

Sublinear Algorithms and Lower Bounds for Metric TSP Cost Estimation

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Abstract

We consider the problem of designing sublinear time algorithms for estimating the cost of minimum metric traveling salesman (TSP) tour. Specifically, given access to a $n \times n$ distance matrix D that specifies pairwise distances between n points, the goal is to estimate the TSP cost by performing only sublinear (in the size of D) queries. For the closely related problem of estimating the weight of a metric minimum spanning tree (MST), it is known that for any $\varepsilon > 0$, there exists an $\tilde{O}(n/\varepsilon^{O(1)})$ time algorithm that returns a $(1 + \varepsilon)$ -approximate estimate of the MST cost. This result immediately implies an $\tilde{O}(n/\varepsilon^{O(1)})$ time algorithm to estimate the TSP cost to within a $(2 + \varepsilon)$ factor for any $\varepsilon > 0$. However, no $o(n^2)$ time algorithms are known to approximate metric TSP to a factor that is strictly better than 2. On the other hand, there were also no known barriers that rule out existence of $(1 + \varepsilon)$ -approximate estimation algorithms for metric TSP with $\tilde{O}(n)$ time for any fixed $\varepsilon > 0$. In this paper, we make progress on both algorithms and lower bounds for estimating metric TSP cost.

On the algorithmic side, we first consider the graphic TSP problem where the metric D corresponds to shortest path distances in a connected unweighted undirected graph. We show that there exists an $\tilde{O}(n)$ time algorithm that estimates the cost of graphic TSP to within a factor of $(2 - \varepsilon_0)$ for some $\varepsilon_0 > 0$. This is the first sublinear cost estimation algorithm for graphic TSP that achieves an approximation factor less than 2. We also consider another well-studied special case of metric TSP, namely, (1, 2)-TSP where all distances are either 1 or 2, and give an $\tilde{O}(n^{1.5})$ time algorithm to estimate optimal cost to within a factor of 1.625. Our estimation algorithms for graphic TSP as well as for (1, 2)-TSP naturally lend themselves to $\tilde{O}(n)$ space streaming algorithms that give an 11/6-approximation for graphic TSP and a 1.625-approximation for (1, 2)-TSP. These results motivate the natural question if analogously to metric MST, for any $\varepsilon > 0$, $(1 + \varepsilon)$ -approximate estimates can be obtained for graphic TSP and (1, 2)-TSP using $\tilde{O}(n)$ queries. We answer this question in the negative – there exists an $\varepsilon_0 > 0$, such that any algorithm that estimates the cost of graphic TSP ((1, 2)-TSP) to within a $(1 + \varepsilon_0)$ -factor, necessarily requires $\Omega(n^2)$ queries. This lower bound result highlights a sharp separation between the metric MST and metric TSP problems.

Similarly to many classical approximation algorithms for TSP, our sublinear time estimation algorithms utilize subroutines for estimating the size of a maximum matching in the underlying graph. We show that this is not merely an artifact of our approach, and that for any $\varepsilon > 0$, any algorithm that estimates the cost of graphic TSP or (1, 2)-TSP to within a $(1 + \varepsilon)$ -factor, can also be used to estimate the size of a maximum matching in a bipartite graph to within an εn additive error. This connection allows us to translate known lower bounds for matching size estimation in various models to similar lower bounds for metric TSP cost estimation.

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1 Introduction

In the metric traveling salesman problem (TSP), we are given n points in an arbitrary metric space with an $n \times n$ matrix D specifying pairwise distances between them. The goal is to find a simple cycle (a TSP tour) of minimum cost that visits all n points. An equivalent view of the problem is that we are given a complete weighted undirected graph $G(V, E)$ where the weights satisfy triangle inequality, and the goal is to find a Hamiltonian cycle of minimum weight. The study of metric TSP is intimately connected to many algorithmic developments, and the polynomial-time approximability of metric TSP and its many natural variants are a subject of extensive ongoing research (see, for instance, [3, 13, 18, 20, 22, 29–33] and references within for some relatively recent developments). In this paper, we consider the following question: can one design sublinear algorithms that can be used to obtain good estimates of the cost of an optimal TSP tour? Since the complete description of the input metric is of size $\Theta(n^2)$, the phrase sublinear here refers to algorithms that run in $o(n^2)$ time.

A standard approach to estimating the metric TSP cost is to compute the cost of a minimum spanning tree (MST), and output two times this cost as the estimate of the TSP cost (since any spanning tree can be used to create a spanning simple cycle by at most doubling the cost). The problem of approximating the cost of the minimum spanning tree in sublinear time was first studied in the graph adjacency-list model by Chazelle, Rubinfeld, and Trevisan [8]. The authors gave an $\tilde{O}(dW/\varepsilon^2)$ -time algorithm to estimate the MST cost to within a $(1 + \varepsilon)$ -factor in graphs where average degree is d , and all edge costs are integers in $[1..W]$. For certain parameter regimes this gives a sublinear time algorithm for estimating the MST cost but in general, this run-time need not be sublinear. Subsequently, in an identical setting as ours, Czumaj and Sohler [11] showed that for any $\varepsilon > 0$, there exists an $\tilde{O}(n/\varepsilon^{O(1)})$ time algorithm that returns a $(1 + \varepsilon)$ -approximate estimate of the MST cost when the input is an n -point metric. This result immediately implies an $\tilde{O}(n/\varepsilon^{O(1)})$ time algorithm to estimate the TSP cost to within a $(2 + \varepsilon)$ factor for any $\varepsilon > 0$. However, no $o(n^2)$ query algorithms are known to approximate metric TSP to a factor that is strictly better than 2. On the other hand, there are also no known barriers that rule out existence of $(1 + \varepsilon)$ -approximate estimation algorithms for metric TSP with $\tilde{O}(n)$ queries for any fixed $\varepsilon > 0$. In this paper, we make progress on both algorithms and lower bounds for estimating metric TSP cost.

On the algorithmic side, we first consider the *graphic TSP* problem, an important case of metric TSP that has been extensively studied in the classical setting – the metric D corresponds to the shortest path distances in a connected unweighted undirected graph [22, 23, 30]. We give the first $\tilde{O}(n)$ time algorithm for graphic TSP that achieves an approximation factor *strictly better* than 2.

Theorem 1. *There is an $\tilde{O}(n)$ time randomized algorithm that estimates the cost of graphic TSP to within a factor of $2 - \varepsilon_0$ for some constant $\varepsilon_0 > 0$.*

On the other hand, if we are willing to allow a higher sublinear time, we can get a better approximation ratio.

Theorem 2. *There is an $\tilde{O}(n^{1.5})$ time randomized algorithm that estimates the cost of graphic TSP to within a factor of $(27/14)$.*

At a high-level, our algorithm is based on showing the following: if a graph G either lacks a matching of size $\Omega(n)$ or has $\Omega(n)$ biconnected components (blocks), then the optimal TSP cost is not too much better than $2n$. Note that a connected unweighted instance of graphic TSP always contains a TSP tour of cost at most $2n$ since the MST cost is $(n - 1)$ on such instances. Conversely, if the graph G has both a large matching and not too many blocks, then we can show that the optimal TSP cost is distinctly better than $2n$. Since we do not know an efficient sublinear algorithm

to estimate the number of blocks in a graph G , we work with another quantity that serves as a proxy for this and can be estimated in $\tilde{O}(n)$ time. The main remaining algorithmic challenge then is to estimate sufficiently well the size of a largest matching. This problem is very important by itself, and has received much attention [17, 24, 25, 28, 36]; please see a detailed discussion of this problem, and relevant recent developments towards the end of this section. Our $\tilde{O}(n)$ query results utilize the recent result of Kapralov *et al.* [17] who give an algorithm to approximate the size of maximum matching to within a constant factor (for some very large constant) in $\tilde{O}(n)$ time in the *pair query* model (is there an edge between a given pair of vertices?). We also show that matching size can be estimated to within a factor of 2 in $\tilde{O}(n^{1.5})$ time, crucial to obtaining the approximation guarantee in Theorem 2.

Our approach for estimating graphic TSP cost in sublinear time also lends itself to an $\tilde{O}(n)$ space streaming algorithm that can obtain an even better estimate of the cost. To our knowledge, no estimate better than a 2-approximation was known previously. In the streaming model, we assume that the input to graphic TSP is presented as a sequence of edges of the underlying graph G . Any algorithm for this model, clearly also works if instead the entries of the distance matrix are presented in the stream – an entry that is 1 corresponds to an edge of G , and it can be ignored otherwise as a non-edge.

Theorem 3. *There is an $O(n)$ space randomized streaming algorithm that estimates the cost of graphic TSP to within a factor of $(11/6)$ in insertion-only streams.*

We next consider another well-studied special case of metric TSP, namely, $(1, 2)$ -TSP where all distances are either 1 or 2 [2, 6, 27], and obtain the following result.

Theorem 4. *There is an $\tilde{O}(n^{1.5})$ time randomized algorithm that estimates the cost of $(1, 2)$ -TSP to within a factor of 1.625.*

Throughout the paper, whenever we refer to a graph associated with a $(1, 2)$ -TSP instance, it refers to the graph G induced by edges of distance 1 in our $\{1, 2\}$ -metric. At a high-level, the idea underlying our algorithm is to analyze the structure of the graph G induced by edges of distance 1. We design an algorithm to estimate the size of a maximal “matching pair” of G which is defined to be the union of a pair of edge-disjoint matchings that is maximal, i.e., that is not a proper subset of another union of edge disjoint matchings. We show that whenever the size of a matching pair is large in a graph G , the TSP cost is distinctly smaller than $2n$, and conversely, if this quantity is not large, the TSP cost is close to $2n$. The main remaining algorithm challenge then is to estimate sufficiently well the size of a maximal matching pair, and we show that this can be done in $\tilde{O}(n^{1.5})$ time.

For $(1, 2)$ -TSP, an $\tilde{O}(n)$ query algorithm that estimates the cost of $(1, 2)$ -TSP to within a factor of 1.75 was claimed in [1] but this result is based on the matching size estimation results of [25]. Unfortunately, as confirmed by the authors [26], there is a problem with the proof of one of the statements in the paper — Observation 3.9 — which is crucial for the correctness of the main result. As a result, the $\tilde{O}(d)$ time result in the neighbor query model as well as the $\tilde{O}(n)$ time result in the adjacency matrix, claimed in [25] can no longer be relied upon, and we have chosen to make this paper independent of these results. It is worth mentioning that if the $\tilde{O}(n)$ -time matching estimation result of [25] can be shown to hold, then the run-time of both Theorems 2 and 4 can be improved to $\tilde{O}(n)$ time.

We note that it is easy to show that randomization is crucial to getting better than a 2-approximation in sublinear time for both graphic TSP and $(1, 2)$ -TSP – see Theorem 21 in Section 6.3. The algorithms underlying Theorems 2 and 4, lend themselves to $\tilde{O}(n)$ space single-pass

streaming algorithms with identical approximation guarantees. These sublinear time algorithms motivate the natural question if analogously to metric MST, there exist sublinear time algorithms that for any $\varepsilon > 0$, output a $(1 + \varepsilon)$ -approximate estimate of TSP cost for graphic TSP and $(1, 2)$ -TSP in $\tilde{O}(n)$ time. We rule out this possibility in a strong sense for both graphic TSP and $(1, 2)$ -TSP.

Theorem 5. *There exists an $\varepsilon_0 > 0$, such that any randomized algorithm that estimates the cost of graphic TSP ($(1, 2)$ -TSP) to within a $(1 + \varepsilon_0)$ -factor, necessarily requires $\Omega(n^2)$ queries.*

This lower bound result highlights a sharp separation between the behavior of metric MST and metric TSP problems. At a high-level, our lower bound is inspired by the work of Bogdanov *et al.* [7] who showed that any query algorithm that for any $\varepsilon > 0$ distinguishes between instances of parity equations (mod 2) that are either satisfiable (Yes) or at most $(1/2 + \varepsilon)$ -satisfiable (No), requires $\Omega(n)$ queries where n denotes the number of variables. However, the query model analyzed in [7] is different from ours (see more details in Section 4). We first show that the lower bound of [7] can be adapted to an $\Omega(n^2)$ lower bound in our model, and then show that instances of parity equations can be converted into instances of graphic TSP (resp. $(1, 2)$ -TSP) such that for some $\varepsilon_0 > 0$, any $(1 + \varepsilon_0)$ -approximation algorithm for graphic TSP (resp. $(1, 2)$ -TSP), can distinguish between the Yes and No instances of the parity equations, giving us the desired result.

Finally, similar to many classical approximation algorithms for TSP, our sublinear time estimation algorithms utilize subroutines for estimating the size of a maximum matching in the underlying graph. We show that this is not merely an artifact of our approach.

Theorem 6. *For any $\varepsilon \in [0, 1/5]$, any algorithm that estimates the cost of an n -vertex instance of graphic TSP or $(1, 2)$ -TSP to within a $(1 + \varepsilon)$ -factor, can also be used to estimate the size of a maximum matching in an n -vertex bipartite graph to within an εn additive error, with an identical query complexity, running time, and space usage.*

This connection allows us to translate known lower bounds for matching size estimation in various models to similar lower bounds for metric TSP cost estimation. In particular, using the results of [5], we can show that there exists an ε_0 such that any randomized single-pass dynamic streaming algorithm for either graphic TSP or $(1, 2)$ -TSP that estimates the cost to within a factor of $(1 + \varepsilon_0)$, necessarily requires $\Omega(n^2)$ space.

We conclude by establishing several additional lower bound results that further clarify the query complexity of approximating TSP cost. For instance, we show that if an algorithm can access an instance of graphic TSP by only querying the edges of the graph (via neighbor and pair queries), then any algorithm that approximates the graphic TSP cost to a factor better than 2, necessarily requires $\Omega(n^2)$ queries. This is in sharp contrast to Theorem 1, and shows that working with the distance matrix is crucial to obtaining sublinear time algorithms for graphic TSP. We also show that even in the distance matrix representation, the task of *finding a tour* that is $(2 - \varepsilon)$ -approximate for any $\varepsilon > 0$, requires $\Omega(n^2)$ queries for both graphic TSP and $(1, 2)$ -TSP.

Matching Size Estimation: As the problem of matching size estimation is intimately connected to metric TSP cost estimation, we briefly review some relevant work here. This line of research primarily assumes that we are given a graph $G(V, E)$ with maximum degree d , that can be accessed via *neighbor queries* [14]: (a) for any vertex v , we can query its degree, and (b) for any vertex v and an integer i , we can learn the i^{th} neighbor of v .

Parnas and Ron [28] initiated the study of matching size estimation in sublinear time and gave an $d^{O(\log(d/\varepsilon))}$ time algorithm that estimates the matching size to within a constant factor plus an additive εn error for any $\varepsilon > 0$. Nguyen and Onak [24] presented a new estimation algorithm and

showed that it can estimate the matching size to within a factor of 2 plus an additive εn error in $2^{O(d)}/\varepsilon^2$ time. We will refer to this approximation guarantee as a $(2, \varepsilon)$ -approximation of matching size. Yoshida *et al.* [36] strongly improved upon the performance guarantee obtained in [24], and showed that a $(2, \varepsilon)$ -approximation to matching size can be accomplished in $O(d^4/\varepsilon^2)$ time (in fact, they obtain the stronger $(2 \pm \varepsilon)$ -approximation guarantee). The analysis of [36] was further improved by Onak *et al.* [25] who showed that the state of the art for $(2, \varepsilon)$ -approximation of matching size. We note that it is known that any $(O(1), \varepsilon)$ -approximate estimate of matching size necessarily requires $\Omega(d)$ queries [28], so the result of [25] is essentially best possible. Unfortunately, as mentioned above, we recently discovered a subtle mistake in the analysis of Onak *et al.* [26]. Consequently, the best known time complexity for obtaining a $(2, \varepsilon)$ -approximate estimate is $\tilde{O}(d^2/\varepsilon^2)$; this weaker result also follows from the work of [25], but does not rely on the incorrect observation in [25].

The difference between a linear dependence versus a quadratic dependence on degree d is however huge in the sublinear time applications when the graph is not very sparse. In particular, while an $\tilde{O}(d)$ query result translates into an $\tilde{O}(n)$ time algorithm in the adjacency matrix model, an $\tilde{O}(d^2)$ query result gives only an $\tilde{O}(n^2)$ time algorithm, which is clearly not useful. Very recently, Kapralov *et al.* [17] gave an alternate approach based on a vertex “peeling” strategy (originally proposed in [28]) that yields an $(O(1), \varepsilon)$ -approximation of matching size in $\tilde{O}(d/\varepsilon^2)$ time. Unfortunately, the constant hidden in the $O(1)$ notation is very large, and efficiently obtaining a $(2, \varepsilon)$ -approximation to matching size remains an important open problem. Meanwhile, by directly building on the work of [36], we obtain an $\tilde{O}(n^{1.5})$ time algorithm for a $(2, \varepsilon)$ -approximation to matching size in the adjacency matrix model, and it is this algorithm that is used in the results of Theorem 2 and Theorem 4.

