Low Risk Sharing with Many Assets[⋆]

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November 2023

Abstract

Classical contributions in international macroeconomics rely on goods-market mechanisms to reconcile the cyclicality of real exchange rates when financial markets are incomplete. However, cross-border trade in one domestic and one foreign-currency-denominated risk-free asset prohibits these mechanisms from breaking the pattern consistent with complete markets. In this paper, we characterize how goods markets drive exchange rate cyclicality, taking into account trade in risk-free and/or risky assets. We show that goods-market mechanisms come back into play, even when there is cross-border trade in two risk-free assets, as long as we allow for empirically plausible heterogeneity in the stochastic discount factors of domestic marginal investors.

[⋆]We thank Giancarlo Corsetti, Valentin Haddad, Sylvain Leduc, Simon Lloyd, and Alan Taylor for helpful comments and suggestions. The views expressed in this article are solely the responsibility of the authors and should not be interpreted as reflecting the views of the Federal Reserve Bank of San Francisco or of anyone else associated with the Federal Reserve System. Remaining errors are our own.

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1. Introduction

From a macroeconomic perspective, exchange rates facilitate the sharing of consumption risk across countries by transforming units of account. From a financial perspective, exchange rates appear in all unhedged cross-border portfolio positions and so are disciplined by a number of arbitrage conditions. These two perspectives can offer contrasting implications for various puzzles in international macroeconomics.

In this paper, we revisit the Backus-Smith condition (Kollmann, 1991; Backus and Smith, 1993), which describes the transmission and sharing of risk across countries in terms of consumption and relative price movements (Obstfeld and Rogoff, 2000). Specifically, when international financial markets are complete, a large class of models admits the following relationship:

$$\left(\frac{C_{t+1}}{C^*_{t+1}} \Big/ \frac{C_t}{C^*_t}\right)^{\sigma} = \frac{\mathcal{E}_{t+1}}{\mathcal{E}_t} \tag{1}$$

where C_t is Home aggregate consumption, C^*_t is Foreign aggregate consumption and $\mathcal{E}_t$ is the real exchange rate where an increase signifies a depreciation of Home currency. The condition above describes perfect risk sharing and determines the cyclicality of real exchange rates. Ex-ante risk-sharing implies ex-post redistribution. With trade in risk-free assets, this redistribution occurs entirely through exchange rate movements. For example, following a positive productivity shock in the Home country, the Home currency depreciates, leading to higher consumption abroad when the Backus-Smith condition is satisfied. Ex-post exchange rates, thus, move to reallocate wealth from Home to Foreign – and so are risky from the perspective of a Home investor. However, in the data, relative consumption and relative prices often move in the same direction, constituting the Backus-Smith *puzzle*.

When markets are incomplete, the condition above needs to hold only in expectation and so may fail ex-post. Classical contributions in international macroeconomics such as Corsetti, Dedola and Leduc (2008) and Benigno and Thoenissen (2008) show that the presence of non-traded risk, which can arise from consumption complementarity across Home and Foreign good varieties, or production of non-traded goods, can help incomplete markets open economy models generate plausible patterns of risk sharing. We refer to these economic mechanisms as goods market mechanisms (as opposed to financial).[1] These models allow cross-border trade in a single risk-free asset (denominated in either currency). In an important contribution, Lustig and Verdelhan (2019) show that in a representative agent model with incomplete markets, prices should always comove negatively with consumption when there is cross-border trade in at least one Home and one Foreign risk-free asset, *regardless* of goods market frictions and other economy specifics — so incomplete market models cannot resolve the puzzle of excessive risk-sharing.[2]

Our first contribution is to highlight the mechanism through which goods markets can reconcile the Backus-Smith puzzle. We show that any model that achieves this resolution must rely on a non-traded component of relative prices which is "safe" from a domestic investor perspective.[3] Having established this mechanism, we show *why* moving from cross-border trade in a single risk-free Home asset to cross-border trade in one Home and additionally just one

[1]Under the *asset market view of the exchange rates*, the Backus-Smith condition offers a characterization of the financial markets. The seminal contributions of Cole and Obstfeld (1991), Tesar (1993), Stockman and Tesar (1995), Lewis (1996), Fitzgerald (2012) among others, show that goods markets can be just as important in determining the comovement of international prices and consumption.

[2]Benigno and Küçük (2012) made a similar observation by introducing international trade in two nominal bonds.

[3]The conditions for goods market mechanism that we derive apply not only to models with consumption or production complementarities described above, but also to models with costly consumer search (Bai and Ríos-Rull, 2015), or global value chain integration (Corsetti, D'Aguanno, Dogan, Lloyd and Sajedi, 2023*b*).

Foreign risk-free asset inhibits *any* goods-market mechanisms from reconciling the patterns of risk-sharing observed in the data.

Our second contribution is to propose a generalization of the representative investor, two-country model to allow heterogeneous marginal investors in the Home country. Both investors price the domestic currency-denominated risk-free asset while only one prices the Foreign currency risk-free asset. We show that within-country heterogeneity in stochastic discount factors (SDFs) can relax the strong implications of representative agent no-arbitrage models even when there is trade in multiple risk-free assets. We find that a necessary condition for within-country heterogeneity to generate empirical risk-sharing is that the domestic incompleteness be large relative to the volatility of exchange rates. Throughout the paper, we assume that the within-country heterogeneity across investors arises only because one investor participates in the Foreign risk-free asset markets in addition to the domestic risk-free asset markets. If the marginal investor who participates in Foreign assets is sufficiently exposed to exchange rate movements but does not insure the other through domestic asset markets, the model can resolve the Backus-Smith puzzle. While we do not take a stance on the model by which the international portfolio is formed, we find that the covariance of SDF differentials within the Home country (non-traded risk) and exchange rates emerges as a sufficient statistic for the comovement of exchange rates with consumption growth.

Finally, we measure the degree of market incompleteness from the data and evaluate the plausibility that the above condition is satisfied. We show that domestic market incompleteness (as measured by the volatility of the difference in investor SDFs within the Home country) will be high if participating in Foreign markets earns excess Sharpe ratios or if the covariance of SDFs within the country is low. Using portfolio return data, we discipline the former by ruling out "good deals" (Cochrane and Saa-Requejo, 2000). We calibrate the covariance using micro data from the literature. For plausible covariances of SDFs, even if we rule out good deals entirely (i.e., domestic equity is the highest Sharpe ratio return), our model can match the facts on international risk sharing with a correlation of non-traded risk and exchange rates of about one-half. Allowing for realistic excess returns from trading in Foreign markets, a correlation of one-third between non-traded risk and exchange rates is sufficient to resolve the Backus-Smith puzzle.

We use a common framework to investigate four cases under (i) financial autarky, (ii) financial trade in a single asset, (iii) trade in Home and Foreign currency-denominated risk-free assets, and (iv) trade in risky assets. Under (i), we show that the positive comovement between relative consumption and relative prices can arise from *uninsured* and *safe* wealth effects, constituting non-traded risk. Under (ii) where there is cross-border trade in a single (Foreign) risk-free asset, while the Foreign household is insuring against these wealth effects, the Home household may not be sufficiently insured depending on the parameters of the specific macro environment. Our main focus is on the case (iii), where there is trade in two *risk-free* assets. Then, households in both countries ex-ante insure these wealth effects – leading to ex-post redistribution and a negative comovement between relative consumption and prices. In this case, exchange rates themselves are effectively traded – see also Chernov, Haddad and Itskhoki (2023). In case (iv), we show that adding trade in risky assets does not necessarily determine the cyclicality of exchange rates, and goods market mechanisms remain powerful in resolving the Backus-Smith puzzle.

Throughout the paper, we maintain a simple structure of incompleteness, illustrated in Figure 1. We assume the Foreign economy has a representative household. The Home market can have an arbitrary amount of heterogeneity. However, we will focus on two limiting SDFs: the marginal Home investor for domestic assets and the marginal Home investor for Foreign assets. Moreover, we focus on the limit where the measure of the marginal Home investor for Foreign assets approaches 0. All these assumptions are made for clarity and simplicity. While we generally maintain a preference-free approach, taking SDFs as given,[4] we provide a grounding economic model evaluated at the limit of financial autarky (as in Corsetti et al., 2008) and full home-bias (as in Itskhoki and Mukhin, 2021) to emphasize the importance of goods-market mechanisms in the economy. The financial autarky limit is particularly instructive because allocations do not depend on the number of assets traded, and we prove the joint log normality of SDFs and exchange rates at the full home-bias limit. Moreover the marginal investors for domestic and Foreign assets in the Home country are identical at the financial autarky limit, so the model coincides with the representative agent economies referenced above.

The contribution of our paper can also be understood in terms of relaxing the assumption of *representative SDF* no-arbitrage models in international macro. In general, assets do not have a unique price under incomplete domestic markets. A very general approach in finance constructs prices using super replication bounds (see Ritchken 1985 and Černý and Hodges 2002), but these are often wide and uninformative. In macro-finance, the literature usually employs a representative agent equilibrium, where the pricing functional is obtained from the marginal utility of the optimized representative agent's consumption (Rubinstein, 1976). We interpret the impossibility result in Lustig and Verdelhan (2019) as a failure of this specific SDF. Instead, we allow for two (or more) SDFs to co-exist within the country, consistent with no arbitrage internationally and reasonable market incompleteness domestically. We discipline

[4]We restrict attention to the time-separable utility functions predominantly used in macro.

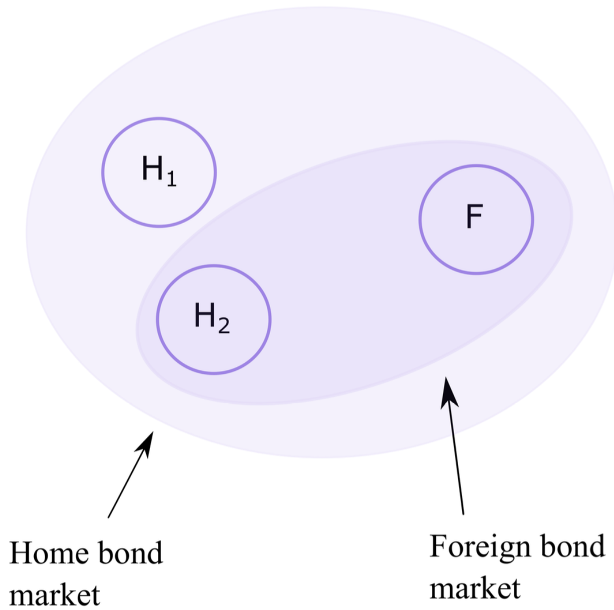

Figure 1: *H_1 and H_2 are "Home" country households, distinguished from F, the Foreign country household, by their consumption basket. The lighter shade circle is the Home bond market, wherein all agents trade freely. The darker shade is the Foreign bond market, wherein only H_2 and F trade.*

this domestic incompleteness by ruling out high Sharpe ratios to be had by participating in Foreign markets – a less strict notion than representative agent no-arbitrage but tighter than incomplete markets no arbitrage.

