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$\boxed{\text{MTH}}$ → Mathmatical Foundatⁿ for CS

Credit = 4

• Lang → mathematical ~~logue~~ logic

• Proposition - or statement, is a declaritive sentence which is either true or false but not both.
  - represented by small letters → (a, b, c .. p, q) called propositional variables

true (T), false (F) called the Truth value

F → { wish, order, Interrogati }

• Compound Proposition - Proposition obtained from the combination of 2 or more Atomic (primary) proposition by means of some word / phrases

• Logical operators (Connectives) - words, phrases that are used to form a compound proposition.

(i) Basic logical Opⁿ

(a) Conjunction - Any 2 propositions p, q combined with the word AND to form a compound prop. is called conjunction of p, q

$\boxed{p \wedge q}$ (or p and q)

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

(same rule) all T $\xrightarrow{\text{then}}$ only T

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eg p : It is cold $\Rightarrow$ $p \wedge q$ = It is cold and it's Raining
q : It is raining

(b) Disjunction $\rightarrow$ OR

$\rightarrow$ $p \vee q$ (p or q)

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

(atleast 1 true) also T

XOR (when 1 is true)

p	q	$p \oplus q$
T	T	F
T	F	T
F	T	T
F	F	F

eg It is cold or it is raining.

(c) Negation: The negation of proposition 'p' is $\rightarrow$ it is not the case that 'p'

- or Not 'p' ($\sim p$)

p	$\sim p$
T	F
F	T

eg "Michael's PC runs linux"

Negation: "It is not the case that "Michael's PC runs linux""

OR "Michael's PC doesn't run linux"

(d). Conditional statement: The compound statement of the form $\rightarrow$ 'if p then q'

Implication $\leftarrow$ also called

- represented as $p \Rightarrow q$ or $p \rightarrow q$

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

Here p : Hypothesis (Premise or Antecedent)
q : Conclusion (consequence)

consist of 2 conditional statement

(e). Biconditional - compound proposition "p"

- Let p, q be prop., the bicond stat $p \leftrightarrow q$ is the prop p iff q

MCQ

$p \leftrightarrow q$

$\Downarrow$ equiv. to

$p \to q$ and $q \to p$

↳ "p if and only if q"

other words

• "p is necessary & sufficient for q"

• "If p then q, and conversely"

• "p iff q"

- represented as → p iff q

or ↳ $p \Leftrightarrow q$ or $p \Longleftrightarrow q$

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

True ~~for~~ → i.e same T.V

(f). Complete Truth Table with all 5 operators

Operators	Connector word	symbol	symbol form
• Conjunction	AND	$\wedge$	$p \wedge q$
• Disjunction	OR	$\vee$	$p \vee q$
• Negation	NOT	$\sim, \neg$	$\sim p$
• Conditional	if ~ then	$\rightarrow$	$p \rightarrow q$
• Biconditional	iff	$\Leftrightarrow$	$p \Leftrightarrow q$

* set	Prop. logic	0s, 1s ~~Boolean~~
$A \cup B$	$p \vee q$	$p + q$
$A \cap B$	$p \wedge q$	$p \cdot q$
$\bar{A}$	$\sim p$	$\bar{p}$

Precedence of logical op^r

op^r	Precedence
$\sim$	1
$\wedge$	2
$\vee$	3
$\rightarrow$	4
$\Leftrightarrow$	5

.) "I + ing

unless = if + not

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* Other connecting words for different conditional statement

$\boxed{p \to q}$

(i) if p then q
(ii) if p, q
(iii) p implies q $(p \Rightarrow q)$
(iv). p only if q
(v). p is sufficient for q
(vi) q if p
(vii) q whenever p
(viii) q follows from p
(ix) q is necessary for p
(x) q unless $\sim q$

which / that

$\sim q \to \sim p$

* Converse, Contrapositive, Inverse

→ ulta → ulta mei ulta → sulte mei ulta

also are conditional statements

$q \to p$ (if q then p) is converse of $p \to q$

$\sim q \to \sim p$ (if $\sim q$ then $\sim p$)

$\sim p \to \sim q$ (if $\sim p$ then $\sim q$)

Q Find converse, CP, Inverse of the conditional statement:
↳ If it rains, the crops will grow

Sol: The given statement is:

If it rains, the crops will grow
which is of the form,

if p then q or if p, q

$\therefore$

or $p \to q$

p : It rains

q : The crops will grow

(i). $\sim p$: It doesn't rain

Negat^n $\sim q$: The crops will not grow

(ii). Converse : The statement $q \rightarrow p$ i.e if q, p

If $\underline{q}$, then $\underline{p}$

(ii) Contrapositive : $\sim q \rightarrow \sim p$

If $\underline{\sim q}$, then $\underline{\sim p}$

(iii). Inverse

$\sim p \rightarrow \sim q$

If $\underline{\sim p} \rightarrow \underline{\sim q}$

$\rightarrow q$ whenever p

Q-2) given conditional statement, "I go to the beach whenever it's a suny summer day"

Note
$\hookrightarrow p \rightarrow$ hypothesis
$q \rightarrow$ conclusion

Ans

p : It's a suny summer day

q : I go to beach

(i) Converse $q \rightarrow p$

p whenever q ✓

simple

(ii) Inverse

$\sim p \rightarrow \sim q$

$\sim q \rightarrow \sim p$ whenever ✓

(ii) Contrapositive $\sim q \rightarrow \sim p$

$\sim p \rightarrow \sim q$ whenever ✓

will be +V+in

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Logical Equivalence/Equal

- 2 compound proposition $P(p,q,r...)$ and $Q(p,q,r...)$ where $p, q, r....$ are propositional variables are called logical equal if they have same truth value for every possible case

- written as → $P \equiv Q$ or $P = Q$

Note : No. of rows in the truth table = 2^n → no. of propositional variable

Tautology, Contradiction and Contingency

- A compound statement $P(p,q,r...)$ is said to be
  (i) Tautology – if it is always true for every possible value of $p, q, r...$ eg $p \vee \sim p$ (T+T) so prove
  (ii) contradiction – if it is alway false eg $p \wedge \sim p$ ''
  (iii) contingency – if it is some true & some false

10 marks

Q Prove that they are logically equal $p \wedge (q \vee r) \equiv (p \wedge q) \vee (q \wedge r)$

Ans No. of propositional variables = 3 (p, q, r)
No. of rows in truth table = $2^3 = 8$

Let $P = p \wedge (q \vee r)$
$Q = (p \wedge q) \vee (p \wedge r)$

$\vee \to$ OR
$\wedge \to$ and

Victor

p	q	r	$q \vee r$	P	$p \wedge q$	$p \wedge r$	Q
T	T	T	T	T	T	T	T
T	T	F	T	T	T	F	T
T	F	T	T	T	F	T	T
T	F	F	F	F	F	F	F
F	T	T	T	F	F	F	F
F	T	F	T	F	F	F	F
F	F	T	T	F	F	F	F
F	F	F	F	F	F	F	F

Here $\to$ 5th and 8th colounms are equal $\to P = Q$

Note :

Two compound proposition P and Q are logically equal if their biconditional statement $P \Leftrightarrow Q$ is a tautology

Q OR easy

~~wheather the following~~

Discuss the equality or inequality of the following proposition

$(p \to \sim q) \vee (q \to \sim r)$ and $(p \to \sim q) \wedge (p \leftrightarrow r)$

So let,

$P = (p \to \sim q) \vee (q \to \sim r)$

$Q = (p \to \sim q) \wedge (p \leftrightarrow r)$

No. of prositional variables = 3 (p, q, r)

