which is more accurate? We have no way of telling from this set of finite data. The only way we could tell would be to take more drillings and figure the area using those numbers. However, these approximations are probably accurate enough for this job.

APPLICATION CONSTRUCTION

EXAMPLE 22.12

Before the parking lot for the building in Example 22.11 is completed it will have to be paved. A photograph of the completed lot is shown in Figure 22.17a. In order to pave this parking lot, its area had to be determined. This area was then multiplied by the thickness of the asphalt to obtain the total volume of the asphalt that was needed.

In order to find the area, the contractor drew a base line through the "middle" of the parking lot and then drew lines perpendicular to the base line every 50 feet. Each of these lines was drawn until it reached the other side of the parking lot, as shown in Figure 22.17b. The following table shows the lengths labeled y (in feet) of each of these lines.

x	0	50	100	150	200	250	300	350	400	450	500	550	600
y	72	104	160	200	200	200	190	190	190	180	180	180	180

Use the data in the table to approximate the area of the parking lot.

SOLUTION We are given that the length of each interval is 50 ft, and so $\Delta x = 50$. As in Example 22.11, we will approximate the lengths at the midpoints by

Figure 22.17a

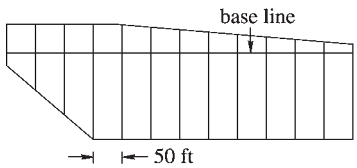


Figure 22.17b

taking the average of the values at each end of the intervals. The next table gives the approximate values at the midpoints of these intervals.

x^*	25	75	125	175	225	275	325	375	425	475	525	575
y^*	88	132	180	200	200	195	190	190	185	180	180	180

As before, we use our formula for approximating the area and obtain

$$\begin{aligned} A &\approx (y_1^* + y_2^* + y_3^* + \cdots + y_{12}^*)\Delta x \\ &= (88 + 132 + 180 + 200 + 200 + 195 + 190 \\ &\quad + 190 + 185 + 180 + 180 + 180)(50) \\ &= (2{,}100)(50) \\ &= 105{,}000 \end{aligned}$$

Thus, we see that the area of this parking lot is approximately 105,000 ft^2.

APPLICATION ENVIRONMENTAL SCIENCE

EXAMPLE 22.13

Table 22.6 gives the change in the annual domestic fuel consumption, in billion gallons, of all motor vehicles in the United States for selected years from 1979 through 2007. Each change is from the immediate prior year. Thus, the change of 5.3 billion gallons given for 1999 means that this was a change of 5.3 billion gallons from 1998. (a) Use the rectangular method to approximate the area under this curve. (b) Use Simpson's rule to approximate the area under this curve. (c) If a total of 119.2 billion gallons of fuel was consumed in 1982, how many gallons were consumed in 2007?

TABLE 22.6 Annual Change in U.S. Motor Vehicle Fuel Consumption, 1979–2007

Year	1979	1981	1983	1985	1987	1989	1991	1993
Change in fuel consumption (billion gal)	3.0	−0.5	2.7	2.6	2.3	1.8	−2.2	4.4

Year	1995	1997	1999	2001	2003	2005	2007
Change in fuel consumption (billion gal)	3.0	3.0	6.0	1.9	1.3	1.3	1.1

SOLUTIONS The given data is for every other year, t, so $\Delta t = 2$ years.

(a) To use the rectangular method we need to find the midpoint of each of these intervals. As in Examples 22.11 and 22.12 we will approximate the lengths at the midpoints by taking the average of the values at the end of each interval. Table 22.7 gives the approximate values, y^*, at the midpoint of these intervals.

EXAMPLE 22.13 (Cont.)

TABLE 22.7 Annual Change in U.S. Motor Vehicle Fuel Consumption, 1979–2007

Year	1980	1982	1984	1986	1988	1990	1992	1994	1996	1998	2000	2002	2004	2006
Consumption (y^*)	1.25	1.1	2.65	2.45	2.05	−0.2	1.1	3.7	3.0	4.5	3.95	1.6	1.3	1.2

Once again we use our formula for approximating the area and obtain

$$\begin{aligned} A &\approx (y_1^* + y_2^* + y_3^* + \cdots + y_{14}^*)\Delta t \\ &= (1.25 + 1.1 + 2.65 + 2.45 + 2.05 - 0.2 + 1.1 + 3.7 + 3.0 \\ &\quad + 4.5 + 3.95 + 1.6 + 1.3 + 1.2)(2) \\ &= (29.65)(2) = 59.3 \end{aligned}$$

Thus, we see that, according to the rectangular approximation method, the total change in fuel consumption was about 59.3 billion gallons.

(b) In Simpson's rule, the width of the interval, h, is the same as the Δx in the rectangular method. Here, since the independent variable is time, we have used t rather than x. The values for y come from the original data in Table 22.6.

$$\begin{aligned} A_s &\approx \frac{\Delta t}{3}(y_0 + 4y_1 + 2y_2 + 4y_3 + \cdots + 2y_{12} + 4y_{13} + y_{14}) \\ &= \frac{2}{3}(3.0 + 4(-0.5) + 2(2.7) + 4(2.6) + 2(2.3) + 4(1.8) \\ &\quad + 2(-2.2) + 4(4.4) + 2(3.0) + 4(3.0) + 2(6.0) \\ &\quad + 4(1.9) + 2(1.3) + 4(1.3) + 1.1) \\ &= \frac{2}{3}(88.3) \approx 58.9 \end{aligned}$$

Thus, we see that, according to Simpson's rule, the total change in fuel consumption was about 43.4 billion gallons.

(c) We are given the information that 119.2 billion gallons of fuel were consumed in 1979. According the rectangular method, $119.2 + 59.3 = 178.5$ billion gallons were consumed in 2007. The result from Simpson's rule predicts that $119.2 + 58.9 = 178.1$ billion gallons were consumed in 2007. The actual consumption in 2007 was 178.1 billion gallons.

EXERCISE SET 22.2

In Exercises 1–10, find the area under the graph of the function from a to b using the method described in this section. The value of n in each problem indicates the number of segments into which you should divide the interval.

1. $f(x) = 3x + 2;\ a = 1,\ b = 7,\ n = 6$

2. $g(x) = 7 - 4x;\ a = -4,\ b = 1,\ n = 5$

3. $h(x) = x^2 + 1;\ a = 0,\ b = 3,\ n = 6$

4. $k(x) = 3x^2 - 2;\ a = 1,\ b = 3,\ n = 4$

5. $j(x) = 4x^2 + 3x - 5; a = 1, b = 5, n = 8$

6. $m(x) = x^3 + 2; a = -1, b = 2, n = 6$

7. $f(x) = 16 - x^2; a = -2, b = 4, n = 8$

8. $g(x) = 4x - x^3; a = 0, b = 2, n = 10$

9. $h(x) = 3x^2 - 2x + 7; a = -5, b = -1, n = 8$

10. $k(x) = x^4 - 3; a = 2, b = 4, n = 8$

In Exercises 11–12, find the approximate area under the curve defined by graphing the sets of experimental data.

11.

x	5	6	7	8	9	10	11	12	13
y	4.2	3.9	3.8	4.0	3.5	3.4	3.9	4.1	4.3

12.

x	1.00	1.25	1.50	1.75	2.00	2.25	2.50
y	16.32	16.48	16.73	16.42	16.38	16.29	16.25

Solve Exercises 13–20.

13. *Business* The marginal profit for a certain type of coat is given by $P'(x) = 78 - 0.8x$ dollars per coat, where x is the number of coats produced and sold weekly. The profit for the first n coats that are produced and sold is determined by finding the area under the graph of P' from $a = 0$ to $b = n$. What is the profit for the first 40 coats produced and sold?

14. *Environmental Science* The level of pollution in San Juan Pedro Bay, due to a sewage spill, is estimated to be $f(t) = \dfrac{1200t}{\sqrt{t^2 + 10}}$ parts per million, where t is the time in days since the spill occurred. Find the total amount of pollution during the first 5 days of the oil spill by using the method described in this section and 10 divisions.

15. *Electronics* The charge on a capacitor in millicoulombs can be estimated by finding the area under the graph of $f(t) = 0.6t^2 - 0.2t^3$ from $t = 1$ to $t = 3$. Estimate the charge on this capacitor by using the method described in this section and eight divisions.

16. *Ecology* A town wants to drain and fill the swamp shown in Figure 22.18.

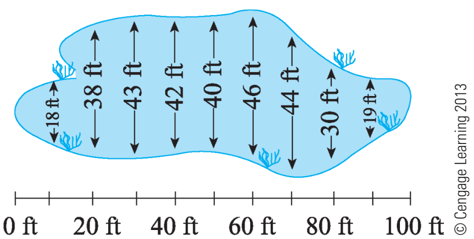


Figure 22.18

(a) What is the surface area of the swamp?

(b) If the swamp has an average depth of 6 ft, how many cubic yards of dirt will it take to fill the "hole" that is left after the swamp is drained?

17. *Transportation* Table 22.8 shows the speed in feet/s (fps) of a Porsche 911 Carrera as it accelerated from a standing start.

TABLE 22.8 Speed of a Porsche 911 Carrera

Time (s)	0	1.5	3.0	4.5	6.0	7.5	9.0	10.5	12.0	13.5	15.0
Speed (fps)	0	34.6	60.0	79.4	95.0	108.3	120.2	131.0	140.4	147.3	150.4

(a) Plot these 11 points on a graph. Connect the points to make a broken-line graph. Make sure that you indicate the units for each axis.

(b) Use the rectangular method to approximate the area under this graph. What unit should be applied to this area?

(c) Use the trapezoidal rule to approximate the area under this graph.

(d) Use Simpson's rule to approximate the area under this graph.

18. ***Transportation*** Table 22.9 shows the speed in feet/s (fps) of a Dodge Viper GTS as it accelerated from a standing start.

TABLE 22.9 Speed of a Dodge Viper GTS

Time (s)	0	1.5	3.0	4.5	6.0	7.5	9.0	10.5	12.0	13.5	15.0
Speed (fps)	0	35.5	66.1	91.9	113.2	130.4	144.3	155.7	165.7	175.6	186.8

(a) Plot these 11 points on a graph. Connect the points to make a broken-line graph. Make sure that you indicate the units for each axis.

(b) Use the rectangular method to approximate the area under this graph. What unit should be applied to this area?

(c) Use the trapezoidal rule to approximate the area under this graph.

(d) Use Simpson's rule to approximate the area under this graph.

19. ***Environmental science*** Table 22.10 shows the atmospheric concentration (in ppm) of carbon dioxide (CO_2) every 5 years from 1960 to 2005.

TABLE 22.10 Atmospheric Concentration of Carbon Dioxide (CO_2) 1960–2005

Year	1960	1965	1970	1975	1980	1985	1990	1995	2000	2005
Concentration (ppm)	317	320	326	331	339	346	354	361	369	380

Use the rectangular method to approximate the total change in atmospheric concentration of carbon dioxide (CO_2) from 1960 to 2005.

20. ***Recreation*** Use the rectangular approximation method to determine the area of the golf green in Figure 22.19.

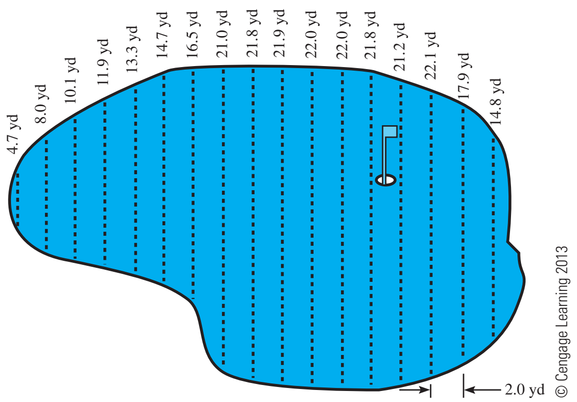


Figure 22.19

[IN YOUR WORDS]

21. Describe how to use the procedures of this chapter to find the area of an irregular-shaped region.

22. What changes would you have to make in the procedures of this chapter if the rectangles were not all the same width?

23. Use the procedures described in Example 22.12 on an irregular-shaped region such as a parking lot or a lake to determine its area.

24. The paragraph following Example 22.11 gave our earlier results to the this problem when we used trapezoidal and Simpson's rules. The trapezoidal rule and the rectangular method gave the same result. Explain why they are the same.

22.3 LIMITS: AN INTUITIVE APPROACH

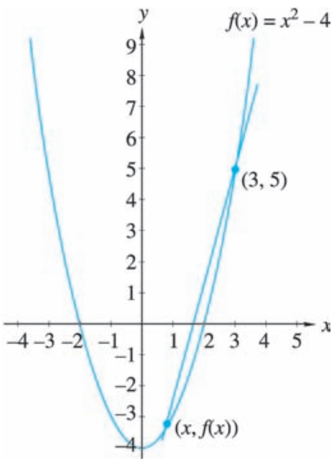


Figure 22.20

In Sections 22.1 and 22.2, we have looked at ways in which we could attempt to solve two of the questions that led to the development of calculus. These two questions are known as the tangent question and the area question. So far we have laid a foundation for answering them. In this section, we will learn about a third part of the foundation—the limit.

To begin our introduction to the limit, we will return to our look at the tangent question. Consider the curve of the function $f(x) = x^2 - 4$. Suppose that we want to find the slope of the tangent to this curve at the point (3, 5). In Section 22.1, we found out how to determine the slope of a secant line to this curve through the point (3, 5). Let $(x, f(x))$ be any point, except the point (3, 5), on the graph of $f(x) = x^2 - 4$. The secant line through these two points is shown in Figure 22.20. The slope of the secant line is given by

$$\overline{m} = \frac{f(x) - 5}{x - 3} = \frac{(x^2 - 4) - 5}{x - 3} = \frac{x^2 - 9}{x - 3}$$

Since the point $(x, f(x))$ is different from (3, 5), we know that $x \neq 3$ and so we can simplify our equation to

$$\overline{m} = \frac{(x + 3)(x - 3)}{x - 3} = x + 3$$

So far so good! Everything that we have done is just like what we did in Section 22.1. But now, we want to look at the problem a little closer. What happens as x gets closer to 3? The values of $\overline{m} = x + 3$ get closer to 6. We put this idea into symbols by writing

$$\overline{m}_{\tan} = \lim_{x \to 3} \frac{x^2 - 9}{x - 3} = \lim_{x \to 3} (x + 3) = 6$$

Since we know that the slope of the tangent line at the point (3, 5) is 6, we are able to determine that the equation of the tangent is

$$y - 5 = 6(x - 3)$$
$$y = 6x - 13$$

This is the basic idea behind a limit. We will now begin to look at a more detailed explanation.

LIMIT OF A FUNCTION

If a function $y = f(x)$ approaches the real number L as x approaches a particular value a, then we say that L is the **limit** of f as x approaches a. The notation for this concept is

$$\lim_{x \to a} f(x) = L.$$

The $\lim_{x \to a} f(x)$ may or may not exist for any function at any particular a. The following examples use numerical calculations to locate limits.

THE NUMERICAL APPROACH TO A LIMIT

To get a better idea of a limit, we will return to Exercises 17–18 from Exercise Set 22.1.

In these two exercises $f(x) = 3x^2 - 5$. We will set up two tables of values. Each table will contain values for x and $\overline{m} = \dfrac{f(x) - f(4)}{x - 4} = \dfrac{f(x) - 43}{x - 4}$. In the first, we will put the values for Exercise 17, and in the second, the values for Exercise 18.

In Exercise 17, all of the values of x were larger than 4, with the first one being the largest, 6, and each one getting smaller. The last value was the smallest, 4.05. We will continue this process by selecting additional numbers larger than 4 (and smaller than 4.05), each one getting closer to 4.

x	6	5	4.5	4.25	4.10	4.05	4.01	4.001	4.0001
$\overline{m}$	30	27	25.5	24.75	24.30	24.15	24.03	24.003	24.0003

The table of values for Exercise 18 contains values of x that are all smaller than 4, with the largest being 3.95. Again, we will continue the process. This time the numbers will all be smaller than 4, each getting increasingly closer to 4.

x	2	3	3.5	3.75	3.9	3.95	3.99	3.999	3.9999
$\overline{m}$	18	21	22.5	23.75	23.70	23.85	23.97	23.997	23.9997

These tables give us a numerical method for approximating the $\lim_{x \to 4} (3x^2 - 5)$. As you can see from the two tables, when x is 4.0001, $m \approx$ 24.0003 and when x is 3.9999, $m \approx 23.9997$. It appears that if we were to pick values of x that were closer to 4, we could then get a value of $f(x) = 3x^2 - 5$ that was closer to 24.

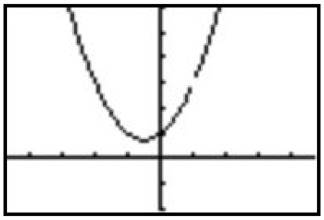
EXAMPLE 22.14

Use a numerical approach to find $\lim_{x \to 1} \dfrac{x^3 - 1}{x - 1}$.

SOLUTION As in the previous explanation, we will construct a table. This time, since we want to find the limits as x gets closer to 1, we will select values of x that keep getting closer to 1. Some of the values selected are less than 1, and some are larger than 1. The results are shown in the following table.

x	0.500	0.900	0.990	0.999	1.001	1.010	1.100	1.500
$\dfrac{x^3-1}{x-1}$	1.75	2.71	2.9701	2.997	3.003	3.0301	3.31	4.75

It appears from the table that $\lim_{x \to 1} \dfrac{x^3 - 1}{x - 1} = 3$.



$[-4.7, 4.7, 1] \times [-2, 16, 1]$

Figure 22.21

This result is confirmed by looking at the graph of $y = \dfrac{x^3 - 1}{x - 1}$, as shown in Figure 22.21. The "hole" in the graph appears to be at (1, 3).

You may remember that one of the factors that we had in Chapter 7 was that $x^3 - 1 = (x - 1)(x^2 + x + 1)$. We can use this knowledge to see that as long as $x \neq 1$, $g(x) = \dfrac{x^3 - 1}{x - 1} = x^2 + x + 1$. This makes it a lot easier to see that as x gets closer to 1, the value of the function gets closer to 3.

This next example will show that this type of simplification does not work for all limits.

EXAMPLE 22.15

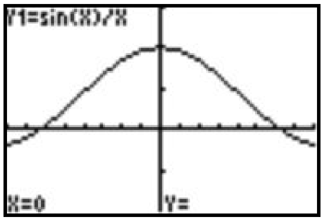
Use a numerical approach to find $\lim\limits_{x \to 0} \dfrac{\sin x}{x}$.

SOLUTION Since we cannot factor an x out of $\sin x$, the simplification method we have been using will not work. But, if we carefully construct a table of values for x and $g(x) = \dfrac{\sin x}{x}$ for values of x near 0, we can get an idea of what this limit will be. The following table shows this. Notice that the values of x are in radians.

x (radians)	0.200	0.100	0.010	−0.010	−0.100	−0.200
$\sin x$	0.1987	0.0998	0.0099	−0.0099	−0.0998	−0.1987
$\dfrac{\sin x}{x}$	0.9933	0.9983	0.9999	0.9999	0.9983	0.9933

From this table of values it appears that

$$\lim_{x \to 0} \frac{\sin x}{x} = 1$$



$[-4, 4, 0.5] \times [-1, 1.5, 0.5]$

Figure 22.22

The graph in Figure 22.22 on a TI-84 seems to reinforce this numerical method that shows $\lim_{x \to 0} \dfrac{\sin x}{x} = 1$ Notice that if you TRACE this curve, there is no y-value when $x = 0$. That is because 0 is not in the domain of the function.

THE GRAPHICAL APPROACH TO A LIMIT

The numerical approach is often a good way to approximate a limit. There are sometimes, however, when a graphical approach will be easier. We will use the graphical approach in the next two examples.

EXAMPLE 22.16

Use a graphical approach to find $\lim_{x\to 1} h(x)$, where

$$h(x) = \begin{cases} 2x - 4 & \text{if } x \neq 1 \\ 5 & \text{if } x = 1 \end{cases}$$

SOLUTION The graph of this function is shown in Figure 22.23. As you can see, as the values of x get close to 1, the values of $h(x)$ get very close to -2. In fact, if we select x close enough to 1, we can get $h(x)$ as close to -2 as we want. We decide, based on our observation, that $\lim_{x\to 1} h(x) = -2$. Notice, however, that when $x = 1$, $h(x) = 5$. Thus, $\lim_{x\to 1} h(x) \neq h(1)$.

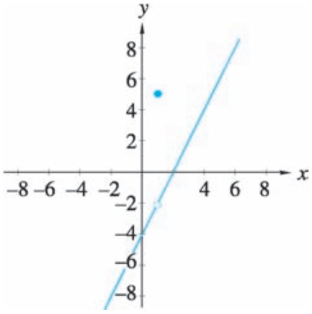


Figure 22.23

NOTE While Example 22.16 is somewhat artificial, you can see that the limit of a function at a given point is not always the same as the value of the function at that point.

The limit of a function at a certain point may not always exist. Until now, all of the functions we have examined have approached a certain number as the values of x got close to a particular value. The next example will show that this does not always happen.

EXAMPLE 22.17

Use a graphical approach to find $\lim_{x\to 0} k(x)$, where

$$k(x) = \begin{cases} 2 & \text{if } x < 0 \\ -4 & \text{if } x = 0 \\ 5 & \text{if } x > 0 \end{cases}$$

SOLUTION The graph of this function is shown in Figure 22.24. As you can see, if the values of x are close to 0 and negative, the value of $k(x)$ is 2. On the other hand, for values of x that are close to 0 and positive, the value of $k(x)$ is 5. There is no specific number that the values of $k(x)$ are near when x is close to 0. We must conclude that the $\lim_{x\to 0} k(x)$ does not exist. The fact that the function is defined when $x = 0$, $k(0) = -4$ has no effect on whether the limit of the function exists at that point.

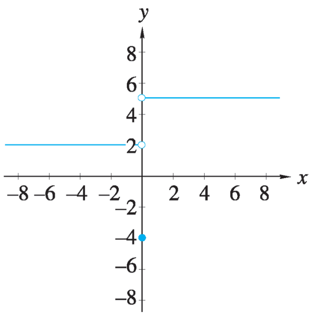


Figure 22.24

We will use the graphical approach to show two basic rules for limits:

RULES FOR LIMITS

Rule 1: $\lim_{x\to c} A = A$, where A and c are real numbers

Rule 2: $\lim_{x\to c} x = c$, where c is a real number

The graph for the first rule is shown in Figure 22.25. As you can see, the value of $f(x)$ is always A and so, as the values of x approach c, $f(x) = A$; thus we have $\lim_{x\to c} f(x) = A$.

The second rule is demonstrated in Figure 22.26. The graph of $f(x) = x$ is a straight line. For any value of c, as x gets close to c, the values of $f(x)$ are just as close to c, since $f(x) = x$. This shows that $\lim_{x\to c} f(x) = c$.

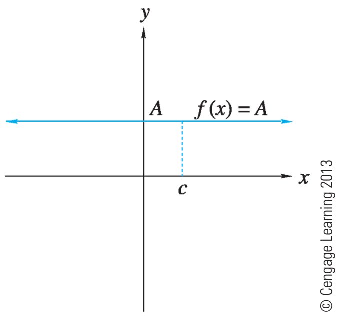


Figure 22.25

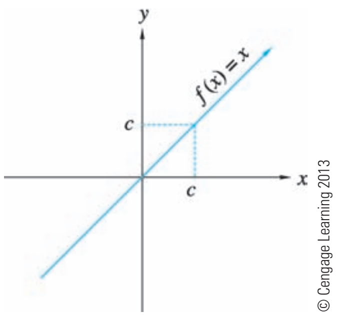


Figure 22.26

EXAMPLE 22.18

Evaluate each of the following limits: (a) $\lim_{x\to 3} 6$, (b) $\lim_{x\to -2} 7$, (c) $\lim_{x\to -5} x$, and (d) $\lim_{x\to 7} x$.

SOLUTIONS The first two, (a) and (b), follow rule 1, $\lim_{x\to c} A = A$. In (a), $A = 6$ so, $\lim_{x\to 3} 6 = 6$, and in (b), $A = 7$ and $\lim_{x\to -2} 7 = 7$.

The last two parts, (c) and (d), use rule 2, $\lim_{x\to c} x = c$. In (c) we have $\lim_{x\to -5} x = -5$ and in (d) $\lim_{x\to 7} x = 7$.

There are four additional rules for limits, as shown in the following box.

RULES FOR LIMITS

Assume that f and g are two functions and $\lim_{x\to c} f(x) = L$ and $\lim_{x\to c} g(x) = M$, where L and M are both real numbers.

Rule 3: Sum or Difference: $\lim_{x\to c}[f(x) \pm g(x)] = \lim_{x\to c} f(x) \pm \lim_{x\to c} g(x)$

$$= L \pm M$$

Rule 4: Product: $\lim_{x\to c}[f(x)g(x)] = \left[\lim_{x\to c} f(x)\right]\left[\lim_{x\to c} g(x)\right]$

$$= LM$$

(Continues)

(Continued)

Rule 5: Quotient: $\lim_{x\to c}\dfrac{f(x)}{g(x)} = \dfrac{\lim_{x\to c} f(x)}{\lim_{x\to c} g(x)} = \dfrac{L}{M}$, if M $\neq 0$

Rule 6: Limits of $[f(x)]^n$: If n is a real number, then

$$\lim_{x\to c}[f(x)]^n = \left[\lim_{x\to c} f(x)\right]^n = L^n, \text{ provided } L^n \text{ is defined and } L \neq 0.$$

EXAMPLE 22.19

Evaluate $\lim_{x\to 2}(3 - x)$.

SOLUTION

$$\begin{aligned} \lim_{x\to 3}(3 - x) &= \lim_{x\to 2} 3 - \lim_{x\to 2} x && \text{(rule 3)} \\ &= 3 - 2 && \text{(rules 1 and 2)} \\ &= 1 \end{aligned}$$

EXAMPLE 22.20

Evaluate $\lim_{x\to 4} x^2$.

SOLUTION

$$\begin{aligned} \lim_{x\to 4} 6x^2 &= \left(\lim_{x\to 4} 6x\right)\left(\lim_{x\to 4} x\right) && \text{(rule 4)} \\ &= 6\left(\lim_{x\to 4} x\right)\left(\lim_{x\to 4} x\right) && \text{(rules 1 and 4)} \\ &= 6 \cdot 4 \cdot 4 && \text{(rule 2)} \\ &= 96 \end{aligned}$$

EXAMPLE 22.21

Evaluate $\lim_{x\to 5}\dfrac{2x + 4}{x - 1}$.

SOLUTION

$$\begin{aligned} \lim_{x\to 5}\frac{2x + 4}{x - 1} &= \frac{\lim_{x\to 5}(2x + 4)}{\lim_{x\to 5}(x - 1)} && \text{(rule 5)} \\ &= \frac{14}{4} = \frac{7}{2} \end{aligned}$$

EXAMPLE 22.22

Evaluate $\lim_{x\to 3} \sqrt[6]{(19x + 7)}$.

SOLUTION

$$
\begin{aligned}
\lim_{x\to 3} \sqrt[6]{(19x + 7)} &= \lim_{x\to 3}(19x + 7)^{1/6} && \text{(algebra of exponents)}\\
&= \left[\lim_{x\to 3}(19x + 7)\right]^{1/6} && \text{(rule 6)}\\
&= 64^{1/6} = 2
\end{aligned}
$$

EXAMPLE 22.23

Evaluate $\lim_{x\to 2} \dfrac{x^2 - 5x + 6}{x^2 - 4}$.

SOLUTION Checking the denominator, we see that $\lim_{x\to 2}(x^2 - 4) = 0$. This means that we cannot use rule 5. But, this does not mean the limit does not exist. If we factor both the numerator and denominator, we see that

$$\frac{x^2 - 5x + 6}{x^2 - 4} = \frac{(x - 2)(x - 3)}{(x - 2)(x + 2)}$$

We are interested in the limits as x approaches 2 and not when x has the value of 2. Thus, the factor $x - 2$ is not 0 for these values and we can cancel this common factor. We then get

$$
\begin{aligned}
\lim_{x\to 2} \frac{x^2 - 5x + 6}{x^2 - 4} &= \lim_{x\to 2} \frac{(x - 2)(x - 3)}{(x - 2)(x + 2)}\\
&= \lim_{x\to 2} \frac{x - 3}{x + 2}\\
&= \frac{2 - 3}{2 + 2} = \frac{-1}{4}
\end{aligned}
$$

CAUTION In Example 22.23, substituting 2 for x in the expression $\dfrac{x^2 - 5x + 6}{x^2 - 4}$ resulted in $\frac{0}{0}$. Whenever substitution results in $\frac{0}{0}$, we must do more work to determine whether a limit exists.

Until now, we have applied the six rules for limits only for the limit of a function as x approaches a specific point. As the next examples show, we often have to be more careful when we find limits at infinity.

EXAMPLE 22.24

Evaluate $\lim_{x\to\infty} \dfrac{1}{x^n}$, $n > 0$.

SOLUTION

$$\lim_{x\to\infty} \frac{1}{x^n} = \left(\lim_{x\to\infty} \frac{1}{x}\right)^n = 0^n = 0$$

Notice from this example that as $x \to \infty$, $\frac{1}{x^2}, \frac{1}{x^3}, \frac{1}{x^4}$, etc., all have limits of 0. Using this with rule 3, we see that

$$\lim_{x\to\infty} \frac{c}{x^n} = \left(\lim_{x\to\infty} c\right)\left(\lim_{x\to\infty} \frac{1}{x^n}\right) = c \cdot 0 = 0$$

Thus, we have the following.

LIMITS AT INFINITY

If c is a constant and n is a positive rational number, then

$$\lim_{x\to\infty} \frac{c}{x^n} = 0$$

We will now show some techniques for finding limits at infinity.

EXAMPLE 22.25

Evaluate $\lim\limits_{x\to\infty} \dfrac{4x^3 + 2x^2 + 5}{5x^3 + x}$.

SOLUTION We will first divide both the numerator and denominator by the largest power of x in the denominator, in this case x^3. This will make each term into a constant or a term with a variable in the denominator and allow us to use the properties for limits at infinity.

$$\lim_{x\to\infty} \frac{4x^3 + 2x^2 + 5}{5x^3 + x} = \lim_{x\to\infty} \frac{\dfrac{4x^3 + 2x^2 + 5}{x^3}}{\dfrac{5x^3 + x}{x^3}}$$

$$= \lim_{x\to\infty} \frac{\dfrac{4x^3}{x^3} + \dfrac{2x^2}{x^3} + \dfrac{5}{x^3}}{\dfrac{5x^3}{x^3} + \dfrac{x}{x^3}}$$

$$= \lim_{x\to\infty} \frac{4 + \dfrac{2}{x} + \dfrac{5}{x^3}}{5 + \dfrac{1}{x^2}}$$

Now, according to the properties for limits at infinity, all the terms with an x in the denominator have a limit of 0. So, we now have

$$\lim_{x\to\infty} \frac{4 + \dfrac{2}{x} + \dfrac{5}{x^3}}{5 + \dfrac{1}{x^2}} = \frac{4 + 0 + 0}{5 + 0} = \frac{4}{5}$$

HINT When you want to find a limit at infinity, divide both the numerator and denominator by the largest power of the variable in the denominator.

In this section, we have tried to give you a foundation for the idea of a limit. The following exercises are designed to further develop this foundation.

EXERCISE SET 22.3

In Exercises 1–8, complete each table and use it to determine the indicated limit.

1.

x	0.9	0.99	0.999	0.9999	1.0001	1.001	1.01	1.1
$f(x) = 3x$								

$\lim_{x\to 1} 3x = ?$

2.

x	2.9	2.99	2.999	2.9999	3.0001	3.001	3.01	3.1
$g(x) = x - 4$								

$\lim_{x\to 3}(x - 4) = ?$

3.

x	-1.1	-1.01	-1.001	-1.0001	-0.9999	-0.999	-0.99	-0.9
$h(x) = x^2 + 2$								

$\lim_{x\to -1}(x^2 + 2) = ?$

4.

x	-2.1	-2.01	-2.001	-2.0001	-1.9999	-1.999	-1.99	-1.9
$k(x) = \dfrac{x^2 - 4}{x + 2}$								

$\lim_{x\to -2} \dfrac{x^2 - 4}{x + 2} = ?$

5.

x	-0.1	-0.01	-0.001	-0.0001	0.0001	0.001	0.01	0.1
$f(x) = \dfrac{\tan x}{x}$								

$\lim_{x\to 0} \dfrac{\tan x}{x} = ?$

6.

x	-0.1	-0.01	-0.001	-0.0001	0.0001	0.001	0.01	0.1
$g(x) = \dfrac{1 - \cos x}{x^2}$								

$\lim_{x\to 0} \dfrac{1 - \cos x}{x^2} = ?$

7.

x	0.9	0.99	0.999	0.9999	1.0001	1.001	1.01	1.1
$h(x) = \dfrac{x}{x - 1}$								

$\lim_{x\to 1} \dfrac{x}{x - 1} = ?$

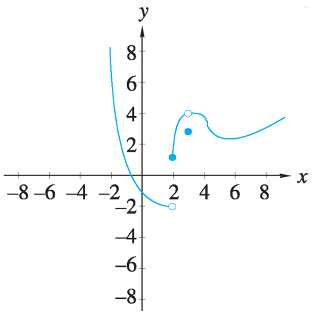
8.

x	-0.1	-0.01	-0.001	-0.0001	0.0001	0.001	0.01	0.1
$k(x) = \dfrac{1}{x^2}$								

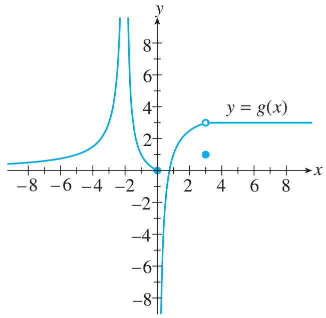
$\lim\limits_{x \to 0} \dfrac{1}{x^2} = ?$

Solve Exercises 9 and 10.

9. For the function f in the given figure, find the following limits or state that the limit does not exist. Give a reason for your answers. **(a)** $\lim\limits_{x \to 0} f(x)$ **(b)** $\lim\limits_{x \to 2} f(x)$ **(c)** $\lim\limits_{x \to 3} f(x)$



10. For the function g in the given figure, find the following limits or state that the limit does not exist. Give reasons for your answers. **(a)** $\lim\limits_{x \to -2} g(x)$ **(b)** $\lim\limits_{x \to 0} g(x)$ **(c)** $\lim\limits_{x \to 3} g(x)$



In Exercises 11–18, sketch the graphs of the functions and find out if $\lim\limits_{x \to c}$ exists at the given value of c or state that the limit does not exist.

11. $f(x) = \begin{cases} 3x - 2 & \text{if } x \neq 1 \\ 0 & \text{if } x = 1 \end{cases} \quad c = 1$

12. $g(x) = \begin{cases} x + 1 & \text{if } x < 0 \\ x^2 + 1 & \text{if } x \geq 0 \end{cases} \quad c = 0$

13. $h(x) = \begin{cases} 5 - 2x & \text{if } x < 0 \\ 0 & \text{if } x = 0 \\ 4x + 5 & \text{if } x > 0 \end{cases} \quad c = 0$

14. $k(x) = \begin{cases} x + 1 & \text{if } x < 1 \\ \text{not defined} & \text{if } x = 1 \\ 2x & \text{if } x > 1 \end{cases} \quad c = 1$

15. $f(x) = \begin{cases} 1 - 3x & \text{if } x < 0 \\ 3x + 1 & \text{if } x \geq 0 \end{cases} \quad c = 0$

16. $g(x) = \begin{cases} 3x + 1 & \text{if } x \leq -1 \\ 1 - 3x & \text{if } x > -1 \end{cases} \quad c = -1$

17. $h(x) = \begin{cases} 3x + 1 & \text{if } x < 0 \\ 4 & \text{if } x = 0 \\ 1 - 3x & \text{if } x > 0 \end{cases} \quad c = 0$

18. $k(x) = \begin{cases} 3x + 1 & \text{if } x < 0 \\ 2 & \text{if } x = 0 \\ x + 1 & \text{if } x > 0 \end{cases} \quad c = 0$

In Exercises 19–40, evaluate the given limit using the rules of limits.

19. $\lim\limits_{x \to 5} 8$

20. $\lim\limits_{x \to -4} x$

21. $\lim\limits_{x \to -6} (x + 7)$

22. $\lim\limits_{x \to 6} (9 - x)$

23. $\lim\limits_{x \to 2} 3x$

24. $\lim\limits_{x \to 4} \left(\frac{3}{2}x - 2\right)$

25. $\lim\limits_{p \to 3} \dfrac{p^2 + 6}{p}$

26. $\lim\limits_{x \to 6} \dfrac{2x^2 - 3}{3 - x}$

27. $\lim\limits_{x \to -1} (2x^2 - 1)^2$

28. $\lim\limits_{t \to 4} (3t^2 + 2)^2$

29. $\lim\limits_{x \to 3} \sqrt{x + 3}$

30. $\lim_{x \to -6} \sqrt{6 - 7x}$

31. $\lim_{x \to 2} \frac{x^2 - 2x}{x - 2}$

32. $\lim_{w \to 3} \frac{w^2 - 1}{w + 1}$

33. $\lim_{x \to -1} \frac{x^2 - 1}{x + 1}$

34. $\lim_{y \to 3} \frac{y^2 - 7y + 12}{y - 3}$

35. $\lim_{h \to 0} \frac{(4 + h)^2 - 4^2}{h}$

36. $\lim_{x \to c} \frac{x^3 - c^3}{x - c}$

37. $\lim_{x \to \infty} \frac{2x^2 + 4}{5x^2 + 3x + 2}$

38. $\lim_{x \to \infty} \frac{6x^3 + 9x + 1}{2x^3 + 4x + 8}$

39. $\lim_{x \to \infty} \frac{7x^4 + 5x^2}{x^5 + 2}$

40. $\lim_{x \to \infty} \frac{4x^5 + 3x}{2x^4 + 1}$

In Exercises 41–48, use a graphing calculator, spreadsheet, or graphing software to guess whether each of the following limits exists. If the limit does exist, estimate its value.

41. $\lim_{x \to 0} \frac{2x^2}{x^2 + \sin x}$

42. $\lim_{x \to 0} \frac{5x^2}{2x^2 + \sin^2 x}$

43. $\lim_{x \to 0} \frac{2x^2}{x^2 + \cos x - 1}$

44. $\lim_{x \to 0} \frac{3x^2}{x^2 + \tan x}$

45. ***Electronics*** Two long parallel wires of equal radii carrying equal currents in opposite directions will exhibit an inductance, in henries, given by

$$L(d) = k \log \frac{d}{0.7788r}$$

where $k = 4.0 \times 10^{-7}\ H$, r is the radius of the wires, and d is the center-to-center separation of the wires. Complete the following table to approximate $\lim_{d \to 0.5} L(d)$ when $r = 0.150$ cm.

d	0.4500	0.4900	0.4990	0.4999	0.5001	0.5010	0.5100	0.5500
$L(d)$								

46. ***Electronics*** The magnetic field B in the shielding wall of a coaxial cable is described by the function

$$B(r) = 2\left(\frac{4 - r^2}{r}\right)$$

where r is the distance from the center of the cable and $0 < r \leq 2$. Complete the following table to approximate $\lim_{d \to 1.5} B(r)$.

d	1.4500	1.4900	1.4990	1.4999	1.5001	1.5010	1.5100	1.5500
$B(r)$								

47. ***Sports technology*** Table 22.11 contains the Olympic results in the women's 200-m dash.

TABLE 22.11 Olympic Results in the Women's 200-m Dash

Year	1948	1952	1956	1960	1964	1968	1972	1976
Time (s)	24.4	23.7	23.4	24.0	23.0	22.5	22.4	22.37

Year	1980	1984	1988	1992	1996	2000	2004	2008
Time (s)	22.03	21.81	21.34	21.81	22.12	21.84	22.05	21.74

(a) Use any method to extrapolate the winning time for the women's 200-m dash in the 2020 Olympics. Describe how you arrived at your answer.

(b) Determine a linear regression curve, L, for this data.

(c) Determine an exponential regression curve, E, for this data.

(d) What is the $\lim_{t\to\infty} L$? What is the $\lim_{t\to\infty} E$?

(e) Discuss the implications and reasonableness of your answers to (d).

48. ***Sports technology*** Table 22.12 contains the year and time of each world record in the men's 100-m dash. Notice that in 1968 and again in 1991 the record was broken twice.

TABLE 22.12 Men's 100-m Dash World Records

Year	1912	1921	1930	1936	1956	1960	1968	1968	1983	1988	1991
Time (s)	10.6	10.4	10.3	10.2	10.1	10.0	9.99	9.95	9.93	9.92	9.90

Year	1991	1994	1996	1999	2005	2006	2006	2007	2008	2008	2009
Time (s)	9.86	9.85	9.84	9.79	9.768	9.763	9.762	9.74	9.72	9.683	9.578

(a) Use any method to extrapolate the world record for the men's 100-m dash in 2020. Describe how you arrived at your answer.

(b) Determine a linear regression curve, L, for this data.

(c) Determine an exponential regression curve, E, for this data.

(d) What is the $\lim_{t\to\infty} L$? What is the $\lim_{t\to\infty} E$?.

(e) Discuss the implications and reasonableness of your answers to (d).

[IN YOUR WORDS]

49. Explain the notation $\lim_{x\to a} f(x) = L$.

50. Suppose that $y = g(x)$ is a function and $\lim_{x\to 2} g(x) = 4$ and $g(2) = -3$.

(a) Sketch a possible graph for g.

(b) Use your graph to explain what is happening to g for values of x close to $x = 2$.

51. Suppose that the function $y = h(x)$ is not defined at $x = 4$. Is it possible for $\lim_{x\to 4} h(x)$ to exist? Explain your answer.

52. Suppose that $\lim_{x\to 1} f(x) = 0$ and $\lim_{x\to 1} g(x) = \infty$.

(a) Give an example of two functions f and g so that $\lim_{x\to 1} [f(x) + g(x)] = \infty$.

(b) Give an example of two functions f and g so that $\lim_{x\to 1} [f(x) \cdot g(x)] = 0$.

(c) Give an example of two functions f and g so that $\lim_{x\to 1} [f(x) \cdot g(x)] = 5$.

53. Suppose that neither $\lim_{x\to 2} f(x)$ nor $\lim_{x\to 2} g(x)$ exists. Give an example of two functions f and g so that $\lim_{x\to 2} [f(x) + g(x)]$ does exist or explain why it is not possible.

22.4 ONE-SIDED LIMITS AND CONTINUITY

In using both the numerical and the graphical approaches to finding $\lim_{x\to c} f(x)$, we studied the behavior of the function f on both sides of c. There are times when it is necessary to investigate the limit on just one side of c. When this is done, we are investigating one-sided limits.

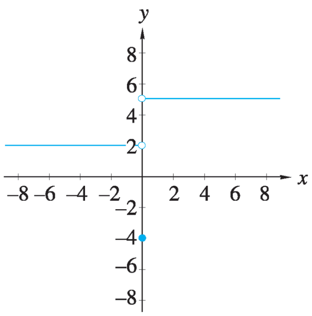


Figure 22.27

There are two types of **one-sided limits**. One, the **left-hand limit,** is written as $\lim_{x \to c^-} f(x)$, and the other, the **right-hand limit**, is written as $\lim_{x \to c^+} f(x)$.

To get a better idea of one-sided limits, we will return to an example we worked earlier. In Example 22.17, we wanted to find $\lim_{x \to 0} k(x)$, where

$$k(x) = \begin{cases} 2 & \text{if } x < 0 \\ -4 & \text{if } x = 0 \\ 5 & \text{if } x > 0 \end{cases}$$

The graph of the function is shown in Figure 22.27.

We know that $\lim_{x \to 0} k(x)$ does not exist, because when x is positive, the value of $k(x)$ is 5, and when x is negative, the value of $k(x)$ is 2. What we have really said is that on the right-hand side of 0 the limit of $k(x)$ is 5 and on the left-hand side, the limit is 2. These are examples of one-sided limits.

In general, the *one-sided limits* of a function $f(x)$ at a point c are the *left-hand limit*, $\lim_{x \to c^-} f(x)$, and the *right-hand limit*, $\lim_{x \to c^+} f(x)$. In the function in Example 22.17, $\lim_{x \to 0^-} k(x) = 2$ and $\lim_{x \to 0^+} k(x) = 5$.

EXAMPLE 22.26

Determine the one-sided limits of $f(x)$ at $x = 3$, if

$$f(x) = \begin{cases} 2x + 1 & \text{if } x < 1 \\ 4 - x & \text{if } 1 \le x \le 3 \\ x & \text{if } x > 3 \end{cases}$$

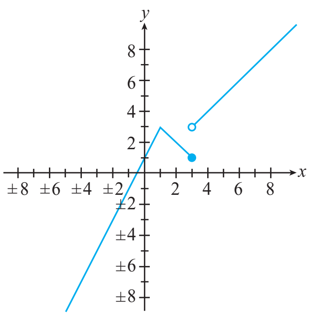


Figure 22.28

SOLUTION The way in which $f(x)$ behaves depends on whether $x < 3$ or $x > 3$, as shown in Figure 22.28. We have different definitions for $f(x)$ for each one-sided limit. When $x < 3$ (but close to 3), f is defined by $f(x) = 4 - x$, and so the left-hand limit is

$$\lim_{x \to 3^-} f(x) = \lim_{x \to 3^-} (4 - x) = 1$$

But, when $x > 3$, f is defined by $f(x) = x$, and so the right-hand is

$$\lim_{x \to 3^+} f(x) = \lim_{x \to 3^+} x = 3$$

One-sided limits provide a way to determine whether or not the limit of a function exists at a point. The limit of a function $f(x)$ will exist at a point c if and only if the right-hand limit of $f(x)$ at c and the left-hand limit of $f(x)$ at c both equal the same real number. Symbolically, this relationship is written as follows.

RELATIONSHIP BETWEEN ONE-SIDED AND TWO-SIDED LIMITS

$\lim_{x \to c} f(x) = L$, if and only if $\lim_{x \to c^-} = L$ and $\lim_{x \to c^+} f(x) = L$, where L is a real number.

EXAMPLE 22.27

Determine the one-sided limits of $f(x)$ at $x = 1$, where

$$f(x) = \begin{cases} 2x + 1 & \text{if } x < 1 \\ 4 - x & \text{if } 1 \leq x \leq 3 \\ x & \text{if } x > 3 \end{cases}$$

SOLUTION This is the same function we studied in Example 22.26. We will take the one-sided limits at $x = 1$. Here, when $x < 1$, f behaves as if $f(x) = 2x + 1$, and we get

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} (2x + 1) = 3$$

When $x > 1$ (but close to 1), we use $f(x) = 4 - x$, with the result that

$$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} (4 - x) = 3$$

Since $\lim_{x \to 1^-} f(x) = \lim_{x \to 1^+} f(x) = 3$, we can say that $\lim_{x \to 1} f(x) = 3$.

We just specified that in order for the limit to exist, both of the one-sided limits must equal the same real number. The next example will show why we must make sure that it is a real number.

EXAMPLE 22.28

Find the one-sided limits of $g(x) = \dfrac{1}{x^2}$, where $x = 0$.

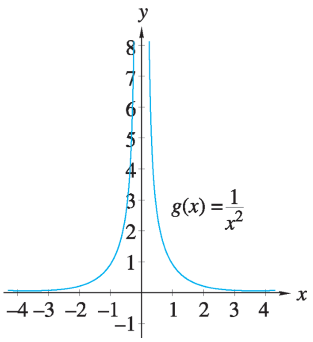


Figure 22.29

SOLUTION We will use a table of values and a graph to give us an indication of these limits. The following table and the graph in Figure 22.29 seem to indicate that as $x \to 0$, from both the left and the right, $g(x)$ increases without bound or without limit.

x	± 1	± 0.5	± 0.1	± 0.01	± 0.001	± 0.0001
$g(x)$	1	4	100	10,000	1,000,000	100,000,000

So, there is no limit for $g(x)$ at $x = 0$. But, because both one-sided limits increase without bound, we say that $g(x)$ becomes positively infinite as x approaches 0. Symbolically, we write

$$\lim_{x \to 0} \frac{1}{x^2} = \infty$$

CAUTION Be careful! The fact that we used an equal sign in this last limit does not mean that the limit exists. In fact, just the opposite is true. The infinity symbol ∞ here means that the limit does not exist and shows this because the one-sided limits increase without bound.

A somewhat different result is shown in the next example, where we look at $\dfrac{1}{x}$ as x gets close to 0.

EXAMPLE 22.29

Find the one-sided limits of $h(x) = \dfrac{1}{x}$ as $x \to 0$.

SOLUTION Again, we will use a table of values and a graph to demonstrate our answer. The table follows and the graph of $h(x)$ is in Figure 22.30.

x	-1	-0.1	-0.01	-0.001	0.001	0.01	0.1	1
$h(x)$	-1	-10	-100	$-1{,}000$	$1{,}000$	100	10	1

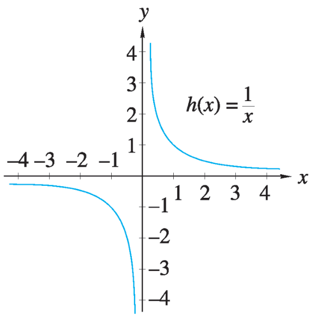


Figure 22.30

As we can see, $\lim\limits_{x \to 0^+} \dfrac{1}{x} = \infty$. But, what happens when x is negative? In this case, the values of $h(x)$ become negatively infinite, so

$$\lim_{x \to 0^-} \frac{1}{x} = -\infty$$

Thus we see two reasons that $\lim\limits_{x \to 0} \dfrac{1}{x}$ does not exist. One reason is that the left-hand and right-hand limits are not equal; the other is that infinity, either positive or negative, is not a real number.

LIMITS AT INFINITY

What happens to $h(x)$ as x gets increasingly larger? The following table gives some indication of what happens when the values of x get larger.

x	100	1,000	10,000	100,000	1,000,000
$h(x) = \frac{1}{x}$	0.01	0.001	0.0001	0.00001	0.000001

This table and the graph in Figure 22.30 indicate that

$$\lim_{x \to \infty} \frac{1}{x} = 0$$

That is, as x gets larger without bound, the value of $\dfrac{1}{x}$ approaches 0. Similarly, as x gets smaller (becomes "more negative") without bound, the value of $\dfrac{1}{x}$ approaches 0. This is written as

$$\lim_{x \to -\infty} \frac{1}{x} = 0$$

These two results are summarized in the following box.

LIMITS AT INFINITY

$$\lim_{x \to \infty} \frac{1}{x} = 0$$

and

$$\lim_{x \to -\infty} \frac{1}{x} = 0$$

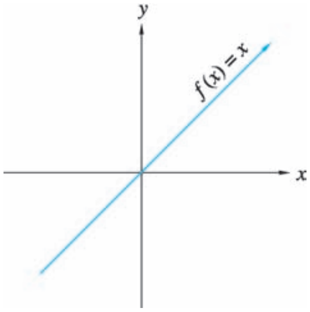


Figure 22.31

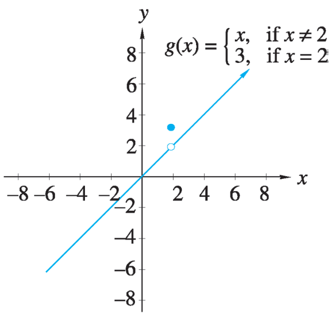


Figure 22.32

CONTINUITY

Let's consider the two functions

$$f(x) = x$$

$$\text{and} \quad g(x) = \begin{cases} x & \text{if } x \neq 2 \\ 3 & \text{if } x = 2 \end{cases}$$

The graphs of these functions appear in Figures 22.31 and 22.32, respectively. Looking at either the graphs or the algebraic definitions, you can see that the two functions are exactly the same, except at one point. At that point, when $x = 2$, there is a "break" in the graph of g, but no "break" in the graph of f. The function g is an example of a function that is not continuous, while f demonstrates a continuous function.

The break in g is used to determine that the function is not continuous. One way of thinking of a continuous function is that it is a function you can draw without lifting your pencil from the paper. Look again at the graphs of f and g. You could draw f without lifting your pencil. You could not draw g without lifting your pencil.

CONTINUITY AT A POINT

As we saw above, we can describe a function as either continuous or noncontinuous by looking at how it is drawn. We can use the idea of limits to provide an algebraic definition of continuity. Look again at the definitions of f and g. What is $\lim_{x\to 2} f(x)$? What are $f(2)$ and $g(2)$?

As you can see, $\lim_{x\to 2} f(x) = f(2) = 2$. We know that f is a continuous function at $x = 2$. But $\lim_{x\to 2} g(x) = 2 \neq g(2) = 3$; thus g is not a continuous function, or is *discontinuous* at $x = 2$. This provides us with a more formal definition:

CONTINUITY AT A POINT

A function f is *continuous* at $x = c$ if and only if the following three conditions are satisfied:

(a) $f(x)$ is defined at $x = c$,

(b) $\lim_{x\to c} f(x)$ exists, and

(c) $\lim_{x\to c} f(x) = f(c)$.

A function that is not continuous at $x = c$ is *discontinuous* at that point.

NOTE A function is discontinuous at $x = c$ if any one of these three conditions are not satisfied.

EXAMPLE 22.30

Discuss the continuity of the function f at $x = 4$, where

$$f(x) = \begin{cases} \dfrac{x^2 - 16}{x - 4} & \text{if } x \neq 4 \\ 8 & \text{if } x = 4 \end{cases}$$

SOLUTION In order to determine if f is continuous, we need to check the three conditions in the definition: (a) The function is defined at $x = 4$, since $f(4) = 8$, (b) $\lim\limits_{x\to 4} \dfrac{x^2 - 16}{x - 4} = \lim\limits_{x\to 4} \dfrac{(x + 4)(x - 4)}{x - 4} = 8$, and (c) $\lim\limits_{x\to 4} \dfrac{x^2 - 16}{x - 4} = 8 = f(4)$.

The three conditions are satisfied, so this function is continuous at $x = 4$.

EXAMPLE 22.31

Discuss the continuity of each of the following functions at the indicated points:

(a) $g(x) = \dfrac{1}{x}$ at $x = 0$

(b) $h(x) = \begin{cases} x - 1 & \text{if } x < 2 \\ x + 1 & \text{if } x \geq 2 \end{cases}$ at $x = 2$

(c) $j(x) = \begin{cases} 0.25x^2 + 2 & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$ at $x = 0$

SOLUTIONS None of these are continuous at the designated points.

(a) $g(x)$ is not defined when $x = 0$, so it does not satisfy the first condition.

(b) $h(x)$ is defined at $x = 2$, but we can see that $\lim\limits_{x\to 2^-} h(x) = 1$ and $\lim\limits_{x\to 2^+} h(x) = 3$, so the $\lim\limits_{x\to 2} h(x)$ does not exist and the second condition is not satisfied.

(c) $j(x)$ is defined at 0, $j(0) = 0$. $\lim\limits_{x\to 0} j(x) = 2$. The third condition is not satisfied, since $\lim\limits_{x\to 0} j(x) = 2 \neq 0 = j(0)$.

The graphs in these three functions are shown in Figures 22.33a, 22.33b, and 22.33c, respectively.

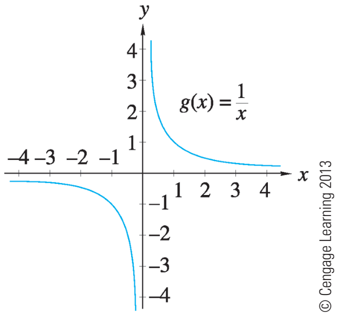


Figure 22.33a

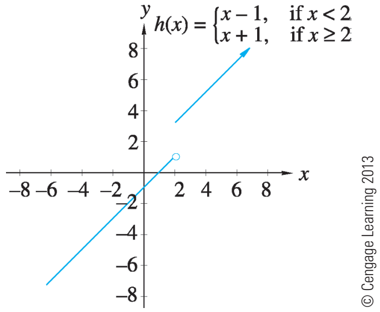


Figure 22.33b

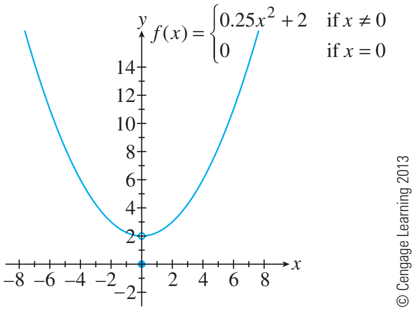


Figure 22.33c

CONTINUITY ON INTERVALS

When we talk about functions, we usually consider them over an interval. An *interval* is a set of all points on the number line between a and b. Because the endpoints are not always included in the intervals, notation has been developed to make these distinctions. We will use the following notations and terms:

INTERVAL NOTATION

$$(a, b): a < x < b$$
$$[a, b): a \leq x < b$$
$$(a, b]: a < x \leq b$$
$$[a, b]: a \leq x \leq b$$

The interval (a, b) is called an *open interval* and does not include either endpoint. The interval $[a, b]$ is a *closed interval* and includes both endpoints. The other two intervals, $[a, b)$ and $(a, b]$, are *half-open intervals*. Because $+\infty$ and $-\infty$ are not numbers, any interval that includes either of these is open on that end. For example $a < x < \infty$, or simply a $< x$, is the interval, (a, ∞) and $-\infty < x \leq b$, or simply $x \leq b$, is the interval $(-\infty, b]$.

The definition of continuity we just gave is for a function continuous at a point. We are usually more interested in the continuity of a function over an interval.

CONTINUITY OVER AN INTERVAL

We say that a function is *continuous on an interval* if it is continuous at each point in the interval.

If you are concerned with the continuity of a function on an interval that includes an endpoint of the interval, then you need to examine the one-sided limits at that endpoint.

EXAMPLE 22.32

The function $k(x) = \sqrt{x - 2}$ is continuous on the interval $[2, \infty)$.
The function $m(x) = \sqrt{3 - x}$ is continuous on the interval $(-\infty, 3]$.
The function $f(x) = \sqrt[3]{x}$ is continuous on $(-\infty, \infty)$.

NOTE All polynomial functions are continuous at every point or on the interval $(-\infty, \infty)$.

Functions that are continuous at all points, such as a polynomial, are said to be *continuous everywhere*, or continuous, and are referred to as **continuous functions**. Rational functions are continuous over their domains.

EXAMPLE 22.33

Find any points of discontinuity for each of the following functions:

(a) $f(x) = \dfrac{x^3 - 3x^2 + 17x - 5}{x^2 - 1}$

(b) $h(x) = \begin{cases} 6 & \text{if } x < 3 \\ x + 3 & \text{if } x > 3 \end{cases}$

SOLUTIONS

(a) Possible points of discontinuity occur when the denominator is 0. The denominator is 0 when x is ± 1, so f is not defined at these two points. By condition (a), if the function is not defined at ± 1, it is not continuous at those two points. These are the only points of discontinuity for f.

(b) The function h is not defined at $x = 3$, so the function is not continuous at this point. It is continuous for all other values of x.

PROPERTIES OF CONTINUOUS FUNCTIONS

The formal definition that we gave for a continuous function was based on a limit. The rules that apply to limits help us develop three properties that apply to continuous functions.

PROPERTIES OF CONTINUOUS FUNCTIONS

Property 1: If f and g are both continuous at a point c, then so are $f + g$, $f - g$, and $f \cdot g$.

Property 2: If f and g are both continuous at c and $g(c) \neq 0$, then $f/g = \dfrac{f}{g}$ is also continuous at c.

Property 3: If f is continuous at $g(c)$ and g is continuous at c, then $f \circ g = f(g(x))$ is continuous at c.

We used property 2 in the solution of Example 22.33a. There, we let f be the polynomial $x^3 - 3x^2 + 17x - 5$ and g the polynomial $x^2 - 1$. Since f and g are both polynomials they are continuous everywhere, so f/g is continuous where $g \neq 0$, namely when $x \neq \pm 1$.

This concludes this section on continuity and completes your foundation for calculus. Limits and continuity will both provide necessary tools as we continue to develop our knowledge of calculus.

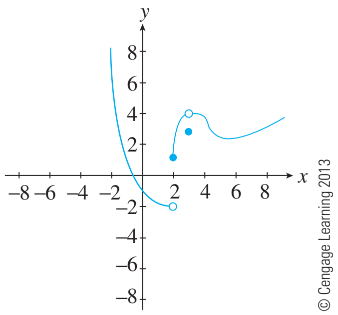
EXERCISE SET 22.4

Solve Exercises 1 and 2.

1. For the function f given in the figure below, find the following limits or state that the limit does not exist.

(a) $\lim_{x \to 2^-} f(x)$ (c) $\lim_{x \to 3^-} f(x)$

(b) $\lim_{x \to 2^+} f(x)$ (d) $\lim_{x \to 3^+} f(x)$

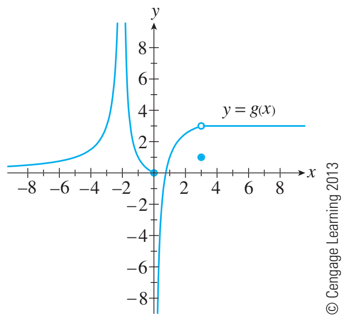


2. For the function g given in the figure below, find the following limits or state that the limit does not exist.

(a) $\lim_{x \to -1^-} g(x)$ (d) $\lim_{x \to 0^+} g(x)$

(b) $\lim_{x \to -1^+} g(x)$ (e) $\lim_{x \to \infty} g(x)$

(c) $\lim_{x \to 0^-} g(x)$ (f) $\lim_{x \to -\infty} g(x)$



In Exercises 3–14, use a table or a graph to determine the one-sided limits.

3. $\lim_{x \to 5^+} (x - 2)$

4. $\lim_{x \to 2^-} (x^2 - 1)$

5. $\lim_{x \to -2^+} \dfrac{x^2 - 4}{x + 2}$

6. $\lim_{x \to -5^-} \dfrac{x^2 + 25}{x + 5}$

7. $\lim_{x \to 2^-} \dfrac{|x - 2|}{x - 2}$

8. $\lim_{x \to 2^+} \dfrac{|x - 2|}{x - 2}$

9. $\lim_{x \to 2^+} \sqrt{x - 2}$

10. $\lim_{x \to -4^+} \sqrt{x + 4}$

11. $\lim_{x \to \infty} \dfrac{9}{3x}$

12. $\lim_{x \to -\infty} \dfrac{-8}{4x}$

13. $\lim_{x \to -\infty} \dfrac{1}{2x + 1}$

14. $\lim_{x \to \infty} \dfrac{-5}{9x - 4}$

In Exercises 15–20, use a graphing calculator, spreadsheet, or graphing software to guess whether each of the following limits exists. If the limit does exist, estimate its value.