Related Literature There is a long-standing literature in incomplete market models in international economies, divided across macroeconomics and finance. Gabaix and Maggiori (2015) and Itskhoki and Mukhin (2021) adopted the notion of market segmentation via an international intermediary to reconcile several puzzles related to exchange rates, including the Backus-Smith puzzle. While heterogeneous marginal investors may reflect a model of intermediation, we specifically investigate the relative dynamics of these SDFs required to reconcile patterns of risk-sharing without relying on model-specific assumptions for preferences, profit sharing or financial frictions.[5] We build on their insights to shed light on the interaction of such market segmentation with many assets traded and real frictions.

Most closely related papers to ours are Benigno and Küçük (2012), Lustig and Verdelhan (2019), Chernov, Haddad and Itskhoki (2023), and Jiang, Krishnamurthy and Lustig (2023*a*). Lustig and Verdelhan (2019) consider multiplicative incomplete market wedges as in Backus, Foresi and Telmer (2001). Benigno and Küçük (2012) and Lustig and Verdelhan (2019) show that introducing a second internationally traded bond can break the ability of international macro models to reconcile the Backus-Smith puzzle. We extend their frameworks beyond the representative agent assumption, generalizing some of their results, and show how domestic market segmentation can retain the role of goods market frictions in explaining the Backus-Smith puzzle as we increase the number of globally traded assets.

Chernov et al. (2023) investigate how different financial market structures and the mix of locally, globally traded, and unspanned risks contribute to different exchange rate puzzles. Models with heterogeneous marginal investors naturally relate to models of intermediation, but our framework can specifically be viewed as a way of breaking global risks into local risks. Relative to their paper, we impose structure on local risks by looking at the relative SDFs of the two Home country investors and investigating their economic dynamics.

Jiang, Krishnamurthy, Lustig and Sun (2023*b*) consider the potential for convenience yields to explain exchange rate puzzles but find these cannot reconcile the Backus-Smith. We show that our heterogeneous investor framework is distinct from a model with convenience yields,

[5] Colacito and Croce (2013) and Farhi and Gabaix (2016) investigate the role of long-run risk and rare disasters respectively in generating observed correlations between cross country asset returns and exchange rate returns. Corsetti et al. (2023*b*) study the role of global value chains for international risk-sharing. Bai and Zhang (2012), Hassan (2013) and Maggiori (2017) study international risk sharing in models with financial frictions, country risk premia, or superior risk-bearing capacity of US intermediaries. Obstfeld (2007), Heathcote and Perri (2014), Engel (2014), Itskhoki and Mukhin (2021) and Maggiori (2022) provide an elaborate discussion of recent international macro literature on exchange rate puzzles.

so our work is complementary. Using a wedge accounting framework, Itskhoki and Mukhin (2023) find that financial shocks can reconcile exchange rate puzzles. Jiang, Krishnamurthy and Lustig (2023*a*) show that Euler equation wedges are necessary to resolve Backus-Smith puzzle. We complement their analysis by structuring these wedges and allowing for multiple within-country SDFs connected by risk-sharing.

From a finance perspective, Bakshi, Cerrato and Crosby (2018) also allow for multiple SDFs considering additive wedges, but their focus is on isolating the spanned and unspanned components and generalizing the results in Brandt, Cochrane and Santa-Clara (2006). Sandulescu, Trojani and Vedolin (2021) extract minimum variance and minimum entropy SDFs and show that the Backus-Smith condition holds with their model-free SDFs. Orłowski, Tahbaz-Salehi, Trojani and Vedolin (2023) extend the result of Lustig and Verdelhan (2019) to allow for varying degrees of financial integration and different market structures with no-arbitrage pricing.

Like us, several papers also try to discipline SDF variation using bounds on Sharpe Ratios, a.k.a good-deal bounds building on the seminal contribution of Cochrane and Saa-Requejo (2000).[6] Unlike these papers, we use these good-deal bounds to check if the within-country market segmentation necessary for resolving the Backus-Smith puzzle is reasonable.

Our work is also related to the broader literature on market segmentation in international macro. Maggiori, Neiman and Schreger (2020) provide evidence of segmentation in international fixed-income markets. Christelis, Georgarakos and Haliassos (2013) explore the determinants of portfolio differences across countries. Cociuba and Ramanarayanan (2019) build a model of endogenously incomplete domestic markets using the framework of Alvarez, Atkeson and Kehoe (2002) and show that the Backus-Smith condition need only hold for households active

[6]See for example, Černỳ (2003), and Boyarchenko, Cerrato, Crosby and Hodges (2014).

in international financial markets. Kollmann (2012) and Chien, Lustig and Naknoi (2020) also model heterogeneous participation and the latter focus on equity market participation. A key difference is that these models assume complete markets internationally. We build on their contributions and study the general case of incomplete international markets.

The rest of the paper is organised as follows. Section 2 provides a characterization of how goods markets drive exchange rate cyclicality and how this mechanism works with trade in risk-free and/or risky assets. Section 3 proposes our generalization of incomplete markets with SDF heterogeneity and derives the minimum bounds necessary for generating empirical risk-sharing patterns with different financial structures. Section 4 imposes discipline on the plausible heterogeneity. Section 5 concludes.

2. Two-country, representative agent, incomplete markets

Consider a two-country model where M_{t+1} denotes the Home representative household's SDF and M^*_{t+1} denotes the Foreign representative household's SDF. Home and Foreign households each trade their respective domestic risk-free real bonds with returns R_t and R^*_t respectively. No-arbitrage pricing implies:[7]

$$\mathbb{E}_t[M_{t+1}] = 1/R_{t+1}, \tag{2}$$

$$\mathbb{E}_t[M^*_{t+1}] = 1/R^*_{t+1} \tag{3}$$

If the Home (Foreign) households also trade the Foreign (home) bond, and the real exchange

[7] E.g. in the case of time-separable CRRA utility $M_{t+1} = \beta \left(\frac{C_{t+1}}{C_t} \right)^{-s}$. Alternatively, in the tradition of Hansen and Jagannathan (1991), the SDFs are simply a risk-return operator implied by the traded assets. See Section 4.

rate at time t is denoted with $\mathcal{E}_t$, then we obtain the following two Euler conditions:

$$\mathbb{E}_t\left[M^*_{t+1}\frac{\mathcal{E}_t}{\mathcal{E}_{t+1}}\right] = 1/R_{t+1}, \tag{4}$$

$$\mathbb{E}_t\left[M_{t+1}\frac{\mathcal{E}_{t+1}}{\mathcal{E}_t}\right] = 1/R^*_{t+1}. \tag{5}$$

We assume SDFs, allocations and prices are jointly log-normal.[8] To close the model without explicitly specifying goods markets, an exchange rate process is needed, which is consistent with equations (2)–(5) above. This problem reduces to finding an exchange rate process that satisfies:[9]

$$cov_t(m^*_{t+1} - m_{t+1}, \Delta e_{t+1}) = var_t(\Delta e_{t+1}) \tag{6}$$

where $x = log(X)$. Naturally, the process corresponding to complete markets ($\Delta e_{t+1} = m^*_{t+1} - m_{t+1}$) is one candidate. More generally, as shown in Backus, Foresi and Telmer (2001), the following process also satisfies equation (6):

$$\Delta e_{t+1} = m^*_{t+1} - m_{t+1} + \eta_{t+1} \tag{7}$$

where η_{t+1} is an incomplete markets wedge which must satisfy certain conditions imposed by asset trade.[10] The Backus-Smith condition (1) restricts the covariance between relative SDFs and exchange rate growth, $cov_t(m^*_{t+1} - m_{t+1}, \Delta e_{t+1})$, to be positive. We refer to this covariance term as the cyclicality of exchange rates.

Combining (2) and (4), with (7) – which implies that the Home bond is internationally

[8]Our results generalize to non log-normal settings using entropy expansions (Lustig and Verdelhan, 2019).
[9]See Appendix A.1 for full derivation
[10]For a closed form solution of the incomplete markets wedge in a two-country open economy model, see Pavlova and Rigobon (2007).

traded – yields:

$$cov_t(m_{t+1}, \eta_{t+1}) = -\mathbb{E}_t[\eta_{t+1}] + \frac{1}{2}var_t(\eta_{t+1}) \tag{8}$$

Combining (3) and (5), with (7) – which implies that the Foreign bond is internationally traded – yields:

$$cov_t(m^*_{t+1}, \eta_{t+1}) = -\mathbb{E}_t[\eta_{t+1}] - \frac{1}{2}var_t(\eta_{t+1}) \tag{9}$$

These two conditions mirror those studied by Lustig and Verdelhan (2019) and bound the joint dynamics of the incomplete market wedge and the SDFs, carrying strong implications for the macro side of the model.

To relate back to international macro models, assuming time-separable, and CRRA preferences, the incomplete markets wedge is related to:

$$\eta_{t+1} = log\underbrace{\left(\frac{P_{t+1}}{P_t}\frac{P^*_t}{P^*_{t+1}}\right)}_{\frac{\mathcal{E}_t}{\mathcal{E}_{t+1}}} - log\underbrace{\left(\frac{C_t}{C_{t+1}}\frac{C^*_{t+1}}{C^*_t}\right)^s}_{\frac{M_{t+1}}{M^*_{t+1}}}$$

where P_t is the Home price level, C_t is aggregate consumption, s is the CRRA coefficient, and terms with asterisks denote the corresponding Foreign objects. The wedge, η, is often interpreted as non-traded risk or the wealth gap, see e.g. Corsetti, Dedola and Leduc (2023*a*).

2.1. International Risk-Sharing with Trade in Risk-free Assets

Having now specified our framework, we illustrate the mechanism through which goods market frictions in incomplete market models can help reconcile the pattern of international risk-sharing.

Proposition 1 (One Int'l Traded Asset, Representative Agent No-Arbitrage).

When only Foreign bonds are internationally traded such that equations (2), (3) and (5) hold, *then* $cov_t(m^*_{t+1} - m_{t+1}, \Delta e_{t+1}) < 0$ *if and only if*

$$cov_t(m_{t+1}, \eta_{t+1}) + \frac{1}{2} var_t(\eta_{t+1}) > var_t(m^*_{t+1} - m_{t+1}) \tag{10}$$

where,

$$cov_t(m_{t+1}, \eta_{t+1}) = cov_t(m_{t+1}, \Delta e_{t+1}) - cov_t(m_{t+1}, m^*_{t+1}) + var_t(m_{t+1}) \tag{11}$$

Proof. See Appendix A.2. □

The RHS of condition (10) is equal to the volatility of the exchange rate growth under complete markets and is strictly positive. The condition is satisfied if either the non-traded risk η_{t+1} leads to relative price fluctuations which are ex-post safe from the perspective of a Home investor, as captured by $cov_t(m_{t+1}, \eta_{t+1}) > 0$ or that the volatility of the non-traded risk is high.[11] Equation (11) shows that non-traded risk results in relative price fluctuations which are particularly *safe* when the Home SDF is very volatile or when international comovement in SDFs is low relative to the comovement of exchange rates and the Home SDF.[12] In Section 2.3, using a two-country model with multiple goods, we investigate the parametric restrictions consistent with this condition.