Now truth table,

$p \rightarrow q$

T	T	T
T	F	F
F	T	T
F	F	T

$p \leftrightarrow q$

T	T	T
T	F	F
F	T	F
F	F	T

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Table-1

p	q	r	$\sim q$	$\sim r$	$p \rightarrow \sim q$	$q \rightarrow \sim r$	P	$p \leftrightarrow r$	Q
T	T	T	F	F	F	F	F	T	F
T	T	F	F	T	F	T	T	F	F
T	F	T	T	F	T	F	T	T	T
T	F	F	T	T	T	T	T	F	F
F	T	T	F	F	T	F	T	F	F
F	T	F	F	T	T	T	T	T	T
F	F	T	T	F	T	T	T	F	F
F	F	F	T	T	T	T	T	T	T

$\therefore P \neq Q$ thus inequal

9 10

Laws of Propositional Logic (with proof)

① Identity Law : $p \wedge T = p$

$p \vee F = p$

{ T → Tautology
F → contradiction

② Domination Law : $p \vee T = T$

$p \wedge F = F$

→ eg

p	T	$p \vee T$
T	T	T
F	T	T

③ Idempotent law : $p \vee p = p$

$p \wedge p = p$

④ Double Negation law : $\sim(\sim p) = p$

⑤ Commutative law : $p \vee q = q \vee p$

$p \wedge q = q \wedge p$

⑥ Associative law : $p \vee (q \vee r) = (p \vee q) \vee r$

$p \wedge (q \wedge r) = (p \wedge q) \wedge r$

⑦ Distributive law : $p \vee (q \wedge r) = (p \vee q) \wedge (p \vee r)$

$p \wedge (q \vee r) = (p \wedge q) \vee (p \wedge r)$

Spill the beans = reveal the secret

(8) De-Morgan's law → eg in New notebook

$$\sim(p \vee q) = \sim p \wedge \sim q$$
$$\sim(p \wedge q) = \sim p \vee \sim q$$

(9). Absorption law

$$p \vee (p \wedge q) = p$$
$$p \wedge (p \vee q) = p$$

(10) Negation law

$$p \vee \sim p = T$$
$$p \wedge \sim p = F$$

Q - with Proof
OR
MCQ
easy

Logical equivalence involving conditional statements

(i). $p \to q = \sim p \vee q$

(ii). $p \to q = \sim q \to \sim p$

(iii). $p \vee q = \sim p \to q$

(iv). $p \wedge q = \sim(p \to \sim q)$

(v). $\sim(p \to q) = p \wedge \sim q$

(vi). $(p \to q) \wedge (p \to r) = p \to (q \wedge r)$

(vii). $(p \to q) \vee (p \to r) = p \to (q \vee r)$

(viii) $(p \to r) \wedge (q \to r) = (p \vee q) \to r$

(ix) $(p \to r) \vee (q \to r) = (p \wedge q) \to r$

MCQ
easy

Logical equivalence involving Biconditional statements

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(i). $p \leftrightarrow q = (p \to q) \wedge (q \to p)$

(ii). $p \leftrightarrow q = \sim p \leftrightarrow \sim q$

(iii). $p \leftrightarrow q = (p \wedge q) \vee (\sim p \wedge \sim q)$

(iv). $\sim (p \leftrightarrow q) = p \leftrightarrow \sim q$

Predicate logic

Note — when the values of variables are not specified, the statements are neither T nor F, once values are assig the stat. P(x) becomes a proposit and has a T.V.

'$x > 5$'

i.e 'x is greater than 5'

x → variable called subject of the statement

greater than 5 → condition on 'x' → predicate

$P(x)$: x is greater than 5 ; $x \in$ Domain

↳ f^n of x — Propositional function

$x = a$: $P(a)$ is a proposition — it's a statement

- Predicate logic is an extension of Propositional logic.
- It adds the concept of predicates and quantifiers to better capture the meaning that can't be expressed by propositional logic.

- Predicate refers to the property that the subject of the statement can have.

In the above eg, $P(x)$: x is greater than 5

P : greater than 5

↳ denotes the predicate "greater than 5" of variable x. (subject)

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• Predicate can be considered as a f^n of subject (variable) x and it tells us the truth value of the statement $P(x)$ at x.

MCQ
• Once a value is assigned to variable x, $P(x)$ become a proposition and has Truth value.

R → Beg/Leaf

Ex $P(x) : x+3 = 7$
$P(3) : 3+3 = 7$ which is False and becomes a $proposit^n$ with truth value F

$P(5) = 5+3 = 7$ " "
$P(4) = 4+3 = 7$ is a proposition with Truth value T

Ex $Q(x, y) : x+y > 2$; $x, y \in I$
$Q(1, 2) : 1+2 > 2$ — True, a $proposit^n$
$Q(0, 1) : 0+1 > 2$ — False, " "

Quantifiers

* (Some) students of section DOC13 are non-serious (OR) intelligent

Quantifier

↳ $disjunct^n$

Quantity ← Quantifiers

p: non serious
q: intelligent
$p \vee q$

• Quantifiers are the words that refers to quantity i.e some, none, many, few, all, every and tells us how many elements are there for that predicate i.e express the extent (to) which a predicate is true over the domain of the subject x.

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• Using a quantifier to create such type of proposition from prop. fⁿ is called Quantification.

Types of Quantificatⁿ

1. **Universal Quantification** - The U.Q of $P(x)$ is the statement "$P(x)$ for all values of x in the domain" or "$P(x)$ is true for all values of x".

- It is denoted by $\forall x \in D$; $P(x)$ is called UQ.

domain or domain of discourse or Universe of discourse

- An element for which $P(x)$ is false is called Counter Example of $\forall x : P(x)$

- The domain must always be be specified when a U.Q is used w/o it the UQ is not defined.

- The notatⁿ $\forall x\ P(x)$ denotes UQ of $P(x)$; $\forall \rightarrow$ Universal Quantifier

Q. Find the truth value of the UQ, $\forall x : P(x)$, where $P(x)$ be the statement $x>5$ and $D \in I$.

Solⁿ
$P(x) : x>5$
Domain : set of Integer
$x \in I$

Real No → Rational, Irrational
Rational → integers, fraction
Irrational → $\sqrt{3}$, π

If $P(4) : 4>5$ (False)
$\therefore$ the Universal Quant $\forall x : P(x)$ is False
i.e $x=4$ is a counterexample for $\forall x\ P(x)$

Q $p(x) : x+1 > x : x \in R$ → True
$\therefore \forall x : P(x)$ is True

Q. $x^2 \geq x$ — (i) $\forall x \in R$ → False counterexample $x = 1/2$
(ii) $\forall x \in I$ → True

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Q. $P(x) : x^2 > 0, \ \forall x \in I$

The T.V of this Quantificat' is <u>False</u> as $x = 0$ is a counterexample.

Existential Qualification

- It's the statement, when there exist an element 'x' in Domain such that $P(x)$ is

- $\exists x : P(x)$ when $\exists$ is called E.Q — there exist / OR for some, for atleast one, there is

- } ~~all~~, some, few, atleast one ← $\exists$ means

Q what is the truth value of E.Q when
$P(x)$ is $x > 5$
$\exists x : P(x)$

Sol^n
$P(x) : x > 5$
$x \in I$ — why not $x = 1, 2, 3, 4, 5$ — False dega as kuch k liye true (some)

take $x = 6$ $P(6) = 6 > 5$ (True)

$\therefore$ $\exists x : P(x)$ is True.

