Incomplete Markets and Exchange Rates*

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Abstract

We show that imperfect risk-sharing *within* countries can reconcile the aggregate cyclicality of exchange rates in a two-country setting, i.e. the Backus-Smith puzzle, as long as exchange rates are sufficiently risky with respect to idiosyncratic states. Leveraging theory mapping household heterogeneity into measurable discount-factor wedges, we use household-level data to provide direct empirical support for our mechanism. We then embed this insight in an equilibrium asset-pricing model and show that the observed exchange rate dynamics require that, even when a country experiences higher consumption growth, idiosyncratic risk remains relatively elevated. In a quantitative exercise, we disentangle distinct roles for market incompleteness both *within* and *across* countries and match key moments of asset prices and exchange rates.

Keywords: Incomplete Markets, Heterogeneous agents, Exchange Rates.

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1. Introduction

We revisit the Backus-Smith condition (Kollmann, 1991; Backus and Smith, 1993) to consider the implications of domestic and international market incompleteness for exchange rate dynamics. This condition is a centerpiece of the international macro-finance literature, serving as a benchmark for risk sharing in flexible-price, representative-agent economies. Furthermore, recent contributions emphasize that a model's ability to reconcile the condition with the data is tightly connected to its success in matching other important moments, such as exchange rate volatility and predictability, and the cross-country correlation in pricing kernels (Chernov, Haddad and Itskhoki, 2024; Jiang, Krishnamurthy and Lustig, 2023).

Under complete financial markets, assuming power utility:

$$\left(\frac{C_{t+1}}{C^*_{t+1}}/\frac{C_t}{C^*_t}\right)^{\gamma} = \frac{\mathcal{E}_{t+1}}{\mathcal{E}_t} \tag{1}$$

where $C_t^{(*)}$ is Home (Foreign) aggregate consumption, $\mathcal{E}_t$ is the real exchange rate where an increase signifies a depreciation of Home currency, and γ is the inverse of the elasticity of intertemporal substitution (EIS). The Backus-Smith condition implies that an exchange rate depreciation coincides with periods of high consumption growth for investors in the Home country. In the data, however, exchange rates tend to appreciate when Home consumption rises, constituting the Backus-Smith *puzzle*.

Our motivation for this paper is driven by two advancements in the literature. First, we extend a canonical two-country framework to allow for imperfect domestic risk sharing—incomplete markets within countries. A large class of incomplete markets models admits an "as-if" representative agent Euler with a discount factor wedge between the pricer's SDF and aggregate consumption growth and we leverage recent advances in the literature (Berger, Bocola and Dovis, 2023) to measure these discount factor wedges directly from household-level data. Second, Lustig and Verdelhan (2019) show that, no-arbitrage restrictions from cross-border trade in at least one Home and one Foreign currency denominated (nominally) risk-free asset imply a strict positive comovement of exchange rates with relative stochastic discount factors (SDFs), even when international financial markets are incomplete and *regardless* of goods markets and other economy specifics, see also Benigno and Küçük (2012).[1] Building on this insight, we develop a distinction between the pricer's SDF relevant for asset prices and aggregate consumption growth– a wedge that arises naturally as a result of imperfect risk-sharing domestically. We show that this distinction is theoretically and quantitatively important for generating realistic exchange rate dynamics, while remaining consistent with domestic asset prices.

We make our case in three steps. First, we analytically derive a condition under which uninsurable risk within countries switches the cyclicality of exchange rates with respect to relative aggregate consumption growth, i.e. the Backus-Smith covariance. Second, we test this condition in reduced form using household level data. Third, we extend an equilibrium model of exchange rates (Lucas, 1982; Backus, Foresi and Telmer, 2001) to incorporate domestically incomplete markets and use it to quantify the mechanism.

Starting from a general heterogeneous agent economy with borrowing constraints, we derive a two-country, consumption-based asset pricing framework with uninsurable idiosyncratic risk within each country. The extended framework ties exchange rates to the pricing kernels of an "as-if" representative agent in each country where an endogenous discount factor wedge fully summarizes the effects of heterogeneity (Nakajima, 2005). As a result, when the as-if representative agent (in each country) prices both Home and Foreign bonds, international no-arbitrage conditions no longer impose a strictly positive co-movement between exchange rates and aggregate consumption growth. Leveraging this framework, we study exchange rates as determined exclusively by these Euler equations to assess the scope for market incompleteness (within and across countries) to reconcile key moments of international aggregates. This focus on Euler equations follows an established tradition in international finance, most recently used in Chernov et al. (2024) and Jiang et al. (2023).[2]

Our key result is that pro-cyclical exchange rates—consistent with the data—are fully compatible with no-arbitrage if and only if cross-country differences in discount-factor wedges

[1]Existing resolutions to the Backus-Smith puzzle based on a representative agent with separable utility (Corsetti, Dedola and Leduc (2008); Benigno and Thoenissen (2008) rely on restricting cross-border trade to a single risk-free asset, as we discuss in Appendix B.3.

[2]In general, asset pricing models solve for the exchange rate at which autarky interest rates are equated across countries, but allocations do not need to satisfy static goods-basket optimization. We explicitly detail wedges, i.e. trade costs (Fitzgerald, 2012), or shocks to home bias (Pavlova and Rigobon, 2007; Gabaix and Maggiori, 2015), required to clear goods markets.

are *sufficiently* negatively correlated with depreciations. The threshold is governed by the ratio of the volatility of exchange rates to the volatility of relative discount factor wedges. This result holds even under frictionless trade in two nominally risk-free bonds and in the presence of additional risky assets. Intuitively, when this condition is met, foreign bonds provide a poor hedge against idiosyncratic domestic risk: they deliver high payoffs precisely when domestic investors face low idiosyncratic risk. Exchange rates therefore become *risky* with respect to the *idiosyncratic* state. As such, our findings link the cyclicality of exchange rates to the classic diversification (or home-bias) puzzle (Baxter and Jermann, 1997; Heathcote and Perri, 2013).

Turning to the data, we test whether exchange rates are indeed risky with respect to the idiosyncratic state. Following Berger et al. (2023), we construct discount-factor ($\beta-$) wedges for the U.S. using household consumption shares from the Consumer Expenditure Survey. We identify high-income, low-net-worth households– those most likely to be unconstrained and to assign the highest prices to bonds– and label them *pricers*. For EIS values ≤ 0.3 (Best, Cloyne, Ilzetzki and Kleven, 2020), we robustly find that the $\beta-$wedge both co-moves sufficiently negatively with real exchange-rate growth and is itself sufficiently volatile for our mechanism to operate. We verify this condition both unconditionally and by constructing conditional moments that control for information observed at time t.

To disentangle the mechanisms underlying our empirical findings, we conduct two additional exercises. First, we show the relevant $\beta-$wedge dynamics are not driven by composition effects (i.e. the identity or characteristics of the pricer as in, e.g., Kollmann (2012)), but instead rely on within-group variation consistent with a risk channel. Second, by contrast, we show that a $\beta-$wedge constructed from measures of permanent idiosyncratic risk (Storesletten, Telmer and Yaron, 2004; Bayer, Luetticke, Pham-Dao and Tjaden, 2019), as was previously the norm in the literature, is not volatile enough to drive exchange rate cyclicality. Our findings therefore point to the importance of transitory risk, such as the possibility of becoming borrowing-constrained.

Furthermore, we confirm our results are robust to allowing for idiosyncratic risk abroad. We construct bilateral discount factor wedges ($\beta - \beta^*$) in two ways. First, where detailed individual consumption growth data is not available, we show how to construct annual wedges using growth in the income shares of high-income individuals (top 2.5%, 5%, 10% of the population) across seven advanced economies from the Global Repository of Income Dynamics (GRID) (Guvenen, Pistaferri and Violante, 2022). Second, in the case of Italy, we can use growth in consumption shares of high-income, low net worth households from the biennial Survey of Household Income and Wealth (SHIW) (Jappelli and Pistaferri, 2010, 2025). In both cases, we verify that the bilateral wedge co-moves sufficiently negatively with real exchange rate growth, confirming the plausibility of our mechanism.

Importantly, across all our exercises we find that the covariance between exchange rate depreciations and the relative valuation of the "as-if" representative agent (the pricer) is positive, in line with international no-arbitrage conditions implied by trade in nominally risk-free bonds (Lustig and Verdelhan, 2019). This holds even though the corresponding covariance constructed from aggregate consumption growth is negative.

Having established the mechanism in the data, we turn to an equilibrium asset pricing model of exchange rates to highlight two additional results. First, we show that the model can generate risky *equilibrium* exchange rates with respect to the idiosyncratic state, as required to resolve the Backus-Smith puzzle, while simultaneously accommodating countercyclical idiosyncratic risk (with respect to domestic aggregate consumption growth), see e.g. Storesletten et al. (2004). Countercyclicality implies that idiosyncratic consumption uncertainty is high—so the pricer's marginal utility growth rises more than that implied by aggregate consumption—precisely when average consumption growth is expected to be low. As a result, to deliver risky equilibrium exchange rates, the model requires that despite higher aggregate consumption growth in the Home country relative to Foreign, the Home $\beta-$wedge falls only modestly: the Home pricer remains concerned about future idiosyncratic risk (and vice versa for Foreign).

Second, we delineate the roles of domestic and international financial market incompleteness. Imperfect risk sharing within countries is necessary to generate exchange-rate cyclicality, but it is not generally sufficient once we allow for additional country-specific factors that imply an imperfect correlation of international stochastic discount factors. In such cases, incomplete international markets are also required. Moreover, international market incompleteness is quantitatively important for matching the volatility of exchange rates, which remains far lower than the volatility of stochastic discount factors when idiosyncratic risk is incorporated and parameters are calibrated to equity prices.

Our calibrated model matches moments from equity and bond markets, as well as the volatility and cyclicality of exchange rates for counter-cyclical idiosyncratic risk (with respect to aggregate consumption growth) and international spanning of 60% for the factor driving aggregate consumption in both countries. In our calibration, allowing for idiosyncratic risk increases exchange rate volatility but does not introduce excess predictability of exchange rates, delivering a near-zero R^2 in a regression of exchange rates on interest rate differentials. As a result, even though we only target the conditional Backus–Smith correlation, the model is also able to deliver a negative unconditional correlation, avoiding the trade-offs imposed by the representative agent model (Chernov et al., 2024; Jiang et al., 2023).

Related Literature Our work builds on literature in both international macro-finance and on incomplete market asset pricing. In the former, important contributions have shown that segmentation (Alvarez, Atkeson and Kehoe, 2002; Hassan, 2013; Sandulescu, Trojani and Vedolin, 2021) or intermediation frictions and UIP shocks (Gabaix and Maggiori, 2015; Itskhoki and Mukhin, 2021) can help resolve various exchange rate puzzles.[3] Recently, Jiang, Krishnamurthy and Lustig (2023); Jiang, Krishnamurthy, Lustig and Sun (2024) confront exchange rate anomalies with exogenous wedges in Euler equations, intended to capture home bias, intermediation, or convenience yields.

We extend these contributions to consider imperfect risk-sharing within countries, which can be represented in the form of discount factor wedges, see e.g. Nakajima (2005); Krueger and Lustig (2010); Werning (2015); Guerrieri and Lorenzoni (2017); Berger et al. (2023) for closed economy applications. In contemporaneous work, Kekre and Lenel (2024) show that *exogenous* permanent discount factor shocks (Stockman and Tesar, 1995) can resolve comovement and predictability puzzles in open economies.

More generally, our model with heterogeneous consumers ties to a large literature investigating whether idiosyncratic risk affects macro-finance aggregates (see, e.g. Mankiw, 1986;

[3] Relatedly, there is a large literature in international macro-finance that studies role of non-separabilities in reconciling various exchange rate puzzles. See eg: Verdelhan (2010); Karabarbounis (2014); Colacito and Croce (2013); Farhi and Gabaix (2016).

Weil, 1992; Guvenen, 2009; Kaplan, Moll and Violante, 2018; Auclert, Rognlie and Straub, 2024; Challe, 2020; Di Tella, Hébert and Kurlat, 2024). In the open economy macro literature, for papers with agent heterogeneity, see Ghironi (2006); Kocherlakota and Pistaferri (2007); Kollmann (2012); Hassan (2013); De Ferra, Mitman and Romei (2020); Auclert, Rognlie, Souchier and Straub (2021); Acharya and Challe (2025); Acharya, Challe and Coulibaly (2025) amongst others. Our contribution relative to these models is to emphasize the riskiness of exchange rates with respect to idiosyncratic states and derive a condition which we directly take to household-level data. There has also been renewed recent interest in the cyclicality of exchange rates. Huang, Kogan and Papanikolaou (2025) connect cyclicality of exchange rates to technological innovation based displacement risk faced by shareholders.

A closely related contribution is Kocherlakota and Pistaferri (2007) who study how particular structures of domestically incomplete markets (but internationally complete) impose restrictions on the comovement of bilateral higher order moments of the consumption distribution and exchange rates.[4] Relative to their contribution, we rely on discount factor wedges, which can capture a wider range of market structures and are measurable in the data, and we emphasize a distinct but complementary role for international market incompleteness.

The rest of the paper is organized as follows. Section 2 details our theoretical framework. Section 3 derives our key condition for imperfect domestic risk-sharing to result in pro-cyclical exchange rates. Section 4 provides evidence for this condition from household-level data for the U.S. and abroad. Section 5 provides analytical results from an equilibrium model for exchange rates and Section 5.1 conducts a quantitative exercise. Section 6 concludes.

2. Two-country, Consumption Based Asset Pricing Model

We begin with a two-country heterogeneous agent framework where heterogeneity within each country can be entirely summarized by a discount factor wedge, capturing a large class

[4] Kocherlakota and Pistaferri (2007) consider two market structures: one with incomplete markets and non-binding borrowing constraints, and second with optimal insurance contracts under private information. Under these two arrangements, all agents in the economy are either on their consumption Euler equation or on an *inverse* Euler equation (for details, see Kocherlakota and Pistaferri, 2009). In contrast, our setting allows for uninsurable risk with potentially binding borrowing constraints more generally. Relatedly, Leduc (2002) using a calibrated open economy model featuring idiosyncratic and aggregate risk investigated the role of idiosyncratic risks in generating currency premia.

of domestically incomplete asset market structures (Nakajima, 2005; Berger et al., 2023). We then operationalize this framework as a consumption-based asset pricing model, widely used in international finance (e.g. Backus et al. (2001); Lustig and Verdelhan (2019)), and investigate the implications for exchange rate cyclicality.

Time is discrete and infinite. There are two types of states, aggregate z_t and idiosyncratic ν_t. Their respective histories are denoted by $z^t = (z_0, z_1, ..., z_t)$ and $\nu^t = (\nu_0, \nu_1, ..., \nu_t)$ and $s^t = (z^t, \nu^t)$ summarizes the joint history.[5] We focus on time-separable CRRA utility. Each agent with joint history s^t derives per-period utility from consumption:

$$u(C(s^t)) = \beta^t \, \frac{C(s^t)^{\,1-\gamma}}{1-\gamma} \tag{2}$$

where β is the constant discount factor, γ is the inverse elasticity of intertemporal substitution, and we abstract from labor supply considerations.[6] We define $M(z^{t+1})$ as the SDF based on Home aggregate consumption (corresponding to a representative agent) and M^*_{t+1} denotes the representative SDF abroad such that $M^{(*)}(z^{t+1}) = \beta \left(\frac{C^{(*)}(z^{t+1})}{C^{(*)}(z^t)}\right)^{-\gamma}$, as in condition (1).

Real exchange rate fluctuations necessitate differentiated consumption bundles across countries.[7] The Home consumption bundle and associated price index are given by

$$C^\nu_t = \left[\alpha^{\frac{1}{\zeta}} c^{\nu\,\frac{\zeta-1}{\zeta}}_{H,t} + (1-\alpha)^{\frac{1}{\zeta}} c^{\nu\,\frac{\zeta-1}{\zeta}}_{F,t}\right]^{\frac{\zeta}{\zeta-1}}, \quad P_t = \left[\alpha p^{\frac{\zeta-1}{\zeta}}_{H,t} + (1-\alpha) p^{\frac{\zeta-1}{\zeta}}_{F,t}\right]^{\frac{\zeta}{\zeta-1}} \tag{3}$$

where ζ is the trade elasticity and α is the measure of home-bias. Foreign quantities and prices are defined symmetrically, with home bias $1-\alpha$. The real exchange rate is given by $\mathcal{E} = P^*/P$.

The individual Home households' budget constraint is given by:

$$\begin{aligned} C(s^t) - I(s^t) \leq R(z^t)B_H(s^{t-1}) - B_H(s^t) + \mathcal{E}_t(z^t)R^*(z^t)B_F(s^{t-1}) - \mathcal{E}(z^t)B_F(s^t) \\ + \int_{j\in J} \tilde{R}^j(z^t)X^j(s^{t-1}) - \int_{j\in J} X^j(s^t), \end{aligned} \tag{4}$$

where $I(s^t) = \frac{P_H(z^t)}{P(z^t)} I_H(s^t, \nu^t) + \frac{P_F(z^t)}{P(z^t)} I_F(s^t, \nu^t)$ is the income drawn by an agent with individual

[5]We use the convention that state vectors subsume Home and Foreign shocks, i.e. $z_t = \{u^z_t, u^{z*}_t\}, \nu_t = \{u^\nu_t, u^{\nu*}_t\}$.

[6]In models with separable utility functions, labor supply considerations don't affect the consumption Euler equations relevant for our analysis, see Berger et al. (2023) for a further discussion.

[7]We abstract from inflation considerations since in our sample of advanced economies inflation is not volatile enough to explain exchange rate anomalies.

history ν^t, $B_H(s^t)(B_F(s^t))$ is the position in Home (Foreign) risk-free bonds and $R^{(*)}(z^t)$ is the corresponding returns and $X^j(s^t)$ denotes risky assets indexed by j held by agent with history ν^t.

Following Berger et al. (2023), we specify borrowing constraints in a general manner:

$$\mathcal{H}(B_H(s^t), B_F(s^t), \{X^j(s^t)\}_{j\in J}) \geq 0, \tag{5}$$

for some vector-valued function $\mathcal{H}$. We make two assumptions: i) we assume $\frac{d\mathcal{H}}{dB} > 0$, i.e. that purchasing risk-free domestic bonds weakly relaxes the borrowing constraint, ii) we consider the limit of zero Foreign liquidity ($B_F \to 0$). The borrowing constraints are analogously defined for Foreign households.

The individuals' SDF $M^{(*)}(s^{t+1})$ is instead defined on individual consumption growth $\frac{C^{(*)}(s^{t+1})}{C^{(*)}(s^t)}$, which is related to aggregate consumption growth as follows:

$$\frac{C(s^{t+1})}{C(s^t)} = \frac{\delta(s^{t+1})}{\delta(s^t)}\frac{C_{t+1}(z^{t+1})}{C_t(z^t)} \tag{6}$$

where $\delta(s^t)$ satisfies the law of large numbers $\int_{\nu^t}\delta(z^t,\nu^t)d\nu^t \;= 1\ \forall z^t$.

We assume that Home and Foreign households can trade domestic and foreign risk-free real bonds with returns $R(z^t)$ and $R^*(z^t)$ (in Foreign currency) respectively. We begin with the case where trade is frictionless (i.e. no borrowing constraints) and relax this later. By no-arbitrage, the household Euler implies:

$$\mathbb{E}_t[M(s^{t+1})] = \frac{1}{R(z^t)},$$

where $\mathbb{E}_t[X(s^{t+1})] = \mathbb{E}[X(s^{t+1})|s^t] = \sum_{s^{t+1}} Pr(s^{t+1}|s^t)X(s^t, s_{t+1})]$ with $Pr(\cdot)$ denoting transition probabilities. Using (6), the Euler for a Home household investing in the Home bond can be expressed as:

$$\mathbb{E}\left[\underbrace{\beta\mathbb{E}\left[\left(\frac{\delta(z^{t+1},\nu^{t+1})}{\delta(z^t,\nu^t)}\right)^{-\gamma}\middle|\nu^t, z^{t+1}\right]}_{\equiv\beta(z^{t+1},\nu^t)} \times \left(\frac{C(z^{t+1})}{C(z^t)}\right)^{-\gamma}\middle|z^t\right] = \frac{1}{R(z^t)} \tag{7}$$

where $\mathbb{E}[X(s^{t+1})|\nu^t, z^{t+1}] = \sum_{\nu^{t+1}} Pr(\nu^{t+1}|\nu^t, z^t)X(v^{t+1}, z^{t+1})$ and $\mathbb{E}[X(z^{t+1},\nu^t)|s^t] = \sum_{z^{t+1}} Pr(z^{t+1}|z^t)X(z^{t+1},\nu^t)$. Intuitively, all agents trading the Home bond agree on its price and we define $\beta(z^{t+1},\nu^t)\times\left(\frac{C(z^{t+1})}{C(z^t)}\right)^{-\gamma}$ as the "pricer's" SDF, sometimes referred to as the "as-if" representative agent, e.g. Werning (2015).

