Sensitivity and Computational Complexity in Financial Networks

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Abstract. Modern financial networks exhibit a high degree of interconnectedness and determining the causes of instability and contagion in financial networks is necessary to inform policy and avoid future financial collapse. In the American Economic Review, Elliott, Golub and Jackson proposed a simple model for capturing the dynamics of complex financial networks. In Elliott, Golub and Jackson's model, each institution in the network can buy underlying assets or percentage shares in other institutions (cross-holdings) and if any institution's value drops below a critical threshold value, its value suffers an additional failure cost. This work shows that even in simple model put forward by Elliott, Golub and Jackson there are fundamental barriers to understanding the risks that are inherent in a network.

First, if institutions are not required to maintain a minimum amount of self-holdings, an ϵ change in investments by a single institution can have an arbitrarily magnified influence on the net worth of the institutions in the system. This sensitivity result shows that if institutions have small self-holdings, then estimating the market value of an institution requires almost perfect information about *every cross-holding in the system*.

Second, we show that even if a regulator has complete information about all cross-holdings in the system, it may be computationally intractable to even estimate the number of failures that could be caused by an arbitrarily small shock to the system.

Together, these results show that any uncertainty in the cross-holdings or values of the underlying assets can be amplified by the network to arbitrarily large uncertainty in the valuations of institutions in the network.

1 Introduction

The recent financial crisis and subsequent bailout have highlighted the need for a better understanding of the dynamics of financial networks. Indeed, the complexity of modern financial networks been blamed for our collective failure to recognize the presence of serious risks in these systems. In this work, we show that even in extremely simple financial networks understanding the risk present in the system is computationally intractable in general, even in the presence of perfect information about all participants in the system. We hope that these insights will help regulators and policymakers to better understand the dynamics of financial networks.

Understanding an individual's risk in a financial network is a difficult task, because an institution's ability to fulfill its outgoing financial obligations is not a local property, *i.e.*, it cannot be understood by examining a single individual in isolation. A institution's ability to make its outgoing payments may depend on whether its incoming payments are made by its debtors, which in turn may depend whether the incoming obligations are made to those institutions, etc. Thus each institution, acting without a global view of the network, cannot effectively understand its risk. Nevertheless, one might hope that a regulator, with a global view of the network could better understand the opportunities and risks inherent in the system.

Even with a global view, the situation remains relatively complex. One complicating factor is the existence of cycles in the financial network, for example institution A may have obligations to institution B which has obligations to institution C, which in turn has obligations to institution A. The existence of these cyclical interdependencies has been put forward as one of the primary mechanisms introducing complexity into financial networks. The interconnectedness of the system makes even the most basic facts nonobvious. For example, the fundamental fact that the market value for institutions in a financial network is uniquely defined requires proof (see e.g. [EN01,GHM12]).

One of the driving forces in the study of financial networks is their ability to magnify risk if an institution, A, defaults on its obligations to institution B, this may cause institution B to default on its obligations to institution C etc. The spread of risk through a financial network is known as *financial contagion* and has been carefully modeled and studied [AG00,EN01,GHM12,AOTS13,GY14,EGJ14].

This work focuses on the quantifying the ability of financial networks to amplify and conceal risk.

2 Our contributions

In this work, we study two questions related to the stability of financial networks. First, we look at how sensitive market valuations can be to small changes in network structure. Second, we examine the computational complexity of determining how far a given network is from a massive failure. Throughout this work we use the network model put forward by Elliott, Golub and Jackson [EGJ14]. In this model, financial institutions own shares of underlying assets as well as shares of each other (cross-holdings). If an institution's market value drops below a certain critical threshold, its value suffers a further discontinuous shock, modeling the effects of a loss of investor confidence, or the failure to pay everyday operating costs. The model is described in detail in Section 4.

Our first result (Theorem 2), shows that financial networks can be highly sensitive to small changes in their link structure. Concretely, we show that if a single institution changes its shareholdings by ϵ the market values of institutions in the system can change by as much as $\epsilon/2r$ where $r \leq 1$ is the minimum amount of self-holdings of the institutions in the network. The minimum self-holdings, r, is a measure of *integration* of the network, where $r = 0$ corresponds to a fully integrated network, and $r = 1$ corresponds to a network with no integration. This result shows that if each institution retains only, say, 5% self-ownership, a change in a single holding by ϵ can result in a 20ϵ change in an institution's market value. This amplification is directly caused by cycles in the network, and in acyclic networks this type of magnification cannot occur (see Lemma 1). Our bounds are essentially tight, and Theorems 1 and 2, show that the true sensitivity is in fact $\Omega(\epsilon/r)$.

This sensitivity magnification has many consequences. First, it means that in order to estimate market values of institutions within the system, all cross-holdings need to be known to an extremely high degree of precision. Investors or regulators who wish to calculate market values can be extremely far off if even a single cross-holding in the network remains unknown to them. Second, because small changes in a single institution's investments can have large effects on market values throughout the system, this indicates a potential for extreme instability in the system as a whole; the small portfolio changes in one institution can have drastically magnified effect market values of other institutions, and so small changes by a myopic institution could topple even seemingly stable institutions. Third, this extreme sensitivity means that calculating market values in a privacy-preserving manner can be extremely difficult [NPH14]. This is the flip-side of the first point, in order to calculate market values for institutions in the system, *all* the interbank holdings need to be known with high precision, thus revealing market prices has the potential to reveal extremely detailed information about each institution's interbank holdings.

Our second result addresses the question of how well a regulator can assess the stability of a network. Suppose a regulator or oversight agency is presented with a network in which every bank is solvent, and suppose the regulator believes that the underlying assets cannot drop in value by more than some fixed amount d. What is the maximum number of failures that can be caused by this drop in asset values? We emphasize that in this scenario, the regulator has complete information about the entire structure of the financial network, and the only uncertainty is in which specific assets may decline in value. We show calculating the number of institutions that can fail in network is as hard as calculated the maximum balanced clique in a bipartite graph (Theorem 3). It is known that if 3-SAT is not in $DTIME\left(2^{n^{3/4+\epsilon}}\right)$ for some ϵ, then there is no polynomial time algorithm for calculating the maximum balanced clique to within a factor of $2^{(\log n)^{\delta}}$ for some $\delta > 0$. Feige and Kogan [FK04] go further, conjecturing that there is no polynomial time algorithm to approximate the maximum balanced bipartite clique to within a factor of n^{δ} for some $\delta > 0$. This would imply that there is no polynomial time algorithm that can even *estimate* the number of failures caused by a drop in asset values to within a factor of n^{δ} in a network with $2n$ institutions.

Unlike our first result, which crucially relies on cycles in the network, this result holds even in acyclic networks: even when there are no cycles it is computationally intractable to estimate the maximum number of failures that could be caused by a fixed drop in asset values. This complexity arises not from cycles, but from the nonlinear dynamics that occur when a bank drops below its critical threshold value.

3 Previous Work

Many different models have been proposed to study financial networks, and specifically models of stability and contagion.

Allen and Gale [AG98] considered a model consisting of depositors and banks. Depositors deposit their money in the banks, and the banks must choose between making *short term* investments or *long term* investments. This model has three time periods, $t = 0, 1, 2$. At $t = 0$, investments are made, at $t = 1$ short term investments pay off, and depositors choose whether to withdraw their money, and at time $t = 2$ long term investments pay off. The banks' investment strategies then depend on the probability that depositors withdraw their money at time $t = 1$. In a follow-up work Allen and Gale [AG00] introduced a network component, whereby banks can exchange deposits with each other in an effort to mitigate risk, and they showed simple contagion effects in this model.

