Pascal's triangle

In mathematics, **Pascal's triangle** is a triangular array of the binomial coefficients that arises in probability theory, combinatorics, and algebra. In much of the Western world, it is named after the French mathematician Blaise Pascal, although other mathematicians studied it centuries before him in India,[1] Persia,[2] China, Germany, and Italy.[3]

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1

A diagram showing the first eight rows of Pascal's triangle, numbered row 0 through row 7.

The rows of Pascal's triangle are conventionally enumerated starting with row $n = 0$ at the top (the 0th row). The entries in each row are numbered from the left beginning with $k = 0$ and are usually staggered relative to the numbers in the adjacent rows. The triangle may be constructed in the following manner: In row 0 (the topmost row), there is a unique nonzero entry 1. Each entry of each subsequent row is constructed by adding the number above and to the left with the number above and to the right, treating blank entries as 0. For example, the initial number in the first (or any other) row is 1 (the sum of 0 and 1), whereas the numbers 1 and 3 in the third row are added to produce the number 4 in the fourth row.

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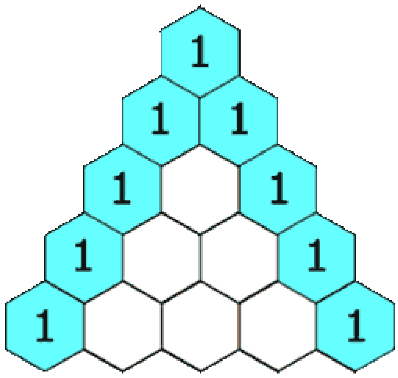
Formula

The entry in the nth row and kth column of Pascal's triangle is denoted $\binom{n}{k}$. For example, the unique nonzero entry in the topmost row is $\binom{0}{0} = 1$. With this notation, the construction of the previous paragraph may be written as follows:

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k},$$

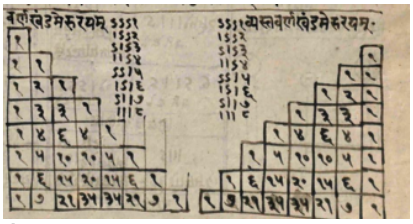
for any non-negative integer n and any integer $0 \leq k \leq n$.[4] This recurrence for the binomial coefficients is known as Pascal's rule.



In Pascal's triangle, each number is the sum of the two numbers directly above it.

Pascal's triangle has higher dimensional generalizations. The three-dimensional version is called *Pascal's pyramid* or *Pascal's tetrahedron*, while the general versions are called *Pascal's simplices*.

History



मेरु प्रस्तार(Meru Prastaara) as used in Indian manuscripts, derived from Pingala's formulae. Manuscript from Raghunath Library J&K; 755 AD

The pattern of numbers that forms Pascal's triangle was known well before Pascal's time. Pascal innovated many previously unattested uses of the triangle's numbers, uses he described comprehensively in the earliest known mathematical treatise to be specially devoted to the triangle, his *Traité du triangle arithmétique* (1654; published 1665).

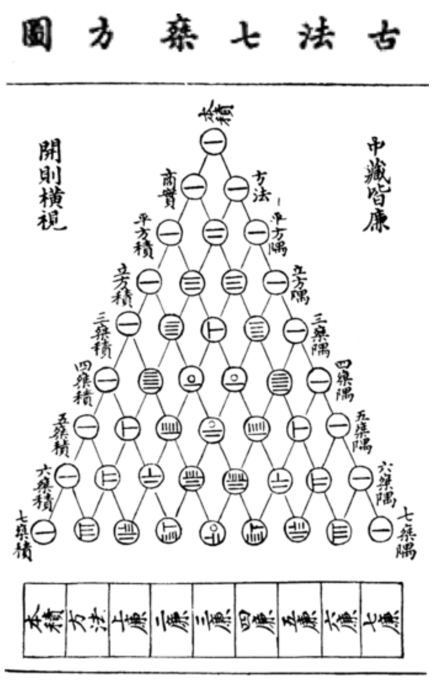
Centuries before, discussion of the numbers had arisen in the context of Indian studies of combinatorics and binomial numbers. It appears from later commentaries that the binomial coefficients and the additive formula for generating them, $\binom{n}{r} = \binom{n-1}{r} + \binom{n-1}{r-1}$, were known to Pingala in or before the 2nd century BC.[5][6] While Pingala's work only survives in fragments, the commentator Varāhamihira, around 505, gave a clear description of the additive formula,[6] and a more detailed explanation of the same rule was given by Halayudha, around 975. Halayudha also explained obscure references to *Meru-prastaara*, the **Staircase of Mount Meru**, giving the first surviving description of the arrangement of these numbers into a triangle.[6][7] In approximately 850, the Jain mathematician Mahāvīra gave a different formula for the binomial coefficients, using multiplication, equivalent to the modern formula $\binom{n}{r} = \frac{n!}{r!(n-r)!}$.[6] In 1068, four columns of the first sixteen rows were given by the mathematician Bhattotpala, who was the first recorded mathematician to equate the additive and multiplicative formulas for these numbers.[6]

At around the same time, the Persian mathematician Al-Karaji (953–1029) wrote a now-lost book which contained the first description of Pascal's triangle.[8][9][10] It was later repeated by the Persian poet-astronomer-mathematician Omar Khayyám (1048–1131); thus the triangle is also referred to as the **Khayyam triangle** in Iran.[11] Several theorems related to the triangle were known, including the binomial theorem. Khayyam used a method of finding nth roots based on the binomial expansion, and therefore on the binomial coefficients.[2]

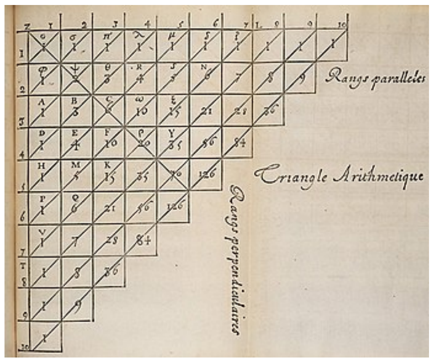
Pascal's triangle was known in China in the early 11th century through the work of the Chinese mathematician Jia Xian (1010–1070). In the 13th century, Yang Hui (1238–1298) presented the triangle and hence it is still called **Yang Hui's triangle** (杨辉三角; 楊輝三角) in China.[12]

