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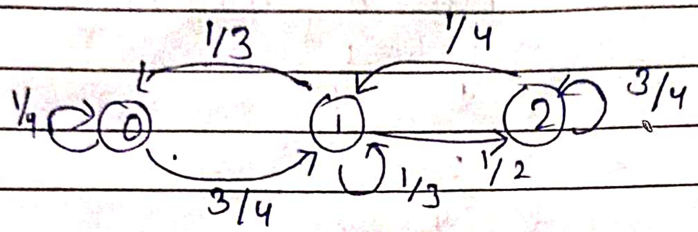
Markov Chain 1.

Q1]

$\rightarrow$

$$P = \begin{array}{c|ccc} & 0 & 1 & 2 \\ 0 & & & \\ 1 & & & \\ 2 & & & \end{array}\quad \begin{bmatrix} 1/4 & 3/4 & 0 \\ 1/3 & 1/3 & 1/3 \\ 0 & 1/4 & 3/4 \end{bmatrix}$$



$\alpha_0 = P[X_0 = 0] = 1/4$

$\alpha_1 = P[X_0 = 1] = 1/2$

$\alpha_2 = P[X_0 = 2] = 1/4$

a) Compute $P[X_0 = 0, X_1 = 1, X_2 = 1]$

$P[X_0 = 0, X_1 = 1, X_2 = 1]$

$= P[X_0 = 0] \cdot P[X_1 = 1 \mid X_0 = 0] \cdot P[X_2 = 1 \mid X_0 = 0, X_1 = 1]$

$= P[X_0 = 0] \quad P_{01}^{(1)} \quad P_{11}^{(1)}$

$= \frac{1}{4} \times \frac{3}{4} \times \frac{1}{3}$

$= \frac{1}{16}$

b) Compute $P(X_2 = 1)$

$$P(X_2 = 1) = \sum_{i=0}^{2} P_{ij}^{(2)} \times \alpha_i$$

$$P(X_2 = 1) = P_{01}^{(2)} \times \alpha_0 + P_{11}^{(2)} \times \alpha_1 + P_{21}^{(2)} \times \alpha_2$$

Now calculate P^2.

$$P^2 = P \times P$$

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$$P^2 = \begin{bmatrix} 5/16 & 7/16 & 1/4 \\ 7/36 & 4/9 & 13/36 \\ 1/12 & 13/48 & 31/48 \end{bmatrix}$$

$$P(X_2=1) = \frac{7}{16}\times\frac{1}{4} + \frac{4}{9}\times\frac{1}{2} + \frac{13}{48}\times\frac{1}{4}$$

$$= \frac{115}{288}$$

$$P(X_2=1) = 0.399$$

c) Compute $P(X_1=1, X_2=1 / X_0=0)$

$$P(X_1=1, X_2=1 / X_0=0)$$
$$= \frac{P(X_1=1, X_2=1, X_0=0)}{P(X_0=0)}$$
$$= \frac{P(X_0=0)\ P_{01}^{(1)}\ P_{11}^{(1)}}{P(X_0=0)}$$
$$= \frac{3}{4}\times\frac{1}{3} = \frac{1}{4}$$

d) Compute $P(X_4=1 / X_2=2)$

$$P(X_4=1 / X_2=2) = P_{21}^{(2)} = \frac{13}{48}$$

e) Compute $P(X_7=0 / X_5=0) = P_{00}^{(2)} = \frac{5}{16}$

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Q2] $P = \begin{bmatrix} p & q \\ q & p \end{bmatrix}$

$[P - \lambda I] = 0$

$$\begin{bmatrix} p-\lambda & q \\ q & p-\lambda \end{bmatrix} = 0$$

$$(p-\lambda)^2 - q^2 = 0$$

$$\lambda^2 - 2p\lambda + (p^2 - q^2) = 0$$

$$\lambda = \frac{2p \pm \sqrt{4p^2 - 4p^2 + 4q^2}}{2}$$

$$= \frac{2p \pm \sqrt{4q^2}}{2}$$

$$= p \pm q$$

$\lambda_1 = p+q$ and $\lambda_2 = p-q$

for $\lambda_1 = p+q$

$$\begin{bmatrix} p-p-q & q \\ q & p-p-q \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -q & q \\ q & -q \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$-qx_1 + q \cdot x_2 = 0$$