15. $\lim_{x \to \infty} \dfrac{2x^2}{x^2 + \sin x}$

16. $\lim_{x \to \infty} \dfrac{5x^2}{2x^2 + \sin^2 x}$

17. $\lim_{x \to 0^-} \dfrac{2x^2}{x^2 - \cos x}$

18. $\lim_{x \to 0^+} \dfrac{2x^2}{x^2 - \tan x}$

19. $\lim_{x \to \infty} \dfrac{5^x - 5}{2^x - 2}$

20. $\lim_{x \to -\infty} \dfrac{5^x - 5}{2^x - 2}$

In Exercises 21–28, determine which functions are continuous at c. If the function is not continuous, give a reason.

21. $f(x) = \begin{cases} 3x+1 & \text{if } x \le 4 \\ x^2 - 3 & \text{if } x > 4 \end{cases}$ $c = 4$

22. $g(x) = \begin{cases} 2x-1 & \text{if } x \le 1 \\ 2x & \text{if } x > 1 \end{cases}$ $c = 1$

23. $h(x) = \begin{cases} 3x+1 & \text{if } x < 1 \\ 3 & \text{if } x = 1 \\ 4x & \text{if } x > 1 \end{cases}$ $c = 1$

24. $j(x) = \begin{cases} 5x-1 & \text{if } x < 1 \\ 4 & \text{if } x = 1 \\ 4x & \text{if } x > 1 \end{cases}$ $c = 1$

25. $g(x) = \begin{cases} x^2 & \text{if } x < -2 \\ 3 & \text{if } x = -2 \\ -4x+2 & \text{if } x > -2 \end{cases}$ $c = -2$

26. $h(x) = \begin{cases} x^2 & \text{if } x \le -2 \\ 2x & \text{if } x > -2 \end{cases}$ $c = -2$

27. $j(x) = \begin{cases} x^3 - 4 & \text{if } x < 2 \\ 5 & \text{if } x = 2 \\ x^2 & \text{if } x > 2 \end{cases}$ $c = 2$

28. $k(x) = \dfrac{x^3 - 1}{x - 1}$ $c = 1$

In Exercises 29–36, find all points of discontinuity.

29. $f(x) = 2x^2 - 5$

30. $g(x) = \dfrac{4}{x - 5}$

31. $h(x) = \dfrac{x^2 + 3x + 2}{x + 1}$

32. $f(x) = \dfrac{x^2 + x - 6}{x^2 + 5x + 6}$

33. $g(x) = \dfrac{x}{x^2 + 1}$

34. $h(x) = \dfrac{x}{x}$

35. $j(x) = \begin{cases} x^2 + 2 & \text{if } x < 2 \\ \sqrt{x - 2} & \text{if } x \ge 2 \end{cases}$

36. $k(x) = \begin{cases} \dfrac{1}{x} & \text{if } x \ne 5 \\ 3 & \text{if } x = 5 \end{cases}$

In Exercises 37–44, determine the intervals over which the given function is continuous.

37. $f(x) = \dfrac{x}{x + 3}$

38. $g(x) = \dfrac{x}{x^2 - 1}$

39. $h(x) = \dfrac{x + 2}{x^2 - 4}$

40. $f(x) = \sqrt{5 - x}$

41. $g(x) = \sqrt{x^2 - 9}$

42. $h(x) = \sqrt{x^2 + 4x + 4}$

43. $j(x) = \dfrac{1}{x}$

44. $k(x) = \dfrac{1}{\sqrt{x}}$

Solve Exercises 45–54.

45. ***Medical technology*** The concentration of a drug in a patient's bloodstream t hours after it was injected is given by

$$C(t) = \frac{0.25t}{t^2 + 5}$$

Find each of the following. **(a)** $C(0.5)$, **(b)** $C(2)$, $\lim\limits_{t \to \infty} C(t)$

46. ***Business*** It has been determined that after t weeks of training, a certain student in a word processing class can keyboard at the rate of $W(t) = 65 + \dfrac{70t^2}{t^2 + 15}$ words per minute. What are the most words per minute that we can expect this student to keyboard; that is, what is $\lim\limits_{t \to \infty} \left(65 + \dfrac{70t^2}{t^2 + 15}\right)$?

47. ***Business*** The cost, in dollars, of an overseas telephone call is given by the function

$$C(t) = \begin{cases} 4.75 & \text{if } 0 < t \leq 3 \\ 0.65t + 2.80 & \text{if } t > 3 \end{cases}$$

where t is the length of the call in minutes. Determine each of the following:

(a) $\lim_{t\to 3^-} C(t)$

(b) $\lim_{x\to 3^+} C(t)$

(c) $\lim_{x\to 3} C(t)$

48. ***Finance*** A certain state income tax schedule can be given by the function

$$T(x) = \begin{cases} 0.04x & \text{if } 0 < x \leq 12{,}500 \\ 0.03x + 125 & \text{if } 12{,}500 < x \leq 35{,}000 \\ 0.02x + 465 & \text{if } x > 35{,}000 \end{cases}$$

where x is the taxable income in dollars, $0 < x$, and $T(x)$ is in dollars. Determine each of the following:

(a) $\lim_{x\to 12{,}500^-} T(x)$

(b) $\lim_{x\to 12{,}500^+} T(x)$

(c) $\lim_{x\to 12{,}500} T(x)$

(d) $\lim_{x\to 35{,}000^-} T(x)$

(e) $\lim_{x\to 35{,}000^+} T(x)$

(f) $\lim_{x\to 35{,}000} T(x)$

49. ***Physics*** The acceleration due to gravity, g, varies with the height above the surface of the earth. The acceleration varies in a different way below the earth's surface. It has been determined that, as a function of r, the distance from the center of the earth, g is described by

$$g(r) = \begin{cases} \dfrac{GMr}{R^3} & \text{for } r < R \\ \dfrac{GM}{r^2} & \text{for } r \geq R \end{cases}$$

where $R \approx 6.371 \times 10^3$ km is the radius of the earth, $M \approx 5.975 \times 10^{24}$ kg is the mass of the earth, and $G \approx 6.672 \times 10^{-11}$ N $\circ$ m^2/kg^2 is the gravitational constant.

(a) Is g a continuous function of r? Explain your answer.

(b) Sketch a graph of g near $r = R$.

50. ***Business*** The cost, in dollars, of an overseas telephone call is given by the function

$$C(t) = \begin{cases} 8.50 & \text{if } 0 < t \leq 3 \\ 0.85t + 5.95 & \text{if } t > 3 \end{cases}$$

where t is the length of the call in minutes. Is C a continuous function at $t = 3$?

51. ***Finance*** A recent federal income tax schedule can be given by the function

$$T(x) = \begin{cases} 0.15x & \text{if } 0 < x \leq 23{,}900 \\ 0.28x - 3{,}107 & \text{if } 23{,}900 < x \leq 61{,}650 \\ 0.33x - 6{,}189.50 & \text{if } 61{,}650 < x \leq 123{,}790 \end{cases}$$

where x is the taxable income in dollars, $0 < x \leq 123{,}790$, and $T(x)$ is in dollars. Determine each of the following:

(a) $\lim_{x\to 23{,}900^-} T(x)$

(b) $\lim_{x\to 23{,}900^+} T(x)$

(c) $\lim_{x\to 23{,}900} T(x)$

(d) Is T continuous at $x = 23{,}900$?

(e) $\lim_{x\to 61{,}650^-} T(x)$

(f) $\lim_{x\to 61{,}650^+} T(x)$

(g) $\lim_{x\to 61{,}650} T(x)$

(h) Is T continuous at $x = 61{,}650$?

52. ***Business*** The local gas company uses the following function for computing its customers' monthly gas bills:

$$C(x) = \begin{cases} 0.47x + 2.95 & \text{if } 0 < x \leq 24 \\ 0.89x - 7.13 & \text{if } x > 24 \end{cases}$$

where x is the number of thermal units (therms) used by the customer and $C(x)$ is the cost in dollars. Determine each of the following:

(a) $\lim_{x\to 24^-} C(x)$

(b) $\lim_{x\to 24^+} C(x)$

(c) $\lim_{x \to 24} C(x)$

(d) Is C continuous at $x = 24$?

53. ***Medical technology*** Two of the rules that have been suggested for adjusting adult drug dosage levels to young children are the ones by Young, Y, and Cowling, C. If d is the recommended adult dosage and a is the age, in years, of the child, then

$$Y(a) = \frac{ad}{a + 12} \quad \text{and} \quad C(a) = \frac{(a + 1)d}{24}$$

(a) Determine $\lim_{a \to 0^+} Y(a)$.

(b) Determine $\lim_{a \to 0^+} C(a)$.

(c) Which dosage seems to be more realistic for a newborn baby? Why?

54. ***Finance*** Table 22.13 contains the federal tax rate for single taxpayers in 2010.

TABLE 22.13 2010 Federal Tax Rates Schedule X—Single

If taxable income is over:	but not over:	The tax is:	of the amount over:
\$0	\$8,375	\$0 + 10%	\$0
\$8,375	\$34,000	\$837.50 + 15%	\$8,375
\$34,000	\$82,400	\$4,681.25 + 25%	\$34,000
\$82,400	\$171,850	\$16,781.25 + 28%	\$82,400
\$171,850	\$373,650	\$41,827.25 + 33%	\$171,850
\$373,650	and greater	\$108,421.25 + 35%	\$373,650

(a) Where do the numbers \$837.50, \$4,681.25, \$16,781.25, \$41,827.25, and \$108,421.25 come from?

(b) We can write a single person's tax liability T as a function of the taxable income x, where

$$T(x) = \begin{cases} 0.10x & \text{if } 0 < x \le 8{,}375 \\ 837.50 + 0.15x & \text{if } 8{,}375 < x \le 34{,}000 \\ 4{,}681.25 + 0.25x & \text{if } 34{,}000 < x \le 82{,}400 \\ 16{,}781.25 + 0.28x & \text{if } 82{,}400 < x \le 171{,}850 \\ 41{,}827.25 + 0.33x & \text{if } 171{,}850 < x \le 373{,}650 \\ 108{,}421.25 + 0.35x & \text{if } x > 373{,}650 \end{cases}$$

where both x and T are in dollars. Is T a continuous function?

[IN YOUR WORDS]

55. The definition of continuity at a point has three conditions that must be satisfied if the function is to be continuous at a specific point.

(a) List each of these conditions.

(b) For each condition, give an example of a function that is not continuous because it does not satisfy that condition but does satisfy any previous conditions.

56. Suppose that f and g are functions and that neither is continuous as $x = 5$. Explain how $f + g$ can be continuous at $x = 5$.

57. **(a)** Describe the differences among $\lim_{x \to a^-} f(x)$, $\lim_{x \to a^+} f(x)$, and $\lim_{x \to a} f(x)$.

(b) Are all three of the limits in (a) ever the same? If so, what does that mean?

(c) Can two of the limits in (a) ever be the same (and the third one different)? If so, what does that mean?

(d) Can all three of the limits in (a) be different? If so, what does that mean?

58. Explain the difference between $\lim_{x \to c} f(x) = \infty$ and $\lim_{x \to \infty} f(x) = c$.

CHAPTER 22 REVIEW

IMPORTANT TERMS AND CONCEPTS

Area under a graph
Average slope
Continuity
 At a point
 Over an interval
Continuous function
Interval notation
Limits
 At infinity
 Left-hand
 One-sided
 Right-hand
Properties of continuous functions
Rules for limits

REVIEW EXERCISES

Find the average slope over the indicated intervals for the given functions in Exercises 1–6.

1. $f(x) = x^2 - 7$ from $x = 0$ to $x = 2$

2. $g(x) = 4x + 5$ from $x = -3$ to $x = -1$

3. $h(x) = 2x^2 + 1$ from $x = -3$ to $x = -2$

4. $j(x) = \dfrac{8}{x + 4}$ from $x = 0$ to $x = \frac{1}{2}$

5. $k(x) = x^2 + 2x$ from $x = -1$ to $x = b$

6. $m(x) = x^2 - 5x$ from $x = 6$ to $x = x_1$

In Exercises 7–10, find the areas under the graph of the given function over the indicated interval. The value of n indicates the number of segments into which each interval should be divided.

7. $f(x) = -(4x + 7)$ over $[-5, -2]$, $n = 6$

8. $g(x) = 2x^2 + 5$ over $[0, 4]$, $n = 8$

9. $h(x) = 3x^2 + 2x - 1$ over $[2, 4]$, $n = 8$

10. $j(x) = 5 - 2x^2$ over $[-1, 1]$, $n = 8$

In Exercises 11–20, use either the graphical or algebraic approach to determine the limit of the given function at the indicated point, or state that the limit does not exist.

11. $\lim_{x \to 1} (3x^2 + 2x - 5)$

12. $\lim_{x \to -2} \dfrac{3x^2 - 12}{x + 2}$

13. $\lim_{x \to 0} \dfrac{3x^2 - 12}{x - 2}$

14. $\lim_{x \to -3} \dfrac{x^2 - 9}{3x + 9}$

15. $\lim_{x \to \infty} \dfrac{6x^2 + 3}{2x^2}$

16. $\lim_{x \to \infty} \dfrac{9x^3 + 2x - 1}{3x^2 + x + 1}$

17. $\lim_{x \to 5^+} \sqrt{x - 5}$

18. $\lim_{x \to 4^-} \dfrac{x + 4}{x^2 - 16}$

19. $\lim_{x \to \infty} \dfrac{x + 4}{2x^2 + 1}$

20. $\lim_{x \to 1} f(x)$

if $f(x) = \begin{cases} x^2 - 1 & \text{if } x < 1 \\ 1 - x & \text{if } x > 1 \end{cases}$

In Exercises 21–24, determine the points of discontinuity for the given function.

21. $f(x) = \dfrac{x}{x-5}$

22. $g(x) = \dfrac{3}{x^3}$

23. $h(x) = \begin{cases} x+5 & \text{if } x < -3 \\ 5-x & \text{if } x \geq -3 \end{cases}$

24. $g(x) = \begin{cases} \dfrac{x}{x+1} & \text{if } x < 1 \\ \dfrac{1}{3-x} & \text{if } x \geq 1 \end{cases}$

In Exercises 25–28, determine the intervals where the given function is continuous.

25. $f(x) = \dfrac{5}{x+7}$

26. $g(x) = \dfrac{x^2-4}{x^2+5x+6}$

27. $h(x) = \dfrac{x+5}{\sqrt{x^2-25}}$

28. $j(x) = \sqrt{\dfrac{3-x}{3+x}}$

CHAPTER 22 TEST

Solve Exercises 1 and 2.

1. Find the average slope of $f(x) = 2x^3 - 3$ from $x = 0$ to $x = 1$.

2. Find the area under the graph of $g(x) = 3x^2 - 2x$ over the interval $[1, 5]$, when it is divided into $n = 4$ subintervals.

In Exercises 3–6, determine the limit of the given function or state that the limit does not exist.

3. $\lim\limits_{x\to 2} (5x^2 - 7)$

4. $\lim\limits_{x\to 2} \left(\dfrac{x^2-4}{3x-6}\right)$

5. $\lim\limits_{x\to 1^+} f(x)$ where $f(x) = \begin{cases} x^3 - 1 & \text{if } x < 1 \\ 4 & \text{if } x = 1 \\ 1+x & \text{if } x > 1 \end{cases}$

6. $\lim\limits_{x\to\infty} \dfrac{5x^3 - 4x + 1}{7x^3 - 2}$

Solve Exercises 7 and 8.

7. Determine the points of discontinuity for $h(x) = \begin{cases} x+7 & \text{if } x < 2 \\ \dfrac{1}{x^2-9} & \text{if } x \geq 2. \end{cases}$

8. Determine the intervals where $j(x) = \dfrac{x^2-9}{x^2-x-12}$ is continuous.

APPENDIX

A THE METRIC SYSTEM

The metric system[1] is becoming more important to workers. Almost every major country uses the metric system. Many industries are converting their manufacturing processes and products to metric specifications. In Canada, and every other major industrial country except the United States, the SI metric system is the official measurement system. Competition in the form of international trade has forced many U.S. companies to convert their products to the metric system. The U.S. Congress passed legislation requiring federal agencies to use the metric system in their business activities.

Older technical service publications were printed using the U.S. customary system, while the latest publications are either in metric or in a combination of both measurement systems. This section gives a brief introduction to the metric system and the different units and their symbols and tells how to convert within the metric system or between the metric and U.S. customary system.

UNITS OF MEASURE

There are seven base units in the metric system and two supplemental units in the SI metric system. All seven base units and the two supplemental units are shown in Table A.1.

All other units are formed from these seven. The base units that you will use most often are those for length (meter), mass or weight (kilogram), time (second), and electric current (ampere). The base unit for temperature is Kelvin, but we will mostly use a variation called the degree Celsius. In addition to the seven base units, there are two supplemental units. We use the supplemental unit for a plane angle (radian).

Each unit can be divided into smaller units or made into larger units. To show that a unit has been made smaller or larger, a prefix is placed in front of the base unit. The prefixes are based on powers of 10. The most common prefixes are shown in Table A.2. For example, 1 kilometer (km) is 1 000 m, 1 milligram (mg) is 0.001 g, 1 megavolt (MV) is 1 000 000 V $= 10^6$ V, and 1 nanosecond (ns) is 0.000 000 001 s $= 10^{-9}$ s.

1 The official name for the metric system is The International System of Units (or, in French, Le Systéme International d'Unités). The official abbreviation for the metric system throughout the world is "SI."

TABLE A.1 SI Metric System Base and Supplemental Units		
BASE UNIT		
Quantity	**Unit**	**Symbol**
Length	meter	m
Mass	kilogram	kg
Time	second	s
Electric current	ampere	A
Temperature	Kelvin	K
Luminous intensity	candela	cd
Amount of substance	mole	mol
SUPPLEMENTAL UNITS		
Quantity	**Unit**	**Symbol**
Plane angle	radian	rad
Solid angle	steradian	sr

TABLE A.2 Metric Prefixes

Multiple	Prefix	Symbol	
1 000 000 000	$= 10^{9}$	giga	G
1 000 000	$= 10^{6}$	mega	M
1 000	$= 10^{3}$	kilo	k
1			
0.01	$= 10^{-2}$	centi	c
0.001	$= 10^{-3}$	milli	m
0.000 001	$= 10^{-6}$	micro	μ
0.000 000 001	$= 10^{-9}$	nano	n

Many quantities, such as area, volume, speed, velocity, acceleration, and force, require a combination of two or more fundamental units. Combinations of these units are referred to as *derived units* and are listed in Table A.3.

Table A.4 shows some of the metric units most commonly used and their uses in various trade and technical areas. This list is not intended to be exhaustive, but to provide you with some idea as to where the different units are used.

WRITING METRICS

You should remember some important rules when you use the metric system.

1. The unit symbols that are used are the same in all languages. This means that the symbols used on Japanese cars are the same as those used on German or American cars.
2. The unit symbols are not abbreviations and a period is not put at the end of the symbol.

TABLE A.3 Derived Units for Common Physical Quantities

Quantity	Derived Unit	Symbol	Alternate Symbol
Acceleration	meter per second squared	m/s^2	
Angular acceleration	radian per second squared	rad/s^2	
Angular velocity	radian per second	rad/s	
Area	square meter	m^2	
Concentration, mass (density)	kilogram per cubic meter	kg/m^3	
	gram per liter	g/L	
Capacitance	farad	F	C/V
Electric current	ampere	A	V/Ω
Electric field strength	volt per meter	V/m	
Electric resistance	ohm	Ω	V/A
Electromotive force (emf)	volt	V	J/C
Force	newton	N	$kg \cdot m/s^2$
Frequency	hertz	Hz	s^{-1}
Illuminance	lux	lx	lm/m^2
Inductance	henry	H	V · s/A
Light exposure	lumen second	lm · s	
Luminance	candela per square meter	cd/m^2	
Magnetic flux	weber	Wb	V · s
Moment of force	newton meter	N · m	
Moment of inertia, dynamic	kilogram meter squared	$kg \cdot m^2$	
Momentum	kilogram meter per second	kg · m/s	
Power	watt	W	J/s
Pressure	pascal	Pa	N/m^2
Quantity of electricity	coulomb	C	
Speed, velocity	meter per second	m/s	
Volume	cubic meter	m^3	
Volume flow rate	cubic meter per second	m^3/s	
	liter per minute	L/min	
Work, energy, quantity of heat	joule	J	N · m

3. Unit symbols are shown in lowercase letters except when the unit is named for a person. Examples of unit symbols that are written in capital letters are joule (J), newton (N), pascal (Pa), watt (W), ampere (A), and coulomb (C). The only exception is the use of L for liter. The symbol L is often used for liter to eliminate any confusion between the lowercase unit symbol (l) and the numeral (1). Some people prefer to use a script ℓ for liter.
4. The symbol is same for both singular and plural (e.g., 1 m and 12 m).
5. Numbers with four or more digits are written in groups of three separated by a space instead of a comma. (The space is optional on four-digit numbers.) This is done because some countries use the comma as a decimal point.

TABLE A.4 Uses of Metric Units in Technical Areas

Quantity	Unit	Symbol	Use
Length	micrometer	μm	paint thickness; surface texture or finish
	millimeter	mm	motor vehicle dimensions; wood; hardware, bolt, and screw dimensions; tool sizes; floor plans
	centimeter	cm	bearing size; length and width of fabric or window; length of weld, channel, pipe, I-beam, or rod
	meter	m	braking distance; turning circle; room size; wall covering; landscaping; architectural drawings; length of pipe or conduit; highway width
	kilometer	km	land distances; maps; odometers
Area	square centimeter	cm^2	piston head surfaces; brake and clutch contact area; glass, tile, or wall covering; area of steel plate
	square meter	m^2	fabric, land, roof, and floor area; room sizes; carpeting; window and/or wall covering
Volume or capacity	cubic centimeter	cm^3	cylinder bore; small engine displacement; tank or container capacity
	cubic meter	m^3	work or storage space; truck body; room or building volume; trucking or shipping space; tank or container capacity; ordering concrete; earth removal
	milliliter	mL	chemicals; lubricants; oils; small liquids; paints;
	liter	L	fuel; large engine displacement; gasoline
Temperature	degree Celsius	°C	thermostats; engine operating temperature; oil or liquid temperature; melting points; welds
Mass or weight	gram	g	tire weights; mailing and shipping packages
	kilogram	kg	batteries; weights; mailing and shipping packages
	metric ton	t	vehicle and load weight; construction material such as sand or cement; crop sales
Bending force, torque, moment of force	newton meter	$N \cdot m$	torque specifications; fasteners
Pressure/vacuum	kilopascal	kPa	gas, hydraulic, oxygen, tire, air, or air hose pressure; manifold pressure compression; tensile strength
Velocity	kilometers per hour	km/h	vehicle speed; wind speed
	meters per second	m/s	speed of air or liquid through a system
Force, thrust, drag	newton	N	pedal; spring; belt; drawbar
Power	watt	W	air conditioner; heater; engine; alternator
Illumination	lumens per square meter	lm/m^2	intensity of light on a given area
Density	milligrams per cubic meter	mg/m^3	industrial hygiene standards for fumes, mists, and dusts
Flow	cubic meters per second	m^3/s	measure of air exchange in a region; exhaust and air exchange system ratings

EXAMPLE A.1

Use 2 473 or 2473 instead of 2,473;
45 689 instead of 45,689;
47 398 254.263 72 instead of 47,398,254.26372.

6. A zero is placed to the left of the decimal point if the number is less than one (0.52 L, not .52 L).
7. Liter and meter are often spelled litre and metre.
8. The units of area and volume are written by using exponents.

EXAMPLE A.2

5 square centimeters is written as 5 cm^2, not as 5 sq cm;
37 cubic meters is written as 37 m^3, not as 37 cu m

CHANGING WITHIN THE METRIC SYSTEM

Changing units of measurement within a system is called *reduction*. A change from one system to another is called *conversion*. We will now discuss reduction in the metric system and then conversion between the metric and U.S. customary system afterward.

The metric prefixes in Table A.2 provide a method to help with metric reduction. Using the information in Table A.2, we construct a metric reduction diagram like the one shown in Figure A.1. Starting on the left with the largest prefix shown in Table A.2, we mark each multiple of 10 until we get to the smallest prefix. Notice that not all multiples are labeled. While there is a prefix for each multiple of 10, we have given you only those that you will need.

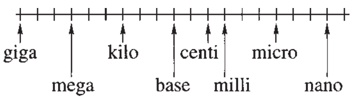


Figure A.1

REDUCTION IN THE SI METRIC SYSTEM

To change from one metric system unit to another:

1. Mark each unit on the reduction scale in Figure A.1.
2. Move the decimal point as many places as you move along the reduction scale according to the following rules:
   (a) To change to a unit farther to the right, move the decimal point to the right.
   (b) To change to a unit farther to the left, move the decimal point to the left.
3. If you are changing between square units, then move the decimal point *twice* as far as indicated in Step 2.
4. If you are changing between cubic units, then move the decimal point *three times* as far as indicated in Step 2.

EXAMPLE A.3

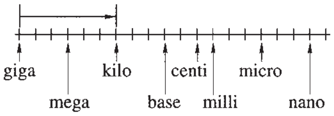


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Figure A.2

Change 7.35 GV to kilovolts.

SOLUTION Since 7.35 GV represents 7.35 gigavolts, we put a mark at the giga point on the reduction scale in Figure A.2. We want to convert GV to kV, so an arrow is drawn from the first mark to the mark labeled "kilo." The arrow points to the right and is 6 units long, so we move the decimal point 6 units to the right. To do this, we need to insert some zeros, with the result

$$7.35\text{ GV} = 7\,350\,000\text{ kV}$$

EXAMPLE A.4

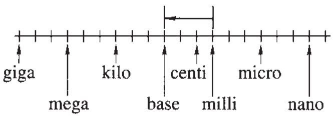

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Figure A.3

Change 9.6 mm to meters.

SOLUTION Since 9.6 mm represents 9.6 millimeters, we put a mark at the milli point on the reduction scale in Figure A.3. We want to convert mm to m, the base unit, so an arrow is drawn from the first mark to the mark labeled "base." The arrow points to the left and is 3 units long, so we move the decimal point 3 units to the left. Again, we need to insert some zeros, with the result

$$9.6\text{ mm} = 0.009\,6\text{ m}$$

EXAMPLE A.5

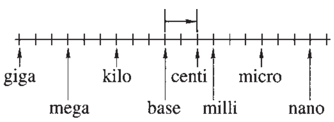

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Figure A.4

Change 3.7 m^2 to square centimeters.

SOLUTION Since 3.7 m^2 represents 3.7 square meters, we put a mark at the base point on the reduction scale in Figure A.4. We want to convert m^2 to cm^2, so the arrow is drawn from the first mark to the mark labeled "centi." The arrow points to the right and is 2 units long. Since we are converting between square units, we double this number, so we move the decimal point 4 units to the right. Again, we need to insert some zeros, with the result

$$3.7\text{ m}^2 = 3\,700\text{ cm}^2$$

To change one set of units into another set of units, you should perform algebraic operations with units to form new units for the derived quantity.

EXAMPLE A.6

Convert a speed of 88.00 km/h to meters per second.

SOLUTION We will use two relationships that will give four possible conversion factors.

$$1\text{ km} = 1000\text{ m} \quad \nearrow \frac{1000\text{ m}}{1\text{ km}} \qquad \searrow \frac{1\text{ km}}{1000\text{ m}}$$

EXAMPLE A.6 (Cont.)

$$1 \text{ h} = 3\,600 \text{ s} \quad \nearrow \frac{3\,600 \text{ s}}{1 \text{ h}} \qquad \searrow \frac{1 \text{ h}}{3\,600 \text{ s}}$$

We write the quantity to be changed, then choose the appropriate conversion factors so that all but two of the units "cancel," leaving the units m/s.

$$88\frac{\cancel{\text{km}}}{\cancel{\text{h}}} \times \frac{1\,000 \text{ m}}{1 \cancel{\text{km}}} \times \frac{1 \cancel{\text{h}}}{3\,600 \text{ s}} \approx 24.44 \text{ m/s}$$

CHANGING BETWEEN THE METRIC AND CUSTOMARY SYSTEMS

Many technicians are often called upon to use both the SI metric and the U.S. customary measurement systems. While it is best not to change from either the metric or the customary system to the other, sometimes a worker has to do so. For example, it is possible that the measuring tools available may be different than the measuring system in which the specifications are given. In such a case, the technician will be required to convert from one system to another. We will show one way to change between the U.S. customary system and the SI metric system. Table A.5 should help you do this.

EXAMPLE A.7

Express 4.3 kg in pounds.

SOLUTION You are changing from the metric (kg) to the customary system.

Look at Table A.5 under the heading, "From Metric to Customary." You are changing a measurement from kg to lb. Look in the column labeled "To change from" until you find the symbol for kilogram (kg). The symbol in the next right-hand column is the symbol for pound (lb).

Now look in the "Multiply by" column just to the right of these two symbols. The number there is 2.205. Multiply the given number of kilograms (4.3) by this number (2.205).

$$4.3 \times 2.205 = 9.4815$$

This means that 4.3 kg is equivalent to 9.4815 lb (or 4.3 kg $\approx$ 9.4815 lb).

EXAMPLE A.8

Express 6.5 fluid ounces in liters.

SOLUTION You are changing from the customary system (fluid ounces) to the metric (liters) system.

Look in Table A.5 under the heading "From Customary to Metric." You are changing a measurement from fluid ounces to liters. Look under the column labeled "To change from" until you find the symbol for fluid ounces—fl oz. The

TABLE A.5 Changing Units Between the Metric and Customary Systems

	FROM METRIC TO CUSTOMARY			FROM CUSTOMARY TO METRIC		
Quantity	**To change from**	**To**	**Multiply by**	**To change from**	**To**	**Multiply by**
Length	μm	mil	0.039 37	mil	μm	25.4
	mm	in.	0.039 37	in.	mm	25.4
	cm	in.	0.393 7	in.	cm	2.54
	m	ft	3.280 8	ft	m	0.304 8
	km	mi	0.621 37	mi	km	1.609 3
Area	cm^2	$in.^2$	0.155	$in.^2$	cm^2	6.451 6
	m^2	ft^2	10.763 9	ft^2	m^2	0.092 9
Volume	cm^3	$in.^3$	0.061	$in.^3$	cm^3	16.387
	m^3	yd^3	1.308	yd^3	m^3	0.7646
	m^3	gal	264.172	gal	m^3	0.003 785
	mL	fl oz	0.033 8	fl oz	mL	29.574
	L	fl oz	33.814	fl oz	L	0.029 6
	L	pt	2.113	pt	L	0.473 2
	L	qt	1.056 7	qt	L	0.946 4
	L	gal	0.264 2	gal	L	3.785 4
Mass or weight	g	oz	0.035 3	oz	g	28.349 5
	kg	lb	2.205	lb	kg	0.453 6
	t	lb	2205	ton	kg	907.2
Bending moment, torque, moment of force	N · m	lbf · in.	8.850 7	lbf · in.	N · m	0.113
	N · m	lbf · ft	0.737 6	lbf · ft	N · m	1.355 8
Pressure, vacuum	kPa	psi	0.145	psi	kPa	6.894 8
Velocity	km/h	mph	0.621 4	mph	km/h	1.609 3
Force, thrust, drag	N	lbf	0.224 8	lbf	N	4.448 2
Power	W	W	1	W	W	1
Temperature	°C	°F	$\frac{9}{5}(°C) + 32$	°F	°C	$\frac{5}{9}(°F - 32)$

fl oz symbol appears twice. Now look in the next right-hand column. Do you see the symbol for liters (L)? It is opposite the second fl oz symbol.

If you look in the "Multiply by" column to the right of these two symbols (fl oz and L), you find 0.029 6. Multiply the number of fluid ounces (6.5) by this number (0.029 6).

$$6.5 \times 0.029\,6 = 0.1924$$

This means that 6.5 fluid ounces is equivalent to 0.192 4 L (or 6.5 fl oz ≈ 0.192 4 L).

EXERCISE SET

In Exercises 1–12, reduce the given unit to the indicated unit.

1. 347 g to kilograms

2. 0.26 km to meters

3. 7.92 kW to watts

4. 2.3 μs to seconds

5. 0.023 85 Ω to milliohms

6. 0.000 235 47 MW to milliwatts

7. 9.72 mm to centimeters

8. 0.35 mm to nanometers

9. 835 000 cm^2 to square meters

10. 2.34 km^2 to m^2

11. 4.35 mm^3 to cubic centimeters

12. 91.52 m^3 to cubic millimeters

Solve Exercises 13–16.

13. *Machine technology* A die is 14 mm long. Express this in centimeters.

14. *Law enforcement* A male suspect is 1.97 m tall. Express this in centimeters.

15. *Environmental science* A wastewater treatment plant has 92 600 kg of sludge. Express this in metric tons.

16. *Medical technology* A technician needs 1 125 mL of a sterile solution. Express this in liters.

In Exercises 17–24, change the given units to the indicated unit. (You may want to consult Table A.3.)

17. 45 m/s to km/h

18. 27 mm/s to km/h

19. *Physics* If a force of 126 N gives a body an acceleration of 9 m/s^2, then the body has a mass of $\frac{126\text{ N}}{9\text{ m/s}^2}$. Convert this to kilograms.

20. *Electricity* An ac electric current that is generated by a source of 120 V and with an impedance of 125 Ω has an effective current of $\frac{120\text{ V}}{125\ \Omega}$. Convert this to amperes.

21. *Electricity* The resistance in a heater is $\frac{84\text{ V}}{8\text{ A}}$. Convert this to ohms.

22. *Physics* A technician determines that the absolute pressure in a tank is 112 cm of mercury. The following computation will convert this pressure to kilopascals: (13 600 kg/m^3) (9.8 m/s^2) (112 cm). Convert this to kilopascals.

23. *Automotive technology* If a 3 250-g piston is held 300 mm above a work surface, then its potential energy relative to the work surface is

$$(3\,250\text{ g})(9.8\text{ m/s}^2)(300\text{ mm})$$

Express this potential energy in joules.

24. *Construction* At the instant a 4-kg sledgehammer reaches a velocity of 26 m/s, it has a kinetic energy of $\frac{1}{2}$ 4 kg$)\left(26\text{ m/s}\right)^2$. Express this kinetic energy in joules.

In Exercises 25–40, convert the given measurement to the indicated unit.

25. 4 m = ________________ ft

26. 16 L = ________________ pt

27. 24 kg = ________________ lb

28. 27.5 N · m = ________________ lbf · ft

29. 105 km/h = ________________ mph

30. 17 in. = ________________ cm

31. 4.5 qt = ________________ L

32. 21 ft^2 = ________________ m^2

33. 5.8lb = ________________ kg

34. 32 psi = ________________ kPa

35. 97 W = ________________ W

36. 98.6°F = ________________ °C

37. *Fire safety* A fire extinguisher contains $2\frac{1}{2}$ gal. Convert this to liters.

38. *Law enforcement* An adult female weighs 120 lb. Convert this to kilograms.

39. *Environmental science* A flow-measuring meter at a wastewater treatment plant recorded 9,660,$\tilde{0}$00 gal in 1 day. Convert this to liters.

40. *Environmental science* A wastewater treatment plant has a daily flow of 59 500 m^3. Convert this to gallons.

ANSWERS TO ODD-NUMBERED EXERCISES

ANSWERS FOR CHAPTER 1

Exercise Set 1.1

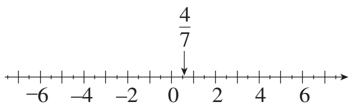
1. Natural number, whole number, integer, rational number, and real number **3.** Irrational number, real number

5. 1800 **7.** −3800 **9.** 328,000 **11.** 433,000

13. 24.37 **15.** 13.74 **17.** 23″ **19.** 15

21. $|0| = 0$

23.



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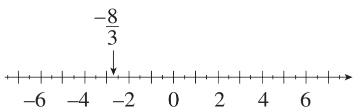
25.



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27. $<$ **29.** $<$ **31.** $>$ **33.** $<$ **35.** $>$ **37.** $<$

39. Commutative law for addition **41.** Commutative law for multiplication **43.** Associative law for addition **45.** Identity element for multiplication

47. Additive inverse **49.** Association law for multiplication

51. -91 **53.** $-\sqrt{2}$ **55.** 2 **57.** $\frac{2}{\sqrt{2}} = \sqrt{2}$

59. $-\frac{1}{5}$ **61.** $\frac{3}{17}$ **63.** 14 **65.** 13 **67.** 1

69. 31 **71.** 1 **73.** -111

75. **(a)** -359 **(b)** -359 **77.** **(a)** -32760 **(b)** -32760

79. 25.3 **81.** -15.48 **83.** 386.2 **85.** $-5, -|4|, -\frac{2}{3}, \frac{-1}{3}, \frac{16}{3}, |-8|$ **87.** **(a)** $\frac{1}{32}$ inch **(b)** $\frac{1}{8}$ inch **(c)** 0.0625 inch

89. 2.6 V **91.** $\frac{5}{7}$ (Sheila), $\frac{2}{3}$ (Hazel), $\frac{3}{5}$ (José), $\frac{9}{16}$ (Robert), and $\frac{4}{9}$ (Lamar) **93.** Mile marker 93 or 221 **95.** $12' \times 14'$ **97.** **(a)** distribution property **(b)** $8(12 + 3) = 8\times 12 + 8 \times 3$

99. **(a)** commutative property for addition **(b)** $257 + 92 = 92 + 257$ **101.** **(a)** Machine 1: 38,927; Machine 2: 28,591; Machine 3: 26,509; Machine 4: 40,425; Machine 5: 31,788 **(b)** 166,240 **103.** 15,750 **111.** Answers depend on the type of calculator used

Exercise Set 1.2

1. 50 **3.** 14 **5.** -9 **7.** -24 **9.** 12 **11.** 15

13. $-\frac{38}{4} = -\frac{19}{2}$ **15.** $\frac{1}{8}$ **17.** $\frac{8}{15}$ **19.** $\frac{3}{20}$ **21.** $\frac{5}{8}$

23. $-\frac{47}{30} = -1\frac{17}{30}$ **25.** $\frac{1}{32}$ **27.** $-\frac{8}{15}$ **29.** $\frac{3}{32}$

31. $-\frac{10}{3}$ **33.** $\frac{6}{5}$ **35.** $-\frac{3}{20}$ **37.** $\frac{49}{25} = 1\frac{24}{25}$ **39.** $\frac{7}{2}$

41. $78\frac{1}{2}$ inches $= 6'6\frac{1}{2}''$ **43.** 57 V **45.** **(a)** $7\frac{33}{40}$ mi **(b)** $11\frac{1}{5}$ mi; $3\frac{3}{10}$ mi **47.** $84'9\frac{1}{2}''$ **49.** $\frac{7}{8}''$

51. **(a)** -21.2 V **(b)** The voltage dropped 21.2 V

53. 626 **55.** 1960 V **57.** $-\$1,135,900$ **59.** 8 wafers

61. 8 in = Rise, 9 in = Run **63.** $8\frac{59}{64}$

65. **(a)** $303\frac{3}{4}$ yards **(b)** 304 yards **67.** **(a)** 1.593 **(b)** 7.399 **(c)** 1.913 **69.** $14' - 7\frac{1}{2}''$ **71.** $19\frac{17''}{32}$

73. 17 **75.** $29\frac{15}{32}$ inches **77.** 625 mg

Exercise Set 1.3

1. 125 **3.** $\frac{3}{2}$ **5.** 16 **7.** $\frac{1}{49}$ **9.** 3^6 **11.** 2^{12} **13.** 2^2

15. 2^6 **17.** x^{20} **19.** $\frac{a^6}{b^3}$ **21.** $\frac{x^3}{4^3}$ **23.** $\frac{a^8b^4}{c^{12}}$ **25.** $\frac{1}{x^7}$

27. $\frac{1}{p^3}$ **29.** $\frac{1}{7^5}$ **31.** $\frac{1}{a^3y^4}$ **33.** $\frac{y^2}{a^4}$ **35.** $\frac{1}{pr^2}$ **37.** $\frac{5^2}{4y^3}$

39. $\frac{y^{15}}{2^3b^6}$ **41.** b^{24} **43.** 5 **45.** 12 **47.** 2 **49.** -3

51. 2 **53.** $\frac{2}{3}$ **55.** 0.2 **57.** -0.1 **59.** 3 **61.** 8.32
63. 5 **65.** $\sqrt[4]{72}$ **67.** 5 **69.** $\frac{1}{2}$ **71.** 7 **73.** $\frac{1}{3}$
75. $5^{1/3}$ **77.** 2 **79.** 8 **81.** 4 **83.** $\frac{1}{4}$ **85.** 81
87. $\dfrac{9}{0.2\sqrt{2}}$ **89.** 0.25 **91.** 51.96 Ω **93.** 8.5 Ω
95. 2.09 m **97.** 1 449 m/s **99.** 8.14 cm **101.** 1.50 A

Exercise Set 1.4

1. Exact **3.** Approximate **5.** Approximate
7. 3 **9.** 3 **11.** 1 **13.** 4 **15.** **(a)** 6.05 **(b)** 6.05
17. **(a)** 5.01 **(b)** 0.027 **19.** **(a)** $27{,}0\bar{0}0$ **(b)** $27{,}0\bar{0}0$
21. **(a)** 86 **(b)** 0.2 **23.** **(a)** 140.070 **(b)** 140.070
25. **(a)** 10 **(b)** 14 **(c)** 14.4 **27.** **(a)** 7 **(b)** 7.0 **(c)** 7.04
29. **(a)** 400 **(b)** $4\bar{0}0$ **(c)** 403 **31.** **(a)** 300 **(b)** 310 **(c)** 305 **33.** **(a)** 10 **(b)** 14 **(c)** 14.4 **35.** **(a)** 7 **(b)** 7.0 **(c)** 7.04 **37.** **(a)** 400 **(b)** $4\bar{0}0$ **(c)** 403
39. **(a)** 300 **(b)** 310 **(c)** 305 **41.** **(a)** 90 **(b)** 89.9 **(c)** 89.899 **43.** **(a)** 240 **(b)** 237.3 **(c)** 237.302
45. **(a)** 440 **(b)** 438.0 **(c)** 437.998 **47.** **(a)** 80 **(b)** 78.7 **(c)** 78.671
49.

	Absolute error	Relative error
Length	-0.28 mm	-1.17%
Width	0.35 mm	4.38 %
Thickness	-0.02 mm	-0.67%

51. 99.37 **53.** 1618 **55.** 1,020 **57.** 17
59. 351.65 m **61.** 924.6 mm **63.** **(a)** 14 770 meters **(b)** 14 765 meters **(c)** 14 765.5 meters

Exercise Set 1.5

1. 4.2×10^4 **3.** 3.8×10^{-4} **5.** 9.80700×10^9
7. 9.70×10^{-5} **9.** 4.3×10^0 **11.** 74×10^3
13. 470×10^{-6} **15.** 9.80700×10^9 **17.** 53.10×10^{-6}
19. 5.6×10^0 **21.** 4500 **23.** 40500000 **25.** 0.000063
27. 72 **29.** 75 000 **31.** 47 500 000 000
33. 0.000 039 20 **35.** 83.15 **37.** 1.5504×10^{16}
39. 2.66×10^{-11} **41.** 2.94×10^5 **43.** 2.2528×10^{-1}
45. 1.2×10^5 **47.** 2.5×10^{-3} **49.** 2×10^4
51. 1.8×10^{-3} **53.** $1.373568 \times 10^{25} \approx 1.38 \times 10^{25}$
55. 2.5×10^{22} **57.** 3.9×10^{14} **59.** 675 (Answers may vary.) **61.** 1.11×10^6 mm **63.** 2.789808×10^{-19} kg
65. One neutron; approximately 1.84×10^3 times heavier **67.** About 282.74×10^3 Ω

Review Exercises

1. **(a)** Integers, rational numbers, real numbers **(b)** Rational numbers, real numbers **(c)** Irrational numbers, real numbers **3.** **(a)** $\frac{3}{2}$ **(b)** $\frac{-1}{8}$ **(c)** -5 **5.** **(a)** Commutative law for addition **(b)** Distributive law **(c)** Additive identity **(d)** Multiplicative inverse
7. -44 **9.** 98 **11.** $-\frac{1}{6}$ **13.** -3 **15.** $-\frac{2}{5}$ **17.** $\frac{8}{3}$
19. $\frac{10}{13}$ **21.** 32 **23.** -64 **25.** 2 **27.** 2^8 **29.** 2^2
31. 4^{15} **33.** 4 **35.** $\dfrac{a}{b}$ **37.** $\dfrac{1}{a^2x^4}$ **39.** **(a)** 2.37 **(b)** 2.37
41. **(a)** both **(b)** 0.7 **43.** **(a)** 7.4 **(b)** 7.35 **(c)** 7.4 **(d)** 7.35 **45.** **(a)** 2.1 **(b)** 2.05 **(c)** 2.1 **(d)** 2.05
47. 3.71×10^{11} **49.** 2.4×10^{-11} **51.** 29.08×10^{15}
53. 75×10^{-24} **55.** 12 **57.** 5 **59.** **(a)** \$57.12 **(b)** \$5,707.10

Chapter 1 Test

1. **(a)** $\frac{4}{3}$ **(b)** $\frac{1}{8}$ **(c)** 6 **3.** $\frac{5}{8}$ **5.** -112 **7.** $\frac{22}{3} = 7\frac{1}{3}$
9. $-10\frac{1}{2}$ **11.** $\dfrac{14}{29}$ **13.** 8 **15.** 25 **17.** $\dfrac{b^3}{a}$
19. 0.000 51 **21.** $\frac{7}{5}$

ANSWERS FOR CHAPTER 2

Exercise Set 2.1

1. 4 is a constant; x and y are variables. **3.** 8 and π are constants; r is a variable. **5.** $3x^3$ and $4x$
7. $(2x^3)(5y)$, $\sqrt{3}ab$, and $-\dfrac{7a}{b}$ **9.** $-1, 5, x$, and $\dfrac{1}{y}$
11. 47 **13.** $\dfrac{\pi a}{4}$, if a is a constant **15.** $3x^2y$ and $17x^2y$
17. $(x + y)^2$ and $5(x + y)^2$ **19.** $a + b - c$ and $-2(a + b - c)$ **21.** $11x$ **23.** $2z$ **25.** $6x + 9x^2$ or $9x^2 + 6x$ **27.** $10w - 7w^2$ or $-7w^2 + 10w$

29. $2ax^2 + a^2x$ **31.** $11xy^2 - 5x^2y$ **33.** $12b$
35. $2a^2 + 5b + 4a$ **37.** $2x^2 + 6x$ **39.** $12y^2 + 14x$
41. $-12b + 6c$ **43.** $7a + 7b$ **45.** $x + y$
47. $5a + 5b - c$ **49.** $6x + 6y$ **51.** $5a + 5b$
53. $6a - 2b + 4c$ **55.** $9x + 7y$ **57.** $-6x + 2y - 13z$
59. $-4x + 3y + 25z$ **61.** $6x + 2y$ **63.** $3y + z$
65. $x + 6y$ **67.** $a - 3b$ **69.** $-70a - 8b$
71. $\frac{13}{6}p = 2\frac{1}{6}p$ **73.** $\dfrac{1}{C_T} = \dfrac{C_2 + C_1}{C_1C_2}$
75. **(a)** N = W − (0.134W + 0.046W + 0.011W + 0.075W + 0.010W + 0.002W + 0.082W)
(b) N = W − (0.36 W) = 0.64 W
77. $42I_a^{\,2} + 70I_b^{\,2}$

Exercise Set 2.2

1. a^3x^3 **3.** $6a^2x^3$ **5.** $-6x^3w^3z$ **7.** $-24ax^4b$
9. $10y - 12$ **11.** $35 - 20w$ **13.** $21xy + 12x$
15. $15t - 5t^2$ **17.** $2a^2 - a$ **19.** $6x^3 - 2x^2 + 8x$
21. $-20y^4 + 8y^3 - 20y^2 + 12y - 24$ **23.** $a^2 + ab + ac + bc$ **25.** $x^3 + 5x^2 + 6x + 30$ **27.** $6x^2 + xy - y^2$
29. $6a^2 - 7ab + 2b^2$ **31.** $2b^2 + 3b - 5$
33. $56a^4b^2 + 3a^2bc - 9c^2$ **35.** $x^2 - 16$ **37.** $p^2 - 36$
39. $a^2x^2 - 4$ **41.** $4r^4 - 9x^2$ **43.** $25a^4x^6 - 16d^2$
45. $\dfrac{9}{16}t^2b^6 - \dfrac{4}{9}p^2a^4f^2$ **47.** $x^2 + 2xy + y^2$
49. $x^2 - 10x + 25$ **51.** $a^2 + 6a + 9$
53. $4a^2 + 4ab + b^2$ **55.** $9x^2 - 12xy + 4y^2$
57. $12x^3 + 40x^2 - 32x$ **59.** $x^2 - y^2 - z^2 + 2yz$
61. $a^3 + 6a^2 + 12a + 8$ **63.** $2n^2 + 6n$
65. $\dfrac{1}{2}(y_2^2 - y_1^2) = \dfrac{1}{2}y_2^2 - \dfrac{1}{2}y_1^2$ **67.** $x = 3t^2 + 15t + 12$
69. $A = \dfrac{1}{2}(4 + x)[(6 + x) + (8 + x)] = x^2 + 11.0 + 28$

Exercise Set 2.3

1. x^4 **3.** $2x^2$ **5.** $3y^2$ **7.** $-3b$ **9.** $11y$ **11.** $-6ay$
13. $18c^2d^2$ **15.** $\dfrac{-3p}{5n^2}$ **17.** $\dfrac{4bcx}{7y}$ **19.** $2a^2 + a$
21. $4b^3 - 2b$ **23.** $6x^2 + 4x$ **25.** $2x^3 - 3$
27. $-6x^3 + 2x$ **29.** $5x + 5y$ **31.** $2x + 3y$
33. $ap - 2$ **35.** $a + 1$ **37.** $-3xy + z$
39. $-b^2x^2 - b^2$ **41.** $x + 1 - y$ **43.** $-\frac{3}{2}xy + 2y^2$
45. $x + 4$ **47.** $x - 1$ **49.** $x - 1$ **51.** $2a + 1$
53. $4y + 2$ **55.** $x - 2$ **57.** $2a - 1$
59. $2x^2 + x - 1 + \dfrac{3}{2x - 1}$ **61.** $r^2 - 3r + \dfrac{5}{r + 2}$
63. $x^3 + 3x^2 + 9x + 27$ **65.** $4x^3 - 3x^2 - x + 6$
67. $x + y$ **69.** $w^2 + wz + z^2$ **71.** $x^2 + y^2$
73. $c^2d^2 + 2cd + 4$ **75.** $x - y$ **77.** $p^2r - 2p + 3r^2$
79. $a + d + 4 + \dfrac{-4}{a - 3d - 1}$ **81.** af
83. $a + b - c$ **85.** $a^2 - a + 1$ **87.** $\dfrac{1}{R_1} + \dfrac{1}{R_2} + \dfrac{1}{R_3}$
89. $V_2 = \dfrac{V_1T_2}{T_1}$ **91.** $8r - 5$

Exercise Set 2.4

1. 39 **3.** 12 **5.** −15 **7.** $\frac{9}{2}$ **9.** −8 **11.** $-\frac{2}{3}$
13. 15 **15.** −24 **17.** 2 **19.** −4 **21.** 4 **23.** 4
25. 7 **27.** $\frac{1}{2}$ **29.** $\dfrac{24}{5}$ **31.** 5 **33.** −6 **35.** 9
37. −6 **39.** $-\frac{40}{3}$ **41.** 6 **43.** 6 **45.** $\dfrac{13}{4}$
47. 14 **49.** −18 **51.** $\frac{24}{7}$ **53.** 78 **55.** $\dfrac{b}{2a}$
57. $-\dfrac{x}{x - 8}$ or $\dfrac{x}{8 - x}$ **59.** $\frac{7}{3}$ **61.** 1 **63.** −3
65. −15 **67.** −2 **69.** $a = -12$ **71.** $a = -2z$
73. 1.38 : 16 **75.** 86 : 1 **77.** 4.5 : 1 **79.** 3.2857 : 1
81. 28 **83.** 79 **85.** 4.267 **87.** 14
89. $C = \frac{5}{9}(F - 32)$

Exercise Set 2.5

1. 10 800 seconds **3.** 1.45 miles **5.** 38.735 cm
7. 60 mph **9.** 9.84 in. **11.** 5.44 cm **13.** 54 : 13
15. 28 : 1 **17.** 1:67 μF **19.** 11.03 cm^3
21. 62 000 000 calories **23.** 3.5 in. **25.** 53.7472 L
27. 1500 A per minute. **29.** **(a)** 44 mm **(b)** 8 mm

Exercise Set 2.6

1. 82 **3.** 60, since no score can be below 60
5. \$1150; \$230 **7.** \$6130 **9.** \$2200 at 7.5% and \$2300 at 6% **11.** 7 hours **13.** 266 miles **15.** 320 km
17. 2.4 hours or 2 hours 24 minutes **19.** $\frac{4}{3}$ hours or 1 hour 20 minutes **21.** 57.5 mL **23.** 468.75 kg of 75% copper; 281.25 kg of 35% copper
25. 350 lb; 8.235 feet from the 500 lb end **27.** 4.5 ft from the left end **29.** 6 in. from right end

31. 30 cc **33.** **(a)** $V = \sqrt{PR}$, where V is the voltage, P the power, and R the resistance **(b)** 23 Ω
35. **(a)** $Z = \sqrt{R^2 + X^2}$ **(b)** 8.9 Ω

Review Exercises

1. $3y$ **3.** $5x - 8$ **5.** $7x^2 + 11$ **7.** $16x + 8$
9. $1 - 3x$ **11.** $9x^2 + x - 2$ **13.** $3a + 9b$ **15.** $a^3 x^3$
17. $27ax^3$ **19.** $20x - 24$ **21.** $8x^2 - 10x$ **23.** $a^2 - 16$
25. $6a^2 - 5ab + b^2$ **27.** $6x^4 - 5x^2 - 6$
29. $x^2 + 4x + 4$ **31.** $30x^3 - 25x^2 - 20x$ **33.** a^3
35. $4a$ **37.** $-9ax^2$ **39.** $18x - 8$ **41.** $x - 4$
43. $x^2 + 3x + 9$ **45.** $x - y$ **47.** $x - 2y$ **49.** 38
51. $\frac{15}{2}$ **53.** 36 **55.** 5 **57.** 4.5 **59.** $-\frac{3}{2}$ **61.** -7
63. 4.5 **65.** 27 **67.** 38 **69.** -12 **71.** $-\frac{1}{5}$ **73.** 2
75. 36.8 ozf ≈ 10.2 N. **77.** 55.0 mph ≈ 1.48 km/min.
79. 160 **81.** 76 **83.** 3.6 hours or 3 hours 36 minutes
85. 35 kg of lead; 63.64% lead **87.** 1538.6 N on the left cable, 1107.4 N on the right cable.
89. 1.6 1b/2 gal is about 95.9 g/L.

Chapter 2 Test

1. $5x^2 - 3x$ **3.** 16.8 **5.** $2x + 8y - 6$ **7.** $4b^2 - 9$
9. $3x^3 + 2x - \frac{1}{2x}$ **11.** $\frac{y^2 - 13y - 8}{(3y + 2)(y - 1)}$
13. $\frac{14}{5}$ **15.** 1.8 qt **17.** 77.153 cm D = 3.175 cm

ANSWERS FOR CHAPTER 3

Exercise Set 3.1

1. $\frac{\pi}{12} \approx 0.2617994$ **3.** $\frac{7\pi}{6} \approx 3.6651914$
5. $\frac{427\pi}{900} \approx 1.4905112$ **7.** $\frac{109\pi}{120} \approx 2.8536133$
9. 240° **11.** 234° **13.** 14.323945° **15.** **(a)** 145° **(b)** $2.5307274 \approx \frac{29\pi}{36}$ **17.** 70° **19.** $x = 55°$, $y = 70°$ **21.** $x = y = 60°$ **23.** 36.87°
25. $A = 23.4$ units²; $P = 23.4$ units **27.** $A =$ 126 units²; $P = 84$ units **29.** $\frac{24}{5} = 4.8$ **31.** 47.5°
33. 120π rad/s **35.** 152° **37.** **(a)** 133.3 m **(b)** 478.5 m²
39. $\sqrt{200} = 10\sqrt{2}$ m ≈ 14.1421 m
41. $\sqrt{5869} \approx 76.6$ ft **43.** **(a)** 5261.54 mm and 8123.08 mm **(b)** 9230.77 mm
45. $\sqrt{119} \approx 10.9$ m **47.** **(a)** 212 m **(b)** 182.8 m
49. 73.4 m **51.** **(a)** Yes **(b)** about 41 ft² **53.** 31.9 ft

Exercise Set 3.2

1. $P = 60$ cm; $A = 225$ cm² **3.** $P = 96.5$ in; $A = 379.5$ in.² **5.** $P = 69$ in; $A = 343.6$ in.²
7. $P = 44$ cm; $A = 168$ cm² **9.** $P = 73.8$ mm; $A = 279.84$ mm² **11.** 444 ft² **13.** 10 000 mm²
15. $P = 3.031$ in; $A = 0.663$ in.² **17.** **(a)** $A = 750$ ft² **(b)** 94.76 ft **19.** 2.1875 in.² or 2.19 in.²
21. 18 638 mm² **23.** **(a)** 309 ft² **(b)** 2085.75 standard-size bricks **(c)** 1792.2 engineered/oversize bricks **(d)** 1390.5 economy-size bricks

Exercise Set 3.3

1. $A = 16\pi$ cm²; $C = 8\pi$ cm **3.** $A = 25\pi$ in.²; $C = 10\pi$ in **5.** $A = 201.64\pi$ mm²; $C = 28.4\pi$ mm
7. $A = 146.41\pi$ mm²; $C = 24.20\pi$ mm
9. **(a)** 576π in.² **(b)** $48\pi \approx 151$ in **11.** 32
13. **(a)** 163.2 mm **(b)** 6935.5 mm² **15.** **(a)** 86.643 in **(b)** 481.52 in.² **17.** **(a)** $A = 2529.876$ in.² **(b)** 197.1 in. **19.** 11.91 mm
21. **(a)** $3\frac{1}{7} - \pi \approx 0.00126$ **(b)** $\frac{3\frac{1}{7} - \pi}{\pi} \approx 0.040$
23. $2\frac{1}{4}$ in. **25.** ≈ 31 420 mm² **27.** **(a)** ≈ 30 670 mm² **(b)** ≈ 69 560 mm²

Exercise Set 3.4

1. 958.75 ft² **3.** 103.425 ft² **5.** 959.17 ft²
7. 104.62 ft² **9.** 333.6 ft² **11.** **(a)** Using the trapezoidal rule the area is approximately 4.54 in.². **(b)** Using Simpson's rule the area is about 4.63 in.². **13.** The area is about 79.4 cm².

Exercise Set 3.5

1. $L = 216\pi \approx 678.6$ in.^{2}, $T = 288\pi \approx 904.8$ in.^{2}, $V = 648\pi \approx 2035.8$ in.^{3} **3.** $L = 136\pi \approx 427.3$ mm^2, $T = 200\pi \approx 628.3$ mm^2, $V = 320\pi \approx 1005.3$ mm^3
5. $L = 735$ in.^{2}, $T = 816.8$ in.^{2}, $V = 859.1$ in.^{3}
7. A sphere has no lateral surface area, $T = 7256\pi \approx 804.26$ cm^2, $V = 682.7\pi \approx 2144.7$ cm^3
9. 5866.7 yd^3 **11.** **(a)** 3600 ft^3 **(b)** 1560 ft^2
13. **(a)** $103\,455\pi \approx 325\,013.5$ mm^3 **(b)** 20 172.79 mm^2 of paper **15.** $1333.3\pi \approx 4188.8$ mm^3
17. 136.1 yd^3 **19.** 57 172 mm^3 **21.** $66\pi \approx 207.3$ in.^{2}
23. 1208.23 m^3 **25.** $48\pi \approx 150.8$ ft^3
27. **(a)** rectangular prism **(b)** 643.5 cm^2 **(c)** 906 cm^2
29. **(a)** 173.36 m^2 **(b)** 693.44 L **31.** 46.6 m^2 × 4327 m = 201638 m^3 **33.** **(a)** 364.7 in.^{3} **(b)** Simpson's rule requires that n be an even number. Here $n = 11$, so we cannot use Simpson's rule.

Exercise Set 3.6

1. $a = 1.5$, $b = 3.75$ **3.** $a = 17$, $b = 42.71$, $c = 47.17$
5. $a = 2.55$, $b = 3.4$, $c = d = 4.25$ **7.** $x = 5.6$, $z = 25.2$ **9.** $a = 8.75$, $b = 14$ **11.** 184,320 in^2, 1280 ft^2 **13.** 5.5 kg **15.** 2143.75 L
17. 145.56 mm^3 **19.** **(a)** 50.27 cm^2 **(b)** 33.51 cm^3
21. width: 49.68″, height: 34.2″ **23.** sodium: 1963 mg, dietary fiber: 211.4 g

Review Exercises

1. $\frac{3}{20}\pi \approx 0.4712$ **3.** 198° **5.** 43° **9.** 3.3 **11.** 30°
13. 33.36 **15.** $A = 294$ units^2; $P = 73$ units
17. $A = 81\pi \approx 254.5$ units^2; $P = 18\pi \approx 56.55$ units
19. $A = 924$ units^2; $P = 154$ units
21. $A = 186.75$ units^2; $P = 67.1$ units
23. $L = 232.32\pi \approx 729.85$ units^2, $T = 525.14\pi \approx 1649.78$ units^2, $V = 1405.54\pi \approx 4415.62$ units^3
25. $L = 2320$ units^2, $T = 3920$ units^2, $V = 11{,}200$ units^3
27. $L = 520\pi \approx 1633.6$ units^2, $T = 930\pi \approx 2921.68$ units^2, $V = 2248.17\pi \approx 7062.84$ units^3
29. Spheres have no lateral surface area, $T = 324\pi \approx 1017.88$ sq units, $V = 972\pi \approx 3053.63$ units^3
31. 430 ft **33.** 160

Chapter 3 Test

1. $\frac{7\pi}{36}$ **3.** 104° **5.** $a = 15$, $b = 50$ **7.** $a = 1.5$ cm
9. $P = 42$ in. **11.** $A = 254.47$ in^2 **13.** $c \approx 12.84$
15. $V \approx 22{,}449.29750$ ft **17.** $x = 54$; $l = 162$ cm, $w = 108$ cm.