Borrowing Constraints. In practice, markets are far from frictionless and borrowing constraints are key for generating sufficient volatility of idiosyncratic risk. Suppose now that agents face constraint (5). We assume there exists at least one agent, within each country, who is unconstrained and this will be the most "patient" consumer who maximally values the risk-free bond. In the presence of borrowing constraints, (7) is replaced by:[8]

$$\mathbb{E}_t\Bigg[\underbrace{\{\max_{\nu^t}\beta(z^{t+1},\nu^t)\}\times\left(\frac{C(z^{t+1})}{C(z^t)}\right)^{-\gamma}}_{\equiv\widehat{M}(z^{t+1},s^t)}\Bigg]=\frac{1}{R(z^t)} \tag{8}$$

Analogously, the aggregate Foreign Euler for the Foreign bond can be expressed as:

$$\mathbb{E}_t[\underbrace{\{\max_{\nu^t}\beta^*(z^{t+1},\nu^t)\}\left(\frac{C^*(z^{t+1})}{C^*(z^t)}\right)^{-\gamma}}_{\equiv\widehat{M}^*_{t+1}(z^{t+1},s^t)}]=\frac{1}{R^*(z^t)} \tag{9}$$

If the *same* Home (Foreign) households also trade the Foreign (Home) bond, denoting the real exchange rate by $\mathcal{E}(z^t)$, such that an increase corresponds to a depreciation of Home currency, then we additionally obtain the following two Euler conditions:

$$\mathbb{E}_t\left[\widehat{M}(z^{t+1},s^t)\,\frac{\mathcal{E}(z^{t+1})}{\mathcal{E}(z^t)}\right]=\frac{1}{R^*(z^t)}, \tag{10}$$

$$\mathbb{E}_t\left[\widehat{M}^*(z^{t+1},s^t)\,\frac{\mathcal{E}(z^t)}{\mathcal{E}(z^{t+1})}\right]=\frac{1}{R(z^t)} \tag{11}$$

The "micro" modules pertaining to idiosyncratic risk (ν^t) and financial market structure, within each country, can be summarized by the discount wedges $\tilde{\beta}^{(*)}(z^{t+1},\nu^t)$ referred to as a β-*wedge*. Crucially, we can measure the $\beta-$ wedge from the micro-data on consumption following Berger et al. (2023).[9]

International Equilibrium and Goods Market Clearing To turn the asset pricing model into an equilibrium model, we need to ensure international goods and asset markets clear. Taking any foreign aggregate endowment process as given, we solve for an exchange rate process such that agents optimally choose not to trade across borders. More generally,

[8] Even in a setting without assumptions on (5), Krusell, Mukoyama and Smith (2011) derive conditions under which it is the most patient –specifically the individual with the most to lose– who prices the assets.

[9] In Section 4 we show that while the $\beta-$ wedge we estimate is sufficiently volatile to rationalize exchange rate dynamics, competing models with permanent risk, derived from a Constantinides and Duffie (1996) economy with integrated markets, cannot deliver sufficient volatility, see e.g. Lettau (2002).

the endowment processes we consider can be interpreted as the wealth of consumers after all gains from trade have been exhausted.[10] The adjusted risk-sharing condition is then given by:

$$\mathbb{E}_t\left[\tilde{\beta}_{t+1}\left(\frac{C_{t+1}}{C_t}\right)^{-\gamma}\frac{\mathcal{E}_{t+1}}{\mathcal{E}_t}\right]=\mathbb{E}_t\left[\tilde{\beta}^*_{t+1}\left(\frac{C^*_{t+1}}{C^*_t}\right)^{-\gamma}\right]. \tag{12}$$

Replacing C_t with $I_t = \int_\nu I_t^\nu$ and C_t^* with $I_t^* = \int_{\nu^*} I_t^{\nu\,*}$, (12) implies equalization of the autarky rates across countries resulting in a no trade equilibrium (Svensson, 1988).

The equilibrium characterized by (12) does not always lie on the static Pareto frontier, i.e. is not consistent with consumption bundle optimization at prices given by (3), but we show it can always be supported with static goods market wedges, isomorphic to home bias shocks (Pavlova and Rigobon, 2007; Gabaix and Maggiori, 2015), described below.[11]

Lemma 1 (Goods Market Clearing) *Given processes for $\{I_t, I_{H,t}, I_{F,t}, \tau_t\}$ and the Pareto Frontier $\{c_H(I_t), c_F(I_t)\}$, goods markets clear if and only if:*

$$\left[I_t^{\frac{\zeta-1}{\zeta}}-\alpha^{\frac{1}{\zeta}}c_H(I_t)^{\frac{\zeta-1}{\zeta}}\right]^{\frac{\zeta}{\zeta-1}}\left(b\hat{I}_{H,t}+(1-b)c_H(I_t)\right)=c_H(I_t)\hat{I}_{F,t}(1-\alpha)^{\frac{1}{\zeta-1}} \tag{13}$$

where $b=(\frac{\alpha}{1-\alpha})^2(1+\tau_t)^\zeta, \tau_t>-1$. In the limit of full home bias $\alpha\to 1$, a zero wedge ($\tau_t=0$) satisfies (13).

Proof: See Appendix A.

3. Exchange Rate Cyclicality

We now investigate the main theoretical implication of our model for cyclicality of exchange rates with respect to relative aggregate consumption growth as well as the relative as-if stochastic discount factors.

We assume $\beta-$wedges, SDFs and prices are jointly log-normal. Moreover, since equations (7)-(11) depend on the joint history s^t and are only uncertain with respect to the future

[10]We follow the zero liquidity approach popular in the heterogeneous agents literature, e.g. Krusell et al. (2011); Werning (2015); Challe (2020); Bilbiie (2024) among others; for its analytical tractability in characterizing SDFs and prices.

[11]While this may sound undesirable, such wedges are non-zero in the data and play a significant role in impeding risk sharing (Fitzgerald, 2012; Bodenstein, Cuba-Borda, Gornemann and Presno, 2024). Such goods markets frictions alone are unable to explain the Backus-Smith puzzle (Lustig and Verdelhan, 2019). Moreover, leading contributions in the literature (e.g. Colacito and Croce, 2013) assume a home bias of 0.97, where Lemma 1 implies $\tau_t \to 0$.

aggregate state z^{t+1}, we drop notation on histories and denote $X(s^t) = X_t, X(z_{t+1}, s^t) = X_{t+1}$, without loss of generality. To close the model, we pin down an exchange rate process consistent with equations (7)–(11) above, which reduces to finding an exchange rate process that satisfies:

$$cov_t(\widehat{m}^*_{t+1} - \widehat{m}_{t+1}, \Delta e_{t+1}) = var_t(\Delta e_{t+1}) \geq 0 \tag{14}$$

where $x = \log(X)$. International no-arbitrage requires that, regardless of the specific structure of international financial markets (i.e. complete or incomplete), exchange rate depreciations coincide with a period where the "pricer's" valuation of returns is low, i.e. exchange rates are risky consistent with international no-arbitrage. We later test this model implication and find evidence in the data of positive comovement between exchange rates and pricers' SDFs, see Section 4. Even stricter, we will calibrate inter-temporal elasticity of substitution while recovering pricers' SDFs from micro data so that condition (14) is satisfied.

Naturally, the process corresponding to complete international financial markets ($\Delta e_{t+1} = \widehat{m}^*_{t+1} - \widehat{m}_{t+1}$) is one candidate to satisfy (14). More generally, as shown in Backus, Foresi and Telmer (2001), the following process also satisfies equation (14):

$$\Delta e_{t+1} = \widehat{m}^*_{t+1} - \widehat{m}_{t+1} + \eta_{t+1} \tag{15}$$

where η_{t+1} is the international incomplete markets wedge which must satisfy certain conditions imposed by asset trade.[12] The wedge, η, is often interpreted as the non-traded component of exchange rate movements or the wealth gap, see e.g. Pavlova and Rigobon (2007); Corsetti et al. (2008); Corsetti, Dedola and Leduc (2023). The special case of a representative agent economy corresponds to the limit $\tilde{\beta}^{(*)}_{t+1} \equiv \log \beta^{(*)}_{t+1} \to \log \beta^{(*)}$ which implies $\widehat{m}^{(*)}_{t+1} = m^{(*)}_{t+1}$. When the economy features a representative agent, condition (14) restricts the covariance between relative consumption growth and exchange rate depreciations to be positive as in (1). In the data, this covariance is negative, hence the Backus-Smith *puzzle*.

The proposition below specifies the conditions for imperfect risk sharing *within* countries to help reconcile the aggregate cyclicality of exchange rates.

Proposition 1 (Two Int'l Traded Assets, Many Agents).

[12] Cross-border trade by the pricer in Home and Foreign bonds yields restrictions (32) and (33) on η_{t+1} reflecting the risk-return trade-off for investors, see Appendix B.1. Using these conditions Lustig and Verdelhan (2019) show that international incompleteness cannot change the sign of the Backus-Smith covariance.

The two-country model with two internationally traded bonds and heterogeneous consumers, characterized by Equations (8), (9), (10) and (11), delivers $cov_t(\Delta c_{t+1} - \Delta c^*_{t+1}, \Delta e_{t+1}) < 0$ *if and only if:*

$$1 \geq -\rho_{\tilde{\beta}_{t+1}-\tilde{\beta}^*_{t+1},\Delta e_{t+1}} \geq \frac{\sigma_t(\Delta e_{t+1})}{\sigma_t(\tilde{\beta}_{t+1} - \tilde{\beta}^*_{t+1})} \tag{16}$$

where $\rho_{\tilde{\beta}_{t+1}-\tilde{\beta}^*_{t+1},\Delta e_{t+1}} \equiv \dfrac{cov_t(\tilde{\beta}_{t+1} - \tilde{\beta}^*_{t+1}, \Delta e_{t+1})}{\sigma_t(\Delta e_{t+1})\sigma_t(\tilde{\beta}_{t+1} - \tilde{\beta}^*_{t+1})}$.

Proof. See Appendix B.1. □

Consider first the limiting case where the Foreign country is populated by a representative agent ($\beta^*_{t+1} = \beta$). Then, Proposition 1 requires that Foreign bonds are a poor hedge for domestic *idiosyncratic* consumption risk– specifically, the $\beta-$wedge is low during periods of depreciation such that the "pricer" values payoffs less at times when the returns on foreign bonds are high. This poor hedging property applies to any foreign currency denominated return and is a possible solution to the *diversification puzzle*, documented in Heathcote and Perri (2013). Intuitively, given that no-arbitrage only requires that exchange rates are counter-cyclical with respect to the pricers' SDFs (14), if exchange rates are sufficiently counter-cyclical with respect to the relative $\beta-$wedge, they can be pro-cyclical with respect to SDFs constructed with aggregate consumption growth.

To illustrate what it means for exchange rates to be risky, in Appendix B.4 we present two tractable models for the "pricer's" kernel, (i) a model where markets are fully integrated and yet agents do not want to trade any assets (Constantinides and Duffie, 1996), and (ii) a two agent model where the investor faces a probability of becoming borrowing constrained (Krusell et al., 2011; Bilbiie, 2024). In (i), Foreign bonds are risky with respect to the $\beta-$wedge if they pay returns (depreciation) when the (cross sectional) volatility of idiosyncratic income risk is low. In (ii), the $\beta-$wedge is risky because the probability of becoming constrained (and receiving a lower income) is lower in periods of depreciation. Additionally allowing for imperfect risk sharing abroad ($\tilde{\beta}^*_{t+1} \neq \log \beta^*$), either Home bonds must also be a poor hedge for foreigners, or if they are safe, they are less so than Foreign bonds are risky for Home households.[13]

[13] Moreover, we note that (16) is a sufficient condition as long as (at least) two risk-free bonds are traded

Limits to International Arbitrage. A prominent literature argues that limits to international arbitrage, e.g. intermediation subject to portfolio constraints Gabaix and Maggiori (2015); Itskhoki and Mukhin (2021); Chernov et al. (2024); Jiang et al. (2023) can explain moments of exchange rates. To compare this mechanism to ours, we consider a variant of our model where both Home and Foreign households are subject to uncertain intermediation shocks ($e^{u^f_{t+1}}$) affecting their returns on foreign portfolios, see Appendix B.2 for details. Allowing for costly intermediation, (16) is replaced by:

$$1 \geq -\rho_{\tilde{\beta}_{t+1}-\tilde{\beta}^*_{t+1},\Delta e_{t+1}} \geq \frac{(1-u^f)\sigma_t(\Delta e_{t+1})}{\sigma_t(\tilde{\beta}_{t+1}-\tilde{\beta}^*_{t+1})} \tag{17}$$

where $u^f \in [0,1]$ is the share of exchange rate volatility stemming from intermediation shocks $var_t(u^f_{t+1}) = u^f \times var_t(\Delta e_{t+1})$. Importantly, the u^f_{t+1} shock is assumed to be orthogonal to other shocks in the economy, hence it does not affect the co-movement of relative consumption and the exchange rate on its own. However, since a fraction of exchange rate volatility now does not stem from $\widehat{m}$ or $\widehat{m}^*$, for a given exchange rate volatility, intermediation shocks dampen the threshold, reinforcing our mechanism.

Representative agent limit ($\tilde{\beta}^{(*)} \to \log \beta^{(*)}$). Finally, before proceeding, we contrast our results to the representative agent limit where the correlation between the relative $\beta-$wedge and exchange rates is zero. In this case, the Backus-Smith covariance can *never* be negative (Lustig and Verdelhan, 2019): condition (16) becomes an impossibility. To get around this stark result, classical contributions restrict attention to the case where only a single bond is internationally traded, i.e. Home or Foreign currency denominated. We investigate this in Appendix B.3.

In the remainder of the paper, we build on condition (16) both empirically and theoretically. In Section 4, we test condition (16) using household-level data on consumption and income. Section 5 solves for equilibrium exchange rates in a two-country Lucas (1982); Cox, Ingersoll and Ross (1985) framework with $\beta-$wedges and we derive conditions required for (16) to be satisfied. We also detail that the completeness of international financial markets *also* matters for exchange rate dynamics.

internationally and we detail a generalization of the framework with trade in risky assets in Appendix B.5.

4. Empirical Evidence

In this section, we measure the discount factor wedges from annual micro-data for U.S. households following Berger et al. (2023), henceforth BBD. Then we assess the plausibility of our mechanism for reconciling international risk sharing patterns by assessing whether condition (16) is met: how risky are exchange rates with respect to the relative $\beta-$wedge in the data? We show robustness of our results by constructing wedges for Italy using biennial consumption panel data, as well as income-based discount factor wedges for seven countries using growth in income shares data.

Wedge measurement. Beginning with the U.S., we use the Consumer Expenditure Survey. We want to measure the discount factor wedge defined in equations (7) and (8) which requires constructing the conditional expectation over idiosyncratic states of marginal utility growth for a currently unconstrained individual facing idiosyncratic risk. Following BBD, we measure this wedge using the cross-sectional average of marginal utility growth of currently unconstrained individuals. First, for each household ν, we define the consumption share $\varphi_t^\nu = \frac{C_t^\nu}{C_t}$ and construct the growth rate of relative marginal utility of a household as $\left(\frac{\varphi_{t+1}^\nu}{\varphi_t^\nu}\right)^{-\gamma}$.[14] Then, at each date t, we collect households with similar levels of income and net worth into distinct groups. For each group of households, we compute a $\beta-$wedge as the average across N_g individuals in group g:

$$\beta_{g,t+1} \equiv \frac{1}{N_g} \sum_{\nu=1}^{N_g} \left(\frac{\varphi_{t+1}^\nu}{\varphi_t^\nu}\right)^{-\gamma} \tag{18}$$

Finally, we take logs and demean such that $\tilde{\beta}_{g,t+1} \equiv \log\left(\beta_{g,t+1}\right) - \overline{\log(\beta_g)}$.

In the presence of borrowing constraints, the theory suggests to select the maximal $\beta-$wedge, see (8). As in BBD, we choose high income, low net worth individuals who are likely to be unconstrained today, but face the possibility of becoming constrained in the future. As such, these households have a high incentive to save for precautionary reasons. We deviate from BBD who assume an EIS of 1 and we construct wedges for $\gamma^{-1} \in [0.3, 0.2, 0.1]$. Best

[14] As in BBD, we use residualized log consumption constructed by partialling out fixed effects for sex, race, education, age of the head of household, and the state of residence. All nominal variables are converted into 2000 dollars with CPI-U.

Table 1: *Summary Statistics for* $\tilde{\beta}_g$

γ^{-1} (EIS)	1	0.3	0.2	0.1
Corr($\tilde{\beta}_g, \Delta \log Y$)	-0.05	-0.60	-0.61	-0.61
Corr($\tilde{\beta}_g, \Delta \log C$)	0.03	-0.51	-0.54	-0.56
$\sigma(\tilde{\beta}_g)$	0.04	0.58	1.29	3.05

et al. (2020) find that $\gamma^{-1} = 0.1$ best fits household-level data and this is further supported by evidence in Di Tella, Hébert, Kurlat and Wang (2023). We use $\gamma^{-1} = 0.3$ in our baseline because the SDFs recovered satisfy the equality in condition (14), and report robustness in Appendix C. For the U.S., we obtain an annual time-series of $\tilde{\beta}_{t+1}$ spanning 1992–2017. Table 1 provides summary statistics on the cyclicality and volatility of this wedge.[15]

Measurement of the $\beta-$wedge requires panel data on household-level consumption growth, income, and a measure of net worth. We proceed in three ways. First, we present results assuming that time-varying idiosyncratic risk in the U.S. only, i.e. $\tilde{\beta}^*_{t+1} = \log \beta$ for the foreign country. We have annual time series of this wedge for the US. Second, we generalize our empirical framework to allow for idiosyncratic risk abroad using (i) a comparable consumption-based wedge for Italy from SHIW data (Jappelli and Pistaferri, 2010), and (ii) cross-country distributional income statistics from the Global Repository of Income Dynamics, provided by Guvenen et al. (2022). Section 4.1 shows similar results are obtained with alternate wedges.

We construct real exchange rates from annual nominal exchange rates (US dollar/ Foreign currency) and consumer price indices using the Macro-History database (Jordà, Schularick and Taylor, 2017) for seventeen advanced economies.

Results. We highlight five findings. First, using U.S. household data on consumption, the constructed $\beta-$wedge is sufficiently negatively correlated with depreciations and sufficiently volatile relative to exchange rates for condition (16) to be satisfied, giving support to our main result. Second, we show that depreciations correlate positively with the relative pricing kernels adjusted for this $\beta-$wedge, consistent with no-arbitrage from international trade in two risk-free assets (14), though they correlate negatively with pricing kernels constructed

[15] Kozliakov, Marin and Singh (2025) revisit Lettau (2002) and show that these measured discount rate wedges generate large enough volatility bounds to match the observed U.S. equity Sharpe ratios.

Table 2: *Empirical Moments: Real Exchange Rate Growth and Berger et al. (2023) Wedge*

ISO	Unconditional		Conditional	
	$-\text{Corr}(\tilde{\beta}, \Delta e)$	Threshold	$-\text{Corr}_t(\tilde{\beta}, \Delta e)$	Threshold$_t$
AUS	0.09	0.23	0.17	0.18
BEL	0.24	0.18	0.30	0.19
CAN	0.33	0.17	0.25	0.15
CHE	-0.01	0.16	0.03	0.14
DEU	0.23	0.18	0.32	0.16
DNK	0.25	0.18	0.40	0.20
ESP	0.33	0.20	0.20	0.17
FIN	0.26	0.21	0.27	0.18
FRA	0.26	0.18	0.31	0.18
GBR	0.37	0.19	0.58	0.17
IRL	0.25	0.15	0.08	0.13
ITA	0.36	0.20	0.28	0.20
JPN	-0.34	0.19	-0.35	0.18
NLD	0.25	0.18	0.22	0.19
NOR	0.26	0.21	0.46	0.22
PRT	0.30	0.19	0.10	0.17
SWE	0.32	0.23	0.31	0.18
AVERAGE	0.22	0.19	0.23	0.17

on aggregate consumption only. Third, we highlight that the correlation with depreciations and variance of the wedge are not driven by a composition effect (i.e. time-varying relative consumption of pricers) but arise because of within-group dispersion capturing risk. Fourth, we show that an alternative construction of the $\beta-$wedge relying on permanent risk (71) is not volatile enough to deliver our results. Last but not least, we leverage international household-level income data and show the constructed bilateral $\beta-$wedges also satisfy (16).

Table 2 reports moments of the data relevant for Proposition 1, namely, the (negative of the) correlation of the wedge with real exchange rate growth and the threshold $\frac{\sigma(\Delta e)}{\sigma(\beta)}$. We report both conditional and unconditional moments. To construct conditional moments we control for date t values of relative log consumption in US and Foreign $(c_t - c_t^*)$, relative short term nominal interest rate $(i_t - i_t^*)$, relative CPI based inflation rates $(\pi_t - \pi_t^*)$, and log bilateral real exchange rate (e_t).

For fifteen of the seventeen bilateral pairs, the exceptions being Japan and Switzerland, we find that bilateral exchange rates are risky with respect to the $\tilde{\beta}$ wedge – i.e. a negative correlation indicating depreciations tend to happen during periods of low valuation by the pricer $(\tilde{\beta})$. Our results are robust to lower degrees of risk aversion (specifically any $\gamma^{-1} \leq 0.33$),

Table 3: *Correlations between Pricing Kernels and Real Exchange Rate Growth*

	Unconditional		Conditional	
ISO	Corr($\Delta c - \Delta c^*, \Delta e$)	Corr($m^* - \widehat{m}, \Delta e$)	Corr$_t$($\Delta c - \Delta c^*, \Delta e$)	Corr$_t$($m^* - \widehat{m}, \Delta e$)
AUS	-0.39	0.07	-0.40	0.14
BEL	-0.30	0.23	-0.44	0.27
CAN	-0.12	0.32	-0.18	0.24
CHE	-0.09	-0.01	-0.23	0.02
DEU	0.08	0.23	-0.03	0.31
DNK	-0.34	0.22	-0.42	0.36
ESP	-0.01	0.33	-0.00	0.21
FIN	-0.38	0.21	-0.32	0.25
FRA	-0.25	0.25	-0.34	0.28
GBR	0.07	0.38	-0.09	0.59
IRL	0.08	0.27	0.52	0.12
ITA	0.01	0.36	0.06	0.28
JPN	0.19	-0.32	0.05	-0.34
NLD	0.19	0.26	-0.12	0.21
NOR	-0.27	0.24	-0.63	0.42
PRT	0.21	0.32	0.09	0.11
SWE	-0.27	0.30	-0.28	0.30
AVERAGE	-0.09	0.21	-0.16	0.22

to alternate conditioning sets, and are also similar from a pooled regression with country fixed effects instead of constructing conditional moments using country level regressions, see Appendix C.1 for details.

Furthermore, while the Backus-Smith correlation is, on average, slightly negative in our sample consistent with the literature, we find that an adjusted condition defined on the pricer's SDF delivers a positive correlation. We construct the as-if representative agent log SDF in the U.S. as implied by the model: $\widehat{m} = -\gamma \Delta c + \tilde{\beta} + \log \beta$ and continue to assume the Foreign SDF is the scaled consumption growth $m^* = -\gamma \Delta c^* + \log \beta^*$. A positive correlation provides support for a framework where there is international no-arbitrage for the pricers' kernels (14).