Acemoglu, Ozdaglar and Tahbaz-Salehi [AOTS13] build on Allen and Gale's 3 time-step model. At time $t = 0$ banks can make a short term investment, a long term investment or loan money to other banks. Long term investments yield a fixed return at $t = 2$ Long term investments that are liquidated at $t = 1$ receive a return that is randomly distributed between two values. Banks whose investments returned the lower of the two values were said to have received a *shock*. Acemoglu, Ozdaglar and Tahbaz-Salehi's considered how two extremal types of networks serve to propagate these shocks. They considered the ring (where each bank only has debts to its two neighbors) and the complete network where each bank's debts are spread to all other banks, as well as all convex combinations of these two. They showed that if the magnitude of the shocks are small then the complete network is more stable and resilient than the ring, and if the magnitudes of the shock are large enough then both the ring and complete networks are the *least* stable networks, thus the complete network exhibits a phase transition, moving from the most stable to the least stable network as the magnitude of the shocks increase.

Eisenberg and Noe [EN01] developed a very simple and appealing network model where each bank has cash reserves and fixed debts to other banks. Eisenberg and Noe's work focused on showing that (under some basic restrictions on the network) there is always a unique clearing vector (indicating how much of its debts each bank pays to its creditors), and they gave a linear program and simple iterative algorithm for calculating this clearing vector and hence the equilibrium valuation of each bank.

Gai and Kapadia [GK10] considered a modification of Eisenberg and Noe's network model, forcing all incoming edges to have the same weight, but allowed banks to have additional illiquid assets. Gai and Kapadia then considered the question of how a single bank failure propagates through a network. For this analysis, they considered different models for generating the underlying graph topology, and plotted contagion effects for different network models characterized by their degree distribution. They found that a single large shock could have devastating effects on the network, but that this was highly dependent on where in the network the shock hit.

Gourieroux, Heam and Monfort [GHM12] considered a model that allows interbank investment via shares (like [EGJ14]) and lending or insurance (like [EN01]). Unlike [EGJ14], they do not introduce discontinuous failure costs. Gourieroux, Heam and Monfort extend Eisenberg and Noe's uniqueness results to show that (under mild constraints on the network) this extended model has a unique equilibrium value for all institutions. They then examined the effects of exogenous shocks on the network (*i.e.*, drops in asset values) using synthetic data and data obtained from the French banking sector.

The notion of failure cascades and contagion have also been studied in the computer science literature by Blume et al. [BEK$^+$11]. Blume et al. considered general cascades in graphs where the edges were unweighted and a node was said to fail if some critical threshold of its neighbors failed. By choosing all edges to have equal weight, and the choosing each institution's failure threshold carefully, the failure model of [EGJ14] can be made to overlap with this general network failure model.

In this work, we use the model of Elliott, Golub and Jackson [EGJ14]. In this model, institutions can own shares in each other, or in "primitive assets" that have intrinsic value outside of the network. The model is explained in detail in the next section. Elliott, Golub and Jackson introduced this model to help analyze and understand contagion effects in networks. Because a cascade of collapse requires an initial failure, Elliott, Golub and Jackson began by showing that the weakest institution can never be made strictly more stable by any fair trade between the institutions. Next they examined the contagion dynamics in a wide class of networks parametrized by "integration" and "diversification." Integration increases as institutions in the network increase their inter-network holdings, *i.e.*, integration increases as the percentage of each institution owned by shareholders external to the network decreases. As integration increases, the institutions fates are more closely tied together. Diversification measures how risk is spread within the network. Diversification increases as institutions increase their *number* of cross-holdings. Neither integration nor diversification have strictly positive or negative effects on network stability, but instead have slightly more complex non-monotonic effects.

The notion of computational complexity has been studied in the context of financial products by Arora et al. who showed that banks can create derivatives that are computational intractable to price accurately [ABBG10,ABBG11]. The result of Arora et al. crucially relies on the information asymmetry between the institution the seller (who creates the derivatives) and the buyer who only sees their resulting composition. Braverman and Pasricha [BP14] show that even in the full information setting pricing compound options is PSPACE complete.

4 Model

We use the model put forward by Elliott, Golub and Jackson [EGJ14]. In this model there are n financial institutions, these can be viewed as countries, banks or private firms, and m underlying assets, that can be viewed as any object or project with intrinsic value. The financial institutions own shares of the underlying assets and each other. While institutions invest in one another, all value in the system is originates from the underlying assets. The price of asset k is denoted by p_k, and we use $D_{ik} \geq 0$ to denote the percentage of asset k owned by institution i. The $n \times m$ matrix of ownership is denoted by $\mathbf{D} = (D_{ik})$.

We define $\mathbf{C} = (C_{ij})$ to be the $n \times n$ matrix indicating the cross-holdings of institutions. Thus institution i owns a C_{ij} fraction of institution j. It will be be useful to view the network of cross-holdings as a directed graph with n nodes representing the financial institutions, and an edge from institution j to i of weight C_{ij} whenever $C_{ij} > 0$. Following [EGJ14], we set $C_{ii} = 0$ for all i. Now, $\sum_i C_{ij}$ is the fraction of institution j that is owned by institutions external to j. The remainder, the amount of self-ownership, is denoted by $\hat{C}_{jj} \stackrel{\text{def}}{=} 1 - \sum_i C_{ij}$. The matrix $\hat{\mathbf{C}}$ will be a diagonal matrix with $\hat{C}_{ii}$ on the diagonal.

As noted by Brioschi, Buzzachi and Colombo [BBC89], this type of model introduces two types of valuations, the *equity valuation* ($\boldsymbol{V}$) and the *market valuation* ($\boldsymbol{v}$). The equity valuation of institution i is denoted by

$$V_i = \underbrace{\sum_k D_{ik} p_k}_{\text{Value of assets held by } i} + \underbrace{\sum_j C_{ij} V_j}_{\text{Values of institutions held by } i} \tag{1}$$

In matrix notation, this becomes $\boldsymbol{V} = \mathbf{D}\boldsymbol{p} + \mathbf{C}\boldsymbol{V}$ which implies $\boldsymbol{V} = (\mathbf{I} - \mathbf{C})^{-1}\mathbf{D}\boldsymbol{p}$. The matrix $\mathbf{I} - \mathbf{C}$ is guaranteed to be invertible because we assume that $\hat{C}_{jj} > 0$, so the column sums of $\mathbf{C}$ are all strictly less than one (see Lemma 2). In fact, the matrix $\mathbf{I} - \mathbf{C}$, is an M-Matrix [PB74], and so $(\mathbf{I} - \mathbf{C})^{-1}$ is an inverse M-Matrix, about which many properties are known [Wil77,Joh82].

This equity valuation significantly overvalues the institutions. In particular, we can see that $\|\boldsymbol{V}\|_1 \geq \|\boldsymbol{p}\|_1$, so the total value of the institutions in the system will (in general) be much larger than the total value of the underlying assets. This occurs because each asset counts towards the equity value of the institution that owns it and also to the institutions that have an equity stake in the asset's owner. The network's inflation of equity values is well-known and validated both theoretically and empirically [FP91,FHT94].