This conditions provides a general characterization for goods market mechanisms developed to resolve the Backus-Smith puzzle in models with consumption or production complementarities (Corsetti et al., 2008; Benigno and Thoenissen, 2008) as well as in models with costly consumer search (Bai and Ríos-Rull, 2015), or global value chain fragmentation (Corsetti et al., 2023*b*), amongst others.

A limitation of Proposition 1, and the models which satisfy it, is that it may exacerbate

[11]Note, that this is consistent with the idea that exchange rate movements exacerbate consumption risk. A low level of consumption implies a high discount factor, and is associated with a depreciation.

[12]Brandt et al. (2006) show that SDFs must comove very strongly to explain the relatively low exchange rate volatility in the data.

other exchange rate puzzles– in particular, that of excess volatility of exchange rates. The RHS of equation (10) is equal to the volatility of exchange rates under complete markets, and models with a low volatility will generally fare better in resolving the cyclicality puzzle.

Corollary 1 shows why introducing a second internationally traded risk-free asset as in Lustig and Verdelhan (2019) undoes these goods-market mechanisms and restores the cyclicality of exchange rates implied by the complete markets.

Corollary 1 (Two Int'l Traded Asset, Representative Agent No-Arbitrage).
As the Home bond also becomes internationally traded without arbitrage, $cov_t(m_{t+1}, \eta_{t+1}) \to$ -$\mathbb{E}_t[\eta_{t+1}] + \frac{1}{2} var_t(\eta_{t+1})$ *as in equation* (8)*, then Condition* (10) *requires* $var_t(\eta_{t+1}) > var_t(m^*_{t+1} - m_{t+1})$ *which is a contradiction, since it would imply* $var_t(\Delta e_{t+1}) < 0$.

Proof. See Appendix A.2. □

Trade in a second internationally traded risk-free asset prevents *any* model from reconciling the Backus-Smith puzzle specifically because this constrains the ex-post *safety* of relative price movements due to non-traded risk. By trading the additional bond investors insure this relative price movement away.

2.2. International Risk Sharing with Trade in Risky Assets

If instead of allowing for trade in both Home and Foreign risk-free assets, we allow for trade in Home and Foreign risky assets, then trade in assets does not necessarily restrict the cyclicality of exchange rates.[13] In practice, few assets traded across borders are risk-free in real terms, so this case is likely to be a better approximation of reality.

[13] Lustig and Verdelhan (2019) derive the restrictions imposed by trade in risky assets in addition to two-risk free bonds. In their environment, trade in risky assets can therefore not break patterns of risk sharing.

In this case, equations (2)–(5) are replaced by:

$$\mathbb{E}_t[M_{t+1}\tilde{R}_{t+1}] = 1, \tag{12}$$

$$\mathbb{E}_t[M_{t+1}\frac{\mathcal{E}_{t+1}}{\mathcal{E}_t}\tilde{R}^*_{t+1}] = 1, \tag{13}$$

$$\mathbb{E}_t[M^*_{t+1}\tilde{R}^*_{t+1}] = 1, \tag{14}$$

$$\mathbb{E}_t[M^*_{t+1}\left(\frac{\mathcal{E}_{t+1}}{\mathcal{E}_t}\right)^{-1}\tilde{R}_{t+1}] = 1, \tag{15}$$

where $\tilde{R}$ and $\tilde{R}^*$ are returns on risky Home and Foreign assets respectively.

Proposition 2 (Risky Assets, Representative Agent No-Arbitrage).

When only risky Home and Foreign assets are internationally traded such that equations (12) *-* (15) *hold, then* $cov_t(m^*_{t+1} - m_{t+1}, \Delta e_{t+1}) < 0$ *if and only if*

$$var_t(\Delta e_{t+1}) + cov_t(\eta_{t+1}, \tilde{r}^*_{t+1} - \tilde{r}_{t+1}) < 0 \tag{16}$$

Proof: See Appendix A.3. □

Proposition 2 shows that trade in risky assets is not sufficient to determine the cyclicality of exchange rates, unless the risky returns are uncorrelated with domestic non-traded risk.

Adding cross-border trade in the Home risk free bond will reimpose equation (8) and cross-border trade in the Foreign risk free bond will reimpose equation (9). Consider the environment in Proposition 1, where only Foreign risk-free bonds are internationally traded. Introducing trade in a Home risky assets does not necessarily recover the strong risk-sharing implications that arise when international trade in a second risk-free asset is allowed.

Corollary 2

When Foreign risk-free bonds are internationally traded such that equations (2), (3) *and* (5) *hold, as well as a Home risky asset is internationally traded such that equations* (12) *and* (15) *hold, then* $cov_t(m^*_{t+1} - m_{t+1}, \Delta e_{t+1}) < 0$ *if and only if*

$$var_t(\Delta e_{t+1}) - cov_t(\eta_{t+1}, \tilde{r}_{t+1}) < 0 \tag{17}$$

Proof. *Additionally* imposing equation (9) implies $cov_t(\eta_{t+1}, \tilde{r}^*_{t+1}) = 0$. The result to Corollary 2 then follows from Proposition 2. See Appendix A.3 for additional steps. □

Corollary 2 also shows that as enough assets become traded, so that we approach complete markets, $\sigma_\eta \to 0$, recovering the impossibility result of Corollary 1.

2.3. An Equilibrium Model in the Autarky Limit

To relate as closely as possible to the prevailing resolutions of the Backus-Smith puzzle in international macroeconomics literature, we specify a model capturing key ingredients in the literature and use it as a basis for constructing the investor SDFs. The representative agent derives per-period utility from consumption:

$$u(C_t) = \beta \frac{C_t^{1-s}}{1-s} \tag{18}$$

where β is the discount factor,[14] and the consumption bundle is given by:

$$C_t = \left[\alpha^{\frac{1}{\phi}} C_{H,t}^{\frac{\phi-1}{\phi}} + (1-\alpha)^{\frac{1}{\phi}} C_{F,t}^{\frac{\phi-1}{\phi}} \right]^{\frac{\phi}{\phi-1}} \tag{19}$$

where ϕ is the trade elasticity and α is the measure of home-bias. The domestic budget constraint is

$$P_t C_t + P_{H,t} Y_{H,t} \leq R_t B_{t-1} - B_t + \mathcal{E}_t R^*_t B^*_t - \mathcal{E}_t B^*_{t-1}$$

[14]The discount factor can be used as a stationarity-inducing device, see Appendix B.

Foreign agents face an analogous maximization. However, Foreign agents only trade in the Foreign denominated bond. Goods market clearing requires:

$$C_{H,t} + C^*_{H,t} = Y_{H,t}; \quad C_{F,t} + C^*_{F,t} = Y^*_{F,t}$$

where endowment process are given by $Y_{H,t} = \rho Y_{H,t-1} + (1-\rho)Y_H + \epsilon_t$, $Y^*_{F,t} = \rho Y^*_{F,t-1} + (1-\rho)Y^*_F + \epsilon^*_t$ and ϵ_t and ϵ^*_t are iid $N(0, \sigma_\epsilon)$.

The corresponding price level is given by:

$$P_t = \left[\alpha P_{H,t}^{\frac{\phi-1}{\phi}} + (1-\alpha)P_{F,t}^{\frac{\phi-1}{\phi}}\right]^{\frac{\phi}{\phi-1}} \tag{20}$$

and P^* is defined symmetrically. The real exchange rate is given by $\mathcal{E} = P^*/P$. We relegate a full description of the model to the Appendix.

Lemma 1 describes the autarky limit for prices and allocations. This is the zero liquidity (Corsetti et al., 2008) and full home-bias limit (Itskhoki and Mukhin, 2021, 2023).

Lemma 1 (Autarky Limit).

In the autarky limit $\alpha \to 1$, $B, B^* \to 0$, the model is summarized by the following equations

$$\begin{aligned}
m_{t+1} &= -s g_{y_{H,t+1}}, \\
m^*_{t+1} &= -s g_{y_{F,t+1}}, \\
\Delta e_{t+1} &= \frac{1}{1-2(1-\phi)}(g_{y_{H,t+1}} - g_{y_{F,t+1}}), \\
\eta_{t+1} &= (g_{y_{H,t+1}} - g_{y_{F,t+1}})\frac{1-s}{1-2(1-\phi)}
\end{aligned}$$

where $g_{y_{t+1}} = y_{t+1} - y_t$. It follows that if $Y_{H,t}, Y_{F,t}$ are normally distributed, then $m_{t+1}, m^*_{t+1}, \eta_{t+1}$ and Δe_{t+1} are jointly log-normally distributed.

Our approximation technique relies on taking the autarky limit for real quantities. There are four reasons why this limit is attractive. First, we prove the joint log normality of SDFs and the exchange rate at the full home-bias limit as $\alpha \to 1$. Second, the allocations and prices are invariant to the number of assets traded.[15] Third, as $\alpha \to 1$, we can evaluate the log-normal expansion of the Euler equations to derive the incomplete markets wedge η_{t+1}. Log-normality is obtained because the Euler equations hold exactly for $\epsilon \to 0$ quantities traded in financial assets. Fourth, the generalized model with investor heterogeneity that we present in Section 3 coincides with the model in Corsetti et al. (2008) at the autarky limit. Hence, one can consider their model in autarky limit as the limit of a two-asset economy with a marginal investor.

In Proposition 3, we derive the implications for Backus-Smith anomaly at the autarky limit.

Proposition 3 (Representative agent Backus-Smith at the autarky limit).

Assuming $var_t(g_{y_{H,t+1}} - g_{y_{F,t+1}}) = var_t(g_{y_{H,t+1}})$, the two-country model at the autarky limit can deliver $cov_t(m^*_{t+1} - m_{t+1}, \Delta e_{t+1}) < 0$ conditional on shocks to $y_{H,t}$ in the following cases

i. *when no assets traded*:

$$\frac{-s}{1-2(1-\phi)} < 0 \tag{21}$$

ii. *with trade in one risk-free asset*:

$$\frac{-s(1-s)}{1-2(1-\phi)} > s^2 - \frac{1}{2}\left[\frac{1-s}{1-2(1-\phi)}\right]^2 \tag{22}$$

Proof. See Appendix A.2. □

As in Corsetti et al. (2008), allowing for a sufficiently low trade elasticity implies that following an increase in Home productivity, demand for Home goods rises so much, that prices must adjust to constrain Foreign consumption of the Home good for markets to clear. A further

[15] While the invariance and the log-normality properties also hold at the complete markets allocation or Cole and Obstfeld (1991) limit, these cannot by construction break the Backus-Smith condition.

interesting point is that, somewhat intuitively, under autarky the ability of the model to match risk sharing depends only on ϕ given that $s > 0$.