Other Related Work: We note here that there is an orthogonal line of research that focuses on computing an approximate solution in near-linear time when the input is presented as a weighted undirected graph, and the metric is defined by shortest path distances on this weighted graph. It is known that in this model, for any $\varepsilon > 0$, there is an $\tilde{O}(m/\varepsilon^2 + n^{1.5}/\varepsilon^3)$ time algorithm that computes a $(3/2 + \varepsilon)$ -approximate solution; here n denotes the number of vertices and m denotes the number of edges [10], and that a $(3/2 + \varepsilon)$ -approximate estimate of the solution cost can be computed in $\tilde{O}(m/\varepsilon^2)$ time [9]. It is not difficult to show that in this access model, even when the input graph is unweighted (i.e. a graphic TSP instance), any algorithm that outputs better than a 2-approximate estimate of the TSP cost, requires $\Omega(n + m)$ time even when $m = \Omega(n^2)$. Hence this access model does not admit sublinear time algorithms that beat the trivial 2-approximate estimate.

Organization: In Section 2, we present our algorithms for graphic TSP (Theorem 1, Theorem 2, and Theorem 3). In Section 3, we present the 1.625-approximation algorithm of $(1, 2)$ -TSP (Theorem 4). In Section 4, we present our lower bound result that rules out possibility of a sublinear-time approximation scheme for both graphic TSP and $(1, 2)$ -TSP (Theorem 5). In Section 5, we present a strong connection between approximating metric TSP cost and estimating matching size (Theorem 6). Finally, in Section 6, we present several additional lower bound results on the complexity of approximating graphic TSP and $(1, 2)$ -TSP cost.

2 Approximation for Graphic TSP Cost

In this section, we exploit well-known properties of biconnected graphs and biconnected components in graphs to give an algorithm that achieves a $(2 - \frac{1}{7c_0})$ -approximation for graphic TSP if we have an efficient algorithm that approximates the maximum matching size within a factor of c_0 . We first relate the cost of the TSP tour in a graph to the costs of the TSP tours in the biconnected

components of the graph. Next we show that if the graph does not have a sufficiently big matching, it does not have a TSP tour whose length is much better than $2n$. We also show that if a graph has too many degree 1 vertices, or vertices of degree 2, both whose incident edges are bridges, then it does not have a TSP tour of cost much better than $2n$. We then establish the converse - a graph that has a good matching and not too many bad vertices (namely, vertices of degree 1 or articulation points of degree 2), then it necessarily has a TSP tour of cost much better than $2n$. We design $\tilde{O}(n)$ time test for the second condition, allowing us to approximate the cost of an optimal graphic TSP tour in sublinear time together with some known techniques for testing the first condition. In what follows, we first present some basic concepts and develop some tools that will play a central role in our algorithms.

2.1 Preliminaries

An unweighted graph $G = (V, E)$, defines a *graphic metric* in V , where the distance between any two vertices u and v is given by the length of the shortest path between u and v . The *graphic TSP* is the Traveling Salesman Problem defined on such a graphic metric. In this paper our goal is to find a non-trivial approximation to the length of the traveling salesman tour in sublinear time in a model where we are allowed to make *distance queries*. In the distance query model, the algorithm can make a query on a pair of vertices (u, v) and get back the answer $d(u, v)$, the distance between u and v in G .

In a connected graph G , an edge e is a *bridge* if the deletion of e would increase the number of connected components of G . A connected graph with no bridge is called a *2-edge-connected graph*. A maximal 2-edge-connected subgraph of G is called a *2-edge-connected component*. The *bridge-block tree* of a graph is a tree such that the vertex set contains the 2-edge-connected components and the edge set contains the bridges in the graph.

A connected graph G is called *2-vertex-connected* or *biconnected* if when any one vertex is removed, the resulting graph remains connected. In a graph which is not biconnected, a vertex v whose removal increases the number of components is called an *articulation point*. It is easy to prove that any biconnected graph with at least 3 vertices does not have degree 1 vertices. A well-known alternate characterization of biconnectivity is that, a graph G is biconnected if and only if for any two distinct edges, there is a simple cycle that contains them.

A *biconnected component* or *block* in a graph is a maximal biconnected subgraph. Any graph G can be decomposed into blocks such that the intersection of any two blocks is either empty, or a single articulation point. Each articulation point belongs to at least two blocks. If a block is a single edge, then we call this block a *trivial block*; otherwise it is a *non-trivial block*. A trivial block is also a bridge in the graph. The size of a block is the number of vertices in the block. The following lemma shows the relationship between the number of blocks and the sum of the sizes of the blocks.

Lemma 2.1. *If a connected graph G has n vertices and k blocks, then the sum of the sizes of the blocks is equal to $n + k - 1$.*

Proof. We prove the lemma by induction on the number k of blocks. The base case is when $k = 1$. In this case, G itself is a block of size n .

For the induction step, we have $k > 1$ and thus the graph has at least one articulation point. Suppose v is an arbitrary articulation point in G . Let V_1, V_2, \dots, V_j be the set of vertices in the connected components of $G \setminus \{v\}$. We have $\sum_{i=1}^j |V_i| = n - 1$. Let G_1, G_2, \dots, G_j be the subgraphs of G induced by $V_1 \cup \{v\}, V_2 \cup \{v\}, \dots, V_j \cup \{v\}$. For any G_i , let k_i be the number of blocks in G_i , we have $\sum_{i=1}^j k_i = k$. By induction hypothesis, the sum of the sizes of blocks in G_i is $|V_i| + 1 + k_i - 1 = |V_i| + k_i$. So the sum of the sizes of blocks in G is $\sum_{i=1}^j |V_i| + k_i = n - 1 + k$. ■

The block decomposition of a graph has a close relationship with the cost of graphic TSP of the graph.

Lemma 2.2 (Lemma 2.1 of [21]). *The cost of the graphic TSP of a connected graph $G = (V, E)$ is equal to the sum of the costs of the graphic TSP of all blocks in the graph.*

Together these two lemmas give us a simple lower bound on the cost of the graphic TSP of a graph G (using the fact that the cost of graphic TSP is at least the number of vertices in the graph).

Lemma 2.3. *If a graph G has n vertices and k blocks, then the cost of graphic TSP of G is at least $n + k - 1$.*

An *ear* in a graph is a simple cycle or a simple path. An ear which is a path is also called an *open ear* and it has two endpoints, whereas for a cycle, one vertex is designated as the endpoint. An *ear decomposition* of a graph is a partition of a graph into a sequence of ears such the endpoint(s) of each ear (except for the first) appear on previous ears and the internal points (the points that are not endpoints) are not on previous ears. A graph G is biconnected if and only if G has an ear decomposition such that each ear but the first one is an open ear [34]. An ear is *nontrivial* if it has at least one internal point. The following lemma upper bounds the cost of graphic TSP of a biconnected graph.

Lemma 2.4 (Lemma 5.3 of [30], also a corollary of Lemma 3.2 of [22]). *Given a 2-vertex-connected graph $G = (V, E)$ and an ear-decomposition of G in which all ears are nontrivial, a graphic TSP tour of cost at most $\frac{4}{3}(|V| - 1) + \frac{2}{3}\pi$ can be found in $O(|V|^3)$ time, where π is the number of ears.*

We now prove an important lemma that gives an upper bound on the cost of graphic TSP in a biconnected graph in terms of the size of a matching in the graph.

Lemma 2.5. *Suppose G is a biconnected graph with at least $n \geq 3$ vertices. If G has a matching M , then the cost of graphic TSP of G is at most $2n - 2 - \frac{2|M|}{3}$.*

Proof. We first find a spanning biconnected subgraph of G that only contains $2n - 2 - M$ edges, then use Lemma 2.4 to bound the cost of graphic TSP.

We construct a spanning biconnected subgraph $G^* = P_0 \cup P_1 \cup \dots$ recursively: P_0 contains a single edge in M . If $G_{i-1} = P_0 \cup P_1 \cup \dots \cup P_{i-1}$ is a spanning subgraph of G , let $G^* = G_{i-1}$ and finish the construction. Otherwise we construct P_i as follows. Let e be an edge in M both whose endpoints are not in G_{i-1} . If there is no such edge, then let e be an arbitrary edge such that at least one of its endpoints is not in G_{i-1} . Let e' be an arbitrary edge in G_{i-1} . By the alternate characterization of biconnectedness, there is a simple cycle C_i that contains both e and e' . Let P_i be the path in C_i that contains e and exactly two vertices in G_{i-1} , which are the endpoints of P_i .

Since P_i contains at least one vertex not in G_{i-1} , the construction always terminates. Note that $P_0 \cup P_1$ is a cycle, and each P_i ($i > 1$) is an open ear of G^* . So, $(P_0 \cup P_1, P_2, \dots)$ is an open ear decomposition of G^* , which means G^* is biconnected.

Now we prove that the number of edges in G^* is at most $2n - 2 - M$. Let n_i be the number of vertices in $G_i \setminus G_{i-1}$. Let G_{-1} be the empty graph, so that $n_0=2$. Let p_i be the number of edges in P_i and m_i be the number of edges e in M such that $e \cap G_i \neq \emptyset$ and $e \cap G_{i-1} = \emptyset$. (Here we view an edge as a 2-vertex set.) Note that $m_0 = 1$. Suppose $G^* = G_k$. Then $\sum_{i=1}^k n_i = n$, $\sum_{i=1}^k p_i$ is the number of edges in G^* and $\sum_{i=1}^k m_i = |M|$. For any $i > 0$, P_i is an open ear whose internal points are not in G_{i-1} . So $n_i = p_i - 1$. If there is an edge $e \in M$ such that $e \cap G_{i-1} = \emptyset$, then P_i contains both endpoints of an edge in M , which means $m_i \leq n_i - 1$. If all edges in M already have an

endpoint in G_{i-1} , $m_i = 0 \leq n_i - 1$. So in both cases, $p_i = n_i + 1 = 2n_i - (n_i - 1) \leq 2n_i - m_i$. Also, $p_0 = 1 = 2n_0 - 2 - m_0$. So the number of edges in G^* is $\sum_{i=0}^k p_i \leq 2n_0 - 2 - m_0 + \sum_{i=1}^k (2n_i - m_i) = 2n - 2 - |M|$.

Since $(P_0 \cup P_1, P_2, P_3, \dots, P_k)$ is an open ear decomposition of G^* , the number of ears in G is k . On the other hand, $\sum_{i=0}^k p_i = 1 + \sum_{i=1}^k (n_i + 1) = n - 1 + k$, we have $n - 1 + k \leq 2n - 2 - |M|$, which means $k \leq n - 1 - |M|$. By Lemma 2.4, the cost of graphic TSP of G^* is at most $\frac{4}{3}(n - 1) + \frac{2}{3}k \leq 2(n - 1) - \frac{2}{3}|M|$.

Since G^* is a subgraph of G that contains all the vertices in G , the cost of graphic TSP of G is at most the cost of graphic TSP of G^* , which is at most $2n - 2 - \frac{2}{3}|M|$. \blacksquare

2.2 Approximation Algorithm for Graphic TSP

In this section, we give the algorithm that approximates the cost of graphic TSP of a graph G within a factor of less than 2.

We call a vertex v a *bad* vertex if v has degree 1 or is an articulation point with degree 2.

For any given $\delta > 0$, the graphic TSP algorithm performs the following two steps.

1. Obtain an estimate $\hat{\alpha}n$ of the size of maximum matching αn .
2. Obtain an estimate $\hat{\beta}n$ of the number of bad vertices βn .

The algorithm then output $\min\{2n, (2 - \frac{2}{7}(\hat{\alpha} - 2\hat{\beta}))n\}$.

To perform the second step in $\tilde{O}(n)$ distance queries and time, we randomly sample $O(\frac{1}{\delta^2})$ vertices. For each sampled vertex, we can obtain the degree with n queries. The following lemma shows that we can also check whether a degree 2 vertex is an articulation point using distance queries in $O(n)$ time. Then by the Chernoff bound, we can approximate the number of bad vertices with additive error $O(\delta n)$ with a high constant probability.

Lemma 2.6. *Suppose a vertex v in a connected graph G has only two neighbors u and w . The following three conditions are equivalent:*

1. v is an articulation point.
2. The edges (u, v) and (v, w) are both bridges.
3. For any vertex $v' \neq v$, $|d(u, v') - d(w, v')| = 2$.

Proof. We first prove the first two conditions are equivalent. If v is an articulation point, then v is in two different blocks. So edge (u, v) and (v, w) are in different blocks, which means v has degree 1 in both blocks. So both blocks are trivial, which means (u, v) and (v, w) are both bridges. If (u, v) and (v, w) are both bridges, then deleting either (u, v) or (v, w) will disconnect u and w , which means deleting v will also disconnect u and w .

Next we prove that the third condition is equivalent to the first two. Suppose v is an articulation point. Since v has degree 2, the graph $G \setminus \{v\}$ has only two components, one containing u and the other containing w . For any vertex $v' \neq v$, without loss of generality, suppose v' is in the same component as u in $G \setminus \{v\}$. Since (u, v) and (v, w) are both bridges in G , any path between v' and w contains u and v . So $d(v', w) = d(v', u) + 2$.

If v is not an articulation point, then u and w are connected in $G \setminus \{v\}$. Let $(u = v_0, v_1, v_2, \dots, v_k = w)$ be the shortest path between u and w in $G \setminus \{v\}$. For any vertex v_i on the path, the distance between v_i and u (resp. w) in $G \setminus \{v\}$ is i (resp. $k - i$). Consider the shortest path between u

and v_i in G . If this path does not contain v , then it is the same as the path in $G \setminus \{v\}$. In this case, $d(u, v_i) = i$. If the shortest path contains v , then v must be the second last vertex on the path and w be the third last one. In this case, $d(u, v_i) = k - i + 2$. So $d(u, v_i) = \min\{i, k - i + 2\}$. Similarly, we also have $d(v_i, w) = \min\{i + 2, k - i\}$. Let $v' = v_{\lfloor k/2 \rfloor}$. Since $|i - (k - i)| \leq 1$, we have $i < k - i + 2$ and $k - i < i + 2$, which means $|d(u, v') - d(w, v')| = |i - (k - i)| \leq 2$. \blacksquare

Next, we prove that if α is small or β is large, the cost of graphic TSP is bounded away from n . The following lemma shows that if the size of maximum matching of a graph is small, then the cost of the graphic TSP is large.

Lemma 2.7. *For any $\varepsilon > 0$, if the maximum matching of a graph G has size at most $\frac{(1-\varepsilon)n}{2}$, then the cost of graphic TSP of G is at least $(1 + \varepsilon)n$.*

Proof. Suppose the optimal TSP tour is $(v_0, v_1, \dots, v_{n-1}, v_n = v_0)$. Since the size of maximum matching in G is at most $\frac{(1-\varepsilon)n}{2}$, there are at most $\frac{(1-\varepsilon)n}{2}$ edges between pairs (v_i, v_{i+1}) where i is even (resp. odd). So there are at least εn pairs of (v_i, v_{i+1}) that have distance at least 2, which means that the optimal cost of TSP tour of G is $\sum_{i=1}^{n-1} d(v_i, v_{i+1}) \geq n + \varepsilon n = (1 + \varepsilon)n$. \blacksquare

The following lemma shows that if β is large, the cost of graphic TSP is large.

Lemma 2.8. *For any $\varepsilon > 0$, if a connected graph G has εn bad vertices, then the cost of graph-TSP of G is at least $(1 + \varepsilon)n - 2$.*

Proof. We first prove by induction on the number of vertices that a graph with k bad vertices has $k - 1$ bridges. The base case is when $n = 2$, the graph has $k = 2$ bad vertices and $1 = k - 1$ bridge.

For the induction step, the graph has n vertices with $n \geq 3$. If G has no degree 1 vertices, then the graph has k articulation points with degree 2. By Lemma 2.6, any edge incident on a degree 2 articulation point is a bridge. So each bad vertex is incident on 2 bridges. On the other hand, a bridge is incident on at most 2 vertices. So there are at least $\frac{2k}{2} = k$ bridges in G . Next, suppose G has degree 1 vertices. Let v be an arbitrary such vertex and let u be its neighbor. Since G is connected and $n \geq 3$, u must have degree at least 2, since otherwise u and v are not connected to other vertices in G . Consider the graph $G \setminus \{v\}$, if u is a bad vertex in G , u has degree 1 in $G \setminus \{v\}$ and is still a bad vertex. So the number of bad vertices in $G \setminus \{v\}$ is $k - 1$. By induction hypothesis, $G \setminus \{v\}$ has at least $k - 2$ bridges. G has at least $k - 1$ bridges since (u, v) is also a bridge.

So G has at least $\varepsilon n - 1$ bridges, and the number of blocks in G is at least $\varepsilon n - 1$. By Lemma 2.3, the cost of graph-TSP of G is at least $n + \varepsilon n - 2 = (1 + \varepsilon)n - 2$. \blacksquare

Finally, the following lemma shows that the cost of graphic TSP is at most $(2 - \frac{2}{7}(\hat{\alpha} - 2\beta))n$.

