

Graph Connectivity and Single Element Recovery via Linear and OR Queries

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Abstract

We study the problem of finding a spanning forest in an undirected, n -vertex multi-graph under two basic query models. One are **Linear** queries which are linear measurements on the incidence vector induced by the edges; the other are the weaker **OR** queries which only reveal whether a given subset of plausible edges is empty or not. At the heart of our study lies a fundamental problem which we call the *single element recovery* problem: given a non-negative vector $x \in \mathbb{R}_{\geq 0}^N$, the objective is to return a single element $x_j > 0$ from the support. Queries can be made in rounds, and our goals is to understand the trade-offs between the query complexity and the rounds of adaptivity needed to solve these problems, for both deterministic and randomized algorithms. These questions have connections and ramifications to multiple areas such as sketching, streaming, graph reconstruction, and compressed sensing.

Our main results are as follows:

- For the single element recovery problem, it is easy to obtain a deterministic, r -round algorithm which makes $(N^{1/r} - 1)$ -queries per-round. We prove that this is tight: any r -round *deterministic* algorithm must make $\geq (N^{1/r} - 1)$ **Linear** queries in some round. In contrast, a 1-round $O(\text{polylog}(N))$ -query *randomized* algorithm is known to exist.
- We design a *deterministic* $O(r)$ -round, $\tilde{O}(n^{1+1/r})$ -OR query algorithm for graph connectivity. We complement this with an $\tilde{\Omega}(n^{1+1/r})$ -lower bound for any r -round deterministic algorithm in the OR-model.
- We design a *randomized*, 2-round algorithm for the graph connectivity problem which makes $\tilde{O}(n)$ -OR queries. In contrast, we prove that any 1-round algorithm (possibly randomized) requires $\tilde{\Omega}(n^2)$ -OR queries. A *randomized*, 1-round algorithm making $\tilde{O}(n)$ -**Linear** queries is already known.

All our algorithms, in fact, work with more natural graph query models which are *special* cases of the above, and have been extensively studied in the literature. These are **Cross** queries (cut-queries) and **BIS** (bipartite independent set) queries. In particular, we show a *randomized*, 1-round (non-adaptive) algorithm for the graph connectivity problem which makes only $\tilde{O}(n)$ -Cross queries.

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1 Introduction

Many modern applications compel algorithm designers to rethink random access to input data, and revisit basic questions in a *query access model* where the input is accessed only via answers to certain kinds of queries. There are many reasons for this ranging from data volume (only snapshots of the data can be accessed) to data ownership (access is restricted via certain APIs).

In this paper, we study algorithms accessing an unknown, undirected multi-graph G on n vertices in the following two basic query models. Think of the graph as an unknown non-negative $\binom{n}{2}$ dimension vector x_G with $\text{supp}(x_G)$ denoting the positive coordinates. With this view, answers to these queries below can be interpreted as *measurements* on this vector.

- **Linear Queries (Linear):** Given any non-negative¹ $\binom{n}{2}$ dimension vector a_G , what is $a_G \cdot x_G$?
- **OR Queries (OR):** Given any subset S of the $\binom{n}{2}$ dimensions, is $\text{supp}(x_G) \cap S$ empty?

Reverting back to the combinatorial nature of graphs, it is perhaps more natural to think of different kinds of queries, and indeed the following two have been extensively studied. These are however special² cases, respectively, of the queries above.

- **Cross-additive Queries (Cross):** Given two disjoint subsets A, B of V , $\text{Cross}(A, B)$ returns the number of edges, including multiplicity, that have one endpoint in A and the other in B .
- **Bipartite Independent Set Queries (BIS):** Given two disjoint subsets A, B of V , $\text{BIS}(A, B)$ returns whether or not there is an edge that has one endpoint in A and the other in B .

The above query models (and similar variants such as additive queries [32], cut-queries [55], edge-detection queries [7, 9]) have a rich literature [1, 5, 9, 17, 18, 32, 45, 50, 55]. Most previous works, however, have focused on either *graph reconstruction* [15, 17, 18, 45], or on *parameter estimation* (e.g., estimating the number of edges [9] or triangles [11]). In this work, however, our goal is to understand the power and limitations of these queries to reveal *structural properties* of the underlying graph. In particular, we study the following basic property.

Problem 1 (Graph connectivity). Given query access to an undirected multigraph on the vertex set $V = \{1, \dots, n\}$, return a *spanning forest*.

It is not too hard to implement the *classic* BFS or DFS traversals to obtain an $\tilde{O}(n)$ -query deterministic algorithm for the above problem in either query model. However, such algorithms are *adaptive*, that is, the queries depend on the answers obtained so far. A much more modern algorithm of Ahn, Guha, and McGregor [2] gives³ an $\tilde{O}(n)$ -Linear query *non-adaptive* but *randomized* algorithm for the problem. This raises the following questions that motivate us

What is the rounds-of-adaptivity versus query-complexity trade-off for deterministic algorithms for Problem 1? Can randomization also help in the OR and BIS models?

It turns out that understanding the complexity of **Problem 1** is closely related to understanding an even more basic problem which we discuss below.

¹Non-negativity is for convenience. A general linear query can be broken into two non-negative queries.

²The Cross (and BIS queries) correspond to $\{0, 1\}$ vectors a_G (and subsets) corresponding to cuts. Indeed, our algorithms work with the weaker queries while our lower bounds will be for the stronger queries. It should also be clear that the Linear (and respectively Cross) queries are at least as strong as OR (resp, BIS) queries.

³Using results in [58], one can also obtain a $\tilde{O}(n)$ -query deterministic algorithm in the Cross-query model.

Single Element Recovery.

Consider a non-negative real-valued vector $x \in \mathbb{R}_{\geq 0}^N$ and suppose we have access to x only via Linear or OR queries where the dimension is now N . We define the following problem which we call the *single-element recovery* problem (following the standard “support-recovery” problem in compressed sensing).

Problem 2 (Single-element recovery). Given a non-negative real-valued vector $x \in \mathbb{R}_{\geq 0}^N$, accessed via either Linear-queries or OR-queries, output *any arbitrary* element⁴ from the support $\text{supp}(x)$.

To see how the above problem relates to [Problem 1](#), consider the vector of possible edges incident to a single vertex. A spanning forest must find an edge incident to this vertex. This corresponds to solving [Problem 2](#) on this vector. The problem is also interesting in its own right, with connections to *combinatorial group testing* [24, 25, 48], *compressed sensing* [20, 23, 36], and *coin-weighing problems*. [12, 32, 44, 56]. While most of these works have focused on recovering the full support, we ask the simpler question of just recovering a single element.

If one allows *randomization*, then one can use ℓ_0 -samplers [37] to solve the above problem using $O(\log^2 N \log(\frac{1}{\delta}))$ Linear queries⁵, *non-adaptively*. In fact, ℓ_0 -samplers return a random element in $\text{supp}(x)$. The parameter δ is the error probability. There have been numerous applications of these (see the table in Figure 1 of [41], for instance), and indeed many applications (including the AGM [2] algorithm alluded to above) need only an arbitrary element in the support. This is precisely what is asked in [Problem 2](#). Furthermore, the upper bound for randomized algorithms is nearly tight [37, 41], and therefore, for randomized algorithms, our understanding is pretty much complete. But what can be said about *deterministically* finding a single support element⁶? This is an important question for it relates to deterministic analogs to the various applications stated above.

It is not too hard to make a couple of observations. One, any *non-adaptive* deterministic algorithm for [Problem 2](#) using Linear-queries can in fact be recursively used to completely recover the whole vector. This implies an $\tilde{\Omega}(N)$ information theoretic lower bound. Two, if one allows more rounds, then one can indeed do better using a binary-search style idea. More precisely, in each round the algorithm partitions the search space into $N^{1/r}$ parts and using $N^{1/r}$ queries finds a non-zero part. In this way in r rounds, one can get an algorithm making $N^{1/r}$ -queries per round. This leads to the following fundamental question which we answer in our paper.

What is the rounds-of-adaptivity versus query-complexity trade-off for deterministic algorithms for [Problem 2](#)?

1.1 Motivation and Perspective

Why should we care about the questions above?

- We think that algorithmic question of computation on graphs via queries is as natural and important as the reconstruction question. Indeed, our study was inspired by trying to understand the power of *cut*-queries to check whether a graph was connected or not; this is an (extremely) special case of submodular function minimization. More recently, this type of “property-testing via queries” question on graphs has been asked for matchings by Nisan [50], and more generally for matrix properties by [58] and [52]. Single element recovery is also as

⁴In the case of OR-queries, we can only return the j with $x_j > 0$

⁵A similar result holds with OR queries as well. See [Section 4](#)

⁶A “deterministic ℓ_0 sampler”, if you allow us the abuse of notation.

natural as whole-vector recovery. Indeed, one can imagine a scenario where recovering a big⁷ subset of the support (diseased blood samples, say) faster and with fewer queries may be more beneficial than reconstructing the whole vector.

- The **Linear** query model is closely connected to *linear sketches* that have found plenty of applications in dynamic streaming; see, e.g. [30, 38, 43]. The single element recovery problem also has connections to the *universal relation* **UR**^C problem in communication complexity, which was studied in [41, 47]. Understanding these questions, therefore, have ramifications to other areas. As a concrete example, one consequence of our results is a deterministic, $O(r)$ -pass dynamic streaming algorithm for graph connectivity in $\tilde{O}(n^{1+1/r})$ space. This was not known before.
- We believe the question of the trade-off between rounds versus query complexity is natural and important, especially in today’s world of massively parallel computing. Such trade-offs are closely related to similar questions in communication complexity, number of passes in streaming algorithms, etc. It is worthwhile building up an arsenal of tools to attack such questions. Indeed, one main contribution of this paper is to show how LP-duality can be used as one such tool.
- Why do we focus on deterministic algorithms? Mainly because, as mentioned above, our understanding of the complexity of randomized algorithms for the problems above is near complete. However, in some applications one may require exponentially low error, or has to deal with an “adversary” (say, the one giving updates to a streaming algorithm) that is not oblivious to the algorithm’s randomness; see, e.g. [10]. This further motivates the study of deterministic algorithms in this context. Furthermore, we need to design lower-bounding techniques which only work against deterministic algorithms, and this is of technical interest.

The above questions on round-versus-query-complexity trade-offs for both graph connectivity and single element recovery can be interpreted as asking bounds on *deterministic adaptive sketching* [3, 39]. The answers to the various queries can be thought of as the sketch. With **Linear** queries, the question is related to linear sketches, which in turn, is closely related to dynamic streaming [30, 38, 43]. For instance, as we state below, one consequence of our results is a deterministic, $O(r)$ -pass dynamic streaming algorithm for graph connectivity in $\tilde{O}(n^{1+1/r})$ space.

1.2 Our Results

Our first result is a tight lower bound for the question on single element recovery. The binary-search style algorithm mentioned above is the best one can do.

Result 1. For the single element recovery with **Linear**-query access, any r -round, deterministic algorithm must make $\geq N^{1/r} - 1$ queries in some round.

We should remind the reader that the above lower bound is for vectors whose domain is non-negative rationals. In particular, it does not hold for Boolean vectors⁸. Moving to the continuous domain allows one to use tools from geometry, in particular duality theory and Caratheodory’s theorem, to prove the tight lower bound. We discuss this in more detail in [Section 1.3](#).

⁷As we show later in [Lemma 5.1](#), algorithmically we can get results when the “single” in single element recovery can be larger.

⁸Indeed, for Boolean vector with **Linear** queries one can recover the whole vector if the query vector has exponentially large coefficients. Even when the coefficients are small ($\{0, 1\}$ even), the vector can be recovered with $O(n/\log n)$ -queries which is information theoretically optimal.

As mentioned above, **Linear** queries are stronger than **OR** queries, and thus the above lower bound holds for **OR** queries as well. The proof for **OR** queries, however, is combinatorial, arguably simpler, and more importantly can be generalized to prove the following lower bound for [Problem 1](#) as well.

Result 2. Any r -round deterministic algorithm for finding a spanning forest, must make $\tilde{\Omega}(n^{1+\frac{1}{r}})$ -**OR** queries. Formal statement in [Theorem 5](#).

As we explain below, the above smooth trade-off between rounds and query complexity is optimal, even when we allow the weaker **BIS**-queries. Algorithmically, we have the following result. We mention that such a result was not known even using **Linear** or **Cross** queries. A similar lower bound as in [Result 2](#) with **Cross**-queries is left open.

Result 3. For any positive integer r , there exists an $O(r)$ -round deterministic algorithm which makes $\tilde{O}\left(n^{1+\frac{1}{r}}\right)$ -**BIS** queries per round, and returns a spanning forest of the graph. Formal statement in [Theorem 7](#).

It is worth remarking that our algorithm with **Linear** queries (which is implied by the weaker **BIS** queries) above also implies an $O(r)$ -pass $\tilde{O}(n^{1+1/r})$ -space *deterministic* algorithm for maintaining a spanning forest in *dynamic* graph streams. As the edge updates arise, one simply updates the answers to the various queries made in each round. This result was not known before.

Finally, we show that for [Problem 1](#), randomization is helpful in decreasing the number of rounds. More precisely, we consider Monte-Carlo algorithms.

Result 4. There exists a 2-round randomized algorithm for graph connectivity which makes $\tilde{O}(n)$ -**OR** queries per round. There exists a 4-round randomized algorithm for graph connectivity which makes $\tilde{O}(n)$ -**BIS** queries per round. Any non-adaptive, randomized algorithm for graph connectivity must make $\tilde{\Omega}(n^2)$ -**OR** queries. Formal statements in [Theorem 8](#) and [Theorem 6](#).

[Table 1](#) summarizes our contributions.

1.3 Technical Overview

In this section we give a technical overview of our results. These highlight the main underlying ideas and will assist in reading the detailed proofs which appear in the subsequent sections.

Overview of Result 1. It is relatively easy to prove an r -round lower bound for single element recovery in the **OR**-query model via an adversary argument (see [Appendix A](#)). At a high level, **OR**-queries only mildly interact with each other and can be easily fooled. **Linear** queries, on the other hand, strongly interact with each other. To illustrate: if we know $x(A)$ and $x(B)$ for $B \subseteq A$, then we immediately know $x(A \setminus B)$. This is untrue for **OR**-queries – if x has a non-zero entry in both A and B , nothing can be inferred about its entries in $A \setminus B$. Indeed, this power manifests itself in the non-adaptive, randomized algorithm using **Cross**-queries; it is important that we can use subtraction. This makes proving lower bounds against **Linear**-queries distinctly harder.

