Incomplete Markets and Exchange Rates[⋆]

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Abstract

We show imperfect risk-sharing *within* countries can reconcile the aggregate cyclicality of exchange rates in a two-country setting, i.e. the Backus-Smith puzzle, as long as exchange rates are sufficiently risky with respect to idiosyncratic states. In equilibrium, these exchange rate dynamics require that, despite a country experiencing higher consumption growth, idiosyncratic risk remains relatively elevated. Furthermore, we identify distinct roles for market incompleteness both *within* and *across* countries, to match key moments of exchange rates. Turning to household-level data, we measure discount factor wedges which capture the effects of imperfect risk sharing, and we provide direct empirical support for the mechanism.

Keywords: Incomplete Markets, Heterogeneous agents, Exchange Rates.

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1. Introduction

We revisit the Backus-Smith condition (Kollmann, 1991; Backus and Smith, 1993) to consider the implications of domestic and international market incompleteness for exchange rate dynamics. This condition is a centrepiece in the literature as a benchmark for international risk-sharing in a flexible-price, representative agent economy and is also closely tied to other important moments (and puzzles) such as the volatility of exchange rates and the correlation of pricing kernels across countries. Under complete financial markets, assuming power utility:

$$\left(\frac{C_{t+1}}{C^*_{t+1}} / \frac{C_t}{C^*_t}\right)^{\gamma} = \frac{\mathcal{E}_{t+1}}{\mathcal{E}_t} \tag{1}$$

where $C_t^{(*)}$ is Home (Foreign) aggregate consumption, $\mathcal{E}_t$ is the real exchange rate where an increase signifies a depreciation of Home currency, and γ is the inverse of the elasticity of intertemporal substitution (EIS). Condition (1) implies that exchange rate depreciations coincide with periods of low marginal utility growth for Home investors and so *exchange rates are said to be counter-cyclical.*[1] In the data, however, exchange rates tend to appreciate when Home consumption rises and marginal utility growth is low, i.e. are pro-cyclical, constituting the Backus-Smith *puzzle*.

Our motivation is driven by two advancements in the literature. First, we extend a canonical two-country framework to allow for imperfect domestic risk sharing—incomplete markets within countries.[2] A large class of incomplete markets models admits an "as-if" representative agent Euler with a discount factor wedge and we leverage recent innovations in the literature (Berger, Bocola and Dovis, 2023) to measures these discount factor wedges directly using household-level data on U.S.

[1]This implication echoes closed-economy complete markets models featuring a correlation of -1 between the representative agent stochastic discount factor (SDF) and the market portfolio, see, e.g. (Duffie, 2001, Ch. 1), also Lettau (2002).

[2]Incomplete domestic markets and heterogeneity within countries has been shown to be important for asset prices (Constantinides and Duffie, 1996), monetary policy (Kaplan, Moll and Violante, 2018) and exchange rates (Kocherlakota and Pistaferri, 2007).

consumption, income and assets, and income data for a panel of advanced economies (Guvenen, Pistaferri and Violante, 2022). Second, it has been shown that existing resolutions to the Backus-Smith puzzle based on the representative agent paradigm rely on allowing cross-border trade *only* in a single risk-free asset. Lustig and Verdelhan (2019) show that, no-arbitrage restrictions from cross-border trade in at least one Home and one Foreign currency denominated (nominally) risk-free asset imply incomplete international financial markets alone cannot resolve the puzzle of excessive risk-sharing *regardless* of goods markets and other economy specifics. So, the literature must move beyond the representative agent. Our takeaway result is that imperfect risk-sharing within countries is a strong confounding factor in the determination of exchange rates.

We generalize the two-country, consumption capital-based asset pricing framework to allow for uninsurable idiosyncratic consumption risk within each country. The extended framework ties exchange rates to the pricing kernels of an "as-if" representative agent in each country, where a discount factor wedge captures a precautionary motive due to idiosyncratic risk. When the as-if representative agent (in each country) prices both Home and Foreign bonds, international no-arbitrage conditions no longer impose a strictly positive co-movement between exchange rates and aggregate consumption growth. Leveraging this framework, we study exchange rates as determined exclusively by these Euler equations, as in Chernov, Haddad and Itskhoki (2024) and Jiang, Krishnamurthy, Lustig and Sun (2024), to assess the scope for market incompleteness (within and across countries) to reconcile key moments of international aggregates.

We analytically derive conditions under which domestic market incompleteness switches the cyclicality of exchange rate depreciations with respect to relative aggregate consumption growth, i.e. the Backus-Smith covariance. Our key result is that procyclical exchange rates, consistent with data, can be obtained if and only if the correlation of cross-country differences in the discount factor wedge with depreciations is sufficiently negative, relative to a threshold, even with frictionless trade in two nominally risk-free bonds (and potentially additional risky assets). The threshold is given by the ratio of the volatility of exchange rates and the volatility of relative discount factor wedges. This condition implies that foreign bonds are a poor hedge for idiosyncratic risk in each country, yielding high returns at a time when domestic investors face low idiosyncratic risk, thus making exchange rates *risky* with respect to the *idiosyncratic* state.

Building on this, we derive the equilibrium exchange rate process in a framework building on Lucas (1982) and Cox, Ingersoll and Ross (1985) where we allow for idiosyncratic risk to be a linear function of the aggregate state.[3] We highlight two additional results. First, the model can deliver risky *equilibrium* exchange rates with respect to the idiosyncratic state, as required to resolve the Backus-Smith puzzle, while also allowing for counter-cyclical idiosyncratic risk (with respect to domestic aggregate consumption growth), see e.g. Storesletten, Telmer and Yaron (2004). Counter-cyclicality implies that idiosyncratic consumption uncertainty is high– the "pricer's" marginal utility growth is expected to rise by more than marginal utility growth from average consumption – at times when average consumption growth is expected to be low. To deliver risky equilibrium exchange rates, our model requires that despite higher aggregate consumption growth in the Home relative to the Foreign country, the Home country experiences a relatively muted fall in their discount factor wedge–the Home pricer remains concerned with the future outlook (and vice versa for higher foreign aggregate consumption growth).

Second, we delineate the effects of domestic and international financial market incompleteness. In particular, we show that while imperfect risk sharing within countries is necessary for resolving exchange rate cyclicality, it is not generally sufficient when there are additional country-specific factors which are necessary to account for an imperfect correlation of international stochastic discount factors. In this case, incomplete financial markets across countries are also required. Moreover, international market incompleteness is necessary to account for the volatility of exchange rates which

[3] We use a discrete time version by Sun (1992), extended to two-countries environment in Backus, Foresi and Telmer (2001) and has been extensively used since, e.g. Lustig and Verdelhan (2019)

is much lower that that of stochastic discount factors when allowing for idiosyncratic risk or calibrating to equity prices. Our calibrated model matches moments from equity and bond market, as well the volatility and cyclicality of exchange rates with international spanning of 60% for the common factor, illustrating a role for both domestic and international market incompleteness.

We provide tractable examples for models of within-country imperfect risk sharing, borrowing from (i) Constantinides and Duffie (1996), and (ii) Krusell, Mukoyama and Smith (2011) and Bilbiie (2024). In the former case, exchange rates are risky with respect to the idiosyncratic state if permanent risk faced by households is low in periods of depreciation (when the Foreign bond pays out). In the latter case, they are risky if the probability of becoming financially constrained is low when exchange rates depreciate. We also motivate our focus on the Backus-Smith condition by showing that it remains the benchmark for welfare analysis, even when there is imperfect risk-sharing within countries. Moreover, we explicitly connect the consumption-based asset pricing models to goods market clearing. In general, asset pricing models solve for the exchange rate at which autarky interest rates are equated across countries, but allocations do not need to satisfy static goods-basket optimization. We detail trade costs (Fitzgerald, 2012), or equivalently shocks to home bias (Pavlova and Rigobon, 2007; Gabaix and Maggiori, 2015), required to clear goods markets.

Turning to the data, our main objective is to test whether exchange rates are indeed "risky" with respect to idiosyncratic state. Following Berger et al. (2023), we construct discount factor wedges based on households' consumption shares from the Consumption Expenditure Survey, identifying high-income, low net worth individuals likely to be unconstrained and assign the highest prices to bonds, i.e. the pricers. The resulting wedges capture the pricing implications of a lack of risk sharing within countries for a large class of incomplete market models. We begin by allowing for imperfect risk sharing only in the U.S., maintaining the representative agent assumption abroad. For a conservative EIS of 0.2 (Best, Cloyne, Ilzetzki and Kleven, 2020), we robustly find that the $\beta-$wedge, on average, co-moves sufficiently negatively with real exchange rate growth in fifteen of seventeen countries and is itself sufficiently volatile, for our mechanism to go through. We confirm support for the sufficient condition both unconditionally and by constructing conditional moments, controlling for variables observed at time t. We analyze our finding further in two ways: i) the $\beta-$wedge dynamics are not driven by composition effects (i.e. the identity or characeristics of the pricer), but rather by within group variation capturing a risk channel; ii) turning to data on permanent idiosyncratic risk from Bayer, Luetticke, Pham-Dao and Tjaden (2019), the constructed $\beta-$wedge is not volatile enough for our mechanism to drive exchange rate cyclicality pointing to the role of transitory risk or the possibility of becoming constrained instead.

To also allow for idiosyncratic risk in a panel of foreign countries where detailed individual consumption growth data is unavailable, we construct discount factor wedges using growth in the income shares of high-income individuals (top 2.5%, 5%, 10% of the population) across seven advanced economies from the Global Repository of Income Dynamics (GRID) (Guvenen et al., 2022), as a proxy for the growth in their consumption shares. While the volatility of these wedges is distorted (much higher) due to the lack of consumption smoothing, we present conditions under which the correlation of bilateral wedges with exchange rates is a valid proxy. We then compare these correlations to a worst-case threshold computed using only U.S. data. Our findings further confirm the relevance of imperfect sharing within countries for the cyclicality of exchange rates.

Finally, using our constructed β wedges (unilateral and bilateral), we find a positive covariance between depreciations and the relative valuation of the "as-if" representative agent, consistent with the international no-arbitrage conditions, even though the Backus Smith covariance is negative.

Related Literature Our work most closely relates to a literature which confronts exchange rate anomalies by allowing for wedges in Euler equations, e.g. Jiang, Krishnamurthy and Lustig (2023); Jiang, Krishnamurthy, Lustig and Sun (2024). Important precursors have shown that segmentation (Alvarez, Atkeson and Kehoe, 2002; Chien, Lustig and Naknoi, 2020; Sandulescu, Trojani and Vedolin, 2021) or intermediation frictions and UIP shocks (Gabaix and Maggiori, 2015; Itskhoki and Mukhin, 2021) can resolve various exchange rate puzzles. We also build on the work of Lustig and Verdelhan (2019), see also Benigno and Küçük (2012), who show that introducing a second internationally traded bond breaks the ability of canonical international macro models to reconcile the Backus-Smith puzzle. Relatedly, Chernov et al. (2024) investigate how different financial market structures and the mix of locally, globally traded, and unspanned risks contribute to different exchange rate puzzles.

We extend these contributions to consider imperfect risk-sharing within countries, which can be represented in the form of discount factor wedges, see e.g. Nakajima (2005); Krueger and Lustig (2010); Werning (2015); Guerrieri and Lorenzoni (2017); Berger et al. (2023) for closed economy applications. In contemporaneous work, Kekre and Lenel (2024) show that *exogenous* permanent discount factor shocks can resolve comovement and predictability puzzles in open economies.

More generally, our model with heterogeneous consumers ties to a large literature investigating whether idiosyncratic risk affects macro-finance aggregates (see, e.g. Mankiw, 1986; Weil, 1992; Guvenen, 2009; Kaplan et al., 2018; Auclert, Rognlie and Straub, 2024; Challe, 2020; Di Tella, Hébert and Kurlat, 2024). In the open economy macro literature, for papers with agent heterogeneity, see Ghironi (2006); Kocherlakota and Pistaferri (2007); Hassan (2013); De Ferra, Mitman and Romei (2020); Kollmann (2012); Auclert, Rognlie, Souchier and Straub (2021); Acharya and Challe (2025) amongst others. Our contribution relative to these models is to emphasize the riskiness of exchange rates with respect to idiosyncratic states and derive a condition which we directly take to household-level data.

A closely related contribution is Kocherlakota and Pistaferri (2007) who show that models with domestically incomplete (but internationally complete) markets only impose restrictions on the comovement of bilateral higher order moments of the consumption distribution and exchange rates.[4] Huang, Kogan and Papanikolaou (2025) connect cyclicality of exchange rates to agent heterogeneity mechanisms through technological innovation based displacement risk faced by shareholders. Acharya, Challe and Coulibaly (2025) study the ability of international real business cycle model with heterogenous agents to resolve exchange rate puzzles.[5]

The rest of the paper is organized as follows. Section 2 derives the condition for imperfect domestic risk-sharing to result in pro-cyclical exchange rates. Section 3 provides analytical results. Section 3.1 presents a quantitative exercise. Section 4 provides tractable examples of models with idiosyncratic risk, characterizes goods markets and discusses welfare. Section 5 presents evidence using household-level data from the U.S. and abroad. Section 6 concludes.

2. Two-country, Consumption Based Asset Pricing Model

Time is discrete and infinite. There are two types of states, aggregate z_t and idiosyncratic ν_t. Their respective histories are denoted by $z^t = (z_0, z_1, ..., z_t)$ and $\nu^t = (\nu_0, \nu_1, ..., \nu_t)$.[6] We use $s^t = (z^t, \nu^t)$ to summarize the joint history. We focus on time-separable CRRA utility. Each agent with joint history s^t derives per-period utility from consumption:

$$u(C(s^t)) = \beta \, \frac{C(s^t)^{\ 1-\gamma}}{1-\gamma} \tag{2}$$

[4] Relatedly, Leduc (2002) using a calibrated open economy model featuring idiosyncratic and aggregate risk investigated the role of these risks in generating currency premia. See also Ramchand (1999) and Tessari (2021).

[5] Relatedly, there is a large literature in international macro-finance that studies role of non-separabilities in reconciling various exchange rate puzzles. See eg: Verdelhan (2010); Karabarbounis (2014); Colacito and Croce (2013); Farhi and Gabaix (2016).

[6] We use the convention that states subsume Home and Foreign shocks, i.e. $z_t = \{u_t^z, u_t^{z*}\}, \nu_t = \{u_t^\nu, u_t^{\nu*}\}$.

where β is the constant discount factor, γ is the inverse elasticity of intertemporal substitution, and we abstract from labour supply considerations.[7] We define $M(z^{t+1})$ as the SDF based on Home aggregate consumption (corresponding to the representative agent case) and M^*_{t+1} denotes the representative SDF abroad such that $M^{(*)}(z^{t+1}) = \beta \left(\frac{C^{(*)}(z^{t+1})}{C^{(*)}(z^t)}\right)^{-\gamma}$, as in condition (1).

Individual Home households' budget constraint is given by:

$$C(s^t) - I(s^t) \leq R(z^t)B_H(s^{t-1}) - B_H(s^t) + \mathcal{E}_t(z^t)R^*(z^t)B_F(s^t) - \mathcal{E}(z^t)B_F(s^{t-1}) \\ + \int_{j\in J} \tilde{R}^j(z^t)X^j(s^t) - \int_{j\in J} X^{i,j}(s^{t-1}), \tag{3}$$

where $I(s^t) = I(z^t, \nu^t)$ is the income drawn by an agent with individual history ν^t, $B_H(s^t)(B_F(s^t))$ is the position in Home (Foreign) risk-free bonds and $R^{(*)}(z^t)$ is the corresponding returns and $X^j(s^t)$ denotes risky assets indexed by j held by agent with history ν^t. Following the international finance literature, we relegate goods market clearing to Section 4.2, where we allow for the possibility that C is a bundle of differentiated goods H and F with prices P_H and P_F, in which case $I(s^t) = \frac{P_H(z^t)}{P(z^t)} I_H(s^t, \nu^t) + \frac{P_F(z^t)}{P(z^t)} I_F(s^t, \nu^t)$.

We specify borrowing constraints in a general manner:

$$\mathcal{H}(B_H(s^t), B_F(s^t), \{X^j(s^t)\}_{j\in J}) \geq 0, \tag{4}$$

for some vector-valued function. We make two assumptions: i) we assume $\frac{d\mathcal{H}}{dB} > 0$, i.e. that purchasing risk-free domestic bonds weakly relaxes the borrowing constraint, ii) we focus on the zero Foreign liquidity ($B_F \to 0$). The borrowing constraints are analogously defined for Foreign households.

Individuals' SDF $M^{(*)}(s^{t+1})$ is instead defined on individual consumption growth $\frac{C^{(*)}(s^{t+1})}{C^{(*)}(s^t)}$, which is related to aggregate consumption growth as follows:

$$\frac{C(s^{t+1})}{C(s^t)} = \frac{\delta(s^{t+1})}{\delta(s^t)} \frac{C_{t+1}(z^{t+1})}{C_t(z^t)} \tag{5}$$

[7] In models with separable utility functions, labour supply considerations don't affect the consumption Euler equations relevant for our analysis, see Berger et al. (2023) for a further discussion.

where $\delta(s^t)$ satisfies the law of large numbers $\int_{\nu^t} \delta(z^t, \nu^t) d\nu^t \ = 1 \ \forall z^t$.

Each Home and Foreign household trades domestic and foreign risk-free real bonds with returns $R(z^t)$ and $R^*(z^t)$ (in Foreign currency) respectively. We begin with the case where trade is frictionless (i.e. no borrowing constraints) and relax this later. By no-arbitrage, the household Euler implies:

$$\mathbb{E}_t[M(s^{t+1})] = \frac{1}{R(z^t)},$$

where $\mathbb{E}_t[X(s^{t+1})] = \mathbb{E}[X(s^{t+1})|s^t] = \sum_{s^{t+1}} Pr(s^{t+1}|s^t)X(s^t, s_{t+1})]$ with $Pr(\cdot)$ denoting transition probabilities. Using (5), the aggregate Euler for Home investors investing in the Home bond can be expressed as:

$$\mathbb{E}\left[\underbrace{\beta\mathbb{E}\left[\left(\frac{\delta(z^{t+1}, \nu^{t+1})}{\delta(z^t, \nu^t)}\right)^{-\gamma} \middle| \nu^t, z^{t+1}\right]}_{\beta(z^{t+1}, \nu^t)} \times \left(\frac{C(z^{t+1})}{C(z^t)}\right)^{-\gamma} \middle| z^t\right] = \frac{1}{R(z^t)} \tag{6}$$

where $\mathbb{E}[X(s^{t+1})|\nu^t, z^{t+1}] = \sum_{\nu^{t+1}} Pr(\nu^{t+1}|\nu^t, z^t)X(v^{t+1}, z^{t+1})$ and $\mathbb{E}[X(z^{t+1}, \nu^t)|s^t] = \sum_{z^{t+1}} Pr(z^{t+1}|z^t)X(z^{t+1}, \nu^t)$. Intuitively, all agents trading the Home bond agree on its price and we define $\beta(z^{t+1}, \nu^t) \times \left(\frac{C(z^{t+1})}{C(z^t)}\right)^{-\gamma}$ as the "pricer's" SDF.[8]

Borrowing Constraints. In practice, markets are far from frictionless and borrowing constraints are key for generating sufficient volatility of idiosyncratic risk. Suppose now that agents face constraint (4). We assume there exists at least one agent who is unconstrained and this will be the most "patient" consumer who maximally values the risk-free bond. In the presence of borrowing constraints, (6) is replaced by:[9]

$$\mathbb{E}_t\left[\underbrace{\{\max_{\nu^t} \beta(z^{t+1}, \nu^t)\} \times \left(\frac{C(z^{t+1})}{C(z^t)}\right)^{-\gamma}}_{\widehat{M}(z^{t+1}, s^t)}\right] = \frac{1}{R(z^t)} \tag{7}$$

[8] This is also sometimes referred to as the "as-if" representative agent, see Werning (2015), Berger et al. (2023).

[9] In a setting without assumptions on (4), Krusell et al. (2011) derive conditions under which it is the most patient –specifically the individual with the most to lose– as opposed to a different agent who prices the assets.

Note that, in the limit where only a few agents are active, this framework relates to models of intermediation such as Hassan (2013); Gabaix and Maggiori (2015); Itskhoki and Mukhin (2021).

The "micro" module pertaining to idiosyncratic risk (ν^t) and financial market structure can be summarized by the discount wedge $\beta(z^{t+1}, \nu^t)$ referred to as a β-*wedge*. Crucially, we can measure the $\beta-$ wedge from the micro-data on consumption following Berger et al. (2023).[10]

Analogously, the aggregate Foreign Euler for the Foreign bondcan be expressed as:

$$\mathbb{E}_t[\underbrace{\{\max_{\nu^t} \beta^*(z^{t+1}, \nu^t)\} M^*(z^{t+1})}_{\widehat{M}^*_{t+1}(z^{t+1}, s^t)}] = \frac{1}{R^*(z^t)} \tag{8}$$

If the *same* Home (Foreign) households also trade the Foreign (Home) bond, denoting the real exchange rate at time t by $\mathcal{E}(z^t)$, such that an increase corresponds to a depreciation of Home currency, then we additionally obtain the following two Euler conditions:

$$\mathbb{E}_t\left[\widehat{M}(z^{t+1}, s^t)\ \frac{\mathcal{E}(z^{t+1})}{\mathcal{E}(z^t)}\right] = \frac{1}{R^*(z^t)}, \tag{9}$$

$$\mathbb{E}_t\left[\widehat{M}^*(z^{t+1}, s^t)\ \frac{\mathcal{E}(z^t)}{\mathcal{E}(z^{t+1})}\right] = \frac{1}{R(z^t)} \tag{10}$$

Next, we assume $\beta-$wedges, SDFs and prices are jointly log-normal. Moreover, since equations (6)-(10) depend on the joint history s^t and are only uncertain with respect to the future aggregate state z^{t+1}, we drop notation on histories and denote $X(s^t) = X_t, X(z_{t+1}, s^t) = X_{t+1}$, without loss of generality. To close the model, we pin down an exchange rate process consistent with equations (6)–(10) above, which reduces to finding an exchange rate process that satisfies:

$$cov_t(\widehat{m}^*_{t+1} - \widehat{m}_{t+1}, \Delta e_{t+1}) = var_t(\Delta e_{t+1}) \geq 0 \tag{11}$$

[10] In Section 5 we show that while the $\beta-$ wedge we estimate is sufficiently volatile to rationalize exchange rate dynamics, competing models with permanent risk and integrated markets cannot deliver sufficient volatility, see e.g. Lettau (2002).

where $x = \log(X)$. International no-arbitrage requires that, regardless of the specific structure of international financial markets (i.e. complete or incomplete), exchange rate depreciations coincide with a period where the "pricer's" valuation of returns is low, i.e. exchange rates are risky consistent with international no-arbitrage. We find evidence in the data of positive comovement between exchange rates and pricers' SDFs, see Section 5.