Lemma 2.9. *If a graph has a matching M of size $\alpha'n$ and the graph has βn bad vertices, the cost of graphic TSP of G is at most $(2 - \frac{2}{7}(\alpha' - 2\beta))n$.*

Proof. Let G_1, G_2, \dots, G_k be the block decomposition of G . Let n_i be the size of G_i . If $|n_i| \geq 3$, by Lemma 2.5, the cost of the graphic TSP of G_i is at most $2n_i - 3$ since any non-empty graph has a matching of size at least 1. If $|n_i| = 2$, then the graphic TSP of G_i is exactly $2 = 2n_i - 2$. Suppose G has ℓ non-trivial blocks. Then by Lemma 2.2 the cost of graphic TSP of G is at most $\sum_{i=1}^k (2n_i - 2) - \ell$, which equals to $2n - 2 - \ell$ by Lemma 2.1.

Let m_i be the size of maximum matching in G_i if G_i is a non-trivial block, and let $m_i = 0$ if G_i is a trivial block. By Lemma 2.5, the cost of the graphic TSP of G_i is at most $2n_i - 2 - \frac{2m_i}{3}$. For any non-trivial block G_i , $M \cap G_i$ is a matching in G_i . So the size of maximum matching in G_i is at least the number of edges in $M \cap G_i$. So by Lemma 2.2 and Lemma 2.1, the cost of graphic TSP of

G is at most $\sum_{i=1}^k (2n_i - 2 - \frac{2}{3}m_i) = 2n - 2 - \frac{2}{3}|M'|$, where M' is the set of edges in M that are not bridges in G . Let B be the number of bridges in G . We have $2n - 2 - \frac{2}{3}|M'| \leq 2n - 2 - \frac{2}{3}(|M| - B)$.

So there are two upper bounds of the graphic TSP of G — $2n - 2 - \ell$ and $2n - 2 - \frac{2}{3}(|M| - B)$. Which bound is better depends on the number of bridges B .

If $B \leq (\frac{4}{7}\alpha' + \frac{6}{7}\beta)n$, the cost of graphic TSP of G is at most

$$2n - 2 - \frac{2}{3}(|M| - B) \leq 2n - \frac{2}{3}(\frac{3}{7}\alpha' - \frac{6}{7}\beta)n = (2 - \frac{2}{7}(\alpha' - 2\beta))n$$

If $B > (\frac{4}{7}\alpha' + \frac{6}{7}\beta)n$, consider the bridge-block tree T of G . T has at least B edges and at least $B+1$ vertices. Since T is a tree, there are at least $\frac{B}{2}$ vertices of degree at most 2. For any vertex v_T of degree at most 2 in T , if the vertex v_T represents a single vertex v in G , then v is either a degree 1 vertex or a degree 2 articulation point in G , otherwise v_T represents a 2-edge-connected component of size at least 2 in G . So There are at least $\frac{B}{2} - \beta n \geq (\frac{2}{7}\alpha' - \frac{4}{7}\beta)n$ 2-edge-connected components of size at least 2. Since any 2-edge-connected component of size at least 2 has no bridge, each such component of G contains at least 1 non-trivial block in G , implying that $\ell \geq \frac{2}{7}(\alpha' - 2\beta)n$. So the cost of graphic TSP of G is at most $2n - 2 - \ell \leq (2 - \frac{2}{7}(\alpha' - 2\beta))n$. ■

We summarize the ideas in this section and prove the following lemma.

Lemma 2.10. *For any $c_0 > 1$ and $\delta > 0$, suppose $\hat{\alpha} \leq \alpha \leq c_0\hat{\alpha} + \delta$ and $\hat{\beta} - \delta \leq \beta \leq \hat{\beta}$. Then $(2 - \frac{2}{7}(\hat{\alpha} - 2\hat{\beta}))n$ is an approximation of the size of graphic TSP within a factor of $2 - \frac{1}{7c_0} + \delta$.*

Proof. Let $\hat{T} = (2 - \frac{2}{7}(\hat{\alpha} - 2\hat{\beta}))n$. Since $\hat{\beta} \geq \beta$ and $\hat{\alpha} \leq \alpha$, by Lemma 2.9, $T \leq \hat{T}$.

Then we prove that $\hat{T} \leq (2 - \frac{1}{7c_0} + \delta)T$. By Lemma 2.7 and Lemma 2.8, $T \geq \max\{(2 - 2\alpha)n, (1 + \beta)n - 2\}$, which means

$$(2 - \frac{1}{7c_0} + \delta)T \geq (2 - \frac{1}{7c_0}) \max\{(2 - 2\alpha)n, (1 + \beta)n\} - 4 + \delta n$$

On the other hand, $\hat{T} \leq (2 - \frac{2}{7}(\frac{\alpha}{c_0} - 2\beta))n + \frac{6}{7}\delta n$ since $c_0\hat{\alpha} + \delta \leq \alpha$ and $\hat{\beta} \leq \beta + \delta$. For sufficient large n , we have $\delta n - 4 \geq \frac{6}{7}\delta n$, so it is sufficient to prove that $\frac{2 - \frac{2}{7}(\frac{\alpha}{c_0} - 2\beta)}{\max\{2 - 2\alpha, 1 + \beta\}} \leq 2 - \frac{1}{7c_0}$ for any $0 \leq \alpha, \beta \leq 1$ and $c_0 \geq 1$.

Let $\gamma = \frac{\alpha}{c_0} - 2\beta$, $1 + \beta = 1 + (\frac{\alpha}{c_0} - \gamma)/2$, so if we fix γ , $\max\{2 - 2\alpha, 1 + \beta\}$ is minimized when $2 - 2\alpha = 1 + (\frac{\alpha}{c_0} - \gamma)/2$. In this case $\alpha = \frac{(2+\gamma)c_0}{4c_0+1}$ and $\max\{2 - 2\alpha, 1 + \beta\} = \frac{4c_0+2}{4c_0+1} - \frac{2c_0}{4c_0+1}\gamma$. If $\gamma \leq \frac{1}{2c_0}$,

$$\begin{aligned} \frac{2 - \frac{2}{7}(\alpha - 2\beta)}{\max\{2 - 2\alpha, 1 + \beta\}} &\leq \frac{2 - \frac{2}{7}\gamma}{\frac{4c_0+2}{4c_0+1} - \frac{2c_0}{4c_0+1}\gamma} = \frac{4c_0+1}{7c_0} - \frac{2 - \frac{4c_0+2}{7c_0}}{\frac{4c_0+2}{4c_0+1} - \frac{2c_0}{4c_0+1}\gamma} \\ &\leq \frac{4c_0+1}{7c_0} + 2 - \frac{4c_0+2}{7c_0} = 2 - \frac{1}{7c_0} \end{aligned}$$

If $\gamma > \frac{1}{2c_0}$, $\frac{2 - \frac{2}{7}(\alpha - 2\beta)}{\max\{2 - 2\alpha, 1 + \beta\}} < \frac{2 - \frac{1}{7c_0}}{1} = 2 - \frac{1}{7c_0}$ since $\beta \geq 0$. So $\hat{T} \leq (2 - \frac{1}{7c_0} + \delta)T$. ■

By Lemma 2.10, we immediately have the following theorem.

Theorem 7. *For any $\delta > 0$ and $c_0 \geq 1$. Given a graph G with maximum matching size αn , suppose there is an algorithm that uses pair queries, runs in t time, and with probability at least $2/3$, outputs an estimate of the maximum matching size $\hat{\alpha}n$ such that $\hat{\alpha} \leq \alpha \leq c_0\hat{\alpha} + \delta$. Then there is an algorithm that approximates the cost of graphic TSP of G to within a factor of $2 - \frac{1}{7c_0} + \delta$, using distance queries, in $t + \tilde{O}(n/\delta^2)$ time with probability at least $3/5$.*

Proof. We first use the algorithm in the assumption to obtain an estimate $\hat{\alpha}n$ of the size of maximum matching αn . The following analysis is based on the event that this algorithm is run successfully, which has probability 2/3.

We then sample $N = \frac{100}{\delta^2}$ vertices. For each sampled vertex v , we first query the distance between v and every vertex in G to obtain the degree of v . If v has degree 2, suppose u and w are the neighbors of v . We query the distance from u and w to every vertex in G . By Lemma 2.6, v is an articulation point if and only if there is no vertex v' such that $|d(u, v') - d(w, v')| \leq 1$. So we can check if v is a bad vertex with $O(n)$ distance queries and time. Suppose there are βn bad vertices in G and $(\hat{\beta} - \delta/2)N$ sampled vertices are bad. By Chernoff bound, the probability that $|\beta - \hat{\beta} + \delta/2| > \delta/2$ is at most $2e^{\frac{\delta^2 N^2}{16}} < 1/15$. We analyze the performance based on the event that $\beta \leq \hat{\beta} \leq \beta + \delta$.

By Lemma 2.10, $(2 - \frac{2}{7}(\hat{\alpha} - 2\hat{\beta}))$ is a $(2 - \frac{1}{7c_0} + \delta)$ approximation of the size of graphic TSP of G . The probability of failure is at most $1/3 + 1/15 = 2/5$. ■

Proof of Theorem 2: The following theorem whose proof appears in Appendix A.1, gives an algorithm for matching size estimation that only uses *pair queries* – given a pair of vertices, is there an edge between them? Note that any pair query can be simulated by a single query to the distance matrix in a graphic TSP instance.

Theorem 8. *For any $\varepsilon > 0$, there is an algorithm that uses pair queries, runs in $\tilde{O}(n^{1.5}/\varepsilon^2)$ time, and with probability 2/3, outputs an estimate of the size of a maximal matching within an additive error εn .*

Substituting the above result in Theorem 7 and using the fact that a maximum matching has size at most twice the size of a maximal matching (setting $c_0 = 2$, and $\delta = \varepsilon$), we obtain Theorem 2.

Proof of Theorem 1: Kapralov et al. [17] give an algorithm that uses $\tilde{O}(d)$ queries (also $\tilde{O}(d)$ time) to approximate the size of maximum matching in a graph with average degree d in the neighbor query model (the approximation ratio is a very large constant). Together with a reduction in [25], this implies a pair query algorithm that uses $\tilde{O}(n)$ queries and time to estimate matching size to a constant factor. Combined with Theorem 7, this implies Theorem 1.

2.3 An $O(n)$ Space $(\frac{11}{6})$ -Approximate Streaming Algorithm for Graphic TSP

We show here that our approach for obtaining a sublinear-time algorithm for graphic TSP can be extended to the insertion-only streaming model to obtain for any $\varepsilon > 0$, an $(\frac{11}{6} + \varepsilon)$ -approximate estimate of the graphic TSP cost using $O(n/\varepsilon^2)$ space, proving Theorem 3. In the streaming model, we assume that the input to graphic TSP is presented as a sequence of edges of the underlying graph G . Any algorithm for this model, clearly also works if instead the entries of the distance matrix are presented in the stream instead – an entry that is 1 corresponds to an edge of G , and it can be ignored otherwise as a non-edge.

Given a stream containing edges of a graph $G(V, E)$, our algorithm performs the following two tasks in parallel:

- Find a maximal matching M in G – let αn denote its size.
- Estimate the number of bridges in the maximal matching M , say βn , to within an additive error of εn .

The algorithm outputs $(2 - \frac{2}{3}(\alpha - \beta))n$ as the estimated cost of graphic TSP of G .

In an insertion-only stream, it is easy to compute a maximal matching M using $O(n)$ space: we start with M initialized to an empty set, and add a new edge (u, v) into the matching M iff neither u nor v are already in M . It is also easy to check if an edge e is a bridge in insertion-only stream with $O(n)$ space. We can do this by maintaining a disjoint-set data structure. Whenever an edge arrives (other than e), we merge the connected components of its endpoints. If there is only one component remaining at the end of the stream, then e is not a bridge, and otherwise, e is a bridge.

To estimate the number of bridges in the maximal matching, we sample $N = 100/\varepsilon^2$ edges in the matching, and run in parallel N tests where each test determines whether or not the sampled edge is a bridge. We use $O(n/\varepsilon^2)$ space in total since we sample $N = O(1/\varepsilon^2)$ edges. Suppose there are $\hat{\beta}$ sampled edges are bridges, then by Chernoff bound, $\hat{\beta}n = \frac{\hat{\beta}|M|}{N}$ is an approximation of βn to within additive error εn with probability at least 9/10.

As stated, this gives us a two-pass algorithm: the first pass for computing the matching M , and the second pass for estimating the number of bridges in M . However, we can do both these tasks in parallel in a single pass as follows: at the beginning of the stream, we start the process of finding connected components of graph G . Whenever an edge e is added to M , if $|M| < N$, then we create a new instance I_e of the connectivity problem that ignores the edge e . This clearly allows us to test whether or not e is a bridge. Once $|M| > N$, then whenever an edge e is added to M , with probability $\frac{N}{|M|}$, we drop uniformly at random an existing instance, say $I_{e'}$ of connectivity, and create a new instance I_e of connectivity that only ignores edge e (we insert back the edge e' into I_e). Since there are at most N instances of connectivity that are running in parallel, the algorithm uses $O(nN) = O(n/\varepsilon^2)$ space.

We now prove that the algorithm gives a good approximation of the cost of graphic TSP.

Lemma 2.11. *If a graph G has a maximal matching M of size αn , and there are βn edges in M that are bridges in G , then the cost of graphic TSP in G is at most $(2 - \frac{2}{3}(\alpha - \beta))n$, and at least $\frac{6}{11}(2 - \frac{2}{3}(\alpha - \beta))n$.*

Proof. Since there are at least $(\alpha - \beta)n$ edges in the matching M that are not a bridge, by Lemma 2.2 and Lemma 2.5, the cost of graphic TSP of G is at most $(2 - \frac{2}{3}(\alpha - \beta))n$.

On the other hand, since M is a maximal matching of G , the size of maximum matching of G is at most $2\alpha n$. By Lemma 2.7, the cost of graphic TSP is at least $(2 - 4\alpha)n$. Graph G also contains at least βn bridges, so by Lemma 2.3, the cost of graphic TSP is also at least $(1 + \beta)n$.

To prove the lemma, it is sufficient to prove that for any $0 \leq \beta \leq \alpha \leq 1$, we have $2 - \frac{2}{3}(\alpha - \beta) \leq \frac{11}{6} \max\{1 + \beta, 2 - 4\alpha\}$. Let $\gamma = \alpha - \beta$. $1 + \beta = 1 + \alpha - \gamma$. So $\max\{1 + \beta, 2 - 4\alpha\} \geq 2 - 4(\frac{1}{5}(1 + \gamma)) = \frac{6}{5} - \frac{4}{5}\gamma$. If $\gamma \leq \frac{1}{4}$, $\frac{2 - \frac{2}{3}(\alpha - \beta)}{\max\{1 + \beta, 2 - 4\alpha\}} \leq \frac{2 - \frac{2}{3}\gamma}{\frac{6}{5} - \frac{4}{5}\gamma} = \frac{5}{6} + \frac{5}{6 - 4\gamma} = \frac{11}{6}$. If $\gamma > \frac{1}{4}$, $2 - \frac{2}{3}(\alpha - \beta) < \frac{11}{6}$, while $\max\{1 + \beta, 2 - 4\alpha\} \geq 1$ since $\beta > 0$. ■

By Lemma 2.11, the expression $(2 - \frac{2}{3}(\alpha - \beta))n$ gives us an 11/6-approximate estimate to the cost of graphic TSP of G . Since we can exactly compute α and approximate β with additive error ε in a single-pass streaming algorithm that uses $O(n/\varepsilon^2)$ space, we have the following theorem:

Theorem 9. *For any $\varepsilon > 0$, there is a single-pass randomized streaming algorithm that estimates the cost of graphic TSP of G to within a factor of $(\frac{11}{6} + \varepsilon)$, in an insertion-only stream, using $O(n/\varepsilon^2)$ space with probability at least 9/10.*

2.4 Extension to the Massively Parallel Computing Model

In the massive parallel computing (MPC) model, the input graph G is partitioned across multiple machines which are able to communicate with one another, and the memory allocated to each

machine is sublinear in the total input size. The computation proceeds in synchronous rounds where in any round, each machine runs a local algorithm on the data assigned to the machine. No communication between machines is allowed during a round. Between the rounds, machines can communicate with each other so long as each machine sends or receives a communication no more than its memory. Any data output from a machine must be computed locally from the data residing on the machine and initially the input data is distributed across machines in an arbitrary manner. The goal is to minimize the total number of rounds.

We extend our algorithms for the query model and the streaming model to the MPC model. In both query model and streaming model, we approximate the size of maximal matching and then approximate the number of articulation points or bridges in the graph to get the upper bounds and lower bounds of the cost of graphic TSP of the graph. The difference is that in streaming model, we can also easily compute a solution to a maximal matching (and not just estimate its size) in contrast to the query model, which results in a better approximating ratio. In general, however, the task of finding an approximate matching can be much harder than approximating the size of the maximum matching.

There have been many works studying the connectivity problem and matching problem in MPC model. Since there is a trade-off between the size of memory and the number of rounds, there are many different “state-of-the-art” results depending on the size of the memory in each machine. So rather than give algorithms for specific tradeoffs, we give two general results that translate various algorithms for the connectivity problem and matching problem to an algorithm for estimating the cost of the graphic TSP problem.

The following two corollaries follow from the proof of Lemma 2.10 and Theorem 9. If a graph G has maximum size $|M^*|$, we say a number $|M|$ is a $(\alpha, \varepsilon n)$ -approximation of $|M^*|$ if $|M| \leq |M^*| \leq \alpha |M| + \varepsilon n$.