When there is trade in assets, condition (10) additionally depends on the inter-temporal elasticity of substitution, s. As a result it is not necessarily true that it is harder for the model to replicate Backus-Smith when there is trade in assets. Figure 2 in Appendix A.4 shows the range of values for which condition (22) is satisfied. Looking at the case (ii), the first term in the RHS is the complete markets exchange rate process. As $s \to 0$, this quantity goes to zero. At the same time, the second term on the RHS which is the volatility of non-traded risk $\sigma^2_{\eta_{t+1}}$ rises. Therefore, the inequality is satisfied for any value of trade elasticity ϕ.

In Appendix B, we use a calibrated two-country open economy model of Corsetti et al. (2008) to capture the mechanisms away from the financial autarky limit and show that the results derived here continue to hold. In particular, Figure 5 illustrates that the volatility of non-traded risk rises by an order of magnitude relative to other components, so the inequality (10) is satisfied and Proposition 1 continues to hold in their baseline calibration.

Finally, it is important to note there is no condition under which the Backus-Smith limit can be met with trade in two risk-free assets (Lustig and Verdelhan, 2019).

3. A model with heterogeneous marginal investors

We now show that investor heterogeneity recovers the ability of the goods-market mechanisms described above to reconcile the Backus-Smith anomaly even when two risk-free assets are internationally traded.

Consider now the case where domestic financial markets are incomplete. The Foreign economy has a representative investor who can frictionlessly buy Home and Foreign risk-free bonds. The Home economy has two investors characterized by SDFs, M and $\hat{M}$. They both participate frictionlessly in the Home risk-free bond market, but only one of the two domestic investors participates in the Foreign risk-free asset market. We assume that the investors who participate in Foreign risk-free bonds are measure zero. This model is characterized by equations (2), (3), (4) and

$$\mathbb{E}_t\left[\hat{M}_{t+1}\frac{\mathcal{E}_{t+1}}{\mathcal{E}_t}\right] = 1/R^*_{t+1}, \tag{23}$$

where we define:

$$\hat{M}_{t+1} = M_{t+1}D_{t+1} \tag{24}$$

where D_{t+1} captures the degree of heterogeneity in the Home country and $D_{t+1} \neq 1$ for at least some t. This can capture a variety of models: $\hat{M}_{t+1}$ may be the intermediaries' SDF in a model akin to Gabaix and Maggiori (2015), but in contrast to the intermediary models, the marginal investor does not have a constant SDF, and we allow their SDFs to comove which is key to our results.

Since we assume that only the exchange rate markets are segmented within the domestic economy, we allow the domestic investors to trade in a Home risk-free bond.[16] Therefore, their marginal utility growth will be equated in expectation:

$$\mathbb{E}_t[M_{t+1}] = \mathbb{E}_t[\hat{M}_{t+1}] \tag{25}$$

Since $\hat{M}_{t+1}$ prices both domestic and Foreign bonds, $\hat{M}_{t+1}$ satisfies all conditions in Lustig and Verdelhan (2019) and will comove with exchange rates according to the following analogue of

[16]From Proposition 2 and Corollary 2, we can generalize our setup to allowing domestic investors to participate in domestic risky asset markets frictionlessly. The main restriction we require is there be domestic segmentation constraining participation in Foreign risk-free asset markets for a large enough measure of Home investors.

equation (1):

$$\mathbb{E}_t\left[\hat{M}_{t+1}\frac{\mathcal{E}_{t+1}}{\mathcal{E}_t}\right] = \mathbb{E}_t\left[M^*_{t+1}\right] \tag{26}$$

From equation (25), we derive the following condition on the heterogeneity:

$$\mathbb{E}_t[d_{t+1}] + \frac{1}{2}var_t(d_{t+1}) + cov_t(m_{t+1}, d_{t+1}) = 0 \tag{27}$$

Critically, equation (27) implies that d_{t+1} cannot be an asset specific discounting factor for the marginal investor – i.e. a convenience yield on specific bonds. Heterogeneity is therefore strictly on the investor, as opposed to the asset side. Additionally, d_{t+1} is non-traded risk, since by equations (24) and (25) it follows that d_{t+1} does not affect domestic asset prices.

Allowing agents to additionally trade in risky assets domestically further restricts heterogeneity, as expected. In particular, building on Corollary 2, we can show if m and $\hat{m}$ trade in $\tilde{r}$, then $cov(d, \tilde{r}) = 0$.[17]

The lemma below shows that the extended model admits the same process for exchange rates but a different set of equilibrium restrictions apply to the wedge η_{t+1}.[18] Specifically, equation (8) is unchanged because the Home bond continues to be traded frictionlessly across markets, but domestic market segmentation with respect to the Foreign bond implies the equation (9) is replaced by:

$$\mathbb{E}_t[d_{t+1}] + \frac{1}{2}var_t(d_{t+1}) + cov_t(m^*_{t+1}, d_{t+1}) + \mathbb{E}_t[\eta_{t+1}] + \frac{1}{2}var_t(\eta_{t+1})$$
$$\cdots + cov_t(m^*_{t+1} + d_{t+1}, \eta_{t+1}) = 0 \tag{28}$$

[17] By analogy to Corollary 2, if domestic agents trade in a complete set of securities, $\sigma_d \to 0$.

[18] Note that potentially an exchange rate process with η_{t+1} replace by $d_{t+1} + \tilde{\eta}_{t+1}$ could be used. Instead of making this assumption, we just allow for different restrictions to apply on η_{t+1}. Our results would be unchanged.

Proposition 4 (Exchange Rates with Heterogeneous Investors).

The model admits the same exchange rate process as shown in equation (7).

Proof. The proof is constructive. Assume equation (7). Using equations (2),(3),(4),and (23) yields equations (8) and (9). Then, we show that equation (6) is satisfied. Detailed proof is relegated to Appendix A.2. □

Moreover, using Proposition 4, and equations (8) and (28), the exchange rate volatility is given by:

$$var_t(\Delta e_{t+1}) = var_t(m^*_{t+1} - m_{t+1}) - 2\log \mathbb{E}_t[D_{t+1}] - 2cov_t(m^*_{t+1} + \eta_{t+1}, d_{t+1}) - var_t(\eta_{t+1}) \quad (29)$$

We next derive restrictions on the dynamics of investor heterogeneity and exchange rates, required to match the patterns of international risk sharing observed in the data.

Proposition 5 (Heterogeneous Marginal Investors).

The two-country model with two internationally traded bonds and heterogeneous Home investors can deliver $cov_t(m^*_{t+1} - m_{t+1}, \Delta e_{t+1}) < 0$ *if and only if*

$$1 \geq \rho_{d_{t+1}, -\Delta e_{t+1}} \geq \frac{\sigma_{\Delta e_{t+1}}}{\sigma_{d_{t+1}}} \quad (30)$$

where $\rho_t\left(d_{t+1}, -\Delta e_{t+1}\right) \equiv \dfrac{cov_t(d_{t+1}, -\Delta e_{t+1})}{\sigma_t(\Delta e_{t+1})\sigma_t(d_{t+1})}$.

Proof: See Appendix A.2. □

The inequality in Proposition 5 describes the joint dynamics of exchange rates and domestic market incompleteness required to break the covariance between SDFs and exchange rates implied by equation (1), when there is trade in both Home and Foreign bonds. First, a necessary condition is that $\frac{\sigma_{\Delta e_{t+1}}}{\sigma_{d_{t+1}}} < 1$– i.e. there is sufficient domestic market incompleteness in the economy relative to the volatility of exchange rates. Since this condition is a critical component for our theory, we evaluate this in the data in Section 4.

Proposition 5 also bounds the sign of the correlation of SDF heterogeneity (non traded risk) and exchange rate appreciation to be positive– as should be expected in theory. Consider the Backus-Smith condition (1) where the Home SDF is replaced by $\hat{M}_{t+1}$. Periods of depreciation $\mathcal{E}_{t+1} > \mathcal{E}_t$ are associated with $\hat{M}_{t+1}$ falling (relatively high $\hat{C}_{t+1}$ is associated with low P_{t+1} due to risk sharing). For relatively stable M_{t+1}, D_{t+1} must fall– signifying C_{t+1} is low relative to $\hat{C}_{t+1}$, ceteris paribus. The sufficient condition is therefore that the marginal investor does not provide enough insurance to the domestic household against exchange rate movements through the domestic asset markets.

To gain concrete understanding of condition (30), we flesh out the financial market structure in the Home economy. The simplest model of heterogeneity consistent with our framework is one where the investor characterized by m_{t+1} and the investor characterized by $\hat{m}_{t+1}$ are identical except the latter participates in financial markets for Foreign assets. Imposing consumption utility structure on the SDFs, $\hat{m}_{t+1} = log(u'(y_{t+1}+w^H_{t+1}+w^F_{t+1}))$ and $m_{t+1} = log(u'(y_{t+1}+w^H_{t+1}))$, where y_{t+1} is the value of the Home country's endowment, w^H_{t+1} is wealth after trade in a set of basis assets (e.g. just the Home bond) and w^F_{t+1} is defined as the residual portfolio wealth after trade in both the set of basis assets and the Foreign bond.[19] Assuming for exposition that m^*_{t+1} does not vary a lot and utility is exponential, equation (1) implies:[20]

$$cov_t(-aw^F_{t+1}, \Delta e_{t+1}) < 0 \tag{31}$$

[19] w^H is the return on the basis asset portfolio which are freely traded by both investors. Note that the autarky limit is where $cov_t(m_{t+1}, -\Delta e_{t+1}) < 0$, requiring $cov_t(y_{t+1}, -\Delta e_{t+1}) > 0$, consistent with Proposition 2. Moreover, at the autarky limit $m_{t+1} \to \hat{m}_{t+1}$.

[20] As is standard in portfolio choice, exponential utility (CARA) allows us to break the individual components by abstracting from wealth effects. Specifically, $u(C) = -e^{-\alpha C}$.

In other words, the marginal investor purchases sufficient insurance ex-ante, that the exchange rate is risky ex-post consistent with redistribution– but this investor does not pass the insurance on to the domestic household through domestic asset markets. It is useful to note that the implied comovement of $\hat{m}_{t+1}$ and m_{t+1} in this framework is given by $\alpha^2 var_t(y_{t+1} + w^H_{t+1}) + \alpha^2 cov_t(y_{t+1} + w^H_{t+1}, w^F_{t+1})$, which will depend on how portfolios are formed and the underlying structure of shocks which we have not specified.[21]

As Corollary 3 below illustrates, the generalised framework nests the representative agent economy.

Corollary 3

As $\sigma_t(d_{t+1}) \to 0$, the model collapses to a representative agent economy and (30) is violated.

Intuitively, heterogeneous marginal investors allow the model with international trade in two risk-free assets to reproduce the Backus-Smith anomaly as long as the volatility of the difference in Home SDFs is sufficiently high, and the covariance between their SDF differences and the exchange rate is sufficiently positive.

Before proceeding to evaluate the plausibility of our mechanism in the data, we revisit the class of models with a single internationally traded asset and ask how the presence of incompleteness (heterogeneous marginal investors in the domestic economy) affects the range of parameters for which they deliver plausible patterns of risk-sharing. Specifically, consider a model where the domestic investor trades in Home bonds only, the domestic marginal investor additionally invests in Foreign bonds, and the the Foreign household trades only in Foreign.