IMP
Note
- When all the element in Domain are listed → $x_1, x_2, x_3 \ldots$
then → (i) the Universal Quant. = Conjunct^n of $P(x_1), P(x_2) \ldots P(x_n)$

$\therefore$ $\forall x : P(x) \cong P(x_1) \wedge P(x_2) \wedge P(x_3) \ldots \wedge P(x)$

(ii) $\exists x : P(x) \cong P(x_1) \vee P(x_2) \ldots \vee P(x_n)$
The Disjunct^n of

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Q eg / what is the Truth value

(i) $\forall x : P(x)$ (ii) $\exists x : P(x)$

when $P(x) : x^2 < 10$ and

Domain consisting of positive integers not exceeding ~~than~~ 5

soln $D = \{1, 2, 3, 4, 5\}$

Truth table

x	$P(x)$	Truth value
1	$P(1) = 1^2 < 10$	T
2	$P(2) = 4 < 10$	T
3	$P(3) = 9 < 10$	T
4	$P(4) = 16 < 10$	F
	$P(5) = 25 < 10$	F

$Q \rightarrow Q(x)$ $x = x + 1$ (real no.)

E.Q → False U.Q → False

Note: The Quantifiers $\forall$, $\exists$ have higher precedence than $\vee, \wedge, \Leftrightarrow, \Rightarrow$ etc.

(i) Truth value of

$\forall x : P(x) \cong T \wedge T \wedge T \wedge F \wedge F = F$

(ii) TV of

$\exists x : P(x) \cong T \vee T \vee T \vee F \vee F = T$

Negation of Qualification

(i) $\sim(\forall x : P(x)) \cong - \forall x : \sim P(x)$

$\cong \exists x : \sim P(x)$

(ii) $\sim(\exists x ; P(x)) \cong \forall x : \sim P(x)$

MCQ

• The Negation of U.Q becomes E.Q of $\sim P(x)$ (negation of predi...)

Note Uniqueness Quantifier -> denoted by $\exists!$ or $\exists_1$, notatⁿ $\exists! x\, P(x)$ states -

there exists a unique x such that $P(x)$ is true

or

(there is exactly one / there is one and only one)

eg $\exists! (x^2-1=0)$ $\therefore x=1$ is a unique real no. so a True statement

* eg "All students of DOC 13 are Intelligent."

Negatⁿ -> There exist some students who are not Intelligent

MCQ *Formulae :-

(1) $\sim(\forall x : P(x) \vee Q(x))$

$= \exists x : \sim(P(x) \vee Q(x))$

$= \exists x : \sim P(x) \wedge \sim Q(x)$

(2) $\sim(\forall x : P(x) \wedge Q(x)) = \exists x : \sim P(x) \vee \sim Q(x)$

(3) $\sim(\exists x : P(x) \vee Q(x)) = \forall x :$

(4) ~~$\sim(\exists x$~~ $: P(x) \wedge Q(x))$

(5) Nested Quantifier - If we use a Quantifier that appears within the scope or domain of another Quantifier is called NQ or Multiple Q.

or

- If we use 2 or more Quantifier at the same time.

- $\forall x, \forall y ; P(x,y)$

$\forall x\, \exists y ; P(x,y)$

$\exists x\, \forall y ; P(x,y)$

$\exists x\, \exists y ; P(x,y)$

MCQ Note : $\exists x\, \forall y ; P(x,y) \neq \forall x\, \exists y ; P(x,y)$

Note → We can never distribute a U.Q over a disjunctⁿ ($\vee$) and nor an E.Q over a conjunctⁿ ($\wedge$)

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Predicate
↑

eg All Rabbits are faster than all tortoise
D_1 (under "All Rabbits"), D_2 (under "all tortoise")

D_1 (Domain 1) → All Rabbits or set of Rabbits
D_2 (" 2) → set of Tortoise

Predicate (P) → faster than

$P(x, y)$: Rabbit x is faster than tortoise y
$x \in D_1$, $y \in D_2$

We can also write,
↓
$$\forall x, \forall y \; ; \; P(x, y)$$
or
$$\forall x \in D_1 \; \forall y \in D_2 \; ; \; P(x, y)$$

eg "There are some Rabbits who are faster than all tortoise
→ Existential-Q (some)

⇒ We can write as :-
$$\exists x \in D_1 \; , \; \forall y \in D_2 \; ; \; P(x, y)$$

Q: (in notebook)

Logical Equivalency (Quantifiers) - The statement involving predicates and Quantifiers are L.E iff they have same t.v, no matter which predicates are used/substituted into the statement and which Domain is used for the statement

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10 marks

imp Q

Argument - Argument means a sequence of statement called 'Premises' that ends with a statement called conclusion.

- If $p_1, p_2 \cdots p_n$ are → Premises
and
q be the → conclusion
then it is denoted as :

$$p_1, p_2, p_3 \cdots p_n \vdash q$$

- A valid argument means that the conclusion of argument must follow from the Truth of the premises of the argument

OR

If all the Premises are True then Concl. also True

MCQ

Note
↳ The argument $p_1, p_2, p_3 \cdots p_n \vdash q$ is Valid if the conditional statement,

$$(p_1 \wedge p_2 \wedge p_3 \cdots p_n) \vdash q \text{ is always True}$$

so it's a Tautology
here ↳ (output always True)

If hypoth. is → false.
then output → always True

Note
↳ If $p_1 \wedge p_2 \wedge p_3 \cdots p_n$ is a contradiction (i.e False)
then again the Argument is valid as output always True

Rules of inference → are basic tools for establishing truth of statements.
→ to deduce new statements from statements we already have. (premises)

templates for constructing valid arguments.

$$(\ldots\ldots) \rightarrow (P) = T$$

Q/ Check the validity of the argument :-

"If it rains, I will drive to university" (S_1), "It rains" (S_2) so "I drive to University" (S_3).

① Make/select propositional variable from above statement

∴ p : It rains
q : I drive to Univ

Trick
" " → Statement

∴ The premises are :- (sequence of statements)

S_1 : If p then q i.e $p \rightarrow q$

S_2 : p (It rains)

S_3 : q (I drive to Univ)

We can also write →

$$\begin{array}{c} S_1 \\ S_2 \\ \hline \therefore S_3 \end{array}$$

• Now prove the validity :- $S_1 \wedge S_2 \rightarrow S_3$ (T)

p	q	S_1 $p\rightarrow q$	S_2 p	$S_1 \wedge S_2$	S_3 (q)	$S_1 \wedge S_2 \rightarrow S_3$
T	T	T	T	T	T	T
T	F	F	T	F	F	T
F	T	T	F	F	T	T
F	F	T	F	F	F	T

∴ $(S_1 \wedge S_2) \rightarrow S_3$ is a tautology
∴ Argument is valid.

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Q "If Shreya works hard, then she will successful", "If she is successfull, then she will be happy" Therefore hardworking leads to happiness

Sol

p : Shreya works hard (Hardworking)
q : she will be successfull
r : she is happy

$S_1 \to$ If p then q
$S_2 \to$ If q then r
$S_3 \to p \to r$

S_1
S_2

S_3

p	q	r	S_1	S_2	S_3	$S_1 \wedge S_2 \to S_3$
T	T	T	T	T	T	T
T	T	F	T	F	F	T
T	F	T	F	T	T	T
T	F	F	F	T	F	T
F	T	T	T	T	T	T
F	T	F	T	F	T	T
F	F	T	T	T	T	T
F	F	F	T	T	T	T

$\therefore$ The Argument is valid. Ans

(i) $\sim p \wedge q \to eq^n$ (1).