Corollary 10. *If there is an algorithm that computes an $(\alpha, \varepsilon n)$ -estimation of the size of maximum matching in MPC model that uses $O(f_1(n, \varepsilon))$ rounds, where each machine has $O(g_1(n, \varepsilon))$ space with probability at least $9/10$, and there is an algorithm that checks if a graph is connected in MPC model that uses $O(f_2(n, \varepsilon))$ rounds, where each machine has $O(g_2(n, \varepsilon))$ space with probability at least $1 - \varepsilon^2/500$. Then there is an algorithm that approximates the size of graphic TSP within a factor of $(\frac{14\alpha-1}{7\alpha} + \varepsilon)$ in $O(f_1(n, \varepsilon) + f_2(n, \varepsilon))$ rounds, where each machine has $\max\{O(g_1(n, \varepsilon), O(g_2(n, \varepsilon)/\varepsilon^2)\}$ space with probability at least $2/3$.*

Proof Sketch. We first run the MPC algorithm that estimates the matching size. Then sample $\frac{100}{\varepsilon^2}$ vertices, and check if any of them are bad vertices so as to estimate the total number of bad vertices in the graph. To check if a vertex is a bad vertex, we first check if it has degree 1 or 2, then check if it is a articulation point by checking the connectedness of the graph when we delete the vertex and all edges incident on it. We can test all sampled vertices simultaneously if each machine has $\Omega(g_2(n, \varepsilon)/\varepsilon^2)$ space. The correctness follows from the same argument as Lemma 2.10. The failure probability of the matching algorithm is at most $1/10$ and the failure probability that we make a mistake on at least one sampled vertex is at most $\frac{100}{\varepsilon^2} \cdot \frac{\varepsilon^2}{500} = 1/5$. So the total probability of failure is at most $1/3$, giving as the desired result. ■

Corollary 11. *Suppose there exists an algorithm that computes an $(\alpha, \varepsilon n)$ -approximation of maximum matching in MPC model using $O(f_1(n, \varepsilon))$ rounds, where each machine has $O(g_1(n))$ space with probability at least $9/10$, and there is an algorithm that checks if a graph is connected in MPC model that uses $O(f_2(n, \varepsilon))$ rounds, where each machine has $O(g_2(n, \varepsilon))$ space with probability at least $1 - \varepsilon^2/500$. Then there is an algorithm that approximates the cost of graphic*

TSP to within a factor of $(\frac{6\alpha-1}{3\alpha} + \varepsilon)$ in $O(f_1(n, \varepsilon) + f_2(n))$ rounds, where each machine has $\max\{O(g_1(n, \varepsilon), O(g_2(n, \varepsilon)/\varepsilon^2)\}$ space with probability at least $2/3$.

Proof Sketch. The proof is similar to the proof of Corollary 10. The difference is that now we can find an approximate matching instead of just estimating the matching size. So we can now sample $\frac{100}{\varepsilon^2}$ edges in the approximate matching and estimate the number of bridges in the matching. The correctness follows from a similar argument as in the proof of Theorem 9. ■

3 (1.625)-Approximation for (1, 2)-TSP Cost in $\tilde{O}(n^{1.5})$ Time

In this section, we give an algorithm that for any $\delta > 0$, approximates the cost of the minimum (1, 2)-TSP to within a factor of $1.625 + \delta$ with $\tilde{O}(n^{1.5}/\delta^2)$ queries. The idea of the algorithm is to approximate the size of a maximal “matching pair” of G . In a graph G , a *matching pair* (M_1, M_2) is a pair of edge-disjoint matchings. A *maximal* matching pair is a matching pair (M_1, M_2) such that for any edge $e \notin M_1 \cup M_2$, neither $M_1 \cup \{e\}$ nor $M_2 \cup \{e\}$ is a matching. The size of a matching pair (M_1, M_2) is the sum of the sizes of M_1 and M_2 . The following lemma shows that the size of any maximal matching pair is lower bounded by the size of maximum matching in the graph.

Lemma 3.1. *Suppose M is a matching in a graph G . Then any maximal matching pair (M_1, M_2) in G has size at least $|M|$.*

Proof. Let X_1 be the set of vertices matched in both M and M_1 , and X_2 be the set of vertices matched in both M and M_2 . We have $|X_1| + |X_2| \leq 2|M_1| + 2|M_2|$ since M_1 and M_2 are both matchings. On the other hand, for any edge $e \in M$, if e is either in M_1 or M_2 , then both of its endpoints are in X_1 or X_2 . If e is neither in M_1 or M_2 , then there are edges $e_1 \in M_1$ and $e_2 \in M_2$ that share an endpoint with e since (M_1, M_2) is a maximal matching pair. So both X_1 and X_2 contain at least one endpoint of e . In both case e 's endpoints appear twice in X_1 and X_2 . So $|X_1| + |X_2| \geq 2|M|$, which means $|M_1| + |M_2| \geq |M|$. ■

We next show that if a graph has a matching pair of large size, then the cost of (1, 2)-TSP is not very large.

Lemma 3.2. *If a graph G with n vertices contains a matching pair (M_1, M_2) of size X , then the cost of (1, 2)-TSP of G is at most $2n - \frac{3}{4}X$.*

Proof. Since M_1 and M_2 are both matchings, $M_1 \cup M_2$ only contains paths and cycles of even length. We delete one edge from each cycle in $M_1 \cup M_2$, resulting in a graph that only contains paths. Since the cycles in $M_1 \cup M_2$ are of even length, the size of any cycle is at least 4. We deleted at most $\frac{1}{4}X$ edges, so G contains a set of vertex disjoint paths (including some of length 0, corresponding to isolated vertices), with total size at least $\frac{3}{4}X$. Construct a TSP tour by ordering the paths arbitrarily, orienting each one, and connecting the end of one path with the start of the next, cyclically. The tour contains at least $\frac{3}{4}X$ edges of weight 1, while the remaining edges are of weight 2. So the cost of the tour is at most $2n - \frac{3}{4}X$. ■

By Lemma 3.1, the maximum matching size is upper bounded by the size of any maximal matching pair. It follows that if the maximum matching size is small, the cost of (1, 2)-TSP is large.

Lemma 3.3. *For any $\varepsilon > 0$, if the maximum matching of a graph G has size at most $\frac{(1-\varepsilon)n}{2}$, then the cost of (1, 2)-TSP of G is at least $(1 + \varepsilon)n$.*

The proof of Lemma 3.3 is similar to the proof of Lemma 2.7 and we omit it here. By Lemma 3.2 and Lemma 3.3, if we can approximate the size of an arbitrary maximal matching pair, then we will get a good approximation of the cost of the $(1, 2)$ -TSP.

Theorem 12. *There is an algorithm that uses pair queries, with probability at least $2/3$, approximates the size of a maximal matching pair to within an additive error of εn in $\tilde{O}(n^{1.5}/\varepsilon^2)$ time.*

The algorithm in Theorem 12 is given in Appendix A.2. With Theorem 12, we can approximate the cost of $(1, 2)$ -TSP in a graph G by the size of a maximal matching pair.

Theorem 13. *For any $\delta > 0$, there is an algorithm that with probability at least $2/3$ estimates the optimal cost of a $(1, 2)$ -TSP instance to within a factor of $(1.625 + \delta)$ using $\tilde{O}(n^{1.5}/\delta^2)$ queries.*

Proof. Let $\varepsilon = \delta/2$. We use the algorithm in Theorem 12 that approximates the size of a maximal matching pair. Suppose the output of the algorithm is \bar{X} . Then, by Theorem 12, there is a maximal matching pair of size X such that $|X - \bar{X}| \leq \varepsilon n$. We output the cost of the $(1, 2)$ -TSP of G to be $\bar{T} = 2n - \frac{3}{4}(\bar{X} - \varepsilon n)$. Suppose the optimal $(1, 2)$ -TSP has cost T . By Lemma 3.2, $T \leq 2n - \frac{3}{4}X \leq 2n - \frac{3}{4}(\bar{X} - \varepsilon n) = \bar{T}$. On the other hand, by Lemma 3.3, the size of maximum matching in G is at least $(2n - T)/2$. So by Lemma 3.1, $X \geq (2n - T)/2$, which means $\bar{X} \geq (2n - T)/2 - \varepsilon n$. So $\bar{T} \leq 2n - \frac{3}{4}(n - T/2 - 2\varepsilon n) < 1.25n + 0.375T + \delta n$. Since T is the cost of $(1, 2)$ -TSP of G , which is at least n , we have $\bar{T} \leq (1.625 + \delta)T$. ■

Remark 3.4. The algorithm can be generalized to insertion-only streaming model. In insertion-only streaming model, we can compute a maximal matching pair as follows: we set M_1 and M_2 as empty set before the stream. Whenever an edge e comes, we first check if there is an edge in M_1 that shares an endpoint with e . If not, then we add e into M_1 . Otherwise, we check if there is an edge in M_2 that shares an endpoint with e . If not, then we add e into M_2 . So we get an algorithm that only uses $O(n)$ space to compute a maximal matching pair. We have the following corollary.

Corollary 14. *There is an insertion-only streaming algorithm that estimates the cost of $(1, 2)$ -TSP of a graph G within a factor of 1.625 using $O(n)$ space.*

4 An $\Omega(n^2)$ Query Lower Bound for Approximation Schemes

In this section, we prove that there exists an $\varepsilon_0 > 0$, such that any query algorithm for graphic or $(1, 2)$ -TSP that returns a $(1 + \varepsilon_0)$ -approximate estimate of optimal cost, requires $\Omega(n^2)$ queries. In order to prove this, we design a new query model for the 3SAT problem and show an $\Omega(n^2)$ query lower bound for 3SAT in this model. We then use a reduction from 3SAT to $(1, 2)$ -TSP in [27] to prove the lower bound for $(1, 2)$ -TSP; with some additional changes, we also get an identical lower bound for graphic TSP.

The idea of proving query lower bound for APX-hard problems by reduction from 3SAT is similar to the idea used in [7], and we follow their general approach. However, in [7], the authors study lower bounds for problems in sparse graphs and hence the query model uses only neighbor queries. So in their query model, the lower bound for 3SAT is $\Omega(n)$. In order to prove an $\Omega(n^2)$ query lower bound in the pair query model, we need to design a new query model for 3SAT.

In the 3SAT problem, we are given a 3CNF instance on n variables, and the goal is to estimate the largest fraction of clauses that can be satisfied by any assignment. The algorithm is allowed to perform only one kind of query: is a variable x present in a clause c ? If the answer is yes, then the algorithm is given the full information about all variables that appear in the clause c . The proof of the next theorem is deferred to Section 4.3.

Theorem 15. *For any $\varepsilon > 0$, any algorithm that with probability at least $2/3$ distinguishes between satisfiable 3CNF instances and 3CNF instances where at most $(7/8 + \varepsilon)$ fraction of clauses can be satisfied, needs $\Omega(n^2)$ queries.*

4.1 Reduction from 3SAT to (1, 2)-TSP

We will utilize an additional property of the hard instances of 3SAT in Theorem 15, namely, each variable occurs the same constant number of times where the constant only depends on ε . We denote the number of variables by n , the number of clauses by m , and the number of occurrences of each variable by k ; thus $m = kn/3$.

We use the reduction in [27] to reduce a 3SAT instance to a (1, 2)-TSP instance. In this reduction, there is a gadget for each variable and for each clause. Each of these gadgets has size at most $L = \Theta(k^2)$. Thus the (1, 2)-TSP contains N vertices where $N \leq L(n + m) = \frac{L(k+3)n}{3}$. Let G_{x_j} be the gadget of variable x_j and G_{c_i} be the gadget of clause c_i . There is a ground graph which is the same for each 3SAT instance. Each variable gadget is connected with the gadgets for clauses that contain that variable. The reduction satisfies the following property. If the 3SAT instance is satisfiable, then the (1, 2)-TSP instance contains a Hamilton cycle supported only on the weight 1 edges. On the other hand, if at most $m - \ell$ clauses can be satisfied in the 3SAT instance, the (1, 2)-TSP cost is at least $N + \lceil \ell/2 \rceil$. Thus there is a constant factor separation between the optimal (1, 2)-TSP cost in the two cases. However, what remains to be shown is that any query algorithm for (1, 2)-TSP can also be directly simulated on the underlying 3SAT instance with a similar number of queries. The theorem below now follows by establishing this simulation.

Theorem 16. *There is a constant ε_0 such that any algorithm that approximates the (1, 2)-TSP cost to within a factor of $(1 + \varepsilon_0)$ needs $\Omega(n^2)$ queries.*

Proof. We consider the following stronger queries for (1, 2)-TSP: for any query (u, v) , if u is in a vertex gadget G_{x_j} and v is in a clause gadget G_{c_i} (or vice versa) and x_j occurs in c_i in the 3SAT instance, then the algorithm is given all the edges incident on G_{c_i} . Otherwise the algorithm just learns if the there is an edge between u and v .

Let $\varepsilon = 1/16$, and let the values of k , L and N correspond to this choice for ε according to the reduction in Section 4.1. Let $\varepsilon_0 = \frac{k}{32(k+3)L}$. Consider the (1, 2)-TSP instance reduced from the 3SAT instance generated by the hard distribution in Theorem 15 with $\varepsilon = 1/16$. If the 3SAT instance is perfectly satisfiable, then the (1, 2)-TSP instance has a Hamilton cycle of cost N . If the 3SAT instance satisfies at most $(15/16)$ -fraction of clauses, then each Hamilton cycle in the (1, 2)-TSP instance has cost at least

$$N + (1/8 - \varepsilon)m/2 = N + (1/8 - \varepsilon)kn/6 \geq (1 + \frac{(1/8 - \varepsilon)k}{2(k+3)L})N = (1 + \varepsilon_0)N$$

For any query (u, v) in the (1, 2)-TSP instance, we can simulate it by at most one query in the corresponding 3SAT instance as follows: if u is in a vertex gadget G_{x_j} and v is in a clause gadget G_{c_i} (or vice versa), then we make a query of x_j and c_i in the 3SAT instance. If the 3SAT query returns YES and the full information of c_i , then we return all the edges incident on G_{c_i} according to the reduction rule and the full information of c_i . If the 3SAT query returns NO or (u, v) are not in a vertex gadget and a clause gadget respectively, we return YES if (u, v) is an edge in the ground graph and NO otherwise.

By Theorem 15, any algorithm that distinguishes a perfectly satisfiable 3SAT instance from an instance where at most $(15/16)$ -fraction of the clauses can be satisfied needs $\Omega(n^2)$ queries. So any algorithm that distinguishes a (1, 2)-TSP instance containing a Hamilton cycle of length N from an instance that has minimum Hamilton cycle of cost $(1 + \varepsilon_0)N$ needs $\Omega(n^2)$ queries.

4.2 $\Omega(n^2)$ Lower Bound for Graphic TSP

We can reduce an instance of $(1, 2)$ -TSP to an instance of graphic TSP by adding a new vertex that is adjacent to all other vertices. By doing so, any pair of vertices in the new graph has a distance at most 2. On the other hand, the cost of graphic TSP in the new graph differs by at most 1 from the cost of $(1, 2)$ -TSP in the old graph. So the $\Omega(n^2)$ query lower bound for $(1, 2)$ -TSP also holds for the graphic TSP problem.

4.3 An $\Omega(n^2)$ Query Lower Bound for the 3SAT Problem

We first prove a lower bound of E3LIN2 problem. E3LIN2 is the problem of deciding the satisfiability of a system of linear equations modulo 2, with three variables per equation.

We consider the following query model: the algorithm can query if an equation contains a variable. If the answer is **YES**, then the algorithm is also given all the variables and the right-hand side of the equation.

Theorem 17. *For any $\varepsilon > 0$, any algorithm that distinguishes between a perfectly satisfiable E3LIN2 instance and an instance that satisfies at most $(1/2 + \varepsilon)$ -fraction of equations needs $\Omega(n^2)$ queries with probability at least $2/3$.*

We start by defining the hard distribution. The distribution is similar to the one in [7], but the query model and therefore the proof are different. Every hard instance has n variables x_1, x_2, \dots, x_n and $m = kn$ equations e_1, e_2, \dots, e_m for some positive integer k . We construct the following two distributions of E3LIN2.

- The distribution \mathcal{D}_{NO} is the distribution of NO-instance, and is generated as follows: We first generate a random permutation $\sigma : [1, 3m] \rightarrow [1, 3m]$. For each $1 \leq i \leq m$, we assign equation e_i the variables $y_1^i = x_{\lceil \frac{\sigma(3i-2)}{3k} \rceil}, y_2^i = x_{\lceil \frac{\sigma(3i-1)}{3k} \rceil}$ and $y_3^i = x_{\lceil \frac{\sigma(3i)}{3k} \rceil}$. The equation e_i is $y_1^i + y_2^i + y_3^i = \pm z_i$ where z_i is chosen to be $+1$ or -1 uniformly randomly.
- The distribution \mathcal{D}_{YES} is the distribution of YES-instance, and is generated as follows: We first assign the variables to each equation with the same process as \mathcal{D}_{NO} . Then we randomly choose an assignment of variables, say A^* . Finally, for each equation e_i , we set $y_1^i + y_2^i + y_3^i = z_i$ where z_i equals the sum of $y_1^i + y_2^i + y_3^i$ according to assignment A^* .