In our proof of [Result 1](#), we use duality theory. To highlight our idea, for simplicity, let's consider a warmup *non-adaptive* problem. The algorithm has to ask $\ll \sqrt{N}$ queries, and on obtaining the response, needs to return a subset $S \subseteq [N]$ of size $\ll \sqrt{N}$ with the guarantee that $\text{supp}(x) \cap S$ is not empty. Note that if this were possible, then there would be a simple 2-round $o(\sqrt{N})$ -algorithm — simply query the individual coordinates of S in the second round. This is what we want to

		Linear Cross		OR BIS	
		Upper Bound	Lower Bound	Upper Bound	Lower Bound
Single Element Recovery	Det	$r, N^{1/r} - 1$	$r, N^{1/r} - 1$	$r, N^{1/r} - 1$	$r, N^{1/r} - 1$
	Rand	$r = 1, O(\log^2 N)$	$r = 1, \Omega(\log^2 N)$ [37]	$r = 1, O(\log^2 N)$	$r = 1, \Omega(\log^2 N)$
Graph Connectivity	Det	$O(r), n^{1+1/r}$?	$O(r), n^{1+1/r}$	$r, \tilde{\Omega}(n^{1+1/r})$
	Rand	$r = 1, \tilde{O}(n)$ [2] $\tilde{O}(n)$ [2, 58]	$r, \Omega(n/\log n)$	$r = 2 4, \tilde{O}(n)$	$r = 1, \tilde{\Omega}(n^2)$

Table 1: *Summary of the state-of-the-art and our results for graph connectivity and single-element recovery problems.* In each cell, we write the number of rounds followed by the query complexity per round. All lower bounds are with respect to the stronger model (Linear and OR). For upper bounds, if there is a discrepancy between the stronger and weaker models, we show this using a | as partition. Bold results are ours. The remaining results are folklore unless a reference is explicitly cited. The ? indicates the main open question of our paper.

disprove. Therefore, given the first round’s $\ll \sqrt{N}$ queries, we need to show there exists responses such that *no matter* which set S of $\ll \sqrt{N}$ size is picked, there exists a feasible $x \in \mathbb{R}_{\geq 0}^N$ which sets all entries in S to 0. Note this is a $\exists \forall \exists$ -statement. How does one go ahead establishing this?

We first observe that for a fixed response \mathbf{a} and a fixed set S , whether or not a feasible $x \in \mathbb{R}_{\geq 0}^N$ exists is asking whether a system of linear inequalities has a feasible solution. Farkas Lemma, or taking the dual, tells us exactly when this is the case. The nice thing about the dual formulation is that the “response” \mathbf{a} becomes a “variable” in the dual program, as it should be since we are trying to find it. To say it another way, taking the dual allows us to assert conditions that the response vector \mathbf{a} must satisfy, and the goal becomes to hunt for such a vector. How does one do that? Well, the conditions are once again linear inequalities, and we again use duality. In particular, we use Farkas Lemma again to obtain conditions certifying the *non-existence* of such an \mathbf{a} . The final step is showing that the existence of this certificate is impossible. This step uses another tool from geometry — Carathéodory’s theorem. Basically, it shows that if a certificate exists, then a *sparse* certificate must exist. And then a simple counting argument shows the impossibility of sparse certificates. This, of course, is an extremely high-level view and for just the warmup problem. In Section 2 we give details of this warmup, an also details of how one proves the general r -round lower bound building on it.

The interested reader may be wondering about the two instantiations of duality (isn’t the dual of the dual the primal?). We point out that duality can be thought of as transforming a \exists statement into a \forall statement: feasibility is a \exists statement, Farkas implies infeasibility is a different \exists statement, and negating we get the original feasibility as a \forall statement. Since we were trying to assert a $\exists \forall \exists$ -statement, the two instantiations of duality hit the two different \exists .

Overview of Result 2. At a high level, the lower bound for Problem 1, the spanning forest problem, boils down to a “direct sum” version of Problem 2, the single element recovery problem. Imagine the graph is an $n \times n$ bipartite graph. Therefore, finding a spanning forest requires us finding an edge incident to *each* of the n vertices on one side. This is precisely solving n -independent versions of Problem 2 in parallel. However, note that a single query can “hit” different instances at once. The question is, as all direct-sum questions are, does this make the problem n -times harder? We do not know the answer for Linear queries and leave this as the main open question of our

work. However, we can show that the simpler, combinatorial proof of [Result 1](#) against OR-queries does have a direct-sum version, and gives an almost tight lower bound for [Problem 1](#). This is possible because OR-queries, as mentioned in the previous paragraph, have only mild interaction between them. We show that this interaction cannot help by more than a $\text{poly}(r)$ -factor. Our proof is an adversary argument, and a similar argument was used recently by Nisan [50] to show that matchings cannot be approximated well by deterministic algorithms with OR-queries. Details of this are given in [Section 3.1](#).

Overview of Result 3. In [Section 4](#), we show some simple, folklore, and known results for single element recovery. We build on these algorithms to obtain our algorithms for [Problem 1](#). With every vertex one associates an unknown vector which is an indicator of its neighborhood. If one applies the r -round binary-search algorithm for the single element recovery problem on each such vector, then in r -rounds with $O(n^{1+1/r})$ -BIS queries, for every vertex one can obtain a single edge incident on it. This alone however doesn't immediately help: perhaps, we only detect $n/2$ edges and get $n/2$ disconnected clusters. Recursively proceeding only gives an $O(r \log n)$ -round algorithm. And we would like no dependence on n .

To make progress, we actually give a more sophisticated algorithm for single element recovery than binary search, which gives more and may be of independent interest. In particular, we describe an algorithm ([Lemma 5.1](#)) for single element recovery which in $O(r)$ rounds, and making $N^{1/r}$ -queries per round, can in fact return as many as $N^{1/4r}$ elements in the support. Once we have this, then for graph connectivity we observe that in $O(r)$ rounds, we get polynomially many edges incident on each vertex. Thus as rounds go on, the number of effective vertices decreases, which allows us to query more aggressively. Altogether, we get an $O(r)$ -round algorithm making only $\tilde{O}(n^{1+1/r})$ -BIS queries. The details of this are described in [Section 5](#).

Overview of Result 4. In the overview of the deterministic algorithm, we had to be a bit conservative in that even after every vertex found k edges (k being 1 or $n^{O(1/r)}$) incident on it, we pessimistically assumed that after this step the resulting graph still has $\Theta(n/k)$ disconnected clusters, and we haven't learned *anything* about the edges across these clusters. In particular, we allow for the situation that the cross-cluster edges can be dense. With randomization, however, we get to sample k *random* edges incident on a vertex. This is where we use the recent result of Holm *et al.* [34] which shows that if the k incident edges are random, then, as long as $k = \Omega(\log n)$, the number of *inter-component* edges between the connected components induced by the sampled edges, is $O(n/k)$. That is the cross-cluster edges are sparse. Therefore, in a single round with $\tilde{O}(n)$ -randomized BIS queries, we can obtain a disconnected random subgraph, but one such that, whp, there exist at most $\tilde{O}(n)$ edges across the disconnected components.

Given the above fact, the algorithm is almost immediate. After round 1, we are in a sparse graph (where nodes now correspond to subsets of already connected vertices). If we were allowed general OR-queries, then a single round with $\tilde{O}(n)$ -OR queries suffices to learn this sparse graph, which in turn, gives us a spanning forest in the original graph. This follows from algorithms for single element recovery when the vector is promised to be sparse (discussed in [Section 4](#)). Unfortunately, these queries may not be BIS-queries; recall that BIS-queries are restricted to ask about edges across two subsets. Nevertheless, we can show how to implement the above idea using 2-extra rounds with only BIS-queries, giving a 4-round algorithm. Details can be found in [Section 6.1](#).

To complement the above, we also prove that even with randomization, one cannot get non-adaptive (1-round) $o(n^2/\log^2 n)$ -query algorithms with OR-queries. Indeed, the family of examples is formed by two cliques (dense graphs) which could have a single edge, or not, that connects them. A single collection of $o(n^2/\log^2 n)$ -OR queries cannot distinguish between these two families. Details

can be found in [Section 3.2](#).

1.4 Related Works

Our work falls in the broad class of algorithm design in the *query access model*, where one has limited access to the input. Over the years there has been a significant amount of work relevant to this paper including in graph reconstruction [[1](#),[4](#),[5](#),[7](#),[12](#),[14](#),[15](#),[18](#),[32](#),[45](#),[53](#)], parameter estimation [[9](#),[11](#),[22](#),[54](#)], minimum cuts [[8](#),[55](#)] sketching and streaming [[2](#),[6](#),[8](#),[28](#),[29](#),[37](#),[40](#),[41](#),[47](#),[58](#)], combinatorial group testing, compressed sensing, and coin weighing [[12](#),[20](#),[23–26](#),[56](#)]. It is impossible to do complete justice, but in [Section 7](#) we give a little more detailed discussion of some of these works and how they fit in with our paper.

1.5 Notation

Throughout the paper, for a positive integer p , $[p]$ denotes the set $\{1, 2, \dots, p\}$. Our randomized algorithms are Monte-Carlo and make a fixed number of queries but fail with some probability. We use “with high probability” or “whp” to denote a failure probability of $\frac{1}{\text{poly}(n)}$ where n is the relevant size parameter. The exponent of the polynomial can be traded off with the constant in the query complexity. Given an undirected multigraph G , and two disjoint subsets S and T of vertices, we use $E(S, T)$ to denote the collection of pairs $(s, t) \in S \times T$ such that there is at least one edge between s and t . We use $\partial(S)$ to denote $E(S, S^c)$. We use $\tilde{O}(f(n))$ to hide $\text{polylog}(f(n))$ -factors.

2 Lower Bound for Single Element Recovery

In this section, we prove the following theorem.

Theorem 1. *Any r -round deterministic algorithm for Single Element Recovery must make $\geq (N^{1/r} - 1)$ -Linear queries in some round.*

As discussed in the introduction, this is not difficult to show for OR-queries (see [Theorem 11](#) in [Appendix A](#)), however, it takes some work to obtain the result for Linear-queries. Following the overview in [Section 1.3](#), we start by describing the lower bound for a simple “trapping problem” problem which illustrates the main ideas. The general proof follows inductively. One piece of notation before we begin: given any subset $S \subseteq [N]$, we use $\mathbf{1}_S$ to denote the N -dimensional indicator vector of the subset S with 1 in the index corresponding to elements in S .

2.1 Warmup: A One Round Lower Bound for a Trapping Problem

In this setting, there are two parameters k and s . The former is an upper bound on the number of non-adaptive (1-round) queries. The objective of the algorithm, after obtaining the answers to the queries, is to find a subset $S \subseteq [N]$ with $|S| \leq s$ such that $x_j > 0$ for *some* $j \in S$. That is, a subset S which traps an element of the support. Note that if such an algorithm exists, then there is a 2-round algorithm for single element recovery making k queries in round 1 and s queries in round 2. We also assume that x is a non-zero vector since otherwise $x([N]) = 0$. Furthermore, by scaling, we assume that $x([N]) = 1$. The main lower bound statement is the following.

Theorem 2. *If $(k + 1)s < N$, then there cannot exist such an algorithm.*

Note that if s divides N , then $k = \frac{N}{s} - 1$ queries indeed do suffice. So the above theorem is tight.

Proof. We let \mathbf{A} denote the $k \times N$ matrix corresponding to the k queries arranged as row vectors. We use $\mathbf{a} \in \mathbb{R}_{\geq 0}^k$ to denote the answers we will give to fool any algorithm. To find this, fix any

subset S with $|S| \leq s$, and consider the following system of inequalities parametrized by the answer vector \mathbf{a} . The only inequalities are the non-negativity constraints.

$$\mathcal{P}(\mathbf{a}; S) = \{x \in \mathbb{R}_{\geq 0}^N : x([N]) = 1 \quad \mathbf{A} \cdot x = \mathbf{a} \quad x(S) = 0\} \quad (\text{P})$$

Note that if $\mathcal{P}(\mathbf{a}; S)$ has a feasible solution, then given the answers \mathbf{a} to its queries, the algorithms *cannot* return the subset S . This is because there is a non-negative x consistent with these answers with S disjoint from its support. In other words, S is *safe* for the lower bound w.r.t. \mathbf{a} . Therefore, if there exists an answer vector \mathbf{a} such that *every* subset $S \subseteq [N]$ with $|S| \leq s$ is safe with respect to \mathbf{a} , that is $\mathcal{P}(\mathbf{a}; S)$ is feasible, then we would have proved our lower bound. We use duality and geometry to prove the existence of this vector (if $(k+1)s < N$).

The first step is to understand when for a *fixed* set S , the system $\mathcal{P}(\mathbf{a}; S)$ is infeasible. This is answered by Farkas Lemma. In particular, consider the following system⁹ of inequalities where the variables are the Lagrange multipliers corresponding to the equalities in (P). We note that the variables are *free*, since $\mathcal{P}(\mathbf{a}; S)$ has only equalities in the constraints. For convenience, we have eliminated the variable corresponding to the subset S and have moved it to the right hand side.

$$\mathcal{C}_S := \left\{ (y^{(0)}, \mathbf{y}) \in \mathbb{R} \times \mathbb{R}^k : y^{(0)} \cdot \mathbf{1}_{[N]} + \mathbf{y} \cdot \mathbf{A} \leq \mathbf{1}_S \right\}$$

Farkas Lemma asserts that $\mathcal{P}(\mathbf{a}; S)$ is *infeasible* if and only if there exists $\mathbf{y} \in \mathcal{C}_S$ such that $y^{(0)} \cdot \mathbf{1} + \mathbf{y} \cdot \mathbf{a} > 0$. Contrapositively, we get that $\mathcal{P}(\mathbf{a}; S)$ is feasible, that is S is safe with respect to \mathbf{a} , iff $y^{(0)} + \mathbf{y} \cdot \mathbf{a} \leq 0$ for *all* $\mathbf{y} \in \mathcal{C}_S$. Since we want an answer \mathbf{a} such that *all* subsets S with $|S| \leq s$ are safe, we conclude that such an answer exists if and only if the following system of linear inequalities has a feasible solution.

$$\mathcal{Q} := \left\{ \mathbf{a} \in \mathbb{R}_{\geq 0}^k : \mathbf{y} \cdot \mathbf{a} \leq -y^{(0)}, \quad \forall S \subseteq [N], |S| \leq s, \quad \forall (y^{(0)}, \mathbf{y}) \in \mathcal{C}_S \right\} \quad (\text{D})$$

In summary, to prove the lower bound, it suffices to show that \mathcal{Q} has a feasible solution, and this solution will correspond to the answers to the queries. Suppose, for the sake of contradiction, \mathcal{Q} is *infeasible*. Then, again by Farkas Lemma (but on a different system of inequalities), there exists multipliers $\lambda_t \geq 0$ corresponding to constraints $(S_t \text{ s.t. } |S_t| \leq s, (y_t^{(0)}, \mathbf{y}_t) \in \mathcal{C}_{S_t})$ for some $t = 1 \dots T$ such that (P1) $\sum_{t=1}^T \lambda_t \mathbf{y}_t \geq \mathbf{0}_k$ where $\mathbf{0}_k$ is the k -dimensional all zero (row) vector, and (P2) $\sum_{t=1}^T \lambda_t y_t^{(0)} > 0$. Note that this time λ_t 's are non-negative since \mathcal{Q} has inequalities in the constraints.