To find an institution's *market value*, we must scale the institution's equity value by the percent stake it has in itself, thus the market value of institution i is $v_i = \hat{C}_{ii} V_i$, so the market values are the solution to the system

$$\boldsymbol{v} = \hat{\mathbf{C}}\boldsymbol{V} = \hat{\mathbf{C}}(\mathbf{I} - \mathbf{C})^{-1}\mathbf{D}\boldsymbol{p} \tag{2}$$

The matrix $\mathbf{C}$ is column sub-stochastic because column i sums to $1 - \hat{C}_{ii}$. The system can also be viewed as a flow, where at each time step money flows between banks according to the link structure of the network (see Appendix A).

5 Sensitivity

Our first result concerns concerns the sensitivity of valuations to small changes in the structure of the network. Suppose a single institution shifts its holdings by a small quantity, ϵ, how much can this small change affect the market valuations in the network? This question is motivated by questions about network stability, and the possibility of privacy-preserving oversight. If small changes in network holdings can lead to large changes in the market values of the institutions, this indicates a fundamental instability in the financial network. Any attempt to calculate market values must know all the cross-holdings to a high degree of accuracy. Additionally, if small changes in interbank holdings can lead to large changes in market values, then any attempt at financial oversight must know all the interbank holdings to a high degree of accuracy in order to predict market values. A high sensitivity also has implications towards privately computing network statistics. Flood et al. [FKOS13] proposed using tools from differential privacy [Dwo06,DMNS06] to provide a means of computing global network characteristics while preserving the privacy of each individual institution's holdings.

First, note that if the total value of the underlying assets is $\|\boldsymbol{p}\|$, then an ϵ change in holdings can easily change the market valuations by $\epsilon\|\boldsymbol{p}\|$ (see Figure 1 in Appendix B for a simple example of this). We show that if the network is acyclic, then this is the largest change possible, but if there are cycles in the network, the sensitivity can be much larger.

Throughout this section, we use r (for "reserve") to measure the fraction of each institution held by investors *outside* the system.[1] We define $r = \min_i \hat{C}_{ii}$. Using the terminology of [EGJ14], r is just a concrete metric of the *integration* of the network, and integration increases as $r \to 0$.

5.1 Sensitivity in acyclic networks

We begin by noting that in the acyclic case, there is a strong bound on each institution's equity valuation, *i.e.*, the equity valuation cannot be too much larger than the market valuation.

Lemma 1. *If the banking network has no cycles, then every institution's equity valuation is at most* $\|\boldsymbol{w}\|_1$ *where* $\boldsymbol{w}$ *is the vector of asset values.*

The proof can be found in Appendix C.

Corollary 1. *If the banking network is acyclic, and one edge changes by at most* ϵ*, then no institution's market value can change by more than* $\epsilon\|\boldsymbol{p}\|_1$*.*

Proof. Since each institution's equity value is at most $\|\boldsymbol{p}\|_1$, an ϵ change in any edge corresponds to an absolute change of at most $\epsilon\|\boldsymbol{p}\|_1$.

[1] The reserve, or self-holdings, can be viewed as the amount of an institution that is not sold, or is held by private shareholders, who retain complete ownership of themselves. These private shareholders buy shares of institutions in the network, but no entity in the network owns shares of the private shareholders.

5.2 Sensitivity in general networks

In this section, we explore how much the market valuations can change when one bank changes its holdings by a small amount *in the presence of cycles in the network graph.* We begin by showing an upper bound on the change in market valuations that depends on the minimum self-ownership ($\hat{C}_{ii}$) of the institutions. For our upper bound, we do not require changes to occur in the holdings of a single bank. Instead, we allow any change in network structure, as long as the total (ℓ_1) change is bounded by ϵ. Formally, this means that we have two network matrices $\mathbf{C}$ and $\tilde{\mathbf{C}}$ such that $\|\mathbf{C}-\tilde{\mathbf{C}}\| \leq \epsilon$, and we would like to bound how much the market valuations can change between these two situations. Because we are showing an upper bound, allowing more general perturbations only strengthens our result.

Theorem 1. *If $\|\mathbf{C}-\tilde{\mathbf{C}}\| < \epsilon$, then $\|\boldsymbol{v}-\tilde{\boldsymbol{v}}\| < \frac{\epsilon}{r}\|\mathbf{D}\boldsymbol{p}\|$, where $r = \min_i(\tilde{\hat{C}}_{ii}, \hat{C}_{ii})$ is the minimum reserve or "self-holdings" of the financial institutions.*

The proof can be found in Appendix D. Note as well, that because $\|\boldsymbol{v}\| = \|\mathbf{D}\boldsymbol{p}\|$, the triangle inequality gives us a trivial upper bound of 2.

The multiplicative bound of ϵ/r in Theorem 1 is much weaker than the bound of ϵ in the acyclic case (Corollary 1), as the minimum self-holdings approaches 0, this difference tends towards infinity. This discrepancy is not a limitation of our proof, but arises as an artifact of the effect that holdings cycles can have on the equity (and market) valuations of the institutions in the network. In Theorem 2 we show that there exist networks where changing a single institution's holdings by ϵ results in a change of $\frac{\epsilon}{2r+\frac{1-r}{2}\epsilon}\|\mathbf{D}\boldsymbol{p}\|$ in one of the institution's market values, where r is the minimum self-holdings in the network.

Theorem 2. *There exist networks where $\|\mathbf{C}-\tilde{\mathbf{C}}\| < \epsilon$, and $\|\boldsymbol{v}-\tilde{\boldsymbol{v}}\| \geq \frac{\epsilon}{2r+\frac{1-r}{2}}\|\mathbf{D}\boldsymbol{p}\|$, where*

$$\boldsymbol{v} = \hat{\mathbf{C}}(\mathbf{I}-\mathbf{C})^{-1}\mathbf{D}\boldsymbol{p}$$

and $r = \min_i(\tilde{\hat{C}}_{ii}, \hat{C}_{ii})$ is the minimum "self-holdings" of the financial institutions.

The proof can be found in Appendix E. The type of equity amplification necessary for the lower bound in Theorem 2 cannot happen in an acyclic network (see Corollary 1).

Comparing the upper and lower bounds we see that they are essentially tight. We have an upper bound of $\max(2, \frac{\epsilon}{r})$ and a lower bound of $\frac{\epsilon}{2r+\frac{1-r}{2}\epsilon}$. If we set $r = \frac{\epsilon}{t}$, then the lower bound becomes

$$\frac{\epsilon}{2r+\frac{(1-r)\epsilon}{2}} \geq \frac{\epsilon}{2r+\frac{\epsilon}{2}} = \frac{t}{2+\frac{t}{2}}$$

and $\lim_{t\to\infty} \frac{t}{2+\frac{t}{2}} = 2$, so we approach the upper bound of 2 given by Theorem 1. On the other hand, if we let $t \to 0$, we get a lower bound of $\frac{t}{2+\frac{t}{2}}$ which approaches $\frac{t}{2} = \frac{\epsilon}{2r}$, so we are within a multiplicative factor of $\frac{1}{2}$ of the upper bound of $\frac{\epsilon}{r}$.