Our final distribution generates an instance from the NO-distribution with probability $1/2$ and an instance from the YES-distribution with probability $1/2$.

If the instance is generated by \mathcal{D}_{YES} , then it is satisfied by the assignment A^* . The following lemma proves that if the instance is generated by \mathcal{D}_{NO} , then with high probability, the at most $(1/2 + \varepsilon)$ -fraction of the equations can be satisfied.

Lemma 4.1. *For any $\varepsilon > 0$, there exists a positive integer k , such that if an instance of E3LIN2 is randomly chosen from \mathcal{D}_{NO} with n variables and $m = kn$ equations, then with probability $9/10$, at most $(1/2 + \varepsilon)$ -fraction of the equations can be satisfied.*

Proof. Let $k = 8/\varepsilon^2$ and so $m = \frac{8n}{\varepsilon^2}$. Fix an assignment A . For each equation e_i , the probability that A satisfies e_i is $1/2$. Since in distribution \mathcal{D}_{NO} , the right hand side of the equations are sampled independently, the event that A satisfies any equation is independent of the event of A satisfying any subset of the other equations. By the Chernoff bound, the probability that A satisfies at least $(1/2 + \varepsilon)$ -fraction of equations is at most $e^{-\frac{\varepsilon^2(m/2)}{4}} \leq e^{-n}$. Taking the union bound over all possible assignments A , the probability that there exists an assignment that satisfies at least $(1/2 + \varepsilon)$ -fraction of equations is at most $2^n \cdot e^{-n} < 1/10$. ■

Now we prove that it is hard to distinguish between the YES and NO instances of this distributions. Define a bipartite graph G_σ associated with the random permutation σ as follows: there are $3m$ vertices on each side of G_σ , there is an edge between the i^{th} vertex on the left and the j^{th} vertex on the right if and only if $\sigma_i = j$. Since σ is chosen uniformly at random, G_σ is a randomly chosen perfect matching. Associate variable x_i with the $(3k(i-1)+1)^{th}$ to the $(3ki)^{th}$ vertices on the left and associate equation e_j with the $(3j-2)^{th}$ to the $(3j)^{th}$ vertex on the right. A variable occurs in an equation if and only if there is an edge between the vertices associate with the variable and the equation.

Fix an algorithm \mathcal{A} , let $\mathcal{E}_{YES}^{\mathcal{A}}$ and $\mathcal{E}_{NO}^{\mathcal{A}}$ be the set of equations given to \mathcal{A} after all the queries to an instance generated by \mathcal{D}_{YES} and \mathcal{D}_{NO} . Denote the knowledge graph $G^{\mathcal{A}}$ as the subgraph of G_σ induced by the equations given to \mathcal{A} and the variables that occur in these equations. The following lemma shows that if an algorithm only discover a small fraction of equations, then the set of equations discovered by the algorithm has the same distribution in the YES and NO cases with some high constant probability.

Lemma 4.2. *For any $k > 0$, there exists a constant δ_0 such that: if $G^{\mathcal{A}}$ contains at most $3\delta_0 n$ edges, then the distributions of $\mathcal{E}_{YES}^{\mathcal{A}}$ and $\mathcal{E}_{NO}^{\mathcal{A}}$ are identical with probability at least $9/10$.*

The proof of Lemma 4.2 is similar to the proof of Theorem 8 in [7]. We prove that the left hand side of the equations in $\mathcal{E}_{YES}^{\mathcal{A}}$ and $\mathcal{E}_{NO}^{\mathcal{A}}$ are independent, and thus the distribution of the right hand side are identical.

Proof. We first prove that there is a constant δ_0 such that with probability at least $9/10$, any set of equations of size $\delta n \leq \delta_0 n$ contains more than $\frac{3}{2}\delta n$ variables. Fix a set of variables V of size $\frac{3}{2}\delta n$. For any equation e , the probability that it contains only the variables in V is $\frac{4.5k\delta n}{3kn} \cdot \frac{4.5k\delta n-1}{3kn-1} \cdot \frac{4.5k\delta n-2}{3kn-2} \leq 4\delta^3$. For any equation e and any set of equations \mathcal{E} that does not contain e , the events that e only contains variables in V and the equations in \mathcal{E} only contain variable in V are negatively correlated. So for any set of equations of size δn , the probability that these equations only contain variables in V is at most $(4\delta^3)^{\delta n} = 4^{\delta n} \delta^{3\delta n}$. Taking the union bound over all possible set of equations of size δn , the probability that one of them only contains variables in V is at most $4^{\delta n} \delta^{3\delta n} \cdot \binom{\delta n}{\delta n} \leq 4^{\delta n} \delta^{3\delta n} \cdot (ek/\delta)^{\delta n} = (4ek)^{\delta n} \delta^{2\delta n}$. We now take the union bound over all sets of variables of size $\frac{3}{2}\delta n$; the probability that there exists a set of equations of size δn which only contains $\frac{3}{2}\delta n$ variables is at most $(4ek)^{\delta n} \delta^{2\delta n} \cdot \binom{n}{1.5\delta n} \leq (4ek)^{\delta n} \delta^{2\delta n} \cdot (\frac{2e}{3\delta})^{1.5\delta n} \leq (3e^{2.5}k)^{\delta n} \delta^{0.5\delta n} = (40k\sqrt{\delta})^{\delta n} \leq (40k\sqrt{\delta_0})^{\delta n}$. Let $\delta_0 < \frac{1}{11(40k)^2}$, and taking union bound over all possible sizes i ranging from 1 to $\delta_0 n$, the probability that any set of equations of size $i \leq \delta_0 n$ contains more than $\frac{3}{2}i$ variables is at least $1 - \sum_{i=1}^{\delta_0 n} (\frac{1}{11})^i \geq 9/10$.

So with probability at least $9/10$, any set of equations with size $i \leq \delta_0 n$ contains more than $\frac{3}{2}i$ variables, which means there is at least one variable that occurs at most once in these equations by the pigeonhole principle. We prove that under this event, the distribution of $\mathcal{E}_{YES}^{\mathcal{A}}$ and $\mathcal{E}_{NO}^{\mathcal{A}}$ are identical if $G^{\mathcal{A}}$ contains at most $3\delta_0 n$ edges.

Notice that the left hand side of $\mathcal{E}_{YES}^{\mathcal{A}}$ and $\mathcal{E}_{NO}^{\mathcal{A}}$ are always identical, we only need to prove that the distributions of the right hand side are identical when $G^{\mathcal{A}}$ has at most $3\delta_0 n$ edges. In this case there are at most $\delta_0 n$ equations in $\mathcal{E}_{YES}^{\mathcal{A}}$ since each equation is associated with 3 vertices. Let the right hand sides of $\mathcal{E}_{YES}^{\mathcal{A}}$ and $\mathcal{E}_{NO}^{\mathcal{A}}$ be vectors b_{YES} and b_{NO} respectively. We prove the distributions of b_{YES} and b_{NO} are identical by induction on the size of b_{YES} (which is also the number of equations in $\mathcal{E}_{YES}^{\mathcal{A}}$).

The base case is when there is no equation in $\mathcal{E}_{YES}^{\mathcal{A}}$ at all (which means the algorithm does not discover any equation). In this case, both b_{YES} and b_{NO} are empty vectors.

In the induction step, $|b_{YES}| = |b_{NO}| > 0$. Since the number of equations is at most $\delta_0 n$, there exists a variable v that only occurs once. Without loss of generality, suppose it occurs in the last equation. Let b'_{YES} and b'_{NO} be the vector obtained by deleting the last entry of b_{YES} and b_{NO} respectively. By induction hypothesis, the distributions of b'_{YES} and b'_{NO} are identical. Moreover, v only occurs in the last equation and only occurs once, the distribution of the last entry of b_{YES} is uniform, independent of the other entries, so is the last entry of b_{NO} . So the distributions of b_{YES} and b_{NO} are identical. ■

Next we prove that in order to discover a constant fraction of equations, we need $\Omega(n^2)$ queries.

Lemma 4.3. *For any $\delta_0 > 0$, there exists a $\delta_1 > 0$ such that: for any algorithm that makes at most $\delta_1 n^2$ queries, G^A contains at most $3\delta_0 n$ edges with probability 9/10.*

The proof of Lemma 4.3 is similar to the proof of Theorem 5.2 in [4] and we will prove it later.

Proof of Theorem 17. For any $\varepsilon > 0$, let k, δ_0, δ_1 be the constant defined in Lemma 4.1, Lemma 4.2, Lemma 4.3 respectively. Consider two instance I_{YES} and I_{NO} generated as follows: we generate the instance I_{YES} by distribution \mathcal{D}_{YES} , then let the left hand side of I_{NO} be the same as the left hand side of I_{YES} , generate the right hand side of I_{NO} uniformly independently for each equation. Since the process of generating the left hand side is the same for \mathcal{D}_{YES} and \mathcal{D}_{NO} , the distribution of I_{NO} is indeed \mathcal{D}_{NO} . By Lemma 4.1, with probability 9/10, the I_{NO} satisfies at most $(1/2 + \varepsilon)$ -fraction of equations. By Lemma 4.3, if an algorithm makes at most $\delta_1 n^2$ queries, then it discovers at most $\delta_0 n$ equations with probability 9/10. Base on this event, by Lemma 4.2, the equations discovered by the algorithm has the same probability of being generated by \mathcal{D}_{YES} and by \mathcal{D}_{NO} . By the union bound, with probability at most 7/10, I_{NO} is an instance that satisfies at most $(1/2 + \varepsilon)$ -fraction of the equations and the algorithm cannot distinguish between I_{YES} and I_{NO} . ■

We use the following standard reduction from Equation to 3SAT in [16]. Given a set of equations \mathcal{E} , we construct a 3CNF formula $\Phi = F(\mathcal{E})$ as follows: For any equation $X_i + X_j + X_k = 1$ in \mathcal{E} , we add four clauses $(X_i \vee X_j \vee X_k)$, $(X_i \vee \bar{X}_j \vee \bar{X}_k)$, $(\bar{X}_i \vee X_j \vee \bar{X}_k)$ and $(\bar{X}_i \vee \bar{X}_j \vee X_k)$ into Φ ; for any equation $X_i + X_j + X_k = 0$ in \mathcal{E} , we add four clauses $(\bar{X}_i \vee X_j \vee X_k)$, $(X_i \vee \bar{X}_j \vee X_k)$, $(X_i \vee X_j \vee \bar{X}_k)$ and $(\bar{X}_i \vee \bar{X}_j \vee \bar{X}_k)$ into Φ . It is clear that if an assignment satisfies an equation in \mathcal{E} , then it also satisfies all of the four corresponding clauses in Φ . Otherwise it satisfies three of the four corresponding clauses. So we have the following lemma.

Lemma 4.4. *For any $\gamma \in (0, 1]$, given a set of equations \mathcal{E} and its corresponding 3CNF formula $\Phi = F(\mathcal{E})$, for any assignment A , A satisfies γ -fraction of equations in \mathcal{E} if and only if A satisfies $(3/4 + \gamma/4)$ -fraction of clauses in Φ .*

Proof of Theorem 15. For any 3CNF formula generated from a E3LIN2 instance \mathcal{E} we consider a stronger type of query model for 3SAT. For any query between a variable and a clause, if the variable occurs in the clause, then the algorithm is not only given the entire clause, but also the other 3 clauses corresponding to the same equation in \mathcal{E} . The new query is equivalent to the query in E3LIN2. ■

4.4 Proof of Lemma 4.3

The proof of Lemma 4.3 is similar to the proof of Theorem 5.2 in [4]. However, the argument from [4] cannot be used in a black-box manner. So, here we present a complete proof. The following lemma from [4] is useful.

Lemma 4.5 (Lemma 5.4 of [4]). *Let $G'(L' \cup R', E')$ be an arbitrary bipartite graph such that $|L'| = |R'| = N$, and each vertex in G' has degree at least $2N/3$. Then for any edge $e = (u, v) \in E'$, the probability that e is contained in a perfect matching chosen uniformly at random in G' is at most $3/N$.*

Denote the vertex sets in bipartite graph G_σ as L and R . We have $|L| = |R| = 3kn$. Suppose whenever a query finds a variable inside an equation, the algorithm is not only given the equation, but also edges incident on the vertices associated with the equation in G_σ . Then, G^A contains exactly those edges that are given to algorithm in response to the queries.

The query process can be viewed as the task of finding $\delta_0 n$ edges in G_σ by the following queries between a variable x_i and an equation e_j : query if there is at least one edge between U and V where $U \subset L$ is the set of vertices associated with x_i and $V \subset R$ is the set of vertices associated with e_j . If so, the algorithm is given all edges incident on the vertices in V . To prove the lemma, we only need to prove that finding $3\delta_0 n$ edges in G_σ in this model needs $\Omega(n^2)$ queries.

For simplicity, we consider the following query model instead: a query asks if there is an edge between a pair of vertices u and v . If so, the algorithm is given all three edges incident on the vertices associated with the same equation as v . Any original query can be simulated by $3k \cdot 3 = 9k$ new queries. So it is sufficient to prove that we need $\Omega(n^2)$ queries in the new model.

We say that an edge (u, v) in G_σ has been *discovered* if the edge is given to the algorithm. After t queries have been made by the algorithm, let $L_U(t) \subseteq L$ and $R_U(t) \subseteq R$ denote the set of undiscovered vertices in L and R respectively. Let $E(t) \subseteq L_U(t) \times R_U(t)$ denote the set of edge slots that have not yet been queried/discovered. Note that by our process for generating G_σ , the undiscovered edges correspond to a random perfect matching between $L_U(t)$ and $R_U(t)$ that is entirely supported on $E(t)$.

We will analyze the performance of any algorithm by partitioning the queries into phases. The first query by the algorithm starts the first phase, and a phase ends as soon as three edges in G_σ have been discovered. Let Z_i be a random variable that denotes the number of queries performed in phase i of the algorithm. Thus we wish to analyze $\mathbb{E} \left[\sum_{i=1}^{\delta_0 n} Z_i \right]$.

For any vertex $w \in (L_U(t) \cup R_U(t))$, we say that the *uncertainty* of w is d if there are at least d edge slots in $E(t)$ that are incident on w .

At time t , we say a vertex $w \in (L_U(t) \cup R_U(t))$ is *bad* if the uncertainty of w is less than $2.5kn$. Note that if at some time t none of the vertices in $(L_U(t) \cup R_U(t))$ are bad then in the next $nk/2$ time steps, the degree of any vertex in $L_U(t) \cup R_U(t)$ in $E(t)$ remains above $2kn$ if there is no successful query. Thus by Lemma 4.5, the probability that any query made during the first $nk/2$ queries in the phase succeeds (in discovery of a new edge in G_σ) is at most $3/(3kn) = 1/(kn)$.

We say a phase is *good* if at the start of the phase, there are no bad vertices, and the phase is *bad* otherwise.

Proposition 4.6. *The expected length of a good phase is at least $nk/4$.*

Proof. If at the start of the phase i , no vertex is bad, then for the next $nk/2$ time steps, the probability of success for any query is at most $1/(kn)$. Thus the expected number of successes (discovery of a new edge in G_σ) in the first $nk/2$ time steps in a phase is at most $1/2$. By Markov's inequality, it then follows that with probability at least $1/2$, there are no successes among the first $nk/2$ queries in a phase. Thus the expected length of the phase is $\geq nk/4$. ■

Note that if all phases were good, then it immediately follows that the expected number of queries to discover $3\delta_0 n$ edges is $\Omega(n^2)$. To complete the proof, it remains to show that most

phases are good. For ease of analysis, we will give the algorithm additional information for free and show that it still needs $\Omega(n^2)$ queries in expectation even to discover the first $3\delta_0 n$ edges in G_σ .

Whenever the algorithm starts a bad phase, we immediately reveal to the algorithm an undiscovered edge (u, v) in G_σ (as well as other two edges incident on the vertices associated with the same equation as v) that is incident on an arbitrarily chosen bad vertex. Thus each bad phase is guaranteed to consume a bad vertex (i.e., make the bad vertex discovered and hence remove it from further consideration). On the other hand, to create a bad vertex w , one of the following two events needs to occur: the number of discovered edges in G_σ plus the number of queries is at least $3kn - 2.5kn = kn/2$.

Since we are restricting ourselves to analyzing the discovery of first $\delta_0 n$ edges in G_σ , any vertex w that becomes bad requires at least $kn/2$ queries incident on it. Thus to create K bad vertices in the first $\delta_0 n$ phases, we need to perform at least $(K \cdot (kn/2))/2$ queries; here the division by 2 accounts for the fact that each query reduces uncertainty for two vertices. It now follows that if the algorithm encounters at least $\delta_0 n/2$ bad phases among the first $\delta_0 n$ phases, then $K \geq \delta_0 n/2$ and hence it must have already performed $\delta_0 kn^2/8$ queries. Otherwise, at least $\delta_0 n/2$ phases among the first $\delta_0 n$ phases are good, implying that the expected number of queries is at least $(\delta_0 n/2) \cdot (nk/4) = \Omega(n^2)$. This completes the proof of Lemma 4.3 with Markov's inequality.