We can focus on the λ_t 's which are *positive* and discard the rest. The next key observation is to *upper bound* the size T of the support. Note that the conditions (P1) and (P2) can be equivalently stated as asserting that the $(k+1)$ -dimensional *cone* spanned by the vectors $(y_t^{(0)}, \mathbf{y}_t)$ contains a non-negative point with first coordinate positive. Caratheodory's theorem (for cones) asserts that any such point can be expressed as a conic combination of at most $(k+1)$ vectors. Therefore, we can assume that $T \leq k+1$.

Now we are almost done. Since $(y_t^{(0)}, \mathbf{y}_t) \in \mathcal{C}_{S_t}$, we have $y_t^{(0)} \cdot \mathbf{1}_{[N]} + \mathbf{y}_t \cdot \mathbf{A} \leq \mathbf{1}_{S_t}$. Taking λ_t combinations and adding, we get (since all $\lambda_t > 0$) that

$$\left(\sum_{t=1}^T \lambda_t y_t^{(0)} \right) \cdot \mathbf{1}_{[N]} + \left(\sum_{t=1}^T \lambda_t \mathbf{y}_t \right) \cdot \mathbf{A} \leq \sum_{t=1}^T \lambda_t \mathbf{1}_{S_t}$$

⁹Here \mathbf{y} and $\mathbf{1}_S$ are *row* vectors. In the general proof, there will be multiple \mathbf{y} 's indexed with super-scripts. All of them are row-vectors. Putting an added $^\top$ would be a notational mess.

Since every $|S_t| \leq s$, the support of the right hand side vector is $\leq sT \leq s(k+1)$. The support of the left hand side vector is $\geq N$. This is because the second summation is a non-negative vector by (P1), and the first has full support. This contradicts $(k+1)s < N$. Hence, \mathcal{Q} has a feasible solution, which in turn means there exists answers \mathbf{a} which foils \mathbf{A} . This proves [Theorem 2](#). ■

2.2 The General r -round Lower Bound

We begin by formally defining what an r -round deterministic algorithm is, and what it means for such an algorithm to successfully solve Single Element Recovery.

Definition 2.1 (r -round deterministic algorithm.). *An r -round deterministic algorithm \mathcal{A} proceeds by making a collection of linear queries $\mathbf{A}^{(1)} \in \mathbb{R}_{\geq 0}^{k_1 \times N}$ and obtains the answer $\mathbf{a}^{(1)} = \mathbf{A}^{(1)} \cdot x$. This is the first round of the algorithm. For $1 < i \leq r$, in the i th round the algorithm makes a collection of linear queries $\mathbf{A}^{(i)} \in \mathbb{R}_{\geq 0}^{k_i \times N}$. This matrix depends on the history $(\mathbf{A}^{(1)}, \mathbf{a}^{(1)}), \dots, (\mathbf{A}^{(i-1)}, \mathbf{a}^{(i-1)})$. Upon making this query it obtains the answer $\mathbf{a}^{(i)} = \mathbf{A}^{(i)} \cdot x$. We call $\Pi_r := ((\mathbf{A}^{(1)}, \mathbf{a}^{(1)}), \dots, (\mathbf{A}^{(r)}, \mathbf{a}^{(r)}))$ the r -round transcript of the algorithm. The output of the deterministic algorithm \mathcal{A} only depends on Π_r .*

A vector $y \in \mathbb{R}_{\geq 0}^N$ is said to be **consistent** with respect to a transcript Π_r if $\mathbf{A}^{(i)} \cdot y = \mathbf{a}^{(i)}$ for all $1 \leq i \leq r$. A transcript $\Pi_r = ((\mathbf{A}^{(1)}, \mathbf{a}^{(1)}), \dots, (\mathbf{A}^{(r)}, \mathbf{a}^{(r)}))$ is **feasible** for the algorithm if there is some vector y consistent with respect to it, and if the algorithm indeed queries $\mathbf{A}^{(i)}$ given the $(i-1)$ -round transcript $((\mathbf{A}^{(1)}, \mathbf{a}^{(1)}), \dots, (\mathbf{A}^{(i-1)}, \mathbf{a}^{(i-1)}))$

Definition 2.2. An r -round deterministic algorithm \mathcal{A} is said to successfully solve Single Element Recovery if for all non-zero $x \in \mathbb{R}_{\geq 0}^N$, upon completion of r -rounds the algorithm \mathcal{A} returns a coordinate $j \in [N]$ with $x_j > 0$. In particular, if the algorithm returns a coordinate j given a feasible transcript Π_r , then every x that is consistent with Π_r must have $x_j > 0$.

For technical reasons, we add a 0th-round for any r -round algorithm. In this round, the query “matrix” $\mathbf{A}^{(0)}$ is the single N -dimensional row with all ones. That is, we ask for the sum of x_j for all $j \in [N]$. We assume that the answer $\mathbf{a}^{(0)}$ is the scalar 1 to capture the fact that the vector x is non-zero.

Next we define the notion of safe subsets with respect to a transcript generated till round i . A safe subset of coordinates are those for which there is a consistent vector x whose support is disjoint from the subset, that is, $x_j = 0$ for all $j \in S$, or equivalently $x(S) = 0$ since $x \geq 0$.

Definition 2.3. Given an i -round transcript $\Pi_i = ((\mathbf{A}^{(0)}, \mathbf{a}^{(0)}), \dots, (\mathbf{A}^{(i)}, \mathbf{a}^{(i)}))$, a subset $S \subseteq [N]$ is **safe** w.r.t. Π_i if the following system of linear inequalities

$$\mathcal{P}(\mathbf{a}^{(\leq i)}; S) := \left\{ x \in \mathbb{R}^N : \begin{cases} \mathbf{A}^{(j)} \cdot x = \mathbf{a}^{(j)} & \forall 0 \leq j \leq i \\ x(S) = 0 \\ x \geq 0 \end{cases} \right\} \quad (\text{Primal})$$

has a feasible solution.

Claim 2.1. If Π_r is a feasible r -round transcript of an algorithm \mathcal{A} such that all singletons are safe w.r.t Π_r , then the algorithm \mathcal{A} cannot be successful in solving Single Element Recovery.

Proof. Given Π_r , the algorithm \mathcal{A} must return some coordinate $j \in [N]$. However $\{j\}$ is safe. That is, there is a feasible solution x to $\mathcal{P}(\mathbf{a}^{(\leq r)}, \{j\})$. Indeed, if x were the input vector, the algorithm would return a coordinate not in the support. ■

Definition 2.4 (Transcript Creation Procedure). *Given an r -round algorithm \mathcal{A} , the transcript creation procedure is the following iterative process. In round i , given the transcript $\Pi_{i-1} := ((\mathbf{A}^{(0)}, \mathbf{a}^{(0)}), \dots, (\mathbf{A}^{(i-1)}, \mathbf{a}^{(i-1)}))$ upon which the algorithm \mathcal{A} queries $\mathbf{A}^{(i)}$, and the transcript creation procedure produces an answer $\mathbf{a}^{(i)}$ such that $\Pi_i = \Pi_{i-1} \circ (\mathbf{A}^{(i)}, \mathbf{a}^{(i)})$ is feasible.*

Our main theorem, which implies [Theorem 1](#), is the following.

Theorem 3 (Transcript Creation Theorem). *Let k_1, \dots, k_r and s_0, s_1, \dots, s_r be positive integers such that $s_0 \leq n - 1$ and $(k_i + 1)s_i \leq s_{i-1}$ for all $i \geq 1$. Then given any r -round algorithm \mathcal{A} making $\leq k_i$ queries in round i , there is a transcript creation procedure to create an r -round transcript such that for all $0 \leq i \leq r$, any subset $S \subseteq [N]$ with $|S| \leq s_i$ is safe with respect to Π_i .*

Corollary 4. *Let k_1, \dots, k_r be any r positive integers with $\prod_{i=1}^r (k_i + 1) < n$. No r -round algorithm \mathcal{A} which makes $\leq k_i$ queries in round i can be successful for the Single Element Recovery problem. In particular, this implies [Theorem 1](#).*

Proof. Set $s_r = 1$, $s_{r-1} = (k_r + 1)$, and in general, $s_i = (k_r + 1)(k_{r-1} + 1) \cdots (k_{i+1} + 1)$. Note that the conditions of [Theorem 3](#) are satisfied. Therefore given any algorithm \mathcal{A} making $\leq k_i$ queries in round i , we can create a r -round transcript such that all singleton sets are safe with respect to Π_r . [Claim 2.1](#) implies \mathcal{A} cannot be successful. ■

2.2.1 Proof of the Transcript Creation Theorem

We start with writing the dual representation of safe sets. Fix a subset $S \subseteq [N]$ and a transcript Π_i . By Farkas lemma we know that the system $\mathcal{P}(\mathbf{a}^{(\leq i)}; S)$ is infeasible only if there exists a infeasibility certificate

$$(\mathbf{y}^{(0)}, \mathbf{y}^{(1)}, \dots, \mathbf{y}^{(i)}) \in \mathbb{R} \times \mathbb{R}^{k_1} \times \cdots \times \mathbb{R}^{k_i} : \sum_{j=0}^i \mathbf{y}^{(j)} \cdot \mathbf{A}^{(j)} \leq \mathbf{1}_S \quad \text{and} \quad \sum_{j=0}^i \mathbf{y}^{(j)} \cdot \mathbf{a}^{(j)} > 0$$

Here $\mathbf{1}_S$ is the n -dimensional indicator vector of the subset S , that is, it has 1 in the coordinates $j \in S$ and 0 otherwise. Taking negations, we get that the system $\mathcal{P}(\mathbf{a}^{(\leq i)}; S)$ is *feasible*, that is $S \subseteq [N]$ is safe w.r.t Π_{i-1} , if and only if the following condition holds

$$S \text{ is safe w.r.t. } \Pi_i \text{ iff } \mathbf{y}^{(\leq i)} \cdot \mathbf{a}^{(\leq i)} := \sum_{j=0}^i \mathbf{y}^{(j)} \cdot \mathbf{a}^{(j)} \leq 0 \quad \text{for all } \mathbf{y}^{(\leq i)} \in \mathcal{C}_S^{(i)} \quad (\text{Dual})$$

where,

$$\mathcal{C}_S^{(i)} := \left\{ \mathbf{y}^{(\leq i)} := (\mathbf{y}^{(0)}, \mathbf{y}^{(1)}, \dots, \mathbf{y}^{(i)}) \in \mathbb{R} \times \mathbb{R}^{k_1} \times \cdots \times \mathbb{R}^{k_i} : \sum_{j=0}^i \mathbf{y}^{(j)} \cdot \mathbf{A}^{(j)} \leq \mathbf{1}_S \right\}$$

We are now ready to prove [Theorem 3](#) via induction on i . The above representation is the dual definition of safe sets, and this definition is what is easy to induct with.

Base Case: $i = 0$. We need to show that any subset $S \subseteq [N]$ of size $|S| \leq s_0 = N - 1$ is safe with respect to the transcript $(\mathbf{A}^{(0)}, \mathbf{a}^{(0)})$. To remind the reader, $\mathbf{A}^{(0)}$ is just the all ones vector and $\mathbf{a}^{(0)}$ is just the scalar 1. Using [\(Dual\)](#), we need to show for any subset $S \subseteq [N]$ with $|S| \leq N - 1$, we must have

$$\mathbf{y}^{(0)} \cdot \mathbf{a}^{(0)} \leq 0 \quad \text{for all } \mathbf{y}^{(0)} \in \mathbb{R} \text{ such that } \mathbf{y}^{(0)} \cdot \mathbf{A}^{(0)} \leq \mathbf{1}_S$$

However, $\mathbf{y}^{(0)} \cdot \mathbf{A}^{(0)}$ is the n -dimensional vector which is $\mathbf{y}^{(0)}$ on all coordinates. Since $|S| \leq n - 1$, there is some coordinate $j \notin S$ such that $\mathbf{1}_S[j] = 0$. Thus, $\mathbf{y}^{(0)} \leq 0$ implying $\mathbf{y}^{(0)} \cdot \mathbf{a}^{(0)} \leq 0$. The base case holds.