By Simplificatⁿ
$\sim p \wedge q$
$\therefore \sim p$ — (5)

(i) $r \to p$ — (2)

By MT using (5)
$r \to p$
$\sim p$
$\sim r$ — (6)

(i) $\sim r \to s$ — (3)

By MP using (6)
$\sim r \to s$
$\sim r$
$\therefore s$ — (7)

(i) $s \to t$ — (4)

By MP using (7)
$s \to t$
s
$\therefore t$

Ans

Show that premises is valid by using laws of inference. – "It's not sunny this afternoon and it's colder than yesterday," "we will go for swimming only if it is sunny," "If we don't go for swimming then we will take a trip," "If we take a trip, we will be at home by sunset". leads ~~to~~ the conclusion "we will be at home by sunset".

Sol

$p \to$ It is sunny $r \to$ we will go for swimming

$q \to$ It is colder than yesterday $s \to$ we will go for a trip

$t \to$ we will be at home by sunset → conclusion

ATQ

$S_1 \to \sim p \wedge q$ — (1)

$S_2 \to r \to p$ — (2)

$S_3 \to \sim r \to s$ — (3)

$S_4 \to s \to t$ — (4)

NOTE → We could have used a Truth table, like previous eg to show that whenever each of the 4 hypothesis is true, the conclusion is also true. But here, there are 5 prop. variables so a difficult task.

∴ We use Rules of inference → Pg-72

↳ to construct an argument to show that our premises lead to the desired conclusions as follows:–

- By Hypothetical syllogism on (3) and (4)

$$\begin{array}{l} \sim r \to s \\ s \to t \\ \hline \therefore \sim r \to t \end{array} \quad -(5)$$

2nd sol^n ↳ in Pg 73 ↳ pdf → go by order

- By simplification on (1)

$$\begin{array}{l} \sim p \wedge q \\ \hline \therefore \sim p \end{array} \quad -(6)$$

- By Modus Tollens on (6) and (2)

∴ Argument is Valid

$$\begin{array}{l} r \to p \\ \sim p \\ \hline \therefore \sim r \end{array} \quad -(7)$$

de

- By Modus Ponens on (5) and (7)

$$\begin{array}{l} \sim r \to t \\ \sim r \\ \hline \therefore t \end{array}$$

→ the conclusion (t) → i.e we will be at home by sunset

Rules of Inference

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(law of detachment)

① Modus Ponens

basis of rule of inference

$$\begin{array}{l} p \\ p \to q \\ \hline \therefore q \end{array}$$

$T^{gy} \Rightarrow p \wedge (p \to q) \to q$

② Modus tollens

$$\begin{array}{l} \sim q \\ p \to q \\ \hline \sim p \end{array}$$

③ Hypothetical syllog

$$\begin{array}{l} p \to q \\ q \to r \\ \hline \therefore p \to r \end{array}$$

④ Disjunctive syllog

$$\begin{array}{l} p \vee q \\ \sim p \\ \hline \therefore q \end{array}$$

⑤ Addit^n

$$\begin{array}{l} p \\ \hline \therefore p \vee q \end{array}$$

⑥ Simplificat^n

$$\begin{array}{l} p \wedge q \\ \hline \therefore p \end{array}$$

⑦ Conjunct^n

$$\begin{array}{l} p \\ q \\ \hline \therefore p \wedge q \end{array}$$

⑧ Resolut^n

$$\begin{array}{l} p \vee q \\ \sim p \vee r \\ \hline \therefore q \vee r \end{array}$$

Here

$\wedge \to$ operat^n

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Introduction to Proof

• Proof - is a valid argument that establish the truth of a mathematical statement called theorem.

Method 1 → Direct Method

The direct method of a conditional statement $p \to q$ is constructive when the first step is the assumption when hypothesis (p) is true and next step are constructive by using Rule of inference and then in the last step it is showing that ~~the~~ conclusion is also true.

Q Using Direct method Proof show that:

"If n is odd integer then n^2 is odd."

Sol^n

The given statement is : "If n is odd integer, then n^2 is odd".

$\forall n \in I, \; p(n) \to q(n)$

where, $p(n) \to n$ is odd

$q(n) \to n^2$ is odd

Now, It is assumed that $p(n)$ is True

$\therefore$ n is odd Integer

$\therefore$ There exist some integer m such that

$n = 2m+1$ → Why? (kyunki tabhi odd hoga)

Now,

$q(n) = n^2$ (Put value)

$= (2m+1)^2 = 4m^2 + 4m + 1$

$= 2(2m^2 + 2m) + 1$

$\Rightarrow 2t + 1$, (where $t = 2m^2 + 2m$) $\in I$ = odd number

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$\therefore$ $q(n)$ is odd number so is also True

Hence the result by Direct method Proof is proved!

$\boxed{\text{Method-2}}$ - Contraposition Method

Proof of the theorem

$$\forall x : p(x) \rightarrow q(x)$$

We will show that the contrapositive statement $\sim q(x) \rightarrow \sim p(x)$ of the conditional statement $p(x) \rightarrow q(x)$ is True

i.e if $\sim q(n)$ is True then $\sim p(n)$ is True

Q. Prove that "if n is Integer and $3n+2$ is odd then n is odd".

Sol^n $\forall n \in I$, $p(n) \rightarrow q(n)$

where,

$p(n)$: $3n+2$ is odd

$q(n)$: n is odd

$\therefore$

$\sim p(n)$: $3n+2$ is not odd

$\sim q(n)$: n is not odd

Let $\sim q(n)$ is true i.e $\Rightarrow$ n is even

There exist an integer 'm' such that $n = 2m$

Now,

$$3n + 2 = 3(2m+2)$$
$$= 6m + 2$$
$$= 2(3m+1) \quad (\text{where } t = 3m+1 \in I)$$
$$= 2t$$

↳ even no

$\therefore$ $3n+2$ is even no
or $3n+2$ is not odd

$\therefore \sim p(n)$ is True.

We know,
If $\sim q(n)$ is true then $\sim p(n)$ is also True

$\therefore$ By the method of Contraposition Proof

If $p(n)$ is true then $q(n)$ is also true

$\therefore$ If $3n+2$ is odd then n is odd

Hence Proved.

$\boxed{\text{Method 3}}$ - Method of Contradiction

To prove → The statement p is True, we can find a Contradiction q such that

$\sim p \rightarrow q$ is True.

Because q is always false and $\sim p \rightarrow q$ is True, we can conclude that $\sim p$ is false then p is True

$\sim p \to q$ is True

$\sim q$ is false

$\sim(\sim p)$ is True

Q P.T $\sqrt{2}$ is Irrational by giving a Proof by contradiction.

Soln Let p be the given proposition:

$\therefore$ $p : \sqrt{2}$ is Irrational

By Method 3, we assume

$\sim p$ is True

i.e

$\sqrt{2}$ is not Irrational is True.

$\therefore$ we have $\sqrt{2}$ is rational.