5 A Reduction from Matching Size to TSP Cost Estimation

In this section, we give a reduction from the problem of estimating the maximum matching size in a bipartite graph to the problem of estimating the optimal $(1, 2)$ -TSP cost. An essentially identical reduction works for graphic TSP cost using the idea described in Section 4.2.

We will denote the size of the largest matching in a graph G by $\alpha(G)$. Given a bipartite graph $G(V, E)$ with n vertices on each side, we construct an instance $G'(V', E')$ of the $(1, 2)$ -TSP problem on $4n$ vertices such that the optimal TSP cost on G' is $5n - \alpha(G)$. Thus for any $\varepsilon \in [0, 1/5)$, any algorithm that can estimate $(1, 2)$ -TSP cost to within a $(1 + \varepsilon)$ -factor, also gives us an estimate of the matching size in G to within an additive error of $5\varepsilon n$.

We will now describe our construction of the graph G' . For clarity of exposition, we will describe G' as the graph that contains edges of cost 1 – all other edges have cost 2. Suppose the vertex set V of G consists of the bipartition $V_1 = \{v_1^1, v_2^1, \dots, v_n^1\}$ and $V_2 = \{v_1^2, v_2^2, \dots, v_n^2\}$. We construct the graph G' as follows: we start with the graph G , then add three sets of vertices V_0 , V_3 and V_4 , such that $V_0 = \{v_1^0, v_2^0, \dots, v_{n/2}^0\}$ with $n/2$ vertices, $V_3 = \{v_1^3, v_2^3, \dots, v_n^3\}$ with n vertices, and $V_4 = \{v_1^4, v_2^4, \dots, v_{n/2}^4\}$ with $n/2$ vertices. The graph G' will only have edges between V_j and V_{j+1} ($j = \{0, 1, 2, 3\}$). We will denote the set of edges between V_j and V_{j+1} as $E_{j,j+1}$. For any vertex $v_i^0 \in V_0$, it connects to v_{2i-1}^1 and v_{2i}^1 in V_1 . $E_{1,2}$ has the same edges as the edges in G . Each vertex $v_i^2 \in V_2$ is connected to vertex v_i^3 in V_3 , that is, vertices in V_2 and V_3 induce a perfect matching (identity matching). Finally, each vertex in V_3 is connected to all the vertices in V_4 . See Figure 1(a) for an illustration.

The lemmas below establish a relationship between matching size in G and $(1, 2)$ -TSP cost in G' .

Lemma 5.1. *Let M be any matching in G . Then there is a $(1, 2)$ -TSP tour T in G' of cost at most $5n - |M|$.*

Proof. Let $f : [n] \rightarrow [n]$ be any bijection from $[n]$ to $[n]$ such that whenever a vertex v_i^1 is matched to a vertex v_j^2 in M , then $f(i) = j$. Consider the following $(1, 2)$ -TSP tour T : each vertex $v_i^0 \in V_0$ connects to v_{2i-1}^1 and v_{2i}^1 in T ; each vertex $v_i^1 \in V_1$ connects to $v_{\lceil (i+1)/2 \rceil}^0$ and $v_{f(i)}^2$ in T . For any

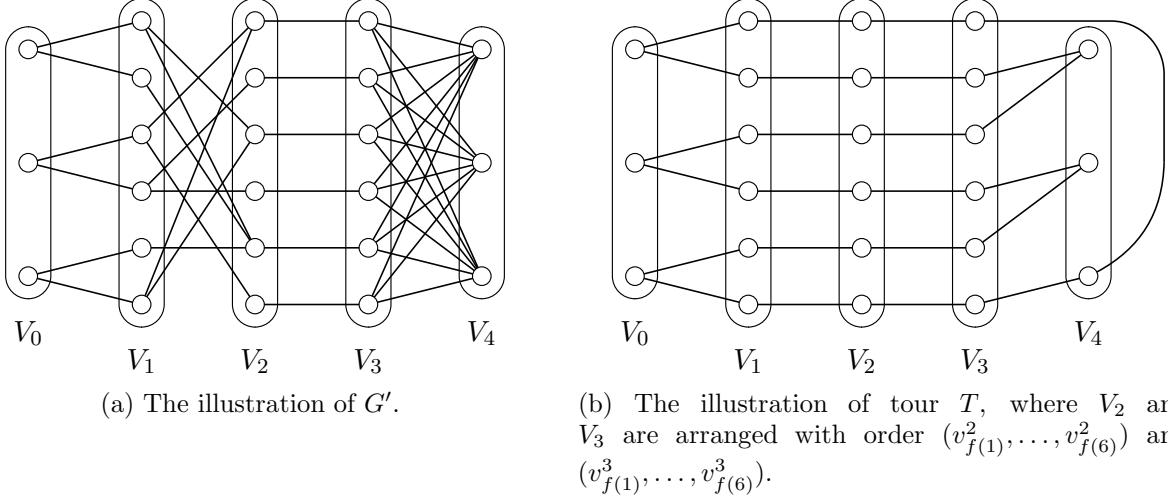


Figure 1: An illustration of the reduction for $n = 6$.

$v_{f(i)}^2 \in V_2$, it connects to v_i^1 and $v_{f(i)}^3$ in T . For any vertex $v_{f(i)}^3 \in V_3$, if $i > 1$, it connects to $v_{f(i)}^2$ and $v_{\lceil i/2 \rceil}^4$ in T ; if $i = 1$, it connects to $v_{f(i)}^2$ and $v_{n/2}^4$ in T . See Figure 1(b) as an illustration. T is clearly a TSP-tour.

All edges in T are also edges in G' except for possibly some edges between V_1 and V_2 . If v_i^1 is matched in M , then $(v_i^1, v_{f(i)}^2)$ is an edge in G' , otherwise it is not in G' and thus has weight 2. So T only has $n - |M|$ weight 2 edges, which means T has cost at most $4n + n - |M| = 5n - |M|$. ■

Lemma 5.2. *For any $(1, 2)$ -TSP tour T in G , T has cost at least $5n - \alpha(G)$.*

To prove Lemma 5.2, we first prove an auxiliary claim.

Claim 5.3. *Suppose $G = (V_1, V_2, E)$ is a bipartite graph which has maximum size $\alpha(G)$. For any 2-degree subgraph H of G , if there are at most X vertices in V_1 has degree 2 in H , then there are at most $\alpha(G) + X$ vertices in V_2 which have degree at least 1 in H . Similarly, if there are at most X vertices in V_2 has degree 2 in H , then there are at most $\alpha(G) + X$ vertices in V_1 which have degree at least 1 in H .*

Proof. If there are at most X vertices in V_1 has degree 2 in H . We construct H' by deleting an arbitrary edge on each degree 2 vertex in V_1 , then construct H'' by deleting an arbitrary edge on each degree 2 vertex in V_2 . Since H'' does not have degree two vertex, it is a matching of G . So the number of degree 1 vertices in V_2 in H'' is at most $\alpha(G)$. On the other hand, any vertex in V_2 which has degree at least 1 in H' also has degree 1 in H'' . So there are at most $\alpha(G)$ vertices of degree at least 1 in V_2 in H' . Furthermore, since there are only X vertices of degree 2 in V_1 in H , we delete at most X edges in H when constructing H' . So H' has at most X more isolate vertices in V_2 than in H , which means H has at most $\alpha(G) + X$ vertices with degree at least 1 in V_2 .

The second part of the claim follows via a similar argument as the first part of the claim. ■

Proof of Lemma 5.2. Let a_{01} be the number of edges in $T \cap E_{0,1}$, $a_{2,3}$ be the number of edges in $T \cap E_{3,4}$. Let G^X be the intersection graph of G and T . Since the vertices in V_0 only connect to the vertices in V_1 in G' , and any vertex in T has degree 2, there are at least $n - a_{01}$ edges incident on V_0 in T are not an edge in G' . On the other hand, since any vertex in V_1 is incident on at most 1 edge in $E_{0,1}$, there are at least a_{01} vertices in V_1 is connected to a vertex in V_0 in T , which

means there are at most $n - a_{01}$ vertices in V_1 has degree 2 in G^X . By Claim 5.3, there are at most $n - a_{01} + \alpha(G)$ vertices in V_2 has edge in G^X . For any isolate vertex in V_2 in T , it has only one edge in G' connecting to V_3 , so this vertex must incident on an edge in T which is not in G' . So there are at least $n - (n - a_{01} + \alpha(G)) = \alpha(G) - a_{01}$ edges incident on V_2 in T which is not in G' .

There are $2n$ edges incident on V_3 in T , but among them, there are only a_{23} edges between V_2 and V_3 which is also in G' , and there are at most n edges between V_3 and V_4 in T since each vertex has degree only 2. So there are at least $2n - n - a_{23} = n - a_{23}$ edges incident on V_3 which is not in G' . On the other hand, since any vertex in V_2 is incident on at most 1 edge in $E_{2,3}$, there are at least a_{23} vertices in V_2 is connected to a vertex in V_3 in T , which means there are at most $n - a_{23}$ vertices in V_2 has degree 2 in G^X . By Claim 5.3, there are at most $n - a_{23} + \alpha(G)$ vertices in V_1 has edge in G^X . For any isolate vertex in V_1 in T , it has only one edge in G' connecting to V_0 , so this vertex must incident on an edge in T which is not in G' . So there are at least $n - (n - a_{23} + \alpha(G)) = a_{23} - \alpha(G)$ edges incident on V_1 in T which is not in G' .

Since any edge has two endpoints, the number of edges in T but not in G' is at least $((n - a_{01}) + (a_{01} - \alpha(G)) + (n - a_{23}) + (a_{23} - \alpha(G))) / 2 = n - \alpha(G)$, which means T has cost at least $4n + n - \alpha(G) = 5n - \alpha(G)$. ■

Corollary 18. *For any $\varepsilon \in [0, 1/5]$, any algorithm that can estimate $(1, 2)$ -TSP cost to within a $(1 + \varepsilon)$ -factor, can be used to estimate the size of a largest matching in a bipartite graph G on $2n$ vertices to within an additive error of $5\varepsilon n$.*

Proof. We use the reduction above to construct a $(1, 2)$ -TSP instance G' on $4n$ vertices. By Lemmas 5.1 and 5.2, the optimal TSP cost for G' is $5n - \alpha(G)$. We now run the $(1 + \varepsilon)$ -approximation algorithm for $(1, 2)$ -TSP on graph G' (note that the reduction can be simulated in each of neighbor query model, pair query model, and the streaming model without altering the asymptotic number of queries used). Suppose the output is C which satisfies $(1 - \varepsilon)(5n - \alpha(G)) \leq C \leq (1 + \varepsilon)(5n - \alpha(G))$, which means $5n - \alpha(G) - 5\varepsilon n < C < 5n - \alpha(G) + 5\varepsilon n$. Let $\hat{\alpha} = 5n - C$, we have $\alpha(G) - 5\varepsilon n < \hat{\alpha} < \alpha(G) + 5\varepsilon n$. ■

6 Additional Lower Bound Results for Approximating Graphic and $(1, 2)$ -TSP Cost

In this section, we prove several additional lower bounds on approximating the costs of graphic TSP and $(1, 2)$ -TSP. Many of these results involve constructing a simple distribution on graphs where some graphs in the support of the distribution have TSP tours of cost close to n while others have cost close to $2n$. We show that no deterministic algorithm can distinguish between these two types of instances, and then invoke Yao's principle [35] to prove lower bounds for randomized algorithms. When the graphs in the distribution have diameter 2, the graphic TSP instances are also instances of the $(1, 2)$ TSP problem. Using this approach we show an $\Omega(n)$ lower bound for both metric TSP and $(1, 2)$ -TSP costs in our query model.

In the standard graph query model allowing both pair queries and neighbor queries, we show a stronger lower bound of $\Omega(\varepsilon^2 n^2)$ for randomized algorithms that estimate the cost of graphic TSP to within a factor of $(2 - \varepsilon)$. This shows that the distance query model is strictly more powerful for estimating graphic TSP cost.

Using Dirac's theorem about the existence of Hamilton Cycles in very dense graphs, we show an $\Omega(n^2)$ lower bound for deterministic algorithms to get any approximation better than 2. For the problem of finding a $(2 - \varepsilon)$ -approximate tour, rather than just estimating its cost, we show an $\Omega(\varepsilon n^2)$ lower bound for both graphic TSP and $(1, 2)$ -TSP.

Finally, we show a space lower bound of $\Omega(\varepsilon n)$ for approximating Graphic TSP to within $2 - \varepsilon$ in the streaming model.

6.1 An $\Omega(n)$ Query Lower Bound for $(2 - \varepsilon)$ -approximating $(1, 2)$ -TSP and Graphic TSP Cost

In this subsection, we show that in our query model, any randomized algorithm that approximates the cost of minimum graphic TSP or $(1, 2)$ -TSP to within a factor of $2 - \varepsilon$ for any ε , we need $\Omega(n)$ queries. As stated above, it suffices to create a distribution over $(n + 1)$ -vertex graphs such that any deterministic algorithm requires $\Omega(n)$ queries to check if the cost of minimum $(1, 2)$ -TSP or graphic TSP is $n + 1$ or $2n$ on this distribution.

The distribution is generated as follows: we start with a “star” graph whose vertices set is $\{v_0, v_1, v_2, \dots, v_n\}$ where v_0 is connected to all other vertices. Then we pick a random permutation π over $[n]$. With probability half, we connect $v_{\pi(i)}$ and $v_{\pi(i+1)}$, for $1 \leq i \leq n - 1$. In this case, the resulting graph is the wheel graph. With probability half we do not join successive vertices in π , and the resulting graph is a star graph. Since v_0 is connected to all other vertices, any two vertices have distance 1 or 2. So graphic TSP and $(1, 2)$ -TSP are the same in this distribution.

Lemma 6.1. *A wheel graph admits a TSP tour of cost $n + 1$ while any TSP tour in a star graph has cost at least $2n$.*

Proof. In a wheel graph, the tour $(v_0, v_{\pi(1)}, v_{\pi(2)}, \dots, v_{\pi(n)}, v_0)$ has cost $n + 1$ since all edges are weight 1. For any tour in a star graph, only the edges incident on v_0 have weight 1. So the cost of the tour is at least $2 + 2(n - 1) = 2n$. ■

Lemma 6.2. *If an algorithm only makes $n/3$ queries, then with probability at least $1/3$, the answer to all these queries is the same in a wheel graph and a star graph.*

Proof. For any query (v_i, v_j) , if one of v_i or v_j is v_0 , then the answer is 1 in both cases. If none of v_i or v_j is v_0 , then the answer is 2 if the graph is the star graph. If the graph is a wheel graph, then the answer is 1 only if i and j are adjacent to each other in π , which has probability at most $2/n$. By union bound, with probability at least $1/3$, the answers of all these queries are the same in both cases. ■

By Lemma 6.1 and Lemma 6.2, we have the following lower bound for graphic TSP and $(1, 2)$ -TSP problem in the distance query model.

Theorem 19. *For any $\varepsilon > 0$, in the distance query model, any algorithm that with probability at least $2/3$ approximates the cost of $(1, 2)$ -TSP or graphic TSP within a factor of $(2 - \varepsilon)$ requires $\Omega(n)$ queries.*

6.2 An $\Omega(\varepsilon^2 n^2)$ Query Lower Bound for $(2 - 2\varepsilon)$ -approximating Graphic TSP in Standard Graph Query Model

In this subsection, if an algorithm for graphic TSP is given only access to the underlying graph G via standard graph queries, namely, pair queries, degree queries and neighbor queries, then any randomized algorithm for approximating graphic TSP cost to within a factor of $2 - \varepsilon$ for any $\varepsilon > 0$, requires $\Omega(\varepsilon^2 n^2)$ queries. Again by Yao’s principle, it suffices to create a distribution over n -vertex graphs such that any deterministic algorithm requires $\Omega(\varepsilon^2 n^2)$ queries to distinguish between graphs where the cost of graphic TSP is n and graphs where the graphic TSP cost is at least $(2 - 2\varepsilon)n - 1$.

We start with a graph G with three parts: a path P with $(1 - 2\epsilon)n$ vertices, and two cliques C_1 and C_2 of size ϵn . Let $u_1, u_2, \dots, u_{\epsilon n}$ be the vertices in C_1 , and $v_1, v_2, \dots, v_{\epsilon n}$ be the vertices in C_2 . Connect all vertices in C_1 to an endpoint of P , and connect all vertices in C_2 to the other endpoint of P . For any vertex u_i (resp. v_i), we say the j^{th} neighbor of u_i (resp. v_i) is u_j (resp. v_j) for any $j \neq i$, and the i^{th} neighbor is the endpoint of P . For any vertex in P , we pick an arbitrary order of its neighbors. With probability half, we change the graph to create a YES case as follows: we pick two different indices i and j from $[\epsilon n]$ randomly. We change the j^{th} neighbor of u_i and v_i to be v_j and u_j respectively and the i^{th} neighbor of u_j and v_j to be v_i and u_i respectively. Otherwise, we do not change the graph and say we are in NO case.

Lemma 6.3. *If we are in the YES case, then the cost of graphic TSP is n . Otherwise, the cost of the graphic TSP is at least $(2 - 2\epsilon)n - 1$.*

Proof. If we are in the YES case, consider the tour that starts at u_i , and goes through the vertices in C_1 in arbitrary order (but not visiting u_j immediately after u_i), then goes to the endpoint of P that connects to all vertices in C_1 , goes through the path P , then visits the vertices in C_2 in arbitrary order ending with v_j (but not visiting v_i right before v_j), and finally goes back to u_i . All edges in this tour have weight 1. So the cost of this tour is n .