Table 3 lists correlations for all seventeen bilateral pairs and the cross-country average. Consistent with our framework, the correlation of relative SDF with exchange rate growth is positive for all pairs with the exception of Japan and Switzerland. In addition, we find that $\text{Cov}(m^* - \widehat{m}, \Delta e) = 0.0134 \approx 0.0121 = Var(\Delta e)$ for our chosen EIS coefficient of 0.3, satisfying the no-arbitrage condition (14) implied by four Euler equations. We find it striking

Table 4: *Decomposition of the Berger et al. (2023) Wedge*

	Jensen's		Composition	
ISO	$-\text{Corr}(\tilde{\beta}^J, \Delta e)$	Threshold	$-\text{Corr}(\tilde{\beta}^C, \Delta e)$	Threshold
AUS	0.06	0.24	0.29	2.05
BEL	0.22	0.19	0.29	1.62
CAN	0.30	0.17	0.37	1.51
CHE	-0.05	0.17	0.32	1.44
DEU	0.21	0.19	0.28	1.61
DNK	0.22	0.18	0.30	1.59
ESP	0.30	0.20	0.35	1.78
FIN	0.23	0.21	0.32	1.84
FRA	0.24	0.19	0.29	1.61
GBR	0.35	0.19	0.28	1.68
IRL	0.22	0.15	0.29	1.33
ITA	0.34	0.20	0.27	1.75
JPN	-0.35	0.19	-0.04	1.69
NLD	0.22	0.19	0.29	1.65
NOR	0.23	0.22	0.36	1.89
PRT	0.28	0.20	0.28	1.71
SWE	0.28	0.23	0.42	2.02
AVERAGE	0.19	0.19	0.29	1.69

that an independently measured discount factor wedge from micro data can reconcile the Backus-Smith puzzle in a way consistent with our general no-arbitrage framework.

Composition vs. Risk. We decompose our main findings further to see if our mechanism is driven by a composition effect (i.e. the changing relative consumption of the pricer) or a risk term captured by within-group dispersion. As shown in BBD, the $\beta-$wedge can be decomposed into following terms:

$$\underbrace{\log \frac{1}{N_g} \sum_{\nu=1}^{N_g} \left(\frac{C_{t+1}^{\nu}/C_t^{\nu}}{C_{t+1}/C_t} \right)^{-\gamma}}_{\tilde{\beta}} = \underbrace{\log \left(\frac{C_{t+1}^{g}/C_t^{g}}{C_{t+1}/C_t} \right)^{-\gamma}}_{\text{Composition term } \tilde{\beta}^C} + \underbrace{\log \frac{1}{N_g} \sum_{\nu=1}^{N_g} \left(\frac{C_{t+1}^{\nu}/C_t^{\nu}}{C_{t+1}^{g}/C_t^{g}} \right)^{-\gamma}}_{\text{Jensen's term } \tilde{\beta}^J},$$

where C^{ν} denotes individual consumption and C^g denotes the group average of consumption. To disentangle the two channels, we construct the $\beta-$wedge using the consumption of an average unconstrained agent (C^g), capturing only the composition channel, and we separately consider the residual Jensen's term. Table 4 reports the correlations of the respective wedges with exchange rates and the thresholds required for Proposition 1 to be satisfied.

While an explanation based solely on composition of spending by households on the Euler equation offers the right sign, the composition based wedge is insufficiently volatile, resulting in high thresholds which cannot be met (final column). Instead, within-group variation in marginal utility growth (detailed in the first two columns) appears to be the main driver behind satisfying condition (16).

Permanent Income Risk The literature on the asset pricing implications of idiosyncratic risk initially focused on settings with no borrowing constraints and permanent income risk, e.g. Constantinides and Duffie (1996) and others. We use a measure of permanent income risk constructed by Bayer et al. (2019) using the Survey of Income Participation Program in the U.S. While we once again find that while the correlation of the permanent income risk based wedge co-moves negatively with exchange rates, the volatility of permanent risk is orders of magnitude too small to satisfy (16) in order to reconcile the Backus-Smith puzzle. This echoes earlier negative findings by Lettau (2002) on the resolution of equity premium puzzle in the U.S with such measures of idiosyncratic risk. We report the results in Appendix C.2.

In sum, these results rule out composition driven wedges and permanent risk–based wedges as mechanisms for satisfying condition (16), directing attention to transitory risk and occasionally binding constraints.

4.1. Foreign Idiosyncratic Risk

A. Shortcomings with $\tilde{\beta}^* = 0$ Until now we have assumed the representative agent Euler equation holds abroad, i.e. foreign idiosyncratic risk plays no role. We highlight two shortcomings of this approach. First, our symmetric two country framework suggests that the relevant condition (16) is on the correlation of the bilateral wedge $(\tilde{\beta} - \tilde{\beta}^*)$ with the real exchange rate. For example, it could be the case that $cov_t(\tilde{\beta}^*, \Delta e) \leq cov_t(\tilde{\beta}, \Delta e) \leq 0$ such that $cov_t(\tilde{\beta} - \tilde{\beta}^*, \Delta e) \geq 0$ despite the evidence we present above, and therefore the Backus-Smith puzzle cannot be resolved. Second, the denominator in the threshold is given by the volatility of the relative $\beta-$ wedge, which could, in principle, be less volatile. While we do not have comparable annual household-level consumption data for other countries to construct an equivalent time-series of $\tilde{\beta}^*$, we discuss below how we construct two comparable bilateral measures using (i) cross-country distributional income growth statistics for eight advanced economies, and (ii) biennial household-level consumption data for Italy over a subsample.

B. Using Global Repository of Income Dynamics We construct a proxy for the bilateral wedge using growth in income shares for high income groups over a balanced sample of 1998–2015 in seven countries (Canada, Denmark, France, Germany, Italy, Norway, and Sweden) against the United States. We use the GRID measure of average residual log earnings growth for 25-55 males to calculate income shares, derived from regressions of log earnings on age dummies. To capture unconstrained agents likely to be the most patient, we present results based on the top 2.5 and top 5 percentile group in each country's income distribution.[16] Details for the construction of wedges are provided in Appendix C.3 and robustness using the top 10 and top 1 percentile groups can be found in Appendix C.3.2. There, we also discuss in detail the shortcomings from using income instead of consumption data.

The first key shortcoming from using income instead of consumption data arises because of consumption smoothing, concern which is even more pertinent when we focus on patient individuals with a higher propensity to save than the average. However, we show that income data can still be used to construct a good proxy for the correlation of the relative (consumption) $\beta-$wedge with depreciation within a class of models characterized by following relationships:

$$\frac{C^{\nu}_{t+1}}{C^{\nu}_{t}} = \theta^{\nu}\frac{I^{\nu}_{t+1}}{I^{\nu}_{t}}, \quad \frac{C^{\nu*}_{t+1}}{C^{\nu*}_{t}} = \theta^{\nu*}\frac{I^{\nu*}_{t+1}}{I^{\nu*}_{t}}, \tag{19}$$

$$\frac{C_{t+1}}{C_{t}} = \theta\frac{I_{t+1}}{I_{t}}, \quad \frac{C^{*}_{t+1}}{C^{*}_{t}} = \theta^{*}\frac{I^{*}_{t+1}}{I^{*}_{t}} \tag{20}$$

where X^{ν}_t, and X_t denote the individual and the economy-average income or consumption. Within this class, it follows that:

$$\rho_{\tilde{\beta}_{t+1}-\tilde{\beta}^*_{t+1},\Delta e_{t+1}} \approx \rho_{\tilde{\beta}^I_{t+1}-\tilde{\beta}^{I*}_{t+1},\Delta e_{t+1}}, \quad \text{if } \frac{\theta^{\nu}}{\theta} \approx \frac{\theta^{\nu*}}{\theta^*}$$

[16]Figure 5 plots the GRID based wedges alongside the wedge constructed by Berger et al. (2023), revealing the two co-move strongly. On average, the country specific wedges we construct are counter-cyclical with respect to own output growth and consumption growth.

Table 5: *Correlation and Thresholds with Real Exchange Rate Growth using GRID data*

Panel A: Correlation and Thresholds for Top 5% and 2.5% Groups

iso	*Unconditional* $-Corr(\tilde{\beta}-\tilde{\beta}^{*},\Delta e)$ Top 5%	Top 2.5%	*Thresh.*	*Conditional* $-Corr_t(\tilde{\beta}-\tilde{\beta}^{*},\Delta e)$ Top 5%	Top 2.5%	$Thresh._t$
CAN	0.46	0.67	0.40	0.52	0.72	0.46
DEU	0.60	0.64	0.39	0.66	0.75	0.30
DNK	0.44	0.50	0.38	0.55	0.61	0.47
FRA	0.17	0.25	0.39	0.35	0.37	0.37
ITA	0.09	0.41	0.39	0.30	0.63	0.44
NOR	0.42	0.32	0.48	0.27	0.21	0.49
SWE	0.49	0.45	0.46	0.47	0.45	0.38
AVERAGE	0.38	0.46	0.41	0.45	0.54	0.41

Panel B: Correlation of Pricing Kernels and Real Exchange Rate Growth

iso	$Corr(\Delta c-\Delta c^{*},\Delta e)$	$Corr(\widehat{m}^{*}-\widehat{m},\Delta e)$ Top 5%	Top 2.5%	$Corr_t(\Delta c-\Delta c^{*},\Delta e)$	$Corr_t(\widehat{m}^{*}-\widehat{m},\Delta e)$ Top 5%	Top 2.5%
CAN	-0.03	0.28	0.51	-0.21	0.45	0.66
DEU	0.19	0.63	0.68	0.12	0.86	0.90
DNK	-0.29	0.26	0.34	-0.61	0.14	0.33
FRA	-0.15	0.15	0.24	-0.21	0.35	0.36
ITA	0.09	0.11	0.40	0.14	0.33	0.61
NOR	-0.38	0.29	0.24	-0.58	0.08	0.10
SWE	-0.06	0.45	0.43	-0.07	0.42	0.42
AVERAGE	-0.09	0.31	0.41	-0.20	0.37	0.48

where $\tilde{\beta}^{I}_{t+1}=-\gamma\log\left(\frac{I^{\nu}_{t+1}/I^{\nu}_{t}}{I_{t+1}/I_{t}}\right)$. See Appendix C.3.1 for a full derivation.

Second, we construct an upper bound for the relevant threshold in (16), which this correlation must exceed. In general, the denominator of this threshold depends on the volatility of the difference in $\beta-$ wedges across countries, but we want to avoid using the volatility of a $\beta-$wedge constructed using income. To this end, we consider a (conservative) lower bound for the denominator – the volatility of the relative $\beta-$wedge– using only the consumption based wedge from BBD. Denote the volatility of the US specific wedge with $\sigma(\tilde{\beta})$, and the correlation between the two wedges by $\rho_{\tilde{\beta},\tilde{\beta}^{*}}\in(-1,1)$. Taking the correlation as given, it can be shown that volatility of the foreign wedge that minimizes volatility of bilateral wedge is $\max\{0, \rho_{\tilde{\beta},\tilde{\beta}^*} \times \sigma(\tilde{\beta})\}$.[17] Then, the volatility for the bilateral wedge is $\sigma(\tilde{\beta})$ when $\rho_{\tilde{\beta},\tilde{\beta}^*} < 0$ and $\sigma(\tilde{\beta})\sqrt{1-\rho^2_{\tilde{\beta},\tilde{\beta}^*}}$ for all $\rho_{\tilde{\beta},\tilde{\beta}^*} \geq 0$. We set $\rho_{\tilde{\beta},\tilde{\beta}^*} = 0.9$, larger than the highest observation for bilateral correlation of GDP growth in the sample ($\rho = 0.85$, GBR/USA), and get an implied volatility of bilateral wedge is about $0.435\sigma(\tilde{\beta})$. We use this implied volatility of the bilateral wedge and the volatility of bilateral exchange rates to construct the thresholds in (16).

Table 5 presents the main results using the GRID dataset. We construct bilateral wedges for the top 5% and top 2.5% income groups, using $\gamma^{-1} = 0.3$ as above. The set of controls for constructing conditional moments is the same as the one used for the consumption based wedge. Panel A reports the correlation of the bilateral wedge with exchange rates alongside the relevant threshold. The correlations are robustly negative and exceed the (conservative) threshold computed using the implied volatility of the bilateral wedge for at least half the countries. Panel B reports the correlation of relative consumption growth with exchange rate, and the correlation of the relative pricers' SDFs with exchange rates. As before, the correlation of relative SDFs with exchange rates is positive for all bilateral pairs, while the correlation is negative when we use relative aggregate consumption growth to construct SDFs.

C. Using Survey of Household Income and Wealth (SHIW) We consider a second bilateral measure by leveraging biennial household-level consumption, income, and net-worth data. We follow the BBD procedure and construct annualized wedge for alternate years starting in 1997 until 2015. We restrict attention to survey years from 1995 onward, which are the first waves for which our extraction delivers a consistent set of income, consumption, wealth and deflator variables across all modules with a sizable panel component. We describe the detailed construction in Appendix C.4. When using $\gamma^{-1} = 0.3$, same as baseline calibration, we find that the constructed wedge for Italy is countercyclical (correlation with $\Delta C_t = -0.59$), but is one-fifth as volatile as the U.S. discount factor wedge ($\sigma(\tilde{\beta}^*) = 0.11$). The bilateral wedge is the difference between the U.S. wedge and the Italy wedge for the same years.

We find that $-Corr(\tilde{\beta} - \tilde{\beta}^*, \Delta e) = 0.57$, exceeding the new threshold of 0.11, so condition (16) in Proposition 1 is satisfied, similar to the case using only U.S data. Moreover, we once

[17] Specifically, we solve $\min_{\sigma^*} Var(\beta - \beta^*) = \sigma^2(\tilde{\beta}) + \sigma^2(\tilde{\beta}^*) - 2\rho_{\tilde{\beta},\tilde{\beta}^*}\sigma(\tilde{\beta})\sigma(\tilde{\beta}^*)$ and derive $\sigma(\tilde{\beta}^*) = \rho_{\tilde{\beta},\tilde{\beta}^*}\sigma$ for all $\rho \geq 0$.

again confirm that $Corr(\widehat{m}^* - \widehat{m}, \Delta e) > 0$ (with respect to the pricer's kernel), consistent with no-arbitrage.

D. Summary We document two robust results across micro datasets where we construct the incomplete markets wedges: (i) the sufficient condition (16) described in Proposition 1 is satisfied, i.e. idiosyncratic risk based wedge is risky with respect to the exchange rate, and (ii) relative SDFs of as-if representative agents are counter-cyclical with exchange rate.

5. A Closed-Form Incomplete Markets Wedge

In this section, we derive *equilibrium* exchange rates (15) and evaluate its relationship to the relative $\beta-$wedge. To do so, we specify laws of motion for aggregate consumption growth and the international incomplete markets wedge as in Lustig and Verdelhan (2019), which builds on Cox et al. (1985, CIR henceforth).[18] We introduce two further ingredients: first, we allow for countercyclical idiosyncratic risk as in Herskovic, Kelly, Lustig and Van Nieuwerburgh (2016); Berger et al. (2023) amongst others; second, using a combination of a common factor and a country-specific factor, we allow for a correlation of SDFs across countries which is positive but below 1, consistent with the international comovement of asset prices and aggregates (Brandt, Cochrane and Santa-Clara, 2006).[19]

Consider a common factor z_t which drives consumption growth globally:

$$z_{t+1} = (1-\rho)\theta + \rho z_t - \sigma_z \sqrt{z_t} u_{t+1}$$

Aggregate consumption growth in the Home country is given by $\Delta c_{t+1} = \sqrt{z_t} u_{t+1}$ while consumption growth abroad is given by $\Delta c^*_{t+1} = \xi^* \sqrt{z_t} u_{t+1} + \sigma_{\nu,t} u^{\nu}_{t+1}$, such that $\xi^* > 1$ ($\xi^* < 1$) corresponds to an environment where the shock u_t has a relatively larger effect on the Foreign (Home) aggregate consumption growth, and u^{ν}_t is a Foreign-specific shock with stochastic volatility $\sigma_{\nu,t}$. Note that $\xi^* = 1$ implies $proj(\Delta c_{t+1} - \Delta c^*_{t+1} | u_{t+1}) = 0$, hence we focus on $\xi^* \lesseqgtr 1$. Both u_{t+1} and u^{ν}_{t+1} are mean zero normal i.i.d innovations with unit

[18] We use the discrete time version by Sun (1992), extended to a two-country environment Backus et al. (2001), which has been extensively used since, see e.g. Lustig and Verdelhan (2019)

[19] The quantitative exercise in Section 5.1 shows that allowing for this shock additionally provides the model with flexibility to match the UIP coefficient in the data and the low R^2 of such a regression.

standard deviations.

In the presence of uninsurable idiosyncratic risk, the relevant pricing kernels are characterized by $\widehat{m}^{(*)}_{t+1}$, defined in (8) and (9), and depend on the $\beta-$wedges $(\tilde{\beta}^{(*)}_{t+1})$ for which we specify linear processes:

$$\tilde{\beta}_{t+1} = \chi_\beta z_t + \phi\sqrt{z_t}u_{t+1}, \tag{21}$$

$$\tilde{\beta}^*_{t+1} = \chi^*_\beta \xi^{*2} z_t + \phi^*(\xi^*\sqrt{z_t}u_{t+1} + \sigma_{\nu,t}u^\nu_{t+1}) \tag{22}$$

We can write the Home and Foreign pricers' SDFs as follows:

$$-\widehat{m}_{t+1} = \hat{\chi} z_t + (\gamma - \phi)\sqrt{z_t}u_{t+1}, \tag{23}$$

$$-\widehat{m}^*_{t+1} = \hat{\chi}^*\xi^{*2} z_t + \chi^*_\nu \sigma^2_{\nu,t} + (\gamma - \phi^*)(\xi^*\sqrt{z_t}u_{t+1} + \sigma_{\nu,t}u^\nu_{t+1}) \tag{24}$$

The drift in the SDF combines the drift in the representative agent valuation and the drift in the discount factor wedge ($\hat{\chi} = \chi_\beta + \chi$). Allowing for drifts improves the model's fit to unconditional moments but does not affect our main results, which concern conditional moments.[20]

Throughout this analysis, we assume counter-cyclical wedges in both countries, i.e. $\phi,\ \phi^* < 0$, and restrict attention to $\gamma - \phi^{(*)} > 0$ such that the pricer's SDF falls in response to an increase in domestic consumption growth. The conditional volatility of the pricers' SDF is given by $var_t(\widehat{m}_{t+1}) = (\gamma - \phi)^2 z_t$, $var_t(\widehat{m}^*_{t+1}) = (\gamma - \phi^*)^2(\xi^{*2} z_t + \sigma^2_{\nu,t})$. We define $\phi^\Delta = \phi - \phi^*\xi^*$ as the relative sensitivity of idiosyncratic risk to the common factor.

Lemma 2 (Equilibrium Incomplete Markets Wedge)

In the model with heterogeneous consumers, satisfying (8)-(11) *and* (15)*, the incomplete*

[20]If $\chi = \chi^* = \chi^*_\nu = 0$, this corresponds to power utility over consumption as in the canonical representative agent Lucas model. Allowing $\chi > 0$ is a reduced form way to capture a connection between the conditional mean and variance of SDFs which may arise due to habit formation or recursive preferences, see e.g. Hassan, Mertens and Wang (2024).

markets wedge is given by:[21]

$$\eta_{t+1} = -\frac{1}{2}(\gamma(1-\xi^*)-\phi^\Delta)\sqrt{(\gamma(1-\xi^*)-\phi^\Delta)^2-\lambda}\ z_t$$
$$+\frac{1}{2}(\gamma-\phi^*)\sqrt{(\gamma-\phi^*)^2-\lambda^\nu}\ \sigma^2_{\nu,t} - \sqrt{(\gamma(1-\xi^*)-\phi^\Delta)^2-\lambda}\sqrt{z_t}u_{t+1}$$
$$+\sqrt{(\gamma-\phi^*)^2-\lambda^\nu}\ \sigma_{\nu,t}u^\nu_{t+1} + \sqrt{\lambda-\kappa}\ \sqrt{z_t}\epsilon_{t+1} + \sqrt{\lambda^\nu-\kappa^\nu}\ \sigma_{\nu,t}\epsilon^\nu_{t+1},$$

where $\lambda \geq \kappa, \lambda^\nu \geq \kappa^\nu$ are parameters governing cross border spanning and

$$\kappa = (\gamma(1-\xi^*)-\phi^\Delta)^2 - (\gamma(1-\xi^*)-\phi^\Delta)\sqrt{(\gamma(1-\xi^*)-\phi^\Delta)^2-\lambda} \geq 0,$$
$$\kappa^\nu = (\gamma-\phi^*)^2 - (\gamma-\phi^*)\sqrt{(\gamma-\phi^*)^2-\lambda^\nu} \geq 0$$

and $var_t(\Delta e_{t+1}) = \kappa z_t + \kappa^\nu \sigma^2_{\nu,t}$.

Proof. See Appendix A. □

where $u^{(\nu)}_{t+1}$ are spanned innovations and $\epsilon^{(\nu)}_{t+1}$ are unspanned innovations. The exposure of the incomplete markets wedge to spanned shocks is given by $\sqrt{(\gamma(1-\xi^*)-\phi^\Delta)^2-\lambda}$ and $\sqrt{\lambda-\kappa}$ is the exposure to unspanned shocks.

To illustrate the effects of incomplete spanning, we parametrize $\lambda = \alpha \times (\gamma(1-\xi^*)-\phi^\Delta)^2$ for $\alpha \in (0,1)$. It can be verified that $\alpha \to 1$ coincides with the complete markets benchmark such that $\lambda_{CM} = \kappa_{CM} = (\gamma(1-\xi^*)-\phi^\Delta)^2$. Instead, $\alpha < 1$ implies increasing levels of unspanned risk. We similarly define $\lambda^\nu = \alpha^\nu(\gamma-\phi^*)^2$ and then $\lambda^\nu_{CM} = \kappa^\nu_{CM} = (\gamma-\phi^*)^2$. When both $\alpha, \alpha^\nu \to 1$, necessitated when there is trade in additional risky assets (see Appendix B.5), the wedge tends to zero $\eta_{t+1} = 0$.

Armed with a closed form expression for the international incomplete markets wedge, we revisit Proposition 1 and moments of exchange rates using an equilibrium exchange, conditional on $u_{t+1} = u^\nu_{t+1} = 1$.