In the west the Pascal's triangle appears for the first time in Arithmetic of Jordanus de Nemore (13th century).[13] The binomial coefficients were calculated by Gersonides in the early 14th century, using the multiplicative formula for them.[6] Petrus Apianus (1495–1552) published the full triangle on the frontispiece of his book on business calculations in 1527.[14] Michael Stifel published a portion of the triangle (from the second to the middle column in each row) in 1544, describing it as a table of figurate numbers.[6] In Italy, Pascal's triangle is referred to as **Tartaglia's triangle**, named for the Italian algebraist Niccolò Fontana Tartaglia (1500–1577), who published six rows of the triangle in 1556.[6] Gerolamo Cardano, also, published the triangle as well as the additive and multiplicative rules for constructing it in 1570.[6]

Pascal's *Traité du triangle arithmétique* (*Treatise on Arithmetical Triangle*) was published in 1655. In this, Pascal collected several results then known about the triangle, and employed them to solve problems in probability theory. The triangle was later named after Pascal by Pierre Raymond de Montmort (1708) who called it "Table de M. Pascal pour les combinaisons" (French: Table of Mr. Pascal for combinations) and Abraham de Moivre (1730) who called it "Triangulum Arithmeticum PASCALIANUM" (Latin: Pascal's Arithmetic Triangle), which became the modern Western name.[15]



Yang Hui's triangle, as depicted by the Chinese using rod numerals, appears in a mathematical work by Zhu Shijie, dated 1303. The title reads "The Old Method Chart of the Seven Multiplying Squares" (Chinese: 古法七乘方圖; the fourth character 椉 in the image title is archaic).



Pascal's version of the triangle

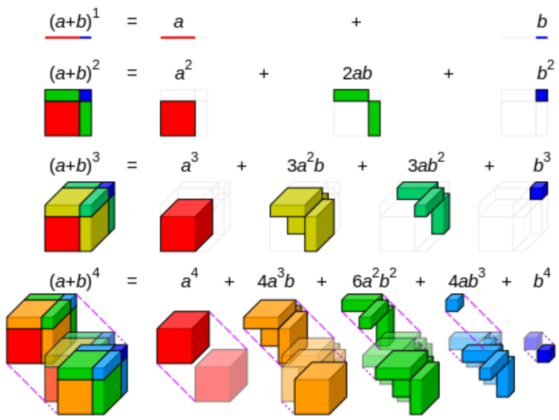
Binomial expansions

Pascal's triangle determines the coefficients which arise in binomial expansions. For example, consider the expansion

$$(x+y)^2 = x^2 + 2xy + y^2 = \mathbf{1}x^2y^0 + \mathbf{2}x^1y^1 + \mathbf{1}x^0y^2.$$

The coefficients are the numbers in the second row of Pascal's triangle: $\binom{2}{0} = 1$, $\binom{2}{1} = 2$, $\binom{2}{2} = 1$.

In general, when a binomial like $x + y$ is raised to a positive integer power of n, we have:



Visualisation of binomial expansion up to the 4th power

$$(x+y)^n = \sum_{k=0}^{n} a_k x^{n-k} y^k = a_0 x^n + a_1 x^{n-1} y + a_2 x^{n-2} y^2 + \ldots + a_{n-1} x y^{n-1} + a_n y^n,$$

where the coefficients a_k in this expansion are precisely the numbers on row n of Pascal's triangle. In other words,

$$a_k = \binom{n}{k}.$$

This is the binomial theorem.

The entire right diagonal of Pascal's triangle corresponds to the coefficient of y^n in these binomial expansions, while the next diagonal corresponds to the coefficient of xy^{n-1} and so on.

To see how the binomial theorem relates to the simple construction of Pascal's triangle, consider the problem of calculating the coefficients of the expansion of $(x+y)^{n+1}$ in terms of the corresponding coefficients of $(x+1)^n$ (setting $y = 1$ for simplicity). Suppose then that

$$(x+1)^n = \sum_{k=0}^{n} a_k x^k.$$

Now

$$(x+1)^{n+1} = (x+1)(x+1)^n = x(x+1)^n + (x+1)^n = \sum_{i=0}^{n} a_i x^{i+1} + \sum_{i=0}^{n} a_i x^i.$$

The two summations can be reorganized as follows:

$$\begin{array}{ccccccccccc}
 & & & & & \binom{0}{0} & & & & & \\
 & & & & \binom{1}{0} & & \binom{1}{1} & & & & \\
 & & & \binom{2}{0} & & \binom{2}{1} & & \binom{2}{2} & & & \\
 & & \binom{3}{0} & & \binom{3}{1} & & \binom{3}{2} & & \binom{3}{3} & & \\
 & \binom{4}{0} & & \binom{4}{1} & & \binom{4}{2} & & \binom{4}{3} & & \binom{4}{4} & \\
\binom{5}{0} & & \binom{5}{1} & & \binom{5}{2} & & \binom{5}{3} & & \binom{5}{4} & & \binom{5}{5}
\end{array}$$

Six rows Pascal's triangle as binomial coefficients

$$\begin{aligned}
\sum_{k=0}^{n} a_k x^{k+1} + \sum_{k=0}^{n} a_k x^k &= \sum_{k=1}^{n+1} a_{k-1} x^k + \sum_{k=0}^{n} a_k x^k \\
&= \sum_{k=1}^{n} a_{k-1} x^k + \sum_{k=1}^{n} a_k x^k + a_0 x^0 + a_n x^{n+1} \\
&= \sum_{k=1}^{n} (a_{k-1} + a_k) x^k + a_0 x^0 + a_n x^{n+1} \\
&= \sum_{k=1}^{n} (a_{k-1} + a_k) x^k + x^0 + x^{n+1}
\end{aligned}$$

(because of how raising a polynomial to a power works, $a_0 = a_n = 1$).

We now have an expression for the polynomial $(x+1)^{n+1}$ in terms of the coefficients of $(x+1)^n$ (these are the a_ks), which is what we need if we want to express a line in terms of the line above it. Recall that all the terms in a diagonal going from the upper-left to the lower-right correspond to the same power of x, and that the a-terms are the coefficients of the polynomial $(x+1)^n$, and we are determining the coefficients of $(x+1)^{n+1}$. Now, for any given $0<k<n+1$, the coefficient of the x^k term in the polynomial $(x+1)^{n+1}$ is equal to $a_{k-1}+a_k$. This is indeed the simple rule for constructing Pascal's triangle row-by-row.

It is not difficult to turn this argument into a proof (by mathematical induction) of the binomial theorem.