6 Bank failures

6.1 Losses caused by failure

The model of [EGJ14] includes a notion of "failure," whereby institutions whose market value drops below a certain critical threshold suffer a further (discontinuous) loss in market value. These discontinuous penalties capture the notion that if an institution cannot pay its operating costs, it may see a further drop in revenues. Similarly, if confidence in the institution is shaken, and its debt rating is downgraded, it may see spike in the cost of capital, and hence see a further drop in value.

These discontinuous penalties are operationalized by a threshold value $\underline{\mathrm{v}}_i$, such that if institution i's market value, v_i drops below $\underline{\mathrm{v}}_i$ then it incurs a failure cost and its market value drops by $\beta_i(\boldsymbol{p})$.

Defining $I_{v_i < \underline{\mathrm{v}}_i}$ to be the indicator variable which is 1 if $v_i < \underline{\mathrm{v}}_i$ and 0 otherwise, and $b_i(\boldsymbol{v}, \boldsymbol{p}) = \beta_i(\boldsymbol{p}) I_{v_i < \underline{\mathrm{v}}_i}$, the market value of the institutions satisfies the equation

$$\boldsymbol{v} = \hat{\mathbf{C}}(\mathbf{I} - \mathbf{C})^{-1}(\mathbf{D}\boldsymbol{p} - \boldsymbol{b}(\boldsymbol{v})) \tag{3}$$

Compare Equation 3 to the linear system given by Equation 2. The introduction of non-linear terms into the model adds significant complexity to the dynamics of the system, and can lead to failure cascades. One of the primary goals of [EGJ14] was to characterize what network features affect the likelihood and severity of failure cascades.

We will show that the introduction of discontinuous failure costs increases the computational complexity of calculating basic network dynamics. In particular calculating the maximum possible number of failures caused by a small number of "seed" failures is computationally intractable.

6.2 The complexity of calculating the maximum number of failures

In this section, we examine the computational complexity of estimating the maximum number of failures that can occur given a small drop in values of the underlying assets.

Our hardness result is based on the hardness of finding a maximum balanced clique in a bipartite graph. This is known as Balanced Complete Bipartite Subgraph (BCBS) problem.

Definition 1 (BCBS). *Given a bipartite graph $G = (V_1, V_2, E)$ with $|V_1| = |V_2| = n$, the* Balanced Complete Bipartite Subgraph *(BCBS) problem is to find the largest integer K such that there exists sets $C_1 \subset V_1$ and $C_2 \subset V_2$ with the properties that $|C_1| = |C_2| = K$, and the induced graph on $C_1 \cup C_2$ is a complete bipartite subgraph of G.*

See Appendix F for a brief review of what is known concerning the hardness of the BCBS problem.

Theorem 3. *For every bipartite graph G on $2n$ nodes, and every $\epsilon > 0$, there is a financial network with $\Omega(n)$ institutions, and a $d > 0$ such that computing the maximum number of institutions that could fail following a shock of $d\epsilon$ in asset prices is as hard as solving the BCBS problem in G.*

The proof can be found in Appendix G. Applying a result of Feige and Kogan [FK04], we obtain the following Corollary.

Corollary 2. *If 3-SAT $\notin DTIME\left(2^{n^{3/4+\epsilon}}\right)$ for some $\epsilon > 0$, then there exists a $\delta > 0$ such that there is no polynomial time algorithm that can calculate the maximum number of failures in a financial network caused by a drop in asset prices of d to within a factor of $2^{(\log n)^{\delta'}}$ for some $\delta' > 0$.*

7 Conclusion

This work highlights two distinct sources of instability in financial networks, instability arising from fluctuations in cross-holdings and instability arising from fluctuations in asset prices. More specifically, we show that there are networks where small fluctuations in cross-holdings or asset prices can have dramatic consequences, and these consequences have numerous implications.

Our first result (Corollary 1, Theorem 1) shows that the effect of small changes in cross-holdings is strongly tied to the integration of the network. In highly integrated networks small changes in cross-holdings can have potentially unbounded effects on market valuations, while in networks with low integration, changes in cross-holdings have more tightly bounded effects on the market values of the institutions in the network. These results can be interpreted in many ways. From a regulatory perspective, in a highly integrated network, any regulator must know the entire cross-holdings network to a very high degree of accuracy in order effectively understand the market values of the institutions. From an institution's perspective, small changes in investment by individual institutions can have their effects greatly magnified throughout the network. From a predictivity perspective, if one wishes to forecast market values into the future, any small forecasting uncertainty in the cross-holdings can have enormous effects on the (predicted) market values of the institutions. From a privacy perspective, institutions cannot maintain any privacy in their investment portfolios without compromising the ability of outsiders (e.g. other institutions, outside investors or regulators) to calculate the market value of the institutions. These problems arise only in highly integrated networks, and they can all be mitigated by imposing a cap on integration. If institutions are required to maintain some fixed percentage of their ownership outside of the network, the sensitivity to changes in cross-holdings can be drastically reduced.

Our second result shows that small changes in the prices of the underlying assets can have unpredictable effects on the number of failures in the system. Specifically, we show that there are networks where it is *computationally infeasible*, even with perfect information about the cross-holdings, to estimate the number of failures that can occur after some small drop in asset prices. This result too can be interpreted from different perspectives. This result implies that a regulator (with perfect information about the network cross-holdings) who believes there may be some bounded fluctuation in asset prices cannot be expected to distinguish a network where these fluctuations cause a small number of failures, from a network where fluctuations of the same magnitude can cause a massive number of failures. Institutions face the same computational challenge. An institution may wonder whether its investment portfolio will protect it from a bounded shock in asset prices, and our results show that even with complete information about the investments of all other agents in the system, it may be infeasible to determine whether a specific portfolio is safe.

Previous works have imposed specific probabilistic models on fluctuations in asset prices, and for a given probabilistic model the number of failures can usually be estimated. But what if the model is incorrect (e.g. the asset prices do not fluctuate independently)? Our results show that there are situations where changes in the distribution of fluctuations (but not their overall magnitude) can have huge and unpredictable effects on the number of failures.

Moving forward, it is an important research question to understand what constraints on the network will allow institutions and regulators to perform these stability analyses.

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Appendix

A A flow model of the system

It is often instructive to create a stochastic matrix representing the system, which models money flowing through the network.

There are n financial institutions, and we introduce n addition entities, the "shareholders." In this model shareholder i owns a $\hat{C}_{ii}$ fraction of institution i and has no other financial ties to the system. This defines a $2n \times 2n$ matrix, $\mathbf{A}$, defined as

$$\mathbf{A} = \left[\begin{array}{c|c} \mathbf{C} & \mathbf{0} \\ \hline \hat{\mathbf{C}} & \mathbf{I} \end{array}\right]$$

The augmented matrix $\mathbf{A}$ is now column stochastic, *i.e.*, its columns sum to 1. The ijth entry of the matrix $\mathbf{A}$ represents the fraction of "agent" j owned by agent i. There are $2n$ agents because there are n financial institutions (agents $1, \ldots, n$) and n collections of shareholders (agents $n+1, \ldots, 2n$). The lower right corner of $\mathbf{A}$ is the identity matrix because the shareholders are completely self-owned. The columns of $\mathbf{A}$ sum to one because for each agent, j, its entire value is owned by the other financial institutions or the external shareholders.