Proposition 2 (Exchange rate cyclicality in equilibrium)

In the model with heterogeneous consumers, satisfying (8)-(11) and (15), using the processes

[21]Square roots refer to the positive root only unless we include the ± ahead. For the z_t factor, while the negative root is a potential solution, conditional on $\phi < 0$, it violates $\lambda > \kappa$ which we require for a real solution.

for SDFs (23) *and* (24), $cov_t(\Delta c_{t+1} - \Delta c^*_{t+1}, \Delta e_{t+1}) < 0$ *if and only if:*

$$(1-\xi^*)\ [(\gamma(1-\xi^*) - \phi^\Delta] \leq -\underbrace{(\gamma - \phi^*)\tilde{\alpha}\frac{\sigma^2_{\nu,t}}{z_t}}_{\geq 0}, \tag{25}$$

where $\tilde{\alpha} = \frac{1-\sqrt{1-\alpha^\nu}}{1-\sqrt{1-\alpha}}$ and $\lim_{\alpha^\nu \to 0} \tilde{\alpha} = 0$.

Proof. See Appendix A. □

Condition (25) ensures that, in equilibrium, the conditional covariance between $\tilde{\beta}_{t+1} - \tilde{\beta}^*_{t+1}$ and Δe_{t+1} is negative and larger than the volatility of exchange rates, i.e. foreign bonds are relatively riskier for the Home consumer. The model is able to deliver a negative aggregate Backus-Smith coefficient and can accommodate counter-cyclical idiosyncratic risk in one or both countries. To understand the underlying mechanism, it is instructive to begin with the limit where the country-specific shock is small ($\sigma^2_\nu \to 0$).

Suppose that $\xi^* > 1$ such that, following $u_{t+1} \uparrow$, Foreign consumption growth outpaces Home consumption growth (i.e. $\Delta c - \Delta c^* \downarrow$) . Then, in order to generate an exchange rate depreciation $\Delta e_{t+1} \uparrow$, we require $\phi^\Delta \leq \gamma(1-\xi^*) \leq 0$: Home idiosyncratic risk is more counter-cyclical than Foreign and the Home $\beta-$wedge falls relative to the Foreign $\beta-$wedge. Intuitively, the relatively higher $\tilde{\beta}^*_{t+1}$ implies Foreign consumers remain concerned about the future leading to a depreciation (15), despite a relative rise in Foreign aggregate consumption growth.

Conversely, if $\xi^* < 1$, Home aggregate consumption rises in relative terms (i.e. $\Delta c - \Delta c^* \uparrow$). For exchange rates to appreciate $\Delta e_{t+1} \downarrow$, the model requires that $\phi^\Delta > \gamma(1-\xi^*) > 0$ (either due to procyclical idiosyncratic risk or because $\phi^*\xi^* < \phi < 0$), such that the Home beta wedge remains high, i.e. valuations of returns remain high due to idiosyncratic risk at a time when losses occur on foreign portfolios. In both cases, the country experiencing faster consumption growth must have a relatively muted fall in the $\beta-$wedge.[22,23]

[22]Contrast this to the representative agent limit (Lustig and Verdelhan, 2019), which coincides with $\phi^\Delta = 0$. The LHS of condition (25) reduces to $\gamma(1-\xi^*)^2$ which is strictly positive, therefore $cov_t(\Delta c_{t+1} - \Delta c^*_{t+1}, \Delta e_{t+1}) > 0$.

[23]Looking to the data, the relevant case is where idiosyncratic risk is counter-cyclical $sign(\phi) = sign(\phi^*) < 0$ and consumption growth is positively correlated across countries ($\xi^* > 0$). Accounting for the equilibrium exchange rate process, this rules out the possibility that both Foreign (Home) bonds are risky with respect to idiosyncratic states for Home (Foreign) households. Instead, the cases above describe under what conditions exchange rates are relatively *riskier* for the Home country.

Cyclicality and International Incompleteness. Away from the limit $\sigma_{\nu,t} \to 0$, the cyclicality of exchange rates also depends on the degree of international financial market completeness. This is because country-specific shocks always induce a (weakly) positive Backus-Smith covariance (maintaining that $\gamma - \phi^{(*)} > 0$).[24] If country specific shocks are sufficiently volatile, Proposition 2 can never be satisfied for values of $\alpha^{\nu} > 0$. Generally, allowing for positive α_{ν} adds to volatility of exchange rates but reduces the ability of the model to resolve the Backus-Smith covariance. Instead, when $\alpha^{\nu} \to 0$, the country-specific shock has zero effect on exchange rates. In contrast, when $\alpha^{\nu} > 0$ more incompleteness ($\alpha \downarrow$) for the common shock (which conditionally satisfies Proposition 1) makes Backus-Smith relatively harder to satisfy. In Section 5.1, we show that the unconditional covariance is also negative in our calibrated model.

The cyclicality and the volatility of exchange rates are closely connected. The next Proposition shows that introducing idiosyncratic risk comes at the cost of further exacerbating exchange rate volatility, making international financial market incompleteness critical.

Proposition 3 (Idiosyncratic Risk and Exchange Rate Volatility)
Exchange rate volatility is higher under internationally complete markets ($\kappa^{CM} > \kappa^{IM}$ and $\kappa^{\nu CM} > \kappa^{\nu IM}$) and, if $\gamma(1-\xi^*) - \phi^{\Delta} > 0$, is also more sensitive to the cyclicality of idiosyncratic risk $\left(\frac{d\kappa^{CM}}{d(-\phi^{\Delta})} > \frac{d\kappa^{IM}}{d(-\phi^{\Delta})} > 0 \text{ and } \frac{d\kappa^{\nu CM}}{d(-\phi^{*})} > \frac{d\kappa^{\nu IM}}{d(-\phi^{*})} > 0\right)$

Proof. See Appendix A. □

In the calibration below, if SDFs are sufficiently volatile to explain (e.g.) stock prices, models with internationally complete markets also struggle to reconcile the low exchange rate volatility observed in the data. Consistent with Brandt et al. (2006), only in the limit where we shut down u^{ν} such that SDFs are (counterfactually) perfectly correlated, can internationally complete markets be consistent with observed exchange rate volatility.

[24] It is important to note there is nothing special about the Foreign country specific shock, and these results will generally apply to any uncorrelated factor driving only the volatility of consumption in one country. In Appendix B.7, we show the same results follow if an uncorrelated shock is added to the Home SDF instead.

5.1. Quantitative Exercise

Calibration The model frequency is monthly. We calibrate z_t using the process for real 3-month U.S. interest rates as in (Backus et al., 2001), over the sample 1991Q1–2017Q4, consistent with our empirical sample. Starting from: $r_t = [\hat{\chi} - \frac{1}{2}(\gamma - \phi)^2]z_t$, we choose $\hat{\chi} = 1 + \frac{1}{2}(\gamma - \phi)^2$ and target an average (annualized) U.S. interest rate of 0.4% so $\mathbb{E}_t[z_{t+1}] = \theta = 0.03\%$ (monthly). Analogous normalizations imply the foreign rate is given by $r_t^* = \xi^{*2} z_t + \tilde{\zeta}\sigma_{\nu,t}^2$ and choose $\tilde{\zeta}$ to target a 1% annual average.[25] We target a correlation of SDFs across countries of 0.8 (Brandt et al., 2006) which implies $\rho_{\widehat{m},\widehat{m}^*} = 0.8 = \theta\xi^*/(\sqrt{\theta}\sqrt{\theta\xi^* + \sigma_\nu})$, from which, given ξ^* discussed below, we back out $\sigma_\nu^2 = 0.031\%$. We choose the volatility of country specific risk $var(\sigma_{\nu,t}^2)$ to match the UIP coefficient. We set $\gamma = 3$ such that our microdata on consumption growth and exchange rates satisfy (14). We calibrate the asymmetry in ϕ and ϕ^*, using Sharpe ratios on equity from the U.S. and abroad. We target a Sharpe ratio of 0.5 for the U.S. based on the S&P 500. Using $(\gamma - \phi)\sqrt{\theta} = 0.5$, we back out $\phi = -4.91$ which is comparable to the U.S. calibration in Acharya, Challe and Dogra (2023). We repeat the exercise abroad with a Sharpe ratio of 0.4, which implies $\phi^* = -0.92$.[26]

Calibrating the degree of international spanning is a challenge. In our baseline, we assume the Foreign shock is unspanned $\alpha^\nu = 0.01$, and we choose $\alpha = 0.6$ to match a conditional exchange rate volatility of 0.1, consistent with the average annualized standard deviation of bilateral exchange rates in our sample. The remaining parameter is ξ^*, which given other parameters must be greater than 1 (see Proposition 2), to deliver a negative Backus-Smith covariance. We choose $\xi^* = 1.29$ which results in a conditional correlation of relative consumption growth and exchange rates depreciation of approximately -0.16 (see Table 3). Table 6 summarizes our parameter calibration and the moments targeted in the model. In sum, the quantitative model is able to match both domestic asset prices, international SDF comovement and moments of exchange rates, while maintaining countercyclical idiosyncratic risk with respect to domestic aggregate states.

[25] We impose $\chi = 1 + \frac{1}{2}\gamma^2$ and $\chi_\beta^{(*)} = \frac{1}{2}\phi^{(*)2} - \gamma\phi^{(*)}$ such that $r_t = z_t$. The foreign interest rate is given by $r_t^* = (\chi^* + \chi_b^*)\xi^{*2} z_t + \chi_\nu^*\sigma_{\nu,t}^2 - \frac{1}{2}(\gamma - \phi^*)\xi^{*2} z_t - \frac{1}{2}(\gamma - \phi^*)\xi^{*2}\sigma_{\nu,t}^2$. We define $\tilde{\zeta} = \chi_\nu^* - \frac{1}{2}(\gamma - \phi^*)^2$ and set $\chi = \chi^*$ to economize on parameters.

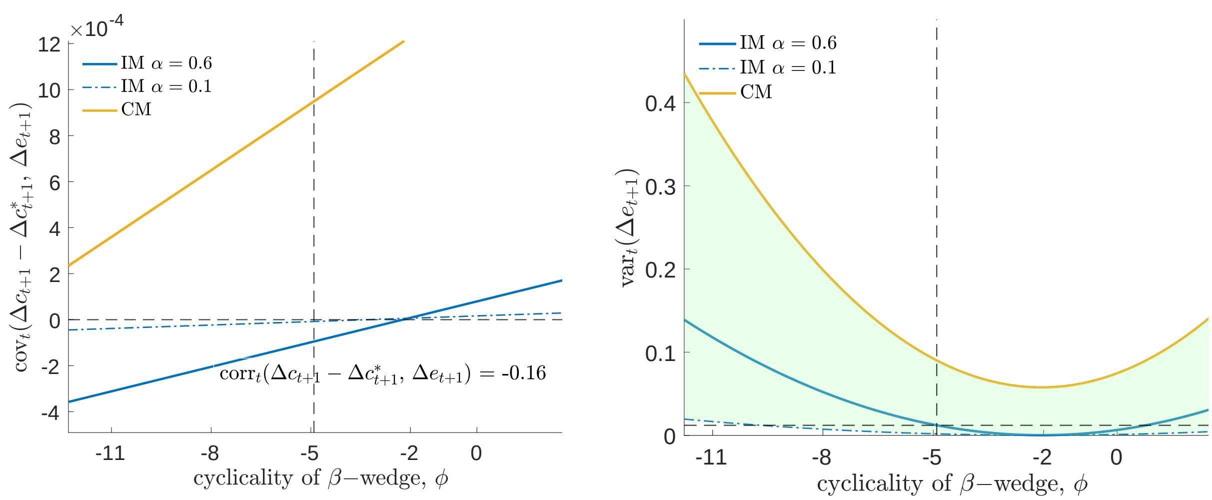
[26] These Sharpe ratios are comparable to the literature and slightly higher than the historical averages in Jordà, Schularick, Taylor and Ward (2019).

Table 6: *Matched Moments*

Parameter	Value	Data (Annualized)
γ	3	$\text{cov}(m^* - m, \Delta e) = \text{var}(\Delta e)$
θ	0.03%	mean real U.S. interest rate (0.4%)
$var(z_t)$	0.43%	s.d of U.S. interest rate (0.95%)
$\tilde{\zeta}$	-0.4157	mean real Foreign interest rate (1%)
σ^2_ν	0.031%	cross-country SDF correlation (0.8)
ϕ	-4.9057	Sharpe ratio (U.S., S&P 500) = 0.5
ϕ^*	-0.9222	Sharpe ratio (RoW) = 0.4
α	0.6	volatility (s.d) of exchange rates (11%)
ξ^*	1.29	Backus–Smith correlation (-0.16)
$var(\sigma^2_{\nu,t})$	0.00053%	$\beta_{UIP} = -1$

Figure 1

(a) *Backus-Smith Covariance* **(b)** *Exchange Rate Volatility*



The solid blue line is our calibration, corresponding to $\alpha = 0.6$. The solid yellow line reflects the complete (international) markets equilibrium, and the dotted blue line corresponds to incomplete markets with spanning $\alpha = 0.1, \alpha^\nu = 0.01$. Horizontal dashed lines reflect the zero-line in the left panel, var(Δe) from the data in the right panel. Vertical dashed lines mark the calibration for ϕ in both panels.

Results. Figure 1 illustrates the main results, plotting the relationship between the Home $\beta-$wedge cyclicality ϕ, the conditional Backus-Smith covariance (Proposition 2) and exchange rate volatility for three levels of spanning – (i) complete markets $\alpha = \alpha^\nu = 1$, (ii) our preferred calibration of intermediate spanning ($\alpha = 0.6, \alpha^\nu = 0.01$) and, for comparison, (iii) very little spanning ($\alpha = 0.1, \alpha^\nu = 0.01$). The left panel shows that for sufficiently counter-cyclical idiosyncratic risk ($\phi << 0$) the Backus-Smith covariance is negative, consistent with

Proposition 2.

The right panel illustrates that the exchange rate volatility under complete international markets is too high when we realistically calibrate to an imperfect correlation of SDFs across countries. This problem is exacerbated further as ϕ becomes increasingly negative. In the model, any level of volatility in the shaded area can be attained by varying α and α^{ν}.

Given ϕ, while spanning of the common factor (higher values of α) leads to a more negative covariance, higher spanning of the Foreign shock contributes to a positive covariance. Figure 3 in Appendix B.6 illustrates comparative statics with respect to spanning of u^{ν} (i.e. varying α^{ν} parameter) and Panel (c) presents results in the correlation space. Larger values of α^{ν} reduce the ability of the model to generate negative Backus-Smith covariance and also increase exchange rate volatility. Consistent with our analysis, international market incompleteness is crucial for both the cyclicality and the volatility of exchange rates when we allow for imperfect risk-sharing within countries.

While we focus on the conditional covariance of (aggregate) SDFs and exchange rates, our model performs well for additional untargeted moments. As we discuss further below, our calibration delivers an R^2_{UIP} of only 0.011% for the UIP regression, consistent with evidence of a 'disconnect'. As such, the unconditional Backus-Smith correlation is also negative (-0.175), see Appendix B.8 for derivations.

Contribution of $\phi < 0$. To conclude, we isolate the marginal contribution of countercyclical idiosyncratic risk ($\phi^{(*)} < 0$) in our calibration, holding all other parameters *fixed*. Table 7 reports key moments of exchange rates and asset prices when we set $\phi = \phi^* = 0$. In our framework, counter-cyclical idiosyncratic risk contributes 62% to the Sharpe ratio at Home (22.5% abroad), consistent with findings in our related work (Kozliakov et al., 2025).

In this environment, countercyclical idiosyncratic risk also raises exchange-rate volatility (holding the degree of international market incompleteness fixed), further illustrating the complementarity between domestic and international incompleteness in shaping our quantitative results. As a result, countercyclical idiosyncratic risk $\phi < 0$ *both* lowers the predictability of exchange rates and contributes to a negative Backus-Smith correlation, resolving the trade-off implied by the representative agent framework for these two moments, as shown in Jiang

Table 7: *Model decomposition*

Moment	Baseline	$\phi, \phi^* \to 0$
s.d.(Δe)	11%	3.6%
Backus–Smith correlation	−0.16	0.185
UIP slope coefficient β	−1.00	2.84
Sharpe ratio (Home)	0.50	0.19
Sharpe ratio (Foreign)	0.40	0.31
R^2_{Fama}	0.11%	2.1%

Note: Moments of real interest rates are unaffected.

et al. (2023). Finally, in normalizing $r_t = z_t$, we assume χ_β is increasing in ϕ. Under this mapping, more counter-cyclical idiosyncratic risk, ceteris paribus, makes the UIP slope more negative, also strengthening the model's ability to account for the Fama puzzle.

6. Conclusion

In this paper, we generalize a two-country no-arbitrage framework beyond the representative agent benchmark to allow imperfect risk sharing both *within* and *across* countries. We show that imperfect risk-sharing *within* countries can reconcile the aggregate cyclicality of exchange rates, i.e. the Backus-Smith puzzle, as long as exchange rates are sufficiently risky with respect to idiosyncratic states. We directly test the condition for idiosyncratic risk using household-level consumption data, with income and net worth data used to identify the relevant 'pricer's' valuation. For countries other than the U.S. and Italy where the micro-data is less rich, we rely only on income data and the ranking of households within the distribution. We robustly find that exchange rates are both risky with respect to the idiosyncratic state, and the idiosyncratic state is sufficiently volatile for our mechanism to be empirically relevant.

Solving for equilibrium exchange rates, we show that the cyclicality of within-country idiosyncratic risk (with respect to aggregate consumption growth) is critical to reconcile the Backus-Smith correlation. Specifically, this requires that, despite a country experiencing relatively high consumption growth, the pricers (or marginal investors) remain concerned about their future prospects, resulting in a co-movement of exchange rates, discount factors and aggregate consumption growth in line with the data. Moreover, we show that international market incompleteness is necessary to maintain pro-cyclical exchange rates when consumption growth is imperfectly correlated across countries (contrasting to results in Lustig and Verdelhan (2019) for representative agent models) and is also needed to reconcile relatively smooth exchange rates with volatile domestic asset prices.

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A. Appendix: Proofs to Lemmas and Propositions

Proof to Lemma 1 Given a level of individual consumption $C(s^t)$, households optimally choose their consumption bundle $\{c_H(s^t), c_F(s^t)\}$. For ease of exposition, since the problem is static we suppress dependence on histories and we denote with a superscript ν individual consumptions $c^\nu = c(z^t, \nu^t)$. The optimal consumption bundle satisfies:

$$\max_{\{c_H^\nu, c_F^\nu\}} \left\{ g^*({c_H^\nu}^*, {c_F^\nu}^*) \quad \text{s.t.} \quad \int_\nu c_H^\nu d\nu + \int_\nu {c_H^\nu}^* d\nu^* = \hat{I}_H, \right. \tag{26}$$
$$\left. \int_\nu c_F^\nu (1+\tau) d\nu + \int_\nu {c_F^\nu}^* d\nu = \hat{I}_F \text{ and } g(c_H^\nu, c_F^\nu) \geq C^\nu \right\},$$

where we define $g(c_H, c_F)$ by (3) and I_H and I_F are world aggregates of good endowments. See Costinot, Lorenzoni and Werning (2014) for a friction-less representative agent framework, and Lloyd and Marin (2024) for a treatment with static wedges. Notice that from first order homogeneity of $g(\cdot)$ it follows that:

$$\frac{g_H(c_H^\nu, c_F^\nu)}{g_F(c_H^\nu, c_F^\nu)} = \frac{g_H(c_H, c_F)}{g_F(c_H, c_F)} = \frac{p_F}{p_H}(1+\tau) \tag{27}$$

From the static problem (26), combining (**??**) with market clearing yields:

$$\frac{c_H^\nu}{c_F^\nu} = \frac{c_H}{c_F} = \left(\frac{\alpha}{1-\alpha}\right)^2 (1+\tau_t)^\zeta \, \frac{\hat{I}_H - c_H}{\hat{I}_F - c_F}, \tag{28}$$

where c_H^ν is individual consumption of the H good, c_H is aggregate consumption of the H good and the first equality follows from the first order homogeneity of the aggregator (3). This condition satisfies both Home and Foreign static problems and market clearing. Rearranging yields the optimal F allocations for the Home households given $\{\hat{I}_H, \hat{I}_F\}$:

$$c_F = \frac{\hat{I}_F c_H}{b\hat{I}_H + (1-b)c_H} \tag{29}$$

where $b = (\frac{\alpha}{1-\alpha})^2(1+\tau_t)^\zeta$.

Next, consider country aggregate income I_t, which determines the exchange rate process by (12). Imposing $C_t = I_t$, the goods-specific allocations implied by the aggregator (3) are given by:

$$(1-\alpha)^{\frac{1}{\zeta-1}} c_F(I) = \left[I^{\frac{\zeta-1}{\zeta}} - \alpha^{\frac{1}{\zeta}} c_H(I)^{\frac{\zeta-1}{\zeta}} \right]^{\frac{\zeta}{\zeta-1}} \tag{30}$$

Combining c_F in (29) and (30) yields:

$$\left[I^{\frac{\zeta-1}{\zeta}} - \alpha^{\frac{1}{\zeta}} c_H(I)^{\frac{\zeta-1}{\zeta}} \right]^{\frac{\zeta}{\zeta-1}} (b\hat{I}_H + (1-b)c_H(I)) = \hat{I}_F c_H(I)(1-\alpha)^{\frac{1}{\zeta-1}} \tag{31}$$

and $\{\tau\} \neq 0$ is generically needed for this to be satisfied away from $\alpha \to 1$. □

Proof to Proposition 1 Condition (14) can be expanded as follows:

$$var_t(\Delta e_{t+1}) = cov_t(m^*_{t+1} - m_{t+1}, \Delta e_{t+1}) + cov_t(\tilde{\beta}^*_{t+1} - \tilde{\beta}_{t+1}, \Delta e_{t+1})$$

Then $cov_t(m^*_{t+1} - m_{t+1}, \Delta e_{t+1}) < 0$ if and only if:

$$var_t(\Delta e_{t+1}) + cov_t(\tilde{\beta}_{t+1} - \tilde{\beta}^*_{t+1}, \Delta e_{t+1}) \leq 0.$$

By the Cauchy-Schwarz inequality:

$$|cov_t(\tilde{\beta}_{t+1} - \tilde{\beta}^*_{t+1}, \Delta e_{t+1})| \leq \sigma_t(\tilde{\beta}_{t+1} - \tilde{\beta}^*_{t+1})\sigma_t(\Delta e_{t+1})$$

Combining the inequalities:

$$\sigma_t(\tilde{\beta}_{t+1} - \tilde{\beta}^*_{t+1})\sigma_t(\Delta e_{t+1}) \geq -cov_t(\tilde{\beta}_{t+1} - \tilde{\beta}^*_{t+1}, \Delta e_{t+1}) \geq var_t(\Delta e_{t+1})$$

Dividing through by $\sigma_t(\tilde{\beta}_{t+1} - \tilde{\beta}^*_{t+1})\sigma_t(\Delta e_{t+1})$ yields the result. □

Restriction on η wedge from cross-border trade in assets. Combining (7) and (10) for the Home pricer and combining (9) and (11) for the foreign pricer, using (15) yields the following conditions:

$$\mathbb{E}_t[\eta_{t+1}] = \frac{1}{2}var_t(\eta_{t+1}) - cov_t(\widehat{m}_{t+1}, \eta_{t+1}) \tag{32}$$

$$-\mathbb{E}_t[\eta_{t+1}] = \frac{1}{2}var_t(\eta_{t+1}) - cov_t(\widehat{m}^*_{t+1}, -\eta_{t+1}) \tag{33}$$

This is a straightforward generalization of the conditions in Lustig and Verdelhan (2019) to our environment with imperfect domestic risk sharing.