Naturally, the process corresponding to complete international financial markets ($\Delta e_{t+1} = \widehat{m}^*_{t+1} - \widehat{m}_{t+1}$) is one candidate to satisfy (11). More generally, as shown in Backus, Foresi and Telmer (2001), the following process also satisfies equation (11):

$$\Delta e_{t+1} = \widehat{m}^*_{t+1} - \widehat{m}_{t+1} + \eta_{t+1} \tag{12}$$

where η_{t+1} is the international incomplete markets wedge which must satisfy certain conditions imposed by asset trade.[11] The wedge, η, is often interpreted as the non-traded component of exchange rate movements or the wealth gap, see e.g. Pavlova and Rigobon (2007); Corsetti, Dedola and Leduc (2008, 2023). The special case of a representative agent economy corresponds to the limit $\tilde{\beta}^{(*)}_{t+1} \equiv \log \beta^{(*)}_{t+1} \to \log \beta^{(*)}$ which implies $\widehat{m}^{(*)}_{t+1} = m^{(*)}_{t+1}$. When international financial markets are complete and the economy features a representative agent, condition (11) restricts the covariance between relative consumption growth and exchange rate depreciations to be positive as in (1). In the data, this covariance is negative, hence the Backus-Smith *puzzle*.

The proposition below specifies the conditions for imperfect risk sharing *within* countries to help reconcile the aggregate cyclicality of exchange rates.

Proposition 1 (Two Int'l Traded Assets, Many Agents).
The two-country model with two internationally traded bonds and heterogeneous consumers, characterized by Equations (7), (8), (9) and (10), delivers $cov_t(\Delta c_{t+1} -$

[11] Cross-border trade by the pricer in Home and Foreign bonds yield restrictions (66) and (67) on η_{t+1} reflecting the risk-return trade-off for investors, see Appendix A.2. Using these conditions Lustig and Verdelhan (2019) show that international incompleteness cannot change the sign of the Backus-Smith covariance.

$\Delta c^*_{t+1}, \Delta e_{t+1}) < 0$ *if and only if:*

$$1 \geq -\rho_{\tilde{\beta}_{t+1}-\tilde{\beta}^*_{t+1},\Delta e_{t+1}} \geq \frac{\sigma_t(\Delta e_{t+1})}{\sigma_t(\tilde{\beta}_{t+1}-\tilde{\beta}^*_{t+1})} \tag{13}$$

where $\rho_{\tilde{\beta}_{t+1}-\tilde{\beta}^*_{t+1},\Delta e_{t+1}} \equiv \dfrac{cov_t(\tilde{\beta}_{t+1}-\tilde{\beta}^*_{t+1},\Delta e_{t+1})}{\sigma_t(\Delta e_{t+1})\sigma_t(\tilde{\beta}_{t+1})}$.

Proof. See Appendix A.2. □

Consider first the limiting case where the Foreign country is populated by a representative agent ($\beta^*_{t+1} = \beta$). Then, Proposition 1 requires that Foreign bonds are a poor hedge for domestic *idiosyncratic* consumption risk– specifically, the $\beta-$wedge is low during periods of depreciation such that the "pricer" values returns less at times when the returns on foreign bonds are high. This poor hedging property applies to any foreign currency denominated return and is a possible solution to *diversification puzzle*, documented in Heathcote and Perri (2013). Intuitively, given that no-arbitrage only requires that exchange rates are counter-cyclical with respect to the pricers' SDFs (11), if exchange rates are sufficiently counter-cyclical with respect to the relative $\beta-$wedge, they can be pro-cyclical with respect to SDFs constructed with aggregate consumption growth.

To illustrate what it means for exchange rates to be risky, in Section 4.1 we present two tractable models for the "pricer's" kernel, (i) a model where markets are fully integrated and yet agents do not want to trade any assets (Constantinides and Duffie, 1996), and (ii) a two agent model where the investor faces a probability of becoming borrowing constrained (Krusell et al., 2011; Bilbiie, 2024). In (i), Foreign bonds are risky with respect to the $\beta-$wedge if they pay returns (depreciation) when the (cross sectional) volatility of idiosyncratic income risk is low. In (ii), the $\beta-$wedge is risky because the probability of becoming constrained (and receiving a lower income) is lower in periods of depreciation. Additionally allowing for imperfect risk sharing abroad ($\tilde{\beta}^*_{t+1} \neq \log \beta^*$), either Home bonds must also a poor hedge for foreigners, or if they are safe, they are less so than Foreign bonds are risky for Home households.[12]

Limits to International Arbitrage. A prominent literature argues that limits to international arbitrage, e.g. intermediation subject to portfolio constraints Gabaix and Maggiori (2015); Itskhoki and Mukhin (2021); Chernov et al. (2024), can explain moments of exchange rates. To compare this mechanism to ours, we consider a variant of our model where both Home and Foreign households are subject to uncertain intermediation costs ($e^{u^f_{t+1}}$) affecting their returns on foreign portfolios, see Appendix A.3 for details. Allowing for costly intermediation, (13) is replaced by:

$$1 \geq -\rho_{\tilde{\beta}_{t+1}-\tilde{\beta}^*_{t+1},\Delta e_{t+1}} \geq \frac{(1-u^f)\sigma_t(\Delta e_{t+1})}{\sigma_t(\tilde{\beta}_{t+1}-\tilde{\beta}^*_{t+1})} \tag{14}$$

where $u^f \in [0,1]$ is the share of exchange rate volatility stemming from intermediation shocks $var_t(u^f_{t+1}) = u^f \times var_t(\Delta e_{t+1})$. Importantly, the u^f_{t+1} shock is assumed to be orthogonal to other shocks in the economy, hence it does not affect the co-movement of relative consumption and the exchange rate on its own. However, since a fraction of exchange rate volatility now does not stem from $\widehat{m}$ or $\widehat{m}^*$, intermediation shocks dampen the threshold required, reinforcing our mechanism.

In Section 5, we find that idiosyncratic risk alone is volatile enough to reconcile the cyclicality of exchange rates even when $u^f = 0$, and since the two mechanisms are reinforcing, we focus on the case without limits to international arbitrage.

In the remainder of the paper, we expand on (13) both theoretically and empirically. In Section 3, we solve for equilibrium exchange rates in a two-country Lucas (1982); Cox et al. (1985) framework with $\beta-$wedges and derive conditions required for condition (13) to be satisfied. We also detail that the completeness of international financial markets *also* matters for exchange rate dynamics. Then, in Section 5, we test condition (13) using household-level data on consumption and income.

[12] Moreover, we note that (13) is a sufficient condition as long as (at least) two risk-free bonds are traded internationally and we detail a generalization of the framework with trade in risky assets in Appendix B.1.

2.1. Digression: Representative agent limit ($\tilde{\beta}^{(*)} \to \log \beta^{(*)}$)

Before proceeding, we contrast our results to the representative agent limit where the correlation between the relative $\beta-$wedge and exchange rates is zero. In this case, the Backus-Smith covariance can *never* be negative (Lustig and Verdelhan, 2019): (13) becomes an impossibility. To get around this stark result, classical contributions restrict attention to the case where only a single bond is internationally traded, i.e. Home or Foreign currency denominated. The corollary below, illustrates why this is helpful.

Corollary 1 (One Int'l Traded Asset, Representative Agent No-Arbitrage).
When only Foreign bonds are internationally traded such that equations (7), (8) *and* (10) *hold, and* $\tilde{\beta}^{(*)} \to \log \beta^{(*)}$, *then* $cov_t(\Delta c_{t+1} - \Delta c^*_{t+1}, \Delta e_{t+1}) < 0$ *if and only if*

$$cov_t(m_{t+1}, \eta_{t+1}) + \log \mathbb{E}_t[e^{\eta_{t+1}}] \geq var_t(m^*_{t+1} - m_{t+1}) \tag{15}$$

where,

$$cov_t(m_{t+1}, \eta_{t+1}) = cov_t(m_{t+1}, \Delta e_{t+1}) - cov_t(m_{t+1}, m^*_{t+1}) + var_t(m_{t+1}) \tag{16}$$

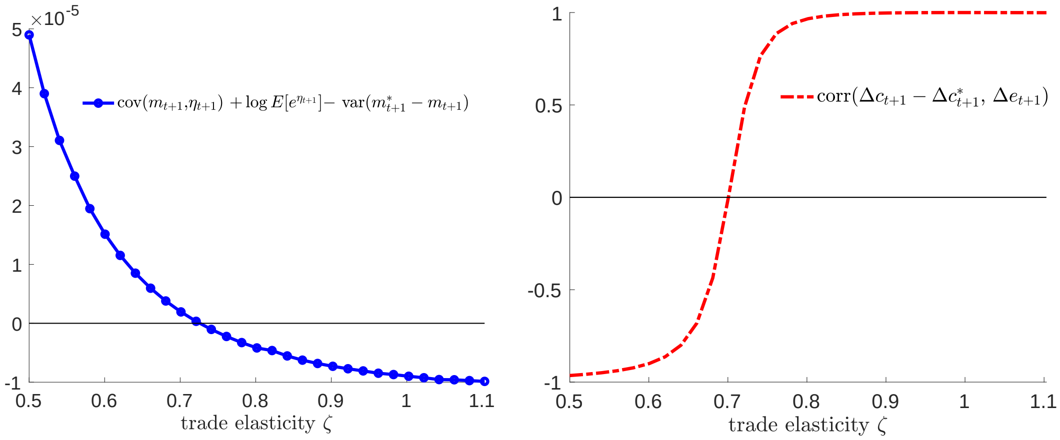
Proof. See Appendix A.1. □

Condition (15) provides a general characterization of mechanisms in the literature developed to resolve the Backus-Smith puzzle, e.g. consumption or production complementarities (Corsetti et al., 2008; Benigno and Thoenissen, 2008) reflected in (16). The RHS of (15) corresponds to the volatility of the exchange rate growth under complete markets, so is strictly positive. Condition (15) is satisfied if either the non-traded component η_{t+1} leads to relative price fluctuations which are sufficiently safe from the perspective of a Home investor ($cov_t(m_{t+1}, \eta_{t+1}) > 0$) or the volatility of the non-traded component is high.[13] Figure 1a plots the condition (15) from simulating

[13] Equation (16) shows that non-traded risk results in relative price fluctuations which are particularly *safe* if the Home SDF is very volatile or international comovement of SDFs is low relative to the comovement of exchange rates and the Home SDF, which is counterfactual (Brandt, Cochrane

Figure 1

(a) *Corollary 1 Statistic* (15) **(b)** *Backus-Smith Correlation*



Calibration: Within the representative agent model, we employ $\beta(C_{t-1}) = \omega C_{t-1}^{-u}$, with $\omega \in (0, 1)$, as the discount factor used as a stationarity-inducing device following Bodenstein (2011), C_t is the aggregate consumption bundle at Home. We set $\omega = 0.96$, $u = 0.005$, EIS coefficient $\gamma^{-1} = 1$, trade elasticity $\zeta \in [0.5, 1.1]$, home-bias $\alpha = 0.6$, $\alpha^* = 1 - \alpha$, persistence of Home endowment shock $\rho = 0.964$, and steady state Home endowment $I_H = 1$. Unconditional moments calculated from second-order simulation with one million draws.

a representative agent, two country, two good endowment version of our economy, allowing for internationally incomplete markets. Figure 1b displays the corresponding Backus-Smith correlation. The correlation is negative for low values of trade elasticity ζ, positive for higher values and intersects 0 at approximately the same value for ζ as Panel (a).[14]

In sum, in the representative agent limit, there is a stark contrast between the economy with one internationally traded asset (Corollary 1) and two assets (Proposition 1) because exchange rate risk becomes spanned when households trade in both Home and Foreign real bonds across borders, see also Chernov et al. (2024). A further limitation of these representative agents models is that they rely on a low volatility of exchange rates (Lustig and Verdelhan, 2019, pp 2241). Allowing for idiosyncratic risk within countries which co-moves with the exchange rate recovers a non-traded

and Santa-Clara, 2006).

[14]While SDFs and prices in the model (away from the Cole and Obstfeld (1991) are log-normally distributed only at the autarky limit, we find that the intersections in Figures (a) and (b) roughly coincide.

component which can deliver pro-cyclicality and high volatility of exchange rates.

3. A Closed-Form Incomplete Markets Wedge

In this section, we evaluate the relationship between the relative $\beta-$wedge — which reflects imperfect risk sharing within countries— and *equilibrium* exchange rates (12). To do so, we specify laws of motion for aggregate consumption growth and the international incomplete markets wedge as in Lustig and Verdelhan (2019), which builds on Cox et al. (1985, CIR henceforth). We introduce two further ingredients: first, we allow for countercyclical idiosyncratic risk as in Herskovic, Kelly, Lustig and Van Nieuwerburgh (2016); Berger et al. (2023) amongst others; second, using a combination of a common factor and a country-specific factor, we allow for a correlation of SDFs across countries which is positive but below 1, consistent with the international comovement of asset prices and aggregates (Brandt et al., 2006).

Consider a common factor z_t which drives consumption growth globally:

$$z_{t+1} = (1-\rho)\theta + \rho z_t + \sqrt{z_t}u_{t+1}$$

Aggregate consumption growth in the Home country is given by $\Delta c_{t+1} = \sqrt{z_t}u_{t+1}$ while consumption growth abroad is given by $\Delta c^*_{t+1} = \xi^* \sqrt{z_t}u_{t+1} + \sigma_\nu u^\nu_{t+1}$, such that $\xi^* > 1$ ($\xi^* < 1$) corresponds to an environment where the shock u_t has a relatively larger effect on the Foreign (Home) aggregate consumption growth, and u^v_t is a Foreign-specific shock. Note that $\xi^* = 1$ implies $proj(\Delta c_{t+1} - \Delta c^*_{t+1}|u_{t+1}) = 0$, hence we focus on $\xi^* \lesseqgtr 1$. Both u_{t+1} and u^ν_{t+1} are mean zero normal i.i.d innovations with unit standard deviations.

In the presence of uninsurable idiosyncratic risk, the relevant pricing kernels are characterized by $\widehat{m}^{(*)}_{t+1}$, defined in (7) and (8), and depend on the $\beta-$wedges ($\tilde{\beta}^{(*)}_{t+1}$) for which we specify linear processes:

$$\tilde{\beta}_{t+1} = \log\beta + \phi\sqrt{z_t}u_{t+1}, \quad \tilde{\beta}^*_{t+1} = \log\beta^* + \phi^*(\xi^*\sqrt{z_t}u_{t+1} + \sigma_\nu u^\nu_{t+1}) \tag{17}$$

We can write the Home and Foreign pricers' SDFs as follows:[15]

$$-\widehat{m}_{t+1} = \log\beta + \chi z_t + (\gamma - \phi)\sqrt{z_t}u_{t+1}, \tag{18}$$

$$-\widehat{m}^*_{t+1} = \log\beta^* + \chi^* z_t + (\gamma - \phi^*)(\xi^*\sqrt{z_t}u_{t+1} + \sigma_\nu u^\nu_{t+1}) \tag{19}$$

Throughout this analysis, we assume countercyclical wedges in both countries, i.e. $\phi,\ \phi^* < 0$, and restrict attention to $\gamma - \phi^{(*)} > 0$ such that the pricer's SDF falls in response to an increase in domestic consumption growth. The conditional volatility of the pricers' SDF is given by $var_t(\widehat{m}_{t+1}) = (\gamma-\phi)^2 z_t$, $var_t(\widehat{m}^*_{t+1}) = (\gamma-\phi^*)^2(\xi^{*2}z_t+\sigma^2_\nu)$. We define $\phi^\Delta = \phi - \phi^*\xi^*$ as the relative sensitivity of idiosyncratic risk to the common factor.

Lemma (Equilibrium Incomplete Markets Wedge)

In the model with heterogeneous consumers, satisfying (7)-(10) and (12), the incomplete markets wedge is given by:[16]

$$\begin{aligned}\eta_{t+1} = &-\frac{1}{2}(\gamma(1-\xi^*) - \phi^\Delta)\sqrt{(\gamma(1-\xi^*) - \phi^\Delta)^2 - \lambda}\ z_t \\ &+\frac{1}{2}(\gamma - \phi^*)\sqrt{(\gamma-\phi^*)^2 - \lambda^\nu}\ \sigma^2_\nu - \sqrt{(\gamma(1-\xi^*) - \phi^\Delta)^2 - \lambda}\sqrt{z_t}u_{t+1} \\ &+\sqrt{(\gamma-\phi^*)^2 - \lambda^\nu}\ \sigma_\nu u^\nu_{t+1} + \sqrt{\lambda - \kappa}\ \sqrt{z_t}\epsilon_{t+1} + \sqrt{\lambda^\nu - \kappa^\nu}\ \sigma_\nu \epsilon^\nu_{t+1},\end{aligned}$$

where $\lambda \geq \kappa, \lambda^\nu \geq \kappa^\nu$ are parameters governing cross border spanning and

$$\kappa = (\gamma(1-\xi^*) - \phi^\Delta)^2 - (\gamma(1-\xi^*) - \phi^\Delta)\sqrt{(\gamma(1-\xi^*) - \phi^\Delta)^2 - \lambda} \geq 0,$$

$$\kappa^\nu = (\gamma-\phi^*)^2 - (\gamma-\phi^*)\sqrt{(\gamma-\phi^*)^2 - \lambda^\nu} \geq 0$$

[15] If $\chi = \chi^* = 0$, this corresponds to power utility over consumption as in the canonical representative agent Lucas model. Allowing $\chi > 0$ is a reduced form way to capture a connection between the conditional mean and variance of SDFs which may arise due to habit formation or recursive preferences, see e.g. Hassan, Mertens and Wang (2024).

[16] Square roots refer to the positive root only unles we include the $\pm$ ahead. For the z_t factor, while the negative root is a potential solution, conditional on $\phi < 0$, it violates $\lambda > \kappa$ which we require for a real solution.

and $var_t(\Delta e_{t+1}) = \kappa z_t + \kappa^{\nu}\sigma_{\nu}^2$.

Proof. See Appendix A.1. $\square$

where $u_{t+1}^{(\nu)}$ are spanned innovations and $\epsilon_{t+1}^{(\nu)}$ are unspanned innovations. The exposure of the incomplete markets wedge to spanned shocks is given by $\sqrt{(\gamma(1-\xi^*)-\phi^{\Delta})^2-\lambda}$ and $\sqrt{\lambda-\kappa}$ is the exposure to unspanned shocks.[17] To illustrate the effects of incomplete spanning, we parametrize $\lambda = \alpha \times (\gamma(1-\xi^*)-\phi^{\Delta})^2$ for $\alpha \in (0,1)$. It can be verified that $\alpha \to 1$ coincides with the complete markets benchmark such that $\lambda_{CM} = \kappa_{CM} = (\gamma(1-\xi^*)-\phi^{\Delta})^2$. Instead, $\alpha < 1$ implies increasing levels of unspanned risk. We similarly define $\lambda^{\nu} = \alpha^{\nu}(\gamma-\phi^*)^2$ and then $\lambda_{CM}^{\nu} = \kappa_{CM}^{\nu} = (\gamma-\phi^*)^2$. When both $\alpha, \alpha^{\nu} \to 1$, necessitated when there is trade in additional risky assets (see Appendix B.1), the wedge tends to zero $\eta_{t+1} = 0$.

Armed with a closed form expression for the international incomplete markets wedge, we revisit moments of exchange rates conditional on $u_{t+1} = u_{t+1}^{\nu} = 1$.

Proposition 2 (Exchange rate cyclicality in equilibrium)
In the model with heterogeneous consumers, satisfying (7)-(10) and (12), using the processes for SDFs (18) and (19), $cov_t(\Delta c_{t+1} - \Delta c_{t+1}^*, \Delta e_{t+1}) < 0$ *if and only if:*

$$(1-\xi^*)\ [(\gamma(1-\xi^*)-\phi^{\Delta}] \leq -\underbrace{(\gamma-\phi^*)\tilde{\alpha}\frac{\sigma_{\nu}^2}{z_t}}_{\geq 0}, \tag{20}$$

where $\tilde{\alpha} = \frac{1-\sqrt{1-\alpha^{\nu}}}{1-\sqrt{1-\alpha}}$ and $\lim_{\alpha^{\nu}\to 0}\tilde{\alpha} = 0$.

Proof. See Appendix A.1. $\square$

Condition (20) ensures that, in equilibrium, the correlation between $\tilde{\beta}_{t+1} - \tilde{\beta}_{t+1}^*$ and Δe_{t+1} is negative and larger than the volatility of exchange rates, i.e. foreign bonds are relatively riskier for the Home consumer. The model is able to deliver a negative aggregate Backus-Smith coefficient and can accommodate counter-cyclical idiosyncratic risk in one or both countries.

[17] Given an incomplete markets wedge η_{t+1}, the SDFs are unique in the span of traded assets, (see Lustig and Verdelhan, 2019, Online Appendix Sec. 5).

It is instructive to begin with the limit where the country-specific shock is small ($\sigma_\nu^2 \to 0$). Suppose that $\xi^* > 1$ such that, following $u_{t+1} \uparrow$, Foreign consumption growth outpaces Home consumption growth (i.e. $\Delta c - \Delta c^* \downarrow$) . Then, in order to generate an exchange rate depreciation $\Delta e_{t+1} \uparrow$, we require $\phi^\Delta \leq \gamma(1-\xi^*) \leq 0$: Home idiosyncratic risk is more counter-cyclical than Foreign and the Home $\beta-$wedge falls relative to the Foreign $\beta-$wedge. Intuitively, the relatively higher $\tilde{\beta}^*_{t+1}$ implies Foreign consumers remain concerned about the future leading to a depreciation (12), despite a relative rise in Foreign aggregate consumption growth.

