Variational Methods for Computer Vision: Exercise Sheet 3

Exercise: November 18, 2015

Part I: Theory

The following exercises should be **solved at home**. You do not have to hand in your solutions, however, writing it down will help you present your answer during the tutorials.

1. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a convex real valued function. A point $\tilde{x} \in \mathbb{R}^n$ is a local minimizer of f if there exists a neighboorhood $\mathcal{N}(\tilde{x})$ such that $f(\tilde{x}) \le f(x),\ \forall x \in \mathcal{N}(\tilde{x})$. A stationary point of f is a point at which the gradient vanishes, hence a point x^* which satisfies the following equation:
$$\nabla f(x^*) = 0.$$
Prove the following statements:
   (a) Every local minimizer of f is a global minimizer.
   (b) Suppose f is additionally differentiable. Every stationary point of f is a global minimizer.

2. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a real valued function. The epigraph of f is the following set:
$$\text{epi } f := \{(u, a) \in \mathbb{R}^n \times \mathbb{R} \mid f(u) \le a\}$$
Prove that f is convex if and only if its epigraph is a convex set.

3. Let $f : \mathbb{R}^n \to \mathbb{R}$ and let $g : \mathbb{R}^n \to \mathbb{R}$ be real valued convex functions. Show whether or not the following functions are convex:
   (a)
$$h(x) := \alpha f(x) + \beta g(x), \text{ where } \alpha, \beta > 0.$$
   (b)
$$h(x) := \max(f(x), g(x))$$
   (c)
$$h(x) := \min(f(x), g(x))$$

4. Let $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ be twice differentiable convex functions. Find the condition on f that assures the function:
$$h(x) := f(g(x))$$
is convex by using the fact that function h is convex if and only if $h(x)'' \ge 0$.

Part II: Practical Exercises

This exercise is to be solved **during the tutorial**.

1. In the lecture we encountered the following cost function for denoising images:

$$E_\lambda(u) = \frac{1}{2}\sum_{i=1}^{N}(f_i - u_i)^2 + \frac{\lambda}{2}\sum_{i=1}^{N}\sum_{j\in\mathcal{N}(i)}(u_i - u_j)^2. \tag{1}$$

   where u is the seeked image, f is the input image and where $\mathcal{N}(i)$ denotes a neighborhood of pixel i. Minimize the above function by solving the linear system of equations which arises from the optimality condition, using the Gauss-Seidel method.

2. To test the denoising capabilities of your method, degrade the input image with Gaussian noise (MATLAB: `help randn`). Does your result depend on the initialization? Explain why/why not. Also explain how the solution depends on the parameter λ.

Matlab-Tutorials:
`http://www.math.utah.edu/lab/ms/matlab/matlab.html`
`http://www.math.ufl.edu/help/matlab-tutorial/`
`http://www.glue.umd.edu/~nsw/ench250/matlab.htm`