Proof to Lemma 2 The individual Euler equation can be expressed as:

$$\mathbb{E}_t\left[\tilde{\beta}_{t+1}\left(\frac{C_{t+1}}{C_t}\right)^{-\gamma}\tilde{R}_{t+1}\right] = 1, \tag{34}$$

for any (risky) asset with return $\tilde{R}_{t+1}$, so that e.g. the return on the foreign nominally risk-free bond is given by $(\mathcal{E}_{t+1}/\mathcal{E}_t)R_{t+1}$. We define

$$\Delta e_{t+1} = \widehat{m}^*_{t+1} - \widehat{m}_{t+1} + \eta_{t+1}$$

Combining (14), (15), (32), and (33), yields:

$$var_t(\eta_{t+1}) = var_t(\widehat{m}^*_{t+1} - \widehat{m}_{t+1}) - var_t(\Delta e_{t+1}) \tag{35}$$

Assume a process for the incomplete markets (IM) wedge:

$$\eta_{t+1} = \Gamma_0 z_t + \Gamma_1\sqrt{z_t}u_{t+1} + \Gamma_2\sqrt{z_t}\epsilon_{t+1} + \Gamma_0^\nu\sigma_\nu^2 + \Gamma_1^\nu\sigma^\nu u^\nu_{t+1} + \Gamma_2^\nu\sigma^\nu\epsilon^\nu_{t+1} \tag{36}$$

where $u^{(\nu)}$ are spanned innovations and $\epsilon^{(\nu)}$ are unspanned innovations. We proceed to determine the coefficients of the IM wedge consistent with no arbitrage. Substituting the pricer's SDF (as in (34)), using (23)-(24) into (35), and denoting $var_t(\Delta e_{t+1}) = \kappa z_t + \kappa^\nu\sigma_\nu^2$:

$$\underbrace{(\Gamma_1^2 + \Gamma_2^2)z_t + (\Gamma_1^{\nu 2} + \Gamma_2^{\nu 2})\sigma_\nu^2}_{var_t(\eta_{t+1})} = \underbrace{\left(\gamma(1-\xi^*) - \phi^\Delta\right)^2 z_t + (\gamma - \phi^*)^2\sigma_\nu^2}_{var_t(\widehat{m}^*_{t+1} - \widehat{m}_{t+1})} - \underbrace{\left(\kappa z_t + \kappa^\nu\sigma_\nu^2\right)}_{var_t(\Delta e_{t+1})}, \tag{37}$$

Solving for the coefficients:

$$\Gamma_1 = \pm\sqrt{(\gamma(1-\xi^*) - \phi^\Delta)^2 - \lambda}, \quad \Gamma_1^\nu = \pm\sqrt{(\gamma - \phi^*)^2 - \lambda^\nu} \tag{38}$$

$$\Gamma_2 = \pm\sqrt{\lambda - \kappa}, \quad \Gamma_2^\nu = \pm\sqrt{\lambda^\nu - \kappa^\nu} \tag{39}$$

where for real solutions, the following restrictions are required:

$$(\gamma(1-\xi^*)-\phi^\Delta)^2 \geq \lambda \geq \kappa, \tag{40}$$

$$(\gamma-\phi^*)^2 \geq \lambda^\nu \geq \kappa^\nu \tag{41}$$

Then, substituting (38)-(39) into (32)-(33) imply:

$$\Gamma_0 = \frac{1}{2}\left\{(\gamma(1-\xi^*)-\phi^\Delta)^2-\kappa\right\} - (\gamma-\phi)\sqrt{(\gamma(1-\xi^*)-\phi^\Delta)^2-\lambda}, \tag{42}$$

$$\Gamma_0^\nu = \frac{1}{2}\left\{(\gamma-\phi^*)^2-\kappa^\nu\right\}, \tag{43}$$

and

$$\Gamma_0 = -\frac{1}{2}\left\{(\gamma(1-\xi^*)-\phi^\Delta)^2-\kappa\right\} - \xi^*(\gamma-\phi^*)\sqrt{(\gamma(1-\xi^*)-\phi^\Delta)^2-\lambda}, \tag{44}$$

$$\Gamma_0^\nu = +\frac{1}{2}\left\{(\gamma-\phi^*)^2-\kappa^\nu\right\} + (\gamma-\phi^*)\sqrt{(\gamma-\phi^*)^2-\lambda^\nu} = 0, \tag{45}$$

respectively. Adding (42) and (44) for z_t and (43) and (45) for σ_ν^2 respectively, this can be rewritten as:

$$\Gamma_0 = -\frac{1}{2}(\gamma-\phi+\xi^*(\gamma-\phi^*))\sqrt{\gamma(1-\xi^*)-\phi^\Delta)^2-\lambda}, \tag{46}$$

$$\Gamma_0^\nu = +\frac{1}{2}(\gamma-\phi^*)\sqrt{(\gamma-\phi^*)^2-\lambda^\nu} \tag{47}$$

Subbing Γ_0 and Γ_0^ν back into (42) and (44) respectively:

$$\kappa = (\gamma(1-\xi^*)-\phi^\Delta)^2 - (\gamma(1-\xi^*)-\phi^\Delta)\sqrt{(\gamma(1-\xi^*)-\phi^\Delta)^2-\lambda} \geq 0, \tag{48}$$

$$\kappa^\nu = (\gamma-\phi^*)^2 - (\gamma-\phi^*)\sqrt{(\gamma-\phi^*)^2-\lambda^\nu} \geq 0 \tag{49}$$

To ensure all square roots are real, we only choose the square root signs which satisfy (40) and (41). As a result, we select positive roots for all except Γ_1 where we select the negative root. This completes the characterization of the η process.

□

Proof to Proposition 2 First consider:

$$cov_t(\widehat{m}_{t+1}^* - \widehat{m}_{t+1}, \Delta e_{t+1}) = var_t(\Delta e_{t+1}) =$$

$$cov_t(m_{t+1}^* - m_{t+1}, \Delta e_{t+1}) + cov_t(\tilde{\beta}_{t+1}^* - \tilde{\beta}_{t+1}, \Delta e_{t+1}) \tag{50}$$

Then, the Backus-Smith covariance is negative if and only if:

$$var_t(\Delta e_{t+1}) + cov_t(\tilde{\beta}_{t+1} - \tilde{\beta}^*_{t+1}, \Delta e_{t+1}) \leq 0 \tag{51}$$

Using the Lemma above:

$$\begin{aligned} &\kappa z_t + \phi^\Delta(\gamma(1-\xi^*) - \phi^\Delta - \sqrt{(\gamma(1-\xi^*) - \phi^\Delta)^2(1-\alpha)}z_t \\ &+\kappa^\nu \sigma_\nu^2 - \phi^* \left(-(\gamma - \phi^*) + \sqrt{(\gamma-\phi^*)^2(1-\alpha^\nu)}\right)\sigma_\nu^2 \leq 0 \end{aligned} \tag{52}$$

We deal with the terms pre-multiplying z_t and σ_ν^2 separately. We substitute κ and κ^ν from the Lemma. Beginning with z_t:

$$\begin{aligned} &(\gamma(1-\xi^*) - \phi^\Delta)^2 - (\gamma(1-\xi^*) - \phi^\Delta)\Gamma_1 + \phi^\Delta(\gamma(1-\xi^*) - \phi^\Delta - \Gamma_1) \\ &= (\gamma(1-\xi^*) - \phi^\Delta)(\gamma(1-\xi^*) - \phi^\Delta - \Gamma_1) + \phi^\Delta(\gamma(1-\xi^*) - \phi^\Delta - \Gamma_1) \\ &= (\gamma(1-\xi^*))(\gamma(1-\xi^*) - \phi^\Delta - \Gamma_1) \\ &= (\gamma(1-\xi^*))(\gamma(1-\xi^*) - \phi^\Delta)\underbrace{(1-\sqrt{1-\alpha})}_{\geq 0} \end{aligned}$$

Then, turning to σ_ν^2:

$$\begin{aligned} &(\gamma - \phi^*)^2 - (\gamma-\phi^*)\sqrt{(\gamma-\phi^*)^2(1-\alpha^\nu)} - \phi^*(-(\gamma-\phi) + \Gamma_1^\nu) \\ &= (\gamma-\phi^*)(\gamma - \phi^* + \phi^*)(1-\sqrt{1-\alpha^\nu}) = \\ &(\gamma-\phi^*)\gamma\underbrace{(1-\sqrt{1-\alpha^\nu})}_{\geq 0} \end{aligned}$$

which is strictly positive maintaining that $\gamma - \phi^* > 0$. Using both terms and multiplying by z_t and σ_ν^2 respectively delivers the Proposition. □

Proof to Proposition 3 We focus on u_{t+1} and u_{t+1}^ν in turn. Consider κ for $\alpha < 1$ and denote this by κ^{IM}:

$$\kappa^{IM} = (\gamma(1-\xi^*) - \phi^\Delta)^2 - (\gamma(1-\xi^*) - \phi^\Delta))^2\sqrt{1-\alpha} \tag{53}$$

Imposing complete markets, $\alpha = 1$, $\lambda^{CM} = (\gamma(1-\xi^*) - \phi^\Delta)^2$ and define this as κ^{CM}:

$$\kappa^{CM} = (\gamma(1-\xi^*) - \phi^\Delta)^2 \tag{54}$$

Since $(\gamma(1-\xi^*)-\phi^\Delta))^2\sqrt{1-\alpha}>0$, $\kappa^{IM}<\kappa^{CM}$. Second, denoting $\kappa^{CM}-\kappa^{IM}=(\gamma(1-\xi^*)-\phi^\Delta))^2\sqrt{1-\alpha}$ and taking the derivative with respect to $-\phi^\Delta$:

$$\frac{d(\kappa^{CM}-\kappa^{IM})}{d(-\phi^\Delta)}=2((\gamma(1-\xi^*)-\phi^\Delta))\sqrt{1-\alpha}>0$$

As long as $(\gamma(1-\xi^*)-\phi^\Delta)>0$, required to satisfy Proposition 2 for $\xi^*>1$, $\phi^\Delta\downarrow$ (or $\phi\downarrow$), $\kappa^{CM}-\kappa^{IM}\uparrow$ delivering the result.

In turn:

$$\kappa^{\nu\ IM}=(\gamma-\phi^*)^2-(\gamma-\phi^*)^2\sqrt{1-\alpha^\nu} \tag{55}$$

and

$$\kappa^{\nu\ CM}=(\gamma-\phi^*)^2 \tag{56}$$

Since $\sqrt{1-\alpha^\nu}>0$, $\kappa^{\nu\ IM}<\kappa^{\nu\ CM}$. Finally:

$$\frac{d(\kappa^{\nu\ CM}-\kappa^{\nu\ IM})}{d(-\phi^*)}=2(\gamma-\phi^*)\sqrt{1-\alpha}>0 \tag{57}$$

which confirms the result. $\square$

B. ONLINE APPENDIX

B.1. Additional Derivations for Section 2.

To find the admissible set of exchange rate processes, consider the log expansions of (7)-(11), assuming joint log normality of SDFs and prices:

$$\mathbb{E}_t[\widehat{m}_{t+1}] + \frac{1}{2}var_t(\widehat{m}_{t+1}) = -r_{t+1}, \quad (58)$$

$$\mathbb{E}_t[\widehat{m}^*_{t+1}] + \frac{1}{2}var_t(\widehat{m}^*_{t+1}) = -r^*_{t+1} \quad (59)$$

$$\mathbb{E}_t[\widehat{m}^*_{t+1}] + \frac{1}{2}var_t(\widehat{m}^*_{t+1}) - \mathbb{E}_t[\Delta e_{t+1}] + \frac{1}{2}var_t(\Delta e_{t+1}) + cov_t(\widehat{m}^*_{t+1}, -\Delta e_{t+1}) = -r_{t+1}, \quad (60)$$

$$\mathbb{E}_t[\widehat{m}_{t+1}] + \frac{1}{2}var_t(\widehat{m}_{t+1}) + \mathbb{E}_t[\Delta e_{t+1}] + \frac{1}{2}var_t(\Delta e_{t+1}) + cov_t(\widehat{m}_{t+1}, \Delta e_{t+1}) = -r^*_{t+1}, \quad (61)$$

where lower case levels denote logs, e.g. $\log(\widehat{M}_{t+1}) = \widehat{m}_{t+1}$ and $\Delta e_{t+1} = e_{t+1} - e_t$. Using (58) and (61), and (59) and (60) respectively, yields:

$$\mathbb{E}_t[\Delta e_{t+1}] + r^*_{t+1} - r_{t+1} = -cov_t(\widehat{m}_{t+1}, \Delta e_{t+1}) - \frac{1}{2}var_t(\Delta e_{t+1}), \quad (62)$$

$$\mathbb{E}_t[\Delta e_{t+1}] + r^*_{t+1} - r_{t+1} = cov_t(\widehat{m}^*_{t+1}, -\Delta e_{t+1}) + \frac{1}{2}var_t(\Delta e_{t+1}) \quad (63)$$

Combining the above yields (14).

B.2. Limits to International Arbitrage

Consider the following aggregate Euler equations capturing within-country idiosyncratic risk, now allowing for shocks to international returns (due to limits to international arbitrage) u_{t+1}:

$$\mathbb{E}_t\left[\widehat{M}_{t+1}\right] R_{t+1} = 1, \quad (64)$$

$$\mathbb{E}_t\left[\widehat{M}^*_{t+1}\right] R^*_{t+1} = 1, \quad (65)$$

$$\mathbb{E}_t\left[\widehat{M}^*_{t+1}\frac{\mathcal{E}_t}{\mathcal{E}_{t+1}}\right] R_{t+1} = \mathbb{E}_t[e^{u^f_{t+1}}], \quad (66)$$

$$\mathbb{E}_t\left[\widehat{M}_{t+1}\frac{\mathcal{E}_{t+1}}{\mathcal{E}_t}\right] R^*_{t+1} = \mathbb{E}_t[e^{u^f_{t+1}}] \quad (67)$$

where we assume intermediation shocks have zero mean ($\mathbb{E}_t[u^f_{t+1}] = 0$). Taking logs, we derive:

$$var_t(\Delta e_{t+1}) - var_t(u^f_{t+1}) + cov_t(\widehat{m}_{t+1} - \widehat{m}^*_{t+1}, \Delta e_{t+1}) = 0$$

Substituting $var_t(u^f_{t+1}) = u^f var_t(\Delta e_{t+1})$ and assuming $u^f \in [0, 1]$, rearranging:

$$var_t(\Delta e_{t+1})(1 - u^f) + cov_t(\widehat{m}_{t+1} - \widehat{m}^*_{t+1}, \Delta e_{t+1}) = 0 \tag{68}$$

Then, repeating the steps in Proposition 1, $cov_t(\Delta c_{t+1} - \Delta c^*_{t+1}, \Delta e_{t+1}) \leq 0$ requires:

$$-cov_t\left(\tilde{\beta}_{t+1} - \tilde{\beta}^*_{t+1}, \Delta e_{t+1}\right) > var_t(\Delta e_{t+1})(1 - u^f)$$

Applying the Cauchy-Schwarz identity and dividing by standard deviations yields condition (17).

B.3. Representative agent limit ($\tilde{\beta}^{(*)} \to \log \beta^{(*)}$)

In the representative agent case, if we restrict attention to the case where only a single bond is internationally traded, i.e. Home or Foreign currency denominated, the Backus-Smith correlation can be flipped even in the representative agent limit.

Corollary 2 (One Int'l Traded Asset, Representative Agent No-Arbitrage).
When only Foreign bonds are internationally traded such that equations (8), (9) and (11) hold, and $\tilde{\beta}^{(*)} \to \log \beta^{(*)}$, then $cov_t(\Delta c_{t+1} - \Delta c^*_{t+1}, \Delta e_{t+1}) < 0$ if and only if

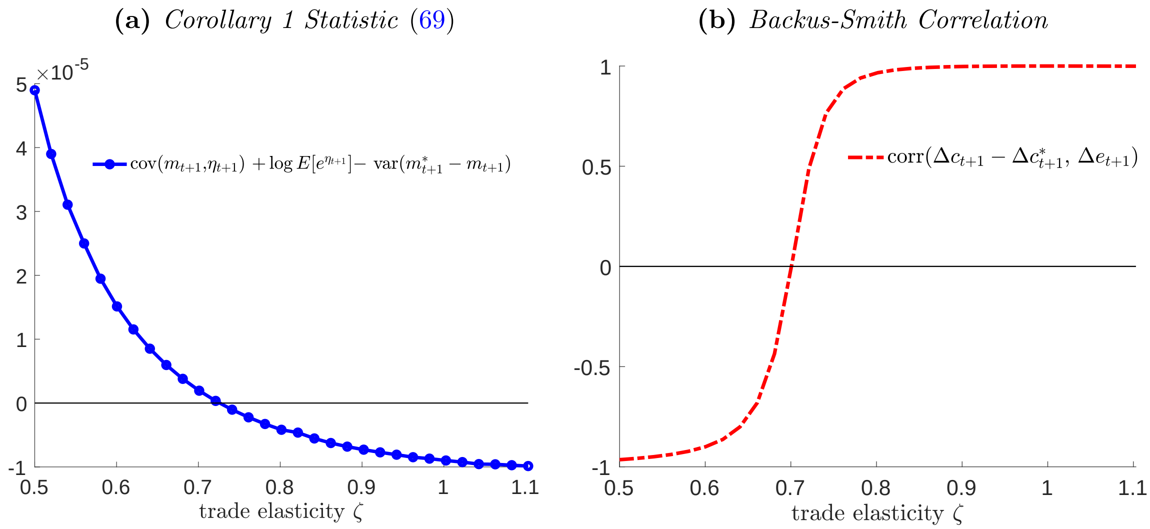
$$cov_t(m_{t+1}, \eta_{t+1}) + \log \mathbb{E}_t[e^{\eta_{t+1}}] \geq var_t(m^*_{t+1} - m_{t+1}) \tag{69}$$

where,

$$cov_t(m_{t+1}, \eta_{t+1}) = cov_t(m_{t+1}, \Delta e_{t+1}) - cov_t(m_{t+1}, m^*_{t+1}) + var_t(m_{t+1}) \tag{70}$$

Condition (69) provides a general characterization of mechanisms in the literature developed to resolve the Backus-Smith puzzle, e.g. consumption or production complementarities (Corsetti et al., 2008; Benigno and Thoenissen, 2008) reflected in (70). The RHS of (69) corresponds to the volatility of the exchange rate growth under complete markets, so is

Figure 2



Calibration: Within the representative agent model, we employ $\beta(C_{t-1}) = \omega C_{t-1}^{-u}$, with $\omega \in (0, 1)$, as the discount factor used as a stationarity-inducing device following Bodenstein (2011), C_t is the aggregate consumption bundle at Home. We set $\omega = 0.96$, $u = 0.005$, EIS coefficient $\gamma^{-1} = 1$, trade elasticity $\zeta \in [0.5, 1.1]$, home-bias $\alpha = 0.6$, $\alpha^* = 1 - \alpha$, persistence of Home endowment shock $\rho = 0.964$, and steady state Home endowment $I_H = 1$. Unconditional moments calculated from second-order simulation with one million draws.

strictly positive. Condition (69) is satisfied if either the non-traded component η_{t+1} leads to relative price fluctuations which are sufficiently safe from the perspective of a Home investor ($cov_t(m_{t+1}, \eta_{t+1}) > 0$) or the volatility of the non-traded component is high.[27] Figure 2a plots condition (69) from simulating a representative agent, two country, two good endowment version of our economy, allowing for internationally incomplete markets. Figure 2b displays the corresponding Backus-Smith correlation. The correlation is negative for low values of trade elasticity ζ, positive for higher values and intersects 0 at approximately the same value for ζ as Panel (a).[28]

In the representative agent limit, there is a stark contrast between the economy with one internationally traded asset (Corollary 2) and two assets (Proposition 1) because exchange rate risk becomes spanned when households trade in both Home and Foreign real bonds across borders, see also Chernov et al. (2024). A further limitation of these representative

[27]Equation (70) shows that non-traded risk results in relative price fluctuations which are particularly *safe* if the Home SDF is very volatile or international comovement of SDFs is low relative to the comovement of exchange rates and the Home SDF, which is counterfactual (Brandt et al., 2006).

[28]While SDFs and prices in the model (away from the Cole and Obstfeld (1991) are log-normally distributed only at the autarky limit, we find that the intersections in Figures (a) and (b) roughly coincide.

agents models is that they rely on a low volatility of exchange rates (Lustig and Verdelhan, 2019, pp 2241). Allowing for idiosyncratic risk within countries which co-moves with the exchange rate recovers a non-traded component which can deliver pro-cyclicality and high volatility of exchange rates.