Conversely, if $\xi^* < 1$, Home aggregate consumption rises in relative terms (i.e. $\Delta c - \Delta c^* \uparrow$). For exchange rates to appreciate $\Delta e_{t+1} \downarrow$, the model requires that $\phi^\Delta > \gamma(1-\xi^*) > 0$ (either due to procyclical idiosyncratic risk or because $\phi^*\xi^* < \phi < 0$), such that the Home beta wedge remains high, i.e. valuations of returns remain high due to idiosyncratic risk at a time when losses occur on foreign portfolios. In both cases, the country experiencing faster consumption growth must have a relatively muted fall in the $\beta-$wedge.[18]

Looking to the data, the relevant case is where idiosyncratic risk is counter-cyclical $sign(\phi) = sign(\phi^*) < 0$ and consumption growth is positively correlated across countries ($\xi^* > 0$). Accounting for the equilibrium exchange rate process, this rules out the possibility that both Foreign (Home) bonds are risky with respect to idiosyncratic states for Home (Foreign) households. Instead, the cases above describe under what conditions exchange rates are relatively *riskier* for the Home country.

The Role for Market Incompleteness. Away from the limit $\sigma_\nu \to 0$, the cyclicality of exchange rates also depends on the degree of international financial market completeness. This is because country-specific shocks always induce a (weakly) positive

[18] Contrast this to the representative agent limit (Lustig and Verdelhan, 2019), which coincides with $\phi^\Delta = 0$. The LHS of condition (20) reduces to $\gamma(1-\xi^*)^2$ which is strictly positive, therefore $cov_t(\Delta c_{t+1} - \Delta c^*_{t+1}, \Delta e_{t+1}) > 0$.

Backus-Smith covariance, maintaining that $\gamma - \phi^{(*)} > 0$.[19] In the limit where the volatility of country specific shocks is large enough, Proposition 2 can never be satisfied for values of $\alpha^{\nu} > 0$. Generally, allowing for positive α_{ν} adds to volatility of exchange rates but reduces the ability of the model to resolve the Backus Smith covariance. Instead, when $\alpha^{\nu} \to 0$, the country-specific shock has zero effect on exchange rates. In all cases, the presence of these shocks, in addition to the common shocks, induces an imperfect correlation of consumption growth across countries.

The cyclicality and the volatility of exchange rates are intricately connected. In the next Proposition we show that introducing idiosyncratic risk comes at the cost of further exacerbating exchange rate volatility.

Proposition 3 (Idiosyncratic Risk and Exchange Rate Volatility)
Exchange rate volatility is higher under internationally complete markets ($\kappa^{CM} > \kappa^{IM}$ *and* $\kappa^{\nu CM} > \kappa^{\nu IM}$) *and, if* $\gamma(1-\xi^*) - \phi^{\Delta} > 0$, *is also more sensitive to the cyclicality of idiosyncratic risk* $\left(\frac{d\kappa^{CM}}{d(-\phi^{\Delta})} > \frac{d\kappa^{IM}}{d(-\phi^{\Delta})} > 0 \text{ and } \frac{d\kappa^{\nu CM}}{d(-\phi^{*})} > \frac{d\kappa^{\nu IM}}{d(-\phi^{*})} > 0\right)$

Proof. See Appendix A.1. □

In the calibration below, if SDFs are sufficiently volatile to explain (e.g.) stock prices, models with internationally complete markets also struggle to reconcile the low exchange rate volatility observed in the data.[20]

3.1. Quantitative Exercise

Calibration We calibrate z_t using the process for U.S. interest rates as in (Backus et al., 2001). Starting from: $r_t = -\log\beta + [\chi - \frac{1}{2}(\gamma-\phi)^2]z_t$, we choose $\chi = -1 + \frac{1}{2}(\gamma-\phi)^2$ to deliver counter-cyclical interest rates needed to match the uncovered interest parity

[19] It is important to note there is nothing special about the Foreign country specific shock, and these results will generally apply to any uncorrelated factor driving only the volatility of consumption in one country. In Appendix A.4, we show the same results follow if an uncorrelated shock is added to the Home SDF instead.

[20] Consistent with Brandt et al. (2006), only in the limit where we shut down u^{ν} such that SDFs are (counterfactually) perfectly correlated, can internationally complete markets be consistent with observed exchange rate volatility.

puzzle (Verdelhan, 2010; Lustig and Verdelhan, 2019). We use average U.S. interest rates of 2.67% over 1991Q1–2017Q12, sample consistent with our later empirical exercises, to calibrate $\mathbb{E}_t[z_{t+1}] = \theta = 0.002225$ (monthly) and select $\log \beta^{(*)}$ to match average interest rates. We target a correlation of SDFs across countries of 0.8 (Brandt et al., 2006) which implies $\rho_{\widehat{m},\widehat{m}^*} = 0.8 = \theta\xi^*/\sqrt{\theta}\sqrt{\theta\xi^* + \sigma_\nu}$, from which, given ξ^* discussed below, we back out $\sigma_\nu^2 = 0.00168$. We take $\gamma = 5$ as a conservative estimate from Best et al. (2020); Di Tella, Hébert, Kurlat and Wang (2023) who find an EIS coefficient between 0.1 and 0.2. We calibrate the asymmetry in $\gamma - \phi$ and $\gamma - \phi^*$, using Sharpe ratios on equity from the U.S. and abroad. We use a Sharpe ratio of 0.5 for the U.S. based on (S&P 500).[21] Using $(\gamma - \phi)\sqrt{\theta} = 0.5$, we back out $\phi = -5.6$ which is comparable to the calibration in Acharya, Challe and Dogra (2023) for the U.S. We repeat the exercise abroad with a Sharpe ratio of 0.4, which implies $\phi^* = -0.848$.

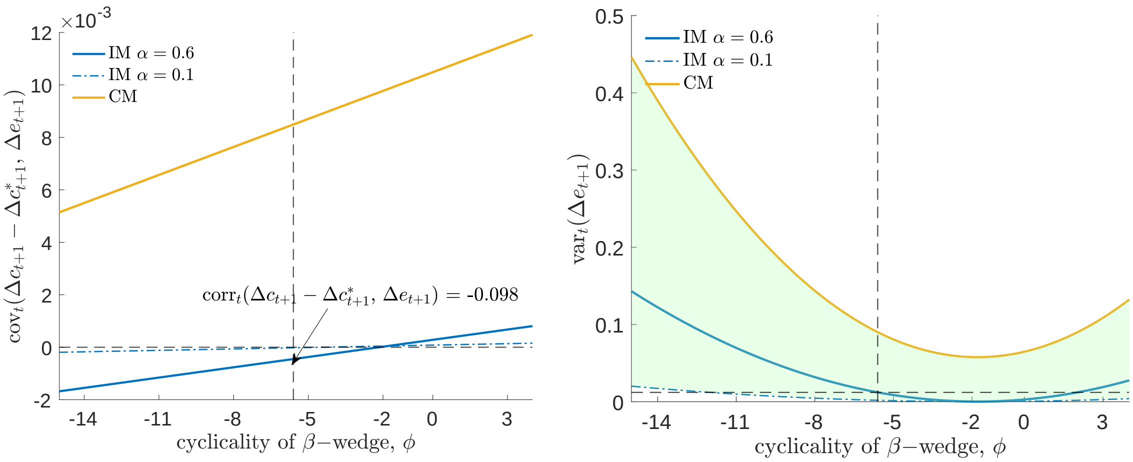
Calibrating the degree of international spanning is a challenge. In our baseline, we assume the Foreign shock is unspanned $\alpha^\nu = 0.01$, and we choose $\alpha = 0.6$ to match an exchange rate volatility of 0.11, the cross-country average standard deviation of bilateral exchange rates in our sample. The remaining parameter is ξ^*, which given other parameters must be greater than 1 (see Proposition 2), to deliver a negative Backus-Smith covariance. We choose a $\xi^* = 1.16$ which results in a correlation of consumption differentials and exchange rates depreciation of approximately -0.1, consistent with the cross-country average of our sample, see Table 3.

Figure 2 illustrates the main results, plotting the relationship between the Home $\beta-$wedge cyclicality ϕ, the Backus-Smith covariance and exchange rate volatility for three levels of spanning. Complete markets $\alpha = \alpha^\nu = 1$, our preferred calibration of intermediate spanning ($\alpha = 0.6, \alpha^\nu = 0.01$) and, for comparison, very little spanning ($\alpha = 0.1, \alpha^\nu = 0.01$). The left panel shows that for sufficiently counter-cyclical idiosyncratic risk ($\phi << 0$) the Backus-Smith covariance is negative, consistent with

[21] These Sharpe ratios are comparable to the literature and slightly higher than the historical averages in Jordà, Schularick, Taylor and Ward (2019).

Figure 2

(a) *Backus-Smith Covariance* (b) *Exchange Rate Volatility*



Solid yellow line reflects complete markets equilibrium, dotted blue line corresponds to incomplete markets with spanning $\alpha = 0.1, \alpha^\nu = 0.01$. *The solid blue line is our calibration, corresponding to* $\alpha = 0.6$. *Horizontal dashed lines reflect the zero-line in the left panel, var*(Δe) *from the data in the right panel. Vertical dashed lines mark the calibration for* ϕ *in both panels.*

Proposition 2. Given ϕ, while spanning of the common factor (higher values of α) leads to more negative values for the covariance, more spanning of the Foreign shock generates a more positive covariance.

The right panel illustrates that the exchange rate volatility under complete international markets is too high when we realistically calibrate to an imperfect correlation of SDFs across countries. This problem is exacerbated further as ϕ becomes increasingly negative. However, any level of volatility in the shaded area can be attained by varying α and α^ν.

Figure 3 in Appendix B.3 illustrates comparative statics with respect to spanning of u^ν (i.e. varying α^ν parameter) and Panel (c) presents results in the correlation space. Larger values of α^ν reduce the ability of the model to generate negative Backus Smith covariance and also increase exchange rate volatility. Consistent with our analysis, international market incompleteness is crucial for both the cyclicality and the volatility of exchange rates when we allow for imperfect risk-sharing within countries.

4. Equilibrium in a Two-Country Two-Good Economy

Before turning to the household-level data, we address three further pertaining to our theoretical analysis. Section 4.1 provides two tractable examples for the equilibrium $\beta-$ wedge and the pricer's SDF. Section 4.2 addresses international goods market clearing. Section 4.3 highlights the relationship of the Backus-Smith condition, international risk sharing and welfare in an environment with imperfect risk-sharing within countries.

4.1. Two Models for the Stochastic Discount Factor

Constantinides and Duffie (1996) The first example we look at considers perfectly integrated financial markets, i.e. there are no borrowing constraints. Following Constantinides and Duffie (1996), we construct a no-trade equilibrium where all agents choose to consume their endowment. Individual endowments are given by $I(s^t) = \delta(s^t)C(z^t)+D(z^t)$, where $D(z^t)$ denotes the aggregate dividend in the economy. We adopt the following process $\frac{\delta(z^{t+1},\nu^{t+1})}{\delta(z^t,\nu^t)} = \exp(\xi(\nu^{t+1})\sqrt{y(z^{t+1})} - y(z^{t+1})/2)$ where $\xi(\nu^{t+1})$ are the uninsurable idiosyncratic shocks, distributed standard normal for all individual histories ν^t, independently from the aggregate state z_t; and $y(z^t)$ is interpreted as cross-sectional volatility of idiosyncratic risk.[22] No-arbitrage pricing requires that any household investing in an arbitrary security with return $\tilde{R}(z^{t+1})$ satisfies:

$$\mathbb{E}\left[\beta e^{y(z^{t+1})\frac{\gamma(\gamma+1)}{2}}\left(\frac{C(z^{t+1})}{C(z^t)}\right)^{-\gamma}\tilde{R}(z^{t+1})\right] = 1 \tag{21}$$

where $\tilde{\beta}_{t+1} = \log\left(\beta e^{y(z^{t+1})\frac{\gamma(\gamma+1)}{2}}\right)$. Since all agents agree on the valuation of assets, standard arguments imply $C(s^t) = I(s^t)$ i.e. agents consume their own endowments and there is no trade in equilibrium. Intuitively, all risk is permanent and assets

[22]The law of large numbers follows from properties of the normal distribution for ξ. Treating y_{t+1} as a constant, and using the moment generating function $M_\xi(h) = e^{h\xi}$ for $h \in \mathbb{R}$, $\mathbb{E}[e^{\xi\sqrt{y}-y/2}] = M_\xi(h)e^{-y/2} = e^0 = 1$.

cannot be used to smooth consumption. An alternative interpretation is that $I(s^t)$ is income after a preliminary round of asset trade has exhausted all gains.

In this environment, a risky foreign return implies that the volatility of permanent risk is low in periods of depreciation, i.e. foreign bonds yield higher return at times when households face low idiosyncratic risk:[23]

$$cov(\tilde{\beta}_{t+1}, \Delta e_{t+1}) < 0 \implies cov(y_{t+1}, \Delta e_{t+1}) < 0 \tag{22}$$

Krusell, Mukoyama & Smith (2011), Bilbiie (2024) The second example, considers a model where households earn high ($v_t = h$) or low ($v_t = l$) levels of income, facing an exogenous probability $1 - s$ of becoming a low type (Krusell et al., 2011) which we allow to be a function of (aggregate) output $s(Y(z^t))$ following Bilbiie (2024). In equilibrium, when households have a low income draw they would like to borrow, but we restrict focus on the zero liquidity limit (binding borrowing constraints).

The first-order condition for the Home saver purchasing a domestic risk free bond is given by:

$$1 = R(z^t)\, E_t\left[\beta\, \frac{s(z^{t+1})\,\left(\frac{C(z^{t+1},h)}{C(z^{t+1})}\right)^{-\gamma} + (1 - s(z^{t+1}))\,\left(\frac{C(z^{t+1},l)}{C(z^{t+1})}\right)^{-\gamma}}{\left(\frac{C(z^t,h)}{C(z^t)}\right)^{-\gamma}} \times \left(\frac{C(z^{t+1})}{C(z^t)}\right)^{-\gamma}\right],$$

where $C(z^{t+1}, l)$ is the consumption of (low type) constrained households and $C(z^{t+1}, h)$ is that of the (high type) saver. The $\beta-$wedge, which premultiplies marginal utility growth from aggregate consumption, arises from the probability of becoming a low type and the difference in the marginal utility of consumption across two states. Because of incomplete domestic markets, marginal utility in the low state is higher than marginal utility in the high state, therefore the saver attaches a premium on the risk-free bond.[24] The saver similarly attaches a premium on foreign bonds, adjusted

[23] In Section 5 we use estimates of permanent risk from Bayer et al. (2019) and find that it is not volatile enough to satisfy (13), highlighting the role of transitory risk or borrowing constraints described in the second example.

[24] One could add further frictions to the model by, e.g., making foreign bonds less liquid in the low state, generating "convenience yield" properties for the domestic bond, see for example Di Tella et al. (2024).

for exchange rate risk (9).

For further illustration, we assume a transfer scheme such that saver's consumption is $C(z^{t+1}, h) = \omega C(z^{t+1})$, and hand-to-mouth agents' consumption is $C(z^{t+1}, l) = (1-\omega)C(z^{t+1})$ with $0.5 < \omega \leq 1$ so that saver's consumption is larger than the hand-to-mouth household's consumption.[25] The $\beta-$wedge then simplifies to: $\tilde{\beta}_{t+1} = \log\left(\beta\left[s(z^{t+1})\left(1-\left(\frac{1-\omega}{\omega}\right)^{-\gamma}\right)+\left(\frac{1-\omega}{\omega}\right)^{-\gamma}\right]\right)$.

In this environment, a risky foreign return $cov(\tilde{\beta}_{t+1}, \Delta e_{t+1}) < 0$ implies that the probability of becoming hand to mouth $1 - s(z^{t+1})$ is low in periods of depreciation when returns to Foreign assets are high:

$$cov(\tilde{\beta}_{t+1}, \Delta e_{t+1}) < 0 \implies cov(s, \Delta e_{t+1}) > 0 \tag{23}$$

4.2. International Equilibrium and Goods Market Clearing

Taking any foreign aggregate endowment process as given, we solve for an exchange rate process such that agents optimally choose not to trade across borders. More generally, the endowment processes we consider can be interpreted as the wealth of consumers after all gains from trade have been exhausted.[26] The adjusted risk-sharing condition is then given by:

$$\mathbb{E}_t\left[\tilde{\beta}_{t+1}\left(\frac{C_{t+1}}{C_t}\right)^{-\gamma}\frac{\mathcal{E}_{t+1}}{\mathcal{E}_t}\right] = \mathbb{E}_t\left[\tilde{\beta}^*_{t+1}\left(\frac{C^*_{t+1}}{C^*_t}\right)^{-\gamma}\right]. \tag{24}$$

Replacing C_t with $I_t = \int_\nu I_t^\nu$ and C_t^* with $I_t^* = \int_{\nu^*} I_t^{\nu\ *}$, (24) implies equalization of the autarky rates across countries resulting in a no trade equilibrium (Svensson, 1988), consistent with the asset pricing model in Section 3.

[25] As well as improving tractability, this transfer scheme also eliminates composition-driven explanations for Backus Smith covariance (e.g. Kollmann, 2012).

[26] We follow the zero liquidity approach popular in the heterogeneous agents literature, e.g. Krusell et al. (2011); Werning (2015); Challe (2020); Bilbiie (2024) among others; for its analytical tractability in characterizing SDFs and prices.

Goods Market Equilibrium. Real exchange rate fluctuations necessitate differentiated consumption bundles across countries. The Home consumption bundle and associated price index are given by

$$C_t^{\nu} = \left[\alpha^{\frac{1}{\zeta}} c_{H,t}^{\nu\,\frac{\zeta-1}{\zeta}} + (1-\alpha)^{\frac{1}{\zeta}} c_{F,t}^{\nu\,\frac{\zeta-1}{\zeta}} \right]^{\frac{\zeta}{\zeta-1}}, \quad P_t = \left[\alpha p_{H,t}^{\frac{\zeta-1}{\zeta}} + (1-\alpha) p_{F,t}^{\frac{\zeta-1}{\zeta}} \right]^{\frac{\zeta}{\zeta-1}} \tag{25}$$

where ζ is the trade elasticity and α is the measure of home-bias. Foreign quantities and prices are defined symmetrically, with home bias $1-\alpha$. The real exchange rate is given by $\mathcal{E} = P^*/P$.

The equilibrium characterized by (24) does not always lie on the static Pareto frontier, i.e. is not consistent with consumption bundle optimization at prices given by (25) (see Appendix B.4.1), but we show it can always be supported with static goods market wedges, isomorphic to home bias shocks (Pavlova and Rigobon, 2007; Gabaix and Maggiori, 2015), described below.[27]

Proposition 4 (Goods Market Clearing)

Given processes for $\{I_t, I_{H,t}, I_{F,t}, \tau_t\}$ and the Pareto Frontier $\{c_H(C_t), c_F(C_t)\}$, goods markets clear if and only if:

$$\left[I_t^{\frac{\zeta-1}{\zeta}} - \alpha^{\frac{1}{\zeta}} c_H(I_t)^{\frac{\zeta-1}{\zeta}} \right]^{\frac{\zeta}{\zeta-1}} \left(b\hat{I}_{H,t} + (1-b) c_H(I_t) \right) = c_H(I_t) \hat{I}_{F,t} (1-\alpha)^{\frac{1}{\zeta-1}} \tag{26}$$

where $b = (\frac{\alpha}{1-\alpha})^2 (1+\tau_t)^{\zeta}, \tau_t > -1$. In the limit of full home bias $\alpha \to 1$, a zero wedge ($\tau_t = 0$) satisfies (26).

Proof. See Appendix A.1. □

[27] While this may sound undesirable, such wedges are non-zero in the data and play a significant role in impeding risk sharing (Fitzgerald, 2012; Bodenstein, Cuba-Borda, Gornemann and Presno, 2024). As illustrated in Section 2, such goods markets frictions alone are unable to explain the Backus Smith puzzle (Lustig and Verdelhan, 2019). Moreover, leading contributions in the literature (e.g. Colacito and Croce, 2013) assume a home bias of 0.97, where Proposition 4 implies $\tau_t \to 0$.

4.3. Backus-Smith, Risk Sharing and Welfare

We conclude our theoretical exposition by discussing the relevance of the Backus-Smith condition as a measure for risk sharing and welfare in an environment featuring i) heterogeneity within countries, ii) static trade wedges. To do so, we compare the decentralized allocation of our model to a planner problem, detailed in Appendix B.4. In general, the Backus-Smith condition shown in equation (1) characterizes the planner's optimal allocation of consumption across countries when $\beta_{t+1} = \beta^*_{t+1} = \beta$ and is the first best outcome when the planner can distribute income uniformly across agents within each country. Trade wedges implied by equation (26) are an additional impediment to risk-sharing, but if they are technological and also faced by the planner, deviations from equation (1) reflect deviations from a constrained efficient allocation. In the case where the planner is unable (or unwilling) to eliminate heterogeneity within countries, the Backus Smith condition (1) is no longer the sufficient measure for risk sharing, see Appendix for details.

5. Empirical Evidence

In this section, we measure the discount factor wedges from the data for U.S. households following Berger et al. (2023), henceforth BBD. Then we assess the plausibility of our mechanism for reconciling international risk sharing patterns by assessing whether condition (13) is met: how risky are exchange rates with respect to the relative $\beta-$wedge in the data?

Wedge measurement. Beginning with the U.S., we use the Consumption Expenditure Survey. We want to measure the discount factor wedge defined in equations (6) and (7) which requires constructing the conditional expectation over idiosyncratic states of marginal utility growth for a currently unconstrained individual facing idiosyncratic risk. Following BBD, we measure this wedge using the cross-sectional

Table 1: *Summary Statistics for* $\tilde{\beta}_g$

γ (Inverse of EIS)	1	5	7.5	10
Corr($\tilde{\beta}_g, \Delta \log Y$)	-0.05	-0.61	-0.60	-0.61
Corr($\tilde{\beta}_g, \Delta \log C$)	0.03	-0.54	-0.55	-0.56
$\sigma(\tilde{\beta}_g)$	0.04	1.29	2.21	3.05

average of marginal utility growth of currently unconstrained individuals. First, for each household ν, we define the consumption share $\varphi_t^\nu = \frac{C_t^\nu}{C_t}$ and construct the growth rate of relative marginal utility of a household as $\left(\frac{\varphi_{t+1}^\nu}{\varphi_t^\nu}\right)^{-\gamma}$.[28] Then, at each date t, we collect households with similar levels of income and net worth into distinct groups. For each group of households, we compute a $\beta-$wedge as the average across N_g individuals in group g:

$$\beta_{g,t+1} \equiv \frac{1}{N_g} \sum_{\nu=1}^{N_g} \left(\frac{\varphi_{t+1}^\nu}{\varphi_t^\nu}\right)^{-\gamma} \tag{27}$$

Finally, we take logs and demean such that $\tilde{\beta}_{g,t+1} \equiv \log(\beta_{g,t+1}) - \overline{\log(\beta_g)}$.