Inductive Case: $i \geq 1$. Assume the conclusion of the theorem holds for all $0 \leq j \leq i - 1$. That is, there is a procedure which has created a transcript $\Pi_{i-1} = ((\mathbf{A}^{(0)}, \mathbf{a}^{(0)}), \dots, (\mathbf{A}^{(i-1)}, \mathbf{a}^{(i-1)}))$ such that every subset $S \subseteq [N]$ with $|S| \leq s_{i-1}$ is safe w.r.t Π_{i-1} . Using (Dual), we can rewrite this as the following statement

$$\text{for all } S \subseteq [N], |S| \leq s_{i-1}, \text{ for all } \mathbf{y}^{(\leq i-1)} \in \mathcal{C}_S^{(i-1)} \text{ we have } \mathbf{y}^{(\leq i-1)} \cdot \mathbf{a}^{(\leq i-1)} \leq 0. \quad (\text{IH})$$

Given Π_{i-1} , the algorithm \mathcal{A} now queries $\mathbf{A}^{(i)}$ in round i . Our goal is to find answers $\mathbf{a}^{(i)} \in \mathbb{R}_{\geq 0}^{k_i}$ such that any subset $S \subseteq [N]$ with $|S| \leq s_i$ is safe w.r.t $\Pi_i = \Pi_{i-1} \circ (\mathbf{A}^{(i)}, \mathbf{a}^{(i)})$. Again referring to (Dual), we need to find $\mathbf{a}^{(i)} \in \mathbb{R}_{\geq 0}^{k_i}$ satisfying the following system of linear inequalities.

$$\mathcal{Q}^{(i)} := \left\{ \mathbf{a}^{(i)} \in \mathbb{R}_{\geq 0}^{k_i} : \mathbf{y}^{(i)} \cdot \mathbf{a}^{(i)} \leq -\left(\mathbf{y}^{(\leq i-1)} \cdot \mathbf{a}^{(\leq i-1)}\right), \forall S \subseteq [N], |S| \leq s_i, \forall \mathbf{y}^{(\leq i)} \in \mathcal{C}_S^{(i)} \right\}$$

Although it may appear that the above system has infinitely many constraints, it suffices to write the constraints for extreme points for the polyhedra $\mathcal{C}_S^{(i)}$'s. To complete the proof, we need to show that $\mathcal{Q}^{(i)}$ is non-empty; if so, we can select any $\mathbf{a}^{(i)} \in \mathcal{Q}^{(i)}$ for completing the transcript creation procedure, and proving the theorem by induction. The next lemma does precisely that; this completes the proof of the theorem. ■ Theorem 3

Lemma 2.2. *The system of inequalities $\mathcal{Q}^{(i)}$ has a feasible solution.*

Proof. For the sake of contradiction, suppose not. Applying Farkas lemma (again), we get the following certificate of infeasibility. There exists the tuples $(\lambda_t > 0, S_t \subseteq [N]$ with $|S_t| \leq s_i, \mathbf{y}_t^{(\leq i)} \in \mathcal{C}_{S_t}^{(i)})$ for $1 \leq t \leq k_i + 1$ such that

$$(P1): \sum_{t=1}^{k_i+1} \lambda_t \mathbf{y}_t^{(i)} \geq \mathbf{0}_{k_i}, \text{ and}$$

$$(P2): \sum_{t=1}^{k_i+1} \lambda_t \left(\mathbf{y}_t^{(\leq i-1)} \cdot \mathbf{a}^{(\leq i-1)} \right) > 0.$$

Since $\mathbf{y}_t^{(\leq i)} \in \mathcal{C}_{S_t}^{(i)}$, we get $\sum_{j=0}^i \mathbf{y}_t^{(j)} \cdot \mathbf{A}^{(j)} \leq \mathbf{1}_{S_t}$ for all $1 \leq t \leq k_i$. Taking the positive λ_t -combinations of these inequalities, we get

$$\sum_{t=1}^{k_i+1} \lambda_t \cdot \left(\sum_{j=0}^i \mathbf{y}_t^{(j)} \cdot \mathbf{A}^{(j)} \right) \leq \sum_{t=1}^{k_i+1} \lambda_t \mathbf{1}_{S_t} \quad (P3)$$

Now, define $\mathbf{w}^{(j)} := \sum_{t=1}^{k_i+1} \lambda_t \mathbf{y}_t^{(j)}$ for $0 \leq j \leq i$. (P1) above implies (Q1): $\mathbf{w}^{(i)} \geq \mathbf{0}_{k_i}$, and (P2) implies (Q2): $\mathbf{w}^{(\leq i-1)} \cdot \mathbf{a}^{(\leq i-1)} > 0$. And finally, (P3) translates to

$$\underbrace{\sum_{j=0}^{i-1} \mathbf{w}^{(j)} \cdot \mathbf{A}^{(j)}}_{\text{Call this } \mathbf{u}_1} + \underbrace{\mathbf{w}^{(i)} \cdot \mathbf{A}^{(i)}}_{\text{Call this } \mathbf{u}_2} \leq \underbrace{\sum_{t=1}^{k_i+1} \lambda_t \mathbf{1}_{S_t}}_{\text{Call this } \mathbf{v}} \quad (Q3)$$

Now we are ready to see the contradiction. First observe that the vector \mathbf{v} has at most $(k_i + 1)s_i$ positive entries since it is the sum of $k_i + 1$ vectors each of support $\leq s_i$. Since $\mathbf{w}^{(i)}$ and $\mathbf{A}^{(i)}$ are

both non-negative, \mathbf{u}_2 is a non-negative vector. This implies that \mathbf{u}_1 must have $\leq (k_i + 1)s_i$ positive entries. From the conditions of the theorem, we get $(k_i + 1)s_i \leq s_{i-1}$. Thus, \mathbf{u}_1 has $\leq s_{i-1}$ positive entries. This in turn implies there exists a scalar θ such that $\theta\mathbf{u}_1 \leq \mathbf{1}_S$ for some subset $S \subseteq [N]$ with $|S| \leq s_{i-1}$. That is,

$$\sum_{j=0}^{i-1} (\theta\mathbf{w}^{(j)}) \cdot \mathbf{A}^{(j)} \leq \mathbf{1}_S \quad \Rightarrow \quad \theta\mathbf{w}^{(\leq i-1)} \in \mathcal{C}_S^{(i-1)}$$

The induction hypothesis (IH) implies $\theta\mathbf{w}^{(\leq i-1)} \cdot \mathbf{a}^{(\leq i-1)} \leq 0$. This contradicts (Q2). This completes the proof of the lemma. ■

3 Lower Bounds for Graph Connectivity with OR-queries.

In this section we establish our lower bounds with OR-queries. In the first subsection, we show that deterministic, r -round algorithms for finding a spanning forest in an n -vertex undirected graph must make $\tilde{\Omega}(n^{1+1/r})$ -OR queries. This builds on an adversary style lower bound for single element recovery with OR-queries. A similar style of argument was used by Nisan [50] to prove lower bounds for finding approximate matchings using OR-queries. It may be instructive to first read this [Theorem 11](#) in [Appendix A](#), for the lower bound for the spanning forest problem is a direct-sum version of that. In the second subsection, we show that non-adaptive (1-round) *randomized* algorithms for graph connectivity with OR-queries must make $\tilde{\Omega}(n^2)$ -queries.

3.1 Lower Bound for Deterministic r -Round Algorithms

The following theorem formalizes [Result 2](#).

Theorem 5. *For any integer $r \geq 1$, any r -round deterministic algorithm in the OR-query access which returns a spanning forest of any given graph $G(V, E)$ must make $\Omega(\frac{n^{1+\frac{1}{r}}}{r^2 \cdot \log n})$ -OR queries.*

Proof. The proof of this result is via an adversary argument. The ADVERSARY creates instances on a bipartite graph $G(U, V, E)$ with n vertices on each side. Let ALG be any deterministic r -round algorithm that is able to find a single edge incident *every* vertex $u \in U$; this is a much weaker condition than finding a spanning forest. We show a strategy for ADVERSARY that forces ALG to make more than $t := t(n, r) = \left(\frac{n^{1+\frac{1}{r}}}{32r^2 \cdot \ln n}\right)$ queries to the graph *in one of its rounds*. This establishes the lower bound in [Theorem 5](#).

For every vertex $u \in U$, let $x_u \in \{0, 1, *\}^n$ denote the incidence vector that the ADVERSARY maintains. $x_u(i) = 0$ implies that the pair (u, i) (for $i \in V$) is *not* an edge in G , $x_u(i) = 1$ implies that (u, i) is an edge in G , and $x_u(i) = *$ implies that the status of (u, i) is still unknown. More precisely, whenever $x_u(i) = *$, given the ADVERSARY responses to the queries made so far, there are two consistent graphs, one containing the edge (u, i) and one not. Note that finding a coordinate $x_u(i) = 1$, which is precisely the single element recovery problem, means that ALG has succeeded in finding an edge incident to vertex u . However, ALG needs to find an $x_u(i) = 1$ for *every* u . Thus, the problem is precisely solving n single element recovery problems *in parallel*, where the queries are allowed to span multiple vectors corresponding to different vertices.

We start with some definitions. For every vertex u , the ADVERSARY maintains a set $A_u \subseteq [n]$ of **active coordinates**. These are precisely the coordinates i for which $x_u(i) = *$. Initially, $A_u = [n]$ for all vertices u , and all x_u 's are all-star. We say a vertex u is an **alive vertex** if $A_u \neq \emptyset$. For alive vertices u , the vector $x_u \in \{0, *\}^n$ has no coordinate set to 1. Otherwise, we call u **dead**, and for a dead vertex $x_u \in \{0, 1\}^n$. That is, the vector corresponding to a dead vertex is completely known.

Every query Q ever made by the algorithm is answered 0 or 1 by ADVERSARY. We call the former query a 0-query, and the latter a 1-query. Note, Q is classified post facto by the answers, and not up front. We call a 1-query Q *explained* if Q contains some $x_u(i)$ which has been already set to 1. Otherwise, a 1-query is unexplained. On the other hand, for every 0-query, the ADVERSARY maintains that every $x_u(i)$ in Q has been set to 0. Note that by definition, every explained 1-query must touch a dead vertex, and every unexplained 1-query Q must intersect A_u for at least one vertex u . We are now ready to state the goal of the ADVERSARY.

Lemma 3.1. *At the end of r rounds, if there exists an alive vertex u such that every unexplained 1-query Q is either disjoint from A_u or $|A_u \cap Q| \geq 2$, then ALG cannot return a j in x_u asserting $x_u(j) = 1$.*

Proof. Suppose ALG does return j asserting $x_u(j) = 1$. Since u was alive, we must have $j \in A_u$ for otherwise $x_u(j) = 0$ since $x_u \in \{0, \star\}^n$. Now, consider the vector \hat{x}_u which sets $\hat{x}_u(i) = 0$ wherever $x_u(i) = 0$, $\hat{x}_u(j) = 0$, and $\hat{x}_u(i) = 1$ for all $i \in A_u \setminus j$. For all other $u' \neq u$, we set $\hat{x}_{u'}(i) = x_{u'}(i)$ for $i \notin A_{u'}$, and $\hat{x}_{u'}(i) = 1$ for all $i \in A_{u'}$. We claim that \hat{x} is consistent with all the responses. This would show that ALG fails. We only need to argue about unexplained 1-queries for all other queries are consistent with the 0, 1-coordinates of x , and thus, \hat{x} . Take such a query Q . If $Q \cap A_u = \emptyset$, then $Q \cap A_{u'} \neq \emptyset$ for some u' , and all $i \in A_{u'}$ is set to $\hat{x}_{u'}(i) = 1$. Otherwise, $|Q \cap A_u| \geq 2$ implying there is some $j' \neq j$ in $Q \cap A_u$. This has been set to $\hat{x}_u(j') = 1$. ■

To achieve the goal, the ADVERSARY maintains the following invariants after every round k .

- (I1.) The number of alive vertices, a_k , is at least $n \cdot (1 - \frac{k}{2r})$.
- (I2.) For every alive vertex u and for every unexplained 1-query Q with $Q \cap A_u \neq \emptyset$, we have $|Q \cap A_u| > n^{1 - \frac{k}{r}}$.

Note that at the beginning, that is after round $k = 0$, the invariants do hold. Furthermore, observe that if the invariants hold for $k = r$, then we get the premise of Lemma 3.1 and the ADVERSARY succeeds in fooling ALG. All that remains is to show how the adversary *answers* the queries made in round $(k + 1)$ (for $0 \leq k \leq r - 1$), and how the various sets are changed so that the invariants are maintained.

Let Q_1, Q_2, \dots, Q_t be the queries made by ALG in round $(k + 1)$. We say Q_ℓ *touches* vertex u if $Q_\ell \cap A_u \neq \emptyset$. Call a query Q_ℓ *broad* if it touches $> 8r \cdot \ln n$ alive vertices. Call Q_ℓ *narrow* otherwise. The next claim shows that a small number alive vertices can “take care of” all broad queries.

Claim 3.2. *There is a subset S of alive vertices with $|S| \leq \frac{n}{4r}$ such that for any broad query Q_ℓ there is some $u \in S$ with $Q_\ell \cap A_u \neq \emptyset$.*

Proof. Consider the following instance of the set cover problem: we have one set $S(u)$ for every alive vertex, and one element $e(Q_\ell)$ for every broad query Q_ℓ . Each set $S(u)$ contains element $e(Q_\ell)$ if and only if $Q_\ell \cap A_u \neq \emptyset$. By setting a weight of $\frac{1}{8r \cdot \ln n}$ on the set $S(u)$ corresponding to each alive vertex u , we obtain a fractional set cover for this instance since, by design, every element (broad query) belongs to at least $8r \cdot \ln n$ many sets (alive vertices). Thus the set cover instance has a fractional set cover of size at most $\frac{n}{8r \cdot \ln n}$ as there are $\leq n$ sets. As the integrality gap of set cover LP is $\ln t \leq 2 \ln n$, there is an integral set cover of size $\leq \frac{n}{4r}$. That is, there exists a set S of $\frac{n}{4r}$ alive vertices such that for every broad Q_ℓ , $Q_\ell \cap A_u \neq \emptyset$ for some $u \in S$. ■

The ADVERSARY does the following: for every $u \in S$, it sets $x_u(i) = 1$ for all $i \in A_u$ and deems u *dead*. By the claim above, this step kills **at most** $\frac{n}{4r}$ alive vertices. It responds 1 to every broad

query Q_ℓ . By the above claim, note that these broad queries are explained 1-queries. Furthermore, if there is any narrow query Q with $Q \cap A_u \neq \emptyset$ for $u \in S$, then ADVERSARY responds to 1 to such queries as well, and these are also explained 1-queries.

Next, the ADVERSARY responds to the remaining narrow queries. If there exists any alive vertex u which is touched by $\geq n^{1/r}$ such narrow queries, then ADVERSARY sets $x_u(i) = 1$ for all $i \in A_u$, deems it dead, and responds 1 to all narrow queries touching this vertex. These 1-queries are also explained. Since every narrow query touches at most $8r \cdot \ln n$ alive vertices, and there are $\leq t = \frac{n^{1+1/r}}{32r^2 \ln n}$ narrow queries to begin with, a counting argument shows that there cannot be more than $\frac{n}{4r}$ vertices which touch more than $n^{1/r}$ narrow queries. Therefore, this step kills **at most** $\frac{n}{4r}$ alive vertices as well. In the remainder of this $(k+1)$ th round, ADVERSARY does not kill any more vertices, and so the total number of vertices killed this round is $\leq \frac{n}{2r}$. Therefore, the number of alive vertices after round $(k+1)$ is $\geq n \cdot (1 - \frac{k}{2r}) - \frac{n}{2r} = n \cdot (1 - \frac{k+1}{2r})$. Thus, Invariant (I1.) holds after round $(k+1)$.

The only unanswered queries left with are narrow queries such that every remaining alive vertex is touched by $\leq n^{1/r} - 1$ of these queries. This is like a single instance of the single element recovery problem, and the remainder of this proof is akin to that of [Theorem 11](#). For every query Q_ℓ and for every alive u with $Q \cap A_u \neq \emptyset$, if $|Q \cap A_u| \leq n^{1-\frac{(k+1)}{r}}$, the ADVERSARY sets $x_u(i) = 0$ for all $i \in Q \cap A_u$, and removes these coordinates from A_u . Since there are $\leq n^{1/r} - 1$ such queries, the total number of vertices removed from A_u is $\leq n^{1-\frac{(k+1)}{r}} \cdot (n^{1/r} - 1) = (n^{1-\frac{k}{r}} - n^{1-\frac{(k+1)}{r}})$. If, for this particular query, all $x_u(i)$ participating in it is set to 0, the ADVERSARY responds 0. Otherwise, it responds 1. In the latter case, the query is an *unexplained* 1-query. This completes the responses of ADVERSARY to all the queries made in this round. We now show that Invariant (I2.) holds.