Initially, the banks are assumed to have some intrinsic value $\mathbf{D}\boldsymbol{p}$, which indicates the value of underlying assets owned by each institution. Without loss of generality, we will assume that all assets are completely owned by the institutions in the network (otherwise, we can simply rescale the value of each asset). This assumption is equivalent to saying that $\mathbf{D}$ is column stochastic.

The $2n \times 2n$ matrix $\mathbf{A}$ allows us to view the market valuations of each institution as the steady state of a dynamical process. Money flows into the system from the underlying assets, and at each time step, the value residing in each financial institution is distributed to its stakeholders according to their stake. This process terminates when all the money in the system (coming from the underlying assets) has been distributed to the external shareholders. Algebraically, this process can be viewed as follows: given a vector $\boldsymbol{W} \in \mathbb{R}^{2n}$, where W_i denotes the value of the underlying assets owned by i, for $i = 1, \ldots, n$ and $W_i = 0$ for $n < i \leq 2n$, then

$$\lim_{t\to\infty} \mathbf{A}^t \boldsymbol{W} = \lim_{t\to\infty} \mathbf{A}^t \left[\begin{array}{c} \boldsymbol{w} \\ \hline \mathbf{0} \end{array}\right] = \left[\begin{array}{c} \mathbf{0} \\ \hline \boldsymbol{v} \end{array}\right] = \left[\begin{array}{c} 0 \\ \vdots \\ 0 \\ \hline v_1 \\ \vdots \\ v_n \end{array}\right] \tag{4}$$

where v_i will denote the market value of institution i, *i.e.*, v_i is the value of institution i owned by its external shareholders. Now, basic linear algebra tells us that

$$\mathbf{A}^t = \left[\begin{array}{c|c} \mathbf{C}^t & \mathbf{0} \\ \hline \hat{\mathbf{C}}\left(\mathbf{C}^{t-1} + \mathbf{C}^{t-2} + \cdots + \mathbf{I}\right) & \mathbf{I} \end{array}\right]$$

Throughout this work, we will use the standard operator norm for a matrix (in the L_1 sense),

$$\|\mathbf{A}\| = \|\mathbf{A}\|_1 = \sup_{\boldsymbol{x}} \frac{\|\mathbf{A}\boldsymbol{x}\|_1}{\|\boldsymbol{x}\|_1}.$$

Because the columns of $\mathbf{A}$ sum to one, we have $\|\mathbf{A}\boldsymbol{x}\|_1 = \|\boldsymbol{x}\|_1$ for any $\boldsymbol{x}$. In particular, this means that

$$\left\| \mathbf{A} \begin{bmatrix} \mathbf{D}\boldsymbol{p} \\ \mathbf{0} \end{bmatrix} \right\|_1 = \|\boldsymbol{p}\|_1 \tag{5}$$

This is the algebraic statement that the total market value of all the institutions is exactly the total value of the underlying assets in the system.

To analyze equation 4 we recall the standard fact about matrix series

Lemma 2. *If $\mathbf{C}$ is a matrix with $\|\mathbf{C}\| < 1$, then $\mathbf{I} - \mathbf{C}$ is invertible and*

$$(\mathbf{I} - \mathbf{C})^{-1} = \sum_{k=0}^{\infty} \mathbf{C}^k$$

Proof. First, note if $\boldsymbol{v} \neq \mathbf{0}$, then $\|\mathbf{C}\boldsymbol{v}\| < \|\boldsymbol{v}\|$, so

$$\|(\mathbf{I} - \mathbf{C})\boldsymbol{v}\| \geq \|\mathbf{I}\boldsymbol{v}\| - \|\mathbf{C}\boldsymbol{v}\| > 0$$

so $\mathbf{I} - \mathbf{C}$ has a trivial kernel, and hence is invertible.

Then, the result follows just as in the scalar case: Letting $\mathbf{S} = \sum_{k=0}^{N} \mathbf{C}^k$, we have $\mathbf{S}(\mathbf{I} - \mathbf{C}) = \mathbf{I} - \mathbf{C}^{N+1}$. Thus

$$\sum_{k=0}^{N} \mathbf{C}^k = (\mathbf{I} - \mathbf{C})^{-1}(\mathbf{I} - \mathbf{C}^{N+1})$$

Letting $N \to \infty$, and noting that $\mathbf{C}^{N+1} \to 0$ gives the result.

This leads to the closed formula used in [EGJ14]:

$$\boldsymbol{v} = \hat{\mathbf{C}}(\mathbf{I} - \mathbf{C})^{-1}\mathbf{D}\boldsymbol{p} \tag{6}$$

Where the $n \times 1$ vector $\boldsymbol{p}$ represents the values of the underlying assets and the vector $\boldsymbol{v}$ represents the market value of the financial institutions (as measured by their external shareholders). Equations 4 and 5 then tell us that $\|\mathbf{D}\boldsymbol{p}\|_1 = \|\boldsymbol{p}\|_1 = \|\boldsymbol{v}\|_1$, which is just the simple statement that the total value of the agents is the same as the total value of the underlying assets, in other words value is never created or destroyed by the network.

B An example change in the network

In this section, we provide a simple network where changing a single connection by ϵ results in a change of $\epsilon\|\boldsymbol{p}\|$ in the final valuation of the institutions. See Figure 1.

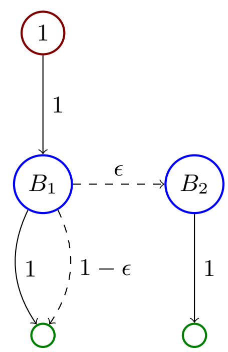

Fig. 1. An ϵ change results in an $\epsilon\|\boldsymbol{p}\|$ change in market values. In this example $\mathbf{D}\boldsymbol{p} = \begin{bmatrix}1\\0\end{bmatrix}$, and $\mathbf{C} = \mathbf{0}$. Thus $\boldsymbol{v} = \begin{bmatrix}1\\0\end{bmatrix}$. Changing $\mathbf{C}$ to $\tilde{\mathbf{C}} = \begin{bmatrix}0 & \epsilon\\0 & 0\end{bmatrix}$ leads to a valuation of $\tilde{\boldsymbol{v}} = \begin{bmatrix}1-\epsilon\\\epsilon\end{bmatrix}$, thus the market valuation of B_2 changes by $\epsilon\|\boldsymbol{p}\|$.

C Proof of Lemma 1

In this section, we prove Lemma 1, which states that in an acyclic network each institution's equity valuation is upper bounded by the total value of the assets in the system. In the language of flow, this translates to the simple statement that in an acyclic graph, no node can receive a flow greater than the total input flow to the graph.

Proof. Organize the financial network into layers, so that each institution only owns shares of institutions at lower layers. By adding fictitious institutions that pass on all of their incoming wealth, we can also ensure that every institution only owns shares in the entities at the preceding level, *i.e.*, institutions at level i only own shares in institutions (or fictitious institutions) at level $i-1$.

Now each institution's book value is the sum of the values on all incoming edges. Since outgoing edges carry a value that is a percentage of book value, the sum of the values on each institution's outgoing edges is at most the sum of the values on its incoming edges. (For real institutions the outgoing sum will be strictly less because the reserve rate $r > 0$, but for the fictitious institutions it can be exactly equal.)