In the presence of borrowing constraints, the theory suggests to select the maximal $\beta-$wedge, see (7). As in BBD, we choose high income, low net worth individuals who are likely to be unconstrained today, but face the possibility of becoming constrained in the future. As such, these households have a high incentive to save for precautionary reasons. We deviate from BBD who assume an EIS of 1 and we construct wedges for $\gamma \in \{5, 7.5, 10\}$. Best et al. (2020) find that $\gamma = 10$ best fits household-level data and this is further supported by evidence in Di Tella et al. (2023). We use $\gamma = 5$ in our baseline and report robustness in Appendix C. For the U.S., we obtain an annual time-series of $\tilde{\beta}_{t+1}$ spanning 1992–2017. Table 1 provides summary statistics on the cyclicality and volatility of this wedge.

Measurement of the $\beta-$wedge requires panel data on household-level consumption growth, income, and a measure of net worth. We are not aware of similar micro-data

[28] As in BBD, we use residualized log consumption constructed by partialling out fixed effects for sex, race, education, age of the head of household, and the state of residence. All nominal variables are converted into 2000 dollars with CPI-U.

for other country from where we can construct similar wedge. We proceed in two ways. First, we present results assuming that time-varying idiosyncratic risk in the U.S. only, i.e. $\tilde{\beta}^*_{t+1} = \log\beta$ for the foreign country. Second, Section 5.1 generalizes to allow for idiosyncratic risk abroad using cross-country distributional income statistics from the Global Repository in Income Dynamics, provided by Guvenen et al. (2022).

We construct real exchange rates from annual nominal exchange rates (US dollar/ Foreign currency) and consumer price indices using the Macro-History database (Jordà, Schularick and Taylor, 2017) for seventeen advanced economies.

Results. We highlight five findings. First, using U.S. household data on consumption, the constructed $\beta-$wedge is sufficiently negatively correlated with depreciations and sufficiently volatile relative to exchange rates for condition (13) to be satisfied, giving support to our main result. Second, we show that depreciations correlate positively with the relative pricing kernels adjusted for this $\beta-$wedge, consistent with no-arbitrage from international trade in two risk-free assets(11), though they correlate negatively with pricing kernels constructed on aggregate consumption only. Third, we highlight that the correlation with depreciations and variance of the wedge are not driven by a composition effect (i.e. time-varying relative consumption of pricers) but arise because of within-group dispersion capturing risk. Fourth, we show that an alternative construction of the $\beta-$wedge relying on permanent risk (21) is not volatile enough to deliver our results. Last but not least, we leverage international household-level income data and show the constructed bilateral $\beta-$wedges also satisfy (13).

Table 2 reports moments of the data relevant for Proposition 1, namely, the (negative of the) correlation of the wedge with real exchange rate growth and the threshold $\frac{\sigma(\Delta e)}{\sigma(\beta)}$. We report both conditional and unconditional moments. To construct conditional moments we control for date t values of relative log consumption in US and Foreign $(c_t - c^*_t)$, relative short term nominal interest rate $(i_t - i^*_t)$, relative CPI based inflation rates $(\pi_t - \pi^*_t)$, and log bilateral real exchange rate (e_t).

Table 2: *Empirical Moments: Real Exchange Rate Growth and Berger et al. (2023) Wedge*

	Unconditional		Conditional	
ISO	$-\text{Corr}(\tilde{\beta}, \Delta e)$	Threshold	$-\text{Corr}_t(\tilde{\beta}, \Delta e)$	Threshold_t
AUS	0.05	0.10	0.07	0.08
BEL	0.24	0.08	0.25	0.09
CAN	0.28	0.08	0.14	0.07
CHE	-0.03	0.07	-0.01	0.06
DEU	0.23	0.08	0.31	0.07
DNK	0.25	0.08	0.36	0.09
ESP	0.32	0.09	0.19	0.08
FIN	0.25	0.09	0.25	0.08
FRA	0.26	0.08	0.27	0.08
GBR	0.36	0.08	0.58	0.08
IRL	0.23	0.07	0.06	0.06
ITA	0.36	0.09	0.23	0.09
JPN	-0.39	0.08	-0.44	0.08
NLD	0.25	0.08	0.17	0.09
NOR	0.23	0.09	0.35	0.09
PRT	0.30	0.09	0.06	0.08
SWE	0.28	0.10	0.24	0.08
AVERAGE	0.20	0.08	0.18	0.08

For fifteen of the seventeen bilateral pairs, the exceptions being Japan and Switzerland, we find that bilateral exchange rates are risky with respect to the $\tilde{\beta}$ wedge – i.e. a negative correlation indicating depreciations tend to happen during periods of low valuation by the pricer ($\tilde{\beta}$). Our results are robust to lower degrees of risk aversion (specifically any $\gamma \geq 3$), to alternate conditioning sets, and are also similar from a pooled regression with country fixed effects instead of constructing conditional moments using country level regressions, see Appendix C.1 for details.

Furthermore, while the Backus-Smith correlation is, on average, slightly negative in our sample consistent with the literature, we find that an adjusted condition defined on the pricer's SDF delivers a positive correlation. We construct the as-if representative agent log SDF in the U.S. as implied by the model: $\widehat{m} = -\gamma\Delta c + \tilde{\beta} + \log\beta$ and continue to assume the Foreign SDF is the scaled consumption growth $m^* = -\gamma\Delta c^* + \log\beta^*$. A positive correlation provides support for a framework where there is international no-arbitrage for the pricers' kernels (11).

Table 3 lists correlations for all seventeen bilateral pairs, and cross-country average

Table 3: *Correlations between Pricing Kernels and Real Exchange Rate Growth*

	Unconditional		Conditional	
ISO	$\text{Corr}(\Delta c - \Delta c^*, \Delta e)$	$\text{Corr}(m^* - \widehat{m}, \Delta e)$	$\text{Corr}_t(\Delta c - \Delta c^*, \Delta e)$	$\text{Corr}_t(m^* - \widehat{m}, \Delta e)$
AUS	-0.39	0.03	-0.40	0.06
BEL	-0.30	0.23	-0.44	0.23
CAN	-0.12	0.27	-0.18	0.14
CHE	-0.09	-0.03	-0.23	-0.02
DEU	0.08	0.23	-0.03	0.30
DNK	-0.34	0.22	-0.42	0.33
ESP	-0.01	0.32	-0.00	0.19
FIN	-0.38	0.22	-0.32	0.23
FRA	-0.25	0.25	-0.34	0.25
GBR	0.07	0.37	-0.09	0.58
IRL	0.08	0.24	0.52	0.09
ITA	0.01	0.35	0.06	0.23
JPN	0.19	-0.37	0.05	-0.44
NLD	0.19	0.26	-0.12	0.17
NOR	-0.27	0.21	-0.63	0.33
PRT	0.21	0.31	0.09	0.07
SWE	-0.27	0.27	-0.28	0.23
AVERAGE	-0.09	0.20	-0.16	0.18

is noted in the last row. Consistent with our framework, the correlation of relative SDF with exchange rate growth is positive for all pairs with the exception of Japan and Switzerland. However, our results fail the stricter test for the theory that the covariance of the relative SDFs is exactly equal to variance of the real exchange rate growth (11), see Table 8, suggesting need for a richer model structure.

Composition vs. Risk. We decompose our main findings further to see if our mechanism is driven by a composition effect (i.e. the changing relative consumption of the pricer) or a risk term captured by within-group dispersion. As shown in BBD, the $\beta-$wedge can be decomposed into following terms:

$$\underbrace{\log \frac{1}{N_g} \sum_{\nu=1}^{N_g} \left(\frac{C^{\nu}_{t+1}/C^{\nu}_t}{C_{t+1}/C_t} \right)^{-\gamma}}_{\tilde{\beta}} = \underbrace{\log \left(\frac{C^{g}_{t+1}/C^{g}_t}{C_{t+1}/C_t} \right)^{-\gamma}}_{\text{Composition term } \tilde{\beta}^C} + \underbrace{\log \frac{1}{N_g} \sum_{\nu=1}^{N_g} \left(\frac{C^{\nu}_{t+1}/C^{\nu}_t}{C^{g}_{t+1}/C^{g}_t} \right)^{-\gamma}}_{\text{Jensen's term } \tilde{\beta}^J},$$

Table 4: *Decomposition of the Berger et al. (2023) Wedge*

	Jensen's		Composition	
ISO	$-\text{Corr}(\tilde{\beta}^J, \Delta e)$	Threshold	$-\text{Corr}(\tilde{\beta}^C, \Delta e)$	Threshold
AUS	0.03	0.10	0.29	1.37
BEL	0.22	0.08	0.29	1.08
CAN	0.25	0.08	0.37	1.00
CHE	-0.06	0.07	0.32	0.96
DEU	0.22	0.08	0.28	1.07
DNK	0.23	0.08	0.30	1.06
ESP	0.30	0.09	0.35	1.19
FIN	0.23	0.09	0.32	1.22
FRA	0.24	0.08	0.29	1.07
GBR	0.35	0.08	0.28	1.12
IRL	0.21	0.07	0.29	0.89
ITA	0.34	0.09	0.27	1.17
JPN	-0.39	0.09	-0.04	1.13
NLD	0.23	0.08	0.29	1.10
NOR	0.20	0.10	0.36	1.26
PRT	0.28	0.09	0.28	1.14
SWE	0.26	0.10	0.42	1.35
AVERAGE	0.18	0.09	0.29	1.13

where C^{ν} denotes individual consumption and C^g denotes the group average of consumption. To disentangle the two channels, we construct the $\beta-$wedge using the consumption of an average unconstrained agent (C^g), capturing only the composition channel, and we separately consider the residual Jensen's term. Table 4 reports the correlations of the respective wedges with exchange rates and the thresholds required for Proposition 1 to be satisfied.

While an explanation based solely on composition of spending by households on the Euler equation offers the right sign, the composition based wedge is insufficiently volatile, resulting in high thresholds which cannot be met (final column). Instead, within-group variation in marginal utility growth (detailed in the first two columns) appears to be the main driver behind satisfying condition (13).

Permanent Income Risk The literature on the asset pricing implications of idiosyncratic risk initially focused on settings with no borrowing constraints and permanent income risk, e.g. Constantinides and Duffie (1996) and others. We use a measure of permanent income risk constructed by Bayer et al. (2019) using the Survey of Income Participation Program in the U.S. While we once again find that while the correlation of the permanent income risk based wedge co-moves negatively with exchange rates, the volatility of permanent risk is orders of magnitude too small to satisfy (13) in order to reconcile the Backus Smith puzzle. This echoes earlier negative findings by Lettau (2002) on the resolution of equity premium puzzle in the U.S with such measures of idiosyncratic risk. We report the results in Appendix C.2.

These results, on the inability of the composition channel and the permanent risk based wedge to satisfy condition (13), narrow our focus to mechanisms based on transitory risk and possibility of becoming constrained.

5.1. Foreign Idiosyncratic Risk

Shortcomings with $\tilde{\beta}^* = 0$ Until now we have assumed the representative agent Euler equation holds abroad, i.e. foreign idiosyncratic risk plays no role. We highlight two shortcomings of this approach. First, our symmetric two country framework suggests that the relevant condition (13) is on the correlation of the bilateral wedge ($\tilde{\beta}-\tilde{\beta}^*$) with the real exchange rate. For example, it could be the case that $cov_t(\tilde{\beta}^*, \Delta e) \leq cov_t(\tilde{\beta}, \Delta e) \leq 0$ such that $cov_t(\tilde{\beta} - \tilde{\beta}^*, \Delta e) \geq 0$ despite the evidence we present above, and therefore the Backus Smith puzzle cannot be resolved. Second, the denominator in the threshold is given by the volatility of the relative $\beta-$ wedge, which could, in principle, be less volatile. While we do not have household-level consumption data for other countries to construct an equivalent time-series of $\tilde{\beta}^*$, we discuss below how we use cross-country distributional income growth statistics to construct a comparable bilateral measure.

Using Global Repository in Income Dynamics We construct a proxy for the bilateral wedge using growth in income shares for high income groups over a balanced sample of 1998–2015 in seven countries (Canada, Denmark, France, Germany, Italy, Norway, and Sweden) against the United States. We use the GRID measure of average residual log earnings growth for 25-55 males to calculate income shares, derived from regressions of log earnings on age dummies. To capture unconstrained agents likely to be the most patient, we present results based on the top 2.5 and top 5 percentile group in each country's income distribution.[29] Details for the construction of wedges are provided in Appendix C.3 and robustness using the top 10 and top 1 percentile groups can be found in Appendix C.3.2.

The first key shortcoming from using income instead of consumption data arises because of consumption smoothing, concern which is even more pertinent when we focus on patient individuals with a higher propensity to save than the average. However, we show that income data can still be used to construct a good proxy the correlation of the relative (consumption) $\beta-$wedge with depreciation within a class of models characterized by following relationships:

$$\frac{C^{\nu}_{t+1}}{C^{\nu}_{t}} = \theta^{\nu}\frac{I^{\nu}_{t+1}}{I^{\nu}_{t}}, \quad \frac{C^{\nu *}_{t+1}}{C^{\nu *}_{t}} = \theta^{\nu *}\frac{I^{\nu *}_{t+1}}{I^{\nu *}_{t}}, \tag{28}$$

$$\frac{C_{t+1}}{C_{t}} = \theta\frac{I_{t+1}}{I_{t}}, \quad \frac{C^{*}_{t+1}}{C^{*}_{t}} = \theta^{*}\frac{I^{*}_{t+1}}{I^{*}_{t}} \tag{29}$$

where X^{ν}_t, and X_t denote the individual and the economy-average income or consumption. Within this class, it follows that:

$$\rho_{\tilde{\beta}_{t+1}-\tilde{\beta}^{*}_{t+1},\Delta e_{t+1}} \approx \rho_{\tilde{\beta}^{I}_{t+1}-\tilde{\beta}^{I*}_{t+1},\Delta e_{t+1}}, \quad \text{if } \tfrac{\theta^{\nu}}{\theta} \approx \tfrac{\theta^{\nu *}}{\theta^{*}}$$

where $\tilde{\beta}^{I}_{t+1} = -\gamma \log\left(\frac{I^{\nu}_{t+1}/I^{\nu}_{t}}{I_{t+1}/I_{t}}\right)$. See Appendix C.3.1 for a full derivation.

Second, we construct an upper bound for the relevant threshold in (13), which this correlation must exceed. In general, the denominator of this threshold depends on

[29] Figure 5 plots the GRID based wedges alongside the wedge constructed by Berger et al. (2023), revealing the two co-move strongly. On average, the country specific wedges we construct are counter-cyclical with respect to own output growth and consumption growth.

the volatility of the difference in $\beta-$ wedges across countries, but we want to avoid using the volatility of a $\beta-$wedge constructed using income. To this end, we consider a (conservative) lower bound for the volatility of the relative $\beta-$wedge using only the consumption based wedge from BBD. Denote the volatility of the US specific wedge with $\sigma(\tilde{\beta})$, and the correlation between the two wedges by $\rho_{\tilde{\beta},\tilde{\beta}^*} \in (-1,1)$. Taking the correlation as given, it can be shown that volatility of the foreign wedge that minimizes volatility of bilateral wedge is $\max\{0, \rho_{\tilde{\beta},\tilde{\beta}^*} \times \sigma(\tilde{\beta})\}$.[30] Then, the volatility for the bilateral wedge of $\sigma(\tilde{\beta})$ when $\rho_{\tilde{\beta},\tilde{\beta}^*} < 0$ and $\sqrt{1-\rho^2_{\tilde{\beta},\tilde{\beta}^*}}$ for all $\rho_{\tilde{\beta},\tilde{\beta}^*} \geq 0$. We set $\rho_{\tilde{\beta},\tilde{\beta}^*} = 0.9$, larger than the highest observation for bilateral correlation of GDP growth in the sample ($\rho = 0.85$, GBR/USA), and get an implied volatility of bilateral wedge is about $0.435\sigma(\tilde{\beta})$. We use this implied volatility of the bilateral wedge and the volatility of bilateral exchange rates to construct the thresholds in (13).

One additional challenge of the GRID dataset is that we can only construct a wedge based on log-income shares which differs from $\log \beta_{t+1}$, defined in (27), due to scaling of marginal utility growth with elasticity of substitution at the individual level. In Appendix C.4, we show that a log-income based wedge in the US data exhibits a weaker correlation with exchange rate than consumption based actual wedge, and is substantially less volatile than correctly-measured consumption based wedge, suggesting that this is again a conservative proxy for the true bilateral wedge correlation.[31]

Results. Table 5 presents the main results using the GRID dataset. We construct bilateral wedges for the top 5% and top 2.5% income groups, using $\gamma = 5$ as above. The set of controls for constructing conditional moments is the same as the one used for the consumption based wedge. Panel A reports the correlation of the bilateral wedge

[30] Specifically, we solve $\min_{\sigma^*} Var(\beta - \beta^*) = \sigma^2(\tilde{\beta}) + \sigma^2(\tilde{\beta}^*) - 2\rho_{\tilde{\beta},\tilde{\beta}^*}\sigma(\tilde{\beta})\sigma(\tilde{\beta}^*)$ and derive $\sigma(\tilde{\beta}^*) = \rho_{\tilde{\beta},\tilde{\beta}^*}\sigma$ for all $\rho \geq 0$.

[31] Indeed, the two effects are offsetting: using income instead of consumption increases the wedge volatility, whereas using differences in log income vs. log of income differences decreases the volatility, such that the GRID based wedge can be more or less volatile relative to a true wedge.

Table 5: *Correlation and Thresholds with Real Exchange Rate Growth using GRID data*

Panel A: Correlation and Thresholds for Top 5% and 2.5% Groups

	Unconditional			*Conditional*		
	-$Corr(\tilde{\beta} - \tilde{\beta}^*, \Delta e)$		*Thresh.*	-$Corr_t(\tilde{\beta} - \tilde{\beta}^*, \Delta e)$		$Thresh._t$
iso	Top 5%	Top 2.5%		Top 5%	Top 2.5%	
CAN	0.46	0.67	0.17	0.52	0.72	0.20
DEU	0.60	0.64	0.17	0.66	0.75	0.13
DNK	0.44	0.50	0.17	0.55	0.61	0.21
FRA	0.17	0.25	0.17	0.35	0.37	0.17
ITA	0.09	0.41	0.17	0.30	0.63	0.20
NOR	0.42	0.32	0.21	0.27	0.21	0.21
SWE	0.49	0.45	0.21	0.47	0.45	0.16
AVERAGE	0.38	0.46	0.18	0.45	0.54	0.18

Panel B: Correlation of Pricing Kernels and Real Exchange Rate Growth

iso	$Corr(\Delta c - \Delta c^*, \Delta e)$	$Corr(\widehat{m}^* - \widehat{m}, \Delta e)$		$Corr_t(\Delta c - \Delta c^*, \Delta e)$	$Corr_t(\widehat{m}^* - \widehat{m}, \Delta e)$	
		Top 5%	Top 2.5%		Top 5%	Top 2.5%
CAN	-0.03	0.28	0.51	-0.21	0.45	0.66
DEU	0.19	0.63	0.68	0.12	0.86	0.90
DNK	-0.29	0.26	0.34	-0.61	0.14	0.33
FRA	-0.15	0.15	0.24	-0.21	0.35	0.36
ITA	0.09	0.11	0.40	0.14	0.33	0.61
NOR	-0.38	0.29	0.24	-0.58	0.08	0.10
SWE	-0.06	0.45	0.43	-0.07	0.42	0.42
AVERAGE	-0.09	0.31	0.41	-0.20	0.37	0.48

with exchange rates alongside the relevant threshold. The correlations are robustly negative and exceed the (conservative) threshold computed using the implied volatility of the bilateral wedge. Panel B reports the correlation of relative consumption growth with exchange rate, and the correlation of the relative pricers' SDFs with exchange rates. As before, the correlation of relative SDFs with exchange rates is positive for all bilateral pairs, while the correlation is negative when we use relative aggregate consumption growth to construct SDFs.

6. Conclusion

We generalize a two-country no-arbitrage framework beyond the representative agent benchmark to allow imperfect risk sharing both *within* and *across* countries. Solving for equilibrium exchange rates, we show that the cyclicality of within-country idiosyncratic risk (with respect to aggregate consumption growth) is critical to reconcile the Backus-Smith correlation. Specifically, this requires that, despite a country experiencing relatively high consumption growth, the pricers (or marginal investors) remain concerned about their future prospects, resulting in a co-movement of exchange rates, discount factors and aggregate consumption growth in line with the data. Solving for the international incomplete markets wedge in closed form, we show that international market incompleteness is necessary to maintain pro-cyclical exchange rates when consumption growth is imperfectly correlated across countries (contrasting to results in Lustig and Verdelhan (2019) for representative agent models) and is also needed to reconcile relatively smooth exchange rates with volatile domestic asset prices.

We directly test the conditions derived for idiosyncratic risk using household-level consumption data, with income and net worth data used to identify the relevant pricer. For countries other than the U.S. where the micro-data is less rich, we rely only on income data and the ranking of households within the distribution. We robustly find that exchange rates are both risky with respect to the idiosyncratic state, and the idiosyncratic state is sufficiently volatile for our mechanism to be empirically relevant.