Since $(a+b)^n = b^n\left(\frac{a}{b}+1\right)^n$, the coefficients are identical in the expansion of the general case.

An interesting consequence of the binomial theorem is obtained by setting both variables x and y equal to one. In this case, we know that $(1+1)^n = 2^n$, and so

$$\sum_{k=0}^{n}\binom{n}{k} = \binom{n}{0}+\binom{n}{1}+\cdots+\binom{n}{n-1}+\binom{n}{n} = 2^n.$$

In other words, the sum of the entries in the nth row of Pascal's triangle is the nth power of 2. This is equivalent to the statement that the number of subsets (the cardinality of the power set) of an n-element set is 2^n, as can be seen by observing that the number of subsets is the sum of the number of combinations of each of the possible lengths, which range from zero through to n.

Combinations

A second useful application of Pascal's triangle is in the calculation of combinations. For example, the number of combinations of n things taken k at a time (called *n choose k*) can be found by the equation

$$\mathbf{C}(n,k) = \mathbf{C}_k^n = {}_nC_k = \binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

But this is also the formula for a cell of Pascal's triangle. Rather than performing the calculation, one can simply look up the appropriate entry in the triangle. Provided we have the first row and the first entry in a row numbered 0, the answer will be located at entry k in row n. For example, suppose a basketball team has 10 players and wants to know how many ways there are of selecting 8. The answer is entry 8 in row 10, which is 45; that is, 10 choose 8 is 45.

Relation to binomial distribution and convolutions

When divided by 2^n, the nth row of Pascal's triangle becomes the binomial distribution in the symmetric case where $p=\frac{1}{2}$. By the central limit theorem, this distribution approaches the normal distribution as n increases. This can also be seen by applying Stirling's formula to the factorials involved in the formula for combinations.

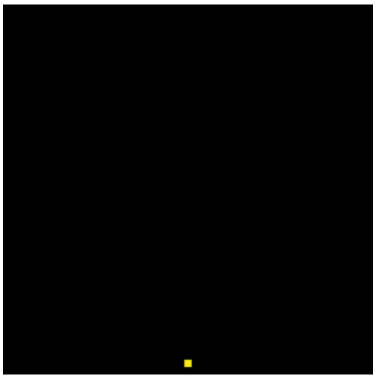
This is related to the operation of discrete convolution in two ways. First, polynomial multiplication exactly corresponds to discrete convolution, so that repeatedly convolving the sequence $\{\ldots,0,0,1,1,0,0,\ldots\}$ with itself corresponds to taking powers of $x+1$, and hence to generating the rows of the triangle. Second, repeatedly convolving the distribution function for a random variable with itself corresponds to calculating the distribution function for a sum of n independent copies of that variable; this is exactly the situation to which the central limit theorem applies, and hence leads to the normal distribution in the limit.

Patterns and properties

Pascal's triangle has many properties and contains many patterns of numbers.

Rows

- The sum of the elements of a single row is twice the sum of the row preceding it. For example, row 0 (the topmost row) has a value of 1, row 1 has a value of 2, row 2 has a value of 4, and so forth. This is because every item in a row produces two items in the next row: one left and one right. The sum of the elements of row n equals to 2^n.



Each frame represents a row in Pascal's triangle. Each column of pixels is a number in binary with the least significant bit at the bottom. Light pixels represent ones and the dark pixels are zeroes.

- Taking the product of the elements in each row, the sequence of products (sequence A001142 in the OEIS) is related to the base of the natural logarithm, *e*.[16][17] Specifically, define the sequence s_n for all $n \geq 0$ as follows$s_n = \prod_{k=0}^{n} \binom{n}{k} = \prod_{k=0}^{n} \frac{n!}{k!(n-k)!}$

  Then, the ratio of successive row products is

$$\frac{s_{n+1}}{s_n} = \frac{(n+1)!^{n+2} \prod_{k=0}^{n+1} \frac{1}{k!^2}}{n!^{n+1} \prod_{k=0}^{n} \frac{1}{k!^2}} = \frac{(n+1)^n}{n!}$$

  and the ratio of these ratios is

$$\frac{s_{n+1} \cdot s_{n-1}}{s_n^2} = \left(\frac{n+1}{n}\right)^n, \; n \geq 1.$$

  The right-hand side of the above equation takes the form of the limit definition of e

$$e = \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n.$$

- π can be found in Pascal's triangle through the Nilakantha infinite series.[18]

$$\pi = 3 + \sum_{n=1}^{\infty} (-1)^{n+1} \frac{\binom{2n+1}{1}}{\binom{2n+1}{2}\binom{2n+2}{2}}$$

- *The value of a row*, if each entry is considered a decimal place (and numbers larger than 9 carried over accordingly), is a power of 11 (11^n, for row n). Thus, in row 2, $\langle 1, 2, 1 \rangle$ becomes 11^2, while $\langle 1, 5, 10, 10, 5, 1 \rangle$ in row five becomes (after carrying) 161,051, which is 11^5. This property is explained by setting $x = 10$ in the binomial expansion of $(x + 1)^n$, and adjusting values to the decimal system. But x can be chosen to allow rows to represent values in *any* base.
  - In base 3: $1\ 2\ 1_3 = 4^2\ (16)$
  - $\langle 1, 3, 3, 1 \rangle \to 2\ 1\ 0\ 1_3 = 4^3\ (64)$
  - In base 9: $1\ 2\ 1_9 = 10^2\ (100)$
  - $1\ 3\ 3\ 1_9 = 10^3\ (1000)$
  - $\langle 1, 5, 10, 10, 5, 1 \rangle \to 1\ 6\ 2\ 1\ 5\ 1_9 = 10^5\ (100000)$

  In particular (see previous property), for $x = 1$ place value remains *constant* (1^{place}=1). Thus entries can simply be added in interpreting the value of a row.

