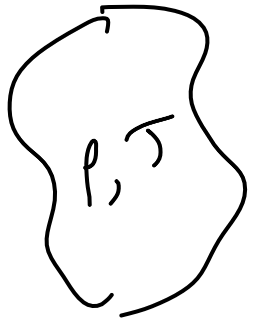
Conservation laws

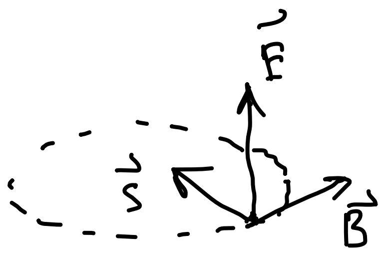
$$\vec{\nabla}\cdot\vec{J} = -\frac{\partial \rho}{\partial t}$$ $\longrightarrow$ local conservation of charge

$$\frac{dW}{dt} = -\frac{dU_{em}}{dt} - \frac{1}{\mu_0}\oint(\vec{E}\times\vec{B})\cdot d\vec{a}$$ $\longrightarrow$ conservation of energy



$$\boxed{\vec{S} = \frac{1}{\mu_0}(\vec{E}\times\vec{B})}$$

$$\frac{dW}{dt} = -\frac{dU_{em}}{dt} - \oint_S \vec{S}\cdot d\vec{a}$$

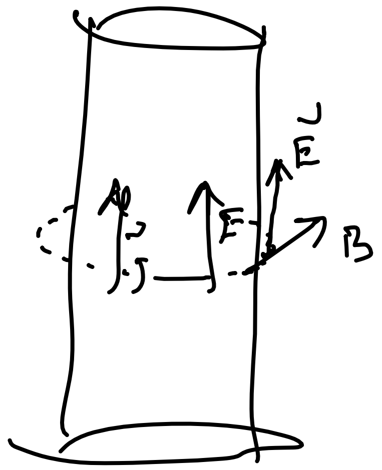


u_{em} , u_{mech}

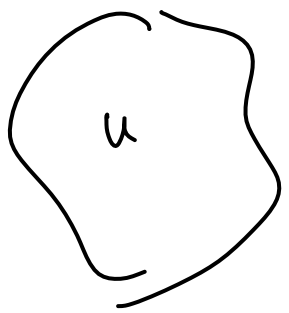
$$\frac{d}{dt}\int(u_{mech} + u_{em})\,d\tau = -\oint \vec{S}\cdot d\vec{a} = -\int \vec{\nabla}\cdot\vec{S}\,d\tau$$

$$\Rightarrow \boxed{\frac{\partial}{\partial t}(u_{mech} + u_{em}) = -\vec{\nabla}\cdot\vec{S}}$$

Ex



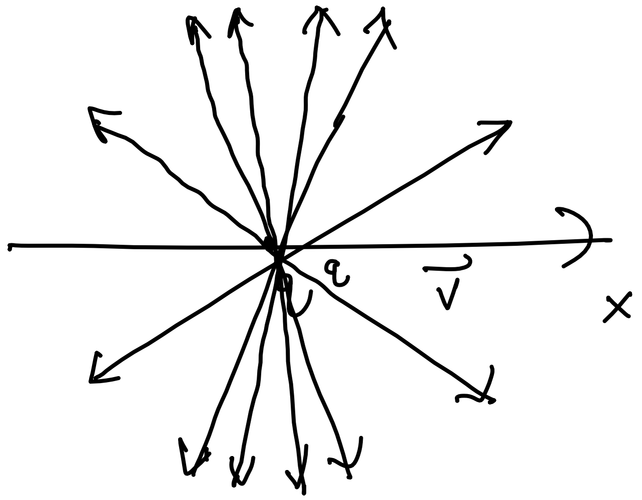
$u = u_{em}$ in free space $u_{mech} = 0$