B.4. Two Models for the Stochastic Discount Factor

Constantinides and Duffie (1996) The first example we look at considers perfectly integrated financial markets, i.e. there are no borrowing constraints. Following Constantinides and Duffie (1996), we construct a no-trade equilibrium where all agents choose to consume their endowment. Individual endowments are given by $I(s^t) = \delta(s^t)C(z^t) + D(z^t)$, where $D(z^t)$ denotes the aggregate dividend in the economy. We adopt the following process $\frac{\delta(z^{t+1},\nu^{t+1})}{\delta(z^t,\nu^t)} = \exp(\xi(\nu^{t+1})\sqrt{y(z^{t+1})} - y(z^{t+1})/2)$ where $\xi(\nu^{t+1})$ are the uninsurable idiosyncratic shocks, distributed standard normal for all individual histories ν^t, independently from the aggregate state z_t; and $y(z^t)$ is interpreted as cross-sectional volatility of idiosyncratic risk.[29] No-arbitrage pricing requires that any household investing in an arbitrary security with return $\tilde{R}(z^{t+1})$ satisfies:

$$\mathbb{E}\left[\beta e^{y(z^{t+1})\frac{\gamma(\gamma+1)}{2}} \left(\frac{C(z^{t+1})}{C(z^t)}\right)^{-\gamma} \tilde{R}(z^{t+1})\right] = 1 \tag{71}$$

where $\tilde{\beta}_{t+1} = \log\left(\beta e^{y(z^{t+1})\frac{\gamma(\gamma+1)}{2}}\right)$. Since all agents agree on the valuation of assets, standard arguments imply $C(s^t) = I(s^t)$ i.e. agents consume their own endowments and there is no trade in equilibrium. Intuitively, all risk is permanent and assets cannot be used to smooth consumption. An alternative interpretation is that $I(s^t)$ is income after a preliminary round of asset trade has exhausted all gains.

Applying this framework to Proposition 1, a risky foreign return implies that the volatility of permanent risk is low in periods of depreciation, i.e. foreign bonds yield higher return at

[29]The law of large numbers follows from properties of the normal distribution for ξ. Treating y_{t+1} as a constant, and using the moment generating function $M_\xi(h) = e^{h\xi}$ for $h \in \mathbb{R}$, $\mathbb{E}[e^{\xi\sqrt{y}-y/2}] = M_\xi(h)e^{-y/2} = e^0 = 1$.

times when households face low idiosyncratic risk:[30]

$$cov(\tilde{\beta}_{t+1}, \Delta e_{t+1}) < 0 \implies cov(y_{t+1}, \Delta e_{t+1}) < 0 \tag{72}$$

Krusell, Mukoyama & Smith (2011), Bilbiie (2024) The second example, considers a model where households earn high ($v_t = h$) or low ($v_t = l$) levels of income, facing an exogenous probability $1 - s$ of becoming a low type (Krusell et al., 2011) which we allow to be a function of (aggregate) output $s(Y(z^t))$ following Bilbiie (2024). In equilibrium, when households have a low income draw they would like to borrow, but we restrict focus on the zero liquidity limit (binding borrowing constraints).

The first-order condition for the Home saver purchasing a domestic risk free bond is given by:

$$1 = R(z^t)\, E_t\left[\beta\, \frac{s(z^{t+1})\,\left(\frac{C(z^{t+1},h)}{C(z^{t+1})}\right)^{-\gamma} + (1 - s(z^{t+1}))\,\left(\frac{C(z^{t+1},l)}{C(z^{t+1})}\right)^{-\gamma}}{\left(\frac{C(z^t,h)}{C(z^t)}\right)^{-\gamma}} \times \left(\frac{C(z^{t+1})}{C(z^t)}\right)^{-\gamma}\right],$$

where $C(z^{t+1}, l)$ is the consumption of (low type) constrained households and $C(z^{t+1}, h)$ is that of the (high type) saver. The $\beta-$wedge, which premultiplies marginal utility growth from aggregate consumption, arises from the probability of becoming a low type and the difference in the marginal utility of consumption across two states. Because of incomplete domestic markets, marginal utility in the low state is higher than marginal utility in the high state, therefore the saver attaches a premium on the risk-free bond.[31] The saver similarly attaches a premium on foreign bonds, adjusted for exchange rate risk (10).

For further illustration, we assume a transfer scheme such that saver's consumption is $C(z^{t+1}, h) = \omega C(z^{t+1})$, and hand-to-mouth agents' consumption is $C(z^{t+1}, l) = (1-\omega)C(z^{t+1})$ with $0.5 < \omega \leq 1$ so that saver's consumption is larger than the hand-to-mouth household's consumption.[32] The $\beta-$wedge then simplifies to: $\tilde{\beta}_{t+1} = \log\left(\beta\left[s(z^{t+1})\left(1 - \left(\frac{1-\omega}{\omega}\right)^{-\gamma}\right) + \left(\frac{1-\omega}{\omega}\right)^{-\gamma}\right]\right)$.

In this environment, a risky foreign return $cov(\tilde{\beta}_{t+1}, \Delta e_{t+1}) < 0$ implies that the probability

[30] In Section 4 we use estimates of permanent risk from Bayer et al. (2019) and find that it is not volatile enough to satisfy (16), highlighting the role of transitory risk or borrowing constraints described in the second example.

[31] One could add further frictions to the model by, e.g., making foreign bonds less liquid in the low state, generating "convenience yield" properties for the domestic bond, see for example Di Tella et al. (2024).

[32] As well as improving tractability, this transfer scheme also eliminates composition-driven explanations for Backus-Smith covariance (e.g. Kollmann, 2012).

of becoming hand to mouth $1 - s(z^{t+1})$ is low in periods of depreciation when returns to Foreign assets are high:

$$cov(\tilde{\beta}_{t+1}, \Delta e_{t+1}) < 0 \implies cov(s, \Delta e_{t+1}) > 0 \tag{73}$$

B.5. Trade in Risky Assets

If instead of allowing for trade in both risk-free assets, we allow for trade in Home and Foreign risky assets, then equations (7)–(11) are replaced by:

$$\mathbb{E}_t[\widehat{M}_{t+1}\tilde{R}_{t+1}] = 1, \tag{74}$$

$$\mathbb{E}_t\left[\widehat{M}_{t+1}\frac{\mathcal{E}_{t+1}}{\mathcal{E}_t}\tilde{R}^*_{t+1}\right] = 1, \tag{75}$$

$$\mathbb{E}_t[\widehat{M}^*_{t+1}\tilde{R}^*_{t+1}] = 1, \tag{76}$$

$$\mathbb{E}_t\left[\widehat{M}^*_{t+1}\left(\frac{\mathcal{E}_{t+1}}{\mathcal{E}_t}\right)^{-1}\tilde{R}_{t+1}\right] = 1, \tag{77}$$

where $\tilde{R}$ and $\tilde{R}^*$ are returns on risky Home and Foreign assets respectively.

Suppose Home and Foreign households trade in Home and Foreign currency denominated risky assets $\tilde{R}_{t+1}$ such that (74)- (77) hold. Assuming joint log normality, the above Euler equations imply:

$$\mathbb{E}_t[\widehat{m}_{t+1}] + \frac{1}{2}var_t(\widehat{m}_{t+1}) + \mathbb{E}_t[\tilde{r}_{t+1}] + \frac{1}{2}var_t(\tilde{r}_{t+1}) + cov_t(\widehat{m}_{t+1}, \tilde{r}_{t+1}) = 0, \tag{78}$$

$$\begin{aligned}\mathbb{E}_t[\widehat{m}_{t+1}] + \frac{1}{2}var_t(\widehat{m}_{t+1}) + \mathbb{E}_t[\tilde{r}^*_{t+1}] + \frac{1}{2}var_t(\tilde{r}^*_{t+1}) + \mathbb{E}_t[\Delta e_{t+1}] + \frac{1}{2}var_t(\Delta e_{t+1}) + \cdots \\ cov_t(\widehat{m}_{t+1}, \tilde{r}^*_{t+1}) + cov_t(\widehat{m}_{t+1}, \Delta e_{t+1}) + cov_t(\Delta e_{t+1}, \tilde{r}^*_{t+1}) = 0,\end{aligned} \tag{79}$$

$$\mathbb{E}_t[\widehat{m}^*_{t+1}] + \frac{1}{2}var_t(\widehat{m}^*_{t+1}) + \mathbb{E}_t[\tilde{r}^*_{t+1}] + \frac{1}{2}var_t(\tilde{r}^*_{t+1}) + cov_t(\widehat{m}^*_{t+1}, \tilde{r}^*_{t+1}) = 0, \tag{80}$$

$$\begin{aligned}\mathbb{E}_t[\widehat{m}^*_{t+1}] + \frac{1}{2}var_t(\widehat{m}^*_{t+1}) - \mathbb{E}_t[\Delta e_{t+1}] + \frac{1}{2}var_t(\Delta e_{t+1}) + \mathbb{E}_t[\tilde{r}_{t+1}] + \frac{1}{2}var_t(\tilde{r}_{t+1}) + \cdots \\ cov_t(\widehat{m}^*_{t+1}, \tilde{r}_{t+1}) + cov_t(\widehat{m}^*_{t+1}, -\Delta e_{t+1}) + cov_t(-\Delta e_{t+1}, \tilde{r}_{t+1}) = 0\end{aligned} \tag{81}$$

Combining (78) - (81)

$$var_t(\Delta e_{t+1}) = cov_t(\widehat{m}_{t+1} - \widehat{m}^*_{t+1}, \Delta e_{t+1}) + cov_t(\eta_{t+1}, \tilde{r}^*_{t+1} - \tilde{r}_{t+1}) \tag{82}$$

Away from the representative agent limit, how many assets would it take to impose full risk-sharing? In practice, many (more than two) assets are traded across borders but few of these are risk-free in real terms, e.g. long-maturity bonds and equity. We extend our main result to a framework with trade in multiple risky assets but when (nominally) risk-free bonds are not available.

Proposition A1 (Many Assets, Many Agents)

When Home and Foreign currency risky assets with returns $\tilde{r}_{t+1}^{(*)}$ *are internationally traded* (74)-(77)*, then* $cov_t(m_{t+1}^* - m_{t+1}, \Delta e_{t+1}) \leq 0$ *if and only if:*

$$-\rho_{\tilde{\beta}_{t+1}-\tilde{\beta}_{t+1}^*,\Delta e_{t+1}} + \frac{cov_t(\eta_{t+1}, \tilde{r}_{t+1}^* - \tilde{r}_{t+1})}{\sigma_t(\tilde{\beta}_{t+1} - \tilde{\beta}_{t+1}^*)\sigma_t(\Delta e_{t+1})} \geq \frac{\sigma_t(\Delta e_{t+1})}{\sigma_t(\tilde{\beta}_{t+1} - \tilde{\beta}_{t+1}^*)} \tag{83}$$

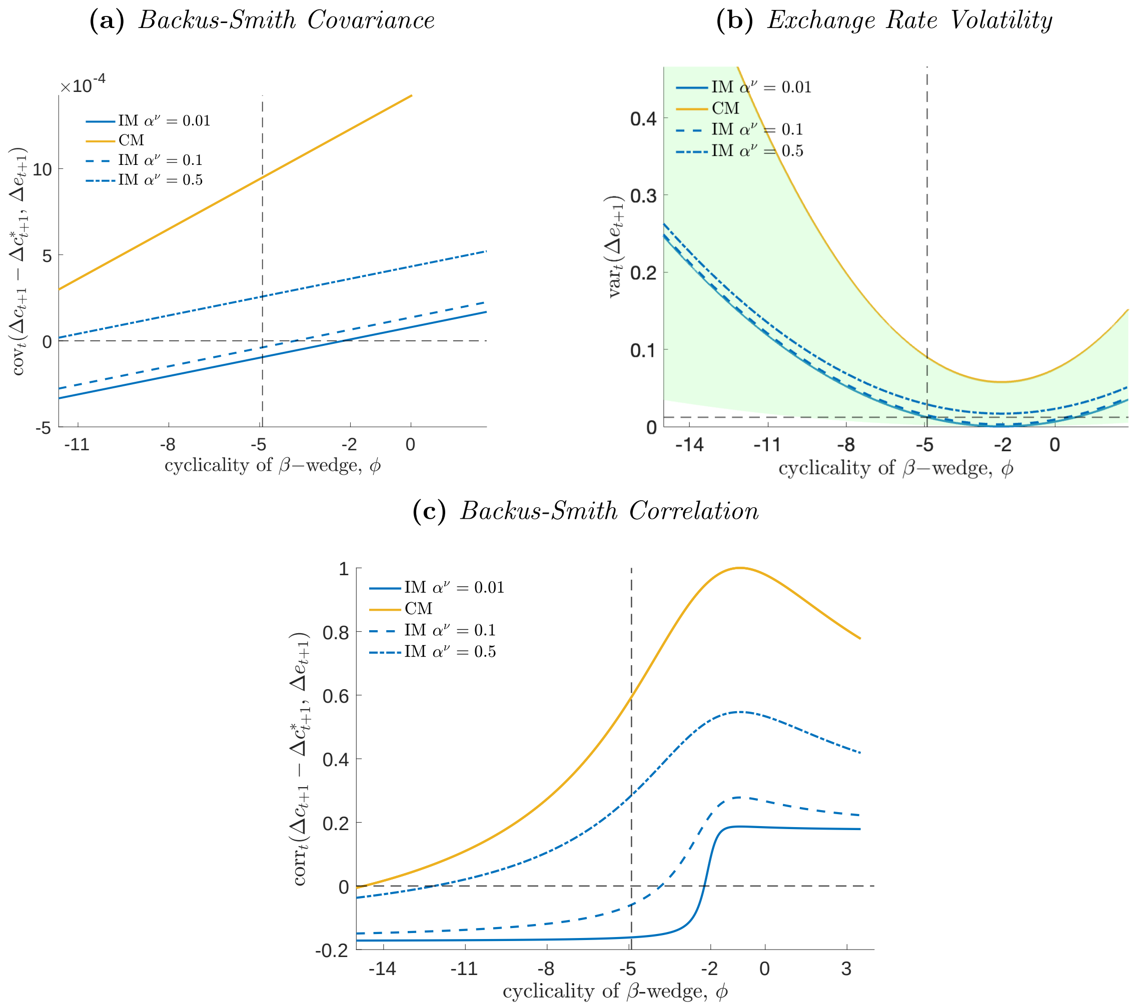
Proof. Rearrange (82) and apply the Cauchy-Schwarz Inequality. □

When only risky assets are traded, exchange rates need not be spanned, allowing further scope for pro-cyclical exchange rates. In particular, pro-cyclicality is recovered if the incomplete markets wedge co-varies positively with the differential return from a risky foreign asset. If, however, we additionally allow for trade in two nominally risk-free assets the new term in Proposition A1 ($cov_t(\eta_{t+1}, \tilde{r}_{t+1}^* - \tilde{r}_{t+1})$ converges to 0. This implies that Proposition A1 in the main body is the necessary and sufficient condition even when there is trade in many assets. Nonetheless, $\rho_{\tilde{\beta}_{t+1}-\tilde{\beta}_{t+1}^*,\Delta e_{t+1}}$ and $\sigma_t(\Delta e_{t+1})$ are themselves changing as the number of traded assets changes, see Section 5. Note also that this is true for any $r^{\widetilde{k(*)}}$ traded, as we detail next.

B.6. Comparative Statics Section 5

Figure 3 provides comparative statics with respect to the spanning of the country-specific factor α^ν. In each panel, the solid blue line shows our baseline calibration of $\alpha^\nu = 0.01$. Spanning of the common shock α is set at baseline value of 0.6 across blue lines. Both the volatility of the exchange rate and the Backus-Smith covariance are increasing in the spanning of the country specific shock, for a given cyclicality of the $\beta-$wedge. Panel (c) shows results in the correlation space.

Figure 3



B.7. Supplement Section 5

Country-specific shock applied to Home SDF. Consider the following pair of SDFs:

$$-\widehat{m}_{t+1} = \beta + \chi z_t + (\gamma - \phi)(\sqrt{z_t} u_{t+1} + \sigma_\nu u^\nu_{t+1}), \tag{84}$$

$$-\widehat{m}^*_{t+1} = \beta^* + \chi^* z_t + (\gamma - \phi^*)(\xi^* \sqrt{z_t} u_{t+1}) \tag{85}$$

where there is a country-specific factor driving Home consumption growth. We repeat our analysis focusing on the projection of moments on u^ν_{t+1}. The process for the incomplete markets wedge continues to be given by (36). Using (37):

$$\underbrace{\Gamma_1^{\nu 2}+\Gamma_2^{\nu 2}}_{var_t(\eta_{t+1}|u_{t+1}^{\nu})}=[(\gamma-\phi)^2-\kappa^{\nu}]\sigma_{\nu}^2 \tag{86}$$

Then:

$$\Gamma_1^{\nu}=\pm\sqrt{(\gamma-\phi)^2-\lambda^{\nu}},\Gamma_2^{\nu}=\pm\sqrt{\lambda^{\nu}-\kappa^{\nu}}$$

Using (32) and (33) we derive:

$$\Gamma_0^{\nu}=\frac{1}{2}((\gamma-\phi)^2-\kappa^{\nu})-(\gamma-\phi)\sqrt{(\gamma-\phi)^2-\lambda^{\nu}}, \tag{87}$$

$$\Gamma_0^{\nu}=0\frac{1}{2}((\gamma-\phi)^2-\kappa^{\nu}) \tag{88}$$

Combining:

$$\kappa^{\nu}=(\gamma-\phi)^2-(\gamma-\phi)^2\sqrt{(\gamma-\phi)^2-\lambda^{\nu}} \tag{89}$$

Finally, we derive

$$cov_t(m_{t+1}^*-m_{t+1},\Delta e_{t+1}|u_{t+1}^{\nu})=(\gamma-\phi)(1-\sqrt{1-\alpha^{\nu}})$$

which is strictly positive maintaining $\gamma-\phi>0$.

B.8. Supplement Section 5.1

Interest Rates Taking a log-normal approximation of (7)-(11), within the CIR-framework this implies:

$$r_t=\left(\hat{\chi}-\frac{1}{2}(\gamma-\phi)^2\right)z_t, \tag{90}$$

$$r_t^*=\left(\hat{\chi}^*-\frac{1}{2}(\gamma-\phi^*)^2\right)\xi^{*2}z_t+\left(\chi_{\nu}^*-\frac{1}{2}(\gamma-\phi^*)^2\right)\sigma_{\nu,t}^2 \tag{91}$$

Assuming $\hat{\chi}=1+\frac{1}{2}(\gamma-\phi)^2$ implies $r_t=z_t$. In turn, we assume $\chi^*=\chi$ and we pursue a similar normalization with χ_{ν}^* such that $\chi_{\nu}^*-\frac{1}{2}(\gamma-\phi^*)^2=\tilde{\zeta}$ and we calibrate $\tilde{\zeta}$ to match foreign mean rates, so:

$$r_t^*=\xi^{*2}z_t+\tilde{\zeta}\sigma_{\nu,t}^2 \tag{92}$$

which further implies a high correlation of r and r^*.

Exchange Rates and the Fama Regression The UIP regression, related to Fama (1984), is often expressed as:

$$\Delta e_{t+1} = \alpha + \beta_{UIP}(r_t - r_t^*) + \epsilon_{t+1} \tag{93}$$

In the above univariate regression:

$$\beta_{UIP} = \frac{cov(\mathbb{E}_{UIP,t}[\Delta e_{t+1}], r_t - r_t^*)}{var(r_t - r_t^*)} \tag{94}$$

Within the CIR framework, using the parametric assumptions above:

$$\beta_{UIP} = \frac{(\hat{\chi} - \hat{\chi}^*\xi^{*2} + \Gamma_0)(1 - \xi^{*2})\sigma_z^2 + (-\chi_\nu^* + \Gamma_0^\nu)(-\tilde{\zeta})\,\mathrm{var}(\sigma_\nu^2)}{(1 - \xi^{*2})^2\sigma_z^2 + (-\tilde{\zeta})^2\,\mathrm{var}(\sigma_{\nu,t}^2)} \tag{95}$$

and the data suggests that this should be close to -1.[33] Then, the R^2 of the UIP regression above is given by:

$$R_{UIP}^2 = \frac{\beta_{UIP}^2 var(r_t - r_t^*)}{var(\Delta e_{t+1})} \tag{96}$$

where:

$$var(\Delta e_{t+1}) = \kappa\mathbb{E}_t[z_t] + \kappa^\nu\mathbb{E}_t[\sigma_{\nu,t}^2] + (\hat{\chi} - \hat{\chi}^*\xi^2 + \Gamma_0)^2 var(z_t) + (-\chi_\nu^* + \Gamma_0^\nu)^2 var(\sigma_{\nu,t}^2) \tag{97}$$

Note that this only coincides with the 'true' R^2 of e.g. Meese and Rogoff (1983) if (95) is the best linear predictor. In general,

$$R^2 = \frac{var(\mathbb{E}_t[\Delta e_{t+1}])}{var(\Delta e_{t+1})} = \frac{(\hat{\chi} - \hat{\chi}^*\xi^2 + \Gamma_0)^2 var(z_t) + (-\chi_\nu^* + \Gamma_0^\nu)^2 var(\sigma_{\nu,t}^2)}{\kappa\mathbb{E}_t[z_t] + \kappa^\nu\mathbb{E}_t[\sigma_{\nu,t}^2] + (\hat{\chi} - \hat{\chi}^*\xi^2 + \Gamma_0)^2 var(z_t) + (-\chi_\nu^* + \Gamma_0^\nu)^2 var(\sigma_{\nu,t}^2)} \tag{98}$$

where the second equality follows from the quantities in the CIR framework. The R^2 tends to one as the conditional volatility tends to zero since this means that at time t, exchange rates are fully predictable. Instead, it tends to zero as the mean of log exchange rate changes is close to zero or constant. In turn, $R^2 = R_{UIP}^2$ as $\mathbb{E}[\sigma_{\nu,t}] = 0$ or $var(\mathbb{E}[\sigma_{\nu,t}]) = 0$.