- Some of the numbers in Pascal's triangle correlate to numbers in Lozanić's triangle.
- The sum of the squares of the elements of row n equals the middle element of row $2n$. For example, $1^2 + 4^2 + 6^2 + 4^2 + 1^2 = 70$. In general form:

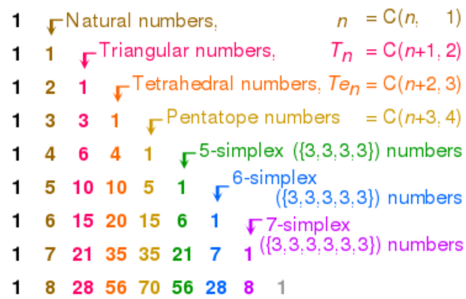
$$\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}.$$

- On any row n, where n is even, the middle term minus the term two spots to the left equals a Catalan number, specifically the $(n/2+1)$th Catalan number. For example: on row 4, $6-1=5$, which is the 3rd Catalan number, and $4/2+1=3$.
- In a row p where p is a prime number, all the terms in that row except the 1s are multiples of p. This can be proven easily, since if $p \in \mathbb{P}$, then p has no factors save for 1 and itself. Every entry in the triangle is an integer, so therefore by definition $(p-k)!$ and $k!$ are factors of $p!$. However, there is no possible way p itself can show up in the denominator, so therefore p (or some multiple of it) must be left in the numerator, making the entire entry a multiple of p.
- *Parity*: To count odd terms in row n, convert n to binary. Let x be the number of 1s in the binary representation. Then the number of odd terms will be 2^x. These numbers are the values in Gould's sequence.[19]
- Every entry in row 2^n-1, $n \geq 0$, is odd.[20]
- *Polarity*: When the elements of a row of Pascal's triangle are added and subtracted together sequentially, every row with a middle number, meaning rows that have an odd number of integers, gives 0 as the result. As examples, row 4 is 1 4 6 4 1, so the formula would be 6 – (4+4) + (1+1) = 0; and row 6 is 1 6 15 20 15 6 1, so the formula would be 20 – (15+15) + (6+6) – (1+1) = 0. So every even row of the Pascal triangle equals 0 when you take the middle number, then subtract the integers directly next to the center, then add the next integers, then subtract, so on and so forth until you reach the end of the row.

Diagonals

The diagonals of Pascal's triangle contain the figurate numbers of simplices:

- The diagonals going along the left and right edges contain only 1's.
- The diagonals next to the edge diagonals contain the natural numbers in order.
- Moving inwards, the next pair of diagonals contain the triangular numbers in order.
- The next pair of diagonals contain the tetrahedral numbers in order, and the next pair give pentatope numbers.



Derivation of simplex numbers from a left-justified Pascal's triangle

$$\begin{aligned} P_0(n) &= P_d(0) = 1, \\ P_d(n) &= P_d(n-1) + P_{d-1}(n) \\ &= \sum_{i=0}^{n} P_{d-1}(i) = \sum_{i=0}^{d} P_i(n-1). \end{aligned}$$

The symmetry of the triangle implies that the n^{th} d-dimensional number is equal to the d^{th} n-dimensional number.

An alternative formula that does not involve recursion is as follows:

$$P_d(n) = \frac{1}{d!}\prod_{k=0}^{d-1}(n+k) = \frac{n^{(d)}}{d!} = \binom{n+d-1}{d}$$

where $n^{(d)}$ is the rising factorial.

The geometric meaning of a function P_d is: $P_d(1) = 1$ for all d. Construct a d-dimensional triangle (a 3-dimensional triangle is a tetrahedron) by placing additional dots below an initial dot, corresponding to $P_d(1) = 1$. Place these dots in a manner analogous to the placement of numbers in Pascal's triangle. To find $P_d(x)$, have a total of x dots composing the target shape. $P_d(x)$ then equals the total number of dots in the shape. A 0-dimensional triangle is a point and a 1-dimensional triangle is simply a line, and therefore $P_0(x) = 1$ and $P_1(x) = x$, which is the sequence of natural numbers. The number of dots in each layer corresponds to $P_{d-1}(x)$.

Calculating a row or diagonal by itself

There are simple algorithms to compute all the elements in a row or diagonal without computing other elements or factorials.

To compute row n with the elements $\binom{n}{0}$, $\binom{n}{1}$, ..., $\binom{n}{n}$, begin with $\binom{n}{0} = 1$. For each subsequent element, the value is determined by multiplying the previous value by a fraction with slowly changing numerator and denominator:

$$\binom{n}{k} = \binom{n}{k-1} \times \frac{n+1-k}{k}.$$

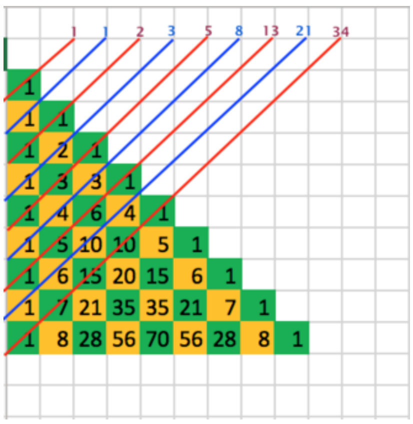
For example, to calculate row 5, the fractions are $\frac{5}{1}$, $\frac{4}{2}$, $\frac{3}{3}$, $\frac{2}{4}$ and $\frac{1}{5}$, and hence the elements are $\binom{5}{0} = 1$, $\binom{5}{1} = 1 \times \frac{5}{1} = 5$, $\binom{5}{2} = 5 \times \frac{4}{2} = 10$, etc. (The remaining elements are most easily obtained by symmetry.)

To compute the diagonal containing the elements $\binom{n}{0}$, $\binom{n+1}{1}$, $\binom{n+2}{2}$, ..., we again begin with $\binom{n}{0} = 1$ and obtain subsequent elements by multiplication by certain fractions:

$$\binom{n+k}{k} = \binom{n+k-1}{k-1} \times \frac{n+k}{k}.$$

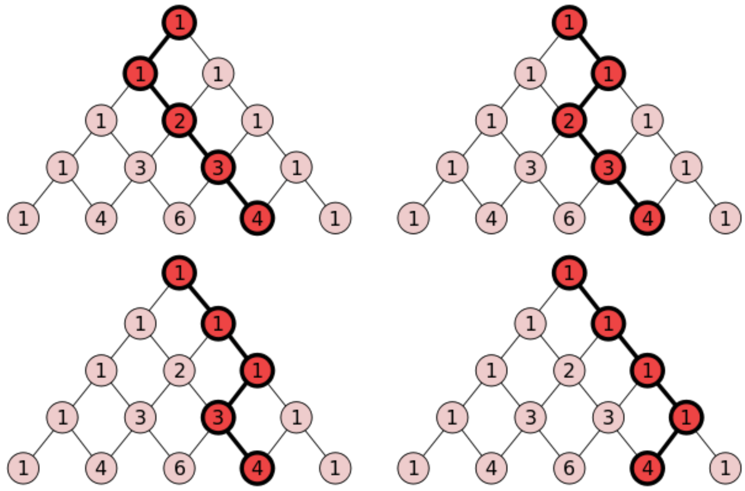
For example, to calculate the diagonal beginning at $\binom{5}{0}$, the fractions are $\frac{6}{1}$, $\frac{7}{2}$, $\frac{8}{3}$, ..., and the elements are $\binom{5}{0} = 1$, $\binom{6}{1} = 1 \times \frac{6}{1} = 6$, $\binom{7}{2} = 6 \times \frac{7}{2} = 21$, etc. By symmetry, these elements are equal to $\binom{5}{5}$, $\binom{6}{5}$, $\binom{7}{5}$, etc.