ANSWERS FOR CHAPTER 4

Exercise Set 4.1

1. Function **3.** Not a function **5.** Function for $x \geq 0$
7. Function for $x \geq \frac{7}{2}$ **9.** Yes, because for each value of x there is just one value of y.
11. No, because when $x = 4$ there are two values of y: -2 and 2, or when $x = 1$, y is 1 or -1.
13. Domain: $\{-3, -2, -1, 0, 1, 2\}$; Range: $\{-7, -5, -3, -1, 1, 3\}$ **15.** Domain: $\{-3, -2, -1, 0, 1, 2, 3\}$; Range: $\{-25, -6, 1, 2, 3, 10, 25\}$
17. Domain and range are both all real numbers.
19. Domain is all real numbers except 5; that is, the domain is $\{x : x \neq 5\}$. Range is all real numbers except 1; that is, the range is $\{y : y \neq 1\}$.
21. Domain is all nonnegative real numbers, $x \geq 0$. Range is all real numbers greater than or equal to -2; that is, the range is $\{y : y \geq -2\}$.
23. Domain is all real numbers except 2 and -3; that is, the domain is $\{x : x \neq 2, x \neq -3\}$.
25. Domain is all real numbers except -1 and 5; that is, the domain is $\{x : x \neq -1, x \neq 5\}$.
27. Domain is all real numbers except -4 and 2; that is, the domain is $\{x : x \neq -4, x \neq 2\}$.
29. Any function with a denominator of $x - 5$ will work as long as the numerator is defined for all real numbers. For example, $y = \dfrac{1}{x - 5}$ is one answer.
31. Any function with a denominator of $(x + 1)(x - 2)$ will work as long as the numerator is defined for all real numbers. For example,

$$y = \frac{1}{(x + 1)(x - 2)} \text{ is one answer.}$$

33. Any function with an even root will work if the quantity under the radical sign is positive. For example, $y = \sqrt{x - 5}$ is one answer.

35. Any function with $x - 4$ in the denominator and with an even root where the quantity under the radical sign is nonnegative when $x \geq -1$. For example, $y = \dfrac{\sqrt{x+1}}{x-4}$ is one answer.

37. -2 **39.** -11 **41.** -13 **43.** $3b + 7$ **45.** 0

47. -6 **49.** $-\frac{46}{25}$ **51.** $x^2 - 15x + 50$

53. $16m^4 - 20m^2$ **55.** 1 **57.** $\frac{2}{17} \approx 0.1176471$

59. $\dfrac{2 - 2x}{4x^2 + 2} = \dfrac{1 - x}{2x^2 + 1}$ **61.** 0 **63.** -40

65. Explicit **67.** Implicit. Solving for y we get an explicit form of $y = \dfrac{x}{1 - x^2}$. **69.** 7 **71.** -3

73. 4 **75.** 49 **77.** -23 **79.** -3 **81.** If $r = 20$, $V = 16{,}000\pi$ ft^3, $S = 2400\pi$ ft^2; if $r = 30$, $V = 36{,}000\pi$ ft^3, $S = 4200\pi$ ft^2 **83.** \$35

85. 4,992 lb/ft^2

Exercise Set 4.2

1. $y = x + 5$ **3.** $y = \dfrac{x + 8}{2} = \frac{1}{2}(x + 8) = \frac{1}{2}x + 4$

5. $y = 3x + 2$ **7.** $y = x^2 + 5$

9.

x	-3	-2	-1	0	1	2	3	4
$f(x)$	0	1	2	3	4	5	6	7
$g(x)$	-10	5	0	-1	5	1	-10	7
$(f+g)(x)$	-10	6	2	2	9	6	-4	14

11.

x	-3	-2	-1	0	1	2	3	4
$f(x)$	0	1	2	3	4	5	6	7
$g(x)$	-10	5	0	-1	5	1	-10	7
$(g-f)(x)$	-10	4	-2	-4	1	-4	-16	0

13.

x	-3	-2	-1	0	1	2	3	4
$f(x)$	0	1	2	3	4	5	6	7
$g(x)$	-10	5	0	-1	5	1	-10	7
$(f/g)(x)$	0	$\frac{1}{5}$	undefined	-3	$\frac{4}{5}$	5	$-\frac{3}{5}$	1

15.

x	-3	-2	-1	0	1	2	3	4
$g(x)$	-10	5	0	-1	5	1	-10	7
$(f \circ g)(x)$	-7	8	3	2	8	4	-7	10

17. The domain and range of f are all real numbers. The domain of g is $\{-3, -2, -1, 0, 1, 2, 3, 4\}$. The range of g is $\{-10, -1, 0, 1, 5, 7\}$. **19. (a)** The domain of f/g is $\{-3, -2, 0, 1, 2, 3, 4\}$. The range of f/g is $\{-3, -\frac{3}{5}, 0, \frac{1}{5}, \frac{4}{5}, 1, 5\}$. **(b)** The domain of g/f is $\{-2, -1, 0, 1, 2, 3, 4\}$. The range of g/f is $\{-\frac{5}{3}, -\frac{1}{3}, 0, \frac{1}{5}, 1, \frac{5}{4}, 5\}$.

21. $x^2 + 3x + 4$ **23.** $x^2 - 3x - 6$ **25.** $3x - x^2 + 6$

27. $3x^3 + 5x^2 - 3x - 5$ **29.** $\dfrac{x^2 - 1}{3x + 5}$ **31.** $\dfrac{3x + 5}{x^2 - 1}$

33. The domains of both f and g all real numbers

35. $9x^2 + 30x + 24$ **37.** $3x^2 + 2$ **39.** $3x^2 + 4x - 1$

41. $2x - 3x^2 - 1$ **43.** $3x^2 - 2x + 1$ **45.** $9x^3 - x$

47. $\dfrac{3x - 1}{3x^2 + x}$ **49.** $\dfrac{3x^2 + x}{3x - 1}$ **51.** $9x^2 + 3x - 1$

53. $27x^2 - 15x + 2$ **55.** The domains of both f and g are all real numbers.

57.

x	1	2	3	4	5	6	7	8
$f(x)$		5	2		7	4	5	9

Any value could be used for $f(1)$ and $f(4)$. Since neither 1 nor 4 are in the range of $gf(1)$ and $f(4)$ will not influence the results for $f \circ g$.

59. (a) $C(n) = 15n + 7500$ **(b)** \$9000 **(c)** \$22,000

61. (a) $P(n) = R(n) - C(N) = 60n - 275$ dollars **(b)** \$2725 **63.** $(P \circ n)(a) = P(n(a)) = 300(7a + 4)^2 - 50(7a + 4)$

65.

x	1	2	3	4	5	6	7
$g(x)$	6	5	7	5	4	4	1

$C(t) = 16{,}250 + 807{,}500\,t - 47{,}500\,t^2$

67. $A(t) = (A \circ V)(t) = 10^5\,[225\sqrt{t}]^{3/4}\,C^{3/4}$
$C = 16430\,(225)^{3/2} t^{3/8} \approx 954{,}496.746\,t^{3/8}$

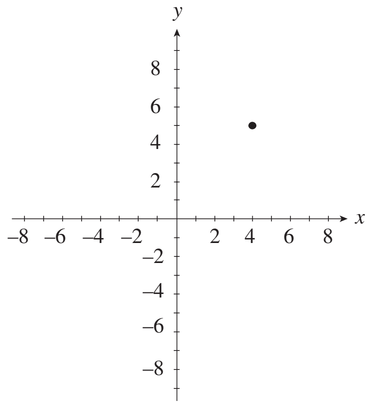
69. (a) $Q = \dfrac{1}{0.015 + 0.0014(0)} \times 1.2\text{ m}^2 \times \left(\dfrac{1.2\text{ m}^2}{4.65\text{ m}}\right)^{2/3} \times \sqrt{0.01} = 3.24\text{ m}^2/\text{s}$

(b) $Q = \dfrac{1}{0.015 + 0.0014(10)} \times \left(\dfrac{1.2\text{ m}^2}{4.65\text{ m}}\right)^{2/3} \times \sqrt{0.01} = 1.68\text{ m}^2/\text{s}$

(c) $Q = \dfrac{c_1}{0.015 + 0.0014(10)} AR^{2/3}\sqrt{s}$

Exercise Set 4.3

1.



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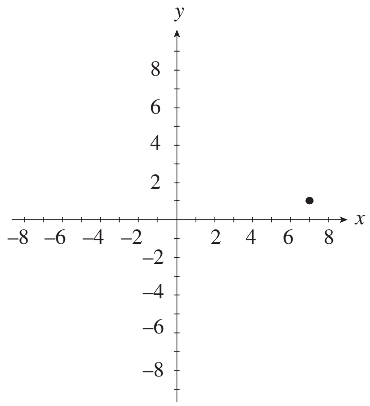
3.



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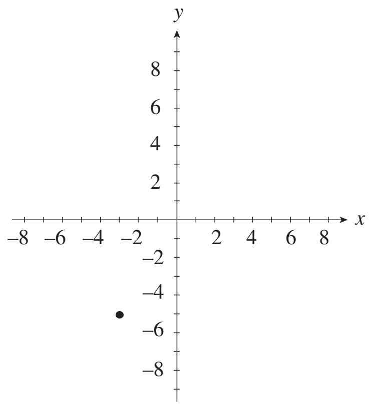

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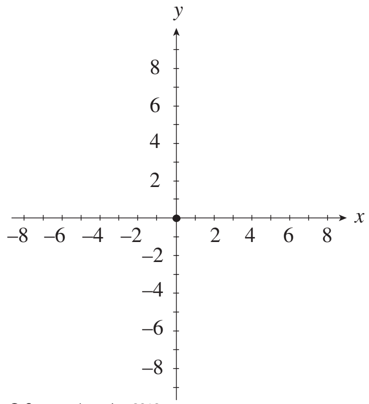
7.



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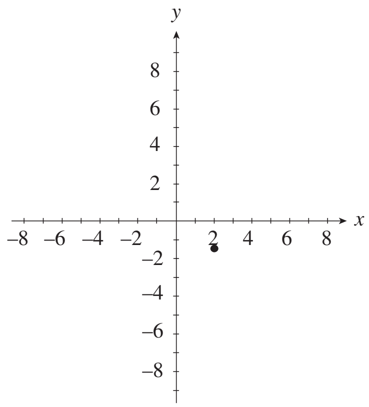
9.



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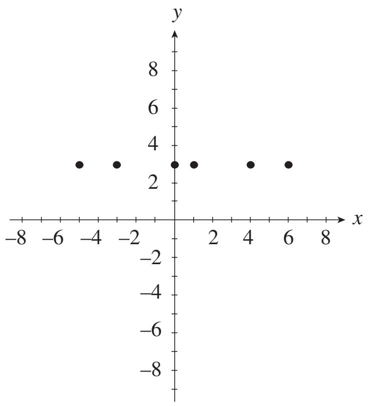
11. $(-1, -4)$

13. They are on a horizontal line and have the same y-coordinate.



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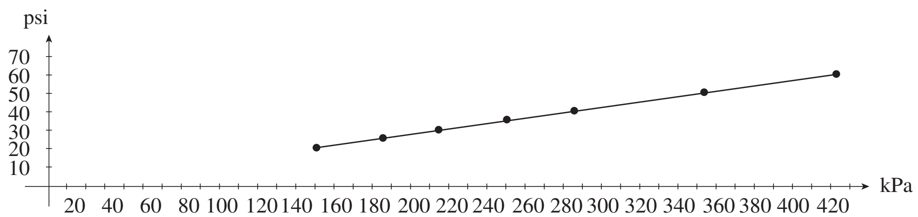
15. The points all lie on the same straight line.



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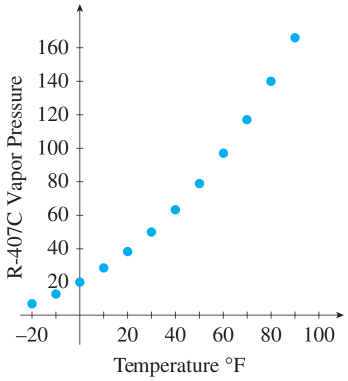
17. On the horizontal line through $y = -2$.

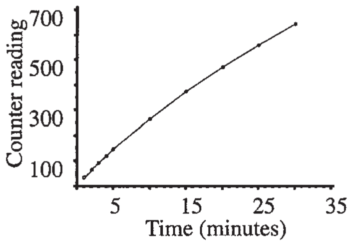
19. On the vertical line $x = -5$ **21.** To the right of the vertical line through $x = -3$. **23.** To the right of the line through the point $(1, 0)$ and below the horizontal line through the point $(0, -2)$.

25. **(a)**



(b) about 70.9 psi **(c)** about 166 psi

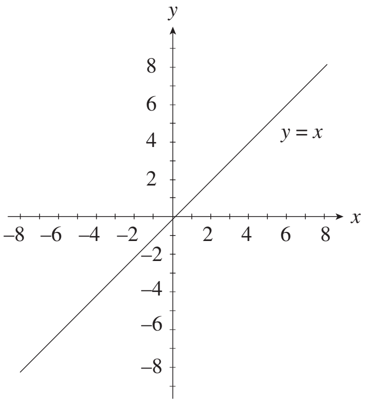
27. **(a)**



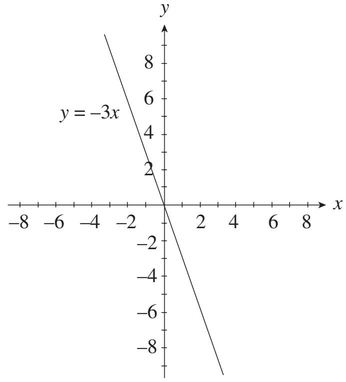
(b) Around 317 **(c)** Around 23.5 min

Exercise Set 4.4

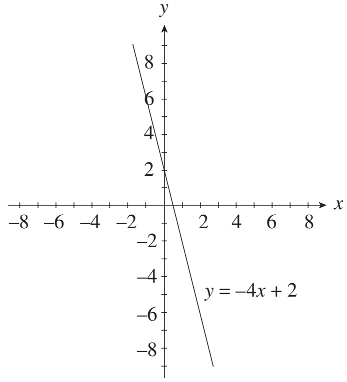
1. **(a)** x-intercept is 0; y-intercept is 0 **(b)** $m = 1$
(c)



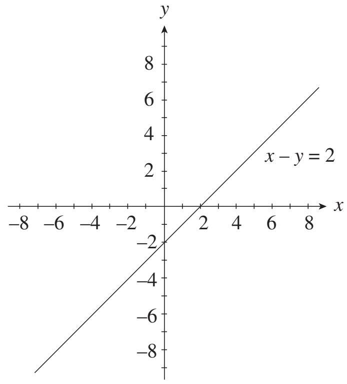
3. **(a)** x-intercept is 0; y-intercept is 0 **(b)** $m = -3$
(c)



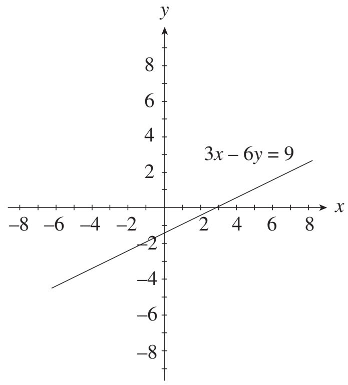
5. **(a)** x-intercept is $\frac{1}{2}$; y-intercept is 2 **(b)** $m = -4$
(c)



7. **(a)** x-intercept is 2; y-intercept is -2 **(b)** $m = 1$
(c)



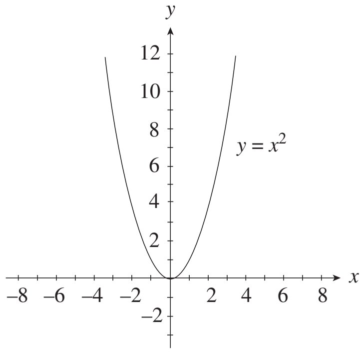
9. **(a)** x-intercept is 3; y-intercept is $-\frac{3}{2}$; **(b)** $m = \frac{1}{2}$
(c)



11. (a)

x	-3	-2	-1	0	1	2	3
y	9	4	1	0	1	4	9

(b)

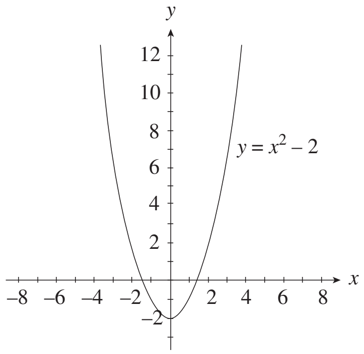


(c) Domain: all real numbers; range: $\{y : y \geq 0\}$

13. **(a)**

x	-3	-2	-1	0	1	2	3
y	7	2	-1	-2	-1	2	7

(b)

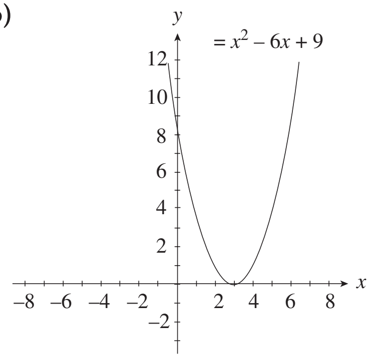


(c) Domain: all real numbers; range: $\{y : y \geq -2\}$

15. **(a)**

x	-3	-2	-1	0	1	2	3	4	5	6
y	36	25	16	9	4	1	0	1	4	9

(b)



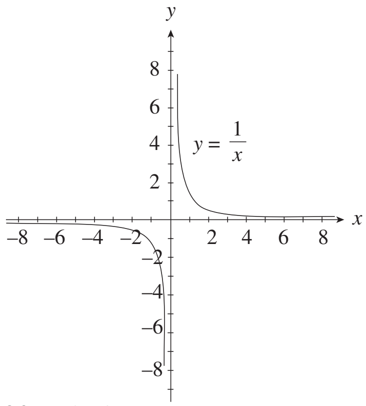
(c) Domain: all real numbers; range: $\{y : y \geq 0\}$

17. **(a)**

x	-3	-2	-1	$-\frac{1}{2}$	$-\frac{1}{4}$	0
y	$-\frac{1}{3}$	$-\frac{1}{2}$	-1	-2	-4	undefined

	$\frac{1}{4}$	$\frac{1}{2}$	1	2	3
	4	2	1	$\frac{1}{2}$	$\frac{1}{3}$

(b)

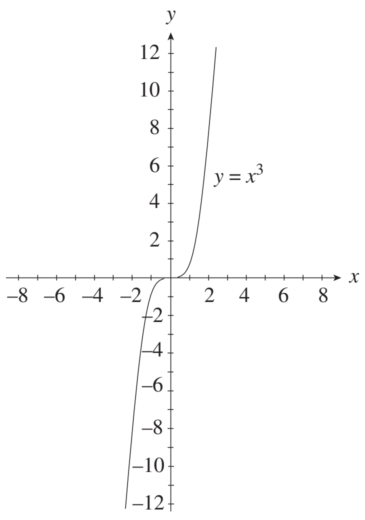


(c) Domain: $\{x : x \neq 0\}$; range: $\{y : y \neq 0\}$

19. **(a)**

x	-3	-2	-1	0	1	2	3
y	-27	-8	-1	0	1	8	27

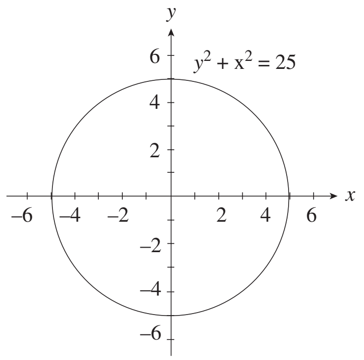
(b)



(c) Domain: all real numbers; range: all real numbers

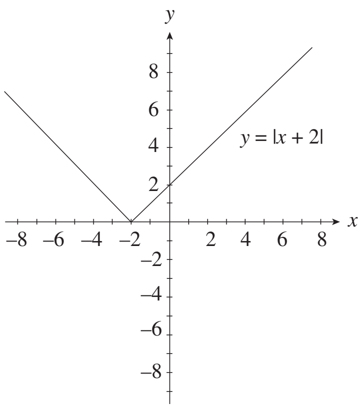
21.

x	0	0	5	-5	3	-3	3	-3	4	4	-4	-4
y	5	-5	0	0	4	4	-4	-4	3	-3	3	-3



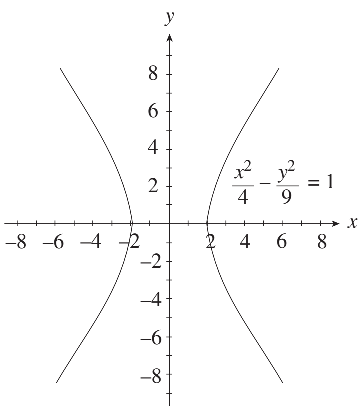
23.

x	-5	-4	-3	-2	-1	0	1	2
y	3	2	1	0	1	2	3	4



25.

x	0	1	-1	2	-2	3	-3	4	-4
y		Undefined	0	0	3.35	3.35	5.2	5.2	



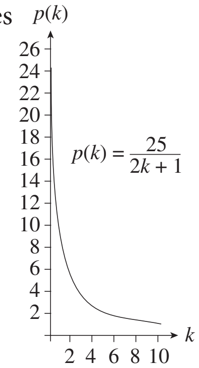
27. Function **29.** Function

31. **(a)**

k	0	1	2	3	4	5
$p(k)$	25.0	8.33	5.00	3.57	2.78	2.27

k	6	7	8	9
$p(k)$	1.92	1.67	1.47	1.31

(b) yes



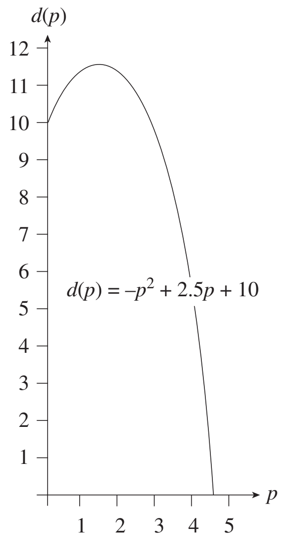
33. **(a)** $0 < p < \$4.65$

(b)

p	0.25	0.50	0.75	1.00
$d(p)$	10.56	11	11.31	11.50

p	1.25	1.50	1.75	2.00
$d(p)$	11.56	11.50	11.31	11

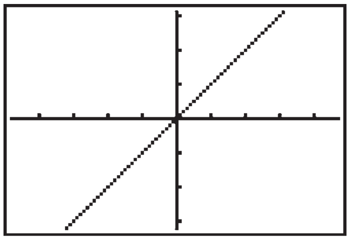
(c)



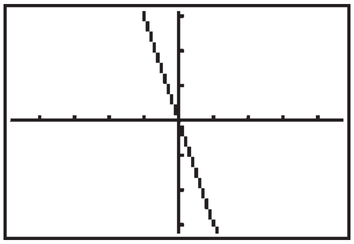
(d) \$1.25

Exercise Set 4.5

1.



3.



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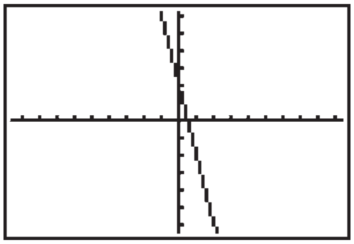

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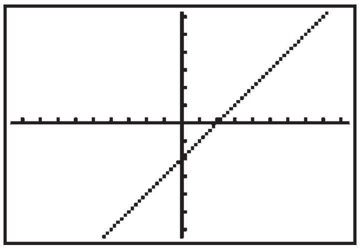
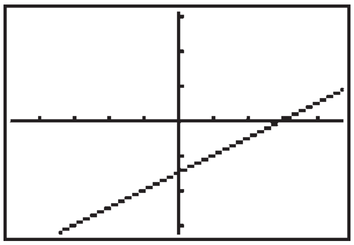
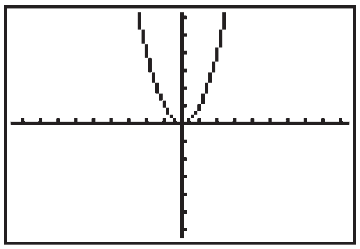


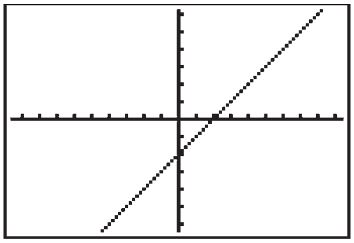
9.



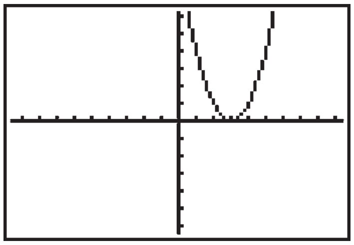
11.



13.



15.



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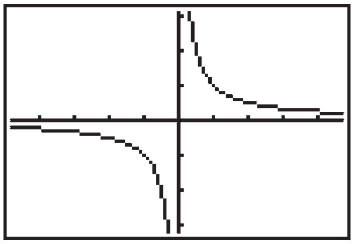
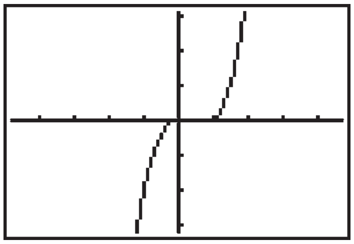


19.



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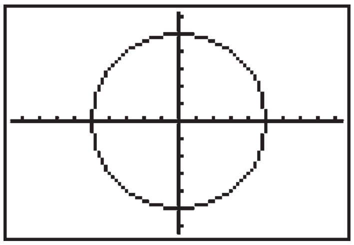
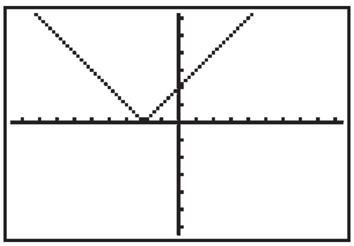
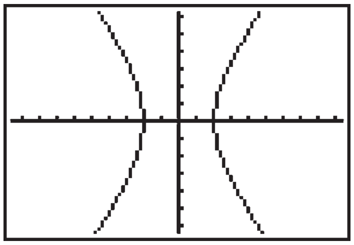


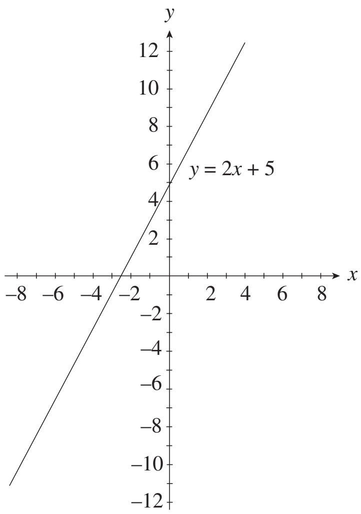
23.



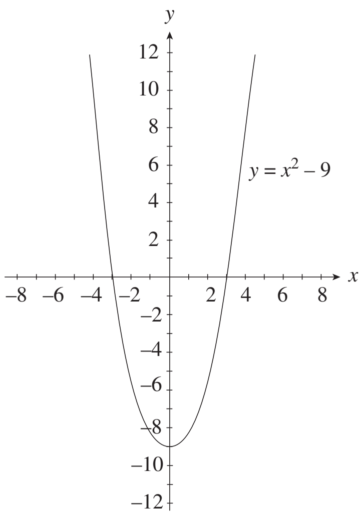
25.



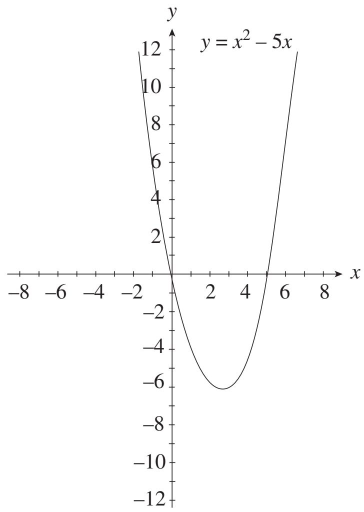
27. The root is $x = -\frac{5}{2}$



29. The roots are $x = -3$ and $x = 3$



31. The roots are $x = 0$ and $x = 5$



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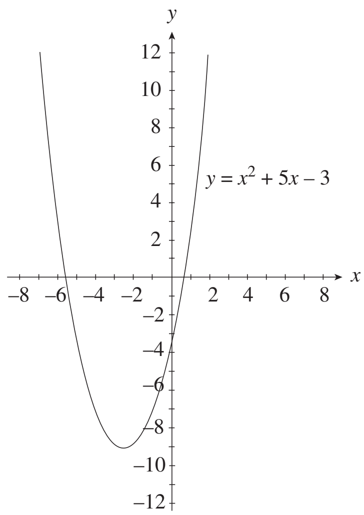
33. The roots are

$$x = -\frac{5}{2} - \frac{\sqrt{37}}{2} \approx -5.5414 \text{ and}$$

$$x = -\frac{5}{2} + \frac{\sqrt{37}}{2} \approx 0.5414$$



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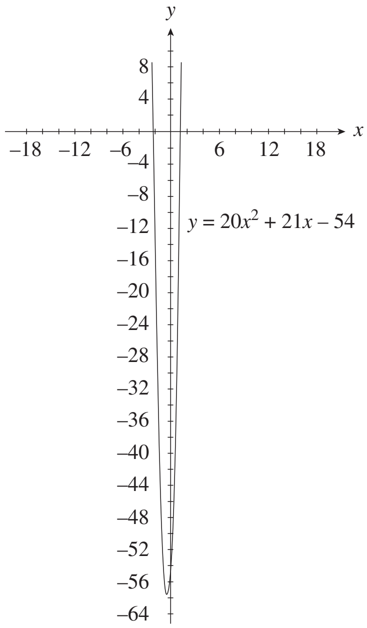
35. The roots are $x = -\frac{9}{4}$ and $x = \frac{6}{5}$



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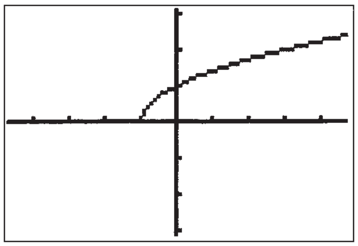
37. The root is $x = -1$.



$[-4.7, 4.7] \times [-3.1, 3.1]$

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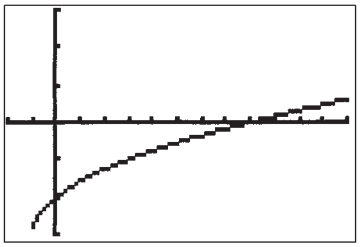
39. The root is $x = 8$.



$[-2, 12] \times [-3, 3]$

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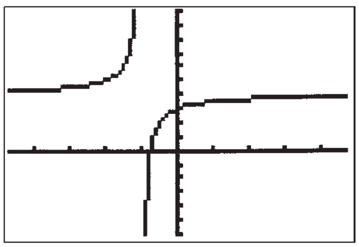
41. The root is $x = -0.75$.



$[-4.7, 4.7] \times [-6, 10]$

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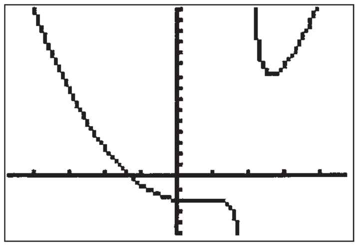
43. The root is $x \approx -1.2695$.



$[-4.7, 4.7] \times [-1, 15]$

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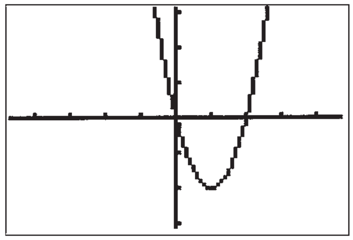
45. **(a)**



$[-9.4, 9.2, 2] \times [-6.2, 6.2, 2]$

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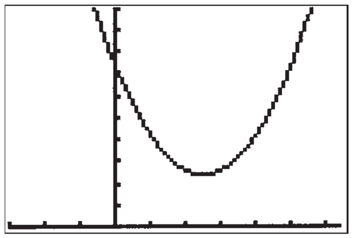
(b) Domain: all real numbers; range: $\{y : y \geq -4\}$

47. **(a)**



$[-6, 12.8, 2] \times [0, 50, 5]$

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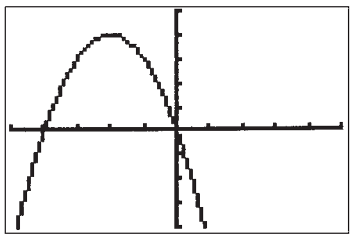
(b) Domain: all real numbers; range: $\{y : y \geq 12\}$

49. **(a)**



$[-5, 5, 1] \times [-4, 5, 1]$

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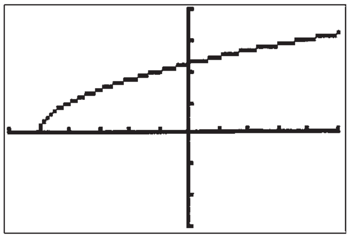
(b) Domain: all real numbers; range: $\{y : y \leq 4\}$

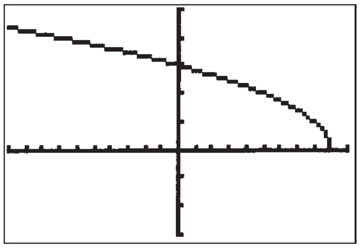
51. **(a)**



$[-6, 5, 1] \times [-3, 4, 1]$

(b) Domain: $\{x : x \geq -5\}$; range: $\{y : y \geq 0\}$

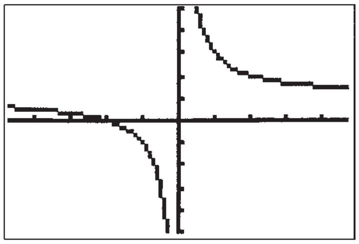
53. **(a)**



$[-10, 10, 1] \times [-3, 5, 1]$

(b) Domain: $\{x : x \leq 9\}$; range: $\{y : y \geq 0\}$

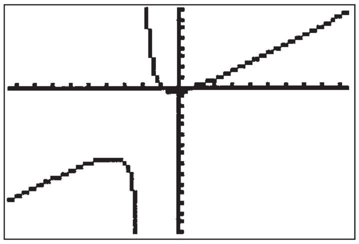
55. **(a)**



$[-4.7, 4.7, 1] \times [-5, 5, 1]$

(b) Domain: $\{x : x \neq 0\}$; range: $\{y : y \neq 1\}$

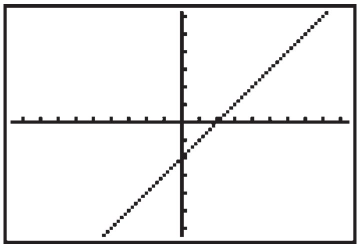
57. **(a)**



$[-9.4, 9.4, 1] \times [-15, 8, 1]$

(b) Domain: $\{x : x \neq -2\}$; range: $\{y : y < -7.47, y > -0.535\}$ (decimal values are approximate)

59. **(a)**



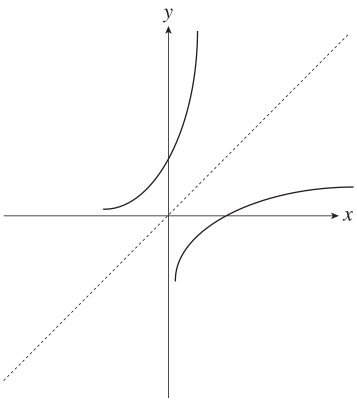
$[-9.4, 9.4, 1] \times [-6.2, 6.2, 1]$

(b) Domain: $\{x : x \neq -2\}$; range: $\{y : y \neq -4\}$

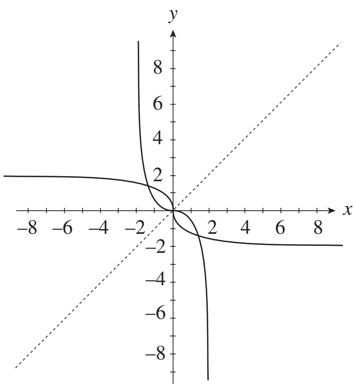
61. Has an inverse function

63. Does not have an inverse function

65.



67.

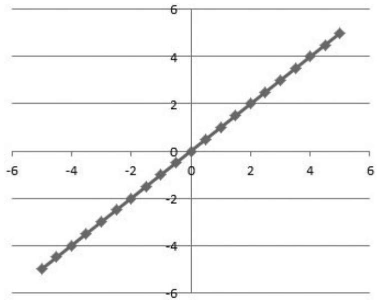


69. **(a)** 30% **(b)** 60% **(c)** 73%

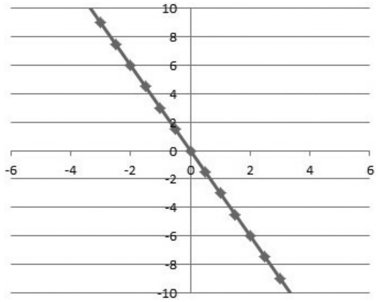
71. No

Exercise Set 4.6

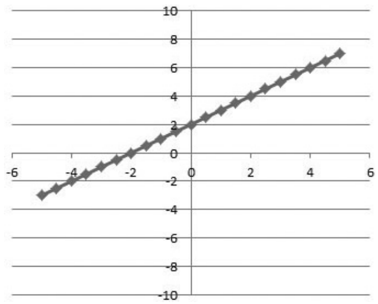
1.



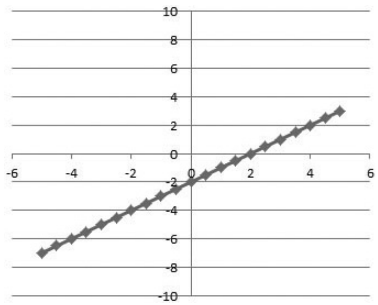
3.



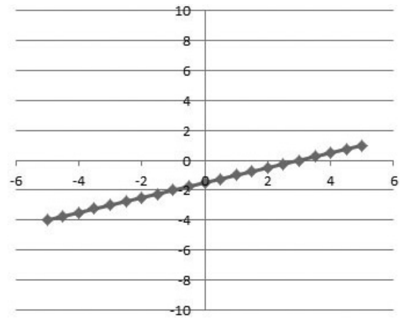
5.



7.



9.



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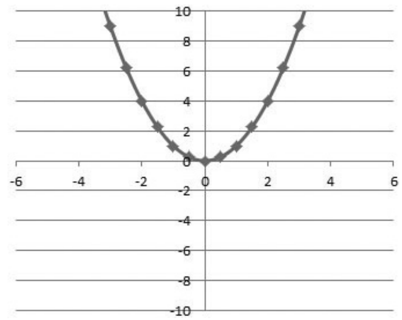
11.



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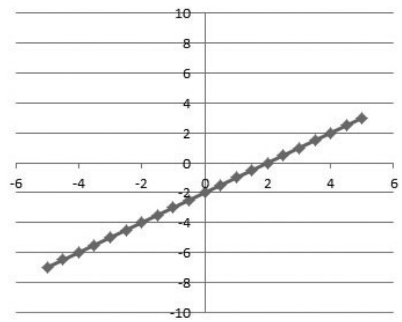
13.



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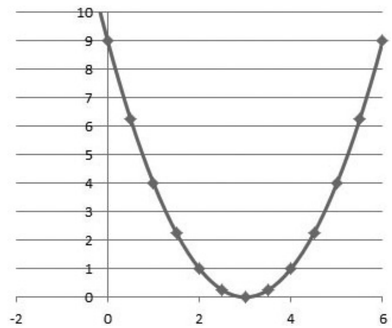
15.



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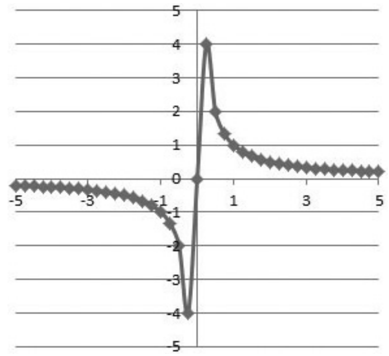
17.



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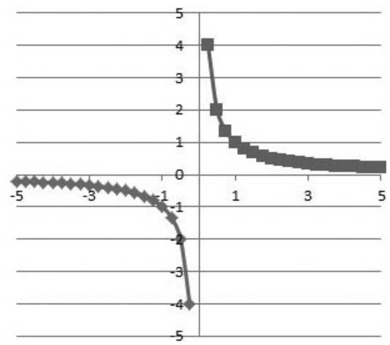
or



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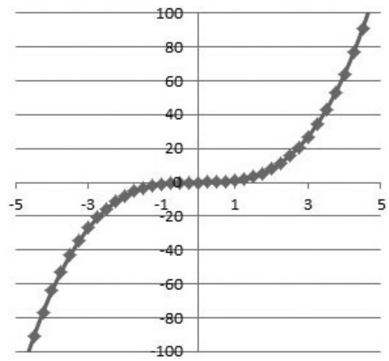
19.



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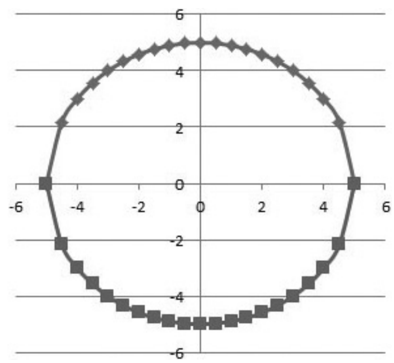
21.



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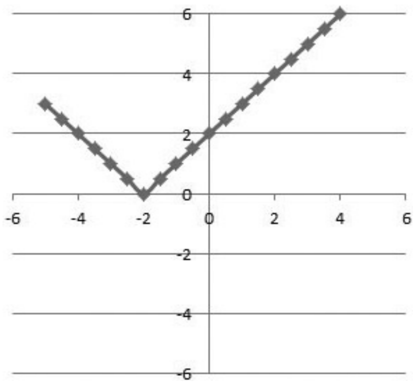
23.



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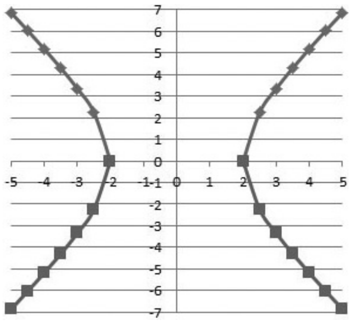
25.



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27. $-\frac{7}{2} = -3.5$ **29.** -4 and 4

31. $-\sqrt{5}$ and $\sqrt{5}$ **33.** -1 and 6

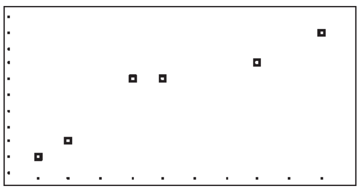
35. $\dfrac{-9 - \sqrt{171}}{10} \approx -2.208$ and $\dfrac{-9 + \sqrt{171}}{10} \approx 0.408$

37. -4 **39.** 3 **41.** 1.5

43. about 2.1409

Exercise Set 4.7

1. (a)



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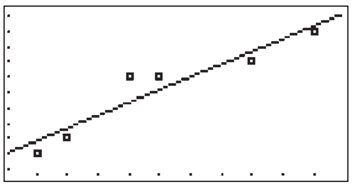
(b) $y = 0.833x - 1.333$

(c) $r \approx 0.943$

(d)



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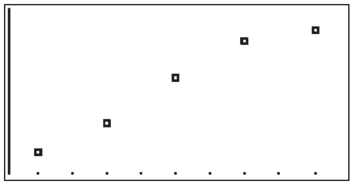
3. (a)



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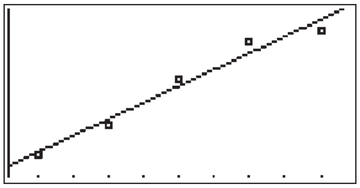
(b) $y = 13.55x + 5.1$

(c) $r \approx 0.986$

(d)



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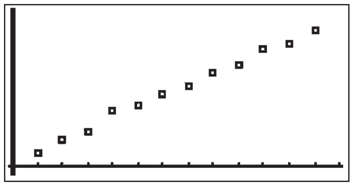
5. (a)



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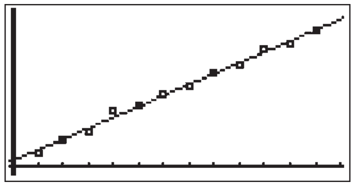
(b) $y = 14.955x + 5.379$

(c) $r \approx 0.996$

(d)



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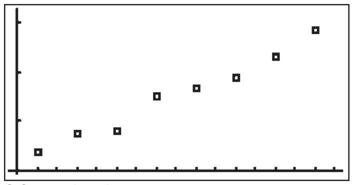
7. (a)



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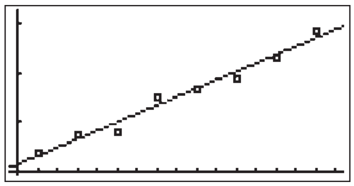
(b) $y = 0.000425x + 0.072$

(c) $r \approx 0.987$

(d)



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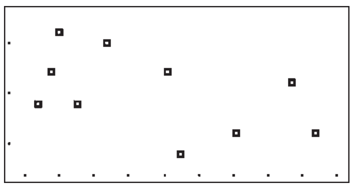
9. (a)



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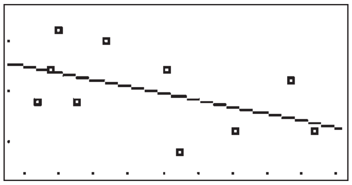
(b) $y = -0.0013x + 0.244$

(c) $r \approx -0.491$

(d)



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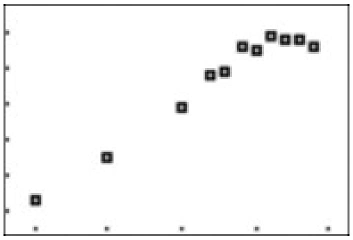
11. **(a)**



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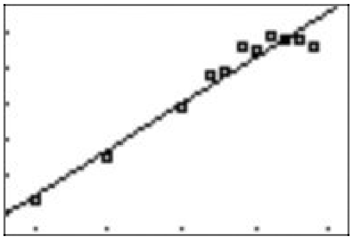
(b) $E(t) \approx 26.1672t - 1417.3603$ billion kWh t years after 1990 **(c)** $r \approx 0.97$

(d)



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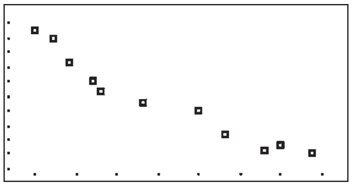
13. **(a)**



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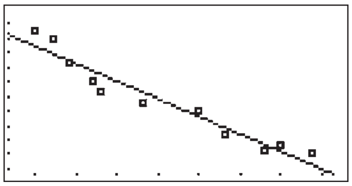
(b) $y = -4.830x + 48.111$

(c) $r \approx -0.966$

(d)



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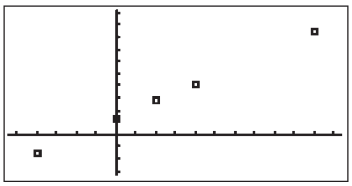
15. **(a)**



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(b) $F = 1.8C + 32$

(c) $m = 1.8$

(d) determine (0, 32)

(e)

Celsius	−40	−20	0	20	25	30	35	40	50	100
Fahrenheit	−40	−4	32	68	77	86	95	104	122	212

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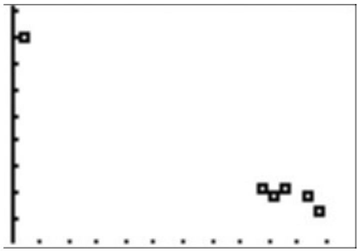
(f) The actual are identical to those in table.

17. **(a)**



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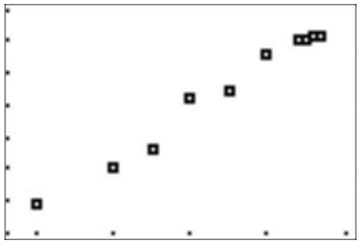
(b) $S(t) \approx -0.0641t + 28.1617$ seconds in year t after 1900 **(c)** -0.064 **(d)** 28.1617 **(e)** about 21.75 seconds in 2000 (actual winning time was 21.98 seconds); about 20.98 seconds in 2012

19. **(a)**



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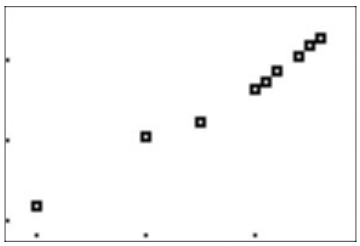
(b) $W(t) \approx 3.8133 - 148.9563$ million tons of municipal solid waste t years after 1900 **(c)** $r \approx 0.99$ **(d)** about 3.8133 **(e)** about -148.9563
(f) 262.9 million tons of municipal solid
(g) 249.6 million tons of municipal solid according to Table 373 of the 2011 *Statistical Abstracts of the United States.*

21. **(a)**



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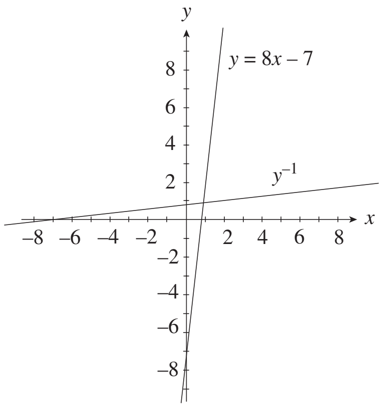
(b) $P(t) \approx 1.9538t - 102.2124$ quadrillion Btu t years after 1900 **(c)** 1.9538 **(d)** -102.2124
(e) 99.0 quadrillion Btu predicted for 2003
(f) 98.5 quadrillion Btu in 2003

Review Exercises

1. (a)



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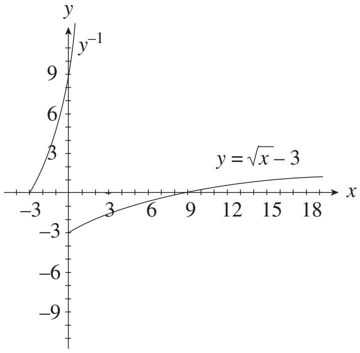
(b) Domain: all real numbers; Range: all real numbers; x-intercept $\frac{7}{8}$; y-intercept -7 **(c)** function **(d)** has an inverse function

3. (a)



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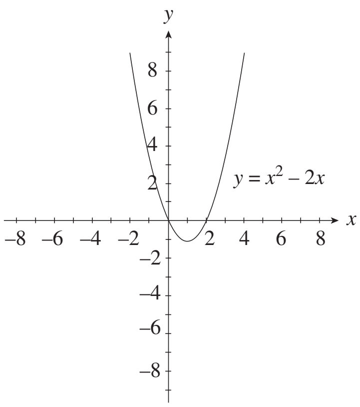
(b) Domain: all non negative real numbers, $x \geq 0$; Range: all real numbers greater than or equal to -3; x-intercept 9; y-intercept -9; **(c)** function **(d)** has an inverse function

5. (a)



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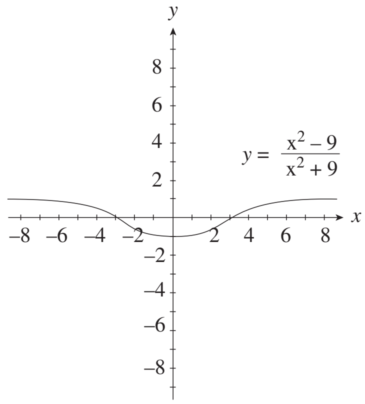
(b) Domain: all real numbers; Range: all real numbers greater than or equal to -1; x-intercepts 0, 2; y-intercept 0 **(c)** function **(d)** does not have an inverse function

7. -12 **9.** 0 **11.** $4a - 20$ **13.** -1

15. 0 **17.** $\frac{7}{25} = 0.28$

19.



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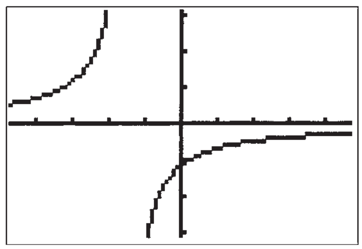
21. 0 **23.** $-20 + \frac{5}{13} = -\frac{255}{13}$

25. 12 **27.** 17 **29.** $\frac{5}{52}$

31. $-\frac{272}{25}$ **33.** -1

35.



$[-9.4, 9.4] \times [-6.2, 6.2]$

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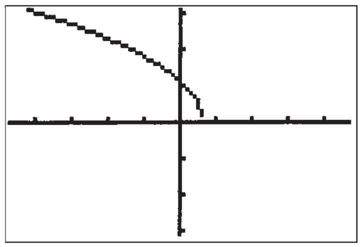
37.



$[-9.4, 9.4] \times [-6.2, 6.2]$

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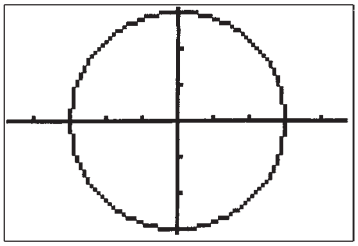
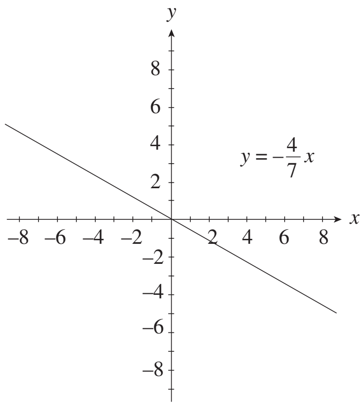
39.

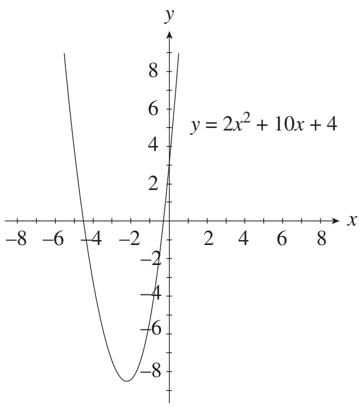


$[-4.7, 4.7] \times [-3.1, 3.1]$

41. $y = -\frac{4}{7}x$; root: $x = 0$



43. $y = 2x^2 + 10x + 4$; roots: $x = -4$ and $x = -1$



45. **(a)** \$29.95, \$17.50, \$15

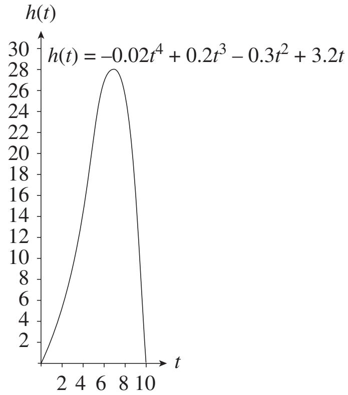
(b) $R(n) = \left(35 - \frac{n}{20}\right)n$

(c) \$3024.95, \$6125, \$6000

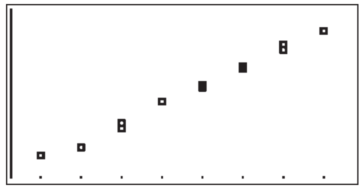
47.

t	0	1	2	3	4	5
$h(t)$	0	3.08	6.48	10.68	15.68	21.00

t	6	7	8	9	10
$h(t)$	25.68	28.28	26.88	19.08	2.00

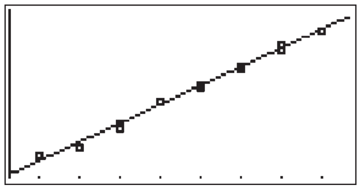


49. **(a)**



(b) The calculator gives a regression line $y \approx 43.9425837x + 18.9393939$, which rounds off to $y = 43.94x + 18.94$.

(c)



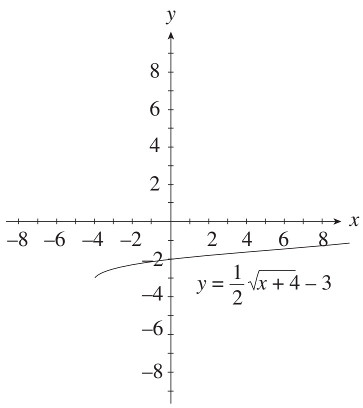
(d) The correlation coefficient is $r \approx 0.996487622$, which rounds off to $y \approx 0.996$.

Chapter 4 Test

1. -19

3. -5

5. **(a)**



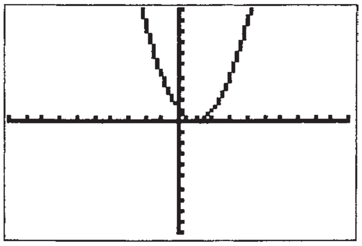
(b) $x \geq -4$ **(c)** $y \geq -2$ **(d)** $x = 32$ **(e)** $y = -2$

7. $\dfrac{3x^2 - x - 70}{x + 5}$

9. $3(x + 5) = 3x + 15$

11. $\dfrac{3x - 20}{3x - 10}$

13.



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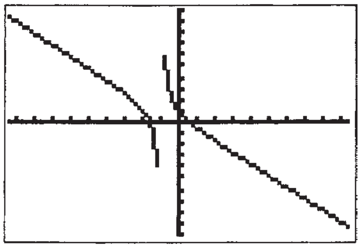
15.



$[-9.4, 9.4] \times [-10, 10]$
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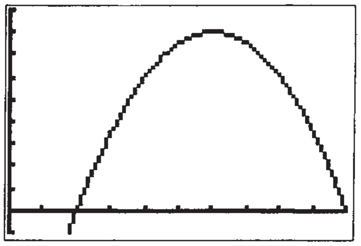
17. **(a)**



$[0, 10, 1] \times [-2, 18, 2]$
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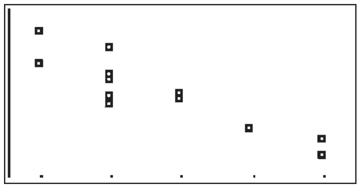
(b) 7 **(c)** 6 ms **(d)** either 4 ms or 8 ms

19. **(a)**



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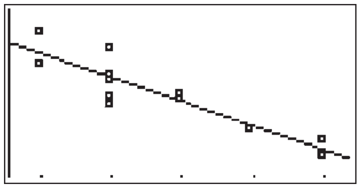
(b) $y = -39.774x + 1063.761$

(c)



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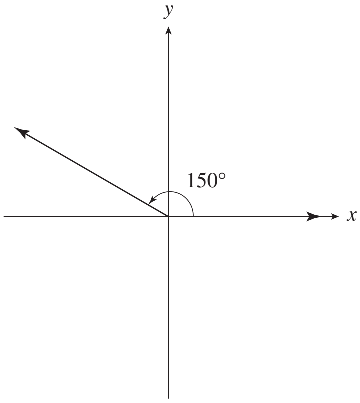
(d) $r = -0.900$

ANSWERS FOR CHAPTER 5

Exercise Set 5.1

1. 1.5705
3. 1.396
5. 2.70475
7. 3.75175
9. 114.592°
11. 85.944°
13. 60°
15. −45°

17. **(c)** 510°, −210°



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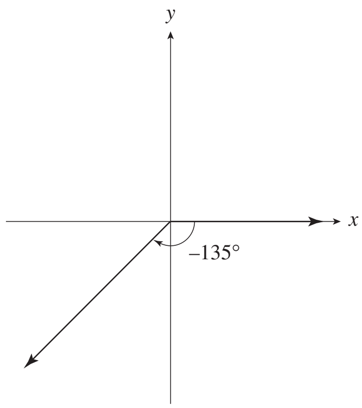
19. **(c)** −495°, 225°



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21. $\sin\theta = \frac{3}{5}$; $\cos\theta = \frac{4}{5}$; $\tan\theta = \frac{3}{4}$; $\csc\theta = \frac{5}{3}$; $\sec\theta\ \frac{5}{4}$; $\cot\theta = \frac{4}{3}$ **23.** $\sin\theta = \frac{-15}{17}$; $\cos\theta = \frac{8}{17}$; $\tan\theta = \frac{-15}{18}$; $\csc\theta = \frac{-17}{15}$; $\sec\theta = \frac{17}{8}$; $\cot\theta = \frac{-8}{15}$

25. $\sin\theta = \dfrac{2}{\sqrt{5}}$; $\cos\theta = \dfrac{1}{\sqrt{5}}$; $\tan\theta = 2$;

$\csc\theta = \dfrac{\sqrt{5}}{2}$; $\sec\theta = \sqrt{5}$; $\cot\theta = \frac{1}{2}$

27. $\sin\theta = -\dfrac{4}{\sqrt{41}}$; $\cos\theta = \dfrac{5}{\sqrt{41}}$; $\tan\theta = \frac{-4}{5}$; $\csc\theta = -\dfrac{\sqrt{41}}{4}$; $\sec\theta = \dfrac{\sqrt{41}}{5}$; $\cot\theta = \frac{-5}{4}$

29. $\sin\theta = \frac{5}{6}$; $\cos\theta = \dfrac{\sqrt{11}}{6}$; $\tan\theta = \dfrac{5}{\sqrt{11}}$; $\csc\theta = \frac{6}{5}$; $\sec\theta = \dfrac{6}{\sqrt{11}}$; $\cot\theta = \dfrac{\sqrt{11}}{5}$

31. $\sin\theta = 0$; $\cos\theta = 1$; $\tan\theta = 0$; $\csc\theta =$ Does not exist; $\sec\theta = 1$; $\cot\theta =$ Does not exist.