If we are in the NO case, then all the edges in the path are bridges. So by Lemma 2.3, the cost of graphic TSP is at least $n + (1 - 2\epsilon)n - 1 = (2 - 2\epsilon)n - 1$. ■

Lemma 6.4. *If an algorithm only makes $\epsilon^2 n^2 / 4$ queries, then with probability at least $1/3$, the answer to these queries are the same in the YES and NO cases.*

Proof. The degree of the vertices are the same in both cases. So, any neighbor query has the same answer. For any pair query or neighbor query, all queries on the vertices in P also have the same answer. For any query on the vertices in the cliques, we say a query is querying a pair of indices (k, ℓ) if it is a pair query between two vertices with indices k and ℓ , or it is a neighbor query that queries the k^{th} (resp. ℓ^{th}) neighbor of u_ℓ or v_ℓ (resp. u_k or v_k). A pair query or a neighbor query has different answers in YES and NO cases only when it is querying the indices i and j that we picked when generating the YES case. Since i and j are chosen randomly, the probability that a query is querying i and j is $\frac{2}{\epsilon n(\epsilon n - 1)}$. If the algorithm only make $\epsilon^2 n^2 / 4$ queries, the probability that there is a query with different answers in YES case and NO case is at most $\frac{\epsilon^2 n^2}{2\epsilon n(\epsilon n - 1)} < 2/3$ by union bound. ■

By Lemma 6.3 and Lemma 6.4, we have the following lower bound for graphic TSP problem in the standard query model.

Theorem 20. *For any $\epsilon > 0$, if an algorithm approximates the cost of graphic TSP within a factor of $(2 - \epsilon)$ with probability at least $2/3$ using only degree queries, neighbor queries and pair queries, then it requires $\Omega(\epsilon^2 n^2)$ queries.*

6.3 An $\Omega(n^2)$ Query Lower Bound for Deterministic Algorithms for $(1, 2)$ -TSP and Graphic TSP

In this subsection, we prove that in our stronger, distance query model, any deterministic algorithm that approximates cost of graphic TSP or $(1, 2)$ -TSP within a factor of $(2 - \epsilon)$ needs $\Omega(\epsilon n^2)$ queries.

We first consider the $(1, 2)$ -TSP problem. We prove that for any $\epsilon n^2 / 5$ queries, even if all the answers are that the distance is 2, the graph could still have a TSP of cost $n + \epsilon n$.

Consider the graph H whose edge set is pairs of vertices that have not been queried. Since there are only $\varepsilon n^2/5$ queries, there are at least $(1 - \varepsilon)n$ vertices that have been queried at most $2n/5 < (1 - \varepsilon)n/2 - 1$ times. These vertices has degree at least $(n - 1) - ((1 - \varepsilon)n/2 - 1) = (1 + \varepsilon)n/2$ in H . Let V_0 be an arbitrary set that contains exactly $(1 - \varepsilon)n$ of these vertices. The subgraph of H induced by V_0 has minimum degree at least $(1 + \varepsilon)n/2 - \varepsilon n = (1 - \varepsilon)n/2 = |V_0|/2$. By the following well-known theorem due to Dirac about the existence of Hamilton cycles in dense graphs, there is a Hamilton cycle in the subgraph of H induced by V_0 .

Lemma 6.5 (Dirac [12]). *Any n -vertex graph G where each vertex has degree at least $n/2$ has a Hamilton cycle.*

So G has a path of length $(1 - \varepsilon)n$ that only contains weight one edges, any TSP tour obtained by expanding this path has length at most $(1 - \varepsilon)n + 2\varepsilon n = (1 + \varepsilon)n$. Thus it is possible that G contains a tour of cost $(1 + \varepsilon)n$ after $\varepsilon n^2/5$ queries.

For graphic TSP problem, we use the same trick as in Section 4, adding a vertex that connects to all other vertices. This results in the same lower bound for graphic TSP as for $(1, 2)$ -TSP.

Theorem 21. *Any deterministic algorithm that approximates the cost of graphic TSP or $(1, 2)$ -TSP to within a factor of $(2 - \varepsilon)$ using distance queries needs $\Omega(\varepsilon n^2)$ queries.*

6.4 An $\Omega(\varepsilon n^2)$ Lower Bound for Finding a $(2 - \varepsilon)$ -Approximate $(1, 2)$ -TSP or Graphic TSP Tour

While our focus in this paper has been on estimating the cost of $(1, 2)$ -TSP or graphic TSP to within a factor that is strictly better than 2, we show here that if the goal were to output an approximate $(1, 2)$ -TSP tour or graphic TSP tour (not just an estimate of its cost), then even with randomization, any algorithm requires $\Omega(\varepsilon n^2)$ distance queries to output a $(2 - \varepsilon)$ -approximate solution for any $\varepsilon > 0$. We start by showing this lower bound for $(1, 2)$ -TSP.

We create a distribution over n -vertex graphs with $(1, 2)$ -TSP cost $(1 + o(1))n$ such that with a large constant probability, any deterministic algorithm requires $\Omega(\varepsilon n^2)$ queries to output a tour that contains at least $3\varepsilon n$ weight-1 edges .

We generate the graph G with n vertices $\{v_1, v_2, \dots, v_n\}$ as follows: we first generate a random permutation $\pi : [n] \rightarrow [n]$. For any $i \neq j$, if $\pi(i) = j$, then v_i and v_j are connected in G .

By construction of G , it consists of vertex disjoint cycles, and each cycle in G corresponds to a cycle in permutation π . Since the expected number of cycles in a random permutation is equal to the n^{th} harmonic number, which is $O(\log n)$ [15], G has a cycle cover with $O(\log n)$ cycles in expectation. By Markov's inequality, the number of cycles in G is $o(n)$ with probability $1 - o(1)$. If we break these cycles into paths and link them in arbitrary order, we obtain a tour of cost at most $n + o(n)$. So the cost of $(1, 2)$ -TSP of G is $(1 + o(1))n$ with probability $1 - o(1)$.

Next, we prove that any algorithm needs $\Omega(n)$ queries to find εn edges. Construct a graph H that only contains a perfect matching such that the i^{th} vertex on the left is matched to the j^{th} vertex on the right if and only if $\pi(i) = j$.

Consider the problem of finding the edges in H by pair queries. Each pair query in G can be simulated by at most 2 pair queries in H . Furthermore, any tour in G corresponding to a perfect matching between the vertices in H . So to prove the lower bound in $(1, 2)$ -TSP, we only need to prove that any algorithm that output a perfect matching between the vertices in H contains at most $3\varepsilon n$ edges in H .

The following lemma follows from the arguments in [4] (also similar to the arguments in Appendix 4.4) about the lower bound for finding edges in a random perfect matching.

Lemma 6.6 (Section 5.2 in [4]). *Any algorithm needs $\Omega(\varepsilon n^2)$ queries to find εn edges in a random perfect matching with sufficiently large constant probability.*

Finally, we prove that if an algorithm only finds εn edges in H , then any output matching contains $\varepsilon n + o(1)$ edges in H with large constant probability. Suppose the algorithm only makes $\varepsilon(1 - \varepsilon)(1 - 2\varepsilon)n^2/3$ queries, then there are at most $\varepsilon(1 - \varepsilon)n$ vertices on the left (resp. right) being queried at least $(1 - \varepsilon)n/4$ times. Let V_0 be the set of vertices v in H such that the edge incident on v is not found by the algorithm and both v and its neighbor are not queried $(1 - 2\varepsilon)n/3$ times. V_0 contains at least $(1 - \varepsilon)^2 n > (1 - 2\varepsilon)n$ vertices. Let H_0 be the subgraph of H induced by V_0 . In H_0 , each vertex v has $\frac{3}{4}|V_0|$ vertices u on the other side such that the algorithm does not query the pair (u, v) .

By Lemma 4.5, each pair of vertices in V_0 contains an edge with probability $O(\frac{1}{n})$. So for any perfect matching between the vertices in H , any edge incident on a vertex in V_0 is also in H with probability only $O(\frac{1}{n})$. So there are $o(n)$ edges incident on the vertices in V_0 that are also edges in H with probability $1 - o(1)$ by Markov's inequality. So the perfect matching contains at most $2\varepsilon n + o(n) < 3\varepsilon n$ edges in H with probability $1 - o(1)$, which implies the same lower bound for (1, 2)-TSP.

For graphic TSP problem, we use the same trick as in Section 4 of adding a vertex that connects to all vertices to prove the same lower bound as for (1, 2)-TSP.

Theorem 22. *Any algorithm that outputs a graphic TSP or (1, 2)-TSP tour within a factor of $(2 - \varepsilon)$ using distance query with sufficiently large constant probability needs $\Omega(\varepsilon n^2)$ queries.*

6.5 An $\Omega(\varepsilon n)$ Lower Bound for $(2 - \varepsilon)$ Approximation of Graphic TSP in the Streaming Model

In this subsection, we prove that any single-pass streaming algorithm that approximates the cost of graphic TSP in insertion-only streams to within a factor of $(2 - \varepsilon)$ with probability at least $2/3$ requires $\Omega(\varepsilon n)$ space.

To prove the lower bound for single-pass streaming algorithm, it is sufficient to prove the lower bound in the one-way communication model. The graphic TSP problem in the communication model is the two-player communication problem in which the edge set E of a graph $G(V, E)$ is partitioned between Alice and Bob, and their goal is to approximate the cost of the graphic TSP of G .

We prove the lower bound by a reduction from the **Index** problem. In **Index**, Alice is given a bit-string $x \in \{0, 1\}^N$, Bob is given an index $k^* \in [N]$, and the goal is for Alice to send a message to Bob so that Bob outputs x_{k^*} . It is well-known that any one-way communication protocol that solves **Index** with probability $2/3$ requires $\Omega(N)$ bits of communication [19].

We use the **Index** problem with size $N = \varepsilon n/4$. We will construct a graph G such that the cost of graphic TSP is at most $n + 2N$ if $x_{k^*} = 1$ and at least $2n - N - 1$ if $x_{k^*} = 0$. Since $N = \varepsilon n/4$, in order to approximate the cost of graphic TSP within a factor of $(2 - \varepsilon)$, Alice and Bob need to be able to check if the cost of graphic TSP is larger than $2n - 2N$ or less than $n + 2N$.

Reduction:

Given an instance of **Index** with size $N = \varepsilon n/4$:

1. Alice and Bob construct the following graph $G(V, E)$ with no communication: The vertex set V is a union of three sets P, U, W , where $P = \{v_1, v_2, \dots, v_{n-2N}\}$, $U = \{u_1, u_2, \dots, u_N\}$ and $W = \{w_1, w_2, \dots, w_N\}$. In Alice's graph, all vertices in P form a path, whose endpoints are

v_1 and v_{n-2N} , for any $i \in [N]$, there is an edge between u_i and w_i . Furthermore, v_1 connects to all u_i such that $x_i = 1$ in her input in the **Index** instance. In Bob's graph, v_1 connects to all w_i for $i \neq k^*$ and w_{k^*} connects to v_{n-2N} instead.

2. Alice and Bob then approximate the cost of graphic TSP of the graph G using the best protocol. Bob outputs $x_{k^*} = 0$ if the cost of graphic TSP is larger than $2n - 2N$ and outputs $x_{k^*} = 1$ otherwise.

The communication cost of this protocol is at most as large as the communication complexity of the protocol used to solve graphic TSP. Now we prove the correctness of the reduction.

Lemma 6.7. *If $x_{k^*} = 1$, then the cost of graphic TSP of G is at most $n + 2N$. If $x_{k^*} = 0$, then the cost of graphic TSP of G is at least $2n - N - 1$.*

Proof. If $x_{k^*} = 1$, consider the tour that first visits the path in P from v_1 to v_{2n-2N} , then visits w_{k^*}, u_{k^*} , then visits w_i, u_i for each $i \neq k^*$ in arbitrary order, and finally goes back to v_1 . Since $x_{k^*} = 1$, u_{k^*} connects to v_1 . Also for each $i \neq k^*$, w_{k^*} connects to v_1 , so for any i and $j \neq k^*$, u_i and w_j have distance at most 3. Since any other edge in the tour has weight 1, the cost of the tour is at most $n + (3 - 1) \cdot N = n + 2N$.

If $x_{k^*} = 0$, both u_{k^*} and w_{k^*} do not connect to v_1 . So u_{k^*}, w_{k^*} and v_{n-2N} form a block in G . For any $i \neq k^*$, both u_i and w_i do not connect to v_{n-2N} . So u_i, w_i and v_1 forms a block in G . Furthermore, all edges in the path are bridges in G . By Lemma 2.3, the cost of graphic TSP of G is at least $n + N + (n - 2N) - 1 = 2n - N - 1$. ■

Theorem 23. *For any $\varepsilon > 0$, any single-pass streaming algorithm that is able to approximate the cost of graphic TSP of an input graph G within a factor of $2 - \varepsilon$ in insertion-only streams with probability at least $2/3$ requires $\Omega(\varepsilon n)$ space.*

Proof. Let Π be any 1/3-error one-way protocol that approximates graphic TSP within a factor of $(2 - \varepsilon)$. By Lemma 6.7, we obtain a protocol for **Index** that errs with probability at most 1/3 and has communication cost at most equal to cost of Π . By the $\Omega(N)$ lower bound on the one-way communication complexity of **Index**, we obtain that communication cost of Π must be $\Omega(N) = \Omega(\varepsilon n)$. The theorem now follows from this argument, as one-way communication complexity lower bounds the space complexity of single pass streaming algorithms. ■

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A Approximating Maximal Matching and Maximal Matching Pair with $\tilde{O}(n^{1.5})$ Pair Queries

In this section, we give algorithms that approximates the size of a maximal matching and a maximal matching pair within a factor of $(1 + \varepsilon)$ with $\tilde{O}(n^{1.5}/\varepsilon^2)$ pair queries. Both algorithms are built on the algorithm in [36] that approximates the size of a maximal independent set.

We first describe the algorithm and result in [36]. Given a graph G with n vertices and m edges, consider the following process that generates a maximal independent set: pick a random permutation π on all vertices. Maintain a set S , initially empty. Consider each vertex in turn, from the lowest rank to the highest rank. For any vertex v , if S contains no neighbor of v , then add v to S . The algorithm $IO^\pi(v)$ (Algorithm 1) checks if a vertex v is inside the maximal independent set generated by π .

ALGORITHM 1: $IO_G^\pi(v)$: Check if a vertex v is in the maximal independent set [36]

```

if  $IO_G^\pi(v)$  has already been computed then
| return the computed answer.
end
Let  $v_1, v_2, \dots, v_t$  be the neighbors of  $v$ , in order of increasing rank.;
 $i \leftarrow 1$ ;
while  $\pi(v_i) < \pi(v)$  do
| if  $IO^\pi(v_i) = \text{TRUE}$  then
| | return FALSE
| end
|  $i \leftarrow i + 1$ .
end
return TRUE

```

Let $T_G(\pi, v)$ be the number of calls to IO_G^π when calling $IO_G^\pi(v)$. The following lemma gives an upper bound on the expected value of $T_G(\pi, v)$ when v and π are chosen randomly.

Lemma A.1 (Theorem 2.1 in [36]). *For any graph G with n vertices and m edges,*

$$\mathbb{E}_{v,\pi}[T_G(\pi, v)] \leq 1 + \frac{m}{n}$$

A.1 Approximating Maximal Matching (Proof of Theorem 8)

Given a graph $G = (V, E)$, let $L(G)$ be the line graph of G , where the vertices in $L(G)$ are the edges in G , and two vertices in $L(G)$ are neighbors if they share a common endpoint in G . Suppose d_{\max}^G is the maximum degree in G , then any vertex in $L(G)$ has degree at most $2d_{\max}^G - 2$. Furthermore, any maximal independent set in $L(G)$ is a maximal matching in G . To approximate a maximal matching in G , it is sufficient to approximate a maximal independent set in $L(G)$.

By Lemma A.1, $\mathbb{E}_{e,\pi}[T_{L(G)}(\pi, e)] = O(d_{\max}^G)$, if an edge e in G (also a vertex in $L(G)$) and a permutation π on the edges in G are chosen randomly. However, d_{\max}^G can be as large as n and the number of edges in G can be as large as n^2 . Thus, to approximate the maximal matching within an additive error εn , we need to sample $\Omega(n)$ edges in G and check if each one is in the maximal matching by calling $IO_{L(G)}^\pi(e)$. So the total number of calls to $IO_{L(G)}^\pi$ can be as large as n^2 .

To reduce the number of calls to IO^π , we design a two-phase algorithm that approximates the size of a maximal matching. In the first phase, we match high degree vertices greedily. In the second phase, we use the process described earlier to approximate the size of a maximal matching in the remaining low-degree graph after the first phase.