Overall patterns and properties



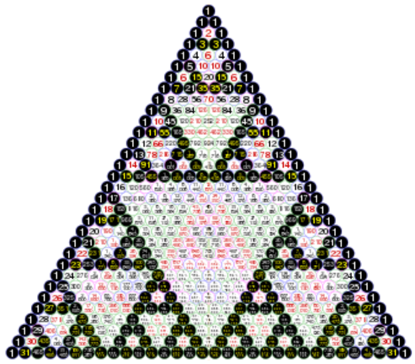
Fibonacci sequence in Pascal's triangle

- The pattern obtained by coloring only the odd numbers in Pascal's triangle closely resembles the fractal called the Sierpinski triangle. This resemblance becomes more and more accurate as more rows are considered; in the limit, as the number of rows approaches infinity, the resulting pattern *is* the Sierpinski triangle, assuming a fixed perimeter. More generally, numbers could be colored differently according to whether or not they are multiples of 3, 4, etc.; this results in other similar patterns.
- In a triangular portion of a grid (as in the images below), the number of shortest grid paths from a given node to the top node of the triangle is the corresponding entry in Pascal's triangle. On a Plinko game board shaped like a triangle, this distribution should give the probabilities of winning the various prizes.

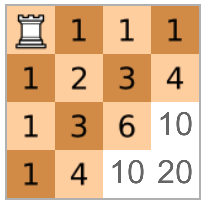


- If the rows of Pascal's triangle are left-justified, the diagonal bands (colour-coded below) sum to the Fibonacci numbers.

1							
1	1						
1	2	1					
1	3	3	1				
1	4	6	4	1			
1	5	10	10	5	1		
1	6	15	20	15	6	1	
1	7	21	35	35	21	7	1



A level-4 approximation to a Sierpinski triangle obtained by shading the first 32 rows of a Pascal triangle white if the binomial coefficient is even and black if it is odd.



Pascal's triangle overlaid on a grid gives the number of distinct paths to each square, assuming only rightward and downward movements are considered.

Construction as matrix exponential

Due to its simple construction by factorials, a very basic representation of Pascal's triangle in terms of the matrix exponential can be given: Pascal's triangle is the exponential of the matrix which has the sequence 1, 2, 3, 4, ... on its subdiagonal and zero everywhere else.

Connections to geometry of polytopes

Pascal's triangle can be used as a lookup table for the number of elements (such as edges and corners) within a polytope (such as a triangle, a tetrahedron, a square and a cube).

$$\exp\begin{pmatrix} . & . & . & . & . \\ 1 & . & . & . & . \\ . & 2 & . & . & . \\ . & . & 3 & . & . \\ . & . & . & 4 & . \end{pmatrix} = \begin{pmatrix} 1 & . & . & . & . \\ 1 & 1 & . & . & . \\ 1 & 2 & 1 & . & . \\ 1 & 3 & 3 & 1 & . \\ 1 & 4 & 6 & 4 & 1 \end{pmatrix}$$
$$e^{counting} = binomial$$

Binomial matrix as matrix exponential. All the dots represent 0.

Number of elements of simplices

Let's begin by considering the 3rd line of Pascal's triangle, with values 1, 3, 3, 1. A 2-dimensional triangle has one 2-dimensional element (itself), three 1-dimensional elements (lines, or edges), and three 0-dimensional elements (vertices, or corners). The meaning of the final number (1) is more difficult to explain (but see below). Continuing with our example, a tetrahedron has one 3-dimensional element (itself), four 2-dimensional elements (faces), six 1-dimensional elements (edges), and four 0-dimensional elements (vertices). Adding the final 1 again, these values correspond to the 4th row of the triangle (1, 4, 6, 4, 1). Line 1 corresponds to a point, and Line 2 corresponds to a line segment (dyad). This pattern continues to arbitrarily high-dimensioned hyper-tetrahedrons (known as simplices).

To understand why this pattern exists, one must first understand that the process of building an n-simplex from an $(n-1)$-simplex consists of simply adding a new vertex to the latter, positioned such that this new vertex lies outside of the space of the original simplex, and connecting it to all original vertices. As an example, consider the case of building a tetrahedron from a triangle, the latter of whose elements are enumerated by row 3 of Pascal's triangle: **1** face, **3** edges, and **3** vertices (the meaning of the final 1 will be explained shortly). To build a tetrahedron from a triangle, we position a new vertex above the plane of the triangle and connect this vertex to all three vertices of the original triangle.

The number of a given dimensional element in the tetrahedron is now the sum of two numbers: first the number of that element found in the original triangle, plus the number of new elements, *each of which is built upon elements of one fewer dimension from the original triangle*. Thus, in the tetrahedron, the number of cells (polyhedral elements) is 0 + 1 = **1**; the number of faces is 1 + 3 = **4**; the number of edges is 3 + 3 = **6**; the number of new vertices is 3 + 1 = **4**. This process of summing the number of elements of a given dimension to those of one fewer dimension to arrive at the number of the former found in the next higher simplex is equivalent to the process of summing two adjacent numbers in a row of Pascal's triangle to yield the number below. Thus, the meaning of the final number (1) in a row of Pascal's triangle becomes understood as representing the new vertex that is to be added to the simplex represented by that row to yield the next higher simplex represented by the next row. This new vertex is joined to every element in the original simplex to yield a new element of one higher dimension in the new simplex, and this is the origin of the pattern found to be identical to that seen in Pascal's triangle. The "extra" 1 in a row can be thought of as the -1 simplex, the unique center of the simplex, which ever gives rise to a new vertex and a new dimension, yielding a new simplex with a new center.

