CS559 Sanjeet Vinod Jain
CWID 20012768

Answer 1
Part 1
Show $E[x] = \mu$

$$LHS = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{\frac{-1}{2\sigma^2}(x-\mu)^2\right\} x\,dx$$

let $t = \frac{x-\mu}{\sigma}$ $\Rightarrow x = \sigma t + \mu$

$\Rightarrow \frac{dt}{dx} = \frac{1}{\sigma} \Rightarrow dx = \sigma dt$

~~Substituting~~ Substituting values we get

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-t^2/2} (\sigma t+\mu)\sigma\,dt \quad \left\{\begin{matrix} x\to-\infty ; t\to-\infty \\ x\to\infty ; t\to\infty \end{matrix}\right\}$$

$$\Rightarrow \frac{\sigma}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} e^{-t^2/2}(\sigma t+\mu)\,dt$$ removed constants

$$\Rightarrow \frac{1}{\sqrt{2\pi}}\left[\underbrace{\int_{-\infty}^{\infty} \sigma t e^{-t^2/2}\,dt}_{1} + \underbrace{\int_{-\infty}^{\infty} \mu e^{-t^2/2}\,dt}_{2}\right]$$

solving 1 $\int_{-\infty}^{\infty} \sigma t e^{-t^2/2}\,dt$

Here t & $e^{-t^2/2}$ are the 2 functions in the integral ~~(lets call them u & v)~~

t is an odd function i.e $f(-t) = -f(t)$

~~Hence the~~ and $e^{-t^2/2}$ is an even function.

$\Rightarrow f(-t) = f(t)$ & $f(-t) = f(t)$

The product of an even & odd function results in an odd function.
and it is also known for an odd function $f(t)$

$$\int_{-\infty}^{\infty} f(t) = 0$$ when $f(t)$ is an odd function.

$\Rightarrow$

$$\int_{-\infty}^{\infty} \sigma t e^{-t^2/2} dt = 0$$

Solving part (2)

$$\int_{-\infty}^{\infty} \mu e^{-t^2/2} dt = \mu \int_{-\infty}^{\infty} e^{-t^2/2} dt$$

- It is known $\int_{-\infty}^{\infty} e^{-ax^2} dx = \sqrt{\frac{\pi}{a}}$ where $a > 0$

$$\Rightarrow \mu \int_{-\infty}^{\infty} e^{-t^2/2} dt = \sqrt{\frac{\pi}{1/2}} = \sqrt{2\pi}$$

putting back values of (1) & (2) in the main equation $\rightarrow$

$$\frac{1}{\sqrt{2\pi}}\left(0 + \mu\sqrt{2\pi}\right) = \frac{\mu \cancel{\sqrt{2\pi}}}{\cancel{\sqrt{2\pi}}} = \mu$$

$$\boxed{\Rightarrow E[x] = \mu}$$

Hence proved

Part 2

Show $E[x^2] = \mu^2 + \sigma^2$

$$LHS = \int_{-\infty}^{\infty} N(x|\mu\ \sigma^2)\, x^2\, dx$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} \exp\left\{-\frac{(x-\mu)^2}{2\sigma^2}\right\} x^2\, dx.$$

let $t = \dfrac{x-\mu}{\sigma}$ then $x = \mu + \sigma t$

$$\Rightarrow \frac{dt}{dx} = \frac{1}{\sigma} \Rightarrow dx = \sigma dt$$

Also $\because x = \mu + \sigma t$ & $dx = \sigma dt$

$$x^2 = (\mu + \sigma t)^2 = \mu^2 + \sigma^2 t^2 + 2\mu\sigma t$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{-t^2/2} (\sigma^2 t^2 + 2\mu\sigma t + \mu^2)\, \sigma dt$$

$$\Rightarrow \frac{\cancel{\sigma}}{\cancel{\sigma}\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-t^2/2} (\sigma^2 t^2 + 2\mu\sigma t + \mu^2)\, dt$$

$$\Rightarrow \frac{1}{\sqrt{2\pi}} \Big[\underbrace{\int_{-\infty}^{\infty} e^{-t^2/2} \sigma^2 t^2\, dt}_{1} + \underbrace{\int_{-\infty}^{\infty} e^{-t^2/2} (2\mu\sigma t)\, dt}_{2} + \underbrace{\int_{-\infty}^{\infty} e^{-t^2/2} \mu^2\, dt}_{3} \Big]$$

Solving ①

$$\int_{-\infty}^{\infty} e^{-t^2/2} \sigma^2 t^2 \, dt = \sigma^2 \int_{-\infty}^{\infty} t^2 e^{-t^2/2} \, dt$$

$$= \sigma^2 \cdot \frac{1}{2}\sqrt{\frac{\pi}{(1/2)^3}} \qquad \left\{ \int_{-\infty}^{\infty} x^2 e^{-ax^2} = \frac{1}{2}\sqrt{\frac{\pi}{a^3}} \right.$$

$$= \sigma^2 \frac{1}{2} \sqrt{8\pi} = \underline{\underline{\sigma^2 \sqrt{2\pi}}}$$

Solving ② $\int_{-\infty}^{\infty} e^{-t^2/2} 2\mu\sigma t \, dt = \left[\int_{-\infty}^{\infty} e^{-t^2} t \, dt\right] 2\mu\sigma$

again we have an odd function (t) multiplied by an even function $e^{-t^2/2}$ which results in an odd function $te^{-t^2/2}$ and its integral from $-\infty$ to ∞ is zero

$$\Rightarrow 2\mu\sigma \int_{-\infty}^{\infty} e^{-t^2/2} t \, dt = \underline{\underline{0}}$$

Solving ③ $\int_{-\infty}^{\infty} \mu^2 e^{-t^2/2} \, dt = \mu^2 \int_{-\infty}^{\infty} e^{-t^2/2} \, dt$

$$= \underline{\underline{\mu^2 \sqrt{2\pi}}} \qquad \left\{ \int_{-\infty}^{\infty} e^{-ax^2} dx = \sqrt{\frac{\pi}{a}} \right.$$

Combining above results to the main equation we get →

$$\Rightarrow E[x^2] = \frac{1}{\cancel{\sqrt{2\pi}}} \left[\sigma^2 \cancel{\sqrt{2\pi}} + 0 + \mu^2 \cancel{\sqrt{2\pi}}\right]$$

$$\Rightarrow E[x^2] = (\mu^2 + \sigma^2)$$ Hence Proved.

~~Q~~ Answer 2

2] 2) $P_x = {}^nC_x\, p^x q^{n-x}$

n = no. of trials
x = no of times for an outcome within n
p = success probability
q = failure —"—

for 5 coins flipped, probability of having 2 heads given as

$P_2 = {}^5C_2\, p^2 q^3$

$= \frac{5!}{3!\,2!} (0.5)^2 (0.5)^3$

$= \frac{5 \times 4}{2} (0.5)^5$

$= 10 (0.5)^5 = 10 \times 0.03125 = 0.3125$

or 31.25%.

{ p & q are 0.5 as odds of heads or no heads for single toss