Variational Methods for Computer Vision: Exercise Sheet 2

Exercise: November 11, 2015

Part I: Theory

The following exercises should be **solved at home**. You do not have to hand in your solutions, however, writing it down will help you present your answer during the tutorials.

1. A multidimensional filter ist called separable, if it can be decomposed in one dimensional filter operations. Prove that the convolution of an image f with a Gaussian kernel K of standard deviation $\sigma > 0$,
$$K(x, y) := \frac{1}{2\pi\sigma^2} \exp\left(-\frac{x^2 + y^2}{2\sigma^2}\right),$$
can be written as the convolution with two one-dimensional filters:
$$k_1(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{x^2}{2\sigma^2}\right) \quad \text{and} \quad k_2(y) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{y^2}{2\sigma^2}\right).$$
Hence:
$$(f * K)(x, y) = ((f * k_1) * k_2)(x, y),$$
Explain why the separability of a filter is a desirable property.

2. Let $f \in C^2(\Omega; \mathbb{R})$ with $\Omega \subset \mathbb{R}^2$ be a real valued function and let $R \in \mathrm{SO}(2)$ be a rotation matrix. Let $\tilde{f}(x) := f(R \cdot x)$ be a rotated version of f. Prove that the magnitude of the gradient and the Laplace operator are rotationally covariant by showing the following identities:

   (a) $R\nabla\tilde{f}(x) = (\nabla f)(R \cdot x)$

   (b) $\|\nabla\tilde{f}(x)\| = \|(\nabla f)(R \cdot x)\|$

   (c) $\Delta\tilde{f}(x) = (\Delta f)(R \cdot x)$

   Reminder: $R \in \mathrm{SO}(2)$ denotes 2×2 matrices with $\det(R) = 1$ and $R^\mathsf{T} R = RR^\mathsf{T} = I$ and can be written as
$$R = \begin{pmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{pmatrix},$$
for some $\alpha \in [0, 2\pi]$.

3. The general diffusion equation can be written as follows
$$\begin{aligned} \partial_t u &= \operatorname{div}(g \cdot \nabla u) && \text{in } \Omega \times [0, \infty), \\ \partial_\nu u &= 0 && \text{on } \partial\Omega \times [0, \infty), \\ u(x, 0) &= u_0(x) && \text{for } x \in \Omega, \end{aligned}$$
where $u \in C^2(\Omega \times \mathbb{R}_0^+; \mathbb{R})$ with $\Omega \subset \mathbb{R}^2$ describes the complete diffusion process and solves the partial differential equation. Prove the following identities:

   (a) *linear homogeneous diffusion*:
$$\operatorname{div}(g \cdot \nabla u) = g\Delta u, \qquad g \in \mathbb{R}.$$

   (b) *linear inhomogeneous diffusion*:
$$\operatorname{div}(g \cdot \nabla u)(x) = g(x)\Delta u(x) + \langle \nabla g(x), \nabla u(x) \rangle, \qquad g \in C^1(\Omega; \mathbb{R}).$$

Part II: Practical Exercises

This exercise is to be solved **during the tutorial**.

1. In the last exercise we implemented an image convolution with the Gaussian kernel using the straightforward definitions. Considering the results from the first theoretical exercise, write a Matlab script that implements a separable convolution with a Gaussian kernel.

2. Download the archive file `vmcv_ex02.zip` from the homepage and unzip it in you home folder. Use the template file `difusion_filter.m` for a nonlinear diffusion filter and complete the missing code at line 58. Test the script on the image `lena.png`.

3. Create a video using the `avifile` command and compare the results.

Matlab-Tutorials:
`http://www.math.utah.edu/lab/ms/matlab/matlab.html`
`http://www.math.ufl.edu/help/matlab-tutorial/`
`http://www.glue.umd.edu/~nsw/ench250/matlab.htm`