Now, the sum of the values coming into layer 1 is the sum of the assets, $\|\boldsymbol{p}\|_1$. Thus the outgoing edges from level 1 to level 2 carry a total weight of at most $\|\boldsymbol{p}\|_1$. Proceeding inductively through the levels, we see that the sum of the values on the incoming edges at level i is at most $\|\boldsymbol{p}\|_1$. Thus the book value of all the institutions on level i is at most $\|\boldsymbol{p}\|_1$ and in particular the book value of any given institution is at most $\|\boldsymbol{p}\|_1$.

D Proof of Theorem 1

In this section, we prove Theorem 1, which states that an ϵ change in holdings can only result in a $\frac{\epsilon}{r}$ multiplicative change in the final valuations of the institutions.

Proof. Let $\tilde{\mathbf{C}} = \mathbf{C} + \mathbf{E}$, and $\hat{\tilde{\mathbf{C}}} = \hat{\mathbf{C}} + \hat{\mathbf{E}}$. By hypothesis $\|\mathbf{E}\| + \|\hat{\mathbf{E}}\| < \epsilon$. Then we have

$$\begin{aligned}
\hat{\tilde{\mathbf{C}}}(\mathbf{I}-\tilde{\mathbf{C}})^{-1} - \hat{\mathbf{C}}(\mathbf{I}-\mathbf{C})^{-1} &= \left[(\hat{\mathbf{C}}+\hat{\mathbf{E}}) - \hat{\mathbf{C}}(\mathbf{I}-\mathbf{C})^{-1}(\mathbf{I}-\tilde{\mathbf{C}})\right](\mathbf{I}-\tilde{\mathbf{C}})^{-1} \\
&= \left[(\hat{\mathbf{C}}+\hat{\mathbf{E}}) - \hat{\mathbf{C}}(\mathbf{I}-\mathbf{C})^{-1}(\mathbf{I}-\mathbf{C}-\mathbf{E})\right](\mathbf{I}-\tilde{\mathbf{C}})^{-1} \\
&= \left[(\hat{\mathbf{C}}+\hat{\mathbf{E}}) - \hat{\mathbf{C}}(\mathbf{I}-(\mathbf{I}-\mathbf{C})^{-1}\mathbf{E})\right](\mathbf{I}-\tilde{\mathbf{C}})^{-1} \\
&= \left[\hat{\mathbf{E}} + \hat{\mathbf{C}}(\mathbf{I}-\mathbf{C})^{-1}\mathbf{E}\right](\mathbf{I}-\tilde{\mathbf{C}})^{-1}
\end{aligned}$$

Now, we notice that

$$\left\|(\mathbf{I}-\tilde{\mathbf{C}})^{-1}\right\| = \left\|\sum_{k=0}^{\infty}\tilde{\mathbf{C}}^k\right\| \le \sum_{k=0}^{\infty}\left\|\tilde{\mathbf{C}}^k\right\| \le \sum_{k=0}^{\infty}(1-r)^k = \frac{1}{r}$$

Because money is never created or destroyed, we have

$$\left\|\hat{\mathbf{C}}(\mathbf{I}-\mathbf{C})^{-1}\boldsymbol{v}\right\|_1 = \|\boldsymbol{v}\|_1$$

Thus we have

$$\begin{aligned}
\left\|\hat{\tilde{\mathbf{C}}}(\mathbf{I}-\tilde{\mathbf{C}})^{-1} - \hat{\mathbf{C}}(\mathbf{I}-\mathbf{C})^{-1}\right\| &= \left\|\left[\hat{\mathbf{E}} + \hat{\mathbf{C}}(\mathbf{I}-\mathbf{C})^{-1}\mathbf{E}\right](\mathbf{I}-\tilde{\mathbf{C}})^{-1}\right\| \\
&\le \left\|\hat{\mathbf{E}}\right\| \cdot \left\|(\mathbf{I}-\tilde{\mathbf{C}})^{-1}\right\| + \left\|\hat{\mathbf{C}}(\mathbf{I}-\mathbf{C})^{-1}\right\| \cdot \|\mathbf{E}\| \cdot \left\|(\mathbf{I}-\tilde{\mathbf{C}})^{-1}\right\|
\end{aligned}$$

Thus we immediately get the bound

$$\left\|\hat{\tilde{\mathbf{C}}}(\mathbf{I}-\tilde{\mathbf{C}})^{-1} - \hat{\mathbf{C}}(\mathbf{I}-\mathbf{C})^{-1}\right\| \le \frac{\epsilon}{r}$$

Since $\left\|\hat{\mathbf{C}}(\mathbf{I}-\mathbf{C})^{-1}\right\| \le 1$, we also have the trivial bound

$$\left\|\hat{\tilde{\mathbf{C}}}(\mathbf{I}-\tilde{\mathbf{C}})^{-1} - \hat{\mathbf{C}}(\mathbf{I}-\mathbf{C})^{-1}\right\| \le 2$$

E Proof of Theorem 2

In this section, we prove Theorem 2 by exhibiting a simple network where increasing one link by ϵ and decreasing another by ϵ leads to a change in valuation of $\frac{\epsilon}{r+\frac{1-r}{2}}\|\mathbf{D}\boldsymbol{p}\|$.

Proof. We exhibit an initial network in Figure 2 and its perturbation in Figure 3.

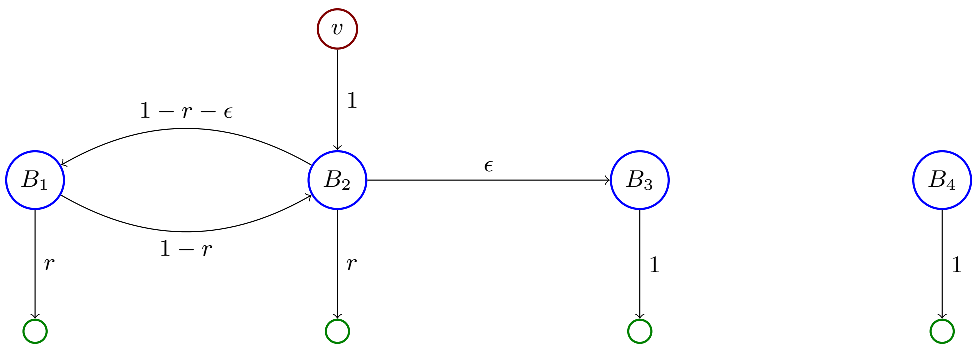

Fig. 2. The initial configuration, banks are in blue, external shareholders are in green, and the asset is shown in red.

In Figure 2, the equity values for the banks satisfy

$$\begin{aligned}
B_1 &= (1-r-\epsilon)B_2 \\
B_2 &= v + (1-r)B_1 \\
B_3 &= \epsilon B_2 \\
B_4 &= 0
\end{aligned}$$

So

$$\begin{aligned}
B_2 &= v + (1-r)B_1 \\
&= v + (1-r)(1-r-\epsilon)B_2 \\
&= v + (1-r-\epsilon-r+r^2+r\epsilon)B_2
\end{aligned}$$

Rearranging gives

$$
\begin{aligned}
B_2 &= v + (1 - r - \epsilon - r + r^2 + r\epsilon)B_2 \\
&\Downarrow \\
v &= (2r + \epsilon - r^2 - r\epsilon)B_2 \\
&\Downarrow \\
B_2 &= \frac{v}{2r + \epsilon - r^2 - r\epsilon} \\
&= \frac{v}{r(2-r) + (1-r)\epsilon} \\
&\geq \frac{1}{2}\left(\frac{v}{r + \frac{1-r}{2}\epsilon}\right)
\end{aligned}
$$

Thus the market valuation of B_3 is ϵB_2 which is at least $\frac{1}{2}\left(\frac{\epsilon v}{r+\frac{1-r}{2}\epsilon}\right)$.