Number of elements of hypercubes

A similar pattern is observed relating to squares, as opposed to triangles. To find the pattern, one must construct an analog to Pascal's triangle, whose entries are the coefficients of $(x+2)^{\text{Row Number}}$, instead of $(x+1)^{\text{Row Number}}$. There are a couple ways to do this. The simpler is to begin with Row 0 = 1 and Row 1 = 1, 2. Proceed to construct the analog triangles according to the following rule:

$$\binom{n}{k} = 2 \times \binom{n-1}{k-1} + \binom{n-1}{k}.$$

That is, choose a pair of numbers according to the rules of Pascal's triangle, but double the one on the left before adding. This results in:

$$\begin{array}{c}
1 \\
1 \quad 2 \\
1 \quad 4 \quad 4 \\
1 \quad 6 \quad 12 \quad 8 \\
1 \quad 8 \quad 24 \quad 32 \quad 16 \\
1 \quad 10 \quad 40 \quad 80 \quad 80 \quad 32 \\
1 \quad 12 \quad 60 \quad 160 \quad 240 \quad 192 \quad 64 \\
1 \quad 14 \quad 84 \quad 280 \quad 560 \quad 672 \quad 448 \quad 128
\end{array}$$

The other way of manufacturing this triangle is to start with Pascal's triangle and multiply each entry by 2^k, where k is the position in the row of the given number. For example, the 2nd value in row 4 of Pascal's triangle is 6 (the slope of 1s corresponds to the zeroth entry in each row). To get the value that resides in the corresponding position in the analog triangle, multiply 6 by $2^{\text{Position Number}} = 6 \times 2^2 = 6 \times 4 = 24$. Now that the analog triangle has been constructed, the number of elements of any dimension that compose an arbitrarily dimensioned cube (called a hypercube) can be read from the table in a way analogous to Pascal's triangle. For example, the number of 2-dimensional elements in a 2-dimensional cube (a square) is one, the number of 1-dimensional elements (sides, or lines) is 4, and the number of 0-dimensional elements (points, or vertices) is 4. This matches the 2nd row of the table (1, 4, 4). A cube has 1 cube, 6 faces, 12 edges, and 8 vertices, which corresponds to the next line of the analog triangle (1, 6, 12, 8). This pattern continues indefinitely.

To understand why this pattern exists, first recognize that the construction of an n-cube from an $(n-1)$-cube is done by simply duplicating the original figure and displacing it some distance (for a regular n-cube, the edge length) orthogonal to the space of the original figure, then connecting each vertex of the new figure to its corresponding vertex of the original. This initial duplication process is the reason why, to enumerate the dimensional elements of an n-cube, one must double the first of a pair of numbers in a row of this analog of Pascal's triangle before summing to yield the number below. The initial doubling thus yields the number of "original" elements to be found in the next higher n-cube and, as before, new elements are built upon those of one fewer dimension (edges upon vertices, faces upon edges, etc.). Again, the last number of a row represents the number of new vertices to be added to generate the next higher n-cube.

In this triangle, the sum of the elements of row m is equal to 3^m. Again, to use the elements of row 4 as an example: $1 + 8 + 24 + 32 + 16 = 81$, which is equal to $3^4 = 81$.

Counting vertices in a cube by distance

Each row of Pascal's triangle gives the number of vertices at each distance from a fixed vertex in an n-dimensional cube. For example, in three dimensions, the third row (1 3 3 1) corresponds to the usual three-dimensional cube: fixing a vertex V, there is one vertex at distance 0 from V (that is, V itself), three vertices at distance 1, three vertices at distance $\sqrt{2}$ and one vertex at distance $\sqrt{3}$ (the vertex opposite V). The second row corresponds to a square, while larger-numbered rows correspond to hypercubes in each dimension.

Fourier transform of sin(x)$^{n+1}$/x

As stated previously, the coefficients of $(x + 1)^n$ are the nth row of the triangle. Now the coefficients of $(x - 1)^n$ are the same, except that the sign alternates from +1 to −1 and back again. After suitable normalization, the same pattern of numbers occurs in the Fourier transform of $\sin(x)^{n+1}/x$. More precisely: if n is even, take the real part of the transform, and if n is odd, take the imaginary part. Then the result is a step function, whose values (suitably normalized) are given by the nth row of the triangle with alternating signs.[21] For example, the values of the step function that results from:

$$\mathfrak{Re}\left(\text{Fourier}\left[\frac{\sin(x)^5}{x}\right]\right)$$

compose the 4th row of the triangle, with alternating signs. This is a generalization of the following basic result (often used in electrical engineering):

$$\mathfrak{Re}\left(\text{Fourier}\left[\frac{\sin(x)^1}{x}\right]\right)$$

is the boxcar function.[22] The corresponding row of the triangle is row 0, which consists of just the number 1.

If n is congruent to 2 or to 3 mod 4, then the signs start with −1. In fact, the sequence of the (normalized) first terms corresponds to the powers of i, which cycle around the intersection of the axes with the unit circle in the complex plane:

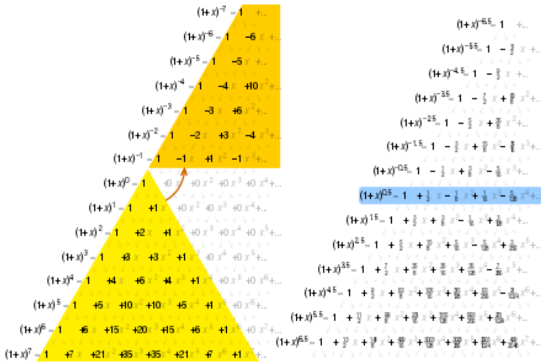
$$+i, -1, -i, +1, +i, \ldots$$

Extensions

Pascal's triangle can be extended to negative row numbers.

First write the triangle in the following form:

n \ m	0	1	2	3	4	5	...
0	1	0	0	0	0	0	...
1	1	1	0	0	0	0	...
2	1	2	1	0	0	0	...
3	1	3	3	1	0	0	...
4	1	4	6	4	1	0	...


Binomial coefficients $C(n, k)$ extended for negative and fractional n, illustrated with a simple binomial. It can be observed that Pascal's triangle is rotated and alternate terms are negated. The case $n = -1$ gives Grandi's series.

Next, extend the column of 1s upwards:

n \ m	0	1	2	3	4	5	...
−4	1						...
−3	1						...
−2	1						...
−1	1						...
0	1	0	0	0	0	0	...
1	1	1	0	0	0	0	...
2	1	2	1	0	0	0	...
3	1	3	3	1	0	0	...
4	1	4	6	4	1	0	...