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A. Appendix

A.1. Proofs to Propositions

Proof to Proposition 1 Condition (11) can be expanded as follows:

$$var_t(\Delta e_{t+1}) = cov_t(m^*_{t+1} - m_{t+1}, \Delta e_{t+1}) + cov_t(\tilde{\beta}^*_{t+1} - \tilde{\beta}_{t+1}, \Delta e_{t+1})$$

Then $cov_t(m^*_{t+1} - m_{t+1}, \Delta e_{t+1}) < 0$ if and only if:

$$var_t(\Delta e_{t+1}) + cov_t(\tilde{\beta}_{t+1} - \tilde{\beta}^*_{t+1}, \Delta e_{t+1}) \leq 0.$$

By the Cauchy-Schwarz inequality:

$$|cov_t(\tilde{\beta}_{t+1} - \tilde{\beta}^*_{t+1}, \Delta e_{t+1})| \leq \sigma_t(\tilde{\beta}_{t+1} - \tilde{\beta}^*_{t+1})\sigma_t(\Delta e_{t+1})$$

Combining the inequalities:

$$\sigma_t(\tilde{\beta}_{t+1} - \tilde{\beta}^*_{t+1})\sigma_t(\Delta e_{t+1}) \geq -cov_t(\tilde{\beta}_{t+1} - \tilde{\beta}^*_{t+1}, \Delta e_{t+1}) \geq var_t(\Delta e_{t+1})$$

Dividing through by $\sigma_t(\tilde{\beta}_{t+1} - \tilde{\beta}^*_{t+1})\sigma_t(\Delta e_{t+1})$ yields the result. □

Proof to Corollary 1 The Backus-Smith covariance $cov_t(m^*_{t+1} - m_{t+1}, \Delta e_{t+1})$ can be rewritten as:

$$cov_t(m^*_{t+1} - m_{t+1}, m^*_{t+1} - m_{t+1}) + cov_t(m^*_{t+1} - m_{t+1}, \eta_{t+1}) \tag{30}$$

$$= var_t(m^*_{t+1} - m_{t+1}) + cov_t(m^*_{t+1}, \eta_{t+1}) - cov_t(m_{t+1}, \eta_{t+1}) \tag{31}$$

At the representative agent limit $\widehat{m}^{(*)}_{t+1} = m^{(*)}_{t+1}$, imposing (67) (international trade in the Foreign asset), but not (66) (international trade in the Home asset) yields:

$$cov_t(m^*_{t+1} - m_{t+1}, \Delta e_{t+1}) = var_t(m^*_{t+1} - m_{t+1}) - \log \mathbb{E}_t[\eta_{t+1}] - cov_t(m_{t+1}, \eta_{t+1}) \leq 0$$

Rearranging delivers the result. Note further that taking the limit $cov_t(m_{t+1}, \eta_{t+1}) \to$ (15) would imply $var_t(\Delta e_{t+1}) < 0$ which cannot be an equilibrium. □

Proof to Lemma The individual Euler equation can be expressed as:

$$\mathbb{E}_t\left[\tilde{\beta}_{t+1}\left(\frac{C_{t+1}}{C_t}\right)^{-\gamma}\tilde{R}_{t+1}\right] = 1, \tag{32}$$

for any (risky) asset with return $\tilde{R}_{t+1}$, so that e.g. the return on the foreign nominally risk-free bond is given by $(\mathcal{E}_{t+1}/\mathcal{E}_t)R_{t+1}$. We define

$$\Delta e_{t+1} = \widehat{m}^*_{t+1} - \widehat{m}_{t+1} + \eta_{t+1}$$

Combining (11), (12), (66), and (67), yields:

$$var_t(\eta_{t+1}) = var_t(\widehat{m}^*_{t+1} - \widehat{m}_{t+1}) - var_t(\Delta e_{t+1}) \tag{33}$$

Assume a process for the incomplete markets (IM) wedge:

$$\eta_{t+1} = \Gamma_0 z_t + \Gamma_1\sqrt{z_t}u_{t+1} + \Gamma_2\sqrt{z_t}\epsilon_{t+1} + \Gamma^\nu_0\sigma^2_\nu + \Gamma^\nu_1\sigma^\nu u^\nu_{t+1} + \Gamma^\nu_2\sigma^\nu\epsilon^\nu_{t+1} \tag{34}$$

where $u^{(\nu)}$ are spanned innovations and $\epsilon^{(\nu)}$ are unspanned innovations. We proceed to determine the coefficients of the IM wedge consistent with no arbitrage. Substituting the pricer's SDF (as in (32)), using (17) into (33), and denoting $var_t(\Delta e_{t+1}) = \kappa z_t + \kappa^\nu \sigma_\nu^2$:

$$\underbrace{(\Gamma_1^2 + \Gamma_2^2)z_t + (\Gamma_1^{\nu 2} + \Gamma_2^{\nu 2})\sigma_\nu^2}_{var_t(\eta_{t+1})} = \underbrace{(\gamma(1-\xi^*) - \phi^\Delta)^2 z_t + (\gamma - \phi^*)^2 \sigma_\nu^2}_{var_t(\widehat{m}^*_{t+1} - \widehat{m}_{t+1})} - \underbrace{\left(\kappa z_t + \kappa^\nu \sigma_\nu^2\right)}_{var_t(\Delta e_{t+1})}, \quad (35)$$

Solving for the coefficients:

$$\Gamma_1 = \pm\sqrt{(\gamma(1-\xi^*) - \phi^\Delta)^2 - \lambda}, \quad \Gamma_1^\nu = \pm\sqrt{(\gamma - \phi^*)^2 - \lambda^\nu} \quad (36)$$

$$\Gamma_2 = \pm\sqrt{\lambda - \kappa}, \quad \Gamma_2^\nu = \pm\sqrt{\lambda^\nu - \kappa^\nu} \quad (37)$$

where for real solutions, the following restrictions are required:

$$(\gamma(1-\xi^*) - \phi^\Delta)^2 \geq \lambda \geq \kappa, \quad (38)$$

$$(\gamma - \phi^*)^2 \geq \lambda^\nu \geq \kappa^\nu \quad (39)$$

Then, substituting (36)-(37) into (66)-(67) imply:

$$\Gamma_0 = \frac{1}{2}\left\{(\gamma(1-\xi^*) - \phi^\Delta)^2 - \kappa\right\} - (\gamma - \phi)\sqrt{(\gamma(1-\xi^*) - \phi^\Delta)^2 - \lambda}, \quad (40)$$

$$\Gamma_0^\nu = \frac{1}{2}\left\{(\gamma - \phi^*)^2 - \kappa^\nu\right\}, \quad (41)$$

and

$$\Gamma_0 = -\frac{1}{2}\left\{(\gamma(1-\xi^*) - \phi^\Delta)^2 - \kappa\right\} - \xi^*(\gamma - \phi^*)\sqrt{(\gamma(1-\xi^*) - \phi^\Delta)^2 - \lambda}, \quad (42)$$

$$\Gamma_0^\nu = +\frac{1}{2}\left\{(\gamma - \phi^*)^2 - \kappa^\nu\right\} + (\gamma - \phi^*)\sqrt{(\gamma - \phi^*)^2 - \lambda^\nu} = 0, \quad (43)$$

respectively. Adding (40) and (42) for z_t and (41) and (43) for σ_ν^2 respectively, this can be rewritten as:

$$\Gamma_0 = -\frac{1}{2}(\gamma - \phi + \xi^*(\gamma - \phi^*))\sqrt{\gamma(1-\xi^*) - \phi^\Delta)^2 - \lambda}, \quad (44)$$

$$\Gamma_0^\nu = +\frac{1}{2}(\gamma - \phi^*)\sqrt{(\gamma - \phi^*)^2 - \lambda^\nu} \quad (45)$$

Subbing Γ_0 and Γ_0^ν back into (40) and (42) respectively:

$$\kappa = (\gamma(1-\xi^*) - \phi^\Delta)^2 - (\gamma(1-\xi^*) - \phi^\Delta)\sqrt{(\gamma(1-\xi^*) - \phi^\Delta)^2 - \lambda^\nu} \geq 0,, \tag{46}$$

$$\kappa^\nu = (\gamma - \phi^*)^2 - (\gamma - \phi^*)\sqrt{(\gamma - \phi^*)^2 - \lambda^\nu} \geq 0 \tag{47}$$

To ensure all square roots are real, we only choose the square root signs which satisfy (38) and (39). As a result, we select positive roots for all except Γ_1 where we select the negative root. This completes the characterization of the η process.

□

Proof to Proposition 2 First consider:

$$\begin{gathered} cov_t(\widehat{m}^*_{t+1} - \widehat{m}_{t+1}, \Delta e_{t+1}) = var_t(\Delta e_{t+1}) = \\ cov_t(m^*_{t+1} - m_{t+1}, \Delta e_{t+1}) + cov_t(\tilde{\beta}^*_{t+1} - \tilde{\beta}_{t+1}, \Delta e_{t+1}) \end{gathered} \tag{48}$$

Then, the Backus-Smith covariance is negative if and only if:

$$var_t(\Delta e_{t+1}) + cov_t(\tilde{\beta}_{t+1} - \tilde{\beta}^*_{t+1}, \Delta e_{t+1}) \leq 0 \tag{49}$$

Using the Lemma above:

$$\begin{aligned} &\kappa z_t + \phi^\Delta(\gamma(1-\xi^*) - \phi^\Delta - \sqrt{(\gamma(1-\xi^*) - \phi^\Delta)^2(1-\alpha)}z_t \\ &+\kappa^\nu \sigma_\nu^2 - \phi^* \left(-(\gamma - \phi^*) + \sqrt{(\gamma-\phi^*)^2(1-\alpha^\nu)}\right)\sigma_\nu^2 \leq 0 \end{aligned} \tag{50}$$

We deal with the terms pre-multiplying z_t and σ_ν^2 separately. We substitute κ and κ^ν from the Lemma. Beginning with z_t:

$$\begin{aligned} &(\gamma(1-\xi^*) - \phi^\Delta)^2 - (\gamma(1-\xi^*) - \phi^\Delta)\Gamma_1 + \phi^\Delta(\gamma(1-\xi^*) - \phi^\Delta - \Gamma_1) \\ &= (\gamma(1-\xi^*) - \phi^\Delta)(\gamma(1-\xi^*) - \phi^\Delta - \Gamma_1) + \phi^\Delta(\gamma(1-\xi^*) - \phi^\Delta - \Gamma_1) \\ &= (\gamma(1-\xi^*))(\gamma(1-\xi^*) - \phi^\Delta - \Gamma_1) \\ &= (\gamma(1-\xi^*))(\gamma(1-\xi^*) - \phi^\Delta)\underbrace{(1-\sqrt{1-\alpha})}_{\geq 0} \end{aligned}$$

Then, turning to σ_ν^2:

$$(\gamma-\phi^*)^2-(\gamma-\phi^*)\sqrt{(\gamma-\phi^*)^2(1-\alpha^\nu)}-\phi^*(-(\gamma-\phi)+\Gamma_1^\nu)$$
$$=(\gamma-\phi^*)(\gamma-\phi^*+\phi^*)(1-\sqrt{1-\alpha^\nu})=$$
$$(\gamma-\phi^*)\gamma\underbrace{(1-\sqrt{1-\alpha^\nu})}_{\geq 0}$$

which is strictly positive maintaining that $\gamma-\phi^*>0$. Using both terms and multiplying by z_t and σ_ν^2 respectively delivers the Proposition. □

Proof to Proposition 3 We focus on u_{t+1} and u_{t+1}^ν in turn. Consider κ for $\alpha<1$ and denote this by κ^{IM}:

$$\kappa^{IM}=(\gamma(1-\xi^*)-\phi^\Delta)^2-(\gamma(1-\xi^*)-\phi^\Delta))^2\sqrt{1-\alpha} \tag{51}$$

Imposing complete markets, $\alpha=1$, $\lambda^{CM}=(\gamma(1-\xi^*)-\phi^\Delta)^2$ and define this as κ^{CM}:

$$\kappa^{CM}=(\gamma(1-\xi^*)-\phi^\Delta)^2 \tag{52}$$

Since $(\gamma(1-\xi^*)-\phi^\Delta))^2\sqrt{1-\alpha}>0$, $\kappa^{IM}<\kappa^{CM}$. Second, denoting $\kappa^{CM}-\kappa^{IM}=(\gamma(1-\xi^*)-\phi^\Delta))^2\sqrt{1-\alpha}$ and taking the derivative with respect to $-\phi^\Delta$:

$$\frac{d(\kappa^{CM}-\kappa^{IM})}{d(-\phi^\Delta)}=2((\gamma(1-\xi^*)-\phi^\Delta))\sqrt{1-\alpha}>0$$

As long as $(\gamma(1-\xi^*)-\phi^\Delta)>0$, required to satisfy Proposition 2 for $\xi^*>1$, $\phi^\Delta\downarrow$ (or $\phi\downarrow$), $\kappa^{CM}-\kappa^{IM}\uparrow$ delivering the result.

In turn:

$$\kappa^{\nu\ IM}=(\gamma-\phi^*)^2-(\gamma-\phi^*)^2\sqrt{1-\alpha^\nu} \tag{53}$$

and

$$\kappa^{\nu\ CM}=(\gamma-\phi^*)^2 \tag{54}$$

Since $\sqrt{1-\alpha^\nu}>0$, $\kappa^{\nu\ IM}<\kappa^{\nu\ CM}$. Finally:

$$\frac{d(\kappa^{\nu\ CM}-\kappa^{\nu\ IM})}{d(-\phi^*)}=2(\gamma-\phi^*)\sqrt{1-\alpha}>0 \tag{55}$$

which confirms the result. $\square$

Proof to Proposition 4 From the static problem (91) (detailed in Appendix B.4.1, combining (93) with market clearing yields, detailed below for convenience:

$$\frac{c_H^\nu}{c_F^\nu} = \frac{c_H}{c_F} = \left(\frac{\alpha}{1-\alpha}\right)^2 (1+\tau_t)^\zeta \, \frac{\hat{I}_H - c_H}{\hat{I}_F - c_F}, \tag{56}$$

where c_H^ν is individual consumption of the H good, c_H is aggregate consumption of the H good and the first equality follows from the first order homogeneity of the aggregator (25). This condition satisfies both Home and Foreign static problems and market clearing. Rearranging yields the optimal F allocations for the Home households given $\{\hat{I}_H, \hat{I}_F\}$:

$$c_F = \frac{\hat{I}_F c_H}{b\hat{I}_H + (1-b)c_H} \tag{57}$$

where $b = (\frac{\alpha}{1-\alpha})^2(1+\tau_t)^\zeta$.

Next, consider country aggregate income I_t, which determines the exchange rate process by (24). Imposing $C_t = I_t$, the goods-specific allocations implied by the aggregator (25) are given by:

$$(1-\alpha)^{\frac{1}{\zeta-1}} c_F(I) = \left[I^{\frac{\zeta-1}{\zeta}} - \alpha^{\frac{1}{\zeta}} c_H(I)^{\frac{\zeta-1}{\zeta}} \right]^{\frac{\zeta}{\zeta-1}} \tag{58}$$

Combining c_F in (57) and (58) yields:

$$\left[I^{\frac{\zeta-1}{\zeta}} - \alpha^{\frac{1}{\zeta}} c_H(I)^{\frac{\zeta-1}{\zeta}} \right]^{\frac{\zeta}{\zeta-1}} (b\hat{I}_H + (1-b)c_H(I)) = \hat{I}_F c_H(I)(1-\alpha)^{\frac{1}{\zeta-1}} \tag{59}$$

and $\{\tau\} \neq 0$ is generically needed for this to be satisfied away from $\alpha \to 1$. $\square$

A.2. Additional Derivations for Section 2.

To find the admissible set of exchange rate processes, consider the log expansions of (6)-(10), assuming joint log normality of SDFs and prices:

$$\mathbb{E}_t[\widehat{m}_{t+1}] + \frac{1}{2}var_t(\widehat{m}_{t+1}) = -r_{t+1}, \quad (60)$$

$$\mathbb{E}_t[\widehat{m}^*_{t+1}] + \frac{1}{2}var_t(\widehat{m}^*_{t+1}) = -r^*_{t+1} \quad (61)$$

$$\mathbb{E}_t[\widehat{m}^*_{t+1}] + \frac{1}{2}var_t(\widehat{m}^*_{t+1}) - \mathbb{E}_t[\Delta e_{t+1}] + \frac{1}{2}var_t(\Delta e_{t+1}) + cov_t(\widehat{m}^*_{t+1}, -\Delta e_{t+1}) = -r_{t+1}, \quad (62)$$

$$\mathbb{E}_t[\widehat{m}_{t+1}] + \frac{1}{2}var_t(\widehat{m}_{t+1}) + \mathbb{E}_t[\Delta e_{t+1}] + \frac{1}{2}var_t(\Delta e_{t+1}) + cov_t(\widehat{m}_{t+1}, \Delta e_{t+1}) = -r^*_{t+1}, \quad (63)$$

where lower case levels denote logs, e.g. $\log(\widehat{M}_{t+1}) = \widehat{m}_{t+1}$ and $\Delta e_{t+1} = e_{t+1} - e_t$. Using (60) and (63), and (61) and (62) respectively, yields:

$$\mathbb{E}_t[\Delta e_{t+1}] + r^*_{t+1} - r_{t+1} = -cov_t(\widehat{m}_{t+1}, \Delta e_{t+1}) - \frac{1}{2}var_t(\Delta e_{t+1}), \quad (64)$$

$$\mathbb{E}_t[\Delta e_{t+1}] + r^*_{t+1} - r_{t+1} = cov_t(\widehat{m}^*_{t+1}, -\Delta e_{t+1}) + \frac{1}{2}var_t(\Delta e_{t+1}) \quad (65)$$

Combining the above yields (11).

Restriction on η wedge from cross-border trade in assets. Combining (6) and (9) for the Home pricer and combining (8) and (10) for the foreign pricer, using (12) yields the following conditions:

$$\mathbb{E}_t[\eta_{t+1}] = \frac{1}{2}var_t(\eta_{t+1}) - cov_t(\widehat{m}_{t+1}, \eta_{t+1}) \quad (66)$$

$$-\mathbb{E}_t[\eta_{t+1}] = \frac{1}{2}var_t(\eta_{t+1}) - cov_t(\widehat{m}^*_{t+1}, -\eta_{t+1}) \quad (67)$$

This is a straightforward generalization of the conditions in Lustig and Verdelhan (2019) to our environment with imperfect domestic risk sharing.

A.3. Limits to International Arbitrage

Consider the following aggregate Euler equations capturing within-country idiosyncratic risk, now allowing for shocks to international returns (due to limits to international arbitrage) u_{t+1}:

$$\mathbb{E}_t\left[\widehat{M}_{t+1}\right]R_{t+1}=1, \tag{68}$$

$$\mathbb{E}_t\left[\widehat{M}^*_{t+1}\right]R^*_{t+1}=1, \tag{69}$$

$$\mathbb{E}_t\left[\widehat{M}^*_{t+1}\frac{\mathcal{E}_t}{\mathcal{E}_{t+1}}\right]R_{t+1}=\mathbb{E}_t[e^{u^f_{t+1}}], \tag{70}$$

$$\mathbb{E}_t\left[\widehat{M}_{t+1}\frac{\mathcal{E}_{t+1}}{\mathcal{E}_t}\right]R^*_{t+1}=\mathbb{E}_t[e^{u^f_{t+1}}] \tag{71}$$

where we assume intermediation shocks have zero mean ($\mathbb{E}_t[u^f_{t+1}]=0$). Taking logs, we derive:

$$var_t(\Delta e_{t+1})-var_t(u^f_{t+1})+cov_t(\widehat{m}_{t+1}-\widehat{m}^*_{t+1},\Delta e_{t+1})=0$$

Substituting $var_t(u^f_{t+1})=u^f var_t(\Delta e_{t+1})$ and assuming $u^f\in[0,1]$, rearrranging:

$$var_t(\Delta e_{t+1})(1-u^f)cov_t(\widehat{m}_{t+1}-\widehat{m}^*_{t+1},\Delta e_{t+1}) \tag{72}$$

Then, repeating the steps in Proposition 1, $cov_t(\Delta c_{t+1}-\Delta c^*_{t+1},\Delta e_{t+1})\leq 0$ requires:

$$-cov_t\left(\tilde{\beta}_{t+1}-\tilde{\beta}^*_{t+1}\right)>var_t(\Delta e_{t+1})(1-u^f)$$

Applying the Cauchy-Schwarz identity and dividing by standard deviations yields condition (14).

A.4. Additional Results for Section 3

Country-specific shock applied to Home SDF. Consider the following pair of SDFs:

$$-\widehat{m}_{t+1}=\beta+\chi z_t+(\gamma-\phi)(\sqrt{z_t}u_{t+1}+\sigma_\nu u^\nu_{t+1}), \tag{73}$$

$$-\widehat{m}^*_{t+1}=\beta^*+\chi^* z_t+(\gamma-\phi^*)(\xi^*\sqrt{z_t}u_{t+1}) \tag{74}$$

where there is a country-specific factor driving Home consumption growth. We repeat our analysis focusing on the projection of moments on u^ν_{t+1}. The process for the incomplete markets wedge continues to be given by (34). Using (35):

$$\underbrace{\Gamma_1^{\nu 2} + \Gamma_2^{\nu 2}}_{var_t(\eta_{t+1}|u_{t+1}^{\nu})} = [(\gamma - \phi)^2 - \kappa^{\nu}]\sigma_{\nu}^2 \tag{75}$$

Then:

$$\Gamma_1^{\nu} = \pm\sqrt{(\gamma-\phi)^2 - \lambda^{\nu}}, \Gamma_2^{\nu} = \pm\sqrt{\lambda^{\nu} - \kappa^{\nu}}$$

Using (66) and (67) we derive:

$$\Gamma_0^{\nu} = \frac{1}{2}((\gamma-\phi)^2 - \kappa^{\nu}) - (\gamma-\phi)\sqrt{(\gamma-\phi)^2 - \lambda^{\nu}}, \tag{76}$$

$$\Gamma_0^{\nu} = 0\frac{1}{2}((\gamma-\phi)^2 - \kappa^{\nu}) \tag{77}$$

Combining:

$$\kappa^{\nu} = (\gamma-\phi)^2 - (\gamma-\phi)^2\sqrt{(\gamma-\phi)^2 - \lambda^{\nu}} \tag{78}$$

Finally, we derive

$$cov_t(m_{t+1}^* - m_{t+1}, \Delta e_{t+1}|u_{t+1}^{\nu}) = (\gamma-\phi)(1 - \sqrt{1-\alpha^{\nu}})$$

which is strictly positive maintaining $\gamma - \phi > 0$.

B. ONLINE APPENDIX

B.1. Trade in Risky Assets

If instead of allowing for trade in both risk-free assets, we allow for trade in Home and Foreign risky assets, then equations (6)–(10) are replaced by:

$$\mathbb{E}_t[\widehat{M}_{t+1}\tilde{R}_{t+1}] = 1, \tag{79}$$

$$\mathbb{E}_t\left[\widehat{M}_{t+1}\frac{\mathcal{E}_{t+1}}{\mathcal{E}_t}\tilde{R}_{t+1}^*\right] = 1, \tag{80}$$

$$\mathbb{E}_t[\widehat{M}_{t+1}^*\tilde{R}_{t+1}^*] = 1, \tag{81}$$

$$\mathbb{E}_t\left[\widehat{M}_{t+1}^*\left(\frac{\mathcal{E}_{t+1}}{\mathcal{E}_t}\right)^{-1}\tilde{R}_{t+1}\right] = 1, \tag{82}$$

where $\tilde{R}$ and $\tilde{R}^*$ are returns on risky Home and Foreign assets respectively.