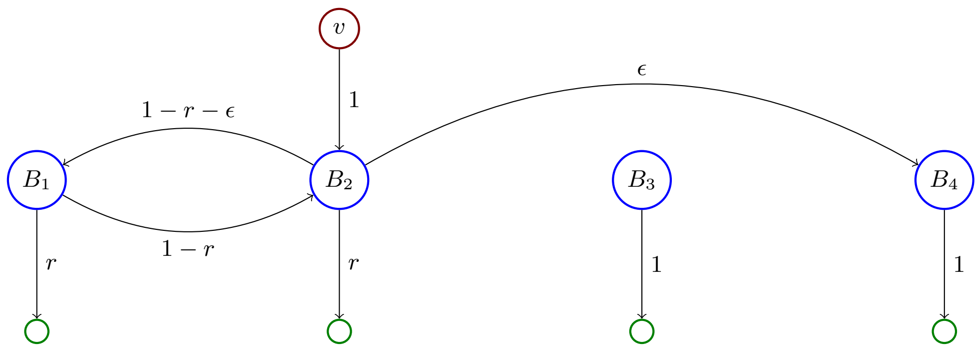

Fig. 3. The perturbed configuration, where one link of weight ϵ has been moved from B_3 to B_4.

If the link from $B_2 \rightarrow B_3$ were moved to $B_2 \rightarrow B_4$ (as in Figure 3) then B_3's value drops to zero and B_4's value increases to ϵB_2.

Thus the change in ℓ_1 norm of the market valuations between the two situations is at least

$$
\frac{\epsilon v}{r + \frac{1-r}{2}\epsilon}
$$

Writing this in matrix notation, we have

$$\mathbf{C} = \begin{bmatrix} 0 & 1-r-\epsilon & 0 & 0 \\ 1-r & 0 & 0 & 0 \\ 0 & \epsilon & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \tilde{\mathbf{C}} = \begin{bmatrix} 0 & 1-r-\epsilon & 0 & 0 \\ 1-r & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & \epsilon & 0 & 0 \end{bmatrix}$$

$$\hat{\mathbf{C}} = \hat{\tilde{\mathbf{C}}} = \begin{bmatrix} r & 0 & 0 & 0 \\ 0 & r & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Note that increasing one link by ϵ and decreasing another by ϵ is actually a change in 2ϵ in the $\|\mathbf{C} - \hat{\mathbf{C}}\|$. Letting $\epsilon' = \frac{\epsilon}{2}$, we have a change of ϵ in $\|\mathbf{C} - \hat{\mathbf{C}}\|$ yields a change of at most $\frac{\epsilon}{2r + \frac{1-r}{2}\epsilon}$. Notice that as $r \to 0$, this is approaches 2, *i.e.*, the resulting valuation is *as far as possible* in terms of the ℓ_1 norm of the market values of the institutions.

F The BCBS problem

There is significant evidence that the BCBS problem is hard to approximate to within a factor of n^δ. Feige showed that for some $\delta > 0$ it is Random 3-SAT hard to approximate BCBS to within a factor of n^δ (Theorem 3 [Fei02]).

Feige and Kogan showed that if BCBS can be approximated to within a factor of $2^{(\log n)^\delta}$ for every $\delta > 0$ then 3-SAT can be solved in time $2^{n^{3/4}+\epsilon}$ for every $\epsilon > 0$ (Theorem 1.3 [FK04]).

Feige and Kogan go on to conjecture that there is no polynomial time algorithm to approximate BCBS to within a factor of n^δ (Conjecture 1.1 [FK04]).

G Proof of Theorem 3

Proof. Let $\ell > 0$ be any integer.

Our starting point is the following hardness result for the BCBS problem. Given an $n \times n$ balanced bipartite graph G, it is hard to decide whether the largest balanced bipartite clique size in G is at least $K \times K$ or at most $K/g \times K/g$ for some gap function g. For instance, $g = 2^{(\log n)^\delta}$ under the assumption that 3-SAT $\notin DTIME\left(2^{n^{3/4+\epsilon}}\right)$ for some $\epsilon > 0$.

Given an $n \times n$ balanced bipartite graph G, we will construct a financial network with $(2+\ell)n$ institutions such that if G has a balanced bipartite subgraph of size K, then a drop in asset prices by $K\epsilon$ can cause at least $(2+\ell)K$ failures. On the other hand, if the largest balanced bipartite subgraph of G is of size $\frac{K}{g}$, a drop in asset prices of $K\epsilon$ can cause at most $K + \frac{K}{g}(\ell+1)$ failures.

This shows that estimating the maximum number of failures induced by a fixed drop in asset prices is at least as hard as estimating the size of the maximum balanced bipartite clique. Without loss of generality, we assume that every vertex in G has degree at least K.

Let D denote the maximum degree of any vertex in G. Let $0 < \epsilon < 1$ be an arbitrary parameter, and let $0 < r < 1$ denote the minimum amount of self-holdings of the institutions in the network we are constructing. (The reduction will hold for any choices of $0 < \epsilon, r < 1$.)

For each node in the graph G, we will associate a financial institution. Let institutions $1, \ldots, n$ correspond to the left-hand nodes of G, and institutions $n+1, \ldots, 2n$ correspond to the right-hand nodes of G.

We will also generate n underlying assets, labelled $a'_i, \ldots, a'_n$ and institution i will complete own asset a'_i for $i = 1, \ldots, n$. Institutions $n+1, \ldots, 2n$ will own none of the underlying assets. All assets will initially be valued at 1. Thus

$$\mathbf{D} = \begin{bmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ & & & 0 & & \\ & & & & \ddots & \\ & & & & & 0 \end{bmatrix}$$

Define $N = \frac{D}{1-r}$ (recall D is the maximum degree of G, and r is an arbitrary parameter that will determine the *integration* of the resulting financial network). We will use $\Gamma(j)$ to denote the neighbors of vertex j in G. Notice that our definition of N ensures that

$$1 - r = \frac{D}{N} \geq \frac{|\Gamma(j)|}{N}$$

for all $j = 1, \ldots, 2n$. This means that if institution j sells an equal $\frac{1}{N}$ stake in itself to all of its neighbors, it will be left with at least an r fraction of self ownership. To operationalize this, we define

$$c_{ij} = \begin{cases} \frac{1}{N} & \text{if } i > j \text{ and } (i,j) \text{ an edge of } G \\ 0 & \text{otherwise} \end{cases}$$

For $1 = 1, \ldots, n$ let $\underline{v}_i = 1 - \frac{|\Gamma(i)|}{N} - \epsilon$, so if institution i's asset drops in value by ϵ then institution i will fail. Let the failure penalty $\beta_i = 1 - \frac{|\Gamma(i)|}{N} - \epsilon$ for $i = 1, \ldots, n$. Thus if asset i drops in price by ϵ, institution i fails and its value immediately drops to 0. For $i = n+1, \ldots, 2n$ let $\underline{v}_i = \frac{|\Gamma(i)| - d}{N}$. Notice that if all assets are initially valued at 1 then

$$v_i = \begin{cases} 1 - \frac{|\Gamma(i)|}{N} & \text{if } 1 \leq i \leq n \\ \\ \frac{|\Gamma(i)|}{N} & \text{if } n < i \leq 2n \end{cases}$$

This financial network has the following properties:

1. If $j > n$ and d of bank j's neighbors fail, *i.e.*, d of the assets a_i $(i \in \Gamma(j))$ drop in value by ϵ, then bank j will fail.