Unconditional Backus-Smith Applying the law of total variance implies:

$$\begin{aligned} cov(\Delta c_{t+1} - \Delta c_{t+1}^*, \Delta e_{t+1}) &= cov_t(\Delta c_{t+1} - \Delta c_{t+1}^*, \Delta e_{t+1}) + \\ (\chi - \chi^*\xi^{*2})var(z_t) + \chi_\nu^{*2} var(\sigma_{\nu,t}) &+ (\chi - \chi^*\xi^{*2})(\chi_b - \chi_b^* + \Gamma_0)var(z_t) + \\ &(-\chi_\nu)\Gamma_0^\nu var(\sigma_{\nu,t}) \end{aligned} \tag{99}$$

[33]New evidence suggests this sign is in fact time varying, see Bussière, Chinn, Ferrara and Heipertz (2022).

where the first term is detailed in the proof to Proposition 2. The consumption differential is given by $\Delta c_{t+1} - \Delta c^*_{t+1}$ The conditional and unconditional volatility of consumption differentials is identical. The unconditional volatility of exchange rates is given by (97).

C. Empirical Results and Robustness

C.1. Berger et al. (2023) wedge

Figure 5 plots the time-series of the consumption based wedge from Berger et al. (2023) dataset, reconstructed for different values of $\gamma \in \{1, 3, 5, 7.5\}$.

Figure 4: *Time-series of Consumption based Wedge*

(a) *Inverse of EIS* $\gamma = 1$ **(b)** *Inverse of EIS* $\gamma = 3$ **(c)** *Inverse of EIS* $\gamma = 5$ **(d)** *Inverse of EIS* $\gamma = 7.5$

Table 8 lists the volatility of bilateral real exchange rate, the (negative of the) correlation of the wedge with real exchange rate growth and the threshold in equation (16) for different values of $\gamma^{-1} \in \{0.1, 0.2\}$ for seventeen advanced economies. Both conditional and unconditional moments are presented in the Table.

There are two main takeaways. First, for fifteen of the seventeen bilateral pairs, the exceptions being Japan and Switzerland, and for different values of γ, we find that the exchange rates are risky with respect to the $\tilde{\beta}$ wedge – the correlation with the bilateral real exchange rate is negative. Second, the threshold is low enough such that the $\tilde{\beta}$ wedge can satisfy the inequality (16) without a need for segmentation shocks i.e. we can set $u = 0$ and still reconcile the sign of the Backus-Smith puzzle. When $\gamma = 1$, exchange rates are still risky with respect to the $\tilde{\beta}$ wedge, but the wedge is not volatile enough and as a result the inequality with the threshold is not met. As the EIS falls, the within group dispersion in marginal utility of consumption share growth is amplified due to a Jensen's term. The within-group dispersion in marginal utility growth proxies for the dispersion in marginal utility growth across idiosyncratic states. As we lower γ^{-1}, the correlation of the wedge with exchange rate growth does not change as much as volatility of the wedge, which directly affects the threshold, making it easier to satisfy the inequality (16).

We can conduct a stronger empirical test of theory and verify if the correlation of difference in as-if representative agent log SDFs with real exchange rate growth is positive. We construct the as-if representative agent log SDF in the U.S. as implied by the model: $\widehat{m} = -\gamma \Delta c + \tilde{\beta} + \log \beta$. Since we have assumed the wedge is constant in Foreign, the Foreign SDF is the scaled consumption growth $m^* = -\gamma \Delta c^* + \log \beta^*$.

Tables 9 lists these correlations (both conditional and unconditional) for all seventeen bilateral pairs for values of $\gamma^{-1} \in \{0.1, 0.2\}$, and cross-country average is noted in the last row. Consistent with theoretical prediction, the correlation of relative SDF with exchange rate growth is positive for all pairs with the exception of Japan and Switzerland.

Overall, we robustly find that (i) exchange rates are risky with respect to the wedge, (ii) inequality (16) is satisfied for values of $\gamma^{-1} \leq \frac{1}{3}$, and (iii) correlation of relative pricing kernels (constructed using the wedge) with exchange rate is positive while correlation of relative consumption growth with exchange rate is negative.

Table 8: *Empirical Moments: Real Exchange Rate Growth and Berger et al. (2023) Wedge*

Panel A $\gamma^{-1} = 0.2$

	Unconditional		Conditional	
ISO	$-\text{Corr}(\tilde{\beta}, \Delta e)$	Threshold	$-\text{Corr}_t(\tilde{\beta}, \Delta e)$	Threshold_t
AUS	0.05	0.10	0.07	0.08
BEL	0.24	0.08	0.25	0.09
CAN	0.28	0.08	0.14	0.07
CHE	-0.03	0.07	-0.01	0.06
DEU	0.23	0.08	0.31	0.07
DNK	0.25	0.08	0.36	0.09
ESP	0.32	0.09	0.19	0.08
FIN	0.25	0.09	0.25	0.08
FRA	0.26	0.08	0.27	0.08
GBR	0.36	0.08	0.58	0.08
IRL	0.23	0.07	0.06	0.06
ITA	0.36	0.09	0.23	0.09
JPN	-0.39	0.08	-0.44	0.08
NLD	0.25	0.08	0.17	0.09
NOR	0.23	0.09	0.35	0.09
PRT	0.30	0.09	0.06	0.08
SWE	0.28	0.10	0.24	0.08
AVERAGE	0.20	0.08	0.18	0.08

Panel B $\gamma^{-1} = 0.1$

	Unconditional			Conditional		
ISO	$\sigma(\Delta e)$	$-\text{Corr}(\tilde{\beta}, \Delta e)$	Threshold	$\sigma_t(\Delta e)$	$-\text{Corr}_t(\tilde{\beta}, \Delta e)$	Threshold_t
AUS	0.13	-0.02	0.04	0.10	-0.02	0.03
BEL	0.10	0.23	0.03	0.09	0.20	0.04
CAN	0.10	0.21	0.03	0.08	0.06	0.03
CHE	0.09	-0.03	0.03	0.08	-0.03	0.03
DEU	0.10	0.23	0.03	0.08	0.29	0.03
DNK	0.10	0.24	0.03	0.09	0.32	0.04
ESP	0.11	0.30	0.04	0.09	0.18	0.03
FIN	0.12	0.24	0.04	0.09	0.21	0.03
FRA	0.10	0.25	0.03	0.09	0.23	0.03
GBR	0.11	0.34	0.04	0.09	0.55	0.03
IRL	0.09	0.19	0.03	0.06	0.05	0.02
ITA	0.11	0.32	0.04	0.10	0.17	0.04
JPN	0.11	-0.41	0.04	0.10	-0.50	0.03
NLD	0.11	0.25	0.03	0.09	0.13	0.04
NOR	0.12	0.19	0.04	0.10	0.26	0.04
PRT	0.11	0.28	0.04	0.09	0.03	0.03
SWE	0.13	0.24	0.04	0.09	0.19	0.03
AVERAGE	0.11	0.18	0.04	0.09	0.14	0.03

Notes: Table lists the negative of the correlation between the US discount factor wedge ($\tilde{\beta}$) and the bilateral real exchange rate growth (Δe), the threshold for exchange rate cyclicality described in equation (16) assuming domestic incomplete markets only within the US ($\tilde{\beta}^* = 0$), and the standard deviation of the real exchange rate growth for pair of seventeen advanced economies. Real exchange rate growth is constructed from Jordà et al. (2017) database. US discount factor wedge is constructed as in Berger et al. (2023) from Consumer Expenditure Survey in the US for $\gamma^{-1} = \{0.2, 0.1\}$ in Panels A, and B. Standard deviation of the discount factor wedge is provided in Table 1. Sample: 1992–2017 (annual).

Table 9: *Correlation between Pricing Kernels and Real Exchange Rate Growth*

Panel A $\gamma^{-1} = 0.2$

	Unconditional		Conditional	
ISO	Corr($\Delta c - \Delta c^{*}, \Delta e$)	Corr($m^{*} - \widehat{m}, \Delta e$)	Corr$_t$($\Delta c - \Delta c^{*}, \Delta e$)	Corr$_t$($m^{*} - \widehat{m}, \Delta e$)
AUS	-0.39	0.03	-0.40	0.06
BEL	-0.30	0.23	-0.44	0.23
CAN	-0.12	0.27	-0.18	0.14
CHE	-0.09	-0.03	-0.23	-0.02
DEU	0.08	0.23	-0.03	0.30
DNK	-0.34	0.22	-0.42	0.33
ESP	-0.01	0.32	-0.00	0.19
FIN	-0.38	0.22	-0.32	0.23
FRA	-0.25	0.25	-0.34	0.25
GBR	0.07	0.37	-0.09	0.58
IRL	0.08	0.24	0.52	0.09
ITA	0.01	0.35	0.06	0.23
JPN	0.19	-0.37	0.05	-0.44
NLD	0.19	0.26	-0.12	0.17
NOR	-0.27	0.21	-0.63	0.33
PRT	0.21	0.31	0.09	0.07
SWE	-0.27	0.27	-0.28	0.23
AVERAGE	-0.09	0.20	-0.16	0.18

Panel B $\gamma^{-1} = 0.1$

	Unconditional		Conditional	
ISO	Corr($\Delta c - \Delta c^{*}, \Delta e$)	Corr($m^{*} - \widehat{m}, \Delta e$)	Corr$_t$($\Delta c - \Delta c^{*}, \Delta e$)	Corr$_t$($m^{*} - \widehat{m}, \Delta e$)
AUS	-0.39	-0.03	-0.40	-0.04
BEL	-0.30	0.22	-0.44	0.18
CAN	-0.12	0.21	-0.18	0.06
CHE	-0.09	-0.03	-0.23	-0.04
DEU	0.08	0.23	-0.03	0.29
DNK	-0.34	0.22	-0.42	0.30
ESP	-0.01	0.31	-0.00	0.18
FIN	-0.38	0.22	-0.32	0.20
FRA	-0.25	0.24	-0.34	0.22
GBR	0.07	0.34	-0.09	0.56
IRL	0.08	0.21	0.52	0.08
ITA	0.01	0.32	0.06	0.17
JPN	0.19	-0.40	0.05	-0.50
NLD	0.19	0.25	-0.12	0.12
NOR	-0.27	0.18	-0.63	0.24
PRT	0.21	0.29	0.09	0.03
SWE	-0.27	0.23	-0.28	0.18
AVERAGE	-0.09	0.18	-0.16	0.13

Notes: Table lists the correlation of relative consumption growth and exchange rate, and correlation of the relative SDF and real exchange rate growth assuming domestic incomplete markets only within the US ($\tilde{\beta}^{*} = 0$). Real exchange rate growth is constructed from Jordà et al. (2017) database. US discount factor wedge is constructed as in Berger et al. (2023) from Consumer Expenditure Survey in the US for $\gamma^{-1} = \{0.2, 0.1\}$ in Panels A, and B. Sample: 1992–2017 (annual). See text for details.

C.1.1 Alternate Conditioning Set

Table 10: *Empirical Moments: Real Exchange Rate Growth and Berger et al. (2023) Wedge with a Large Conditioning Set*

ISO	$\sigma_t(\Delta e)$	$-\text{Corr}_t(\tilde{\beta}, \Delta e)$	Threshold_t
AUS	0.08	0.30	0.23
BEL	0.07	0.22	0.22
CAN	0.07	0.40	0.17
CHE	0.06	0.38	0.16
DEU	0.08	0.65	0.22
DNK	0.08	0.49	0.21
ESP	0.08	0.50	0.24
FIN	0.08	0.34	0.19
FRA	0.09	0.39	0.22
GBR	0.07	0.60	0.20
IRL	0.05	-0.02	0.18
ITA	0.08	0.37	0.21
JPN	0.09	-0.37	0.21
NLD	0.08	0.31	0.24
NOR	0.09	0.31	0.25
PRT	0.08	0.49	0.25
SWE	0.09	0.37	0.23
AVERAGE	0.08	0.34	0.21

We conduct robustness with a larger conditioning set to construct moments of Δe_{t+1}, and $\tilde{\beta}_{t+1}$. We control for date t values of log consumption in US and Foreign (c_t, c_t^*), short term nominal interest rate (i_t, i_t^*), level of long term interest rate in each country, log CPI in each country, and log bilateral real exchange rate (e_t). Relative to the baseline, we allow each country variables to have a separate loading in the conditioning set. Table 10 shows that the correlation remains sufficiently negative to meet the threshold. Results are shown for $\gamma^{-1} = 0.3$.

C.1.2 Panel Fixed Effects Regressions

Table 11: *Panel Fixed Effects Empirical Moments: Real Exchange Rate Growth and Berger et al. (2023) Wedge*

ISO	$\sigma_t(\Delta e)$	$-\text{Corr}_t(\tilde{\beta}, \Delta e)$	Threshold_t
AUS	0.12	0.10	0.21
BEL	0.09	0.24	0.17
CAN	0.09	0.24	0.17
CHE	0.09	0.10	0.15
DEU	0.09	0.26	0.17
DNK	0.09	0.24	0.17
ESP	0.10	0.18	0.18
FIN	0.11	0.24	0.20
FRA	0.09	0.27	0.17
GBR	0.10	0.45	0.17
IRL	0.09	0.02	0.16
ITA	0.10	0.24	0.17
JPN	0.10	-0.37	0.17
NLD	0.09	0.26	0.17
NOR	0.11	0.29	0.20
PRT	0.10	0.19	0.18
SWE	0.11	0.35	0.20
AVERAGE	0.10	0.19	0.18

We now construct the conditional moments by residualizing the beta wedge and real exchange rate using a panel fixed effects regression. In the baseline exercise reported in the main text, the residuals were constructed from each country's own regression, allowing both the intercept and slope to be different across bilateral pair of countries. We continue to set $\gamma^{-1} = 0.3$.

Table 11 reports the results. The conditional correlation of the beta wedge with exchange rate is sufficiently negative that it meets the threshold.[34]

[34] We briefly note why the conditional correlation estimated here may differ from the conditional correlation in the main text. Fixed effect regression based residual correlations emphasize the co-movement of deviations relative to common global relationships. If the true underlying relationships differ across countries (heterogeneous effects), imposing common slopes reduces correlation or even distorts it, because residuals become mixed with systematic differences in responsiveness.

C.2. Permanent income risk based wedge

Bayer et al. (2019) construct a measure of very persistent income risk using Survey of Income Participation Program (SIPP). In the Constantinides and Duffie (1996) model, the discount factor wedge $\tilde{\beta}^{CD}$ is a function of the cross-sectional variance of the permanent income process y^2. Namely:

$$\tilde{\beta}^{CD} = \frac{\gamma\,(\gamma+1)}{2} y^2$$

Using $\gamma^{-1} = 0.3$, and the variance of income risk series estimated in Bayer et al. (2019), we reconstruct the beta wedge assuming only permanent income risk. We keep the sample fixed at annual 1992–2013 (last year for which Bayer et al. (2019) series is available).

Table 12: *Empirical Moments: Real Exchange Rate Growth and Bayer et al. (2019) Wedge*

ISO	$\sigma(\Delta e)$	$-\mathrm{Corr}(\tilde{\beta}^{\mathrm{CD}}, \Delta \mathrm{e})$	$\sigma(\tilde{\beta}^{CD})$	Threshold
AUS	0.13	0.22	0.07	1.88
BEL	0.12	0.08	0.07	1.66
CAN	0.09	0.42	0.07	1.22
CHE	0.12	-0.09	0.07	1.74
DEU	0.12	0.10	0.07	1.68
DNK	0.12	0.07	0.07	1.64
ESP	0.12	0.11	0.07	1.68
FIN	0.12	0.09	0.07	1.76
FRA	0.12	0.09	0.07	1.64
GBR	0.13	0.36	0.07	1.78
IRL	0.09	-0.13	0.07	1.34
ITA	0.12	0.08	0.07	1.72
JPN	0.13	-0.37	0.07	1.80
NLD	0.12	0.10	0.07	1.69
NOR	0.12	0.30	0.07	1.73
PRT	0.12	0.11	0.07	1.63
SWE	0.13	0.21	0.07	1.84
AVERAGE	0.12	0.11	0.07	1.67

Table 12 reports the results for condition (16) with the $\beta-$wedge constructed as shown in equation (71) in Section B.4. While the discount factor wedge measured only with permanent income risk also comoves negatively with exchange rate, the volatility of this wedge is of two orders of magnitude lower than the discount factor wedge constructed from Berger et al.

(2023) with $\gamma^{-1} = 0.3$ (see Table 1 for summary statistics of the Berger et al. (2023) wedge). Consequently, relying solely on permanent income risk based wedge is insufficient to explain the Backus-Smith cyclicality puzzle.

C.3. Bilateral Wedge from GRID

There is another major difference in the way the wedge is calculated using the GRID data relative to the Berger et al. (2023). The data in the GRID is on $\frac{1}{N_g}\sum_{\nu=1}^{N_g} \log\left(\frac{I^{\nu}_{t+1}}{I^{\nu}_t}\right)$, where I^{ν} is individual with idiosyncratic history ν_t's income. We can therefore only construct $\frac{1}{N_g}\sum_{i=1}^{N_g} \log\left(\frac{I^{\nu}_{t+1}}{I^{\nu}_t}\right)^{-\gamma}$. The true wedge and the proxy wedge are related by a Jensen's term:

$$\underbrace{\log \frac{1}{N_g}\sum_{\nu=1}^{N_g}\left(\frac{I^{\nu}_{t+1}/I_{t+1}}{I^{\nu}_t/I_t}\right)^{-\gamma}}_{\text{True wedge}} = \underbrace{\frac{1}{N_g}\sum_{\nu=1}^{N_g}\log\left(\frac{I^{\nu}_{t+1}/I_{t+1}}{I^{\nu}_t/I_t}\right)^{-\gamma}}_{\text{Proxy Wedge}} + \text{Jensen gap}$$

where I_t is the average income in the economy. We can decompose either of the wedges into within group term and a composition term:

$$\log \frac{1}{N_g}\sum_{\nu=1}^{N_g}\left(\frac{I^{\nu}_{t+1}/I_{t+1}}{I^{\nu}_t/I_t}\right)^{-\gamma} = \log\left(\frac{I^{g}_{t+1}/I_{t+1}}{I^{g}_t/I_t}\right)^{-\gamma} + \log \frac{1}{N_g}\sum_{\nu=1}^{N_g}\left(\frac{I^{\nu}_{t+1}/I^{g}_{t+1}}{I^{\nu}_t/I^{g}_t}\right)^{-\gamma}$$

$$\frac{1}{N_g}\sum_{\nu=1}^{N_g}\log\left(\frac{I^{\nu}_{t+1}/I_{t+1}}{I^{\nu}_t/I_t}\right)^{-\gamma} = \log\left(\frac{I^{g}_{t+1}/I_{t+1}}{I^{g}_t/I_t}\right)^{-\gamma} + \frac{1}{N_g}\sum_{\nu=1}^{N_g}\log\left(\frac{I^{\nu}_{t+1}/I^{g}_{t+1}}{I^{\nu}_t/I^{g}_t}\right)^{-\gamma}$$

where I^g_t denotes average income of the group g. Both the true and the proxy wedges have identical composition terms capturing change in powered income shares of groups in the economy. Both also capture a within group dispersion of income shares across households but a Jensen's gap arises– the true wedge captures the convexity component from averaging before taking logs. Taking a second order approximation of the wedge, it can be shown that the Jensen gap term increases in γ since the degree of convexity goes up with γ. There are no a priori reasons to believe that the Jensen gap term would not affect the correlation of the wedge with exchange rate. However, when we computed the correlation of $\beta-$wedges computed with different EIS parameters for the U.S., we found that the correlation was relatively stable as we increased γ.

We measure bilateral wedges using micro statistics from the Global Repository of Income

Dynamics (Guvenen et al., 2022, GRID henceforth). We focus on a sample of advanced economies for which we can obtain the longest panel to construct the bilateral wedge against the U.S. The annual data sample covers Canada, Denmark, France, Germany, Italy, Norway, Sweden, the UK, and the US spanning 1998–2015.

Variables used in our wedge construction are residual log earnings growth for top 1 percent, 2.5 percent, 5 percent, 10 percent of each country's log earnings distribution in year t, and the mean residual log earnings growth for that year. These statistics are computed for Male in age groups 25–55 population of each country. GRID constructs residual log earnings growth from first regressing log earnings on age dummies for each year t and then taking average of growth rate.[35]

Denote with $\overline{\log I_{gt}}$ the average (residual) log earnings for the group g, and $\overline{\log I_t}$ as the average (residual) log earnings in the country at time t. Then the income share at date t for a group g is defined as $\varphi^I_{gt} \equiv \exp\left(\overline{\log I_{gt}} - \overline{\log I_t}\right)$. The discount factor wedge for a group g is then constructed as:[36]

$$\beta^I_{gt} = \left(\frac{\varphi^I_{gt+1}}{\varphi^I_{gt}}\right)^{-\gamma} \tag{100}$$

where γ is the inverse of the intertemporal elasticity of substitution. We de-mean each country's wedge for (i) partially addressing measurement error as in Berger et al. (2023), and (ii) comparison across countries.[37] The bilateral wedge is then given by the difference in the de-meaned country specific wedges for group g:

$$\Delta\beta_{gt} \equiv \beta^I_{gt} - \beta^{*I}_{gt} \tag{101}$$

C.3.1 When can we use correlation of income wedge to proxy correlation of consumption wedge.

Ideally, we would use micro-data on consumption for each country and construct bilateral wedge distances. Instead, we need to rely on micro-data for income. The two key concerns are i) the correlation of consumption shares differs from the correlation of growth shares, ii)

[35] In unreported results, we verified the results are robust to using statistics for all genders, age 25–55 population.

[36] The data in GRID is averages at the percentile level. This proxy construction is thus different from Berger et al. (2023) in that we are taking average income shares to construct a wedge, whereas Berger et al. (2023) construct average of the individual wedges.