Suppose Home and Foreign households trade in Home and Foreign currency denominated risky assets $\tilde{R}_{t+1}$ such that (79)- (82) hold. Assuming joint log normality, the above Euler equations imply:

$$\mathbb{E}_t[\widehat{m}_{t+1}] + \frac{1}{2}var_t(\widehat{m}_{t+1}) + \mathbb{E}_t[\tilde{r}_{t+1}] + \frac{1}{2}var_t(\tilde{r}_{t+1}) + cov_t(\widehat{m}_{t+1}, \tilde{r}_{t+1}) = 0, \quad (83)$$

$$\mathbb{E}_t[\widehat{m}_{t+1}] + \frac{1}{2}var_t(\widehat{m}_{t+1}) + \mathbb{E}_t[\tilde{r}^*_{t+1}] + \frac{1}{2}var_t(\tilde{r}^*_{t+1}) + \mathbb{E}_t[\Delta e_{t+1}] + \frac{1}{2}var_t(\Delta e_{t+1}) + \cdots$$
$$cov_t(\widehat{m}_{t+1}, \tilde{r}^*_{t+1}) + cov_t(\widehat{m}_{t+1}, \Delta e_{t+1}) + cov_t(\Delta e_{t+1}, \tilde{r}^*_{t+1}) = 0, \quad (84)$$

$$\mathbb{E}_t[\widehat{m}^*_{t+1}] + \frac{1}{2}var_t(\widehat{m}^*_{t+1}) + \mathbb{E}_t[\tilde{r}^*_{t+1}] + \frac{1}{2}var_t(\tilde{r}^*_{t+1}) + cov_t(\widehat{m}^*_{t+1}, \tilde{r}^*_{t+1}) = 0, \quad (85)$$

$$\mathbb{E}_t[\widehat{m}^*_{t+1}] + \frac{1}{2}var_t(\widehat{m}^*_{t+1}) - \mathbb{E}_t[\Delta e_{t+1}] + \frac{1}{2}var_t(\Delta e_{t+1}) + \mathbb{E}_t[\tilde{r}_{t+1}] + \frac{1}{2}var_t(\tilde{r}_{t+1}) + \cdots$$
$$cov_t(\widehat{m}^*_{t+1}, \tilde{r}_{t+1}) + cov_t(\widehat{m}^*_{t+1}, -\Delta e_{t+1}) + cov_t(-\Delta e_{t+1}, \tilde{r}_{t+1}) = 0 \quad (86)$$

Combining (83) - (86)

$$var_t(\Delta e_{t+1}) = cov_t(\widehat{m}_{t+1} - \widehat{m}^*_{t+1}, \Delta e_{t+1}) + cov_t(\eta_{t+1}, \tilde{r}^*_{t+1} - \tilde{r}_{t+1}) \quad (87)$$

Away from the representative agent limit, how many assets would it take to impose full risk-sharing? In practice, many (more than two) assets are traded across borders but few of these are risk-free in real terms, e.g. long-maturity bonds and equity. We extend our main result to a framework with trade in multiple risky assets but when (nominally) risk-free bonds are not available.

Proposition A1 (Many Assets, Many Agents)

When Home and Foreign currency risky assets with returns $\tilde{r}^{(*)}_{t+1}$ are internationally traded (79)-(82), then $cov_t(m^*_{t+1} - m_{t+1}, \Delta e_{t+1}) \leq 0$ if and only if:

$$-\rho_{\tilde{\beta}_{t+1}-\tilde{\beta}^*_{t+1}, \Delta e_{t+1}} + \frac{cov_t(\eta_{t+1}, \tilde{r}^*_{t+1} - \tilde{r}_{t+1})}{\sigma_t(\tilde{\beta}_{t+1} - \tilde{\beta}^*_{t+1})\sigma_t(\Delta e_{t+1})} \geq \frac{\sigma_t(\Delta e_{t+1})}{\sigma_t(\tilde{\beta}_{t+1} - \tilde{\beta}^*_{t+1})} \quad (88)$$

Proof. Rearrange (87) and apply the Cauchy-Schwarz Inequality. □

When only risky assets are traded, exchange rates need not be spanned, allowing further scope for pro-cyclical exchange rates. In particular, pro-cyclicality is recovered if the incomplete markets wedge co-varies positively with the differential return from a risky foreign asset. If, however, we additionally allow for trade in two nominally risk-free assets the new term in Proposition A1 ($cov_t(\eta_{t+1}, \tilde{r}^*_{t+1} - \tilde{r}_{t+1})$ converges to 0. This implies that Proposition A1 in the main body is the necessary and sufficient condition even when there is trade in many assets. Nonetheless, $\rho_{\tilde{\beta}_{t+1}-\tilde{\beta}^*_{t+1}, \Delta e_{t+1}}$ and $\sigma_t(\Delta e_{t+1})$ are themselves changing as the number of traded assets changes, see Section 3. Note also that this is true for any $r^{\widetilde{k(*)}}$ traded, as we detail next.

B.2. Degree of Market Completeness and Risk-Sharing

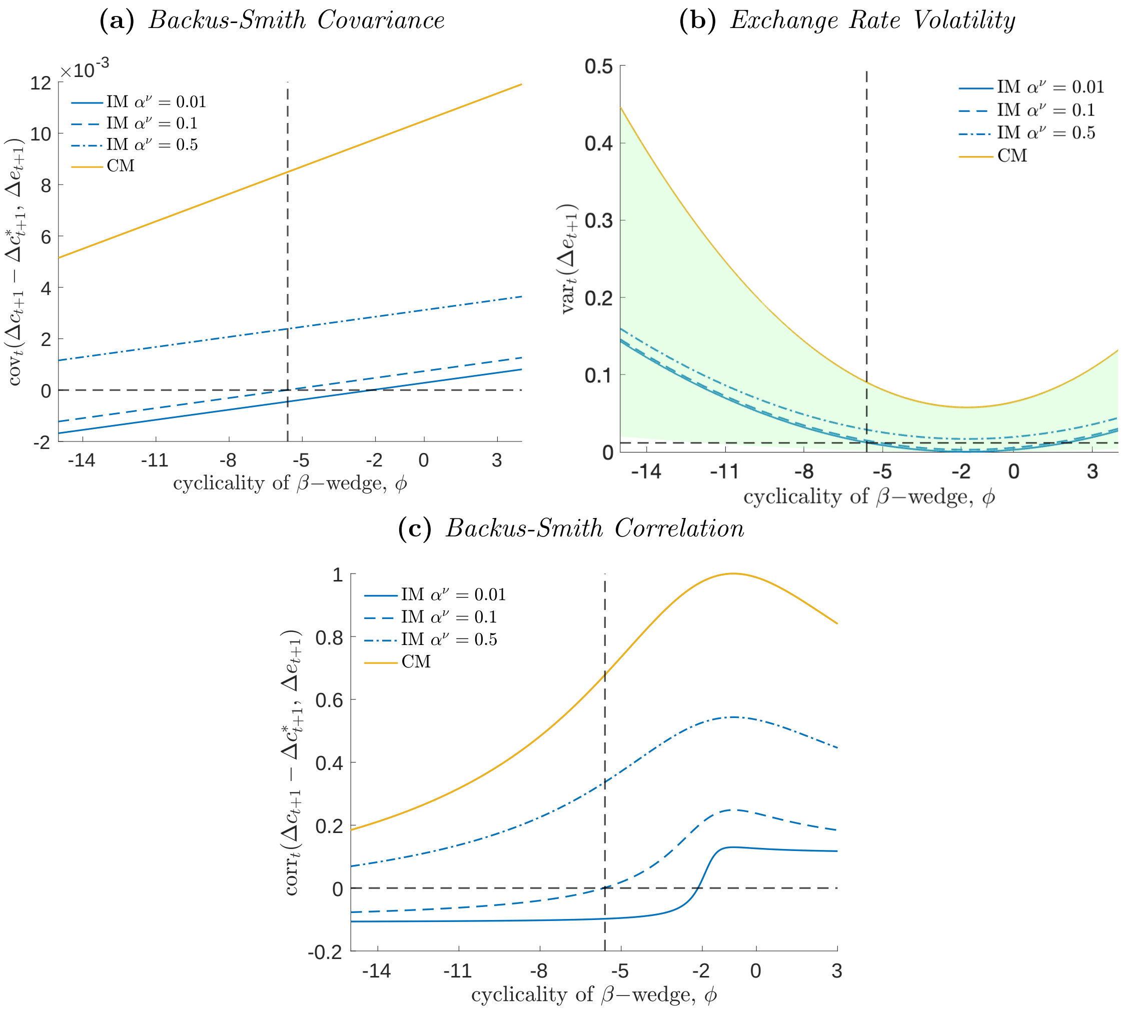
To illustrate that two nominally risk-free assets suffice to span exchange rates, regardless of market completeness, we turn to a framework in the tradition of Lucas (1978):

$$\Delta c_{t+1} = \sum_{k=1}^{N} g_{y_{k,t+1}}, \tag{89}$$

$$m_{t+1} = -\gamma \Delta c_{t+1}, \tag{90}$$

where $g_{y_{k,t+1}} = y_{k,t+1} - y_{k,t} \sim i.i.d\ \mathcal{N}(\mu_{y_k}, \sigma_{y_k})$ denotes the growth rate of k-th productive unit that comprises the consumption good. Corresponding variables for the Foreign economy are denoted with an asterisk. Start with the case of $N = 1$ productive units, discussed in (Lustig and Verdelhan, 2019, Sec III.A) and in Appendix B.1. Frictionless international trade in Home and Foreign risk-free bonds (66)- (67) and additional trade in a Home and a Foreign risky asset (a claim on $g_{y_{k,t+1}}, g^*_{y_{k,t+1}}$ respectively) will imply that the incomplete markets wedge η_{t+1} is orthogonal to $g_{y_{k,t+1}}, g^*_{y_{k,t+1}}$, and it then follows that the only equilibrium is $\eta_{t+1} = 0$– i.e. markets are complete. When $N > 1$, additional risky claims need to traded to complete the market. However, for any N, frictionless international trade in just the Home and the Foreign real bonds ensures that the sign of the relationship between the pricers' SDFs and exchange rate depreciations is positive.

Figure 3



B.3. Additional results for Section 3

Figure 3 provides comparative statics with respect to the spanning of the country-specific factor α^ν. In each panel, the solid blue line shows our baseline calibration of $\alpha^\nu = 0.01$. Spanning of the common shock α is set at baseline value of 0.6 across blue lines. Both the volatility of the exchange rate and the Backus Smith covariance are increasing in the spanning of the country specific shock, for a given cyclicality of the $\beta-$wedge. Panel (c) shows results in the correlation space.

B.4. Efficiency Conditions and Planning Problem

B.4.1 Optimal Consumption Bundle with Heterogeneous Agents

Given a level of individual consumption $C(s^t)$, households optimally choose their consumption bundle $\{c_H(s^t), c_F(s^t)\}$. For ease of exposition, since the problem is static we suppress dependence on histories and we denote with a superscript ν individual consumptions $c^\nu = c(z^t, \nu^t)$. The optimal consumption bundle satisfies:

$$\max_{\{c_H^\nu, c_F^\nu\}} \Bigg\{ g^*(c_H^{\nu\,*}, c_F^{\nu\,*}) \quad \text{s.t.} \quad \int_\nu c_H^\nu d\nu + \int_\nu c_H^{\nu\,*} d\nu^* = \hat{I}_H, \tag{91}$$
$$\int_\nu c_F^\nu (1+\tau) d\nu + \int_\nu c_F^{\nu\,*} d\nu = \hat{I}_F \text{ and } g(c_H^\nu, c_F^\nu) \geq C^\nu \Bigg\},$$

where we define $g(c_H, c_F)$ by (25) and I_H and I_F are world aggregates of good endowments. See Costinot, Lorenzoni and Werning (2014) for a friction-less representative agent framework, and Lloyd and Marin (2024) for a treatment with static wedges. Notice that from first order homogeneity of $g(\cdot)$ it follows that:

$$\frac{g_H(c_H^\nu, c_F^\nu)}{g_F(c_H^\nu, c_F^\nu)} = \frac{g_H(c_H, c_F)}{g_F(c_H, c_F)} = \frac{p_F}{p_H}(1+\tau) \tag{92}$$

B.4.2 Planner's Problem, Risk Sharing and Exchange Rate Cyclicality

Consider the following planning problem where we assume i) equal weights for each agent within each country and ii) the planner potentially faces costs to allocate good F to Home households. As a result of (ii), if costs are positive, the allocation is constrained efficient.

The planning problem is given as follows:

$$\max_{\{\delta^\nu, \delta^{\nu *}, c_H, c_F, c_H^*, c_F^*\}} \mu \int_\nu \frac{1}{1-\gamma} (\underbrace{\delta^\nu g(c_H^\nu, c_F^\nu)}_{C^\nu})^{1-\gamma} d\nu + (1-\mu) \int_{\nu^*} \frac{1}{1-\gamma} (\delta_t^{\nu *} g^*(c_H^{\nu *}, c_F^{\nu *}))^{1-\gamma} d\nu,$$
$$\text{s.t.} \quad \int_\nu c_H^\nu d\nu + \int_\nu c_H^{*\,\nu} d\nu = \hat{I}_H, \qquad \int_\nu c_F^\nu d\nu (1+\tau^{plan}) + \int_\nu c_F^{*\,\nu} d\nu = \hat{I}_F,$$
$$\int_\nu \delta^\nu d\nu = 1, \quad \int_\nu \delta^{*\,\nu} d\nu = 1$$

where $g(\cdot)$ is characterized by (25) and the constraints have associated multipliers λ_H, λ_F and $\lambda^\delta, \lambda^{\delta *}$. There are three associated efficiency conditions.

The first reveals that the planner wishes to equate marginal utility for every agent within countries:

$$\mu \left(C^\nu\right)^{-\gamma} C = \lambda^\delta, \quad (1-\mu) \left(C^{*\nu}\right)^{-\gamma} C = \lambda^{*\delta}$$

Since λ^δ and $\lambda^{*\delta}$ do not depend on inidividual histories ν, the planner chooses $\delta^\nu = \delta$ which must imply from the law of large numbers $\delta = 1$, and analogously $\delta^* = 1$.

The second condition relates relative goods consumption across the two countries:

$$\frac{c_H}{c_F} = \left(\frac{\alpha}{1-\alpha}\right)^2 (1+\tau_t^{plan})^\zeta \, \frac{c_H^*}{c_F^*}, \tag{93}$$

which coincides with the decentralized allocation if $\tau^{plan} = \tau$.

The final condition concerns the role of real exchange rates. Equating the first order conditions with respect to c_H and c_H^* implies:

$$\frac{\int_\nu \left(\delta^\nu\right)^{1-\gamma} d\nu}{\int_\nu \left(\delta^{\nu *}\right)^{1-\gamma} d\nu} \left(\frac{C}{C^*}\right)^{-\gamma} = \frac{1-\mu}{\mu} \, \frac{1}{\mathcal{E}} \tag{94}$$

where we have derived the real exchange rate as the ratio of multipliers as follows $\frac{dC^*}{dC} = \frac{1}{\mathcal{E}}$.[32,33] Condition (94) illustrates how domestic market incompleteness affects the international transmission of risk. Assuming $\mu = 1-\mu$, (94) differs from (1) because, absent borrowing constraints:

$$\int_\nu \left(\delta^\nu\right)^{1-\gamma} d\nu \neq \int_\nu \left(\delta^\nu\right)^{-\gamma} d\nu \tag{96}$$

reflecting a Jensen's term which arises because the planner cares about inequality. To

[32]This equation follows because:

$$\left(\frac{C}{C^*}\right)^{-\gamma} = \frac{1-\mu}{\mu}\frac{g_H^*}{g_H} = \frac{1-\mu}{\mu}\frac{g_F^*}{g_F}(1+\tau) \tag{95}$$

and $\frac{g_F^*}{g_F} = \frac{\frac{dC^*}{dc_F^*}}{\frac{dC}{dc_F}} = \frac{\frac{dC^*}{dC}}{\frac{dc_F^*}{dc_F}} = \frac{\frac{dC^*}{dC}}{\frac{dY_F - c_F(1+\tau)}{dc_F}} = -\frac{dC^*}{dC}(1+\tau)^{-1}$.

[33]Away from the flex price limit, Aguiar, Itskhoki and Mukhin (2025) argue that, due to pricing to market frictions, the real exchange rate does not correspond to the ratio of multipliers, and therefore the Backus-Smith condition is not a sufficient condition for risk-sharing.

see this, we can rewrite the above as

$$\int_\nu (\delta^\nu)^{1-\gamma}\, d\nu = \int_\nu (\delta^\nu)^{-\gamma}\,(\delta^\nu d\nu) = \mathbb{E}_\delta[(\delta^\nu)^{-\gamma}]$$

reflecting that the planner weights the discount factor adjustment $(\delta^\nu)^{-\gamma}$ by the dispersion of consumptions across agents.

Moreover, in the presence of borrowing constraints (4), the decentralized equilibrium exchange rate process reflects the maximal beta wedge (i.e. that of the most patient investors), not the average:

$$\frac{\int_\nu \left(\delta_{t+1}^\nu\right)^{-\gamma} d\nu}{\int_\nu \left(\delta_t^\nu\right)^{-\gamma} d\nu} \neq \max_\nu \beta_{t+1}^\nu$$

yielding an additional inefficiency of the decentralized equilibrium.

To recover the first best outcome and the Backus-Smith condition (1), substitute $\delta_t^\nu = 1\ \forall \nu$, and evaluate (94) for $\mu = 1/2$. In general, comparing the planner conditions with the decentralized allocation reveals two impediments to risk-sharing, τ and δ^ν. If trade wedges are technological $\tau = \tau^{plan}$, then idiosyncratic risk would be the only reason why risk sharing is low.

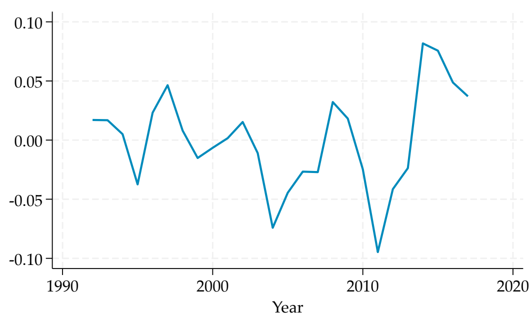
C. Empirical Results and Robustness

C.1. Berger et al. (2023) wedge

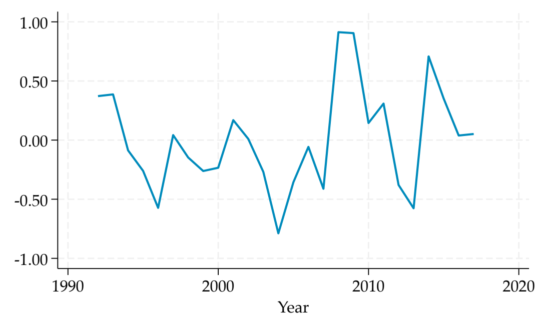
Figure 5 plots the time-series of the consumption based wedge from Berger et al. (2023) dataset, reconstructed for different values of $\gamma \in \{1, 3, 5, 7.5\}$.

Figure 4: *Time-series of Consumption based Wedge*

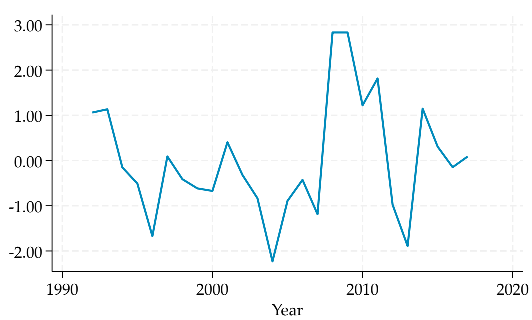
(a) *Inverse of EIS* $\gamma = 1$



(b) *Inverse of EIS* $\gamma = 3$



(c) *Inverse of EIS* $\gamma = 5$



(d) *Inverse of EIS* $\gamma = 7.5$

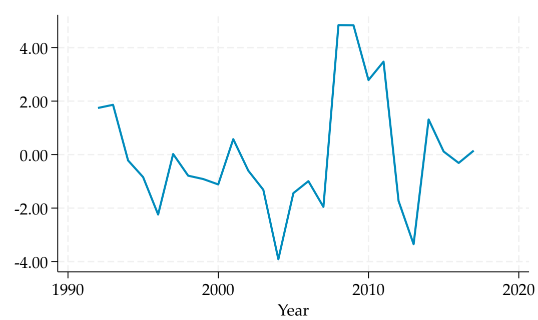


Table 6 lists the volatility of bilateral real exchange rate, the (negative of the) correlation of the wedge with real exchange rate growth and the threshold in equation (13) for different values of $\gamma \in \{3, 7.5, 10\}$ for seventeen advanced economies. Both conditional and unconditional moments are presented in the Table.