Fix any alive vertex u which remains alive after round $(k+1)$. Fix any unexplained 1-query Q (which could also be from a previous round) which intersects A_u . If Q is from round $(k+1)$, then by the description of the ADVERSARY strategy, $|Q \cap A_u| > n^{1-\frac{k+1}{r}}$ for otherwise, the ADVERSARY would have removed these coordinates from A_u . If Q is from a previous round, then since (I2.) held after round k , we get that $|Q \cap A_u| > n^{1-\frac{k}{r}}$ before round $(k+1)$. For every vertex u that remains alive, we know that the ADVERSARY removes $\leq (n^{1-\frac{k}{r}} - n^{1-\frac{(k+1)}{r}})$ vertices from A_u . Therefore, after round $(k+1)$, we still have $|Q \cap A_u| > n^{1-\frac{k}{r}} - (n^{1-\frac{k}{r}} - n^{1-\frac{(k+1)}{r}}) = n^{1-\frac{k+1}{r}}$. Thus, Invariant (I2.) is maintained.

In sum, this shows how ADVERSARY can answer all the queries Q_1, \dots, Q_t in round $(k+1)$ such that both invariants are maintained. Thus, after round r , the ADVERSARY can maintain the premise of [Lemma 3.1](#). This in turn, proves [Theorem 5](#). ■

3.2 $\tilde{\Omega}(n^2)$ -Lower Bound for Randomized Non-adaptive Algorithms

The following theorem is a formalization of the lower bound result stated in [Result 4](#).

Theorem 6. *Any 1-round (non-adaptive) randomized algorithm which makes less than $\frac{n^2}{2916 \log^2 n}$ -OR queries on a graph, cannot infer whether the graph is connected or not with probability $\geq \frac{7}{16}$.*

The $7/16$ can be made arbitrarily close to $1/2$ by making the constant 2916 larger; we omit these details. To prove the above theorem, by Yao's minimax theorem, it suffices to give a distribution \mathcal{D} over n -vertex graphs such that any *deterministic* collection of $\leq \frac{n^2}{2916 \log^2 n}$ -OR queries fails on this distribution with probability at least $7/16$. To describe \mathcal{D} , we describe how a graph is generated in two steps. In the first step, we assign each vertex in V to either L or R with equal probability. We then insert all possible edges among vertices in L , as well as among vertices in R – that is,

the graphs induced by L and R are cliques. Let $G_1(V, E_1)$ denote the graph at this stage. Then with probability $1/2$, we output this graph as the final graph (a No instance), and with probability $1/2$, we sample one of the $|L| \cdot |R|$ edge slots connecting vertices in L to vertices in R , uniformly at random, insert this edge (call it e), and output the resulting graph as the final graph (a Yes instance). Let $G(V, E)$ denote the final graph.

Lemma 3.3. *Let \mathcal{Q} be any fixed set of $\frac{n^2}{2916 \log^2 n}$ OR queries. Then with probability at least $7/8$, the answers to queries in \mathcal{Q} are the same on the graphs $G_1(V, E_1)$ and $G(V, E)$.*

Proof. We partition \mathcal{Q} into two sets of queries, namely, a set \mathcal{Q}_1 that contains queries of size at least $36 \log^2 n$ (long queries), and a set \mathcal{Q}_2 containing the remaining queries (short queries). We first claim that with probability at least $1 - 1/n$, the response to *all* long queries is 1 in G_1 (and hence, since G is a supergraph of G_1 , also in G). To see this, fix any long OR-query Q with $|Q| \geq 36 \log^2 n$. Let us consider these as possible edges F in the n -vertex graph. Note that there must exist a set $S \subseteq V$ of with $|S| \geq 6 \log n$ such that every edge in F is incident to some vertex of S . The reason is that t vertices can contain at most $\binom{t}{2} \leq t^2$ edges. Therefore, there exists a set $F' \subseteq F$ with $|F'| \geq 3 \log n$ such that F' induces an acyclic subgraph. Now, the probability that every edge in F' has one end-point in L and other in R , is at most $(1/2)^{3 \log n} = 1/n^3$. This means with probability at least $1 - \frac{1}{n^3}$, at least one edge of F' (and thus F) must actually have both endpoints in either L or R , implying that edge is in G_1 . Therefore, the response to Q is 1. Taking union bound over all long queries in \mathcal{Q}_1 , we conclude that with probability at least $1 - \frac{1}{n}$, all queries in \mathcal{Q}_1 must be answered 1 in G_1 (and hence G_2). We refer to this event as \mathcal{E}_1 .

We now analyze the behavior of short queries. Any query $Q \in \mathcal{Q}_2$ that evaluates to 1 on G_1 , continues to be evaluated so in G since we do not remove any edges in going from G_1 to G . So it suffices to show that every short query that evaluates to 0 in G_1 , also evaluates to 0 in G , whp. We first observe that with probability at least $1 - \frac{1}{n}$, the number of edge slots between L and R , that is the quantity $|L| \cdot |R|$, is at least $n^2/9$. This follows from a simple application of Chernoff bounds - each of the sets L and R , are of size at least $n/3$ with probability at least $1 - \frac{1}{n}$, implying that $|L||R|$, is at least $n^2/9$ with probability at least $1 - \frac{1}{n}$. We will refer to this event as \mathcal{E}_2 .

From here on, we condition on the simultaneous realization of both events \mathcal{E}_1 and \mathcal{E}_2 . Now fix a short query $Q \in \mathcal{Q}_2$ that evaluates to 0 in G_1 . Since the Yes-instance chooses an edge slot e among the $|L||R|$ edge slots uniformly at random, the probability that e appears in Q , is at most $\frac{36 \log^2 n}{n^2/9} = \frac{324 \log^2 n}{n^2}$. Thus, the expected number of queries in \mathcal{Q}_2 that contains the edge slot e is at most $|\mathcal{Q}_2| \cdot \frac{324 \log^2 n}{n^2} \leq \frac{1}{9}$. That is, the probability that some query in \mathcal{Q}_2 contains the edge slot e is at most $1/9$.

Putting together, the probability that the set of queries \mathcal{Q} have different responses on graphs G_1 and G is at most $1/9 + 2/n$ which is at most $1/8$ for sufficiently large n . \blacksquare

We can now complete the proof of [Theorem 6](#) as follows. By [Lemma 3.3](#), any deterministic algorithm that performs less than $\frac{n^2}{2916 \log^2 n}$ OR queries sees the same answers on Yes and No instances generated from distribution \mathcal{D} with probability at least $7/8$. Thus any deterministic algorithm must err with probability at least $7/16$ in distinguishing between the Yes and No instances of \mathcal{D} . As mentioned above, Yao's minimax lemma implies the theorem.

4 Algorithms Warmup: Algorithms for Single Element Recovery

In this section, we state some simple and/or well known algorithms for single element recovery. We will be using these as subroutines for our algorithms for graph connectivity as well. We state most

of our algorithms in the weaker OR-query model remarking what advantage, if any, Linear queries may provide.

Lemma 4.1. *Let $x \in \mathbb{R}_{\geq 0}^N$ be a non-zero, non-negative vector, and let r be a positive integer. There exists an r -round deterministic algorithm $\text{BinarySearch}_r(x)$ which makes $(N^{1/r} - 1)$ -OR queries per round, and returns a coordinate j with $x_j > 0$. If Linear queries are allowed, then one can recover x_j as well.*

Proof. (Sketch) Divide $[N]$ into $N^{1/r}$ blocks each of size $N^{1-1/r}$, and run OR query on each block but the last, taking $N^{1/r} - 1$ queries in all. If one of them evaluates to 1, recurse on that for the next $r - 1$ rounds. Otherwise, recurse on the last block. ■

Next, we state a standard result from the combinatorial group testing and coin-weighing literature [20, 26, 35, 42, 51] which says that if the support of x is known to be small, then there exist efficient *one-round* deterministic algorithms to recover the complete vector.

Lemma 4.2. *[35, 51] Let $x \in \mathbb{R}_{\geq 0}^N$ be a non-zero vector, and let d be any positive integer. There exists a 1-round (non-adaptive) deterministic algorithm $\text{BndSuppRec}(x, d)$ which makes $O(d^2 \log N)$ -OR queries and (a) either asserts $\text{supp}(x) > d$, or (b) recovers the full support of x . With Linear queries, the number of queries reduces to $O(d \log N)$.*

Proof. (Sketch) We give a very high level sketch only for the sake of completeness. For the case of $d = 1$, take the $O(\lceil \log N \rceil \times N)$ matrix A where column i is the number i represented in binary. Then Ax (the “OR product”) points to the unique element in the support. To see the *existence* of a deterministic procedure for larger d , one can proceed by the probabilistic method. If one samples each coordinate with probability $1/d$, then with constant probability the vector restricted to this sample has precisely support 1 for which the above “ $d = 1$ ” algorithm can be used to recover it. Repeating this $O(d \log N)$ times leads to error probability which swamps the union bound over $\leq N^d$ possible sets, implying the existence of a deterministic scheme. Finally, another $O(d)$ arises since we need to recover all the $\leq d$ coordinates. All this can be made explicit by using ideas from error correcting codes; we point the interested reader to [35, 51] for the details. ■

Next we move to randomized algorithms. Here ideas from F_0 -estimation [6, 28] and ℓ_0 -sampling [19, 29, 37] give the following algorithms.

Lemma 4.3. *Let $x \in \mathbb{R}_{\geq 0}^N$ be a non-zero vector. There exists a 1-round (non-adaptive) randomized algorithm $\text{RandSuppSamp}(x)$ which makes $O(\log^2 N \log(\frac{1}{\delta}))$ -OR queries and returns a random $j \in \text{supp}(x)$ with probability $\geq 1 - \delta$.*

Proof. (Sketch) Suppose we knew the support $\text{supp}(x) = d$. Then, we sample each $j \in [N]$ with probability $1/d$ to get a subset $R \subseteq [N]$. With constant probability $\text{supp}(x \cap R) = 1$ and, conditioned on that, it contains a random $j \in \text{supp}(x)$. Therefore, running the algorithm $\text{BndSuppRec}(x \cap R, 1)$ asserted in Lemma 4.2, we can find a random $j \in \text{supp}(x)$ with constant probability. Repeating this $O(\log(1/\delta))$ times gives the desired error probability. Since we don’t know $\text{supp}(x)$, we run for various powers of 2 in 1 to n ; we are guaranteed success in at least one of the scales. ■

Lemma 4.4. *(Theorem 7 in [13], also in [21, 27]) Let $x \in \mathbb{R}_{\geq 0}^N$ be a non-zero vector. There exists a 1-round (non-adaptive) randomized algorithm $\text{SuppEst}(x)$ which makes $O(\log N \cdot \log(1/\delta))$ -OR-queries and returns an estimate \tilde{s} of the support which satisfies $\frac{\text{supp}(x)}{3} \leq \tilde{s} \leq 3\text{supp}(x)$ with probability $\geq 1 - \delta$.*

5 Deterministic Algorithm for Graph Connectivity

In this section, we prove the following theorem which formalizes [Result 3](#).

Theorem 7. *Let r be any fixed positive integer. There exists an $35r$ -round deterministic algorithm $\text{DetGraphConn}(G)$ which makes at most $O(n^{1+\frac{1}{r}} \log n)$ -BIS-queries on an undirected multigraph G , and returns a spanning forest of G .*

We start by establishing some simple subroutines which we need.

5.1 Simple Subroutines

We begin by strengthening the simple algorithm BinarySearch asserted in [Lemma 4.1](#). While in r -rounds with $O(N^{1/r})$ -OR queries $\text{BinarySearch}_r(x)$ recovers a single element in the support, one can in fact get many more elements from the support. This result may be of independent interest.

Lemma 5.1. *Let $x \in \mathbb{R}_+^N$ be a non-zero, non-negative vector, and let r be a positive integer, and let $c < r$. There exists a $\lceil 2r/c \rceil$ -round deterministic algorithm $\text{DetFindMany}_{r,c}(x)$ which makes $O(N^{c/r} \log N)$ -OR queries per round, and returns $\min(N^{c/4r}, \text{supp}(x))$ distinct coordinates from $\text{supp}(x)$.*

Proof. In the first round, we partition the range $[N]$ into $N^{c/2r}$ blocks of size $N^{1-c/2r}$ each. Let these blocks be B_1, \dots, B_k with $k = N^{c/2r}$. For each $i \in [k]$, we run the algorithm $\text{BndSuppRec}(x \cap B_i, N^{c/4r})$ asserted in [Lemma 4.2](#). The total number of queries used here is $O(N^{c/2r} \cdot (N^{c/4r})^2 \log N) = O(N^{c/r} \log N)$.

At the end of this round, either we recover $\text{supp}(x \cap B_i)$ for each block, and thus recover $\text{supp}(x)$, and we are done. Or, there is at least one block of size $N^{1-c/2r}$ which is guaranteed to contain $\geq N^{c/4r}$ elements in its support. We call this the heavy block of round 1. Next, we now proceed to recover $N^{c/4r}$ elements from this heavy block of round 1.

In the second round, we partition the indices of this heavy block again into $N^{c/2r}$ blocks of size $N^{1-2c/2r}$ each, and run BndSuppRec again on this block with $d = N^{c/4r}$. Once again, either we recover the entire support of the heavy block (which is guaranteed to contain at least $N^{c/4r}$ elements) and we are done. Or find a block of size $N^{1-2c/2r}$ that contains at least $N^{c/4r}$ elements in its support – this is the heavy block of round 2 – and we now proceed to recover $N^{c/4r}$ elements in the heavy block of round 2.

We continue in this manner, and after $\lceil 2r/c \rceil - 1$ rounds, either we have already recovered at least $N^{c/4r}$ elements in the support of x , or have identified a heavy block of size $N^{1-(\frac{2r}{c}-1)\cdot\frac{c}{2r}} = N^{c/2r}$ that contains at least $N^{c/4r}$ elements in the support of x . In the final round, we can simply probe each entry completing the proof. ■

Remark 5.2. The trade-off between the number of queries and number of elements recovered is not tightly established for the purpose of what we need in the graph connectivity algorithm. For instance, using the same idea as above, in 2 rounds one can actually recover $\min(N^{1/4}, \text{supp}(x))$ coordinates making $O(N^{3/4})$ -queries per round.

Next, we give an algorithm to find edges between two disjoint sets of vertices using BIS-queries.