33. $\sin\theta = 0$; $\cos\theta = -1$; $\tan\theta = 0$; $\csc\theta =$ Does not exist; $\sec = -1$; $\cot\theta =$ Does not exist

35. $\sin\theta = \frac{4}{5}$; $\cos\theta = \frac{3}{5}$; $\tan\theta = \frac{4}{3}$; $\csc\theta = \frac{5}{4}$; $\sec\theta = \frac{5}{3}$; $\cot\theta = \frac{3}{4}$ **37.** $\sin\theta = \frac{21}{29}$; $\cos\theta = \frac{-20}{29}$; $\tan\theta = \frac{-21}{20}$; $\csc\theta = \frac{29}{21}$; $\sec\theta = \frac{-29}{20}$; $\cot\theta = \frac{-20}{21}$ **39.** $\sin\theta = -\dfrac{\sqrt{15}}{8}$; $\cos\theta = \frac{-7}{8}$; $\tan\theta = \dfrac{\sqrt{15}}{7}$; $\csc\theta = -\dfrac{8}{\sqrt{15}}$; $\sec\theta = \frac{-8}{7}$; $\cot\theta = \dfrac{7}{\sqrt{15}}$

Exercise Set 5.2

1. $\sin\theta = \frac{5}{13}$; $\cos\theta = \frac{12}{13}$; $\tan\theta = \frac{5}{12}$; $\csc\theta = \frac{13}{5}$; $\sec\theta = \frac{13}{12}$; $\cot\theta = \frac{12}{5}$ **3.** $\sin\theta = \frac{3}{5}$; $\cos\theta = \frac{4}{5}$; $\tan\theta = \frac{3}{4}$; $\csc\theta = \frac{5}{3}$; $\sec\theta = \frac{5}{4}$; $\cot\theta = \frac{4}{3}$

5. $\sin\theta = \dfrac{1.4}{\sqrt{7.25}}$; $\cos\theta = \dfrac{2.3}{\sqrt{7.25}}$; $\tan\theta = \frac{14}{23}$; $\csc\theta = \dfrac{\sqrt{7.25}}{1.4}$; $\sec\theta = \dfrac{\sqrt{7.25}}{2.3}$; $\cot\theta = \frac{23}{14}$ **7.** $\sin\theta = \frac{15}{17}$; $\cos\theta = \frac{8}{17}$; $\tan\theta = \frac{15}{8}$; $\csc\theta = \frac{17}{15}$; $\sec\theta = \frac{17}{8}$; $\cot\theta = \frac{8}{15}$ **9.** $\sin\theta = \frac{3}{5}$; $\cos\theta = \frac{4}{5}$; $\tan\theta = \frac{3}{4}$; $\csc\theta = \frac{5}{3}$; $\sec\theta = \frac{5}{4}$; $\cot\theta = \frac{4}{3}$ **11.** $\sin\theta = \frac{20}{29}$; $\cos\theta = \frac{21}{29}$; $\tan\theta = \frac{20}{21}$; $\csc\theta = \frac{29}{20}$; $\sec\theta = \frac{29}{21}$; $\cot\theta = \frac{21}{20}$ **13.** $\sin\theta = 0.866$; $\cos\theta = 0.5$; $\tan\theta = 1.732$; $\csc\theta = 1.155$; $\sec\theta = 2$; $\cot\theta = 0.577$ **15.** $\sin\theta = 0.085$; $\cos\theta = 0.996$; $\tan\theta = 0.085$; $\csc\theta = 11.765$; $\sec\theta = 1.004$; $\cot\theta = 11.718$ **17.** 0.3189593 **19.** 0.3307184 **21.** 4.1760011 **23.** 0.1298773 **25.** 0.247404 **27.** 0.7291147 **29.** 0.1417341 **31.** 4.7970857 **33.** 22.61 *m* **35.** 4.92 V

Exercise Set 5.3

1. 47.05° **3.** 77.92° **5.** 32.97° **7.** 73.00° **9.** 61.68° **11.** 12.12° **13.** $B = 73.5°$, $b = 24.6$, $c = 25.7$ **15.** $B = 17.4°$, $a = 19.1$, $b = 5.9$ **17.** $B = 47°$, $a = 32.3$, $c = 47.3$ **19.** $B = 0.65$, $b = 4.9$, $c = 8.2$ **21.** $B = 1.42$, $a = 2.7$, $b = 17.8$ **23.** $B = 0.16$, $a = 246.6$, $c = 249.8$ **25.** $A = 0.54$, $B = 1.03$, $c = 17.5$ **27.** $A = 1.03$, $B = 0.54$, $a = 15.4$ **29.** $A = 0.73$, $B = 0.84$, $b = 22.4$ **31.** 32° **33.** 1147.3 m **35.** $F_y = 5$ N, $F_x = 8.7$ N **37.** 9.62° **39.** 117.2 m from the intersection; 276.2 m from service station **41.** 61.56 ft **43.** 23.9°

Exercise Set 5.4

1. 87° **3.** 43° **5.** $\frac{\pi}{8}$ **7.** 1.36 **9.** 158° **11.** 209° **13.** 1.02 **15.** 4.11 **17.** Quadrant II **19.** IV **21.** III **23.** II **25.** I, II **27.** II, IV **29.** I, IV **31.** III, IV **33.** Quadrants I and II **35.** Quadrants I and IV **37.** IV **39.** positive **41.** positive **43.** negative **45.** negative **47.** 0.6820 **49.** −2.3559 **51.** −0.2830 **53.** 0.6755 **55.** −0.1853 **57.** 4.9131 **59.** 0.8241 **61.** −0.0454 **63.** 0.9989 **65.** −1.8479 **67.** 0.9090 **69.** 30.0°; 150.0° **71.** 153.4°; 333.4° **73.** 76.6°; 283.4° **75.** 189.4°; 350.6° **77.** 0.85; 2.29 **79.** 1.95; 5.09 **81.** 0.23; 2.91 **83.** 1.19; 5.09 **85.** 57.1° **87.** 76.6° **89.** 18.7° **91.** −32.6° **93.** 1.91 **95.** 1.28 rad **97.** 1.25 rad **99.** 0.24 rad **101.** 24.26 mA **103.** 440.46 mm **105.** 4,546 ft^2 **107.** −0.6898 rad ≈ −39.52° **109.** 43.57°

Exercise Set 5.5

1. 861.525 m **3.** 2261.9 in/m or 37.7 in/s **5.** 29.45 cm **7.** 220.80 cm^2 **9.** 0.168 m **11.** 24 776.5 mm/min. or 412.9 mm/sec; 24 776.5 mm **13.** 9.515°; 657.66 miles **15.** 83 (The actual answer is 83.77, but this would be 83 bytes.) **17.** 5100 **19.** 8.80 m/sec **21.** 7393.3 km **23.** 30° **25.** **(a)** $\frac{\pi}{21600}$ rad/sec ≈ 0.000 145 444 rad/s **(b)** about 2.175843801 mi/s **27.** **(a)** about 3.54″ **(b)** about 3.64″

29. $B = \sin^{-1}\left(\dfrac{r \sin A}{C}\right)$

31. (a) 13 ft-9 in.
(b) 22.5°

Review Exercises

1. $\frac{\pi}{3} \approx 1.047$ **3.** 5.672 **5.** -2.007 **7.** 135° **9.** 123.19° **11.** $-246.94°$ **13.** (a) I (b) 60° (c) 420° and $-300°$ **15.** (a) IV (b) 35° (c) 685° and $-35°$ **17.** (a) III (b) 65° (c) 245° and $-475°$ **19.** (a) II (b) $\frac{\pi}{4}$ (c) $\frac{11\pi}{4}$ and $\dfrac{-5\pi}{4}$ **21.** (a) II, (b) 0.99, (c) 8.43 and -4.13 **23.** (a) II, (b) 1.17, (c) 1.97 and -10.59 **25.** $\sin\theta = -4/5$; $\cos\theta = 3/5$; $\tan\theta = -4/3$; $\csc\theta = -5/4$; $\sec\theta = 5/3$; $\cot\theta = -3/4$ **27.** $\sin\theta = 21/29$, $\cos\theta = -20/29$, $\tan\theta = -21/20$, $\csc\theta = 29/21$, $\sec\theta = -29/20$, $\cot\theta = -20/21$ **29.** $\sin\theta = 1/\sqrt{50}$, $\cos\theta = 7/\sqrt{50}$, $\tan\theta = 1/7$, $\csc\theta = \sqrt{50}$, $\sec\theta = \sqrt{50}/7$, $\cot\theta = 7$ **31.** $\sin\theta = 8/17$, $\cos\theta = 15/17$, $\tan\theta = 8/15$, $\csc\theta = 17/8$, $\sec\theta = 17/15$, $\cot\theta = 15/8$ **33.** $\sin\theta = 4/5$, $\cos\theta = 3/5$, $\tan\theta = 4/3$, $\csc\theta = 5/4$, $\sec\theta = 5/3$, $\cot\theta = 3/4$ **35.** $\sin\theta = 84/91$, $\cos\theta = 35/91$, $\tan\theta = 84/35$, $\csc\theta|=|91/84$, $\sec\theta = 91/35$, $\cot\theta = 35/84$ **37.** $\cos\theta = 24/27.2$, $\tan\theta = 12.8/24$, $\csc\theta = 27.2/12.8$, $\sec\theta = 27.2/24$, $\cot\theta = 24/12.8$ **39.** $\sin\theta = 3.12/4$, $\cos\theta = 2.5/4$, $\tan\theta = 3.12; 2.5$, $\csc\theta = 4/3.12$, $\cot\theta = 2.5/3.12$ **41.** $\tan\theta = 0.577$, $\csc\theta = 2$, $\sec\theta = 1.155$, $\cot\theta = 1.732$ **43.** 0.7071 **45.** 0.6619 **47.** -0.6663 **49.** -0.0585 **51.** 30.0°; 150.0° **53.** 138.6°; 221.4° **55.** 2.09; 4.19 **57.** 2.38; 5.52 **59.** 60.0° **61.** $-45.0°$ **63.** 22.6° **65.** 22.1° **67.** 6.3 *A* **69.** $\theta = 41.3°$; $V = 66.6$ V **71.** 581.2 ft **73.** 66,705 miles/hr. **75.** 23.0° **77.** 204.2 feet **79.** $F_y = 2891.3$ lb; $F_x = 1972.3$ lb.

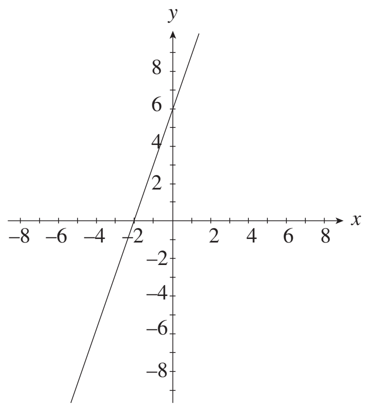
Chapter 5 Test

1. $\frac{5\pi}{18} \approx 0.8725$ **3.** **(a)** III **(b)** 57° **5.** **(a)** $\frac{9}{23.4} \approx 0.38462$ **(b)** $\frac{9}{21.6} \approx 0.41667$ **7.** **(a)** 0.79864 **(b)** 2.47509 **(c)** -1.66164 **(d)** -0.49026 **9.** 179.15 ft

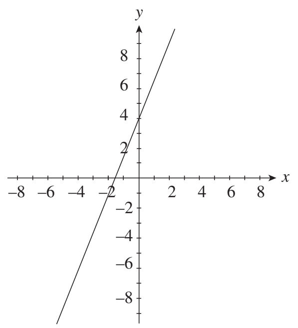
ANSWERS FOR CHAPTER 6

Exercise Set 6.1

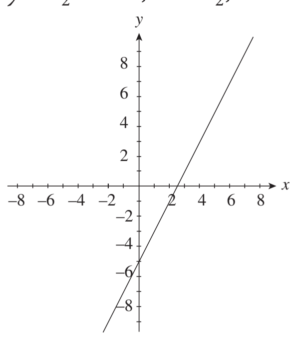
1. 3 **3.** $-\frac{5}{4}$ **5.** $\frac{10}{7}$ **7.** $m = 0$ **9.** Undefined **11.** $y - 3 = 4(x - 5)$ **13.** $y + 5 = \frac{2}{3}(x - 1)$ **15.** $y = -5$ **17.** $y = -\frac{5}{3}(x - 2)$ **19.** $x = 7$ **21.** $y - 2 = \frac{3}{4}(x + 3)$ or $y - 5 = \frac{3}{4}(x - 1)$ **23.** $y = 3$ **25.** $y \approx 0.5095x + 9.038$ **27.** $y \approx 1.9711x + 4393$ **29.** $y = 2x + 4$ **31.** $y = 5x - 3$ **33.** $y = 3x + 6$; $m = 3$, $b = 6$



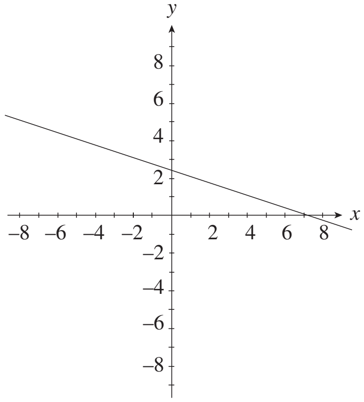
35. $y = \frac{5}{2}x + 4$; $m = \frac{5}{2}$, $b = 4$



37. $y = \frac{5}{2}x - 5$; $m = \frac{5}{2}$, $b = -5$



39. $y = -\frac{1}{3}x + \frac{7}{3}$; $m = -\frac{1}{3}$, $b = \frac{7}{3}$



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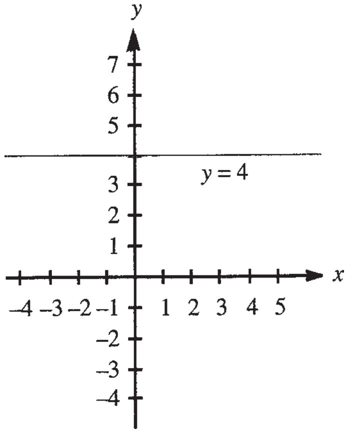
41. about $-19.29°$

43. about $32.01°$

45. **(a)** $m = 0$
(b)



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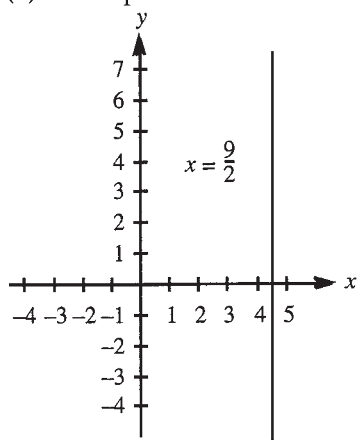
47. **(a)** The slope is undefined.
(b)



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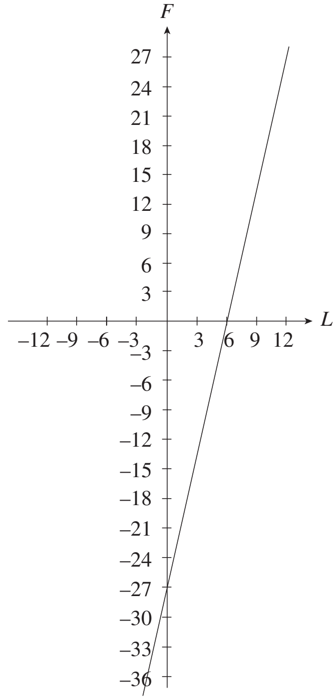
49. $m = \frac{1780\pi}{60} = \frac{89\pi}{3}$

51. **(a)** $F = kL - kL_0$ **(b)** $F = 4.5L - 27$
(c)



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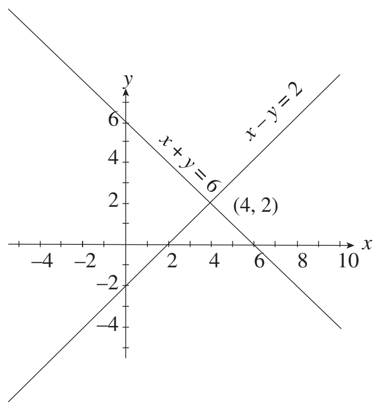
53. 9.89° toward the south

55. about 16.03°

57. **(a)** $P(t) \approx 0.0801t - 1.3257$ quadrillion British thermal units per year t years after 1900 **(b)** about 4.58°

Exercise Set 6.2

1. (4, 2)



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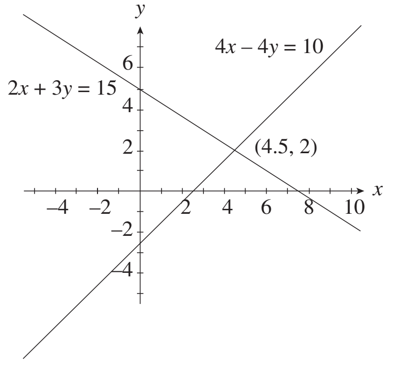
3. (4.5, 2)



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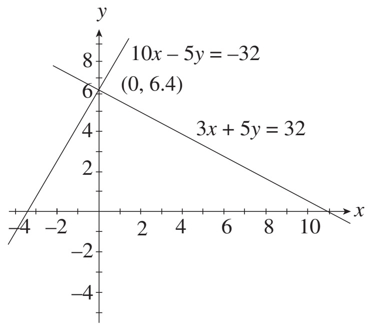
5. $\left(0, \frac{32}{5}\right) = (0, 6.4)$



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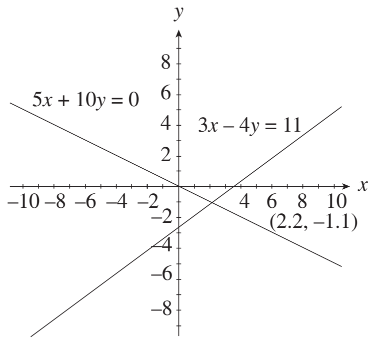
7. $(2.1, -1.1)$



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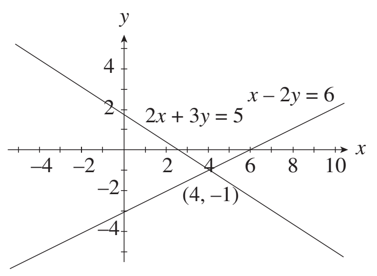
9. $(3, 5)$ **11.** $(-4, 6)$ **13.** $(8, -2)$ **15.** $(3.3, -1.2)$
17. $(2.6, -3.4)$ **19.** $(-2.1, 5.3)$ **21.** $(7, 2)$
23. $(-2, 1)$ **25.** $(-3, 3)$ **27.** $(1.5, -6.5)$
29. $(9.2, -4.6)$ **31.** $(4, -1)$



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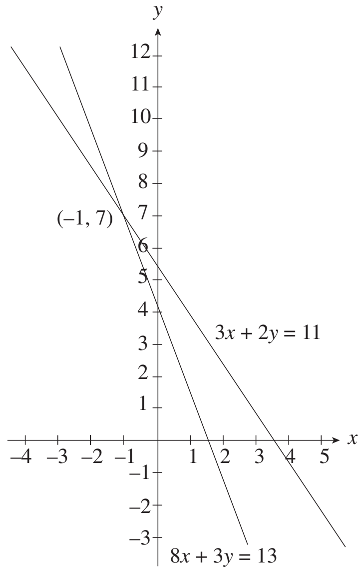
33. $(-1, 7)$



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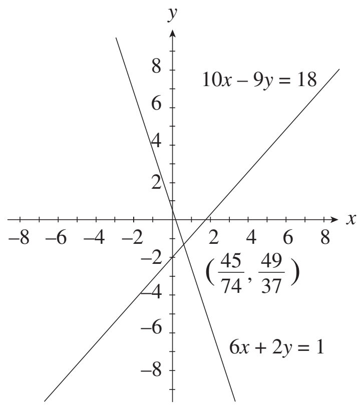
35. $(0.608, -1.324)$



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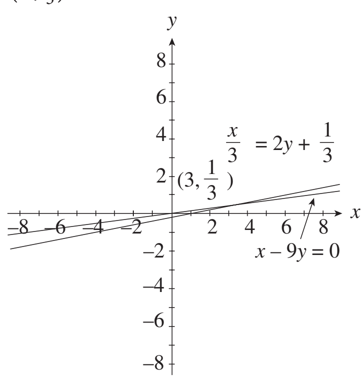
37. $\left(3, \frac{1}{3}\right)$



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39. $(3.1, -2.7)$ **41.** length = 13 km, width = 5 km
43. length = 16.875 km, width = 5.625 km
45. 6 250 L of 5%; 3 750 L of 13% gasohol
47. 6.4 kg, 4 m **49.** $I_1 = 1$ A, $I_2 = 1$ A, $I_3 = 0$ A
51. $I_1 = 0.224$ A, $I_2 = 1.052$ A, $I_3 = 0.828$ A

Exercise Set 6.3

1. $x = 2, y = -1, z = 4$ **3.** $x = -3, y = -2, z = 4$ **5.** $x = 2, y = -1, z = 4$ **7.** $x = -3, y = -2, z = 4$ **9.** $x = 1.3, y = 2.7, z = -2$ **11.** $x = 1.25, y = 3.75, z = -5.5$ **13.** $x = -6, y = 2, z = 5$ **15.** $I_1 = 0.875$ A, $I_2 = 2.125$ A, $I_3 = 1.25$ A **17.** $2a + 6b - c = 40$; $a + 7b - c = 50$; $a = -4$, $b = 6$, $c = -12$ **19.** 4 large trucks, 2 medium, 3 small **21.** 40 lb of the 10−12−15 fertilizer, 240 lb of the 10−0−5 fertilizer, and 120 lb of the 30−6−15 fertilizer

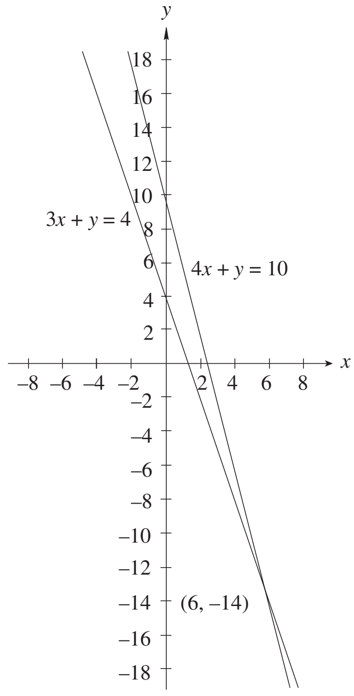
Exercise Set 6.4

1. -14 **3.** 13 **5.** 25 **7.** -10 **9.** (a) 2, (b) $\begin{vmatrix} 3 & -4 \\ -2 & 1 \end{vmatrix} = -5$, (c) 5 **11.** (a) 1 (b) -13 (c) 13 **13.** (a) -2 (b) -1 (c) -1 **15.** 0, rows 1 and 3 are identical. **17.** 0, row 3 is a multiple of row 1. **19.** 501 **21.** -120 **23.** 252.865 83 **25.** $-0.057\,525$ **27.** 0 **29.** Approximately -321.545 **31.** 0 **33.** $x = 1, y = 5$ **35.** $x = -2, y = 4$ **37.** $x = 3.522\,8, y = 1.316\,9$ **39.** no solution, inconsistent system **41.** $x = -7.876\,8, y = -17.689\,1$ **43.** no solution, inconsistent system **45.** $x = -2, y = 4$ **47.** $x = 1.282\,5, y = -1.015, z = 0.14$ **49.** $x = -2.1, y = 0.17, z = -4.35, w = 9.58$ **51.** $I_1 = \frac{-144}{-114} = \frac{24}{19}, I_2 = \frac{168}{-114} = -\frac{28}{19}, I_3 = \frac{-24}{-144} = \frac{4}{19}$. **53.** 18.18% metal A, 9.09% metal B, and 72.72% metal C **55.** $F_A = -1{,}000$ N, $F_B = 2{,}800$ N, $F_C = 600$ N **57.** $a = 0.05$, $b = 1$ and $c = 0$

Review Exercises

1. 27 **3.** -67 **5.** 10 **7.** -32 **9.** (a) $\frac{-2}{7}$ (b) $y - 1 = \frac{-2}{7}(x - 5)$ or $y - 3 = \frac{-2}{7}(x + 2)$ **11.** (a) $\frac{-5}{11}$ (b) $y - 4 = \frac{-5}{11}(x + 2)$ or $y + 1 = \frac{-5}{11}(x - 9)$ **13.** $y = 2x - 3$ **15.** $y = \frac{4}{5}x + \frac{8}{5}$ **17.** $y \approx 0.1809x - 7.7983$ **19.** 27.00°

21. $x = 6, y = -14$



23. $x = 3, y = -4$ **25.** $x = \frac{19}{11} \approx 1.727\,3$, $y = -\frac{1}{11} \approx -0.090\,9$ **27.** $x = 3, y = 5$ **29.** $x = 1, y = 2$ **31.** $x = 5.593\,75, y = 3.718\,75, z = -10.406\,25$ **33.** $x = 2, y = 1$ **35.** $x = \frac{57}{49} \approx 1.163\,3$, $y = \frac{-71}{49} \approx -1.449\,0$ **37.** $x = 2, y = -1, z = 1$ **39.** 30 pounds of Colombian Supreme and 20 pounds of Mocha Java **41.** 24 large, 32 middle size, and 10 small offices

Chapter 6 Test

1. (a) $\frac{4}{9}$ (b) $y - 2 = \frac{4}{9}(x + 4)$ or $y - 6 = \frac{4}{9}(x - 5)$ or, in slope-intercept form, $y = \frac{4}{9}x + \frac{34}{9}$ **3.** $\approx .5681x - 10.8404$ **5.** 34 **9.** $\left(\frac{5}{2}, \frac{3}{4}\right)$ **11.** $x_1 = 110, x_2 = -123, x_3 = -65$, and $x_4 = -51$ **13.** $I_1 = \frac{20}{11}$ A, $I_2 = \frac{25}{11}$ A, and $I_3 = -\frac{5}{11}$ A

ANSWERS FOR CHAPTER 7

Exercise Set 7.1

1. $3p + 3q$ **3.** $15x - 3xy$ **5.** $p^2 - q^2$
7. $4x^2 - 36p^2$ **9.** $r^2 + 2rw + w^2$ **11.** $4x^2 + 4xy + y^2$
13. $\frac{4}{9}x^2 + \frac{16}{3}xb + 16b^2$ **15.** $4p^2 - 3pr + \frac{9}{16}r^2$
17. $a^2 + 5a + 6$ **19.** $x^2 - 3x - 10$ **21.** $6a^2 + 5ab + b^2$
23. $6x^2 - 7x - 20$ **25.** $a^3 + 3a^2b + 3ab^2 + b^3$
27. $x^3 + 12x^2 + 48x + 64$ **29.** $8a^3 - 12a^2b + 6ab^2 - b^3$
31. $27x^3 - 54x^2y + 36xy^2 - 8y^3$ **33.** $m^3 + n^3$
35. $r^3 - t^3$ **37.** $8x^3 + b^3$ **39.** $27a^3 - d^3$
41. $3a^2 + 12a + 12$ **43.** $5rt + 2r^2 + \dfrac{r^3}{5t}$
45. $x^4 - 36$ **47.** $9x^4 - y^4$ **49.** 0
51. $x^3 - 3x^2 - 9x + 27$ **53.** $r^3 - 3r^2t + 3rt^2 - t^3$
55. $125 + 27x^3$ **57.** $(x + y)^2 - 2(x + y)(w + z) + (w + z)^2$ **59.** $(x + y)^2 - z^2 = x^2 + 2xy + y^2 - z^2$
61. $z^2 = R^2 + {x_L}^2 - 2x_Lx_C + {x_C}^2$
63. $a_c = \dfrac{(2t^2 - t)^2}{r} = \dfrac{4t^4 - 4t^3 + t^2}{r}$
65. $d = \left(\dfrac{P}{3EI}\right)(l_1^3 - l_2^3)$ **67.** $L = L_0 + \alpha L_0 T$

Exercise Set 7.2

1. $6(x + 1)$ **3.** $6(2a - 1)$ **5.** $2(2x - y + 4)$
7. $5(x^2 + 2x + 3)$ **9.** $5(2x^2 - 3)$ **11.** $2x(2x + 3)$
13. $7b(by + 4)$ **15.** $ax(3 + 6x - 2) = ax(1 + 6x)$
17. $2ap(2p + 3aq + 4q^2)$ **19.** $(a - b)(a + b)$
21. $(x - 2)(x + 2)$ **23.** $(y - 9)(y + 9)$
25. $(2x - 3)(2x + 3)$ **27.** $(3a^2 - b)(3a^2 + b)$
29. $(5a - 7b)(5a + 7b)$ **31.** $(12 - 5b^2)(12 + 5b^2)$
33. $5(a - 5)(a + 5)$ **35.** $7(2a - 3b^2)(2a + 3b^2)$
37. $(a - 3)(a + 3)(a^2 + 9)$
39. $16(x - 2y)(x + 2y)(x^2 + 4y^2)$ **41.** $2\pi r(r + h)$
43. $\frac{1}{2}d(v_2 - v_1)(v_2 + v_1)$ **45.** $\dfrac{(\omega_f - \omega_0)(\omega_f + \omega_0)}{2\theta}$
47. $W = \frac{1}{2}I(\omega_2 - \omega_1)(\omega_2 + \omega_1)$
49. $\Delta A = 45\pi\,(2r + 45\pi)$

Exercise Set 7.3

1. $b^2 - 4ac = 113$; $\sqrt{113} \approx 10.6$; does not factor using rational numbers **3.** $b^2 - 4ac = 196$; $\sqrt{196} = 14$; factors
5. $b^2 - 4ac = 169$; $\sqrt{169} = 13$; factors
7. $(x + 2)(x + 5)$ **9.** $(x - 3)(x - 9)$
11. $(x - 2)(x - 25)$ **13.** $(x - 2)(x + 1)$
15. $(x - 5)(x + 2)$ **17.** $(r + 5)^2$ **19.** $(a + 11)^2$
21. $(f - 15)^2$ **23.** $(6y - 1)(y - 1)$
25. $(7t + 2)(t + 1)$ **27.** $(7b + 1)(b - 5)$
29. $(4e - 1)(e + 5)$ **31.** $(3u + 4)(u + 2)$
33. $(9t + 2)(t - 3)$ **35.** $(3x - 1)(2x + 5)$
37. $(5a + 3)(3a - 5)$ **39.** $(5e + 3)(3e + 5)$
41. $(5x - 2)(2x - 3)$ **43.** $3(r - 7)(r + 1)$
45. $7t^2(7t - 1)(t - 2)$ **47.** $(3x + 2y)(2x - 5y)$
49. $(2a + b)(4a - 9b)$ **51.** $(a - b)(a^2 + ab + b^2)$
53. $(2x - 3)(4x^2 + 6x + 9)$
55. $i = 0.7(t^2 - 3t - 4) = 0.7(t - 4)(t + 1)$
57. $(0.009n^2 - 3)(n - 2000) = 0.0001(n^2 - 30{,}000)(n - 2000)$
59. **(a)** $4x^2 + 32x - 36 = 0$; **(b)** $4(x + 9)(x - 1)$
61. $V = x(18 - 2x)(18 - 1.5x)$

Exercise Set 7.4

1. $\frac{35}{40}$ **3.** $\dfrac{ax}{ay}$ **5.** $\dfrac{3ax^3y}{3a^2x}$ **7.** $\dfrac{4(x + y)}{x^2 - y^2} = \dfrac{4x + 4y}{x^2 - y^2}$
9. $\dfrac{(a + b)^2}{a^2 - b^2} = \dfrac{a^2 + 2ab + b^2}{a^2 - b^2}$ **11.** $\frac{19}{12}$
13. $\dfrac{x}{4}$ **15.** $\dfrac{4}{x - 3}$ **17.** $\dfrac{x - 4}{x + 4}$ **19.** $\dfrac{x}{3}$ **21.** $\dfrac{x + 3}{x^2 + 5}$
23. $\dfrac{2m - m^2}{3 + 6m^2}$ **25.** $\dfrac{x}{x - 3}$ **27.** $\dfrac{2b}{3(b + 5)}$
29. $\dfrac{z + 3}{z - 3}$ **31.** $\dfrac{x + 1}{x + 4}$ **33.** $\dfrac{2x + 1}{x + 5}$ **35.** $\dfrac{y(2y + 1)}{y - 1}$
37. $\dfrac{y^2 + xy + x^2}{2}$ **39.** $x - y$ **41.** $p = \dfrac{2wh}{s + 1}$
43. $\Delta V = \dfrac{V_1T_2}{T_1} + \dfrac{V_2T_1}{T_2} = \dfrac{V_1T_2^2 + V_2T_1^2}{T_1T_2}$

Exercise Set 7.5

1. $\dfrac{10}{xy}$ **3.** $\dfrac{12x^2}{5y^3}$ **5.** $\dfrac{3y}{7x}$ **7.** $\dfrac{8x^3}{21y}$ **9.** $\dfrac{5}{6xy}$ **11.** $\dfrac{5a^3d}{2b}$
13. $\dfrac{8y^2}{25x^2}$ **15.** $\dfrac{y^3}{5p^2}$ **17.** $4y$ **19.** $5(a - b)$

21. $\dfrac{3(x^2-100)}{4(x+5)}$ **23.** $\dfrac{(2x-1)(x+3)}{-3x}$

25. $a+2$ **27.** $\dfrac{1}{4a}$ **29.** $\dfrac{1}{x+1}$ **31.** $\dfrac{4}{y}$

33. $\dfrac{x-1}{3(x+2)}$ **35.** $\dfrac{(x-2y)(x-5y)(x-4y)}{(x+4y)(x+2y)(x-7y)}$

37. $\dfrac{3x-5}{x+3}$ **39.** $\dfrac{x+y}{x-y}$ **41.** $\dfrac{x^3+y^3}{4(x^3-y^3)}$

43. $\dfrac{x+5}{3x}$ **45.** $\dfrac{3}{x+3}$ **47.** $\dfrac{a^2+aa'+a'^2}{a(a+a')}$

49. $\dfrac{4\pi ne^2wv}{(v-w^2)(mv^2-\pi ne^2)}$

Exercise Set 7.6

1. 1 **3.** $\frac{2}{3}$ **5.** $\frac{5}{6}$ **7.** $\frac{2}{15}$ **9.** $\dfrac{6}{x}$ **11.** $\dfrac{1}{a}$ **13.** $\dfrac{5x}{y}$

15. $-\dfrac{3r}{2t}$ **17.** $\dfrac{3+x}{x+2}$ **19.** $\dfrac{t-2}{t+1}$ **21.** $\dfrac{2y}{x+2}$

23. $\dfrac{7}{a+b}$ **25.** $\dfrac{2y+3x}{xy}$ **27.** $\dfrac{ad-4b}{bd}$

29. $\dfrac{7x-3}{x(x^2-1)}$ **31.** $\dfrac{6-2x}{(x^2-1)(x+1)}$

33. $\dfrac{x^2+8x-10}{(x^2-36)(x-5)}$ **35.** $\dfrac{11-3x}{(x-3)(x^2-4)}$

37. $\dfrac{1-x-8x^2}{x(3x-1)(x-4)}$ **39.** $\dfrac{3x^2-8x-5}{(x^2-1)(x+4)}$

41. $\dfrac{6y+13-y^2}{(y-2)(y+1)(y+4)}$

43. $\dfrac{2x^4-x^3+13x^2+2x-4}{(x^2+3)(x-1)^2(x+2)}$ **45.** $\dfrac{x+2}{x-3}$

47. $\dfrac{x(x-1)}{x+1}$ **49.** $\dfrac{x^2-2xy-y^2}{x^2+y^2}$ **51.** $\dfrac{x+3}{x+2}$

53. $\dfrac{t(t-1)}{t^2+1}$ **55.** $\dfrac{x^2+y^2}{2x}$ **57.** $\dfrac{R_1+R_2}{R_1R_2}$

59. $\dfrac{C_1C_2+C_1C_3+C_2C_3}{C_1C_2C_3}$ **61.** $P=\dfrac{Ak^2r^2}{kr^2+L^2}$

63. $\dfrac{V_1R_2R_3+V_2R_1R_3+V_3R_1R_2}{R_2R_3+R_1R_3+R_1R_2}$

Review Exercises

1. $5x^2-5xy$ **3.** $x^3-6x^2y+12xy^2-8y^3$
5. $2x^2-9x-18$ **7.** x^4-25 **9.** $8+12x+6x^2+x^3$
11. $9(1+y)$ **13.** $7(x-3)(x+3)$ **15.** $(x-6)(x-5)$
17. $(x+8)(x-2)$ **19.** $(2x+3)(x-3)$

21. $\dfrac{x}{3y}$ **23.** $\dfrac{x-3}{x+3}$ **25.** $\dfrac{x^2-xy+y^2}{x+y}$ **27.** $\dfrac{3xy}{7}$

29. $\dfrac{4}{x}$ **31.** $\dfrac{7x}{y}$ **33.** $\dfrac{7x^2-27x+2}{(x+2)^2(x-5)^2}$

35. $\dfrac{-2x^2}{y^2-x^2}$ **37.** $\dfrac{2(x^2+36)(x+1)}{(x+2)(x^2-36)}$ **39.** $\dfrac{(x-6)^2}{(x+6)^2}$

41. $\dfrac{y-x}{y+x}$ **43.** -1 **45.** $2x^2-1$

Chapter 7 Test

1. $x^2+2x-15$ **3.** $-15x^3+6x^2+20x-8$
5. $2(x-8)(x+8)$ **7.** $(5x-7)(2x+3)$

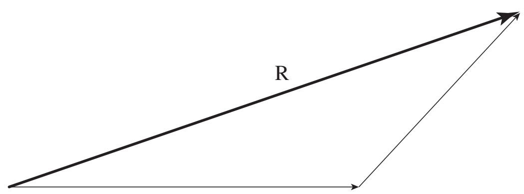
9. $\dfrac{x-5}{x+1}$ **11.** $\dfrac{3x(x-1)}{(x+2)^2}=\dfrac{3x^2-3x}{x^2+4x+4}$

13. $\dfrac{x^2-2x+6}{x-5}$ **15.** $\dfrac{(x+1)^2}{x(x+2)}$ **17.** $\dfrac{2r_1r_2}{r_1+r_2}$

ANSWERS FOR CHAPTER 8

Exercise Set 8.1

1.



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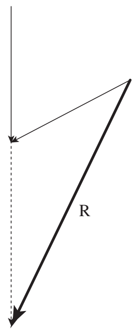

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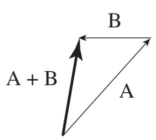

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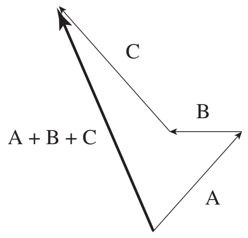

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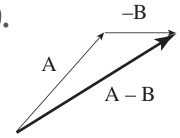

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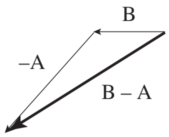

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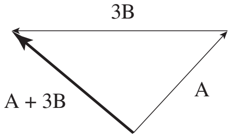

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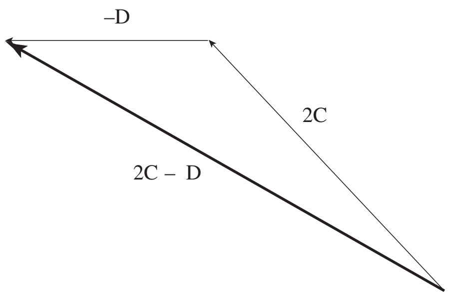
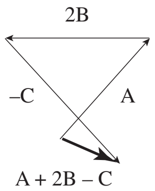


17.



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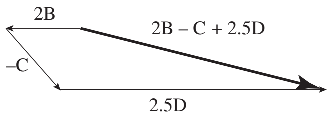


21. $\mathbf{P}_x = 5.176$; $\mathbf{P}_y = 19.319$

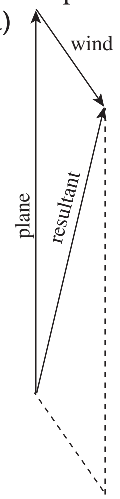
23. $\mathbf{P}_x = 4.688$; $\mathbf{P}_y = -17.793$

25. $\mathbf{V}_x = 9.372$; $\mathbf{V}_y = 2.687$ **27.** $A = 15$ **29.** $C = 17$

31. 13 km/h **33.** $\mathbf{V}_x \approx 295$ N, $\mathbf{V}_x \approx 930$ N

35. parallel to the ramp ≈ 12.8 lb, perpendicular to the ramp ≈ 152.5 lb **37.** $V = 17.5$ V, $\phi \approx 30.96°$

39. **(a)**



(b) velocity about 111.18 mph

41. **(a)** $20\sqrt{101} \approx 201$ lb **(b)** 5.7°

Exercise Set 8.2

1. $R = 12.5, \theta_R = 180°$ **3.** $R = 34.8, \theta_R = 270°$

5. $R = 73, \theta_R = 131.11°$ **7.** $-89.7, \theta_R = 24.79°$

9. $A = 65, \theta_A = 59.49°$ **11.** $C = 12.5, \theta_C = 20.61°$

13. $E = 6.5, \theta_E = 14.25°$ **15.** $G = 15.1, \theta_G = 56.31°$

17. $R = 12.33, \theta_R = 32.03°$ **19.** $R = 53, \theta_R = 178.11°$

21. $R = 88.50, \theta_R = 223.83°$ **23.** $R = 22.49, \theta_R = 349.18°$ **25.** $R = 13.04, \theta_R = 150.30°$

27. $R = 63.58, \theta_R = 4.6$ rad **29.** $R = 13.13, \theta_R = 64.38°$

31. $A = 1213$ N, $B = 1832$ N, $C = 2107$ N

33. Tension in the cable: 390.5 lb, compression in the boom: 311.9 lb

35. ground speed ≈ 488.3 mph, course ≈ 68.2°, drift angle ≈ 5.2°

37. 324.9 V **39.** **(a)** 46.8 mph **(b)** 220.1 mph

Exercise Set 8.3

1. $R = 114.50, \theta = 14.49°$ if 70-lb force has direction 0° **3.** **(a)** 20.48 lb **(b)** 14.34 lb **(c)** vertical = 16.07 horizontal = 19.15

5. 1616.8 N forward, 525.33 N sideward

7. 372.16 mi/hr in the compass direction 173.83°

9. 1112.62 kg perpendicular, 449.53 kg parallel

11. 119.11 lb horizontal, 154.40 lb vertical

13. 6 A, 38.66° **15.** 10.68 A, −59.46°

17. 31.24 A, $\theta = -39.81°$ **19.** 55.84, $\theta = 6.52°$

21. 126.24 m/s at an angle of −18.09°

23. 43.1 in. × 13.5 in. or $3'7\frac{1}{10}'' \times 1'1\frac{1}{2}''$

Exercise Set 8.4

1. $a = 7.59, b = 22.77, C = 75.3°$ **3.** $A = 67.1°, C = 15.9°, c = 4.22$ **5.** $C = 73.31°, B = 20.37, b = 6.69$ **7.** no solution, not a triangle

9. $B = 57.31°, C = 77.69°, c = 22.5$; or $B = 122.69, C = 12.31°, c = 4.91$ **11.** $A = 148.8°, b = 20.17, c = 23.74$ **13.** $B = 64.62°, A = 79.78°, a = 21.13$; or $B = 115.38°, A = 29.02°, a = 10.42$

15. no solution **17.** $B = 0.44, A = 1.33, a = 19.52$

19. $B = 1.35, a = 90.67, c = 193.98$

21. From B to C is 305.5 m; From A to C is 369.6 m

23. 34.3 miles **25.** 294.77 m **27.** 444 ft 4 in.

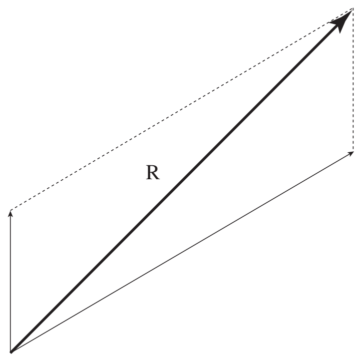
29. 8.49 cm **31.** 79.50 mm

Exercise Set 8.5

1. $c = 11.31, A = 33.6°, B = 104.1°$
3. $b = 68.02, A = 40.1°, C = 26.2°$
5. $A = 45°, B = 23.2°, C = 111.8°$
7. $b = 103.25, A = 38.16°, C = 49.40°$
9. $A = 30.34°, B = 44.11°, C = 105.55°$
11. $c = 11.77, A = 0.56, B = 1.76$
13. $A = 0.70$ rad, $b = 74.52$, $C = 0.46$ rad
15. $A = 0.96, B = 0.59, C = 1.59$
17. $b = 68.56$, $A = 0.925$ rad, $C = 0.590$ rad
19. $A = 1.59$ rad, $B = 0.83$ rad, $C = 0.72$ rad
21. 5.61 in^2 **23.** 513.1 m^2 **25.** 10 426 km
27. 92.25 ft **29.** 30.98 N, $\theta = 39.36°$
31. 76 kΩ **33.** downhill cable: 252.3 ft, uphill cable: 219.7 ft **35.** 27,912 ft^2 $\approx$ 0.64 acre

Review Exercises

1.



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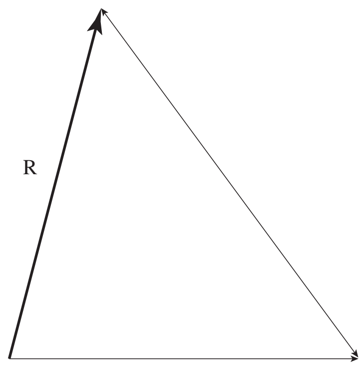
3.



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5. $P_x = -13.68, P_y = 32.22$
7. $P_x = -23.29, P_y = 3.09$
9. $R = 17.89, \theta_R = 333.43°$
11. $A_x = 36.71, A_y = 9.84$
13. $C_x = 6.12, C_y = 18.41$ **15.** $R = 50.41, \theta_R = 20.69°$
17. $R = 72.76$, $\theta_R = 3.13$ rad **19.** $A = 25.7°$, $B = 97.5°$, $C = 56.8°$ **21.** $A = 145.93°$, $a = 146.17$, $C = 14.47°$ **23.** $B = 0.12$, $C = 2.90$, $c = 258.06$ **25.** $A = 0.52$, $B = 0.41$, $c = 109.27$
27. 2 831.175 kg at 83.97° **29.** 66.47 lb. parallel; 107.53 lb. perpendicular **31.** 195.47 m

Chapter 8 Test

1. $V_x = -21.34, V_y = 41.88$ **3.** $R = 59.69$, $\theta_R = 93.86°$ **5.** $a = 9.38$ **7.** 49.59 in.

ANSWERS FOR CHAPTER 9

Exercise Set 9.1

1. 0.3 **3.** 10 **5.** $-\frac{1}{7}$ **7.** 9 **9.** -3 **11.** $\frac{15}{8}$
13. $\frac{9}{2}$ **15.** 15 **17.** -9 **19.** $\frac{2}{5}$ **21.** 1 **23.** $\frac{30}{37}$
25. $-\frac{3}{7}$ **27.** -6 **29.** 1 **31.** $\dfrac{rt}{r-t}$
33. $\dfrac{R_1R_2R_3}{R_1R_2 + R_1R_3 + R_2R_3}$ **35.** $\dfrac{V - 2\pi r^2}{2\pi r}$
37. $\dfrac{R_1 f(n-1)}{R_1 - f(n-1)}$ **39.** $C = \dfrac{dR_1R_2}{(9 \times 10^9)(R_1 - R_2)}$
41. $f = \dfrac{r_1r_2}{(n-1)(r_2 - r_1)}$ **43.** $D = d + \dfrac{h}{2}$
45. $3\frac{3}{4}$ h or 3 h 45 min **47.** 3 hours
49. $h = \frac{28}{5} = 5$ hours 36 minutes
51. 12 h **53.** $f_a = \dfrac{f(V + V_L)}{V - V_s}$
55. $R_1 = \dfrac{R_2R_3(V_1 - V)}{VR_3 + VR_2 - V_2R_3 - V_3R_2}$

Exercise Set 9.2

1. $R = kl$ **3.** $A = kd^2$ **5.** $IN = k$
7. $r = kt, k = \frac{2}{3}$ **9.** $d = kr^2, k = \frac{1}{2}$
11. 9 **13.** 180 m/s **15.** 435.6 neutrons

17. **(a)** 0.085 **(b)** 30.55 m^3 **19.** 106.67 Ω **21.** 160 psi **23.** 0.075 A **25.** 142.22 lb **27.** $m = 16$ kg

Exercise Set 9.3

1. $K = kmv^2$ **3.** $f = \dfrac{kv}{l}$ **5.** $E = \dfrac{klI}{d^2}$

7. $a = kbc$; $k = 2.5$ **9.** $u = \dfrac{kv}{w}$; $k = 8$

11. $r = kst$; $k = \frac{1}{3}$, $r = 4$ **13.** $a = kxy^2$; $k = 0.12$, $a = 162$ **15.** $p = \dfrac{kr\sqrt{t}}{4}$; $k = 1$, $p = 36$

17. **(a)** $A = kbh$; $k = \frac{1}{2}$ **(b)** $A = 25$ **19.** 22.36 V

21. 17.14 m **23.** 77.5 Btu/hr **25.** 1.25 ft^3

Exercise Set 9.4

1. ± 3 **3.** $-3, 2$ **5.** $-1, 12$ **7.** $2, -4$ **9.** $0, 5$

11. $3, 4$ **13.** $-2, \frac{7}{2}$ **15.** $\frac{3}{2}, 4$ **17.** $3, -\frac{1}{3}$ **19.** $\frac{5}{2}, \frac{7}{2}$

21. $\frac{5}{3}, -\frac{7}{2}$ **23.** $\frac{1}{5}, \frac{3}{2}$ **25.** $-\frac{3}{2}, \frac{20}{3}$ **27.** $-1, 3$

29. $\frac{6}{5}, -\frac{2}{5}$ **31.** $2, -\frac{1}{2}$ **33.** $-12, 7$ **35.** $-3, 7$

37. $-1, 24$ **39.** $\frac{10}{7}, 5$ **41.** $7, 24$ **43.** $-\frac{1}{7}, 1.5$

45. 4 s **47.** width 8 cm, length 13 cm **49.** 5 m and 12 m **51.** 3.00 in. **53.** 12.0 m

55. Pipe A: 12 h, pipe B: 6 h **57.** 15.00 in.

Exercise Set 9.5

1. $-2, -4$ **3.** -2 **5.** $\dfrac{3 \pm \sqrt{29}}{2}$

7. $-k \pm \sqrt{k^2 - c}$ **9.** $-4, 1$ **11.** $2, -\frac{1}{3}$

13. $-1, \frac{1}{7}$ **15.** $-1, \frac{7}{2}$ **17.** $-2, \frac{4}{3}$ **19.** $-\frac{2}{3}, -\frac{2}{3}$

21. $\dfrac{3 \pm \sqrt{17}}{4}$ **23.** $\dfrac{-5 \pm \sqrt{17}}{2}$ **25.** $\dfrac{-3 \pm \sqrt{15}}{2}$

27. $1 \pm \sqrt{8}$ **29.** $\pm\dfrac{\sqrt{6}}{2}$

31. no real roots because the discriminant is -48

33. no real roots because the discriminant is $-\frac{647}{81}$

35. $\dfrac{-0.2 \pm \sqrt{0.064}}{0.02}$ **37.** $5 \pm \sqrt{13}$

39. $\dfrac{2 \pm \sqrt{1.6}}{2.4}$ **41.** $\dfrac{-\sqrt{3} \pm \sqrt{87}}{6}$

43. no real root because the discriminant is negative

45. $\dfrac{-2 \pm \sqrt{13}}{3}$ **47.** Approximately 7.78 s

49. Approximately 11.86 s **51.** approximately 16.38 cm wide and 24.57 cm long

53. 2.25 inches wide, 7.5 inches deep, or 3.75 inches wide and 4.5 inches deep

55. \$20 **57.** 1062.1 Ω **59.** $\dfrac{-13 + 3\sqrt{46}}{2} \approx 3.67$ in.

61. 35.6 in. **63.** 3.5 s

65. either 20 or 370 objects were sold

67. There are two possible answers. In one, each field is 300 ft × 450 ft; and in the other, each field measures 600 ft × 225 ft.

69. Answers will vary **71.** **(a)** $-2, \frac{1}{3}$

Exercise Set 9.6

1.

x	-3	-2	-1	0	1	2	3	4
$f(x)$	-0.5	0	0.5	1	0	-1	-2	-3

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3.

x	-3	-2	-1	0	1	2	3	4	5
$h(x)$	3	$\sqrt{8}$	$\sqrt{7}$	$\sqrt{6}$	$\sqrt{5}$	2	9	10	11

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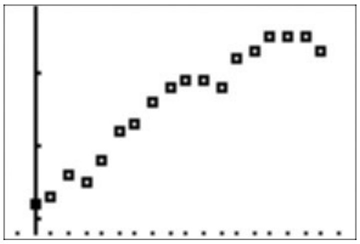
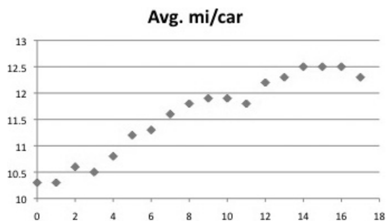
5. **(a)** $f(-3) = 9$ **(b)** $f(0) = 5$ **(c)** $f(5) = 6$

7. **(a)** $h(-2) = 6$ **(b)** $h(0) = 0$ **(c)** $h(4) = 1$

9. **(a)** TI-84:



or, with a spreadsheet:



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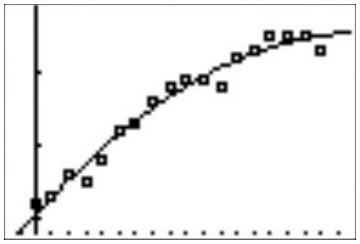
(b) $M(t) \approx -0.0060t^2 + 0.2516t + 10.1014$ average 1000 miles a car is driven t years after 1990

(c) TI-84:



or, with a spreadsheet:

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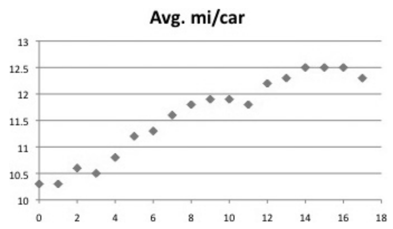
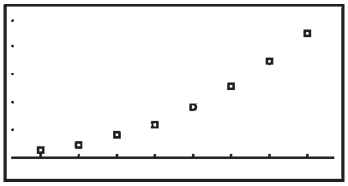
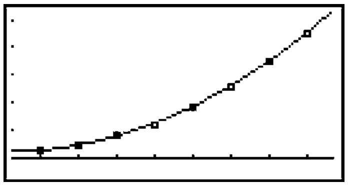


11. **(a)**

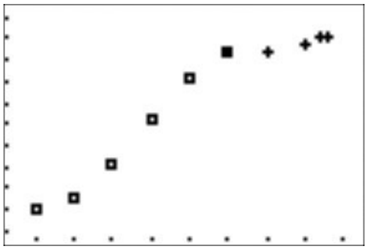


(b) $D(s) \approx 0.071s^2 - 0.274s + 19.089$ feet at s miles per hour

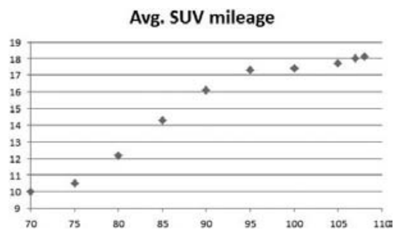
(c)



13. **(a)** TI-84:



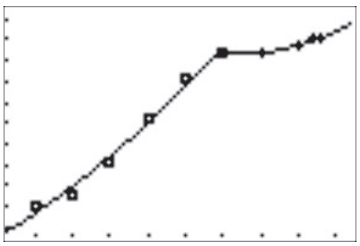
or, with a spreadsheet:



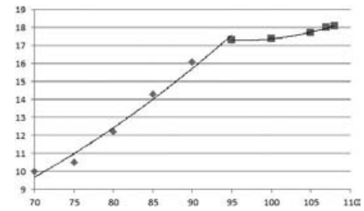
(b) $E(t) \approx \begin{cases} 0.0028t^2 - 0.1431t + 6.04 & \text{if } t \le 95 \\ 0.0059t^2 - 1.1364t + 72.0169 & \text{if } t \ge 95 \end{cases}$

average SUV miles per gallon t years after 1900

(c) TI-84:

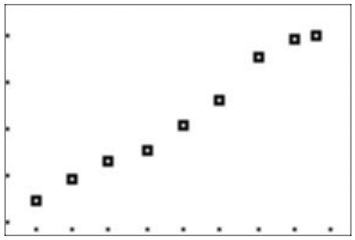


or, with a spreadsheet:

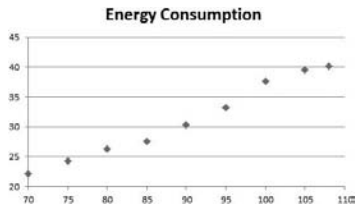


(d) about 2032

15. **(a)** TI-84:



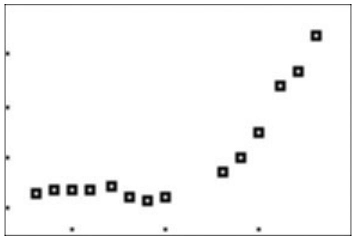
or, with a spreadsheet:



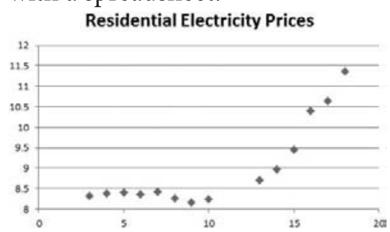
(b) $C_Q(t) \approx 0.0036t^2 - 0.1374t + 14.1987$ quadrillion Btus t years after 1970

(c) $C_L(t) \approx 0.5018t - 13.7886$ quadrillion Btus t years after 1970 **(d)** $C_Q(115) \approx 45.70$ quadrillion Btu, $C_L(115) \approx 43.92$ quadrillion Btu

17. **(a)** TI-84:



or, with a spreadsheet:



(b) $P(t) \approx 0.0277t^2 - 0.4136t + 9.6478$ dollars/kWh t years after 1990 **(c)** about \$11.80\$/kWh **(d)** One possible answer is

$$P(t) \approx \begin{cases} -0.170t^2 + 0.1812t + 7.9245 & \text{if } t \le 8 \\ 0.0447t^2 - 0.8456t + 12.1671 & \text{if } t \ge 8 \end{cases}$$

dollars/kWh t years after 1990, **(e)** about \$12.23/kWh

Review Exercises

1. 6 **3.** -2 **5.** $\dfrac{2 \pm \sqrt{10}}{3}$ **7.** $\dfrac{A - 2lw}{2(l + w)}$

9. $\frac{pq}{p+q}$ **11.** 1, 7 **13.** 1, 10 **15.** $-7, -8$

17. $1, \frac{3}{2}$ **19.** $-2, \frac{5}{6}$ **21.** $\pm\frac{3}{2}$ **23.** $-5, 1$

25. $\dfrac{-19 \pm 9\sqrt{5}}{2}$ **27.** $\dfrac{-5 \pm \sqrt{193}}{6}$ **29.** $4 \pm \sqrt{11}$

31. $1, -\frac{5}{2}$ **33.** $\dfrac{-1 \pm \sqrt{13}}{3}$ **35.** $\dfrac{4 \pm \sqrt{6}}{5}$

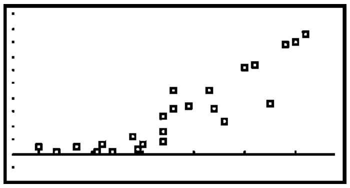
37. no answer because the discriminant is negative

39. $\dfrac{3 \pm \sqrt{17}}{4}$ **41.** **(a)** $f(-3) = 8$ **(b)** $f(0)$ is not defined **(c)** $f(5) = \pi$ **43.** 0.0899 A **45.** 3 sec

47. **(a)** 12 m/s **(b)** 3.16 mm diameter

49. $k \approx 8.99 \times 10^9$

51. **(a)**



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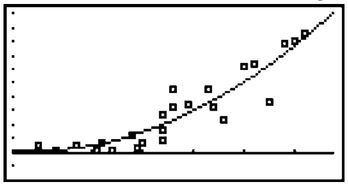
(b) Corn yield of inoculate plants is $I(r) \approx 2.8460r^2 - 10.2118r + 9.7163$ percent of plants infected based on Stewart's wilt rating of r

(c)



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(d) The percent of plants infected with Stewart's wilt were higher for hybrids with a higher rating.

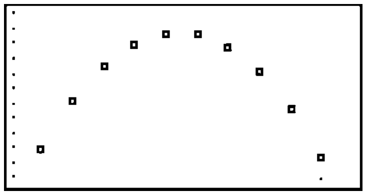
Chapter 9 Test

1. 4 **3.** $-1, 5$ **5.** double root, $\frac{3}{4}$ **7.** $\dfrac{-3 \pm \sqrt{5}}{2}$

9. $x = -10$ and $x = 3$ **11.** **(a)** $f(-2) = 0$ **(b)** $f(0) = 2$ **(c)** $f(2) = 0$ **(d)** $f(5) = 20$

13. $D = \dfrac{\sqrt{v + 5}}{3H}$ **15.** $v = \dfrac{2s - at^2}{2t}$

17. **(a)**



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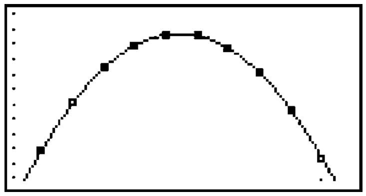
(b) The height of the ball is $h(t) \approx -15.9955t^2 + 86.5568 + 0.045$ feet t seconds after it was thrown

(c)



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(d) $h(1.69) \approx 100.5$ ft

ANSWERS FOR CHAPTER 10

Exercise Set 10.1

1. **(a)** Amplitude: 2 **(b)** Period: 4π **(c)** Frequency: $\dfrac{1}{4\pi}$ cycle **3.** **(a)** Amplitude: 1.5 **(b)** Period: 4π **(c)** Frequency: $\dfrac{1}{4\pi}$ cycle

5. **(a)** Amplitude: 1.5 **(b)** Period: 2 **(c)** Frequency: 0.5 cycle **7.** **(a)** Amplitude: 2 **(b)** Period: 2π **(c)** Frequency: $\dfrac{1}{2\pi}$ cycle

9. Period $= \pi$, amplitude $= 3$, frequency $= \frac{1}{\pi}$

11. Period $= 2\pi$, amplitude $= 2$, frequency $= \frac{1}{2\pi}$

13. Period $= 1$, amplitude $= 8$, frequency $= 1$

15. Period $= \frac{\pi}{2}$, amplitude $= \frac{1}{2}$, frequency $= \frac{2}{\pi}$

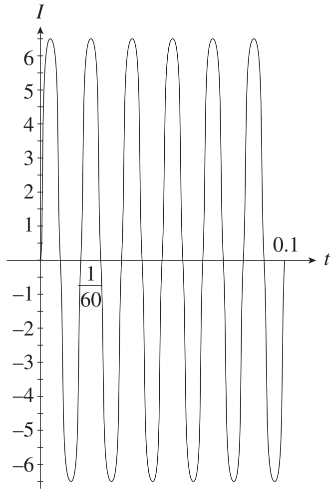
17. Period $= 4\pi$, amplitude $= \frac{1}{3}$, frequency $= \frac{1}{4\pi}$

19. Period 8π, amplitude $= 3$, frequency $= \frac{1}{8\pi}$

21. Period 2π, amplitude $= \frac{1}{2}$, frequency $= \frac{1}{2\pi}$

23. **(a)** Amplitude: 6.5 A, period: $\frac{1}{60}$, frequency: 60

(b)



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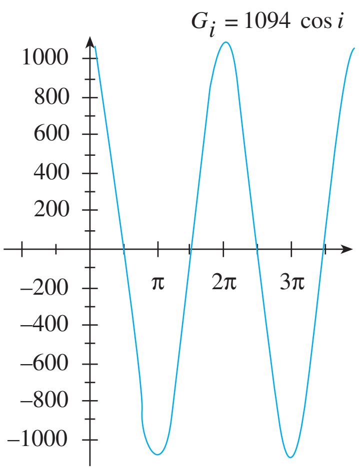
25. **(a)** Amplitude: 1094, period: 2π, frequency: $\frac{1}{2\pi}$

(b)



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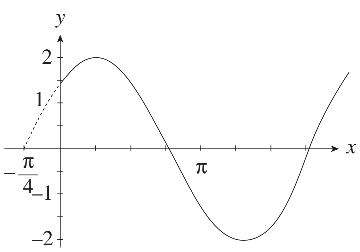
27. ≈ 0.000000149 or 1.49×10^{-7} sec.