There are two main takeaways. First, for fifteen of the seventeen bilateral pairs, the exceptions being Japan and Switzerland, and for different values of γ, we find that the exchange rates are risky with respect to the $\tilde{\beta}$ wedge – the correlation with the

Table 6: *Empirical Moments: Real Exchange Rate Growth and Berger et al. (2023) Wedge*

Panel A $\gamma = 3$

	Unconditional			Conditional		
ISO	$\sigma(\Delta e)$	$-\text{Corr}(\tilde{\beta}, \Delta e)$	Threshold	$\sigma_t(\Delta e)$	$-\text{Corr}_t(\tilde{\beta}, \Delta e)$	Threshold_t
AUS	0.13	0.11	0.30	0.10	0.20	0.24
BEL	0.10	0.25	0.24	0.09	0.32	0.25
CAN	0.10	0.35	0.22	0.08	0.29	0.20
CHE	0.09	0.01	0.21	0.08	0.05	0.19
DEU	0.10	0.24	0.24	0.08	0.33	0.21
DNK	0.10	0.26	0.23	0.09	0.43	0.26
ESP	0.11	0.33	0.26	0.09	0.21	0.22
FIN	0.12	0.27	0.27	0.09	0.29	0.23
FRA	0.10	0.27	0.24	0.09	0.33	0.23
GBR	0.11	0.37	0.25	0.09	0.58	0.22
IRL	0.09	0.26	0.20	0.06	0.09	0.17
ITA	0.11	0.37	0.26	0.10	0.31	0.27
JPN	0.11	-0.32	0.25	0.10	-0.31	0.24
NLD	0.11	0.25	0.24	0.09	0.25	0.25
NOR	0.12	0.27	0.28	0.10	0.50	0.29
PRT	0.11	0.31	0.25	0.09	0.12	0.23
SWE	0.13	0.33	0.30	0.09	0.34	0.24
AVERAGE	0.11	0.23	0.25	0.09	0.25	0.23

Panel B $\gamma = 7.5$

$\gamma = 7.5$	Unconditional			Conditional		
ISO	$\sigma(\Delta e)$	$-\text{Corr}(\tilde{\beta}, \Delta e)$	Threshold	$\sigma_t(\Delta e)$	$-\text{Corr}_t(\tilde{\beta}, \Delta e)$	Threshold_t
AUS	0.13	0.01	0.06	0.10	0.01	0.05
BEL	0.10	0.24	0.05	0.09	0.22	0.05
CAN	0.10	0.24	0.04	0.08	0.09	0.04
CHE	0.09	-0.03	0.04	0.08	-0.02	0.04
DEU	0.10	0.24	0.05	0.08	0.30	0.04
DNK	0.10	0.24	0.05	0.09	0.34	0.05
ESP	0.11	0.31	0.05	0.09	0.19	0.04
FIN	0.12	0.25	0.05	0.09	0.23	0.05
FRA	0.10	0.26	0.05	0.09	0.25	0.05
GBR	0.11	0.35	0.05	0.09	0.56	0.04
IRL	0.09	0.21	0.04	0.06	0.05	0.03
ITA	0.11	0.34	0.05	0.10	0.20	0.05
JPN	0.11	-0.40	0.05	0.10	-0.48	0.04
NLD	0.11	0.25	0.05	0.09	0.15	0.05
NOR	0.12	0.20	0.06	0.10	0.29	0.05
PRT	0.11	0.29	0.05	0.09	0.04	0.05
SWE	0.13	0.26	0.06	0.09	0.21	0.05
AVERAGE	0.11	0.19	0.05	0.09	0.15	0.05

Continued on next page

[Continued] Table 6

Panel C $\gamma = 10$

ISO	Unconditional $\sigma(\Delta e)$	$-\text{Corr}(\tilde{\beta}, \Delta e)$	Threshold	Conditional $\sigma_t(\Delta e)$	$-\text{Corr}_t(\tilde{\beta}, \Delta e)$	Threshold_t
AUS	0.13	-0.02	0.04	0.10	-0.02	0.03
BEL	0.10	0.23	0.03	0.09	0.20	0.04
CAN	0.10	0.21	0.03	0.08	0.06	0.03
CHE	0.09	-0.03	0.03	0.08	-0.03	0.03
DEU	0.10	0.23	0.03	0.08	0.29	0.03
DNK	0.10	0.24	0.03	0.09	0.32	0.04
ESP	0.11	0.30	0.04	0.09	0.18	0.03
FIN	0.12	0.24	0.04	0.09	0.21	0.03
FRA	0.10	0.25	0.03	0.09	0.23	0.03
GBR	0.11	0.34	0.04	0.09	0.55	0.03
IRL	0.09	0.19	0.03	0.06	0.05	0.02
ITA	0.11	0.32	0.04	0.10	0.17	0.04
JPN	0.11	-0.41	0.04	0.10	-0.50	0.03
NLD	0.11	0.25	0.03	0.09	0.13	0.04
NOR	0.12	0.19	0.04	0.10	0.26	0.04
PRT	0.11	0.28	0.04	0.09	0.03	0.03
SWE	0.13	0.24	0.04	0.09	0.19	0.03
AVERAGE	0.11	0.18	0.04	0.09	0.14	0.03

Notes: Table lists the negative of the correlation between the US discount factor wedge ($\tilde{\beta}$) the bilateral real exchange rate growth (Δe), the threshold for exchange rate cyclicality described in equation (13) assuming domestic incomplete markets only within the US ($\tilde{\beta}^* = 0$), and the standard deviation of the real exchange rate growth for pair of seventeen advanced economies. Real exchange rate growth is constructed from Jordà et al. (2017) database. US discount factor wedge is constructed as in Berger et al. (2023) from Consumer Expenditure Survey in the US for $\gamma = \{3, 7.5, 10\}$ in Panels A, B, and C. Standard deviation of the discount factor wedge is provided in Table 1. Sample: 1992–2017 (annual).

bilateral real exchange rate is negative. Second, starting from values of $\gamma = 3$, the threshold is low enough such that the $\tilde{\beta}$ wedge can satisfy the inequality (13) without a need for segmentation shocks i.e. we can set $u = 0$ and still reconcile the sign of the Backus Smith puzzle. When $\gamma = 1$, exchange rates are still risky with respect to the $\tilde{\beta}$ wedge, but the wedge is not volatile enough and as a result the inequality with the threshold is not met. As the EIS increases, the within group dispersion in marginal utility of consumption share growth is amplified due to a Jensen's term. The within-group dispersion in marginal utility growth proxies for the dispersion in marginal utility growth across idiosyncratic states. As we increase γ, the correlation of the wedge with exchange rate growth does not change as much as volatility of the wedge, which directly affects the threshold, making it easier to satisfy the inequality (13).

Table 7: *Correlation between Pricing Kernels and Real Exchange Rate Growth*

Panel A $\gamma = 3$

	Unconditional		Conditional	
ISO	$\text{Corr}(\Delta c - \Delta c^*, \Delta e)$	$\text{Corr}(m^* - \widehat{m}, \Delta e)$	$\text{Corr}_t(\Delta c - \Delta c^*, \Delta e)$	$\text{Corr}_t(m^* - \widehat{m}, \Delta e)$
AUS	-0.39	0.08	-0.40	0.17
BEL	-0.30	0.23	-0.44	0.28
CAN	-0.12	0.34	-0.18	0.28
CHE	-0.09	-0.00	-0.23	0.03
DEU	0.08	0.23	-0.03	0.32
DNK	-0.34	0.22	-0.42	0.38
ESP	-0.01	0.33	-0.00	0.21
FIN	-0.38	0.21	-0.32	0.26
FRA	-0.25	0.25	-0.34	0.30
GBR	0.07	0.38	-0.09	0.59
IRL	0.08	0.29	0.52	0.15
ITA	0.01	0.36	0.06	0.31
JPN	0.19	-0.30	0.05	-0.30
NLD	0.19	0.27	-0.12	0.24
NOR	-0.27	0.25	-0.63	0.46
PRT	0.21	0.32	0.09	0.13
SWE	-0.27	0.31	-0.28	0.33
AVERAGE	-0.09	0.22	-0.16	0.24

Panel B $\gamma = 7.5$

$\gamma = 7.5$	Unconditional		Conditional	
ISO	$\text{Corr}(\Delta c - \Delta c^*, \Delta e)$	$\text{Corr}(m^* - \widehat{m}, \Delta e)$	$\text{Corr}_t(\Delta c - \Delta c^*, \Delta e)$	$\text{Corr}_t(m^* - \widehat{m}, \Delta e)$
AUS	-0.39	-0.01	-0.40	-0.00
BEL	-0.30	0.23	-0.44	0.20
CAN	-0.12	0.23	-0.18	0.08
CHE	-0.09	-0.03	-0.23	-0.03
DEU	0.08	0.23	-0.03	0.29
DNK	-0.34	0.23	-0.42	0.31
ESP	-0.01	0.32	-0.00	0.19
FIN	-0.38	0.22	-0.32	0.21
FRA	-0.25	0.25	-0.34	0.23
GBR	0.07	0.36	-0.09	0.57
IRL	0.08	0.23	0.52	0.08
ITA	0.01	0.34	0.06	0.20
JPN	0.19	-0.39	0.05	-0.48
NLD	0.19	0.26	-0.12	0.14
NOR	-0.27	0.19	-0.63	0.27
PRT	0.21	0.30	0.09	0.05
SWE	-0.27	0.25	-0.28	0.20
AVERAGE	-0.09	0.19	-0.16	0.15

(Continued on next page)

Table 7 (Continued)

Panel C $\gamma = 10$

	Unconditional		Conditional	
ISO	Corr($\Delta c - \Delta c^*, \Delta e$)	Corr($m^* - \widehat{m}, \Delta e$)	Corr$_t$($\Delta c - \Delta c^*, \Delta e$)	Corr$_t$($m^* - \widehat{m}, \Delta e$)
AUS	-0.39	-0.03	-0.40	-0.04
BEL	-0.30	0.22	-0.44	0.18
CAN	-0.12	0.21	-0.18	0.06
CHE	-0.09	-0.03	-0.23	-0.04
DEU	0.08	0.23	-0.03	0.29
DNK	-0.34	0.22	-0.42	0.30
ESP	-0.01	0.31	-0.00	0.18
FIN	-0.38	0.22	-0.32	0.20
FRA	-0.25	0.24	-0.34	0.22
GBR	0.07	0.34	-0.09	0.56
IRL	0.08	0.21	0.52	0.08
ITA	0.01	0.32	0.06	0.17
JPN	0.19	-0.40	0.05	-0.50
NLD	0.19	0.25	-0.12	0.12
NOR	-0.27	0.18	-0.63	0.24
PRT	0.21	0.29	0.09	0.03
SWE	-0.27	0.23	-0.28	0.18
AVERAGE	-0.09	0.18	-0.16	0.13

Notes: Table lists the correlation of relative consumption growth and exchange rate, and correlation of the relative SDF and real exchange rate growth assuming domestic incomplete markets only within the US ($\tilde{\beta}^* = 0$). Real exchange rate growth is constructed from Jordà et al. (2017) database. US discount factor wedge is constructed as in Berger et al. (2023) from Consumer Expenditure Survey in the US for $\gamma = \{3, 7.5, 10\}$ in Panels A, B, and C. Sample: 1992–2017 (annual). See text for details.

We can conduct a stronger empirical test of theory and verify if the correlation of difference in as-if representative agent log SDFs with real exchange rate growth is positive. We construct the as-if representative agent log SDF in the U.S. as implied by the model: $\widehat{m} = -\gamma \Delta c + \tilde{\beta} + \log \beta$. Since we have assumed the wedge is constant in Foreign, the Foreign SDF is the scaled consumption growth $m^* = -\gamma \Delta c^* + \log \beta^*$.

Tables 7 lists these correlation (both conditional and unconditional) for all seventeen bilateral pairs for values of $\gamma \in \{3, 7.5, 10\}$, and cross-country average is noted in the last row. Consistent with theoretical prediction, the correlation of relative SDF with exchange rate growth is positive for all pairs with the exception of Japan and Switzerland.

Overall, we robustly find that (i) exchange rates are risky with respect to the wedge, (ii) inequality (13) is satisfied for values of $\gamma >= 3$, and (iii) correlation of relative pricing kernels (constructed using the wedge) with exchange rate is positive while correlation of relative consumption growth with exchange rate is negative.

Finally, for completeness, Table 8 notes the covariance between pricing kernels and real exchange rate growth for the baseline case of $\gamma = 5$ reported in the main text. Volatility of exchange rate is reported in Table 7. As can be seen, the covariance of the as-if rep agent pricing kernels is not equal to the volatility of exchange rate.

Table 8: *Covariances between Pricing Kernels and Real Exchange Rate Growth*

$\gamma = 5$	Unconditional (100 ×)		Conditional (100 ×)	
ISO	$\text{Cov}(\Delta c - \Delta c^*, \Delta e)$	$\text{Cov}(m^* - \widehat{m}, \Delta e)$	$\text{Cov}_t(\Delta c - \Delta c^*, \Delta e)$	$\text{Cov}_t(m^* - \widehat{m}, \Delta e)$
AUS	-0.06	0.57	-0.04	0.75
BEL	-0.03	3.07	-0.03	1.95
CAN	-0.01	3.38	-0.01	1.29
CHE	-0.01	-0.42	-0.02	-0.15
DEU	0.01	3.19	-0.00	3.11
DNK	-0.06	2.90	-0.06	2.96
ESP	-0.00	4.69	-0.00	2.13
FIN	-0.11	3.35	-0.03	2.47
FRA	-0.02	3.37	-0.03	2.63
GBR	0.01	5.07	-0.01	6.08
IRL	0.02	2.58	0.05	0.65
ITA	0.00	5.18	0.01	2.42
JPN	0.03	-5.25	0.01	-5.22
NLD	0.03	3.58	-0.01	1.46
NOR	-0.04	3.33	-0.07	3.77
PRT	0.05	4.48	0.02	0.72
SWE	-0.05	4.51	-0.02	2.57
AVERAGE	-0.02	2.80	-0.02	1.74

Notes: Table lists the 100 × covariance of relative consumption growth and exchange rate, and 100 × covariance of the relative SDF and real exchange rate growth assuming domestic incomplete markets only within the US ($\tilde{\beta}^* = 0$). Real exchange rate growth is constructed from Jordà et al. (2017) database. US discount factor wedge is constructed as in Berger et al. (2023) for $\gamma = 5$ from Consumer Expenditure Survey in the US. Sample: 1992–2017 (annual). See text for details.

C.1.1 Alternate Conditioning Set

Table 9: *Empirical Moments: Real Exchange Rate Growth and Berger et al. (2023) Wedge with a Large Conditioning Set*

ISO	$\sigma_t(\Delta e)$	$-\text{Corr}_t(\tilde{\beta}, \Delta e)$	Threshold_t
AUS	0.08	0.18	0.10
BEL	0.07	0.18	0.09
CAN	0.07	0.33	0.08
CHE	0.06	0.29	0.07
DEU	0.08	0.68	0.10
DNK	0.08	0.49	0.09
ESP	0.08	0.49	0.11
FIN	0.08	0.36	0.09
FRA	0.09	0.34	0.10
GBR	0.07	0.67	0.09
IRL	0.05	-0.13	0.08
ITA	0.08	0.32	0.10
JPN	0.09	-0.48	0.09
NLD	0.08	0.24	0.11
NOR	0.09	0.16	0.10
PRT	0.08	0.49	0.12
SWE	0.09	0.28	0.10
AVERAGE	0.08	0.29	0.09

We conduct robustness with a larger conditioning set to construct moments of Δe_{t+1}, and $\tilde{\beta}_{t+1}$. We control for date t values of log consumption in US and Foreign (c_t, c_t^*), short term nominal interest rate (i_t, i_t^*), level of long term interest rate in each country, log CPI in each country, and log bilateral real exchange rate (e_t). Relative to the baseline, we allow each country variables to have a separate loading in the conditioning set. Table 9 shows that the correlation remains sufficiently negative to meet the threshold. Results are shown for $\gamma = 5$.

C.1.2 Panel Fixed Effects Regressions

Table 10: *Panel Fixed Effects Empirical Moments: Real Exchange Rate Growth and Berger et al. (2023) Wedge*

ISO	$\sigma_t(\Delta e)$	$-\text{Corr}_t(\tilde{\beta}, \Delta e)$	Threshold_t
AUS	0.12	0.05	0.09
BEL	0.09	0.21	0.07
CAN	0.09	0.17	0.08
CHE	0.09	0.07	0.07
DEU	0.09	0.23	0.07
DNK	0.09	0.21	0.08
ESP	0.10	0.15	0.08
FIN	0.11	0.20	0.09
FRA	0.09	0.24	0.08
GBR	0.10	0.43	0.07
IRL	0.09	0.00	0.07
ITA	0.10	0.20	0.08
JPN	0.10	-0.46	0.07
NLD	0.09	0.23	0.08
NOR	0.11	0.22	0.09
PRT	0.10	0.16	0.08
SWE	0.11	0.27	0.09
AVERAGE	0.10	0.15	0.08

We now construct the conditional moments by residualizing the beta wedge and real exchange rate using a panel fixed effects regression. In the baseline exercise reported in the main text, the residuals were constructed from each country's own regression, allowing both the intercept and slope to be different across bilateral pair of countries. We continue to set $\gamma = 5$.

Table 10 reports the results. The conditional correlation of the beta wedge with exchange rate is sufficiently negative that it meets the threshold.[34]

[34] We briefly note why the conditional correlation estimated here may differ from the conditional correlation in the main text. Fixed effect regression based residual correlations emphasize the co-movement of deviations relative to common global relationships. If the true underlying relationships differ across countries (heterogeneous effects), imposing common slopes reduces correlation or even distorts it, because residuals become mixed with systematic differences in responsiveness.

C.2. Permanent income risk based wedge

Bayer et al. (2019) construct a measure of very persistent income risk using Survey of Income Participation Program (SIPP). In the Constantinides and Duffie (1996) model, the discount factor wedge $\tilde{\beta}^{CD}$ is a function of the cross-sectional variance of the permanent income process y^2. Namely:

$$\tilde{\beta}^{CD} = \frac{\gamma\,(\gamma+1)}{2} y^2$$

Using $\gamma = 5$, and the variance of income risk series estimated in Bayer et al. (2019), we reconstruct the beta wedge assuming only permanent income risk. We keep the sample fixed at annual 1992–2013 (last year for which Bayer et al. (2019) series is available).

Table 11: *Empirical Moments: Real Exchange Rate Growth and Bayer et al. (2019) Wedge*

ISO	$\sigma(\Delta e)$	$-\text{Corr}(\tilde{\beta}^{\text{CD}}, \Delta\text{e})$	$\sigma(\tilde{\beta}^{CD})$	Threshold
AUS	0.13	0.19	0.03	4.89
BEL	0.12	0.06	0.03	4.33
CAN	0.09	0.39	0.03	3.19
CHE	0.12	-0.09	0.03	4.54
DEU	0.12	0.09	0.03	4.38
DNK	0.12	0.06	0.03	4.26
ESP	0.12	0.11	0.03	4.38
FIN	0.12	0.09	0.03	4.59
FRA	0.12	0.07	0.03	4.28
GBR	0.13	0.31	0.03	4.65
IRL	0.09	-0.10	0.03	3.49
ITA	0.12	0.07	0.03	4.48
JPN	0.13	-0.39	0.03	4.69
NLD	0.12	0.08	0.03	4.41
NOR	0.12	0.25	0.03	4.52
PRT	0.12	0.09	0.03	4.25
SWE	0.13	0.17	0.03	4.80
AVERAGE	0.12	0.09	0.03	4.36

Table 11 reports the results for the condition (13) with the $\beta-$wedge constructed as shown in equation (21) in Section 4.1. While the discount factor wedge measured only with permanent income risk also comoves negatively with exchange rate, the volatility of this wedge is of two orders of magnitude lower than the discount factor wedge constructed from Berger et al. (2023) with $\gamma = 5$ (see Table 1 for summary statistics of the Berger et al. (2023) wedge). Consequently, relying solely on permanent income risk based wedge is insufficient to explain the Backus Smith cyclicality puzzle.

C.3. Bilateral Wedge from GRID

There is another major difference in the way the wedge is calculated using the GRID data relative to the Berger et al. (2023). The data in the GRID is on $\frac{1}{N_g}\sum_{\nu=1}^{N_g}\log\left(\frac{I^{\nu}_{t+1}}{I^{\nu}_t}\right)$, where I^{ν} is individual with idiosyncratic history ν_t's income. We can therefore only construct $\frac{1}{N_g}\sum_{i=1}^{N_g}\log\left(\frac{I^{\nu}_{t+1}}{I^{\nu}_t}\right)^{-\gamma}$. The true wedge and the proxy wedge are related by a Jensen's term:

$$\underbrace{\log\frac{1}{N_g}\sum_{\nu=1}^{N_g}\left(\frac{I^{\nu}_{t+1}/I_{t+1}}{I^{\nu}_t/I_t}\right)^{-\gamma}}_{\text{True wedge}} = \underbrace{\frac{1}{N_g}\sum_{\nu=1}^{N_g}\log\left(\frac{I^{\nu}_{t+1}/I_{t+1}}{I^{\nu}_t/I_t}\right)^{-\gamma}}_{\text{Proxy Wedge}} + \text{Jensen gap}$$

where I_t is the average income in the economy. We can decompose either of the wedges into within group term and a composition term:

$$\log\frac{1}{N_g}\sum_{\nu=1}^{N_g}\left(\frac{I^{\nu}_{t+1}/I_{t+1}}{I^{\nu}_t/I_t}\right)^{-\gamma} = \log\left(\frac{I^{g}_{t+1}/I_{t+1}}{I^{g}_t/I_t}\right)^{-\gamma} + \log\frac{1}{N_g}\sum_{\nu=1}^{N_g}\left(\frac{I^{\nu}_{t+1}/I^{g}_{t+1}}{I^{\nu}_t/I^{g}_t}\right)^{-\gamma}$$

$$\frac{1}{N_g}\sum_{\nu=1}^{N_g}\log\left(\frac{I^{\nu}_{t+1}/I_{t+1}}{I^{\nu}_t/I_t}\right)^{-\gamma} = \log\left(\frac{I^{g}_{t+1}/I_{t+1}}{I^{g}_t/I_t}\right)^{-\gamma} + \frac{1}{N_g}\sum_{\nu=1}^{N_g}\log\left(\frac{I^{\nu}_{t+1}/I^{g}_{t+1}}{I^{\nu}_t/I^{g}_t}\right)^{-\gamma}$$

where I^g_t denotes average income of the group g. Both the true and the proxy wedges have identical composition terms capturing change in powered income shares of groups in the economy. Both also capture a within group dispersion of income shares across households but a Jensen's gap arises– the true wedge captures the convexity component from averaging before taking logs. Taking a second order approximation of the wedge, it can be shown that the Jensen gap term increases in γ since the degree of convexity goes up with γ. There are no a priori reasons to believe that the Jensen gap term would not affect the correlation of the wedge with exchange rate. However, when we computed the correlation of $\beta-$wedges computed with different EIS parameters for the U.S., we found that the correlation was relatively stable as we increased γ. In Table 15, we show the correlation with exchange rate when measured using log-income based wedge and the consumption based wedge for $\gamma = 5$. We discuss this issue further in Section C.4.

We measure bilateral wedges using micro statistics from the Global Repository of Income Dynamics (Guvenen et al., 2022, GRID henceforth). We focus on a sample of advanced economies for which we can obtain the longest panel to construct the bilateral wedge against the U.S. The annual data sample covers Canada, Denmark, France, Germany, Italy, Norway, Sweden, the UK, and the US spanning 1998–2015.