Lemma 5.3. *Let A and B be two disjoint sets of vertices with at least one edge between them. There exists a $2r$ -round deterministic algorithm $\text{DetFindEdge}_r(A, B)$ which makes $O(|A|^{1/r} + |B|^{1/r})$ -BIS queries per round, and returns an edge (a, b) with $a \in A$ and $b \in B$.*

Proof. Consider the $|B|$ dimensional vector x where x_b indicates the number of edges from a vertex $b \in B$ to vertices in A . We can simulate an OR-query in this vector using a BIS-query in the graph — for any subset $S \subseteq B$, $\text{OR}(S)$ on x has the same answer as $\text{BIS}(A, S)$. Therefore, using Lemma 4.1, in r -rounds and $|B|^{1/r}$ -BIS-queries, we can find a coordinate $b^* \in B$ with $x_{b^*} > 0$. That is, there is an edge between b^* and some vertex in A .

We can find one such vertex $a \in A$ to which b^* has an edge, again as above. We define the $|A|$ -dimensional vector y where y_a indicates the number of edges from b^* to a . Once again, the OR-query on y can be simulated using a BIS query on the graph — for any subset $S \subseteq A$, $\text{OR}(S)$ on y is the same as $\text{BIS}(S, \{b^*\})$. ■

5.2 The Connectivity Algorithm

Now we give the $O(r)$ -round deterministic algorithm to find a spanning forest. First, we need the following simple claim.

Claim 5.4. *Let $G(V, E)$ be an arbitrary connected multigraph graph on n vertices, and let D be an arbitrary integer in $\{0, 1, \dots, (n - 1)\}$. Let V_L denote all vertices in V whose degree is at most D , and let $V_H = V \setminus V_L$. Let $E' \subseteq E$ be an arbitrary set of edges that satisfies the following property: for each vertex $u \in V_L$, the set E' contains all edges incident on u , and for every each vertex $v \in V_H$, the set E' contains D arbitrary edges incident on v . Then the graph $G' = (V, E')$ contains at most $\lfloor n/D \rfloor$ connected components.*

Proof. Suppose G' has $K \geq \frac{n}{D}$ connected components. Thus, there must exist some component C with $\leq D$ vertices. Firstly, that C must have some vertex $v \in V_H$. If not, then since vertices in V_L have all their edges in G also in G' , this component would be disconnected in G which contradicts G 's connectedness. Secondly, observe that this leads to a contradiction: v has degree at least D in G' , and since there are at most $D - 1$ other vertices in C , one of v 's neighbor in G' must lie outside C . This contradicts that C is a connected component. ■

We are now ready to describe the algorithm $\text{DetGraphConn}(G)$. For simplicity, assume G is connected and our goal is to find a spanning tree. Subsequently, we explain how to modify the algorithm to find a spanning forest of a general graph. The algorithm proceeds in $O(\log r)$ phases starting with phase 0. The input to phase i is a partition $\Pi_i = (S_1, \dots, S_p)$ of the vertices. Each S_j in Π_i is guaranteed to be a connected in the graph G . Π_0 is the trivial partition of n singletons. Given Π_i , we define the graph $G_i = (\Pi_i, \mathcal{E}_i)$ where \mathcal{E}_i is the collection of *pseudo-edges* between components: we have a pseudo-edge $(S_a, S_b) \in \mathcal{E}_i$ if and only if there exists some edge in G between a vertex $u \in S_a$ and a vertex $v \in S_b$. Thus, G_0 is indeed the original graph. Note that by our assumption that G is connected, all the G_i 's are connected. We will be collecting pseudo-edges which will imply the connected components; we initialize this set \mathcal{F} to empty set. We will maintain the following invariant for a phase: $|\Pi_i| \leq n^{1-\frac{4^i-1}{r}}$; this is certainly true for $i = 0$. Next, we describe a phase i .

1. For each $S \in \Pi_i$, we construct a vector x indexed by all sets in $\Pi_i \setminus S$ where x_T indicates whether there is a pseudo-edge (S, T) in G_i . Next, we run the algorithm $\text{DetFindMany}_{r,c}(x)$ asserted in Lemma 5.1 to either find all pseudo-edges incident on S , or at least $n^{4^i/4r}$ of them. To do so, we set c such that $N^{c/4r} = n^{4^i/4r}$, where N is the dimension of x . That is, $N = |\Pi_i| - 1$. Indeed, we should set $c = \theta \cdot 4^i$ where $N^\theta = n$. Note, $\theta \geq 1$. Also note that the OR-queries on x can be simulated using BIS-queries on the original graph G . This is because we are looking at edges between S and a union of a subset of parts in $\Pi_i \setminus S$.

The number of rounds is $\lceil \frac{2r}{c} \rceil \leq \lceil \frac{2r}{4^i} \rceil$. The number of BIS-queries per round is $O(N^{c/r} \log N) = O(n^{4^i/r} \log n)$ per subset $S \in \Pi_i$. And thus, the total number of queries made is $N \cdot O(n^{4^i/r} \log n) \leq |\Pi_i| \cdot O(n^{4^i/r} \log n) \leq n^{1-\frac{4^i-1}{r}} \cdot O(n^{4^i/r} \log n) = O(n^{1+1/r} \cdot \log n)$.

2. Let $\mathcal{E}'_i \subseteq \mathcal{E}_i$ be the pseudo-edges obtained from the previous step. Let \mathcal{F}'_i be an arbitrary spanning forest of \mathcal{E}'_i . We add all these edges to the collection \mathcal{F} . Note, \mathcal{F}'_i is a collection of $\leq |\Pi_i|$ pseudo-edges.
3. Applying [Claim 5.4](#) to the graph G_i , adding the pseudo edges in \mathcal{E}'_i reduces the number of connected components to at most $|\Pi_i|/n^{\frac{4^i}{4r}}$. We now repeat the above two steps 11 more times *sequentially*, and each time the number of connected components multiplicatively drops by $n^{\frac{4^i}{4r}}$. Thus, after the 12 sub-phases we end up with the partition Π_{i+1} of connected components, with $|\Pi_{i+1}| \leq |\Pi_i|/n^{\frac{12 \cdot 4^i}{4r}} \leq n^{1-\frac{4^i-1}{r}} \cdot n^{-\frac{12 \cdot 4^i}{4r}} = n^{1-\frac{4^{i+1}-1}{r}}$, as desired. The second inequality follows from the invariant before phase $(i+1)$ started.

To summarize, Phase i performs $O(\frac{r}{4^i})$ -rounds and makes $O(n^{1+1/r} \log n)$ -BIS queries per round. We run phase 0 to $L = O(\log r)$, till we get $|\Pi_L| \leq \sqrt{n}$. After than we run a clean up phase.

4. *Clean-up Phase.* Once $|\Pi_L| = O(\sqrt{n})$, for each pair (S, T) in $\Pi_L \times \Pi_L$, we make a single BIS-query to detect if the pseudo-edge $(S, T) \in \mathcal{E}_L$. The total number of queries is $O(n)$. We add an arbitrary spanning tree of \mathcal{E}_L to the set \mathcal{F} . At this point, \mathcal{F} lets us know the structure of connectivity via pseudo-edges. The next step is to recover the actual graph edges.
5. *Tree Building Phase.* Note that the total number of pseudo-edges in \mathcal{F} is $< n - 1$. For each $(S, T) \in \mathcal{F}$, we now desire to find an edge (s, t) in the graph where $s \in S$ and $t \in T$. Note that once we do this, we have the spanning tree the graph. This can be done in $2r$ more rounds using the algorithm `DetFindEdger(S, T)` using $O(|S|^{1/r} + |T|^{1/r})$ -BIS queries per round. Therefore, the total number of queries per round of this phase is $O(n) \cdot O(n^{1/r}) = O(n^{1+1/r})$.

The number of rounds is $\sum_{i=1}^{O(\log r)} \frac{24r}{4^i} + 1 + 2r \leq 35r$.

This ends the description of the algorithm when G is connected. If G had more than one connected component, then one can recognize the connected components as the algorithm progresses. More precisely, if the algorithm is processing the partition $\Pi_i = (S_1, \dots, S_p)$ and find that S_i has no edges coming out of it, then by the invariant that S_i is connected, the algorithm can discard this component and proceed on the remaining graph as if it were connected. The analysis becomes better as the effective number of vertices decrease but the number of available queries don't. This completes the proof of [Theorem 7](#).

6 Randomized Algorithms for Graph Connectivity

In the first subsection, we give a 2-round randomized algorithm which makes $\tilde{O}(n)$ -OR queries and returns a spanning forest of G . Recall, [Theorem 6](#) proved a $\tilde{\Omega}(n^2)$ -lower bound for 1-round randomized algorithms using OR-queries. We also show how to implement the 2-round algorithm in 4-rounds using only BIS-queries. In the second subsection we give a non-adaptive (1-round), randomized algorithm which makes $\tilde{O}(n)$ Cross-queries and returns a spanning forest of G . Previously, this result was known to hold only with Linear-queries [2].

6.1 OR-queries

We prove the following theorem.

Theorem 8. *There exists a 2-round randomized algorithm $\text{RandGraphConnOR}(G)$ which makes at most $O(n \log^5 n)$ -OR-queries and returns a spanning forest of G with high probability. There exists a 4-round randomized algorithm $\text{RandGraphConnBIS}(G)$ which makes at most $O(n \log^5 n)$ -BIS-queries and returns a spanning forest of G with high probability.*

Before we dive into the algorithm, we first establish some simple subroutines which are implied by the algorithms for single element recovery described in [Section 4](#).

6.1.1 Simple Subroutines

We begin with a subroutine for a 1-round randomized algorithm for support recovery.

Lemma 6.1. *Let $x \in \mathbb{R}_{\geq 0}^N$ be a non-zero vector and let M be an upper bound on $\text{supp}(x)$. There exists a 1-round (non-adaptive) randomized algorithm $\text{RandSuppRec}(x)$ which makes $O(M \log^3 N)$ -OR queries and recovers $\text{supp}(x)$ whp.*

Proof. This follows by running $O(M \log N)$ copies of the algorithm $\text{RandSuppSamp}(x)$ from [Lemma 4.3](#) with $\delta = 1/2$. Whp, we would get $\Theta(M \log N)$ random samples from $\text{supp}(x)$, and by a coupon collector argument, whp these will contain the full support. ■

The above lemma, for instance, implies that if a graph has $\leq m$ edges, then there is a 1-round randomized algorithm to reconstruct the graph using $O(m \log^3 n)$ OR-queries. Note, these are not necessarily BIS-queries, and we do need little extra work to handle BIS-queries. In particular, we need the following lemmas.

Lemma 6.2 (A corollary of [Lemma 4.3](#)). *Let v be a vertex, S be a subset of vertices not containing v , and s be a positive integer. There exists a 1-round (non-adaptive) randomized algorithm $\text{RandEdges}(v, S, s)$ which makes $O(s \log^3 n)$ BIS-queries and, whp, returns s random edges from v to S , uniformly at random with repetition. If there are no edges from v to S , the algorithm says so. We use $\text{RandEdges}(v, s)$ to denote $\text{RandEdges}(v, V \setminus v, s)$.*

Lemma 6.3 (A corollary of [Lemma 4.4](#)). *Let A be a subset of vertices in a graph. There is a 1-round randomized algorithm $\text{DegEst}(A)$ which makes $O(\log^2 n)$ -BIS-queries and returns an estimate d of $|E(A, A^c)|$ which satisfies $\frac{|E(A, A^c)|}{3} \leq d \leq 3|E(A, A^c)|$ whp.*

Lemma 6.4 (A corollary of [Lemma 6.1](#)). *Let A be a subset of vertices and let d be an upper bound on the number of edges in $E(A, A^c)$. There exists a 1-round randomized algorithm $\text{FindNbrs}(A)$ which makes $O(d \log^3 n)$ BIS-queries and finds the subset $B \subseteq A^c$ of vertices which has at least one edge to some vertex in A , whp. That is, it finds the vertex neighbors of A .*

Proof. For the first part, let x be the $|A^c|$ dimensional vector with x_b indicating the number of edges between $b \in A^c$ and A . Note that an $\text{OR}(x, S)$ query on this vector for $S \subseteq A \times A^c$ can be simulated using an BIS-query on the graph as well. Note $\text{supp}(x) \leq d$ (it could be much smaller). Using the algorithm asserted in [Lemma 6.1](#), we can find $\text{supp}(x)$, that is, all $b \in A^c$ which have some neighbor in A . This is precisely what this lemma asserts.

6.1.2 The Connectivity Algorithm

Below we give an algorithm which runs 2-rounds with OR-queries, and in 4-rounds with BIS-queries.

1. In Round 1, every vertex $v \in V$ whp samples $s = O(\log^2 n)$ -edges incident to it using the algorithm `RandEdges`(v, s) asserted in [Lemma 6.2](#). This requires $O(n \log^5 n)$ -BIS queries.

Let $\Pi := (S_1, \dots, S_p)$ be the connected components formed by these edges. Due to [Lemma 6.5](#), whp we have that $E_{\text{Cross}}(\Pi) = O(n \log n)$.

- 2a. If we had OR-queries available, then in Round 2 we next apply the algorithm `RandSuppRec`(x) asserted by [Lemma 6.1](#) with $M = O(n \log n)$, on the vector x which indexed by (u, v) for vertex pairs across different components of Π . This requires $O(n \log^4 n)$ OR-queries. We obtain the support of this vector x , that is the set $E_{\text{Cross}}(\Pi)$, whp. These edges, along with the edges sampled in Round 1, gives the spanning forest of the graph. This completes the description of `RandGraphConnOR`(G).
- 2b. If we did not have OR-queries but only BIS-queries, then in Round 2, for each S_i , $i \in [p]$, we estimate an upper bound d_i on $|E(S_i, S_i^c)|$ using the algorithm `DegEst`(S_i) asserted in [Lemma 6.3](#). This round requires $O(n \log^2 n)$ BIS-queries since $p \leq n$.

3. In Round 3, for each $i \in [p]$, we use the algorithm `FindNbrs`(S_i) asserted in [Lemma 6.4](#) to find the vertices $V_i \subseteq S_i^c$ which have at least one edge to at least one node in S_i . The number of BIS-queries needed is $O(\sum_i d_i \cdot \log^3 n) = O(n \log^4 n)$, since $\sum_{i=1}^p d_i = O(|E_{\text{Cross}}(\Pi)|) = O(n \log n)$.

From the information obtained after Round 3, we can figure out *pseudo-edges* $\tilde{E} \subseteq [p] \times [p]$ where $(i, j) \in \tilde{E}$ if there is a vertex in S_j which has an edge to S_i , or vice-versa. To use the terms defined above, $(i, j) \in \tilde{E}$ if $V_i \cap S_j$ or $V_j \cap S_i$ is non-empty. Note that if there is no edge $(i, j) \in \tilde{E}$, there is no edge from any vertex in S_i to any vertex in S_j in the original graph.