Exercise Set 10.2

1. Amplitude $= 2$, period $= 2\pi$, phase shift $= -\frac{\pi}{4}$ or $\frac{\pi}{4}$ left, vertical displacement $= 0$



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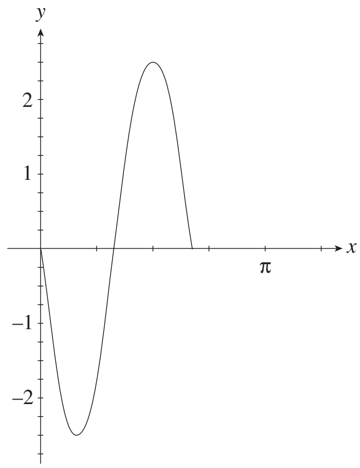
3. Amplitude $= 2.5$, period $= \frac{2\pi}{3}$, phase shift $= \frac{\pi}{3}$ or $\frac{\pi}{3}$ to the right, vertical displacement $= 0$



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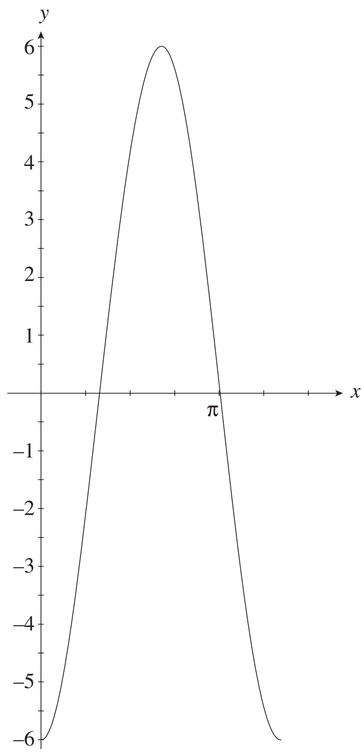
5. Amplitude $= 6$, period $= 240° = \frac{4\pi}{3}$, phase shift $= -\dfrac{180°}{1.5} = -120° = -\dfrac{2\pi}{3}$ to the left, vertical displacement $= 0$



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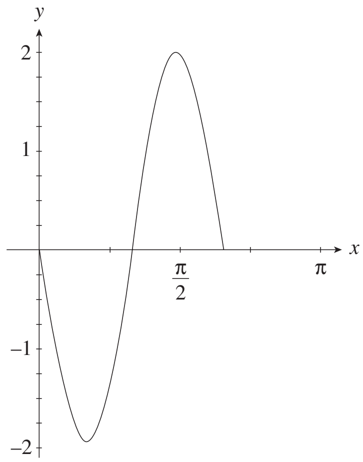
7. Amplitude $= 2$, period $= 120° = \frac{2\pi}{3}$, phase shift $= \frac{\pi}{3} = 30°$ vertical displacement $= 0$



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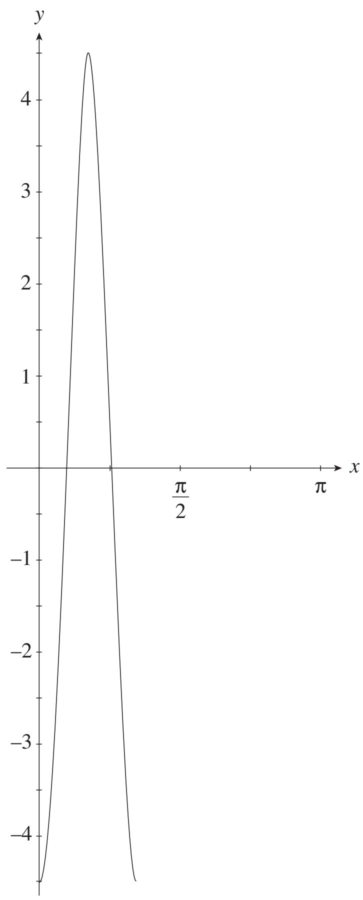
9. Amplitude $= 4.5$, period $= \frac{\pi}{3}$, phase shift $= \frac{4}{3}$, vertical displacement $= 0$



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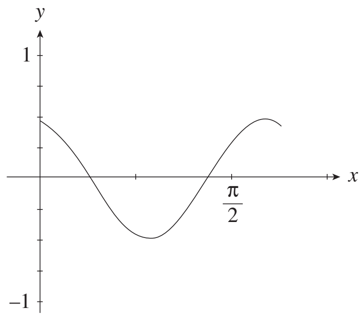
11. Amplitude $= 0.5$, period $= 2$, phase shift $= -\frac{1}{8}$, vertical displacement $= 0$



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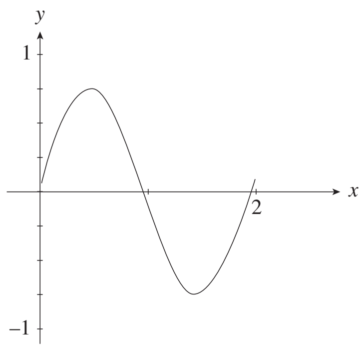
13. Amplitude $= 0.75$, period $= 2$, phase shift $= -\frac{\pi}{3}$, vertical displacement $= 0$



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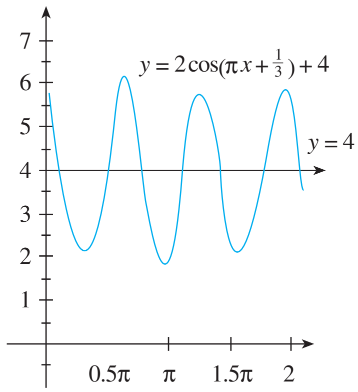
15. Amplitude $= 2$, period $= 2$, phase shift $= -\dfrac{1}{3\pi}$, vertical displacement $= 4$



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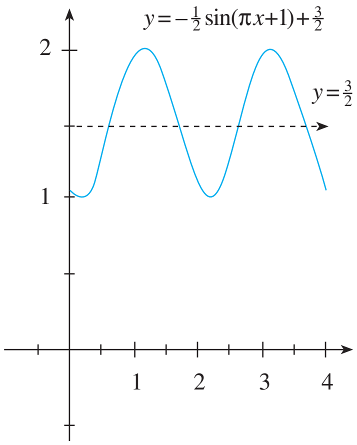
17. Amplitude $= \frac{1}{2}$, period $= 2$, phase shift $= -\dfrac{1}{\pi}$, vertical displacement $= \frac{3}{2}$



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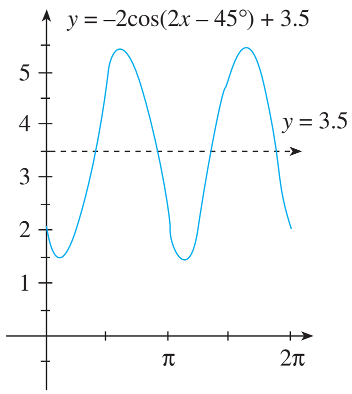
19. Amplitude $= 2$, period $= 180° = \pi$, phase shift $= 22.5°$, vertical displacement $= 3.5$



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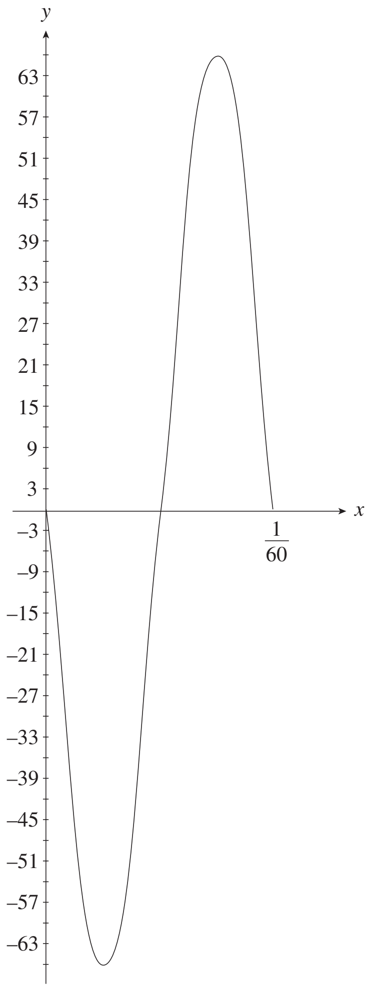
21. **(a)** Amplitude: 65; period $\frac{1}{60}$; phase shift: $-\frac{1}{240}$

(b)



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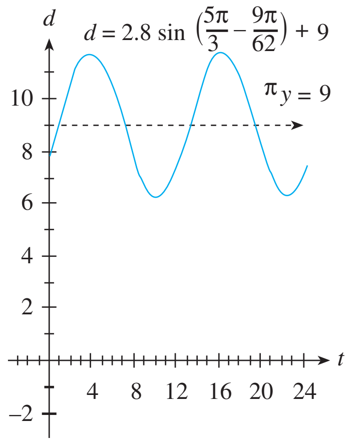
23. **(a)** $d = 2.8\sin\left(\frac{\pi}{6.2}t - \frac{0.9\pi}{6.2}\right) + 9 = 2.8\sin\left(\frac{5\pi}{31}t - \frac{9\pi}{62}\right) + 9$

(b)



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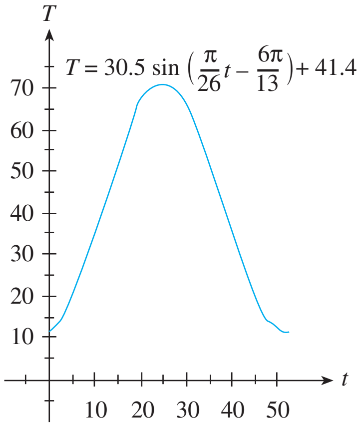
25. **(a)** $T = 30.5\sin\left(\frac{\pi}{26}t - \frac{6\pi}{13}\right) + 41.5$

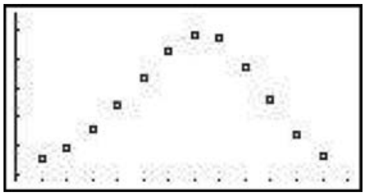
(b)



27. **(a)** A = 1.25 **(b)** Period: 1 year **(c)** Vertical displacement: 5 hours **(d)** Horizontal displacement: 6 months = .5 years **(e)** $Y = 1.25\sin(x + 0.5) + 1$

29. **(a)** $A = 2.5$ **(b)** 1 year **(c)** 10 **(d)** 0 **(e)** $y = 2.5\sin(x) + 10$

31. **(a)**

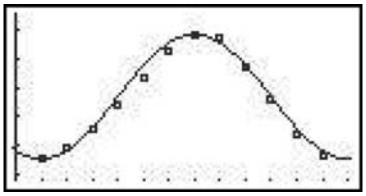


Many things in nature are cyclic and our experience tells us that temperatures tend to occur in annual cycles.

(b) $A = 21.75$ and $D = 36.95$

(c) Period is 12 months, so $B = \dfrac{\pi}{6}$ and $C = -\dfrac{2\pi}{3}$ $y = 21.75\sin\left(\dfrac{\pi}{6}x - \dfrac{2\pi}{3}\right) + 36.95$

(d)

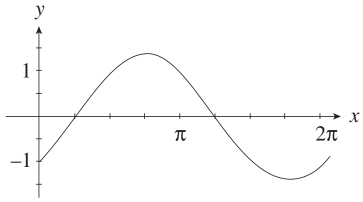


(e) $y \approx 21.2215\sin(0.5509x - 2.3218) + 36.8330$

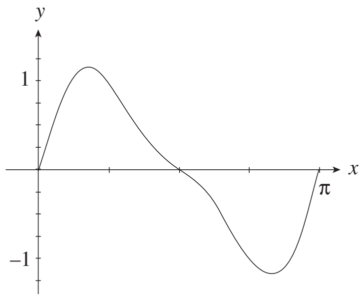
(f) $y \approx 21.75\sin(0.5236x - 2.0944) + 36.95$

Exercise Set 10.3

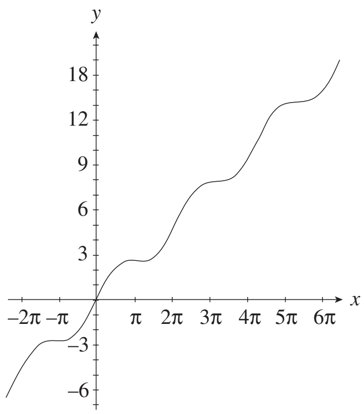
1.



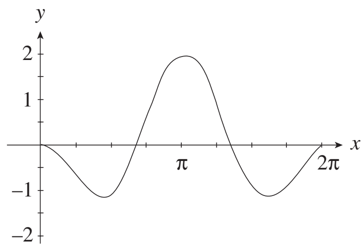
3.



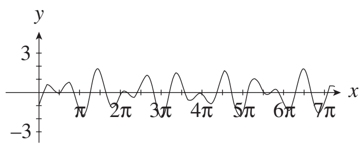
5.



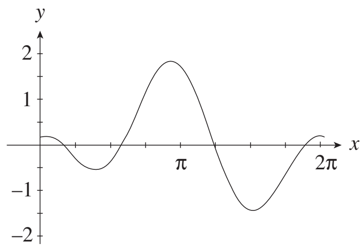
7.



9.



11.



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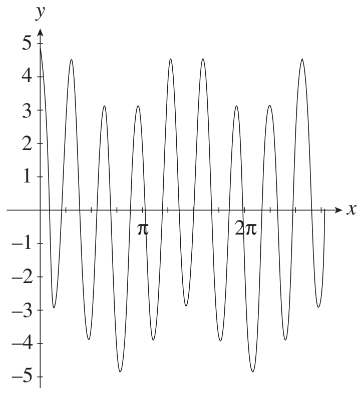
13.



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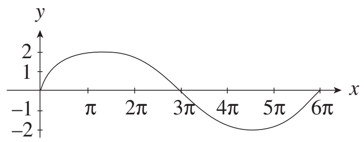
15.



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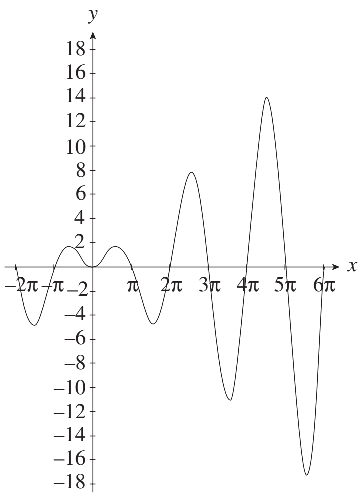
17.



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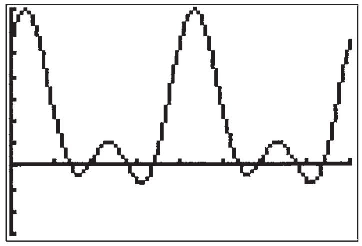
19. **(a)** Amplitude (maximum): 5, period: π **(b)** viewing window: one cycle, $[0, \pi, \frac{\pi}{4}] \times [-3, 7, 1]$; two cycles, $[0, 2\pi, \frac{\pi}{4}] \times [-3, 7, 1]$

(c)



$\left[0, 2\pi, \frac{\pi}{4}\right] \cdot [-3, 7, 1]$

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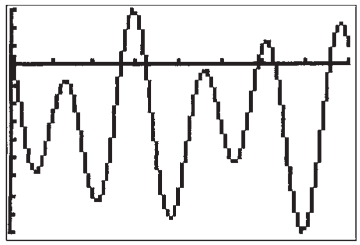
21. **(a)** Amplitude (maximum): 6, period: 2π **(b)** viewing window: one cycle, $[0, 2\pi, \frac{\pi}{4}] \times [-9, 3, 1]$; two cycles, $[0, 4\pi, \frac{\pi}{4}] \times [-9, 3, 1]$

(c)



$\left[0, 2\pi, \frac{\pi}{4}\right] \cdot [-9, 3, 1]$

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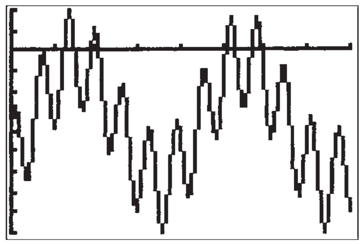
23. **(a)** Amplitude (maximum): 5.5, period: none, not a periodic function **(b)** viewing window: $[0, 4\pi, \frac{\pi}{2}] \times [-9, 2, 1]$

(c)



$\left[0, 4\pi, \frac{\pi}{2}\right] \cdot [-9, 2, 1]$

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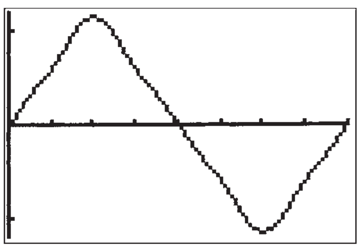
25.



$\left[0, 2\pi, \frac{\pi}{4}\right] \cdot [0, 3, 1]$

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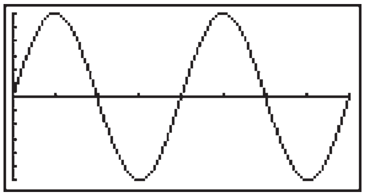
27. $P = 120 \cos \omega t\,(5 \sin \omega t)$

$P = 300 \cos \omega t \sin \omega t$

(a) π

(b) $A = 300$

(c)



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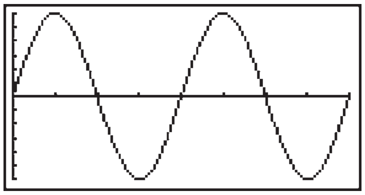
(d)



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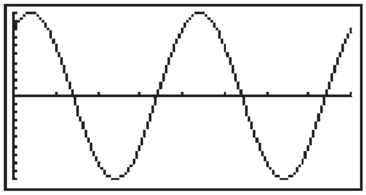
(e) They appear to be identical.

29. **(a)**



$[0, 2\pi/5.\pi/10]\times[-10, 10, 1]$

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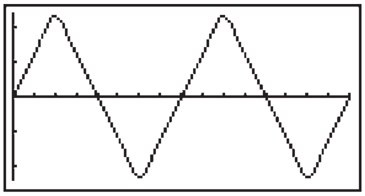
(b) $A = 14$

(c) Period $= 2\pi/5$

31. **(a)** Make sure that you key in $\dfrac{\sin 3x}{3^2}$ as $(\sin(3x))/3^2$ or $(\sin(3x))/9$.



$[0, 4\pi,\pi/4] \times [-1.2, 1.2, 0.5]$

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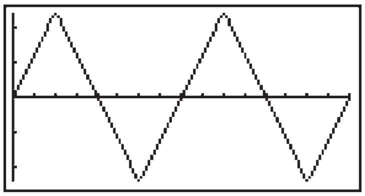
(b)



$[0, 4\pi,\pi/4] \times [-1.2, 1.2, 0.5]$

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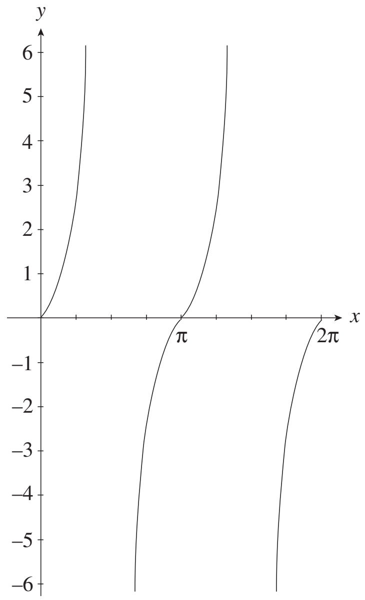
Exercise Set 10.4

1.



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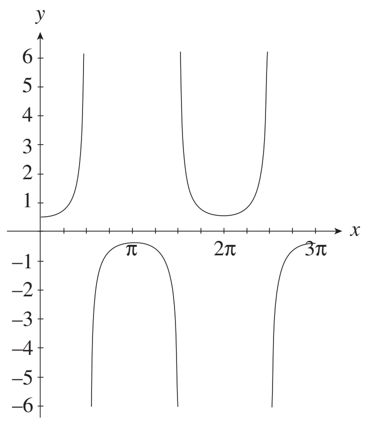
3.



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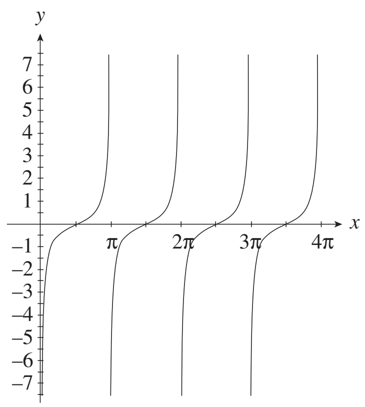
5.



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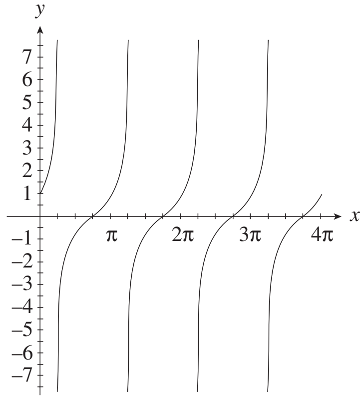
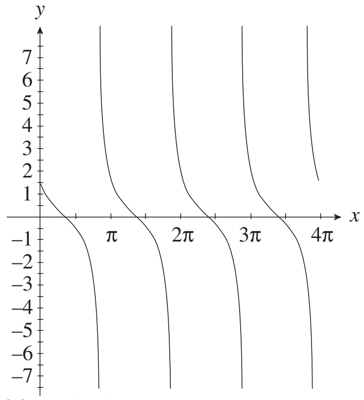
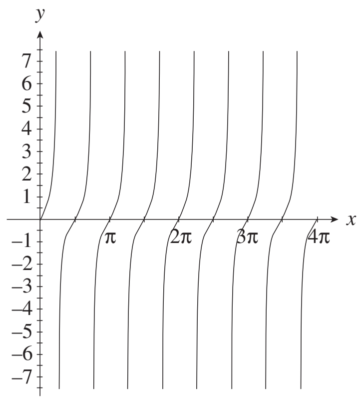
7.



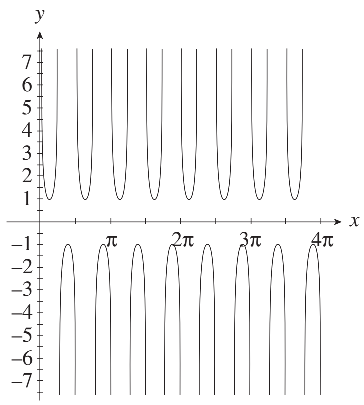
9.



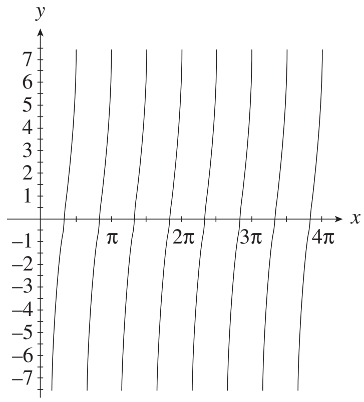
11.



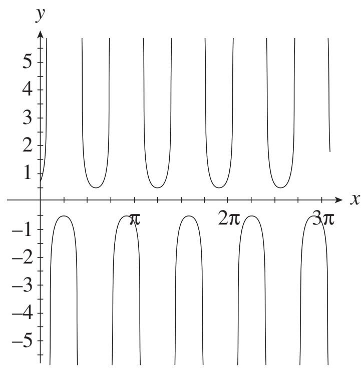
13.



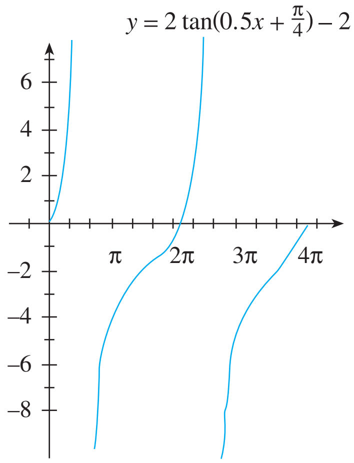
15.



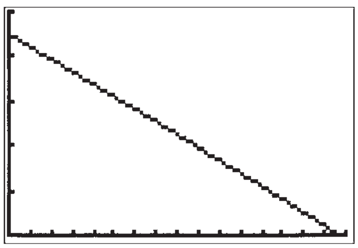
17.



19. **(a)** $[0, 4\pi, \frac{\pi}{2}] \times [-8, 6, 2]$
(b) $y = 2\tan(0.5x + \frac{\pi}{4}) - 2$

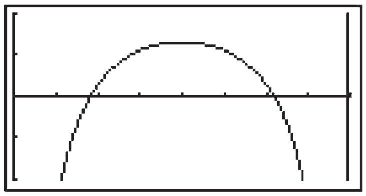


21. **(a)** about 1.68 cm **(b)** 15.71°
(c)



[0, 16] x [0, 5]

23.



[0, 2π, π/4] x [−1, 1, 0.5]

Exercise Set 10.5

1. Phasor A: Phasor A is used as a reference. Since it has a peak amplitude of 3.0, this represents graph of $Y_A = 3.0 \sin x$.

Phasor B: Phasor B is at an angle of 45° to Phasor A and has a peak amplitude of 2.0; therefore it is graphed as $Y_B = 2.0 \sin(x + 45°)$.

3. Phasor A: Phasor A is used as a reference. Since it has a peak amplitude of 2.75, this represents graph of $Y_A = 2.75 \sin x$.

Phasor B: Phasor B is at an angle of 35° to Phasor A and has a peak amplitude of 2.0; therefore it is graphed as $Y_B = 2.0 \sin(x + 35°)$.

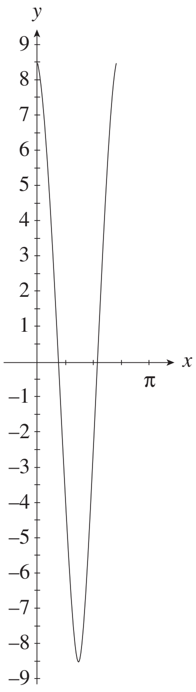
Phasor C: Phasor C is at an angle of −65° to Phasor A and has a peak amplitude of 1.8; therefore it is graphed as $Y_B = 1.8 \sin(x - 65°)$.

5. $y = 10 \sin 8\pi t$

7. $y = 0.8 \sin \frac{\pi}{3} t$

9. -2.79 ft/sec^2

11. **(a)** 8.5 cm **(b)** $\frac{\pi}{2.8}$ **(c)** $\frac{2.8}{\pi}$

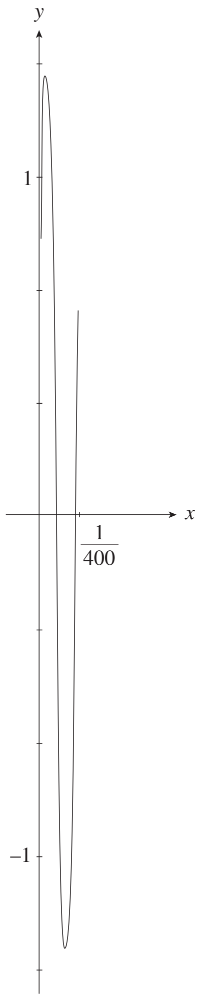


13. amplitude 10 A, period $\frac{1}{60}$ sec/cycle, $f = 60$ Hz, $\omega = 120\pi$ rad/sec **15.** $y = 6.8 \sin(160\pi t + \pi/3)$

17. $V = 220 \sin\left(80\pi t - \frac{\pi}{3}\right)$

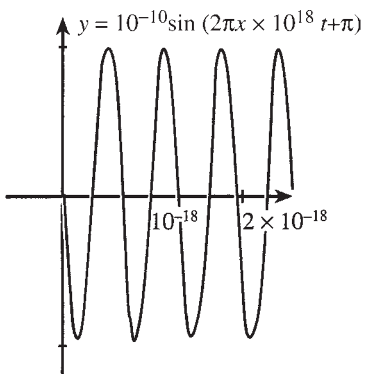
19. $V = 220 \cos\left(80\pi t + \frac{\pi}{5}\right)$

21. $f = 10^{23}$, amp $= 10^{-12}$ **23.**



25. **(a)** Period: $\dfrac{2\pi}{2\pi \times 10^{18}} = 10^{-18}$ Hz, amplitude: 10^{-10} m

(b)

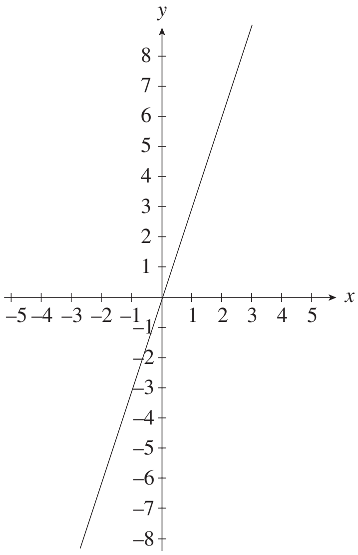


Exercise Set 10.6

1.

t	-3	-2	-1	0	1	2	3;
x	-3	-2	-1	0	1	2	3
y	-9	-6	-3	0	3	6	9

$y = 3x$

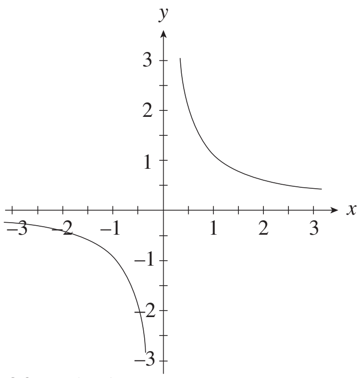


3.

t	-3	-2	-1	0	1	2	3;
x	-3	-2	-1	0	1	2	3
y	$-\frac{1}{3}$	$-\frac{1}{2}$	-1	*	1	$\frac{1}{2}$	$\frac{1}{3}$

$y = \frac{1}{x}$

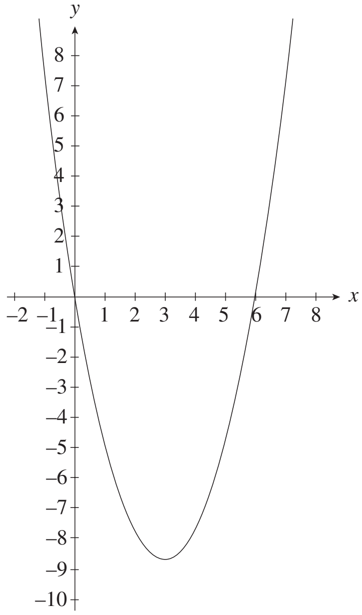
*Not defined



5.

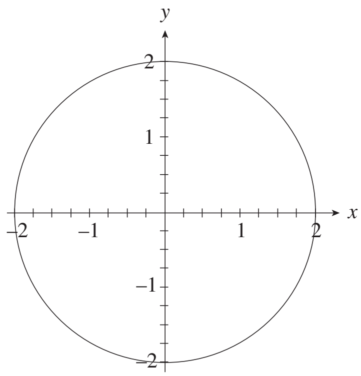
t	-5	-4	-3	-2	-1	0	1	2	3	4	5;
x	8	7	6	5	4	3	2	1	0	-1	-2
y	16	7	0	-5	-8	-9	-8	-5	0	7	16

$y = (x-3)^2 - 9 = x^2 - 6x$



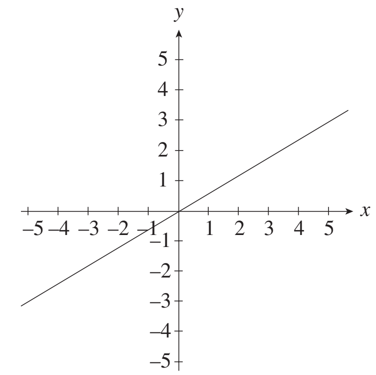
7.

t	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π	$\frac{5\pi}{4}$	$\frac{3\pi}{2}$	$\frac{7\pi}{4}$
x	0	$\sqrt{2}$	2	$\sqrt{2}$	0	$-\sqrt{2}$	-2	$-\sqrt{2}$
y	2	$\sqrt{2}$	0	$-\sqrt{2}$	-2	$-\sqrt{2}$	0	$\sqrt{2}$



9.

t	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π	$\frac{5\pi}{4}$	$\frac{3\pi}{2}$	$\frac{7\pi}{4}$	2π
x	2	$\frac{5\sqrt{2}}{2}$	5	$\frac{5\sqrt{2}}{2}$	0	$-\frac{5\sqrt{2}}{2}$	-5	$-\frac{5\sqrt{2}}{2}$	0
y	0	$\frac{3\sqrt{2}}{2}$	3	$\frac{3\sqrt{2}}{2}$	0	$-\frac{3\sqrt{2}}{2}$	-3	$-\frac{3\sqrt{2}}{2}$	0



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11.

t	-5	-4	-3	-2	-1	0	1	2	3	4	5
x	-5.96	-4.76	-2.86	-1.09	-0.16	0	0.16	1.09	2.86	4.76	5.96
y	0.72	1.65	1.99	1.41	0.46	0	0.46	1.41	1.99	1.65	0.72

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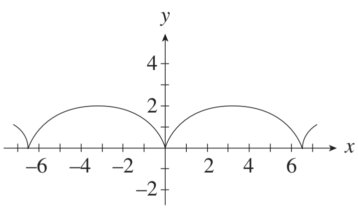

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13.

t	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{5\pi}{4}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{7\pi}{4}$	$\frac{11\pi}{6}$	2π
x	1	$\frac{2}{\sqrt{3}}$	$\sqrt{2}$	2	*	-2	$-\sqrt{2}$	$-\frac{2}{\sqrt{3}}$	-1	$-\frac{2}{\sqrt{3}}$	$-\sqrt{2}$	-2	*	2	$\sqrt{2}$	$\frac{2}{\sqrt{3}}$	1
y	*	4	$2\sqrt{2}$	$\frac{4}{\sqrt{3}}$	2	$\frac{4}{\sqrt{3}}$	$2\sqrt{2}$	4	*	-4	$-2\sqrt{2}$	$-\frac{4}{\sqrt{3}}$	-2	$-\frac{4}{\sqrt{3}}$	$-2\sqrt{2}$	-4	*

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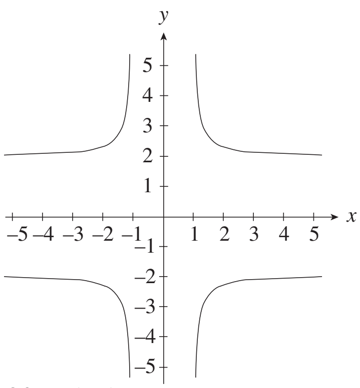
*Not defined



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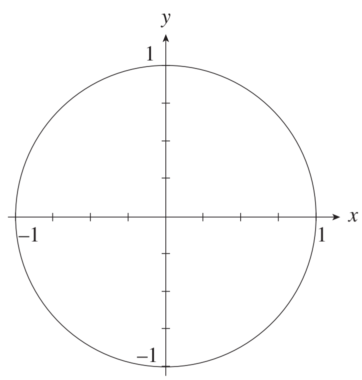
15. **(a)** 1 : 1;



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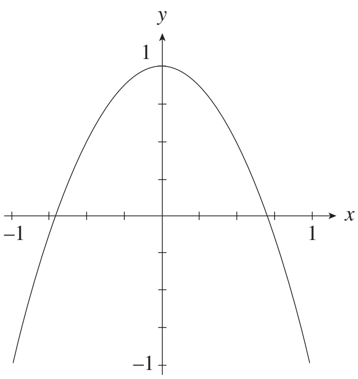
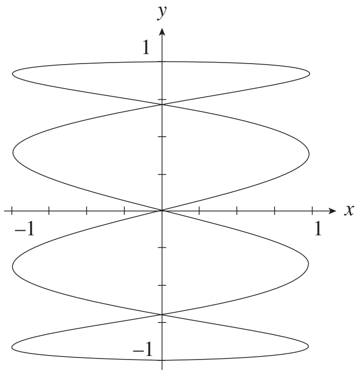
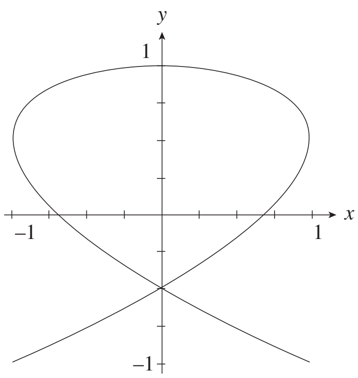
17. **(a)** 2 : 1;



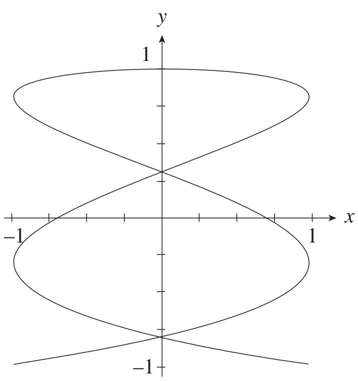
19. **(a)** 1 : 4;



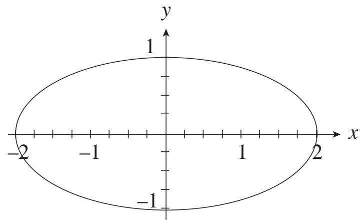
21. **(a)** 2 : 3;



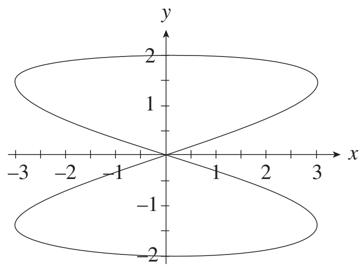
23. **(a)** 2 : 5;



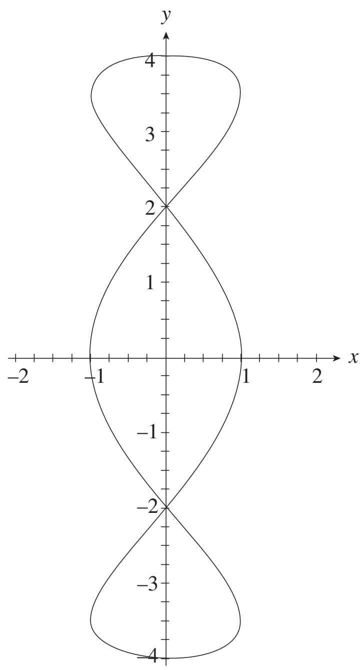
25.



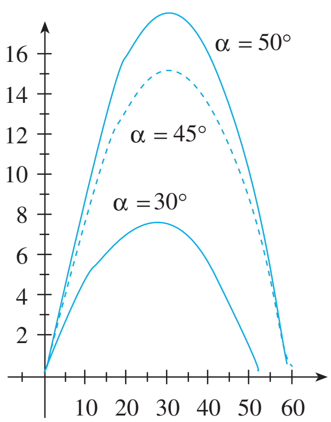
27.



29.

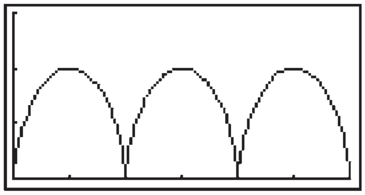


31. **(a)**



(b) $y = x \tan \alpha - \dfrac{gx^2}{2V^2}\sec^2\alpha$ **(c)** $x = \dfrac{V^2 \sin 2\alpha}{g}$

33. **(a)**



[0, 6π, π/2] x [0, 3, 1]

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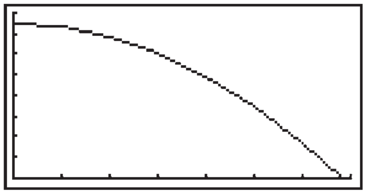
(b) Amplitude: 1.8 ft, period: $1.8\,\pi \approx 5.65$, frequency: $\frac{1}{1.8\pi} \approx 0.18$ sec.

35. **(a)** $\begin{cases} x = 100t \\ y = -16t^2 + 7.5 \end{cases}$

(b)



[0, 70, 10] x [0, 8, 1]

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(c) $x = 140t$

$58 = 100t$

$t = \dfrac{58}{100} = 0.58$ seconds

(d) $y = -16t^2 + 7.5$

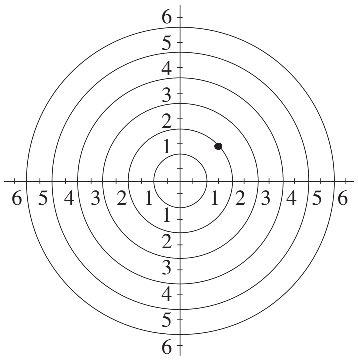
$y = -16(0.58)^2 + 7.5$

$y = 2.1$ ft.

Yes

Exercise Set 10.7

1.



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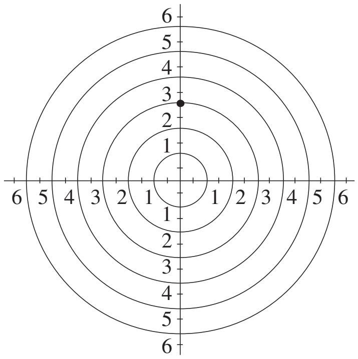
3.



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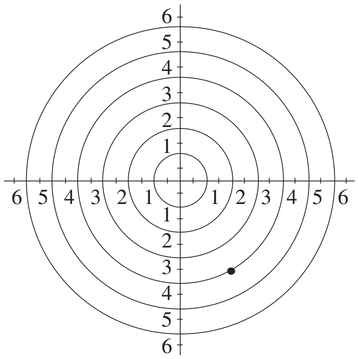
5.



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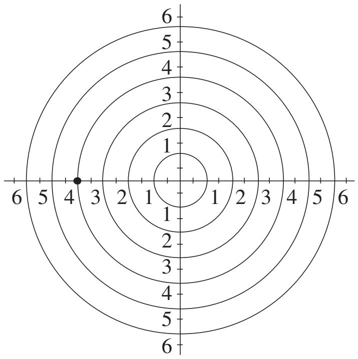
7.



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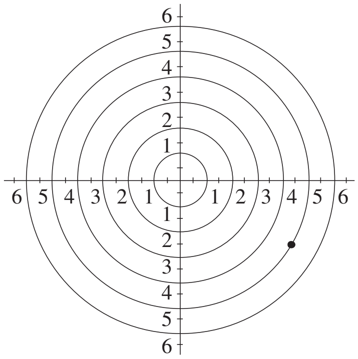
9.



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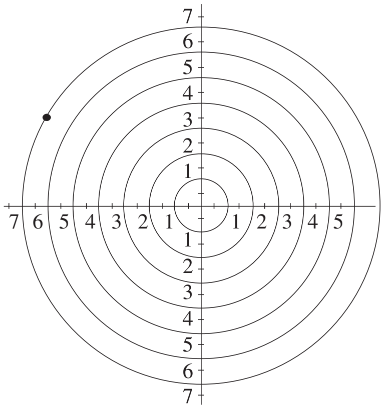
11.



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13. $(2, 3.46)$ **15.** $(-1.414, 1.414)$ or $(-\sqrt{2}, \sqrt{2})$

17. $(-5.64, -2.05)$ **19.** $(0.80, 1.83)$ **21.** $(-2.95, 0.52)$

23. $(4.81, 3.59)$

25. $(5.66, 45°) \approx (4\sqrt{2}, 45°) = (4\sqrt{2}, \frac{\pi}{4})$

27. $(5, 35.87°) = (5, 0.64)$

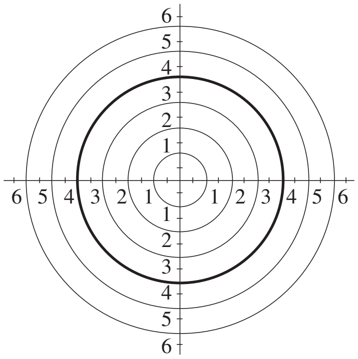
29. $(29, 133.60°) = (29, 2.332)$

31. $(5, 126.87°) = (5, 2.214)$

33. $(12.21, 235.01°) = (12.21, 4.102)$

35. $(9.22, 77.47°) = (9.22, 1.352)$

37.



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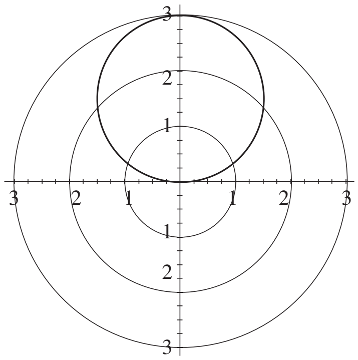
39.



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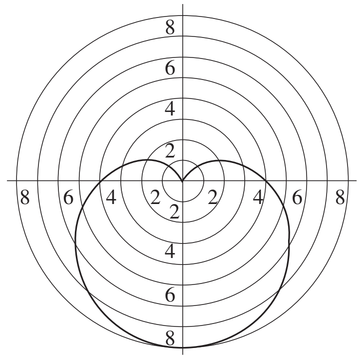
41.



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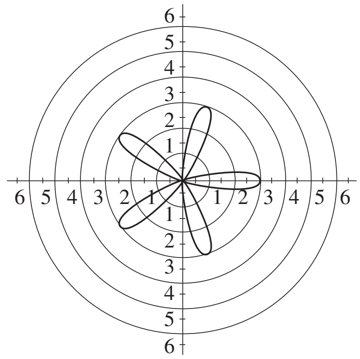
43.



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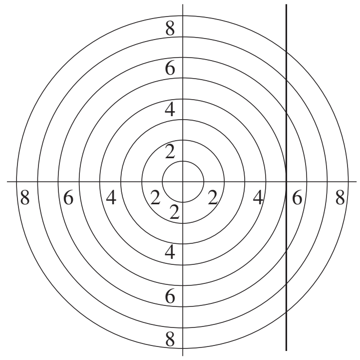
45.



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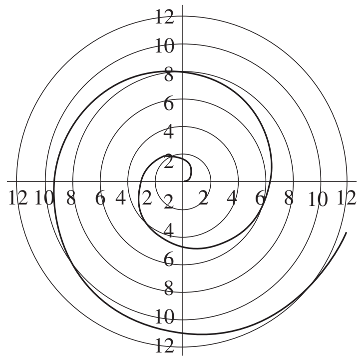
47.



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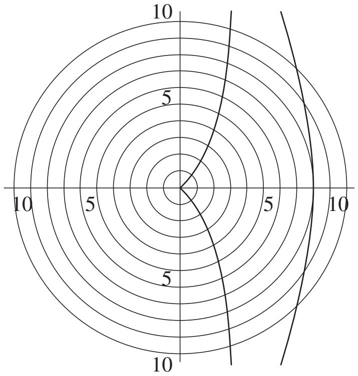
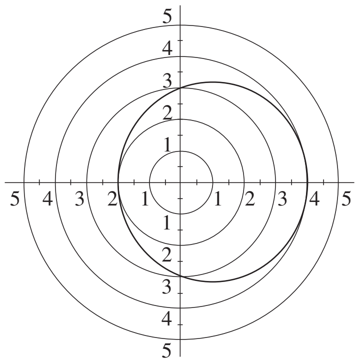
49.



51.



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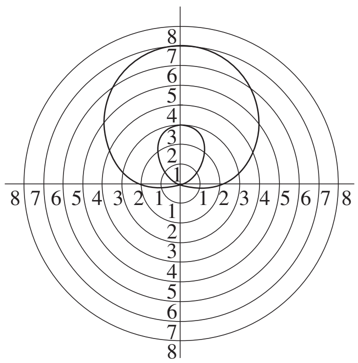

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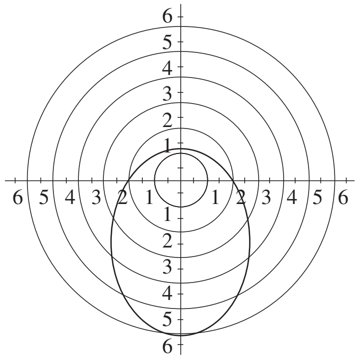
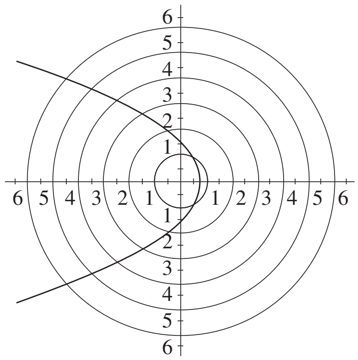
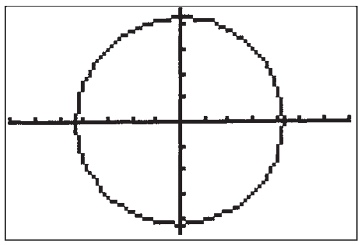


57.

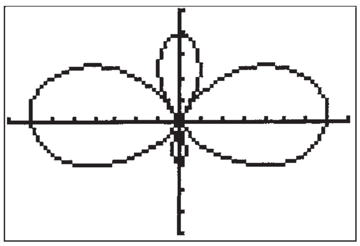


59.



$[-7050, 7050, 1000] \times [-4650, 4650, 1000]$

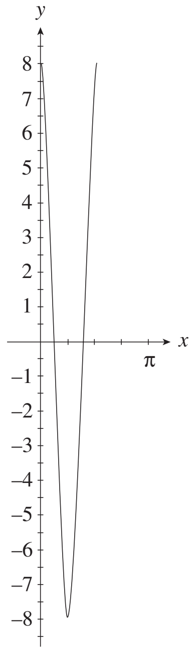
61.



$[-8, 8, 1] \times [-5, 5, 1]$

Review Exercises

1. period $\frac{\pi}{2}$, amplitude 8, frequency $\dfrac{2}{\pi}$, displacement 0



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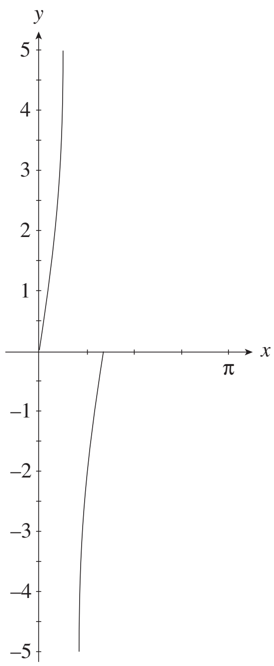
3. period $\frac{\pi}{3}$, amplitude ∞, frequency $\dfrac{3}{\pi}$, displacement 0



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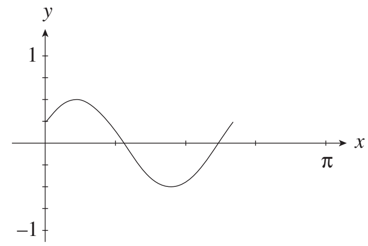
5. period $\frac{2\pi}{3}$, amplitude $\frac{1}{2}$, frequency $\dfrac{3}{2\pi}$, displacement $\frac{-\pi}{9}$



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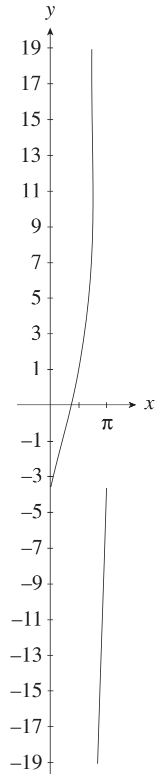
7. period π, amplitude ∞, frequency $\dfrac{1}{\pi}$, displacement $\frac{\pi}{4}$



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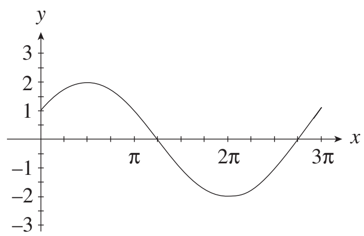
9. period 3π, amplitude 2, frequency $\dfrac{1}{3\pi}$, displacement $\frac{\pi}{4}$



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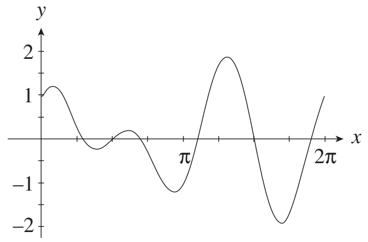
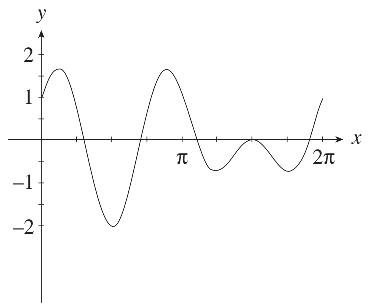
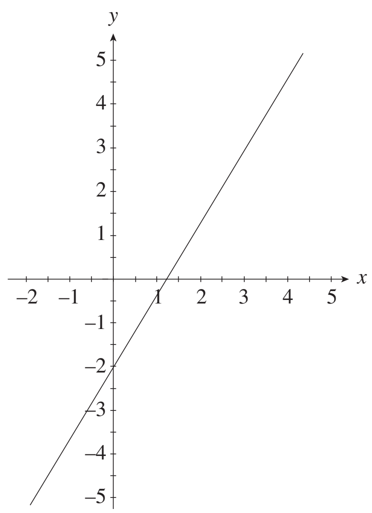
11.



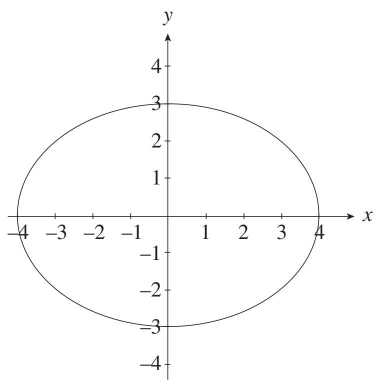
13.



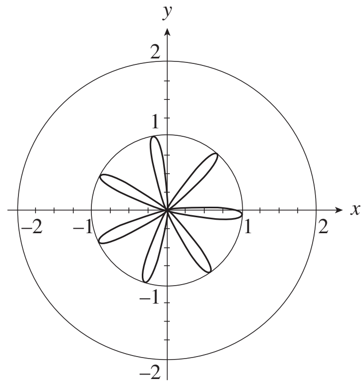
15.



17.



19.

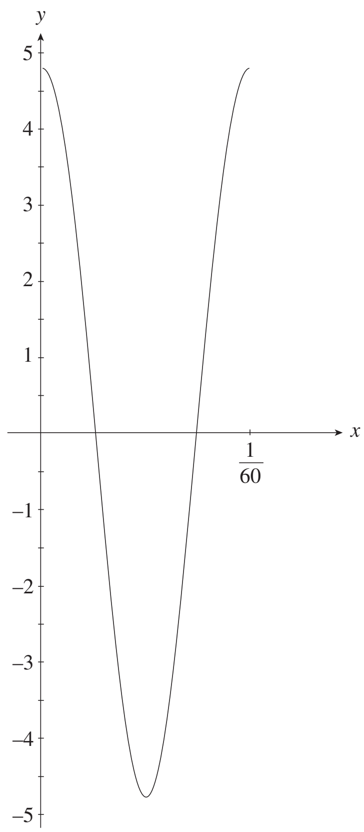


21. Amplitude: 1.75
Period: 4π
Frequency: $1/4\pi$

23. $Y_A = 3.0 \sin x$
$Y_B = 2.0 \sin(x + 75°)$

25. $y = 85 \sin 10\pi t$

27. $I = 4.8 \sin(120\pi t + \frac{\pi}{2})$

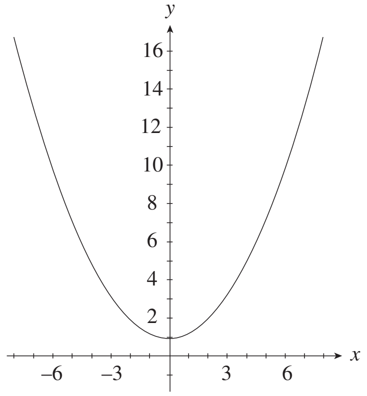


29. (*f*) Model: $y = 5.193 \sin(0.514x + 1.031) + 5.672$

Chapter 10 Test

1. period $\frac{2\pi}{5}$, amplitude 3, frequency $\frac{5}{2\pi}$, displacement 0 period $\frac{2\pi}{3}$, amplitude 2.4, frequency $\frac{3}{2\pi}$, displacement $\frac{\pi}{12}$ period $\frac{\pi}{2}$, amplitude ∞, frequency $\frac{2}{\pi}$, displacement $-\frac{\pi}{10}$

3.



5. (1.147, 1.638)
7. $y = 5.7 \sin 120\pi t$
9. $Y_A = 1.5 \sin x$
$Y_B = 1.25 \sin(x - 20°)$

ANSWERS FOR CHAPTER 11

Exercise Set 11.1

1. 5 **3.** 4 **5.** 1/5 **7.** 1/2 **9.** 9 **11.** 25 **13.** 1/8 **15.** −1/2 **17.** 1/2 **19.** 32 **21.** 3^7 **23.** 7^4 **25.** x^{10} **27.** y^2 **29.** 9^{10} **31.** x^{21} **33.** x^5y^5 **35.** $1/a^5b^5$ **37.** x^8 **39.** $1/x^6$ **41.** x^2 **43.** $r^{7/4}$ **45.** $a^{5/6}$ **47.** $d^{5/12}$ **49.** b^3/a^3 **51.** r^2/s^3 **53.** y^6 **55.** a^5/b^{12} **57.** $1/yb^{13/2} = \frac{1}{yb^{13/2}}$ **59.** p^3/x^2 **61.** $\frac{y^6}{z^2}$ **63.** $\frac{2y^2}{3x}$ **65.** $\frac{y^{19/15}}{x^{31/15}}$ **67.** $\frac{t^{11/6}x^{1/6}}{6}$ **69.** $\frac{4y^{11}}{27x^{10}}$ **71.** 4.0993852 **73.** 25.368006 **75.** 3.5162154 **77.** 4.5606226 **79.** −0.2253777406 **81.** 221.1125 m $\approx$ 221 m **83.** $p = 905\,146.3$ N/m^2 $T = 91\,238.747$ K **85.** $1 - \left(\frac{p_1}{p_2}\right)^{-7/2} = 1 - \left(\frac{p_2}{p_1}\right)^{7/2}$

Exercise Set 11.2

1. $2\sqrt[3]{2}$ **3.** $\sqrt[3]{5}$ **5.** y^4 **7.** $a\sqrt[5]{a^2}$ **9.** $xy^3\sqrt{y}$ **11.** $a\sqrt[4]{ab^3}$ **13.** $2x\sqrt[3]{x}$ **15.** $3x\sqrt{3xy}$ **17.** −2 **19.** ab^3 **21.** $pq^2r\sqrt[4]{r^3}$ **23.** $2x/3$ **25.** xy^2/z **27.** $\frac{2x\sqrt[3]{2y^2}}{z^2}$ **29.** $8xy^2\sqrt{x}/3z^2p = \frac{8xy^2\sqrt{x}}{3z^2p}$ **31.** $4\sqrt{3}/3 = \frac{4\sqrt{3}}{3}$ **33.** $3\sqrt[3]{2}/2 = \frac{3\sqrt[3]{2}}{2}$ **35.** $5\sqrt{2x}/2x = \frac{5\sqrt{2x}}{2x}$ **37.** $3\sqrt[4]{8z}/4z = \frac{3\sqrt[4]{8z}}{4z}$ **39.** $2\sqrt[3]{2y}/x = \frac{2\sqrt[3]{2y}}{x}$ **41.** $-2x\sqrt[3]{by}/3bz = \frac{-2x\sqrt[3]{by}}{3bz}$ **43.** $2 \times 10^2 = 2000$ **45.** $50\sqrt{10}$ **47.** $2\sqrt{10} \times 10^3 = 2000\sqrt{10}$ **49.** $\sqrt[3]{12.5} \times 10^3$ **51.** $\sqrt{\frac{x^2 + y^2}{xy}} = \frac{\sqrt{x^3y + xy^3}}{xy}$ **53.** $a + b$ **55.** $\sqrt{\frac{b + a^2}{a^2b}} = \frac{\sqrt{b^2 + a^2b}}{ab}$ **57.** $\sqrt[18]{27x^2}$ **59.** $\sqrt[21]{-624.2x^{15}y^{10}z}$ **61.** $f = \frac{\sqrt{T\mu}}{2L\mu} = \frac{1}{2L\mu}\sqrt{T\mu}$

63. $z = \dfrac{1}{\sqrt{\dfrac{R^2+x^2}{R^2x^2}}} = \sqrt{\dfrac{R^2x^2}{R^2+x^2}} = \dfrac{Rx\sqrt{R^2+x^2}}{R^2+x^2}$

$= \dfrac{Rx}{R^2+x^2}\sqrt{R^2+x^2}$ **65.** $1 + \dfrac{\sqrt{m^2+2h}}{m}$ or

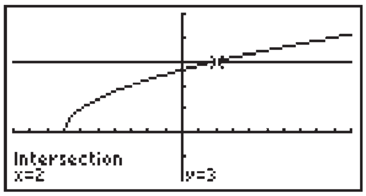
$\dfrac{m+\sqrt{m^2+2h}}{m}$

Exercise Set 11.3

1. $7\sqrt{3}$ **3.** $5\sqrt[3]{9}$ **5.** $8\sqrt{3}+4\sqrt{2}$ **7.** $3\sqrt{5}$
9. $-\sqrt{7}$ **11.** $5\sqrt{15/3} = \frac{5}{3}\sqrt{15}$ **13.** $-\sqrt{2}$
15. $3x\sqrt{xy}$ **17.** $(2q+p^2)\sqrt[3]{3p^2q}$ **19.** $\dfrac{x^2-y^2}{x^2y^2}\sqrt{xy}$
21. $2\sqrt{3ab}/3$ **23.** $\sqrt{40} = 2\sqrt{10}$ **25.** $x\sqrt{15}$
27. $8x\sqrt{x}$ **29.** $3/4$ **31.** $\sqrt{2x}+2$ **33.** $x+2\sqrt{xy}+y$ **35.** $\dfrac{\sqrt[3]{5\cdot 7^2}}{7} = \dfrac{\sqrt[3]{245}}{7}$ **37.** $a-b$
39. $\sqrt[6]{x^5}$ **41.** $\sqrt[12]{3^3 49^4 x^{11}} = \sqrt[12]{155{,}649{,}627x^{11}}$ **43.** 4
45. $\sqrt[3]{2b}/2$ **47.** $\sqrt[6]{\dfrac{x^4}{3}} = \dfrac{\sqrt[6]{243x^4}}{3}$
49. $2\sqrt{\dfrac{2a^5}{b^5}} = 2\dfrac{\sqrt{2a^5b^7}}{b}$ **51.** $\sqrt[6]{2}$ **53.** $.\dfrac{x-\sqrt{5}}{x^2-5}$
55. $\dfrac{(\sqrt{5}-\sqrt{3})^2}{2} = 4-\sqrt{15}$
57. $2x\sqrt{x^2-1}/(x^2-1), x \neq \pm 1$
59. $-\dfrac{\sqrt{x^2-y^2}+\sqrt{x^2+xy}}{y}, y \neq 0$ **61.** $-\dfrac{b}{a}$
63. **(a)** $R = \dfrac{x^{3/2}}{x+1}$, **(b)** $4.259\ \Omega$ **65.** $\dfrac{c}{a}$

Exercise Set 11.4

1. 22 **3.** 22.5 **5.** no solution **7.** 25 **9.** 16
11. 10 **13.** 2 **15.** $\frac{2829}{296} \approx 14.43$ **17.** no solution
19. no solution
21. **(a)**



[−10, 10, 1] x [−2. 5, 1]
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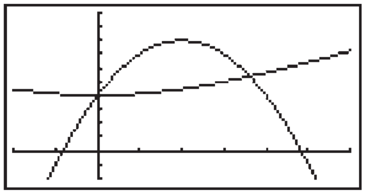
(b) $x = 2$
23. **(a)**



[−2, 6, 1] x [−2, 10, 1]
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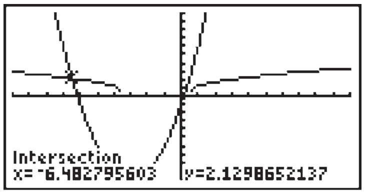
(b) $x = 0$ and $x \approx 3.6151$
25. **(a)**



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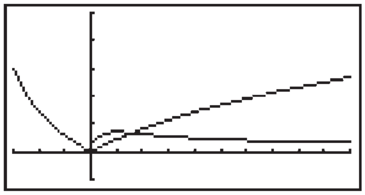
(b) $x \approx -6.4828$
27. **(a)**



[−3, 10, 1] x [−1, 5, 1]
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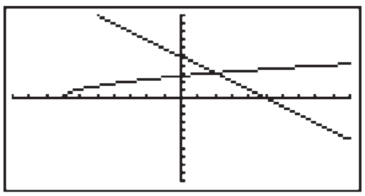
(b) $x = 0$ and $x \approx 1.5874$
29. $R = \dfrac{v^2}{\mu_s g}$ **31.** $I_A = 1.419$; $I_B = 4.691$ **33.** 1,183.36 ft

Review Exercises

1. 7 **3.** 125 **5.** 1/27 **7.** 5^{13} **9.** 2^{32}
11. x^8y^{12} **13.** y^5a^5 **15.** $x^{5/6}$ **17.** $xy^8/a = \dfrac{xy^8}{a}$
19. -3 **21.** $ab^2\sqrt{b}$ **23.** $\dfrac{-2xy^2}{z}\sqrt[3]{z^2}$ **25.** $\dfrac{\sqrt{a^3-b^3}}{ab}$
27. $8\sqrt{5}$ **29.** $8\sqrt{6}-2\sqrt{3}$ **31.** $(2+3a)\sqrt[4]{a^3b}$
33. $\dfrac{(a^2-b^2)\sqrt{7ab}}{7ab}$ **35.** $\sqrt{15}$ **37.** $x\sqrt{14}$
39. $a-2b$ **41.** $\dfrac{\sqrt[3]{a}}{2}$ **43.** 82 **45.** no solution
47. 18.14 **49.** 2
51. **(a)**



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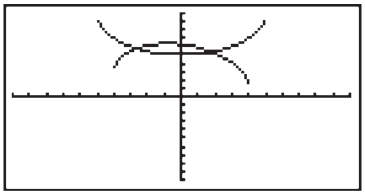
(b) $x = 2$

53. **(a)**



(b) $x = -2.6516$ and $x \approx 1.8219$

55. **(a)** 28.72 mg **(b)** 16.49 mg

57. $v = \sqrt{2rt^2 + 4t + 5}$

Chapter 11 Test

1. 8^{-2} **3.** $\dfrac{x^5}{y^{15}}$ **5.** -4 **7.** $x^2y\sqrt{y}$ **9.** $x + 3$

11. $2\sqrt{3x}$ **13.** $\dfrac{\sqrt[3]{2y^2}}{y}$ **15.** 2 **17.** 27

19. **(a)** $F = \dfrac{7.12 \times 10^5 d^3}{\ell}$ **(b)** $F = 246{,}067.2$ kg

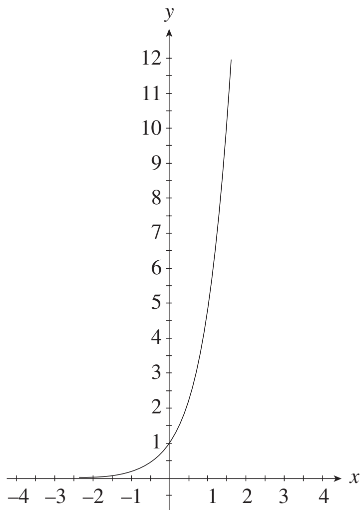
ANSWERS FOR CHAPTER 12

Exercise Set 12.1

1. 4.7288 **3.** 5.6164 **5.** 3.4154

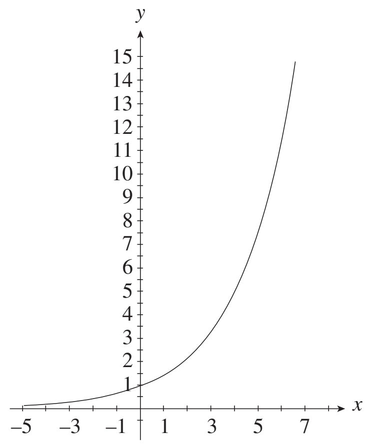
7.

x	-3	-2	-1	0	1	2	3	4	5
$f(x)$	$\frac{1}{64}$	$\frac{1}{16}$	$\frac{1}{4}$	1	4	16	64	256	1024



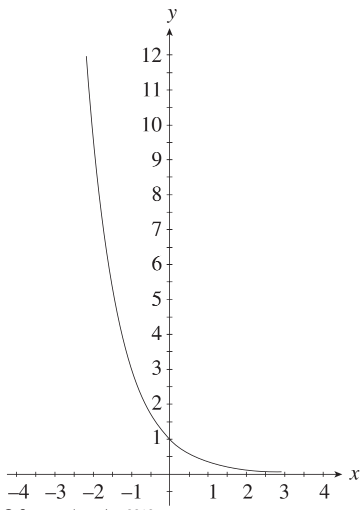
9.

x	-3	-2	-1	0	1	2	3	4	5
$h(x)$	$\frac{8}{27}$	$\frac{4}{9}$	$\frac{2}{3}$	1	1.5	2.25	3.375	5.0625	7.59375



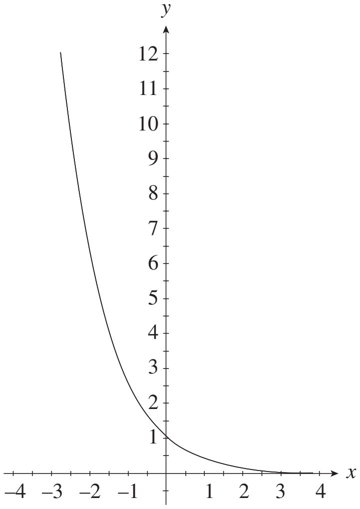
11.

x	-3	-2	-1	0	1	2	3	4	5
$f(x)$	27	9	3	1	$\frac{1}{3}$	$\frac{1}{9}$	$\frac{1}{27}$	$\frac{1}{81}$	$\frac{1}{243}$



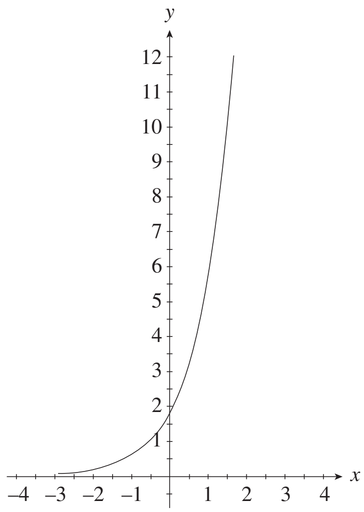
13.

x	-3	-2	-1	0	1	2	3	4	5
$h(x)$	13.824	5.760	2.4	1	0.417	0.174	0.072	0.030	0.0126



15.

x	-3	-2	-1	0	1	2	3	4
$f(x)$	0.064	0.192	0.577	1.732	5.196	15.588	46.765	140.296

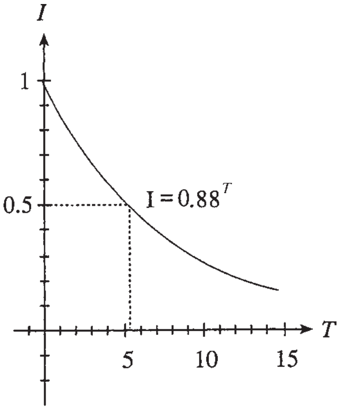


17. **(a)** \$1338.23 **(b)** \$1343.92 **(c)** \$1346.86 **(d)** \$1348.85

19. \$2.12 **21.** **(a)** 40,650 **(b)** 650,400 **(c)** 1 h **(d)** 3 h

23. **(a)** 400 **(b)** $\frac{2}{5} = 0.4$ **(c)** 3,600 **25.** **(a)** Here $T = 1.75$, and so $I = 0.88^{1.75} \approx 0.80$.