Now the rule:

$$\binom{n}{m} = \binom{n-1}{m-1} + \binom{n-1}{m}$$

can be rearranged to:

$$\binom{n-1}{m} = \binom{n}{m} - \binom{n-1}{m-1}$$

which allows calculation of the other entries for negative rows:

n \ m	0	1	2	3	4	5	...
−4	1	−4	10	−20	35	−56	...
−3	1	−3	6	−10	15	−21	...
−2	1	−2	3	−4	5	−6	...
−1	1	−1	1	−1	1	−1	...
0	1	0	0	0	0	0	...
1	1	1	0	0	0	0	...
2	1	2	1	0	0	0	...
3	1	3	3	1	0	0	...
4	1	4	6	4	1	0	...

This extension preserves the property that the values in the mth column viewed as a function of n are fit by an order m polynomial, namely

$$\binom{n}{m} = \frac{1}{m!}\prod_{k=0}^{m-1}(n-k) = \frac{1}{m!}\prod_{k=1}^{m}(n-k+1).$$

This extension also preserves the property that the values in the nth row correspond to the coefficients of $(1+x)^n$:

$$(1+x)^n = \sum_{k=0}^{\infty}\binom{n}{k}x^k \quad |x|<1$$

For example:

$$(1+x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + \cdots \quad |x|<1$$

When viewed as a series, the rows of negative n diverge. However, they are still Abel summable, which summation gives the standard values of 2^n. (In fact, the n = -1 row results in Grandi's series which "sums" to 1/2, and the n = -2 row results in another well-known series which has an Abel sum of 1/4.)

Another option for extending Pascal's triangle to negative rows comes from extending the *other* line of 1s:

n \ *m*	−4	−3	−2	−1	0	1	2	3	4	5	...
−4	1	0	0	0	0	0	0	0	0	0	...
−3		1	0	0	0	0	0	0	0	0	...
−2			1	0	0	0	0	0	0	0	...
−1				1	0	0	0	0	0	0	...
0	0	0	0	0	1	0	0	0	0	0	...
1	0	0	0	0	1	1	0	0	0	0	...
2	0	0	0	0	1	2	1	0	0	0	...
3	0	0	0	0	1	3	3	1	0	0	...
4	0	0	0	0	1	4	6	4	1	0	...

Applying the same rule as before leads to

n \ *m*	−4	−3	−2	−1	0	1	2	3	4	5	...
−4	1	0	0	0	0	0	0	0	0	0	...
−3	−3	1	0	0	0	0	0	0	0	0	...
−2	3	−2	1	0	0	0	0	0	0	0	...
−1	−1	1	−1	1	0	0	0	0	0	0	..
0	0	0	0	0	1	0	0	0	0	0	...
1	0	0	0	0	1	1	0	0	0	0	...
2	0	0	0	0	1	2	1	0	0	0	...
3	0	0	0	0	1	3	3	1	0	0	...
4	0	0	0	0	1	4	6	4	1	0	...

This extension also has the properties that just as

$$\exp\begin{pmatrix} . & . & . & . & . \\ 1 & . & . & . & . \\ . & 2 & . & . & . \\ . & . & 3 & . & . \\ . & . & . & 4 & . \end{pmatrix} = \begin{pmatrix} 1 & . & . & . & . \\ 1 & 1 & . & . & . \\ 1 & 2 & 1 & . & . \\ 1 & 3 & 3 & 1 & . \\ 1 & 4 & 6 & 4 & 1 \end{pmatrix},$$

we have

$$\exp\begin{pmatrix} . & . & . & . & . & . & . & . & . & . \\ -4 & . & . & . & . & . & . & . & . & . \\ . & -3 & . & . & . & . & . & . & . & . \\ . & . & -2 & . & . & . & . & . & . & . \\ . & . & . & -1 & . & . & . & . & . & . \\ . & . & . & . & 0 & . & . & . & . & . \\ . & . & . & . & . & 1 & . & . & . & . \\ . & . & . & . & . & . & 2 & . & . & . \\ . & . & . & . & . & . & . & 3 & . & . \\ . & . & . & . & . & . & . & . & 4 & . \end{pmatrix} = \begin{pmatrix} 1 & . & . & . & . & . & . & . & . & . \\ -4 & 1 & . & . & . & . & . & . & . & . \\ 6 & -3 & 1 & . & . & . & . & . & . & . \\ -4 & 3 & -2 & 1 & . & . & . & . & . & . \\ 1 & -1 & 1 & -1 & 1 & . & . & . & . & . \\ . & . & . & . & . & 1 & . & . & . & . \\ . & . & . & . & . & 1 & 1 & . & . & . \\ . & . & . & . & . & 1 & 2 & 1 & . & . \\ . & . & . & . & . & 1 & 3 & 3 & 1 & . \\ . & . & . & . & . & 1 & 4 & 6 & 4 & 1 \end{pmatrix}$$

Also, just as summing along the lower-left to upper-right diagonals of the Pascal matrix yields the Fibonacci numbers, this second type of extension still sums to the Fibonacci numbers for negative index.

Either of these extensions can be reached if we define

$$\binom{n}{k} = \frac{n!}{(n-k)!k!} \equiv \frac{\Gamma(n+1)}{\Gamma(n-k+1)\Gamma(k+1)}$$

and take certain limits of the gamma function, $\Gamma(z)$.

See also

- Bean machine, Francis Galton's "quincunx"
- Bell triangle
- Bernoulli's triangle
- Binomial expansion
- Euler triangle
- Floyd's triangle
- Gaussian binomial coefficient
- Hockey-stick identity
- Leibniz harmonic triangle
- Multiplicities of entries in Pascal's triangle (Singmaster's conjecture)
- Pascal matrix
- Pascal's pyramid
- Pascal's simplex
- Proton NMR, one application of Pascal's triangle
- (2,1)-Pascal triangle
- Star of David theorem
- Trinomial expansion
- Trinomial triangle