Let \tilde{F} be an arbitrary spanning forest in \tilde{E} . Note the connected components of \tilde{F} are also the connected components of G . What remains is to find the edges of G (we only have pseudo-edges now) to connect up the connected components. We use the next round to find the actual graph edges.

4. In Round 4, for every pseudo-edge $(i, j) \in \tilde{F}$, we know either $V_i \cap S_j \neq \emptyset$ or $V_j \cap S_i \neq \emptyset$ and we know which is the case. Suppose $V_i \cap S_j \neq \emptyset$. Then, let w_{ij} be a vertex in $V_i \cap S_j$. Let W be all such vertices collected as we go over all the pseudo-edges $(i, j) \in \tilde{F}$. Note $|W| \leq k - 1 < n$.

For each vertex $w_{ij} \in W$ in parallel, we use the algorithm `RandEdges`($w_{ij}, S_i, 1$) asserted in [Lemma 6.2](#) with to get a graph edge from w_{ij} to a vertex $u \in S_i$. Once we get all such edges, we would have obtained a spanning forest of G whp. The number of queries is $O(|W| \log^3 n)$ which is $O(n \log^3 n)$. This completes the description of `RandGraphConnBIS`(G).

Note that every step in the above description uses randomized subroutines which succeed whp. To prove [Theorem 8](#) via union bound, we need to ensure that only a polynomially many events occur. Indeed this is the case; the total number of events is $O(n)$. [Lemma 6.5](#) completes the proof of [Theorem 8](#).

Lemma 6.5. *Let G be an undirected graph on n vertices. Suppose every vertex $v \in G$ samples $O(\log^2 n)$ edges with repetition, and let $\Pi = (S_1, \dots, S_p)$ be the resulting connected components. Then, whp, $|E_{\text{cross}}(\Pi)| = O(n \log n)$.*

Proof. We use the following strong theorem recently proved by Holm *et al.* [34] in FOCS 2019. The only difference is that the process in [34] is without repetition. But it can be easily modified for the with-repetition case, and we sketch a proof below.

Theorem 9 (Theorem 1.2, Corollary 2.22 of [34]). *Let G be an arbitrary undirected n -vertex graph and let $k \geq c \log n$, where c is a large enough constant. Let G' be a random subgraph of G where every vertex v independent samples a subset of $\min(k, \deg(v))$ edges incident on it, each subset equally likely. Then the expected number of edges in G' that connect different connected components of G is $O(n/k)$. Furthermore, there exists a constant b such that the probability that the number of edges in G' that connect different connected components of G exceeds $\ell \cdot bn/k$ is at most $2^{-\ell}$.*

Fix a vertex v . Let d be its degree. Let $k = c \log n$ with c as in the above theorem. Note that if $d \leq 2c \log n = 2k$, then by a coupon collector argument, whp, our with repetition experiment will sample all the edges. We may assume, therefore, $d > 2k$, then again a coupon collector style argument shows that whp we will obtain at least k distinct edges. Furthermore, by symmetry, every subset of k distinct edges are going to be equally likely. Therefore, we can apply the above theorem which implies the lemma (set $\ell = \log^2 n$).

For the interested reader, we include a self contained proof (a weaker version of [34] theorem) of the lemma in Appendix B. ■

6.2 Linear Queries

In this section we prove the following theorem.

Theorem 10. *There exists a 1-round (non-adaptive) randomized algorithm $\text{RandGraphConnCross}(G)$ which makes at most $O(n \log^4 n)$ -Cross-queries and returns a spanning forest of G with high probability.*

Our main tool is a data structure, which we call **Partition-Sampler**, which takes input a single partition $\Pi := (S_1, \dots, S_p)$ of the vertex set, and, with high probability, for each $i \in [p]$ returns an edge uniformly at random from $E(S_i, S_i^c)$. The key feature of **Partition-Sampler** is that it can be constructed *non-adaptively*. That is, we next show how to create the **Partition-Sampler** by making $O(n \log^3 n)$ Linear (indeed, Additive) queries to the graph.

Remark 6.6. It is important to assert that the above guarantee provided by **Partition-Sampler** in response to a given query holds for answering *exactly one query partition*. In other words, asking multiple queries from **Partition-Sampler** adaptively would ensure that the returned solution is wrong with probability close to one. Therefore, if one desires to find random edges for k different partitions, it is necessary to construct k independent copies of **Partition-Sampler**.

Notation. For any set $S \subseteq V$, we use $\partial(S)$ to denote the set of pairs in the cut $(S, V \setminus S)$ which have edges between them. We let $w(\partial(S))$ denote the number of these edges. If $S = \{v\}$ for some $v \in V$, we slightly abuse the notation and use $\partial(v)$ instead of $\partial(\{v\})$. Additionally, we use $\partial^+(v)$ (resp. $\partial^-(v)$) to denote the set of edges that are go to lexicographically larger (resp. smaller) vertices in $\partial(v)$.

Preprocessing. Let $L := \{2^i \mid 0 \leq i \leq \log \binom{n}{2}\}$ and $k = 100 \log n$. For any $\ell \in L$, sample k sets $T_1^\ell, \dots, T_k^\ell$, each chosen by picking each of the $\binom{n}{2}$ vertex-pairs independently with probability $1/\ell$. For any set T_i^ℓ for $\ell \in L$ and $i \in [k]$, sample k random subsets of T_i^ℓ denoted by $T_{i,1}^\ell, \dots, T_{i,k}^\ell$ each chosen by picking each element in T_i^ℓ w.p. $1/2$ independent of other elements. This step can be done without any interaction with the graph G .

Initialization. For any vertex $v \in V$, any set $T_{i,j}^\ell$ for $\ell \in L$, and $i, j \in [k]$, let:

$$\begin{aligned} t_i^\ell(v) &= w\left(\partial^+(v) \cap T_i^\ell\right) - w\left(\partial^-(v) \cap T_i^\ell\right), \\ t_{i,j}^\ell(v) &= w\left(\partial^+(v) \cap T_{i,j}^\ell\right) - w\left(\partial^-(v) \cap T_{i,j}^\ell\right). \end{aligned}$$

It is easily verified that for any vertex v , the scalar $t_{i,j}^\ell(v)$ can be computed using 2-Cross-queries. For each vertex v , one recognizes the set W of vertices $u \in V \setminus v$ such that the pair $(u, v) \in T_{i,j}^\ell$; one then queries $\text{Cross}(\{v\}, W)$ to get $w\left(\partial^+(v) \cap T_{i,j}^\ell\right)$. The Partition-Sampler algorithm simply calculates $t_{i,j}^\ell(v)$ for all vertices v and all choice of parameters ℓ, i, j in the initialization step. This takes $O(n \log^3 n)$ -Cross queries.

Query Response. Given a partition Π , and for all sets $S \in \Pi$,

1. For all $\ell \in L, i \in [k], j \in [k]$, compute $t_i^\ell(S) := \sum_{v \in S} t_i^\ell(v)$ and $t_{i,j}^\ell(S) := \sum_{v \in S} t_{i,j}^\ell(v)$.
2. *Level test:* Find (ℓ^*, i^*) such that for all $j \in [k]$, $t_{i^*,j}^{\ell^*}(S) \in \{0, t_{i^*}^{\ell^*}(S)\}$. If no such choice exists, return \perp and terminate.
3. *Edge test:* Find a pair $e \in T_{i^*}^{\ell^*}$ such that for all $j \in [k]$, whenever $e \in T_{i^*,j}^{\ell^*}$, $t_{i^*,j}^{\ell^*}(S) = t_{i^*}^{\ell^*}(S)$. If no such edge exists, return \perp and terminate.
4. Return the pair e found in the last test, and assert this is in $E(S, S^c)$.

In the following, we prove the correctness of Partition-Sampler.

Lemma 6.7 (Level Test Correctness). *For parameters (ℓ^*, i^*) chosen by level test, $|T_{i^*}^{\ell^*} \cap \partial(S)| = 1$ with high probability.*

Proof. We first prove that the probability that the level test outputs \perp is at most $1/n^{O(1)}$. Let $\ell \in L$ be such that $\ell \leq |\partial(S)| < 2\ell$. For this choice of ℓ and by randomness in choice of T_i^ℓ for $i \in [k]$,

$$\mathbb{P}\left(|T_i^\ell \cap \partial(S)| = 1\right) = \Omega(1).$$

As a result, with high probability, there exists some $i \in [k]$, such that $|T_i^\ell \cap \partial(S)| = 1$. It can be verified that such choice of (ℓ, i) pass the level test because there is only one edge in $\partial(S)$ that can contribute to the value of $t_i^\ell(S)$ (the other edges will cancel out each other's contribution). Clearly, such a choice for (ℓ, i) satisfies the requirement in the lemma statement. We now prove that any (ℓ, i) such that $|T_i^\ell \cap \partial(S)| \neq 1$, would pass the test only with probability $1/n^{O(1)}$; this, together with a union bound on $\text{polylog}(n)$ choices for (ℓ, i) finalizes the proof.

If $|T_i^\ell \cap \partial(S)| = 0$, clearly (ℓ, i) cannot pass the first step of the level test. Hence, in the following, we assume that $|T_i^\ell \cap \partial(S)| \geq 2$. For any $j \in [k]$, we prove that,

$$\mathbb{P}\left(t_{i,j}^\ell(S) \notin \{0, t_i^\ell(S)\}\right) = \Omega(1),$$

which immediately finalizes the proof as we repeat this process $k = O(\log n)$ times.

Let e_1 and e_2 be two edges in $T_i^\ell \cap \partial(S)$. Suppose we fix the assignment of every edge in $\partial(S)$ in $T_{i,j}^\ell$ except for e_1 and e_2 . There is always one choice of e_1 and e_2 that ensures that $t_{i,j}^\ell(S) \notin \{0, t_i^\ell(S)\}$. Hence, w.p. at least $1/4$, $t_{i,j}^\ell(S) \notin \{0, t_i^\ell(S)\}$, finalizing the proof. ■

Lemma 6.8 (Edge Test Correctness). *Conditioned on $|T_{i^*}^{\ell^*} \cap \partial(S)| = 1$, the edge e returned by the edge test is the single edge in $T_{i^*}^{\ell^*} \cap \partial(S)$ with high probability.*

Proof. Fix $e^* = T_{i^*}^{\ell^*} \cap \partial(S)$. It is immediate that $t_{i^*,j}^{\ell^*} = t_{i^*}^{\ell^*}$ iff $e^* \in T_{i^*,j}^{\ell^*}$. Now fix any other edge $e \neq e^* \in T_{i^*}^{\ell^*}$. The probability that e appears in all $T_{i^*,j}^{\ell^*}$ in which e^* also appears is at most $1/2^k = 1/n^{O(1)}$. A union bound on all possible edges finalizes the proof. ■

The correctness of Partition-Sampler now follows immediately from Lemma 6.7 and Lemma 6.8.

6.2.1 The Connectivity Algorithm

The algorithms is simply as follows:

1. Create $t := O(\log n)$ Partition-Sampler data structure D_1, \dots, D_t by querying the graph non-adaptively.
2. Define $S_1^0 = \{v_1\}, \dots, S_n^0 := \{v_n\}$.
3. For $i = 1$ to t steps:
 - (a) Query D_i with sets $(S_1^{i-1}, \dots, S_{n'}^{i-1})$ to obtain an edge from each $\partial(S_j^{i-1})$.
 - (b) Let $S_j^i := S_j^{i-1} \cup S_{j'}^{i-1}$ whenever the edge sampled for S_j^{i-1} by D_i is incident on a vertex in $S_{j'}^{i-1}$.

Since the algorithm finds an edge out of every connected cluster, and the number of clusters drop by a factor 2 in each round, in $O(\log n)$ -rounds the algorithm finds a spanning forest of G , whp. Once again, the total number of bad events is $O(n)$, and thus the whp holds due to a union bound over the whp assertions in Lemma 6.7 and Lemma 6.8. The total number of Cross-queries is in the creation of the $O(\log n)$ Partition-Sampler data structures. This completes the proof of Theorem 10.

7 Related Work

Graph Reconstruction via Cross Queries. As mentioned in the Introduction, most of the work in the literature on Cross-queries in graphs has focused on reconstructing the graph. Starting with the work of Grebinski and Kucherov [32], a long line [4, 5, 12, 14, 15, 18, 45, 53] of work culminated in a randomized, adaptive, *polynomial time* $O(\frac{m \log n}{\log m})$ -Cross query algorithm due to Choi [17] to reconstruct the graph. Interestingly, there is a non-adaptive, deterministic algorithm with the same number of queries [12, 18], however neither are the queries explicit, nor is there an efficient algorithm known to reconstruct the graph from the answers. To our knowledge, obtaining a polynomial time, non-adaptive algorithm with optimal query complexity is an open problem. Having said that, it is rather straightforward (see [53], for instance) to obtain an efficient $O(m \log n)$ -Cross query non-adaptive, deterministic algorithm.

One recent work using Cross-queries which is similar in spirit to our paper, is one by Rubinstein, Schramm, and Weinberg [55]. They give a randomized, $O(1)$ -round¹⁰ algorithm which makes $\tilde{O}(n)$ -Cross queries in a *simple* undirected graph and returns the global minimum cut. Their result, although related, is rather incomparable. For one, it uses more rounds than our algorithm, two, it runs on simple graphs, and lastly, it is unclear whether their algorithm can return a spanning tree. Our algorithm, on the other hand, cannot find the minimum cut. Indeed, for general multigraphs, a recent result of Assadi, Chen, and Khanna [8] shows that any $O(1)$ -round algorithm which finds the *exact* minimum cut must make $\tilde{\Omega}(n^2)$ queries.

¹⁰They don't specify the number of rounds, but our guess is 3

Parameter Estimation via **IS and **BIS** Queries.** Beame et al. [9] considered the problem of estimating the number of edges in a graph given **BIS**-queries. They gave a randomized algorithm which obtained an $(1 + \varepsilon)$ -approximation making $\text{poly}(\log n, \frac{1}{\varepsilon})$ -**BIS** queries. This improved upon a earlier result of Dell and Lapinskas [22] who gave a $\tilde{O}(n)$ -query algorithm.

Building upon [9], very recently Bhattacharya et al. [11] give a polylogarithmic query approximation to estimate the number of triangles (using a stronger query model). To our knowledge, we don't know of an explicit reference to graph reconstruction using these queries; however, it is not hard to obtain a 2-round, randomized algorithm making $O(m \log n)$ -**BIS** queries (assuming we know m); indeed, our algorithm in [Section 6.1](#) does that.