(b)



(c) Locate 0.50 on the I-axis. Draw a horizontal line from there until it intersects the graph of $I = 0.88^T$. From there draw a vertical line toward the T-axis. It intersects the T-axis near 5.42, and so we conclude that a thickness of 5.42 mm will produce an intensity of 0.50.

27. **(a)** $D(0) = 25$ mg **(b)** 17.5% **(c)** 3.65 mg **(d)** 16.73 hr. or about 16 hr 44 min

29. **(a)** $S_e(t) \approx 10.0299 \times 1.1258^t$ dollars, t years after 1970 **(b)** $S_q(t) \approx 2038.0643t^2 - 289627.4514t + 10286696.6571$ dollars, t years after 1970 **(c)** $S_e(110) \approx$ \$4,576,436, $S_q(110) \approx$ \$3,088,255, **(d)** Exponential model: 51.8%, Quadratic model: 2.4% **(e)** \$3,930,000 **(f)** \$4,880,000

31. **(a)** $T(t) = 69.3297(0.9703^t) + 22$°C minutes after the water was removed from the heat source **(b)** 33.4°C **(c)** $\approx$ 141 minutes or 2 hrs and 21 min.

33. **(a)** $S(t) \approx 114.3245 \times 1.1177^t$ million subscribers t years after 2000 **(b)** 279 million subscribers **(c)** 3.3% **(d)** 607 million subscribers **(e)** No, because this is more than the population of the United States.

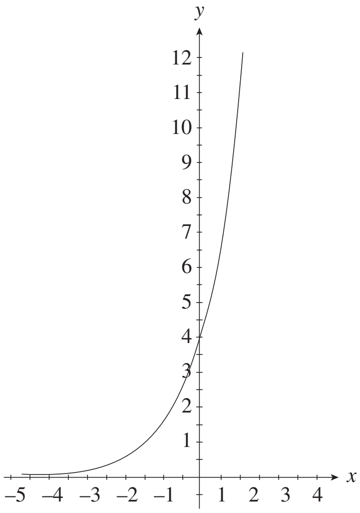
Exercise Set 12.2

1. 20.085 537 **3.** 104.584 986 **5.** 0.018 316

7. 0.063 928

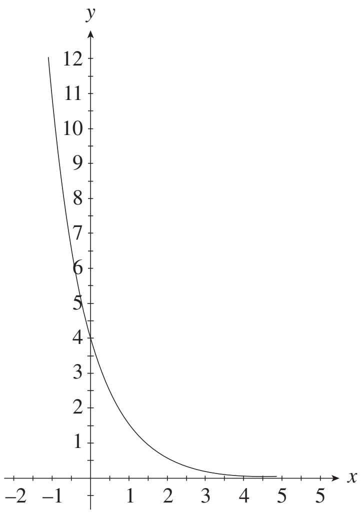
9.

x	-2	-1	0	1	2	3	4
$f(x)$	0.5413	1.4715	4	10.8731	29.5562	80.3421	218.3926



11.

x	-4	-3	-2	-1	0	1	2
$h(x)$	218.3926	80.3421	29.5562	10.8731	4	1.4715	0.5413



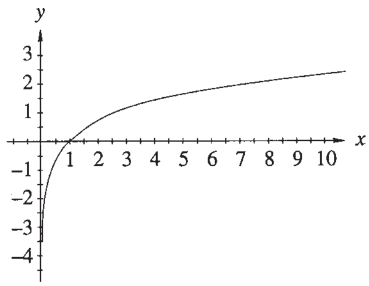
13. **(a)** 125 mg **(b)** 85.91 mg **(c)** 0.31 mg **15.** **(a)** 8660 **(b)** 25,981 **(c)** 260,980 **17.** **(a)** 280,510 **(b)** 393,430 **19.** 110.6°F **21.** 110.26 min or about 1 hour 50 minutes **23.** 0.0066 C **25.** 25.5 μCi **27.** **(a)** 0.05 A **(b)** 0.048 A **(c)** 0.046 A **29.** 23.8 units

Exercise Set 12.3

1. $6^3 = 216$ **3.** $4^2 = 16$ **5.** $(\frac{1}{7})^2 = \frac{1}{49}$
7. $2^{-5} = \frac{1}{32}$ **9.** $9^{7/2} = 2{,}187$ **11.** $\log_5 625 = 4$
13. $\log_2 128 = 7$ **15.** $\log_7 343 = 3$ **17.** $\log_5 \frac{1}{125} = -3$
19. $\log_4 128 = \frac{1}{128}$ **21.** 1.609 437 912

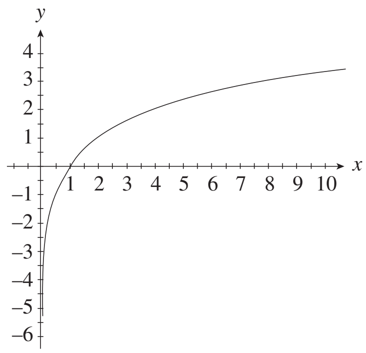
23. 1.558 355 122 **25.** 0.6020599913
27. 1.102776615
29. 1.292 029 674 **31.** 1.125 708 821
33.

x	0.1	0.5	1	2	4	6	8	10
$\ln x$	-2.30	-0.69	0	0.69	1.39	1.79	2.08	2.30



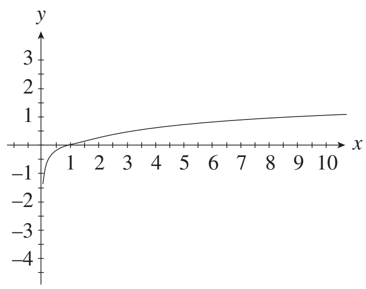
35.

x	0.1	0.5	1	2	4	6	8	10
$\log_2 x$	-3.32	-1	0	1	2	2.58	3	3.32



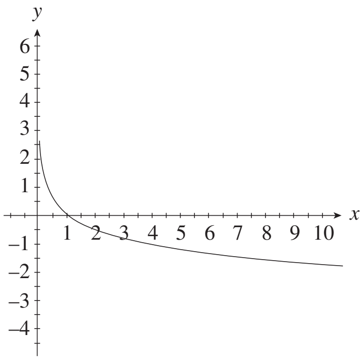
37.

x	0.1	0.5	1	2	4	6	8	10
$\log_{12} x$	-0.93	-0.28	0	0.28	0.56	0.72	0.84	0.93



39.

x	0.1	0.5	1	2	4	6	8	10
$\log_{1/4} x$	1.66	0.5	0	-0.5	-1	-1.29	-1.5	-1.66



41. **(a)** $I = I_0\,10^{\beta/10}$ **(b)** 50 dB **(c)** 90 dB

43. 3,884 ft^3/acre **45.** $E = 10^{9+1.5M_w}$

Exercise Set 12.4

1. $\log 2 - \log 3$ **3.** $\log 2 + \log 7$

5. $\log 2 + \log 2 + \log 3$

7. $\log 2 + \log 3 + \log 5 + \log 5 - \log 7$ **9.** $\log 2 + \log x$

11. $\log 2ax - \log 3y$ **13.** $\log 22$ **15.** $\log \frac{11}{3}$

17. $\log 12$ **19.** $\log \frac{4x}{y}$ **21.** $\log(5^2 \cdot 5^3) = \log 4000$

23. $\log \frac{2}{3} \cdot \frac{6}{7} = \log \frac{4}{7}$ **25.** 0.9030 **27.** 1.0791

29. 1.1761 **31.** 1.5741 **33.** 2.3010 **35.** 3.6990

37. 1.5476 **39.** 0.2851 **41.** 6

43. $\log\left(\frac{I_2\sqrt{R_2}}{I_1\sqrt{R_1}}\right)^{20} = \log\left(\frac{I_2^{20}R_2^{10}}{I_1^{20}R_1^{10}}\right)$

Exercise Set 12.5

1. 2.0922 **3.** 0.5746 **5.** -2.4763 **7.** 2.6513

9. 5.5 **11.** 2.6094 **13.** $10^{2.3} \approx 199.53$ **15.** $10^{17} + 5$

17. $\sqrt{\frac{1}{2}e^9} = \frac{\sqrt{2}\,e^{4.5}}{2} \approx 63.6517$ **19.** $\frac{1}{18} \approx 0.056$

21. 101 **23.** 4.040404 **25.** 11.55 yr **27.** 138.63 yr

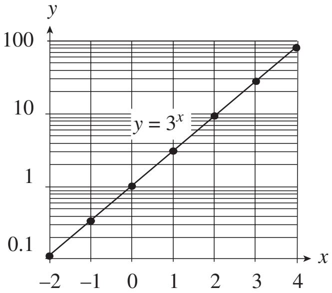
29. 5.5% **31.** 8.15% **33.** 5.6 **35.** 4418.07 m

37. 0.104 s **39.** 0.1354 seconds

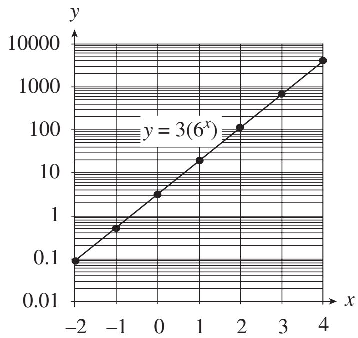
41. $t = \frac{RC}{-2}\ln\left(1 - \frac{2E}{U^2C}\right)$

Exercise Set 12.6

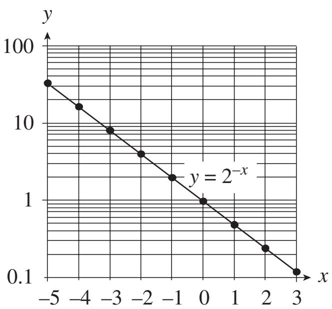
1.



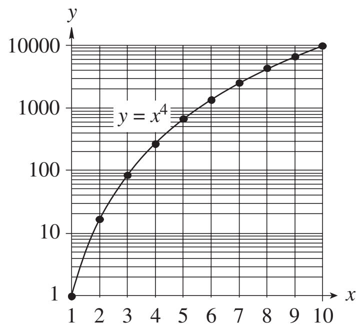
3.



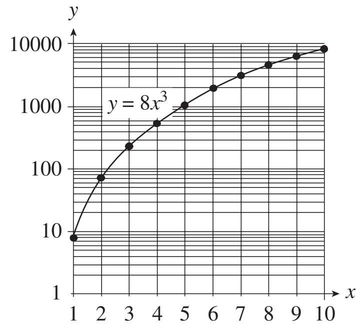
5.



7.



9.



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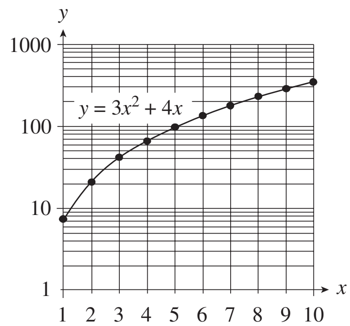

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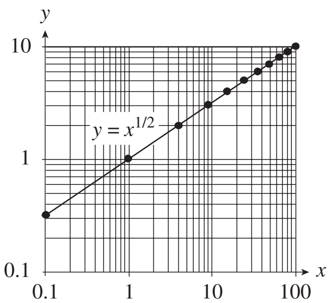

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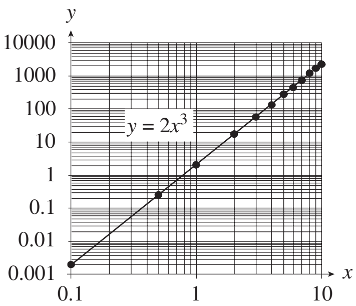

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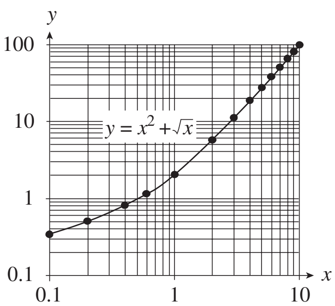

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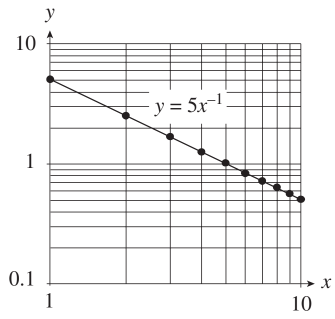
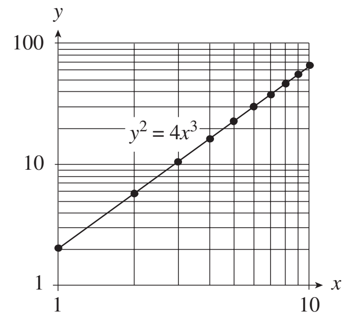
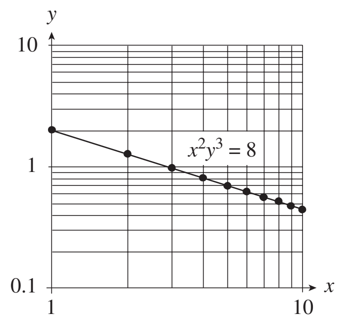


21.



23.



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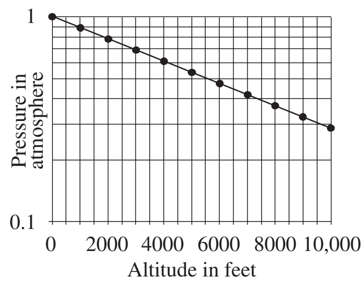
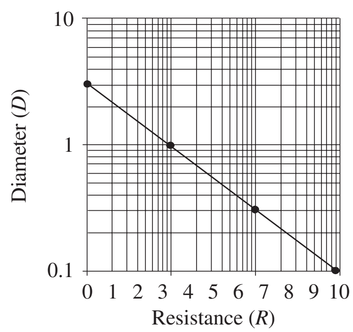
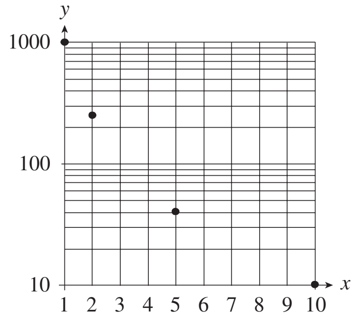


27.



29.



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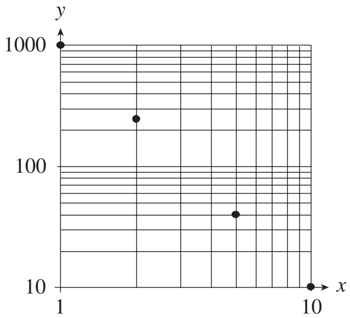
$y = 1000\,x^{-2}$



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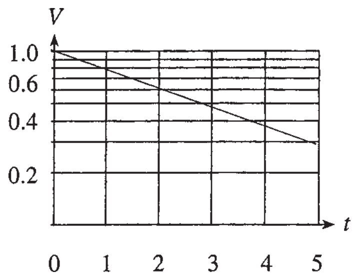
31.



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33. $V(t) \approx 214.6375\,(0.8901^t)$

Review Exercises

1. 148.413 16 **3.** 106.697 74 **5.** 0.903 09

7. 4.398 146

9.

x	-2	-1	0	1	2	3	4
$5e^x$	0.677	1.839	5	13.59	36.95	100.43	272.99

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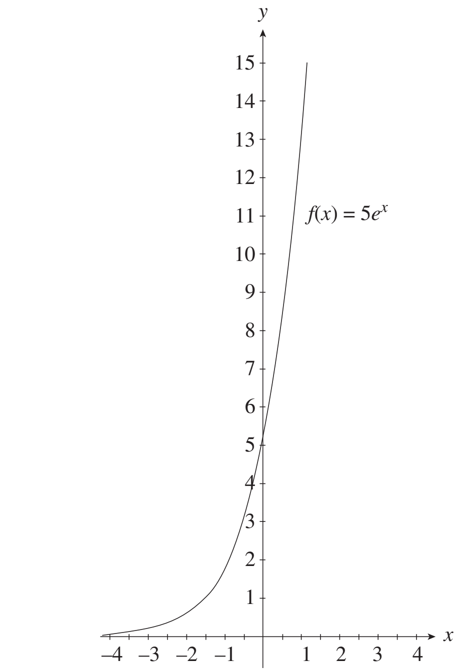

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11.

x	-3	-2	-1	0	1	2
5^x	0.008	0.04	0.2	1	5	25

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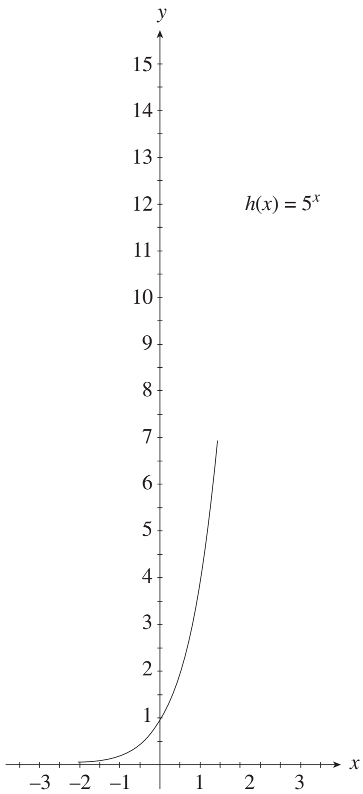

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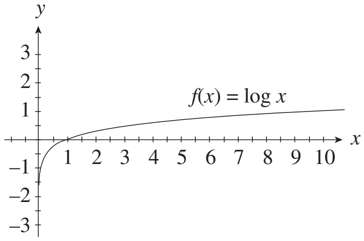
13.

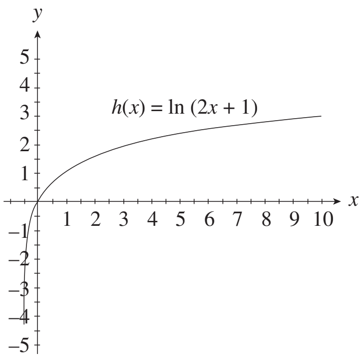
x	0	1	2	3	4	5
$\log x$	not defined	0	0.301	0.477	0.602	0.699

x	6	7	8	9	10
$\log x$	0.778	0.845	0.903	0.959	1



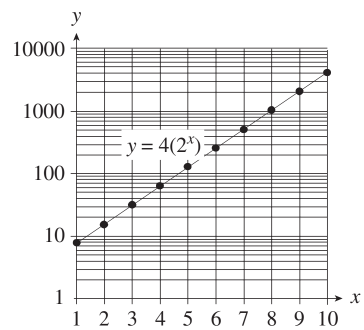
15.

x	$-\frac{1}{2}$	0	1	2	3	4	5	6	7	8	9	10
$\ln(2x + 1)$	not defined	0	1.099	1.609	1.946	2.197	2.398	2.565	2.708	2.833	2.944	3.045

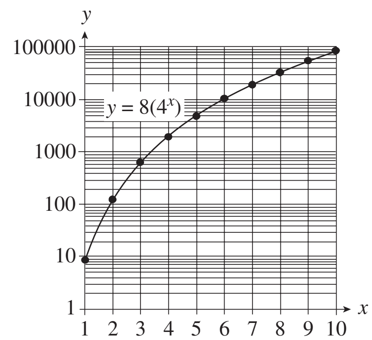


17. $\log 3 - \log 4$ **19.** $\log 5 + \log x$ **21.** $3(\ln 4 + \ln x)$
23. $\frac{1}{2}(\log 6 + \log 5)$ **25.** $\log 45$ **27.** $\log x^5$
29. $\log x^8$ **31.** 2.404 **33.** 0.480 **35.** $e^{9.1} \approx 8955.29$
37. $0.5 \times 10^{50} - 0.5$

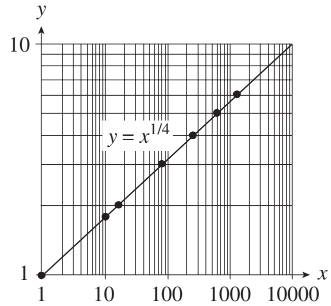
39.



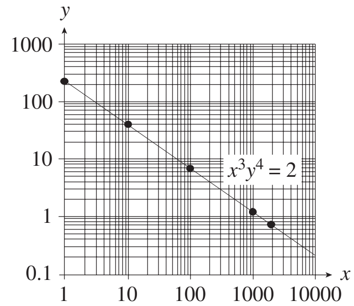
41.



43.



45.



47. **(a)** \$1414.79 in the first bank (compounded semiannually)
(b) \$1059.98 in the second bank (compounded monthly)
(c) \$1419.07 in the third bank (compounded continuously)
(d) 19.80 years

49. **(a)** 10,000 **(b)** 156,250 **51.** $y = 0.4(6.2764)^x$

53. **(a)** $I = I_0 \cdot 10^{\frac{6-M}{2.5}}$ **(b)** $10^{5.8} \approx 630{,}957.34$ times

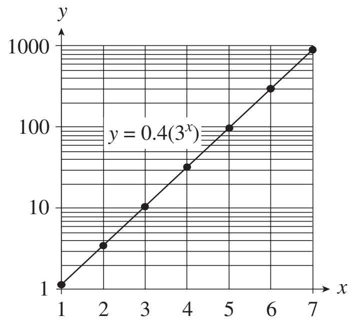
55. **(a)** $B(t) \approx 255.6676(1.0494^t)$ billion dollars t years after 1990 **(b)** $B(20) \approx 670.0729$ billion dollars **(c)** $B_L(t) \approx 15.5198t + 251.7790$ billion dollars t years after 1990
(d) $B_L(20) \approx 562.1741$ billion dollars

Chapter 12 Test

1. 200.33681 **3.** 4.281 515 453
5. log 5 + log 13 + log x

7. $\frac{1}{5}$ (log 5 + 3 log x) − log 9 − log x **9.** $\log \dfrac{17}{x^2}$
11. $e^{17.2} \approx 29\,502\,925.92$ **13.** 3.236
15. $10^5 - 2 \approx 99\,998$
17.

x	1	2	3	4	5	6	7
$0.4(3^x)$	1.2	3.6	10.8	32.4	97.2	291.6	874.8



19. 7,200 ft^3/acre **21.** 92.07 yr

ANSWERS FOR CHAPTER 13

Exercise Set 13.1

1. $\frac{1}{4}$ **3.** $\frac{1}{13}$ **5.** $\frac{3}{13}$ **7.** $\frac{1}{4}$ **9.** $\frac{13}{51}$ **11.** $\frac{1}{6}$
13. $\frac{1}{2}$ **15.** $\frac{1}{36}$ **17.** $\frac{11}{36}$
19.

Die 1\Die 2	1	2	3	4	5	6
1	(1, 1)	(1, 2)	(1, 3)	(1, 4)	(1, 5)	(1, 6)
2	(2, 1)	(2, 2)	(2, 3)	(2, 4)	(2, 5)	(2, 6)
3	(3, 1)	(3, 2)	(3, 3)	(3, 4)	(3, 5)	(3, 6)
4	(4, 1)	(4, 2)	(4, 3)	(4, 4)	(4, 5)	(4, 6)
5	(5, 1)	(5, 2)	(5, 3)	(5, 4)	(5, 5)	(5, 6)
6	(6, 1)	(6, 2)	(6, 3)	(6, 4)	(6, 5)	(6, 6)

21. $\frac{1}{36}$ **23.** 0 **25.** $\frac{1}{6}$ **27.** $\frac{1}{6}$ **29.** $\frac{1}{1{,}296}$ **31.** $\frac{4}{49}$ **33.** $\frac{4}{7}$
35. **(a)** $\frac{9}{16} = 56.25\%$ **(b)** $\frac{1}{16} = 6.25\%$ **(c)** $\frac{3}{8} = 37.5\%$
37. **(a)** 0.56 **(b)** 0.06 **(c)** 0.38
39. 0.000 625 **41.** $\frac{1}{10} = 0.10$

Exercise Set 13.2

1.

Number	2	3	4	6	7
Frequency	2	1	4	2	1
Relative frequency (%)	20	10	40	20	10

3.

Number	73	74	77	80	82	83	84	87	89	92	94	96	100
Frequency	1	1	1	1	1	1	2	1	1	1	3	5	1
Relative frequency (%)	5	5	5	5	5	5	5	10	5	5	15	25	5

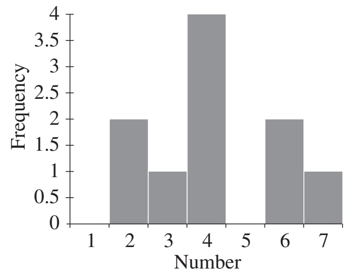
5.

Number	1–2	3–4	5–6	7–8
Frequency	2	5	2	1
Relative frequency (%)	20	50	20	10

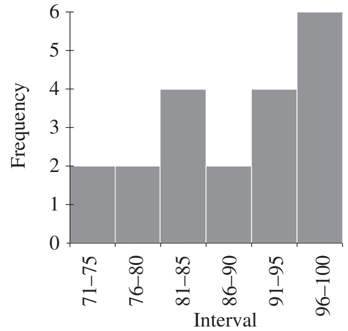
7.

Number	71–75	76–80	81–85	86–90	91–95	96–100
Frequency	2	2	4	2	4	6
Relative frequency (%)	10	10	20	10	20	30

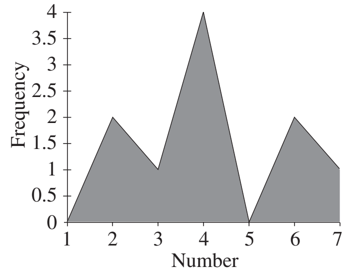
9.



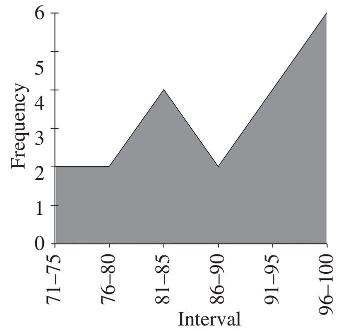
11.



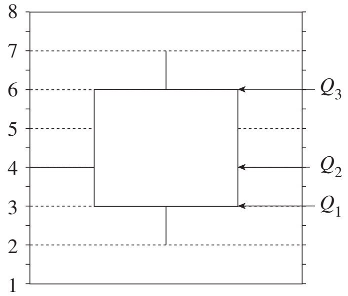
13.



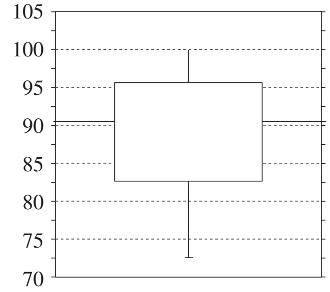
15.



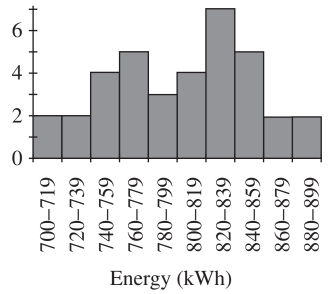
17. Mean: 4.2; Median: 4; Mode: 4; $Q_1 = 3$; $Q_2 = 4$; $Q_3 = 6$



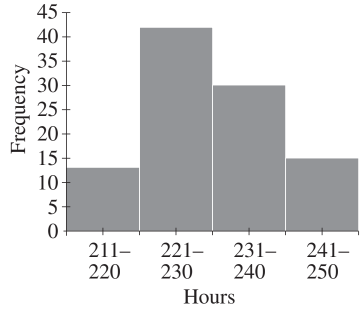
19. Mean: 88.35; Median: 90.5; Mode: 96; $Q_1 = 82.5$; $Q_2 = 90.5$; $Q_3 = 96$



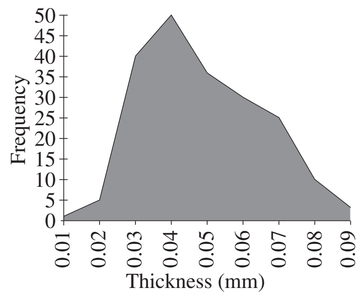
21.



23.



25.



27. $\bar{x}$ = mean

$$= \frac{0.01 \times 1 + 0.02 \times 5 + 0.03 \times 40 + 0.04 \times 50 + 0.05 \times 36 + 0.06 \times 30 + 0.07 \times 25 + 0.08 \times 10 + 0.09 \times 3}{200};$$

$\bar{x} = 0.04865$; median $= 0.05$; mode $= 0.04$; $Q_1 = 0.04$; $Q_2 = 0.05$; $Q_3 - = 0.06$

29.

mA	5.24	5.26	5.27	5.28	5.29	5.31	5.34	5.35	5.39	5.42	5.43	5.44	5.45	5.46	5.47
Frequency	1	2	2	1	2	3	4	3	2	2	1	1	1	3	2

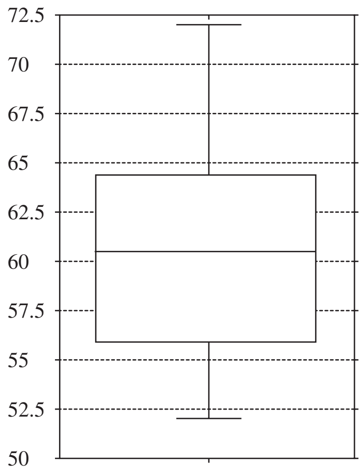
31. $\bar{x} = \frac{160.76}{30} = 5.3587$; median $= 5.345$; mode $= 5.34$

33. **(a)** Sorting the data from low to high, we get

52	54	55	57	59	59
62	63	64	65	67	72

The sum of the data is 729; hence the mean is $\frac{729}{12} = 60.75$. The median is halfway between the 6th and 7th numbers, 59 and 62, so the median is $\frac{59 + 62}{2} = \frac{121}{2} = 60.5$. The mode is 59. Q_1 is halfway between the 3rd and 4th numbers, 55 and 57, so $Q_1 = \frac{55 + 57}{2} = \frac{112}{2} = 56$. $Q_2 =$ median $= 60.5$. $Q_3 = \frac{64 + 65}{2} = \frac{129}{2} = 64.5$

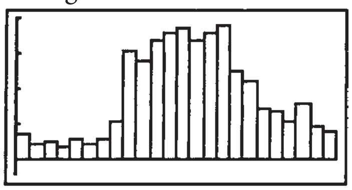
(b)



35. Mean $= 659.25$, Median $= 626$, Mode $= 892$, $Q_1 = 528$, $Q_2 = 626$, $Q_3 = 800$

37. Mean $= 25{,}905.65$, Median $= 19{,}478$, Mode: None, $Q_1 = 15{,}122$, $Q_2 = 19{,}478$, $Q_3 = 38{,}646$

39. Histogram:

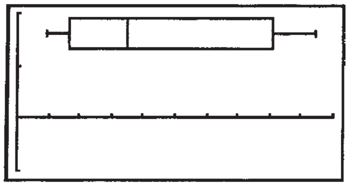


$[0, 24, 1] \times [-100, 1000, 250]$

Broken-line graph:

$[0, 24, 1] \times [-100, 1000, 250]$

Box plot: Set the box plot to get the Xlist from yStat and the Freq from fStat.



$[0, 1000, 100] \times [-1, 2, 1]$

Exercise Set 13.3

1. 2.84 **3.** 6.62 **5.** 1.69 **7.** 8.23 **9.** **(a)** 5 **(b)** 10 **11.** **(a)** 15 **(b)** 20 **13.** **(a)** 0.0217 **(b)** The number of data points between 9.9997 and 10.0431 is 390. **(c)** The number of data points between these two quantities is 480.

15. **(a)** 0.01164 **(b)** Number of points is 156 **(c)** Number of data points is 197

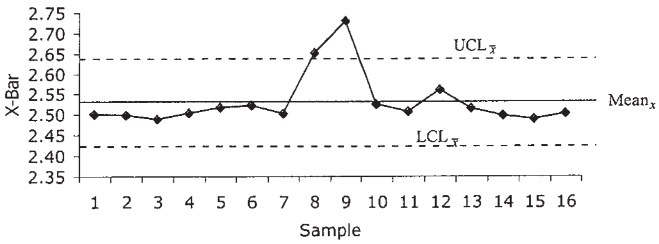
17. **(a)** $\bar{x} = 60.75$ $s = 5.85$ **(b)** 66.7% **(c)** 100%

19. **(a)** $\bar{x} = 659.25$ $s = 174.67$ **(b)** 58.3% **(c)** 95.8%

21. **(a)** $\bar{x} = 25{,}905.65$ $s = 13{,}280.32$ **(b)** 74.2% within one standard deviation **(c)** 93.5% within two standard deviations **23.** **(a)** mean: 458.58; standard deviation: 306.64 **(b)** One standard deviation is a range from 151.94 to 765.22. The number of hits within this range is 4152. There are a total of 11006 hits. Therefore, the percent within 1 standard deviation is 4152/11006, which is approximately 38%. **(c)** 100%

Exercise Set 13.4

1. Central line: $y = 254.37$, $UCL_{\bar{x}} = 254.66155$, $LCL_{\bar{x}} = 254.07845$
3. Central line: $y = 16.4$, $UCL_{\bar{x}} = 24.3178$, $LCL_{\bar{x}} = 8.4822$
5. Central line: $y = 453.0$, $UCL_{\bar{x}} = 459.6906$, $LCL_{\bar{x}} = 446.3094$
7. Central line: $y = 22.25$, $UCL_{\bar{x}} = 25.775$, $LCL_{\bar{x}} = 18.725$
9. Central line: $y = 2.53$, $UCL_R = 2.644$, $LCL_R = 2.416$
11.



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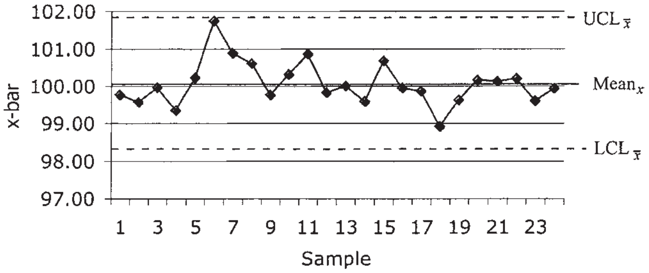
13. Central line: $y = 100.04$, $UCL_R = 101.797$, $LCL_R = 98.283$
15.



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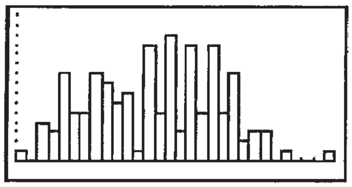
17. Central line: $y = 0.041$, $UCL_p = 0.162$, $LCL_p = 0$
19. Central line: $y = 0.036$, $UCL_p = 0.161$, $LCL_p = 0$
21. about 0.000667 mm

Review Exercises

1. $\frac{1}{2}$ **3.** $\frac{5}{8}$ **5.** $\frac{1}{3}$ **7.** $\frac{1}{36}$ **9.** $\left(\frac{23}{25}\right)^3 = \frac{12167}{15625}$
11. $Q_1 = 299.7$; $Q_2 = 300.1$; $Q_3 = 300.45$
13. 300.1 **15.** 0.931 **17.** 90.9%
19.



$[4.36, 4.392, 0.001] \times [-1, 15, 1]$

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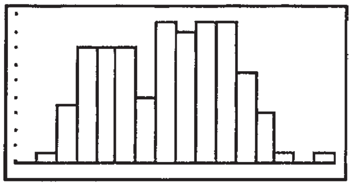

$[4.36, 4.392, 0.001] \times [-1, 18, 2]$

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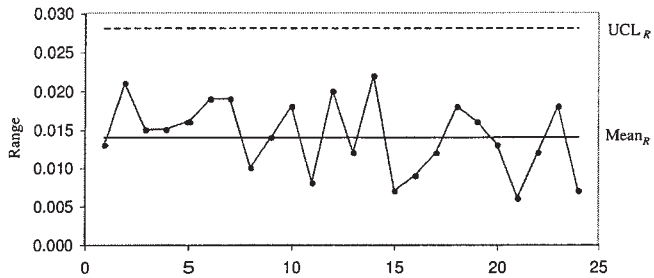
21. $Q_1 = 4.3695$, $Q_2 = 4.375$, $Q_3 = 4.3795$
23. 4.376 in. **25.** $s \approx 0.006$ **27.** $\approx 97.92\%$
29. $y = 0.014$, $UCL_R = 0.028$, $LCL_R = 0$
31.



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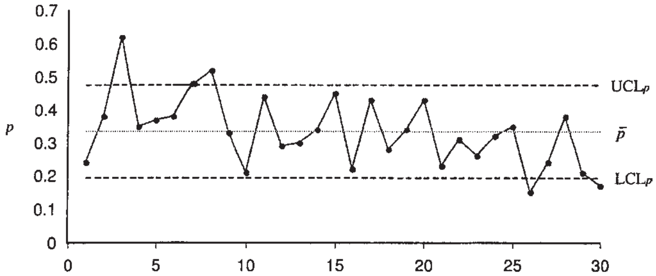
33. $\sigma = 0.0005$ in.
35. $s \approx 10.74$
37.



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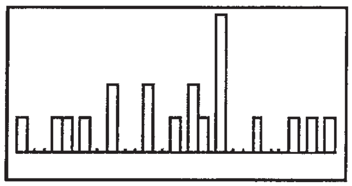
Chapter 13 Test

1. $\frac{1}{26}$
3.



$[3598.2, 3601.7, 0.1] \times [-0.5, 4.02, 1]$

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5. $Q_1 = 3599.2$, $Q_2 = 3600.1$, $Q_2 = 3600.4$
7. 3,600.4 rpm
9. Solution $s \approx 0.94$
11. 100%
13. $y = 2.22$, $UCL_R \approx 5.07$, $LCL_R = 0$

ANSWERS FOR CHAPTER 14

Exercise Set 14.1

1. $5j$ **3.** $0.2j$ **5.** $5j\sqrt{3}$ **7.** $-6j\sqrt{5}$ **9.** $\frac{3}{4}j$
11. $-3j$ **13.** -11 **15.** -18 **17.** -10 **19.** -42
21. $-5j$ **23.** $-j$ **25.** $\frac{2}{3}j$ **27.** $\frac{3}{20}j$ **29.** $-\frac{5}{8}$
31. $0.5j$ **33.** $-2j\sqrt{8.1} = -6j\sqrt{0.9}$ **35.** $-2j\sqrt{15}$
37. $-14j$ **39.** $x = 7, y = -2$ **41.** $x = 10, y = -3$
43. $x = 2, y = -2$ **45.** $x = -\frac{1}{2}, y = -\frac{5}{4}$ **47.** $x = 3.5, y = -4.3$ **49.** $8 + 10j$ **51.** $-10 - 5j$
53. $2 + 3j$ **55.** $-6 + 8j$ **57.** $3 - 4j$ **59.** $1 - 2j$
61. $2 + 3j\sqrt{2}$ **63.** $2 - \frac{1}{2}j\sqrt{6}$ **65.** $\frac{1}{2} - \frac{2}{9}j$
67. $\frac{5}{8} - \frac{2}{3}j$ **69.** $3.6 - 4.5j$ **71.** $7 - 2j$ **73.** $6 + 5j$
75. 19 **77.** $8j$ **79.** $\sqrt{2} - 7.3j$ **81.** $-\frac{1}{2} + \frac{3}{2}j$, $-\frac{1}{2} - \frac{3}{2}j$ **83.** $3j, -3j$ **85.** $-\frac{3}{4} + \frac{j\sqrt{47}}{4}$, $-\frac{3}{4} - \frac{j\sqrt{47}}{4}$ **87.** $-\frac{1}{5} \pm \frac{2j\sqrt{6}}{5}$, $-\frac{1}{10} - \frac{j\sqrt{6}}{5}$
89. $6.25 + 49.61j\ \Omega$ or $6.25 - 49.61j\ \Omega$
91. $R = \pm\sqrt{\frac{X}{B}} - X^2$

Exercise Set 14.2

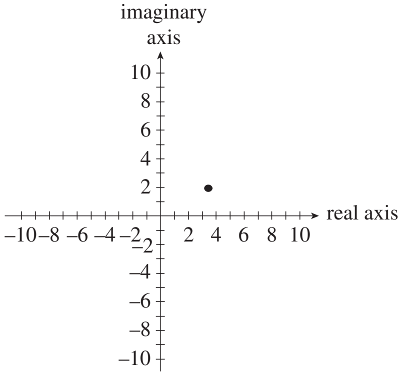
1. $-1 + 7j$ **3.** $5 - 2j$ **5.** $7 - j$
7. $2 - \sqrt{5} + 11j$ **9.** $8 + 2j$ **11.** $3 - 6j$
13. $7 - 2j$ **15.** $-3 + 6j$ **17.** $10 - 45j$
19. $7 + 11j$ **21.** $36 + 8j$ **23.** $-126 + 48j$
25. 9 **27.** $-3 + 4j$ **29.** $48 - 14j$ **31.** 29
33. $1 - 5j$ **35.** $-3j$ **37.** $2j$ **39.** $\frac{1}{13} + \frac{5}{13}j$
41. $-\sqrt{2} - j$ **43.** $4.059 - 0.765j$ **45.** -4
47. $-0.5 + \frac{j\sqrt{3}}{2}$ **49.** $(a + bj) + (a - bj) = 2a$, which is a real number. **51.** $(a + bj)(a - bj) = a^2 + b^2$, which is a real number.
53. $10.53 - 4.21j$ V **55.** $5.08 + 2.88j$ V
57. $5.84 + 1.16j\ \Omega$ **59.** $0.4 - 0.3j\ \Omega$
61. $0.74 + 1.12j$A

63. The following program will work on a TI-82 graphing calculator.

```
Disp "COMPLEX"
: ClrHome
: Disp "ENTER THE FIRST"
: Disp "NUMBER AS A + BJ"
: Prompt A, B
: ClrHome
: Disp "ENTER THE SECOND"
: Disp "NUMBER AS C + DJ"
: Prompt C, D
: ClrHome
: Disp "THE SUM IS"
: Disp "     REAL"
: Disp A + C
: Disp "     IMAGINARY"
: Disp B + D
: Disp " "
: Disp "PRESS <ENTER>"
: Pause
: ClrHome
: Disp "THE DIFFERENCE"
: Disp "IS REAL"
: Disp A - C
: Disp "     IMAGINARY"
: Disp B - D
: Disp " "
: Disp "PRESS <ENTER>"
: Pause
: ClrHome
: Disp "THE PRODUCT IS"
: Disp "     REAL"
: Disp AC - BD
: Disp "     IMAGINARY"
: Disp BC + AD
: Disp " "
: Disp "PRESS <ENTER>"
: Pause
: ClrHome
: Disp "THE QUOTIENT IS"
: Disp "     REAL"
: Disp (AC + BD) / (C^2 + D^2)▶Frac
: Disp "     IMAGINARY"
: Disp (BC - AD) / (C^2 + D^2)▶Frac
```

Exercise Set 14.3

1. $2\sqrt{3} + 2j$



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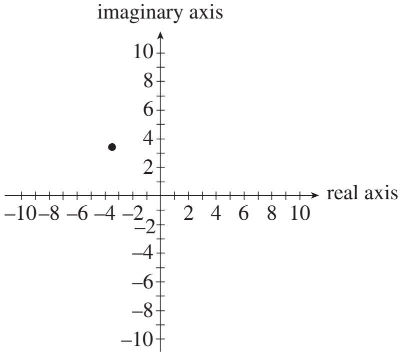
3. $-\dfrac{5\sqrt{2}}{2} + \dfrac{5j\sqrt{2}}{2} \approx -3.5355 + 3.5355j$



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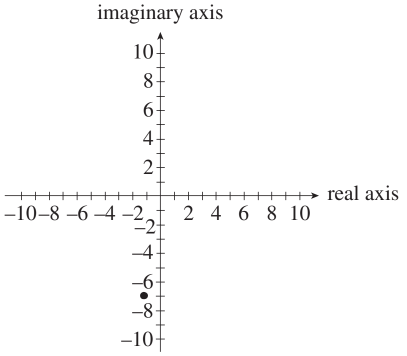
5. $-1.2155 - 6.8937j$



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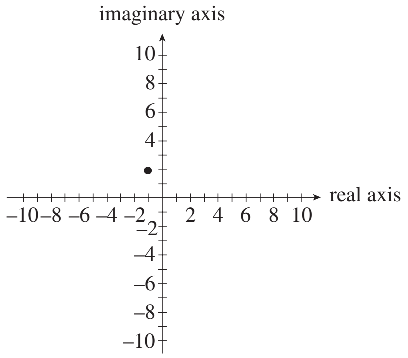
7. $-0.8452 + 1.8126j$



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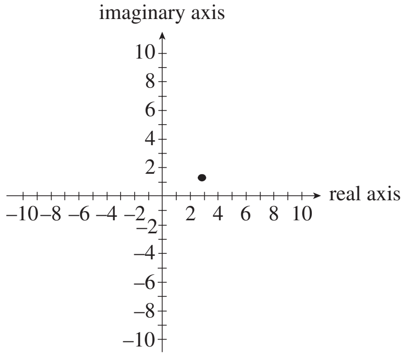
9. $2.7189 + 1.2679j$



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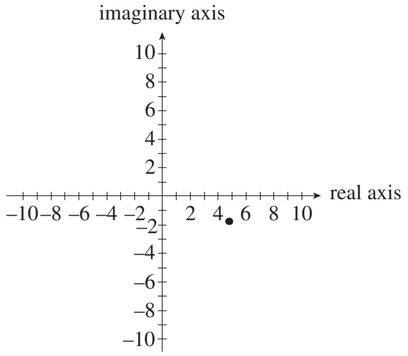
11. $4.6985 - 1.7101j$



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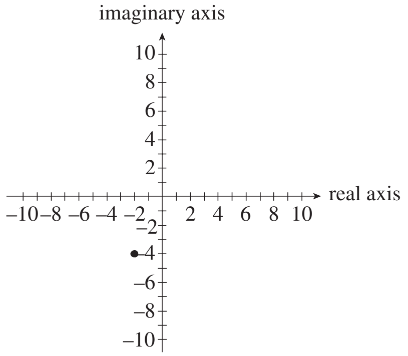
13. $-1.9018 - 4.0784j$



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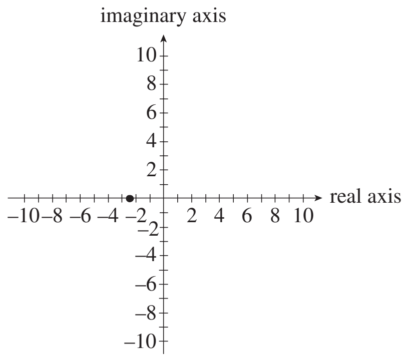
15. -2.5



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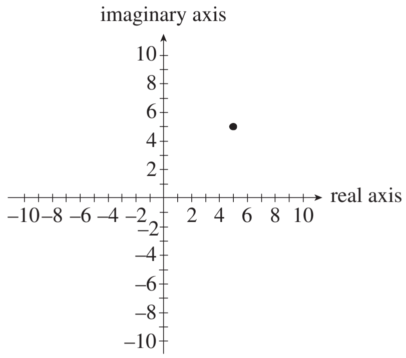
17. $7.071 \angle 45^\circ$



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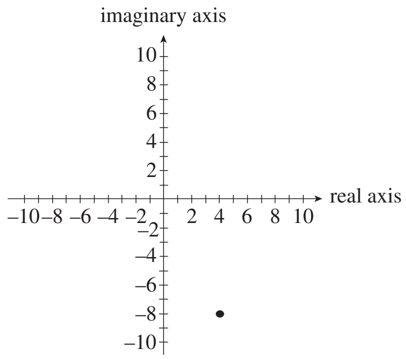
19. $8.944 \angle 296.6^\circ$



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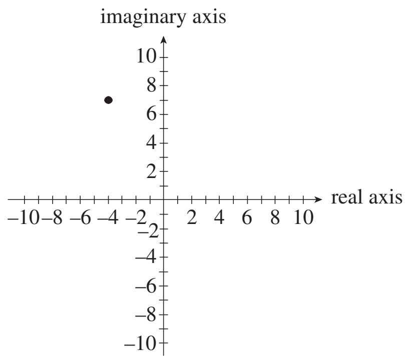
21. $8.0623 \angle 119.7^\circ$



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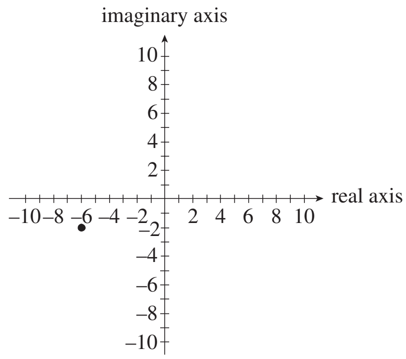
23. $6.3246 \angle 198.4^\circ$



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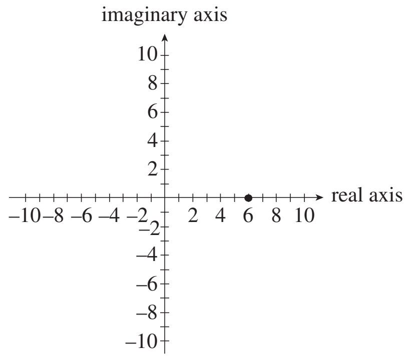
25. $6 \angle 0^\circ$



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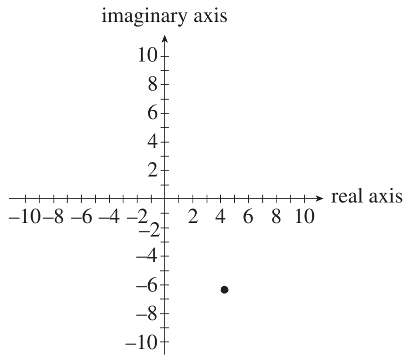
27. $7.5717 \angle 303.7^\circ$



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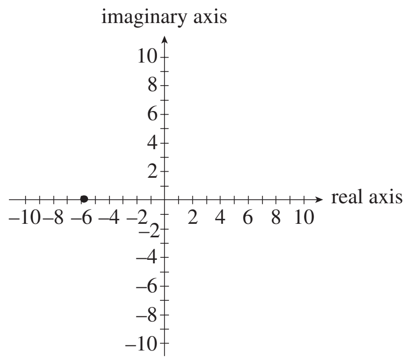
29. $5.8034 \angle 178.0^\circ$



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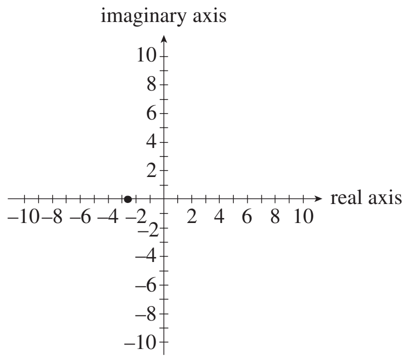
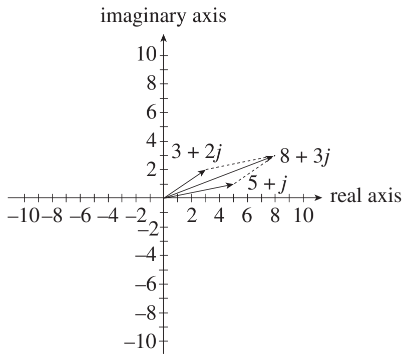
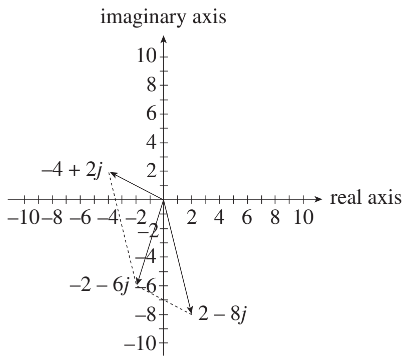
31. $2.7 \underline{/180°}$



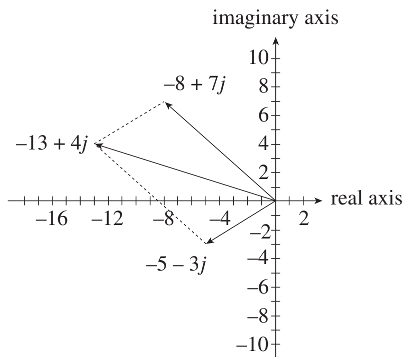
33.



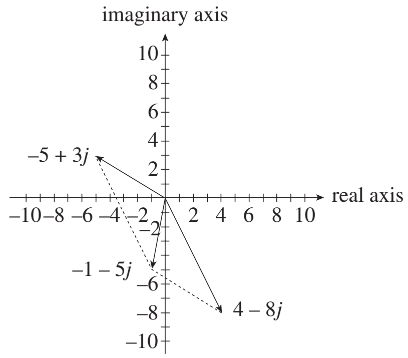
35.



37.



39.