$x_2 = 1$

$x_1 = 1$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

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for $\lambda_2 = p-q$

$$\begin{pmatrix} p-p+q & q \\ q & p-p+q \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} q & q \\ q & q \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$qx_1 + qx_2 = 0$$
$$qx_1 + qx_2 = 0$$
$$x_2 = 1, \quad x_1 = -1$$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$$X = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

$$X^{-1} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} 1/2 & 1/2 \\ -1/2 & 1/2 \end{bmatrix}$$

$$y^{(1)} = \begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix} \qquad y^{(2)} = \begin{pmatrix} -1/2 \\ 1/2 \end{pmatrix}$$

$$P^n = \Sigma \lambda_i^n x_i y_i'$$
$$= (p+q)^n \begin{pmatrix} 1 \\ 1 \end{pmatrix} \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \end{bmatrix} + (p-q)^n \begin{pmatrix} -1 \\ 1 \end{pmatrix} \begin{bmatrix} -1/2 & 1/2 \end{bmatrix}$$
$$= (p+q)^n \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix} + (p-q)^n \begin{bmatrix} 1/2 & -1/2 \\ -1/2 & 1/2 \end{bmatrix}$$

we apply limit $n \to \infty$ then we get

$$\lim P^n = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}$$

Q 3.)

→ Let X_n = no of machine in operation at the end of n^{th} day.

P (One machine breakdown) = α

P (One machine repaired) = β

$P_{00} = P(X_1 = 0 \mid X_0 = 0) = 1-\beta$

$P_{01} = P(X_1 = 1 \mid X_0 = 0) = \beta$

$P_{10} = P(X_1 = 0 \mid X_0 = 1) = \alpha(1-\beta)$

$P_{11} = P(X_1 = 1 \mid X_0 = 1) = \alpha\beta + (1-\alpha)(1-\beta)$

$P_{02} = P(X_1 = 2 \mid X_0 = 0) = 0$

$P_{12} = P(X_1 = 2 \mid X_0 = 1) = \beta(1-\alpha)$

$P_{20} = P[X_1 = 0 \mid X_0 = 2] = \alpha^2$

$P_{21} = P(X_1 = 1 \mid X_0 = 2) = \alpha(1-\alpha) + \alpha(1-\alpha)$

$P_{22} = P(X_1 = 2 \mid X_0 = 2) = (1-\alpha)(1-\alpha) = (1-\alpha)^2$

∴ One step transition Matrix be.

$$P = \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{bmatrix} 1-\beta & \beta & 0 \\ \alpha(1-\beta) & \alpha\beta+(1-\alpha)(1-\beta) & \beta(1-\alpha) \\ \alpha^2 & \alpha(1-\alpha)+\alpha(1-\alpha) & (1-\alpha)^2 \end{bmatrix} \end{array}$$

Q 4.)

→ $$P(X(t) = n) = \frac{(at)^{n-1}}{(1+at)^{n+1}} \qquad n = 1, 2, 3, \ldots$$

$$= \frac{at}{1+at} \qquad n = 0$$

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$$E(X(t)) = \sum_{n=0}^{\infty} n \; P(X(t)=n)$$

$$= 0\left(\frac{at}{1+at}\right) + \sum_{n=1}^{\infty} n \frac{(at)^{n-1}}{(1+at)^{n+1}}$$

$$= \frac{(at)^0}{(1+at)^2} + \frac{2at}{(1+at)^3} + \frac{3(at)^2}{(1+at)^4} + \dots$$

$$= \frac{1}{(1+at)^2} + 2\left(\frac{at}{(1+at)^3}\right) + 3\left(\frac{at}{(1+at)^2}\right)^2 + \dots$$

$$= \frac{1}{(1+at)^2}\left[1 + 2\left(\frac{at}{(1+at)}\right) + 3\left(\frac{at}{(1+at)}\right)^2 + \dots\right]$$