In the first phase, we run Algorithm 2, which returns a partial matching M and a vertex set S that contains all vertices not matched in M . The algorithm works as follows: at first, M is empty and S is the vertex set V . We consider all vertices one by one in arbitrary order. When considering v , if v is not matched in M , then we sample $c_0 = 100\sqrt{n} \log n$ vertices from S , and check if some neighbor of v is among them. If so, we add v and one of its neighbors into M and delete these two vertices from S . The algorithm only uses $\tilde{O}(n^{1.5})$ pair queries since for each vertex we only check if it is a neighbor of $\tilde{O}(\sqrt{n})$ vertices. We prove that with high probability, the subgraph of G induced by S has degree at most \sqrt{n} .

ALGORITHM 2: Match high degree vertices

```

 $S \leftarrow V, M \leftarrow \emptyset, c_0 \leftarrow 100\sqrt{n} \log n.;$ 
for  $v \in V$  do
    if  $v \notin S$  then
        continue;
    end
    Sample  $c_0$  vertices  $u_1, u_2, \dots, u_{c_0}$  from  $S$ .;
    for  $i \leftarrow 1 \dots c_0$  do
        if  $(v, u_i)$  is an edge in  $G$  then
             $M \leftarrow M \cup \{(v, u_i)\}, S \leftarrow S \setminus \{v, u_i\}$  ;
            break;
        end
    end
end
Output  $S$  and  $M$ .

```

Lemma A.2. *The subgraph of G induced by S has degree at most \sqrt{n} with probability at least $1 - o(\frac{1}{n})$.*

Proof. v is unmatched in M only if when running the loop on v , none of the $c_0 = 100\sqrt{n} \log n$ vertices sampled are neighbors of v . The probability that this happens is at most $o(\frac{1}{n^2})$ by Chernoff bound. Taking the union bound on all vertex v , with probability at least $1 - o(\frac{1}{n})$, all vertices in S have at most \sqrt{n} neighbors in S . ■

After the first phase, we only need to approximate the size of a maximal matching in the subgraph G_S of G induced by S , which has degree at most \sqrt{n} . There are at most $n\sqrt{n}$ edges in G_S . So in order to approximate the size of a maximal matching we only need to sample $O(\sqrt{n})$ edges and run $IO_{L(G_S)}^\pi$ on them. The expected total number of calls to $IO_{L(G_S)}^\pi$ is $O(\sqrt{n} \cdot \sqrt{n}) = O(n)$ by Lemma A.1 since the maximum degree in G_S is \sqrt{n} . The following lemma shows that we can simulate this process using $\tilde{O}(n^{1.5})$ pair queries.

Lemma A.3. *There is an algorithm that approximates the size of a maximal independent set in $L(G_S)$ within an additive error εn that uses $\tilde{O}(n^{1.5}/\varepsilon^2)$ pair queries with probability at least $2/3$.*

We first prove an auxiliary claim.

Claim A.4. *Suppose there are two sets of objects S_1 and S_2 such that $\frac{|S_1|}{|S_2|} < \sqrt{n}$ and $|S_1| + |S_2| < n^2$. If we pick a rank permutation π on all objects in $S_1 \cup S_2$, then with probability $1 - o(\frac{1}{n^2})$, any set of $10\sqrt{n} \log n$ successive objects in π contains an object in S_2 .*

Proof. Let $m_0 = |S_1| + |S_2|$ and $m_1 = |S_1|$. For any set of $10\sqrt{n} \log n$ indices, the probability that there is no object in S_2 on these indices in π is

$$\frac{m_1}{m_0} \cdot \frac{m_1 - 1}{m_0 - 1} \cdots \frac{m_1 - 10\sqrt{n} \log n + 1}{m_0 - 10\sqrt{n} \log n + 1} < \left(\frac{m_1}{m_0}\right)^{10\sqrt{n} \log n} < \left(1 - \frac{1}{2\sqrt{n}}\right)^{10\sqrt{n} \log n} = o\left(\frac{1}{n^4}\right)$$

By taking the union bound on all possible successive indices, the probability that any successive $10\sqrt{n} \log n$ objects in rank π contains an object in S_2 is at least $1 - o(\frac{1}{n^2})$. ■

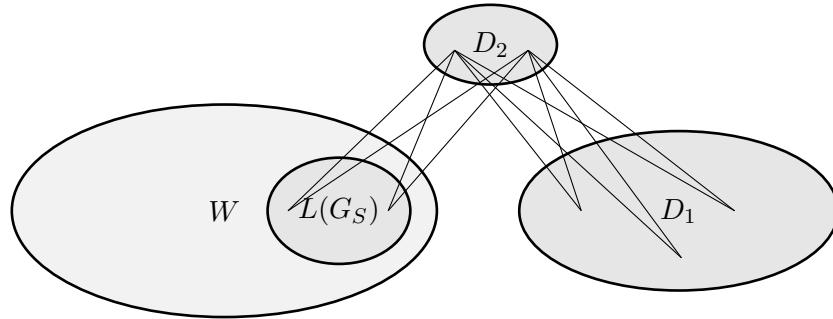


Figure 2: Illustration of H

Proof of Lemma A.3. Suppose E_S is the set of edges of G_S . Let W be the set of pairs of vertices in G . By definition, $|W| = n(n-1)/2$. We construct a graph H as follows: the vertex set of H is $W \cup D_1 \cup D_2$ where $|D_1| = n\sqrt{n}$ and $|D_2| = \sqrt{n}$. Each vertex in D_2 is connected to each vertex in $D_1 \cup E_S$. For any two vertices in E_S , they are connected if and only if they have a common endpoint in G_S . All vertices in $W \setminus E_S$ are isolated vertices. See Figure 2 as an illustration.

Let π be a random permutation on the vertices in H . If the lowest rank among $E_S \cup D_1 \cup D_2$ is a vertex in $E_S \cup D_1$, which has probability at least $\frac{|D_1|}{|D_1|+|D_2|} > 1 - \frac{1}{n}$, no vertex in D_2 is inside the maximal independent set of H generated by π . In this case, the size of maximal independent set is the size of a maximal independent set of $L(G_S)$ plus $|D_1| = n\sqrt{n}$ plus the size of $W \setminus E_S$. Thus, to prove the lemma, it is sufficient to approximate the size of E_S and the size of a maximal independent set of H both with additive error $\varepsilon n/2$.

To approximate the size of E_S , we sample $M_0 = \frac{100n\sqrt{n}}{\varepsilon^2}$ pairs of vertices in G and query if there is an edge between them. Suppose \bar{X}_0 sampled pairs are edges in G_S , then by Chernoff bound, $\left| \frac{\bar{X}_0 \cdot |W|}{M_0} - |E_S| \right| < \varepsilon n/2$ with probability at least $1 - \frac{1}{2\sqrt{n}} = 1 - o(\frac{1}{n})$ since $|W| < n^2$. Thus, we can approximate the size of E_S with additive error $\varepsilon n/2$ by making $O(n\sqrt{n}/\varepsilon^2)$ pair queries with probability $1 - o(\frac{1}{n})$.

Let V_H be the set of vertices in H . By definition, $|V_H| = n(n-1)/2 + n\sqrt{n} + \sqrt{n}$. We use Algorithm 1 to approximate the size of the maximal independent set of H generated by π . We sample $M_1 = \frac{100n}{\varepsilon^2}$ vertices in V_H , and call IO_H^π on them. Suppose IO_H^π returns TRUE \bar{X}_1 times and the size of maximal independent set of H generated by π is X_1 , then by Chernoff bound, $\left| \frac{\bar{X}_1 \cdot |V_H|}{M_1} - X_1 \right| < \varepsilon/2$ with probability $9/10$ since $|V_H| < n^2$.

Now we bound the number of calls to IO_H^π . The number of vertices in H is $n(n-1)/2 + n\sqrt{n} + \sqrt{n} = \Omega(n^2)$. Since G_S has degree at most \sqrt{n} , $L(G_S)$ also has degree $O(\sqrt{n})$ and $|E_S| < n\sqrt{n}$. So the number of edges in $L(G_S)$ and the number of edges incident on D_2 are both $O(n^2)$, which means H has $O(n^2)$ edges. By Lemma A.1, if we randomly choose a vertex $v \in V_H$, $\mathbb{E}_{v,\pi}[T_H(v, \pi)] = O(1)$. So the total number of calls to IO_H^π is $O(n/\varepsilon^2)$ in expectation.

Finally, we describe how to simulate $IO_H^\pi(v)$ by pair queries in G , and bound the number of queries. There are three kinds of vertices in H , the vertices in D_1 , D_2 or W . If $v \in D_1$, the neighbors of v are all vertices in D_2 . We do not need any query to figure out the neighbors of v . If $v \in D_2$, then the neighbors of v are all vertices in $E_S \cup D_1$. We simulate $IO_H^\pi(v)$ as follows: we consider all vertices in $W \cup D_1$ one by one from lowest rank to highest rank. When considering vertex u , if the rank of u is larger than v then return TRUE, otherwise we use at most one pair query to check if $u \in E_S \cup D_1$, if so, we run $IO_H^\pi(u)$, otherwise do nothing and continue to the next u . Since $|W \cup D_1| < n^2$ and $|E_S \cup D_1| \geq |D_1| = n\sqrt{n}$, by Claim A.4, with probability $1 - o(\frac{1}{n^2})$, we use at most $10\sqrt{n} \log n$ pair queries between successive recursive calls to IO_H^π . If $v \in W$, we first use one pair query to check if $v \in E_S$. If $v \notin E_S$, we can directly output TRUE. If $v \in E_S$, let v_i and v_j be the endpoints of v in G_S . We consider all vertices in D_2 and all vertices in W that contains either v_i or v_j (there are $2n-1$ of them). When considering vertex u , if the rank of u is larger than v then return TRUE. Otherwise, we use at most one pair query to check if $u \in E_S \cup D_1$, if so, we run $IO_H^\pi(u)$. Otherwise do nothing and continue to the next u . Since there are at most $2n-1+\sqrt{n}$ vertices we considered and at least $|D_2| = \sqrt{n}$ of them are v 's neighbors, the number of pair queries between successive calls to IO_H^π is at most $10\sqrt{n} \log n$ with probability $1 - o(\frac{1}{n^2})$ by Claim A.4. By taking union bound on all v , we use $\tilde{O}(n^{1.5}/\varepsilon^2)$ pair queries in total in expectation with probability $1 - o(1)$. By Markov's inequality, we use $\tilde{O}(n^{1.5}/\varepsilon^2)$ queries in total with probability at least $9/10$.

The failure probability is at most $1/10 + \frac{1}{n} + o(\frac{1}{n}) < 1/3$ by union bound. ■

Note that the algorithm in Lemma A.3 needs to sample the rank of all possible vertex pairs. So it requires $O(n^2)$ time although it only uses $\tilde{O}(n^{1.5})$ queries. However, we can use the “permutation generation on the fly” technique in [25] (see Section 4 in [25] for details) to generate the rank of a pair of vertices only when the algorithm needs it. By doing this, the algorithm in Lemma A.3 can work in $\tilde{O}(n^{1.5})$ time.

Lemma A.5. *There is an algorithm that approximates the size of a maximal independent set in $L(G_S)$ within an additive error εn that uses $\tilde{O}(n^{1.5}/\varepsilon^2)$ time with probability at least $2/3$.*

Theorem 8 follows from Lemma A.2 and Lemma A.5.

A.2 Approximating Maximal Matching Pair (Proof of Theorem 12)

In this section, we generalize the idea in Section A.1 to an algorithm that approximates a maximal matching pair in a graph G . The algorithm has two phases. In the first phase, we greedily match the high degree vertices. In the second phase, we construct an auxiliary graph H such that approximation of a maximal independent set of H gives an approximation of a maximal matching pair of the graph G .

In the first phase, we run Algorithm 3, which returns two partial matchings M_1, M_2 such that $M_1 \cap M_2 = \emptyset$, and two sets S_1, S_2 that contain all vertices not matched in M_1 and M_2 respectively. The algorithm works similarly to Algorithm 2. For any vertex v and $j \in \{1, 2\}$, if $v \in S_j$, we sample $c_0 = 100\sqrt{n} \log n$ vertices from S_j and check if there is a neighbor of v . Suppose there is a sampled u such that (v, u) is an edge and it is not in M_{3-j} . Then we add this edge into M_j and delete these two vertices from S_j . Like Algorithm 2, Algorithm 3 also only uses $\tilde{O}(n^{1.5})$ queries. We prove that the subgraphs of G induced by S_1 and S_2 both have maximum degree at most \sqrt{n} with high probability.

ALGORITHM 3: Match high degree vertices into a matching pair

```

 $S_1 \leftarrow V, S_2 \leftarrow V, M_1 \leftarrow \emptyset, M_2 \leftarrow \emptyset, c_0 \leftarrow 100\sqrt{n} \log n.;$ 
for  $v \in V$  do
    if  $v \in S_1$  then
        Sample  $c_0$  vertices  $u_1, u_2, \dots, u_{c_0}$  from  $S_1.;$ 
        for  $i \leftarrow 1 \dots c_0$  do
            if  $(v, u_i)$  is an edge in  $G$  and  $(v, u_i) \notin M_2$  then
                 $M_1 \leftarrow M_1 \cup \{(v, u_i)\}, S_1 \leftarrow S_1 \setminus \{v, u_i\} ;$ 
                break;
            end
        end
    end
    if  $v \in S_2$  then
        Sample  $c_0$  vertices  $u_1, u_2, \dots, u_{c_0}$  from  $S_2.;$ 
        for  $i \leftarrow 1 \dots c_0$  do
            if  $(v, u_i)$  is an edge in  $G$  and  $(v, u_i) \notin M_1$  then
                 $M_2 \leftarrow M_2 \cup \{(v, u_i)\}, S_2 \leftarrow S_2 \setminus \{v, u_i\} ;$ 
                break;
            end
        end
    end
end
Output  $S_1, S_2, M_1$  and  $M_2.$ 

```

Lemma A.6. *The subgraphs of G induced by S_1 and S_2 both have degree at most \sqrt{n} with probability at least $1 - o(\frac{1}{n})$.*

The proof of Lemma A.6 is similar to the proof of Lemma A.2 and we omit it here.

In the second phase, we construct a graph H_P such that any maximal independent set in H_P with M_1 and M_2 represent a maximal matching pair of G . Let G_{S_1} be the subgraph of G induced by S_1 without the edges in M_2 , and G_{S_2} be the subgraph of G induced by S_2 without the edges in

M_1 . Let $L(G_{S_1})$ and $L(G_{S_2})$ be the line graph of G_{S_1} and G_{S_2} respectively. The graph H_P contains a copy of $L(G_{S_1})$ and a copy of $L(G_{S_2})$; furthermore, for any pair of vertices in $L(G_{S_1})$ and $L(G_{S_2})$ that represent the same edge in G , we add an edge between them.

Lemma A.7. *For any maximal independent set I in H_P , suppose I_1 is the set of edges in G that are represented by a vertex in I inside the copy of $L(G_{S_1})$ and I_2 is the set of edges in G that represented by a vertex in I inside the copy of $L(G_{S_2})$. $(M_1 \cup I_1, M_2 \cup I_2)$ is a maximal matching pair of G .*

Proof. Since $L(G_{S_1})$ and $L(G_{S_2})$ are line graphs, I_1 and I_2 are both matchings. Furthermore, any pair of vertices in H that represent the same edge have an edge in H between them, so $I_1 \cap I_2 = \emptyset$. Additionally, G_{S_1} and G_{S_2} contains no edges in $M_1 \cup M_2$, G_{S_1} (resp. G_{S_2}) contains no vertices in M_1 (resp. M_2). $M_1 \cup I_1$ and $M_2 \cup I_2$ are both matchings and $(M_1 \cup I_1) \cap (M_2 \cup I_2) = \emptyset$, which means $(M_1 \cup I_1, M_2 \cup I_2)$ is a matching pair of G .

For any edge e in G which is not in $M_1 \cup I_1 \cup M_2 \cup I_2$, we prove that there are two edges $e_1 \in M_1 \cup I_1$ and $e_2 \in M_2 \cup I_2$ that each share an endpoint with e . If e is not an edge in G_{S_1} , then e contains an endpoint that is matched in M_1 since $e \notin M_2$, which means there is an edge $e_1 \in M_1$ that shares a common endpoint with e . If e an edge in G_{S_1} , suppose v_1^e is the vertices in H that represents e in the copy of $L(G_{S_1})$. Since $e \notin I_1$, $v_1^e \notin I$. There is a vertex $u \in I$ that connects to v_1^e in H . Since $e \notin M_2$, u is not in the copy of $L(G_{S_2})$, which means u is inside the copy of $L(G_{S_1})$. Suppose e_1 is the edge represented by u , by definition of I_1 and $L(G_{S_1})$, $e_1 \in I_1$ and e_1 shares an endpoint with e .

Following a similar argument, there is also an edge $e_2 \in M_2 \cup I_2$ that shares an endpoint with e . ■

By Lemma A.7, to approximate the size of a maximal matching pair of G , it is sufficient to approximate the size of a maximal independent set of H_P . The following lemma follows from a similar argument as the prove of Lemma A.3 and Lemma A.5.

Lemma A.8. *There is an algorithm that approximates the size of a maximal independent set in H_P within an additive error εn that uses $\tilde{O}(n^{1.5}/\varepsilon^2)$ pair queries and time with probability at least 2/3.*

Theorem 12 follows from Lemma A.6 and Lemma A.8.