41. $8.0 \underline{/-54.1°} = 8.0 \underline{/305.9°}$ A

43. $119.6 \underline{/157.3°}$ V **45.** $1.61 - 1.92j$ A

Exercise Set 14.4

1. $3e^{3\pi/2\,j} \approx 3e^{4.7\,j}$ **3.** $2e^{1.05j}$ **5.** $1.3e^{5.7j}$

7. $3.1e^{0.44j}$ **9.** $10e^{0.64j}$ **11.** $15e^{2.21j}$

13. 5 cis 0.5 $= 4.3879 + 2.3971\,j$

15. 2.3 cis 4.2 $= -1.1276 - 2.0046\,j$

17. $12e^{5j}$ **19.** $2.4e^{5.9j}$ **21.** $28e^{3.72j}$ **23.** $4e^{2j}$

25. $4.25e^{1.5j}$ **27.** $81e^{8j}$ **29.** $39.0625e^{6j}$ **31.** $2e^{3j}$

33. $2.5e^{2.1j}$ **35.** $61.4e^{0.4031j}$ V

37. In exponential form $Z = 135e^{-0.9163j}\ \Omega$ or $135e^{5.37j}$; in rectangular form $Z = 82.2 - 107.1\,j\ \Omega$

39. $V = 80e^{0.7029j}$ V **41.** $I = 46e^{-0.7937j}$ A

43. $3.4e^{1.074j}\ \Omega$

Exercise Set 14.5

1. $15(\cos 69° + j \sin 69°)$ **3.** $10(\cos 4.1 + j \sin 4.1)$

5. $4(\cos 60° + j \sin 60°)$ **7.** $4.5 \underline{/111°}$ **9.** $12 \underline{/8.0}$

11. $15625(\cos 144° + j \sin 144°)$

13. $1124.864(\cos 10.26 + j \sin 10.26)$

15. $3.4^4 \underline{/21.2} = 113.6336$ **17.** $12e^{3.8j}$ **19.** $0.125e^{0.9j}$

21. $1, -0.5 + 0.866j, -0.5 - 0.866j$

23. $1.732 - j, 2j, -1.732 - j$

25. $1.8478 - 0.7654\,j, 0.7654 + 1.8478\,j, -1.8478 + 0.7654\,j, -0.7654 - 1.8478\,j$

27. $1.0586 + 0.1677\,j, 0.1677 + 1.0586\,j, -0.9550 + 0.5866\,j, -0.7579 - 0.7579\,j$, 0,4866 $- 0.9550\,j$

29. $0.866 - 0.5\,j, j, -0.866 - 0.5\,j$

31. $1.9319 + 0.5176\,j, 0.5176 + 1.9319\,j, -1.4142 + 1.4142\,j, -1.9319 - 0.5176\,j, -0.5176 - 1.9319\,j, 1.4142 - 1.4142\,j$ **33.** $108 \underline{/19°}$

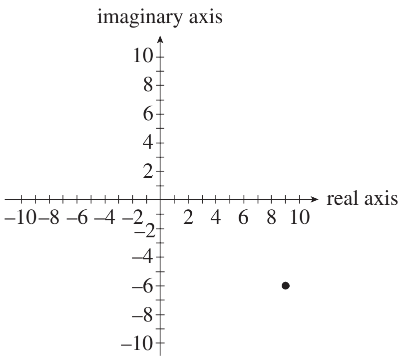
35. $80 \angle -36.87°$ V **37.** $Z = -0.0448 + 0.1536\,j = 0.16 \angle 106.3°\ \Omega$
39. **(a)** $0.2137 \angle -20.56°$ S **(b)** about $0.200 - 0.075\,j$ S

Exercise Set 14.6

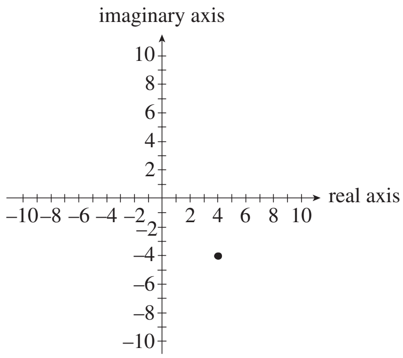
1. **(a)** $3 - 2\,j$ **(b)** $5 + j$ **3.** **(a)** $1 + 2\,j$ **(b)** $1.8 - 0.6\,j$ **5.** **(a)** $6.803 \angle 53.26°$ **(b)** $1.6611 \angle 32.44°$
7. **(a)** $8.208 \angle 0.449$ **(b)** $1.226 \angle 0.45$ **9.** **(a)** $12 + 5\,j$ **(b)** $1.9207 + 0.2387j$ **11.** **(a)** $6.595 \angle -2.95°$ **(b)** $0.916 \angle 29.32°$ **13.** $-13 - 84\,j$ V **15.** 2 V
17. $2.8 \angle 23.7°\ \Omega$ **19.** $X_L = 75.40\ \Omega$, $X_C = 66.31\ \Omega$, $Z = 39.07\ \Omega$, $\phi = 13.45°$ **21.** $X_L = 150.80\ \Omega$, $X_C = 44.21\ \Omega$, $Z = 108.45\ \Omega$, $\phi = 79.37°$
23. $X_L = 12.5\ \Omega$, $X_C = 100\ \Omega$, $Z = 91.87\ \Omega$, $\phi = -72.26°$ **25.** **(a)** $77.62 \angle 14.93°$ **(b)** 271.67 V
27. **(a)** $\sqrt{10} \angle 18.43°$ **(b)** 9.01 V **29.** $2 - 9\,j$
31. 22.51 Hz **33.** $0.16 - 0.12\,j$

Review Exercises

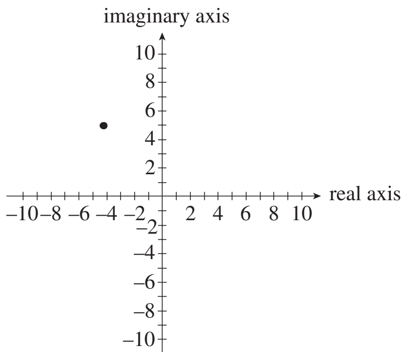
1. $7j$ **3.** $3\sqrt{6}j$ or $3\,j\sqrt{6}$ **5.** -6 **7.** $9 - 3\,j$
9. $-1 + 4\,j$ **11.** $-36 + 8\,j$ **13.** 25
15. $10.82 \angle -33.69°$



17. $5.657 \angle -45°$



19. $-4.331 + 4.847\,j$



21. -10 **23.** $-4{,}782{,}969\,j$ **25.** $-26.080 + 6.988\,j$ **27.** 4096 **29.** $0.866 - 0.5\,j$, j, $-0.866 - 0.5\,j$ **31.** $2 + 3.464\,j$, $-2 - 3.464\,j$
33. $18.385e^{0.391j}$ **35.** $6740.6e^{2.702j}$
37. magnitude: 7.433, direction: $-10.86°$
39. $X_L = 113.10\ \Omega$, $X_C = 53.05\ \Omega$, $Z = 81.43$, $\phi = 47.51°$

Chapter 14 Test

1. $4\,j\sqrt{5}$ **3.** $-4\sqrt{3} + 4\,j$ **5.** $-13 + 8\,j$
7. $-\frac{38}{25} - \frac{9}{25}j = -1.52 + 0.36\,j$ **9.** $\frac{4}{3}j$
11. $2.1941 + 2.0460\,j$ **13.** **(a)** $9 - j\ \Omega$ **(b)** $\frac{118}{41} - \frac{4}{41}j\ \Omega \cong 2.878 + 0.098\,j$
15. $0.28 + 1.04\,j\ \Omega$

ANSWERS FOR CHAPTER 15

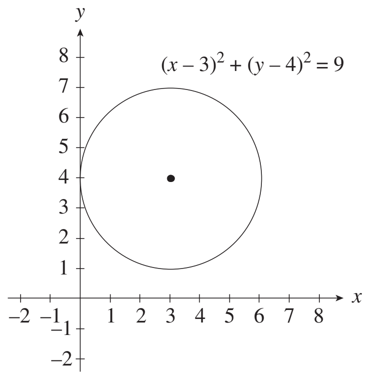
Exercise Set 15.1

1. $\sqrt{194}$ **3.** $\sqrt{65}$ **5.** $\sqrt{277}$ **7.** 7
9. $\left(\frac{9}{2}, -\frac{5}{2}\right)$ **11.** $\left(1, -\frac{11}{2}\right)$ **13.** $\left(\frac{15}{2}, -6\right)$ **15.** $\left(\frac{3}{2}, 5\right)$
17. $L \approx 78.65$ **19.** $L \approx 27.71$ **21.** $L \approx 4.63$, $L \approx 4.64$
23. $L \approx 3.79$, $L \approx 3.81$ **25.** $-\frac{13}{5}$ **27.** $-\frac{1}{8}$ **29.** $\frac{14}{9}$
31. 0 **33.** $y = -\frac{13}{5}x + \frac{46}{5}$ **35.** $y = -\frac{1}{8}x - \frac{43}{8}$
37. $y = \frac{14}{9}x - \frac{53}{3}$ **39.** $y = 5$ **41.** 1

43. approximately 0.1511352 **45.** 68.19859° **47.** 153.43495° **49.** $-\frac{1}{3}$ **51.** 2 **53.** $y + 5 = 6(x - 2)$ or $y = 6x - 17$ **55.** $y + 2 = -\frac{5}{2}(x - 4)$ *or* $y = -\frac{5}{2}x + 8$ **57.** $y + 4 = 1.732(x + 2)$ or $y = 1.732x - 0.536$ **59.** $y = 0.364x + 3$ **61.** $2x - 3y + 11 = 0$ **63.** $5x - 2y = 22$ **65.** $m = -\frac{3}{2}$, y-intercept $= 6$, x-intercept $= 4$ **67.** $m = \frac{1}{3}$, y-intercept $= -3$, x-intercept $= 9$ **69.** $v = 2.6 + 0.4t$ **71.** 0.8 Ω **73.** **(a)** $C = 1{,}225 + 1.25n$ **(b)** \$26,225 **75.** **(a)** 362.08 ppm **(b)** 441.28 ppm **(c)** Answer varies with year. **77.** about 328.7 m

Exercise Set 15.2

1. $(x-2)^2 + (y-5)^2 = 9; x^2 + y^2 - 4x - 10y + 20 = 0$ **3.** $(x + 2)^2 + y^2 = 16; x^2 + y^2 + 4x - 12 = 0$ **5.** $(x + 5)^2 + (y + 1)^2 = \dfrac{25}{4}; x^2 + y^2 + 10x + 2y + 19.75 = 0$ **7.** $(x-2)^2 + (y+4)^2 = 1; x^2 + y^2 - 4x + 8y + 19 = 0$ **9.** $C = (3, 4), r = 3$



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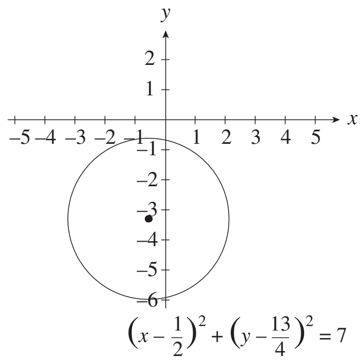
11. $C = \left(-\frac{1}{2}, -\dfrac{13}{4}\right), r = \sqrt{7}$



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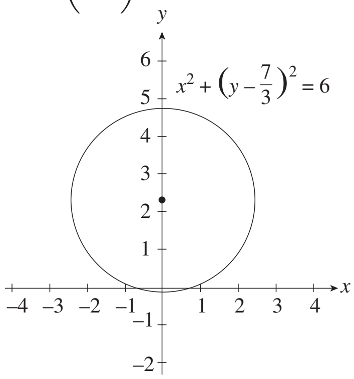
13. $C = \left(0, \frac{7}{3}\right), r = \sqrt{6}$



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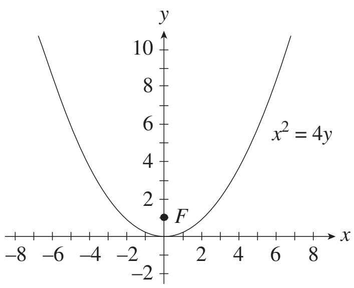
15. Circle, $C = (-2, 3), r = 3$ **17.** Circle, $C = (-5, 3), r = 9$ **19.** Not a circle since $r^2 = -1$ **21.** Circle, $C = (-3, 0), r = 5$ **23.** Circle, $C = \left(-\frac{5}{2}, \frac{9}{2}\right), r = 6$ **25.** Not a circle since $r^2 = -1.3$ **27.** $x^2 + y^2 = 310.03^2$ (Assume circle is at the center of a coordinate system.) **29.** $(x - 1.495 \times 10^8)^2 + y^2 = 1.478 \times 10^{11}; x^2 + y^2 + 4x - 12 = 0$ **31.** $x^2 + (y - 0.9)^2 = 0.55^2$ **33.** **(a)** $x^2 + (y - 5)^2 = 13^2$ **(b)** 28 cm

Exercise Set 15.3

1. $F = (0, 1)$, directrix: $y = -1$, opens upward



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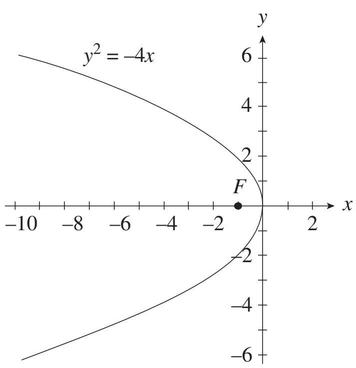
3. $F = (-1, 0)$, directrix: $x = 1$, opens left



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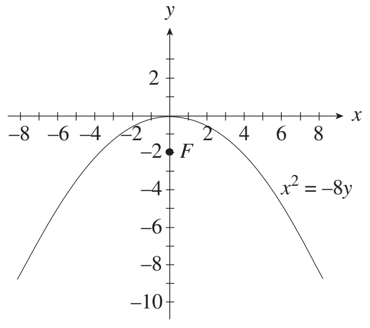
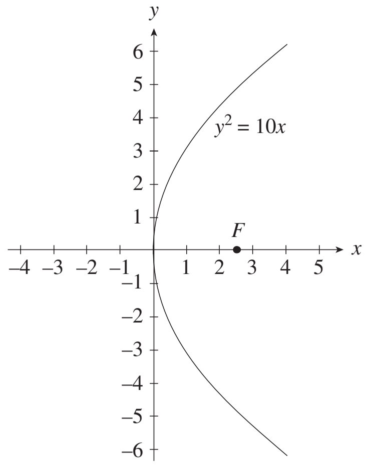
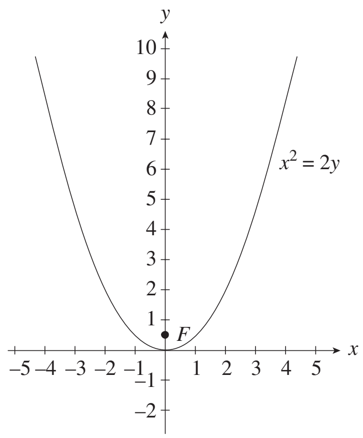
5. $F = (0, -2)$, directrix: $y = 2$, opens downward



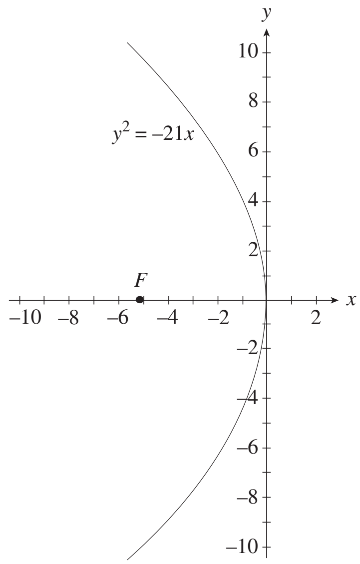
7. $F = \left(\frac{5}{2}, 0\right)$, directrix: $x = -\frac{5}{2}$, opens right



9. $F = (0, \frac{1}{2})$, directrix: $x = -\frac{1}{2}$, opens upward



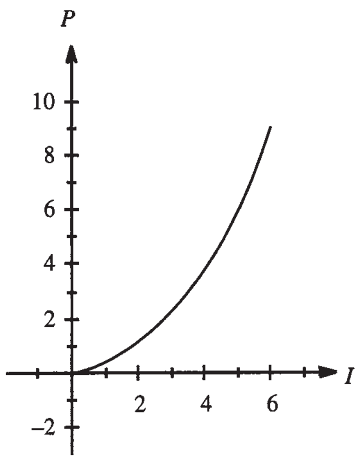
11. $F = \left(-\frac{21}{4}, 0\right)$, directrix: $x = -\frac{21}{4}$, opens left



13. $x^2 = 16y$ **15.** $y^2 = -24x$ **17.** $y^2 = 8x$

19. $y^2 = 6x$ **21.** 27.78 m **23.** $\frac{6.25}{6} \approx 1.042$ m from vertex

25.



27. 3.125 ft

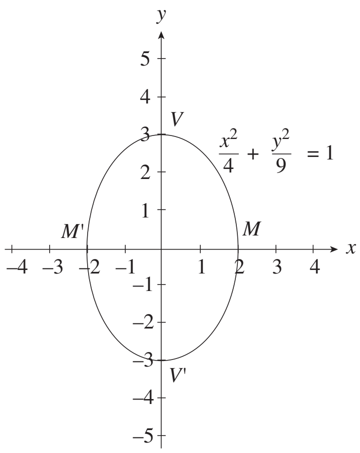
Exercise Set 15.4

1. $\frac{x^2}{36} + \frac{y^2}{20} = 1$ **3.** $\frac{x^2}{12} + \frac{y^2}{16} = 1$

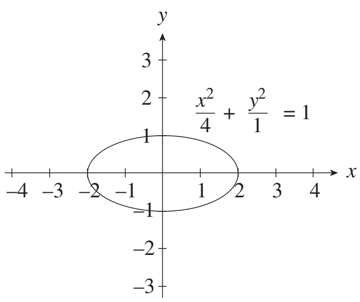
5. $\frac{x^2}{25} + \frac{y^2}{16} = 1$ **7.** $\frac{x^2}{16} + \frac{y^2}{9} = 1$

9. $\frac{x^2}{9} + \frac{y^2}{16} = 1$

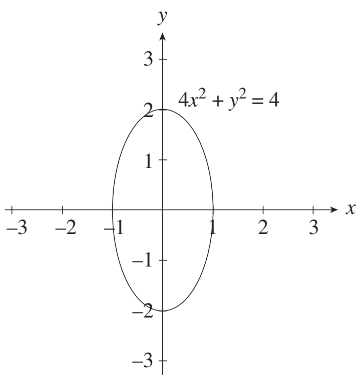
11. $V = (0, 3), V' = (0, -3), F = (0, \sqrt{5})$, $F' = (0, -\sqrt{5}), M = (2, 0), M' = (-2, 0)$



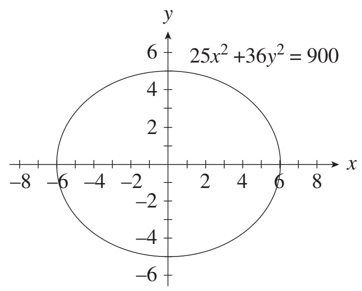
13. $V = (2, 0), V' = (-2, 0), F = (\sqrt{3}, 0)$, $F' = (-\sqrt{3}, 0), M = (0, 1), M' = (0, -1)$



15. $V = (0, 2), V' = (0, -2), F = (0, \sqrt{3})$, $F' = (0, -\sqrt{3}), M = (1, 0), M' = (-1, 0)$



17. $V = (6, 0), V' = (-6, 0), F = (\sqrt{11}, 0)$, $F' = (-\sqrt{11}, 0), M = (0, 5), M' = (0, -5)$



19. 2.992×10^8 km **21.** major axis $= \sqrt{200} \approx 14.14$; minor axis $= 10$ **23.** 0.9999253

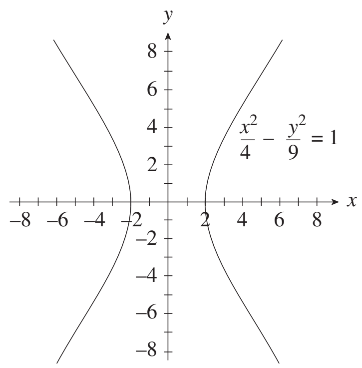
25. $8\sqrt{2} \approx 11.3$ in.

Exercise Set 15.5

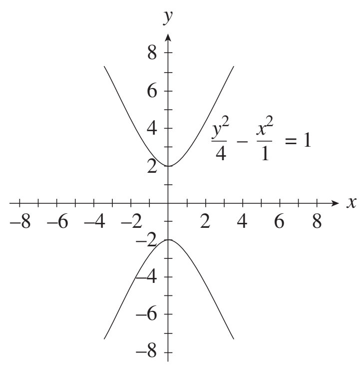
1. $\frac{x^2}{16} - \frac{y^2}{20} = 1$ **3.** $\frac{y^2}{9} - \frac{x^2}{16} = 1$

5. $\frac{x^2}{9} - \frac{y^2}{16} = 1$ **7.** $\frac{x^2}{7} - \frac{y^2}{9} = 1$

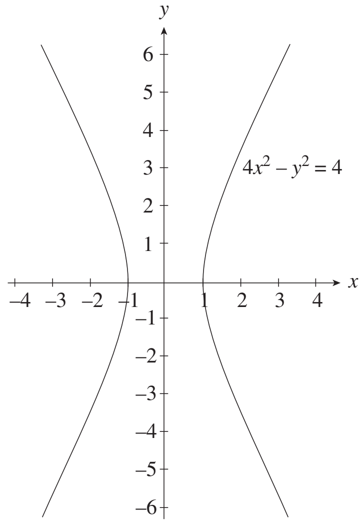
9. $V = (2, 0), V' = (-2, 0), F = (\sqrt{13}, 0)$, $F' = (-\sqrt{13}, 0), M = (0, 3), M' = (0, -3)$



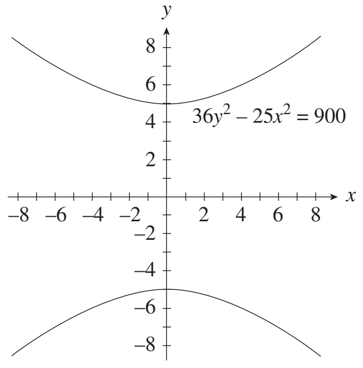
11. $V = (0, 2), V' = (0, -2), F = (0, \sqrt{5})$, $F' = (0, -\sqrt{5}), M = (1, 0), M' = (-1, 0)$



13. $V = (1, 0)$, $V' = (-1, 0)$, $F = (\sqrt{5}, 0)$, $F' = (-\sqrt{5}, 0)$, $M = (0, 2)$, $M' = (0, -2)$

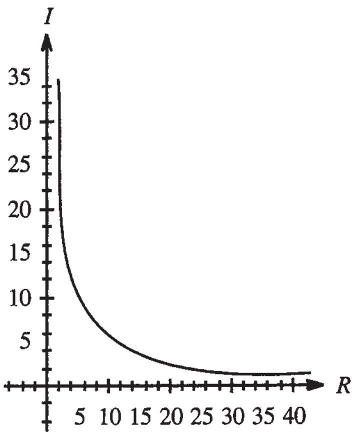


15. $V = (0, 5)$, $V' = (0, -5)$, $F = (0, \sqrt{61})$, $F' = (0, -\sqrt{61})$, $M = (6, 0)$, $M' = (-6, 0)$



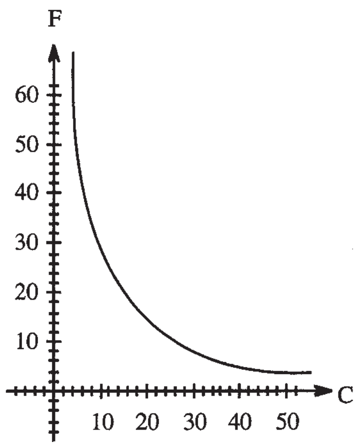
17. $\dfrac{x^2}{49} - \dfrac{y^2}{61.25} = 1$ **19.** $\sqrt{2}$

21.



23. $\dfrac{x^2}{1764} - \dfrac{y^2}{337.5} = 1$

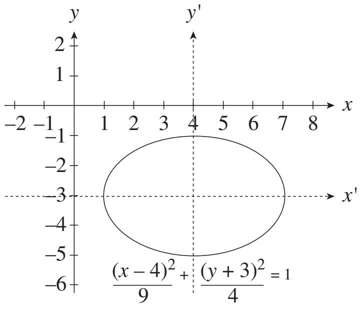
25.



Exercise Set 15.6

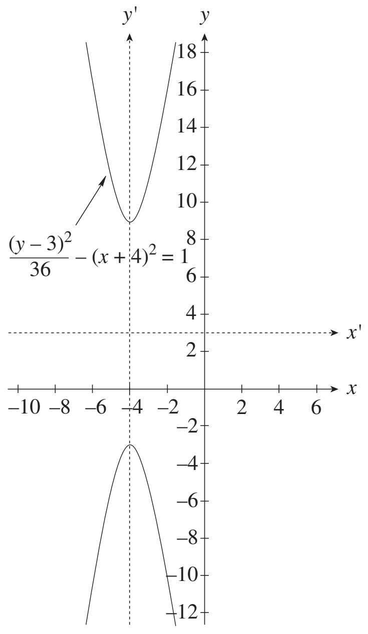
1. ellipse;

	$x'y'$-system	xy-system
center	$(0', 0')$	$(4, -3)$
vertices	$(\pm 3', 0')$	$(7, -3)$, $(1, -3)$
foci	$(\pm\sqrt{5}', 0')$	$(\sqrt{5} + 4, -3)$, $(-\sqrt{5} + 4, -3)$
endpoints of minor axis	$(0', \pm 2')$	$(4, -1)$, $(4, -5)$



3. hyperbola;

	$x'y'$-system	xy-system
center	$(0', 0')$	$(-4, 3)$
vertices	$(0', \pm 6)$	$(-4, 9)$, $(-4, -3)$
foci	$(0', \pm\sqrt{37}')$	$(-4, \sqrt{37} + 3)$, $(-4, -\sqrt{37} + 3)$
endpoints of conjugate axis	$(\pm 2', 0')$	$(-3, 3)$, $(-5, 3)$



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5. hyperbola;

	$x'y'$-system	xy-system
center	$(0', 0')$	$(-5, 3)$
vertices	$(\pm 2', 0')$	$(3, 0), (-7, 0)$
foci	$(2 \pm \sqrt{26}', 0')$	$(2\sqrt{26} - 5, 0)$, $(-2\sqrt{26} - 5, 0)$
endpoints of conjugate axis	$(0', \pm 10')$	$(-5, 10)$, $(-5, -10)$

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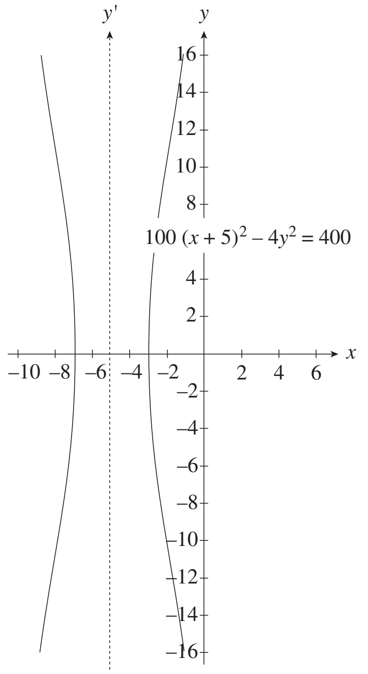

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7. ellipse;

	$x'y'$-system	xy-system
center	$(0', 0')$	$(-2, \frac{3}{2})$
vertices	$(0', \pm 3')$	$(-2, \frac{7}{2}), (-2, -\frac{1}{2})$
foci	$(0', \pm\sqrt{3}')$	$(-2, \sqrt{3} + \frac{3}{2})$, $(-2, -\sqrt{3} + \frac{3}{2})$
endpoints of conjugate axis	$(\pm 1', 0')$	$(-1, \frac{3}{2}), (-3, \frac{3}{2})$

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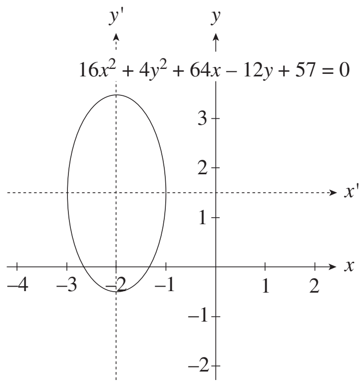

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9. ellipse;

	$x'y'$-system	xy-system
center	$(0', 0')$	$(5, 2)$
vertices	$(0', \pm 5')$	$(5, 7), (5, -3)$
foci	$(0', \pm\sqrt{21}')$	$(5, 2 + \sqrt{21})$, $(5, 2 - \sqrt{21})$
endpoints of conjugate axis	$(\pm 2', 0')$	$(7, 2), (3, 2)$

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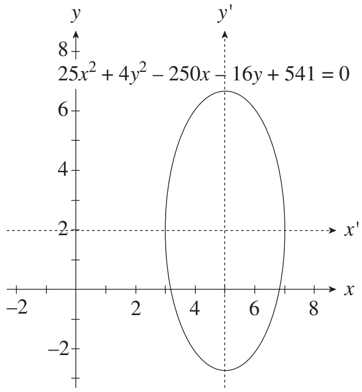

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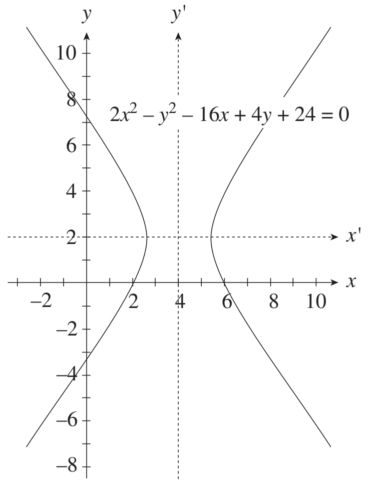
11. hyperbola;

	$x'y'$-system	xy-system
center	$(0', 0')$	$(4, 2)$
vertices	$(\pm\sqrt{2}', 0')$	$(4 + \sqrt{2}, 2)$, $(4 - \sqrt{2}, 2)$
foci	$(\pm\sqrt{6}', 0')$	$(4 + \sqrt{2}, 2)$, $(4 - \sqrt{2}, 2)$
endpoints of conjugate axis	$(0', \pm 2')$	$(4, 0)$, $(4, 4)$



13. $(x - 2)^2 = 32(y + 3)$

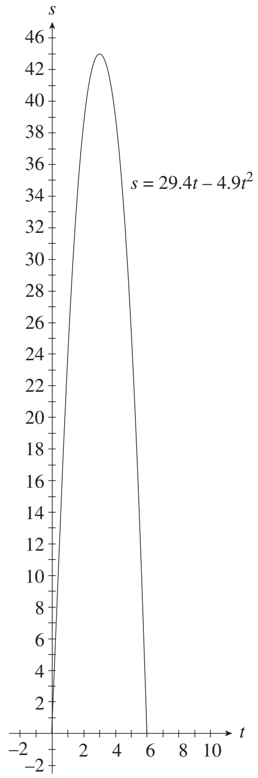
15. $\dfrac{(x - 4)^2}{36} + \dfrac{(y + 3)^2}{20} = 1$

17. $\dfrac{(y - 2)^2}{9} - \dfrac{(x + 3)^2}{16} = 1$

19. $(y - 1)^2 = -16(x + 5)$

21. $\dfrac{(x + 4)^2}{36} + \dfrac{(y - 1)^2}{100} = 1$

23. This is a parabola with the equation $-4.9(t - 3)^2 = s - 44.1$. Maximum height is when $t = 3$ and $s = 44.1$ m.



25. If the transverse axis passes through A and B and the conjugate axis passes through the midpoint of $\overline{AB}$, then A is at $(-500, 0)$ and B at $(500, 0)$. The plane is at $(216, 1073.054)$ and lies on the hyperbola $\dfrac{x^2}{8100} - \dfrac{y^2}{241\,900} = 1$

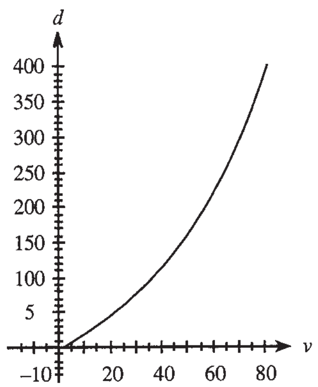
27. **(a)** $x^2 = -200(y - 0.02)$ **(b)** 2.00 cm

29. **(a)** $24(y - 125)^2 - x^2 = 15{,}000$, upper branch

(b) $\left(\dfrac{1{,}200 + 120\sqrt{5}}{19}, \dfrac{2{,}400 + 240\sqrt{5}}{19}\right) \approx (77.3, 154, 6)$

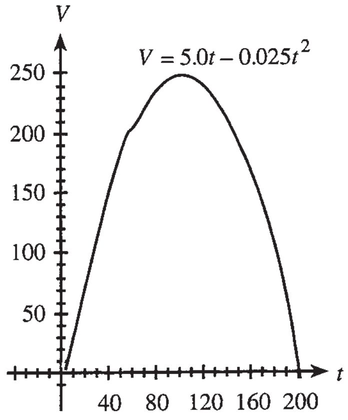
31. **(a)** $y = 2\sqrt{0.0625 - 0.25x^2}$ and $y = -2.5\sqrt{0.1 - 0.4x^2}$

(b)



33. **(a)** $C = 1.14x - 0.01x^2$

(b)



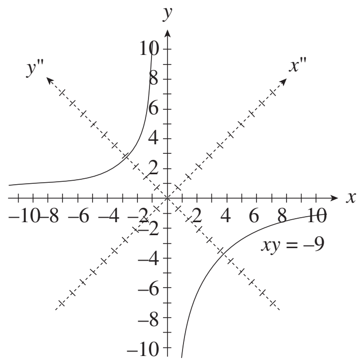
(c) 57, **(d)** \$32.49

Exercise Set 15.7

1. **(a)** discriminant is 1; curve is a hyperbola **(b)** 45°

(d) $\dfrac{y''^2}{18} - \dfrac{x''^2}{18} = 1$

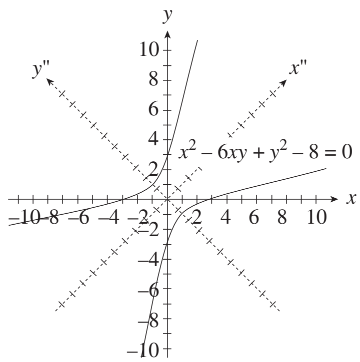
(e)



3. **(a)** discriminant is 32; curve is a hyperbola **(b)** 45°

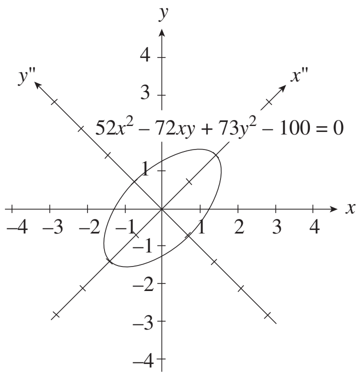
(d) $\dfrac{y''^2}{2} - \dfrac{x''^2}{4} = 1$

(e)



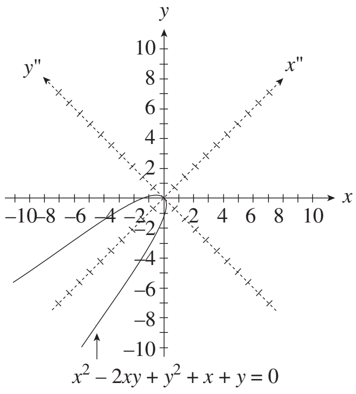
5. **(a)** discriminant is $-10{,}000$; curve is an ellipse

(b) 36.870° **(d)** $\dfrac{x''^2}{4} + y''^2 = 1$



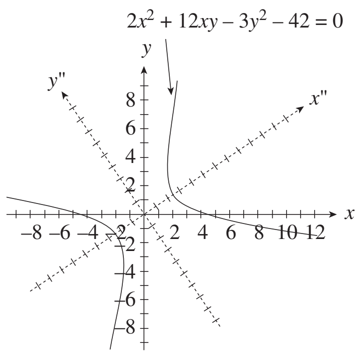
7. **(a)** discriminant is 0; curve is a parabola **(b)** 45°

(d) $y''^2 = -\dfrac{\sqrt{2}}{2}x''$

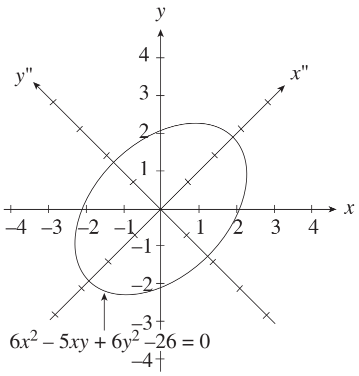


9. **(a)** discriminant is 168; curve is a hyperbola

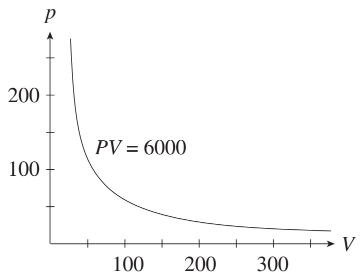
(b) 33.690° **(d)** $\dfrac{x''^2}{7} - \dfrac{y''^2}{6} = 1$



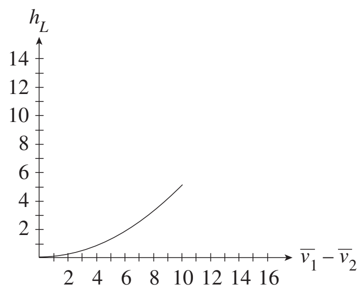
11. **(a)** discriminant is -119; curve is an ellipse
(b) $45°$ **(d)** $7x''^2 + 17y''^2 = 52$



13.

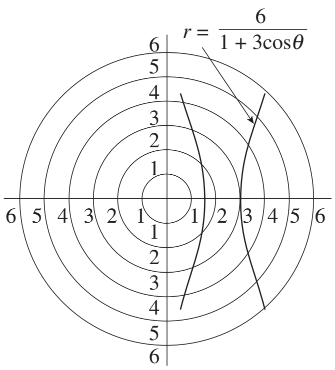


15. Parabola

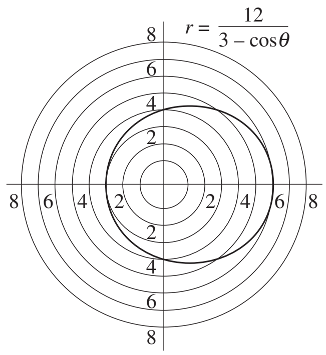


Exercise Set 15.8

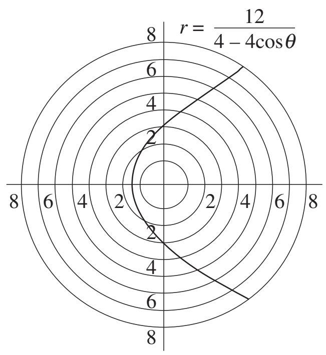
1. hyperbola



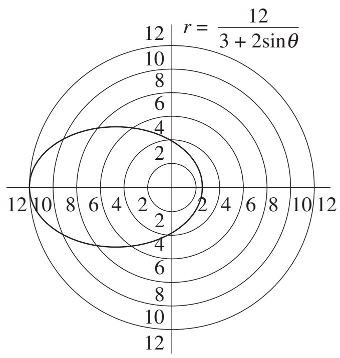
3. ellipse



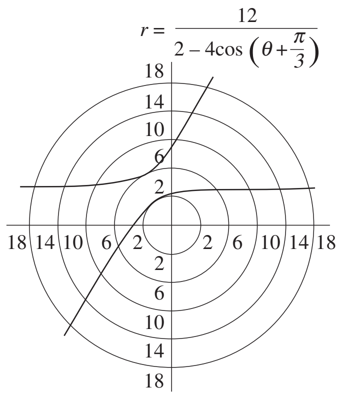
5. parabola



7. ellipse



9. hyperbola rotated $\frac{\pi}{3}$ radians



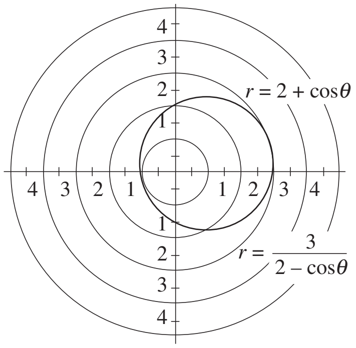
11. $r = \dfrac{6}{1 - \frac{3}{2}\cos\theta} = \dfrac{12}{2 - 3\cos\theta}$

13. $r = \dfrac{5}{1 - \sin\theta}$

15. $r = \dfrac{\frac{2}{3}}{1 - \frac{2}{3}\cos\theta} = \dfrac{2}{3 - 2\cos\theta}$

17. Greatest distance: 4.335×10^7 miles; shortest distance: 2.854×10^7 miles

19.



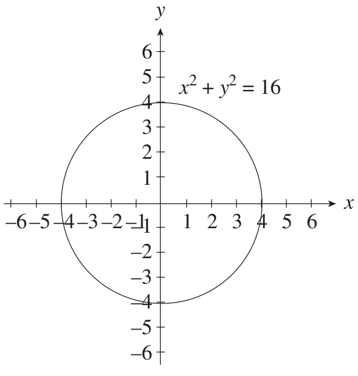
Review Exercises

1. **(a)** 5 **(b)** $\left(\frac{1}{2}, 7\right)$ **(c)** $-\frac{4}{3}$ **(d)** $y - 5 = -\frac{4}{3}(x - 2)$ or $4x + 3y = 23$ **3.** **(a)** $\sqrt{104} = 2\sqrt{26}$ **(b)** (2, 1) **(c)** 5 **(d)** $y + 4 = 5(x - 1)$ or $y - 5x = -10$

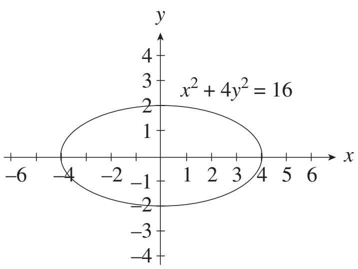
5. line in #1: $\frac{3}{4}$; line in #2: $\frac{12}{5}$; line in #3: $-\frac{1}{5}$; line in #4: 1

7. $y - 5 = -2(x + 3)$ or $y + 2x = -1$

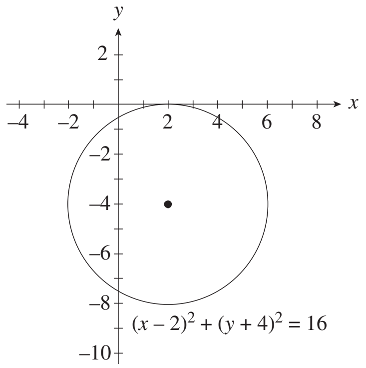
9.



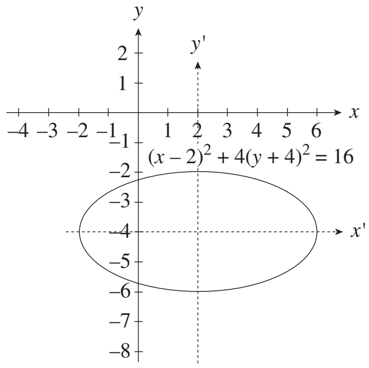
11.



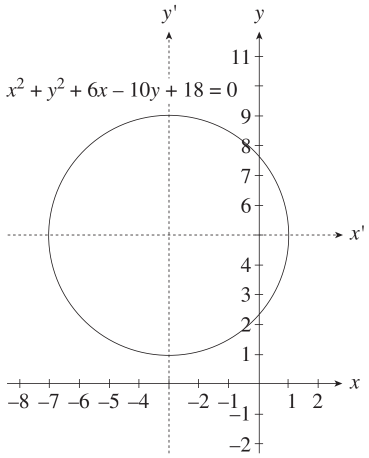
13.



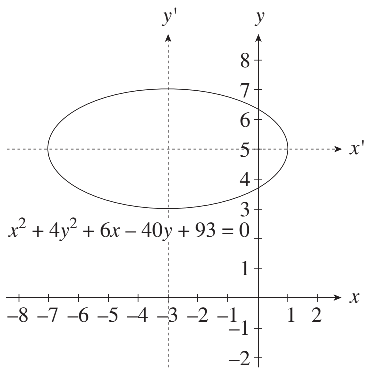
15.



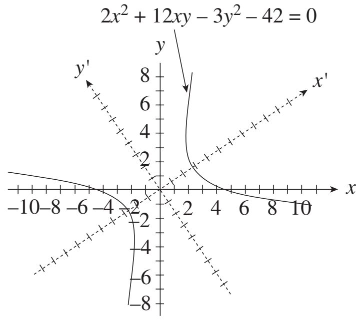
17.



19.



21.



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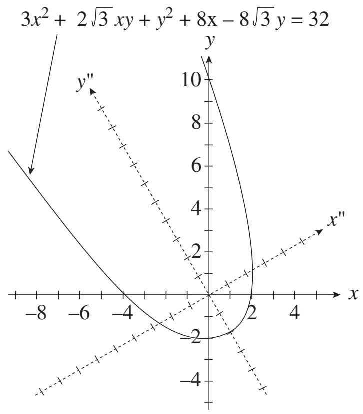
23.



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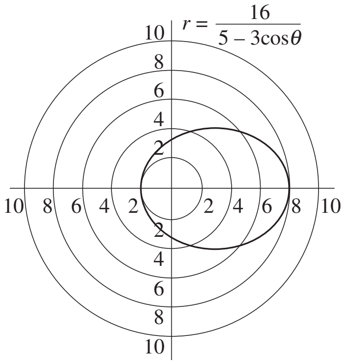
25.



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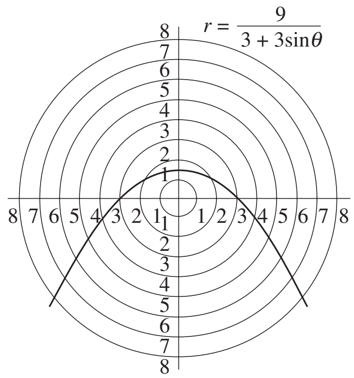
27.



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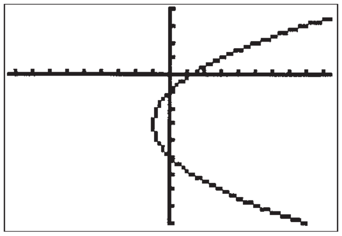
29. **(a)**



[–9.4, 9.4] x [–9, 4]

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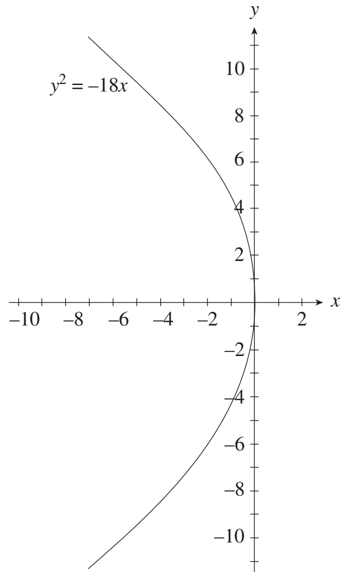
(b) Parabola

(c) Horizontal axis, opens to the right, vertex: $(-1, -3)$, directrix: $x = -2$, and focus: $(0, -3)$.

31. $\dfrac{x^2}{50^2} + \dfrac{y^2}{195.84} = 1$ **33.** If the x-axis passes through A and B and the y-axis through the midpoint of $\overline{AB}$, then B has the coordinates $(200, 0)$ and $A = (-200, 0)$. The plane is at $(91.13, -50)$. The plane lies on the hyperbola $\dfrac{x^2}{6400} - \dfrac{y^2}{33\,600} = 1$. **35.** about 82.25 ft

Chapter 15 Test

1. Focus $(-4.5, 0)$; directrix $x = 4.5$



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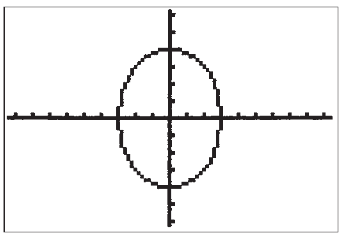
3. **(a)**



[–9.4, 9.4] x [–6.2, 6.2]

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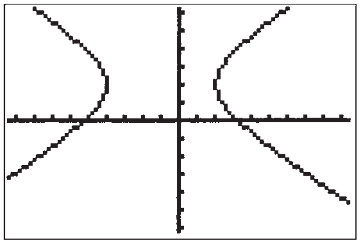
(b) Ellipse

(c) Center: $(0, 0)$, Vertices: $(-3, 0)$ and $(3, 0)$, Foci: $(-\sqrt{7}, 0)$ and $(\sqrt{7}, 0)$; endpoints of minor axis: $(0, 4)$ and $(0, -4)$

5. $y + \frac{13}{2} = -\frac{2}{3}(x - 3)$ or $6y + 4x + 27 = 0$

7. **(a)**



[–9.4, 9.4] x [–6.2, 6.2]

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(b) Hyperbola **(c)** Center: $(-1, 2)$, Vertices: $(-4, 2)$ and $(2, 2)$, Foci: $(-1 - \sqrt{13}, 2)$ and $(-1 + \sqrt{13}, 2)$; endpoints of minor axis: $(-1, 4)$ and $(-1, 0)$ **9.** **(a)** $\frac{\pi}{4}$ **(b)** an ellipse

11. 75 ft above its lowest point

ANSWERS FOR CHAPTER 16

Exercise Set 16.1

1. $7259 = (7 \times 10^3) + (2 \times 10^2) + (5 \times 10^1) + (9 \times 10^0)$

3. $12{,}953 = (1 \times 10^4) + (2 \times 10^3) + (9 \times 10^2) + (5 \times 10^1) + (3 \times 10^0)$

5. $28.137 = (2 \times 10^1) + (8 \times 10^0) + (1 \times 10^{-1}) + (3 \times 10^{-2}) + (7 \times 10^{-3})$

7. $321.794 = (3 \times 10^2) + (2 \times 10^1) + (1 \times 10^0) + (7 \times 10^{-1}) + (9 \times 10^{-2}) + (4 \times 10^{-3})$

9. $1101_2 = (1 \times 2^3) + (1 \times 2^2) + (1 \times 2^0)$

11. $1\,1101.011_2 = (1 \times 2^4) + (1 \times 2^3) + (1 \times 2^2) + (1 \times 2^0) + (1 \times 2^{-2}) + (1 \times 2^{-3})$

13. $1101_2 = 13$ **15.** $1111_2 = 15$ **17.** $101\,0011_2 = 83$

19. $10\,1101\,0111_2 = 727$ **21.** $1\,1101.011_2 = 29.375$

23. $101\,1110.01101_2 = 94.40625$ **25.** $97 = 110\,0001_2$

27. $54 = 11\,0110_2$ **29.** $35 = 10\,0011_2$ **31.** $0.25 = 0.01_2$

33. $0.95 = 0.1\overline{1100}_2$ **35.** $12.75 = 1100.11_2$

37. $144.875 = 1001\,0000.111_2$ **39.** $196.59375 = 1100\,0100.10011_2$ **41.** **(a)** 63 **(b)** 47 **(c)** 174 **(d)** 10.5

43. 182.9.165.174 **45.** 105.25.181.162

47. 10010110.01001000.10110100.10001100

49. 01101011.11001111.00111001.01000111

Exercise Set 16.2

1. 110_2 **3.** $101\,0000_2$ **5.** $1000\,1000_2$ **7.** $1110\,1000_2$

9. $11\,1100\,0101_2$ **11.** $11\,1010.1001_2$ **13.** 237

15. 87,271 **17.** 47.62 **19.** **(a)** 7042 **(b)** 259

21. **(a)** 9173 **(b)** 1549 **23.** **(a)** 96,343 **(b)** 48,796

25. **(a)** 872.72 **(b)** 269.88 **27.** **(a)** 963.73 **(b)** 558.106

29. **(a)** 905.236 **(b)** 378.506 **31.** 0011_2 **33.** $10\,0111_2$

35. 10.101_2 **37.** **(a)** 0111_2 **(b)** 100_2 **39.** **(a)** $01\,1100_2$ **(b)** 10001_2 **41.** **(a)** $101\,0011_2$ **(b)** $10\,1001_2$

43. **(a)** $1010\,0010_2$ **(b)** $111\,1001_2$ **45.** **(a)** 10.101_2 **(b)** 10.01_2 **47.** **(a)** 1010.101_2 **(b)** 110.0101_2

49. **(a)** 1010.101_2 **(b)** 100.111_2 **51.** **(a)** 11011.0011_2 **(b)** 10000.1101_2

Exercise Set 16.3

1. $(3 \times 8^2) + (2 \times 8^1) + (4 \times 8^0)$ **3.** $(6 \times 8^3) + (3 \times 8^2) + (4 \times 8^0)$ **5.** 212 **7.** 3268

9. 3,092.375 **11.** 3,886.921875 **13.** 231_8

15. 1560_8 **17.** 0.57_8 **19.** 1243.35_8 **21.** $1\,0101\,0111_2$

23. $0101\,0100\,0100_2$ **25.** $1\,1101\,1101.0000\,10_2$

27. $1010\,0001\,1100.0111\,0101_2$ **29.** 13_8 **31.** $547\,474_8$

33. 553.67_8 **35.** 6527.315_8 **37.** Do your math homework.

39. 124 150 145 040 163 164 165 144 170 040 157 146 040 155 141 164 162 151 143 145 163 040 151 163 040 151 156 103 150 141 160 164 145 162 040 061 071 056

Exercise Set 16.4

1. 600_8 **3.** $62\ 5255_8$ **5.** 377.045_8 **7.** 4433.566_8
9. 246_8 **11.** $5\ 0454_8$ **13.** 25.410_8 **15.** **(a)** 5125_8 **(b)** 344_8 **17.** **(a)** $70\ 1334_8$ **(b)** $5\ 7656_8$
19. **(a)** 7152.265_8 (b) 1015.525_8
21. **(a)** $77\ 7710.2257_8$ **(b)** $14\ 0173.4757_8$

Exercise Set 16.5

1. $(10 \times 16^2) + (5 \times 16^1) + (13 \times 16^0)$
3. $(2 \times 16^4) + (A \times 16^3) + (3 \times 16^1) + (7 \times 16^0) + (B \times 16^{-1}) = (2 \times 16^4) + (10 \times 16^3) + (3 \times 16^1) + (7 \times 16^0) + (11 \times 16^{-1})$ **5.** 291
7. 2653 **9.** 12,229 **11.** 179.9375 **13.** 932.734375
15. 55,266.5546875 **17.** $18B_{16}$ **19.** $7F6_{16}$
21. $CDE2_{16}$ **23.** 0.5_{16} **25.** $0.D_{16}$ **27.** $0.B4_{16}$
29. $0.4\overline{147AE}_{16}$ **31.** $2D5.9\overline{B851E}_{16}$
33. $0011\ 1001\ 0101_2$ **35.** $1010\ 0111\ 0011\ 1100_2$
37. $B4_{16}$ **39.** $BD1_{16}$ **41.** 32.1.203.186.0.0.66.122.1 71.205.67.33.50.87.150.82

Exercise Set 16.6

1. $70A_{16}$ **3.** $D4\ 7AF9_{16}$ **5.** $88D\ 0011_{16}$
7. $15B\ 19C4_{16}$ **9.** $258B_{16}$ **11.** $CBA\ ED77_{16}$
13. **(a)** $E6FF_{16}$ **(b)** 8800_{16} **15.** **(a)** $E7\ D2A7_{16}$ **(b)** $2E\ D3A8_{16}$ **17.** **(a)** $F07\ A4D9_{16}$ **(b)** $CC\ AC53_{16}$ **19.** **(a)** $FF58\ C987_{16}$ **(b)** $74F3\ D6BE_{16}$ **21.** **(a)** $C3B6.B76_{16}$ **(b)** $3925.0E9_{16}$
23. **(a)** FEF1.F830 **(b)** $4F9E.8353_{16}$
25. **(a)** $FBA8.D4B_{16}$ **(b)** $7191.65B_{16}$
27. **(a)** $FF716.845F_{16}$ **(b)** $5AE4B.337F_{16}$
29. Black = #000000, Green = #008000, Silver = #C0C0C0, Lime = #00FF00, Gray = #808080, Olive = #808000, White = #FFFFFF, Yellow = #FFFF00, Maroon = #800000, Navy = #000080, Red = #FF0000, Blue = #0000FF, Purple = #800080, Teal = #008080, Fuchsia = #FF00FF, and Aqua = #00FFFF

Chapter 16 Review

1. $8531 = (8 \times 10^3) + (5 \times 10^2) + (3 \times 10^1) + (1 \times 10^0)$
3. $53.194 = (5 \times 10^1) + (3 \times 10^0) + (1 \times 10^{-1}) + (9 \times 10^{-2}) + (4 \times 10^{-3})$
5. $1\ 0011_2 = (1 \times 2^4) + (1 \times 2^1) + (1 \times 2^0)$
7. $11.1001_2 = (1 \times 2^1) + (1 \times 2^0) + (1 \times 2^{-1}) + (1 \times 2^{-4})$
9. $7452_8 = (7 \times 8^3) + (4 \times 8^2) + (5 \times 8^1) + (2 \times 8^0)$
11. $502.16_8 = (5 \times 8^2) + (0 \times 8^1) + (2 \times 8^0) + (1 \times 8^{-1}) + (6 \times 8^{-2})$ or $(5 \times 8^2) + (2 \times 8^0) + (1 \times 8^{-1}) + (6 \times 8^{-2})$
13. $A72_{16} = (A \times 16^2) + (7 \times 16^1) + (2 \times 16^0) = (10 \times 16^2) + (7 \times 16^1) + (2 \times 16^0)$
15. $7F.0D_{16} = (7 \times 16^1) + (F \times 16^0) + (D \times 16^{-2}) = (7 \times 16^1) + (15 \times 16^0) + (D \times 16^{-2})$
17. **(a)** $1\ 0110\ 0101\ 0100_2$ **(b)** $1\ 3124_8$ **(c)** 1654_{16}
19. **(a)** $10\ 0000.\overline{1100}_2$ **(b)** 40.63146_8 **(c)** $20.CCD_{16}$
21. $AD.68_{16}$ **23.** $1111\ 0100.0011\ 11_2$ **25.** $1\ 0010_2$
27. 10001.0011_2 **29.** $1\ 016C_{16}$ **31.** $B2B.3D87_{16}$
33. **(a)** 110_2 **(b)** 11_2 **35.** **(a)** $10\ 0110.0110_2$ **(b)** $1\ 0001.1011_2$ **37.** **(a)** 8464_{16} **(b)** $27F6_{16}$
39. **(a)** $F3\ 2869.59_{16}$ **(b)** $86\ CC8F.0A_{16}$
41. 244.73.173.140 **43.** 115 141 164 150 145 155 141 164 151 143 163 072 040 124 150 145 040 161 165 145 145 156 040 157 146 040 164 150 145 040 163 143 151 145 156 143 145 163 056

Chapter 16 Test

1. $3057.26 = (3 \times 10^3) + (5 \times 10^1) + (7 \times 10^0) + (2 \times 10^{-1}) + (6 \times 10^{-2})$
3. **(a)** $1\ 0110\ 0100\ 0101.101_2$ **(b)** 13105.5_8 (c) $1645.A_{16}$
5. 171.1_8 **7.** 100.101_2 **9.** $10\ 0010_2$ **10.** $562D3_{16}$
11. 101.0111_2 **13.** $433E.CDE_{16}$

ANSWERS FOR CHAPTER 17

Exercise Set 17.1

1. 9 **3.** 3 **5.** 130 **7.** 0 **9.** 0 **11.** 11 **13.** 0
15. 79 **17.** yes **19.** yes **21.** yes **23.** yes
25. $Q(x) = x^4 + 3x^3 - 8x^2 - 24x + 3$; $R(x) = 18$
27. $Q(x) = 5x^2 - 8x + 24$; $R(x) = -63$
29. $Q(x) = 8x^4 + 4x^3 - 2x^2 + 6x + 1$; $R(x) = \frac{1}{2}$
31. $Q(x) = 2x^3 - 3x^2 - 4$; $R(x) = 0$

Exercise Set 17.2

1. $1, \dfrac{-5+\sqrt{85}}{10}, \dfrac{-5-\sqrt{85}}{10}$ **3.** $\frac{1}{3}, 3, -3$

5. $2, -2, j, -j$ **7.** $3, -3, -1+2j, -1-2j$

9. $-1, -1, 2, 3$ **11.** $-1, -1, -3, 1$

13. $-\frac{1}{2}, \frac{1}{3}, -2+j, -2-j$

15. $1, \frac{2}{3}, \dfrac{-1+j\sqrt{3}}{2}, \dfrac{-1-j\sqrt{3}}{2}$

17. $1, 1, 1, -1+j\sqrt{3}, -1-j\sqrt{3}$

19. $3, -2, \frac{2}{3}, \dfrac{-1+\sqrt{5}}{2}, \dfrac{-1-\sqrt{5}}{2}$

21. $j, -j, 3j, -3j, 7, -1$

Exercise Set 17.3

1. 0.5641

3. $x_1 = -3.4321$
$x_2 = 1.3315$
$x_3 = 0.520$
$x_4 = -0.4202$

5. $x_1 = -3.3539$
$x_2 = 1.8774$
$x_3 = 0.4765$

7. $x_1 = 1.3532$
$x_2 = -1$

9. $x_1 = 2.0946$
$x_2 = -1$

11. $x_1 = -1.6573$
$x_2 = 1$

13. $x_1 = -1$
$x_2 = -1.4333$
$x_3 = 0.2381$

15. $x_1 = 1.4142$
$x_2 = -1.4142$
$x_3 = -0.5$
$x_4 = -0.25$

17. 6.22731 years, or about 6 years 83 days

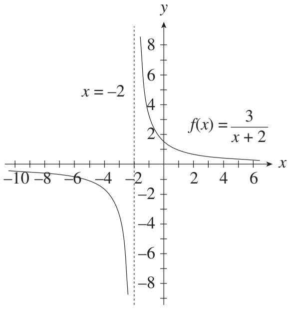
19. **(a)** 63.87006 cm **(b)** $521\,101\pi$ cm^3

21. The approximate roots are ± 0.3536, ± 0.8654, ± 1.4639, ± 1.7321, and ± 2.2323

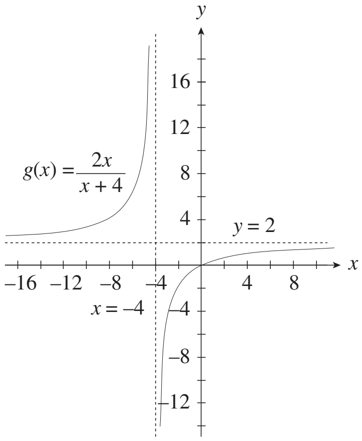
23. 1 in. or 2 in. **25.** 9 feet **27.** $R = 2$ **29.** 3.5 cm

Exercise Set 17.4

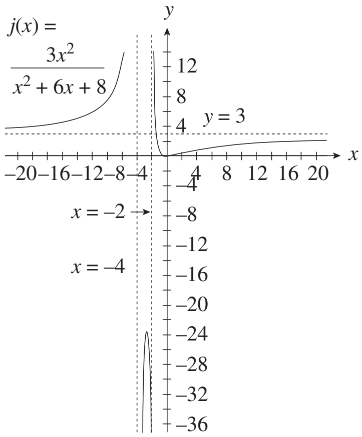
1. vertical asymptote at $x = -2$; horizontal asymptote at $y = 0$



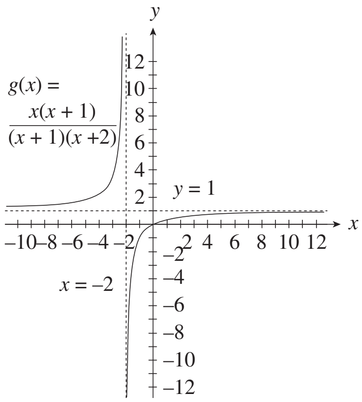
3. vertical asymptote at $x = -4$; horizontal asymptote at $y = 2$



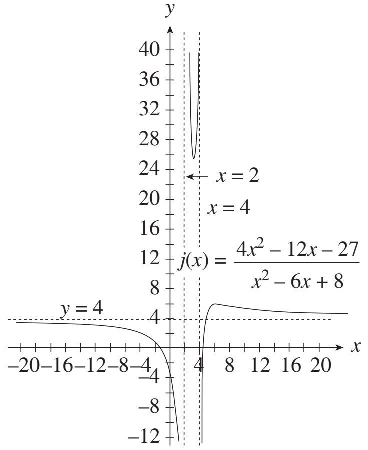
5. vertical asymptotes at $x = -4$ and $x = -2$; horizontal asymptote at $y = 3$



7. vertical asymptote at $x = -2$; horizontal asymptote at $y = 1$



9. vertical asymptote at $x = 4$; horizontal asymptote at $y = 2$



11. $x = -2$ or $x = 3$ **13.** 1 **15.** -5 **17.** $\frac{21}{8}$ **19.** 3 **21.** 4.49 hours **23.** \$73/set

Review Exercises

1. 30, degree $= 2$ **3.** -13, degree $= 3$
5. $Q(x) = 4x - 18$; $R(x) = 51$
7. $Q(x) = 9x^3 + 4x^2 + 6x - 1$; $R(x) = 0$
9. 1, 1, 1 **11.** $-1, -1, -1, 3$
13. $-3.044, -0.328, 0.548, 1.824$
15. $-1, -1, j, j, -j, -j$
17. vertical asymptotes at $x = -4$ and $x = 1$; horizontal asymptote at $y = 0$; solution: $x = 0$
19. vertical asymptotes at $x = 2$ and $x = -2$; horizontal asymptote at $y = 0$; solution: $x = 18$
21. 1 (100%)

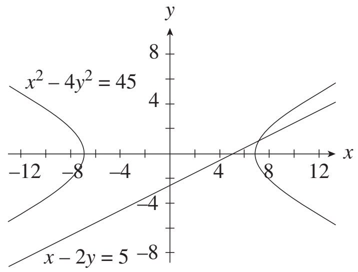
Chapter 17 Test

1. 5 **3.** Quotient: $x^3 - 2x^2 - 9x - 10$; Remainder: 15
5. $1, -2, -\frac{2}{3}$ **7.** vertical asymptotes: $x = -3$ and $x = -2$; horizontal asymptote: $y = 3$; solutions: $x = 1$ and $x = -\frac{1}{3}$ **9.** 2.48 yr

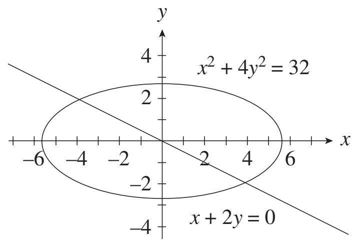
ANSWERS FOR CHAPTER 18

Exercise Set 18.1

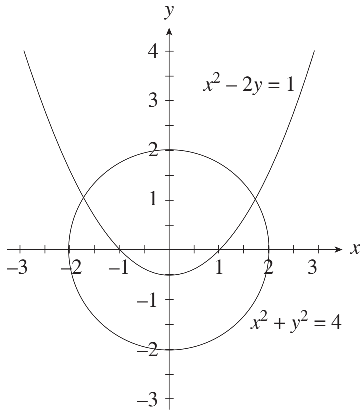
1. $(7, 1)$



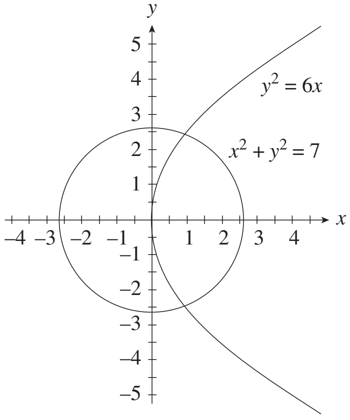
3. $(4, -2), (-4, 2)$



5. $(\sqrt{3}, 1), (-\sqrt{3}, 1)$



7. $(1, \sqrt{6}), (1, -\sqrt{6}), (-7, j\sqrt{42}), (-7, -j\sqrt{42})$



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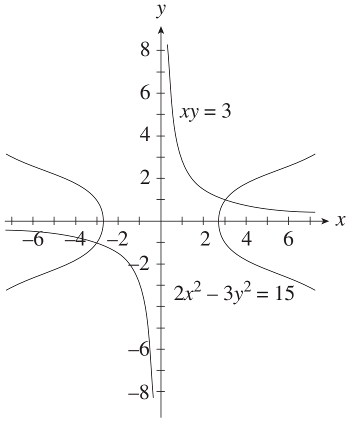
9. $(3, 1), (-3, -1), \left(j\sqrt{\tfrac{3}{2}}, -3j\sqrt{\tfrac{2}{3}}\right), \left(-j\sqrt{\tfrac{3}{2}}, 3j\sqrt{\tfrac{2}{3}}\right)$



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11. 42.08 mph for the first part of the trip; 32.08 mph for the last part.