Corollary 4

Allowing for heterogeneous marginal investors and trade in only the Foreign bond implies

[21] Corsetti, Dedola and Leduc (2014) discipline portfolios using the data and show there is low-risk sharing when there is trade in one international nominal risk-free bond, and trade in international equities.

$cov_t(m^*_{t+1} - m_{t+1}, \Delta e_{t+1}) < 0$ *if :*

$$cov_t(m_{t+1}, \eta_{t+1}) + cov_t(d_{t+1}, \Delta e_{t+1}) > var_t(m^*_{t+1} - m_{t+1}) - \frac{1}{2} var_t(\eta_{t+1}) \tag{32}$$

Proof. See Appendix A.2. □

Interestingly, if $cov_t(\Delta e_{t+1}, d_{t+1}) < 0$, this restricts the set of equilibria where the model with one internationally traded bond can reconcile the Backus-Smith puzzle. In this case, there are two sources of non-traded risk which could potentially work in opposite directions. A negative value for $cov_t(\Delta e_{t+1}, d_{t+1})$ attenuates the pass-through of exchange rate risk to domestic non-participants' SDF through risk-sharing within the Home country (27).[22]

4. How much heterogeneity?

In this section, we make a first pass at evaluating the plausibility of the conditions under which the model with heterogeneous marginal investors reproduces a correct pattern of risk-sharing. We begin by estimating $\hat{M}_{t+1}$ and m_{t+1} in the spirit of Hansen and Jagannathan (1991).

Measuring m and $\hat{m}$ To evaluate $var_t(\hat{m}_{t+1}), var_t(\hat{m}_{t+1})$, we additionally assume investors trade in Home equity. For the domestic investor:

$$\mathbb{E}_t[M_{t+1} R^e_{t+1}] = 1 \tag{33}$$

where R^e_{t+1} is the return on equity. Then, we use the Hansen and Jagannathan (1991) bounds to back out a measure for $var_t(m_{t+1})$.

[22]Note that Proposition 4 relates to a model with two internationally traded bonds– here we assume the covariance is the same sign in the model with a single internationally traded bond. In the Corsetti et al. (2008) economy, $cov_t(\Delta e_{t+1}, \hat{m}_{t+1})$ is not necessarily restricted in the same way.

$$\frac{\sqrt{var(M_{t+1})}}{\mathbb{E}_t[M_{t+1}]}\rho_{-M_{t+1},R^e_{t+1}} = sup\left|\frac{\mathbb{E}_t[R^e_{t+1}] - R_{t+1}}{\sqrt{var_t(R^e_{t+1})}}\right|,$$
$$\frac{\sqrt{var_t(M_{t+1})}}{\mathbb{E}_t[M_{t+1}]} \geq sup\left|\frac{\mathbb{E}_t[R^e_{t+1}] - R_{t+1}}{\sqrt{var_t(R^e_{t+1})}}\right| \tag{34}$$

So, assuming $\mathbb{E}_t[M_{t+1}] = 1$ and rearranging yields:

$$var_t(M_{t+1}) \geq sup\left(\underbrace{\frac{\mathbb{E}_t[R^e_{t+1}] - R_{t+1}}{\sqrt{var_t(R^e_{t+1})}}}_{\mathbb{E}_t[SR_{t+1}]}\right)^2 \tag{35}$$

The right hand side of the condition above is the squared Sharpe ratio. To maximize the RHS, we choose a high return to variance domestically-traded asset such as equity.

We do not measure $var_t(\hat{m}_{t+1})$ directly. Rather, we leverage the concept of "Good-Deal Bounds" of Cochrane and Saa-Requejo (2000), and ask: what additional Sharpe ratio can the domestic investor earn, by participating in the Foreign markets (like $\hat{m}$)?

Lemma 2 (Limits on heterogeneity and no good deals).

We consider equilibria where we rule out good deals where the Sharpe ratio is $K \geq 1$ times the maximal domestic Sharpe Ratio. Then:

$$(K-1)var_t(m_{t+1}) \geq -2\mathbb{E}_t[d_{t+1}]$$

Proof. The volatility of $\hat{m}_{t+1}$ is given by:

$$var_t(\hat{m}_{t+1}) = var_t(m_{t+1}) + var_t(d_{t+1}) + 2cov_t(m_{t+1}, d_{t+1}) \tag{36}$$

However, since the investors in the Home country share risk, $cov_t(m_{t+1}, d_{t+1})$ is pinned down by equation (27). The result follows by substituting $var_t(\hat{m}_{t+1}) = Kvar_t(m_{t+1})$ in equation (36) and imposing within country risk-sharing equation (27). □

The case of $K \leq 1$ corresponds to a world where the maximal Sharpe ratio available to the investor who can access Foreign markets $(\hat{m}_{t+1})$ is no higher than that of the domestic asset investor (m_{t+1}). Risk-sharing within the domestic country then implies $var_t(m_{t+1}) = var_t(\hat{m}_{t+1})$.

Measuring d We now measure the amount of heterogeneity and incompleteness in the domestic economy. Specifically, we look for a plausible values for $\sigma_{d_{t+1}}$. A sufficiently high value makes it more plausible that our generalized model resolves the Backus-Smith puzzle even when there is trade in two risk-free assets.

Lemma 3 (Domestic market incompleteness and no good deals).

Assume now that there are no good-deals, such that $var_t(\hat{m}_{t+1}) \leq K var_t(m_{t+1})$. *Then*:

$$var_t(d_{t+1}) \leq var(m_{t+1}) \left[1 + K(1 - \frac{2}{\sqrt{K}} \rho_t(\hat{m}_{t+1}, m_{t+1})) \right] \tag{37}$$

Proof. Consider:

$$var_t(d_{t+1}) = var(\hat{m}_{t+1}) + var(\hat{m}_{t+1}) - 2cov_t(\hat{m}_{t+1}, m_{t+1}),$$

$$var_t(d_{t+1}) = var(\hat{m}_{t+1}) + var(\hat{m}_{t+1}) - 2\rho_t(\hat{m}_{t+1}, m_{t+1})\sigma_t(\hat{m}_{t+1})\sigma_t(m_{t+1})$$

Then,

$$var_t(d_{t+1}) \leq (K+1)var(m_{t+1}) - 2\rho_{\hat{m}_{t+1}, m_{t+1}} \sqrt{K} \sigma_t^2(m_{t+1}), \tag{38}$$

Rearranging yields the result. □

Finally, we evaluate the above expression. First, we use standard values from the literature. We take a Sharpe ratio of 0.5 annually implying $var(m_{t+1}) = 0.5$ as in Lustig and Verdelhan

(2019). This is on the conservative side, since the gross Sharpe ratio on the S&P 500 is just above 0.6 from time-series momentum strategies (Babu, Levine, Ooi, Pedersen and Stamelos, 2020). Secondly, $\rho_t(\hat{m}_{t+1}, m_{t+1})$ is the correlation between the two SDFs of domestic investors. Zhang (2021) measures correlations between SDFs of various agents (domestic and foreign). They find a value of 0.5 for the correlation between the domestic and the stockholder SDFs, and a value of 0.21 for within country correlation between the stockholders' and the non stockholders' SDFs. A lower correlation would provide a better fit for our model as can be seen from Lemma 4. So in order to be conservative, we set the the correlation between the two SDFs of domestic investors to the higher value of 0.5.

For deriving the no good-deal bounds, we first use $K = 1$, ruling out the possibility that there are high Sharpe ratios to be had in markets. In this case,

$$var_t(d_{t+1}) = var_t(m_{t+1}) = 0.5$$

From Proposition 4, what matters then is the ratio $\frac{\sigma_{\Delta e_{t+1}}}{\sigma_{d_{t+1}}}$. In the data, $var_t(\Delta e_{t+1}) = 0.11$, see e.g. Lustig and Verdelhan (2019), Lloyd and Marin (2023). As a result, for reconciling the Backus-Smith anomaly, our model requires that the correlation of heterogeneity with exchange rate growth be sufficiently low, where the threshold is given by

$$\rho^{K=1}_{d_{t+1},\Delta e_{t+1}} \leq -\frac{0.33}{0.7} \tag{39}$$

Next, we evaluate a more empirically realistic scenario. We leverage the finding in Barroso and Santa-Clara (2015) that carry trade exposure can double the Sharpe ratio of a diversified stock-bond portfolio, i.e. $K \leq 2$.[23] This would imply $var_t(d_{t+1}) = 0.89$ and therefore the

[23] There is substantial variation in the maximum annualized Sharpe ratio documented in the literature. Jordà and Taylor (2012), Asness, Moskowitz and Pedersen (2013), and Burnside, Cerrato and Zhang (2020) find strategies with Sharpe ratio as high as 2.42, 1.59, and 3.73 respectively. Lustig, Roussanov and Verdelhan

threshold correlation between exchange rate growth and SDF heterogeneity is now:

$$\rho^{K=2}_{d_{t+1},\Delta e_{t+1}} \leq -\frac{0.33}{0.89} \tag{40}$$

Finally, as robustness, if we maintain $K = 2$ but allow for more insurance within countries, such that $\rho_t(\hat{m}_{t+1}, m_{t+1}) = 0.75$. In this case, the bound becomes tighter $\rho_{d_{t+1},\Delta e_{t+1}} \leq -\frac{0.33}{0.66}$.

5. Conclusion

A classical strand of the literature in international macroeconomics has focused on formulating goods-market mechanisms which generate a negative relationship between consumption growth and depreciation– the opposite sign to that implied by the Backus-Smith condition– as long as financial markets are incomplete. We show that any model which achieves this resolution must rely on a non-traded component to relative prices which is "safe" from a domestic investor perspective. However, Lustig and Verdelhan (2019) determine that any two-country model with a representative agent and frictionless trade in Home and Foreign currency denominated risk-free bonds recovers the exchange rate cyclicality implied by complete markets. We show this is because international trade in these two assets make exchange rate movements fully insurable ex-ante, resulting in redistribution which makes ex-post exchange rate movements risky.

We propose a generalization of the model, beyond the representative agent, where we consider heterogeneous marginal investors in the Home country. This can be interpreted as a specific case of incomplete markets where multiple SDFs can exist. We characterize the relative SDF dynamics between the two Home investors that are necessary to reconcile the

(2011), Menkhoff, Sarno, Schmeling and Schrimpf (2012), and Hassan and Mano (2019) find currency trade strategies with Sharpe ratio of 0.99, 0.95, and 0.69 respectively.

Backus-Smith puzzle, whilst allowing for trade in multiple assets. We show that a necessary condition is existence of sufficient heterogeneity in SDFs domestically, and we show that even ruling out good deals available from participating in markets, such heterogeneity is plausible. A sufficient condition is for the marginal investor for assets to purchase sufficient insurance from abroad– but not pass it on through domestic bond markets. Since this depends on the structure of shocks and how portfolios are formed, we leave specific modelling of the portfolio choice problem leading to such non-transmission for future research.

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A. Appendix

A.1. Additional Derivations for Section 2.