$$= \frac{1}{(1+at)^2}\left(\frac{1}{\left(1-\frac{at}{1+at}\right)^2}\right)$$

$$E(X(t)) = \frac{1}{(1+at)^2} \frac{1}{\left(\frac{1+at-at}{1+at}\right)^2}$$

$$= \frac{1}{(1+at)^2} (1+at)^2$$

$$E(X(t)) = 1$$

$$E(X(t))^2 = \sum_{n=1}^{\infty} n^2 \; P(X(t)=n)$$

$$= \sum_{n=1}^{\infty} n^2 \times \frac{(at)^{n-1}}{(1+at)^{n+1}}$$

$$= \sum_{n=1}^{\infty} (n(n-1)+n) \; P(n)$$

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$$= \sum_{n=1}^{\infty} n(n-1)P(n) + \sum_{n=1}^{\infty} nP(n)$$

$$= \sum_{n=1}^{\infty} n(n-1)\cdot P(n) + 1$$

$$= \sum n(n-1) \frac{(at)^{n-1}}{(1+at)^{n+1}} + 1$$

$$= \frac{at}{(1+at)^3} \sum_{n=2}^{\infty} n(n-1) \left(\frac{at}{1+at}\right)^{n-2} + 1$$

$$= \frac{at}{(1+at)^3} \sum_{n=2}^{\infty} \frac{\delta^2}{\delta x^2} x^n + 1 \qquad \left(\because x = \frac{at}{1+at}\right)$$

$$= \frac{at}{(1+at)^3} \frac{\delta^2}{\delta^2 x} \sum_{n=2}^{\infty} x^2 + 1$$

$$= \frac{at}{(1+at)^3} \frac{\delta^2}{\delta x^2} \left[x^2 + x^3 + \cdots\right] + 1$$

$$= \frac{at}{(1+at)^3} \frac{\delta^2}{\delta x^2} x^2 \left[1 + x + x^2 + \cdots\right] + 1$$

$$= \frac{at}{(1+at)^3} \frac{\delta^2}{\delta x^2} x^2 \frac{1}{(1-x)} + 1$$

$$= \frac{at}{(1+at)^3} \frac{\delta}{\delta x} \left[\frac{(1-x)2x + x^2}{(1-x)^2}\right] + 1$$

$$= \frac{at}{(1+at)^3} \frac{\delta}{\delta x} \left[\frac{2x - 2x^2 + x^2}{(1-x)^2} \right] + 1$$

$$= \frac{at}{(1+at)^3} \frac{\delta}{\delta x} \left[\frac{2x - x^2}{(1-x)^2} \right] + 1$$

$$= \frac{at}{(1+at)^3} \frac{\delta}{\delta x} \left[\frac{-1+1+2x-x^2}{(1-x)^2} \right] + 1$$

$$= \frac{at}{(1+at)^3} \frac{\delta}{\delta x} \left[\frac{1-(1-2x+x^2)}{(1-x)^2} \right] + 1$$

$$= \frac{at}{(1+at)^3} \frac{\delta}{\delta x} \left[\frac{1-(1-x)^2}{(1-x)^2} \right] + 1$$

$$= \frac{at}{(1+at)^3} \frac{\delta}{\delta x} \left[\frac{1}{(1-x)^2} - 1 \right] + 1$$

$$= \frac{at}{(1+at)^3} \left(\frac{2}{(1-x)^3} \right) + 1$$

$$= \frac{at}{(1+at)^3} \left(\frac{2}{\left(1 - \frac{at}{1+at}\right)^3} \right) + 1$$

$$E(x(t))^2 = \frac{at}{(1+at)^3} \, 2(1+at)^3 + 1$$

$$E(x(t))^2 \quad 1+2at$$

$$V(x(t)) = E(x(t))^2 - (E(x(t)))^2$$

$$= 1+2at - 1$$

$$V(x(t)) = 2at$$

Variance depends on t $\therefore$ It is not stationary

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Q5)
⟹ i) Water level in a tank at time $t \geq 0$

Time - time $(t \geq 0)$ - continuous
State - water level in a tank - Continuous
i.e Continuous time Continuous state space.

ii) Number of customers in a shop at time $t \geq 0$
Time - time $(t \geq 0)$ - Continuous
State Space - No of customers - Discrete

i.e Continuous time Discrete state space

iii) Number of breakdowns of a machinery in each week.
Time - no of week - Discrete
State space - no of breakdowns - Discrete
i.e Discrete time Discrete state space.

iv) Water level in the tank at the end of each hour.
Time - no of hour - Discrete
State space - water level in tank - Continuous.

i.e Discrete time continuous state space.