13. One is 40 Ω and the other is 60 Ω

15. Either $v_1 = 3$ m/s and $v_2 = 5.5$ m/s or $v_1 = 6$ m/s and $v_2 = 3.5$ m/s

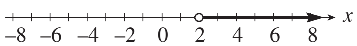
17. $F = 67.87$; $\ell = 2.36$ **19.** $x = 10.16$ mm, $y = 4.57$ mm **21.** $L = 1.336\, v^2 x$ **23.** $g = 32$, $n = 13$

Exercise Set 18.2

1. $x + 5 < 15$ **3.** $3x < 30$ **5.** $\dfrac{-x}{2} > -5$

7. $x^2 < 100$

9. $x > 2$



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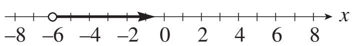
11. $x > -6$



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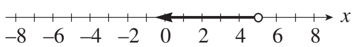
13. $x < 5$



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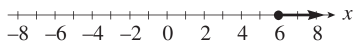
15. $x \geq 6$



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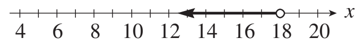
17. $x < 18$



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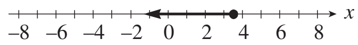
19. $x \leq 3\frac{1}{2}$



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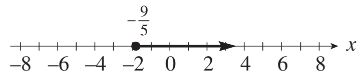
21. $x \geq -\frac{9}{5}$



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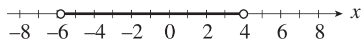
23. $-6 < x < 4$



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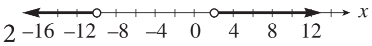
25. $x < -10$ or $x > 2$



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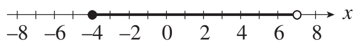
27. This is a contradictory inequality.

29. $-4 \leq x < 7$



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31. This is a contradictory inequality

33. 6357 km $\leq r \leq$ 6378 km

35. $+0.10° \leq c \leq +1.10°$

37. 266.67 Ω $< R <$ 1333.33 Ω **39.** 43 or more

41. **(a)** $|\epsilon| < 0.6$ **(b)** $|P - 9.0| < 0.6$ **(c)** $8.4 < P < 9.6$ W

43. At least 70,001 copies must be sold.

Exercise Set 18.3

1. $x < -1$ or $x > 3$ **3.** $-4 \leq x \leq 1$

5. $-1 < x < 1$ **7.** $-3 \leq x < 5$

9. $x \leq -1$ or $x \geq 2$ **11.** $x < 2$ or $x > 3$

13. $-3 < x < -\frac{1}{2}$ **15.** $-\frac{1}{2} < x < 1$

17. Absolute inequality—all real numbers satisfy this inequality.

19. $x < -3$ or $-1 < x < 2$

21. $-4 < x < -3$ or $x > \frac{5}{2}$ **23.** $x < -4$

25. $-2 < x < 0$ or $x > 2$

27. Contradictory inequality. No real numbers solve this inequality. **29.** $x < -1$ or $2 < x < 5$

31. $-1 < x < 0$ or $x > 2$ **33.** $x \leq -\frac{1}{3}$ or $\frac{1}{2} < x \leq 3$

35. $x > 9$ or $-1 < x < 1$ **37.** $x > 1$

39. $\dfrac{-3 - \sqrt{17}}{2} \leq x \leq \dfrac{-3 + \sqrt{17}}{2}$

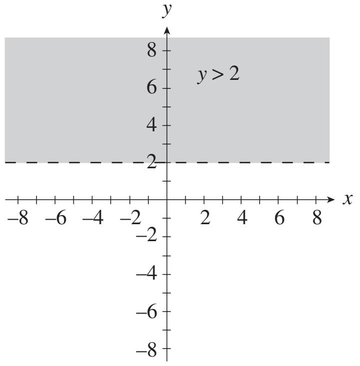
41. 1 m $< d < \sqrt{6}$ m **43.** $x < 0.123$ or $x > 0.977$

45. **(a)** $1\ s < t < 3\ s$ **(b)** $t > 4.75\ s$

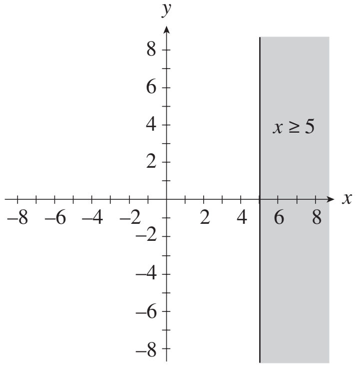
47. $D \geq 35.6$ in.

Exercise Set 18.4

1.



3.



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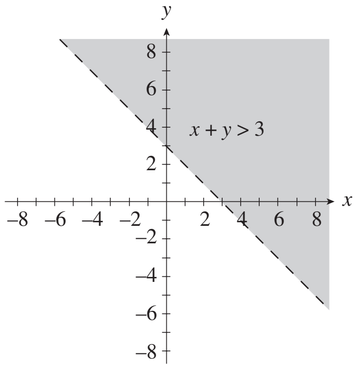

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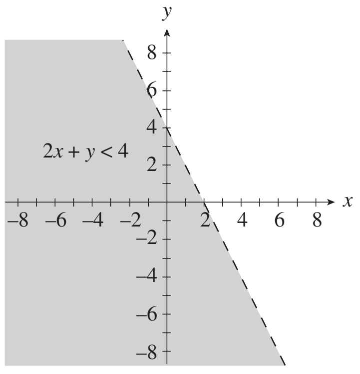
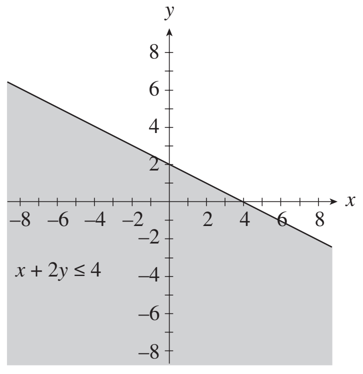


9.



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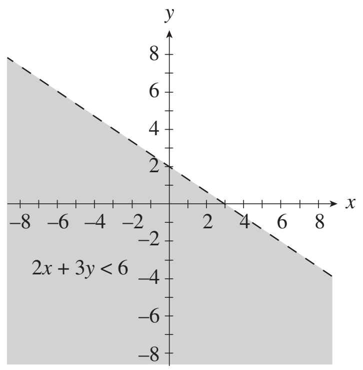

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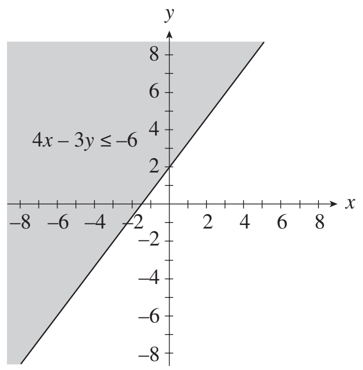

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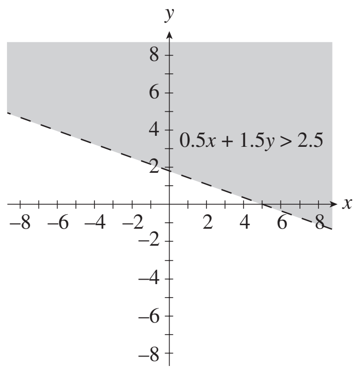

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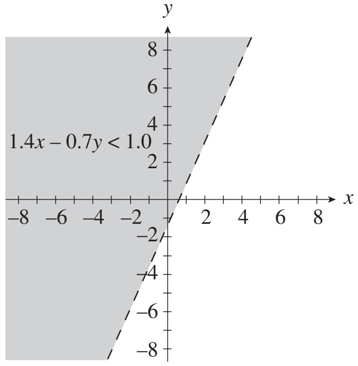

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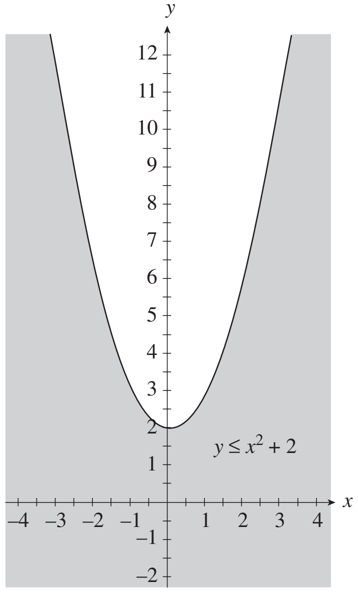

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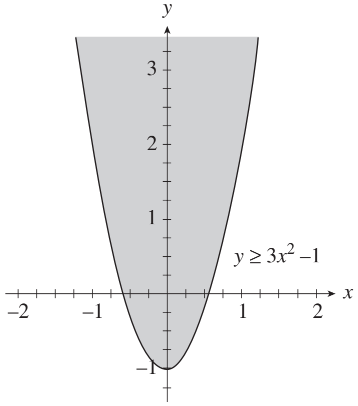

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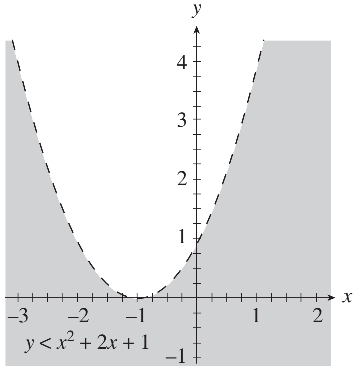

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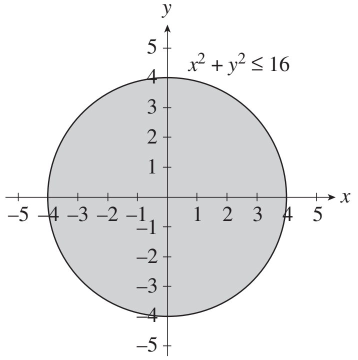

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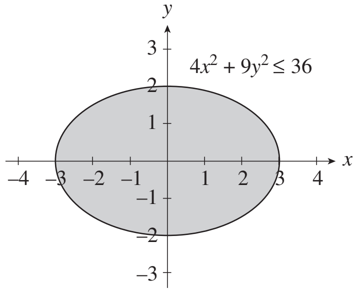

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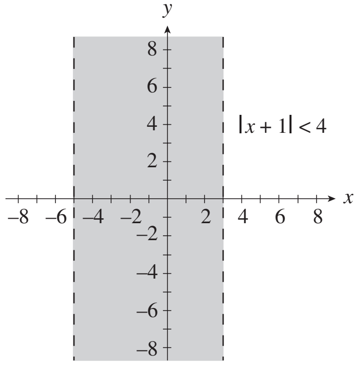

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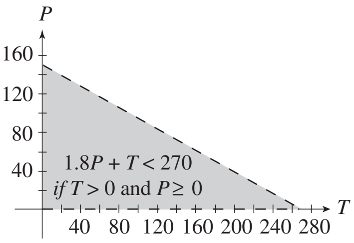

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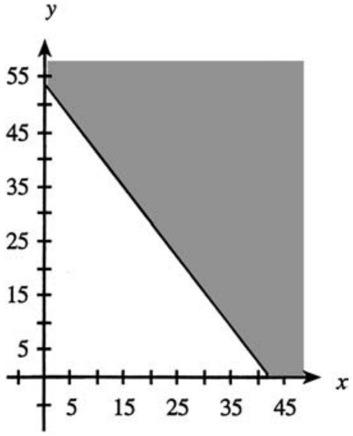
33.



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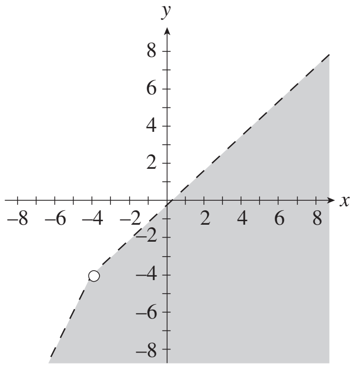
Exercise Set 18.5

1.



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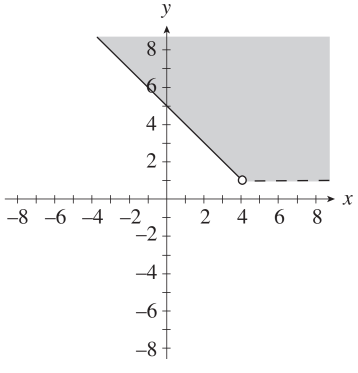
3.



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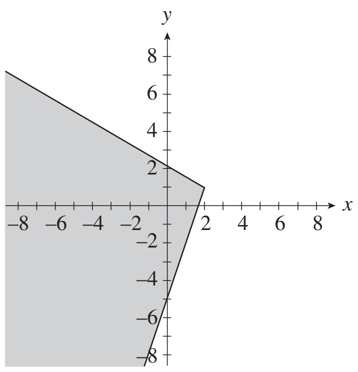
5.



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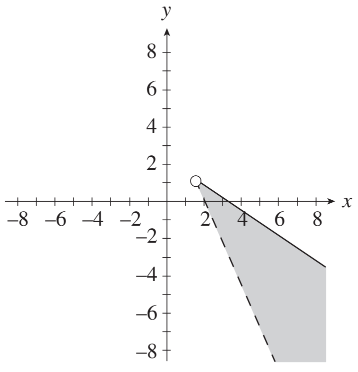
7.



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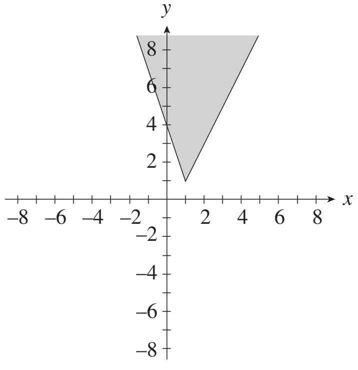
9.



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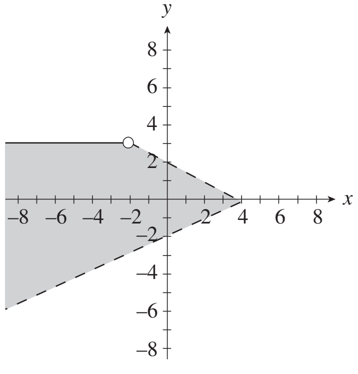
11.



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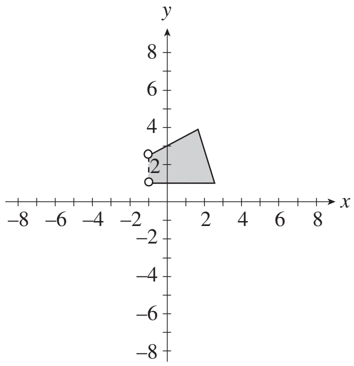
13.



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15. (4, 3) max is 11 at (4, 3) **17.** max is 400 at any point on $3x + 2y = 80$ where $20 \leq x \leq \frac{80}{3}$.

19. max is 216 at (12, 12). **21.** 0 of x and 400 of y will produce the most profit: \$3,200. \$3,008.

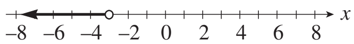
23. 120 of B and none of A for a profit of \$1140.00.

25. 37 home loans and 7 commercial loans

27. Buy 10,000 small and 5,000 large, for a total cost of \$2,100.

Review Exercises

1. $x < -3$



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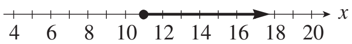
3. $x \geq 11$



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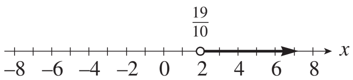
5. $x > 1.9$



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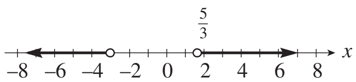
7. $x < -3$ or $x > \frac{5}{3}$



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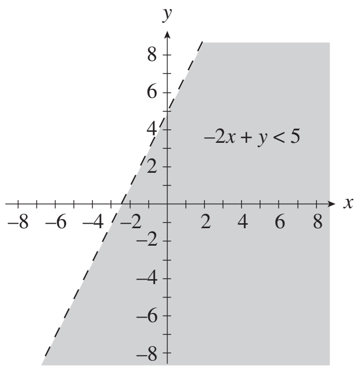

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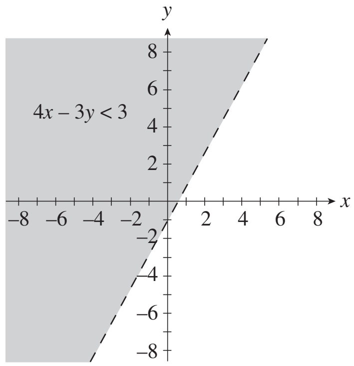

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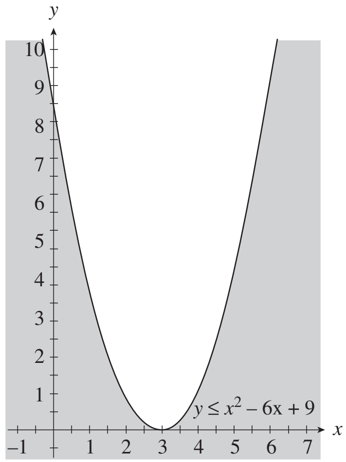

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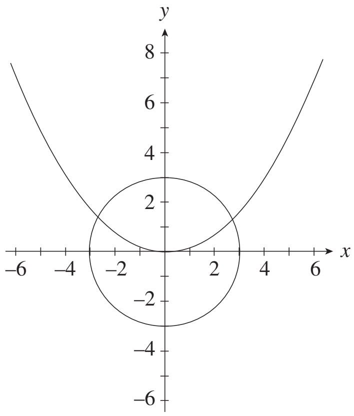

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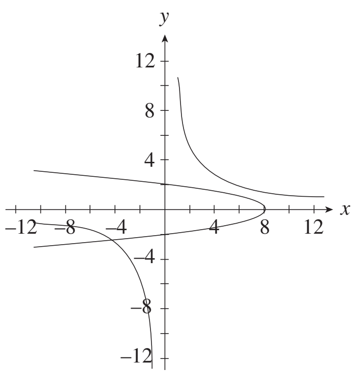

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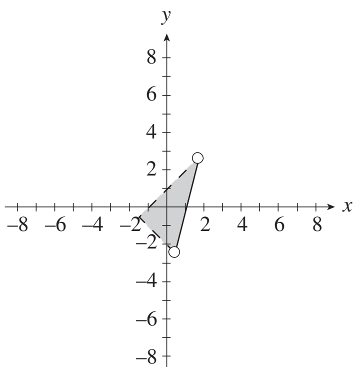

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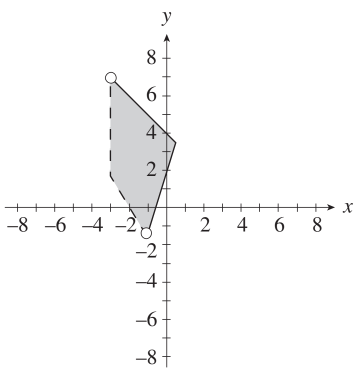
21.



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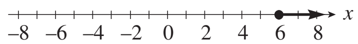
23. $\left(1\frac{1}{6}, 3\frac{2}{3}\right)$

25. $R_1 + R_2 \le 5\ \Omega$

27. 6 of C and 8 of S for a profit of \$20,460.

Chapter 18 Test

1. $x > 6$



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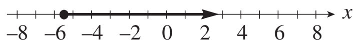
3. $x \ge -\frac{11}{2}$



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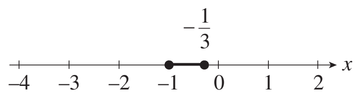
5. $-1 \le x \le -\frac{1}{3}$



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7. **(a)** 5 tablets of brand x and 4 of brand Y; **(b)** 44 ¢

ANSWERS FOR CHAPTER 19

Exercise Set 19.1

1. 2×3 **3.** 3×6 **5.** 4×2

7. $a_{11} = 1$; $a_{24} = 13$; $a_{21} = 8$; $a_{32} = 6$

9. $x = 12, y = -6, z = 2, w = 5$

11. $\begin{bmatrix} -3 & 6 & 6 & -8 \\ 12 & 9 & 17 & 0 \\ 9 & 0 & 5 & 10 \end{bmatrix}$ **13.** $\begin{bmatrix} 2 & -1 & 4 \\ 5 & 9 & -2 \end{bmatrix}$

15. $\begin{bmatrix} -1 & 7 & 4 \\ 8 & 1 & 15 \end{bmatrix}$ **17.** $\begin{bmatrix} 12 & 9 & 6 \\ 15 & 0 & 21 \end{bmatrix}$

19. $\begin{bmatrix} -5 & 16 & 8 \\ 12 & 0 & 25 \end{bmatrix}$ **21.** $\begin{bmatrix} -22 & -1 & -2 \\ -9 & 2 & -5 \end{bmatrix}$

23. $\begin{bmatrix} -3 \\ -7 \\ 3 \\ -5 \end{bmatrix}$ **25.** $\begin{bmatrix} 21 \\ 27 \\ -12 \\ 30 \end{bmatrix}$ **27.** $\begin{bmatrix} 26 \\ 24 \\ -11 \\ 35 \end{bmatrix}$

29. $\begin{bmatrix} 0 \\ -22 \\ 9 \\ 25 \end{bmatrix}$ **31.** $\begin{bmatrix} 27 & 12 & 15 \\ 18 & 15 & 16 \end{bmatrix}$

or 27 of computer chip A, 12 of computer chip B, 15 EPROMS, 18 keyboards, 15 motherboards, and 16 disk drives.

33. $\begin{bmatrix} 135 & 82 \\ 65 & 118 \end{bmatrix}$

35.
$$\begin{array}{ll} \text{Chip type:} & \begin{array}{cccc} 286 & 386 & 486 & 586 \end{array} \\ \begin{array}{l} \text{Warehouse A} \\ \text{Warehouse B} \\ \text{Warehouse C} \end{array} & \begin{bmatrix} 297 & 2541 & 3531 & 292 \\ 1232 & 4653 & 3436 & 83 \\ 352 & 3439 & 3017 & 1113 \end{bmatrix} \end{array}$$

Exercise Set 19.2

1. $\begin{bmatrix} 26 & 9 \\ -20 & -23 \end{bmatrix}$ **3.** $[-23]$

5. $\begin{bmatrix} -3 & 13 & 8 \\ -2 & 16 & 14 \end{bmatrix}$ **7.** $\begin{bmatrix} 5 & 17 & 5 \\ 44 & 67 & 37 \\ -9 & -6 & -7 \end{bmatrix}$

9. $\begin{bmatrix} -12 & 14 & 11 \\ 88 & 18 & 23 \end{bmatrix}$ **11.** $\begin{bmatrix} 7 & 16 \\ 17 & 38 \end{bmatrix}$

13. $\begin{bmatrix} -8 & 2 \\ -12 & 2 \end{bmatrix}$ **15.** $\begin{bmatrix} -68 & 18 \\ -160 & 42 \end{bmatrix}$

17. $\begin{bmatrix} -1 & 18 \\ 5 & 40 \end{bmatrix}$ **19.** $\begin{bmatrix} 42 & 96 \\ 102 & 228 \end{bmatrix}$ **21.** $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, No

23. Warehouse A: \$283,025, warehouse B: \$354,995, and warehouse C: \$364,415

25. $A^2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0+1 & 0+0 \\ 0+0 & 1+0 \end{bmatrix}$

$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$. Similarly, $B^2 = C^2 = I$.

27. **(a)** The commutator of A and B is

$AB - BA = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} - \begin{bmatrix} -i & 0 \\ 0 & i \end{bmatrix} = 2\begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$

$= 2i\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = 2iC$. Similar methods are used for (b) and (c).

29. **(a)** $\begin{bmatrix} -1 & -2 \\ -5 & -4 \\ 2 & -7 \\ 1 & 5 \end{bmatrix}$ **(b)** $\begin{bmatrix} -2 & 1 \\ -4 & 5 \\ -7 & 2 \\ 5 & -1 \end{bmatrix}$

(c) $\begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ x_3 & y_3 \\ x_4 & y_4 \\ x_5 & y_5 \end{bmatrix}\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$

Exercise Set 19.3

1. $\begin{bmatrix} 1 & -1 \\ -5 & 6 \end{bmatrix}$ **3.** $\begin{bmatrix} -1.5 & 2 \\ 4 & -5 \end{bmatrix}$

5. $\begin{bmatrix} -1 & -1.5 \\ -1.5 & -2 \end{bmatrix}$ **7.** $\begin{bmatrix} 0.6 & -2 \\ -0.8 & 3 \end{bmatrix}$

9. $\begin{bmatrix} 0 & 1 & 0 \\ -\frac{1}{3} & 0 & 0 \\ 0 & 0 & \frac{1}{4} \end{bmatrix}$ **11.** Singular matrix because column 2 is twice column 1.

13. $\begin{bmatrix} 1 & 3 & -2 \\ -1 & -4 & 3 \\ 0 & -4 & 5 \end{bmatrix}$

15. Singular matrix because row 2 is -2 times row 1

17. $\begin{bmatrix} 0.5 & 0 & 0 \\ -0.5 & 0.5 & 0 \\ 0 & -0.5 & 0.5 \end{bmatrix}$ **19.** $\begin{bmatrix} 3 & 3 & -1 \\ -2 & -2 & 1 \\ -4 & -5 & 2 \end{bmatrix}$

21. All answers should check.

Exercise Set 19.4

1. $x = -1, y = 2$ **3.** $x = -1.5, y = 2.5$

5. No solution **7.** $x = 1.5, y = -4, z = 3.5$

9. $x = 5.5, y = -3.5, z = -6.5$

11. $x = -2, y = 5$ **13.** $x = -5, y = 7$

15. $x = 4.2, y = -2.4$ **17.** $x = -2, y = 4, z = 1$

19. $x = -3.4, y = 2.2, z = 1.5$

21. $I_1 = 6.05, I_2 = 5.26, I_3 = 0.79$

23. $I_1 = 1.22, I_2 = 2.37, I_3 = 1.64$

25. $A = \frac{200}{11} \approx 18.18, B = \frac{100}{11} \approx 9.09, C = \frac{800}{11} \approx 72.73$

27. $F_A = -1{,}000$ N, $F_B = 2{,}800$ N, and $F_C = 600$ N.

29. $a = 0.05$, $b = 1$, and $c = 0$.

Review Exercises

1. $\begin{bmatrix} 4 & 4 & 2 & 7 \\ 9 & 7 & 3 & 4 \\ 9 & 5 & -8 & -3 \end{bmatrix}$ **3.** $\begin{bmatrix} 1 & 5 & 1 & 5 \\ 1 & 8 & -3 & 4 \\ 5 & 7 & -6 & 3 \end{bmatrix}$

5. $\begin{bmatrix} 8 & 3 & 4 & 4 \\ 3 & 14 & -14 & 8 \\ 18 & 35 & -16 & 24 \end{bmatrix}$ **7.** $\begin{bmatrix} 3 & 8 \\ 4 & 13 \\ 1 & 2 \end{bmatrix}$

9. $\begin{bmatrix} 12 & 15 & 3 \\ 8 & 10 & 2 \\ -4 & -5 & -1 \end{bmatrix}$ **11.** $\begin{bmatrix} -2.5 & -1.5 \\ 2 & 1 \end{bmatrix}$

13. $\begin{bmatrix} -0.5 & 0.125 & 0 \\ 0 & 0.25 & 0 \\ 0.5 & -0.125 & 1 \end{bmatrix}$ **15.** $x = 2.5, y = -6.4$

17. $x = 2, y = -3, z = 5$ **19.** $T = 86.47, a = 1.23$

21. $x'' = -x, y'' = -y$

23. $M = \begin{bmatrix} 1 - \dfrac{d}{f_2} & \dfrac{d}{f_1 f_2} - \dfrac{1}{f_1} - \dfrac{1}{f_2} \\ d & 1 - \dfrac{d}{f_1} \end{bmatrix}$

Chapter 19 Test

1. **(a)** $\begin{bmatrix} 7 & 5 & -7 \\ 19 & -6 & 6 \end{bmatrix}$ **(b)** $\begin{bmatrix} 26 & 10 & -6 \\ 42 & -18 & -2 \end{bmatrix}$

3. **(a)** $CD = \begin{bmatrix} -7 & 24 \\ -11 & 16 \end{bmatrix}$ **(b)** $DC = \begin{bmatrix} 2 & 3 \\ -46 & 7 \end{bmatrix}$

5. $x = 1, y = -2, z = 3$

7. $I_1 = \frac{20}{11}$ A, $I_2 = \frac{25}{11}$ A, $I_3 = -\frac{5}{11}$ A

ANSWERS FOR CHAPTER 20

Exercise Set 20.1

1. $\frac{1}{2}; \frac{1}{3}; \frac{1}{4}; \frac{1}{5}; \frac{1}{6}; \frac{1}{7}$ **3.** 4; 9; 16; 25; 36; 49
5. $-1; 2; -3; 4; -5; 6$ **7.** $\frac{1}{3}; \frac{3}{5}; \frac{5}{7}; \frac{7}{9}; \frac{9}{11}; \frac{11}{13}$
9. $1; \frac{2}{3}; \frac{4}{9}; \frac{8}{27}; \frac{16}{81}; \frac{32}{243}$ **11.** $0; \frac{1}{9}; \frac{1}{4}; \frac{9}{25}; \frac{4}{9}; \frac{25}{49}$
13. 1; 2; 6; 24; 120; 720 **15.** 5; 8; 11; 14; 17; 20
17. 2; 2; 4; 64; 16,777,216; 1.329228×10^{36}
19. 1; 2; 9; 262,144; $5^{262,144}$; $6^{(5^{262,144})}$ **21.** $1; \frac{1}{2}; \frac{1}{2}; \frac{1}{4}; \frac{1}{8}; \frac{1}{32}$
23. $0; 2; 2; 0; -2; -2$ **25.** 18.75 ft or 18 ft 9 in
27. **(a)** \$30,400; **(b)** \$35,900

Exercise Set 20.2

1. Arithmetic sequence with $d = 8$; $a_9 = 65$
3. Geometric sequence with $r = \frac{-1}{2}$; $a_{10} = \frac{1}{64}$
5. Geometric sequence with $r = -4$; $a_6 = (-1)(-4)^5 = 1024$ **7.** Arithmetic sequence with $d = -5$; $a_8 = -32$ **9.** Neither; can't tell from the given information what a_{10} will be.
11. Arithmetic sequence with $d = 0.4$; $a_9 = 3.6$
13. Arithmetic sequence with $d = \frac{-1}{6}$; $a_8 = -\frac{1}{2}$
15. Geometric sequence with $r = 0.1$; $a_6 = 0.00005$
17. Arithmetic sequence with $d = 2$; $a_9 = 17 + \sqrt{3}$
19. Neither; $a_6 = 1.2$, $a_n = -1^{n-1}\,[4.3 - (1.1)(n - 1)]$
21. -51 **23.** 243 **25.** 20 **27.** 9
29. -3.5 **31.** $\frac{1}{8}$ miles **33.** 12.3 cm; 51
35. **(a)** about 15.63 miles **(b)** about 43.05 miles.

Exercise Set 20.3

1. $\frac{3}{2} + \frac{3}{4} + \frac{3}{8} + \frac{3}{16}$ **3.** $1 - \frac{3}{2} + 3 - \frac{27}{4}$ **5.** 0
7. 91 **9.** 225 **11.** $\dfrac{1{,}593{,}342}{67{,}925} \approx 23.457$
13. Arithmetic; $S_{10} = 165$ **15.** Geometric; $S_8 = 255$
17. Arithmetic; $S_{12} = 192$ **19.** Arithmetic; $S_{20} = 57.5$
21. Geometric; $S_{16} = 0.5625$ **23.** Arithmetic; $S_{14} = 114.8$ **25.** Geometric; $S_{15} = 17920.253$
27. Geometric; $S_8 = 9840$ **29.** 392.32m when it hits the fourth time, 634.10m when it hits the tenth time **31.** 104 ft **33.** \$2540.47; \$3.99
35. **(a)** \$3,662,000 **(b)** \$7,682,000

Exercise Set 20.4

1. Converges; $\frac{1}{3}$ **3.** Converges; 0.6
5. Converges; 4 **7.** Converges; 1.389
9. Converges; 0.0333… **11.** Converges; 0.555…
13. Diverges; $r = 5/4$ **15.** Diverges; $r = 10$
17. Converges; 2 **19.** Converges; 2.4142
21. Diverges; $r = \sqrt{5}$ **23.** 4/9 **25.** $\frac{19}{33}$ **27.** $\frac{13{,}520}{9999}$
29. $\frac{20{,}986}{3330} = \frac{10{,}493}{1665}$ **31.** 45 m **33.** 1,250 revolutions

Exercise Set 20.5

1. $a^4 + 4a^3 + 6a^2 + 4a + 1$
3. $81x^4 - 108x^3 + 54x^2 - 12x + 1$
5. $\dfrac{x^6}{64} + \frac{3}{16}x^5d + \frac{15}{16}x^4d^2 + \frac{5}{2}x^3d^3 + \frac{15}{4}x^2d^4 + 3xd^5 + d^6$
7. $\dfrac{a^5}{32} - \dfrac{5a^4}{4b} + 20\dfrac{a^3}{b^2} - 160\dfrac{a^2}{b^3} + 640\dfrac{a}{b^4} - \dfrac{1024}{b^5}$
9. $a^7 + 7a^6b + 21a^5b^2 + 35a^4b^3 + 35a^3b^4 + 21a^2b^5 + 7ab^6 + b^7$ **11.** $t^8 - 8t^7a + 28t^6a^2 - 56t^5a^3 + 70t^4a^4 - 56t^3a^5 + 28t^2a^6 - 8ta^7 + a^8$
13. $64a^6 - 192a^5 + 240a^4 - 160a^3 + 60a^2 - 12a + 1$
15. $x^{14}y^7 + \frac{7}{2}x^{12}y^6a + \frac{21}{4}x^{10}y^5a^2 + \frac{35}{8}x^8y^4a^3 + \frac{35}{16}x^6y^3a^4 + \frac{21}{32}x^4y^2a^5 + \frac{7}{64}x^2ya^6 + \dfrac{a^7}{128}$
17. $x^{12} + 12x^{11}y + 66x^{10}y^2 + 220x^9y^3$
19. $1 + 2a + 3a^2 + 4a^3$ **21.** $1 - \dfrac{b}{3} + \dfrac{2b^2}{9} - \dfrac{14b^3}{81}$
23. 1.464 **25.** 1.049 **27.** 1.008 **29.** 0.983
31. $3003x^{20}y^5$ **33.** $126720x^8y^4$ **35.** $1001a^{10}b^4$
37. **(a)** $1 + \dfrac{v^2}{2c^2} + \dfrac{3v^4}{8c^4}$ **(b)** $mc^2 + \dfrac{mv^2}{2} + \dfrac{3mv^4}{8c^2}$
39. $r^3 + \frac{3}{2}\ell^2r + \dfrac{3\ell^4}{8r}$

Review Exercises

1. $\frac{1}{3}; \frac{1}{4}; \frac{1}{5}; \frac{1}{6}; \frac{1}{7}; \frac{1}{8}$ **3.** $\frac{-1}{3}; \frac{1}{10}; \frac{-1}{21}; \frac{1}{36}; \frac{-1}{55}; \frac{1}{78}$
5. $1; \frac{1}{2}; \frac{1}{6}; \frac{1}{24}; \frac{1}{120}; \frac{1}{720}$ **7.** 1; 3; 4; 7; 11; 18
9. Arithmetic sequence, $d = -3$, $a_{10} = -17$
11. Arithmetic sequence, $d = 4$, $a_7 = 23$

13. Geometric sequence, $r = -\frac{1}{6}$, $a_{10} = -\frac{-}{3359232}$ ≈ -0.0000003. **15.** -24

17. **(a)** $(a)\frac{4}{3} + \frac{4}{9} + \frac{4}{27} + \frac{4}{81}$ **(b)** $2 + \frac{3}{2} + \frac{4}{3} + \frac{5}{4}$

19. 39 **21.** Arithmetic series; $S_{10} = 265$

23. Geometric series; $S_{14} = 1.4999997$

25. Converges; $S = \frac{2}{3}$ **27.** Diverges

29. Converges; $S \approx 1.3660$ **31.** $\frac{185}{999} = \frac{5}{27}$

33. $a^5 + 10a^4 + 40a^3 + 80a^2 + 80a + 32$

35. $\frac{x^6}{64} - \frac{9x^5y^2}{16} + \frac{135x^4y^4}{16} - \frac{135x^3y^6}{2} + \frac{1215x^2y^8}{4} - 729xy^{10} + 729y^{12}$ **37.** $32{,}768x^{15} + 245{,}760x^{14}y + 850{,}160x^{13}y^2 + 1{,}863{,}680x^{12}y^3$

39. $1 + 5x + 15x^2 + 35x^3$ **41.** $2{,}480{,}640x^{14}y^6$

43. 1.004 **45.** \$727.76 **47.** 65.76 cm

49. 19.834 sec

Chapter 20 Test

1. $-5, 5, \frac{5}{3}, 1, \frac{5}{7}, \frac{5}{9}$ **3.** arithmetic sequence, $d = -2.5$, $a_{10} = -7.5$ **5.** 25 **7.** $\frac{435}{999} = \frac{145}{333}$

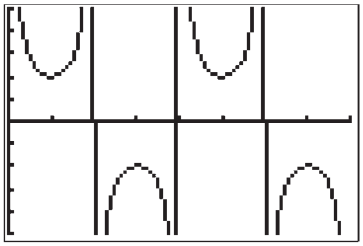
9. 123,000

ANSWERS FOR CHAPTER 21

Exercise Set 21.1

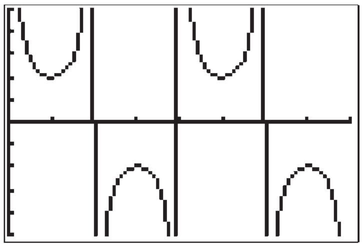
1-29. All answers are proofs and will not be displayed in this section.

31. This identity is true based on the graphs of $y = 2 \csc 2x$ and $y = \sec x \csc x$ shown below.



$y = 2\csc 2x$

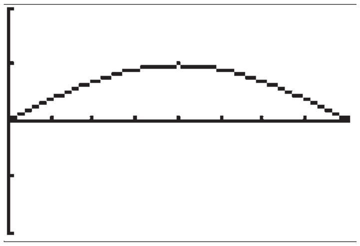
$\left[0, 2\pi, \frac{\pi}{4}\right] \cdot [-5, 5, 1]$



$y = 2\sec x \csc 2x$

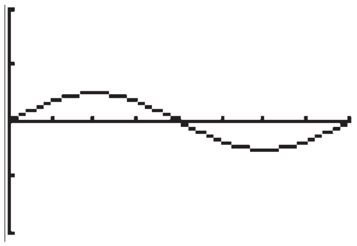
$\left[0, 2\pi, \frac{\pi}{4}\right] \cdot [-5, 5, 1]$

33. This identity is not true based on the graphs of $y = \sin \frac{1}{2}x$ and $y = \frac{1}{2}\sin x$ shown below.



$y = \sin \frac{1}{2}x$

$\left[0, 2\pi, \frac{\pi}{4}\right] \cdot [-2, 2, 1]$



$y = \sin \frac{1}{2}x$

$\left[0, 2\pi, \frac{\pi}{4}\right] \cdot [-2, 2, 1]$

35-43. All answers are proofs and will not be displayed in this section.

Exercise Set 21.2

1. $\frac{\sqrt{6}-\sqrt{2}}{4}$ **3.** $\frac{\sqrt{3}}{2}$

5. $\frac{\sqrt{6}-\sqrt{2}}{\sqrt{6}+\sqrt{2}} = \frac{10-4\sqrt{3}}{4} = \frac{5-2\sqrt{3}}{2}$

7. $\frac{1}{2}$ **9.** $\cos x$ **11.** $-\sin x$ **13.** $-\cos x$

15. $\sin x$ **17.** $\frac{21+\sqrt{105}}{32} \approx 0.97647$

19. $\frac{21+\sqrt{105}}{7\sqrt{7}-3\sqrt{15}} \approx 4.52768$

21. $\frac{7\sqrt{7}+3\sqrt{15}}{32} \approx 0.94185$ **23.** I

25. $\frac{\sqrt{105}-21}{32} \approx -0.33603$

27. $\frac{\sqrt{105}-21}{7\sqrt{7}+3\sqrt{15}} \approx -0.35678$

29. $\frac{7\sqrt{7}-3\sqrt{15}}{32} \approx 0.21566$ **31.** IV

33. $\sin 60° = \frac{\sqrt{3}}{2}$ **35.** $\cos 44°$ **37.** $\cos\alpha$

39. $\cos^2 x - \sin^2 x$

41-59. All answers are proofs and will not be displayed in this section.

Exercise Set 21.3

1. $\sqrt{\frac{2+\sqrt{3}}{4}} \approx x0.96593$

3. $\sqrt{\frac{2-\sqrt{3}}{4}} \approx 0.25882$

5. $\sqrt{\frac{2+\sqrt{3}}{4}} \approx 0.96593$

7. $\sqrt{\frac{1+\sqrt{\frac{2+\sqrt{3}}{4}}}{2}} = \sqrt{\frac{2+\sqrt{2+\sqrt{3}}}{4}} \approx 0.99145$

9. $\frac{\sqrt{2}}{2+\sqrt{2}} \approx 0.41421$ **11.** $\sqrt{\frac{2-\sqrt{3}}{4}} \approx 0.25882$

13. $\frac{1}{4}(\sqrt{3}\sqrt{2+\sqrt{2+\sqrt{3}}} - \sqrt{2-\sqrt{2+\sqrt{3}}}) \approx$ 0.79335

15. $\sin 2x = -\frac{336}{625}$; $\cos 2x = \frac{527}{625}$; $\tan 2x = -\frac{336}{527}$; $\sin\frac{x}{2} = \frac{7\sqrt{2}}{10}$; $\cos\frac{x}{2} = \frac{\sqrt{2}}{10}$; $\tan\frac{x}{2} = 7$

17. $\sin 2x = \frac{840}{841}$; $\cos 2x = -\frac{41}{841}$; $\tan 2x = -\frac{840}{41}$; $\sin\frac{x}{2} = \sqrt{\frac{9}{58}}$; $\cos\frac{x}{2} = \sqrt{\frac{49}{58}}$; $\tan\frac{x}{2} = \frac{3}{7}$

19. $\sin 2x = \frac{840}{1369}$; $\cos 2x = \frac{-1081}{1369}$; $\tan 2x = -\frac{840}{1081}$; $\sin\frac{x}{2} = \sqrt{\frac{49}{74}}$; $\cos\frac{x}{2} = -\sqrt{\frac{25}{74}}$; $\tan\frac{x}{2} = -\frac{7}{5}$

21-41. All answers are proofs and will not be displayed in this section.

Exercise Set 21.4

1. 90°, 270° **3.** 30°, 210° **5.** 228.59°, 311.41°
7. 51.34°, 231.34° **9.** 90°, 270° **11.** 180°
13. 0°, 90°, 180° **15.** 0°, 90°, 180°, 270°
17. 30°, 150°, 210°, 330° **19.** 15°, 75°, 195°, 255°
21. 45°, 225° **23.** 0°, 40°, 60°, 80°, 120°, 160°, 180°, 200°, 240°, 280°, 300°, 320° **25.** 45°, 135°, 225°, 315°
27. 0°, 135°, 180°, 315° **29.** 30°, 90°, 150°
31. 20.87° **33.** 0.001 328 *s* **35.** 0, π

Review Exercises

1-9. All answers are proofs and will not be displayed in this section. **11.** 139.11°, 319.11°
13. 90°, 120°, 240°, 270° **15.** 0°, 90°, 180°, 270°
17. 0°, 180° **19.** 218.17°, 321.83° **21.** $-\frac{120}{169}$
23. $\sqrt{25/26} \approx 0.98058$ **25.** $\frac{-21}{221} \approx -0.09502$
27. $\frac{220}{221} \approx 0.99548$ **29.** $\frac{140}{221} \approx 0.63348$
31. 90°, 210°, 330°

Chapter 21 Test

1. $-\frac{33}{65}$ **3.** $-\frac{63}{65}$ **5.** $-\frac{24}{25}$ **7.** $\frac{2}{\sqrt{5}}$ **9.** $4\sin 12x$

11. This answer is a proof and will not be displayed in this section. **13.** $2r(1-\cos\theta)$

ANSWERS FOR CHAPTER 22

Exercise Set 22.1

1. 9 **3.** 7 **5.** 28 **7.** −1/2 **9.** 1

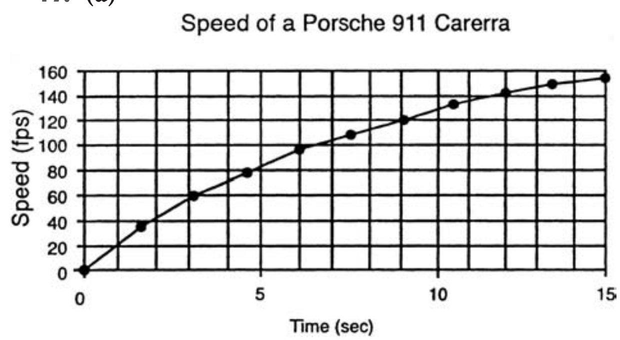
11. $\dfrac{x_1^2 + x_1 - 2}{x_1 - 1} = x_1 + 2$ **13.** $2x_1 + h$

15. **(a)** 29 liters/min **(b)** 39 L/m **(c)** 34 L/m

17. **(a)** 30 **(b)** 27 **(c)** 25.5 **(d)** 24.75 **(e)** 24.3 **(f)** 24.15

19. **(a)** $\dfrac{116 \text{ feet}}{1 \text{ second}}$ **(b)** feet per second **(c)** $s(t) = 5.89993\,t^2 + 21.19379\,t - 9.40381$ **(d)** 124.44 feet per second **(e)** 126.80 feet per second **21.** **(a)** 1.9 **(b)** ppm per year **(c)** $C(t) = 0.02091\,t^2 - 1.93364\,t + 3.58.2$ **(d)** 2.35 ppm per year **(e)** 2.44 ppm per year **23.** **(a)** 36.5 mi **(b)** 43.8 mph

17. **(a)**



(b) 1487.1 feet
(c) 1487.1 feet
(d) 1492 feet

19. 343.83 parts per million per year

Exercise Set 22.2

1. 84 **3.** 11.9375 **5.** 181 **7.** 72.28125
9. 175.75 **11.** 30.85 **13.** $2,480 **15.** 1.206 Ω

Exercise Set 22.3

1.

x	0.9	0.99	0.999	0.9999	1.0001	1.001	1.01	1.1
$f(x) = 3x$	2.7	2.97	2.997	2.9997	3.0003	3.003	3.03	3.3

$\lim_{x\to 1} 3x = 3$

3.

x	−1.1	−1.01	−1.001	−1.0001	−0.9999	−0.999	−0.99	−0.9
$h(x) = x^2 + 2$	3.21	3.0201	3.002	3.0002	2.9999	2.998	2.9801	2.81

$\lim_{x\to -1} (x^2 + 2) = 3$

5.

x	−0.1	−0.01	−0.001	−0.0001	0.0001	0.001	0.01	0.1
$f(x) = \dfrac{\tan x}{x}$	1.0033467	1.0000333	1.0000003	1	1	1.0000003	1.0000333	1.0033467

$\lim_{x\to 0} \dfrac{\tan x}{x} = 1$

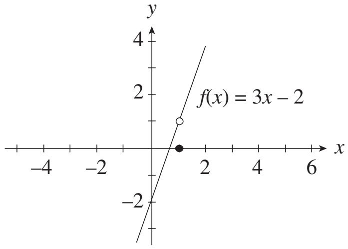
7.

x	0.9	0.99	0.999	0.9999	1.0001	1.001	1.01	1.1
$h(x) = \dfrac{x}{x-1}$	−9	−99	−999	−9999	10001	1001	101	11

$\lim_{x\to 1} \dfrac{x}{x-1}$ Does not exist

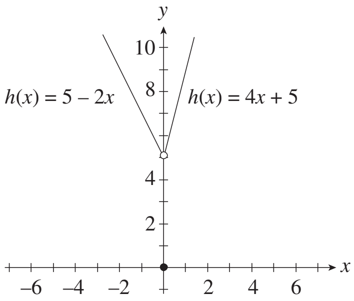
9. **(a)** 0, both sides of graph near $x = -1$ converge on 0 **(b)** Does not exist, graph goes toward -2 when $x < 2$ and goes toward 1 for $x > 2$ **(c)** 4, both sides of graph near $x = 3$ converge on 4.

11.



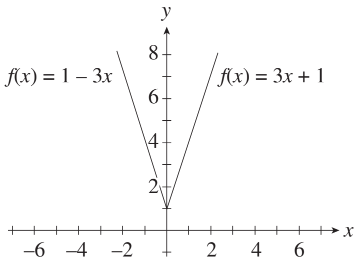
$\lim_{x \to 1} f(x)$ Does not exist

13.



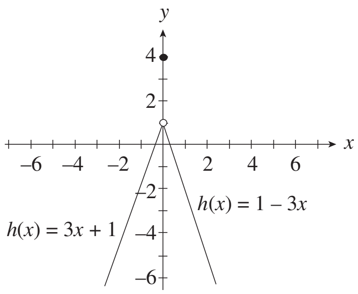
$\lim_{x \to 1} h(x)$ Does not exist

15.



$\lim_{x \to 1} g(x) = 1$

17.



$\lim_{x \to 1} h(x)$ Does not exist

19. 8 **21.** 1 **23.** 6 **25.** 5 **27.** 1 **29.** $\sqrt{6}$
31. 2 **33.** -2 **35.** 8 **37.** $\frac{2}{5}$ **39.** 0 **41.** 0

43. 4

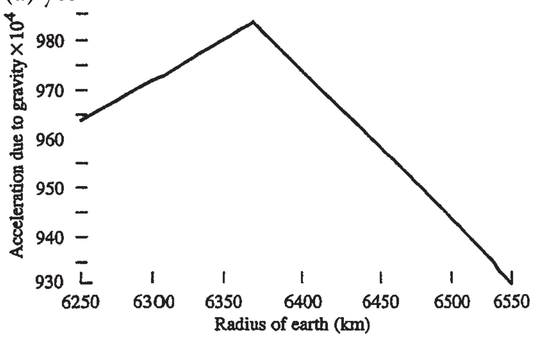
45.

(d)	$L(d)$
0.45	2.34278×10^{-7}
0.49	2.49072×10^{-7}
0.499	2.49072×10^{-7}
0.5001	2.52616×10^{-7}
0.5010	2.52928×10^{-7}
0.5100	2.56021×10^{-7}
0.5500	2.69138×10^{-7}

47. **(a)** Between 20.81 and 20.88 seconds for the winning runner to finish the women's 200 meter dash in the 2020 Olympics. **(b)** $L(t) = -0{:}0411t + 25.7342$ seconds t years after 1900 **(c)** $E(t) = 25.9079(0.9982t)$ seconds t years after 1900 **(d)** $\lim_{t \to \infty} L(t) = -\infty$ and $\lim_{t \to \infty} E(t) = 0$
(e) Neither answer is reasonable over the long term.

Exercise Set 22.5

1. **(a)** -2 **(b)** 1 **(c)** 4 **(d)** 4 **3.** 3 **5.** -4
7. -1 **9.** 0 **11.** 0 **13.** 0 **15.** 2 **17.** 0
19. $-\infty$ **21.** Continuous
23. Not continuous, $\lim_{x \to 1-} h(x) = 4 \neq 3 = h(1)$
25. Not continuous, $\lim_{x \to -2-} g(x) = 4 \neq g(-2)$
27. Not continuous, $\lim_{x \to 2-} j(x) = 0 = j(2)$
29. None **31.** -1 **33.** None **35.** 2
37. $(-\infty, -3), (-3, \infty)$ **39.** $(-\infty, -2), (-2, 2), (2, \infty)$
41. $(-\infty, -3], [3, \infty)$ **43.** $(-\infty, 0), (0, \infty)$
45. **(a)** 0.02381 **(b)** 0.0556 **(c)** 0
47. **(a)** \$4.75 **(b)** \$4.75 **(c)** \$4.75
49. **(a)** yes
(b)



51. **(a)** \$3,585 **(b)** \$3,585 **(c)** \$3,585 **(d)** Yes
(e) \$14,155 **(f)** \$14,155 **(g)** \$14,155 **(h)** Yes

53. **(a)** $\lim_{a \to 0^+} Y(a = 0)$

(b) $\lim_{a \to 0^+} C(a) = \frac{a + d}{24} = \frac{d}{24}$

(c) In both cases, the dosage approaches 0. This makes sense, since one would not give a baby a dosage of an adult medicine.

Review Exercises

1. 2 **3.** -10 **5.** $b + 1$ **7.** 21 **9.** 65.96875
11. 0 **13.** 6 **15.** 3 **17.** 0 **19.** 0 **21.** 5
23. -3 **25.** $(-\infty, -7)(-7, \infty)$
27. $(-\infty, -5), (5, \infty)$

Chapter 22 Test

1. 2 **3.** 13 **5.** 2 **7.** $x = 2$ and $x = 3$

INDEX OF APPLICATIONS

This index categorizes the application problems in this book alphabetically. After each topic, the index lists the text page number followed by the problem number in parentheses. For example, if you are looking for a problem on accounting, you would turn to page 738 and look for problem number 74.

INDEX

Properties of Exponents

1. $b^m b^n = b^{m+n}$
2. $(b^m)^n = b^{mn}$
3. $(ab)^n = a^n b^n$
4. $\left(\dfrac{a}{b}\right)^m = \dfrac{a^m}{b^m}, b \neq 0$
5. $\dfrac{b^m}{b^n} = b^{m-n}, b \neq 0$
6. $b^0 = 1, b \neq 0$
7. $b^{-n} = \dfrac{1}{b^n}, b \neq 0$

Properties of Radicals

1. $\sqrt[n]{ab} = \sqrt[n]{a}\sqrt[n]{b}$
2. $\sqrt[n]{\dfrac{a}{b}} = \dfrac{\sqrt[n]{a}}{\sqrt[n]{b}}$
3. $(\sqrt[n]{b})^n = b^{n/n} = b$
4. $\sqrt[n]{b} = b^{1/n}$
5. $\sqrt[m]{\sqrt[n]{b}} = \sqrt[mn]{b}$

Properties of Logarithms

1. $\log_b xy = \log_b x + \log_b y$
2. $\log_b \dfrac{x}{y} = \log_b x - \log_b y$
3. $\log_b x^p = p \log_b x$
4. $\log_b 1 = 0$
5. $\log_b b = 1$
6. $\log_b b^n = n$

Quadratic Formula

If $ax^2 + bx + c = 0$ and $a \neq 0$, then $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

Binomial Formula

$$(x+y)^n = x^n + \binom{n}{1}x^{n-1}y + \binom{n}{2}x^{n-2}y^2 + \cdots + \binom{n}{n-1}xy^{n-1} + y^n$$

Special Products and Factors

$a(x + y) = ax + ay$

$(x + y)(x - y) = x^2 - y^2$

$(x \pm y)^2 = x^2 \pm 2xy + y^2$

$(x \pm y)^3 = x^3 \pm 3x^2y + 3xy^2 \pm y^3$

$x^3 + y^3 = (x + y)(x^2 - xy + y^2)$

$x^3 - y^3 = (x - y)(x^2 + xy + y^2)$

Basic Operations on Complex Numbers

$(a + bj) \pm (c + dj) = (a \pm c) + (b \pm d)j$

$(a + bj)(c + dj) = (ac - bd) + (ad + bc)j$

$\dfrac{(a + bj)}{(c + dj)} = \dfrac{(ac + bd)}{c^2 + d^2} + \dfrac{(bc - ad)}{c^2 + d^2}j$

$|a + bj| = \sqrt{a^2 + b^2}$

Properties of Inequalities

If a, b, and c are real numbers, and

If $a < b$, then $a + c < b + c$

If $a < b$ and $c > 0$, then $ac < bc$

If $a < b$ and $c < 0$, then $ac > bc$

If $a < b$ and $n > 0$, then $a^n < b^n$ and $\sqrt[n]{a} < \sqrt[n]{b}$

Properties of Absolute Value

If x and a are real numbers, $a > 0$, $|x| = \begin{cases} -x \text{ if } x < 0 \\ x \text{ if } x \geq 0 \end{cases}$

If $|x| < a$, then $-a < x < a$

If $|x| > a$, then $x < -a$ or $x > a$

Variation

If k is a constant

Direct: $y = kx$

Indirect: $y = \dfrac{k}{x}$ or $yx = k$

Joint: $y = kxz$

DeMoivre's Formula

For any complex number $z = r \text{ cis } \theta = r \angle\theta = re^{j\theta}$, and n an integer, then $z^n = (re^{j\theta})^n = r^n e^{jn\theta} = r^n \angle n\theta$

$$r^n \text{ cis } (n\theta) = r^n (\cos n\theta + j \sin n\theta)$$

Roots of a Complex Number

If $z = r(\cos\theta + j\sin\theta)$, then the nth roots of z are given by the formula

$$w_k = \sqrt[n]{r}\left[\cos\left(\frac{\theta}{n} + \frac{360^\circ \cdot k}{n}\right) + j\sin\left(\frac{\theta}{n} + \frac{360^\circ \cdot k}{n}\right)\right]$$

where $k = 0, 1, 2, \ldots n - 1$

Some Trigonometric Identities

Reciprocal Identities

$$\sin\theta = \frac{1}{\csc\theta} \quad \cos\theta = \frac{1}{\sec\theta} \quad \tan\theta = \frac{1}{\cot\theta}$$

$$\csc\theta = \frac{1}{\sin\theta} \quad \sec\theta = \frac{1}{\cos\theta} \quad \cot\theta = \frac{1}{\tan\theta}$$

Pythagorean Identities

$\sin^2\theta + \cos^2\theta = 1$

$\tan^2\theta + 1 = \sec^2\theta$

$1 + \cot^2\theta = \csc^2\theta$

Quotient Identities

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \quad \cot\theta = \frac{\cos\theta}{\sin\theta}$$

Sum and Difference Identities

$\sin(\theta \pm \phi) = \sin\theta\cos\phi \pm \cos\theta\sin\phi$

$\cos(\theta \pm \phi) = \cos\theta\cos\phi \mp \sin\theta\sin\phi$

$$\tan(\theta \pm \phi) = \frac{\tan\theta + \tan\phi}{1 \mp \tan\theta\tan\phi}$$

Double-Angle Identities

$\sin 2\theta = 2\sin\theta\cos\theta$

$$\begin{aligned}\cos 2\theta &= \cos^2\theta - \sin^2\theta \\ &= 2\cos^2\theta - 1 \\ &= 1 - 2\sin^2\theta\end{aligned}$$

$$\tan 2\theta = \frac{2\tan\theta}{1 - \tan^2\theta}$$

Half-Angle Identities

$$\sin\frac{\theta}{2} = \pm\sqrt{\frac{1 - \cos\theta}{2}}$$

$$\cos\frac{\theta}{2} = \pm\sqrt{\frac{1 + \cos\theta}{2}}$$

$$\begin{aligned}\tan\frac{\theta}{2} &= \pm\sqrt{\frac{1 - \cos\theta}{1 + \cos\theta}} \\ &= \frac{\sin\theta}{1 + \cos\theta} \\ &= \frac{1 - \cos\theta}{\sin\theta}\end{aligned}$$

Sine Law

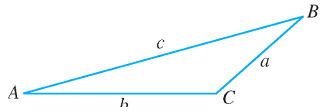
$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

Cosine Law

$a^2 = b^2 + c^2 - 2bc\cos A$

$b^2 = a^2 + c^2 - 2ac\cos B$

or $c^2 = a^2 + b^2 - 2ab\cos C$



Cosine Law (Alternative Version)

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

$$\cos B = \frac{a^2 + c^2 - b^2}{2ac}$$

$$\text{or } \cos C = \frac{a^2 + b^2 - c^2}{2ab}$$