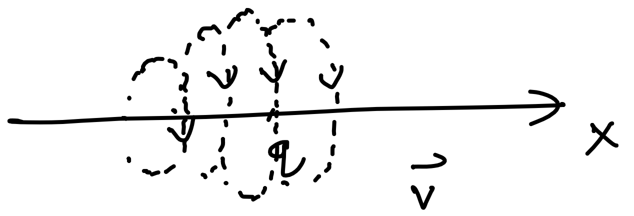


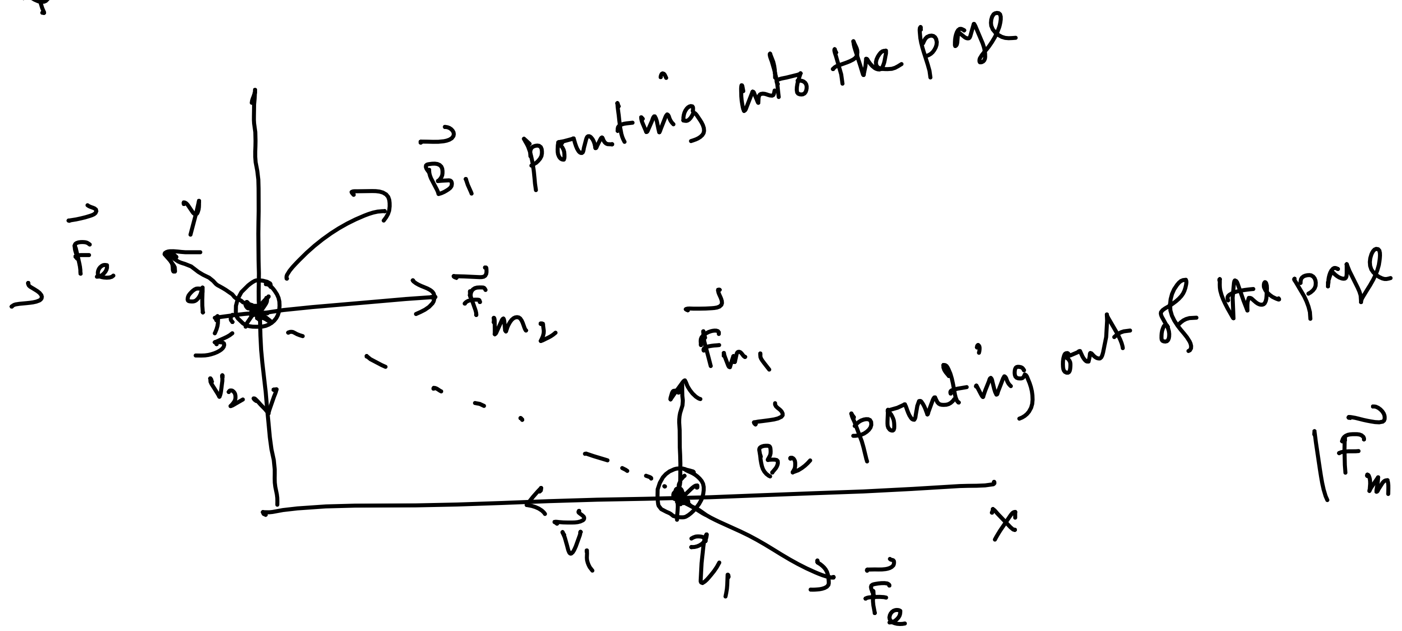
$$\boxed{\frac{\partial u}{\partial t} = -\vec{\nabla}\cdot\vec{S}}$$ $\longrightarrow$ $\vec{S}$ describes flow of energy in the same way as $\vec{J}$ describes flow of charge

$\longrightarrow$ Poynting vector

Conservation of linear momentum







$|\vec{F}_{m_1}| = |\vec{F}_{m_2}|$ but directions not opposite

Apparent violation of Newton's third law $\rightarrow$ apparent violation of conservation of linear momentum.

Field themselves carry momentum?

Maxwell's stress tensor

Total electromagnetic force on charges in volume V

$$\vec{F} = \int_V (\vec{E} + \vec{v}\times\vec{B})\rho\, d\tau = \int_V (\rho\vec{E} + \vec{J}\times\vec{B})\, d\tau$$

Force per unit volume $\vec{f} = \rho\vec{E} + \vec{J}\times\vec{B}$

$$\vec{f} = \epsilon_0(\vec{\nabla}\cdot\vec{E})\vec{E} + \left(\frac{1}{\mu_0}\vec{\nabla}\times\vec{B} - \epsilon_0\frac{\partial\vec{E}}{\partial t}\right)\times\vec{B}$$

$$\frac{\partial}{\partial t}(\vec{E}\times\vec{B}) = \vec{E}\times\frac{\partial\vec{B}}{\partial t} + \frac{\partial\vec{E}}{\partial t}\times\vec{B}$$

$$\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}$$

$$\Rightarrow \frac{\partial\vec{E}}{\partial t}\times\vec{B} = \frac{\partial}{\partial t}(\vec{E}\times\vec{B}) - \vec{E}\times\frac{\partial\vec{B}}{\partial t}$$

$$= \frac{\partial}{\partial t}(\vec{E}\times\vec{B}) - \vec{E}\times(\vec{\nabla}\times\vec{E})$$

$$\Rightarrow \vec{f} = \epsilon_0(\vec{\nabla}\cdot\vec{E})\vec{E} + \frac{1}{\mu_0}(\vec{\nabla}\times\vec{B})\times\vec{B} - \epsilon_0\frac{\partial}{\partial t}(\vec{E}\times\vec{B}) - \epsilon_0\vec{E}\times(\vec{\nabla}\times\vec{E})$$

$$= \epsilon_0\left[(\vec{\nabla}\cdot\vec{E})\vec{E} - \underline{\vec{E}\times(\vec{\nabla}\times\vec{E})}\right] - \frac{1}{\mu_0}\underline{\vec{B}\times(\vec{\nabla}\times\vec{B})} - \epsilon_0\frac{\partial}{\partial t}(\vec{E}\times\vec{B})$$