Apart from **BIS**-query, another similar query model is the **IS** model which takes input a subset A and says whether there is any edge with both endpoints in A . This has also been called the *edge-detecting model* [1, 7]. We should stress here that the **IS** queries are significantly weaker than **BIS** queries. On the one hand, one can simulate **IS** queries using $O(\log n)$ nonadaptive **BIS** queries. On the other hand, it is known, for instance, that $O(\log n)$ -**BIS** queries can estimate the degree ([Lemma 6.3](#)), but $\tilde{\Omega}(n)$ -**IS** queries are needed to estimate the same [9]. For our problem at hand (of finding a spanning forest), one can use the construction in [Section 3.2](#) to show any (randomized, adaptive) algorithm for finding a spanning forest needs to make $\tilde{\Omega}(n^2)$ -**IS** queries.

Sketching, Streaming, and other Access Models. Linear queries are more famous as **Linear** sketches. Over the past two decades, a huge amount of literature has amassed on linear sketching; we refer the readers to surveys [33, 60] and the references within. Here, we mention the works most relevant to our paper.

Ahn, Guha, and McGregor [2] were the first to give a **Linear** sketch for connectivity. As mentioned in the overview, their work implies a non-adaptive, randomized $\tilde{O}(n)$ -**Linear** query algorithm for finding a spanning forest. Our result (??) can be thought of as a special class of linear sketch for connectivity. One of the main applications of linear sketches arises in dynamic streaming algorithms. In a dynamic stream, objects are inserted and deleted in a stream, and the algorithm has to maintain a certain structure in bounded space. The AGM result [2] immediately implied a $\tilde{O}(n)$ -space randomized, one-pass dynamic stream algorithm. We are, in fact, unaware of any results on *deterministic* dynamic stream algorithms for maintaining a spanning forest. Our result ([Result 3](#)) implies an $O(n^{1+1/r})$ -space algorithm in $O(r)$ passes, which in turn implies a $O(\log n)$ -pass semi-streaming ($\tilde{O}(n)$ -space) algorithm.

Recently, Nelson and Yu [47] proved an $\Omega(n \log^3 n)$ -lower bound on the space requirement for the single-pass dynamic spanning forest problem. This, for instance, proves a lower bound of $\Omega(n \log^2 n)$ on the number of **Cross**-queries required by any non-adaptive algorithm (the information theoretic lower bound is only $\Omega(n)$).

Finally, we mention that many other query-access models have been proposed in the literature, and this approach is instructive and important to understand the power and limitations of algorithms. We mention one such recent work by Sun et al. [58] which consider querying an unknown *matrix* via matrix-vector queries. The paper studies multiple objectives arising from linear algebra, statistics, and most relevant to us, graph problems. For instance, the matrix could be the adjacency or edge-incidence matrix of an unknown graph. This model is (way) stronger than even the **Linear**-query model as every query returns $\tilde{O}(n)$ -bits of information. For connectivity, [58] show that if the matrix is edge-incidence, then results [40] on dynamic spectral sparsifiers imply $O(\text{polylog}(n))$ queries suffice, while with an adjacency matrix, $\tilde{\Omega}(n)$ -queries are required.

Combinatorial Group Testing, Compressed Sensing, and Coin Weighing. The single element recovery problem is closely related to all these three deep fields. In group testing, we are

given OR-query access to a vector as in Single Element Recovery, but the objective is to recover the whole support. This field started with the work of [24] out of a very practical application in World War II (we refer to the book [25] and various references within for historical perspectives), but it has since had numerous applications in fields as diverse as DNA screening [48] to multiaccess communication (MAC) protocols [59]. It is known that if there are d elements in the support, then $O(d^2 \min(\log(N/d), \log_d^2 N))$ non-adaptive, deterministic queries suffice [26, 35, 42, 51] and $\Omega(d^2 \log_d N)$ -queries are needed for a non-adaptive algorithm. Closing this gap is an outstanding open question in the combinatorial group testing community. On the other hand, with adaptive algorithms can solve this problem with $O(d \log(N/d))$ -queries, and this is necessary. We are, however, unaware of any work understanding the trade-off with rounds of adaptivity. We refer the interested reader to the book [25] and surveys [48] and lecture notes [49].

Compressed Sensing and Coin Weighing problems are closer to the single element recovery problem with Linear queries. In the coin weighing problem [57], one is given N coins out of a collection of coins of two distinct weights w_0 and w_1 , a spring scale (as opposed to a balance-scale), and the objective is to determine the weight of each coin with minimal number of weighings. This is same as given a 0, 1-vector in N dimensions, recover it using Linear queries. There is a slew of work (we simply point the reader to the references in [12]) on these problems. It is known that with no other assumption, $\frac{2n}{\log n}$ is the correct answer for non-adaptive algorithms [44, 46]. When it is known that there are $\leq d$ instances of one coin, then the best non-adaptive algorithm makes $O(d \log n)$ -queries [44] while the best non-adaptive algorithm makes $O(\frac{d \log N/d}{\log d})$ -queries [12]. When the vector is an arbitrary non-negative vector, then the problem falls in the realm of (non-negative) compressed sensing. One of the main problems in compressed sensing is given a d -sparse (or close to d -sparse) N -dimensional vector, can it be recovered (or approximately recovered) from linear measurements. It is now well known that $O(d \log(N/d))$ non-adaptive measurements suffice [16, 20, 23], and this is tight. There are various nuanced results in what it means by close to sparse and approximation, and we point the reader to the survey [31] and the references within for a deeper picture.

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A Simple Lower Bound for Single Element Recovery with OR-queries

We provide a simple proof for the single element recovery problem with OR-queries. Note that since OR-queries are weaker than Linear-queries, [Theorem 1](#) already implies this, but this proof is arguably simpler (or rather, more direct, or to use the language from the proof of [Theorem 1](#), it works in the *primal* space).

Theorem 11. *Any r -round deterministic algorithm for Single Element Recovery must make $\geq (N^{1/r} - 1)$ -OR queries in some round.*

Proof. Suppose, for the sake of contradiction, there is an algorithm which makes $< N^{1/r} - 1$ queries every round. The proof is via an adversary argument. Given an algorithm ALG, the ADVERSARY maintains a vector $x \in \{0, \star\}^N$ and responds queries consistent with x ; the \star indicates that the ADVERSARY has not committed to the value of x on that coordinate. At the end of r -rounds, the algorithm returns a coordinate j . Our goal is to reveal $x \in \{0, 1\}^N$ at that point such that $x_j = 0$ and x is consistent with the transcript so far. If we are successful, then the algorithm cannot be correct on all inputs. We call this *fooling* the algorithm.

The ADVERSARY maintains a set of active vertices A . Initially $A = [N]$. For every $i \in A$, we have $x_i = \star$; for every $i \notin A$, we have $x_i = 0$. Every query Q made will be responded either 0 or 1. The former are called 0-queries, and ADVERSARY will maintain $x_i = 0$ for all i in a 0-query. All others are 1-queries. The adversary maintains the following invariant.

- (I.) After round $k \geq 1$, for every 1-query Q with $Q \cap A \neq \emptyset$, we have $|Q \cap A| > N^{1-k/r}$.

Note that at the beginning, that is after round $k = 0$, the above invariant holds vacuously.

Consider an arbitrary round $1 \leq k \leq r$. Let Q_1, Q_2, \dots, Q_t be the queries in this round, with $t < N^{1/r} - 1$. Call Q_r *small* if $|Q_r \cap A| \leq N^{1-k/r}$. The ADVERSARY responds 0 to every small query and (a) sets $x_i = 0$ for $i \in Q_r \cap A$, and (b) removes these elements from A . Note that each query can lead to a drop of $\leq N^{1-k/r}$ in the size of $|A|$. Since A has diminished in size, this may make some other query small, and the ADVERSARY repeats this process till no small queries remain. For all such queries, the ADVERSARY responds 1. This completes the description of the responses, and now let us show that the invariant holds. Indeed, fix any 1-query Q . If this query Q is from round k , then the invariant holds by the description of the ADVERSARY process: otherwise, Q would be small and therefore a 0-query. If Q is from a previous round, then since the invariant held after round $(k-1)$, we know that before round k , $|Q \cap A| > N^{1-(k-1)/r}$. After round k , the set A decreases, but by at most $N^{1-k/r} \cdot (N^{1/r} - 1)$ since there are $< N^{1/r} - 1$ queries in all (perhaps all queries are small). Thus, after round k , we still have $|Q \cap A| > N^{1-(k-1)/r} - N^{1-k/r} \cdot (N^{1/r} - 1) = N^{1-\frac{k}{r}}$.

To complete the proof of the theorem, note that at the end of r rounds, we are in a situation where there is a subset of elements A and every query Q ever made by the algorithm is either $Q \cap A = \emptyset$ and we have responded 0, or $|Q \cap A| \geq 2$ and we have responded 1. This is trouble for the algorithm. Suppose the algorithm returns $j \in [N]$ claiming that $x_j > 0$. Well, consider the vector x which is $x_i = 0$ for all $j \cup [N] \setminus A$ and $x_i = 1$ otherwise. We claim this is consistent with every query — if $Q \cap A = \emptyset$, then we respond 0 as we should, and if $|Q \cap A| \geq 2$, then we respond 1. Since $|Q \cap A| \geq 2$, there must exist some element of $A \setminus j$ in Q . That is, x has at least one endpoint in Q . Therefore, we are consistent. This proves the algorithm's behavior is incorrect completing the proof of this theorem. ■

B Self contained proof of Lemma 6.5

Lemma B.1. Let G be an undirected graph on n vertices. Suppose every vertex $v \in G$ samples $O(\log^2 n)$ edges with repetition, and let $\Pi = (S_1, \dots, S_p)$ be the resulting connected components. Then, whp, $|E_{\text{cross}}(\Pi)| = O(n \log n)$.

Proof. We analyze the above by deferring the $O(\log^2 n)$ edges per vertex over $L = O(\log n)$ phases. Let E^R be the collection of sampled edges initialized to \emptyset . We also maintain a set \tilde{E} of edges in the graph initialized to \emptyset . These will be edges we will “give up” on. We use this simple fact.

Fact B.2. Let Π denote the connected components induced by E_R and Π' the connected components induced by $E^R \cup \tilde{E}$. Then, $E_{\text{cross}}(\Pi) \subseteq E_{\text{Cross}}(\Pi') \cup \tilde{E}$. In particular, $|E_{\text{cross}}(\Pi)| \leq |E_{\text{cross}}(\Pi')| + |\tilde{E}|$.

In each phase, we will either sample $O(\log n)$ edges on every vertex, or we will add $O(n)$ edges to \tilde{E} . Let E_t^R be the collection of sampled edges in the first $t - 1$ phases. Let Π_t be the partition of the vertices formed by the connected components of $E_t^R \cup \tilde{E}$. We call a component C in Π_t *big* if $|E(C, C^c)| > |C|$, and *small* otherwise. Big components have the following useful property.

Claim B.3. Let C be a big component. Now suppose every vertex $v \in C$ samples $O(\log n)$ edges incident to it independently with replacement. Then whp, an edge in $E(C, C^c)$ is sampled.

Proof. Let \mathcal{E} be the event that we *don't* sample an edge from $E(C, C^c)$. Rename the vertices in C as v_1, \dots, v_ℓ . Let α_i denote the number of edges v_i has to C^c , that is, $\alpha_i := |E(v_i, C^c)|$. Thus, we have $\sum_{i=1}^\ell \alpha_i > \ell$. The probability that v_i *doesn't* sample an edge in $E(v_i, C^c)$ in the t th phase is $\leq \left(1 - \frac{\alpha_i}{\alpha_i + \ell}\right)^{O(\log n)}$ since v_i has at most $\alpha_i + \ell$ edges incident on it. Therefore, $\mathbb{P}[\mathcal{E}] = \prod_{i=1}^\ell \left(1 - \frac{\alpha_i}{\alpha_i + \ell}\right)^{O(\log n)}$. If any of the $\alpha_i > \ell$, we get that one of the product terms in the RHS becomes $< \frac{1}{\text{poly}(n)}$. If all $\alpha_i \leq \ell$, then $\mathbb{P}[\mathcal{E}^{(t)}] = \prod_{i=1}^\ell \left(1 - \frac{\alpha_i}{\alpha_i + \ell}\right)^{O(\log n)}$ which is at most $\prod_{i=1}^\ell \left(1 - \frac{\alpha_i}{2\ell}\right)^{O(\log n)} \leq e^{-\frac{O(\log n)}{\ell} \sum_{i=1}^\ell \alpha_i} \leq 1/\text{poly}(n)$ since α_i 's sum to $> \ell$. ■

Let $n^{(t)} = |\Pi_t|$, and let $n_b^{(t)}$ and $n_s^{(t)}$ denote the number of big and small components, respectively. Note that, the total number of edges in $\tilde{E}^{(t)} := \cup_{C \in \Pi_t: \text{small}} |E(C, C^c)| = O(n)$. Among the $n_b^{(t)}$ big components, some of these have edges to small components, and some of these don't. Let $n_{b,1}^{(t)}$ be the first number and $n_{b,2}^{(t)}$ be the second. If $n_{b,1}^{(t)} < n_{b,2}^{(t)}$, then, we *coarsen* Π_t to Π_{t+1} by adding all the edges of $\tilde{E}^{(t)}$ to \tilde{E} ; it is as if we are committing to these edges being in the final E_{cross} . The number of connected components in Π_{t+1} is $\leq n_{b,1}^{(t)} + \theta$ where θ is the collection of “new” components. Note each such new component must contain a big component among to type 2 and a small component. Thus, $\theta \leq \min(n_{b,2}^{(t)}, n_s)$. Therefore, since $n_{b,1}^{(t)} \leq n_{b,2}^{(t)}$, we get that $n^{(t+1)} \leq \frac{2}{3}n^{(t)}$. On the other hand, if $n_{b,1}^{(t)} \geq n_{b,2}^{(t)}$, then we sample $O(\log n)$ -edges per vertex. By [Claim B.3](#), the number of big components will then drop to $n_b^{(t+1)} \leq \frac{1}{2}n_{b,1}^{(t)} + n_{b,2}^{(t)} \leq \frac{3}{4}n_b^{(t)}$.

Therefore, in each phase either the number of components or the number of big components drop by a constant factor. Two small components can't merge to give a big component. Therefore, in $O(\log n)$ -phases, we end up either with a connected graph, or with a graph with all small components. We end the process in either case; note that in both cases if Π_L is the final partition, we have $E_{\text{cross}}(\Pi_L) = O(n)$. Now, we use [Fact B.2](#). Since Π_L is induced by $E^R \cup \tilde{E}$, we get that

$|E_{\text{cross}}(\Pi)| \leq |E_{\text{cross}}(\Pi_L)| + |\tilde{E}| = O(n \log n)$. This is because, the number of phases is $O(\log n)$ and in each phase $|\tilde{E}|$ gets $O(n)$ edges. ■