If There exists integers p, q where $q \neq 0$ and $(p, q) = 1$ such that

$$\sqrt{2} = \frac{p}{q}$$

$$\Rightarrow 2 = \frac{p^2}{q^2} \quad \text{(On Sq.)}$$

$$2q^2 = p^2 \quad \text{--- (1)}$$

$\Rightarrow p^2$ is even

$\Rightarrow p$ is also even

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$\therefore$ There exists integer $m \in I$ such that,

$$\boxed{p = 2m} \quad - (2)$$

From eqn (1) using eqn (2)

$\therefore$ $(2m)^2 = 2q^2$

$4m^2 = 2q^2$

$q^2 = 2m^2$

$\Rightarrow$ q^2 is even

$\therefore$ q is also even

$\therefore$ $\exists$ integer n such that

$$q = 2n \quad - (3)$$

From (2) and (3) both p, q are even

$\therefore$ $(p, q) = 2$, which contradicts the given hypothesis $(p, q) = 1$

$\therefore$ Our assumption is wrong/false.

$\therefore$ $\sim p$ leads to a contradiction (False)

$\therefore$ p is True

Hence $\sqrt{2}$ is Irrational. Hence Proved

SET

No. of subset of a set $A = 2^{n(A)}$

$\{\because n(A) = \text{No. of elements in } A\}$

ϕ = Null / Empty / void set $= \{\ \}$

ϕ is subset of every set; $\phi \subseteq A$

① Union → A ∪ B or A or B

$A \subseteq A \cup B$

$B \subseteq A \cup B$

② Intersection → $A \cap B$ as A and B

i.e. common element

③ Complement → $\bar{A} = X - A$

④ A-B → from A remove the element of B if exist
and
B-A → from B remove the element of A if exist

* Cartesian Product of sets

Let A, B are non-empty sets

$A \times B = \{(x, y) : x \in A, y \in B\}$

$B \times A = \{(b, a)$ or $(y, x) : b \in B, a \in A\}$

$\boxed{A \times B \neq B \times A}$

• eg $A = \{1, 2, 3\}$ and $B = \{a, b\}$

$A \times B = \{1, 2, 3\} \times \{a, b\}$

$= \{(1, a), (1, b), (2, a), (2, b), (3, a), (3, b)\}$

$A \times B \neq B \times A$

HP.

$B \times A = \{(a, 1), (a, 2), (a, 3), (b, 1), (b, 2), (b, 3)\}$

• Note → $\boxed{n(A \times B) = n(A) \cdot n(B) = n(B \times A)}$

→ No. of subsets of $A \times B = 2^{n(A \times B)}$

$= 2^{n(A) \cdot n(B)}$

RELATION

$x \xrightarrow{R} y$

• 'R' is a relation from set A to set B

• $R : A \to B$

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$(x,y) \in R$ or $x R y$ or x is related to y

$\therefore R \subseteq A \times B$

Types (i) **Reflexive** – A relation R on set 'A' is reflexive if $\forall n \in A, (n,n) \in R$ i.e nRn

– i.e every element of set A is related to itself.

eg $A = \{1, 2, 3\}$

$R = \{(1,1), (1,2), (1,3), (2,2), (2,3), (2,1), (3,3)..\}$

$\therefore (1,1), (2,2), (3,3) \in R$

(ii) **Non-Reflexive** – A relation 'R' on ~~relation~~ set A is NR if some element of set A is related to itself

– i.e $n \in A$, nRn $((n,n) \in R)$

eg $A = \{1, 2, 3\}$

$R = \{(1,1), (1,2), (2,2), (2,3), (2,1), (3,1), (3,2)\}$

here $(1,1), (2,2) \in R$

(iii) **Irreflexive** – A relatn 'R' on set A is called Irreflexive if No element are related to itself

$\forall n \in A, (n,n) \notin R$

eg $A = \{1, 2, 3\}$

$R = \{(1,2), (1,3), (2,3), (3,1)\}$

(iv), (v), (vi), (vii), (viii) + POSET

BTO (4 pages)

Relatⁿ (PTO → 4 pages)

(iv). Symmetric → A relation 'R' on set A is symmetric if

$(x, y) \in R \Rightarrow (y, x) \in R \quad \forall\, x, y \in A$

eg

$R = \{(1,1), (1,2), (1,3), (2,1), (2,2), (2,3), (3,1), (3,2)\}$

(v) Assymetric → " " if

$(x, y) \in R \Rightarrow (y, x) \notin R \quad \forall\, (x, y) \in A$

(vi) Anti-symmetric → " " if

$(x, y) \in R$ and $(y, x) \in R \Rightarrow x = y$

xRy and $yRx \Rightarrow x = y$

(vii). Transitive Relatⁿ → " " if

$(x, y), (y, z) \in R \Rightarrow (x, z) \in R$

$xRy,\ yRz \Rightarrow xRz$

(viii). Equivalence → " " if R is reflexive, Symmetric, Transitive

eg

$R = \{(1,1), (1,2), (1,3), (2,1), (2,2), (2,3), (3,1), (3,2), (3,3)\}$

1 4

4 1

check for

(i) Reflexive → Yes

as $(1,1), (2,2), (3,3) \in R$

(ii) symmetric → Yes

as $(1,2), (2,1) \in R$

$(2,3), (3,2) \in R$

(iii) Transitive → Yes

as $(1,2), (2,1) \Rightarrow (1,1) \in R$

$(1,2), (2,3) \Rightarrow (1,3) \in R$

$(1,3), (3,2) \Rightarrow (1,2) \in R$

---- etc

$\therefore$ R is a Equival. Relaⁿ

Ans

POSET(starting)
$\boxed{*}$
P.T.O
4 pages

$\boxed{\text{POSET}}$ - stands for 'Partially Ordered Relation'

- A relation 'R' on set A is called a POSET if R is
  - Reflexive
  - Antisymm
  - Transitive

- denoted by (A, R)

Q. Show that the Relation (i) '$\leq$' (ii) '$\geq$' is a partially ordering Relat" on the set of Integers (Z)

OR

kuch - sense nhi bana

Show that (i) $(Z, \leq)$ (ii) $(Z, \geq)$ is a Poset for Z → integer

Ans.

Z : set of integers

Relation R : '$\leq$'

$$R = \{ (x, y) \; ; \; x \leq y \; ; \; x, y \in Z \}$$

or xRy

(i) Reflexive ⇒ $\forall x \in Z$, $x \leq x$

$\therefore \forall x \in Z$, xRx or $(x, x) \in R$

∴ Reflexive.

(ii). Antisymmetric ⇒ $x, y \in Z$ and $x \leq y, \; y \leq x \Rightarrow x = y$

$\therefore xRy, \; yRx \Rightarrow x = y$

∴ R is Antisymmetric.

(iii) Transitive $\rightarrow$ $\forall\, x, y, z \in Z$ if $x \leq y$, $y \leq z \Rightarrow x \leq z$

i.e xRy, $yRz \Rightarrow xRz$

$\therefore$ R is Transitive as well.

Hence $\rightarrow$ R is $\left\{ \begin{array}{l} R \\ \text{Anti-symm} \\ T \end{array} \right.$ $\therefore$ R is a POSET i.e $(Z, \leq)$ is a Poset

(*) The partial ordered Relation is often denoted by : $\leq$

i.e $x R y$

or $x \leq y$ means x precedes y

and $x < y$ means x strictly precede y

* **Comparable Elements** - 2 elements x, y of a POSET are said to be comparable if either $x \leq y$ or $y \leq x$

* **Incomparable Element** - if neither $x \leq y$ nor $y \leq x$ then called not comparable.

Q. In Poset $(\mathbb{Z}^+, /)$ set, relatⁿ the element 3, 9 are comparable

if $3/9$ ✓ $9/3$ ✓

but

2, 7 are not comparable as $2 \nmid 7$ and $7 \nmid 2$.