[37] Results are robust to using the non-deamened wedges.

the volatility of consumption shares is significantly lower than the volatility of income shares. Consider the following statistical model capturing the idea of consumption smoothing:

$$\frac{C^{\nu}_{t+1}}{C^{\nu}_{t}} = \theta^{i}\frac{I^{\nu}_{t+1}}{I^{\nu}_{t}}, \quad \frac{C^{\nu*}_{t+1}}{C^{\nu*}_{t}} = \theta^{\nu*}\frac{I^{\nu*}_{t+1}}{I^{\nu*}_{t}}, \tag{102}$$

$$\frac{C_{t+1}}{C_{t}} = \theta\frac{I_{t+1}}{I_{t}}, \quad \frac{C^{*}_{t+1}}{C^{*}_{t}} = \theta^{*}\frac{I^{*}_{t+1}}{I^{*}_{t}} \tag{103}$$

where $\theta^{(*)}, \theta^{i(*)} \leq 1$ such that consumption growth is less volatile than income growth, and for high income individuals on their Euler equations $\theta^{\nu(*)} < \theta^{(*)}$, i.e. they engage in more consumption smoothing than the aggregate. As long as $\theta^{\nu}/\theta \approx \theta^{\nu*}/\theta^{*}$, such that consumption smoothing of the marginal investor relative to the aggregate is similar across countries, then:

$$\begin{aligned} cov\left(\tilde{\beta}_{t+1} - \tilde{\beta}^{*}_{t+1}, \Delta e_{t+1}\right) &= cov\left(-\gamma\log\left(\frac{\theta^{\nu}}{\theta}\frac{I^{\nu}_{t+1}/I^{\nu}_{t}}{I_{t+1}/I_{t}}\right) + \gamma\log\left(\frac{\theta^{\nu*}}{\theta^{*}}\frac{I^{\nu*}_{t+1}/I^{\nu*}_{t}}{I^{*}_{t+1}/I^{*}_{t}}\right), \Delta e_{t+1}\right) \\ &\approx \frac{\theta^{i}}{\theta}cov\left(\tilde{\beta}^{y}_{t+1} - \tilde{\beta}^{y*}_{t+1}, \Delta e_{t+1}\right) \end{aligned}$$

where $\tilde{\beta}^{y}_{t+1} = -\gamma\log\left(\frac{I^{\nu}_{t+1}/I^{\nu}_{t}}{I_{t+1}/I_{t}}\right)$.

Critically, in the correlation space, the ratio θ^{ν}/θ does not affect scaling. Specifically, when $\theta^{\nu}/\theta \approx \theta^{\nu*}/\theta^{*}$, then:

$$\rho_{\tilde{\beta}_{t+1}-\tilde{\beta}^{*}_{t+1},\Delta e_{t+1}} = \frac{cov_t\left(\tilde{\beta}_{t+1} - \tilde{\beta}^{*}_{t+1}, \Delta e_{t+1}\right)}{\sigma_t(\tilde{\beta}_{t+1} - \tilde{\beta}^{*}_{t+1})\sigma_t(\Delta e_{t+1})} \approx \frac{cov_t\left(\tilde{\beta}^{y}_{t+1} - \tilde{\beta}^{y*}_{t+1}, \Delta e_{t+1}\right)}{\sigma_t(\tilde{\beta}^{y}_{t+1} - \tilde{\beta}^{y*}_{t+1})\sigma_t(\Delta e_{t+1})} = \rho_{\tilde{\beta}^{y}_{t+1}-\tilde{\beta}^{y*}_{t+1},\Delta e_{t+1}}$$

C.3.2 Summary Statistics and Additional Results with GRID dataset

Table 13: *Summary Statistics for GRID data set*

Panel A: Log-income based Country-specific Wedge

	Top 10%		Top 5%		Top 2.5%		Top 1%	
iso	$\sigma(\beta^I_{gt})$	$\text{Corr}(\beta^I_{gt}, \widehat{Y}_t)$	$\sigma(\beta^I_{gt})$	$\text{Corr}(\beta^I_{gt}, \widehat{Y}_t)$	$\sigma(\beta^I_{gt})$	$\text{Corr}(\beta^I_{gt}, \widehat{Y}_t)$	$\sigma(\beta^I_{gt})$	$\text{Corr}(\beta^I_{gt}, \widehat{Y}_t)$
CAN	0.04	0.03	0.09	0.09	0.10	0.06	0.10	-0.04
DEU	0.06	-0.36	0.13	-0.27	0.13	-0.21	0.10	-0.24
DNK	0.06	0.30	0.14	-0.07	0.15	-0.16	0.15	-0.26
FRA	0.07	-0.19	0.18	-0.29	0.24	-0.28	0.27	-0.27
ITA	0.13	0.33	0.33	0.10	0.37	-0.00	0.32	-0.04
NOR	0.07	0.52	0.12	0.43	0.18	0.32	0.28	0.30
SWE	0.06	-0.53	0.16	-0.58	0.22	-0.60	0.25	-0.60
USA	0.06	-0.17	0.09	-0.03	0.12	-0.01	0.13	-0.07
AVERAGE	0.07	-0.01	0.15	-0.08	0.19	-0.11	0.20	-0.15

Panel B: Log-Income based Bilateral Wedge

	$\sigma(\Delta e_t)$	$\sigma(\Delta\beta_{gt})$			
iso		Top 10%	Top 5%	Top 2.5%	Top 1%
CAN	0.09	0.03	0.04	0.05	0.06
DEU	0.10	0.10	0.16	0.17	0.17
DNK	0.10	0.08	0.14	0.16	0.16
FRA	0.10	0.08	0.14	0.16	0.18
ITA	0.11	0.13	0.26	0.25	0.27
NOR	0.11	0.09	0.14	0.22	0.35
SWE	0.13	0.06	0.11	0.16	0.19
AVERAGE	0.10	0.08	0.14	0.17	0.20

Notes: Panel A reports country-specific summary statistics for the standard deviation of the wedge, $\sigma(\beta^I_{gt})$, and its correlation with output growth, $\text{Corr}(\beta^I_{gt}, \widehat{Y}_t)$, across four percentile groups. Panel B lists the standard deviation of the bilateral real exchange rate (Δe) and the bilateral discount factor wedges constructed for different groups β^I_{gt}. Wedges are constructed using residual log earnings data for Male ages between 25–55 in GRID. EIS, $\gamma^{-1} = 0.3$. Real exchange rate growth is constructed from Jordà et al. (2017) database. Sample: 1998–2015 (annual). See text for details.

Table 14: *Correlation, Thresholds, and Covariances with Real Exchange Rate Growth*

Panel A: Correlation and Thresholds for Top 10% and 1% Groups

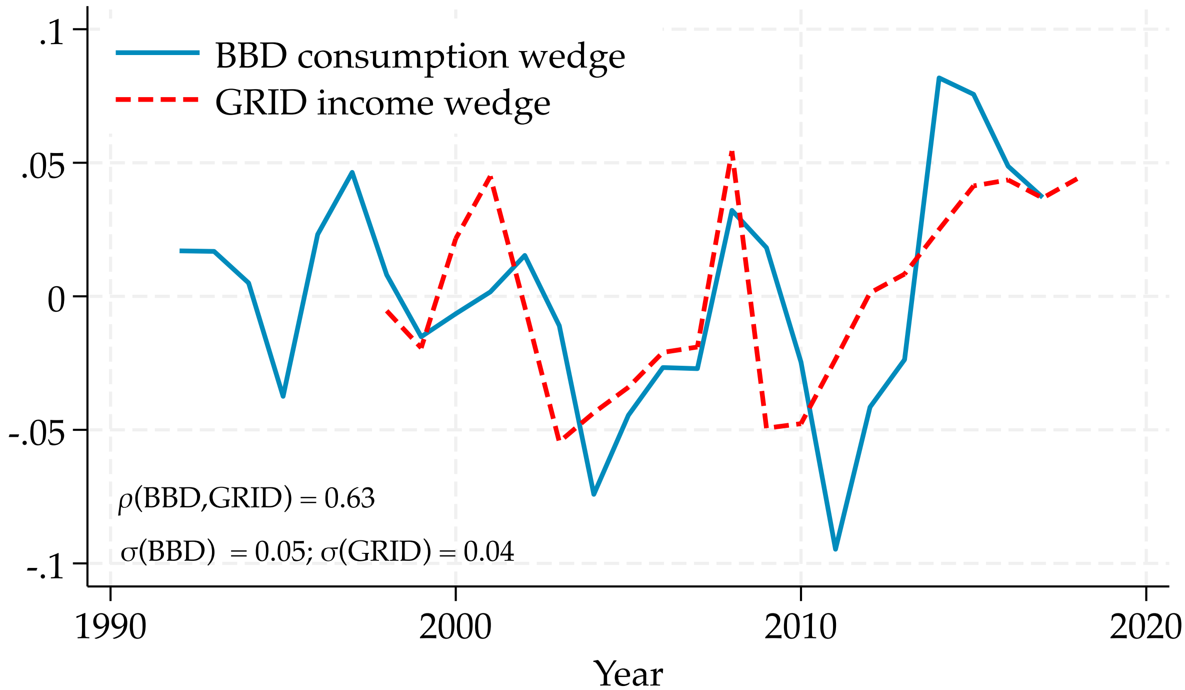
	Unconditional			*Conditional*		
	-Corr($\tilde{\beta} - \tilde{\beta}^*, \Delta e$)		*Thresh.*	-Corr$_t$($\tilde{\beta} - \tilde{\beta}^*, \Delta e$)		*Thresh.*$_t$
iso	Top 10%	Top 1%		Top 10%	Top 1%	
CAN	0.30	0.66	0.40	0.33	0.63	0.46
DEU	0.54	0.57	0.39	0.63	0.88	0.30
DNK	0.29	0.52	0.38	0.57	0.57	0.47
FRA	0.18	0.13	0.39	0.31	0.46	0.37
ITA	0.18	0.52	0.39	0.36	0.63	0.44
NOR	0.29	0.23	0.48	0.06	0.17	0.49
SWE	0.42	0.40	0.46	0.49	0.45	0.38
AVERAGE	0.31	0.43	0.41	0.39	0.54	0.41

Panel B: Correlation of Pricing Kernels and Real Exchange Rate Growth

iso	Corr($\Delta c - \Delta c^*, \Delta e$)	Corr($\widehat{m}^* - \widehat{m}, \Delta e$)		Corr$_t$($\Delta c - \Delta c^*, \Delta e$)	Corr$_t$($\widehat{m}^* - \widehat{m}, \Delta e$)	
		Top 10%	Top 1%		Top 10%	Top 1%
CAN	-0.03	0.13	0.55	-0.21	0.06	0.54
DEU	0.19	0.53	0.59	0.12	0.61	0.89
DNK	-0.29	0.06	0.38	-0.61	-0.38	0.36
FRA	-0.15	0.14	0.11	-0.21	0.23	0.42
ITA	0.09	0.19	0.51	0.14	0.42	0.62
NOR	-0.38	0.09	0.18	-0.58	-0.20	0.10
SWE	-0.06	0.32	0.39	-0.07	0.37	0.41
AVERAGE	-0.09	0.21	0.39	-0.20	0.16	0.48

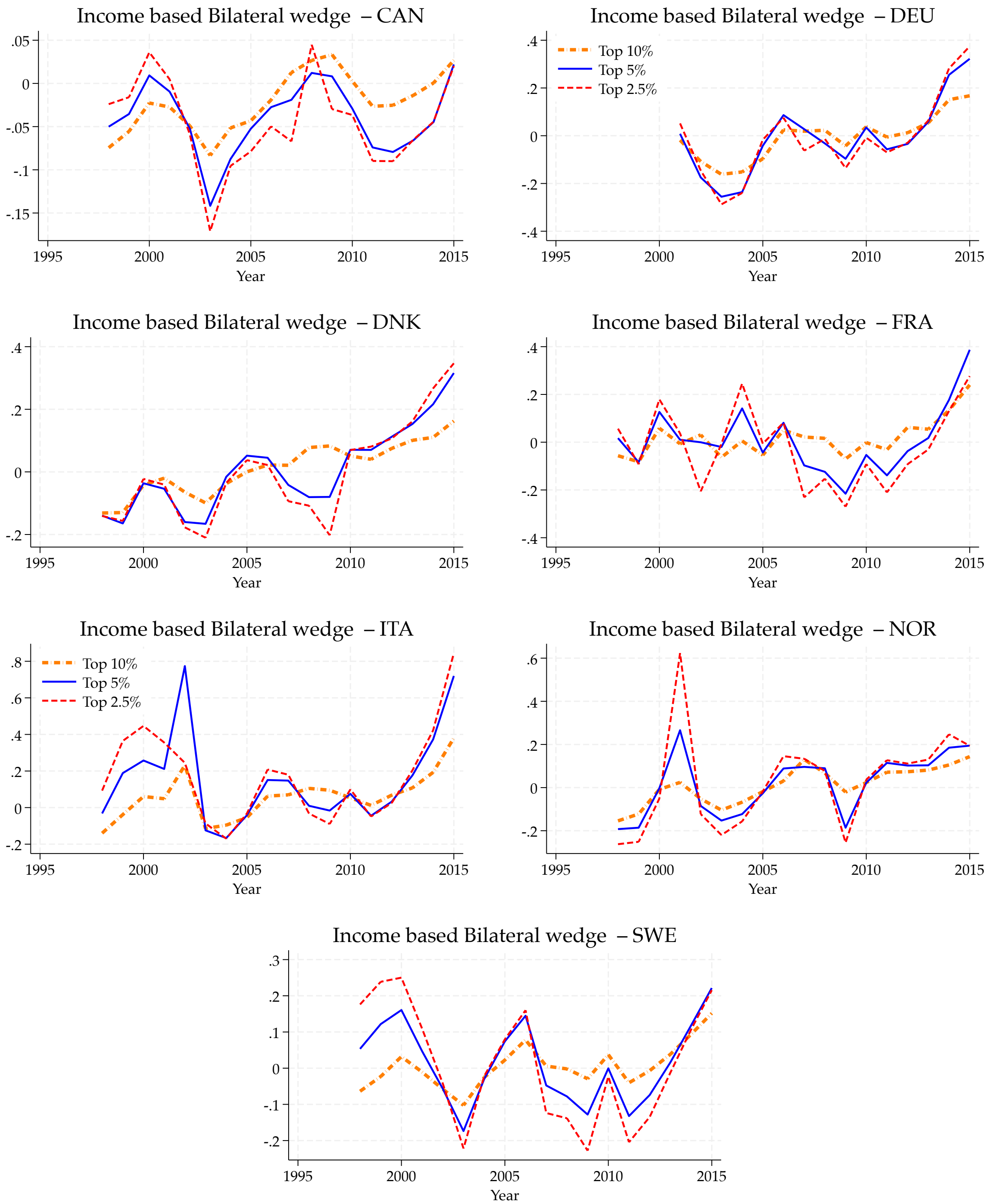
C.3.3 Comparison with GRID based wedge with Consumption based wedge

Figure 5: *Consumption based Wedge and GRID's Income Based Wedge for the US (2.5 percentile)*



Notes: Figure plots the time-series of consumption based wedge in solid blue line, measured from Consumer expenditure survey for the US, and the log-income based wedge for the US measured from the GRID dataset in dashed red line. EIS is set at 1 in constructing both wedges.

Figure 6: *Time-series plots of the bilateral wedges*



Notes: Figure plots the time-series of the bilateral income based wedges for US/Foreign country bilateral pairs for different percentile groups. EIS, γ^{-1} is set to 0.3. Wedges are constructed from the GRID dataset.

C.3.4 Conditional Summary Statistics with GRID dataset

Table 15: *Summary Statistics for Country-specific Wedges (Conditional)*

Panel A: Log-income based Country-specific Wedge

	Top 10%		Top 5%		Top 2.5%		Top 1%	
iso	$\sigma_t(\beta^I_{gt})$	$\text{Corr}_t(\beta^I_{gt}, \widehat{Y}_t)$	$\sigma_t(\beta^I_{gt})$	$\text{Corr}_t(\beta^I_{gt}, \widehat{Y}_t)$	$\sigma_t(\beta^I_{gt})$	$\text{Corr}_t(\beta^I_{gt}, \widehat{Y}_t)$	$\sigma_t(\beta^I_{gt})$	$\text{Corr}_t(\beta^I_{gt}, \widehat{Y}_t)$
CAN	0.03	0.17	0.07	0.12	0.08	0.08	0.08	0.02
DEU	0.04	-0.40	0.10	-0.30	0.11	-0.26	0.09	-0.29
DNK	0.02	-0.53	0.09	-0.66	0.10	-0.70	0.11	-0.70
FRA	0.04	-0.80	0.11	-0.82	0.14	-0.77	0.11	-0.60
ITA	0.06	-0.08	0.15	-0.22	0.14	-0.41	0.16	-0.39
NOR	0.04	-0.25	0.09	-0.11	0.17	-0.07	0.28	0.02
SWE	0.05	-0.72	0.10	-0.68	0.13	-0.68	0.14	-0.65
USA	0.05	-0.32	0.08	-0.12	0.11	-0.06	0.13	-0.10
AVERAGE	0.04	-0.37	0.10	-0.35	0.12	-0.36	0.14	-0.34

Panel B: Log-Income based Bilateral Wedge

	$\sigma_t(\Delta e_t)$	$\sigma_t(\Delta\beta_{gt})$			
		Top 10%	Top 5%	Top 2.5%	Top 1%
CAN	0.10	0.02	0.03	0.04	0.05
DEU	0.08	0.04	0.09	0.11	0.10
DNK	0.09	0.02	0.07	0.09	0.11
FRA	0.09	0.04	0.11	0.14	0.14
ITA	0.09	0.05	0.15	0.14	0.19
NOR	0.11	0.05	0.11	0.19	0.29
SWE	0.09	0.04	0.10	0.13	0.14
AVERAGE	0.09	0.04	0.09	0.12	0.15

Notes: Table reports conditonal moments. Panel A reports country-specific summary statistics for the standard deviation of the wedge, $\sigma(\beta^I_{gt})$, and its correlation with output growth, $\text{Corr}(\beta^I_{gt}, \widehat{Y}_t)$, across four percentile groups. Panel B lists the standard deviation of the bilateral real exchange rate (Δe) and the bilateral discount factor wedges constructed for different groups β^I_{gt}. Wedges are constructed using residual log earnings data for Male ages between 25–55 in GRID. EIS, $\gamma^{-1} = 0.3$. Real exchange rate growth is constructed from Jordà et al. (2017) database. Sample: 1998–2015 (annual). See text for details.

C.4. Italian household data and construction of the SHIW wedge

For Italy we use the Bank of Italy *Survey of Household Income and Wealth* (SHIW) data. The SHIW is a nationally representative survey of the resident population, designed along the same lines as the Italian Labour Force Survey. Sampling is organized in two stages: municipalities are stratified by region and population size, and then households are randomly selected from municipal registry offices. Each cross-sectional wave contains about 8,000 households. Interviews are conducted in the first months of the calendar year, and flow variables (income and consumption) refer to the preceding calendar year, while wealth and debt are measured as end-of-year stocks. The unit of observation is the household, defined as all individuals residing in the same dwelling and related by blood, marriage, or adoption; cohabiting partners are also treated as a household.

From 1980 onwards the SHIW collects detailed information on disposable income and non-durable consumption; starting in the late 1980s the wealth module is complete and distinguishes real assets (housing, business wealth, durables) and financial assets (deposits, bonds, mutual funds, stocks, etc.), as well as mortgage and non-mortgage liabilities. Monetary variables are not top-coded. The survey has an important panel dimension: a fraction of households is re-interviewed in subsequent waves, so that by the 2000s more than half of the sample in a given wave is composed of panel households. This structure allows us to observe the same households in consecutive waves and to construct consumption, income, and net-worth growth rates over approximately two-year horizons.

Our sample construction follows closely Jappelli and Pistaferri (2025), which builds on Jappelli and Pistaferri (2010). We restrict attention to survey years from 1995 onward, which are the first waves for which our extraction delivers a consistent set of income, consumption, wealth and deflator variables across all modules. All monetary variables are deflated by the SHIW consumption deflator so that all amounts are expressed in constant prices. Throughout, we use the SHIW sampling weights when forming cross-sectional moments.

Net disposable income is defined as

$$Y_{ht} \equiv Y_{ht}^{\ell} + Y_{ht}^{m} + Y_{ht}^{\tau} + Y_{ht}^{c},$$

where Y^{ℓ} is labor income, Y^{m} self-employment and business income, Y^{τ} pensions and transfers, and Y^{c} income from capital and financial assets. Non-durable consumption C_{ht} is taken from the SHIW non-durable consumption aggregate after deflation. Net worth W_{ht} is the sum of real assets, including housing and business wealth) and financial assets minus total debt . We also use information on household composition, region (north/centre/south), gender, marital status, age and education of the head.

In line with Jappelli and Pistaferri (2025), we purge composition effects from the cross-sectional distributions of consumption, income and wealth by residualizing their logs on basic demographics separately in each survey wave. We then construct "composition-adjusted" log variables by adding back only the estimated intercept. Exponentiating and scaling by household size yields per-capita, composition-adjusted variables. To obtain a coherent panel, we group households by their original SHIW identifier and keep only those that are observed in at least two consecutive waves. For each such household we construct lagged per-capita income and consumption, and define an "initial income" measure as the within-household mean of lagged income. We then drop observations with missing lags so that, in each wave t, we observe both current and lagged consumption for the same household.

Constructing incomplete markets wedge. The SHIW data is biennial after 1998, but the first interval in our sample is the three-year gap between 1995 and 1998. We therefore treat the horizon h_t as 3 years for the 1995–1998 pair and 2 years for all subsequent adjacent waves. Assuming CRRA preferences with coefficient of relative risk aversion γ and abstracting from time discounting, the relative stochastic discount factor (SDF) of household j with respect to the SHIW aggregate over horizon h_t is, as in BBD, $\beta^{ITA}_{g,t+1} = \frac{1}{N_g}\sum_{h=1}^{N_g}\left(\frac{g^C_{t,t+h_t}}{g^{C(j)}_{t,t+h_t}}\right)^{\gamma/h_t}$, where $g^C_{t,t+h_t}$ and $g^{C(j)}_{t,t+h_t}$ are aggregate and household-level consumption growth respectively.

Following our baseline construction for the U.S., we focus on "pricers" that are likely to be unconstrained today but face a non-trivial probability of becoming constrained in the future. In each pair of SHIW waves we partition households into groups g based on their position in the joint distribution of current income and net worth. We focus on high-income, low-net-worth households, who are most likely to be unconstrained today but to face binding borrowing constraints with positive probability in the future. Our wedge construction yields an annualized biennial series for the Italian discount-factor wedge starting with the 1998 wave (constructed from the 1995–1998 interval, using $h_t = 3$) and continuing biennially thereafter. This gives us a sufficiently long time series of Italian wedges for years 1997–2015 to study how the bilateral wedge co-moves with real exchange rates over different regimes, and to show that the main patterns we document with a U.S-only wedge are robust to using a fully bilateral consumption-based measure constructed from Italian and US micro data.