2. If $j > n$ and the total drop in value of bank j's neighbors is less than $\frac{d}{N}$ then bank j will not fail.

Now, we examine the properties of this system when the assets are allowed to drop in price by total amount $d\epsilon$. A drop of $d\epsilon$ can always cause d institutions on the left-hand side of the network to fail, simply by dropping the value of each of their assets by ϵ.

What happens if the price drop is not concentrated among exactly d assets? Let t denote the number of assets that drop in value by at least ϵ. If $t < d$, then at least $t\epsilon$ from the "shock budget" of $d\epsilon$ was used to lower the price of these t assets, which leaves a budget of $(d-t)\epsilon$ remaining. Now, consider how much this drop can affect one of a right-hand institutions, j. Even if this drop is concentrated entirely among the left-hand neighbors of j, the drop j feels is at most

$$\frac{1}{N}\left(t + (d-t)\epsilon\right) < \frac{1}{N}\left(t + (d-t)\right) = \frac{d}{N}$$

Thus institution j cannot fail, since the failure of a right-hand institution requires a drop in value of at least $\frac{d}{N}$. This means that in this case exactly $t < d$ institutions fail. Since we are interested in the *maximum* number of failures that can arise from a drop in asset value of $d\epsilon$, we can, without loss of generality, assume that *exactly* d assets drop in value by ϵ, causing *exactly* d failures among the left-hand institutions.

Now, suppose there is a biclique of size K in G. If $d = K$, then causing d failures among the left-hand members of this biclique will cause $d = K$ failures among the right-hand members.

On the other hand, suppose the largest biclique is of size $\frac{K}{g}$. If the failure of d left-hand institutions causes the failure of P right-hand institutions, then each of the P failed institutions on the right must be connected to each of the d failed institutions on the left. Thus there must be a biclique of size $\min(d, P) = P$.

Thus in the "yes" case (G has a biclique of size K) we can cause at least K right hand failures with a failure budget of $K\epsilon$. In the "no" case (the largest biclique in G is of size ${}^{K}/_{g}$) the maximum number of failures of right-hand institutions is bounded by $\frac{K}{g}$.

Now, to amplify this discrepancy, we add a chain of ℓ institutions connected to each right hand institution. Thus for every right-hand institution, b_i, it will have a chain of institutions $b_i^{(1)}, \ldots, b_i^{(\ell)}$ where $b_i^{(j)}$ owns a $1-r$ fraction of $b_i^{(j-1)}$ and has no other holdings. Thus if b_i fails, then $b_i^{(1)}$ through $b_i^{(\ell)}$ fail as well (see Figure 4).

This new network has $(\ell+2)n$ banks, and has the following properties. Given a failure budget of $K\epsilon$, if the largest balanced clique in G is a $K \times K$, a drop in value by $K\epsilon$ causes at most $(2+\ell)K$, but if the largest biclique is of size $\frac{K}{g}$, then a drop in asset values of at most $K\epsilon$ can cause at most $K + (\ell+1)\frac{K}{g} = (g+\ell+1)\frac{K}{g}$ failures.

Note that when the gap, $g = n^{\delta}$, choosing $\ell = poly(n)$, we obtain a gap of $((\ell+2)n)^{\delta'}$ for some $\delta' < \delta$.

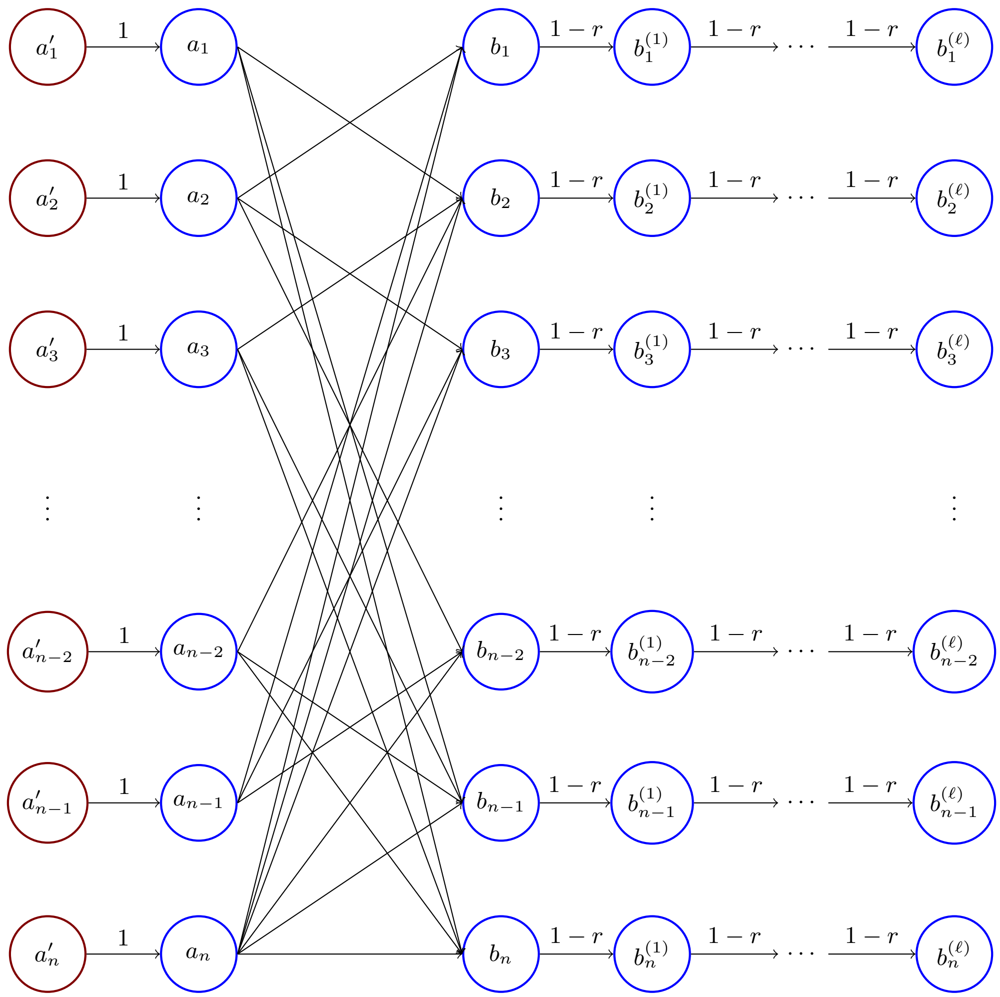

Fig. 4. There are n assets, $\{a'_i\}$, shown in red. Each asset a'_i is fully owned by institution a_i. Institutions $\{a_i\}$, $\{b_i\}$ correspond to a hard instance of balanced bipartite clique. Institutions $\left\{b_i^{(j)}\right\}$ serve to amplify the failures that occur in the first level.