* **Total Ordering Relatⁿ**

A relation R is called a TOR if, R is a partially Order Relation and also satisfy Dichotomy law.

i.e every ~~element~~ pair of the elements of A are comparable

$\forall x, y \in A$ either $x \leq y$ or $y \leq x$

* **Immediate → Predecessor**
  ↳ Successor

If $(A, \leq)$ and $x, y \in A$ and no element lies b/w x, y.
then x is called Immediate P of y
or
y is called Immediate S of x

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(10) # **Hasse Diag** — A graphical representation of Poset in which all the arrowheads are understood to be pointing in Upward direction.

Vimp

- How to draw

Let (A, R) → poset

$\boxed{\text{step 1}}$ → write R in the Roster form

$\boxed{\text{step 2}}$ → Draw the Dia-graph of R (directed fig.)

~~eg~~ If $(x, y) \in R$

$x \rightarrow y$

$\boxed{\text{Step 3}}$ → Delete all the loops from the diagraph
or → remove the elements which comes from Reflexive R.

eg For a Reflexive relatⁿ $(x, x \in R)$

$x \rightarrow x$

or

x → loop

$\boxed{\text{Step 4}}$ → Eliminate all the edges that ~~are~~ ~~are~~ comes from Transitive property of a relatⁿ

eg $x \rightarrow y$ → edge

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Step 5 → Replace the circle with dots and also omit the arrows.

Q Draw the Hasse diagram of the Poset $(S, \preceq)$ where

$S = \{3, 4, 6, 12, 24, 48, 72\}$ and the $\preceq$ is defined as $x \preceq y$ if x divides y or x is a factor of y.

Sol Given, set $S = \{3, 4, 6, 12, 24, 48, 72\}$ and

Relation $R(\preceq)$ where,

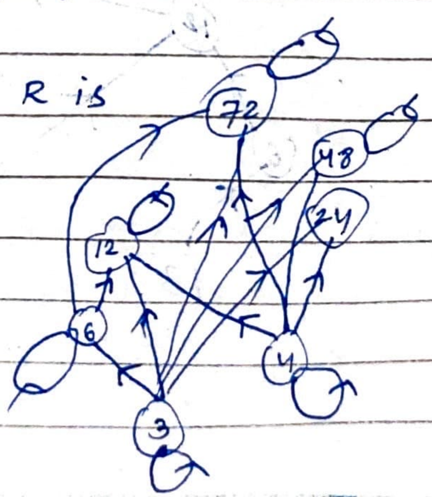
$$R = \{(x, y) ; x \preceq y, x, y \in S\}$$
$$= \{(x, y) ; x \text{ divides } y, x, y \in S\}$$

Step 1

$R = \{(3,3), (3,6), (3,12), (3,24), (3,48), (3,72),$
$(4,4), (4,12), (4,24), (4,48), (4,72),$
$(6,6), (6,12), (6,24), (6,48), (6,72),$
$(12,12), (12,24), (12,48), (12,72)$
$(24,24), (24,48), (24,72),$
$(48,48), (72,72)\}$

Step 3
(Remove loops) eg $(3,3), (4,4)$...

Step 2 The Digraph for R is



similar as
join all

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If $(a,b)(b,c) \Rightarrow (a,c)$

~~Step~~ 3 + 4

a. • To find the edges that comes from Transitive property are:-

↳ Don't use reflexive relation

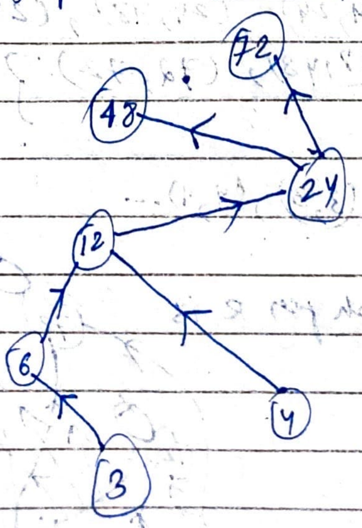
• i.e (3,3), (4,4) . . . .

∴ (3,6) (6,12) = (3,12)
(3,6) (6,24) = (3,24)
(3,6) (6,48) = (3,48)
(3,6) (6,72) = (3,72)
(6,12) (12,24) = (6,24)
(6,12) (12,48) = (6,48)
(6,12) (12,72) = (6,72)
(4,12) (12,24) = (4,24)
(4,12) (12,48) = (4,48)
(4,12) (12,72) = (4,72)
(12,24) (24,48) = (12,48)
(12,24) (24,72) = (12,72)

Transitive edges

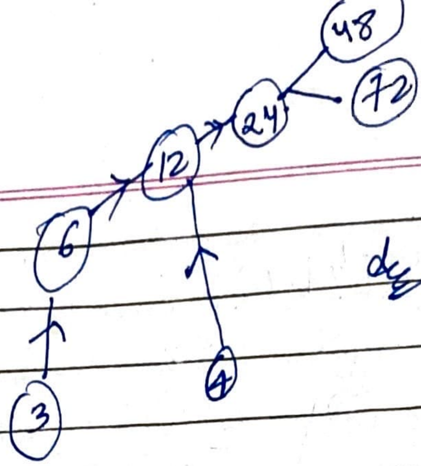
From the Digraph remove loops and Transitive edges

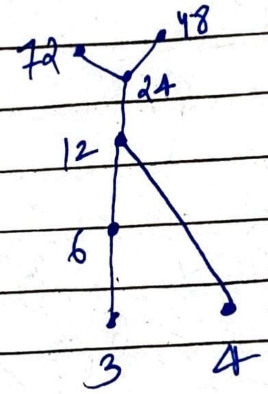
Final



2 elements at base $<$ 3, 4

(seedha banao)





Note: If m be a positive Integer (natural no) then D_m denotes → the set of all divisors of m
↳ set no.

means
eg D_{15} → all divisor of 15 ka set.
→ { 1, 3, 5, 15 }

Q. Draw the Hasse diag of the closet of D_{36}, '/'
↳ operation

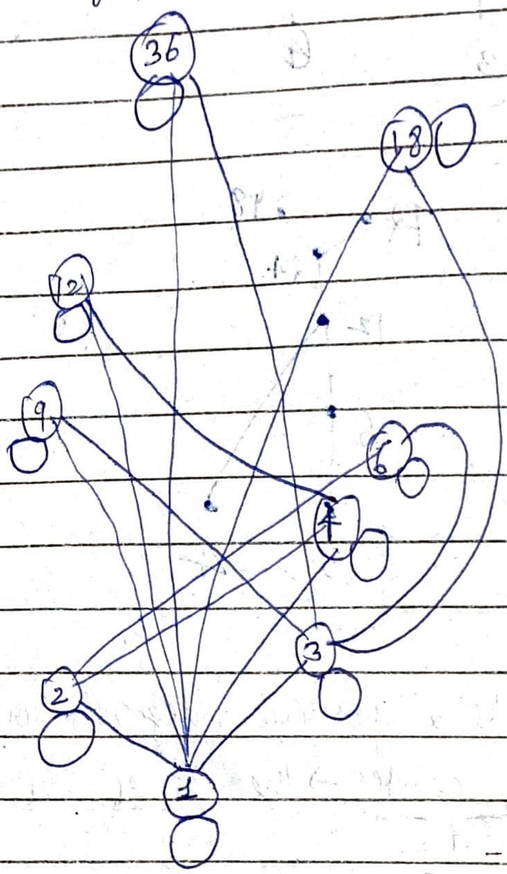
Ans Given D_{36} = set of all divisors of 36
= { 1, 2, 3, 4, 9, 12, 18, 36 }

and
The Relation 'R' be defined as $R = \{(x,y) : x/y \ x,y \in D_{36}\}$

step 1 (Roster form)

R = { (1,1), (1,2), (1,3), (1,4), (1,6), (1,9), (1,12), (1,18), (1,36),
(2,2), (2,4), (2,6), (2,12), (2,18), (2,36)
(3,3), (3,6), (3,9), (3,12), (3,18), (3,36)
(4,12), (4,4), (4,36)
(6,6), (6,12), (6,18), (6,36)
(9,9) (9,18), (9,36)
(12,12) (12,36) (18,18) (18,36) (36,36) }

Step 2 → Draw the diagraph for R :



loops are → (1,1), (2,2), (3,3), (4,4), (9,9), (6,6), ...