To find the admissible set of processes, consider the log expansions of the above conditions, assuming joint log normality:

$$\mathbb{E}_t[m_{t+1}] + \frac{1}{2}var_t(m_{t+1}) = -r_{t+1}, \tag{41}$$

$$\mathbb{E}_t[m^*_{t+1}] + \frac{1}{2}var_t(m^*_{t+1}) = -r^*_{t+1} \tag{42}$$

$$\mathbb{E}_t[m^*_{t+1}] + \frac{1}{2}var_t(m^*_{t+1}) - \mathbb{E}_t[\Delta e_{t+1}] + \frac{1}{2}var_t(\Delta e_{t+1}) + cov_t(m^*_{t+1}, -\Delta e_{t+1}) = -r_{t+1}, \tag{43}$$

$$\mathbb{E}_t[m_{t+1}] + \frac{1}{2}var_t(m_{t+1}) + \mathbb{E}_t[\Delta e_{t+1}] + \frac{1}{2}var_t(\Delta e_{t+1}) + cov_t(m_{t+1}, \Delta e_{t+1}) = -r^*_{t+1}, \tag{44}$$

where lower case levels denote logs, e.g. $log(M_{t+1}) = m_{t+1}$ and $\Delta e_{t+1} = e_{t+1} - e_t$. Using (41) and (44), and (42) and (43) respectively, yields:

$$\mathbb{E}_t[\Delta e_{t+1}] + r^*_{t+1} - r_{t+1} = -cov_t(m_{t+1}, \Delta e_{t+1}) - \frac{1}{2}var_t(\Delta e_{t+1}), \tag{45}$$

$$\mathbb{E}_t[\Delta e_{t+1}] + r^*_{t+1} - r_{t+1} = cov_t(m^*_{t+1}, -\Delta e_{t+1}) + \frac{1}{2}var_t(\Delta e_{t+1}) \tag{46}$$

A.2. Proofs to Propositions

Proof to Proposition 1 The Backus-Smith condition is related to the covariance $cov_t(m^*_{t+1} - m_{t+1}, \Delta e_{t+1})$ which can be rewritten as:

$$cov_t(m^*_{t+1} - m_{t+1}, m^*_{t+1} - m_{t+1}) + cov_t(m^*_{t+1} - m_{t+1}, \eta_{t+1}) \tag{47}$$

$$= var_t(m^*_{t+1} - m_{t+1}) + cov_t(m^*_{t+1}, \eta_{t+1}) - cov_t(m_{t+1}, \eta_{t+1}) \tag{48}$$

Imposing (9) (international trade in the Foreign asset), but not (8) (international trade in the Home asset) as is done in Lustig and Verdelhan (2019), assuming $\mathbb{E}_t[\eta_{t+1}] = 0$, and rearranging yields the result. □

Proof to Corollary 1 The volatility of the exchange rate is given by:

$$var_t(\Delta e_{t+1}) = var(m^*_{t+1} - m_{t+1}) + var_t(\eta_{t+1}) + 2cov_t(m^*_{t+1}, \eta_{t+1}) - 2cov_t(m_{t+1}, \eta_{t+1})$$

Imposing (8) and (9):

$$var_t(\Delta e_{t+1}) = var(m^*_{t+1} - m_{t+1}) - var_t(\eta_{t+1})$$

Taking the limit $cov_t(m_{t+1}, \eta_{t+1}) \to$ (10) would imply $var_t(\Delta e_{t+1}) < 0$ which cannot be an equilibrium. □

Proof to Proposition 3 From Section A.4 below:

$$cov_t(m_{t+1}, \eta) = -\frac{s(1-s)}{1-2(1-\phi)} var_t(g_{y_{H,t+1}}), \tag{49}$$

$$var_t(m_{t+1} - m^*_{t+1}) = s^2 var_t(g_{y_{H,t+1}} - g_{y_{F,t+1}}), \tag{50}$$

$$\frac{1}{2} var_t(\eta_{t+1}) = \frac{1}{2}\left[\frac{1-s}{1-2(1-\phi)}\right]^2 var_t(g_{y_{H,t+1}} - g_{y_{F,t+1}}) \tag{51}$$

Assuming $var_t(g_{y_{H,t+1}} - g_{y_{F,t+1}}) = var_t(g_{y_{H,t+1}})$ (i.e no covariance and conditioning on H shocks) implies that Proposition 1 amounts to:

$$\frac{-s(1-s)}{1-2(1-\phi)} > s^2 - \frac{1}{2}\left[\frac{1-s}{1-2(1-\phi)}\right]^2 \tag{52}$$

$\square$

Proof to Proposition 4 We begin by deriving the condition that must be satisfied by an exchange rate progress satisfying no-arbitrage in the generalized model. Combining (2), (3), (4),(23), (27) yields:

$$\mathbb{E}_t[\Delta e_{t+1}] + \frac{1}{2}var_t(\Delta e_{t+1}) + cov_t(m_{t+1}, \Delta e_{t+1}) + cov_t(d_{t+1}, \Delta e_{t+1}) + r^*_{t+1} - r_{t+1} = 0, \tag{53}$$

$$-\mathbb{E}_t[\Delta e_{t+1}] + \frac{1}{2}var_t(\Delta e_{t+1}) + cov_t(m^*_{t+1}, -\Delta e_{t+1}) - r^*_{t+1} + r_{t+1} = 0 \tag{54}$$

Combining the above, the restriction that must be satisfied by any exchange rate process which admits no arbitrage is therefore:

$$var_t(\Delta e_{t+1}) = cov_t(m^*_{t+1} - m_{t+1}, \Delta e_{t+1}) - cov_t(d_{t+1}, \Delta e_{t+1}) \tag{55}$$

Assuming $\Delta e_{t+1} = m^*_{t+1} - m_{t+1} + \eta_{t+1}$:

$$var_t(\Delta e_{t+1}) = var_t(m^*_{t+1} - m_{t+1}) + var_t(\eta_{t+1}) + 2cov_t(m^*_{t+1} - m_{t+1}, \eta_{t+1}) \tag{56}$$

Using equations (8) and (28), we can express the covariance term as

$$\begin{aligned} cov_t(m^*_{t+1} - m_{t+1}, \eta_{t+1}) = -\mathbb{E}_t[d_{t+1}] - \frac{1}{2}var_t(d_{t+1}) - cov_t(m^*_{t+1}, d_{t+1}) \\ -cov_t(d_{t+1}, \eta_{t+1}) - var_t(\eta_{t+1}) \end{aligned} \tag{57}$$

Using equations (27) and (57), we can simplify equation (56) :

$$var_t(\Delta e_{t+1}) = var_t(m^*_{t+1} - m_{t+1}) + var_t(\eta_{t+1}) + cov_t(m^*_{t+1} - m_{t+1}, \eta_{t+1}) + \cdots$$

$$\left\{cov_t(m_{t+1}, d_{t+1}) - cov_t(m^*_{t+1}, d_{t+1}) - cov_t(d_{t+1}, \eta_{t+1}) - var_t(\eta_{t+1})\right\},$$

$$var_t(\Delta e_{t+1}) = var_t(m^*_{t+1} - m_{t+1}) + cov_t(m^*_{t+1} - m_{t+1}, \eta_{t+1}) - cov_t(d_{t+1}, \Delta e_{t+1}) \tag{58}$$

so equation (55) is satisfied. □

Proof to Proposition 5 The covariance can be rewritten as:

$$cov_t(m^*_{t+1} - m_{t+1}, \Delta e_{t+1}) = var_t(m^*_{t+1} - m_{t+1}) + cov_t(m^*_{t+1}, \eta_{t+1}) - cov_t(m_{t+1}, \eta_{t+1}),$$

$$cov_t(m^*_{t+1} - m_{t+1}, \Delta e_{t+1}) = var_t(m^*_{t+1} - m_{t+1}) - \cdots$$

$$\left\{\mathbb{E}_t[d_{t+1}] + \frac{1}{2}var_t(d_{t+1}) + cov_t(m^*_{t+1} + \eta, d_{t+1}) + \mathbb{E}_t[\eta_{t+1}] + \frac{1}{2}var_t(\eta_{t+1})\right\} + \cdots$$

$$\left\{\mathbb{E}_t[\eta_{t+1}] - \frac{1}{2}var_t(\eta_{t+1})\right\}$$

Simplifying:

$$cov_t(m^*_{t+1} - m_{t+1}, \Delta e_{t+1}) = var_t(m^*_{t+1} - m_{t+1}) - \cdots$$

$$\left\{\log \mathbb{E}_t[D_{t+1}] + cov_t(m^*_{t+1} + \eta, d_{t+1})\right\} - var_t(\eta_{t+1}) \tag{59}$$

where $\log \mathbb{E}_t[D_{t+1}] = \mathbb{E}_t[d_{t+1}] + \frac{1}{2}var_t(d_{t+1})$. Using (29), this can be rewritten as:

$$cov_t(m^*_{t+1} - m_{t+1}, \Delta e_{t+1}) = var_t(\Delta e_{t+1}) + log\mathbb{E}_t D_{t+1} + cov_t(m_{t+1} - \eta_{t+1}), d^*_{t+1}) \tag{60}$$

The model can reconcile Backus-Smith if and only if:

$$var_t(\Delta e_{t+1}) + log\mathbb{E}_t[D_{t+1}] + cov_t(m^*_{t+1} + \eta_{t+1}, d_{t+1}) \leq 0 \tag{61}$$

Additionally, using equation (27) we get:

$$cov_t(d_{t+1}, -\Delta e_{t+1}) \geq var_t(e_{t+1}) \tag{62}$$

Finally, the Cauchy Schwarz identity implies:

$$cov_t(d_{t+1}, -\Delta e_{t+1}) \leq \sqrt{\sigma^2_{d_{t+1}} \sigma^2_{\Delta e_{t+1}}} \tag{63}$$

Combining the inequalities and dividing by $\sigma(d)$ yields the result. □

Proof to Corollary 4 With heterogeneous marginal investors in the domestic country, when only the Foreign bond is traded across borders, the relevant Euler equations are (2), (3),(23) and (25). Using (48) but replacing (9) by (28) yields the result. □