References

1. Maurice Winternitz, *History of Indian Literature*, Vol. III
2. Coolidge, J. L. (1949), "The story of the binomial theorem", *The American Mathematical Monthly*, **56** (3): 147–157, doi:10.2307/2305028 (https://doi.org/10.2307%2F2305028), JSTOR 2305028 (https://www.jstor.org/stable/2305028), MR 0028222 (https://www.ams.org/mathscinet-getitem?mr=0028222).
3. Peter Fox (1998). *Cambridge University Library: the great collections* (https://books.google.com/books?id=xxlgKP5thL8C&pg=PA13). Cambridge University Press. p. 13. ISBN 978-0-521-62647-7.
4. The binomial coefficient $\binom{n}{k}$ is conventionally set to zero if k is either less than zero or greater than n.
5. A. W. F. Edwards. *Pascal's arithmetical triangle: the story of a mathematical idea.* JHU Press, 2002. Pages 30–31.
6. Edwards, A. W. F. (2013), "The arithmetical triangle", in Wilson, Robin; Watkins, John J. (eds.), *Combinatorics: Ancient and Modern*, Oxford University Press, pp. 166–180.
7. Alexander Zawaira; Gavin Hitchcock (2008). *A Primer for Mathematics Competitions* (https://books.google.com/books?id=A21T73sqZ3AC&pg=PA237). Oxford University Press. p. 237. ISBN 978-0-19-156170-2.
8. Selin, Helaine (2008-03-12). *Encyclopaedia of the History of Science, Technology, and Medicine in Non-Western Cultures* (https://books.google.com/books?id=kt9DIY1g9HYC&q=al+karaji+pascal%27s+triangle&pg=PA132). Springer Science & Business Media. p. 132. Bibcode:2008ehst.book.....S (https://ui.adsabs.harvard.edu/abs/2008ehst.book.....S). ISBN 9781402045592.
9. The Development of Arabic Mathematics Between Arithmetic and Algebra - R. Rashed (https://books.google.fr/books?id=vSkClSvU_9AC&pg=PA62&redir_esc=y#v=onepage&q&f=false/) "Page 63"
10. Sidoli, Nathan; Brummelen, Glen Van (2013-10-30). *From Alexandria, Through Baghdad: Surveys and Studies in the Ancient Greek and Medieval Islamic Mathematical Sciences in Honor of J.L. Berggren* (https://books.google.com/books?id=kAjABAAAQBAJ&q=al+karaji+binomial+theorem&pg=PA54). Springer Science & Business Media. p. 54. ISBN 9783642367366.
11. Kennedy, E. (1966). *Omar Khayyam. The Mathematics Teacher 1958*. National Council of Teachers of Mathematics. pp. 140–142. JSTOR i27957284 (https://www.jstor.org/stable/i27957284).
12. Weisstein, Eric W. (2003). *CRC concise encyclopedia of mathematics*, p. 2169. ISBN 978-1-58488-347-0.
13. Hughes, Barnabas (1 August 1989). "The arithmetical triangle of Jordanus de Nemore". *Historia Mathematica*. **16** (3): 213–223. doi:10.1016/0315-0860(89)90018-9 (https://doi.org/10.1016%2F0315-0860%2889%2990018-9).
14. Smith, Karl J. (2010), *Nature of Mathematics* (https://books.google.com/books?id=Di0HyCgDYq8C&pg=PA10), Cengage Learning, p. 10, ISBN 9780538737586.
15. Fowler, David (January 1996). "The Binomial Coefficient Function". *The American Mathematical Monthly*. **103** (1): 1–17. doi:10.2307/2975209 (https://doi.org/10.2307%2F2975209). JSTOR 2975209 (https://www.jstor.org/stable/2975209). See in particular p. 11.
16. Brothers, H. J. (2012), "Finding e in Pascal's triangle", *Mathematics Magazine*, **85**: 51, doi:10.4169/math.mag.85.1.51 (https://doi.org/10.4169%2Fmath.mag.85.1.51), S2CID 218541210 (https://api.semanticscholar.org/CorpusID:218541210).
17. Brothers, H. J. (2012), "Pascal's triangle: The hidden stor-*e*", *The Mathematical Gazette*, **96**: 145–148, doi:10.1017/S0025557200004204 (https://doi.org/10.1017%2FS0025557200004204).
18. Foster, T. (2014), "Nilakantha's Footprints in Pascal's Triangle", *Mathematics Teacher*, **108**: 247, doi:10.5951/mathteacher.108.4.0246 (https://doi.org/10.5951%2Fmathteacher.108.4.0246)
19. Fine, N. J. (1947), "Binomial coefficients modulo a prime", *American Mathematical Monthly*, **54** (10): 589–592, doi:10.2307/2304500 (https://doi.org/10.2307%2F2304500), JSTOR 2304500 (https://www.jstor.org/stable/2304500), MR 0023257 (https://www.ams.org/mathscinet-getitem?mr=0023257). See in particular Theorem 2, which gives a generalization of this fact for all prime moduli.
20. Hinz, Andreas M. (1992), "Pascal's triangle and the Tower of Hanoi", *The American Mathematical Monthly*, **99** (6): 538–544, doi:10.2307/2324061 (https://doi.org/10.2307%2F2324061), JSTOR 2324061 (https://www.jstor.org/stable/2324061), MR 1166003 (https://www.ams.org/mathscinet-getitem?mr=1166003). Hinz attributes this observation to an 1891 book by Édouard Lucas, *Théorie des nombres* (p. 420).
21. For a similar example, see e.g. Hore, P. J. (1983), "Solvent suppression in Fourier transform nuclear magnetic resonance", *Journal of Magnetic Resonance*, **55** (2): 283–300, Bibcode:1983JMagR..55..283H (https://ui.adsabs.harvard.edu/abs/1983JMagR..55..283H), doi:10.1016/0022-2364(83)90240-8 (https://doi.org/10.1016%2F0022-2364%2883%2990240-8).
22. Karl, John H. (2012), *An Introduction to Digital Signal Processing* (https://books.google.com/books?id=9Dv1PClLZWIC&pg=PA110), Elsevier, p. 110, ISBN 9780323139595.

External links

- "Pascal triangle" (https://www.encyclopediaofmath.org/index.php?title=Pascal_triangle), *Encyclopedia of Mathematics*, EMS Press, 2001 [1994]
- Weisstein, Eric W. "Pascal's triangle" (https://mathworld.wolfram.com/PascalsTriangle.html). *MathWorld*.
- The Old Method Chart of the Seven Multiplying Squares (http://www.york.ac.uk/depts/maths/histstat/images/triangle.gif) *(from the Ssu Yuan Yü Chien of Chu Shi-Chieh, 1303, depicting the first nine rows of Pascal's triangle)*
- Pascal's Treatise on the Arithmetic Triangle (https://web.archive.org/web/20040803130916/http://www.lib.cam.ac.uk/RareBooks/PascalTraite/) *(page images of Pascal's treatise, 1654; summary (https://web.archive.org/web/20040803233048/http://www.lib.cam.ac.uk/RareBooks/PascalTraite/pascalintro.pdf))*