Step 3+4 → To find the edges that comes from Transitive properties are : and Remove them

(1,2) (2,4) = (1,4)
(1,2) (2,6) = (1,6)
(1,2) (2,12) = (1,12)
(1,2) (2,18) = (1,18)
(1,2) (2,36) = (1,36) as
(1,3) (3,6) → already (1,6) ✗
(1,3) (3,9) = (1,9)
~~(1,3) (3,~~
(2,4) (4,12) = (2,12)
(2,4) (4,36) = (2,36)

(1,4)(4,12) = (1,12) ✗

~~(2,3)(3,6) (2,6)~~

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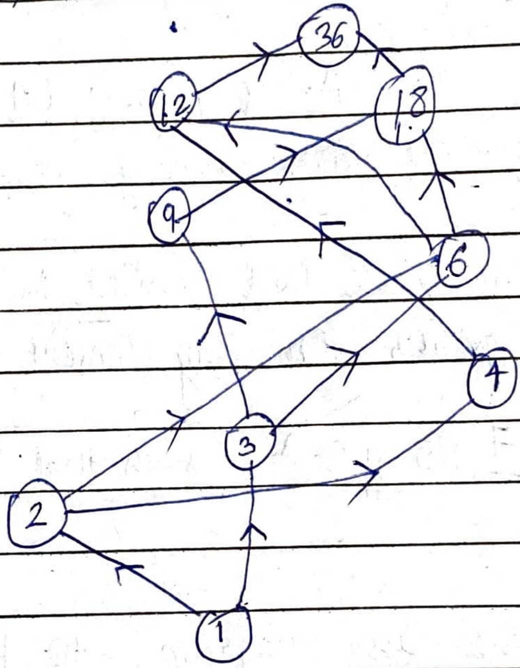
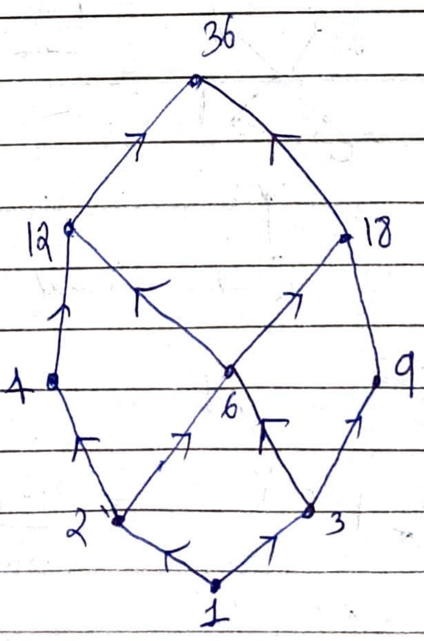
(2,6) (6,18) = (2,18)
(3,6) (6,12) = (3,12)
(3,6) (6,18) = (3,18)
(3,6) (6,36) = (3,36)
(4,12) (12,36) = (4,36)
(6,12) (12,36) = (6,36)
(9,18) (18,36) = (9,36)

∴ Final



∴

∴ 36 → greatest element
1 → smallest element.



the required H.D.

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* ~~Lattice~~ - ~~A Poset~~

* Maximal and Minimal element of POSET

- An element of a Poset is called Maximal if it is not less than any other element of the Poset

i.e

$x \in A$ of poset $(A \leq)$ is maximal if there exists NO $y \in A$ such that $x \leq y$

- An element of a Poset is called Minimal if it is not greater than any element

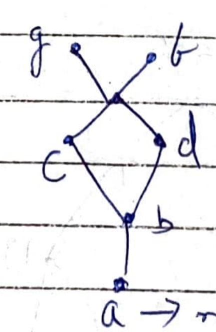
i.e

$\exists$ No $y \in A$ such that $y \leq x$

Note ↳ In the Hasse diagram, the top and bottom elements are the maximal and minimal elements

~~for eg~~

eg

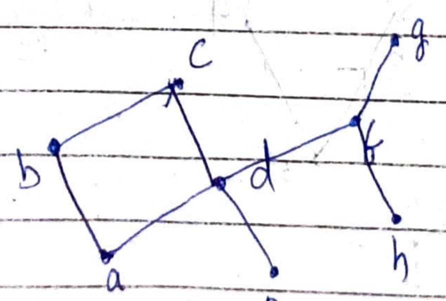


f, g → maximal

↳ not greatest (imp)

a → minimal

eg



Max → c, g (no element after

Minimal → a, e, h (" " before

* Greatest and least element

- An element $x \in A$ is called greatest element of the Poset if $\forall a \in A$, $a \leq x$
  ↓ preceed (comes before x)

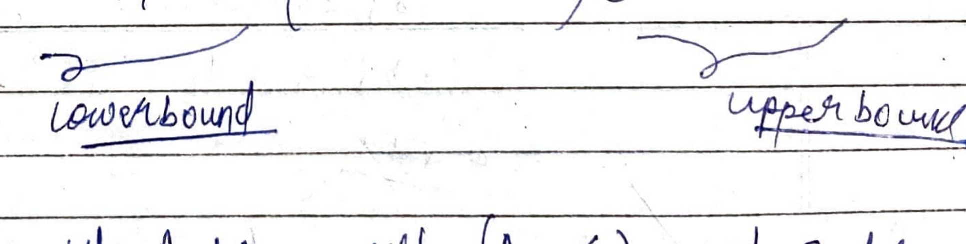
eg previous Q, 36 → greatest element as all element comes before 36.

- An element $x \in A$ is called least if $\forall a \in A$, $x \leq a$
  ↓ preceed

$\boxed{\text{Note}}$ ↳ The greatest and least element if exists are Unique.

* Upper Bound and least Upper Bound

eg



- Upper bound (UB) - Let A be a poset $(A, \leq)$ and S be a subset of set A, then an element $x \in A$ is called UB of S if $\forall a \in S$, $a \leq x$
  ↓ preceed (comes before)

- Least UB (lub) or Supremum - " " " " called lub of S if $\forall a \in S$ $x \leq y$ for all UB y of S

*Lower Bound and Greatest Lowerbound

• Let A . . . . .