A.3. Trade in Risky Assets

Suppose Home and Foreign households trade in Home and Foreign currency denominated risky assets $\tilde{R}_{t+1}$ such that (12)- (15) hold. Assuming joint log normality, the above Euler equations imply:

$$\mathbb{E}_t[m_{t+1}] + \frac{1}{2}var_t(m_{t+1}) + \mathbb{E}_t[\tilde{r}_{t+1}] + \frac{1}{2}var_t(\tilde{r}_{t+1}) + cov_t(m_{t+1}, \tilde{r}_{t+1}) = 0, \tag{64}$$

$$\mathbb{E}_t[m_{t+1}] + \frac{1}{2}var_t(m_{t+1}) + \mathbb{E}_t[\tilde{r}^*_{t+1}] + \frac{1}{2}var_t(\tilde{r}^*_{t+1}) + \mathbb{E}_t[\Delta e_{t+1}] + \frac{1}{2}var_t(\Delta e_{t+1}) + \cdots$$

$$cov_t(m_{t+1}, \tilde{r}^*_{t+1}) + cov_t(m_{t+1}, \Delta e_{t+1}) + cov_t(\Delta e_{t+1}, \tilde{r}^*_{t+1}) = 0, \tag{65}$$

$$\mathbb{E}_t[m^*_{t+1}] + \frac{1}{2}var_t(m^*_{t+1}) + \mathbb{E}_t[\tilde{r}^*_{t+1}] + \frac{1}{2}var_t(\tilde{r}^*_{t+1}) + cov_t(m^*_{t+1}, \tilde{r}^*_{t+1}) = 0, \tag{66}$$

$$\mathbb{E}_t[m^*_{t+1}] + \frac{1}{2}var_t(m^*_{t+1}) - \mathbb{E}_t[\Delta e_{t+1}] + \frac{1}{2}var_t(\Delta e_{t+1}) + \mathbb{E}_t[\tilde{r}_{t+1}] + \frac{1}{2}var_t(\tilde{r}_{t+1}) + \cdots$$

$$cov_t(m^*_{t+1}, \tilde{r}_{t+1}) + cov_t(m^*_{t+1}, -\Delta e_{t+1}) + cov_t(-\Delta e_{t+1}, \tilde{r}_{t+1}) = 0 \tag{67}$$

Combining (64) and (65):

$$\mathbb{E}_t[\Delta e_{t+1}] + \frac{1}{2}var_t(\Delta e_{t+1}) + \mathbb{E}_t[\tilde{r}^*_{t+1}] + \frac{1}{2}var_t(\tilde{r}^*_{t+1}) - \mathbb{E}_t[\tilde{r}_{t+1}] - \frac{1}{2}var_t(\tilde{r}_{t+1}) + \cdots$$

$$cov_t(m_{t+1}, \Delta e_{t+1}) + cov_t(\Delta e_{t+1}, \tilde{r}^*_{t+1}) + cov_t(m_{t+1}, \tilde{r}^*_{t+1} - \tilde{r}_{t+1}) = 0 \tag{68}$$

Combining (66) and (67):

$$-\mathbb{E}_t[\Delta e_{t+1}] + \frac{1}{2}var_t(\Delta e_{t+1}) + \mathbb{E}_t[\tilde{r}_{t+1}] + \frac{1}{2}var_t(\tilde{r}_{t+1}) - \mathbb{E}_t[\tilde{r}^*_{t+1}] - \frac{1}{2}var_t(\tilde{r}^*_{t+1}) + \cdots$$
$$cov_t(m^*_{t+1}, -\Delta e_{t+1}) + cov_t(-\Delta e_{t+1}, \tilde{r}_{t+1}) - cov_t(m^*_{t+1}, \tilde{r}^*_{t+1} - \tilde{r}_{t+1}) = 0 \tag{69}$$

Together, the above conditions yield:

$$var_t(\Delta e_{t+1}) = cov_t(m^*_{t+1} - m_{t+1}, \Delta e_{t+1}) - cov_t(\Delta e_{t+1} - m^*_{t+1} + m_{t+1}, \tilde{r}^*_{t+1} - \tilde{r}_{t+1}) \tag{70}$$

Assuming the exchange rate process is given by $\Delta e_{t+1} = m^*_{t+1} - m_{t+1} + \eta_{t+1}$ this condition reduces to:

$$var_t(\Delta e_{t+1}) = var_t(m^*_{t+1} - m_{t+1}) + cov_t(m^*_{t+1} - m_{t+1}, \eta_{t+1}) - cov_t(\eta_{t+1}, \tilde{r}^*_{t+1} - \tilde{r}_{t+1}) \tag{71}$$

Imposing the exchange rate process, we can derive restrictions to the incomplete market wedge analogous to equations (8) and (9). Then, doing a log expansion from combining equations (15), (64), and the exchange rate process, we get:

$$cov_t(m_{t+1}, \eta_{t+1}) = -\mathbb{E}_t[\eta_{t+1}] + \frac{1}{2}var_t(\eta_{t+1}) - cov_t(\tilde{r}_{t+1}, \eta_{t+1}) \tag{72}$$

Additionally, equations (13) and (66) imply:

$$cov_t(m^*_{t+1}, \eta_{t+1}) = -\mathbb{E}_t[\eta_{t+1}] - \frac{1}{2}var_t(\eta_{t+1}) - cov_t(\tilde{r}^*_{t+1}, \eta_{t+1}) \tag{73}$$

The volatility of the exchange rate is given by:

$$var_t(\Delta e_{t+1}) = var_t(m^*_{t+1} - m_{t+1}) + var_t(\eta_{t+1}) + 2cov_t(m^*_{t+1} - m_{t+1}, \eta_{t+1}) = \cdots$$
$$var_t(m^*_{t+1} - m_{t+1}) + cov_t(m^*_{t+1} - m_{t+1}, \eta_{t+1}) - cov_t(\eta_{t+1}, \tilde{r}^*_{t+1} - \tilde{r}_{t+1}) \tag{74}$$

which verifies (70), so the exchange rate process is admissible.

Proof to Proposition 2: Using (70) and imposing $\Delta e_{t+1} = m^*_{t+1} - m_{t+1} + \eta_{t+1}$:

$$var_t(\Delta e_{t+1}) = cov_t(m^*_{t+1} - m_{t+1}, \Delta e_{t+1}) - cov_t(\eta_{t+1}, \tilde{r}^*_{t+1} - \tilde{r}_{t+1}) \tag{75}$$

In that case, $cov_t(m^*_{t+1} - m_{t+1}, \Delta e_{t+1}) < 0$ if and only if $var_t(\Delta e_{t+1}) + cov_t(\eta_{t+1}, \tilde{r}^*_{t+1} - \tilde{r}_{t+1}) <$ 0. $\square$

Proof to Corollary 2: Next, suppose we reintroduce trade in risk-free bonds. Then (8) and (9) hold. In particular, introducing a Home internationally trade risk-free bond implies:

$$cov_t(\tilde{r}_{t+1}, \eta_{t+1}) = 0 \tag{76}$$

Introducing a Foreign internationally trade risk-free bond implies:

$$cov_t(\tilde{r}^*_{t+1}, \eta_{t+1}) = 0 \tag{77}$$

$\square$

A.4. An equilibrium model

To fix ideas, we present an equilibrium two-country, two-good, endowment model solved under the assumption of financial autarky. This allows us to express SDFs and prices as functions of exogenous variables. Financial autarky is not a restrictive assumption for us since we are interested in the sign of covariances when there is trade in assets, and Euler equations apply even in the ϵ liquidity limit. However, to attain joint normality of SDFs and prices, we further need to assume the limit of full home-bias, as in e.g. Itskhoki and Mukhin (2021).

Starting with the static conditions:

$$C = \left[\alpha^{\frac{1}{\phi}} C_H^{\frac{\phi - 1}{\phi}} + (1 - \alpha)^{\frac{1}{\phi}} C_F^{\frac{1}{\phi}} \right]^{\frac{\phi}{\phi - 1}} \tag{78}$$

Relative demand for goods requires:

$$\frac{C_F}{C_H} = \frac{1-\alpha}{\alpha} ToT^{-\phi}, \tag{79}$$

$$\frac{C_F^*}{C_H^*} = \frac{1-\alpha^*}{\alpha^*} ToT^{-\phi}, \tag{80}$$

where τ denotes the terms of trade.Market clearing requires:

$$C_H + C_H^* = Y_H \tag{81}$$

$$C_F + C_F^* = Y_F \tag{82}$$

The real exchange rate is given by:

$$\mathcal{E}_= \frac{P^*}{P} \tag{83}$$

and the terms of trade:

$$ToT = \frac{P_F}{P_H} \tag{84}$$

The law of one price holds for each good but not the aggregate basket unless $\alpha = \alpha^*$.

Under financial autarky, $PC = P_H Y_H$ and $P^*C^* = P_F Y_F$. Combining this with relative demand yields:

$$C_H + ToT\ C_F = Y_H, \tag{85}$$

$$C_F = \frac{1-\alpha}{\alpha}\left(\frac{1}{ToT}\right)^{\phi} C_H, \tag{86}$$

$$C_H\left[1 + ToT^{1-\phi}\left(\frac{1-\alpha}{\alpha}\right)\right] = Y_H, \tag{87}$$

$$C_H = Y_H\left[1 + ToT^{1-\phi}\left(\frac{1-\alpha}{\alpha}\right)\right]^{-1} \tag{88}$$

$$C_H = Y_H\left[\frac{\alpha}{\alpha + ToT^{1-\phi}(1-\alpha)}\right] \tag{89}$$

For :

$$C_H^* ToT^{-1} + C_F^* = Y_F, \tag{90}$$

$$C_F^* = \frac{1-\alpha^*}{\alpha^*}\left(\frac{1}{ToT}\right)^{\phi} C_H^*, \tag{91}$$

$$C_H^*\left[ToT^{-1} + ToT^{-\phi}\left(\frac{1-\alpha^*}{\alpha^*}\right)\right] = Y_F, \tag{92}$$

$$C_H^* = Y_F\left[\frac{\alpha^* ToT^{-1} + ToT^{-\phi}1-\alpha^*}{\alpha^*}\right]^{-1}, \tag{93}$$

$$C_H^* = Y_F\left[\frac{\alpha^*}{\alpha^* ToT^{-1} + ToT^{-\phi}(1-\alpha^*)}\right] \tag{94}$$

Balanced trade, and the law of one price, requires $\tau_t C_F = C_{H^*}$ in every period. Using relative demand:

$$ToT_t = \frac{\alpha^* \frac{P_t^*}{p_{H,t}}^{\phi} C_t^*}{(1-\alpha)\frac{P_t}{p_{F,t}}^{\phi} C_t} \tag{95}$$

Using autarky again:

$$ToT_t = \frac{\alpha^* \frac{P_t^*}{p_{H,t}}^{\phi} \frac{p_{F,t}^*}{P_t^*} Y_{F,t}^*}{(1-\alpha)\frac{P_t}{p_{F,t}}^{\phi} \frac{p_{H,t}}{P_t} Y_{H,t}} \tag{96}$$

Imposing $\alpha^* = (1-\alpha)$:

$$ToT_t = \underbrace{\frac{P_t^*}{P_t}^{\phi-1}}_{\mathcal{E}_t^{\phi-1}} \frac{Y_{F,t}^*}{Y_{H,t}} \underbrace{\frac{p_{F,t}}{p_{H,t}}^{1+\phi}}_{\tau_t^{1+\phi}} \tag{97}$$

So:

$$ToT_t^{-\phi} = \mathcal{E}_t^{\phi-1} \frac{Y_{F,t}^*}{Y_{H,t}} \tag{98}$$

Using a first order approximation and $q = (2\alpha - 1)\tau$,

$$\tau = \frac{y_H - y_F}{1 - 2\alpha(1-\phi)}, \tag{99}$$

$$\Delta e = (2\alpha - 1) * (g_{y_H} - g_{y_F}) \tag{100}$$

We take a limit of $\alpha \to 1$ implying:

$$c_t = c_{H,t} = y_{H,t} \tag{101}$$

$$c_t^* = c_{F_t}^* = y_{F,t} \tag{102}$$

We can then construct:

$$\eta_{t+1} = (g_{y_{H,t+1}} - g_{y_{F,t+1}}) \frac{1-s}{1-2\alpha(1-\phi)} \tag{103}$$

using (29).