Variables used in our wedge construction are residual log earnings growth for top 1 percent, 2.5 percent, 5 percent, 10 percent of each country's log earnings distribution in year t, and the mean residual log earnings growth for that year. These statistics are computed for Male in age groups 25–55 population of each country. GRID constructs residual log earnings growth from first regressing log earnings on age dummies for each year t and then taking average of growth rate.[35]

Denote with $\overline{\log I_{gt}}$ the average (residual) log earnings for the group g, and $\overline{\log I_t}$ as the average (residual) log earnings in the country at time t. Then the income share at date t for a group g is defined as $\varphi^I_{gt} \equiv \exp\left(\overline{\log I_{gt}} - \overline{\log I_t}\right)$. The discount factor wedge for a group g is then constructed as:[36]

$$\beta^I_{gt} = \left(\frac{\varphi^I_{gt+1}}{\varphi^I_{gt}}\right)^{-\gamma} \tag{97}$$

where γ is the inverse of the intertemporal elasticity of substitution. We de-mean

[35] In unreported results, we verified the results are robust to using statistics for all genders, age 25–55 population.

[36] The data in GRID is averages at the percentile level. This proxy construction is thus different from Berger et al. (2023) in that we are taking average income shares to construct a wedge, whereas Berger et al. (2023) construct average of the individual wedges.

each country's wedge for (i) partially addressing measurement error as in Berger et al. (2023), and (ii) comparison across countries.[37] The bilateral wedge is then given by the difference in the de-meaned country specific wedges for group g:

$$\Delta\beta_{gt} \equiv \beta^{I}_{gt} - \beta^{*I}_{gt} \tag{98}$$

C.3.1 When can we use correlation of income wedge to proxy correlation of consumption wedge.

Ideally, we would use micro-data on consumption for each country and construct bilateral wedge distances. Instead, we need to rely on micro-data for income. The two key concerns are i) the correlation of consumption shares differs from the correlation of growth shares, ii) the volatility of consumption shares is significantly lower than the volatility of income shares. Consider the following statistical model capturing the idea of consumption smoothing:

$$\frac{C^{\nu}_{t+1}}{C^{\nu}_{t}} = \theta^{i}\frac{I^{\nu}_{t+1}}{I^{\nu}_{t}}, \quad \frac{C^{\nu*}_{t+1}}{C^{\nu*}_{t}} = \theta^{\nu*}\frac{I^{\nu*}_{t+1}}{I^{\nu*}_{t}}, \tag{99}$$

$$\frac{C_{t+1}}{C_{t}} = \theta\frac{I_{t+1}}{I_{t}}, \quad \frac{C^{*}_{t+1}}{C^{*}_{t}} = \theta^{*}\frac{I^{*}_{t+1}}{I^{*}_{t}} \tag{100}$$

where $\theta^{(*)}, \theta^{i(*)} \leq 1$ such that consumption growth is less volatile than income growth, and for high income individuals on their Euler equations $\theta^{\nu(*)} < \theta^{(*)}$, i.e. they engage in more consumption smoothing than the aggregate. As long as $\theta^{\nu}/\theta \approx \theta^{\nu*}/\theta^{*}$, such that consumption smoothing of the marginal investor relative to the aggregate is similar across countries, then:

$$\begin{aligned} cov\left(\tilde{\beta}_{t+1} - \tilde{\beta}^{*}_{t+1}, \Delta e_{t+1}\right) &= cov\left(-\gamma\log\left(\frac{\theta^{\nu}}{\theta}\frac{I^{\nu}_{t+1}/I^{\nu}_{t}}{I_{t+1}/I_{t}}\right) + \gamma\log\left(\frac{\theta^{\nu*}}{\theta^{*}}\frac{I^{\nu*}_{t+1}/I^{\nu*}_{t}}{I^{*}_{t+1}/I^{*}_{t}}\right), \Delta e_{t+1}\right) \\ &\approx \frac{\theta^{i}}{\theta} cov\left(\tilde{\beta}^{y}_{t+1} - \tilde{\beta}^{y*}_{t+1}, \Delta e_{t+1}\right) \end{aligned}$$

where $\tilde{\beta}^{y}_{t+1} = -\gamma\log\left(\frac{I^{\nu}_{t+1}/I^{\nu}_{t}}{I_{t+1}/I_{t}}\right)$.

Critically, in the correlation space, the ratio θ^{ν}/θ does not affect scaling. Specifically,

[37] Results are robust to using the non-deamened wedges.

when $\theta^{\nu}/\theta \approx \theta^{\nu *}/\theta^{*}$, then:

$$\rho_{\tilde{\beta}_{t+1}-\tilde{\beta}^{*}_{t+1},\Delta e_{t+1}} = \frac{cov_t\left(\tilde{\beta}_{t+1}-\tilde{\beta}^{*}_{t+1},\Delta e_{t+1}\right)}{\sigma_t(\tilde{\beta}_{t+1}-\tilde{\beta}^{*}_{t+1})\sigma_t(\Delta e_{t+1})} \approx \frac{cov_t\left(\tilde{\beta}^{y}_{t+1}-\tilde{\beta}^{y*}_{t+1},\Delta e_{t+1}\right)}{\sigma_t(\tilde{\beta}^{y}_{t+1}-\tilde{\beta}^{y*}_{t+1})\sigma_t(\Delta e_{t+1})} = \rho_{\tilde{\beta}^{y}_{t+1}-\tilde{\beta}^{y*}_{t+1},\Delta e_{t+1}}$$

C.3.2 Summary Statistics and Additional Results with GRID dataset

Table 12: *Summary Statistics for GRID data set*

Panel A: Log-income based Country-specific Wedge

	Top 10%		Top 5%		Top 2.5%		Top 1%	
iso	$\sigma(\beta^I_{gt})$	$\text{Corr}(\beta^I_{gt}, \widehat{Y}_t)$	$\sigma(\beta^I_{gt})$	$\text{Corr}(\beta^I_{gt}, \widehat{Y}_t)$	$\sigma(\beta^I_{gt})$	$\text{Corr}(\beta^I_{gt}, \widehat{Y}_t)$	$\sigma(\beta^I_{gt})$	$\text{Corr}(\beta^I_{gt}, \widehat{Y}_t)$
CAN	0.06	0.03	0.13	0.09	0.15	0.06	0.15	-0.04
DEU	0.10	-0.36	0.19	-0.27	0.19	-0.21	0.15	-0.24
DNK	0.10	0.30	0.20	-0.07	0.23	-0.16	0.22	-0.26
FRA	0.11	-0.19	0.28	-0.29	0.35	-0.28	0.41	-0.27
ITA	0.20	0.33	0.50	0.10	0.55	-0.00	0.48	-0.04
NOR	0.10	0.52	0.18	0.43	0.28	0.32	0.43	0.30
SWE	0.09	-0.53	0.23	-0.58	0.33	-0.60	0.37	-0.60
USA	0.09	-0.17	0.14	-0.03	0.18	-0.01	0.19	-0.07
AVERAGE	0.11	-0.01	0.23	-0.08	0.28	-0.11	0.30	-0.15

Panel B: Log-Income based Bilateral Wedge

	$\sigma(\Delta e_t)$	$\sigma(\Delta\beta_{gt})$			
iso		Top 10%	Top 5%	Top 2.5%	Top 1%
CAN	0.09	0.05	0.06	0.08	0.09
DEU	0.10	0.14	0.24	0.26	0.26
DNK	0.10	0.13	0.20	0.24	0.24
FRA	0.10	0.12	0.21	0.24	0.28
ITA	0.11	0.19	0.39	0.38	0.40
NOR	0.11	0.13	0.22	0.34	0.52
SWE	0.13	0.09	0.17	0.24	0.28
AVERAGE	0.10	0.12	0.21	0.25	0.30

Notes: Panel A reports country-specific summary statistics for the standard deviation of the wedge, $\sigma(\beta^I_{gt})$, and its correlation with output growth, $\text{Corr}(\beta^I_{gt}, \widehat{Y}_t)$, across four percentile groups. Panel B lists the standard deviation of the bilateral real exchange rate (Δe) and the bilateral discount factor wedges constructed for different groups β^I_{gt}. Wedges are constructed using residual log earnings data for Male ages between 25–55 in GRID. Inverse of EIS, $\gamma = 5$. Real exchange rate growth is constructed from Jordà et al. (2017) database. Sample: 1998–2015 (annual). See text for details.

Table 13: *Correlation, Thresholds, and Covariances with Real Exchange Rate Growth*

Panel A: Correlation and Thresholds for Top 10% and 1% Groups

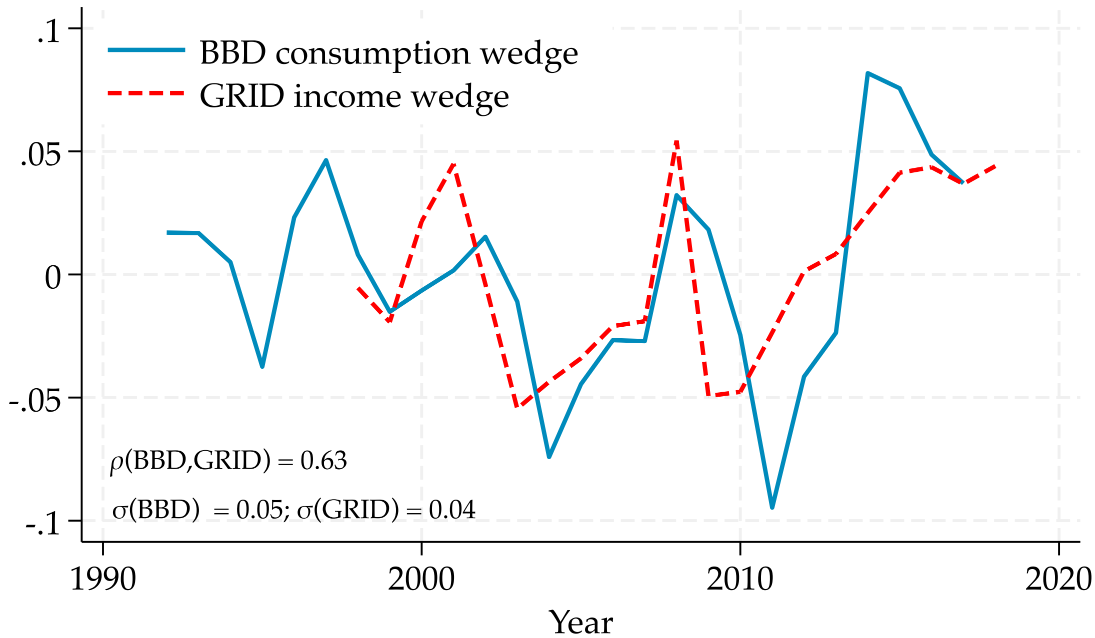
	Unconditional			*Conditional*		
	-Corr($\tilde{\beta} - \tilde{\beta}^{*}, \Delta e$)		*Thresh.*	-Corr$_t$($\tilde{\beta} - \tilde{\beta}^{*}, \Delta e$)		*Thresh.$_t$*
iso	Top 10%	Top 1%		Top 10%	Top 1%	
CAN	0.30	0.66	0.17	0.33	0.63	0.20
DEU	0.54	0.57	0.17	0.63	0.88	0.13
DNK	0.29	0.52	0.17	0.57	0.57	0.21
FRA	0.18	0.13	0.17	0.31	0.46	0.17
ITA	0.18	0.52	0.17	0.36	0.63	0.20
NOR	0.29	0.23	0.21	0.06	0.17	0.21
SWE	0.42	0.40	0.21	0.49	0.45	0.16
AVERAGE	0.31	0.43	0.18	0.39	0.54	0.18

Panel B: Correlation of Pricing Kernels and Real Exchange Rate Growth

iso	Corr($\Delta c - \Delta c^{*}, \Delta e$)	Corr($\widehat{m}^{*} - \widehat{m}, \Delta e$)		Corr$_t$($\Delta c - \Delta c^{*}, \Delta e$)	Corr$_t$($\widehat{m}^{*} - \widehat{m}, \Delta e$)	
		Top 10%	Top 1%		Top 10%	Top 1%
CAN	-0.03	0.13	0.55	-0.21	0.06	0.54
DEU	0.19	0.53	0.59	0.12	0.61	0.89
DNK	-0.29	0.06	0.38	-0.61	-0.38	0.36
FRA	-0.15	0.14	0.11	-0.21	0.23	0.42
ITA	0.09	0.19	0.51	0.14	0.42	0.62
NOR	-0.38	0.09	0.18	-0.58	-0.20	0.10
SWE	-0.06	0.32	0.39	-0.07	0.37	0.41
AVERAGE	-0.09	0.21	0.39	-0.20	0.16	0.48

C.3.3 Comparison with GRID based wedge with Consumption based wedge

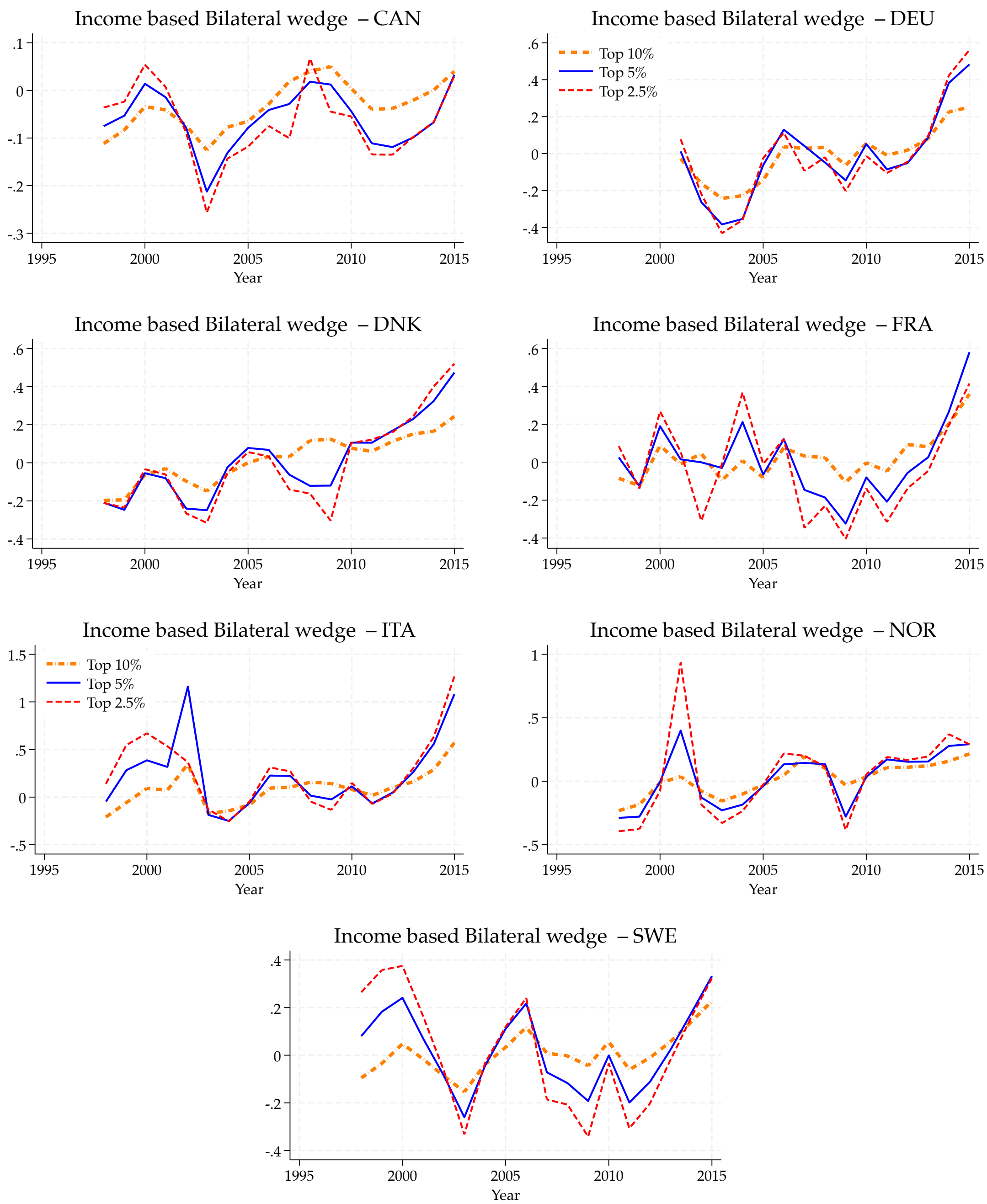
Figure 5: *Consumption based Wedge and GRID's Income Based Wedge for the US (2.5 percentile)*



Notes: Figure plots the time-series of consumption based wedge in solid blue line, measured from Consumption expenditure survey for the US, and the log-income based wedge for the US measured from the GRID dataset in dashed red line. EIS is set at 1 in constructing both wedges.

C.3.4 Time-series plots of the bilateral wedges

Figure 6: *Bilateral Wedges*



Notes: Figure plots the time-series of the bilateral income based wedges for US/Foreign country bilateral pairs for different percentile groups. Inverse of the EIS, γ is set to 5. Wedges are constructed from the GRID dataset.

C.3.5 Conditional Summary Statistics with GRID dataset

Table 14: *Summary Statistics for Country-specific Wedges (Conditional)*

Panel A: Log-income based Country-specific Wedge

	Top 10%		Top 5%		Top 2.5%		Top 1%	
iso	$\sigma_t(\beta^I_{gt})$	$\text{Corr}_t(\beta^I_{gt}, \widehat{Y}_t)$	$\sigma_t(\beta^I_{gt})$	$\text{Corr}_t(\beta^I_{gt}, \widehat{Y}_t)$	$\sigma_t(\beta^I_{gt})$	$\text{Corr}_t(\beta^I_{gt}, \widehat{Y}_t)$	$\sigma_t(\beta^I_{gt})$	$\text{Corr}_t(\beta^I_{gt}, \widehat{Y}_t)$
CAN	0.05	0.17	0.10	0.12	0.12	0.08	0.12	0.02
DEU	0.06	-0.40	0.14	-0.30	0.16	-0.26	0.14	-0.29
DNK	0.03	-0.53	0.13	-0.66	0.15	-0.70	0.17	-0.70
FRA	0.07	-0.80	0.17	-0.82	0.21	-0.77	0.17	-0.60
ITA	0.08	-0.08	0.23	-0.22	0.21	-0.41	0.24	-0.39
NOR	0.05	-0.25	0.13	-0.11	0.25	-0.07	0.42	0.02
SWE	0.07	-0.72	0.15	-0.68	0.20	-0.68	0.21	-0.65
USA	0.08	-0.32	0.13	-0.12	0.17	-0.06	0.20	-0.10
AVERAGE	0.06	-0.37	0.15	-0.35	0.19	-0.36	0.21	-0.34

Panel B: Log-Income based Bilateral Wedge

	$\sigma_t(\Delta e_t)$	$\sigma_t(\Delta\beta_{gt})$			
		Top 10%	Top 5%	Top 2.5%	Top 1%
CAN	0.10	0.02	0.05	0.07	0.08
DEU	0.08	0.05	0.14	0.16	0.14
DNK	0.09	0.03	0.10	0.13	0.17
FRA	0.09	0.06	0.16	0.21	0.21
ITA	0.09	0.08	0.22	0.20	0.28
NOR	0.11	0.08	0.16	0.28	0.44
SWE	0.09	0.07	0.15	0.20	0.21
AVERAGE	0.09	0.06	0.14	0.18	0.22

Notes: Table reports conditonal moments. Panel A reports country-specific summary statistics for the standard deviation of the wedge, $\sigma(\beta^I_{gt})$, and its correlation with output growth, $\text{Corr}(\beta^I_{gt}, \widehat{Y}_t)$, across four percentile groups. Panel B lists the standard deviation of the bilateral real exchange rate (Δe) and the bilateral discount factor wedges constructed for different groups β^I_{gt}. Wedges are constructed using residual log earnings data for Male ages between 25–55 in GRID. Inverse of EIS, $\gamma = 5$. Real exchange rate growth is constructed from Jordà et al. (2017) database. Sample: 1998–2015 (annual). See text for details.

C.4. Consumption-based wedge vs log income-based wedge in CEX

We construct log income based wedge in the CEX data:

$$\tilde{\beta}^{\ln \text{I}} = \frac{1}{N_g} \sum_{\nu=1}^{N_g} \log \left(\frac{I^\nu_{t+1}/I^\nu_t}{I_{t+1}/I_t} \right)^{-\gamma}$$

Table 15: *Empirical Moments: Real Exchange Rate Growth and log income based Wedge*

		Consumption based		Log-Income based	
ISO	$\sigma(\Delta e)$	$-\text{Corr}(\tilde{\beta}, \Delta e)$	Threshold	$-\text{Corr}(\tilde{\beta}^{\ln y}, \Delta e)$	Threshold
AUS	0.13	0.05	0.10	0.17	0.83
BEL	0.10	0.24	0.08	0.03	0.66
CAN	0.10	0.28	0.08	0.28	0.61
CHE	0.09	-0.03	0.07	0.03	0.58
DEU	0.10	0.23	0.08	-0.02	0.66
DNK	0.10	0.25	0.08	0.05	0.64
ESP	0.11	0.32	0.09	0.18	0.72
FIN	0.12	0.25	0.09	0.20	0.75
FRA	0.10	0.26	0.08	0.04	0.66
GBR	0.11	0.36	0.08	0.22	0.68
IRL	0.09	0.23	0.07	0.15	0.54
ITA	0.11	0.36	0.09	0.28	0.71
JPN	0.11	-0.39	0.08	-0.17	0.69
NLD	0.11	0.25	0.08	0.02	0.67
NOR	0.12	0.23	0.09	0.27	0.77
PRT	0.11	0.30	0.09	0.04	0.70
SWE	0.13	0.28	0.10	0.28	0.82
AVERAGE	0.11	0.20	0.08	0.12	0.69

We find that the correlation of this wedge with exchange rate is similar to that of the true consumption based wedge. However, volatility of this wedge is of an order of magnitude smaller than the true wedge's volatility. Standard deviation of log income based wedge is 0.16, while that of the consumption based wedge is 1.29. While income based wedge should be more volatile than consumption based wedge, the volatility of log income based wedge is smaller than an actual income based wedge because of a missing Jensen's term as discussed in Section C.3. It turns out, on net, the log income based wedge is much less volatile than the true consumption based wedge. As a result, the threshold is not satisfied with the log-income wedge. The correlation of log income based wedge with exchange rate (0.12) is similar to the observed correlation of consumption based wedge with exchange rate (0.20).