. called L.B of S if $\forall a \in S$,

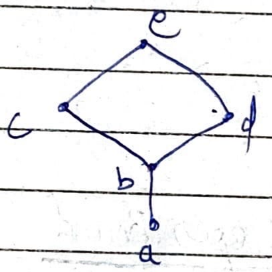
$x \leq a$

• and is called glb or Infimum if $y \leq x$, for all lower bound y of S

MCQ $\boxed{\text{Note}}$

↳ lub and glb of a set if exists, is always Unique

for eg



Here $S = \{a, b, c, d, e\}$

Q Find UB of $\{a,b\}$ = b, c, d, e

As First, Find individual UB

$\{a\}$ = a, b, c, d, e

$\{b\}$ = b, c, d, e

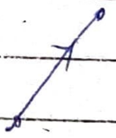
$\Rightarrow \{a,b\}$ = b, c, d, e

↓ a and b

use it

Q Trick → If directly connected of 2 elements

b → lub



a → glb

Trick ← Q Find lub of $\{a,b\}$ = b

lub ↑

glb ↓

glb of $\{a,b\}$ = a

Q UB for $\{b\}$ = b, c, d, e

→ lb

glb lub

Note → For single element can consider equality

→ For single element, glb lub are same

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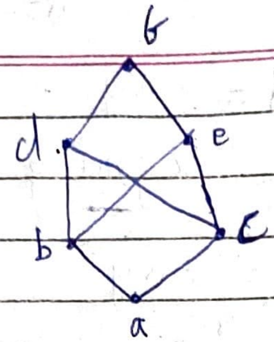
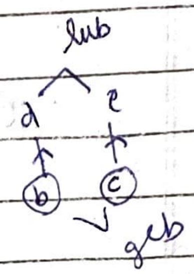
Q



b/d c f
c d e f



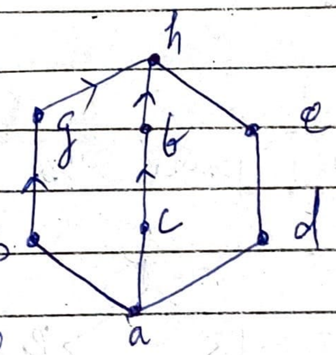
(i) UB {b, c} = d, e, f Trick = directly conn

(ii) ~~lub {b, c}~~ = Doesn't exist, as not unique as we get 2 lub < d e

(iii) LB {d, e} = b, c, a

(iv) glb {d, e} = Doesn't exist as < b c

Q



i) lub {b, c} = h

UB{b} = b, g, h
UB{c} = c, f, h

(ii) lub {b, c, d} = h

UB {d} = d, e, h

(iii) glb {b, f} = a

(b, a) (f, c, a)

Imp

Note (1) For a Poset (A, ≤) if elements of set A are Natural No and the Partial Order relatⁿ (≤) is "divide" (or "factor of")

then {lub {x, y} = LCM {x, y} and glb {x, y} = HCF {x, y}
lub {x, y, z} = LCM {x, y, z}}

② lub $\{x, y\}$ = sup $\{x, y\}$ = $x \vee y$
'x joint y'

and

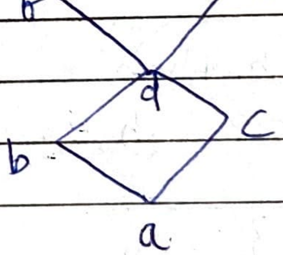
glb $\{x, y\}$ = Inf $\{x, y\}$ = $x \wedge y$
'x meet y'

Q-10 # **Lattice** (Done)

vgmp A Poset $(A, \leq)$ is said to be lattice if every pair of elements in A has Unique < lub / glb
$\forall x, y \in A$, Sup $\{x, y\}$ and Inf $\{x, y\}$ exist in A.

Q Find wheather the poset who Hasse Diag is a lattice or not ?

H.D →



Ans

④ Let's find the Closure Table for Sup | lub and Inf | glb

(i) For Supremum | lub ($\vee$) (upward direct^n)

V	a	b	c	d	e	f	g
a	a	b	c	d	e	f	g
b	b	b	d	d	g	f	g
c	c	d	c	d	e	f	g
d	d	d	d	d	e	f	g
e	e	e	e	e	e	g	g
f	f	f	f	f	g	f	g
g	g	g	g	g	g	g	g

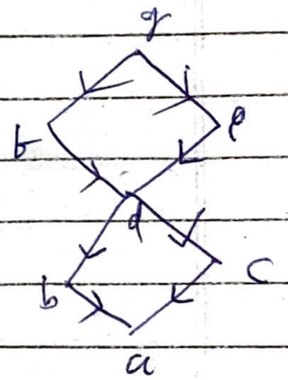
be < b → d e g b
e → e g

c d

(ii) For Inf / glb ($\wedge$). (Downward direc^n)

$\wedge$	a	b	c	d	e	f	g
a	a	a	a	a	a	a	a
b	a	b	a	b	b	b	b
c	a	a	c	c	c	c	c
d	a	b	c	d	d	d	d
e	a	b	c	d	e	d	e
f	a	b	c	d	d	f	f
g	a	b	c	d	e	f	g

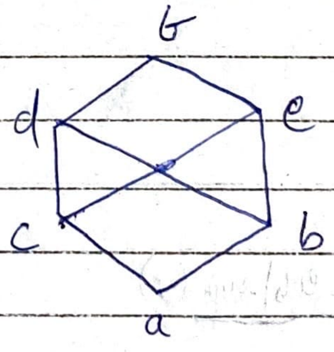
rough dig



∴ From the table we can conclude that the sup and Inf of every pair of Element exist.

∴ The Poset $(A, \leq)$ is a lattice.

Q 52)

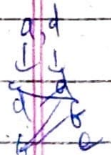


Find whether the Poset $(A, \leq)$ is a lattice or not.

∴ $A = \{a, b, c, d, e, f\}$

(i) closure Table for lub / sup ($\vee$)

$\vee$	a	b	c	d	e	f
a	a	b	c	~~a~~ d	~~a~~ e	~~a~~ f
b	b	b	–	d	e	f
c	c	–	c			
d				d		
e					e	
f						f



$a \to a, c, b, d$

$d \to d, f$

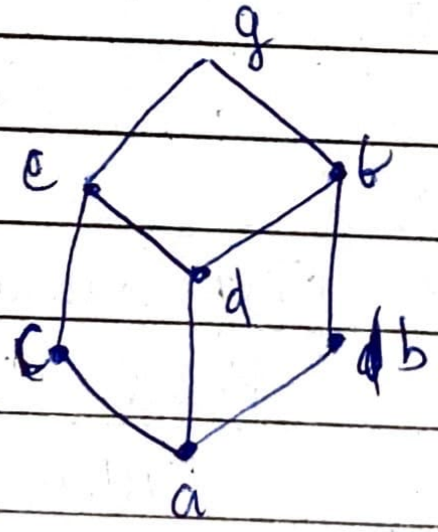
now

∴ Not a lattice

∴ $b \vee c \to d, e$

2 lub

I ↦ b, e d, g e



is a lattice

i) For lub/sup (v)

d → d e f g
e → e g

v	a	b	c	d	e	f	g
a	a	b	c	d	e	f	g
b	b	b	g	f	g	f	g
c	c	g	c	e	e	g	g
d	d	f	e	d	e	f	g
e	e	g	e	e	e	g	g
f	f	f	g	f	g	f	g
g	g	g	g	g	g	g	g

(ii) For glb / Inf ($\wedge$) (Downward direct)

$\wedge$	a	b	c	d	e	f	g
a	a	a	a	a	a	a	a
b	a	b	a	a	a	b	b
c	a	a	c	a	c	a	c
d	a	a	a	d	d	d	d
e	a	a	c	d	e	d	e
f	a	b	a	d	d	f	f
g	a	b	c	d	e	f	g

a

b → glb

Here don't consider upward direct

$\therefore$ A lattice, as each pair has Unique $\Leftrightarrow$ lub, glb

$\boxed{\text{Unit 3}}$