Next, we show our approximated equilibrium model delivers the Backus-Smith puzzle and its resolution. In particular,

$$cov_t(m_{t+1}^* - m_{t+1}, \Delta e_{t+1}) = cov_t\left(-s(g_{y_{F,t+1}} - g_{y_{H,t+1}}), \frac{1}{1-2(1-\phi)}(g_{y_{H,t+1}} - g_{y_{F,t+1}}\right) < 0 \tag{104}$$

if $\phi < \frac{1}{2}$ – consistent with Corsetti et al. (2008). However, we are able to go a step further and explain why the mechanism goes through in the one-traded asset case. Notice that with no trade in assets– the coefficient of risk aversion s does not feature.

The figure below illustrates the range of parameters for which Proposition 2 is satisfied.

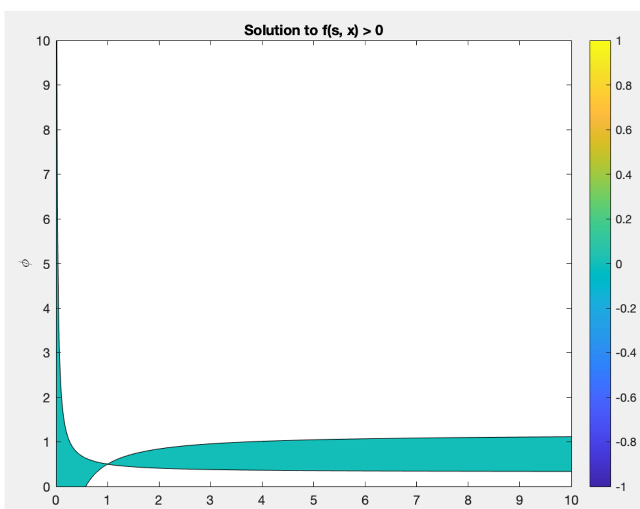

Figure 2: *Shaded region reflects parameters for which the model can satisfy the empirical Backus Smith correlation, at the limit of financial autarky and full home-bias.*

B. A calibrated two country open economy model

Below, we present a simple version of the endowment economy from Corsetti et al. (2008). Since only one bond is internationally traded, only equations (2) - (4) hold, so we refer to this model as the 3-Euler equation model.

The representative agent derives utility from consumption:

$$u(C_t) = \beta(C_t)\frac{C_t^{1-s}}{1-s} \tag{105}$$

where the consumption bundle is given by:

$$C_t = \left[\alpha^{\frac{1}{\phi}} C_{H,t}^{\frac{\phi-1}{\phi}} + (1-\alpha)^{\frac{1}{\phi}} C_{F,t}^{\frac{\phi-1}{\phi}}\right]^{\frac{\phi}{\phi-1}} \tag{106}$$

where ϕ is the trade elasticity. To ensure stationarity we use Uzawa (endogenous) discount factors, see Bodenstein (2011),

$$\beta(C_t) = \omega(C_{t-1})^{-u} \tag{107}$$

Home agents receive an endowment of their domestic good. They also invest in their domestic bonds and "an international bond" which pays in units of Home aggregate consumption and is zero in net supply. The Home agent faces the following budget constraint:

$$P_t C_t - P_{H,t} Y_{H,t} \leq R_t B_{t-1} - B_t + \mathcal{E}_t (R_t^* B_{t-1}^F - B_t^F) \tag{108}$$

Foreign agents face an analogous maximization but purchase only the Foreign bond.

Goods market clearing requires:

$$C_{H,t} + C_{H,t}^* = Y_{H,t} C_{F,t} + C_{F,t}^* = Y_{F,t}^*$$

where $Y_{H,t} = \rho Y_{H,t-1} + (1-\rho)Y_H + \epsilon$, $Y_{F,t}^* = \rho Y_{F,t-1}^* + (1-\rho)Y_F^* + \epsilon_t$. Bond market clearing requires:

$$B_t = 0,$$

$$B_t^* + B_t^F = 0$$

Returning to the financial side of the model, the Home agents' inter-temporal allocation satisfies:

$$\mathbb{E}_t[M_{t+1}R_{t+1}] = 1, \tag{109}$$

$$\mathbb{E}_t\left[M_{t+1}\frac{\mathcal{E}_{t+1}}{\mathcal{E}_t}R^*_{t+1}\right] = 1, \tag{110}$$

whereas Foreign agents face:

$$\mathbb{E}_t[M^*_{t+1}R^*_{t+1}] = 1 \tag{111}$$

The international risk sharing condition in the model is given by:

$$\begin{gathered}\mathbb{E}_t[M_{t+1}\frac{\mathcal{E}_{t+1}}{\mathcal{E}_t}] = \mathbb{E}_t\left[M^*_{t+1}\right] \leftrightarrow \\ \mathbb{E}_t\left[\beta\left(\frac{C_{t+1}}{C_t}\right)^{-s}\frac{\mathcal{E}_{t+1}}{\mathcal{E}_t}\right] = \mathbb{E}_t\left[\beta\left(\frac{C^*_{t+1}}{C^*_t}\right)^{-s}\right]\end{gathered} \tag{112}$$

Critically, if the Foreign risk-free bond was also traded then the second risk-sharing condition below would also need to be satisfied:

$$\mathbb{E}_t\left[\beta\left(\frac{C_{t+1}}{C_t}\right)^{-s}\right] = \mathbb{E}_t\left[\beta\left(\frac{C^*_{t+1}}{C^*_t}\right)^{-s}\frac{\mathcal{E}_t}{\mathcal{E}_{t+1}}\right] \tag{113}$$

Notice that (112) and (113) are the same if approximated to first order, but in general will imply significantly different results.

B.1. Quantitative results

We consider the following calibration: $\omega = 0.96, s = 1, \phi \in \{0.5, 1, 2\}, \alpha = 0.85, \alpha^* = 1 - \alpha, \rho = 0.96, Y_H = 1, u = 0.01$. We also contrast the model to the complete markets case, where (112) is replaced by (1).

Figure 3 below illustrates the Cole-Obstfeld result. The one bond economy perfectly approximates the complete markets allocation for $\phi = 1$. Specifically, $var(\eta_t) = 0$ and $corr_t(m^*_{t+1} - m_{t+1}, \Delta e_{t+1}) = 1$.

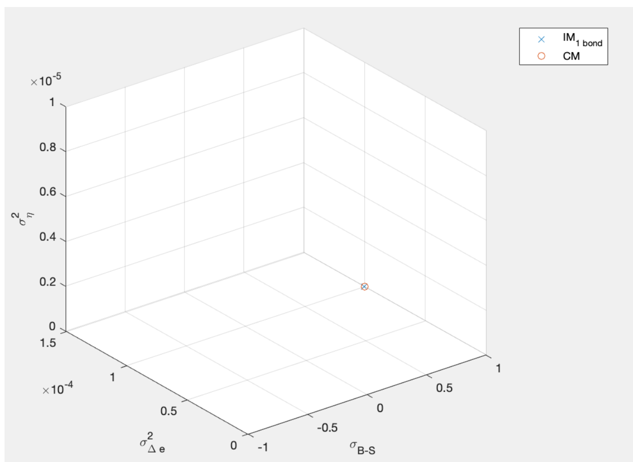

Figure 3: *Cole-Obstfeld parameterization.*

In this instance, financial markets are indeed irrelevant. Figure 4, contrasts the pattern of transmission, the volatility of the exchange rate and, critically, the volatility of non-traded risk for $\phi = 0.5$. The Backus-Smith correlation is significantly negative, the volatility of exchange rates rises and the volatility of the IM wedge rises.

Financial markets here matter – incompleteness allows the model to reconcile the data, but introducing a second internationally traded bond kills the result.

Finally, we evaluate what drives the negative Backus-Smith coefficient in the 3 Euler model,

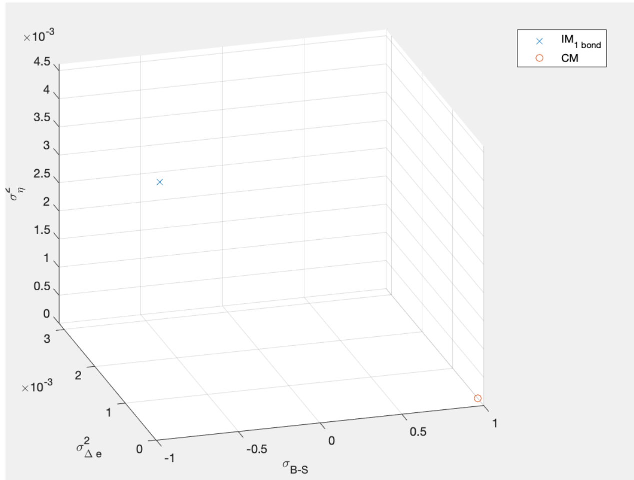

Figure 4: $\phi = 0.5$.

in view of conditions (10). Figure 3 evaluates the various quantities.

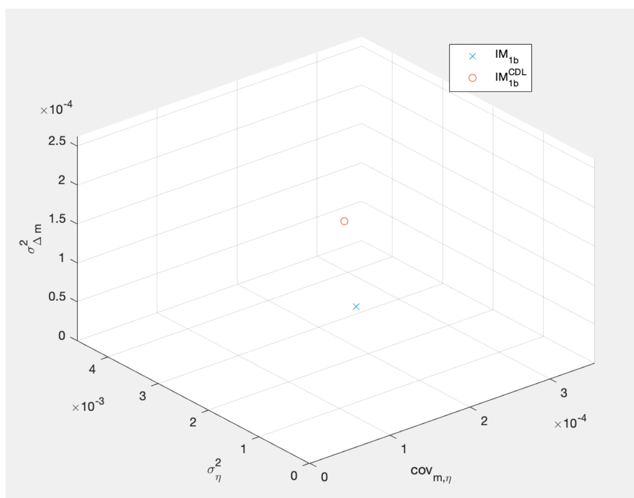

Figure 5: $\phi = 0.5$. *Evaluating Proposition 3.*

Lowering the trade elasticity, raises $cov_t(m_{t+1}, \eta_{t+1}$, lowers $var_t(m^*_{t+1} - m_{t+1})$ and increases $var_t(\eta_{t+1})$, all consistent with condition (10) being violated, so that $\rho^{BS} < 0$. However, the rise in $var_t(\eta_{t+1})$ is order of magnitude larger and therefore drives the result. Consistent with the description of the mechanism in Corsetti et al. (2008), the low trade elasticity prevents an increase in demand for Foreign goods following a Home income shock, therefore Home consumption rises without a fall in the Home price – increasing the volatility of the incomplete markets wedge (or non traded risk).