[Use $\vec{\nabla}(\vec{A}\cdot\vec{B}) = \vec{A}\times(\vec{\nabla}\times\vec{B}) + \vec{B}\times(\vec{\nabla}\times\vec{A}) + (\vec{A}\cdot\vec{\nabla})\vec{B} + (\vec{B}\cdot\vec{\nabla})\vec{A}$]

$$\vec{\nabla}(E^2) = 2(\vec{E}\cdot\vec{\nabla})\vec{E} + 2\vec{E}\times(\vec{\nabla}\times\vec{E})$$

$$\Rightarrow \vec{E}\times\vec{\nabla}\times\vec{E} = \frac{1}{2}\vec{\nabla}(E^2) - (\vec{E}\cdot\vec{\nabla})\vec{E}, \quad \vec{B}\times(\vec{\nabla}\times\vec{B}) = \frac{1}{2}\vec{\nabla}(B^2) - (\vec{B}\cdot\vec{\nabla})\vec{B}$$

$$\vec{f} = \epsilon_0 \left[(\vec{\nabla}\cdot\vec{E})\vec{E} + (\vec{E}\cdot\vec{\nabla})\vec{E} - \frac{1}{2}\vec{\nabla}(E^2) \right] + \frac{1}{\mu_0}\left[\underbrace{(\vec{\nabla}\cdot\vec{B})}_{0}\vec{B} + (\vec{B}\cdot\vec{\nabla})\vec{B} - \frac{1}{2}\vec{\nabla}(B^2) \right] - \epsilon_0 \frac{\partial}{\partial t}(\vec{E}\times\vec{B})$$

added term

$$T_{ij} = \epsilon_0 \left[E_i E_j - \frac{1}{2}\delta_{ij} E^2 \right] + \frac{1}{\mu_0}\left[B_i B_j - \frac{1}{2}\delta_{ij} B^2 \right]$$

$\delta_{ij} = 1$ only when $i = j$

$= 0$ when $i \neq j$

→ Maxwell's stress tensor

→ Nine elements $i, j = x, y, z$

$$T_{xx} = \epsilon_0 \left[E_x^2 - \frac{1}{2}E^2 \right] + \frac{1}{\mu_0}\left[B_x^2 - \frac{1}{2}B^2 \right]$$

$$T_{xy} = \epsilon_0 E_x E_y + \frac{1}{\mu_0} B_x B_y$$

$$\vdots$$

$$(\vec{\nabla}\cdot\overleftrightarrow{T})_j = \sum_{x,y,z} \nabla_i T_{ij}$$

$$\boxed{\vec{f} = \vec{\nabla}\cdot\overleftrightarrow{T} - \epsilon_0\mu_0 \frac{\partial \vec{S}}{\partial t}}$$

$$\vec{S} = \frac{1}{\mu_0}(\vec{E}\times\vec{B})$$

Total force on charges in V,

$$\vec{F} = \int_V (\vec{\nabla}\cdot\overleftrightarrow{T})\, d\tau - \epsilon_0\mu_0 \int \frac{\partial \vec{S}}{\partial t}\, d\tau$$

$$\boxed{\vec{F} = \oint_S \overleftrightarrow{T}\cdot d\vec{a} - \epsilon_0\mu_0 \frac{d}{dt}\int_V \vec{S}\, d\tau}$$

$\overleftrightarrow{T} \rightarrow$ force per unit area acting on the surface enclosing V

$$\begin{pmatrix} T_{xx} & T_{xy} & T_{xz} \\ T_{yx} & T_{yy} & T_{yz} \\ T_{zx} & T_{zy} & T_{zz} \end{pmatrix} \rightarrow$$

diagonal elements $\rightarrow$ pressure

off-diagonal elements $\rightarrow$ shear

$$\vec{F} = \frac{d\vec{p}_{mech}}{dt}$$

$$\Rightarrow \frac{d\vec{p}_{mech}}{dt} = \underbrace{-\epsilon_0\mu_0 \frac{d}{dt}\int_V \vec{S}\, d\tau}_{} + \underbrace{\oint \overleftrightarrow{T}\cdot d\vec{a}}_{}$$

$\vec{p}_{mech} \rightarrow$ momentum of charges in volume V

$$\vec{p} = \mu_0\epsilon_0 \int \vec{S}\, d\tau = \int \underbrace{\mu_0\epsilon_0 \vec{S}}\, d\tau$$

$\rightarrow$ momentum stored in the field

$\rightarrow$ momentum per unit time flowing through the surface enclosing V

$\vec{g} = \mu_0\epsilon_0 \vec{S}$ = field momentum density

in empty space $\frac{d\vec{p}_{mech}}{dt} = 0$

$$-\frac{d}{dt}\int \vec{g}\, d\tau = -\oint \overleftrightarrow{T}\cdot d\vec{a} = -\oint \vec{\nabla}\cdot\overleftrightarrow{T}\, d\tau$$

$$\Rightarrow \boxed{\vec{\nabla}\cdot-\overleftrightarrow{T} = -\frac{\partial \vec{g}}{\partial t}}$$